ϕ506 MidTerm Exam Preview

Fri. 10/15/93

The ϕ506 MidTerm Exam will be given on Mon. 10/18/93, @ 11AM–1PM, in AJM 221.

This exam will be open-book, open-notes... with the following restrictions on the reference material you bring to the exam:

1. The "book" can be a copy of Davydov, or some other single QM text of your choice.

2. The "notes" can be your copies of class notes as they have been handed out, or notes/summaries in your own handwriting. Your problem solutions and/or solution keys are also OK.

3. One math reference (CRC Tables) is permitted. A dictionary, too.

The exam will be worth 150pts, and will consist of 5 problems, with general descriptions as follow...

1. Uses of the uncertainty relations to estimate QM system energies.

2. Constants-of-the-motion for stationary states.

3. Nature of the minimum energy in a bound state problem.

4. The QM version of the classical work-energy theorem.

5. Using $|\Psi(x)|^2$ as a position probability density.

May your value of $\langle \text{GOOD} | \text{LUCK} \rangle$ be (+)ve definite.

Dick Robiscoe

ϕ 506 MidTerm Exam　　　　　　　　Mon. 18 Oct. 1993

This exam is open-book, open-notes, and is worth 150 points. There are 5 problems w/ individual point values as marked. For each problem, $\boxed{\text{box}}$ your answer, $\boxed{\text{number}}$ your solution pages consecutively, write your $\boxed{\text{name}}$ on the cover sheet, and $\boxed{\text{staple}}$ the pages together before handing them in.

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1 [25 pts]. Use the position-momentum uncertainty relation (in 1D) to estimate the minimum (total) energy for a particle of mass m that is:

(A) confined to a box of length l;

(B) sitting on a table, and subjected to gravity (of acceleration g).

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2 [30 pts]. A dynamical quantity Q (may or may not be an operator) does <u>not</u> depend explicitly on time. Show that in a QM "stationary state", described by a wavefunction ψ_n (as defined by Schrödinger's eigenvalue eqtn: $\mathcal{H}\psi_n = E_n\psi_n$, E_n = cnst energy), the expectation value $\langle Q\rangle$ is a constant of the motion.

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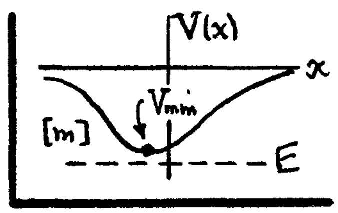
3 [35 pts]. Consider a 1D bound state problem, w/ the potential $V(x)$ vanishing as $|x|\to\infty$, and going through an absolute minimum $V_{min} < 0$ near $x = 0$. A particle of mass m bound in this well has an energy $E < 0$. Show that it is <u>not</u> possible to have $E < V_{min}$.



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4 [35 pts]. For a 1D QM system with Hamiltonian $\mathcal{H} = (p^2/2m) + V(x)$, find an expression for the time rate-of-change of kinetic energy: $\frac{d}{dt}\langle K\rangle$, w/ $K = p^2/2m$. Relate your result to the classical work-energy theorem: $\frac{d}{dt}K = (\text{force})\times(\text{velocity})$.

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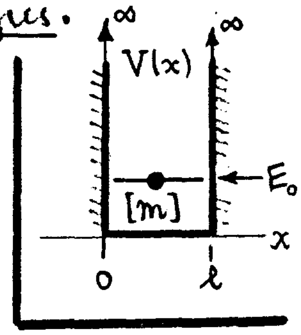
5 [25 pts]. A particle moves along the x-axis, in a QM state described by the wavefunction: $\psi(x) = \mathcal{N}(a^2 - x^2)$, for $|x| \leq a$; $\psi(x) \equiv 0$, for $|x| > a$. $\mathcal{N}$ & a are positive cnsts. What is the probability that a measurement of the particle's position will locate it somewhere in the range $(-)a/2$ to $(+)a/2$?

ϕ506 MidTerm Solutions (Oct. '93).

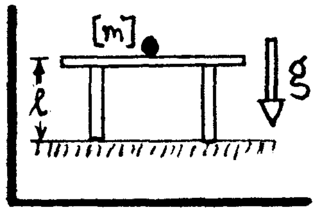
① [25pts]. Uncertainty relation estimates of ground-state energies.



(A) A 1D QM "box" of length ℓ can be described by a potential: $V(x)=0$, over $0<x<\ell$; $V(x)=\infty$, for $x\leq 0$ & $x\geq\ell$ (per CLASS NOTES, EXAMPLE on pp. Prop. 6-7). The contained particle of mass m is never found at $x\leq 0$ or $x\geq \ell$, but moves freely in $0<x<\ell$. The confinement to within $\Delta x<\ell$ must be accompanied by momentum components of size $\Delta p\sim\hbar/\Delta x\gtrsim\hbar/\ell$, so m's K.E. in the box is

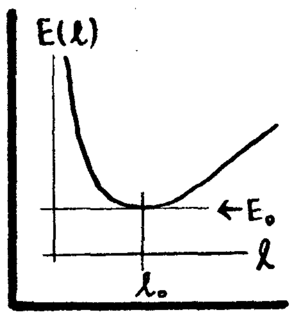
$$\boxed{E_0\sim(\Delta p)^2/2m\gtrsim\hbar^2/2m\ell^2} \tag{1}$$

Since the P.E. in the box is $V=0$, then E_0 is also m's minimum total energy -- to within numerical factors. The actual min. energy is (per Eq. (19), p. Prop. 7) $E_0=\pi^2(\hbar^2/2m\ell^2)$.



(B) To be on the table, m must be localized to within a vertical distance $\Delta x\sim\ell$, the height of the table. The momentum components accompanying this localization are $\Delta p\sim\frac{\hbar}{\ell}$, so m's total energy (K.E. + gravitational P.E.) at position ℓ is //

$$E(\ell)\sim\frac{1}{2m}(\hbar/\ell)^2+mg\ell\ . \tag{2}$$



$E(\ell)$ goes through a min. E_0 @ ℓ such that

$\rightarrow\ \partial E(\ell)/\partial\ell=0\Rightarrow\ \ell=(\hbar^2/m^2g)^{1/3}=\ell_0$;

So// $\boxed{E_0=E(\ell_0)\sim\frac{3}{2}(\hbar^2mg^2)^{1/3}}$, is required min. total energy. (3)

Notice that the assumed height of the table has dropped out of E_0 -- we get the same result no matter what table height/localization we assume at the outset.

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② [30 pts]. Condition for a constant-of-the-motion for a stationary state.

1. Since Q does not depend explicitly on time (i.e. $\partial Q/\partial t = 0$), then the QM equation-of-motion [CLASS NOTES, p. Prop. 16, Eq. (14)] prescribes that ...

$$\rightarrow \frac{d}{dt}\langle Q\rangle = \frac{i}{\hbar}\langle[\mathcal{H}, Q]\rangle. \tag{1}$$

Then Q is a cnst-of-the-motion if and only if it commutes with the system Hamiltonian, i.e. $[\mathcal{H}, Q] = 0$. This was duly noted in Eq. (6) of p. Prop. 14 of CLASS NOTES.

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2. In the n^{th} stationary state of a QM system specified by $\mathcal{H}\psi_n = E_n\psi_n$, the expectation value in Eq. (1) is

$$\rightarrow \langle[\mathcal{H}, Q]\rangle_n = \langle\psi_n|\mathcal{H}Q - Q\mathcal{H}|\psi_n\rangle$$

$$= \langle\psi_n|\mathcal{H}(Q\psi_n)\rangle - \langle\psi_n|Q(\overbrace{\mathcal{H}\psi_n}^{=E_n\psi_n})\rangle. \tag{2}$$

In the 2nd integral, $\mathcal{H}\psi_n = E_n\psi_n$ as noted, and the cnst E_n comes out of the integral, i.e. $\langle\psi_n|Q(\mathcal{H}\psi_n)\rangle = E_n\langle\psi_n|Q\psi_n\rangle$. In the 1st integral in (2), use the fact that $\mathcal{H}$ is self-adjoint (i.e. $\langle f|\mathcal{H}g\rangle = \langle\mathcal{H}f|g\rangle$) to write: $\langle\psi_n|\mathcal{H}(Q\psi_n)\rangle = \langle\mathcal{H}\psi_n|Q\psi_n\rangle = E_n\langle\psi_n|Q\psi_n\rangle$, since E_n is real. Then, in Eq. (2), we get...

$$\rightarrow \langle[\mathcal{H}, Q]\rangle_n = E_n\langle\psi_n|Q\psi_n\rangle - E_n\langle\psi_n|Q\psi_n\rangle = 0. \tag{3}$$

i.e.// $[\mathcal{H}, Q] = 0$, generally, in any stationary state (for $\partial Q/\partial t = 0$);

so// $\frac{d}{dt}\langle Q\rangle = 0$, by Eq. (1), and Q is a cnst-of-the-motion. (4)

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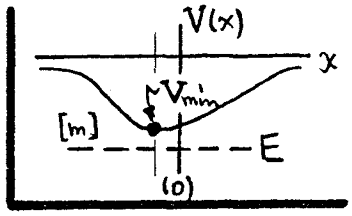
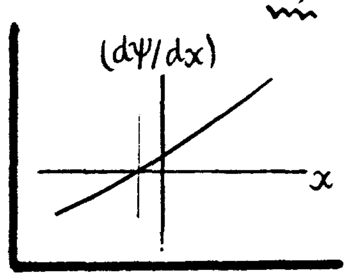
(3) [35 pts]. A lower limit on the energy for a particle bound in a 1D well.

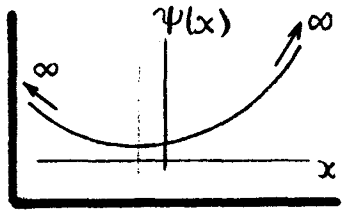


1. If the bound state energy $E < V_{min}$ (absolute minimum), then $E < V(x)$ over the entire range. Schrödingers Eqtn, written as...

$$\rightarrow \frac{1}{\psi}\frac{d}{dx}\left(\frac{d\psi}{dx}\right) = \frac{2m}{\hbar^2}\left[V(x) - E\right] > 0\text{; for all } x; \qquad (1)$$



then prescribes that $(d\psi/dx)$ increases over the entire range of x. Consequently, $(d\psi/dx)$ vs. x, and ψ vs. x, appear as sketched at right for this situation.



2. Clearly, when $E < V_{min}$, the wavefcn ψ for this state will diverge at the extremities of the well: $\psi \to \infty$ as $|x| \to \infty$. Among other unpleasantries, this would result in an inscrutable global probability: $\int_{-\infty}^{+\infty} |\psi(x)|^2 dx \to \infty$. The standard (and reasonable) requirement in QM is that $\psi(x)$ vanishes as $|x| \to \infty$ for a finite well, and that $\int_{-\infty}^{+\infty} |\psi|^2 dx \to$ finite. So the assumption $E < V_{min}$ is not reasonable, and must be thrown out. We conclude, on this basis...

$$\boxed{\text{Any bound state must have } E > V_{min} \text{ (i.e. } |E| < |V_{min}|\text{)}.} \qquad (2)$$

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3. Another argument against $E < V(x)$ everywhere is that -- since $E = K + V$ -- we would have a situation where the kinetic energy: $K = p^2/2m < 0$, everywhere, so that expectation values of the momentum p would be imaginary everywhere. Then any planewave component $\psi \sim \exp\left(\frac{i}{\hbar}px\right)$ of the system wavefcn would either grow or decrease exponentially, and the bound particle could not exhibit a stable motion anywhere in its range.

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④ [35 pts]. QM version of the work-energy theorem.

1. QMly, $K = p^2/2m$ is a operator (w/ $p = -i\hbar\,\partial/\partial x$) that does not depend explicitly on time. So, by the QM eqtn-of-motion (CLASS NOTES, p. Prop. 16, Eq. (14)), we write: $\frac{d}{dt}\langle K\rangle = \frac{i}{\hbar}\langle[\mathcal{H}, K]\rangle$. But $\mathcal{H} = K + V$, and since K commutes with itself, the equation of interest is...

$$\rightarrow \frac{d}{dt}\langle K\rangle = \frac{i}{\hbar}\langle[V, K]\rangle = \frac{i}{2m\hbar}\langle[V, p^2]\rangle. \qquad (1)$$

So we need the commutator $[V, p^2]$, w/ $V = V(x)$ & $p = (\hbar/i)\frac{\partial}{\partial x}$.

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2. A straightforward expansion of the RHS of the eqtn...

$$\rightarrow [V, p^2] = Vp^2 - p^2 V = [V, p]p + p[V, p], \qquad (2)$$

establishes (2) as an identity. Now we need just: $[V, p] = i\hbar(\partial V/\partial x)$, by Eq. (11A), p. Prop. 15. With that, we can write...

$$\rightarrow [V, p^2] = i\hbar\{(\partial V/\partial x)p + p(\partial V/\partial x)\}, \qquad (3)$$

w/ $p = (\hbar/i)\frac{\partial}{\partial x}$ still an operator (acting on everything to the right).

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3. Now use (3) in (1). Put $-(\partial V/\partial x) = F(x)$, the force acting on m, and put $v = p/m$ = the velocity operator for m. Then...

$$[V, p^2] = -i\hbar m\{Fv + vF\},$$

So
$$\boxed{\frac{d}{dt}\langle K\rangle = \tfrac{1}{2}\langle Fv + vF\rangle}; \quad v = \left(\frac{\hbar}{im}\right)\frac{\partial}{\partial x} \text{ velocity operator}, \quad F = F(x) \text{ acting force.} \qquad (4)$$

(4) is the desired QM result for how m's K.E. changes in time. The classical result is: $\frac{d}{dt}K = Fv$, which closely resembles (4). "All" that happens in QM is that $Fv \rightarrow \frac{1}{2}(Fv + vF)$ becomes a symmetrized product, and $v \rightarrow$ operator.

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5 [25 pts]. Calculating an explicit position probability with a given $\psi(x)$.

1. Since $|\psi(x)|^2 dx \propto$ probability that the particle is in dx at x, then the probability of finding the particle in $-b \leqslant x \leqslant +b$ is $\propto \int_{-b}^{+b} |\psi(x)|^2 dx$. When $b = a$, this probability can be set $= 1$ ($\Rightarrow$ 100% chance of finding the particle in $-a \leqslant x \leqslant +a$... since $\psi \equiv 0$ for $x > |a|$). For $b = \frac{1}{2}a$, the probability is:

$$\rightarrow P(\tfrac{1}{2}) = p(\tfrac{1}{2})/p(1); \quad \text{w/} \quad p(\epsilon) = \int_{-\epsilon a}^{+\epsilon a} |\psi(x)|^2 dx, \quad 0 \leqslant \epsilon \leqslant 1. \tag{1}$$

The division by $p(1)$ here saves an explicit evaluation of $\mathcal{N}$.

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2. For $\psi(x) = \mathcal{N}(a^2 - x^2)$, $|x| \leqslant a$, the integral in (1) is...

$$\rightarrow p(\epsilon) = \mathcal{N}^2 \int_{-\epsilon a}^{+\epsilon a} (a^2 - x^2)^2 dx = \mathcal{N}^2 \left\{ a^4 \int_{-\epsilon a}^{\epsilon a} dx - 2a^2 \int_{-\epsilon a}^{\epsilon a} x^2 dx + \int_{-\epsilon a}^{\epsilon a} x^4 dx \right\}$$

$$= \mathcal{N}^2 \left\{ 2\epsilon a^5 - \frac{4}{3}\epsilon^3 a^5 + \frac{2}{5}\epsilon^5 a^5 \right\}$$

$$= \mathcal{N}^2 \cdot 2\epsilon a^5 \left\{ 1 - \frac{2}{3}\epsilon^2 + \frac{1}{5}\epsilon^4 \right\}. \tag{2}$$

So// $\underline{\underline{p(1) = \left(\frac{16}{15}\right)\mathcal{N}^2 a^5}}$, $\underline{\underline{p(\tfrac{1}{2}) = \left(\frac{203}{240}\right)\mathcal{N}^2 a^5}}$. (3)

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3. If we needed the norm cnst $\mathcal{N}$, we'd set $p(1) = 1 \Rightarrow \mathcal{N} = \sqrt{(15/16a^5)}$. As it is, $\mathcal{N}$ divides out of the problem, and the desired probability of finding the particle in $|x| \leqslant \frac{1}{2}a$, per Eq. (1), is...

$$\boxed{P(1/2) = p(1/2)/p(1) = 203/256 = 0.793}. \tag{4}$$

Conversely, the particle is found in $\frac{a}{2} \leqslant x \leqslant a$ only 0.207 of the time. The par- is found in $-\epsilon a \leqslant x \leqslant +\epsilon a$ a percentage: $P(\epsilon) = \frac{15}{8}\epsilon\left\{1 - \frac{2}{3}\epsilon^2 + \frac{1}{5}\epsilon^4\right\}$. This probability is $\simeq 50\%$ when $\epsilon = 0.281$; the particle spends half its time exploring only 28% of its allowed range.