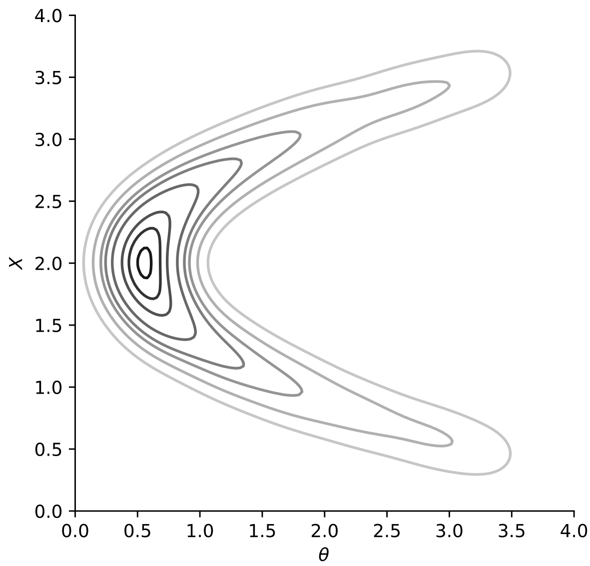
Please submit your solutions via Ilias. The submission is not a formal requirement for passing the exam but doing the exercises will be very helpful to do so. Submissions should be a single PDF document (note that Jupyter notebooks can and should also be downloaded as PDFs, and not only submitted as .ipynb files).

1. **Intuition question** Consider two random variables θ and x. Their joint distribution $p(\theta, x)$ is shown below. For all sketches, please label the θ or x-axis with a least two ticks. You do not have to label the vertical axis.



   (a) Are θ and x statistically independent? Explain your answer.
   (b) Are θ and x correlated? Explain your answer.
   (c) Sketch the conditional distribution $p(\theta|x)$ for $x = 3$.
   (d) Sketch the marginal distribution of x, i.e. $p(x)$.
   (e) Sketch the mean of $p(\theta|x)$ as a function of x. In other words, sketch the expectation $\mathbb{E}[\theta|x]$ as a function of x.

   Now, assume that the marginal distribution of θ, i.e. $p(\theta)$, is the prior distribution, and the conditional distribution $p(x|\theta)$ is the likelihood.

   In a single plot, sketch the following two:

   (f) The prior.
   (g) The posterior distribution as a function of θ, for $x = 3$.

   Finally, sketch:

   (h) The likelihood as a function of θ, for $x = 3$.

2. **Theory question** Consider the Gaussian random variable $\boldsymbol{w} \in \mathbb{R}$ with probability density function $p(\boldsymbol{w}) = \mathcal{N}(\boldsymbol{w}; \boldsymbol{\mu}, \Sigma)$ where $\boldsymbol{\mu} \in \mathbb{R}^F$ and symmetric positive definite $\Sigma \in \mathbb{R}^{F \times F}$. You have access to data $\boldsymbol{y} \in \mathbb{R}^N$ assumed to be generated from $\boldsymbol{w}$ through a linear map $\Phi \in \mathbb{R}^{F \times N}$ according to the likelihood

$$p(\boldsymbol{y}|\boldsymbol{w}) = \mathcal{N}(\boldsymbol{y}; \Phi^T \boldsymbol{w}, \Lambda),$$

   where $\Lambda \in \mathbb{R}^{N \times N}$ is symmetric positive definite. What is:

(a) the pdf of the *marginal* $p(\boldsymbol{y}) = \int p(\boldsymbol{y}|\boldsymbol{w})p(\boldsymbol{w})d\boldsymbol{w}$?

(b) the pdf of the *posterior* $p(\boldsymbol{w}|\boldsymbol{y})$?

3. **Coding Question** We aim to estimate the probability of people wearing glasses θ. We are given a binary vector which, for N randomly selected people, indicates whether they wear glasses (1) or not (0). For example:

$$\text{Observations} = [1, 1, 0, 0, 0, 0, 0, 0, 0, 0] \tag{1}$$

Upfront, we assume that the probability of people wearing glasses (θ) is uniformly distributed in $[0, 1]$. We model this uniform distribution as a Beta distribution $B(\alpha, \beta)$.

Task 1: What values of α and β make the Beta distribution uniform in $[0, 1]$? Implement the distribution (using, e.g. ‘pytorch.distributions‘) and plot its density in $[0, 1]$.

Given θ, the number of people who wear glasses in a population is a Binomial distribution (in other words, the likelihood is a Binomial distribution). Since the Beta distribution is the conjugate prior to the Binomial distribution, we can update the posterior in closed-form, and the posterior will be a Beta distribution with updated values for α and β. Assuming that x out of the N observed people wore glasses, the update is:

$$p(\theta|x) = B(\alpha + x, \beta - x + N)$$

Task 2: Implement the update rule for α and β.

Task 3: Visualize the posterior distribution given the uniform prior in $[0, 1]$ and having observed a single person that wears glasses.

Task 4: Use the posterior distribution obtained in Task 3 as prior and perform a new update, assuming that a second person has also worn glasses. Visualize the posterior.

Task 5: Iteratively repeat the procedure from Task 4 for all 10 observations defined above.

Task 6: Instead of updating the posterior ”one step at a time” as done in Tasks 3-6, you can also perform the update ”at once” (two out of 10 people wear glasses) with the update rule implemented in Task 2. Perform the update given all observations in one step and compare your result to Task 5.