12.5. Equations of lines and planes

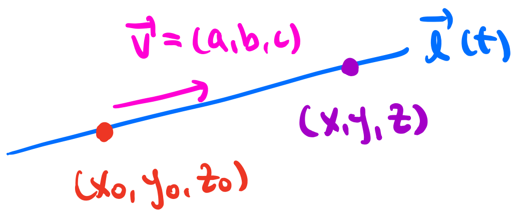
Note In order to specify a line in $\mathbb{R}^3$, you need one of the following data:

- two points on the line
- a point and a direction vector.

Thm The line through the point (x_0, y_0, z_0) with a direction vector $\vec{v} = (a, b, c)$ is parametrized by

$$\vec{\ell}(t) = (x_0 + at,\ y_0 + bt,\ z_0 + ct),\quad t \in \mathbb{R}.$$

$$= (x_0, y_0, z_0) + t(a, b, c)$$



Explanation:

$\vec{v}$ is parallel to $(x - x_0, y - y_0, z - z_0)$

$\Rightarrow t(a,b,c) = (x - x_0, y - y_0, z - z_0)$

Rmk There are (infinitely) many different ways to parametrize a given line, depending on the choice of a point and a direction vector.

e.g. The x-axis is parametrized by

$$\vec{\ell_1}(t) = (t, 0, 0) = (0,0,0) + t(1,0,0)$$

$$\vec{\ell_2}(t) = (1+2t, 0, 0) = (1,0,0) + t(2,0,0)$$

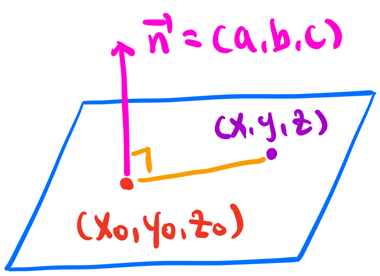
Note In order to specify a plane in $\mathbb{R}^3$, you need one of the following data:

- three points on the plane
- a point and a normal vector

$\text{normal} \;\|\; \text{perpendicular}$

Thm The plane through the point (x_0, y_0, z_0) with a normal vector $\vec{n} = (a, b, c)$ is given by the following equation:

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$



Explanation:

$\vec{n}$ is orthogonal to $(x - x_0, y - y_0, z - z_0)$

$\Rightarrow (a, b, c) \cdot (x - x_0, y - y_0, z - z_0) = 0$

Rmk (1) The above equation can also be written as $ax + by + cz = d$ with $d = -ax_0 - by_0 - cz_0$.

(2) The coefficients of the equation depend on the choice of a normal vector.

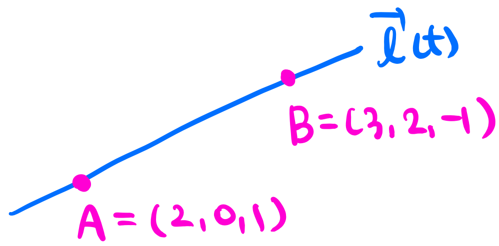
e.g. The xy-plane is given by

$$\begin{cases} z = 0 \iff 0 \cdot x + 0 \cdot y + 1 \cdot z = 0 & (\vec{n} = (0,0,1)) \\ 3z = 0 \iff 0 \cdot x + 0 \cdot y + 3 \cdot z = 0 & (\vec{n} = (0,0,3)) \end{cases}$$

Ex Parametrize the following lines.

(1) The line through $A=(2,0,1)$ and $B=(3,2,-1)$

Sol A direction vector is $\overrightarrow{AB}=(1,2,-2)$



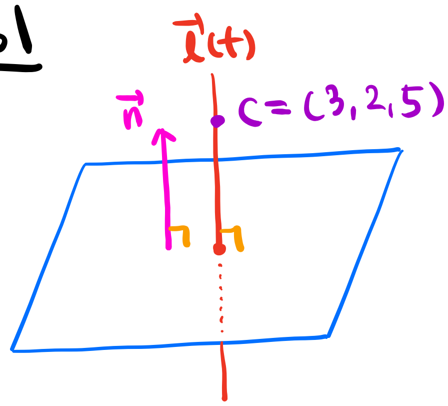
$$\rightsquigarrow \boxed{\vec{l}(t)=(2+t,\ 0+2t,\ 1-2t),\quad t\in\mathbb{R}}$$

Note You can use the point $B=(3,2,-1)$ to get a different parametrization:

$$\vec{l}(t)=(3+t,\ 2+2t,\ -1-2t),\quad t\in\mathbb{R}$$

(2) The line through $C=(3,2,5)$ which is perpendicular to the plane $2x-y+3z=5$.

Sol



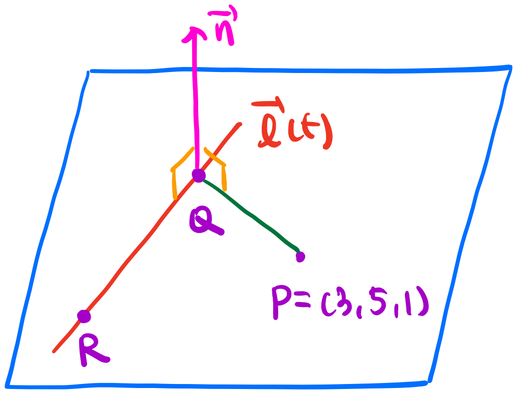
A direction vector is given by a normal vector

$\vec{n}=(2,-1,3)$

$$\rightsquigarrow \boxed{\vec{l}(t)=(3+2t,\ 2-t,\ 5+3t),\quad t\in\mathbb{R}}$$

Ex Find an equation of the plane which passes through $P = (3,5,1)$ and contains the line $\vec{\ell}(t) = (4-t,\ 2t-1,\ -3t)$.

Sol



Choose two points on $\vec{\ell}(t)$:

$Q = \vec{\ell}(0) = (4,-1,0)$, $R = \vec{\ell}(1) = (3,1,-3)$.

A normal vector $\vec{n}$ should be perpendicular to both $\overrightarrow{QP}$ and $\overrightarrow{QR}$

$$\Rightarrow \vec{n} = \overrightarrow{QP} \times \overrightarrow{QR} = (-1,6,-1) \times (-1,2,-3)$$

$$= (-16,-2,4)$$

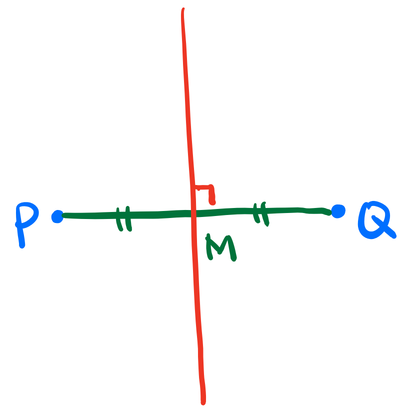
$$\rightsquigarrow \boxed{-16(x-3) - 2(y-5) + 4(z-1) = 0}$$

Note You can also use the point Q or R to get the same equation.

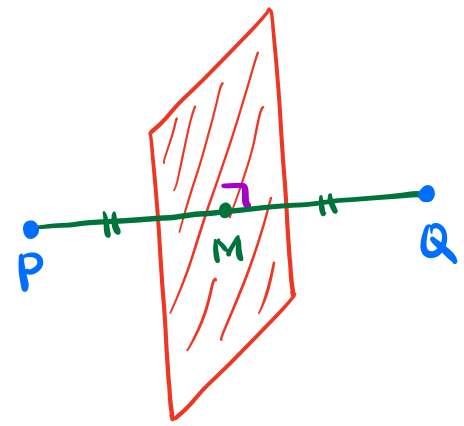
Ex Find the set of all points which are equidistant from $P=(2,1,3)$ and $Q=(4,-3,1)$.

Sol We get hints from a similar problem in $\mathbb{R}^2$.

- In $\mathbb{R}^2$:



- In $\mathbb{R}^3$:



The midpoint between P and Q is

$$M=\left(\frac{2+4}{2},\ \frac{1-3}{2},\ \frac{3+1}{2}\right)=(3,-1,2)$$

A normal vector is given by $\overrightarrow{PQ}=(2,-4,2)$.

$$\rightsquigarrow \boxed{2(x-3)-4(y+1)+2(z-2)=0}$$

Rmk You can also use the distance formula.

Distance from P = Distance from Q

$$\Rightarrow \sqrt{(x-2)^2+(y-1)^2+(z-3)^2}=\sqrt{(x-4)^2+(y+3)^2+(z-1)^2}$$

You can simplify this equation to get the same answer.