4) To look briefly at the <u>fields</u> in this representation, write...

$$\phi(\mathbf{r}) = \sum_{l,m} \phi_{lm}(\mathbf{r}) \;, \quad \phi_{lm}(\mathbf{r}) = \left(\frac{4\pi}{2l+1}\right) \frac{q_{lm}}{r^{l+1}}\, Y_{lm}(\theta,\varphi) \;;$$

so// $$\mathbb{E}(\mathbf{r}) = \sum_{l,m} \mathbb{E}_{lm}(\mathbf{r}) \;, \quad \mathbb{E}_{lm} = -\nabla\phi_{lm} \;; \tag{13}$$

w/
$$\left.\begin{cases} (E_{lm})_r = -\dfrac{\partial}{\partial r}\phi_{lm} = +\left(\dfrac{4\pi}{2l+1}\right)\dfrac{q_{lm}}{r^{l+2}}\left[(l+1)Y_{lm}(\theta,\varphi)\right], \\ (E_{lm})_\theta = -\dfrac{1}{r}\dfrac{\partial}{\partial\theta}\phi_{lm} = -\left(\dfrac{4\pi}{2l+1}\right)\dfrac{q_{lm}}{r^{l+2}}\left[\dfrac{\partial}{\partial\theta}Y_{lm}(\theta,\varphi)\right], \\ (E_{lm})_\varphi = -\dfrac{1}{r\sin\theta}\dfrac{\partial}{\partial\varphi}\phi_{lm} = -\left(\dfrac{4\pi}{2l+1}\right)\dfrac{q_{lm}}{r^{l+2}}\left[\dfrac{im}{\sin\theta}Y_{lm}(\theta,\varphi)\right]. \end{cases}\right\} \tag{14}$$

<u>NOTE</u>: all the $\mathbb{E}_{lm}$ components fall off as $1/r^{l+2}$ [$E(\text{monopole}) \sim 1/r^2$, $E(\text{dipole}) \sim 1/r^3$, $E(\text{q'pole}) \sim 1/r^4$, etc.], but they do <u>not</u> share the same ∡lar dependence. E.g.

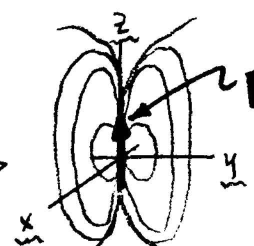
$$(E_{lm})_\theta \;\propto\; a_{lm}\, e^{-i\varphi}\, Y_{l,m+1} - b_{lm}\, e^{i\varphi}\, Y_{l,m-1} \,. \tag{15}$$

monopole field is $E = q\,\dfrac{\mathbf{r}}{r^3}$

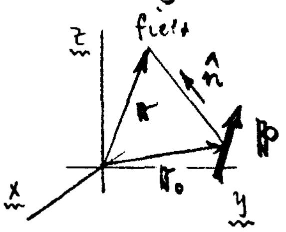
As a specific example, the <u>$l=1$ dipole field</u> is...

$$\left.\begin{cases} E_r = \dfrac{2p}{r^3}\cos\theta \\ E_\theta = \dfrac{p}{r^3}\sin\theta \\ E_\varphi = 0 \end{cases}\right\} \quad \mathbf{p} = (0,0,p) \text{ along } z\text{-axis only} \tag{16}$$

(no φ-dependence, by symmetry)



More generally...

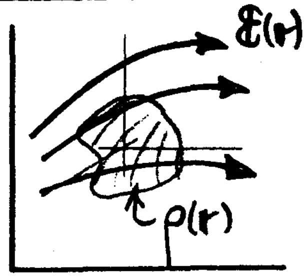


$$\boxed{\mathbb{E}(\mathbf{r}) = \frac{1}{R^3}\left[3\hat{n}(\hat{n}\cdot\mathbf{p}) - \mathbf{p}\right]} \tag{17}$$

w/ $\mathbb{R} = \mathbf{r} - \mathbf{r}_0$ & $\hat{n} = \hat{R}$

On RHS of $\mathbb{E}(\text{dipole})$, can (should) put $(-)\frac{4\pi}{3}\delta(\mathbb{R})$, to take care of singularity @ $\mathbf{r} = \mathbf{r}_0$. See Jk^n Eq (4.20).

5) A major place where the multipole expansion of $\phi(\mathbf{r})$ is actually used is in atomic/nuclear physics, where the charge distribution $\rho(\mathbf{r})$ is that of the atom/nucleus, and $\rho(\mathbf{r})$ is acted on by an external field $\mathcal{E}(\mathbf{r})$. Of interest is how the atom/nucleus energy changes in the presence of $\mathcal{E}$. Evidently, this goes as:



NOTE: factor is 1, not 1/2.

$$\rightarrow W(\mathcal{E}) = \frac{1}{1}\int \rho(\mathbf{r})\,\varphi(\mathbf{r})\,d^3x \quad \varphi = \text{ptl. due to external field acting on distrib}^{n}\ \rho. \qquad (18)$$

Assume $\varphi(\mathbf{r})$ does not vary rapidly over dim^{n}s of $\rho(\mathbf{r})$ [obviously true for external $\mathcal{E}$ applied to an atom/nucleus]. Then, via Taylor...

$$\varphi(\mathbf{r}) = \varphi(0) + (\mathbf{r}\cdot\nabla)\varphi(0) + \frac{1}{2}\sum_{i,j=1}^{3} x_i x_j (\partial^2\varphi/\partial x_i\,\partial x_j)_0 + \cdots \qquad (19)$$

... use $\nabla\varphi = -\mathcal{E}$, external field ...

$$\xrightarrow{\text{so}} \varphi(\mathbf{r}) = \varphi(0) - \mathbf{r}\cdot\mathcal{E}(0) - \frac{1}{2}\sum_{i,j} x_i x_j (\partial\mathcal{E}_j/\partial x_i)_0 + \cdots \qquad (20)$$

Here, $\mathbf{r}$ ranges over the (small) size of the atomic/nuclear $\rho(\mathbf{r})$, while $\mathcal{E}(0)$ & $(\partial\mathcal{E}_j/\partial x)_0$ are evaluated at its center. To make contact with the multipole expansion we've used, we note...

$\nabla\cdot\mathcal{E} = 0$, for external field $\Rightarrow$ add $\frac{1}{6}r^2\,\nabla\cdot\mathcal{E}(0)$ to (20)...

$$\rightarrow \varphi(\mathbf{r}) = \varphi(0) - \mathbf{r}\cdot\mathcal{E}(0) - \frac{1}{6}\sum_{i,j}(3x_i x_j - r^2\delta_{ij})(\partial\mathcal{E}_j/\partial x_i)_0 \qquad (21)$$

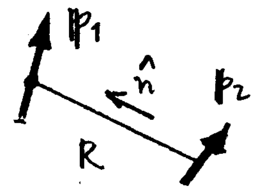
The leading terms in the interaction energy for chg. distrib^n ρ in ext^l fld $\mathcal{E}$:

$$\boxed{W = \underbrace{q\,\varphi(0)}_{\text{monopole (Coulomb)}} - \underbrace{\mathbf{p}\cdot\mathcal{E}(0)}_{\text{dipole (Stark term)}} - \underbrace{\frac{1}{6}\sum_{i,j} Q_{ij}(\partial\mathcal{E}_j/\partial x_i)_0}_{\text{quadrupole coupling}} + \cdots} \qquad (22)$$

Multipoles & Dielectrics (cont'd)

Eq.(22) can be used <u>within</u> the atomic/nuclear $\rho(\mathbf{r})$. For example...

$$W(\text{dipole } \mathbf{p}_1) = -\mathbf{p}_1 \cdot \mathcal{E}(0);$$



... let $\mathcal{E}(0)$ be due to an (internal) dipole $\mathbf{p}_2$...

i.e., $\mathcal{E}(0) = \frac{1}{R^3}\left[3\hat{n}(\hat{n}\cdot\mathbf{p}_2) - \mathbf{p}_2\right]$,

So,

$$\boxed{W\left(\begin{matrix}\text{dipole-dipole}\\ \text{interaction}\end{matrix}\right) = \frac{1}{R^3}\left[\mathbf{p}_1\cdot\mathbf{p}_2 - 3(\hat{n}\cdot\mathbf{p}_1)(\hat{n}\cdot\mathbf{p}_2)\right].} \tag{23}$$

In its magnetic form ($\mathbf{p} \to$ magnetic dipole), this form of interaction energy W governs all the fine & hyperfine structure terms in atoms.