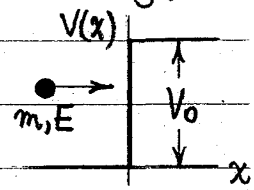
Physics 505 Final Exam

16 December 1970

All 10 questions are of equal weight. Time limit is 3 hours.

① Use the Bohr-Sommerfeld quantization rule, $\oint p(x)\,dx = nh$, to
Prob. #② calculate the allowed energy levels of a ball of mass m bouncing
φ507 Final elastically in a vertical direction in a uniform gravitational
(May 1992) field of acceleration g.

Prob. #⑱ ② Solve the one-dimensional Schrodinger equation for a particle (mass m)
φ506 (Aut '93) in an attractive delta-function potential: $V(x) = -C\,\delta(x)$, C = const.
Show that there is only one bound state, and calculate its energy.



x③ A particle of mass m and energy $E>0$ moves along the
Prob. #⑤ x-axis and encounters a step-function potential at the
φ506 MidTerm origin: $V(x)=0$ for $x<0$, $V(x)=V_0$ for $x>0$. Calculate the reflection coef-
(10/24/94) ficient R for the encounter, and sketch R vs. E for $E<V_0$ and $E>V_0$.

✓④ Consider the operator $\Lambda = a^\dagger a$, where a and $a^\dagger$ obey the anti-commutation
Prob. #④ rule: $aa^\dagger + a^\dagger a = 1$. Assume there exists a set of orthonormal eigenstates
φ506 Final. $|\lambda\rangle$ such that $\Lambda|\lambda\rangle = \lambda|\lambda\rangle$. By calculating $a|\lambda\rangle$ and $a^\dagger|\lambda\rangle$ explicitly,
(Dec. '93) Show that in fact there are only two eigenstates of Λ. What are the
allowed eigenvalues?

x⑤ A particle is in a one-dimensional harmonic oscillator potential. At
time $t=0$, it is completely localized at the origin: $\Psi(x,0) \propto \delta(x)$.
What is the probability, at some later time t, of finding the particle
in the n^{th} eigenstate of the oscillator?

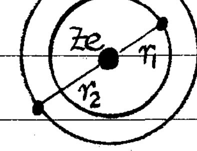
Prob. #① ϕ506 Final (Dec. 1993)

✓⑥ Let the stationary states of a system be specified by eigenfunctions $u_\alpha(x)$, with energy eigenvalues E_α (i.e. if H = system Hamiltonian: $Hu_\alpha = E_\alpha u_\alpha$) If p is the momentum operator (for a particle of mass m), and x is the conjugate position, prove the identity

$$\langle u_\alpha | p | u_\beta \rangle = \frac{im}{\hbar}(E_\alpha - E_\beta)\langle u_\alpha | x | u_\beta \rangle .$$

Prob. #⑧ ϕ506 (Aut. 93)

⑦ An ion consists of two electrons (mass m, charge e) bound to a nucleus of charge Ze. Because of the mutual electrostatic repulsion between the electrons, they tend to stay on opposite sides of the nucleus, maintaining an average separation $r_1 + r_2$, where r_1 and r_2 are the electron-nuclear distances.

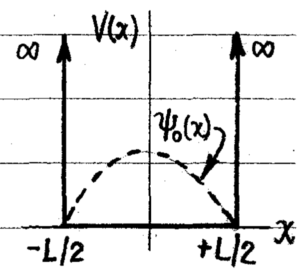


a) Write down an expression for the total energy E of the system, including an approximate term representing the electron-electron repulsion. Label the individual contributions to E.

b) By using the uncertainty relations, estimate the ground state energy E_0 of the ion (Hint: the contributions from the two electrons enter E in an entirely equivalent and symmetric way).

Prob. #② ϕ506 Final (Dec. 1993)

⑧ A particle is in the ground state of an infinitely deep one-dimensional square well potential of width L as shown. Explicitly calculate the probability distribution function for the various values of the particle momentum in this state. Sketch a rough graph of this function vs. the momentum, clearly indicating the zeroes and maxima. What is the most probable value of the momentum?



Prob. #9
ϕ507 (Win. '92)

(9) Given a complete set of eigenstates $\psi_m(x)$, and arbitrary operators A and B, use the closure relation to establish the identity

$$\langle k|AB|\ell\rangle = \sum_m \langle k|A|m\rangle\langle m|B|\ell\rangle,$$

where $\langle k|Q|\ell\rangle$ is the matrix element $\int \psi_k^*(x)\{Q\}\psi_\ell(x)\,dx$, etc.

(10) By considering the Fourier pair

$$\psi(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\varphi(k,t)\,e^{+ikx}\,dk\ ,\quad \varphi(k,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\psi(x,t)\,e^{-ikx}\,dx$$

Show directly that if ψ satisfies the Schrodinger equation in configuration space, namely

$$i\hbar\frac{\partial}{\partial t}\psi(x,t) = \left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)\right]\psi(x,t)$$

then φ satisfies the counterpart momentum space equation

$$i\hbar\frac{\partial}{\partial t}\varphi(k,t) = \left[\frac{\hbar^2 k^2}{2m} + V\!\left(i\frac{\partial}{\partial k}\right)\right]\varphi(k,t)$$

Clearly state the assumptions which must be made concerning the behavior of ψ and the potential function V.

Solutions to Phys. 505 Final

16 Dec 70

0 "Use the Bohr-Sommerfeld quantization rule, $\oint p\,dx = nh$, to calculate the allowed energy levels of a ball (mass m) bouncing elastically in a vertical direction in a uniform gravitational field (acceleration g)."

Prob. # 2 ϕ507 Final (May 1992)

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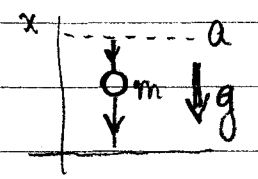


Potential is: $V(x) = mgx$

Total energy: $E = mga$, where a = max. hgt.

$$\oint p\,dx = 2\int_0^a \sqrt{2m(E-V(x))}\,dx = 2\sqrt{2m^2 g}\int_0^a (a-x)^{\frac{1}{2}}dx = nh$$

But $d(a-x)^{\frac{3}{2}} = \frac{3}{2}(a-x)^{\frac{1}{2}}(-dx) \Rightarrow (a-x)^{\frac{1}{2}}dx = -\frac{2}{3}d(a-x)^{\frac{3}{2}}$

$$\therefore \int_0^a (a-x)^{\frac{1}{2}}dx = \frac{2}{3}(a-x)^{\frac{3}{2}}\Big|_a^0 = \frac{2}{3}a^{\frac{3}{2}} = \frac{2}{3}\left(\frac{E}{mg}\right)^{\frac{3}{2}}$$

So we have ...

$$2\sqrt{2m^2 g}\ \frac{2}{3}\left(\frac{E}{mg}\right)^{\frac{3}{2}} = nh \Rightarrow E_n = \left(9mg^2n^2h^2/32\right)^{\frac{1}{3}}$$

2 "Solve the one-dimensional Schrodinger equation for a particle of mass m in an attractive delta-function potential $V(x) = -C\delta(x)$, C = cnst. Show that there is only one bound state, and calculate its energy."

Prob # 28 ϕ506 (Aut. 93)

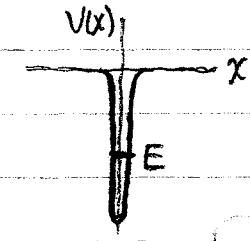
X X X

$$\frac{d}{dx}\left(\frac{d\psi}{dx}\right) + \frac{2m}{\hbar^2}\left(E + C\,\delta(x)\right)\psi = 0 . \quad \text{Integrate } \int_{-\epsilon}^{+\epsilon} dx$$

$$\therefore \left[\psi'(+\epsilon) - \psi'(-\epsilon)\right] + \frac{2m}{\hbar^2} E \int_{-\epsilon}^{+\epsilon} \psi\, dx + \frac{2m}{\hbar^2} C\, \psi(0) = 0$$

As $\epsilon \to 0$, term in E vanishes, and we get discontinuity condition...

$$\psi'(0+) - \psi'(0-) = -\frac{2m}{\hbar^2} C\, \psi(0)$$



For bound state, set $E = -\hbar^2 k^2/2m$. Solutions outside the well are

$\psi(x) = A e^{+kx}$ for $x<0$ & $\psi(x) = B e^{-kx}$ for $x>0$

Continuity in ψ at $x=0 \Rightarrow B \equiv A$. The ψ' discontinuity gives

$$-kA - kA = -\frac{2m}{\hbar^2} C A \Rightarrow k = mC/\hbar^2$$

$$\therefore E = -\tfrac{1}{2} m C^2/\hbar^2 \quad \text{is only bound state}$$

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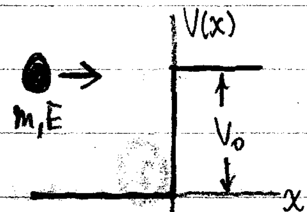
(5) "A particle of mass m and energy $E>0$ moves along the x-axis and encounters a step-function potential at the origin: $V(x) = 0$ for $x<0$, and $V(x) = V_0$ for $x>0$. Calculate the reflection coefficient R for the encounter, and sketch a graph of R vs. E for both $E<V_0$ and $E>V_0$."



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Define: $k = \sqrt{\frac{2m}{\hbar^2} E}$, $K = \sqrt{\frac{2m}{\hbar^2}(E - V_0)}$ $\begin{cases} \text{if } E > V_0,\ K \text{ is real} \\ \text{if } E < V_0,\ K \to i|K| . \end{cases}$

$x<0 \ldots$ unit incident ampl. $x>0\ldots$

$$\psi(x) = e^{ikx} + Be^{-ikx} \quad \Big\} \quad \psi(x) = Ce^{+i\kappa x}$$

ψ cont. at $x=0 \Rightarrow 1+B = C$ $\quad\Big\|\quad$ $1+B = C$

ψ' cont. at $x=0 \Rightarrow ik(1-B) = i\kappa C$ $\quad\Big\|\quad$ $1-B = \frac{\kappa}{k}C$

$$\therefore\ 2 = \left(1+\frac{\kappa}{k}\right)C \quad \text{or} \quad C = \frac{2k}{k+\kappa}$$

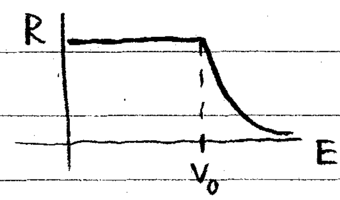
and $B = C-1 = \dfrac{k-\kappa}{k+\kappa} = \dfrac{\sqrt{E}-\sqrt{E-V_0}}{\sqrt{E}+\sqrt{E-V_0}}$

Define : $\mu = \sqrt{1-\frac{V_0}{E}}$. Then $B = \dfrac{1-\mu}{1+\mu}$

Refl. coefficient is : $R = |B|^2 = \left|\dfrac{1-\mu}{1+\mu}\right|^2$

$E > V_0 \Rightarrow \mu$ is real, and $R<1$

$E < V_0 \Rightarrow \mu = i|\mu|$ is imag. and $R \equiv 1$



4 "Consider the operator $\Lambda = a^\dagger a$, where a and $a^\dagger$ obey the anti-commutation rule $aa^\dagger + a^\dagger a = 1$. Assume there exist a set of orthonormal eigenstates $|\lambda\rangle$ such that $\Lambda|\lambda\rangle = \lambda|\lambda\rangle$. By calculating $a|\lambda\rangle$ and $a^\dagger|\lambda\rangle$ explicitly, show that in fact there are only two eigenstates of Λ. What are the allowed eigenvalues of Λ?"

x x x

$$a|\lambda\rangle = (aa^\dagger + a^\dagger a)a|\lambda\rangle = a\Lambda|\lambda\rangle + \Lambda a|\lambda\rangle = (\lambda+\Lambda)a|\lambda\rangle$$

$$\therefore\ \Lambda(a|\lambda\rangle) = (1-\lambda)(a|\lambda\rangle) \Rightarrow a|\lambda\rangle = A|1-\lambda\rangle$$

To get the cnst A, take

$$\langle a\lambda | a\lambda\rangle = |A|^2 \underbrace{\langle 1-\lambda | 1-\lambda\rangle}_{1} \quad \text{or} \quad |A|^2 = \langle\lambda | a^\dagger a | \lambda\rangle = \lambda$$

→ Thus have: $a|\lambda\rangle = \sqrt{\lambda}\,|1-\lambda\rangle$ for step-down operator

Treating $a^\dagger$ similarly...

$$a^\dagger|\lambda\rangle = a^\dagger(aa^\dagger + a^\dagger a)|\lambda\rangle = \Lambda a^\dagger|\lambda\rangle + a^\dagger\Lambda|\lambda\rangle = (\Lambda+\lambda)a^\dagger|\lambda\rangle$$

$$\therefore\ \Lambda(a^\dagger|\lambda\rangle) = (1-\lambda)a^\dagger|\lambda\rangle \Rightarrow a^\dagger|\lambda\rangle = B|1-\lambda\rangle$$

$$\langle a^\dagger\lambda | a^\dagger\lambda\rangle = |B|^2 \underbrace{\langle 1-\lambda | 1-\lambda\rangle}_{1}, \text{ or } |B|^2 = \langle\lambda | \underbrace{aa^\dagger}_{1-\Lambda} | \lambda\rangle = (1-\lambda)$$

→ Thus have: $a^\dagger|\lambda\rangle = \sqrt{1-\lambda}\,|1-\lambda\rangle$ for step-up operator.

Both a & $a^\dagger$ generate only state $|1-\lambda\rangle$ from λ. Thus Λ has only two eigenstates, with eigenvalues λ & $1-\lambda$.

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5 "A particle is in a one-dimensional harmonic oscillator potential. At time $t=0$, it is completely localized at the origin: $\Psi(x,0) \propto \delta(x)$. What is the relative probability, at some later time t, of finding the particle in the n^{th} eigenstate of the oscillator?"

X X X

General wavefcn at $t>0$ is

$$\Psi(x,t) = \sum_n c_n \psi_n(x) e^{-\frac{i}{\hbar}E_n t} \quad \begin{cases} \psi_n = n^{th} \text{ eigenfcn} \\ E_n = n^{th} \text{ eigenenergy} \end{cases}$$

where: $c_n = \int \psi_n^*(x)\,\Psi(x,0)\,dx \propto \psi_n^*(0)$

Prob. of finding n^{th} state at time t is

$$P_n = |C_n|^2 \propto |\psi_n(0)|^2$$

But : $\psi_n(x) = \left(\frac{\alpha/\sqrt{\pi}}{2^n n!}\right)^{\frac{1}{2}} e^{-\frac{m\omega}{2\hbar}x^2} H_n(\alpha x)$, $\alpha = \sqrt{\frac{m\omega}{\hbar}}$

$$\therefore P_n \propto \frac{\alpha/\sqrt{\pi}}{2^n n!} H_n^2(0)$$

But $H_n(0) = (-1)^{\frac{n}{2}} n!/\left(\frac{n}{2}\right)!$ for n even, and $H_n(0) = 0$ for n odd

$$\therefore P_n \propto \frac{\alpha/\sqrt{\pi}}{2^n n!}\left(\frac{n!}{(n/2)!}\right)^2 = \frac{\alpha/\sqrt{\pi}}{2^n} n!/\left(\left(\frac{n}{2}\right)!\right)^2, \quad n = 0, 2, 4, 6, \ldots$$

This is required answer. We note that P_n decreases with increasing n – which makes sense. This is seen by Stirling's approximation...

$$n! \simeq \sqrt{2\pi n}\, n^n e^{-n} \Rightarrow P_n \propto \frac{\alpha}{\pi}\sqrt{\frac{2}{n}} \to 0 \text{ for } n \to \infty.$$

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5 "A system is in a so-called stationary state, described by a wave function $\psi(x,t) = u(x)\exp(-\frac{i}{\hbar}Et)$, with discrete energy eigenvalue E. Show that the expectation value of the momentum p in this state is identically zero. (~~Hint: $\langle p \rangle = m\, d\langle x \rangle/dt$~~)."

x x x

For stationary state $\psi = u e^{-\frac{i}{\hbar}Et}$, $Hu = Eu$, and the Hamiltonian H is indpt of time. Then

$$\langle p \rangle = m\frac{d}{dt}\langle x \rangle = \frac{im}{\hbar}\langle [H, x] \rangle$$

$$= \frac{im}{\hbar}\langle \psi | Hx - xH | \psi \rangle$$

Since H is time indpt, and Hermitian, we have

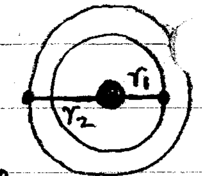
$$\langle p \rangle = \frac{im}{\hbar}\left[\langle u|Hx|u\rangle - \langle u|xH|u\rangle \right]$$

$$= \frac{im}{\hbar}\left[\langle \underbrace{Hu}_{=E^*u}|x|u\rangle - \langle u|x|\underbrace{Hu}_{=Eu}\rangle \right]$$

$$= \frac{im}{\hbar}(E^* - E)\langle u|x|u\rangle$$ But E is real $\therefore E^* = E$

$\therefore \langle p \rangle \equiv 0$ QED

7 "An ion consists of two electrons (mass m, charge e) bound to a nucleus of charge Ze. Because of the mutual electrostatic repulsion between the electrons, they tend to stay on opposite sides of the nucleus, maintaining an average separation $r_1 + r_2$, where r_1 and r_2 are electron-nuclear distances.



a) Write down an expression for the total energy E of the system, including an approximate term representing the electron-electron repulsion. Label the individual contributions to E.

b) By using the uncertainty relations, estimate the ground state energy E_0 of the ion (Hint: the contributions from the two electrons enter E in an entirely symmetric and equivalent way).

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a) $$E = \underbrace{\frac{1}{2m}(p_1^2 + p_2^2)}_{e \text{ kinetic energy}} - \underbrace{Ze^2\left(\frac{1}{r_1} + \frac{1}{r_2}\right)}_{e\text{-}Ze \text{ Coulomb interaction}} + \underbrace{\frac{e^2}{r_1 + r_2}}_{\text{electrostatic } e\text{-repulsion}}$$, p_1 & p_2 are e momenta

b) $p_1 \sim \hbar/r_1$ and $p_2 \sim \hbar/r_2$, by uncertainty relations. Then have

$$E(r_1,r_2) \sim \frac{\hbar^2}{2m}\left(\frac{1}{r_1^2}+\frac{1}{r_2^2}\right) - Ze^2\left(\frac{1}{r_1}+\frac{1}{r_2}\right) + \frac{e^2}{r_1+r_2}$$

Want to minimize E w.r.t. r_1 & r_2. We note ...

$$\frac{\partial E}{\partial r_1} \sim -\frac{\hbar^2}{m}\frac{1}{r_1^3} + Ze^2\frac{1}{r_1^2} - \frac{e^2}{(r_1+r_2)^2}$$

Now r_1 & r_2 contribute to E and $\partial E/\partial r_i$ in completely equivalent ways. We thus claim E is minimized for some $r_2 = r_1$. Then

$$\frac{\partial E}{\partial r_1} \sim -\frac{\hbar^2}{m}\frac{1}{r_1^3} + (Z-\tfrac{1}{4})e^2\frac{1}{r_1^2} = 0 \Rightarrow r_1 = \frac{\hbar^2}{m}\Big/(Z-\tfrac{1}{4})e^2$$

and: $$E_0 = \frac{\hbar^2}{m}\frac{1}{r_1^2} - 2Ze^2\frac{1}{r_1} + \frac{e^2}{2r_1} = -(Z-\tfrac{1}{4})^2\frac{me^4}{\hbar^2}$$

Note: we used $p_{1,2} \sim \alpha\hbar/r_{1,2}$ with $\alpha = 1$. For general α, we would have gotten $E_0 = -(Z-\frac{1}{4})^2 me^4/(\alpha\hbar)^2$. Thus for $\alpha = 1/2$, the calculated E_0 would be $4\times$ as large. Etc.

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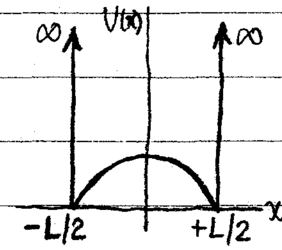
8 "A particle is in the ground state of an infinitely deep one-dimensional square well potential of width L as shown. Explicitly calculate the probability distribution function for the various values of the particle momentum in this state. Sketch a rough graph of this function vs. the momentum, clearly indicating the zeroes and maxima, etc."



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Normalized gnd state wfcn (corresponding to $E_0 = \frac{(\hbar k_0)^2}{2m}$, $k_0 = \frac{\pi}{L}$) is

$\psi_0(x) = \sqrt{\frac{2}{L}}\cos\left(\frac{\pi x}{L}\right)$ over $-L/2 \le x \le +L/2$, $\psi_0(x) = 0$ for $|x| > L/2$

The desired momentum spectrum fcn, with $k = p/\hbar$, is given by

$$\varphi(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\psi_0(x)\,e^{-ikx}\,dx = \frac{1}{\sqrt{\pi L}}\int_{-L/2}^{+L/2}\cos\left(\frac{\pi x}{L}\right)e^{-ikx}\,dx$$

$$= \frac{1}{\sqrt{\pi L}}\,\frac{1}{2}\int_{-L/2}^{+L/2}\left[e^{+i\left(\frac{\pi}{L}-k\right)x} + e^{-i\left(\frac{\pi}{L}+k\right)x}\right]dx$$

But: $\displaystyle\int_{-L/2}^{+L/2} e^{+i\alpha x}\,dx = \frac{1}{i\alpha}e^{+i\alpha x}\Big|_{-L/2}^{+L/2} = \frac{2}{\alpha}\sin\left(\alpha\frac{L}{2}\right)$ and same for $-\alpha$

$$\therefore\ \varphi(k) = \frac{1}{\sqrt{\pi L}}\left\{\frac{1}{\frac{\pi}{L}-k}\sin\left(\frac{\pi}{L}-k\right)\frac{L}{2} + \frac{1}{\frac{\pi}{L}+k}\sin\left(\frac{\pi}{L}+k\right)\frac{L}{2}\right\}$$

$$= \frac{1}{\sqrt{\pi L}}\left\{\frac{L}{\pi-kL}\underbrace{\sin\left(\frac{\pi}{2}-\frac{kL}{2}\right)}_{\to\ \cos\frac{kL}{2}} + \frac{L}{\pi+kL}\underbrace{\sin\left(\frac{\pi}{2}+\frac{kL}{2}\right)}_{\to\ \cos\frac{kL}{2}}\right\}$$

$$= \sqrt{\frac{L}{\pi}}\left\{\frac{1}{\pi-kL}+\frac{1}{\pi+kL}\right\}\cos\frac{kL}{2} = \frac{\sqrt{4\pi L}}{\pi^2-(kL)^2}\cos\frac{kL}{2}$$

Momentum prob. distribution $\Big\}$ $$|\varphi(k)|^2 = \frac{4\pi L}{(\pi^2-k^2L^2)^2}\cos^2\frac{kL}{2}$$

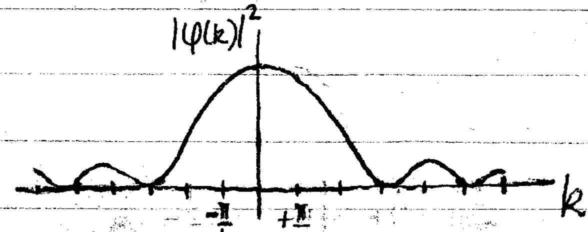
Note: $|\varphi(0)|^2 = \frac{4}{\pi^3}L \simeq 0.129\,L$ $\qquad k^2L^2 \simeq \pi^2 - 2\pi\epsilon$

What happens as $kL \to \pi$? Suppose $kL = \pi - \epsilon$. Then

$\pi^2 - k^2L^2 \simeq 2\pi\epsilon$, as $\epsilon \to 0$

$\cos\frac{kL}{2} = \cos\left(\frac{\pi}{2}-\frac{\epsilon}{2}\right) = \sin\frac{\epsilon}{2} \simeq \frac{\epsilon}{2}$ $\Big\}$ $\displaystyle |\varphi(k=\tfrac{\pi}{L})|^2 \underset{\epsilon\to 0}{\simeq} \frac{4\pi L}{(2\pi\epsilon)^2}\left(\frac{\epsilon}{2}\right)^2 = \frac{1}{4\pi}L \simeq 0.079\,L$

Required sketch looks like ...



6 "Suppose the stationary states of a system are described by the eigenfunctions $u_\alpha(x)$, with energy eigenvalues E_α (i.e. with H the system Hamiltonian, $H u_\alpha = E_\alpha u_\alpha$). If p is the momentum operator (for a particle of mass m), and x is the conjugate position, prove the identity

$$\langle u_\alpha | p | u_\beta \rangle = \frac{im}{\hbar}(E_\alpha - E_\beta)\langle u_\alpha | x | u_\beta \rangle$$

x x x

$$\langle \alpha | p | \beta \rangle = m\frac{d}{dt}\langle \alpha | x | \beta \rangle = m\frac{i}{\hbar}\langle \alpha | [H, x] | \beta \rangle \quad \left\{ \begin{array}{l} \text{by QM} \\ \text{eqtn of motion} \end{array} \right.$$

$$= \frac{im}{\hbar}\left[\langle \alpha | Hx | \beta \rangle - \langle \alpha | xH | \beta \rangle\right]$$

$$= \frac{im}{\hbar}\left[\langle \underbrace{H\alpha}_{= E_\alpha \alpha} | x | \beta \rangle - \langle \alpha | x | \underbrace{H\beta}_{E_\beta \beta} \rangle\right] \quad \left\{ \begin{array}{l} \text{since } H \text{ is} \\ \text{Hermitian} \end{array} \right.$$

$$\langle \alpha | p | \beta \rangle = \frac{im}{\hbar}(E_\alpha - E_\beta)\langle \alpha | x | \beta \rangle \qquad \underline{\underline{\text{QED}}}$$

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9 "Given a complete set of eigenstates $\psi_m(x)$, and arbitrary operators A and B, use the closure relation to establish the identity

$$\langle k | AB | \ell \rangle = \sum_m \langle k | A | m \rangle \langle m | B | \ell \rangle$$

where $\langle k | Q | \ell \rangle$ is the matrix element $\int \psi_k^*(x)\{Q\}\psi_\ell(x)\,dx$, etc."

x x x

$$\langle k | AB | \ell \rangle = \int \psi_k^*(x) A \{B\psi_\ell(x)\}\,dx$$

Now $B\psi_\ell(x)$ = some fcn $\phi_\ell(x)$, which we can write as

$$\phi_\ell(x) = \int \delta(x-x')\,\phi_\ell(x')\,dx'$$

i.e. $$B\psi_\ell(x) = \int \delta(x-x')\,B'\psi_\ell(x')\,dx'$$

$$= \int \underbrace{\left(\sum_m \psi_m(x)\psi_m^*(x')\right)}_{=\delta(x-x'),\ \text{by closure}} B'\psi_\ell(x')\,dx'$$

$$\therefore \langle k|AB|\ell\rangle = \int \psi_k^*(x)\,A\left[\int \sum_m \psi_m(x)\,\psi_m^*(x')\,B'\psi_\ell(x')\,dx'\right]dx$$

$$= \sum_m \int \psi_k^*(x)\,A\,\psi_m(x)\,dx \times \int \psi_m^*(x')\,B'\,\psi_\ell(x')\,dx'$$

$$= \sum_m \langle k|A|m\rangle\langle m|B|\ell\rangle$$

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10 "By considering the Fourier pair

$$\psi(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\varphi(k,t)\,e^{+ikx}\,dk,\quad \varphi(k,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\psi(x,t)\,e^{-ikx}\,dx,$$

Show directly that if ψ satisfies the Schrodinger equation in configuration space, namely

$$i\hbar\frac{\partial}{\partial t}\psi(x,t) = \left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)\right]\psi(x,t),$$

then φ satisfies the counterpart momentum space equation

$$i\hbar\frac{\partial}{\partial t}\varphi(k,t) = \left[\frac{\hbar^2k^2}{2m} + V\left(i\frac{\partial}{\partial k}\right)\right]\varphi(k,t).$$

Clearly state the assumptions which must be made concerning the behaviour of ψ and the potential function V.

x x x

Taking the time derivative of φ directly

$$i\hbar \frac{\partial \varphi}{\partial t} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \left(i\hbar \frac{\partial \psi}{\partial t} \right) e^{-ikx} dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \left[-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V(x) \right] \psi e^{-ikx} dx$$

Partial integrate 1^{st} term twice ...

$$\int_{-\infty}^{+\infty} \left(\frac{\partial^2 \psi}{\partial x^2} \right) e^{-ikx} dx = \underbrace{\left(\frac{\partial \psi}{\partial x} \right)}_{\to 0} e^{-ikx} \Big|_{-\infty}^{+\infty} - (-ik) \int_{-\infty}^{+\infty} \left(\frac{\partial \psi}{\partial x} \right) e^{-ikx} dx$$

$$= +ik \left[\underbrace{\psi}_{\to 0} e^{-ikx} \Big|_{-\infty}^{+\infty} - (-ik) \int_{-\infty}^{+\infty} \psi e^{-ikx} dx \right] = -k^2 \int_{-\infty}^{+\infty} \psi e^{-ikx} dx$$

$$\therefore i\hbar \frac{\partial \varphi}{\partial t} = + \frac{\hbar^2 k^2}{2m} \varphi + \underbrace{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} V(x) \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \varphi(k',t) e^{+ik'x} dk' \right] e^{-ikx} dx}_{I}$$

$$I = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left[\int_{-\infty}^{+\infty} e^{+ik'x} V(x) e^{-ikx} dx \right] \varphi(k',t) dk'$$

Suppose $V(x) = \sum_n a_n x^n$. Note $x e^{-ikx} = i \frac{\partial}{\partial k} e^{-ikx}$

$$\therefore V(x) e^{-ikx} = \sum_n a_n \left(i \frac{\partial}{\partial k} \right)^n e^{-ikx} = V\left(i \frac{\partial}{\partial k} \right) e^{-ikx}$$

$$\Rightarrow I = \int_{-\infty}^{+\infty} \left[V\left(i \frac{\partial}{\partial k} \right) \underbrace{\frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{i(k'-k)x} dx}_{\delta(k'-k)} \right] \varphi(k',t) dk'$$

$$= V\left(i \frac{\partial}{\partial k} \right) \int_{-\infty}^{+\infty} \delta(k'-k) \varphi(k',t) dk' = V\left(i \frac{\partial}{\partial k} \right) \varphi(k,t)$$

$$\therefore i\hbar \frac{\partial}{\partial t} \varphi(k,t) = \left[+ \frac{\hbar^2 k^2}{2m} + V\left(i \frac{\partial}{\partial k} \right) \right] \varphi(k,t) \qquad \underline{\underline{\text{QED}}}$$

Assumptions: 1) ψ & $\partial \psi / \partial x$ vanish at ∞

2) V expansible in power series