Exam 1 review – Winter 2020 exam

Math 215 — First Midterm

February 13, 2020

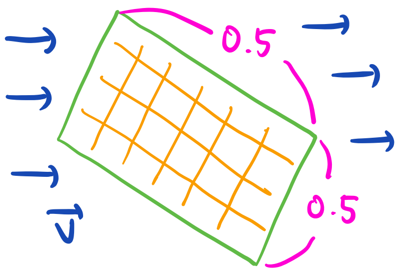
First 3 Letters of Last Name: ☐☐☐ First initial: ☐ UM Id#: ____________

Instructor: ______________________________ Section: ____________

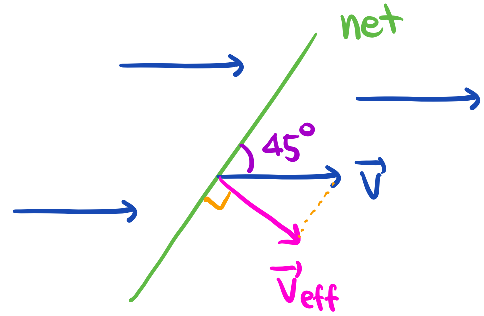
1. **Do not open this exam until you are told to do so.**
2. This exam has 13 pages including this cover. There are 9 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
3. Do not separate the pages of this exam, other than the formula sheet at the end of the exam. If they do become separated, write your name on every page and point this out to your instructor when you hand in the exam.
4. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
5. The true or false questions are the only questions that do not require you to show your work. For all other questions show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it. Include units in your answer where that is appropriate.
6. You may use no aids (e.g., calculators or notecards) on this exam.
7. **Turn off all cell phones**, remove all headphones, and **place any watch you are using on the desk in front of you**.

Problem	Points	Score
1	4	
2	15	
3	10	
4	15	
5	10	
6	10	
7	16	
8	10	
9	10	
Total	100	

1. [4 points] Trout are swimming up the Blackfoot River at forty meters per hour (relative to an observer on the bank of the river) to spawn, and their density is four fish per meter cubed. Gwynn loves to fish along the Blackfoot River. The river runs parallel to the x-axis, its surface is parallel to the plane $z = 0$, and the net they are using has a rectangular opening that is one half meter wide by one half meter long with normal vector $\mathbf{n}$. Gwynn puts her net into the river so that $\mathbf{n} \cdot \mathbf{j} = 0$ and its opening is at an angle of forty-five degrees to the surface of the water. Approximately how many fish does Gwynn catch in ten minutes?



3-dim'l view



2-dim'l view on the xz-plane

$\vec{v}$: the velocity of trout.

$\vec{v}_{eff}$: the "effective" velocity of trout through the net

The effective speed of trout through the net is

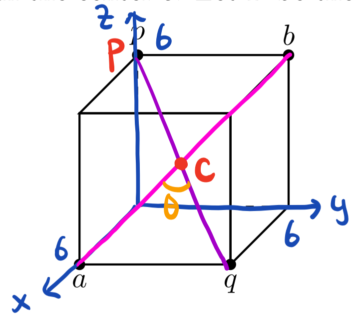
$$|\vec{v}_{eff}| = |\vec{v}| \cos 45^\circ = 40 \cdot \frac{\sqrt{2}}{2} = 20\sqrt{2} \ (m/h)$$

The area of the net is $\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4} \ (m^2)$.

The number of trout passing through the net in 10 minutes $(= \frac{1}{6}$ hour$)$ is

$$\underbrace{\frac{1}{4}}_{\text{area}} \cdot \underbrace{4}_{\text{density}} \cdot \underbrace{20\sqrt{2}}_{\text{speed}} \cdot \underbrace{\frac{1}{6}}_{\text{time}} = \boxed{\frac{10}{3}\sqrt{2}}$$

2. [15 points] The sides of the cube below have length six. The line segments ab and pq intersect at the center of the cube, let's call the center c. Let T be the triangle with vertices a, c, and q.



a. [8 points] What is the area of T?

$a = (6,0,0)$, $q = (6,6,0)$, $b = (0,6,6)$

c is the midpoint of $ab \Rightarrow c = (3,3,3)$.

$\overrightarrow{ca} = (3,-3,-3)$, $\overrightarrow{cq} = (3,3,-3)$

$\overrightarrow{ca} \times \overrightarrow{cq} = (18,0,18)$

$$\text{Area}(T) = \frac{1}{2}\left|\overrightarrow{ca} \times \overrightarrow{cq}\right| = \frac{1}{2}\sqrt{18^2+0^2+18^2} = \boxed{9\sqrt{2}}$$

b. [7 points] If θ is the angle of T at c, then what is $\cos(\theta)$?

$\overrightarrow{ca} \cdot \overrightarrow{cq} = 3\cdot 3 + (-3)\cdot 3 + (-3)\cdot(-3) = 9$.

$|\overrightarrow{ca}| = \sqrt{3^2+(-3)^2+(-3)^2} = \sqrt{27}$.

$|\overrightarrow{cq}| = \sqrt{3^2+3^2+(-3)^2} = \sqrt{27}$.

$$\cos\theta = \frac{\overrightarrow{ca}\cdot\overrightarrow{cq}}{|\overrightarrow{ca}||\overrightarrow{cq}|} = \frac{9}{\sqrt{27}\cdot\sqrt{27}} = \boxed{\frac{1}{3}}$$

3. [10 points] Indicate if each of the following is true or false by circling the correct answer.

a. [2 points] The vector equation

$$\langle x, y, z\rangle \times \langle 1, 1, 1\rangle = \langle 0, 1, 0\rangle$$

has a solution.

True $\boxed{\text{False}}$

b. [2 points] Suppose $y = mx + b$ is the equation of a line ℓ in $\mathbb{R}^2$. The line ℓ can be parameterized by $\mathbf{r}(t) = \langle 0, b\rangle + t\langle 1, m\rangle$.

$\boxed{\text{True}}$ False

c. [2 points] Suppose two space curves C_1 and C_2 are parameterized by $\mathbf{r}_1(t) = \langle 1 + t, 2 + 4t, -3 - 3t\rangle$ and $\mathbf{r}_2(s) = \langle -2s^3, -2 - 8s^3, 6s^3\rangle$, respectively. The space curves C_1 and C_2 are lines that are equal to each other.

$\boxed{\text{True}}$ False

d. [2 points] Suppose the plane P is given by the equation $ax + by + cz + d = 0$. Suppose $p_0 = (x_0, y_0, z_0)$ and $p_1 = (x_1, y_1, z_1)$ are points in $\mathbb{R}^3$ not on P. If $\langle a, b, c\rangle \cdot \langle x_0 - x_1, y_0 - y_1, z_0 - z_1\rangle > 0$, then p_1 and p_0 are on the same side of P.

True $\boxed{\text{False}}$

e. [2 points] If $\mathbf{u}$ and $\mathbf{v}$ are two vectors in $\mathbb{R}^3$ with $\mathbf{u} \cdot \mathbf{v} = 0$, then the curve parameterized by $\mathbf{r}(t) = \cos(t)\mathbf{u} + \sin(t)\mathbf{v}$ is a circle.

True $\boxed{\text{False}}$

* Explanations on the next page

(a) Note : $\vec{v} \times \vec{w}$ is perpendicular to both $\vec{v}$ and $\vec{w}$.

If $(x, y, z) \times (1,1,1) = (0,1,0)$ has a solution,

then $(0,1,0)$ must be perpendicular to $(1,1,1)$

However, they are not : $(0,1,0) \cdot (1,1,1) = 1 \neq 0$.

$\Rightarrow$ The statement is false

(b) $\vec{r}(t) = (0, b) + t(1, m) = (t, b+mt)$

$\Rightarrow y = mt + b = mx + b$.

$\Rightarrow$ The statement is true.

(c) $\vec{r_1}(t) = (1+t, 2+4t, -3-3t)$

$\Rightarrow C_1$ is a line with a direction vector $\vec{v_1} = (1, 4, -3)$.

Set $u = s^3 \Rightarrow \vec{r_2}(u) = (-2u, -2-8u, 6u)$

$\Rightarrow C_2$ is a line with a direction vector $\vec{v_2} = (-2, -8, 6)$

$\vec{v_2} = -2\vec{v_1}$: C_1 and C_2 are parallel.

$\vec{r_1}(-1) = \vec{r_2}(0) = (0, -2, 0)$

$\Rightarrow C_1$ and C_2 pass through the same point.

$\Rightarrow C_1$ and C_2 are the same lines

$\Rightarrow$ The statement is true.

(d) The plane $ax+by+cz+d=0$ has a normal vector

$\vec{n} = (a, b, c)$.

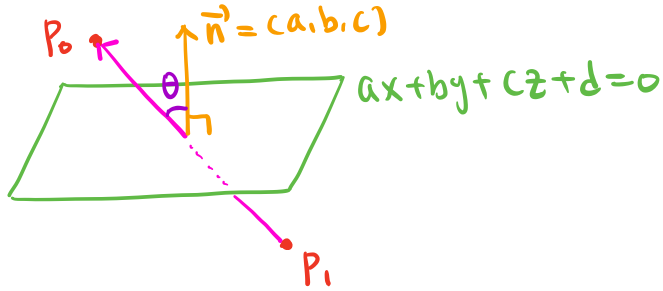
$\overrightarrow{P_1P_0} = (x_0-x_1, y_0-y_1, z_0-z_1)$

θ: the angle between $\vec{n}$ and $\overrightarrow{P_1P_0}$.

$(a,b,c)\cdot(x_0-x_1, y_0-y_1, z_0-z_1) > 0 \Rightarrow \vec{n}\cdot\overrightarrow{P_1P_0} > 0$

$$\Rightarrow \cos\theta = \frac{\vec{n}\cdot\overrightarrow{P_1P_0}}{|\vec{n}||\overrightarrow{P_1P_0}|} > 0 \Rightarrow \theta < \frac{\pi}{2}.$$

However, P_0 and P_1 may not be on the same side of the plane:



$\Rightarrow$ The statement is false

(e) Take $\vec{u} = (2,0,0)$, $\vec{v} = (0,1,0)$ $\Rightarrow \vec{u}\cdot\vec{v} = 0$.

$\vec{r}(t) = \cos t\,\vec{u} + \sin t\,\vec{v} = (2\cos t, \sin t, 0)$

$\Rightarrow x^2 + 4y^2 = 4\cos^2 t + 4\sin^2 t = 4$, $z = 0$

$\Rightarrow \vec{r}(t)$ parametrizes the ellipse $x^2+4y^2=1$

on the xy-plane

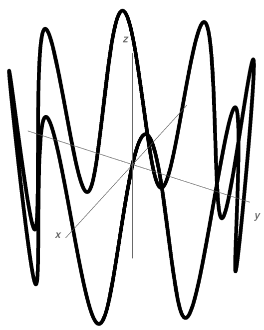
$\Rightarrow$ The statement is false.

Note The statement is true if $|\vec{u}| = |\vec{v}|$.

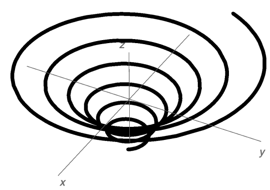
4. [15 points] If possible, match each of the parametric equations below with an appropriate graph. If there is no match, write **none**.

*Explanations are on the next page

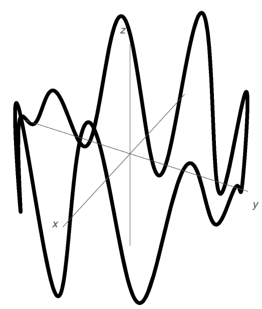
- $\mathbf{r}(t) = \langle \sin(t), \cos(t), \sin(8t) \rangle$ ___i___
- $\mathbf{r}(t) = \langle \sin(t), \cos(t), \sin(8t)\cos(t) \rangle$ ___iii___
- $\mathbf{r}(t) = \langle e^t \cos(20t), e^t \sin(20t), t \rangle$ ___ii___
- $\mathbf{r}(t) = \langle e^t \cos(20t), t, e^t \sin(20t) \rangle$ ___vi___
- $\mathbf{r}(t) = \langle \sin(t), \cos(t), 1 \rangle$ ___iv___



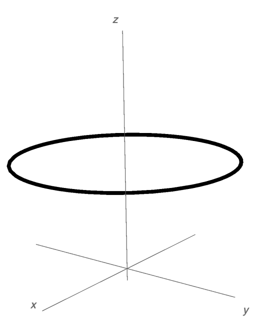
(i)



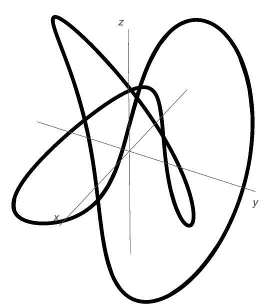
(ii)



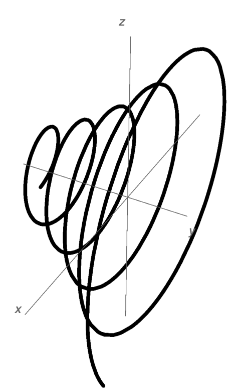
(iii)



(iv)



(v)



(vi)

General tips :

(1) Find a surface which the curve lies on by finding a relation between the coordinate functions

(2) Find a range for some coordinate functions

(3) Check whether some coordinate functions are increasing, decreasing, or periodic.

- $\vec{r}(t) = (\sin(t), \cos(t), \sin(8t))$.

$\Rightarrow x^2+y^2 = \sin^2(t) + \cos^2(t) = 1$

$\Rightarrow$ The curve lies on the cylinder $x^2+y^2=1$ with $z = \sin(8t)$ sinusoidally oscillating between -1 and 1.

$\Rightarrow$ Match : i.

- $\vec{r}(t) = (\sin(t), \cos(t), \sin(8t)\cos(t))$

$\Rightarrow x^2+y^2 = \sin^2(t) + \cos^2(t) = 1$.

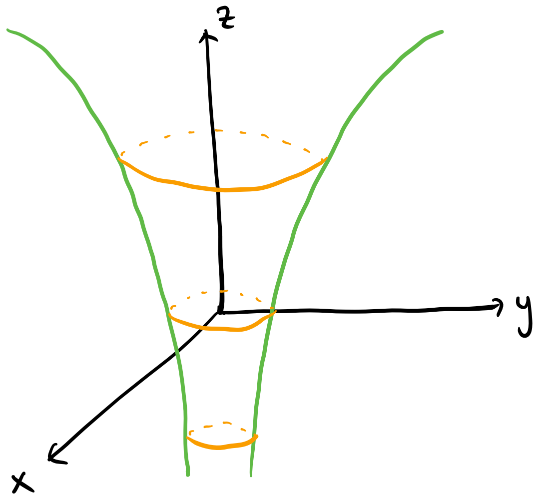
$\Rightarrow$ The curve lies on the cylinder $x^2+y^2=1$ with $z = \sin(8t)\cos(t)$ non-sinusoidally oscillating between -1 and 1.

$\Rightarrow$ Match : iii.

- $\vec{r}(t) = (e^t\cos(20t), e^t\sin(20t), t)$

  $\Rightarrow x^2+y^2 = e^{2t}\cos^2(20t) + e^{2t}\sin^2(20t) = e^{2t} = e^{2z}$.

  $\Rightarrow$ The curve lies on the surface $x^2+y^2 = e^{2z}$



$x=0 \Rightarrow y^2 = e^{2z} \Rightarrow y = \pm e^z$.

$z=k \Rightarrow x^2+y^2 = e^{2k}$

$\rightsquigarrow$ a circle.

$\Rightarrow$ Match: ii.

- $\vec{r}(t) = (e^t\cos(20t), t, e^t\sin(20t))$

  $\Rightarrow$ The curve is the same as the previous one with the y and z coordinates swapped.

  $\Rightarrow$ Match: vi.

- $\vec{r}(t) = (\sin(t), \cos(t), 1)$

  $\Rightarrow x^2+y^2 = \sin^2(t) + \cos^2(t) = 1,\ z=1$

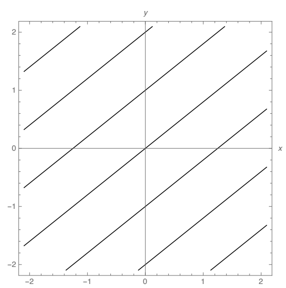
  $\Rightarrow$ The curve is a circle on the plane $z=1$

  $\Rightarrow$ Match: iv.

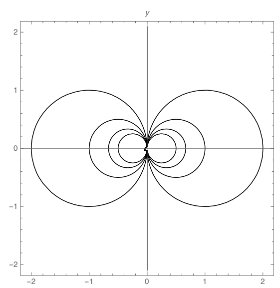
5. [10 points] If possible, match each of the nine sets of level curves below with the appropriate function. If there is no match, write **none**.

- $\cos(2(x-y))$ vii (or v)
- $x/(x^2+y^2)$ ii
- $\cos(x^2+y^2)$ ix
- y^3-x vi
- $\sin(2(x-y))$ v (or vii)
- $4x+5y$ iii
- $4x-5y$ i
- $x/(1+x^2+y^2)$ viii
- y^2+x iv

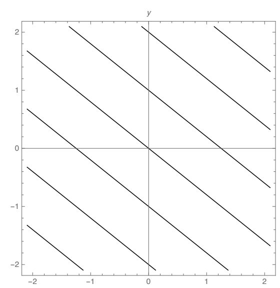
*Explanations are on the next page



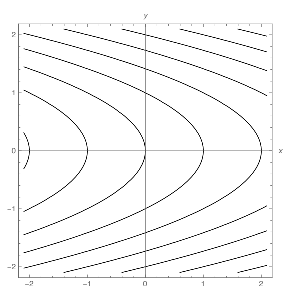
(i)



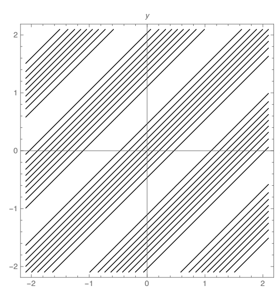
(ii)



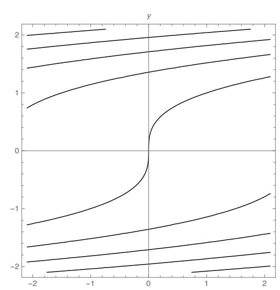
(iii)



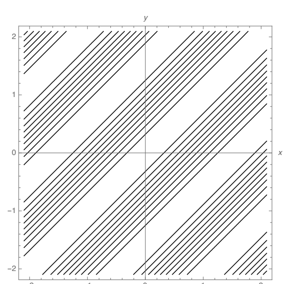
(iv)



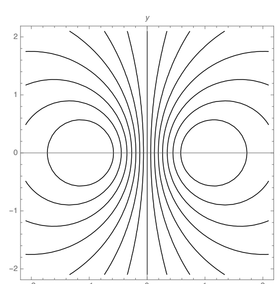
(v)



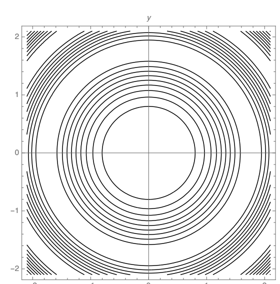
(vi)



(vii)



(viii)



(ix)

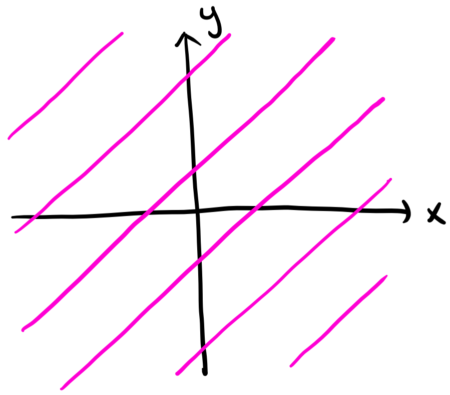
*The contour maps for cos(2(x−y)) and sin(2(x−y)) are indistinguishable unless levels are specified.

Note Level curves at different levels cannot intersect

Idea : Look at the level 0 and find a contour map which it fits in.

- $\cos(2(x-y)) = 0 \Rightarrow 2(x-y) = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \cdots$

$\Rightarrow x = y \pm \frac{\pi}{4}, y \pm \frac{3\pi}{4}, \cdots$

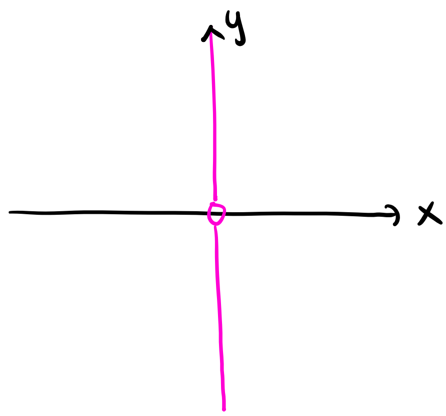


$\Rightarrow$ Match : vii (or v)

* lines on (i) appear to have slopes slightly less than 1.

- $\frac{x}{x^2+y^2} = 0 \Rightarrow x = 0$: the y-axis except (0,0)

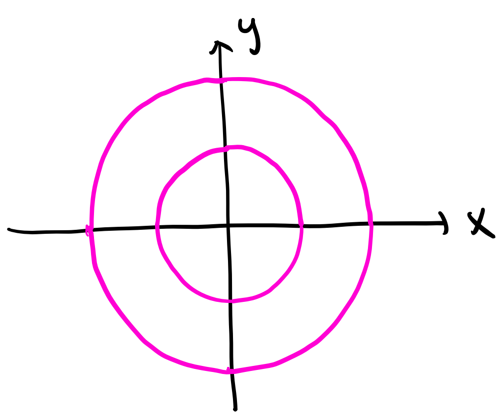
not on domain.



$\Rightarrow$ Match : ii

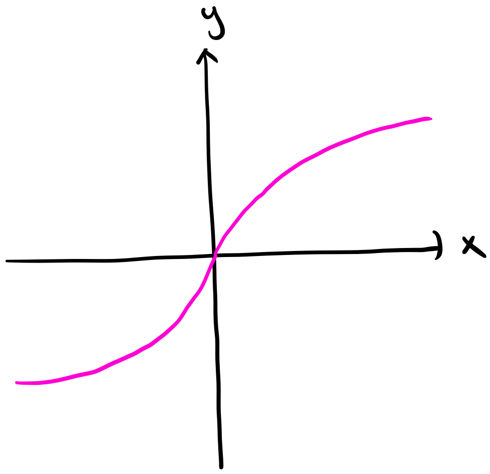
* The contour map (viii) has the origin (0,0) on the domain.

- $\cos(x^2+y^2) = 0 \Rightarrow x^2+y^2 = \frac{\pi}{2}, \frac{3\pi}{2}, \cdots$



$\Rightarrow$ Match : ii.

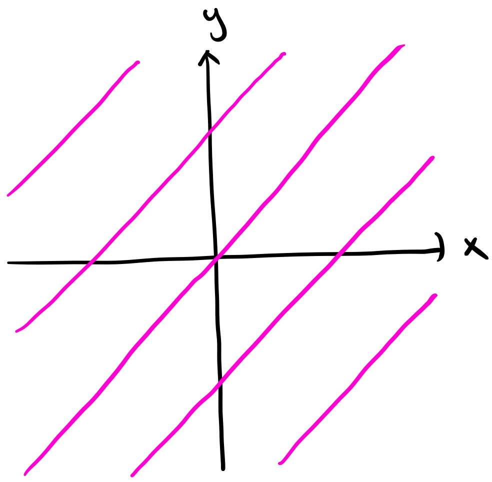
- $y^3 - x = 0 \Rightarrow x = y^3$.



$\Rightarrow$ Match : vi.

- $\sin(2(x-y)) = 0 \Rightarrow 2(x-y) = 0, \pm\pi, \pm 2\pi, \cdots$

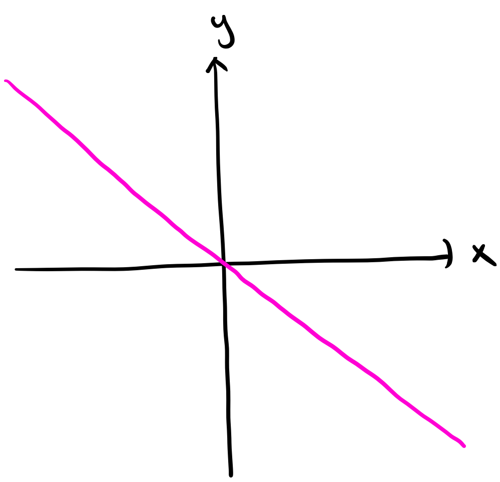
$\Rightarrow x = y, y \pm \frac{\pi}{2}, y \pm \pi, \cdots$



$\Rightarrow$ Match : v (or vii).

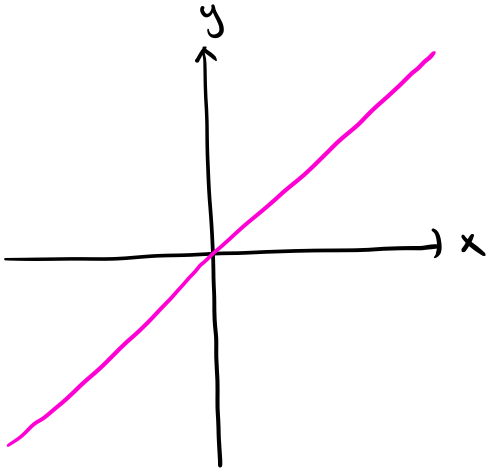
* lines on (i) appear to have slopes slightly less than 1.

- $4x + 5y = 0 \Rightarrow y = -\frac{4}{5}x$



$\Rightarrow$ Match : iii.

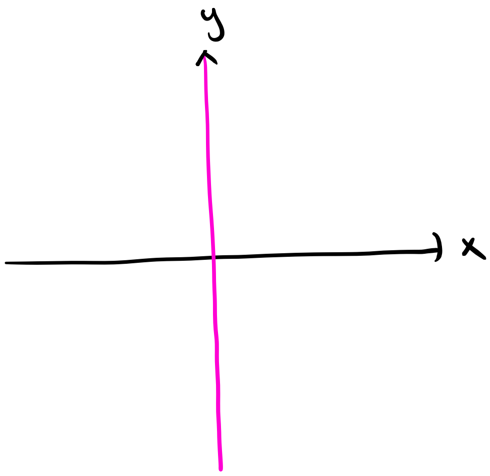
• $4x-5y=0 \Rightarrow y=\frac{4}{5}x$



$\Rightarrow$ Match : i.

* lines on (v) appear to have slope equal to 1

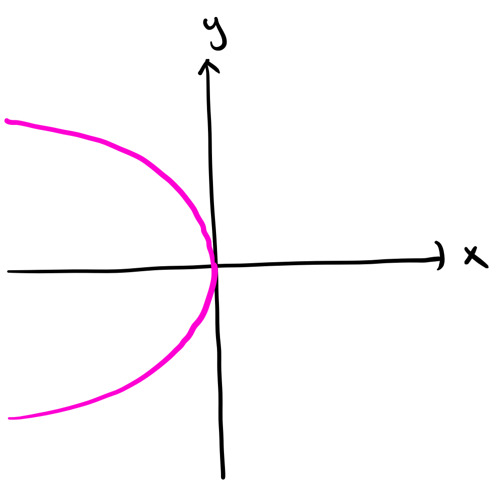
• $\frac{x}{1+x^2+y^2}=0 \Rightarrow x=0$: the y-axis.



$\Rightarrow$ Match : viii

* The contour map (ii) does not have the origin on the domain

• $y^2+x=0 \Rightarrow x=-y^2$



$\Rightarrow$ Match : iv.

6. [10 points] Suppose $g(x, y) = x + \ln(5x^2 - 4y^2)$.

a. [4 points] Find an equation for the tangent plane to the surface given by the equation $z = g(x, y)$ at the point $(1, 1, 1)$.

$$g_x = 1 + \frac{10x}{5x^2 - 4y^2} \rightsquigarrow g_x(1,1) = 11.$$

$$g_y = -\frac{8y}{5x^2 - 4y^2} \rightsquigarrow g_y(1,1) = -8.$$

The tangent plane equation is

$$z = g(1,1) + g_x(1,1)(x-1) + g_y(1,1)(y-1)$$

$$\Rightarrow \boxed{z = 1 + 11(x-1) - 8(y-1)}$$

b. [4 points] Find the linearization $L_g(x, y)$ of the function $g(x, y)$ at the point $(1, 1)$.

$$L_g(x,y) = \boxed{1 + 11(x-1) - 8(y-1)}$$

* The linear approximation is given by the z-coordinate of the tangent plane

c. [2 points] Use the linear approximation $L_g(x, y)$ to estimate $g(1.1, 1.1)$.

$$g(1.1, 1.1) \approx L_g(1.1, 1.1)$$

$$= 1 + 11(1.1-1) - 8(1.1-1)$$

$$= \boxed{1.3}$$

7. [16 points] Find or estimate, depending on the type of data provided, the partial derivative in the x direction at the point $(0,0)$ **and** the partial derivative in the y direction at the point $(0,0)$ for each of the following functions.

a. [4 points] For a function f given by the formula $f(x,y) = y^2\cos(1+x-y^2x)$

$$f_x = -y^2 \sin(1+x-y^2x)\cdot(1-y^2)$$

$$f_y = 2y\cos(1+x-y^2x) - y^2\sin(1+x-y^2x)\cdot(-2xy)$$

product rule

$$\Rightarrow \boxed{f_x(0,0)=0,\ f_y(0,0)=0}$$

b. [4 points] For a function g described by the data in the table below.

$x \backslash y$	-2	-1	0	1	2
-2	6	9	9	9	10
-1	12	16	18	19	20
0	20	22	25	27	30
1	28	36	43	47	48
2	35	49	55	61	66

$g_x(0,0) \approx$ change rate from $(-1,0)$ to $(1,0)$

$$= \frac{g(1,0)-g(-1,0)}{1-(-1)} = \frac{43-18}{2} = \boxed{12.5}$$

$g_y(0,0) \approx$ change rate from $(0,-1)$ to $(0,1)$

$$= \frac{g(0,1)-g(0,-1)}{1-(-1)} = \frac{27-22}{2} = \boxed{2.5}$$

c. [4 points] For the function

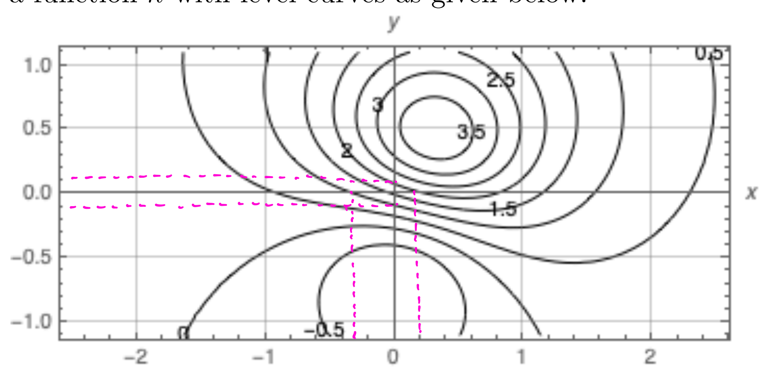
$$m(x,y) = \begin{cases} x(2x^3 - 4xy^2 - 4y^3)/(2x^3 + y^2) & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

$$m_x(0,0) = \lim_{h\to 0} \frac{m(h,0) - m(0,0)}{h} = \lim_{h\to 0} \frac{m(h,0)}{h}$$

$$= \lim_{h\to 0} \frac{h(2h^3 - 0 - 0)}{(2h^3+0)h} = \lim_{h\to 0} \frac{2h^4}{2h^4} = \boxed{1}$$

$$m_y(0,0) = \lim_{h\to 0} \frac{m(0,h) - m(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = \boxed{0}$$

d. [4 points] For a function h with level curves as given below.



$h_x(0,0) \approx$ change rate from $(-0.3, 0)$ to $(0.2, 0)$

$$= \frac{h(0.2,0) - h(-0.3,0)}{0.2-(-0.3)} = \frac{2-1}{0.5} = \boxed{2}$$

$h_y(0,0) \approx$ change rate from $(0,-0.1)$ to $(0,0.1)$

$$= \frac{h(0,0.1) - h(0,-0.1)}{0.1-(-0.1)} = \frac{2-1}{0.2} = \boxed{5}$$

8. [10 points] The trajectory of a particle is given by $\mathbf{r}(t) = \langle \sqrt{3}t^2, 2t^3, \sqrt{6}t^2 \rangle$ for $0 \le t \le \sqrt{8}$. Let C denote the corresponding space curve.

a. [5 points] Find an equation for the tangent line to C at the point $(4\sqrt{3}, 16, 4\sqrt{6})$

The tangent vector: $\vec{r}\,'(t) = (2\sqrt{3}t, 6t^2, 2\sqrt{6}t)$

At C: $\vec{r}(t) = (4\sqrt{3}, 16, 4\sqrt{6})$

$\Rightarrow (\sqrt{3}t^2, 2t^3, \sqrt{6}t^2) = (4\sqrt{3}, 16, 4\sqrt{6}) \rightsquigarrow t = 2$.

The tangent vector is $\vec{r}\,'(2) = (4\sqrt{3}, 24, 4\sqrt{6})$

$\Rightarrow \vec{\ell}(t) = \boxed{(4\sqrt{3} + 4\sqrt{3}t, 16 + 24t, 4\sqrt{6} + 4\sqrt{6}t)}$

b. [5 points] How long is C?

$$\text{Arc length} = \int_0^8 |\vec{r}\,'(t)|\, dt$$

$$|\vec{r}\,'(t)| = |(2\sqrt{3}t, 6t^2, 2\sqrt{6}t)|$$

$$= \sqrt{12t^2 + 36t^4 + 24t^2}$$

$$= \sqrt{36t^2 + 36t^4} = 6t\sqrt{t^2+1}$$

$$\Rightarrow \text{Arc length} = \int_0^8 6t\sqrt{t^2+1}\, dt$$

$$(u = t^2 + 1 \Rightarrow du = 2t\, dt)$$

$$= \int_1^9 3u^{1/2}\, du = 2u^{3/2} \Big|_{u=1}^{u=9}$$

$$= \boxed{52}$$

9. [10 points] In this problem all coordinates are measured in meters and time is measured in seconds. At time $t = 0$ a ladybug, named Sam, is at position $(1, 1, 1)$ and is flying with constant velocity $\langle 1, 2, 3\rangle$ meters per second A sensor placed at $(3, 6, 7)$ can detect ladybug motion that occurs within a sphere of radius 7 meters. Does the sensor detect Sam? If so, at what time is Sam *last* detected by the sensor?

$\vec{r}(t)$: position at time t.

$\Rightarrow$ $\begin{cases} \text{initial position } \vec{r}(0) = \langle 1,1,1\rangle \\ \text{velocity } \vec{r}\,'(t) = \langle 1,2,3\rangle \end{cases}$

$$\vec{r}(t) = \vec{r}(0) + \int_0^t \vec{r}\,'(u)\,du = \langle 1,1,1\rangle + \int_0^t \langle 1,2,3\rangle\,du$$

$$= \langle 1,1,1\rangle + \langle t, 2t, 3t\rangle = \langle 1+t, 1+2t, 1+3t\rangle.$$

Sam is detected by the sensor when his path intersects with the sphere.

The sphere equation is $(x-3)^2 + (y-6)^2 + (z-7)^2 = 49$.

$\Rightarrow (1+t-3)^2 + (1+2t-6)^2 + (1+3t-7)^2 = 49$.

$\rightsquigarrow t^2 - 4t + 4 + 4t^2 - 20t + 25 + 9t^2 - 36 + 36 = 49$

$\rightsquigarrow 14t^2 - 60t + 16 = 0$

$\Rightarrow t = \frac{7}{2}, 4$

quadratic formula or factorization

$\Rightarrow$ Sam is $\boxed{\text{last detected by the sensor at } t = 4.}$

You may use this page for scratch work.

This sheet will not be graded. Do not turn it in.

- $\sin^2(x) + \cos^2(x) = 1$, $\cos(2x) = \cos^2(x) - \sin^2(x)$, $\sin(2x) = 2\sin(x)\cos(x)$
- $\sin^2(x) = \dfrac{1-\cos(2x)}{2}$, $\cos^2(x) = \dfrac{1+\cos(2x)}{2}$
- $\cos(\pi/3) = 1/2$, $\sin(\pi/3) = \sqrt{3}/2$, $\cos(\pi/4) = \sqrt{2}/2$, $\sin(\pi/4) = \sqrt{2}/2$, $\cos(\pi/6) = \sqrt{3}/2$, $\sin(\pi/6) = 1/2$, $\cos(0) = 1$, $\sin(0) = 0$.
- $\frac{d}{dx}\sin(x) = \cos(x)$, $\frac{d}{dx}\cos(x) = -\sin(x)$.
- Volume of the parallelepiped determined by the vectors $\mathbf{v_1} = \langle a, b, c\rangle$, $\mathbf{v_2} = \langle d, e, f\rangle$, and $\mathbf{v_3} = \langle g, h, i\rangle$ is $|\mathbf{v_1} \cdot (\mathbf{v_2} \times \mathbf{v_3})| =$ absolute value of $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$
- Distance from a point (a, b, c) to a plane $Ax + By + Cz + D = 0$ is $\frac{|Aa+Bb+Cc+D|}{\sqrt{A^2+B^2+C^2}}$.
- The circumference of a circle of radius a is $2\pi a$.
- The area of a disk of radius a is πa^2.
- The volume of a right circular cylinder of radius a and height h is $\pi a^2 h$.
- The curvature of the curve given by the parametric equation $\mathbf{r}(t)$ is $\kappa(t) = \frac{|\mathbf{r}'(t)\times\mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}$.
- $\int \sin^2(u)\,du = \frac{u}{2} - \frac{\sin(2u)}{4} + C$ $\quad\quad$ $\int \cos^2(u)\,du = \frac{u}{2} + \frac{\sin(2u)}{4} + C$
- $\int \ln(u)\,du = u\ln(u) - u + C$
- The volume of a right circular cylinder of radius a and height h is $\pi a^2 h$.
- The volume of a sphere of radius a is $\frac{4\pi a^3}{3}$.
- The surface area of a sphere of radius a is $4\pi a^2$.
- The volume of a cone with base radius a and height b is $\frac{1}{3}\pi a^2 b$.
- Polar coordinates $x = r\cos(\theta)$, $y = r\sin(\theta)$.
- Cylindrical coordinates $x = r\cos(\theta)$, $y = r\sin(\theta)$, $z = z$.
- Spherical coordinates $x = \rho\cos(\theta)\sin(\phi)$, $y = \rho\sin(\theta)\sin(\phi)$, $z = \rho\cos(\phi)$.
- Green's Theorem:
$$\oint_{\partial D} P dx + Q dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA.$$
- Stokes' Theorem:
$$\oint_{\partial S} \vec{F} \cdot d\vec{r} = \iint_S \text{curl}\vec{F} \cdot d\vec{S}$$
- Divergence Theorem:
$$\iint_{\partial E} \vec{F} \cdot d\vec{S} = \iiint_E (\text{div}\vec{F}) dV.$$