ϕ507 MidTerm Exam $\left\{\begin{array}{l}\text{in class}\\ \text{2hr. limit}\end{array}\right.$ Mon. 22 Mar. 1993

This exam is open-book, open-notes, and is worth 160 pts. total. For each of the 4 problems, $\boxed{\text{box}}$ the answer on your solution sheet. $\boxed{\text{Number}}$ your solution pages, put your $\boxed{\text{name}}$ on p.1, and $\boxed{\text{staple}}$ pages together before handing them in.

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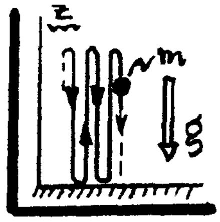
① [40 pts.]. A ball of mass m bounces vertically along the z-axis in a uniform gravitational field w/ acceleration g = cnst. The motion is perfectly elastic (i.e. the plane at $z=0$ is perfectly reflecting).



(A) Use the Bohr-Sommerfeld rule to find m's allowed energy levels E_n.

(B) Find m's classical bounce frequency ω (i.e. $\omega = 2\pi/\text{bounce period}$). Show, as $n \to$ large, that the spacing between adjacent energy levels ($\Delta n = 1$) is $\Delta E_n = \hbar\omega$.

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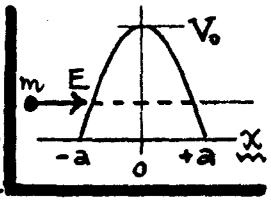
② [40 pts.]. A particle of mass m and energy $E>0$ is incident along the x-axis on a 1D parabolic barrier w/ potential: $V(x) = V_0\left[1-\left(\frac{x}{a}\right)^2\right]$.



(A) If $E \ll V_0$, find the transmission coefficient $T(E)$ for m penetrating the barrier. Show that: $T(E) = \exp\left[-\frac{1}{U}(V_0 - E)\right]$, and specify the cnst U.

(B) Suppose a particle beam impinges on $V(x)$. If the particle energies are uniformly distributed over $\bar{E} \pm \frac{1}{2}\Delta E$, find the fraction of the beam that penetrates $V(x)$.

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③ [40 pts.]. If the scattering potential has the periodicity property that -- for $\mathbf{a}$ a constant vector: $V(\mathbf{r}+\mathbf{a}) = V(\mathbf{r})$, show that in the first Born approximation the scattering of an incident particle vanishes unless: $\mathbf{q}\cdot\mathbf{a} = 2n\pi$, w/ $n = 0, 1, 2, \ldots$ Here: $\mathbf{q} = \mathbf{k}(\text{before}) - \mathbf{k}(\text{after})$, is the usual momentum transfer.

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④ [40 pts.]. Consider low energy scattering (S-wave only) from a hard sphere potential: $V(r) = \begin{cases} -V_0, & 0 \leq r \leq a; \\ 0, & r \geq a. \end{cases}$ The projectile energy $E \ll V_0$. By requiring continuity at $r=a$ for the interior ($r<a$) and exterior ($r>a$) wavefunction and its derivative, find an expression determining the S-wave phase shift $\delta_0(k)$, w/ $k = \sqrt{2mE/\hbar^2}$. Solve for $\delta_0(k)$ (approximately) when both ka & $\delta_0 \ll 1$. Find the scattering amplitude $f_k(\theta)$, and write expressions for both the differential & total cross-sections $d\sigma/d\Omega$ & σ.

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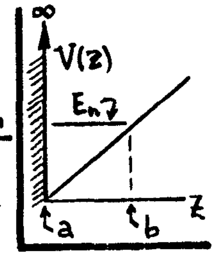

① [40 pts.]. Bohr-Sommerfeld quantization of an elastically bouncing ball.

1) For the ball of mass m, the potential is: $V(z) = \begin{cases} mgz, & z>0 \\ \infty, & z<0 \end{cases}$, and

(A) if its total energy is E, the turning points are at $a=0$ (the elastic surface) and $b = E/mg$ (where $E = V(z)$). The Bohr-Sommerfeld rule [Class notes, p. WKB 18, Eq. (52)] is*

$$(n+\tfrac{1}{2})\pi\hbar = \int_a^b [2m(E-V(z))]^{\frac{1}{2}} dz = \sqrt{2mE}\int_0^{E/mg} [1-(mg/E)z]^{\frac{1}{2}} dz, \quad (1)$$

where: $n = 0, 1, 2, \ldots$ Define $y = (mg/E)z$ and do the integral...

$$(n+\tfrac{1}{2})\pi\hbar = \sqrt{2mE}\left(\frac{E}{mg}\right)\int_0^1 (1-y)^{\frac{1}{2}} dy = \sqrt{\frac{2}{mg^2}}\, E^{3/2}\cdot\frac{2}{3};$$

... Solve for E: $\boxed{E_n = \left[\frac{9}{8} mg^2 \pi^2 \hbar^2\right]^{1/3} (n+\tfrac{1}{2})^{2/3}}$, $n = 0, 1, 2, \ldots$ (2)

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2) Classically, the ball reaches max height $b = E/mg$, and takes time $t =$

(B) $\sqrt{2b/g}$ to fall to the surface. The bouncing period is $T = 2\times$ this, and the natural frequency of bouncing is...

$$\rightarrow \omega = 2\pi/T = 2\pi/2\sqrt{2b/g} = \pi\sqrt{\frac{mg^2}{2E}}, \quad \text{so}\ \frac{1}{\sqrt{E}} = \omega\sqrt{\frac{2}{\pi^2 mg^2}}. \quad (3)$$

When $n \rightarrow$ large, E_n in (2) is $\sim$ continuous fcn of n, and we may differentiate

$$\rightarrow \frac{dE_n}{dn} = \left[\frac{9}{8} mg^2\pi^2\hbar^2\right]^{1/3}\cdot\frac{2}{3}\Big/(n+\tfrac{1}{2})^{\frac{1}{3}} = \frac{2}{3}\left[\frac{9}{8} mg^2\pi^2\hbar^2\right]^{1/2}\frac{1}{\sqrt{E_n}}.$$

... combine with (3) above $\Rightarrow$ $\underline{dE_n/dn = \hbar\omega_n}$. (4)

Here we've taken the quantized frequency: $\underline{\omega_n = \pi\sqrt{mg^2/2E_n}}$, after Eq. (2). From (4), we get the spacing $\Delta E_n = dE_n$ between adjacent levels $\Delta n = dn = 1$, as

$\boxed{\Delta E_n = \hbar\omega_n}$. (5) This SHO-like behavior ($\sim$ equally spaced levels in the neighborhood of E_n, $n \rightarrow$ large) is a general feature of WKB energies.

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* The presence of the ∞ wall at $z=0$ modifies the BS rule a bit: $(n+\frac{1}{2})$ becomes $(n+\frac{3}{4})$.

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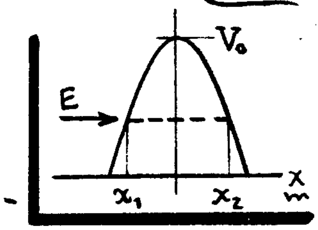

② [40 pts.]. QM tunneling through a parabolic barrier.

(A) **1.** $T(E) = \exp\left\{-\frac{2}{\hbar}\int_{x_1}^{x_2}\sqrt{2m[V(x)-E]}\,dx\right\}$, is the transmission coefficient [from class notes, p. WKB 23, Eq. (11)]. For $V(x) = V_0[1-(x/a)^2]$, the turning points [w/ $E = V(x)$] are at: $x_{1,2} = \mp a\sqrt{1-(E/V_0)}$, symmetric, and so...

$$T(E) = \exp\left\{-\frac{2}{\hbar}\cdot 2\int_0^{x_2}\sqrt{2m\left[V_0\left(1-\frac{x^2}{a^2}\right)-E\right]}\,dx\right\},\quad x_2 = a\sqrt{1-(E/V_0)}$$

$$= \exp\left\{-\frac{4}{\hbar}\sqrt{2mV_0}\,\frac{1}{a}\int_0^{x_2}\sqrt{x_2^2-x^2}\,dx\right\} \leftarrow \text{let } u = x/x_2$$

$$= \exp\left\{-\frac{4}{\hbar}\sqrt{\frac{2m}{V_0}}\,a(V_0-E)\int_0^1\sqrt{1-u^2}\,du\right\}. \quad (1)$$

But $\int_0^1\sqrt{1-u^2}\,du = \frac{1}{2}\sin^{-1}(1) = \frac{\pi}{4}$. Then the transmission coefficient is

$$\boxed{T(E) = \exp\{-(V_0-E)/U\}},\ \text{w/}\ U = \frac{1}{\pi}\sqrt{(\hbar^2/2ma^2)V_0}. \quad (2)$$

This WKB estimate will be ~ good if $E \ll V_0$ (not too close to top of barrier).

(B) **2.** For a particle beam with energies uniformly distributed in $\bar{E} \pm \frac{1}{2}\Delta E$, the probability of finding a particle in range dE at energy E is $dE/\Delta E$. So long as the maxm energy $\bar{E} + \frac{1}{2}\Delta E < V_0$, the overall probability of this particle penetrating the barrier is: $T(E)\cdot dE/\Delta E$, w/ $T(E)$ in Eq. (2). Over the entire beam energy spread, $\bar{E} - \frac{1}{2}\Delta E \leqslant E \leqslant \bar{E} + \frac{1}{2}\Delta E$, the penetration probability -- which is the same as the fractional transmission -- is

$$\rightarrow P(E) = \int_{E_1}^{E_2} T(E)\,dE/\Delta E = \frac{1}{\Delta E}\int_{E_1}^{E_2} e^{-\left(\frac{V_0-E}{U}\right)}dE,\ \text{w/}\ E_{1,2} = \bar{E} \mp \tfrac{1}{2}\Delta E$$

$$= \left(e^{-V_0/U}\right)\frac{U}{\Delta E}\int_{y_1}^{y_2} e^y\,dy,\ \text{w/}\ y_{1,2} = \frac{1}{U}\left(\bar{E} \mp \tfrac{1}{2}\Delta E\right)$$

or

$$\boxed{P(E) = \left[\frac{\sinh(\Delta E/2U)}{\Delta E/2U}\right]e^{-\frac{1}{U}(V_0-E)}} \quad (3)$$

When the spread $\Delta E \to 0$, the [ ] $\to 1$; we recover $T(E)$ of (2).

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(3) [40 pts.]. Scattering from a periodic potential: $V(\mathbf{r}+\mathbf{a}) = V(\mathbf{r})$.

1. In Born Approxn, the diff'l scattering cross-section is [Class notes, p. ScT 12, Eq (28)]:

$$\rightarrow \frac{d\sigma}{d\Omega} = \left(\frac{m}{2\pi\hbar^2}\right)^2 |\tilde{V}(\mathbf{q})|^2, \text{ w/ } \underline{\underline{\tilde{V}(\mathbf{q})}} = \int_\infty V(\mathbf{r}') e^{i\mathbf{q}\cdot\mathbf{r}'} d^3x' \;\; \Big\lceil \underline{\mathbf{q} = \mathbf{k}_{before} - \mathbf{k}_{after} = \text{momentum transfer.}} \quad (1)$$

The required scattering periodicity (i.e. scattering only at $\mathbf{q}\cdot\mathbf{a} = 2n\pi$) must be a feature of the Fourier transform $\tilde{V}(\mathbf{q})$ of a periodic $V(\mathbf{r})$.

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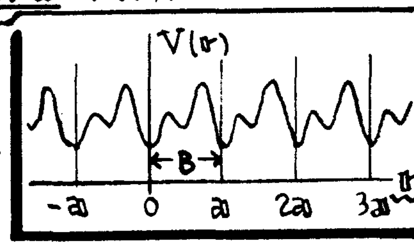
2. A periodic $V(\mathbf{r})$ is defined in a basic interval B (i.e. $0 \leqslant \mathbf{r} \leqslant \mathbf{a}$, symbolically); it is zero outside B, but repeats itself so that $V(\mathbf{r}+\lambda\mathbf{a}) = V(\mathbf{r})$, for $\lambda = 0, \pm1, \pm2, \ldots$ In fact we can represent such a fcn by the ∞ sum...



$$\underline{V(\mathbf{r}) = \sum_{\lambda=-\infty}^{\lambda=+\infty} V(\mathbf{r}+\lambda\mathbf{a}).} \quad (2)$$

For this representation, it is easy to show that: $V(\mathbf{r}+\mathbf{a}) = V(\mathbf{r})$, so the periodicity condition is OK.

Using Eq (2) for $\tilde{V}(\mathbf{q})$ in (1):

$$\rightarrow \tilde{V}(\mathbf{q}) = \sum_{\lambda=-\infty}^{\lambda=+\infty} \int_\infty V(\mathbf{r}'+\lambda\mathbf{a}) e^{i\mathbf{q}\cdot\mathbf{r}'} d^3x' \;\; \Big\lceil \text{change integration variables to } \mathbf{r} = \mathbf{r}'+\lambda\mathbf{a}; \text{ note: } V(\mathbf{r}) \text{ vanishes outside interval B.}$$

$$= \sum_{\lambda=-\infty}^{\lambda=+\infty} e^{-i\lambda\mathbf{q}\cdot\mathbf{a}} \int_B V(\mathbf{r}) e^{i\mathbf{q}\cdot\mathbf{r}} d^3x = \underline{\tilde{V}_B(\mathbf{q})\, S(\mathbf{q}\cdot\mathbf{a})}, \quad (3)$$

$$\text{w/ } S(\phi) = \sum_{\lambda=-\infty}^{\lambda=+\infty} e^{-i\lambda\phi} = \sum_{\lambda=0}^{\infty} (e^{i\phi})^\lambda + \sum_{\lambda=0}^{\infty} (e^{-i\phi})^\lambda - 1, \text{ w/ } \phi = \mathbf{q}\cdot\mathbf{a} \quad (4)$$

$\tilde{V}_B(\mathbf{q})$ is $V(\mathbf{r})$'s Fourier Transform over its basic interval; the sum $S(\phi) \Rightarrow$ periodicity.

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3. Clearly, $S(\phi) \rightarrow \infty$ when $\phi = 2n\pi$ ($n = 0, 1, 2, \ldots$), for then it is an ∞ series of ones. When $\phi \neq 2n\pi$, use the geometric series $\left[\sum_{\lambda=0}^{N} r^\lambda = (1-r^{N+1})/(1-r)\right]$ to sum Eq. (4):

$$\rightarrow S(\phi) = \lim_{N\to\infty} \left\{ \frac{1-e^{i(N+1)\phi}}{1-e^{i\phi}} + \frac{1-e^{-i(N+1)\phi}}{1-e^{-i\phi}} - 1 \right\} = \lim_{N\to\infty} \left\{ \frac{\cos N\phi - \cos(N+1)\phi}{1-\cos\phi} \right\}$$

$$\text{or/ } \underline{S(\phi) = \lim_{N\to\infty} \left\{ \sin[(N+\tfrac{1}{2})\phi] / \sin\tfrac{\phi}{2} \right\},} \quad (5)$$

When $\phi \neq 2n\pi$, $S(\phi)$ is well-behaved, but tends to zero because of the rapidly oscillating numerator. Then, indeed:

$$\boxed{\frac{d\sigma}{d\Omega} \propto |\tilde{V}_B(\mathbf{q})\, S(\mathbf{q}\cdot\mathbf{a})|^2 \cong 0, \text{ unless } \mathbf{q}\cdot\mathbf{a} = 2n\pi.} \quad (6)$$

The Born Approxn $\Rightarrow$ "strong" (∞) scattering when $\mathbf{q}\cdot\mathbf{a} = 2n\pi$.

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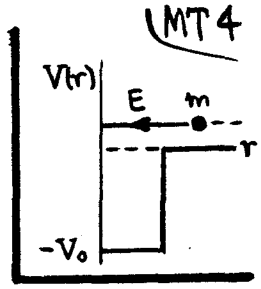

④ [40 pts.]. S-wave (low energy) scattering from a hard sphere.

1. The radial eqtn for $l=0$ is: $[d^2/dr^2 + k^2 - \frac{2m}{\hbar^2}V(r)]\,v_{k0}(r) = 0$; see class notes, p. PW3, Eq. (7). For this case, for the two regions:

$$\underline{\text{exterior } (r \geq a)}: \left(\frac{d^2}{dr^2} + k^2\right) v_{k0}(r) = 0, \;\text{w/}\; k = \sqrt{\frac{2mE}{\hbar^2}} \Rightarrow \underline{\underline{v_{k0}(r) = A\sin(kr + \delta_0)}};$$

$$\underline{\text{interior } (r \leq a)}: \left(\frac{d^2}{dr^2} + K^2\right) v_{k0}(r) = 0, \;\text{w/}\; K^2 = K_0^2 + k^2,\; K_0 = \sqrt{2mV_0/\hbar^2} \Rightarrow \underline{\underline{v_{k0}(r) = B\sin Kr}}. \quad (1)$$

A & B are cnsts. $v_{k0}(r<a)$ is chosen to be well-behaved as $r \to 0$ ($\frac{1}{r}v_{k0}$ must be finite at $r=0$). $v_{k0}(r>a)$ continues $v_{k0}(r<a)$ across the boundary at $r=a$, and it involves the S-wave phase shift $\delta_0 = \delta_0(k)$ of interest.

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2. Continuity in v & v' at $r=a$ $\left[\text{i.e. } \left(\frac{v'}{v}\right)\Big|_{r=a+\epsilon} = \left(\frac{v'}{v}\right)\Big|_{r=a-\epsilon}, \text{ as } \epsilon \to 0\right]$ implies:

$$\boxed{k\,\mathrm{ctn}(ka + \delta_0) = K\,\mathrm{ctn}\,Ka}, \;\text{w/}\; \underline{k} = \sqrt{\frac{2mE}{\hbar^2}},\; \underline{K} = \sqrt{K_0^2 + k^2} \;\&\; \underline{K_0} = \sqrt{\frac{2mV_0}{\hbar^2}}. \quad (2)$$

This expression determines $\delta_0(k)$. When ka & δ_0 are both small, it is approximately (since $\mathrm{ctn}\,x \simeq 1/x$, as $x \to 0$)...

$$\rightarrow k/(ka + \delta_0) \simeq K\,\mathrm{ctn}\,Ka \Rightarrow \underline{\underline{\delta_0(k) \simeq ka\left[\left(\frac{\tan Ka}{Ka}\right) - 1\right]}}. \quad (3)$$

The case $Ka \to \pi/2$ implies $\delta_0(k)$ becomes very large; this is "resonance" scattering, which we shall not consider here.

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3. The various quantities required, for S-wave scattering, are (from class notes):

Scattering amplitude [p. PW4, Eq. (16)]: $\underline{f_k(\theta) \simeq \frac{1}{k}e^{i\delta_0}\sin\delta_0 \simeq a\left[\left(\frac{\tan Ka}{Ka}\right) - 1\right]}$; (4A)

differential cross-section [p. PW4, Eq. (17)]: $\underline{\frac{d\sigma}{d\Omega} = |f_k(\theta)|^2 \simeq a^2\left[\left(\frac{\tan Ka}{Ka}\right) - 1\right]^2}$; (4B)

total cross-section [p. PW5, Eq. (18)]: $\underline{\sigma \simeq (4\pi/k^2)\sin^2\delta_0 \simeq 4\pi a^2\left[\left(\frac{\tan Ka}{Ka}\right) - 1\right]^2}$. (4C)

The scattering is isotropic (no θ-dependence), with a <u>weak</u> E-dependence (in k).
