Please submit your solutions via Ilias. The submission is not a formal requirement for passing the exam but doing the exercises will be very helpful to do so. Submissions should be a single PDF document (note that Jupyter notebooks can and should also be downloaded as PDFs, and not only submitted as .ipynb files).

1. **Exercise** For each of the following, prove or give a counterexample
   (a) If $P(a|b,c) = P(b|a,c)$, then $P(a|c) = P(b|c)$
   (b) If $P(a|b,c) = P(a)$, then $P(b|c) = P(b)$
   (c) If $P(a|b) = P(a)$, then $P(a|b,c) = P(a|c)$

2. **Exercise**
   (a) Show that conditional probabilities satisfy the axioms of probability (nonnegativity, normalization and countable additivity).
   (b) We have two biased coins, a gold and a bronze. In any given toss, the probability of heads for the gold coin is 0.99, whereas for the bronze coin is 0.01. We choose one of the two at random, with equal probability, and proceed with two independent tosses. Let B be the event that the bronze coin was selected and H_i be the event that the ith toss resulted in heads.
      i. Are H_1 and H_2 conditionally independent given B?
      ii. Are H_1 and H_2 independent?
   (c) You and your friend are candidates in a TV show, and you are told that there is a cash-prize behind two of three doors (with equal probabilities for each). Each of you, in turn, chooses a single door and the show-master immediately opens it (thus the second person cannot choose the same door).
      i. If you choose before your friend, what is the probability that you get a prize?
      ii. If you choose first and get a prize, what is the probability that your friend gets a prize?
      iii. If you choose first and don't get a prize, what is the probability that your friend gets a prize?
      iv. Is it in your best interest if your friend chooses first?
      v. If you choose first, what is the probability that you get a prize, given that your friend gets a prize?

3. **Exercise** In the past year there was a lion spotted in Kleinmachnow, Berlin. Given the following random variables

$$\begin{aligned} Z &= \text{Zoo misses a lion} \\ L &= \text{Lion present} \\ B &= \text{Boar present} \\ S &= \text{Lion reported} \end{aligned}$$

   and probabilities

$$\begin{aligned} P(Z=1) &= 10^{-3} \\ P(B=1) &= 10^{-1} \\ P(L=1|Z=1) &= 0.95 \\ P(L=1|Z=0) &= 10^{-4} \end{aligned}$$

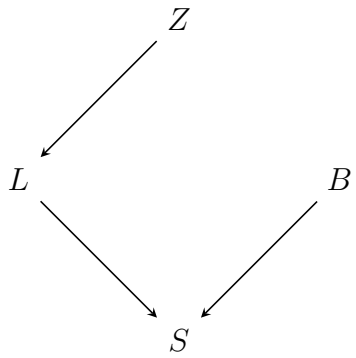

Figure 1: Lion DAG.

Suppose the false positive rate for reporting a lion is $l = 10^{-3}$; the rate for reporting a lion in the case of a boar $\alpha_b = 0.05$ and reporting a lion when a lion is present $\alpha_L = 0.95$.

(a) Implement and plot the corresponding directed graphical model in a jupyter notebook (for a hint see Figure). You will want to explore *igraph* or *networkx*. For networkx checkout d-seperation.

(b) With

i. $P(S = 0|L = 0, B = 0) = 1 - l$
ii. $P(S = 0|L = 0, B = 1) = (1 - l)(1 - \alpha_B)$
iii. $P(S = 0|L = 1, B = 0) = (1 - l)(1 - \alpha_L)$
iv. $P(S = 0|L = 1, B = 1) = (1 - l)(1 - \alpha_L)(1 - \alpha_B)$

compute

i. $P(L = 1|S = 1)$
ii. $P(L = 1|S = 1, Z = 0)$
iii. $P(B = 1|S = 1, Z = 0)$

(c) Plot $P(L = 1|S = 1, Z = 0)$ over (varying) l.

Note: Some of the questions are adapted from textbooks, which will be credited and revealed afterwards.