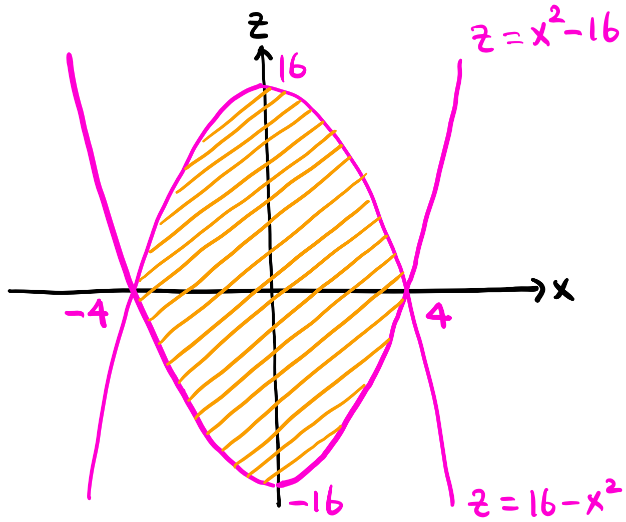
# Exam 2 Review - Winter 2019 exam

1. This problem has three parts.

(a) (2 points) Sketch the curves $z = 16 - x^2$ and $z = x^2 - 16$ in the $x$-$z$ plane. Determine the points of intersection of the two curves.




Intersections:

$z = x^2 - 16$ and $z = 16 - x^2$

$\Rightarrow x^2 - 16 = 16 - x^2 \Rightarrow x^2 = 16$

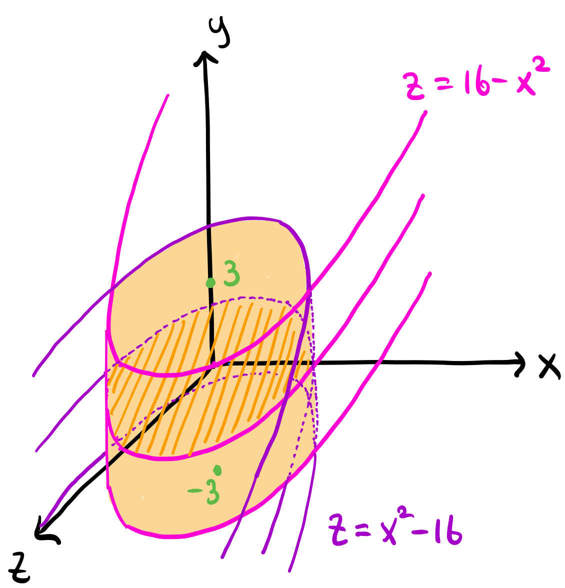
$\Rightarrow x = \pm 4,\ z = 0.$

(b) (5 points) Find the area bounded by the curves $z = 16 - x^2$ and $z = x^2 - 16$ in the $x$-$z$ plane.

The region is given by $-4 \le x \le 4,\ x^2 - 16 \le z \le 16 - x^2$

$$\text{Area} = \int_{-4}^{4} \int_{x^2-16}^{16-x^2} 1 \, dz \, dx = \int_{-4}^{4} 32 - 2x^2 \, dx = 32x - \frac{2}{3}x^3 \Big|_{x=-4}^{x=4} = \boxed{\frac{512}{3}}$$

(c) (3 points) Find the volume bounded by the surfaces $z = 16 - x^2$, $z = x^2 - 16$, $y = 3$, and $y = -3$.




The solid is a cylinder with height 6 and base area $\frac{512}{3}$

$\Rightarrow$ Volume $= \frac{512}{3} \cdot 6 = \boxed{1024}$

Note You can also compute the volume by a triple integral.

$$\text{Volume} = \int_{-3}^{3} \int_{-4}^{4} \int_{x^2-16}^{16-x^2} 1 \, dz \, dx \, dy$$

2. Let $f(x, y, z) = \frac{x^3+y^3+z^3}{3} - xyz$.

(a) (4 points) Find $\nabla f$ and evaluate it at $(x, y, z) = (1, 2, 3)$.

$$\nabla f = (f_x, f_y, f_z) = \boxed{(x^2 - yz,\ y^2 - xz,\ z^2 - xy)}$$

$$\nabla f\,(1, 2, 3) = \boxed{(-5, 1, 7)}$$

(b) (3 points) Find the unit vector $\mathbf{u}$ along which the directional derivative $D_{\mathbf{u}}f$ is maximum at $(x, y, z) = (1, 2, 3)$.

The direction of the maximum directional derivative is given by $\nabla f\,(1, 2, 3)$

$$\Rightarrow \vec{u} = \frac{\nabla f\,(1,2,3)}{|\nabla f(1,2,3)|} = \frac{(-5, 1, 7)}{\sqrt{(-5)^2 + 1^2 + 7^2}} = \boxed{\frac{1}{\sqrt{75}}(-5, 1, 7)}$$

(c) (3 points) Find a unit vector $\mathbf{u}$ such that the directional derivative $D_{\mathbf{u}}f$ is zero.

Set $\vec{u} = (a, b, c)$

$D_{\vec{u}}f\,(1, 2, 3) = \nabla f\,(1, 2, 3) \cdot \vec{u} = (-5, 1, 7) \cdot (a, b, c) = -5a + b + 7c$.

$D_{\vec{u}}f\,(1, 2, 3) = 0 \Rightarrow -5a + b + 7c = 0$.

Take $\vec{u}$ to be the unit vector of $(1, -2, 1)$

$$\Rightarrow \vec{u} = \frac{(1, -2, 1)}{\sqrt{1^2 + (-2)^2 + 1^2}} = \boxed{\frac{1}{\sqrt{6}}(1, -2, 1)}$$

<u>Note</u> You can take $\vec{u}$ to be any unit vector with

$-5a + b + 7c = 0$

e.g. $\vec{u} = \frac{1}{\sqrt{26}}(1, 5, 0)$ or $\frac{1}{5\sqrt{2}}(0, -7, 1)$.

3. Evaluate the triple integral $\int\int\int_B xyz\, dV$ for the following two choices of $B$:

(a) (5 points) $B$ is the cube $0 \le x \le 1$, $0 \le y \le 1$, $0 \le z \le 1$.

$$\iiint_B xyz\,dV = \int_0^1\int_0^1\int_0^1 xyz\,dz\,dy\,dx = \int_0^1\int_0^1 \frac{xyz^2}{2}\Big|_{z=0}^{z=1} dy\,dx$$

$$= \int_0^1\int_0^1 \frac{xy}{2}\,dy\,dx = \int_0^1 \frac{xy^2}{4}\Big|_{y=0}^{y=1} dx$$

$$= \int_0^1 \frac{x}{4}\,dx = \frac{x^2}{8}\Big|_{x=0}^{x=1} = \boxed{\frac{1}{8}}$$

<u>Note</u> You can also compute the integral as follows:

$$\iiint_B xyz\,dV = \int_0^1\int_0^1\int_0^1 xyz\,dz\,dy\,dx = \left(\int_0^1 x\,dx\right)\left(\int_0^1 y\,dy\right)\left(\int_0^1 z\,dz\right)$$

$$= \frac{x^2}{2}\Big|_{x=0}^{x=1}\cdot\frac{y^2}{2}\Big|_{y=0}^{y=1}\cdot\frac{z^2}{2}\Big|_{z=0}^{z=1} = \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2} = \frac{1}{8}$$

(b) (5 points) $B$ is the region $0 \le x \le y \le z \le 1$.

The bounds are $0 \le z \le 1$, $0 \le y \le z$, $0 \le x \le y$.

$$\iiint_B xyz\,dV = \int_0^1\int_0^z\int_0^y xyz\,dx\,dy\,dz = \int_0^1\int_0^z \frac{x^2yz}{2}\Big|_{x=0}^{x=y} dy\,dz$$

$$= \int_0^1\int_0^z \frac{y^3z}{2}\,dy\,dz = \int_0^1 \frac{y^4z}{8}\Big|_{y=0}^{y=z} dz$$

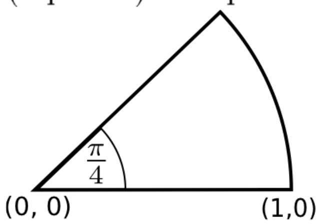
$$= \int_0^1 \frac{z^5}{8}\,dz = \frac{z^6}{48}\Big|_{z=0}^{z=1} = \boxed{\frac{1}{48}}$$

<u>Note</u> You can also write the integral as follows:

$$\iiint_B xyz\,dV = \int_0^1\int_0^y\int_y^1 xyz\,dy\,dx\,dz = \int_0^1\int_x^1\int_y^1 xyz\,dz\,dy\,dx.$$

4. In both parts, the density is assumed to be $\rho(x, y) = x$.

(a) (5 points) The plot below



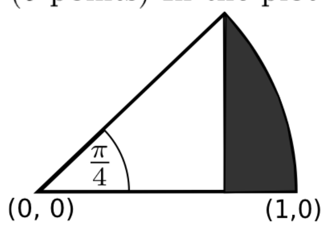

shows the sector of a circle of radius 1. The angle at the center is $\frac{\pi}{4}$ or $45^o$ as indicated. Find the mass of the sector.

In polar coordinates, the domain D is given by $0 \le \theta \le \frac{\pi}{4}$, $0 \le r \le 1$.

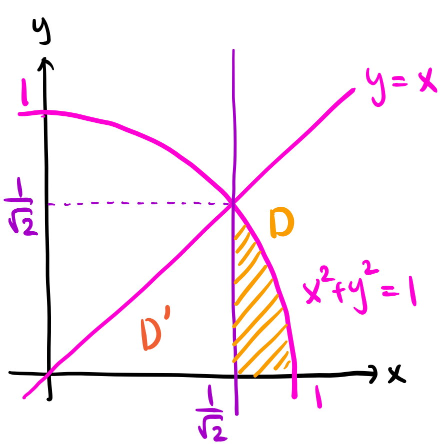
$$m_D = \iint_D \rho(x,y)\, dA = \iint_D x\, dA = \int_0^{\pi/4}\int_0^1 r\cos\theta \cdot \underbrace{r}_{\text{Jacobian}}\, dr\, d\theta$$

$$= \int_0^{\pi/4} \frac{r^3}{3}\cos\theta \Big|_{r=0}^{r=1} d\theta = \int_0^{\pi/4} \frac{1}{3}\cos\theta\, d\theta = \frac{1}{3}\sin\theta \Big|_{\theta=0}^{\theta=\pi/4} = \boxed{\frac{\sqrt{2}}{6}}$$

(b) (5 points) In the plot below




a part of the sector is shaded. Find the mass of the shaded region.




Intersection : $x^2+y^2=1$ and $y=x$

$\Rightarrow x^2+x^2=1 \Rightarrow x=y=\frac{1}{\sqrt{2}}$.

D : the shaded region

D' : the complement of D in the sector

D' is given by $0 \le x \le \frac{1}{\sqrt{2}}$, $0 \le y \le x$

$$\Rightarrow m_{D'} = \iint_{D'} \rho(x,y)\, dA = \int_0^{1/\sqrt{2}}\int_0^x x\, dy\, dx = \int_0^{1/\sqrt{2}} x^2\, dx = \frac{x^3}{3}\Big|_{x=0}^{x=1/\sqrt{2}} = \frac{\sqrt{2}}{12}$$

$$m_D + m_{D'} \underset{(a)}{=} \frac{\sqrt{2}}{6} \Rightarrow m_D = \frac{\sqrt{2}}{6} - \frac{\sqrt{2}}{12} = \boxed{\frac{\sqrt{2}}{12}}$$

<u>Note</u> In polar coordinates, D is given by $0 \le \theta \le \frac{\pi}{4}$, $\frac{1}{\sqrt{2}\cos\theta} \le r \le 1$

5. In all the three parts, $x$ and $y$ are assumed to be positive with $xy = 4$.

(a) (3 points) Find the point $(x, y)$, with $x, y > 0$, that is an extremum of $f(x, y) = \frac{x^2+y^2}{2}$ subject to $g(x, y) = xy = 4$.

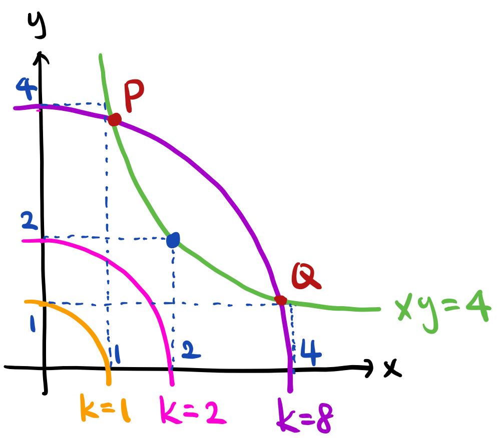
Solve $\nabla f = \lambda \nabla g$ and $g = 4$.

$\Rightarrow (x, y) = \lambda (y, x)$ and $xy = 4$

$$\Rightarrow \begin{cases} x = \lambda y \rightsquigarrow x^2 = \lambda xy \\ y = \lambda x \rightsquigarrow y^2 = \lambda xy \end{cases}$$

$$\Rightarrow x^2 = y^2 \underset{x,y>0}{\Rightarrow} x = y \underset{xy=4}{\Rightarrow} (x, y) = \boxed{(2,2)}$$

(b) (3 points) Sketch the curve $g(x, y) = xy = 4$ as well as the level curves of $f(x, y)$ with $f(x, y)$ being $\frac{1}{2}, 2, 8$, respectively, in the first quadrant of the $x$-$y$ plane. Is the extremum of part (a) a minimum or a maximum? Explain.




Level $k$: $f(x,y) = k \Rightarrow \frac{x^2+y^2}{2} = k \Rightarrow x^2+y^2 = 2k$

$$\begin{cases} k = \frac{1}{2}: \ x^2+y^2 = 1 \\ k = 2: \ x^2+y^2 = 4 \\ k = 8: \ x^2+y^2 = 16. \end{cases}$$

The curve $xy = 4$ intersects the level curve at 8.

$f(2,2) = 4 < 8$.

$\Rightarrow (2,2)$ cannot be a maximum on the curve $xy = 4$.

$\Rightarrow (2,2)$ must be $\boxed{\text{a minimum}}$

(c) (4 points) Find the minimum value of $\frac{x^3}{3} + \frac{2y^{3/2}}{3}$ subject to $xy = 4$ and $x, y$ positive.

Set $f(x,y) = \dfrac{x^3}{3} + \dfrac{2y^{3/2}}{3}$ and $g(x,y) = xy - 4$.

Solve $\nabla f = \lambda \nabla g$ and $g = 0$.

$\Rightarrow (x^2, y^{1/2}) = \lambda (y, x)$ and $xy - 4 = 0$

$$\Rightarrow \begin{cases} x^2 = \lambda y \rightsquigarrow x^3 = \lambda xy \\ y^{1/2} = \lambda x \rightsquigarrow y^{3/2} = \lambda xy \end{cases}$$

$\Rightarrow x^3 = y^{3/2} \Rightarrow x = y^{1/2}$

$xy - 4 = 0 \Rightarrow y^{1/2} \cdot y - 4 = 0 \Rightarrow y^{3/2} = 4 \Rightarrow y = 4^{2/3}, \ x = 4^{1/3}.$

The minimum value is $f(4^{1/3}, 4^{2/3}) = \boxed{4}$

<u>Note</u> You can also solve this by removing the constraint.

$xy = 4 \Rightarrow y = \dfrac{4}{x}.$

$$\frac{x^3}{3} + \frac{2y^{3/2}}{3} = \frac{x^3}{3} + \frac{2}{3}\left(\frac{4}{x}\right)^{3/2} = \frac{x^3}{3} + \frac{16}{3x^{3/2}} = \frac{x^3}{3} + \frac{16}{3}x^{-3/2}$$

The minimum occurs at the critical point.

$$\frac{d}{dx}\left(\frac{x^3}{3} + \frac{16}{3}x^{-3/2}\right) = x^2 - 8x^{-5/2}$$

$$x^2 - 8x^{-5/2} = 0 \Rightarrow x^{9/2} = 8 \Rightarrow x = 4^{1/3} \Rightarrow \frac{x^3}{3} + \frac{16}{3}x^{-3/2} = 4$$

The minimum value is 4.

6. Consider the surface given by $x^2 + y^2 + 3z^2 = 3$ and the plane $x + y + z = 1$.

(a) (3 points) Explain why the plane intersects the surface.

The point $(0,0,1)$ lies on the surface $x^2+y^2+3z^2=1$ and the plane $x+y+z=1$.

$\Rightarrow$ $\boxed{\text{The surface and the plane intersect.}}$

(b) (7 points) Suppose the curve along which the plane intersects the surface is $C$. Find the maximum and minimum values of $z$ for points on $C$.

The constraints are $x^2+y^2+3z^2=1$ and $x+y+z=1$.

$x+y+z=1 \rightsquigarrow x=1-y-z$.

$x^2+y^2+3z^2=1 \rightsquigarrow (1-y-z)^2+y^2+3z^2=1$.

We find the extreme values of $z$ subject to the constraint $(1-y-z)^2+y^2+3z^2=1$.

Set $f(y,z)=z$ and $g(y,z)=(1-y-z)^2+y^2+3z^2-1$.

$\nabla f = \lambda \nabla g \Rightarrow (0,1) = \lambda(-2(1-y-z)+2y, -2(1-y-z)+6z)$

$\Rightarrow 0 = \lambda(4y+2z-2),\ 1=\lambda(2y+8z-2)$.

$\lambda \neq 0$ by the second equation

$\Rightarrow 0 = 4y+2z-2 \Rightarrow z = 1-2y$.

$(1-y-z)^2+y^2+3z^2=1 \Rightarrow (1-y-1+2y)^2+y^2+3(1-2y)^2=1$

$\Rightarrow 14y^2-12y=0 \Rightarrow y=0, \frac{6}{7}$

$\Rightarrow (y,z)=(0,1), (\frac{6}{7}, -\frac{5}{7})$

$\Rightarrow$ $\boxed{\text{Maximum} = 1,\ \text{minimum} = -\frac{5}{7}}$

7. The surface $x^2 + y^2 + z^2 = 1$ is the sphere, and as you know, the area of the upper hemisphere with $z > 0$ is $2\pi$. In this problem, you will find areas of portions of that surface and your answers should therefore be less than $2\pi$.

(a) (4 points) Find the area of the upper hemisphere inside the cylindrical surface $x^2 + y^2 = \frac{1}{4}$.

The domain D is given by $x^2+y^2 \le \frac{1}{4}$.

In polar coordinates, D is given by $0 \le \theta \le 2\pi$, $0 \le r \le \frac{1}{2}$.

$x^2+y^2+z^2 = 1$ and $z > 0 \Rightarrow z = \sqrt{1-x^2-y^2}$

$$\text{Area} = \iint_D \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA.$$

$$\frac{\partial z}{\partial x} = -\frac{x}{\sqrt{1-x^2-y^2}}, \quad \frac{\partial z}{\partial y} = -\frac{y}{\sqrt{1-x^2-y^2}}$$

$$\Rightarrow \text{Area} = \iint_D \sqrt{1+\frac{x^2}{1-x^2-y^2}+\frac{y^2}{1-x^2-y^2}} = \iint_D \frac{1}{\sqrt{1-x^2-y^2}}\, dA.$$

$$= \int_0^{2\pi}\int_0^{1/2} \frac{\overset{\text{Jacobian}}{r}}{\sqrt{1-r^2}}\, dr\, d\theta \underset{u=1-r^2}{=} \int_0^{2\pi}\int_1^{3/4} u^{-1/2}\cdot\left(-\frac{1}{2}\right) du\, d\theta$$

$$= \int_0^{2\pi} -u^{1/2}\Big|_{u=1}^{u=3/4} d\theta = \int_0^{2\pi} 1-\frac{\sqrt{3}}{2}\, d\theta = \boxed{\pi(2-\sqrt{3})}$$

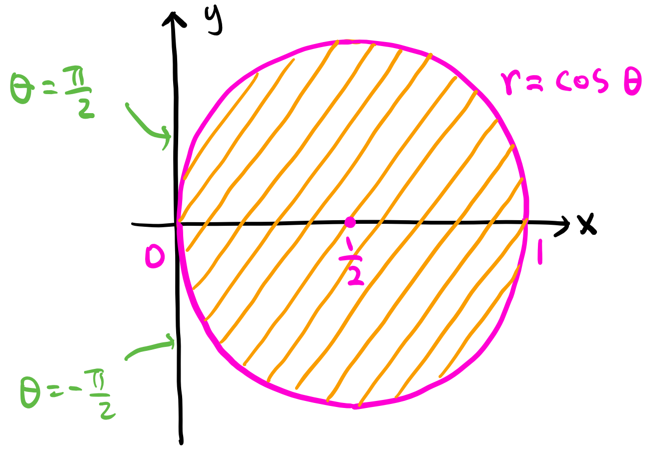
Note You can find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ using implicit differentiation.

The sphere is a level surface of $f(x,y,z) = x^2+y^2+z^2$.

$$\Rightarrow \frac{\partial z}{\partial x} = -\frac{f_x}{f_z} = -\frac{x}{z}. \quad \frac{\partial z}{\partial y} = -\frac{f_y}{f_z} = -\frac{y}{z}.$$

(b) (6 points) Find the area of the upper hemisphere inside the cylindrical surface $\left(x-\frac{1}{2}\right)^2+y^2=\frac{1}{4}$.

The domain $D$ is given by $(x-\frac{1}{2})^2+y^2 \le \frac{1}{4}$.




In polar coordinates,

$(x-\frac{1}{2})^2+y^2=\frac{1}{4} \rightsquigarrow x^2-x+y^2=0$

$\rightsquigarrow x = x^2+y^2 \rightsquigarrow r\cos\theta = r^2$

$\rightsquigarrow r=\cos\theta$

$D$ is given in polar coordinates by $-\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$, $0\le r\le\cos\theta$.

$$\Rightarrow \text{Area} = \iint_D \sqrt{1+\frac{x^2}{1-x^2-y^2}+\frac{y^2}{1-x^2-y^2}} = \iint_D \frac{1}{\sqrt{1-x^2-y^2}}\,dA.$$

Jacobian

$$= \int_{-\pi/2}^{\pi/2}\int_0^{\cos\theta} \frac{r}{\sqrt{1-r^2}}\,dr\,d\theta = \int_{-\pi/2}^{\pi/2}\int_1^{\sin^2\theta} u^{-1/2}\left(-\frac{1}{2}\right)du\,d\theta$$

$$= \int_{-\pi/2}^{\pi/2} -u^{1/2}\Big|_{u=1}^{u=\sin^2\theta}\,d\theta = \int_{-\pi/2}^{\pi/2} 1-|\sin\theta|\,d\theta$$

$$= 2\int_0^{\pi/2} 1-\sin\theta\,d\theta = 2(\theta+\cos\theta)\Big|_{\theta=0}^{\theta=\pi/2} = \boxed{2\left(\frac{\pi}{2}-1\right)}$$

$$= \int_{-\pi/2}^{\pi/2} \int_{1}^{\sin^2\theta} u^{-1/2} \cdot \left(-\frac{1}{2}\right) du\, d\theta$$

$$= \int_{-\pi/2}^{\pi/2} -u^{1/2} \Big|_{u=1}^{u=\sin^2\theta} d\theta$$

$$= \int_{-\pi/2}^{\pi/2} 1 - |\sin\theta|\, d\theta$$

$$= 2\int_{0}^{\pi/2} 1 - \sin\theta\, d\theta$$

$$= 2\left(\theta + \cos\theta\right)\Big|_{\theta=0}^{\theta=\pi/2}$$

$$= \boxed{2\left(\frac{\pi}{2} - 1\right)}$$