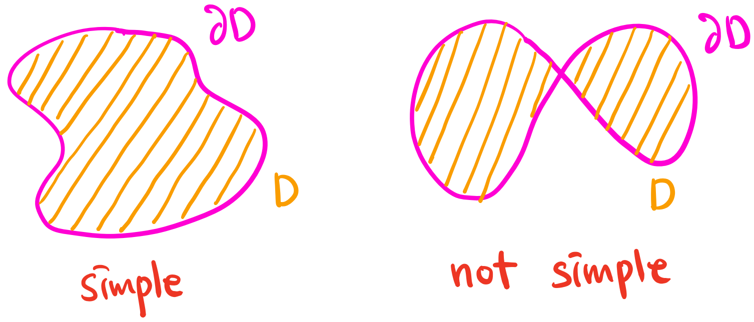
16.4. Green's theorem

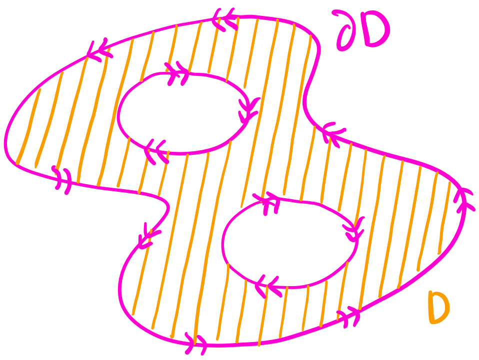
Def Let D be a domain in $\mathbb{R}^2$ with boundary ∂D.

(1) ∂D is simple if it has no self-intersections.

e.g.



(2) ∂D is positively oriented if it travels in a way that the interior of D lies on the left side



$\Rightarrow$ Outer boundary : counterclockwise

Inner boundary : clockwise.

Thm (Green's theorem)

Let $\vec{F} = (P, Q)$ be a differentiable vector field on a domain D.

If the boundary ∂D is simple and positively oriented, then

$$\int_{\partial D} \vec{F} \cdot d\vec{r} = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \, dA$$

On the formula sheet.

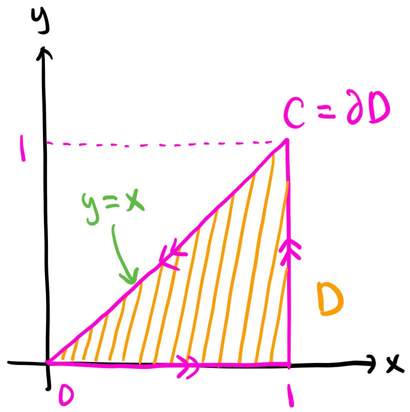
Note Green's theorem is useful for computing $\int_C \vec{F} \cdot d\vec{r}$ when

- C is (almost) a loop in $\mathbb{R}^2$
- $\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y}$ is easy to integrate

Ex Consider the vector field $\vec{F}(x,y) = (x^4 + 4xy^2,\ 3x^2 - 7y^5)$.

Find $\int_C \vec{F} \cdot d\vec{r}$ where C is the triangular curve with vertices at $(0,0)$, $(1,0)$, and $(1,1)$, oriented counterclockwise.

Sol



D : the region enclosed by C

$\Rightarrow \partial D = C$ is positively oriented.

D is given by $0 \le x \le 1$, $0 \le y \le x$.

$P = x^4 + 4xy^2$, $Q = 3x^2 - 7y^5$

$$\Rightarrow \frac{\partial P}{\partial y} = 8xy, \quad \frac{\partial Q}{\partial x} = 6x$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_{\partial D} \vec{F} \cdot d\vec{r} \underset{\text{Green's thm}}{=} \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA$$

$$= \int_0^1 \int_0^x 6x - 8xy\, dy\, dx = \int_0^1 6xy - 4xy^2 \Big|_{y=0}^{y=x} dx$$

$$= \int_0^1 6x^2 - 4x^3\, dx = 2x^3 - x^4 \Big|_{x=0}^{x=1} = \boxed{1}$$

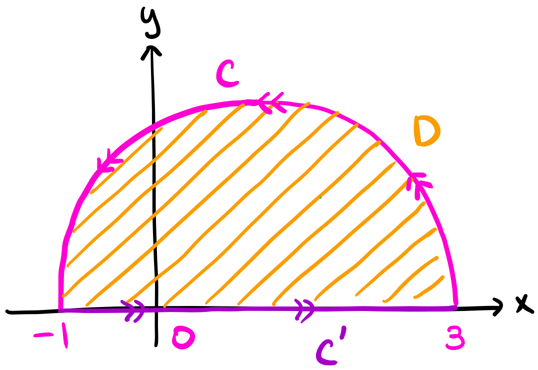
Note This solution is very simple compared to a direct computation of the integral over each line segment using parametrizations.

Ex Consider the vector field $\vec{F}(x,y) = (2x^3+y,\ 2x-3y^4)$

Find $\int_C \vec{F}\cdot d\vec{r}$ where C is the upper half of the circle

$(x-1)^2+y^2=4$ with counterclockwise orientation.

Sol



C': the line segment from $(-1,0)$ to $(3,0)$

D: the region enclosed by C and C'.

$\Rightarrow \partial D = C + C'$ is positively oriented.

$$P = 2x^3+y,\quad Q = 2x-3y^4 \Rightarrow \frac{\partial P}{\partial y} = 1,\quad \frac{\partial Q}{\partial x} = 2$$

$$\int_{\partial D}\vec{F}\cdot d\vec{r} = \int_C \vec{F}\cdot d\vec{r} + \int_{C'}\vec{F}\cdot d\vec{r}$$

$$\Rightarrow \int_C \vec{F}\cdot d\vec{r} = \int_{\partial D}\vec{F}\cdot d\vec{r} - \int_{C'}\vec{F}\cdot d\vec{r} \quad (*)$$

$$\int_{\partial D}\vec{F}\cdot d\vec{r} \underset{\text{Green's thm}}{=} \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA = \iint_D 1\, dA = \text{Area}(D) = \frac{1}{2}\pi\cdot 2^2 = 2\pi$$

C' is parametrized by $\vec{r}(t) = (t, 0)$ with $-1 \le t \le 3$.

$$\vec{F}(\vec{r}(t)) = (2t^3+0,\ 2t-3\cdot 0^4) = (2t^3, 2t)$$

$$\vec{r}'(t) = (1, 0)$$

$$\Rightarrow \vec{F}(\vec{r}(t))\cdot\vec{r}'(t) = 2t^3\cdot 1 + 2t\cdot 0 = 2t^3.$$

$$\Rightarrow \int_{C'}\vec{F}\cdot d\vec{r} = \int_{-1}^{3}\vec{F}(\vec{r}(t))\cdot\vec{r}'(t)\,dt = \int_{-1}^{3} 2t^3\,dt = \frac{t^4}{2}\Big|_{t=-1}^{t=3} = 41$$

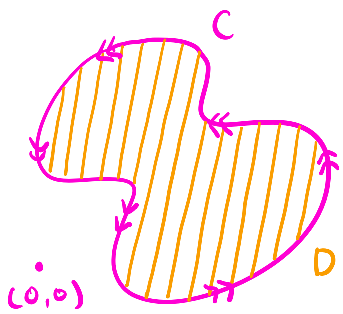
$$\int_C \vec{F}\cdot d\vec{r} \underset{(*)}{=} \boxed{2\pi - 41}$$

☆ **Ex** Consider the vortex field $\vec{V}(x,y) = \left(-\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2}\right)$.

Let C be a simple loop in $\mathbb{R}^2$, oriented counterclockwise.

(1) Find $\int_C \vec{V} \cdot d\vec{r}$ when C does not enclose the origin.

Sol



D : the region enclosed by C

$\Rightarrow \partial D = C$ is positively oriented.

$\vec{V}$ is defined on D.

(D does not contain the origin)

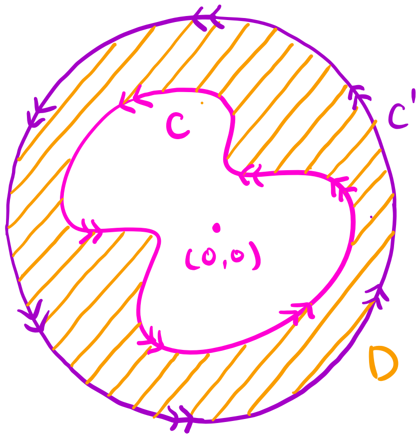
$$P = -\frac{y}{x^2+y^2}, \quad Q = \frac{x}{x^2+y^2} \underset{\text{Lecture 32}}{\Rightarrow} \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

$$\int_C \vec{V} \cdot d\vec{r} = \int_{\partial D} \vec{V} \cdot d\vec{r} \underset{\text{Green's thm}}{=} \iint_D \underbrace{\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}}_{=0} \, dA = \boxed{0}$$

Note You can get the same answer using the fundamental theorem for line integrals. In fact, since D is simply connected, the vortex field $\vec{V}$ is conservative on D by the relation $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$.

(2) Find $\int_C \vec{V}\cdot d\vec{r}$ when C encloses the origin.

Sol



* We can't consider the region enclosed by C since $\vec{V}$ is not defined at the origin.

C' : a circle centered at the origin which encloses C with counterclockwise orientation.

D : the region bounded by C and C'

$\Rightarrow \partial D = -C + C'$ is positively oriented

(C is negatively oriented)

$\vec{V}$ is defined on D. (D does not contain the origin)

$$\int_{\partial D} \vec{V}\cdot d\vec{r} = -\int_C \vec{V}\cdot d\vec{r} + \int_{C'} \vec{V}\cdot d\vec{r}$$

$$\Rightarrow \int_C \vec{V}\cdot d\vec{r} = -\int_{\partial D} \vec{V}\cdot d\vec{r} + \int_{C'} \vec{V}\cdot d\vec{r} \quad (*)$$

$$\int_{\partial D} \vec{V}\cdot d\vec{r} \underset{\text{Green's thm}}{=} \iint_D \underbrace{\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}}_{=0} dA = 0$$

$$\Rightarrow \int_C \vec{V}\cdot d\vec{r} \underset{(*)}{=} \int_{C'} \vec{V}\cdot d\vec{r} \underset{\text{Lecture 31}}{=} \boxed{2\pi}$$