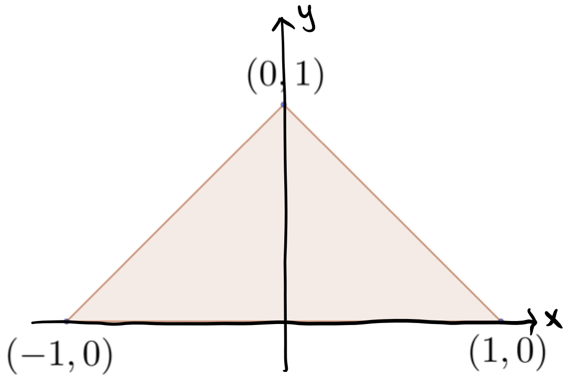
Final exam review - Winter 2018 exam

1. Consider the triangle with vertices at $(1, 0)$, $(-1, 0)$, and $(0, 1)$:



a) (2 points) Find the area of the triangle.

The triangle has base 2 and height 1 $\Rightarrow$ Area $= \frac{1}{2} \cdot 2 \cdot 1 = \boxed{1}$

b) (4 points) Suppose the density (mass per unit area) is $\rho(x, y) = x^2$. Find the mass of the triangle.

The left side and the right side of the triangle are given by the lines $x = y - 1$ and $x = 1 - y$.

$\Rightarrow$ The trianglular domain D is given by $0 \le y \le 1$, $y - 1 \le x \le 1 - y$.

$$m = \iint_D \rho(x,y,z)\, dA = \int_0^1 \int_{y-1}^{1-y} x^2 \, dx\, dy = \int_0^1 \frac{x^3}{3}\Big|_{x=y-1}^{x=1-y} dy = \int_0^1 \frac{2}{3}(1-y)^3\, dy$$

$$\underset{u=1-y}{=} \int_1^0 \frac{2}{3} u^3 \cdot (-1)\, du = -\frac{u^4}{6}\Big|_{u=1}^{u=0} = \boxed{\frac{1}{6}}$$

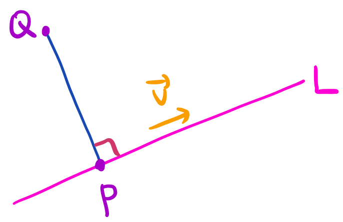
c) (4 points) Suppose the density is $\rho(x, y) = y^2$. Find the mass of the triangle.

$$m = \iint_D \rho(x,y,z)\, dA = \int_0^1 \int_{y-1}^{1-y} y^2 \, dx\, dy = \int_0^1 2(1-y)y^2\, dy = \left(\frac{2}{3}y^3 - \frac{1}{2}y^4\right)\Big|_{y=0}^{y=1} = \boxed{\frac{1}{6}}$$

<u>Note</u> In both parts, the density functions are even with respect to x. Since the domain D is symmetric about the y-axis, the mass is twice the mass of the part in the first quadrant. Hence you can compute $m = 2\int_0^1 \int_0^{1-y} \rho(x,y,z)\, dx\, dy$.

2. The line $(x, y, z) = (1 - 2t, 2t - 1, t)$, with t as the parameter, is denoted by L. The point Q is $(1, 2, 3)$.

a) (5 points) Find a point P on L such that the line from P to Q is orthogonal (perpendicular) to L.



L has a direction vector $\vec{v} = (-2, 2, 1)$

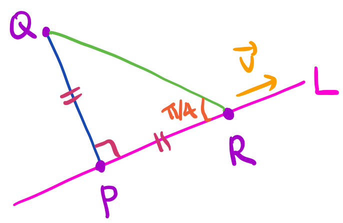
$\vec{PQ}$ is orthogonal to $\vec{v} \Rightarrow \vec{PQ} \cdot \vec{v} = 0$

$P = (1-2t, 2t-1, t) \Rightarrow \vec{PQ} = (2t, 3-2t, 3-t)$.

$\vec{PQ} \cdot \vec{v} = 0 \Rightarrow (2t, 3-2t, 3-t) \cdot (-2, 2, 1) = 0 \Rightarrow 9 - 9t = 0 \Rightarrow t = 1$.

$\Rightarrow P = (1 - 2\cdot 1, 2\cdot 1 - 1, 1) = \boxed{(-1, 1, 1)}$

b) (5 points) Find a point R on L such that the line from R to Q makes an angle equal to 45^o or $\pi/4$ with L.



$\dfrac{|\vec{PQ}|}{|\vec{PR}|} = \tan\dfrac{\pi}{4} = 1 \Rightarrow |\vec{PQ}| = |\vec{PR}|$

$\vec{PQ} = (2, 1, 2) \Rightarrow |\vec{PQ}| = \sqrt{2^2 + 1^2 + 2^2} = 3$

$\vec{PR}$ is parallel to $\vec{v}$ with $|\vec{PR}| = |\vec{PQ}| = 3$.

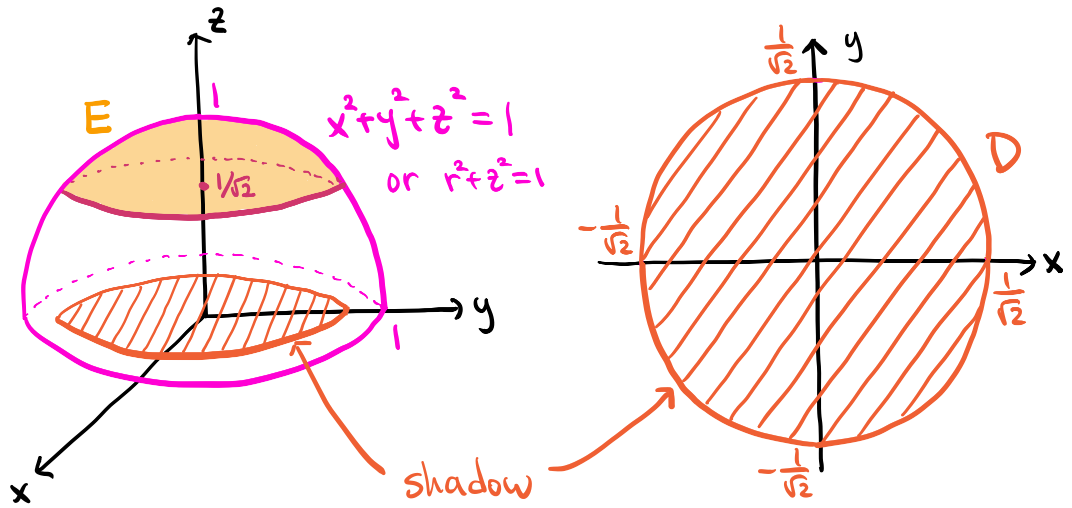
$|\vec{v}| = \sqrt{(-2)^2 + 2^2 + 1^2} = 3 \Rightarrow \vec{PR} = \pm\vec{v} = \pm(-2, 2, 1)$.

$\Rightarrow R = P \pm \vec{PR} = (-1, 1, 1) \pm (-2, 2, 1) = \boxed{(-3, 3, 2) \text{ or } (1, -1, 0)}$

<u>Note</u> Alternatively, since $\vec{QR}$ and $\vec{v}$ form an angle of $\frac{\pi}{4}$ radians, you can set $R = (2u, 3-2u, 3-u)$ from the line equation and use the dot product formula $\vec{QR} \cdot \vec{v} = |\vec{QR}||\vec{v}|\cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}|\vec{QR}||\vec{v}|$ to solve for u. This method does not make use of the point P, but involves some algebraic computations.

3. Consider the solid sphere $x^2 + y^2 + z^2 \leq 1$. * The answers on the archive are incorrect.

a) (3 points) Find the volume of the sphere above the plane $z = \frac{1}{\sqrt{2}}$.



E : the solid sphere above the plane $z = \frac{1}{\sqrt{2}}$.

In cylindrical coordinates : $x^2+y^2+z^2=1 \Rightarrow r^2+z^2=1 \Rightarrow z=\sqrt{1-r^2}$ $(z \geq 0)$

Intersection : $z=\frac{1}{\sqrt{2}}$ and $z=\sqrt{1-r^2} \Rightarrow \frac{1}{\sqrt{2}}=\sqrt{1-r^2} \Rightarrow \frac{1}{2}=1-r^2 \Rightarrow r=\frac{1}{\sqrt{2}}$.

The shadow on the xy-plane is given by $0 \leq \theta \leq 2\pi$, $0 \leq r \leq \frac{1}{\sqrt{2}}$

For each point on the shadow : $\frac{1}{\sqrt{2}} \leq z \leq \sqrt{1-r^2}$.

$\Rightarrow$ E is given by $0 \leq \theta \leq 2\pi$, $0 \leq r \leq \frac{1}{\sqrt{2}}$, $\frac{1}{\sqrt{2}} \leq z \leq \sqrt{1-r^2}$.

$$\Rightarrow \mathrm{Vol}(E) = \iiint_E 1\,dV = \int_0^{2\pi}\int_0^{1/\sqrt{2}}\int_{1/\sqrt{2}}^{\sqrt{1-r^2}} 1\cdot \overset{\text{Jacobian}}{r}\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^{1/\sqrt{2}} \left(\sqrt{1-r^2}-\frac{1}{\sqrt{2}}\right) r\,dr\,d\theta$$

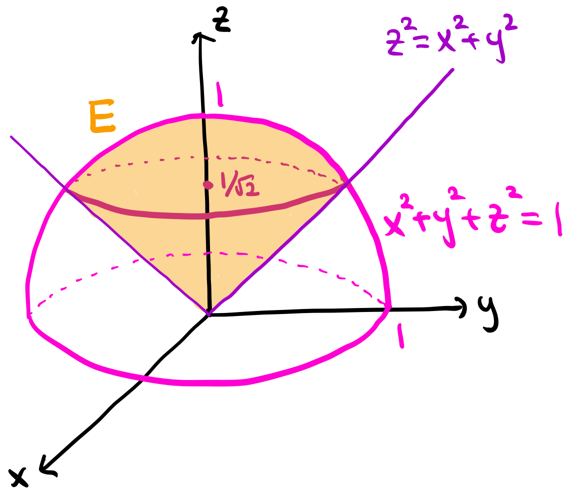
$$\underset{u=1-r^2}{=} \int_0^{2\pi}\int_1^{1/2} \left(u^{1/2}-\frac{1}{\sqrt{2}}\right)\left(-\frac{1}{2}\right)du\,d\theta = \int_0^{2\pi} \left(-\frac{u^{3/2}}{3}+\frac{\sqrt{2}}{4}u\right)\Bigg|_{u=1}^{u=1/2} d\theta$$

$$= \int_0^{2\pi} \frac{1}{3} - \frac{5}{12\sqrt{2}}\,d\theta = \boxed{\frac{2\pi}{3} - \frac{5\pi}{6\sqrt{2}}}$$

Note You can also use the bounds $0 \leq z \leq \frac{1}{\sqrt{2}}$, $0 \leq \theta \leq 2\pi$, $0 \leq r \leq \sqrt{1-z^2}$

obtained by the cross section method.

b) (3 points) Find the volume inside the sphere and above the cone $z^2 = x^2 + y^2$, $z \geq 0$.



Intersection : $z^2 = x^2+y^2$ and $x^2+y^2+z^2=1$

$\Rightarrow z^2+z^2=1 \Rightarrow z = \frac{1}{\sqrt{2}}$ $(z \geq 0)$

The solid E consists of the two parts:

- E_a is the part above the plane $z = \frac{1}{\sqrt{2}}$
- E_b is the part below the plane $z = \frac{1}{\sqrt{2}}$

The intersection circle is of radius $\frac{1}{\sqrt{2}}$.

$$\mathrm{Vol}(E_a) \underset{(a)}{=} \frac{2\pi}{3} - \frac{5\pi}{6\sqrt{2}}, \quad \mathrm{Vol}(E_b) = \underbrace{\frac{1}{3}\cdot\pi\left(\frac{1}{\sqrt{2}}\right)^2\cdot\frac{1}{\sqrt{2}}}_{\text{volume of cone}} = \frac{\pi}{6\sqrt{2}}$$

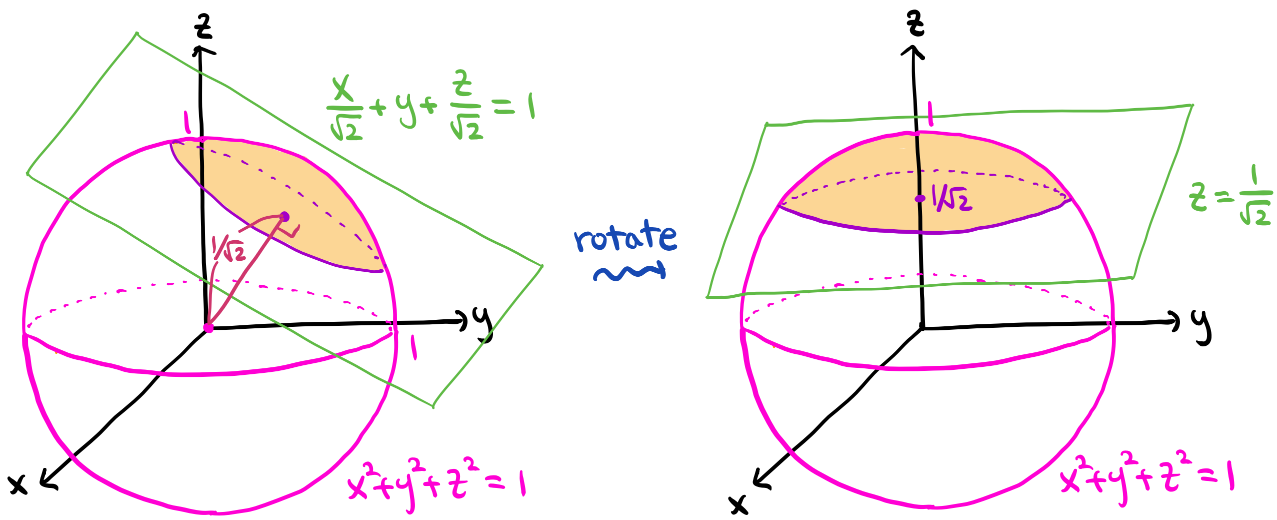
$$\Rightarrow \mathrm{Vol}(E) = \mathrm{Vol}(E_a) + \mathrm{Vol}(E_b) = \frac{2\pi}{3} - \frac{5\pi}{6\sqrt{2}} + \frac{\pi}{6\sqrt{2}} = \boxed{\frac{2\pi}{3} - \frac{2\pi}{3\sqrt{2}}}$$

c) (4 points) The plane $\frac{x}{\sqrt{2}} + y + \frac{z}{\sqrt{2}} = 1$ cuts the solid sphere into two pieces. Find the volumes of both the pieces.

The distance from $(0,0,0)$ to the plane $\frac{x}{\sqrt{2}} + y + \frac{z}{\sqrt{2}} - 1 = 0$ is

$$\frac{\left|\frac{0}{\sqrt{2}} + 0 + \frac{0}{\sqrt{2}} - 1\right|}{\sqrt{(1/\sqrt{2})^2 + 1^2 + (1/\sqrt{2})^2}} = \frac{1}{\sqrt{2}}$$

$\Rightarrow$ We can rotate the plane $\frac{x}{\sqrt{2}} + y + \frac{z}{\sqrt{2}} = 1$ to the plane $z = \frac{1}{\sqrt{2}}$.



E_1 : the part above the plane $z = \frac{1}{\sqrt{2}} \Rightarrow \mathrm{Vol}(E_1) \underset{(a)}{=} \boxed{\frac{2\pi}{3} - \frac{5\pi}{6\sqrt{2}}}$

E_2 : the part below the plane $z = \frac{1}{\sqrt{2}} \Rightarrow \mathrm{Vol}(E_2) = \underbrace{\frac{4\pi}{3}\cdot 1^3}_{\text{volume of sphere}} - \mathrm{Vol}(E_1) = \boxed{\frac{2\pi}{3} + \frac{5\pi}{6\sqrt{2}}}$

4. Consider the surface $2z^2 = x^4 + y^2$.

a) (2 points) Find the gradient of $f(x, y, z) = x^4 + y^2 - 2z^2$ at $(x, y, z) = (1, 1, 1)$.

$$\nabla f = (f_x, f_y, f_z) = (4x^3, 2y, -4z) \Rightarrow \nabla f(1,1,1) = \boxed{(4, 2, -4)}$$

b) (2 points) Find the tangent plane to the surface at the point $(1, 1, 1)$.

$2z^2 = x^4 + y^2 \Rightarrow x^4 + y^2 - 2z^2 = 0$: a level surface of $f(x,y,z) = x^4 + y^2 - 2z^2$

$\nabla f(1,1,1) = (4, 2, -4)$ is a normal vector

$\Rightarrow$ The tangent plane at $(1,1,1)$ is given by $\boxed{4(x-1) + 2(y-1) - 4(z-1) = 0}$

c) (3 points) Find the area of the tangent plane inside the cylinder $x^2 + y^2 = 1$.

$4(x-1) + 2(y-1) - 4(z-1) = 0 \Rightarrow 4x + 2y - 4z = 2 \Rightarrow z = x + \frac{y}{2} - \frac{1}{2}$.

The surface S is parametrized by $\vec{r}(x,y) = (x, y, x + \frac{y}{2} - \frac{1}{2})$

The domain D is given by $x^2 + y^2 \le 1$.

$$\text{Area}(S) = \iint_S 1\, dS = \iint_D |\vec{r}_x \times \vec{r}_y|\, dA$$

$\vec{r}_x = (1, 0, 1)$, $\vec{r}_y = (0, 1, \frac{1}{2}) \Rightarrow \vec{r}_x \times \vec{r}_y = (-1, -\frac{1}{2}, 1)$

$$\Rightarrow \text{Area}(S) = \iint_D \sqrt{(-1)^2 + (-\tfrac{1}{2})^2 + 1^2}\, dA = \iint_D \frac{3}{2}\, dA = \frac{3}{2}\text{Area}(D) = \frac{3}{2} \cdot \underbrace{\pi \cdot 1^2}_{\text{Area of disk}} = \boxed{\frac{3\pi}{2}}$$

d) (3 points) Find the area of the tangent plane inside the elliptical cylinder $\frac{x^2}{9} + \frac{z^2}{16} = 1$.

$4(x-1) + 2(y-1) - 4(z-1) = 0 \Rightarrow 4x + 2y - 4z = 2 \Rightarrow y = -2x + 2z + 1$.

The surface S is parametrized by $\vec{r}(x,z) = (x, -2x + 2z + 1, z)$.

The domain D is given by $\frac{x^2}{9} + \frac{z^2}{16} \le 1$.

$$\text{Area}(S) = \iint_S 1\, dS = \iint_D |\vec{r}_x \times \vec{r}_z|\, dA$$

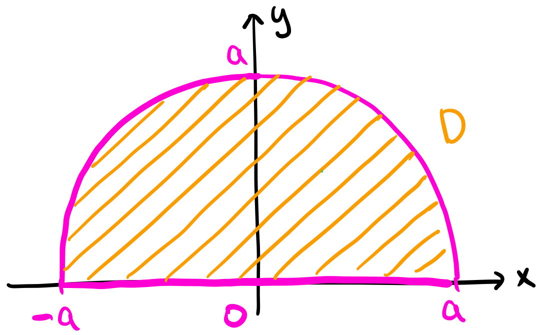
$\vec{r}_x = (1, -2, 0)$, $\vec{r}_z = (0, 2, 1) \Rightarrow \vec{r}_x \times \vec{r}_z = (-2, -1, 2)$

$$\Rightarrow \text{Area}(S) = \iint_D \sqrt{(-2)^2 + (-1)^2 + 2^2}\, dA = \iint_D 3\, dA = 3\,\text{Area}(D) = 3 \cdot \underbrace{\pi \cdot 3 \cdot 4}_{\text{Area of ellipse}} = \boxed{36\pi}$$

<u>Note</u> On the exam, you can use the fact that the area of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} \le 1$ is $ab\pi$. For an explanation, see Fact 1 in the Final exam facts note.

5. The density (mass per unit area) is $\rho(x, y) = 1$ throughout this problem.

a) (1 point) Consider the semi-circular plate $x^2 + y^2 \leq a^2$ with $y \geq 0$. Find the mass of this plate.



The domain D is a semidisk of radius a.

$$m = \iint_D \rho(x,y)\,dA = \iint_D 1\,dA = \text{Area}(D)$$

$$= \frac{1}{2}\underbrace{\pi a^2}_{\text{Area of disk}} = \boxed{\frac{\pi a^2}{2}}$$

b) (4 points) Find the y-coordinate of the center of mass of the plate of part (a).

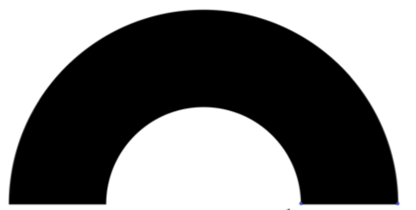
$$\bar{y} = \frac{1}{m}\iint_D y\rho(x,y)\,dA = \frac{2}{\pi a^2}\iint_D y\,dA$$

In polar coordinates, D is given by $0 \leq \theta \leq \pi$, $0 \leq r \leq a$.

$$\Rightarrow \bar{y} = \frac{2}{\pi a^2}\int_0^\pi \int_0^a r\sin\theta \cdot \underbrace{r}_{\text{Jacobian}}\,dr\,d\theta = \frac{2}{\pi a^2}\int_0^\pi \frac{r^3}{3}\sin\theta \Big|_{r=0}^{r=a} d\theta = \frac{2}{\pi a^2}\int_0^\pi \frac{a^3}{3}\sin\theta\,d\theta$$

$$= -\frac{2}{\pi a^2}\cdot\frac{a^3}{3}\cos\theta\Big|_{\theta=0}^{\theta=\pi} = \boxed{\frac{4a}{3\pi}}$$

c) (5 points) Now consider the following half-annular plate (center origin, inner radius $\frac{1}{2}$, outer radius 1):



Find the y-coordinate of the center of mass of this plate.

Let D denote the domain.

$$m = \iint_D \rho(x,y)\,dA = \iint_D 1\,dA = \text{Area}(D) = \underbrace{\frac{1}{2}\pi\cdot 1^2}_{\text{outer semidisk}} - \underbrace{\frac{1}{2}\pi\cdot\left(\frac{1}{2}\right)^2}_{\text{inner semidisk}} = \frac{3\pi}{8}$$

$$\bar{y} = \frac{1}{m}\iint_D y\rho(x,y)\,dA = \frac{2}{\pi a^2}\iint_D y\,dA$$

In polar coordinates, the domain D is given by $0 \leq \theta \leq \pi$, $\frac{1}{2} \leq r \leq 1$.

$$\Rightarrow \bar{y} = \frac{8}{3\pi}\int_0^\pi \int_{1/2}^1 r\sin\theta \cdot \underbrace{r}_{\text{Jacobian}}\,dr\,d\theta = \frac{8}{3\pi}\int_0^\pi \frac{r^3}{3}\sin\theta \Big|_{r=1/2}^{r=1} d\theta = \frac{8}{3\pi}\int_0^\pi \frac{7}{24}\sin\theta\,d\theta$$

$$= -\frac{8}{3\pi}\cdot\frac{7}{24}\cos\theta\Big|_{\theta=0}^{\theta=\pi} = \boxed{\frac{14}{9\pi}}$$

6. Consider the vector field $\mathbf{F} = (3x^2 + 3y)\mathbf{i} + (3x + 3y^2)\mathbf{j}$.

a) (5 points) Find a potential function $f(x, y)$ such that $\mathbf{F} = \nabla f$.

$\vec{F} = (3x^2+3y, 3x+3y^2) \Rightarrow P = 3x^2+3y, \ Q = 3x+3y^2$

$\frac{\partial P}{\partial y} = 3, \ \frac{\partial Q}{\partial x} = 3 \Rightarrow \frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y} \Rightarrow \vec{F}$ is conservative.

$\vec{F} = \nabla f \Rightarrow P = f_x, \ Q = f_y$.

$\int P\,dx = \int 3x^2+3y\,dx = x^3 + 3xy$

$\int Q\,dy = \int 3x+3y^2\,dy = 3xy + y^3$

$\Rightarrow f(x,y) = \boxed{x^3 + 3xy + y^3}$

(To find $f(x,y)$, we collect all terms without duplicates)

Note You get a different potential function by adding a constant.

b) (5 points) Evaluate the line integral

$$\int_C \mathbf{F} \cdot d\mathbf{r},$$

where the curve C is given by $\mathbf{r}(t) = t^2 \cos t\,\mathbf{i} + t^4 \sin t\,\mathbf{j}$, $0 \le t \le \pi$.

$\int_C \vec{F}\cdot d\vec{r} \underset{\substack{\uparrow \\ \text{Fund. thm}}}{=} f(\vec{r}(\pi)) - f(\vec{r}(0))$

$\vec{r}(\pi) = (-\pi^2, 0) \Rightarrow f(\vec{r}(\pi)) = f(-\pi^2, 0) = -\pi^6$

$\vec{r}(0) = (0,0) \Rightarrow f(\vec{r}(0)) = f(0,0) = 0$.

$\Rightarrow \int_C \vec{F}\cdot d\vec{r} = -\pi^6 - 0 = \boxed{-\pi^6}$

7. Each part of this problem asks you to evaluate a line integral over a closed curve. The curve is always assumed to be counterclockwise.

a) (3 points) Evaluate $\int_C \frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy$ with C being $x^2+y^2=1$.

This problem considers an integral of the vortex field

$$\vec{V}(x,y) = \left(-\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2}\right)$$

The curve C is a circle centered at (0,0), oriented counterclockwise.

$$\Rightarrow \int_C \vec{V}\cdot d\vec{r} = \boxed{2\pi}$$ (See Lecture 31)

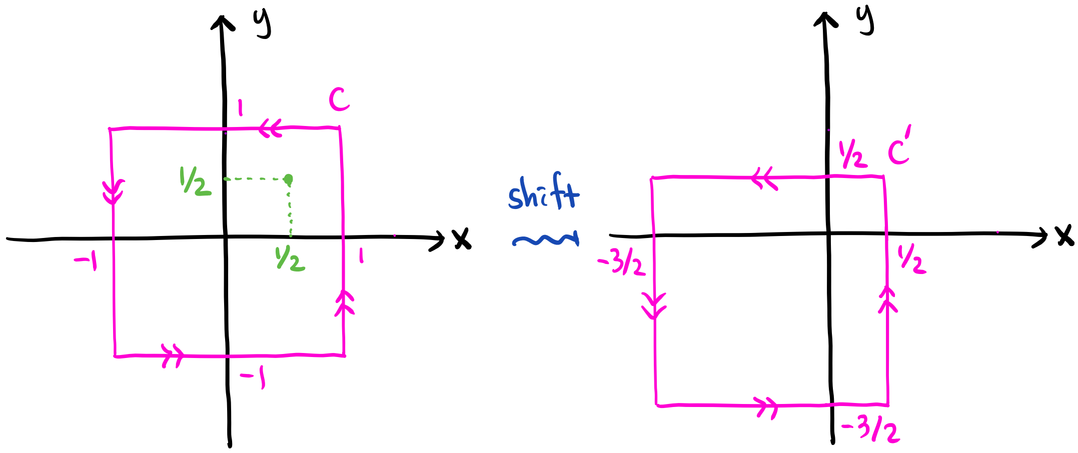
b) (3 points) Evaluate $\int_C \frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy$ with C being $\frac{(x-1)^2}{2} + (y-2)^2 = 1$.

At (0,0) : $\frac{(x-1)^2}{2} + (y-2)^2 = \frac{(0-1)^2}{2} + (0-2)^2 > 1$.

$\Rightarrow$ The curve C does not enclose (0,0)

$$\Rightarrow \int_C \vec{V}\cdot d\vec{r} = \boxed{0}$$ (See Lecture 33)

c) (4 points) Evaluate $\int_C \frac{-2(y-\frac{1}{2})}{(x-\frac{1}{2})^2+(y-\frac{1}{2})^2}dx + \frac{2(x-\frac{1}{2})}{(x-\frac{1}{2})^2+(y-\frac{1}{2})^2}dy$ with C being the square joining the four points $(\pm 1, \pm 1)$.



$$\int_C 2\vec{V}(x-\tfrac{1}{2}, y-\tfrac{1}{2})\cdot d\vec{r} = \int_{C'} 2\vec{V}(x,y)\cdot d\vec{r} = 2\int_{C'} \vec{V}\cdot d\vec{r}$$

C' encloses (0,0) with counterclockwise orientation.

$$\Rightarrow \int_{C'} \vec{V}\cdot d\vec{r} = 2\pi$$ (See Lecture 33)

$$\Rightarrow \int_C \vec{V}\cdot d\vec{r} = 2\cdot 2\pi = \boxed{4\pi}$$

8. The flux of the vector field $\mathbf{F}$ out of the surface S is given by $\int\int_S \mathbf{F}\cdot\mathbf{n}\,dS$, where $\mathbf{n}$ is the outward normal. In each part below, $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$ is the position vector.

a) (2 points) Find the flux of $\mathbf{F} = \frac{\mathbf{r}}{|\mathbf{r}|^2}$ out of the surface $x^2 + y^2 + z^2 = 1$.

Let S be the sphere $x^2+y^2+z^2=1$.

The unit normal vector of S is $\vec{n} = (x, y, z) = \vec{r}$. (See Lecture 37)

$$\Rightarrow \vec{F}\cdot\vec{n} = \frac{\vec{r}}{|\vec{r}|^2}\cdot\vec{r} = \frac{\vec{r}\cdot\vec{r}}{|\vec{r}|^2} = \frac{|\vec{r}|^2}{|\vec{r}|^2} = 1 \text{ on } S.$$

$$\Rightarrow \iint_S \vec{F}\cdot d\vec{S} = \iint_S \vec{F}\cdot\vec{n}\,dS = \iint_S 1\,dS = \text{Area}(S) = 4\pi\cdot 1^2 = \boxed{4\pi}$$

b) (2 points) Find the flux of $\mathbf{F} = \frac{\mathbf{r}}{|\mathbf{r}|^3}$ out of the surface $x^2 + y^2 + z^2 = 1$.

$$\vec{r} = (x, y, z) \Rightarrow |\vec{r}| = \sqrt{x^2+y^2+z^2}$$

$$\Rightarrow \vec{F} = \left(\frac{x}{(x^2+y^2+z^2)^{3/2}}, \frac{y}{(x^2+y^2+z^2)^{3/2}}, \frac{z}{(x^2+y^2+z^2)^{3/2}}\right) : \text{the inverse square field.}$$

The surface S is a sphere centered at $(0,0,0)$, oriented outward.

$$\Rightarrow \iint_S \vec{F}\cdot d\vec{S} = \boxed{4\pi} \text{ (see Lecture 37)}$$

<u>Note</u> You can also solve part (b) using the unit normal vector $\vec{n}$.

c) (3 points) Find the flux of $\mathbf{F} = (x - z^2x)\mathbf{i} + (z^2y + y)\mathbf{j} + z\,\mathbf{k}$ out of the surface $x^2 + y^2 + z^2 = 1$.

S : the sphere $x^2+y^2+z^2=1$ with outward orientation

E : the solid ball $x^2+y^2+z^2 \le 1$

$\Rightarrow \partial E = S$ is oriented outward.

$$\vec{F} = (x - z^2x,\ z^2y + y,\ z) \Rightarrow \text{div}(\vec{F}) = 3$$

$$\Rightarrow \iint_S \vec{F}\cdot d\vec{S} = \iint_{\partial E} \vec{F}\cdot d\vec{S} \underset{\text{div thm}}{=} \iiint_E \text{div}(\vec{F})\,dV = \iiint_E 3\,dV = 3\,\text{Vol}(E) = 3\cdot\underbrace{\frac{4\pi}{3}\cdot 1^3}_{\text{volume of sphere}} = \boxed{4\pi}$$

d) (3 points) Find the flux of $\mathbf{F} = \frac{\mathbf{r}}{|\mathbf{r}|^3}$ out of the surface $\left(x - \frac{1}{6}\right)^2 + \frac{1}{2}\left(y - \frac{1}{3}\right)^2 + \left(z + \frac{1}{5}\right)^2 = 1$.

$\vec{F}$ is the inverse square field as in part (b).

$$\text{At } (0,0,0): \left(x - \tfrac{1}{6}\right)^2 + \tfrac{1}{2}\left(y - \tfrac{1}{3}\right)^2 + \left(z + \tfrac{1}{5}\right)^2 = \left(0 - \tfrac{1}{6}\right)^2 + \tfrac{1}{2}\left(0 - \tfrac{1}{3}\right)^2 + \left(0 + \tfrac{1}{5}\right)^2 < 1.$$

$\Rightarrow$ The surface S encloses $(0,0,0)$ with outward orientation.

$$\Rightarrow \iint_S \vec{F}\cdot d\vec{S} = \boxed{4\pi} \text{ (see Lecture 39)}$$

9. Let $\mathbf{F} = yz\,\mathbf{i} + 2xz\,\mathbf{j} + 3xy\,\mathbf{k}$ be the vector field for this problem. The surface S is $x^2 + y^2 = 1$ with $-1 \le z \le 1$. The surface is a cylinder. The surface is obviously open at the top or the bottom. As always, the position vector is $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$.

a) (3 points) The bottom boundary of the surface denoted by C_1 is given by $x^2 + y^2 = 1$ and $z = -1$. The orientation is counterclockwise in the x-y plane. Find the circulation $\int_{C_1} \mathbf{F} \cdot d\mathbf{r}$.

C_1 is parametrized by $\vec{r}_1(t) = (\cos t, \sin t, -1)$ with $0 \le t \le 2\pi$.

$$\int_{C_1} \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \vec{F}(\vec{r}_1(t)) \cdot \vec{r}_1{}'(t)\,dt$$

$$\vec{F} = (yz, 2xz, 3xy) \Rightarrow \vec{F}(\vec{r}_1(t)) = (-\sin t, -2\cos t, 3\cos t \sin t).$$

$$\vec{r}_1{}'(t) = (-\sin t, \cos t, 0) \Rightarrow \vec{F}(\vec{r}_1(t)) \cdot \vec{r}_1{}'(t) = \sin^2 t - 2\cos^2 t.$$

$$\Rightarrow \int_{C_1} \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \sin^2 t - 2\cos^2 t\,dt = \int_0^{2\pi} \frac{1-\cos(2t)}{2} - (1+\cos(2t))\,dt$$

$$= \int_0^{2\pi} -\frac{1+3\cos(2t)}{2}\,dt = -\left(\frac{t}{2} + \frac{3\sin(2t)}{4}\right)\Bigg|_{t=0}^{t=2\pi} = \boxed{-\pi}$$

b) (2 points) The top boundary of the surface denoted by C_2 is given by $x^2 + y^2 = 1$ and $z = 1$. The orientation is counterclockwise in the x-y plane. Find the circulation $\int_{C_2} \mathbf{F} \cdot d\mathbf{r}$.

C_2 is parametrized by $\vec{r}_2(t) = (\cos t, \sin t, 1)$ with $0 \le t \le 2\pi$.

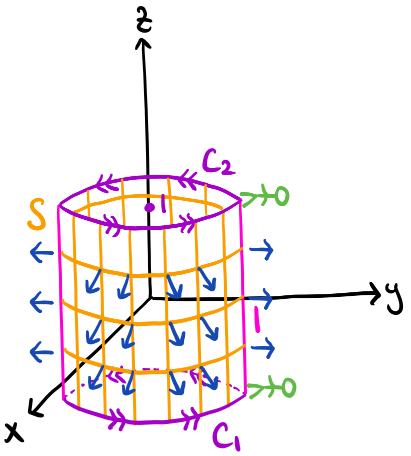
$$\int_{C_2} \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \vec{F}(\vec{r}_1(t)) \cdot \vec{r}_2{}'(t)\,dt$$

$$\vec{F} = (yz, 2xz, 3xy) \Rightarrow \vec{F}(\vec{r}_2(t)) = (\sin t, 2\cos t, 3\cos t \sin t).$$

$$\vec{r}_1{}'(t) = (-\sin t, \cos t, 0) \Rightarrow \vec{F}(\vec{r}_1(t)) \cdot \vec{r}_1{}'(t) = -\sin^2 t + 2\cos^2 t.$$

$$\Rightarrow \int_{C_2} \vec{F} \cdot d\vec{r} = \int_0^{2\pi} -\sin^2 t + 2\cos^2 t\,dt = -\int_{C_1} \vec{F} \cdot d\vec{r} \underset{(a)}{=} \boxed{\pi}$$

c) (5 points) Suppose the unit normal $\mathbf{n}$ to the cylindrical surface S points outward away from the axis of the cylinder. Calculate the flux $\int\int_S \operatorname{curl} \mathbf{F} \cdot \mathbf{n}\,dS$.



$\partial S = C_1 - C_2$ is positively oriented

(C_2 is negatively oriented)

*The right hand rule does not apply here.

$$\iint_S \operatorname{curl}(\vec{F}) \cdot d\vec{S} \underset{\text{Stokes' thm}}{=} \int_{\partial S} \vec{F} \cdot d\vec{s} = \int_{C_1} \vec{F} \cdot d\vec{r} - \int_{C_2} \vec{F} \cdot d\vec{r}$$

$$\underset{(a),(b)}{=} -\pi - \pi = \boxed{-2\pi}$$