

# Exam 1 review - Fall 2016 exam

1. Consider the planes  $x + y + z = 1$  and  $x - y + z = 2$ .

- (a) (5 points) Recall that the angle between two planes is by definition equal to the angle between their normals. If  $\theta$  is the angle between the two given planes, find  $\cos \theta$ .

Normal vectors are  $\vec{n}_1 = (1, 1, 1)$ ,  $\vec{n}_2 = (1, -1, 1)$

$$\cos \theta = \frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1| |\vec{n}_2|} = \frac{1 \cdot 1 + 1 \cdot (-1) + 1 \cdot 1}{\sqrt{1^2 + 1^2 + 1^2} \cdot \sqrt{1^2 + (-1)^2 + 1^2}} = \boxed{\frac{1}{3}}$$

Note The statement of the problem is slightly ambiguous, as you can also get  $\cos \theta = -\frac{1}{3}$  by considering  $-\vec{n}_1$  instead of  $\vec{n}_1$ .

- (b) (5 points) Find a plane that is orthogonal to the two given planes and which goes through the point  $(2, 1, 3)$ .

A normal vector  $\vec{n}$  should be orthogonal to both  $\vec{n}_1$  and  $\vec{n}_2$

$$\Rightarrow \vec{n} = \vec{n}_1 \times \vec{n}_2 = (1, 1, 1) \times (1, -1, 1) = (2, 0, -2)$$

The plane equation is given by

$$2(x-2) + 0 \cdot (y-1) - 2(z-3) = 0$$

Note The answer can be given in many other forms, such as  $x - z + 1 = 0$ .

\*This problem is not under the coverage of Exam 1 this semester.

2. (10 points) Consider  $f(x, y) = x^2 + \frac{y^2}{4}$ . Let  $P$  be the point with  $(x, y) = (1, 2)$ . Find the gradient  $\nabla f$ , the direction (unit vector) of fastest increase, the direction (unit vector) of fastest decrease, and the direction in which the function neither increases nor decreases at the point  $P$  ( $(x, y) = (1, 2)$ ).

$$\nabla f = (f_x, f_y) = (2x, \frac{y}{2}) \Rightarrow \nabla f(1, 2) = \boxed{(2, 1)}$$

The direction of fastest increase is given by  $\nabla f(1, 2)$ .

Its unit vector is

$$\frac{\nabla f(1, 2)}{|\nabla f(1, 2)|} = \frac{(2, 1)}{\sqrt{2^2 + 1^2}} = \boxed{\frac{1}{\sqrt{5}} (2, 1)}$$

The direction of fastest decrease is given by  $-\nabla f(1, 2)$ .

Its unit vector is

$$-\frac{\nabla f(1, 2)}{|\nabla f(1, 2)|} = -\frac{(2, 1)}{\sqrt{2^2 + 1^2}} = \boxed{-\frac{1}{\sqrt{5}} (2, 1)}$$

Let  $\vec{u} = (a, b)$  be the unit vector along which the directional derivative of  $f$  is zero at  $P = (1, 2)$

$$D_{\vec{u}} f(1, 2) = \nabla f(1, 2) \cdot \vec{u} = (2, 1) \cdot (a, b) = 2a + b$$

$$D_{\vec{u}} f(1, 2) = 0 \Rightarrow 2a + b = 0 \Rightarrow b = -2a$$

$$\vec{u} = \pm \frac{(1, -2)}{|(1, -2)|} = \pm \frac{(1, -2)}{\sqrt{1^2 + (-2)^2}} = \boxed{\pm \frac{1}{\sqrt{5}} (1, -2)}$$

3. (10 points) Let  $f(x, y) = F(x^2 + y^2) + G(xy)$ , where  $F$  and  $G$  are functions of a single variable. Assume  $F(2) = 1$ ,  $F'(2) = 2$  and  $G(-1) = -1$ ,  $G'(-1) = -2$ . Find  $f(x, y)$ ,  $\partial f / \partial x$  and  $\partial f / \partial y$  at  $(x, y) = (1, -1)$ .

$$f(1, -1) = F(2) + G(-1) = 1 - 1 = \boxed{0}$$

$$\begin{aligned} f_x &= \frac{\partial}{\partial x} (F(x^2 + y^2) + G(xy)) = \frac{\partial}{\partial x} F(x^2 + y^2) + \frac{\partial}{\partial x} G(xy) \\ &= F'(x^2 + y^2) \cdot 2x + G'(xy) \cdot y \end{aligned}$$

$$f_x(1, -1) = F'(2) \cdot 2 + G'(-1) \cdot (-1) = 2 \cdot 2 + (-2) \cdot (-1) = \boxed{6}$$

$$\begin{aligned} f_y &= \frac{\partial}{\partial y} (F(x^2 + y^2) + G(xy)) = \frac{\partial}{\partial y} F(x^2 + y^2) + \frac{\partial}{\partial y} G(xy) \\ &= F'(x^2 + y^2) \cdot 2y + G'(xy) \cdot x \end{aligned}$$

$$f_y(1, -1) = F'(2) \cdot (-2) + G'(-1) \cdot 1 = 2 \cdot (-2) + (-2) \cdot 1 = \boxed{-6}$$

### \*Bonus exercise

If you are also given  $F''(2) = 3$  and  $G''(-1) = -3$ ,

find  $f_{xy}(1, -1)$ .

$$\begin{aligned} \underline{\text{Sol}} \quad f_{xy} &= \frac{\partial f_x}{\partial y} = \frac{\partial}{\partial y} (F'(x^2 + y^2) \cdot 2x + G'(xy) \cdot y) \\ &= F''(x^2 + y^2) \cdot 2y \cdot 2x + G''(xy) \cdot x \cdot y + G'(xy) \cdot 1 \\ \Rightarrow f_{xy}(1, -1) &= F''(2) \cdot (-4) + G''(-1) \cdot (-1) + G'(-1) \cdot 1 \\ &= 3 \cdot (-4) + (-3) \cdot (-1) + (-2) \cdot 1 = \boxed{-11} \end{aligned}$$

4. Consider the plane  $x + 2y + 2z = 4$ .

(a) (5 points) Find the distance from the plane to the origin.

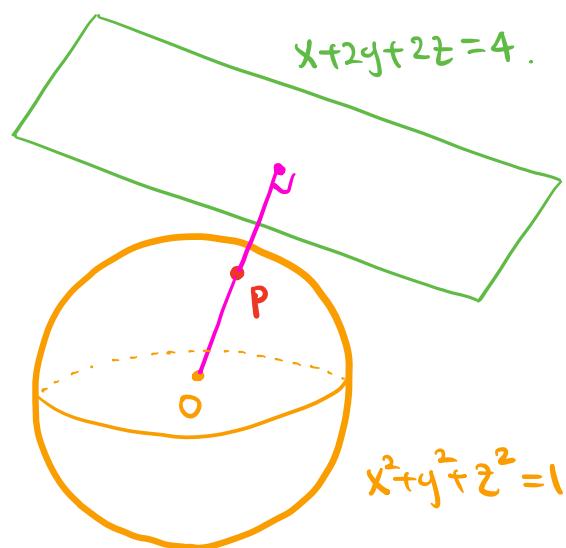
The distance from the plane  $x + 2y + 2z - 4 = 0$  to

the origin  $(0, 0, 0)$  is equal to

$$\frac{|0 + 2 \cdot 0 + 2 \cdot 0 - 4|}{\sqrt{1^2 + 2^2 + 2^2}} = \boxed{\frac{4}{3}}$$

(b) (5 points) Find the point on the sphere  $x^2 + y^2 + z^2 = 1$  which is closest to the plane.

Let  $P$  be the point on the sphere which is closest to the plane.



$\overrightarrow{OP}$  must be a normal vector of the plane  $x + 2y + 2z = 4$ .

$\Rightarrow \overrightarrow{OP}$  is parallel to

$$\vec{n} = (1, 2, 2)$$

Also  $\overrightarrow{OP}$  represents a radius

of the sphere  $x^2 + y^2 + z^2 = 1$ .

$$\Rightarrow |\overrightarrow{OP}| = 1$$

Hence  $\overrightarrow{OP}$  is the unit vector in the direction of  $\vec{n}$

$$\Rightarrow \overrightarrow{OP} = \frac{\vec{n}}{|\vec{n}|} = \frac{(1, 2, 2)}{\sqrt{1^2 + 2^2 + 2^2}} = \frac{1}{3}(1, 2, 2)$$

$$\Rightarrow P = \boxed{\left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right)}$$

5. (10 points) The pressure  $P$ , volume  $V$ , and temperature  $T$  of one mole of an ideal gas satisfy  $PV = RT$ , where  $R$  is a constant. Suppose  $R$  is measured using  $R = PV/T$ .

- (a) (5 points) Find the differential  $dR$ .

$$dR = \frac{\partial R}{\partial P} dP + \frac{\partial R}{\partial V} dV + \frac{\partial R}{\partial T} dT$$

$$= \boxed{\frac{V}{T} dP + \frac{P}{T} dV - \frac{PV}{T^2} dT}$$

- (b) (5 points) If the percentage errors in the measurement of  $P$ ,  $V$ , and  $T$  are 1%, 2%, and 3%, respectively, find the maximum percentage error in  $R$ .

The errors are represented by the differentials.

The error  $dR = \frac{V}{T} dP + \frac{P}{T} dV - \frac{PV}{T^2} dT$  attains the

maximum with  $dP = 0.01P$ ,  $dV = 0.02V$ ,  $dT = -0.03T$ .

$$\Rightarrow dR = \frac{V}{T} 0.01P + \frac{P}{T} 0.02V - \frac{PV}{T^2} (-0.03T)$$

$$= 0.01 \cdot \frac{PV}{T} + 0.02 \cdot \frac{PV}{T} + 0.03 \frac{PV}{T}$$

$$= 0.06 \cdot \frac{PV}{T} = 0.06 R.$$

The maximum percentage error in  $R$  is

$$\frac{dR}{R} \cdot 100 = \frac{0.06R}{R} \cdot 100 = \boxed{6\%}$$

6. Consider the surface  $xyz = 1$  in  $\mathbb{R}^3$ . Let  $P$  be the point  $(x, y, z) = (2, 1, \frac{1}{2})$ .

(a) (5 points) Find the normal to the surface at the point  $P$ .

The surface is a level surface of  $f(x, y, z) = xyz$ .

$$\nabla f = (f_x, f_y, f_z) = (yz, zx, xy).$$

A normal vector at  $P = (2, 1, \frac{1}{2})$  is given by

$$\nabla f(2, 1, \frac{1}{2}) = (\frac{1}{2}, 1, 2)$$

(b) (5 points) Find the tangent plane to the surface at the point  $P$ .

The tangent plane to the surface at  $P = (2, 1, \frac{1}{2})$  is

$$\frac{1}{2}(x-2) + 1 \cdot (y-1) + 2(z - \frac{1}{2}) = 0$$

Note You can solve this problem using the method from section 14.4.

$$xyz = 1 \rightsquigarrow z = \frac{1}{xy}$$

$\Rightarrow$  The surface is the graph of  $g(x, y) = \frac{1}{xy}$ .

$$g_x = -\frac{1}{x^2y} \Rightarrow g_x(2, 1) = -\frac{1}{4}.$$

$$g_y = -\frac{1}{xy^2} \Rightarrow g_y(2, 1) = -\frac{1}{2}$$

The tangent plane at  $P = (2, 1, \frac{1}{2})$  is given by

$$z = g(2, 1) + g_x(2, 1)(x-2) + g_y(2, 1)(y-1)$$

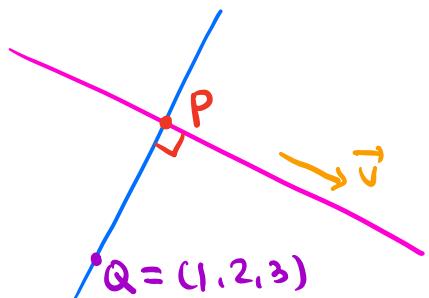
$$\Rightarrow z = \frac{1}{2} - \frac{1}{4}(x-2) - \frac{1}{2}(y-1)$$

7. An unknown line through the point  $(1, 2, 3)$  intersects the given line  $\mathbf{r}(t) = (t+3, 3-t, t+2)$  at right angles

- (a) (7 points) Find the point of intersection of the two lines.

We find the value of  $t$  at the intersection point  $P$ .

The given line has a direction vector  $\vec{v} = (1, -1, 1)$ .



$$\text{Set } Q = (1, 2, 3)$$

$\Rightarrow \overrightarrow{PQ}$  is perpendicular to  $\vec{v}$

$$\Rightarrow \overrightarrow{PQ} \cdot \vec{v} = 0$$

$$\Rightarrow (-t-2, t-1, -t+1) \cdot (1, -1, 1) = 0$$

$$\rightsquigarrow -3t = 0 \rightsquigarrow t = 0$$

$$\Rightarrow P = \vec{r}(0) = \boxed{(3, 3, 2)}$$

- (b) (3 points) Find the parametric equation of the unknown line.

The unknown line passes through  $P = (3, 3, 2)$  with a direction vector  $\overrightarrow{PQ} = (-2, -1, 1)$ .

$\Rightarrow$  The unknown line is parametrized by

$$\vec{l}(t) = \boxed{(3-2t, 3-t, 1+t)}$$

Note The answer can be given in many other forms,  
such as  $\vec{l}(t) = (1+2t, 2+t, 3-t)$ .

8. In this problem, take  $g = 10 \text{ m/s}^2$ . A projectile is fired at an angle  $\alpha$ ,  $0 < \alpha < 90^\circ$  on level ground at a speed of 120 m/s. Let  $\mathbf{r}(t)$  be its position as a function of time.

- (a) (3 points) Take  $\mathbf{r}(0) = (0, 0)$ ,  $\mathbf{r}'(0) = 120(\cos \alpha, \sin \alpha)$ , and  $\mathbf{r}''(t) = (0, -g)$ , and calculate  $\mathbf{r}(t)$ . Wind resistance is ignored.

$$\begin{aligned}\vec{r}'(t) &= \vec{r}'(0) + \int_0^t \vec{r}''(u) du = 120(\cos \alpha, \sin \alpha) + \int_0^t (0, -10) du \\ &= 120(\cos \alpha, \sin \alpha) + (0, -10t) \\ &= (120 \cos \alpha, 120 \sin \alpha - 10t) \\ \vec{r}(t) &= \vec{r}(0) + \int_0^t \vec{r}'(u) du \\ &= (0, 0) + \int_0^t (120 \cos \alpha, 120 \sin \alpha - 10u) du \\ &= \boxed{(120 \cos \alpha t, 120 \sin \alpha t - 5t^2)}\end{aligned}$$

- (b) (3 points) Express the range of the projectile as a function of  $\alpha$ .

The range is given by the x-coordinate when the projectile hits the ground.

$$y=0 \Rightarrow 120 \sin \alpha t - 5t^2 = 0 \Rightarrow t = \cancel{0}, \frac{24 \sin \alpha}{5}$$

$$\Rightarrow \text{The range is } 120 \cos \alpha \cdot \frac{24 \sin \alpha}{5} = \boxed{2880 \sin \alpha \cos \alpha}$$

- (c) (4 points) If the range is 720 m, find the two possible values of  $\alpha$ . You may use the trig identity  $\sin 2\alpha = 2 \sin \alpha \cos \alpha$ .

$$720 = 2880 \sin \alpha \cos \alpha = 1440 \sin(2\alpha)$$

$$\Rightarrow \sin(2\alpha) = \frac{1}{2} \Rightarrow 2\alpha = \frac{\pi}{6}, \frac{5\pi}{6} \Rightarrow \alpha = \boxed{\frac{\pi}{12}, \frac{5\pi}{12}}$$