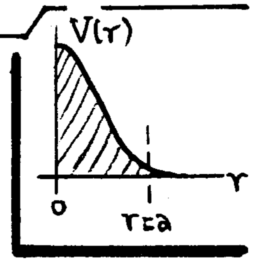
3. On the RHS of (31), the v_{kl} are generally complicated fcns of the δ_l. But in the lowest order of approx$^{\underline{n}}$, we can take: $\underline{v_{kl}(x) \simeq [\cos\delta_l]\cdot kx\, j_l(kx)}$, i.e. a free particle wavefcn (comp. w/ Eq.(9)). This Ansatz essentially amounts to a first Born Approx$^{\underline{n}}$ to the δ_l's. Using this approx$^{\underline{n}}$ in (31), while taking $r\to\infty$...

$$\boxed{\tan\delta_l(k) \simeq -(2m/\hbar^2 k)\int_0^\infty [kx\, j_l(kx)]^2\, V(x)\, dx}. \tag{32}$$

The upper limit $x\to\infty$ means distances at which $V(x)\to$ negligible. The coefficient out in front ($\propto 1/k$) implies $\delta_l(k)\to 0$ at high energy. The approx$^{\underline{n}}$ in (32) should be good when $V(x)$ is "weak", so $|\delta_l(k)| \ll 1$.



4. In (32), suppose V is well-localized, so that $V(r)$ is ~ negligible beyond some $r\sim a$, and suppose the particle energy is low enough so that $ka \ll 1$. Then $j_l(z) \simeq z^l/(2l+1)!!$ is in its asymptotic regime wherever V is appreciable. With r as integration variable, the integral cuts off @ $r=a$, where $V\to 0$. For such a "<u>hard-core</u>" $V(r)$, then:

$$\rightarrow \tan\delta_l(k) \simeq -\frac{2mk}{\hbar^2}\int_0^a \left[r\cdot\frac{(kr)^l}{(2l+1)!!}\right]^2 V(r)\, dr,$$

or

$$\underline{\underline{\tan\delta_l(k) \simeq -\frac{2m}{\hbar^2}\cdot\frac{(ka)^{2l+1}}{[(2l+1)!!]^2}\cdot\int_0^a (r/a)^{2l+1}[r\,V(r)]\, dr,\quad ka\ll 1.}} \tag{33}$$

If $V(r)\sim V_0 =$ cnst over $0 \lesssim r \sim a$, then (33) yields...

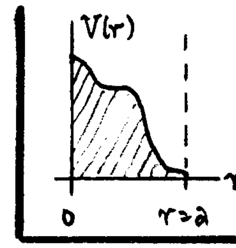
$$\left.\begin{array}{l} \left\|\delta_l(k)/\delta_0(k) \simeq \dfrac{3}{2l+3}(ka)^{2l}/[(2l+1)!!]^2,\ \text{for } ka\ll 1; \right. \\ \left\| \text{w/ } \delta_0(k) = -\tfrac{1}{3}[V_0/(\hbar^2/2ma^2)]\, ka \leftarrow \text{S-wave } (l=0) \text{ phase shift.}\right. \end{array}\right\} \tag{34}$$

Such phase shifts as in Eqs. (33) & (34) obey the general features of Eqs. (21), (22) & (25), namely: $\underline{\delta_l(k)\to 0, \text{ when } k\to 0}$; $\underline{\delta_l(k)\equiv 0, \text{ if } V\equiv 0}$; and $\underline{\delta_l(k)\to 0, \text{ when } l\to\infty}$. They also grow smaller very rapidly: $\delta_1/\delta_0 \simeq \frac{1}{15}(ka)^2$, $\delta_2/\delta_0 \simeq \frac{1}{525}(ka)^4$, etc. As $k\to 0$, the only <u>important</u> phase shift is the S-wave ($l=0$).

Hard-core scattering: the exterior solution.

5) We can treat "hard-core" scattering in a different way -- which does not depend on integral approxns as in Eqs. (33) & (34). Suppose we have a "hard-core" potential:

$$\rightarrow V = V(r), \text{ for } 0 \leq r \leq a \text{ ; } V = 0, \text{ for } r > a. \tag{35}$$



This V is of a kind often seen in nuclear physics: it could be the strong interaction between a nucleus (as target) and a nucleon (as projectile). For $r<a$, the radial wavefcn $u_{kl}(r) = rR_{kl}(r)$ obeys Eq. (26); for $r>a$, since $V=0$, the radial eqtn is that of a free particle, i.e. [Eq. (27)]:

$$\left[\left[\frac{d^2}{dr^2} + k^2 - \frac{l(l+1)}{r^2}\right] r R_{kl}(r) = 0, \text{ for } r>a \ \left(\text{w/ } k^2 = 2mE/\hbar^2\right). \tag{36}\right.$$

So long as $r>0$, acceptable solutions to this eqtn are <u>both</u> types of spherical fcns, viz. $j_l(kr)$ & $n_l(kr)$; see Eq. (25), p. free 5. We thus take the linear combn:

$$\rightarrow \underline{R_{kl}(r) = Aj_l(kr) + Bn_l(kr)}, \text{ for } r>a, \text{ w/ } A \,\&\, B = \text{cnsts}. \tag{37}$$

Recalling the asymptotic behavior of these fcns (p. free 4, (21) & (22); p. free 6, Eq. (26)):

$$\left[\left[j_l(x) \simeq \begin{cases} x^l/(2l+1)!!, & \text{for } x \ll l; \\ \frac{1}{x}\sin\left(x - \frac{l}{2}\pi\right), & \text{for } x \gg l. \end{cases} \right\| \ n_l(x) \simeq \begin{cases} -(2l-1)!!/x^{l+1}, & \text{for } x \ll l; \\ -\frac{1}{x}\cos\left(x - \frac{l}{2}\pi\right), & \text{for } x \gg l. \end{cases}\right. \tag{38}$$

we can -- from (37) -- write the asymptotic behavior of $R_{kl}(r)$ as $r \to \infty$...

$$\rightarrow R_{kl}(r) \rightarrow \frac{1}{r}\left[\frac{A}{k}\sin\left(kr - \frac{l\pi}{2}\right) - \frac{B}{k}\cos\left(kr - \frac{l\pi}{2}\right)\right], \text{ as } r \to \infty. \tag{39}$$

The phase shifts $\delta_l(k)$ made their appearance when -- in Eq. (9) -- we claimed we could equally well write the asymptotic form for $R_{kl}(r)$ as...

$$\rightarrow R_{kl}(r) \rightarrow \frac{1}{r}\sin\left(kr - \frac{l\pi}{2} + \delta_l\right), \text{ as } r \to \infty;$$

$$= \frac{1}{r}\left[(\cos\delta_l)\sin\left(kr - \frac{l\pi}{2}\right) + (\sin\delta_l)\cos\left(kr - \frac{l\pi}{2}\right)\right]. \tag{40}$$

Identifying (40) & (39), we see: $A = k\cos\delta_l$, $B = -k\sin\delta_l$. Hence, in (37)...

$$\rightarrow \underline{R_{kl}(r) = \mathcal{N}k\left[(\cos\delta_l)j_l(kr) - (\sin\delta_l)n_l(kr)\right]}, \text{ for all } r \geq a. \tag{41}$$

Phase shifts for hard-core scattering.

In (41), $\mathcal{N}$ is a norm cnst, and the solution must hold for all exterior points $r > a$, where $V = 0$. The solution for interior points $(r < a)$ will depend on the detailed behavior of $V(r)$ in that region; it is a generally unsolvable problem.

BUT... even w/o an explicit form for $V(r)$ @ $0 \leq r < a$, we can write a continuity eqtn at $r = a$, which must hold no matter what $V(r)$ is. Namely, we must have the wavefcn $R_{kl}(r)$ and its derivative $R'_{kl}(r) = \frac{d}{dr} R_{kl}(r)$ both continuous at $r = a$. So the ratio must be finite, i.e.

$$\frac{R'_{kl}(a)}{R_{kl}(a)} = k\left[\frac{(\cos\delta_l)\, j'_l(ka) - (\sin\delta_l)\, n'_l(ka)}{(\cos\delta_l)\, j_l(ka) - (\sin\delta_l)\, n_l(ka)}\right] = \Gamma_l(ka) \quad \text{↖ } \Gamma_l = \text{some finite \# @ } r = a. \tag{42}$$

This eqtn can be solved for the phase shifts δ_l in terms of Γ_l as...

$$\boxed{\tan\delta_l(k) = [k j'_l(ka) - \Gamma_l(ka) j_l(ka)] / [k n'_l(ka) - \Gamma_l(ka) n_l(ka)]}. \tag{43}$$

The problem of finding the $\delta_l(k)$ is thereby reduced to finding (or estimating) the logarithmic derivative: $\Gamma_l(kr) = \frac{d}{dr} \ln R_{kl}(r)$ at the boundary, $r = a$.

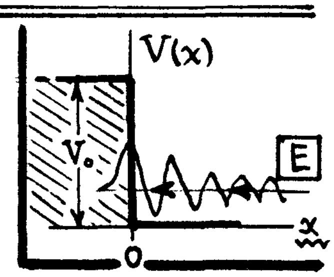
Without knowing $V(r)$ explicitly even yet, we can make a useful approx^{n} to (43) for low-energy scattering. For $k \ll \Gamma_l(ka)$,★ Eq. (43) yields...

$$\rightarrow \tan\delta_l(k) \simeq [j_l(ka)/n_l(ka)] - \cancel{\mathcal{O}(k/\Gamma)}^{\nearrow 0}, \quad \text{for } k \ll \Gamma_l. \tag{44}$$

Use the asymptotic forms in Eq. (38) for $ka \to 0$. Then (44) gives...

$$\boxed{\delta_l(k) \simeq -C_l (ka)^{2l+1}, \text{ w/ } C_l = (2l+1)/[(2l+1)!!]^2}, \text{ for } ka \ll l. \tag{45}$$

★ For scattering from a step-fcn potential @ energy $E < V_0$, as sketched at right, it is ~ easy to compare the log derivative $\Gamma = \frac{1}{\psi}(d\psi/dx)\big|_{x=0}$ to $k = \sqrt{2mE/\hbar^2}$. Result: $\underline{\Gamma/k = \sqrt{(V_0 - E)/E}}$. Evidently $\Gamma \gg k$ when $V_0 \gg E$. This condition justifies Eq. (44).



Hard-core scattering cross-sections σ & $d\sigma/d\Omega$ for S & P waves.

This result ¶ is similar to the Born approxn in Eq.(33), but is somewhat cruder (it contains no explicit mention of V). However, the phase shifts in (45) do obey the general conditions of Eqs.(21),(22) & (25): $\delta_\ell(k) \to 0$ for $k \to 0$, or $V \to 0$ ($a \to 0$), or $\ell \to \infty$. The condition $ka \ll \ell$ means that V's range: $a \ll \ell/k = b$, the impact parameter. So "a" is small for all but scattering at $\ell = 0$; this "hard-core" approximation really amounts to considering that S-wave scattering dominates.

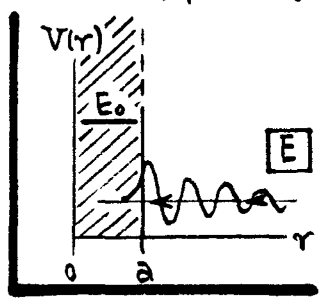
Finally, with (45), the partial-wave cross-sections of Eq.(19) are...

$$\sigma_\ell(k) \simeq \frac{4\pi}{k^2}(2\ell+1)\,\delta_\ell^2(k) = 4\pi a^2 \cdot \left[(2\ell+1)\,C_\ell^2\,(ka)^{4\ell}\right], \tag{46}$$

so// $\sigma_0(k) = 4\pi a^2$ {S-wave}, $\sigma_1(k) = 4\pi a^2 \cdot \frac{1}{3}(ka)^4$ {P-wave}, etc.

The S-wave cross-section $\sigma_0(k)$ is indpt of energy, and is just $4\times$ the geometrical cross-section πa^2 [the "4" is due to diffraction effects; Sakurai, p. 408].

The total cross-section is approximated at low energy by...



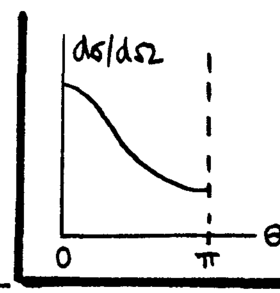
$$\sigma(E) \simeq \sigma_0(k) + \sigma_1(k) \simeq 4\pi a^2\left[1 + \frac{1}{3}(E/E_0)^2\right], \tag{47}$$

w/ $E \ll E_0 = \hbar^2/2ma^2$ ~ typical particle energy in V.

In the same approxn, the differential scattering cross-section [Eq.(17)] is...

$$\frac{d\sigma}{d\Omega} \simeq \frac{1}{k^2}\left|\sum_{\ell=0}^{\infty}(2\ell+1)\,\delta_\ell(k)\,P_\ell(\cos\theta)\right|^2 \simeq \frac{1}{k^2}\left[\delta_0(k) + 3\,\delta_1(k)\cos\theta\right]^2$$

$$\text{n//}\quad \frac{d\sigma}{d\Omega} \simeq a^2\left[\underbrace{1}_{\text{S-wave}} + \underbrace{(E/E_0)\cos\theta}_{\text{P-wave}}\right]^2,\quad E \ll E_0. \tag{48}$$



NOTE: it is an interference between S & P waves which gives $d\sigma/d\Omega$ a weak dependence on scattering $\measuredangle\,\theta$.

In (45), we should be a bit careful for $\ell = 0$. Using the exact forms for j_0 & n_0:
$\tan\delta_0(k) \simeq j_0(ka)/n_0(ka) = -\left(\frac{\sin ka}{ka}\right)/\left(\frac{\cos ka}{ka}\right) = -\tan ka$, so: $\delta_0(k) \simeq -ka$ "exactly". This corresponds to $C_0 = 1$ in Eq.(45), and: $\sigma_0(k) \simeq 4\pi a^2$, in Eq.(46).