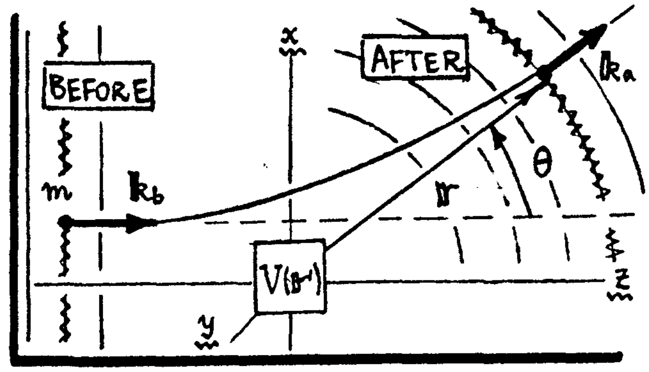
Recollection of the scattering amplitude : $A \rightarrow f_k(\theta)$ for sph. symmetry.

Partial Wave Method of Scattering Analysis [Ref. Davydov Sec. 109].



1) In our previous analysis of scattering via Born approximation, we concluded early on that if the incoming wave were a free-particle planewave $\phi_b(\mathbf{r}) = e^{i\mathbf{k}_b\cdot\mathbf{r}}$, then the scattered wave ψ would consist of ϕ_b plus a <u>spherical</u> <u>wave</u> $\frac{1}{r}e^{ikr}$ generated by ϕ_b's encounter with the scattering potential V. In particular:

$$\psi(\mathbf{r}) = \phi_b(\mathbf{r}) + A(\mathbf{k}_b \rightarrow \mathbf{k}_a)\,\frac{e^{ikr}}{r}, \quad k\hat{\mathbf{r}} = \mathbf{k}_a \text{ (final state momentum)};$$

$$A(\mathbf{k}_b \rightarrow \mathbf{k}_a) = -\frac{m}{2\pi\hbar^2}\int_\infty d^3x'\, e^{-i\mathbf{k}_a\cdot\mathbf{r}'}\, V(\mathbf{r}')\,\psi(\mathbf{r}') . \qquad (1)$$

see Eq. (11), NOTES: p. ScT 6.

A is the "scattering amplitude", and for non-spherically symmetric V's, it may depend on the azimuthal ∡ φ (in xy plane) as well as the scattering ∡ θ. For spherically symmetric V's, A can at most depend on θ and the magnitude of the momentum: $k = \sqrt{2mE/\hbar^2}$; then it is generally written as: $A(\mathbf{k}_b \rightarrow \mathbf{k}_a) = f_k(\theta)$, so that...

$$\rightarrow \psi(\mathbf{r}) = \phi_b(\mathbf{r}) + f_k(\theta)\,\frac{e^{ikr}}{r} \qquad (2)$$

$f_k(\theta)$ is the "scattering amplitude" for a spherically symmetric pot$^{\underline{l}}$ $V = V(r)$.

Furthermore, we saw [Eq. (14), p. ScT 7] that for <u>elastic</u> <u>scattering</u> ($^{\text{w}}|\mathbf{k}_{after}| = |\mathbf{k}_{before}|$), the differential scattering cross-section $\frac{d\sigma}{d\Omega}$ could be expressed in terms of the scattering amplitude... for $\psi(\mathbf{r})$ of Eq. (2):

$$\boxed{d\sigma/d\Omega = |f_k(\theta)|^2}. \qquad (3)$$

We have a prescription for $f_k(\theta)$ from Eq. (1), but it does not display cons$^{\underline{n}}$ of ∡ momentum. We shall now show that $f_k(\theta)$ can be written in terms which <u>do</u> display ∡ momentum states explicitly. This display is the Partial Wave Method.

Expression of $f_k(\theta)$ in a series of ∡ momentum eigenstates.

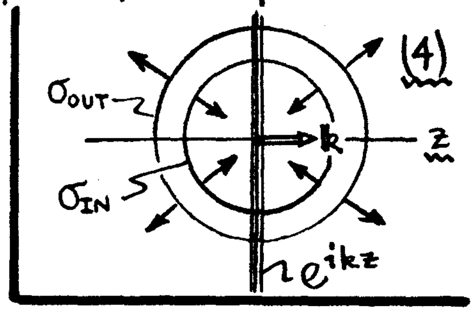
2) There are two ways of arguing that $f_k(\theta)$ in Eq. (2) can be written in terms of ∡ momentum eigenstates. As follows...

① <u>By argument from modified free-particle states.</u>

Start from Eq. (36), p. free 7, for asymptotic form of free-particle planewave:

$$\rightarrow e^{ikz} \underset{r\to\infty}{\simeq} \frac{1}{2i}\sum_{l=0}^{\infty}(2l+1)\left[\underbrace{\left(\frac{e^{+ikr}}{kr}\right)}_{\text{outgoing sph. wave}\ \}\ \sigma_{OUT}} - e^{il\pi}\underbrace{\left(\frac{e^{-ikr}}{kr}\right)}_{\text{incoming sph. wave}\ \}\ \sigma_{IN}}\right]P_l(\cos\theta), \quad (4)$$

$(z = r\cos\theta)$



For no scattering at all (no interaction as $r\to 0$) σ_{IN} must collapse toward the origin and ultimately become part of σ_{OUT}. But if a scattering interaction takes place as $r \to 0$ (via a finite range $V(r)$, $\lim_{r\to\infty} rV = 0$), then <u>$\sigma_{IN} \to \sigma'_{OUT}$</u>, where σ'_{OUT} shows some residual effect of V: $\sigma'_{OUT} \neq \sigma_{OUT}$. Now -- if we are to conserve particles: <u>$|\sigma'_{OUT}|^2 = |\sigma_{OUT}|^2$</u> -- then the effect of V can at most be to shift the <u>phase</u> between σ'_{OUT} (scattered) & σ_{OUT} (free), i.e.

$$\rightarrow \sigma'_{OUT} = (e^{2i\delta_l})\,\sigma_{OUT} \quad (5)$$

$\surd$ phase shift $\delta_l = \delta_l(k)$ is real, and depends on ∡ momentum l & wave # k; $|\sigma'_{OUT}|^2 = |\sigma_{OUT}|^2 \Rightarrow$ particle cons'n.

Then, for a scattering encounter, the free-particle planewave in (4) is modified, as:

$$e^{ikz} \to \psi(r,\theta) = \frac{1}{2i}\sum_{l=0}^{\infty}(2l+1)\left[e^{2i\delta_l}\left(\frac{e^{+ikr}}{kr}\right) - e^{il\pi}\left(\frac{e^{-ikr}}{kr}\right)\right]P_l(\cos\theta);$$

write: $e^{2i\delta_l} = 1 + (e^{2i\delta_l} - 1) = 1 + 2i\,e^{i\delta_l}\sin\delta_l$,

So, scattered wave } $\psi(r,\theta) = e^{ikz} + \underbrace{f_k(\theta)\frac{e^{ikr}}{r}}$,

$$\text{where: } \boxed{f_k(\theta) = \frac{1}{k}\sum_{l=0}^{\infty}(2l+1)\left[e^{i\delta_l}\sin\delta_l\right]P_l(\cos\theta)} \quad (6)$$

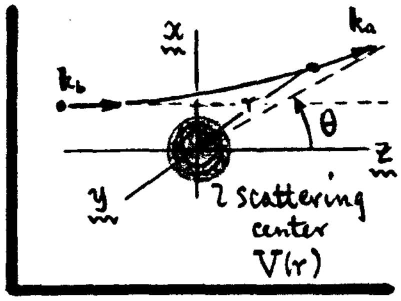
$\surd$ SCATTERING AMPLITUDE in terms of l-waves.

In this view, the scattering is wholly determined in terms of the (so-far not known) phase shifts δ_l. <u>NOTE</u>: for no scattering at all, the δ_l's all $\equiv 0$, so that $f_k(\theta) \equiv 0$.

Second argument for expression of $f_k(\theta)$ in a series of l-waves.

The second way of showing that $f_k(\theta)$ can be expressed as in Eq. (6) goes like...

② By argument from asymptotic forms of scattered states.



The scattered wavefcn ψ, for a central potential $V(r)$ with finite range $[\lim_{r\to\infty} rV(r)=0]$ must be (exactly) of form:

$$\left[\begin{array}{l}\psi(r,\theta)=\sum_{l=0}^{\infty}\frac{c_l}{r}v_{kl}(r)P_l(\cos\theta),\\ \text{w/}\ \left[\frac{d^2}{dr^2}+k^2-\frac{2m}{\hbar^2}V(r)-\frac{l(l+1)}{r^2}\right]v_{kl}(r)=0.\end{array}\right\}\text{exact problem} \tag{7}$$

(the incident wave moves with $\mathbf{k}_b \parallel z$-axis). The $\{c_l\}$ are cnsts. Generally, we can't solve the radial eqtn for the $v_{kl}(r)$. But we do know a limiting case:

→ If $V(r)\equiv 0$: $v_{kl}(r)\propto r j_l(kr)\xrightarrow[r\to\infty]{}\sin(kr-\frac{l}{2}\pi)$, for free particle. (8)

When $V(r)\neq 0$ [but: $\lim_{r\to\infty} rV(r)=0$]†, v_{kl} will still be "free" as $r\to\infty$; it cannot change its r-dependence, but may show a dependence on $\cos(kr-\frac{l}{2}\pi)$. So:

→ $v_{kl}(r)\to\sin(kr-\frac{l}{2}\pi+\delta_l)$, for scattered particle, when $V(r)\neq 0$. (9)

The "phase shifts" $\delta_l=\delta_l(k)$ account for residual distortions by $V(r)$. The asymptotic form of the scattered wave in Eq. (7) is then...

$$\psi(r,\theta)\underset{r\to\infty}{\simeq}\sum_{l=0}^{\infty}\frac{c_l}{r}\sin(kr-\tfrac{l}{2}\pi+\delta_l)P_l(\cos\theta). \tag{10}$$

Now, assume that this scattered wave looks (asymptotically) like ψ of Eq. (2):

$$\psi(r,\theta)\underset{r\to\infty}{\simeq}e^{ikr\cos\theta}+f_k(\theta)\frac{e^{ikr}}{r}, \tag{11}$$

where the incoming planewave e^{ikz} is given by Eq. (4) [w/ $z=r\cos\theta$]. Equate (11) with (10), and insert the planewave expansion for e^{ikz}. Result is...

$$\to f_k(\theta)e^{ikr}=\sum_{l=0}^{\infty}\left[c_l\sin(kr-\tfrac{l}{2}\pi+\delta_l)-\tfrac{1}{k}(2l+1)i^l\sin(kr-\tfrac{l}{2}\pi)\right]P_l(\cos\theta). \tag{12}$$

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† This condition harks back to the Born Approxn--see Eq. (21), p. SCT9. For reasonable applicability, the integral $\int_\infty d^3x\,[V(r)/r]=4\pi\int_0^\infty dr\,[rV(r)]$ must be finite. Hence: $\lim_{r\to\infty} rV(r)=0$.

Second argument for $f_k(\theta) \propto \sum_l (l\text{-waves})$. [cont'd]

This last expression is semi-opaque, but it actually fixes both the expansion coefficients $\{c_l\}$ and the scattering amplitude $f_k(\theta)$. For, if we use: $i^l = e^{i\frac{l\pi}{2}}$, and expand: $\sin x = \frac{1}{2i}(e^{ix} - e^{-ix})$, we find that Eq. (12) can be written:

$$\rightarrow f_k(\theta)\, e^{ikr} = \frac{e^{ikr}}{2i}\sum_{l=0}^{\infty}\overbrace{\left[c_l\, e^{i(\delta_l - \frac{l}{2}\pi)} - \left(\frac{2l+1}{k}\right)\right]}^{①} P_l(\cos\theta) -$$

$$- \frac{e^{-ikr}}{2i}\sum_{l=0}^{\infty}\overbrace{\left[c_l\, e^{-i\delta_l} - \left(\frac{2l+1}{k}\right)e^{i\frac{l}{2}\pi}\right]}^{②} e^{i\frac{l}{2}\pi} P_l(\cos\theta). \qquad (13)$$

Now the coefficients of the $e^{\pm ikr}$ terms on both sides of this eqtn must be equal (why?). Since there are no e^{-ikr} terms on the LHS, then...

$$[②] \equiv 0 \Rightarrow \underline{\underline{c_l = \left(\frac{2l+1}{k}\right)e^{i(\delta_l + \frac{l}{2}\pi)}}}. \qquad (14)$$

This fixes the c_l, as advertised. Now if we use these c_l in [①] in (13)...

$$[①] = \frac{2l+1}{k}\left(e^{2i\delta_l} - 1\right) = \left(\frac{2l+1}{k}\right)\cdot 2i\, e^{i\delta_l}\sin\delta_l, \qquad (15)$$

then, using this result in (13), and cancelling e^{ikr} on both sides, we get:

$$\boxed{f_k(\theta) = \frac{1}{k}\sum_{l=0}^{\infty}(2l+1)\left[e^{i\delta_l}\sin\delta_l\right]P_l(\cos\theta)} \qquad (16)$$

SCATTERING AMPLITUDE in terms of l-waves.

This expression for $f_k(\theta)$ is the same as we derived in Eq. (6). So the scattering amplitude as an l-wave expansion can be justified equally well by considering slightly perturbed free-particle states, or mostly-free perturbed states. In both cases, we get $f_k(\theta)$ per Eq. (16) -- wholly determined by the hypothesized "phase shifts" $\delta_l = \delta_l(k)$.

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3) Assuming we can calculate, or measure, the phase shifts δ_l for a given scattering potential $V(r)$, the problem is ~ solved. E.g. by Eqs. (3) & (16):

$$\underline{\underline{d\sigma/d\Omega = |f_k(\theta)|^2 = \frac{1}{k^2}\left|\sum_{l=0}^{\infty}(2l+1)\left[e^{i\delta_l}\sin\delta_l\right]P_l(\cos\theta)\right|^2.}} \qquad (17)$$