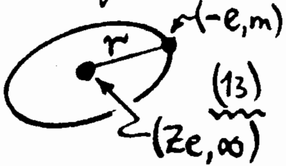
H-atom Radial Equation (Davydov ¶ 38) & The Saga of the $F(a;b;z)$'s.

3) To get down to cases with the H-like atom, put $V(r) = -Ze^2/r$ in Eq.(3), so...

$$\left[\!\left[\frac{d^2R}{dr^2} + \frac{2\mu}{\hbar^2}\left[E + \frac{Ze^2}{r} - \frac{l(l+1)\hbar^2}{2\mu r^2}\right]R = 0\right]\!\right] \begin{cases} l = 0, 1, 2, \ldots \\ \psi = \frac{1}{r} R Y_{lm} \end{cases} \tag{13}$$



It is traditional (and useful) to write this equation in dimensionless variables, adopting the following "atomic units" of length and energy...

Nonrel. H-atom

LENGTH = Bohr radius : $\underline{a_0} = \hbar^2/me^2 = 0.529 \times 10^{-8}$ cm (m=electron mass);

ENERGY = Hartree : $\underline{E_0} = e^2/a_0 = \alpha^2 mc^2 = 27.1$ eV (2x H-atom ionization);

Here : $\underline{\alpha} = e^2/\hbar c = 1/137.036$ (±1 ppm) is the fine-structure cnst. (14)

In these units, we define dimensionless variables ρ & ϵ, and rewrite (13)...

$$\left[\begin{array}{l} \rho = r/a_0, \text{ radial position in units of Bohr radii,} \\ \epsilon = E/E_0, \text{ electron energy in units of Hartrees;} \\ \left\{\frac{d^2}{d\rho^2} + \left[2\epsilon + \frac{2Z}{\rho} - \frac{l(l+1)}{\rho^2}\right]\right\} R(\rho) = 0. \end{array}\right. \tag{15}$$

We are not including reduced mass corrections [in (14), we've already let $\mu = \frac{mM}{m+M} \to m$, as for an ∞ly heavy nucleus], and -- in addition to ignoring electron & nuclear spins [meaning there are no magnetic energies in the problem] -- we assume the nucleus is a point, so the Coulomb potential holds all the way down to $\rho = 0$.

For bound states $\epsilon < 0$, and we define : $\underline{\kappa^2 = -2\epsilon = 2|E|/E_0}$. Asymptotics are :

$$\left.\begin{array}{l} \underline{\rho \to \infty} \Rightarrow \left\{\frac{d^2}{d\rho^2} - \kappa^2\right\} R(\rho) \simeq 0, \text{ so } R(\rho) \propto e^{\pm\kappa\rho}. \\ \qquad \text{For } R(\infty) \to 0, \text{ choose only : } \underline{R(\rho) \propto e^{-\kappa\rho}, \text{ as } \rho \to \infty}. \\ \underline{\rho \to 0} \Rightarrow \left\{\frac{d^2}{d\rho^2} - \frac{l(l+1)}{\rho^2}\right\} R(\rho) \simeq 0, \text{ so } R(\rho) \propto \rho^{l+1}, \text{ or } \rho^{-l}. \\ \qquad \text{For } R(0) = 0, \text{ choose only : } \underline{R(\rho) \propto \rho^{l+1}, \text{ as } \rho \to 0}. \end{array}\right\} \tag{16}$$

Factor out the asymptotic dependences, and rewrite the radial eqtn (15) as...

$$\left[\begin{array}{l} R(\rho) = \text{cnst} \cdot \rho^{l+1} e^{-\kappa\rho} f(\rho), \\ \text{w/ } \rho \frac{d^2 f}{d\rho^2} + [2(l+1) - 2\kappa\rho] \frac{df}{d\rho} + [2Z - 2\kappa(l+1)] f = 0. \end{array}\right. \tag{17}$$