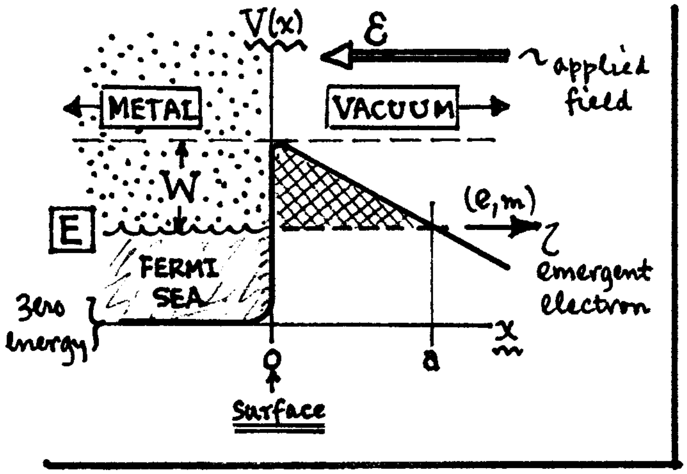
WKB : Field Emission from a Metal Surface.

4) As an example of the use of Eq.(11) for T, we look at "field emission", where electrons are pulled out of a metal surface by application of a strong external electric field $\mathcal{E}$. The appropriate energy diagram is...



E = highest energy of an electron in Fermi sea.

W = "work function" of metal ($W = e\varphi$). This is the height of the barrier.

When the external field $\mathcal{E}$ is applied, the total external (vacuum) potential may be written:

$$V(x) = E + W - e\mathcal{E}x, \text{ for } x > 0. \tag{12}$$

Near the metal's surface ($x=0$), $V(x)$ is modified by surface irregularities (N.B. "irregularity" is a synonym for surface science). We assume this region is small compared to the total barrier width a, which is found from...

$$\rightarrow @\ x = a:\ V(x) - E = W - e\mathcal{E}x = 0 \Rightarrow \underline{\underline{a = W/e\mathcal{E}}}. \tag{13}$$

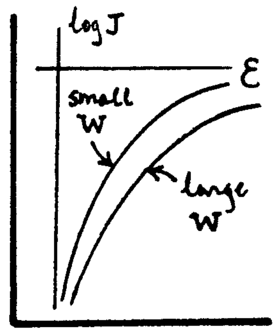
Most of the emitted e's in fact come from the top of the Fermi sea, and the emission current density J will be proportional to the probability that the e's tunnel through the indicated barrier. According to Eq.(11)...

$$J \propto T \simeq \exp\left\{-\frac{2}{\hbar}\int_0^a \sqrt{2m[V(x)-E]}\,dx\right\} = \exp\left\{-\frac{2}{\hbar}\sqrt{2m}\int_0^{W/e\mathcal{E}} \sqrt{W - e\mathcal{E}x}\,dx\right\}$$

by

$$\boxed{J \propto \exp\left\{-\frac{4}{3}\left(\frac{\sqrt{2me}}{\hbar}\right)\varphi^{3/2}/\mathcal{E}\right\}},\quad \varphi = W/e \text{ [work fcn in volts]} \tag{14}$$

So, for field emission, the prediction is: $\log J = -(\text{const})\cdot\varphi^{3/2}/\mathcal{E}$, as sketched at right. This result agrees semi-quantitatively with exptal data [see, e.g., p. 24 of Kaminsky "Atomic & Ionic Impact Phenomena on Metal Surfaces" (Academic Press, 1965)].

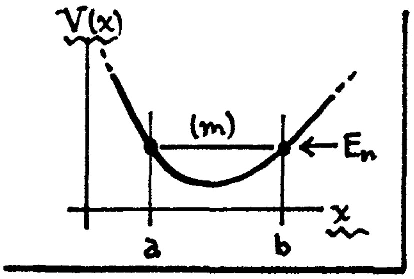


WKB: More complicated problems

Other Applications of the WKB Method to QM

We have now solved two prototype QM problems by using the WKB approximation, viz.

Ⓐ Bound states of a 1D potential well.

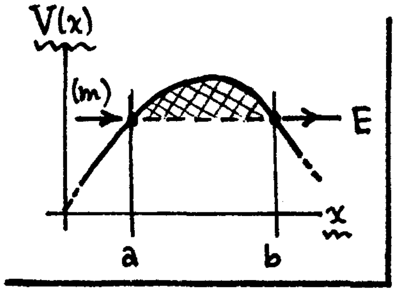


WKB (Bohr-Sommerfeld) Energy Quantization:

$$\boxed{\int_a^b \sqrt{2m[E_n - V(x)]}\, dx = (n+\tfrac{1}{2})\pi\hbar}, \quad n = 0,1,2,\ldots \qquad (1)$$

(approx^{n} good for: $k_{AV}(b-a) \gg 1$; limit of ψ_{WKB} validity).

Ⓑ Tunneling thru a 1D potential barrier.



WKB barrier transmission coefficient:

$$\boxed{T = \exp\left\{-\frac{2}{\hbar}\int_a^b \sqrt{2m[V(x)-E]}\, dx\right\}}. \qquad (2)$$

(approx^{n} good for: $K_{AV}(b-a) \gg 1$; limit of ψ_{WKB} validity).

Combinations of these problems (i.e. $V(x) = \bigvee + \bigwedge + \bigvee + \ldots$) provide a rich variety of QM models. We shall now survey a few such models.

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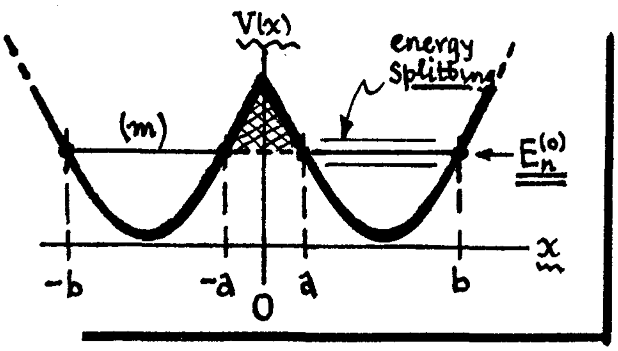
1) First we look at a double (or multiple) well.



The well of type Ⓐ above is reflected thru the origin to form a symmetric "double well" as sketched. The wells are coupled -- in the sense that the energy levels E_n of the RH well depend in part on the presence of the LH well. Specifically, coupling is provided by QM tunneling back & forth thru the potential barrier between $-a$ & $+a$. This tunneling has a novel effect on the well energies: each energy $E_n^{(0)}$ is split.

WKB: Double Well. Energy Bands.

Details are left to a problem (# 9). Results are as follows...

1. Let $E_n^{(0)}$ be the n^{th} energy level of either well alone, calculated from...

$$\rightarrow \int_a^b \sqrt{2m[E_n^{(0)} - V(x)]}\, dx = (n + \tfrac{1}{2})\pi\hbar, \text{ for one well.} \tag{3a}$$

2. Let $\omega_n^{(0)}$ be the oscillation frequency for the particle in one well...

$$\left[\begin{array}{l}\text{natural}\\ \text{oscillation}\\ \text{period}\end{array}\right\} \frac{2\pi}{\omega_n^{(0)}} = 2\int_a^b \frac{dx}{p_n^{(0)}(x)/m}, \quad \text{w/ } p_n^{(0)}(x) = \sqrt{2m[E_n^{(0)} - V(x)]}. \tag{3b}$$

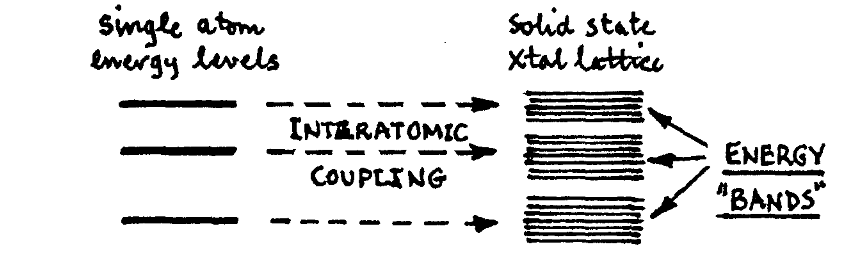
For m rattling about in one well, this is the time elapsed between successive presentations to the barrier.

3. The energy level $E_n^{(0)}$ in Eq. (3a) is split as follows...

$$E_n^{(0)} \xrightarrow[\text{tunnel}]{\text{QM}} E_n = E_n^{(0)} \pm \Delta E_n, \text{ i.e. } E_n^{(0)} \longrightarrow \begin{cases} E_n^{(0)} + \Delta E_n \\ E_n^{(0)} - \Delta E_n \end{cases}$$

$$\text{w/ } \boxed{\Delta E_n = \underbrace{\left(\frac{\hbar\omega_n^{(0)}}{2\pi}\right)}_{\text{presentation frequency}} \underbrace{\exp\left\{(-)\int_{-a}^{+a}\sqrt{(2m/\hbar^2)[V(x) - E_n^{(0)}]}\, dx\right\}}_{\text{barrier penetration probability}}}. \tag{3c}$$

The splitting of energy levels induced by coupling is a general feature of QM systems (recall fs & hfs splitting in H-atom). Such splitting is dramatically illustrated in the solid state (i.e. for an electron interacting w/ a crystal lattice):



Each atomic level -- coupled to many other atoms in the lattice -- splits into a large # of perturbed levels to form an energy band.

The energy band structure could be estimated in the present context by model at right... (augmented Kronig - Penney model).