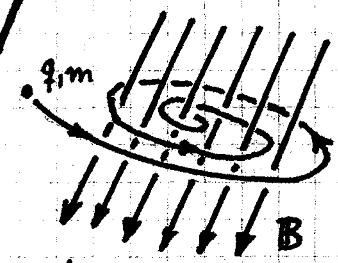
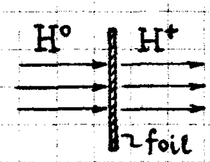
ϕ520 Midterm Exam (in class, 2hr.) Sun. 24 Feb. 1991

This exam is open-book, open-notes, and is worth 100 points. In your solutions: box the answer to each problem, number the pages consecutively, put your name on p.1, and staple the pages together before handing them in.

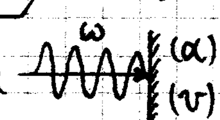


(1) [20pts.] A high-energy cosmic ray particle (charge q, mass m, and total energy $\mathcal{E}_0 = \gamma_0 mc^2$) enters a region of space where there is a uniform, static magnetic field $\mathbb{B}$ which is $\perp$ q's motion. The particle begins a circular orbit which decays due to radiative loss. The orbit becomes an inward spiral, and q ends up essentially at rest in the field. Calculate the total energy radiated by q during this orbit.

(2) [30 pts.] A non-relativistic particle of charge Q & mass M is traveling along at velocity $v_0 \ll c$ when it slams into a small block of material and stops. Inside the material, Q is decelerated according to the law: $\frac{dv}{dt} = -v/\tau$, $\tau =$ cnst. Find the full frequency-angle spectrum ($d^2I/d\omega\, d\Omega$) of the radiation emitted by Q, starting from Jackson's Eq. (14.65). Work to the lowest order in $\beta_0 = \frac{v_0}{c} \ll 1$, and clearly state your assumptions. Sketch the $\measuredangle$ dependence of the spectrum; also sketch $d^2I/d\omega\, d\Omega$ vs. frequency ω.



(3) [25 pts.] A high-energy beam of neutral hydrogen atoms H^0 passes through a thin foil target. The beam loses little energy, but the foil strips the H^0's of their electrons (which stop in the target). Downstream from the foil, the beam is predominantly protons, H^+. Discuss the problem of calculating the radiation from the foil target. It is important to note that you are dealing with a continuous beam, not individual charges.



(4) [25 pts]. An EM planewave at frequency ω penetrates a metal surface and propagates inside the metal according to a 1D wave equation: $u_{xx} - \alpha u_t - \frac{1}{v^2} u_{tt} = 0$. Here, u is any component of the wave's $\mathbb{E}$ field, and α is a constant (at low ω) proportional to the metal's conductivity. If α is "large", find the characteristic depth to which the wave propagates before becoming "extinct" for all practical purposes.

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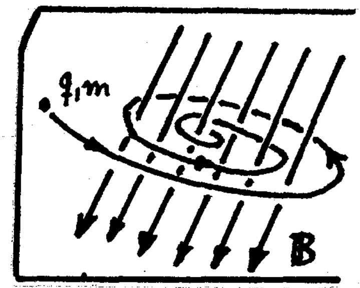
① [20 pts]. Radiation from cosmic ray trapped in a $\mathbb{B}$-field



1. The accounting procedure is ...

initial energy : $\mathcal{E}_0 = \gamma_0 mc^2$;

final energy : $\mathcal{E} = \gamma mc^2 \big|_{\gamma=1} = mc^2$, q at rest

energy loss ; $\Delta\mathcal{E} = \mathcal{E}_0 - \mathcal{E} = (\gamma_0 - 1)mc^2$. (1)

No other energy loss mechanism is measured, so this $\Delta\mathcal{E}$ goes entirely into radiation. Thus

$$\boxed{\mathcal{E}(\text{radiation}) = \Delta\mathcal{E} = (\gamma_0 - 1)mc^2.} \tag{2}$$

2. Problem can also be done in a highly overprepared fashion, as follows.

Jk$^{\text{n}}$ Eq. (14.46) : $P = \frac{2}{3}(q^2/c^3)\gamma^4|\dot{\mathbf{v}}|^2$ ∫ radiative power loss for circular motion

$\dot{\mathbf{v}}$ provided by Lorentz force : $\gamma m\dot{\mathbf{v}} = \frac{q}{c}\mathbf{v}\times\mathbb{B}$ ∫ $\gamma \simeq$ cnst over one orbit

Combine first two lines $\Rightarrow$ $P = \frac{2}{3}(q^4B^2/m^2c^3)(\gamma^2 - 1)$.

But $P = -\frac{d\mathcal{E}}{dt}$, with $\mathcal{E} = \gamma mc^2$. Then γ obeys :

$$\longrightarrow -mc^2\frac{d\gamma}{dt} = \frac{2}{3}(q^4B^2/m^2c^3)(\gamma^2 - 1) \tag{3}$$

Can solve for $\gamma = \gamma(t)$ and $\mathcal{E}(t) = \gamma(t)mc^2$, etc. Find that

$$\longrightarrow \mathcal{E}(\text{radiation}) = \int_0^\infty P\,dt = \Delta\mathcal{E} = (\gamma_0 - 1)mc^2, \tag{4}$$

just as above.

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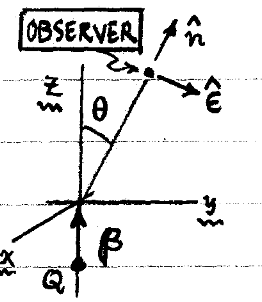


② [30 pts]. Do a ~ full Bremsstrahlung problem (at $\beta \ll 1$).

1. Let Q travel along the z-axis. Assume β & $\dot{\beta}$ are collinear, and $\beta \ll 1$ (non-relativistic). Jackson's Eq. (14.65) is then:

$$\rightarrow \left.\frac{d^2 I}{d\omega\, d\Omega}\right|_{\hat{\epsilon}} = \frac{Q^2}{4\pi^2 c}\left|\hat{\epsilon}\cdot\int_{-\infty}^{\infty}[\hat{n}\times(\hat{n}\times\dot{\beta})]\, e^{i\omega(t-\frac{1}{c}z(t)\cos\theta)}\,dt\right|^2, \qquad (1)$$

for the frequency-angle spectrum of radiation with polarization $\hat{\epsilon}$ at observer. $\hat{\epsilon}$ is $\perp$ $\hat{n}$, and -- since the radiation pattern must be cylindrically symmetric about the z-axis -- we can locate both $\hat{n}$ and $\hat{\epsilon}$ in the yz-plane. Then we have: $\hat{\epsilon}\cdot[\hat{n}\times(\hat{n}\times\dot{\beta})] = -\hat{\epsilon}\cdot\dot{\beta} = +\dot{\beta}\sin\theta$ (the other polarization gives zero). The stopping law $dv/dt = -v/\tau$ gives $\dot{\beta} = -\frac{1}{\tau}\beta$, so Eq. (1) reduces to:

$$\rightarrow \frac{d^2 I}{d\omega\, d\Omega} = \frac{Q^2\sin^2\theta}{4\pi^2 c\tau^2}\left|\int_0^{\infty}\beta(t)\, e^{i\omega(t-\frac{1}{c}z(t)\cos\theta)}\,dt\right|^2 \qquad (2)$$

where deceleration begins at time $t=0$.

2. The mechanics problem for Q's stopping is straightforwardly solved...

$$\left[\begin{array}{l} \dfrac{dv}{dt} = -\dfrac{v}{\tau} \Rightarrow v(t) = v_0 e^{-t/\tau}, \quad v_0 = v(0) = \text{initial velocity;} \\[2ex] \dfrac{dz}{dt} = v(t) \Rightarrow z(t) = v_0\tau\left[1-e^{-t/\tau}\right], \quad \text{stopping begins at } z=0. \end{array}\right\} \qquad (3)$$

The integral in Eq. (2) is then...

$$\rightarrow J(\omega) = \int_0^{\infty}\beta(t)\, e^{i\omega(\cdots)}\,dt \leftarrow \text{let } \beta_0 = v_0/c \ll 1\ldots$$

$$= \beta_0\int_0^{\infty} e^{-(\frac{1}{\tau}-i\omega)t}\left\{e^{-i\omega\tau\beta_0\cos\theta[1-e^{-t/\tau}]}\right\}dt$$

to lowest order in β_0, this factor just $\equiv 1$.

$$\simeq \beta_0\tau/(1-i\omega\tau); \qquad (4)$$

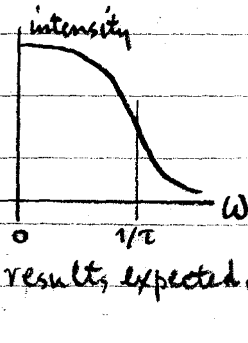
so//

$$\frac{d^2 I}{d\omega\, d\Omega} = \frac{Q^2\sin^2\theta}{4\pi^2 c\tau^2}\left|\frac{\beta_0\tau}{1-i\omega\tau}\right|^2,$$

or//

$$\boxed{\frac{d^2 I}{d\omega\, d\Omega} = \left(\frac{Q^2\beta_0^2\sin^2\theta}{4\pi^2 c}\right)\frac{1}{1+\omega^2\tau^2}}. \qquad (5)$$

The spectrum has (expected) dipolar angular distribution. Frequency spectrum is ~flat as $\omega\to 0$, and negligibly small at freqs. $\omega > 1/\tau$. Both results expected.



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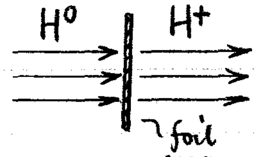


(3) [25 pts]. Discuss radiation from a beam-stripping foil.

1. Any radiation will come from the <u>electrons</u> which are suddenly stopped in the foil -- they decelerate from velocity v (H^0 beam velocity) to velocity 0 in a characteristic time $\Delta t \sim d/v$, where d is the foil thickness. So the electron deceleration is $a = \Delta v/\Delta t \sim v^2/d$. The total charge being decelerated in each Δt is: $\underline{\Delta Q = I\Delta t \sim Id/v}$, where I is the beam current.

2. The protons passing through the foil lose little energy, so they are negligibly decelerated and do not contribute significant radiation -- the protons are just spectators. On the other hand, the electrons are "destroyed" (stopped) in the foil -- this is in effect the <u>inverse</u> of beta decay, where electrons are "created" (emitted) by an atomic nucleus. Jackson treats β-decay in his Sec. (15.6), and shows that the frequency-angle spectrum at low freqs is: $d^2I/d\omega d\Omega = \frac{e^2}{4\pi^2 c}\left|\hat{\epsilon}\cdot\beta/(1-\hat{n}\cdot\beta)\right|^2$. This is for emission of a <u>single</u> electron. One might be tempted to just replace the charge e here by $\Delta Q \sim Id/v$ to get the low-ω spectrum for stripping.

3. But we note that Jk^{n}'s Eq. (15.63) does <u>not</u> hold in its stated form for a <u>succession</u> of charges ΔQ, which radiate over periods $0\to\Delta t$, $\Delta t\to 2\Delta t$, $2\Delta t\to 3\Delta t$, etc. That is because there is a phase factor $e^{i\phi}$ in the $|\quad|^2$, which appears in the partial integration between Jk^{n} (14.65) & (14.67). For a succession (i.e. beam) of ΔQ's...

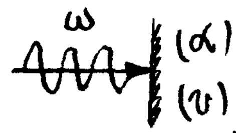
$$\rightarrow \frac{d^2I}{d\omega d\Omega}\bigg|_{\omega\to 0} = \frac{(\Delta Q)^2}{4\pi^2 c}\left|\sum_{\text{beam}}\left(\frac{\hat{\epsilon}\cdot\beta}{1-\hat{n}\cdot\beta}\right)e^{i\phi}\right|^2, \quad \phi = \omega\left(t - \frac{1}{c}\hat{n}\cdot\mathbf{r}(t)\right). \qquad (1)$$

If all the ΔQ's are stopped similarly, then the phases ϕ will differ only by the emission times $n\Delta t$, $n = 1, 2, \ldots, \infty$ (for a CW beam). The low-ω spectrum is:

$$\rightarrow (d^2I/d\omega\, d\Omega)_{\omega\to 0} = [(\Delta Q)^2/4\pi^2 c]\left|\hat{\epsilon}\cdot\beta/(1-\hat{n}\cdot\beta)\right|^2\left|\sum_n e^{in\omega\Delta t}\right|^2 \qquad (2)$$

The additional factor: $\left|\sum_{n=1}^{N} e^{in\omega\Delta t}\right|^2 = \left(\sin\frac{N}{2}\omega\Delta t/\sin\frac{1}{2}\omega\Delta t\right)^2$ generally vanishes for $\omega > 0$ and $N\to\infty$; it $\equiv N^2$ only for $\omega \equiv 0$. So Jackson's $(d^2I/d\omega d\Omega)_{\omega\to 0} \to 0$ for beam stripping, and a treatment is needed for the $\omega > 0$ part of the spectrum.

④ [25 pts]. EM wave propagation in a metal.



1. The planewave $u(x,t) = e^{i(kx-\omega t)}$ propagates in the metal according to $u_{xx} - \alpha u_t - (1/v^2)u_{tt} = 0$. By direct substitution...

$$\longrightarrow -k^2 + i\alpha\omega + (\omega^2/v^2) = 0, \text{ so } k = \frac{\omega}{v}\sqrt{1 + i(\alpha v^2/\omega)}. \qquad (1)$$

The (+) ve square root is chosen so that $k \geq 0$ when $\alpha \to 0$; this means the rightward traveling wave continues to the right.

2. If $\alpha \to$ "large" (and ω is not too big), write k in Eq. (1) as...

$$k = \frac{\omega}{v}\left[i\left(\frac{\alpha v^2}{\omega}\right)\right]^{\frac{1}{2}}\sqrt{1 - i(\omega/\alpha v^2)} \simeq \sqrt{i}\,(\alpha\omega)^{\frac{1}{2}}\left[1 - \tfrac{1}{2}i(\omega/\alpha v^2)\right]. \qquad (2)$$

$$\ldots \text{but } \sqrt{i} = (e^{i\frac{\pi}{2}})^{\frac{1}{2}} = e^{i(\pi/4)} = \frac{1}{\sqrt{2}}(1+i)\ldots$$

$$\text{so } k \simeq \sqrt{\frac{\alpha\omega}{2}}(1+i)\left[1 - \tfrac{1}{2}i(\omega/\alpha v^2)\right], \text{ for } \alpha \to \text{large};^{\dagger}$$

$$\text{or } \left[\!\left[\, k = k_R + i k_I \,\Big/ \begin{array}{l} k_R = \sqrt{\frac{\alpha\omega}{2}}\left[1 + \frac{1}{2}(\omega/\alpha v^2)\right], \\ k_I = \sqrt{\frac{\alpha\omega}{2}}\left[1 - \frac{1}{2}(\omega/\alpha v^2)\right]. \end{array} \right]\!\right] \qquad (3)$$

3. Put k of Eq. (3) into the planewave (in the metal) to get

$$\longrightarrow u(x,t) = \left[e^{-k_I x}\right] e^{i(k_R x - \omega t)}. \qquad (4)$$

The factor in front attenuates to ~ negligible values at distances Δx such that $k_I \Delta x \sim 1$. The characteristic penetration depth is thus

$$\boxed{\Delta x \sim 1/k_I = \sqrt{2/\alpha\omega}\left[1 + \tfrac{1}{2}(\omega/\alpha v^2)\right]}. \qquad (5)$$

With $\alpha = 4\pi\mu\sigma/c^2$, it is easy to show $\Delta x \equiv \delta$, Jackson's "skin depth" of Eq. (7.77).

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† From class notes (2/12/91): $\alpha = 4\pi\mu\sigma/c^2$. In Eq. (3), $\alpha \to$ "large" means $\alpha v^2 \gg \omega$. With $v = c/\sqrt{\mu\epsilon}$, this translates to: $4\pi\sigma \gg \epsilon\omega$, as a condition on conductivity σ.