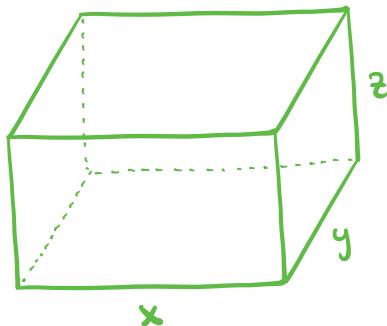


Exam 2 Review - Fall 2017 exam

1. (10 points) A rectangular box without a lid has volume 4 m^3 . Find the minimum possible surface area. No points will be awarded for guessing.



$$\begin{aligned} & \text{Width } x, \text{ length } y, \text{ height } z \\ \Rightarrow & \left\{ \begin{array}{l} \text{volume} = xyz = 4 \\ \text{surface area} = xy + 2yz + 2zx \end{array} \right. \end{aligned}$$

Set $f(x, y, z) = xy + 2yz + 2zx$ and $g(x, y, z) = xyz - 4$.

Solve $\nabla f = \lambda \nabla g$ and $g=0$

$$\Rightarrow (y+2z, x+2z, 2x+2y) = \lambda(yz, zx, xy) \text{ and } xyz - 4 = 0.$$

$$\Rightarrow \begin{cases} y+2z = \lambda yz \rightsquigarrow xy + 2xz = \lambda xyz \\ x+2z = \lambda zx \rightsquigarrow xy + 2yz = \lambda xyz \\ 2x+2y = \lambda xy \rightsquigarrow 2xz + 2yz = \lambda xyz \end{cases}$$

$$\Rightarrow xy + 2xz = xy + 2yz = 2xz + 2yz$$

$$\Rightarrow xy = 2yz = 2zx \Rightarrow \frac{xyz}{xy} = \frac{xyz}{2yz} = \frac{xyz}{2zx} \Rightarrow z = \frac{x}{2} = \frac{y}{2}$$

$$xyz = 4 \Rightarrow x = 2, y = 2, z = 1.$$

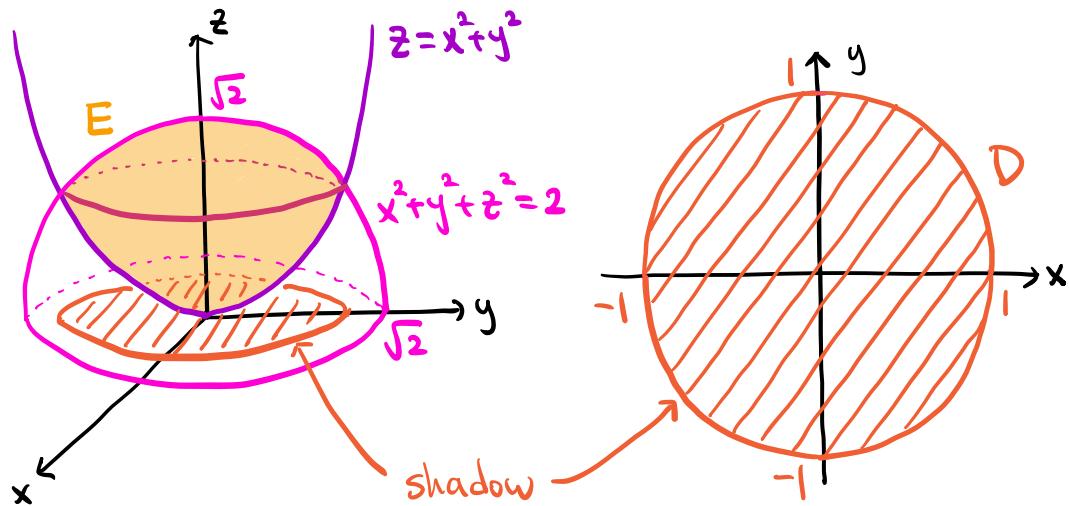
The minimum surface area is $f(2, 2, 1) = \boxed{12 \text{ m}^2}$

Note Alternatively, you can remove the constraint $xyz = 4$

by writing $z = \frac{4}{xy}$ and find the minimum of

$xy + 2yz + 2zx = xy + 2y \cdot \frac{4}{xy} + 2 \cdot \frac{4}{xy} \cdot x$ on the open domain given by $x > 0$ and $y > 0$.

2. (10 points) Find the volume of the region bounded above by the spherical surface $x^2 + y^2 + z^2 = 2$ and below by the paraboloid surface $z = x^2 + y^2$.



In cylindrical coordinates:

$$x^2 + y^2 + z^2 = 2 \rightsquigarrow r^2 + z^2 = 2 \rightsquigarrow z = \sqrt{2 - r^2}$$

$$z = x^2 + y^2 \rightsquigarrow z = r^2$$

$$\text{Intersection: } z = \sqrt{2 - r^2} \text{ and } z = r^2 \Rightarrow z = \sqrt{2 - z}$$

$$\Rightarrow z^2 = 2 - z \Rightarrow z = 1, \cancel{z=0} \Rightarrow r=1.$$

The shadow on the xy -plane: $0 \leq \theta \leq 2\pi, 0 \leq r \leq 1$.

For each point on the shadow: $r^2 \leq z \leq \sqrt{2 - r^2}$.

$$\Rightarrow 0 \leq \theta \leq 2\pi, 0 \leq r \leq 1, r^2 \leq z \leq \sqrt{2 - r^2}.$$

$$\text{Volume} = \iiint_E 1 \, dV = \int_0^{2\pi} \int_0^1 \int_{r^2}^{\sqrt{2-r^2}} 1 \cdot r \, dz \, dr \, d\theta$$

Jacobian

$$= \int_0^{2\pi} \int_0^1 (\sqrt{2-r^2} - r^2) r \, dr \, d\theta = \int_0^{2\pi} \int_2^1 (u^{1/2} - 2 + u) (-\frac{1}{2}) \, du \, d\theta$$

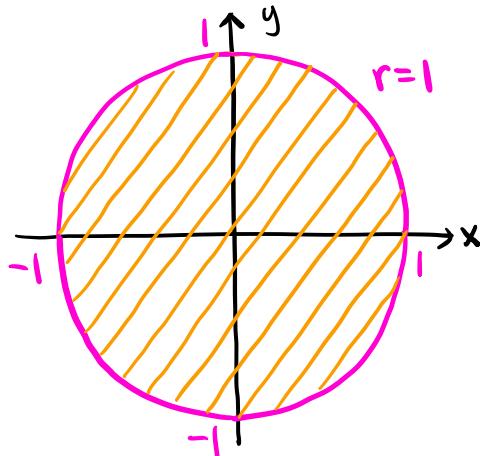
\uparrow
 $u = 2 - r^2$

$$= \int_0^{2\pi} -\frac{u^{3/2}}{3} + u - \frac{u^2}{4} \Big|_{u=2}^{u=1} \, d\theta = \int_0^{2\pi} \frac{8\sqrt{2} - 7}{12} \, d\theta = \boxed{\frac{\pi(8\sqrt{2} - 7)}{6}}$$

3. Find the area of the loops, which are given by

- a) (3 points) $r = 1$ and $0 \leq \theta < 2\pi$,

The loop is the circle of radius 1 and center $(0,0)$.



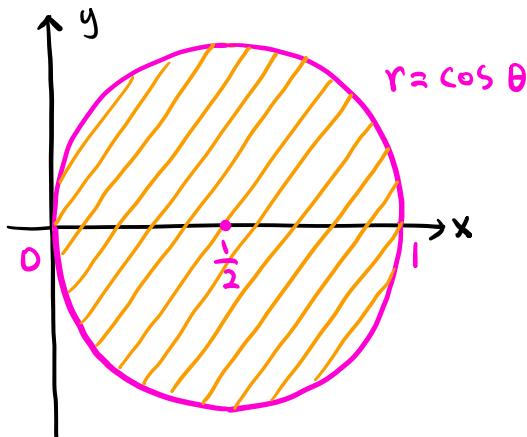
$$\Rightarrow \text{The area is } \pi \cdot 1^2 = \boxed{\pi}$$

- b) (7 points) $r = \cos \theta$ and $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$,

in polar coordinates. You may use the formula $\cos^2 \theta = \frac{1+\cos 2\theta}{2}$.

$$\begin{aligned} r = \cos \theta &\rightsquigarrow r^2 = r \cos \theta \rightsquigarrow x^2 + y^2 = x \\ &\rightsquigarrow x^2 - x + \frac{1}{4} + y^2 = \frac{1}{4} \rightsquigarrow (x - \frac{1}{2})^2 + y^2 = \frac{1}{4}. \end{aligned}$$

The loop is the circle of radius $\frac{1}{2}$ and center $(\frac{1}{2}, 0)$



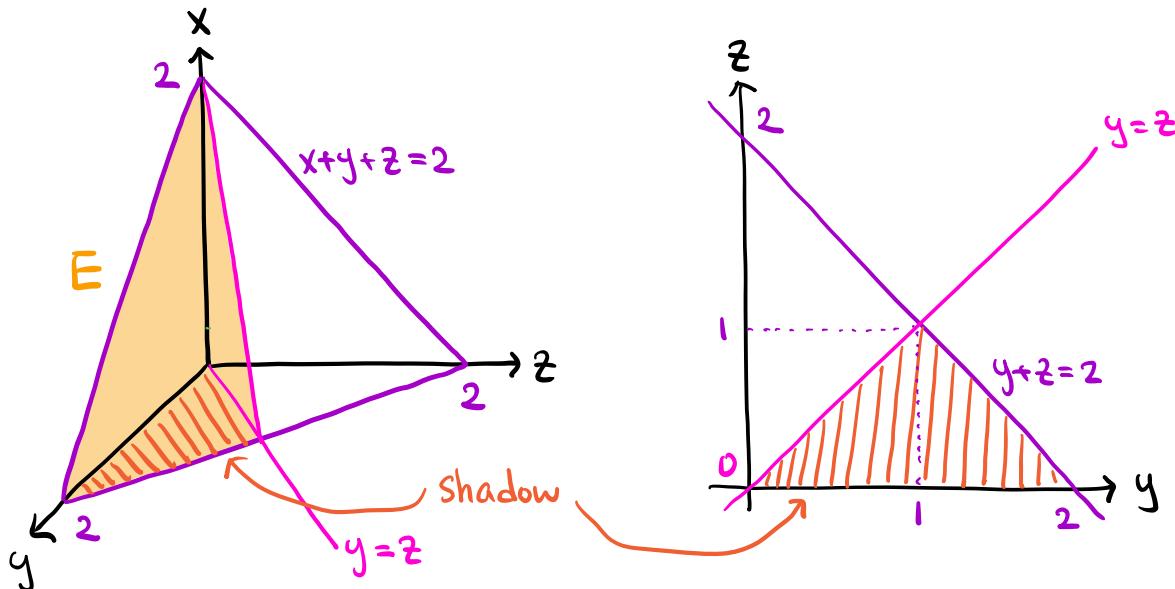
$$\Rightarrow \text{The area is } \pi \cdot (\frac{1}{2})^2 = \boxed{\frac{\pi}{4}}$$

Note In each part, you can also compute the area using a double integral.

4. (10 points) Find the volume of the tetrahedron whose faces are given by the planes $x = 0$, $z = 0$, $y = z$, $x + y + z = 2$. Express your answer as a simple fraction.

$x + y + z = 2$ is a plane

$x\text{-intercept} = 2$, $y\text{-intercept} = 2$, $z\text{-intercept} = 2$.



The base on the yz -plane has area $\frac{1}{2} \cdot 2 \cdot 1 = 1$.

The height of the tetrahedron is 2

$$\Rightarrow \text{Volume} = \frac{1}{3} \cdot 1 \cdot 2 = \boxed{\frac{2}{3}}$$

Note You can also compute the volume by a triple integral.

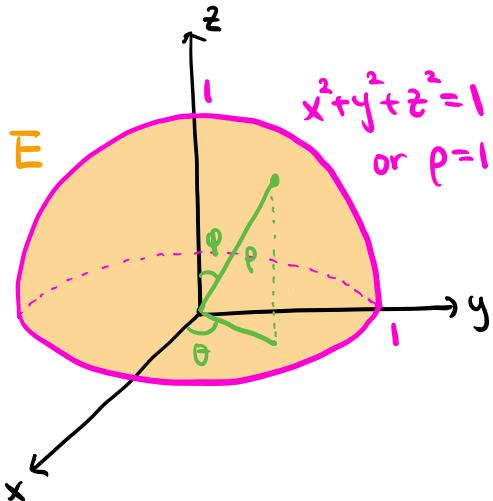
The base on the yz -plane : $0 \leq z \leq 1$, $z \leq y \leq 2-z$.

For each point on the base : $0 \leq x \leq 2-y-z$

$$\Rightarrow \text{Volume} = \int_0^1 \int_0^{2-z} \int_0^{2-y-z} 1 dx dy dz$$

5. Each of the three problems below is about the center of mass of the solid hemisphere $0 \leq x^2 + y^2 + z^2 \leq 1$ and $z \geq 0$.

- a) (4 points) Assume the density is $\rho(x, y, z) = r$, where r is the distance from the origin. Find the z -coordinate of the center of mass. Express your answer as a simple fraction.



In spherical coordinates :

$$x^2 + y^2 + z^2 = 1 \rightsquigarrow \rho^2 = 1 \rightsquigarrow \rho = 1$$

φ is maximized on the xy -plane

\Rightarrow The solid E is given by

$$0 \leq \theta \leq 2\pi, 0 \leq \varphi \leq \frac{\pi}{2}, 0 \leq \rho \leq 1$$

$$m = \iiint_E \rho(x, y, z) dV = \iiint_E \sqrt{x^2 + y^2 + z^2} dV$$

$$= \int_0^{2\pi} \int_0^{\pi/2} \int_0^1 \rho \cdot \rho^2 \sin \varphi \underbrace{d\rho d\varphi d\theta}_{\text{Jacobian}} = \int_0^{2\pi} \int_0^{\pi/2} \frac{\rho^4}{4} \sin \varphi \Big|_{\rho=0}^1 d\varphi d\theta$$

$$= \int_0^{2\pi} \int_0^{\pi/2} \frac{1}{4} \sin \varphi d\varphi d\theta = \int_0^{2\pi} -\frac{1}{4} \cos \varphi \Big|_{\varphi=0}^{\pi/2} d\theta = \int_0^{2\pi} \frac{1}{4} d\theta = \frac{\pi}{2}$$

$$\bar{z} = \frac{1}{m} \iiint_E z \rho(x, y, z) dV = \frac{2}{\pi} \iiint_E z \sqrt{x^2 + y^2 + z^2} dV$$

$$= \frac{2}{\pi} \int_0^{2\pi} \int_0^{\pi/2} \int_0^1 \rho^2 \cos \varphi \cdot \rho^2 \sin \varphi \underbrace{d\rho d\varphi d\theta}_{\text{Jacobian}}$$

$$= \frac{2}{\pi} \int_0^{2\pi} \int_0^{\pi/2} \frac{\rho^5}{5} \cos \varphi \sin \varphi \Big|_{\rho=0}^1 d\varphi d\theta$$

$$= \frac{2}{\pi} \int_0^{2\pi} \int_0^{\pi/2} \frac{1}{5} \cos \varphi \sin \varphi d\varphi d\theta = \frac{2}{\pi} \int_0^{2\pi} \frac{1}{5} u du d\theta \quad \begin{matrix} \uparrow \\ u = \sin \theta \end{matrix}$$

$$= \frac{2}{\pi} \int_0^{2\pi} \frac{1}{10} u^2 \Big|_{u=0}^1 d\theta = \frac{2}{\pi} \int_0^{2\pi} \frac{1}{10} d\theta = \boxed{\frac{2}{5}}$$

- b) (Hint, 0 points). If the density is $\rho(x, y, z) = 1$, the z -coordinate of the center of mass is $\bar{z} = \frac{3}{8}$, which you may assume. Your answer to part (a) must be greater than $3/8$.

$$m = \iiint_E \rho(x, y, z) dV = \iiint_E 1 dV = \text{vol}(E) = \frac{1}{2} \cdot \frac{4\pi}{3} \cdot 1^3 = \frac{2\pi}{3}$$

volume of sphere

$$\bar{z} = \frac{1}{m} \iiint_E z \rho(x, y, z) dV = \frac{3}{2\pi} \iiint_E z dV = \frac{3}{8}$$

- c) (6 points) Suppose the density is $\rho(x, y, z) = 2r + 3$. Find the z -coordinate of the center of mass. It can be helpful to use your answer to (a) and the information in (b).

$$\begin{aligned} m &= \iiint_E \rho(x, y, z) dV = \iiint_E 2\sqrt{x^2+y^2+z^2} + 3 dV \\ &= \underbrace{2 \iiint_E \sqrt{x^2+y^2+z^2} dV}_{\text{mass in (a)}} + \underbrace{3 \iiint_E 1 dV}_{\text{mass in (b)}} \\ &= 2 \cdot \frac{\pi}{2} + 3 \cdot \frac{2\pi}{3} = 3\pi. \end{aligned}$$

$$\begin{aligned} \bar{z} &= \frac{1}{m} \iiint_E z \rho(x, y, z) dV = \frac{1}{3\pi} \iiint_E z(2\sqrt{x^2+y^2+z^2} + 3) dV \\ &= \underbrace{\frac{1}{3\pi} \left(2 \iiint_E z \sqrt{x^2+y^2+z^2} dV + 3 \iiint_E z dV \right)}_{m\bar{z} \text{ in (a)}} \\ &= \frac{1}{3\pi} \left(2 \cdot \frac{\pi}{2} \cdot \frac{2}{5} + 3 \cdot \frac{2\pi}{3} \cdot \frac{3}{8} \right) = \boxed{\frac{23}{60}} \end{aligned}$$

Note You can directly compute the integrals in (c).

6. The two problems below are about surface areas.

- a) (3 points) Find the surface area of $z = x^2 + y^2$ above the region $D: 0 \leq x^2 + y^2 \leq 1$.

The surface is the graph of $f(x,y) = 2 - x^2 - y^2$.

In polar coordinates, D is given by $0 \leq \theta \leq 2\pi$, $0 \leq r \leq 1$

$$\text{Area} = \iint_D \sqrt{1 + f_x^2 + f_y^2} dA = \iint_D \sqrt{1 + 4x^2 + 4y^2} dA$$

$$= \int_0^{2\pi} \int_0^1 \sqrt{1+4r^2} \cdot r \, dr \, d\theta \stackrel{\substack{\text{Jacobian} \\ \uparrow}}{=} \int_0^{2\pi} \int_1^5 u^{\frac{1}{2}} \cdot \frac{1}{8} du \, d\theta$$

$$u = 1+4r^2$$

$$= \int_0^{2\pi} \frac{1}{12} u^{\frac{3}{2}} \Big|_{u=1}^{u=5} d\theta = \int_0^{2\pi} \frac{1}{12} (5^{\frac{3}{2}} - 1) d\theta = \boxed{\frac{\pi}{6} (5^{\frac{3}{2}} - 1)}$$

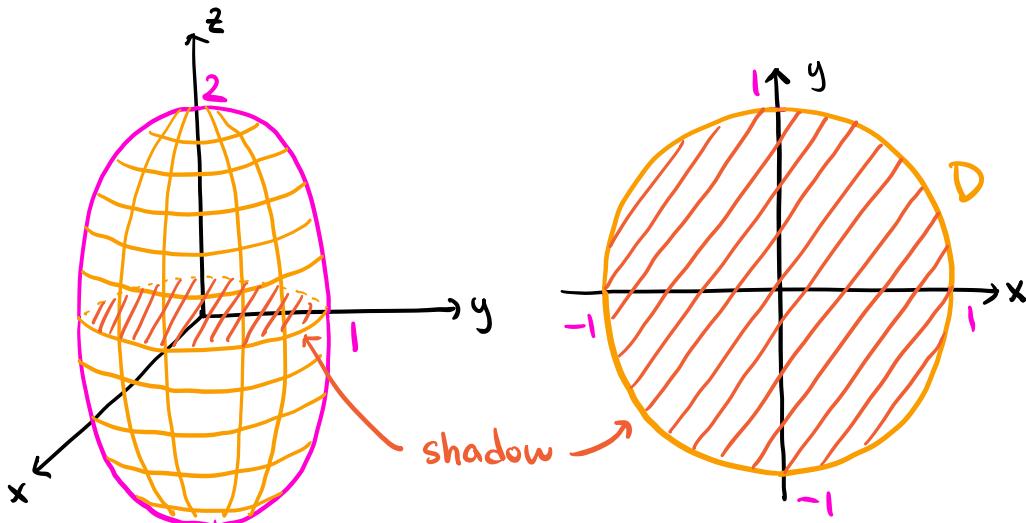
- b) (7 points) The area of the ellipsoidal surface

$$x^2 + y^2 + \frac{z^2}{4} = 1$$

is equal to

$$C \int_0^1 \frac{r(1+3r^2)^{1/2}}{(1-r^2)^{1/2}} dr$$

for a constant C . Find the constant C .



The shadow D on the xy-plane is given by $x^2 + y^2 \leq 1$.

In polar coordinates, D is given by $0 \leq \theta \leq 2\pi$, $0 \leq r \leq 1$

The area of the ellipsoid is twice the area of the upper half.

$$\Rightarrow \text{Area} = 2 \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} dA$$

The ellipsoid is a level curve of $f(x, y, z) = x^2 + y^2 + \frac{z^2}{4}$.

$$\Rightarrow \begin{cases} \frac{\partial z}{\partial x} = -\frac{f_x}{f_z} = -\frac{2x}{z/2} = -\frac{4x}{z} \\ \frac{\partial z}{\partial y} = -\frac{f_y}{f_z} = -\frac{2y}{z/2} = -\frac{4y}{z} \end{cases}$$

$$\text{Area} = 2 \iint_D \sqrt{1 + \frac{16x^2}{z^2} + \frac{16y^2}{z^2}} dA = 2 \iint_D \sqrt{\frac{16x^2 + 16y^2 + z^2}{z^2}} dA$$

$$x^2 + y^2 + \frac{z^2}{4} = 1 \rightsquigarrow z^2 = 4 - 4x^2 - 4y^2$$

$$\Rightarrow \text{Area} = 2 \iint_D \sqrt{\frac{16x^2 + 16y^2 + 4 - 4x^2 - 4y^2}{4 - 4x^2 - 4y^2}} dA$$

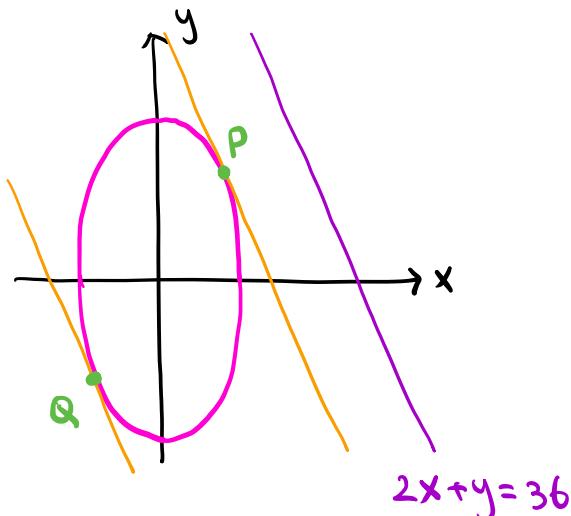
$$= 2 \int_0^{2\pi} \int_0^1 \sqrt{\frac{16r^2 + 4 - 4r^2}{4 - 4r^2}} \cdot r dr d\theta \quad \text{Jacobian}$$

$$= 2 \int_0^1 \int_0^{2\pi} \sqrt{\frac{1+3r^2}{1-r^2}} d\theta dr = 4\pi \int_0^1 \sqrt{\frac{1+3r^2}{1-r^2}} dr$$

$$\Rightarrow C = \boxed{4\pi}$$

7. (10 points) Find the points on the ellipse $\frac{x^2}{4} + \frac{y^2}{16} = 2$ that are closest to and furthest from the straight line $2x + y = 36$ in the x - y plane.

**very
tricky!**



P: the point on the ellipse closest to the line $2x+y=36$.

Q: the point on the ellipse furthest from the line $2x+y=36$.

The ellipse is a level curve of $f(x,y) = \frac{x^2}{4} + \frac{y^2}{16}$.

At P and Q, the tangent line to the ellipse must be parallel to the line $2x+y=36$.

$\Rightarrow \nabla f$ must be perpendicular to the line $2x+y=36$.

$$2x+y=36 \rightsquigarrow y=36-2x$$

\Rightarrow The direction vector of the line $2x+y=36$ is $\vec{v} = (1, -2)$

$$\nabla f \cdot \vec{v} = 0 \Rightarrow \left(\frac{x}{2}, \frac{y}{8} \right) \cdot (1, -2) = 0 \Rightarrow \frac{x}{2} - \frac{y}{4} = 0 \Rightarrow y = 2x$$

$$\frac{x^2}{4} + \frac{y^2}{16} = 2 \Rightarrow \frac{x^2}{4} + \frac{(2x)^2}{16} = 2 \Rightarrow \frac{x^2}{2} = 2 \Rightarrow x = \pm 2$$

$$\Rightarrow (x, y) = (2, 4), (-2, -4)$$

$$\Rightarrow \boxed{P = (2, 4), Q = (-2, -4)}$$