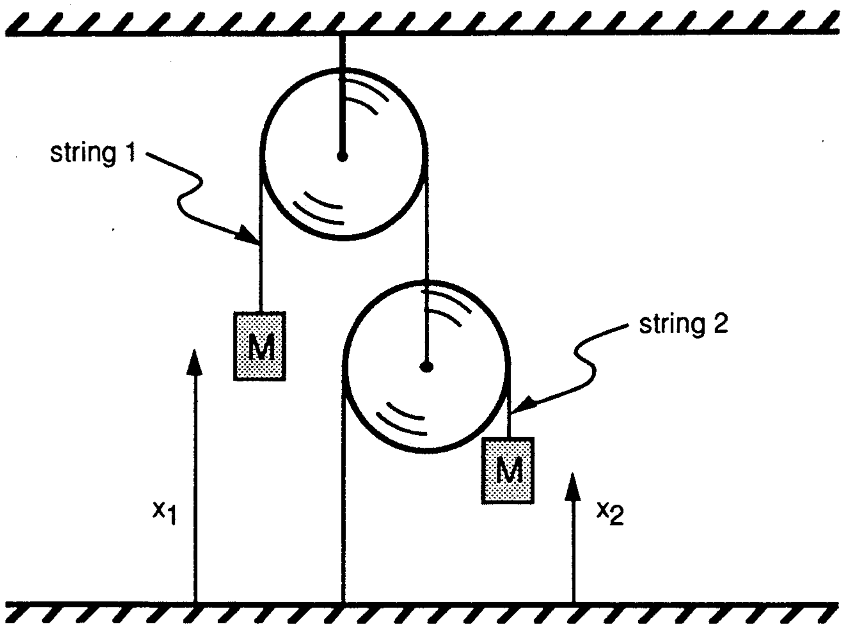
DEPARTMENT OF PHYSICS

M.S./PH.D. QUALIFYING/COMPREHENSIVE EXAMINATION

June 1993

Answer each of the following questions, all of which carry equal weight. Begin your answer to each question on a new sheet of paper; solutions to different questions must **not** appear on the same sheet. Each sheet of paper must be labeled with your name and the problem number in the upper right hand corner. When more than one sheet is submitted for a problem, be sure the pages are ordered properly.

1. A pulley system is set up as shown below with equal masses on the ends of the strings. The pulleys (of course) are massless and frictionless, and the strings are massless and inextensible.



A. Find the Lagrangian for the two-mass system.

B. Find the equation of constraint between x_1 and x_2.

C. Find the equation of motion for each mass.

D. If M = 2 kg, find the tension in string 1 and the tension in string 2.

E. The system starts at rest at t = 0. Find the displacement Δx_2 after 3 seconds have elapsed. (You may assume that x_2 remains positive during this time.)

I

A. $L = T - V$

$$= \frac{1}{2}M\dot{x}_1^2 + \frac{1}{2}M\dot{x}_2^2 - Mgx_1 - Mgx_2$$

B. $dx_2 = -2dx_1$

$x_2 = -2x_1 + C$, where C is a constant.

C. Using the equation of constraint, the Lagrangian becomes:

$$L = \frac{5}{2}M\dot{x}_1^2 + Mgx_1 - MgC$$

The equation of motion for mass 1 is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) - \frac{\partial L}{\partial x_1} = 0$$

$$\ddot{x}_1 = \frac{1}{5}g$$

From the equation of constraint,

$$\ddot{x}_2 = -2\ddot{x}_1 = -\frac{2}{5}g$$

D. String 1:

$$T_1 - Mg = +M\left(\frac{1}{5}g\right)$$

$$T_1 = \frac{6}{5}Mg = \underline{23.5\,N}$$

D. con't

String 2:

$$T_2 - Mg = M\left(-\tfrac{2}{5}g\right)$$

$$T_2 = \tfrac{3}{5} Mg = \underline{11.8\,N}$$

E. $$\Delta x_2 = \cancel{\dot{x}_{20}}^{\,0}\Delta t + \tfrac{1}{2}\ddot{x}_2 \Delta t^2$$

$$= -\frac{g}{5}(3\,\text{sec})^2 = \underline{-17.6\,m}$$

2. The excerpt below is from an article that appeared in the Seattle P-I in March 1991:

> **Magnetic fields to blame, widow claims**
> SEATTLE (AP) - The widow of a former Seattle City Light worker yesterday filed what could be an unprecedented pension claim, saying her husband died from on-the-job exposure to electromagnetic fields. Her claim cites a recent study showing that City Light workers were exposed to fields measuring as high as 204 milligauss, nearly 100 times the amount linked to cancers in young children living near power lines.

After reading the article, a junior high student wants to investigate the effect of ELF (extremely low frequency) magnetic fields on growing bacteria colonies. You are asked to guide her in planning a safe and reasonable experiment. After discussing the project with you, she decides to construct an apparatus capable of subjecting the bacteria to an oscillating magnetic field with a peak value of approximately 20 gauss, about 100 times the value to which the worker mentioned in the article was exposed.

Design a coil configuration and circuit that the student can use for constructing the apparatus from the following list of available equipment:

(1) a shallow glass Petri dish, 8 cm diameter.

(2) a cylindrical plastic tube, 10 cm outer diameter, 9 cm inner diameter, 50 cm long

(3) 850 m of #18 insulated copper wire
(0.10 cm diameter, 1.02Ω /1000m, current capacity 2A)

(4) a transformer with a primary/secondary turns ratio of 7:1.

(5) a 50Ω (10W) resistor

(6) an AC outlet (120V, 60 Hz) with a 15A circuit breaker.

> $\mu_o = 4\pi \times 10^{-7}$ T m/A
>
> 1T = 10^4 gauss

In the answers to the questions below, calculations to 10% are adequate.

1. Coil configuration

   a. Sketch and describe quantitatively a coil configuration that can be made from the components on hand. It should provide a fairly uniform 60 Hz magnetic field of about 20 gauss over the bacteria sample in the Petri dish.

   b. Calculate the magnetic field at the sample as a function of the current *i*.

   c. Calculate the value of the self-inductance, *L*, for your coil.

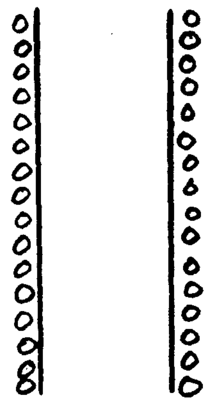
2. Circuit

   a. Sketch a schematic diagram of the full circuit from the AC outlet to the coil.

   b. Calculate the peak value of the current in the coil and the peak value of the magnetic field.

   c. Calculate the *average* power dissipated in the circuit.

II 1. Coil configuration



diameter of wire = 0.10 cm

outer diameter of tube : 10 cm

length of tube : 50 cm

# of turns / layer = $50\text{cm}/0.10\text{cm} = 500$

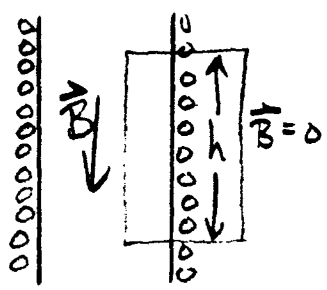
length of wire in first layer : $500(\pi d) = 500(10.1\text{cm}\,\pi)$
$= \underline{159\text{m}}$

length of wire available = 850m , so no more than 5 layers.

$$\left.\begin{array}{l} \text{for 2}^{nd}\text{ layer : } 500(10.3\text{cm}\,\pi) = 162\text{ m} \\ \text{for 3}^{rd}\text{ layer : } 500(10.5\text{cm}\,\pi) = 165\text{ m} \\ \text{for 4}^{th}\text{ layer : } 500(10.7\text{cm}\,\pi) = 168\text{ m} \\ \text{for 5}^{th}\text{ layer : } 500(10.9\text{cm}\,\pi) = 172\text{ m} \end{array}\right\} \text{826m for 5 layers}$$

∴ $\boxed{\text{Length of wire in coil} \approx 826\text{m, consisting of 5 layers at 500 turns / layer.}}$

b)



Ampere's Law:

$\oint \vec{B}\cdot d\vec{l} = Bh = \mu_0 n h i$

where n = # turns / m

$B = \mu_0 n i$

$= 4\pi \times 10^{-7}\,\text{Tm/A}\left(\frac{5 \times 500}{0.5\text{m}}\right) i$

$= 20\pi i \times 10^{-4}\,\text{T}$

$\boxed{B = 20\pi i \text{ Gauss}}$

1c) $L = \frac{N\Phi_B}{i}$, where N = # of turns
Φ_B = flux set up in each turn by i

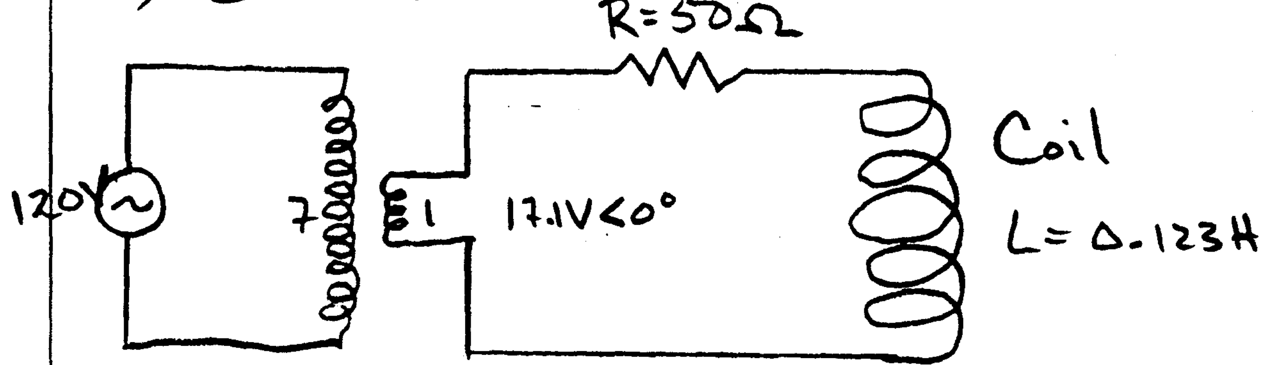
N = (# of turns/layer)(# of layers) = 2500

$\Phi_B = BA$, where $A = \pi r^2 = \pi\left(\frac{0.1m}{2}\right)^2$

$$L = \frac{N\Phi_B}{i} = \frac{2500 \times [20\pi i \times 10^{-4} T] \times [\pi (\frac{0.1m}{2})^2]}{i}$$

$$\boxed{L = \frac{5}{4}\pi^2 \times 10^{-2} H = 0.123\ H = 123\ mH}$$

2a) Circuit



b) $R_{wire} = \rho l = \frac{1.02\Omega}{1000m}(826m) = 0.83\ \Omega$

$R_t = 50.0\Omega + 0.83\Omega = 50.8\Omega$

$X_L = \omega L = 2\pi(60Hz)(0.123\ H) = 46.4\ \Omega$

$Z = R_t + jX_L = 50.8\Omega + j\,46.4\ \Omega$

$= \sqrt{(50.8\Omega)^2 + (46.4\Omega)^2} \angle \tan^{-1}\frac{46.4}{50.8}$

$Z = 68.8\ \Omega \angle 42.4°$

2b) cont.

$$i_{rms} = \frac{17.1\,V \angle 0°}{68.8\,\Omega \angle 42.4°} = 0.249\,A \angle -42.4°$$

$$\boxed{i_{peak} = \sqrt{2}\,|i_{rms}| = 0.351\,A}$$

$$B_{peak} = 20\pi\, i_{peak} \text{ Gauss}$$

$$= 20\pi(0.351) \text{ Gauss}$$

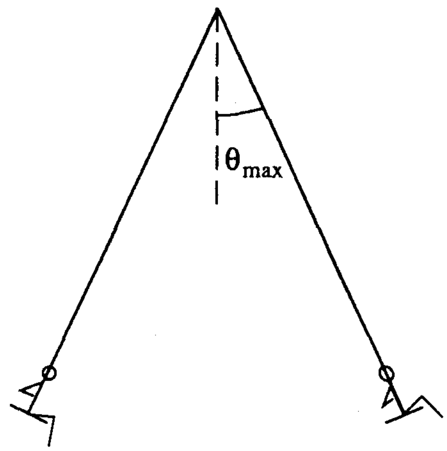
$$\boxed{B_{peak} = 22 \text{ Gauss}}$$

c) $P_R = i_{rms}^2 R_t = (0.249\,A)^2(50.8\,\Omega) = \underline{3.1\,\text{Watt}}$

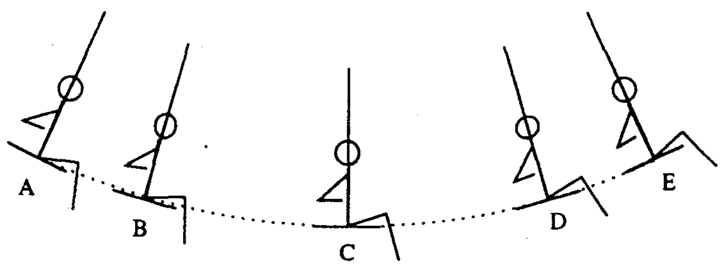
Power dissipated in resistor and wire of coil is less than 10W and current in circuit is less than 2A. Student should be safe.

3. A child swings back and forth on a swing. The mass of the child is M; the mass of the swing is m; the total mass of the child-swing system is M_c; the distance from the point of suspension to the center of mass is L. The different conditions in Parts A and B of this problem result in different motions.

A. The child sits on the swing and remains seated throughout the motion. The swing moves with constant angular amplitude, rising in front and in back to the same maximum angle, θ_{max}.



1. Below is a stroboscopic representation of one-half period of the motion.



a. Using a dot to represent the child-swing system at the five labelled positions, draw a similar diagram on your paper. Indicate with a vector at each dot the direction and magnitude of the velocity.

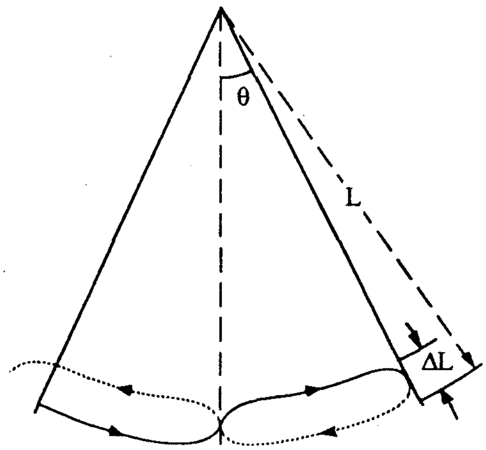
b. On a separate diagram, show the approx. **direction** (not the magnitude) of the acceleration at each dot. Explain the reasoning used to construct the vectors.

2. Draw separate free-body diagrams (i.e., diagrams in which forces acting on the body are represented as vectors) when the swing is in the vertical position at point C: (1) for the child and (2) for the swing.

a. For each force on your diagrams, specify the agent that exerts the force and the object on which the force is exerted.

b. For each force on your diagrams, identify the reaction force and specify the agent that exerts that reaction force and the object on which that reaction force acts.

B. The child alternately stands up and squats down on the swing. When the swing passes the vertical position (going forwards and backwards), the child quickly stands up from a squatting position, raising the center of mass of the child-swing system by a distance ΔL. He remains standing as the swing ascends to its highest point (front and back). The child then quickly squats down, lowering the center of mass to the original location, and remains squatting as the swing descends (forwards and backwards).



Show that, if no dissipative forces are present, the mechanical energy of the child-swing system increases with each oscillation and can be expressed as $E_n = E_o\,\beta^{2n}$, where β is a constant.

> Assume that: (1) the times involved when the child changes positions are sufficiently short compared to the period of the motion and (2) $\Delta L \ll L$. You may find it helpful to break the problem into the following steps:

1. Find the change in mechanical energy of the system during the first half-period of one oscillation.

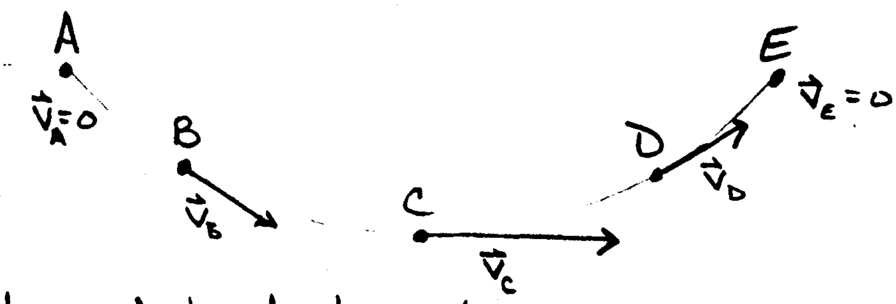
2. Show that the energy E_1 after the first period of oscillation can be expressed as $E_1 = E_o\,\beta^2$ and find β.

3. Show that the expression given for E_n represents the energy after n periods. Explain your reasoning.

III

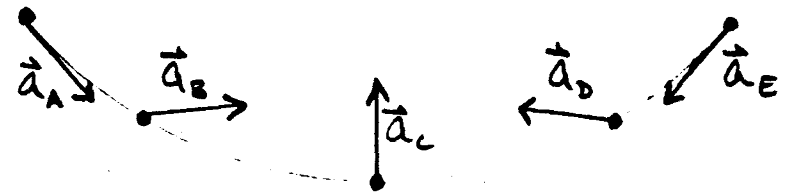
A.

1a.)



$\vec{v}$ is tangent to the trajectory.

b.)



point A: From operational def of $\vec{a}$:

$$\vec{a} = \frac{\Delta\vec{v}}{\Delta t}$$

$\vec{a}$ must be in direction of $\Delta\vec{v}$ for positive Δt. If $\vec{v}_A = 0$ and $\vec{v}_{A+\Delta t}$ is tangent to trajectory, then $\Delta\vec{v}$ must be tangent to trajectory.

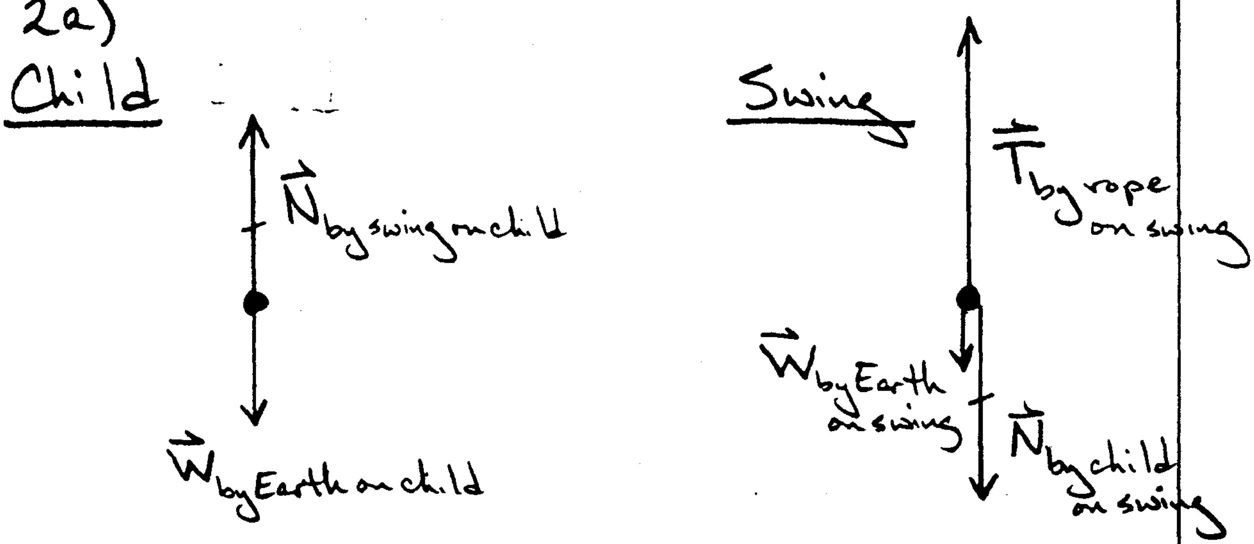
B: Swing is speeding up <u>and</u> changing direction
$\therefore \measuredangle$ between $\vec{a}_B$ and $\vec{v}_B$ is $> 0°$ and $< 90°$.

C: Swing neither speeding up nor slowing down
$\therefore \measuredangle$ between $\vec{a}_C$ and $\vec{v}_C$ is $90°$.

D: Swing is slowing down <u>and</u> changing dir.
$\therefore \measuredangle$ between $\vec{a}_D$ and $\vec{v}_D$ is $> 90°$ and $< 180°$

E: Same reasoning as A.

2a)



b) <u>3rd Law force-pairs</u>

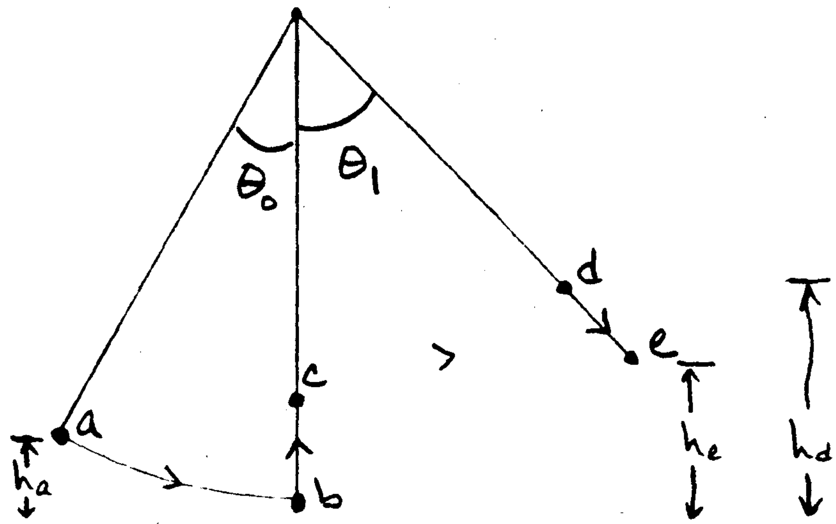
$\vec{N}_{\text{by swing on child}}$: $\vec{N}_{\text{by child on swing}}$

$\vec{W}_{\text{by Earth on child}}$: $\vec{W}_{\text{by child on Earth}}$

$\vec{T}_{\text{by rope on swing}}$: $\vec{T}_{\text{by swing on rope}}$

$\vec{W}_{\text{by Earth on swing}}$: $\vec{W}_{\text{by swing on Earth}}$

B.



$$h_a = L(1-\cos\theta_0)$$
$$h_e = L(1-\cos\theta_1)$$
$$h_d = (L-\Delta L)(1-\cos\theta_1) = h_e(1-\Delta L/L)$$

$$\mathcal{E}_a = mgh_a$$

Energy conserved from $a \rightarrow b$:

$$\mathcal{E}_b = \tfrac{1}{2}mv_b^2 = \mathcal{E}_a$$

Angular momentum conserved from $b \rightarrow c$:

$$mLv_b = m(L-\Delta L)v_c$$

$$v_c = v_b \frac{1}{1-\Delta L/L}$$

$$\mathcal{E}_c = \tfrac{1}{2}mv_c^2 = \tfrac{1}{2}mv_b^2\left(\frac{1}{1-\Delta L/L}\right)^2$$
$$\approx \mathcal{E}_b(1+2\Delta L/L)$$
$$= \mathcal{E}_a(1+2\Delta L/L)$$

Energy is conserved from $c \to d$:

$$\mathcal{E}_d = mgh_d = \mathcal{E}_c$$

Over 1/2 period the mechanical energy changes by a factor of:

$$\frac{\mathcal{E}_e}{\mathcal{E}_a} = \frac{\mathcal{E}_e}{\mathcal{E}_d}\frac{\mathcal{E}_d}{\mathcal{E}_a} = \frac{\mathcal{E}_e}{\mathcal{E}_d}\frac{\mathcal{E}_c}{\mathcal{E}_a}$$

$$= \left[\frac{mgh_e}{mgh_d}\right](1 + 2\Delta L/L)$$

$$= \frac{(1 + 2\Delta L/L)}{(1 - \Delta L/L)} \approx 1 + 3\Delta L/L$$

The process then repeats itself. During each half period the energy increases by a factor of $\beta \equiv 1 + 3\Delta L/L$.

After one period: $\mathcal{E}_1 = (1 + 3\Delta L/L)^2 \mathcal{E}_0$

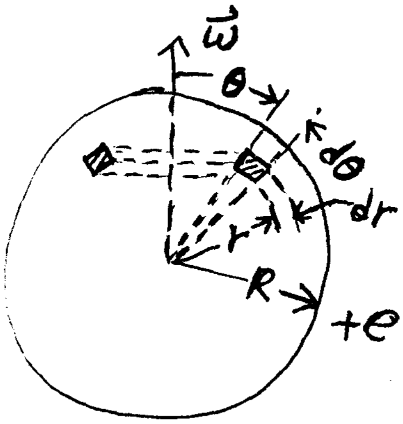
After n periods: $\mathcal{E}_n = (1 + 3\Delta L/L)^{2n} \mathcal{E}_0$.

4. E & M

A model "proton" has its charge $+e$ uniformly distributed throughout the volume of a sphere of radius R. This sphere rotates about its polar axis at angular velocity ω. Find the magnetic moment $\vec{m}$ for this proton.

E&M Problem 4 Solution

Proton Magnetic Moment



Given $d\vec{m} = \vec{A}dI$ for a planar current loop, this solid sphere of charge $+e$ uniformly distributed within radius R and spinning at angular velocity ω will have magnetic moment

$\vec{m} = \int \vec{A}\,dI$. For the hatched current loop above, $\vec{A} = \hat{z}\pi r^2 \sin^2\theta$ and $dI = \frac{dq}{T} = f\,dq = \frac{\omega}{2\pi}dq$,

where $dq = \rho\,dV$, $\rho = \frac{e}{\frac{4}{3}\pi R^3}$, and

$dV = 2\pi r\sin\theta\, r\,d\theta\,dr = 2\pi r^2 \sin\theta\,dr\,d\theta$.

Altogether, $dI = \frac{\omega}{2\pi}\frac{3e}{4\pi R^3}2\pi r^2\sin\theta\,dr\,d\theta$, and

$d\vec{m} = \vec{A}dI = \hat{z}\pi r^2\sin^2\theta\,\frac{3\omega e}{4\pi R^3}r^2\sin\theta\,dr\,d\theta$, so

$$\vec{m} = \hat{z}\int_0^R dr\,\frac{3\omega e r^4}{4R^3}\int_0^\pi \sin^3\theta\,d\theta.$$

Now $\int_0^\pi \sin^3\theta\,d\theta = \int_0^\pi(\sin\theta - \cos^2\theta\sin\theta)\,d\theta$

$= \left| -\cos\theta + \frac{\cos^3\theta}{3}\right|_0^\pi = \frac{2}{3} + \frac{2}{3} = \frac{4}{3}$, so

$$\boxed{\vec{m} = \frac{1}{5}\hat{z}\,\omega e R^2}$$

5. <u>Quantum Mechanics</u>

(a) Show that the true ground state energy E_o of a system satisfies

$$E_o \leq \langle E \rangle$$

where $\langle E \rangle$ is the expectation value of the energy in an arbitrary state Ψ (that is, $\langle E \rangle = \dfrac{\langle \psi | \hat{H} | \psi \rangle}{\langle \psi | \psi \rangle}$, where $\hat{H}$ is the system Hamiltonian).

(b) Use a trial wavefunction of the form e^{-ax^2} to find an upper bound on the ground state of a particle of mass m in the 1-d anharmonic potential λx^4. Express your answer in terms of the "natural" unit of energy $\left(\dfrac{\hbar^4 \lambda}{m^2}\right)^{1/3}$.

5 - Quantum Mechanics Solution

(a) Expand ψ in eigenfunctions of $\hat{A}$:

$$\psi = \sum_{j=0}^{\infty} c_j \varphi_j \qquad \text{(or the continuum analog)}$$

Then $\langle E \rangle = \sum_{j=0}^{\infty} |c_j|^2 E_j$ " " "

where $\hat{A}\varphi_j = E_j \varphi_j$ and we take $\sum_j |c_j|^2 = 1$

Equivalently; $\langle E \rangle = E_0 + (|c_0|^2 - 1)E_0 + |c_1|^2 E_1 + |c_2|^2 E_2 + \ldots$

$$= E_0 + |c_1|^2(E_1 - E_0) + |c_2|^2(E_2 - E_0) + \ldots$$

$$= E_0 + \sum_{j=1}^{\infty} \underbrace{|c_j|^2}_{\geq 0} \underbrace{(E_j - E_0)}_{\geq 0} \geq E_0$$

(b) $E_0 \leq \dfrac{\langle \psi | \hat{H} | \psi \rangle}{\langle \psi | \psi \rangle} = \dfrac{\int_{-\infty}^{\infty} e^{-ax^2}\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \lambda x^4\right) e^{-ax^2}}{\int_{-\infty}^{\infty} e^{-2ax^2}\, dx}$

denominator $= \sqrt{\pi/2a}$

numerator $= \int_{-\infty}^{\infty} dx\, e^{-2ax^2}\left[\left(-\frac{\hbar^2}{2m}\right)(4a^2x^2 - 2a) + \lambda x^4\right]$

$$= \left(-\frac{\hbar^2}{2m}\right)\left(4a^2 \cdot \frac{1}{2} \cdot \frac{\sqrt{\pi}}{(2a)^{3/2}} - 2a\sqrt{\frac{\pi}{2a}}\right) + \lambda \cdot \frac{3}{4} \cdot \frac{\sqrt{\pi}}{(2a)^{5/2}}$$

$$= \sqrt{\pi/2a}\left(\frac{\hbar^2 a}{2m} + \frac{3}{16}\frac{\lambda}{a^2}\right)$$

So $E \leq \dfrac{\hbar^2 a}{2m} + \dfrac{3}{16}\dfrac{\lambda}{a^2}$

Now ~~w/~~ we minimize the r.h.s. w/ resp. to a:

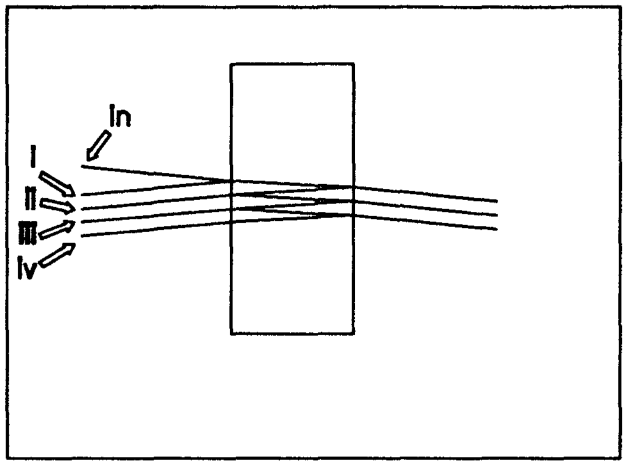
$$\frac{d}{da}(\text{r.h.s}) = \frac{\hbar^2}{2m} - \frac{3}{8}\frac{\lambda}{a^3} = 0 \implies a = \left(\frac{3}{4}\frac{\lambda m}{\hbar^2}\right)^{1/3}$$

At this value of a, the r.h.s $= \dfrac{\hbar^2}{2m}\left(\dfrac{3}{4}\dfrac{\lambda m}{\hbar^2}\right)^{1/3} + \dfrac{3}{16}\lambda\left(\dfrac{4}{3}\dfrac{\hbar^2}{\lambda m}\right)^{2/3}$

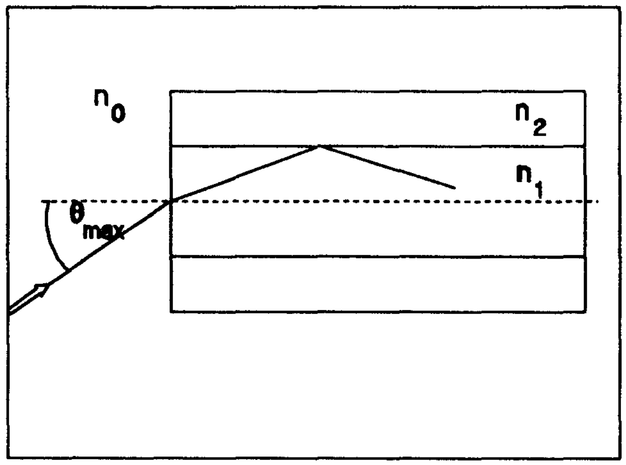
$$\text{So} \quad \underline{E_0 \leq \left(\frac{3}{4}\right)^{4/3}\left(\frac{\hbar^4 \lambda}{m^2}\right)^{1/3}}$$

6. <u>Optics</u>

a) Consider the reflection of a beam that is incident on a plate of glass from air at near normal incidence as shown. If the glass has an index of refraction of 1.5, find the intensity of the reflected beams marked by i,ii,iii and iv. You may assume that the beams are slightly separated so that interference between the beams is not important.



b) The Numerical Aperture (NA) of an optical fiber is defined by NA = $n_o \sin(\theta_{max})$, where n_o is the index for the medium outside the entrance of the fiber and θ_{max} is the maximum angle that can enter the fiber and propagate with guiding. If the fiber is a step index fiber with a index of n_1 in the core and an index of n_2 in the cladding surrounding the core, find an expression for NA. Calculate θ_{max} for the case where the fiber is entered from air and n_1=1.62 and n_2=1.52.

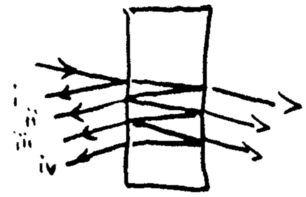


c) Estimate the radius a of the core of an optical fiber that is needed for the optical fiber to support single mode propagation of a beam at a wavelength λ.

Optics solution to Problem 6

a) The reflection at an interface for normal incidence is

$$R = \left(\frac{n_2 - n_1}{n_2 + n_1}\right)^2 .$$



Since air has $n \approx 1$ and the glass has $n \approx 1.5$,

$$R = \left(\frac{1.5-1}{1.5+1}\right)^2 = (.2)^2 = .04$$

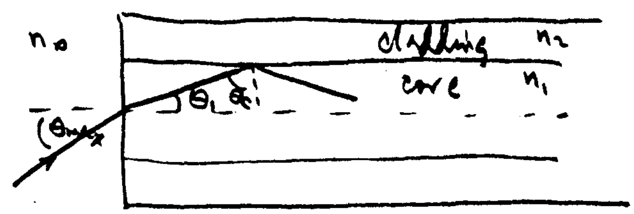
This will be the reflection at both interfaces. Note $T = 1 - R = (1 - 0.04)$

Thus at

$$\boxed{\begin{array}{ll} \text{i)} & I = 0.04 = 4 \times 10^{-2} \\ \text{ii)} & I = (1-0.04)(0.04)(1-0.04) = 0.0368 = 3.7 \times 10^{-2} \\ \text{iii)} & I = (1-0.04)^2 (0.04)^3 = 5.9 \times 10^{-5} \\ \text{iv)} & I = (1-0.04)^2 (0.04)^5 = 9.4 \times 10^{-8} \end{array}}$$

b) Use Snell's law

$$n_0 \sin\theta_{max} = n_1 \sin\theta_1$$



at θ_{max}, we are at the critical angle for the reflection between the core and cladding

Thus $\theta_c = \pi/2 - \theta_1$ with $\sin\theta_c = \frac{n_2}{n_1}$

$$\rightarrow \sin(\pi/2 - \theta_1) = \frac{n_2}{n_1} = \cos(\theta_1) = (1 - \sin^2\theta_1)^{1/2}$$
$$= \left(1 - \frac{n_0^2 \sin^2\theta_{max}}{n_1^2}\right)^{1/2}$$

Thus

$$\frac{n_0^2 \sin^2\theta_{max}}{n_1^2} = 1 - \left(\frac{n_2}{n_1}\right)^2 \longrightarrow n_0 \sin\theta_{max} = (n_1^2 - n_2^2)^{1/2}$$

$$\boxed{NA} = n_0 \sin\theta_{max} = \boxed{(n_1^2 - n_2^2)^{1/2}}$$

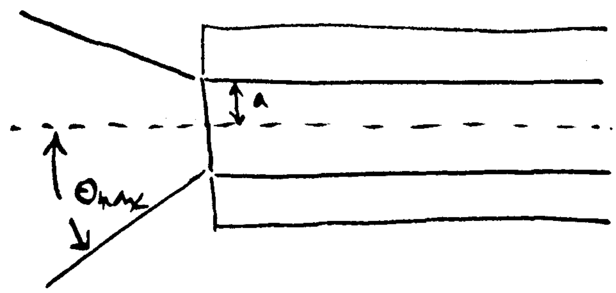
For $n_0 \approx 1$ in air, $n_1 = 1.62$, $n_2 = 1.52$

$$\boxed{\theta_{max}} = \sin^{-1}\left[\left((1.62)^2 - (1.52)^2\right)^{1/2}\right] = \boxed{34^\circ}$$

Optics (cont) Solution

c) To make a single mode fiber, we need the core to be small enough to produce loss to the cladding for higher order modes.

To estimate this, we can make "a" small enough so θ_{max} is the same as the diffraction of a single mode from a small hole of diameter $2a$.

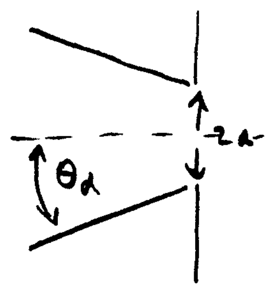


$$\sin\theta_{max} = (n_1^2 - n_2^2)^{1/2}$$

from b) w/ $n_0 = 1$

For diffraction from a circular aperture (dia = 2a)

$$\sin\theta_d = \frac{1.22\,\lambda}{(\text{dia})} = \frac{1.22\,\lambda}{2a}$$



Note: [$\theta_d \approx \frac{\lambda}{\text{dia}}$ would be sufficient for estimate]

Thus for single mode fiber, we require

$$\frac{1.22\,\lambda}{2a} \geq \sin\theta_{max} = (n_1^2 - n_2^2)^{1/2} = NA$$

or $\boxed{a \leq \frac{1.22\,\lambda}{2\,(NA)}}$

Note: Actual requirement when done exactly is

$$a < \frac{2.403\,\lambda}{2\pi\,(NA)}$$

Thus estimate is close.

7. Quantum Mechanics

A spinless particle of mass m bounces elastically between two infinite plane walls separated by a distance L. The particle is in its lowest possible energy state.

(a) What is the energy of this state?

(b) The separation between the walls is increased very slowly (i.e., adiabatically) to 2L. What is the change in the energy of the particle?

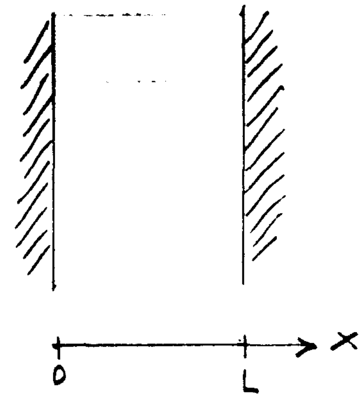
(c) We assume that instead of (b), the separation between the walls is increased very rapidly (this is the opposite limit of the adiabatic approximation, and is called the sudden approximation).

    (i) Evaluate the change in the mean value of the energy of the particle.

    (ii) Evaluate the probability that the particle is left in its lowest possible energy state.

Quantum Mechanics Problem 7

Solution



(a) $\psi_n(x) = A_n \sin \frac{n\pi}{L} x$

$$E_n = \frac{\hbar^2}{2m}\left(\frac{n\pi}{L}\right)^2$$

Ground state: $n = 1$

$$E_{GS}^{(L)} = \frac{\hbar^2}{2m}\left(\frac{\pi}{L}\right)^2 = \frac{\hbar^2\pi^2}{2mL^2}$$

(b) Adiabatic change $L \to 2L$. The particle remains at all times in the lowest energy state of the well as its width goes from L to $2L$. Thus,

$$E^{(2L)} = E_{GS}^{(2L)} = \frac{\hbar^2}{2m}\left(\frac{\pi}{2L}\right)^2 = \frac{\hbar^2\pi^2}{8mL^2}$$

$\therefore$

$$\Delta E = E_{GS}^{(2L)} - E_{GS}^{(L)} = -\frac{3}{4}\left(\frac{\hbar^2\pi^2}{2mL^2}\right)$$

$$= -\frac{3}{4} E_{GS}^{(L)}$$

(c) Sudden change $L \to 2L$. Immediately after the the width of the well is doubled, the particle is in the state

$$\psi(x) = \left(\frac{2}{L}\right)^{1/2} \sin\frac{\pi}{L}x \,, \qquad 0 < x < L$$
$$= 0 \qquad\qquad L < x < 2L$$

(i) It is clear that this state is not an eigen-state of the new potential well. However, it is also clear that the mean value of $\hat{H}$ does not change, i.e.,

$$\langle \psi | \hat{H} | \psi \rangle = E_{GS}^{(L)} .$$

(ii) The ground state of the new well has the (normalized) wave function

$$\phi(x) = \left(\frac{2}{2L}\right)^{1/2} \sin\frac{\pi}{2L}x \qquad 0 < x < 2L$$

The amplitude for the particle to be found in the ground state of the new well is

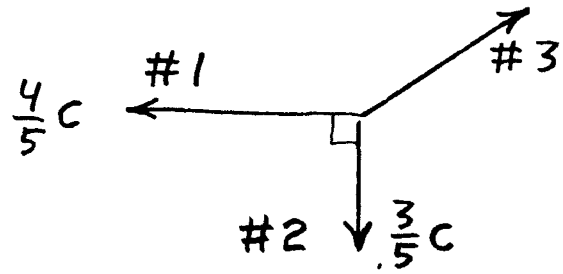
$$\langle \psi | \phi \rangle = \frac{\sqrt{2}}{L}\int_0^L dx \, \sin\frac{\pi}{L}x \; \sin\frac{\pi}{2L}x$$
$$= \frac{4\sqrt{2}}{3\pi} .$$

The corresponding probability is

$$|\langle \psi | \phi \rangle|^2 = \frac{32}{9\pi^2} .$$

8. Relativity

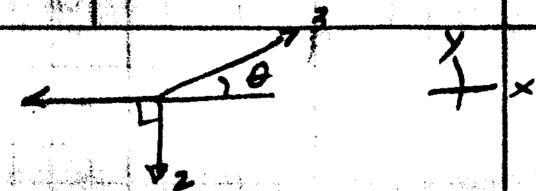
A particle of rest mass M_0 is at rest in the laboratory when it decays into three identical particles, each of rest mass m_0. Two of the particles have velocities at right angles to each other, and of the size shown:



(a) Find the magnitude and direction of the third particle's velocity.

(b) Find $\frac{3m_0}{M_0}$.

#8 Relativity - solution



(a) $p_{3x} = -p_1 \Rightarrow \quad \dfrac{v_{3x}}{\sqrt{1-v_3^2/c^2}} = \dfrac{4}{5}\dfrac{c}{\sqrt{1-\frac{16}{25}}} = \dfrac{4}{3}c$

$p_{3y} = -p_2 \Rightarrow \quad \dfrac{v_{3y}}{\sqrt{1-v_3^2/c^2}} = \dfrac{3}{5}\dfrac{c}{\sqrt{1-9/25}} = \dfrac{3}{4}c$

So $\dfrac{v_3^2}{1-v_3^2/c^2} = \left(\dfrac{16}{9}+\dfrac{9}{16}\right)c^2$

$v_3^2 = \dfrac{\frac{16}{9}+\frac{9}{16}}{1+\frac{16}{9}+\frac{9}{16}}c^2 \Rightarrow \underline{v_3 = 0.837\,c}$

$\theta = \tan^{-1}\dfrac{p_{3y}}{p_{3x}} = \dfrac{3/4\; m_0 c}{4/3\; m_0 c} = \dfrac{9}{16} \Rightarrow \theta = \tan^{-1}\dfrac{9}{16}$

$= \underline{29.4^\circ}$

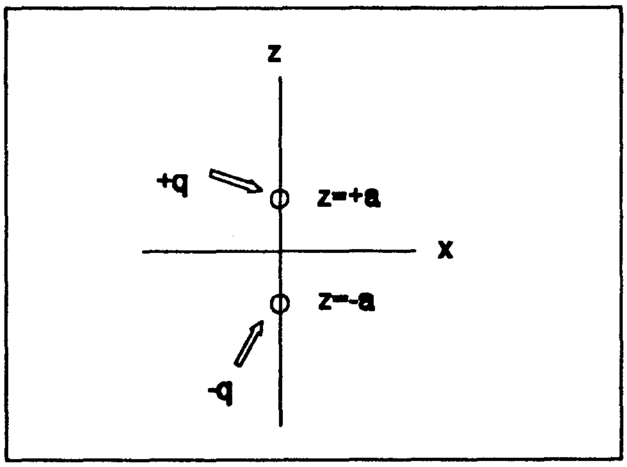
(b) $M_0 c^2 = \gamma_1 m_0 c^2 + \gamma_2 m_0 c^2 + \gamma_3 m_0 c^2$

$M_0 = m_0\left(\dfrac{5}{3}+\dfrac{5}{4}+\sqrt{1+\dfrac{16}{9}+\dfrac{9}{16}}\right) = m_0(.2108)$

$\dfrac{3m_0}{M_0} = \underline{0.632}$

9. E & M

a) Consider two charges of equal and opposite sign on the z axis as shown. Find the electric field at the origin.



b) Now find the electric field due to the charges at a point $r=(r,\theta,\phi)$ when $r \gg a$.

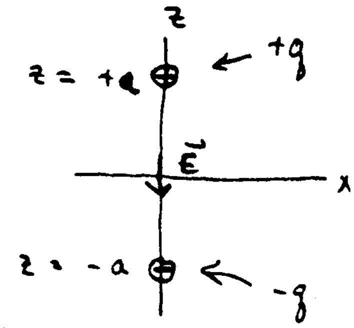
c) Now let the position of the positive charge oscillate as $z=a\cos(\omega t)$ and that of the negative charge oscillate as $z=-a\cos(\omega t)$. Find the vector potential **A** when $r \gg a$.

d) Explain in reasonable detail how you would use your answer in c) to find the electric field in the far field where $\lambda \ll r$.

EM — solutions to Problem 9

a) For one point charge $\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{r}$

Thus at origin $\boxed{\vec{E} = -\frac{2}{4\pi\epsilon_0}\frac{q}{a^2}\hat{z}}$



b) When $r \gg a$, easiest method is to use dipole approximation

$$\phi = \frac{1}{4\pi\epsilon_0}\frac{\vec{p}\cdot\hat{r}}{r^2} \qquad \text{with } \vec{p} = (2a)q\,\hat{z}$$

$$\to \phi = \frac{2aq}{4\pi\epsilon_0}\frac{\cos\theta}{r^2}$$

Now to find $\vec{E}$, use $\vec{E} = -\nabla\phi = -\hat{r}\frac{\partial}{\partial r}\phi - \hat{\theta}\frac{1}{r}\frac{\partial}{\partial\theta}\phi$

$$\boxed{\vec{E} = \frac{(2aq)}{4\pi\epsilon_0}\frac{1}{r^3}\left[2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right]}$$

c) To find $\vec{A}$, we use $\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\int\frac{\vec{J}(\vec{r}',t')}{R}\,d\tau'$

For point charges, $\vec{J} = \rho\,\vec{v}$ $\qquad t' = t - R/c$

Thus for one point charge $\vec{A} = \frac{\mu_0}{4\pi}\int\frac{\rho\,\vec{v}\,d\tau'}{R} = \frac{\mu_0}{4\pi}\frac{q\vec{v}}{R}$

Now since $+q$ charge oscillates as $z = a\cos\omega t = \mathrm{Re}\, a\,e^{-i\omega t}$

$$\vec{v} = -i\omega a\,e^{-i\omega t}\hat{z} \quad \to \vec{A} = \frac{\mu_0}{4\pi}q\frac{(-i\omega a)\,e^{-i\omega t'}}{R}\hat{z}$$

For two charges opposite in sign and position, $\vec{A}$ doubles

Also for $r \gg a$, $\vec{R} = \vec{r} - \vec{r}' \to \vec{r}$

Thus $\boxed{\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\frac{2q(-i\omega a)}{r}\,e^{-i\omega t}\,e^{+i\frac{\omega}{c}r}\,\hat{z}}$

EM cont Solution * note: soln for d) shown completely but only description was requested.

d) To find $\vec{E}$, use $\boxed{\vec{A} \to \vec{B} \to \vec{E}}$ so we don't need to solve for ϕ.

$$\boxed{\vec{B} = \nabla \times \vec{A}} = \left[\frac{\mu_0}{4\pi} (2qa)(-i\omega)\, e^{-i\omega t} \right] \nabla \times \left[\frac{e^{ikr}}{r} \hat{z} \right] \qquad k = \frac{\omega}{c}$$

Since we need curl in spherical coordinates, write $\hat{z} = \hat{r}\cos\theta - \hat{\theta}\sin\theta$

$$\nabla \times \left[\frac{e^{ikr}}{r} \cos\theta\, \hat{r} \right] = -\frac{\hat{\phi}}{r} \frac{\partial}{\partial\theta}[\,] = -\frac{\hat{\phi}}{r} \frac{e^{ikr}}{r} \left[\frac{\partial}{\partial\theta} \cos\theta \right]$$

$$= +\hat{\phi}\, \frac{e^{ikr}}{r^2} \sin\theta$$

$$\nabla \times \left[-\frac{e^{ikr}}{r} \sin\theta\, \hat{\theta} \right] = +\frac{\hat{\phi}}{r} \frac{\partial}{\partial r} \{ r [\,] \} = -\frac{\hat{\phi}}{r} \sin\theta \frac{\partial}{\partial r} \left[\cancel{r} \frac{e^{ikr}}{\cancel{r}} \right]$$

$$= -\frac{\hat{\phi}}{r} \sin\theta\, e^{ikr} (ik)$$

$$\to \vec{B} = \left[\frac{\mu_0}{4\pi} (2qa)(-i\omega)\, e^{-i\omega t} \right] \hat{\phi} \sin\theta\, e^{ikr} \left[\cancel{\frac{1}{r^2}} - \frac{ik}{r} \right]$$

for $\lambda \ll r$, neglect $\frac{1}{r^2}$ term

$$\to \vec{B} = \frac{\mu_0}{4\pi} (2qa)(-i\omega) \left[e^{ikr - i\omega t} \right] \frac{\sin\theta}{r} (-ik)\, \hat{\phi}$$

$$\boxed{\vec{B} = -\frac{\mu_0 k\omega (2qa)}{4\pi} \frac{\sin\theta}{r} e^{ikr - i\omega t}\, \hat{\phi}}$$

real part goes as $\cos(kr - \omega t)$

Now to find $\vec{E}$ use $\nabla \times \vec{B} = \frac{1}{c^2} \frac{\partial}{\partial t} \vec{E} = \frac{1}{c^2}(-i\omega)\vec{E}$

$$\boxed{\vec{E} = \frac{c^2 \nabla \times \vec{B}}{-i\omega}} = \left[\frac{-ic^2 \mu_0 k (2qa)}{4\pi} e^{-i\omega t} \right] \nabla \times \left[\sin\theta \frac{e^{ikr}}{r} \hat{\phi} \right]$$

$$\nabla \times \left[\sin\theta \frac{e^{ikr}}{r} \hat{\phi} \right] = \frac{\hat{r}}{r\sin\theta} \frac{\partial}{\partial\theta} \left[\sin\theta \sin\theta \frac{e^{ikr}}{r} \right] - \frac{\hat{\theta}}{r} \frac{\partial}{\partial r} \left[\cancel{r} \sin\theta \frac{e^{ikr}}{\cancel{r}} \right]$$

$$= \frac{\hat{r}}{r^2} 2\cos\theta\, e^{ikr} - \frac{\hat{\theta}}{r} \sin\theta\, e^{ikr} (ik)$$

EM (cont) solutions

d) (cont)

$$\vec{E} = \left[\frac{-i c^2 \mu_0 k \,(2qa)\, e^{-i\omega t}}{4\pi}\right] e^{ikr} \left[\frac{2\cos\theta\, \hat{r}}{r^2} - \frac{ik \sin\theta\, \hat{\theta}}{r}\right]$$

for $\lambda \ll r$, drop $\frac{1}{r^2}$ term

$$\rightarrow \vec{E} = -\frac{c^2 \mu_0 k^2 (2qa)}{4\pi} \frac{\sin\theta}{r}\, \hat{\theta}\; e^{ikr - i\omega t}$$

note: $c^2 \mu_0 = \frac{1}{\epsilon_0 \mu_0} \mu_0 = \frac{1}{\epsilon_0}$

$$\rightarrow \boxed{\vec{E} = -\frac{k^2 p_0}{4\pi\epsilon_0} \frac{\sin\theta}{r}\, \hat{\theta}\; e^{ikr - i\omega t}}$$

dipole moment $p_0 = 2qa$

(Real part is $\frac{-k^2 p_0}{4\pi\epsilon_0} \frac{\sin\theta}{r}\, \hat{\theta}\, \cos(kr - \omega t)$)

10. Solid State

Conductivity and Hall Effect

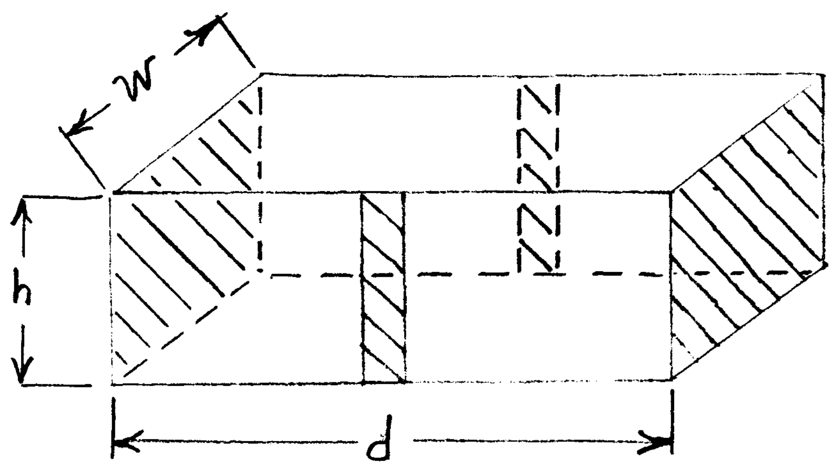
A parallelepiped sample is supplied with electrodes as shown.

(a) Draw a diagram showing how you would measure the dc electrical conductivity σ of the sample, giving a formula for σ in terms of meter readings and sample dimensions.

(b) Draw a diagram for measuring the Hall effect, showing clearly the directions of the applied magnetic and electric fields and the two possible directions of the induced Hall electric field.

(c) Explain why it is possible to obtain the current carrier sign from a Hall experiment but not from a conductivity experiment.

(d) Derive an expression for Hall voltage in terms of applied fields, sample dimensions, and carrier properties (charge q, concentration n, and mobility μ which is drift velocity divided by applied electric field).

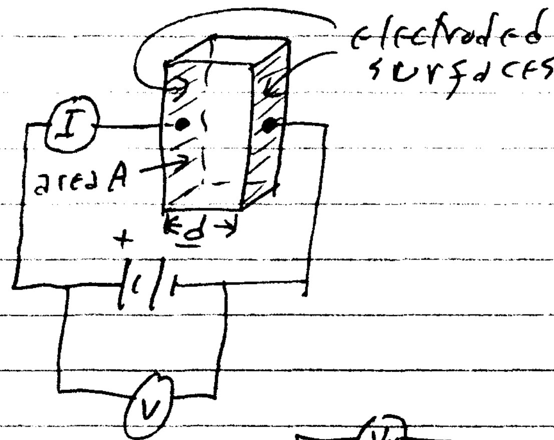


Solid State Problem 10 Solution

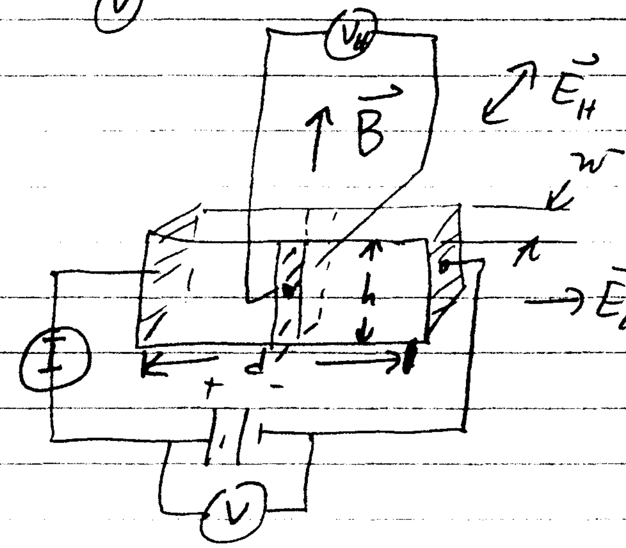
(a) $R = \rho d / A$

$= V/I$

$$\boxed{\sigma = \frac{1}{\rho} = \frac{d}{A}\,\frac{I}{V}}$$



(b)



(c) In the σ experiment, I is in decreasing V direction regardless of carrier sign.
In the Hall experiment, the magnetic force $q\vec{v}\times\vec{B}$ is in the same direction regardless of carrier sign, so the carriers pile up on the same side, and give a Hall field whose direction depends on sign of q.

(d) $V_H = E_H w \qquad q\vec{E}_H + q\vec{v}\times\vec{B} = 0$

$\vec{E}_H = -\mu \vec{E}_A \times \vec{B}$, sign of μ depends on sign of q $\qquad |V_H| = w\mu|\vec{E}_a \times \vec{B}|$

11. Atomic Physics

a) Sodium has 11 electrons. Give the electron configuration for the ground state. (As an example, the configuration for He is $(1s)^2$).

b) Give the spectroscopic notation for the ground state and the next two excited states. (As an example of the notation, the ground state of He is $(1s)^2\ {}^1S_0$).

c) Now place the atom in a magnetic field. Draw a diagram showing (and labeling by m_j) the resulting levels in the magnetic field for the three states found in b).

d) Estimate the magnetic field necessary to obtain a splitting of the levels in the ground state comparable to kT at room temperature. Is this a magnetic field obtainable in the laboratory?

Some constants:
m_e=9.109 x 10^{-31}kg
k=8.617 x 10^{-5}eV/K
h=6.626 x 10^{-34}joule-s
e=1.602 x 10^{-19}coul
μ_B=5.788 x 10^{-9}eV/G

Atomic Physics   solutions to Problem 11

a) 11 electrons for Na give the following configuration for the ground state

$$\boxed{(1s)^2\,(2s)^2\,(2p)^6\,(3s)}$$

b) thus the ground state will be an S state (ie $l=0$)
since it is a one electron system, it will be a "doublet"
ie $2S+1 = 2(\tfrac{1}{2})+1 = 2$

The value of j will be $\vec{J} = \vec{l} + \vec{s} = 0 + \vec{\tfrac{1}{2}} = \tfrac{1}{2}$

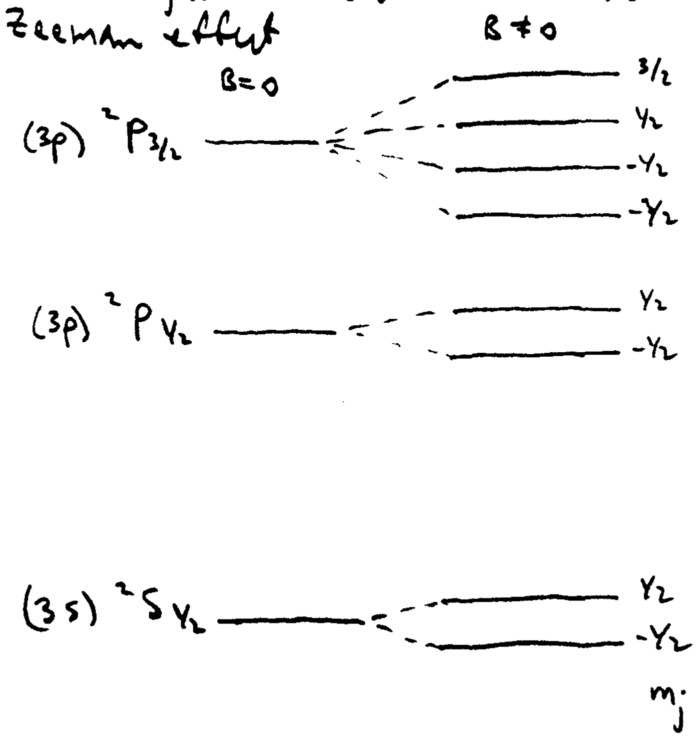
Thus the ground state will be $\boxed{(3s)\ {}^2S_{1/2}}$

The next two excited states will come from the next level (3p) which forms a true doublet.

Now $\vec{J} = \vec{l} + \vec{s} = \vec{1} + \vec{\tfrac{1}{2}} = 3/2,\ 1/2$

Thus the next two states are $\rightarrow$ $\boxed{\begin{array}{l}(3p)\ {}^2P_{3/2}\\ (3p)\ {}^2P_{1/2}\end{array}}$

c) In a magnetic field these states will split due to the Zeeman effect

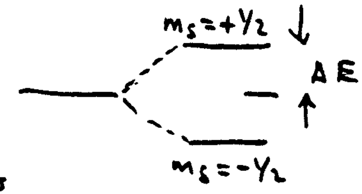


Atomic Physics — Solutions

d) Since $l=0$ in the ground state, the Zeeman splitting will only be due to the magnetic moment of the spin.

Thus the level shift ΔE will be

$$\Delta E = -\vec{\mu}\cdot\vec{B} = +g_s \mu_B m_s B$$



$$g_s = 2 \qquad \mu_B = 5.788\times10^{-9}\ \mathrm{eV/G}$$

Splitting is $2\Delta E$

$$2\Delta E = 2 g_s \mu_B m_s B = 2(2)\,\mu_B\left(\tfrac{1}{2}\right)B$$
$$= 1.58\times10^{-8}\ \mathrm{eV/G}\ \ B$$

Now at room temp ($T = 300°K$)

$$kT = (8.617\times10^{-5}\ \mathrm{eV/K})(300\mathrm{K}) = 2.58\times10^{-2}\ \mathrm{eV}$$

Thus for splitting comparable to kT

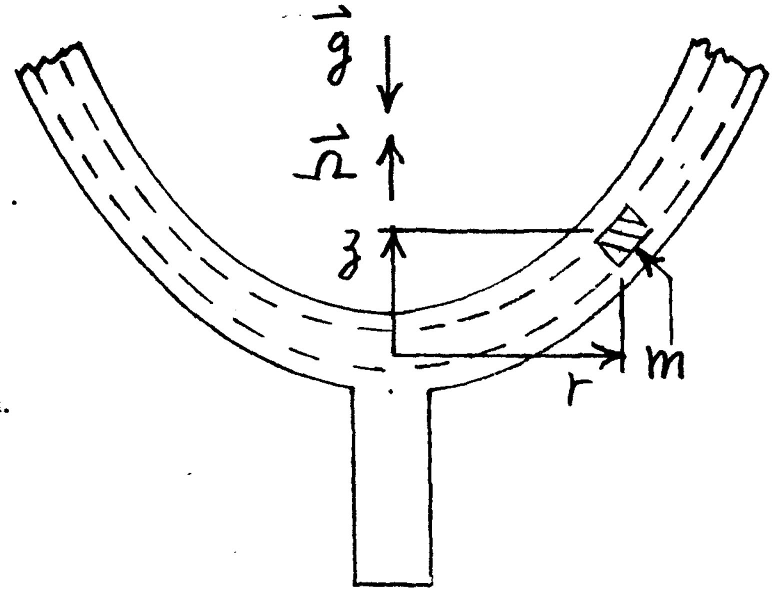
need $$B = \frac{kT}{2\mu_B} = \frac{2.58\times10^{-2}\ \mathrm{eV}}{1.58\times10^{-8}\ \mathrm{eV/G}} = \boxed{1.63\times10^{6}\ \mathrm{G}}$$

This is a very large field. Laboratory fields are limited to ~10 kG.

Thus 10^6 G is not obtainable in the laboratory.

12. Mechanics

A bead of mass m is free to slide frictionlessly inside a tube bent into a parabolic shape described by $z = \frac{1}{2}ar^2$. The tube is constrained to rotate at uniform angular velocity Ω about the vertical z axis.



(a) Write the Lagrangian for this system.

(b) Write the Lagrange equation for this system.

(c) Find the critical angular velocity Ω_c, above which the bead will slide out toward $r \to \infty$.

(d) Find the frequency ω of small oscillations of the bead for $\Omega < \Omega_c$.

Mechanics Problem 12 Solution

(a) $L = T - V = \frac{1}{2}m(\dot{r}^2 + \Omega^2 r^2 + \dot{z}^2) - mgz$

but $z = \frac{1}{2}ar^2$ so

$$\boxed{L = \frac{1}{2}m(\dot{r}^2 + \Omega^2 r^2 + a^2 r^2 \dot{r}^2) - \frac{1}{2}mgar^2}$$

(b) $$\frac{d}{dt}\frac{\partial L}{\partial \dot{r}} - \frac{\partial L}{\partial r} = \frac{d}{dt}\left[m\dot{r}(1 + a^2 r^2)\right] - mr\left[\Omega^2 - ga + a^2\dot{r}^2\right]$$

$$= m\ddot{r}(1 + a^2 r^2) + 2ma^2 r\dot{r}^2 - ma^2 r\dot{r}^2 + mr(ga - \Omega^2)$$

$$= \boxed{m\ddot{r}(1 + a^2 r^2) + ma^2 r\dot{r}^2 + mr(ga - \Omega^2) = 0}$$

(c) Keep only terms in Lagrange equation to 1st order in r and its derivatives, and see if small-oscillation solution exists about some $r_{eq} < \infty$. This first-order equation is

$$m\ddot{r} + mr(ga - \Omega^2) = 0.$$

Let $r = r_{eq} + r_0 e^{i\omega t}$ and plug in:

$$-m\omega^2 r_0 e^{i\omega t} + m\left[r_{eq} + r_0 e^{i\omega t}\right](ga - \Omega^2) = 0$$

so $r_{eq} = 0$ unless $\Omega = \boxed{\Omega_c = \sqrt{ga}}$ in which case any r gives neutral equilibrium.

Mechanics Problem 12 Solution - cont.

(d) For small oscillations about $r_{eq}=0$, the above first-order equation is

$$\boxed{\omega^2 = ga - \Omega^2}$$

which gives a real frequency $\sqrt{ga-\Omega^2}$ for $\Omega < \Omega_c = \sqrt{ga}$. For $\Omega > \Omega_c$, $\omega = \pm i\sqrt{\Omega^2 - ga}$ which when plugged in gives

$$r = r_0 e^{i\omega t} = r_0 e^{\sqrt{\Omega^2-ga}\,t}$$

if the $\omega = -i\sqrt{\Omega^2-ga}$ root is chosen.

The $\omega = +i\sqrt{\Omega^2-ga}$ root gives

$$r = r_0 e^{-\sqrt{\Omega^2-ga}\,t}$$

for which the bead exponentially slows down as it approaches the bottom; this is also a physically meaningful solution, if $\Omega > \Omega_c$.

13. Mathematical Physics

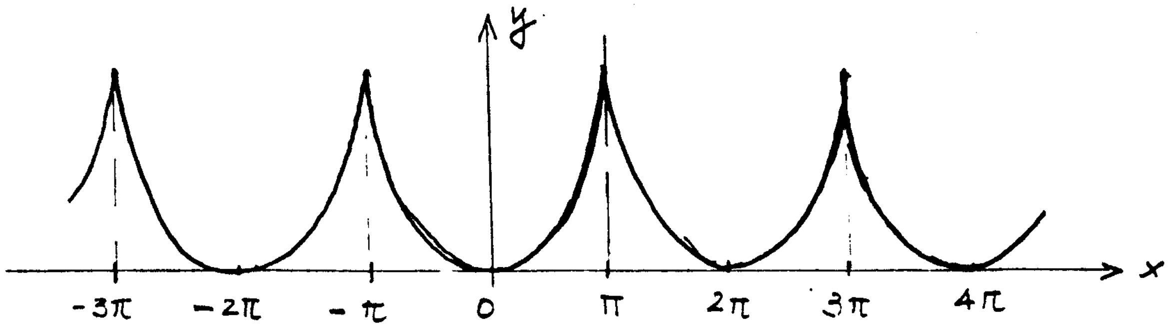
Expand $f(x) = x^2$, with $-\pi \leq x \leq \pi$, in a Fourier series. (Evaluate the coefficients of the Fourier series in closed form, and write down the first four or five terms.)

Mathematical Physics Problem 13 Solution

Solution.

First we note that $f(x)$ is even $\Rightarrow$ the Fourier series consists of cosines only.

The graph of $f(x)$ together with its periodic extension is



$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \qquad (1)$$

$$b_n \equiv 0$$

From Eq. (1) $\Rightarrow$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} dx\, f(x) = \frac{2}{\pi}\int_0^{\pi} dx\, x^2 = \frac{2}{3}\pi^2$$

$$\underset{n\geq 1}{a_n} = \frac{1}{\pi}\int_{-\pi}^{\pi} dx\, f(x)\cos nx$$

$$= \frac{2}{\pi}\int_0^{\pi} dx\, x^2 \cos nx$$

Integrating by parts

$$a_n = -\frac{4}{\pi n}\int_0^\pi dx\; x \sin nx =$$

$$= \frac{4}{\pi n^2}\left(x\cos nx\right)_0^\pi - \frac{4}{\pi n^2}\int_0^\pi dx\; \cos nx$$

$$= \frac{4}{n^2}\cos n\pi$$

$\therefore$

$$a_n = (-1)^n \frac{4}{n^2}, \qquad n = 1, 2, 3, \ldots$$

Thus, for $-\pi \le x \le \pi$:

$$x^2 = \frac{\pi^2}{3} - 4\left(\cos x - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \cdots\right)$$

14. Statistical Mechanics

A "one-dimensional" gas is composed of N non-interacting spin-0 bosons with energy-momentum relation $\varepsilon = pc$, moving in a "box" of length L. Use the fact that the density of states $g(\varepsilon)$ in this case is given by $g(\varepsilon) = \frac{L}{2\pi\hbar c}$

(a) Write the total particle number N as an integral over the Bose-Einstein distribution function.

(b) Evaluate the integral of part (a) to express the chemical potential μ as a function of N, T and L.

(c) One may find a 1st-order approximation to the internal energy U near T=0 by setting the activity $\exp(\mu/k_B T)$ equal to 1. Under this approximation, what is the T-dependence of U at low T?

14 Statistical Mechanics Solution

(a) $$N = \int_0^\infty d\epsilon\; \underset{\substack{\uparrow \\ \text{density} \\ \text{of states}}}{g(\epsilon)}\; \underbrace{\frac{1}{e^{(\epsilon-\mu)/kT}-1}}_{\text{Bose distribution}} = \frac{L}{2\pi\hbar c}\int_0^{\infty}\frac{d\epsilon}{e^{(\epsilon-\mu)/kT}-1}$$

(b) let $x = \frac{\epsilon-\mu}{kT}$ $\qquad d\epsilon = kT\,dx$

then $$N = \frac{LkT}{2\pi\hbar c}\int_{-\mu/kT}^{\infty}\frac{dx}{e^x-1} = \frac{LkT}{2\pi\hbar c}\ln(1-e^{-x})\Big]_{-\frac{\mu}{kT}}^{\infty}$$

$$= \frac{-LkT}{2\pi\hbar c}\ln\left(1-e^{\mu/kT}\right)$$

So $$\mu = kT\ln\left\{1-e^{-\frac{N2\pi\hbar c}{LkT}}\right\}$$

(c) $$U = \int_0^\infty d\epsilon\;\epsilon\, g(\epsilon)\frac{1}{e^{\frac{\epsilon-\mu}{kT}}-1}$$

Setting $e^{\mu/kT}=1$,

$$U \cong \frac{L}{2\pi\hbar c}\int_0^\infty d\epsilon\,\frac{\epsilon}{e^{\epsilon/kT}-1} = (kT)^2\frac{L}{2\pi\hbar c}\int_0^\infty\frac{dx\,x}{e^x-1}$$

So the T-dependence is all in the prefactor,

and $\qquad U \propto T^2$

15. <u>Quantum Mechanics</u>

A beam of particles (mass m, momentum $\vec{k}$) is scattered from a fixed, spherically symmetric potential of the form

$$V(r) = \frac{C}{r^2} e^{-ar}$$

where a and C are constants.

(a) Find the differential scattering cross-section $\frac{d\sigma}{d\Omega}$ for scattering from V(r) in the Born approximation.

(b) Under what circumstances - that is, for what range of $|\vec{k}|$ - might you expect the Born approximation to be valid?

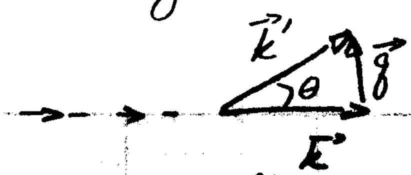
(c) Evaluate the differential cross-section for forward scattering.

(a) Born approx.

$$\frac{d\sigma}{d\Omega} = \left| \frac{m}{2\pi\hbar^2} \int d^3r \; V(r)\, e^{i\vec{q}\cdot\vec{r}} \right|^2$$

where $\vec{q}$ = scattering wavevector

i.e. $q = 2k \sin \theta/2$



$$\int d^3r \; V(r)\, e^{i\vec{q}\cdot\vec{r}} = \frac{2\pi}{iq} \int_0^\infty r\,dr\, V(r)\; 2i \sin qr$$

$$= \frac{4\pi}{q} C \int_0^\infty \frac{dr}{r} e^{-ar} \sin qr = \frac{4\pi}{q} C \int_0^\infty \frac{dx}{x} e^{-\frac{a}{q}x} \sin x$$

Let $I(\alpha) = \int_0^\infty \frac{dx}{x} e^{-\alpha x} \sin x$

- to evaluate, note $I(0) = \int_0^\infty \frac{dx}{x} \sin x = \frac{\pi}{2}$ (in CRC)

and $\frac{dI}{d\alpha} = -\int_0^\infty dx\, e^{-\alpha x} \sin x = -\frac{1}{2i}\left(\frac{1}{\alpha - i} - \frac{1}{\alpha + i}\right)$

$$= -\frac{1}{1+\alpha^2}$$

$$I(\alpha) = I(0) + \int_0^\alpha d\alpha' \left(\frac{-1}{1+\alpha'^2}\right)$$

$$= \frac{\pi}{2} - \arctan \alpha$$

So

$$\frac{d\sigma}{d\Omega} = \left| \frac{m}{2\pi\hbar^2} \frac{4\pi}{q} C \left(\frac{\pi}{2} - \arctan \frac{a}{q}\right)\right|^2$$

$$= \frac{4m^2C^2}{\hbar^4 q^2} \left(\pi/2 - \arctan \frac{a}{q}\right)^2$$

(b) The Born approximation is valid for "large" energies — that is, $\frac{\hbar^2 k^2}{2m}$ large compared with any characteristic energy of $V(r)$. But out of C and a we can construct only the energy Ca^2. So insist

$$\frac{\hbar^2 k^2}{2m} \gg Ca^2$$

$$\text{or } k \gg \sqrt{\frac{2mCa^2}{\hbar^2}}$$

(c) We have $q = 2k \sin \theta/2 \to 0$ for forward scattering. Thus we need to use

$$\arctan a/q \cong \pi/2 - q/a + \ldots$$

and then

$$\frac{d\sigma}{d\Omega} \cong \frac{4m^2C^2}{\hbar^4 q^2}\left(\frac{\pi}{2} - \frac{\pi}{2} + \frac{q}{a} \ldots\right)^2$$

$$\cong \left(\frac{2mC}{\hbar^2}\right)^2 \frac{1}{a^2}$$