A COMPREHENSIVE GUIDE ON LINE/SURFACE INTEGRALS IN MATH 215

Line integrals:

(I) If you integrate a scalar function f over a curve C, you should use the definition

$$\int_C f\ ds = \int_a^b f(\vec{r}(t))|\vec{r}\,'(t)|\ dt.$$

(II) If you integrate a vector field $\vec{F}$ over a curve C, there are several possibilities.

(1) If the vector field is conservative (i.e., has a potential function f), then you should always use the Fundamental theorem.

(2) If the curve C is a loop, then there are several subcases.

(a) If $\vec{F}$ is two dimensional and defined everywhere inside C, then you should use Green's theorem.

(b) If $\vec{F}$ is two dimensional but undefined at some points inside C with C not being a circle centered at the origin, then the best way is probably to use Green's theorem by choosing a large circle enclosing C (c.f. Homework 8 question 9(a)).

(c) If $\vec{F}$ is three dimensional with $\text{curl}(\vec{F})$ being easy to compute (or mentioned in the problem), then you should try to use Stokes' theorem.

(3) If $\vec{F} = (P, Q)$ is two dimensional with simple $\partial Q/\partial x - \partial P/\partial y$, then you should think about using Green's theorem by considering an additional curve (c.f., Homework 8 question 10).

(4) If none of the above applies, then you should use the definition

$$\int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)dt.$$

Surface integrals:

(I) If you integrate a scalar function f over a surface S, there are two possibilities:

(1) If S is not a sphere, then you should use the definition

$$\iint_S f\ dS = \iint_D f(\vec{r}(u,v))|\vec{r}_u \times \vec{r}_v|\ dA.$$

(2) If S is a sphere $x^2+y^2+z^2 = R^2$, then you can use the unit normal vector $\vec{n} = (x/R, y/R, z/R)$ and find a vector field $\vec{F}$ with $\vec{F} \cdot \vec{n} = f$ to convert it to a vector field integral $\iint_S \vec{F} \cdot d\vec{S}$ and use the divergence theorem (c.f. Homework 10 question 7).

(II) If you integrate a vector field $\vec{F}$ over a surface S, there are several possibilities.

(1) If you integrate the curl of a vector field, then you should use Stokes' theorem.

(2) If the surface S is a boundary surface, there are several subcases.

(a) If $\vec{F}$ is defined everywhere inside S, then you should use the divergence theorem.

(b) If $\vec{F}$ is undefined at some points inside S with S not being a sphere centered at the origin, then the best way is probably to use the divergence theorem by choosing a large sphere enclosing S (c.f. Homework 10 question 5).

(c) If $\vec{F}$ is undefined at some points inside S with S being a sphere $x^2 + y^2 + z^2 = R^2$, it's usually best to compute $\iint \vec{F} \cdot \vec{n}\ dS$ with $\vec{n} = (x/R, y/R, z/R)$ (c.f. Homework 10 question 2(a)).

(3) If the surface S is flat and parallel to one of the xy, yz, or zx planes, then it's often best to compute $\iint \vec{F} \cdot \vec{n}\ dS$.

(4) If you can make S into a boundary surface by closing the top or the base, etc, then you should think about using the divergence theorem after adding an appropriate surface (c.f. Homework 10 question 6).

(5) If none of the above applies, then you should use the definition

$$\iint_S \vec{F} \cdot\ d\vec{S} = \iint_D \vec{F}(\vec{r}(u,v)) \cdot (\vec{r}_u \times \vec{r}_v)\ dA.$$