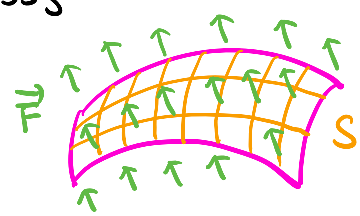
16.7. Surface integrals : vector fields

☆ **Def** Given an orientable surface S parametrized by $\vec{r}(u,v)$
not important
on a domain D, the surface integral (or flux) of
a vector field $\vec{F}$ over S is

$$\iint_S \vec{F}\cdot d\vec{S} := \iint_D \vec{F}(\vec{r}(u,v))\cdot(\vec{r}_u \times \vec{r}_v)\, dA$$

Note (1) If $\vec{F}$ is a velocity field of a fluid, then the flux
$\iint_S \vec{F}\cdot d\vec{S}$ is the rate of flow across S.



(2) The orientation of S is determined by the direction
of normal vectors

oriented upward   oriented downward

☆ (3) The surface integral of a vector field depends on
the orientation of the surface

$$\iint_S \vec{F}\cdot d\vec{S} = -\iint_{-S} \vec{F}\cdot d\vec{S}$$

where $-S$ is the surface S with the opposite orientation.

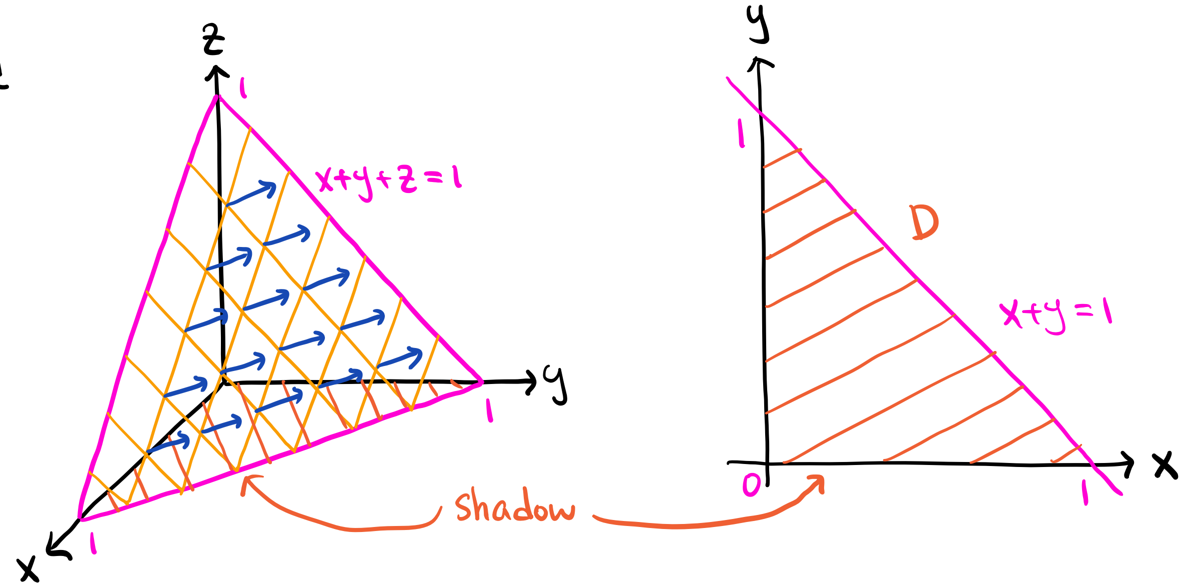
(4) If $\vec{n}$ denotes the unit normal vector of S, then

$$\iint_S \vec{F}\cdot d\vec{S} = \iint_S \underbrace{\vec{F}\cdot\vec{n}}_{\text{scalar}}\, dS$$

* This is useful if S is a sphere or a flat surface.

Ex Let S be the part of the plane $x+y+z=1$ that lies in the first octant. Find the upward flux of the vector field $\vec{F}(x,y,z) = (z, x^2, x+y)$ across S.

Sol



$x+y+z=1 \rightsquigarrow z = 1-x-y$.

S is parametrized by $\vec{r}(x,y) = (x, y, 1-x-y)$

The domain D is given by $0 \le x \le 1,\ 0 \le y \le 1-x$.

$\vec{r}_x = (1, 0, -1),\ \vec{r}_y = (0, 1, -1)$

$\Rightarrow \vec{r}_x \times \vec{r}_y = (1, 1, \underset{\overset{\vee}{0}}{1})$: oriented upward

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_D \vec{F}(\vec{r}(x,y)) \cdot (\vec{r}_x \times \vec{r}_y)\, dA$$

$\vec{F}(\vec{r}(x,y)) \cdot (\vec{r}_x \times \vec{r}_y) = (1-x-y, x^2, x+y) \cdot (1,1,1) = x^2+1$.

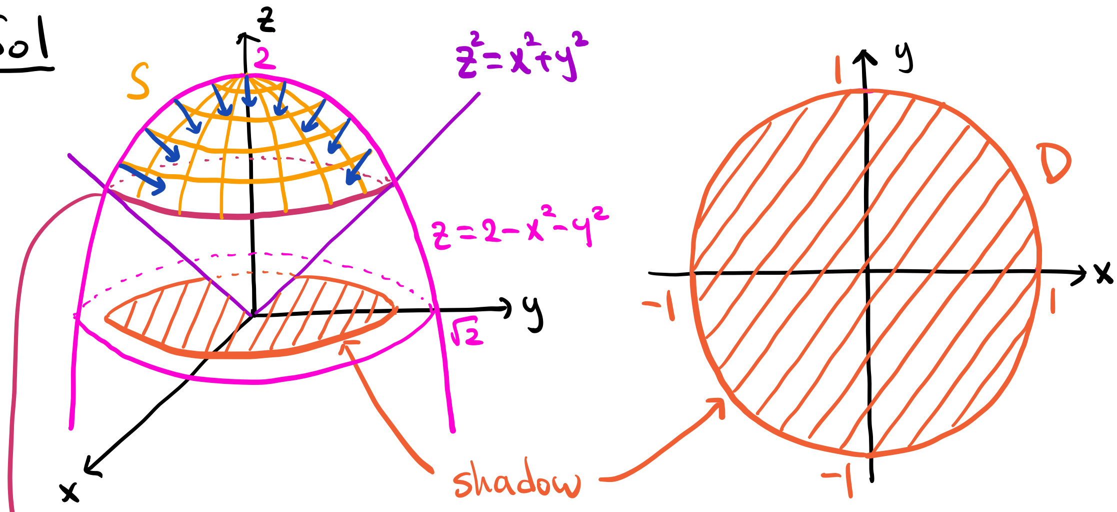
$$\Rightarrow \iint_S \vec{F} \cdot d\vec{S} = \int_0^1 \int_0^{1-x} x^2+1\, dy\, dx = \int_0^1 (1-x)(x^2+1)\, dx$$

$$= \int_0^1 -x^3+x^2-x+1\, dx = \left(-\frac{x^4}{4} + \frac{x^3}{3} - \frac{x^2}{2} + x\right)\Bigg|_{x=0}^{x=1}$$

$$= \boxed{\frac{7}{12}}$$

Ex Let S be the part of the paraboloid $z = 2 - x^2 - y^2$ that lies in the cone $z^2 = x^2 + y^2$. Find the downward flux of the vector field $\vec{F}(x,y,z) = (y, -x, z)$ across S.

Sol



Intersection : $z = 2 - x^2 - y^2$ and $z^2 = x^2 + y^2$

$\Rightarrow z = 2 - z^2 \Rightarrow z = 1, \cancel{-2} \Rightarrow x^2 + y^2 = 1$
$\quad z \geq 0$

S is parametrized by $\vec{r}(x,y) = (x, y, 2 - x^2 - y^2)$

The domain D is given by $x^2 + y^2 \leq 1$.

$\vec{r}_x = (1, 0, -2x)$, $\vec{r}_y = (0, 1, -2y)$

$\Rightarrow \vec{r}_x \times \vec{r}_y = (2x, 2y, \underset{\substack{\vee \\ 0}}{1})$: oriented upward

$$\iint_S \vec{F} \cdot d\vec{S} = \underset{\substack{\uparrow \\ \text{opposite orientation}}}{-} \iint_D \vec{F}(\vec{r}(x,y)) \cdot (\vec{r}_x \times \vec{r}_y)\, dA$$

$\vec{F}(\vec{r}(x,y)) \cdot (\vec{r}_x \times \vec{r}_y) = (y, -x, 2 - x^2 - y^2) \cdot (2x, 2y, 1) = 2 - x^2 - y^2$.

In polar coordinates, D is given by $0 \leq \theta \leq 2\pi$, $0 \leq r \leq 1$.

$$\iint_S \vec{F} \cdot d\vec{S} = -\iint_D 2 - x^2 - y^2 \, dA = -\int_0^{2\pi} \int_0^1 (2 - r^2)\, \overset{\text{Jacobian}}{r}\, dr\, d\theta$$

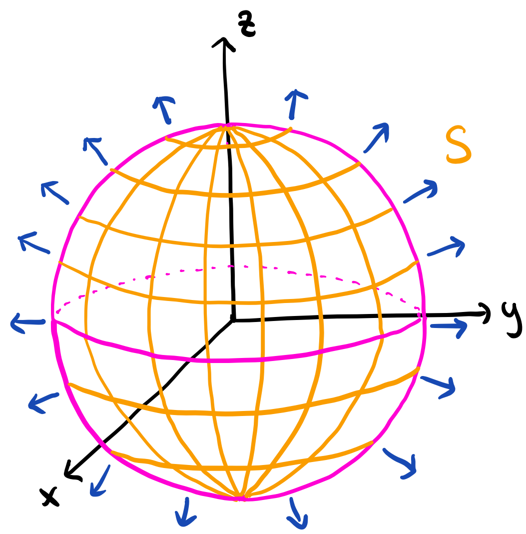
$$= -\int_0^{2\pi} r^2 - \frac{r^4}{4} \Big|_{r=0}^{r=1} d\theta = -\int_0^{2\pi} \frac{3}{4}\, d\theta = \boxed{-\frac{3\pi}{2}}$$

★ **Ex** Consider the inverse square field

$$\vec{F}(x,y,z) = \left(\frac{x}{(x^2+y^2+z^2)^{3/2}}, \frac{y}{(x^2+y^2+z^2)^{3/2}}, \frac{z}{(x^2+y^2+z^2)^{3/2}} \right)$$

Find $\iint_S \vec{F} \cdot d\vec{S}$ where S is a sphere centered at the origin, oriented outward.

Sol



Let R be the radius of S

$\Rightarrow$ S is given by $x^2+y^2+z^2 = R^2$

$\rightsquigarrow$ a level surface of

$f(x,y,z) = x^2+y^2+z^2$

$\Rightarrow$ A normal vector of S is

$\nabla f = (2x, 2y, 2z)$

$$\boxed{\begin{array}{l}\text{The unit normal vector of } S \text{ is given by} \\ \vec{n} = \dfrac{\nabla f}{|\nabla f|} = \dfrac{(2x,2y,2z)}{\sqrt{4x^2+4y^2+4z^2}} = \dfrac{(2x,2y,2z)}{2R} = \left(\dfrac{x}{R}, \dfrac{y}{R}, \dfrac{z}{R}\right)\end{array}}$$

$\vec{n}$ is oriented outward (pointing away from the origin)

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_S \vec{F} \cdot \vec{n}\, dS$$

$$\vec{F} \cdot \vec{n} = \frac{x^2}{R(x^2+y^2+z^2)^{3/2}} + \frac{y^2}{R(x^2+y^2+z^2)^{3/2}} + \frac{x^2}{R(x^2+y^2+z^2)^{3/2}}$$

$$= \frac{x^2+y^2+z^2}{R(x^2+y^2+z^2)^{3/2}} \underset{\uparrow}{=} \frac{R^2}{R \cdot R^3} = \frac{1}{R^2}$$

$x^2+y^2+z^2 = R^2$ on S

$$\Rightarrow \iint_S \vec{F} \cdot d\vec{S} = \iint_S \frac{1}{R^2}\, dS = \frac{1}{R^2} \text{Area}(S) = \frac{1}{R^2} \cdot \underbrace{4\pi R^2}_{\text{Area of sphere}} = \boxed{4\pi}$$