14.8. Lagrange multipliers

☆ **Thm** (Method of Lagrange multipliers)

Given a differentiable function $f(x,y,z)$ with a constraint $g(x,y,z)=0$, the extreme values can be found as follows if they exist.

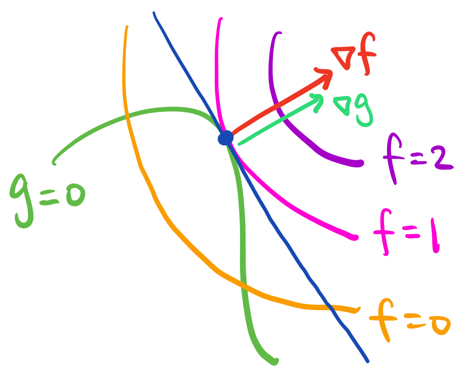
Step 1. Solve the equations $\nabla f = \lambda \nabla g$ and $g=0$.

Step 2. Evaluate $f(x,y,z)$ at all solutions.

Step 3. Compare all values from Step 2.

$\begin{cases} \text{maximum} = \text{the largest of these values} \\ \text{minimum} = \text{the smallest of these values} \end{cases}$

* Explanation :



At extrema, level curves (or surfaces) of f and g must be tangent.

$\Rightarrow \nabla f$ and ∇g are parallel

$\Rightarrow \nabla f = \lambda \nabla g$

Note (1) In Math 215, you can always assume that extreme values exist. In general, however, there may be no extreme values.

(2) Step 1 often involves heavy algebra.

Ex Find the minimum value of x^2+y^2+z on the plane

$2x+2y+z=9$

Sol 1 (Lagrange multipliers)

Constraint : $2x+2y+z=9 \rightsquigarrow 2x+2y+z-9=0$

Set $f(x,y,z)=x^2+y^2+z$ and $g(x,y,z)=2x+2y+z-9$.

Solve $\nabla f=\lambda\nabla g$ and $g=0$

$\Rightarrow (2x,2y,1)=\lambda(2,2,1)$ and $2x+2y+z-9=0$

$\rightsquigarrow 2x=2\lambda,\ 2y=2\lambda,\ 1=\lambda,\ 2x+2y+z=9$

$\Rightarrow \lambda=1,\ x=1,\ y=1,\ z=9-2x-2y=5$

The minimum value is $f(2,2,5)=\boxed{7}$

Sol 2 (Removing the constraint)

$2x+2y+z=9 \rightsquigarrow z=9-2x-2y$

$\Rightarrow x^2+y^2+z=x^2+y^2+9-2x-2y$.

We find the minimum of $h(x,y)=x^2+y^2+9-2x-2y$ on $\mathbb{R}^2$.

Since $\mathbb{R}^2$ is open, the minimum is at a critical point.

$\Rightarrow \nabla h=(0,0) \Rightarrow (2x-2,2y-2)=0 \Rightarrow x=1,\ y=1$.

$\Rightarrow$ The minimum is $h(1,1)=\boxed{5}$

Note It's not always possible to remove the constraint in this way.

Ex Find the shortest distance from the origin to the surface $2x + 4y + z^2 = 20$.

Sol Distance from the origin is $\sqrt{x^2+y^2+z^2}$.

We find the minimum of $x^2+y^2+z^2$ subject to the constraint $2x+4y+z^2-20=0$

Set $f(x,y,z) = x^2+y^2+z^2$ and $g(x,y,z) = 2x+4y+z^2-20$

Solve $\nabla f = \lambda \nabla g$ and $g=0$

$\Rightarrow (2x, 2y, 2z) = \lambda(2, 4, 2z)$ and $2x+4y+z^2-20=0$

$\rightsquigarrow 2x = 2\lambda,\ 2y = 4\lambda,\ 2z = 2\lambda z,\ 2x+4y+z^2=20$.

Case 1 $z=0$: $x=\lambda,\ y=2\lambda,\ 2x+4y=20$

$\Rightarrow 2\lambda + 4\cdot 2\lambda = 20 \Rightarrow 10\lambda = 20 \Rightarrow \lambda = 2$

$\Rightarrow x=2,\ y=4,\ z=0$.

Case 2 $z \neq 0$: $2z = 2\lambda z \Rightarrow \lambda = 1$.

$2x = 2\lambda = 2 \Rightarrow x=1,\quad 2y = 4\lambda = 4 \Rightarrow y=2$

$2x+4y+z^2=20 \Rightarrow 10+z^2=20 \Rightarrow z=\pm\sqrt{10}$

$\Rightarrow x=1,\ y=2,\ z=\pm\sqrt{10}$.

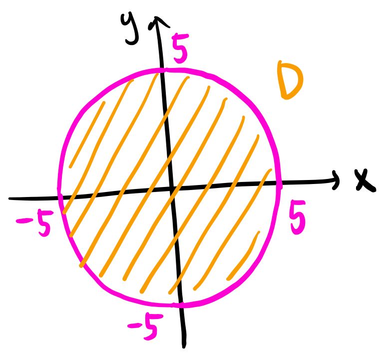
$f(2,4,0) = 20,\quad f(1,2,\sqrt{10}) = f(1,2,-\sqrt{10}) = 15$

$\Rightarrow$ The minimum of $f(x,y,z) = x^2+y^2+z^2$ is 15.

$\Rightarrow$ The shortest distance is $\boxed{\sqrt{15}}$

Ex Find the extreme values of $6x+8y$ on the domain given by $x^2+y^2 \leq 25$.

Sol



D is closed and bounded.

We apply the Extreme value theorem with $f(x,y) = 6x+8y$

Step 1. Evaluate $f(x,y)$ at all critical points.

$\nabla f = (6,8) \neq (0,0) \Rightarrow$ no critical points.

Step 2. Find the extrema of $f(x,y)$ on the boundary.

Set $g(x,y) = x^2+y^2-25 \Rightarrow g(x,y)=0$ on the boundary.

Solve $\nabla f = \lambda \nabla g$ and $g=0$

$\Rightarrow (6,8) = \lambda(2x, 2y)$ and $x^2+y^2-25=0$

$\rightsquigarrow 6 = 2\lambda x, \quad 8 = 2\lambda y, \quad x^2+y^2 = 25$

$\lambda \neq 0 \Rightarrow x = \dfrac{3}{\lambda}, \quad y = \dfrac{4}{\lambda}.$ (*)

$x^2+y^2 = 25 \rightsquigarrow \dfrac{9}{\lambda^2} + \dfrac{16}{\lambda^2} = 25 \rightsquigarrow 25 = 25\lambda^2 \rightsquigarrow \lambda = \pm 1.$

$\lambda = 1 \overset{(*)}{\Rightarrow} x=3, \ y=4, \quad \lambda=-1 \overset{(*)}{\Rightarrow} x=-3, \ y=-4.$

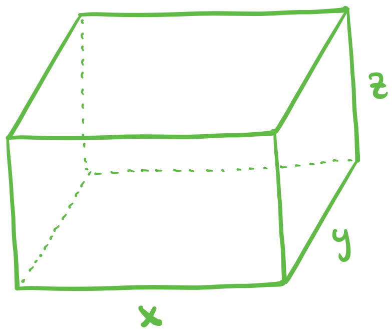
$f(3,4) = 50, \quad f(-3,-4) = -50$

Step 3. Compare all values from steps 1 and 2.

$\boxed{\text{Maximum} = 50, \quad \text{minimum} = -50}$

Ex Find the minimum surface area of a rectangular box with volume 1.

Sol



Width x, length y, height z

$$\Rightarrow \begin{cases} \text{Volume} = xyz = 1 \\ \text{Surface area} = 2xy + 2yz + 2zx \end{cases}$$

Set $f(x,y,z) = 2xy + 2yz + 2zx$ and $g(x,y,z) = xyz - 1$.

Solve $\nabla f = \lambda \nabla g$ and $g = 0$

$\Rightarrow (2y+2z,\ 2x+2z,\ 2x+2y) = \lambda(yz, zx, xy)$ and $xyz - 1 = 0$.

$\rightsquigarrow 2y+2z = \lambda yz,\ 2x+2z = \lambda zx,\ 2x+2y = \lambda xy,\ xyz = 1$.

$\Rightarrow \lambda xyz = x(2y+2z) = y(2x+2z) = z(2x+2y)$

$\Rightarrow 2xy + 2xz = 2xy + 2yz = 2xz + 2yz$

$\Rightarrow xy = yz = zx \Rightarrow \dfrac{xyz}{xy} = \dfrac{xyz}{yz} = \dfrac{xyz}{zx} \Rightarrow x = y = z$.

$xyz = 1 \Rightarrow x = y = z = 1$.

The minimum is $f(1,1,1) = \boxed{6}$

Note Alternatively, you can remove the constraint $xyz = 1$ by writing $z = \dfrac{1}{xy}$ and considering

$$2xy + 2yz + 2zx = 2xy + 2y \cdot \frac{1}{xy} + \frac{2}{xy} \cdot x = 2xy + \frac{2}{x} + \frac{2}{y}$$

on the domain $x, y > 0$.