# 15.1. Double integrals over rectangles

Recall: For a function $f(x)$ defined on $[a,b]$, the integral

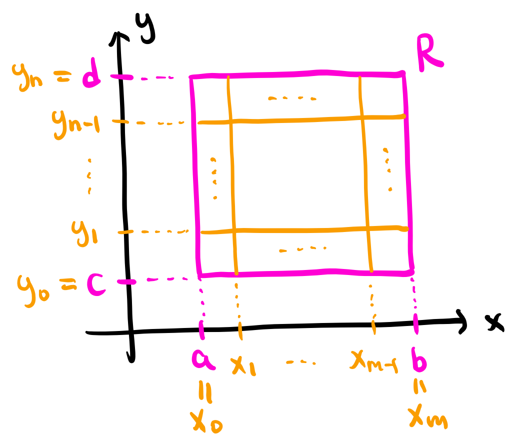
$\int_a^b f(x)dx$ is defined as the limit of Riemann sums.

**Def** Let $f(x,y)$ be a function defined on a rectangle

$R = [a,b]\times[c,d] := \{(x,y)\in\mathbb{R}^2 : a\le x\le b,\ c\le y\le d\}$.

(1) If $R$ is divided into equal subrectangles $R_{ij}$

each with area $\Delta A$ and a sample point $(x_i^*, y_j^*)$,




$$R_{ij} = [x_{i-1}, x_i]\times[y_{j-1}, y_j]$$

$$\Delta A = \frac{1}{mn}(b-a)(d-c).$$

the sum $\sum_{i=1}^{m}\sum_{j=1}^{n} f(x_i^*, y_j^*)\,\Delta A$ is called a <u>Riemann sum</u>.

(2) The <u>integral</u> of $f(x,y)$ on $R$ is given by

$$\iint_R f(x,y)\,dA := \lim_{m,n\to\infty}\sum_{i=1}^{m}\sum_{j=1}^{n} f(x_i^*, y_j^*)\,\Delta A.$$

**Thm** (Fubini's theorem)

If $f(x,y)$ is a continuous function on $R=[a,b]\times[c,d]$,

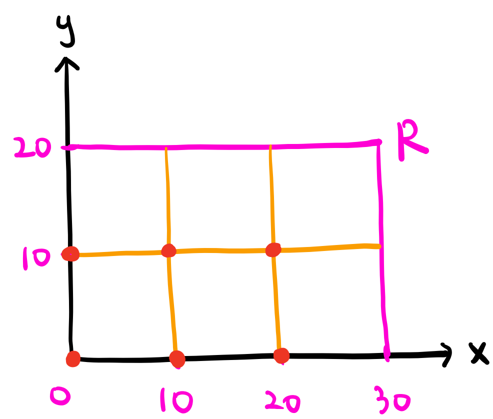
$$\iint_R f(x,y)\,dA = \int_a^b\int_c^d f(x,y)\,dy\,dx = \int_c^d\int_a^b f(x,y)\,dx\,dy.$$

**Ex** A 20-feet by 30-feet swimming pool is filled with water. The depth is measured at 5-feet intervals as follows:

| | 0 | 10 | 20 | 30 |
|---|---|---|---|---|
| 0 | 2 | 4 | 7 | 8 |
| 10 | 2 | 6 | 10 | 10 |
| 20 | 2 | 2 | 3 | 4 |

Estimate the volume of water.

**Sol** Use a Riemann sum with 10-feet intervals.




Each subrectangle has area

$\Delta A = 10 \cdot 10 = 100$.

For each subrectangle, we take the lower left vertex to be the sample point.

Set $d(x,y)$ to be the depth at $(x,y)$

$d(0,0) = 2$, $d(10,0) = 4$, $d(20,0) = 7$,

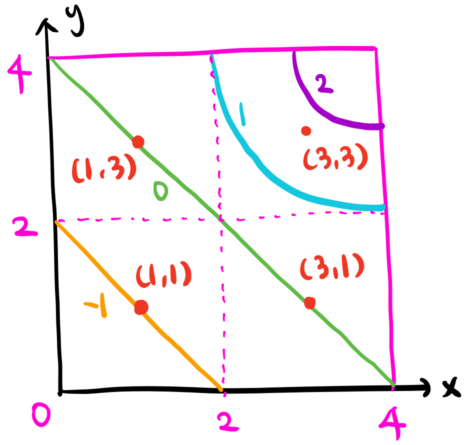
$d(0,10) = 2$, $d(10,10) = 6$, $d(20,10) = 10$

$\Rightarrow$ Riemann sum $= (2+4+7+2+6+10) \cdot 100 = 3100$

$\Rightarrow$ Volume $\approx \boxed{3100 \ (ft^3)}$

**Note** You can use different sample points.

**Ex** A contour map of $f(x,y)$ on $R = [0,4] \times [0,4]$ is given as follows:




Use the midpoint rule with $m = n = 2$ to estimate the integral $\iint_R f(x,y)\,dA$.

**Sol** The midpoint rule chooses sample points to be the midpoints of the subrectangles.

We divide $R$ into 4 equal subrectangles, each with area $\Delta A = 2 \cdot 2 = 4$.

The sample points are $(1,1), (1,3), (3,1), (3,3)$.

$f(1,1) = -1$, $f(1,3) = 0$, $f(3,1) = 0$, $f(3,3) \approx 1.6$.

Riemann sum $\approx (-1 + 0 + 0 + 1.6) \cdot 4 = 2.4$.

$\Rightarrow \iint_R f(x,y)\,dA \approx \boxed{2.4}$

Ex Evaluate $\iint_R \frac{x}{1+xy}\,dA$ where $R = [0,3] \times [0,2]$.

Sol $\iint_R \frac{x}{1+xy}\,dA = \int_0^3 \int_0^2 \frac{x}{1+xy}\,dy\,dx$

$(u = 1+xy \Rightarrow du = x\,dy)$

$$= \int_0^3 \int_1^{1+2x} \frac{1}{u}\,du\,dx$$

$$= \int_0^3 \ln(u) \Big|_{u=1}^{u=1+2x} dx$$

$$= \int_0^3 \ln(1+2x)\,dx$$

$(v = 1+2x \Rightarrow dv = 2dx)$

$$= \int_1^7 \ln(v) \cdot \frac{1}{2}\,dv$$

$$= \frac{1}{2}\left(v\ln(v) - v\right) \Big|_{v=1}^{v=7}$$

$$= \boxed{\frac{1}{2}\left(7\ln(7) - 6\right)}$$

Note You can instead evaluate $\int_0^2 \int_0^3 \frac{x}{1+xy}\,dx\,dy$.

However, this integral is more difficult to compute than the one we used above.