16.9. The divergence theorem

☆☆ **Thm** (The divergence theorem)

Let $\vec{F}$ be a differentiable vector field on a solid E. If the boundary ∂E is oriented outward, then

$$\iint_{\partial E} \vec{F}\cdot d\vec{S} = \iiint_E \operatorname{div}(\vec{F})\, dV.$$

Note (1) The divergence theorem is extremely useful for computing the flux over a boundary surface.

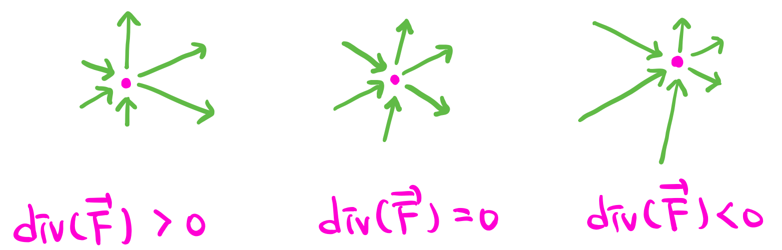
* Its usage is very similar to the usage of Green's theorem for line integrals.

(2) A boundary surface is also called a closed surface.

e.g. spheres, ellipsoids, ⋯

(3) Intuitively, the divergence measures the net flow out of each point:

- $\operatorname{div}(\vec{F}) > 0 \Rightarrow$ outflux > influx "a source"
- $\operatorname{div}(\vec{F}) = 0 \Rightarrow$ outflux = influx
- $\operatorname{div}(\vec{F}) < 0 \Rightarrow$ outflux < influx "a sink".

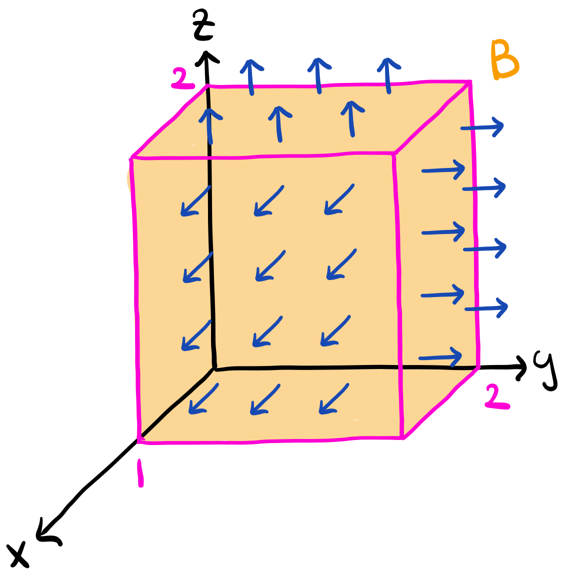


Ex Consider the rectangular box $B = [0,1] \times [0,2] \times [0,2]$.

Find the outward flux of $\vec{F}(x,y,z) = (xy^2 + z,\ e^{xz},\ xyz - y^2 z)$

across the boundary ∂B.

Sol



$$\iint_{\partial B} \vec{F} \cdot d\vec{S} \underset{\text{div. thm}}{=} \iiint_B \operatorname{div}(\vec{F})\, dV.$$

$P = xy^2 + z,\quad Q = e^{xz},\quad R = xyz - y^2 z$

$\Rightarrow \operatorname{div}(\vec{F}) = P_x + Q_y + R_z = y^2 + 0 + xy - y^2 = xy.$

$$\iint_{\partial B} \vec{F} \cdot d\vec{S} = \int_0^1 \int_0^2 \int_0^2 xy\, dz\, dy\, dx = \int_0^1 \int_0^2 2xy\, dy\, dx$$

$$= \int_0^1 xy^2 \Big|_{y=0}^{y=2} dx = \int_0^1 4x\, dx = 2x^2 \Big|_{x=0}^{x=1} = \boxed{2}$$

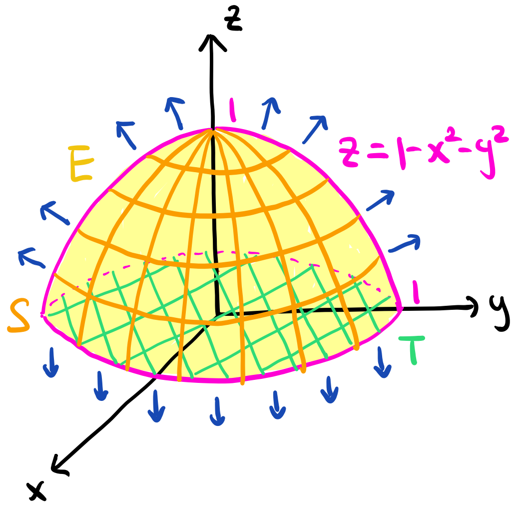
Note This solution is very simple compared to a direct computation of the integral over each face of B using parametrizations.

Ex Consider the vector field $\vec{F}(x,y,z) = (y^2 - xz,\ x^3 - yz,\ z^2 - 1)$

Find $\iint_S \vec{F} \cdot d\vec{S}$ where S is the paraboloid $z = 1 - x^2 - y^2$

with $z \geq 0$, oriented upward.

Sol



T : the disk with $x^2+y^2 \leq 1$ and $z=0$, oriented downward

E : the solid bounded by S and T.

$\Rightarrow \partial E = S + T$ is oriented outward.

$P = y^2 - xz,\ Q = x^3 - yz,\ R = z^2 - 1$

$\Rightarrow \operatorname{div}(\vec{F}) = P_x + Q_y + R_z = -z - z + 2z = 0.$

$$\iint_{\partial E} \vec{F} \cdot d\vec{S} = \iint_S \vec{F} \cdot d\vec{S} + \iint_T \vec{F} \cdot d\vec{S}$$

$$\Rightarrow \iint_S \vec{F} \cdot d\vec{S} = \iint_{\partial E} \vec{F} \cdot d\vec{S} - \iint_T \vec{F} \cdot d\vec{S} \quad (*)$$

$$\iint_{\partial E} \vec{F} \cdot d\vec{S} \underset{\text{div. thm}}{=} \iiint_E \underbrace{\operatorname{div}(\vec{F})}_{=0} dV = 0$$

The unit normal vector of T is $\vec{n} = (0,0,-1)$

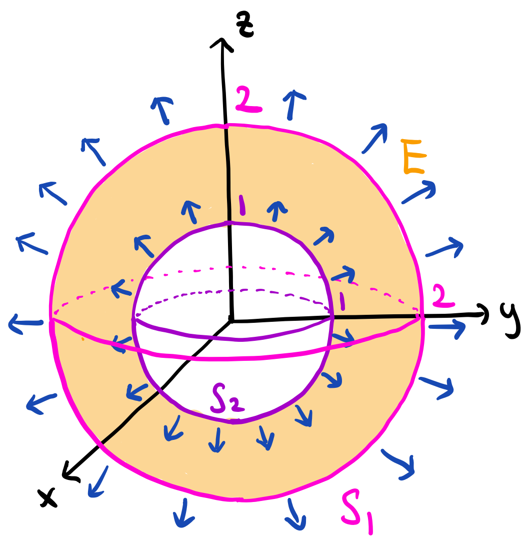
$$\Rightarrow \vec{F} \cdot \vec{n} = z^2 - 1 \underset{z=0 \text{ on } T}{=} -1 \quad \text{on } T$$

$$\Rightarrow \iint_T \vec{F} \cdot d\vec{S} = \iint_T \vec{F} \cdot \vec{n}\, dS = \iint_T -1\, dS = -\text{Area}(T) = -\pi \cdot 1^2 = -\pi.$$

$$\iint_S \vec{F} \cdot d\vec{S} = 0 - (-\pi) = \boxed{\pi}$$

Ex Consider the solid E bounded by the spheres S_1 and S_2 respectively given by $x^2+y^2+z^2=1$ and $x^2+y^2+z^2=4$, both oriented away from the origin. For a vector field $\vec{F}$ with $\iiint_E \text{div}(\vec{F})\,dV = 10$ and $\iint_{S_2} \vec{F}\cdot d\vec{S} = 16$, find $\iint_{S_1} \vec{F}\cdot d\vec{S}$.

Sol



$\partial E = -S_1 + S_2$ is oriented outward

(S_1 is oriented inward with respect to E)

$$\iint_{\partial E} \vec{F}\cdot d\vec{S} = -\iint_{S_1} \vec{F}\cdot d\vec{S} + \iint_{S_2} \vec{F}\cdot d\vec{S}$$

$$\Rightarrow \iint_{S_1} \vec{F}\cdot d\vec{S} = \iint_{S_2} \vec{F}\cdot d\vec{S} - \iint_{\partial E} \vec{F}\cdot d\vec{S}$$

$$\iint_{\partial E} \vec{F}\cdot d\vec{S} \underset{\text{div. thm}}{=} \iiint_E \text{div}(\vec{F})\,dV = 10$$

$$\Rightarrow \iint_{S_1} \vec{F}\cdot d\vec{S} = 16 - 10 = \boxed{6}$$

Note This problem is very similar to the "smiley face problem" from Lecture 34. Intuitively, we have

net flux = outflux − influx

$$\Rightarrow \iiint_E \text{div}(\vec{F})\,dV = \iint_{S_2} \vec{F}\cdot d\vec{S} - \iint_{S_1} \vec{F}\cdot d\vec{S}$$

$$\Rightarrow \iint_{S_1} \vec{F}\cdot d\vec{S} = \iint_{S_2} \vec{F}\cdot d\vec{S} - \iiint_E \text{div}(\vec{F})\,dV = 16 - 10 = 6.$$

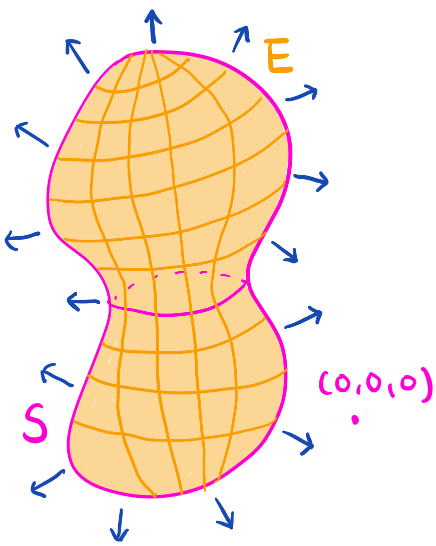
★ Ex Consider the inverse square field

$$\vec{F}(x,y,z) = \left(\frac{x}{(x^2+y^2+z^2)^{3/2}}, \frac{y}{(x^2+y^2+z^2)^{3/2}}, \frac{z}{(x^2+y^2+z^2)^{3/2}} \right)$$

Let S be a boundary surface with outward orientation.

(1) Find $\iint_S \vec{F} \cdot d\vec{S}$ when S does not enclose the origin.

Sol



E : the solid enclosed by S

$\Rightarrow \partial E = S$ is oriented outward.

$\vec{F}$ is defined on E.

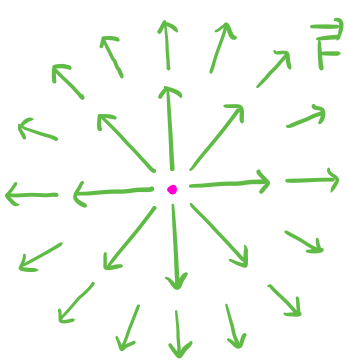
(E does not contain the origin)

$\operatorname{div}(\vec{F}) = 0$

Lecture 34

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_{\partial E} \vec{F} \cdot d\vec{S} \underset{\text{div. thm}}{=} \iiint_E \underbrace{\operatorname{div}(\vec{F})}_{=0} \, dV = \boxed{0}$$

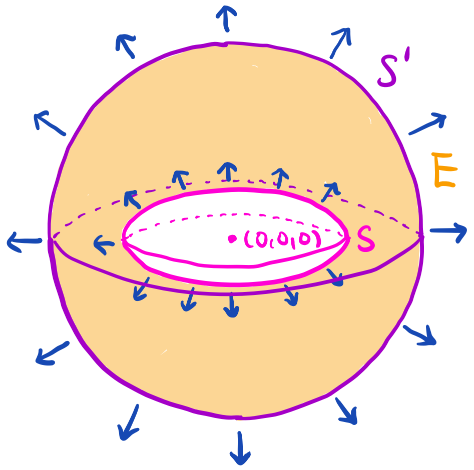
Note Intuitively, the inverse square field $\vec{F}$ has no sources or sinks in the domain.



* The only source is at the origin where $\vec{F}$ is not defined.

(2) Find $\iint_S \vec{F}\cdot d\vec{S}$ when S encloses the origin.

Sol



* We can't consider the solid bounded by S since $\vec{F}$ is not defined at the origin.

S' : a sphere centered at the origin which encloses S with outward orientation.

E : the solid bounded by S and S'.

$\Rightarrow \partial E = -S + S'$ is oriented outward.

(S is oriented inward with respect to E)

$\vec{F}$ is defined on E. (E does not contain the origin)

$$\iint_{\partial E} \vec{F}\cdot d\vec{S} = -\iint_S \vec{F}\cdot d\vec{S} + \iint_{S'} \vec{F}\cdot d\vec{S}$$

$$\Rightarrow \iint_S \vec{F}\cdot d\vec{S} = \iint_{S'} \vec{F}\cdot d\vec{S} - \iint_{\partial E} \vec{F}\cdot d\vec{S} \quad (*)$$

$$\iint_{\partial E} \vec{F}\cdot d\vec{S} \underset{\text{div. thm}}{=} \iiint_E \underbrace{\operatorname{div}(\vec{F})}_{=0} dV = 0$$

$$\Rightarrow \iint_S \vec{F}\cdot d\vec{S} \underset{(*)}{=} \iint_{S'} \vec{F}\cdot d\vec{S} \underset{\text{Lecture 37}}{=} \boxed{4\pi}$$

<u>Note</u> This example is a mathematical presentation of Gauss's law for electromagnetic fields.