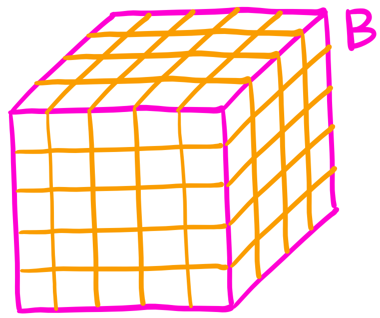
15.6. Triple integrals - definitions and properties

Def Let $f(x,y,z)$ be a function defined on a box

$B = [a,b] \times [c,d] \times [e,f]$

$:= \{(x,y,z) \in \mathbb{R}^3 : a \le x \le b,\ c \le y \le d,\ e \le z \le f\}$.

(1) If B is divided into equal subboxes B_{ijk} each with volume ΔV and a sample point $(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*)$,



the sum $\displaystyle\sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} f(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*)\,\Delta V$ is called a Riemann sum.

(2) The integral of $f(x,y,z)$ on B is given by

$$\iiint_B f(x,y,z)\,dV := \lim_{l,m,n\to\infty} \sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} f(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*)\,\Delta V.$$

Thm (Fubini's theorem for triple integrals)

If $f(x,y,z)$ is continuous on $B = [a,b] \times [c,d] \times [e,f]$,

$$\iiint_B f(x,y,z)\,dV = \int_a^b \int_c^d \int_e^f f(x,y,z)\,dz\,dy\,dx$$

$$= \int_e^f \int_c^d \int_a^b f(x,y,z)\,dx\,dy\,dz$$

$$= \cdots$$

Prop Let E be a solid with density $\rho(x,y,z)$

(1) Its volume is $\mathrm{Vol}(E) = \iiint_E 1\, dV$.

(2) Its mass is $m = \iiint_E \rho(x,y,z)\, dV$.

(3) Its center of mass is $(\bar{x}, \bar{y}, \bar{z})$ with

$$\begin{cases} \bar{x} = \dfrac{1}{m} \iiint_E x\rho(x,y,z)\, dV. \\ \bar{y} = \dfrac{1}{m} \iiint_E y\rho(x,y,z)\, dV. \\ \bar{z} = \dfrac{1}{m} \iiint_E z\rho(x,y,z)\, dV. \end{cases}$$

Prop (Triple integrals and symmetry)

Let $f(x,y,z)$ be a function on a solid E.

(1) If E is symmetric about the xy-plane while $f(x,y,z)$ is odd respect to z, then $\iint_E f(x,y,z)\, dV = 0$.

(2) If E is symmetric about the yz-plane while $f(x,y,z)$ is odd respect to x, then $\iint_E f(x,y,z)\, dV = 0$.

(3) If E is symmetric about the xz-plane while $f(x,y,z)$ is odd respect to y, then $\iint_E f(x,y,z)\, dV = 0$.

Ex Consider the box $B = [-2,2] \times [-1,1] \times [0,3]$ with density $\rho(x,y,z) = z$.

(1) Find the mass.

Sol $m = \iiint_B \rho(x,y,z)\,dV = \int_{-2}^{2}\int_{-1}^{1}\int_{0}^{3} z\,dz\,dy\,dx$

$$= \int_{-2}^{2}\int_{-1}^{1} \left.\frac{z^2}{2}\right|_{z=0}^{z=3} dy\,dx = \int_{-2}^{2}\int_{-1}^{1} \frac{9}{2}\,dy\,dx$$

$$= \int_{-2}^{2} 2\cdot\frac{9}{2}\,dx = 4\cdot 2\cdot\frac{9}{2} = \boxed{36}$$

(2) Find the center of mass.

Sol B is symmetric about the yz, xz planes.

$$\bar{x} = \frac{1}{m}\iiint_B x\rho(x,y,z)\,dV = \frac{1}{m}\iiint_B \underbrace{xz}_{\text{odd w.r.t. } x}\,dV = 0$$

$$\bar{y} = \frac{1}{m}\iiint_B y\rho(x,y,z)\,dV = \frac{1}{m}\iiint_B \underbrace{yz}_{\text{odd w.r.t. } y}\,dV = 0$$

$$\bar{z} = \frac{1}{m}\iiint_B z\rho(x,y,z)\,dV = \frac{1}{m}\iiint_B z^2\,dV$$

$$= \frac{1}{36}\int_{-2}^{2}\int_{-1}^{1}\int_{0}^{3} z^2\,dz\,dy\,dx = \frac{1}{36}\int_{-2}^{2}\int_{-1}^{1} \left.\frac{z^3}{3}\right|_{z=0}^{z=3} dy\,dx$$

$$= \frac{1}{36}\int_{-2}^{2}\int_{-1}^{1} 9\,dy\,dx = \frac{1}{36}\int_{-2}^{2} 2\cdot 9\,dx$$

$$= \frac{1}{36}\cdot 4\cdot 2\cdot 9 = 2$$

$\Rightarrow$ The center of mass is $\boxed{(0,0,2)}$

<u>Ex</u> Consider the solid

$$E = \{(x,y,z) \in \mathbb{R}^3 : 0 \le x \le 2-y-z,\ 0 \le y \le 1,\ y \le z \le 2-y\}.$$

(1) Find the mass of E with density $\rho(x,y,z) = y$.

<u>Sol</u> $m = \iiint_E \rho(x,y,z)\, dV$.

The outermost integral should have constant bounds.

$\Rightarrow$ dy on the outermost integral.

dz on the next integral.

($\because$ y is constant for the inner double integral)

dx on the innermost integral.

$$\Rightarrow m = \int_0^1 \int_y^{2-y} \int_0^{2-y-z} y \, dx\, dz\, dy$$

$$= \int_0^1 \int_y^{2-y} y(2-y-z)\, dz\, dy$$

$$= \int_0^1 y\left(2z - yz - \frac{z^2}{2}\right)\Big|_{z=y}^{z=2-y} dy$$

$$= \int_0^1 2y^3 - 4y^2 + 2y \, dy$$

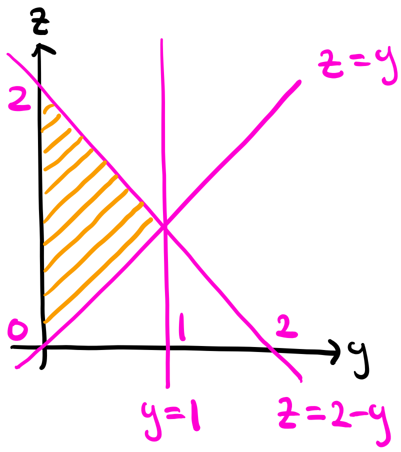
$$= \boxed{\frac{1}{6}}$$

(2) Sketch the solid E.

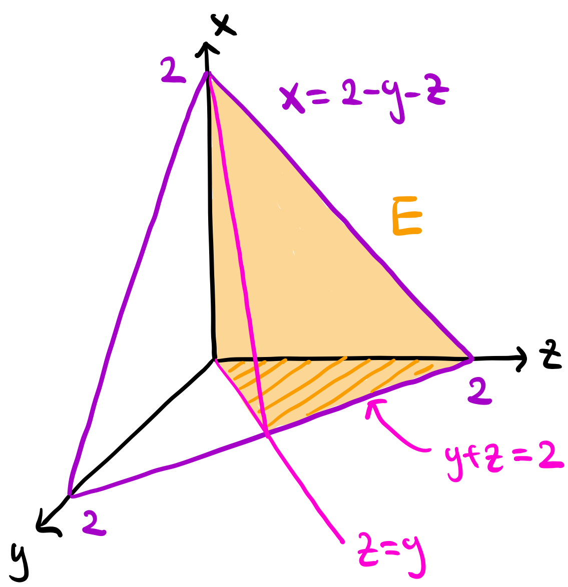
<u>Sol</u> Idea: Put the innermost variable on the vertical axis and sketch the domain for the other two variables.

Domain on the yz-plane is given by

$0 \le y \le 1$ and $y \le z \le 2-y$.



$0 \le x \le 2-y-z \Rightarrow E$ lies between the yz-plane and the surface $x = 2-y-z$.



$x = 2-y-z \rightsquigarrow x+y+z=2$

This is a plane with

x-intercept $= 2$

y-intercept $= 2$

z-intercept $= 2$