

12.5. Continued

Prop Two planes are parallel if and only if their normal vectors are parallel

e.g. The planes $x+y+z=3$ and $2x+2y+2z=7$ are parallel

$$\left(\begin{array}{l} \text{normal vectors: } \vec{n}_1 = (1, 1, 1), \vec{n}_2 = (2, 2, 2) \\ \leadsto \vec{n}_1 \text{ and } \vec{n}_2 \text{ are parallel} \end{array} \right)$$

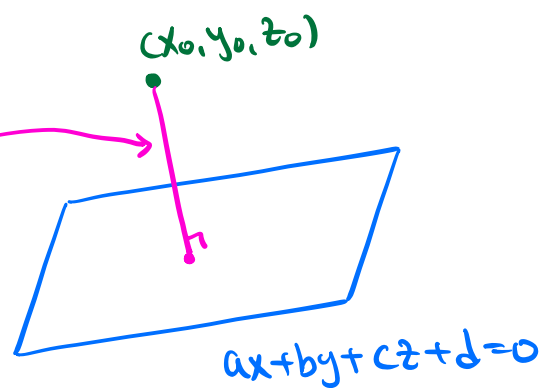
The planes $x-y+z=2$ and $3x-z=10$ are not parallel.

$$\left(\begin{array}{l} \text{normal vectors: } \vec{n}_1 = (1, -1, 1), \vec{n}_2 = (3, 0, -1) \\ \leadsto \vec{n}_1 \text{ and } \vec{n}_2 \text{ are not parallel} \end{array} \right)$$

Thm The distance from a point (x_0, y_0, z_0) to the plane $ax+by+cz+d=0$ is equal to

$$\frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

On the formula sheet.



Ex Find the two planes which are parallel to the plane $3x - 2y + 6z = 0$ and 2 units away from it.

Sol The planes have a normal vector $\vec{n} = (3, -2, 6)$

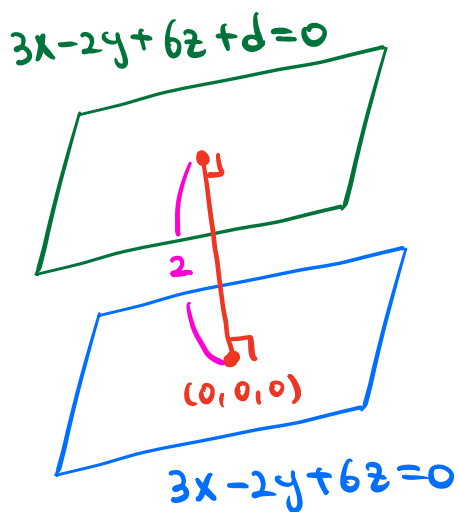
\Rightarrow The plane equation must be

$$3x - 2y + 6z + d = 0 \text{ for some } d.$$

* To find the constant d , we choose a point on the given plane and consider the distance to the other plane.

$$\text{Set } x=0, y=0 : 3x - 2y + 6z = 0 \leadsto 6z = 0 \leadsto z = 0.$$

$\Rightarrow (0, 0, 0)$ is on the plane $3x - 2y + 6z = 0$.



Distance from $(0, 0, 0)$ to the plane $3x - 2y + 6z + d = 0$ is 2.

$$\Rightarrow \frac{|3 \cdot 0 - 2 \cdot 0 + 6 \cdot 0 + d|}{\sqrt{3^2 + (-2)^2 + 6^2}} = 2.$$

$$\leadsto \frac{|d|}{7} = 2 \leadsto |d| = 14 \leadsto d = \pm 14.$$

\Rightarrow The planes are

$$3x - 2y + 6z \pm 14 = 0$$

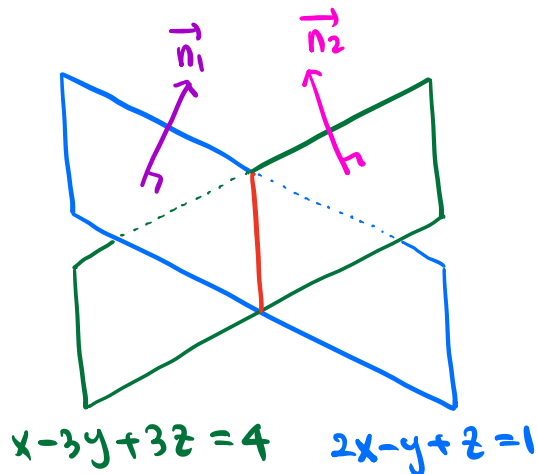
Ex Describe the intersection of the planes

$$2x - y + z = 1 \quad \text{and} \quad x - 3y + 3z = 4.$$

Sol Normal vectors are $\vec{n}_1 = (2, -1, 1)$, $\vec{n}_2 = (1, -3, 3)$

\Rightarrow The planes are not parallel.

\Rightarrow The intersection is a line.



The line is perpendicular to both \vec{n}_1 and \vec{n}_2 .

\Rightarrow A direction vector is

$$\begin{aligned}\vec{n}_1 \times \vec{n}_2 &= (2, -1, 1) \times (1, 3, -3) \\ &= (0, 7, 7)\end{aligned}$$

To find a point on the line, we solve the plane equations together by setting one of x, y, z to be 0.

* If the direction vector has a zero component, you should not set the corresponding variable to be 0.

$$z = 0 \Rightarrow 2x - y = 1 \quad \text{and} \quad x - 3y = 4 \quad \leadsto x = 1, y = 1.$$

$\Rightarrow (1, 1, 0)$ is on the line.

\leadsto The intersection is $\vec{r}(t) = (1 + 0 \cdot t, 1 + 7t, 0 + 7t)$

Note On this line, the x -coordinate is always 1.

So you cannot find a point by setting $x = 0$.

12.6. Cylinders and quadric surfaces

Def (1) A quadric surface is a surface given by a degree 2 equation.

e.g. $x^2 + y^2 + z^2 = 1 \rightsquigarrow$ a sphere.

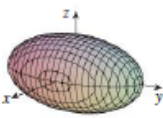
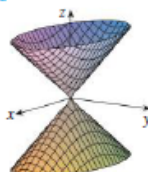
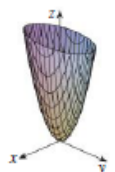
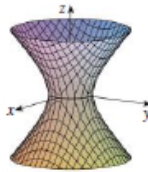
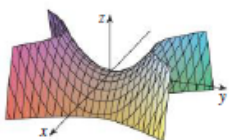
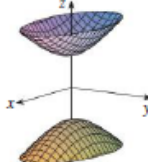
$x^2 + y^2 = 4 \rightsquigarrow$ a circular cylinder.

$z = x^2 + y^2 \rightsquigarrow$ a circular paraboloid.

(2) A cylinder is a surface made of parallel identical cross sections.

e.g. circular cylinders, prisms, ...

* On the textbook, you can find a table of quadric surfaces with their equations and graphs.

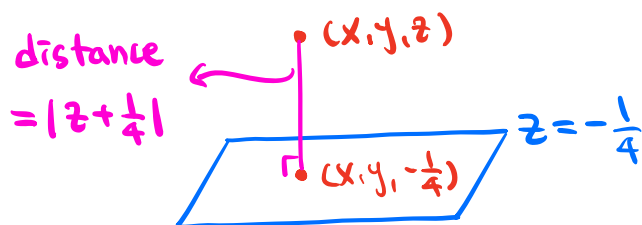
Surface	Equation	Surface	Equation
<p>Ellipsoid</p> 	$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ <p>All traces are ellipses. If $a = b = c$, the ellipsoid is a sphere.</p>	<p>Cone</p> 	$\frac{z^2}{c^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ <p>Horizontal traces are ellipses. Vertical traces in the planes $x = k$ and $y = k$ are hyperbolas if $k \neq 0$ but are pairs of lines if $k = 0$.</p>
<p>Elliptic Paraboloid</p> 	$\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ <p>Horizontal traces are ellipses. Vertical traces are parabolas. The variable raised to the first power indicates the axis of the paraboloid.</p>	<p>Hyperboloid of One Sheet</p> 	$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ <p>Horizontal traces are ellipses. Vertical traces are hyperbolas. The axis of symmetry corresponds to the variable whose coefficient is negative.</p>
<p>Hyperbolic Paraboloid</p> 	$\frac{z}{c} = \frac{x^2}{a^2} - \frac{y^2}{b^2}$ <p>Horizontal traces are hyperbolas. Vertical traces are parabolas. The case where $c < 0$ is illustrated.</p>	<p>Hyperboloid of Two Sheets</p> 	$-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ <p>Horizontal traces in $z = k$ are ellipses if $k > c$ or $k < -c$. Vertical traces are hyperbolas. The two minus signs indicate two sheets.</p>

Ex Describe each set using an equation and a graph.

(1) The set of all points which are equidistant from the point $(0, 0, \frac{1}{4})$ and the plane $z = -\frac{1}{4}$.

Sol Distance from $(0, 0, \frac{1}{4})$ is $\sqrt{x^2 + y^2 + (z - \frac{1}{4})^2}$

Distance from the plane $z = -\frac{1}{4}$ is $|z + \frac{1}{4}|$.

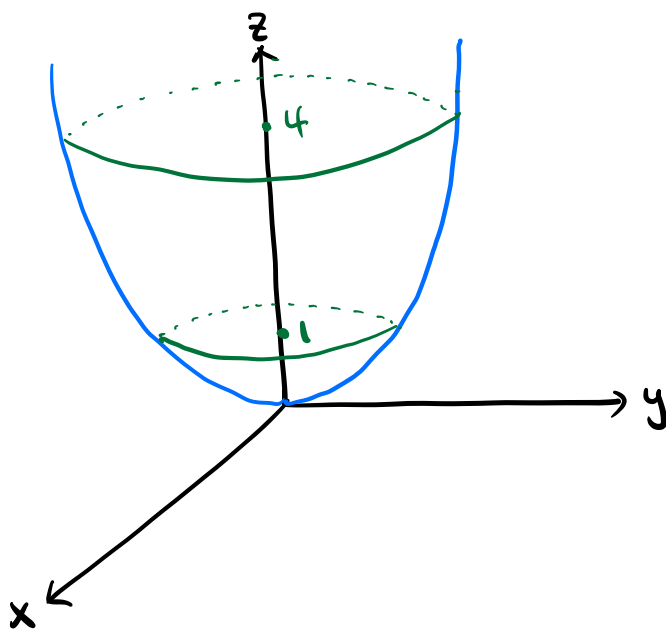


$$\Rightarrow \sqrt{x^2 + y^2 + (z - \frac{1}{4})^2} = |z + \frac{1}{4}|$$

$$\leadsto x^2 + y^2 + (z - \frac{1}{4})^2 = (z + \frac{1}{4})^2$$

$$\leadsto x^2 + y^2 + \cancel{z^2} - \frac{1}{2}z + \frac{1}{16} = \cancel{z^2} + \frac{1}{2}z + \frac{1}{16}$$

$$\leadsto \boxed{z = x^2 + y^2} : \text{a circular paraboloid}$$



$$x=0 \Rightarrow z = y^2$$

\leadsto a parabola

$$z=1 \Rightarrow 1 = x^2 + y^2$$

\leadsto a circle

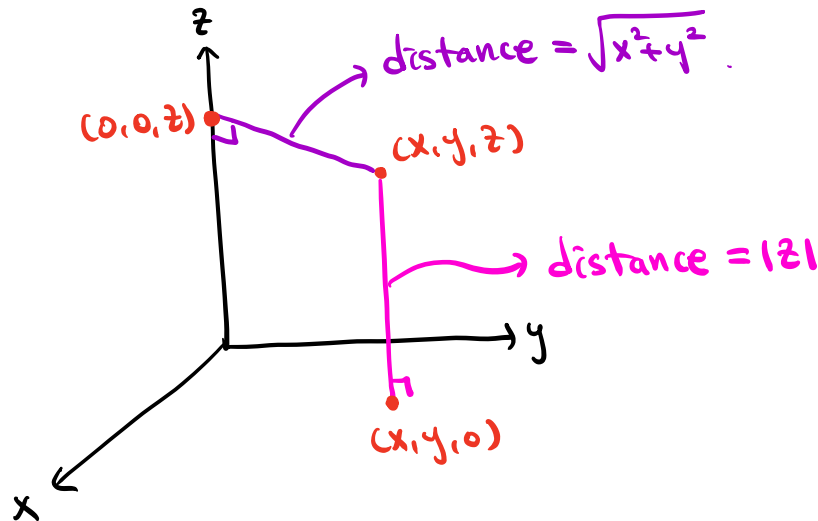
$$z=4 \Rightarrow 4 = x^2 + y^2$$

\leadsto a circle

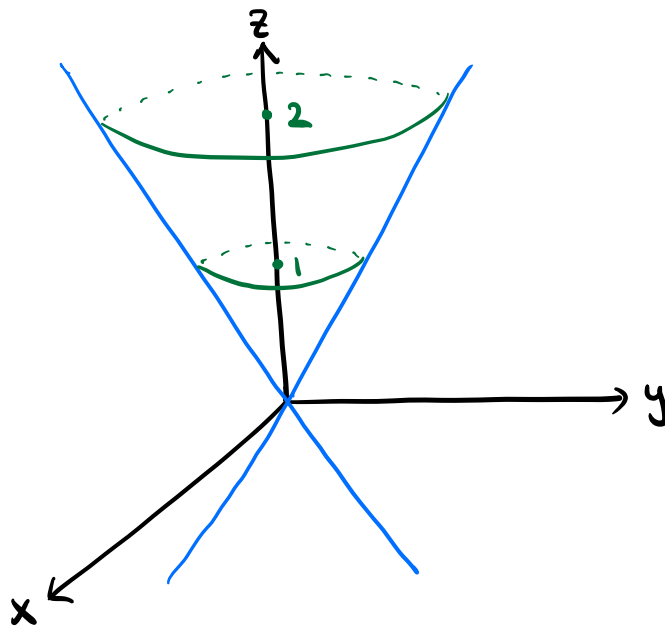
(2) The set of all points which are equidistant from the xy -plane and the z -axis.

Sol Distance from the xy -plane is $|z|$.

Distance from the z -axis is $\sqrt{x^2 + y^2}$.



$$\Rightarrow |z| = \sqrt{x^2 + y^2} \leadsto \boxed{z^2 = x^2 + y^2} : \text{a circular cone.}$$



$$x=0 \Rightarrow z^2 = y^2 \Rightarrow z = \pm y$$

\leadsto two lines

$$z=1 \Rightarrow 1 = x^2 + y^2$$

\leadsto a circle

$$z=2 \Rightarrow 4 = x^2 + y^2$$

\leadsto a circle