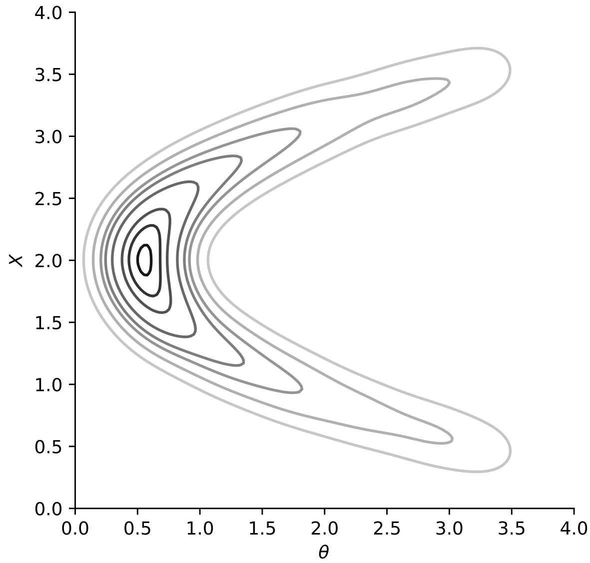
Please submit your solutions via Ilias. The submission is not a formal requirement for passing the exam but doing the exercises will be very helpful to do so. Submissions should be a single PDF document (note that Jupyter notebooks can and should also be downloaded as PDFs, and not only submitted as .ipynb files).

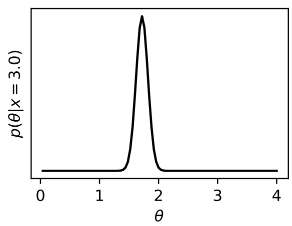
1. **Intuition question** Consider two random variables θ and x. Their joint distribution $p(\theta, x)$ is shown below. For all sketches, please label the θ or x-axis with a least two ticks. You do not have to label the vertical axis.



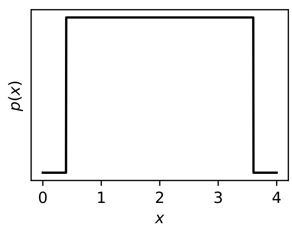
   (a) They are not independent.

   (b) They are uncorrelated.

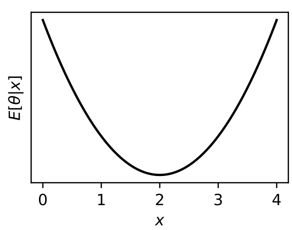
   (c)



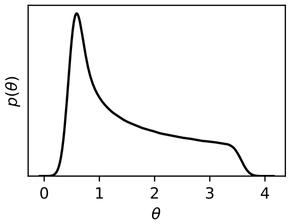
   (d)



   (e)

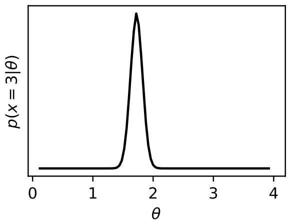


(f)



(g) Same as (c)

(h)



2. **Theory question** Consider the Gaussian random variable $\boldsymbol{w} \in \mathbb{R}$ with probability density function $p(\boldsymbol{w}) = \mathcal{N}(\boldsymbol{w}; \boldsymbol{\mu}, \Sigma)$ where $\boldsymbol{\mu} \in \mathbb{R}^F$ and symmetric positive definite $\Sigma \in \mathbb{R}^{F \times F}$. You have access to data $\boldsymbol{y} \in \mathbb{R}^N$ assumed to be generated from $\boldsymbol{w}$ through a linear map $\Phi \in \mathbb{R}^{F \times N}$ according to the likelihood

$$p(\boldsymbol{y}|\boldsymbol{w}) = \mathcal{N}(\boldsymbol{y}; \Phi^T \boldsymbol{w}, \Lambda),$$

where $\Lambda \in \mathbb{R}^{N \times N}$ is symmetric positive definite. What is:

(a) the pdf of the *marginal* $p(\boldsymbol{y}) = \int p(\boldsymbol{y}|\boldsymbol{w}) p(\boldsymbol{w}) d\boldsymbol{w}$?

(b) the pdf of the *posterior* $p(\boldsymbol{w}|\boldsymbol{y})$?

Solution:

(a) By application of the marginalization formula for two Gaussian RVs:

$$p(y) = \mathcal{N}(y; \Phi^\top \mu, \Lambda + \Phi^\top \Sigma \Phi)$$

(b) By application of the conditioning formula for two Gaussian RVs:

$$\begin{aligned} p(w|y) &= \mathcal{N}(w; \mu_w, \Sigma_w) \\ \mu_w &= \mu + \Sigma\Phi(\Phi^\top \Sigma \Phi + \Lambda)^{-1}(y - \Phi^\top \mu) \\ \Sigma_w &= \Sigma - \Sigma\Phi(\Phi^\top \Sigma \Phi + \Lambda)^{-1} \Phi^\top \Sigma. \end{aligned}$$