DEPARTMENT OF PHYSICS

2006 COMPREHENSIVE EXAM

Answer each of the following questions, all of which carry equal weight. Begin your answer to each question on a new sheet of paper; solutions to different questions must not appear on the same sheet. Each sheet of paper must be labeled with your name and the problem number in the upper right hand corner. When more than one sheet is submitted for a problem, be sure the pages are ordered properly.

PHYSICAL CONSTANTS

Quantity	Symbol	Value
acceleration due to gravity	g	9.8 m s^{-2}
gravitational constant	G	6.67×10^{-11} N m^2 kg^{-2}
permittivity of vacuum	e_0	8.85×10^{-12} C^2 N^{-1} m^{-2}
permeability of vacuum	μ_0	$4\pi \times 10^{-7}$ N A^{-2}
speed of light in vacuum	c	3.00×10^8 m s^{-1}
elementary charge	e	1.602×10^{-19} C
mass of electron	m_e	9.11×10^{-31} kg
mass of proton	m_p	1.673×10^{-27} kg
Planck constant	h	6.63×10^{-34} J s
Avogadro constant	N_A	6.02×10^{23} mol^{-1}
Boltzmann constant	k	1.38×10^{-23} J K^{-1}
molar gas constant	R	8.31 J mol^{-1} K^{-1}
standard atmospheric pressure		1.013×10^5 Pa

Problem #1

A ball with mass 1.0 kg falls from the height h = 1.0 meter on a vertical spring with the spring constant k_1=100 N m^{-1}.

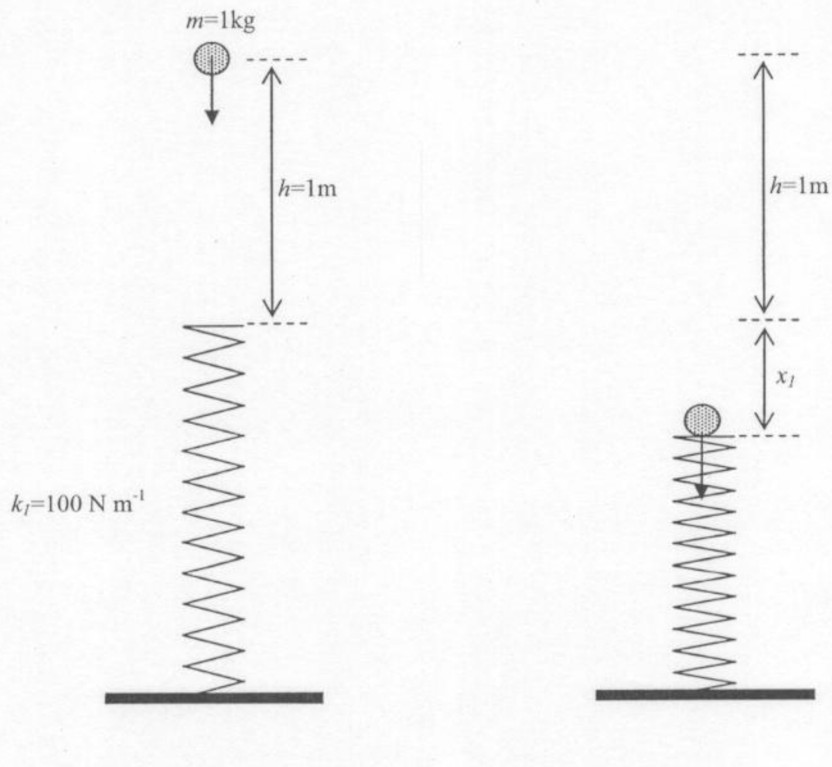
(a) What is the maximum compression distance of the spring (shown as x_1 in the figure) if the motion is totally without friction?

(b) Suppose the ball sticks to the spring when the two come into the contact. What is the oscillation frequency of the system?

(c) What time does it take for the spring to reach maximum compression from the moment of the contact with the ball?

(d) Suppose the whole system is immersed in a viscous liquid. The friction force acting on the ball is proportional to the velocity, $F_f = \beta\dot{x}$, where β= 100 N s m^{-1}.

Estimate the time it takes for the ball to reach the point of contact. What is approximately the velocity of the ball at that point? What is the oscillation frequency of the system?



1. Classical mechanics

(a) Maximum compression distance

$$\frac{1}{2} D_1 x_1^2 = mg(h + x_1)$$

$$\frac{1}{2} D_1 x_1^2 - mg x_1 - mgh = 0$$

$$x_1^2 - \frac{2mg}{D_1} x_1 - \frac{2mgh}{D_1} = 0$$

$$x_1 = \frac{mg}{D_1} \pm \sqrt{\left(\frac{mg}{D_1}\right)^2 + \frac{2mgh}{D_1}}$$

$$= \frac{0.5\,kg \cdot 9.8\,\frac{m}{s^2}}{10^3\,N.m^{-1}} \pm \sqrt{\left(\frac{0.5 \cdot 9.8}{10^3}\,m\right)^2 - \frac{2 \cdot 0.5 \cdot 9.8 \cdot}{10^3}\,m^2}$$

$$= 4.9 \cdot 10^{-3}\,m \pm \sqrt{2.4 \cdot 10^{-5} + 9.8 \cdot 10^{-3}}\,m =$$

$$= 4.9 \cdot 10^{-3}\,m + 9.9 \cdot 10^{-2} = 0.104\,m$$

(b) Maximum velocity: $\dot{v} = a = 0$; $\frac{1}{2} m v_0^2 = mgh$

$$mg = D_1 \cdot x_2 \Rightarrow x_2 = \frac{mg}{D_1} = 4.9 \cdot 10^{-3}\,m$$

$$\frac{1}{2} g t_1^2 = h \Rightarrow t_1 = \sqrt{2h/g} = \sqrt{2 \cdot 1m / 9.8\,m \cdot s^{-2}} = 0.452\,s$$

$$\frac{d^2x}{dt^2} m = mg - x D_1 \Rightarrow \frac{d^2x}{dt^2} + \underbrace{\frac{D_1}{m}}_{a} x = \underbrace{g}_{b}$$

$$\left.\begin{aligned} &\ddot{x} + ax - b = 0 \\ &x(0) = 0 \\ &\dot{x}(0) = v_0 = \sqrt{2gh} \end{aligned}\right\} \begin{aligned} &x(t) = \frac{b}{a} + C_1 \cos\sqrt{a}\,t + C_2 \sin\sqrt{a}\,t \\ &x(0) = \frac{b}{a} + C_1 = 0 \Rightarrow C_1 = -\frac{b}{a} \\ &\dot{x}(0) = \sqrt{a}\, C_2 \cos(\sqrt{2} \cdot 0) = \sqrt{2gh} \Rightarrow C_2 = \sqrt{2gh/a} \end{aligned}$$

$$x(t) = \frac{b}{a} - \frac{b}{a} \cos\sqrt{a} \cdot t + \sqrt{\frac{2gh}{a}} \sin\sqrt{a}\, t$$

$$\frac{mg}{D_1} = \frac{mg}{D_1}\left(1 - \cos\sqrt{\frac{D_1}{m}} \cdot t\right) + \sqrt{\frac{2ghm}{D_1}} \sin\sqrt{\frac{D_1}{m}}\, t$$

$$\cos\sqrt{\frac{D_1}{m}} \cdot t = \sqrt{\frac{2hD_1}{mg}} \sin\sqrt{\frac{D_1}{m}}\, t$$

$$\left(1 - \sin^2\sqrt{\frac{D_1}{m}}\, t\right) = \frac{2hD_1}{mg} \sin^2\sqrt{\frac{D_1}{m}}\, t$$

$$\sin\sqrt{\frac{D_1}{m}}\, t = \frac{1}{\sqrt{1 + \frac{2hD_1}{mg}}} = \frac{1}{\sqrt{1 + \frac{2 \cdot 1m \cdot 10^3 N \cdot m^{-1}}{0{,}5\, kg\; 9{,}8\ m/s^2}}}$$

$$= \frac{1}{\sqrt{1 + 4 \cdot 10^2}} \approx 0.05$$

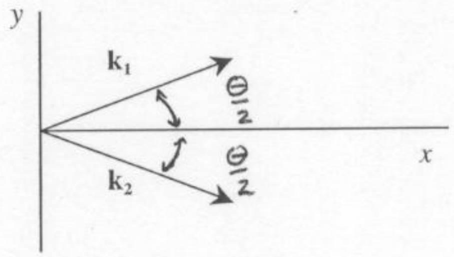
$$t_2 = \sqrt{\frac{1\, kg}{10^3 N \cdot m^{-1}}} \cdot \text{Arc Sin } 0{,}05 \approx \sqrt{10^{-3}} \cdot 0{,}05 = \underline{\underline{1{,}6 \cdot 10^{-3}\ s}}$$

$$t = t_1 + t_2 = \underline{\underline{0{,}453\ s}}$$

1b.

PROBLEM #2

Consider two coherent plane light waves having $|\mathbf{k}_1| = |\mathbf{k}_2|$ and $\mathbf{E}_{01} = \mathbf{E}_{02} = \mathbf{E}_0$. The directions of the $\mathbf{k}_1$ and $\mathbf{k}_2$ vectors are given below. Assume that the waves are polarized perpendicular to the page.



a) Calculate the intensity distribution of the wave interference pattern (the time-averaged irradiance) as a function of spatial coordinates x and y. Describe the interference pattern in words and determine the spacing spacing between fringes, D_F , of the resulting interference pattern. [Note that the pattern is independent of the coordinate z for the $\mathbf{k}_1$ and $\mathbf{k}_2$ vectors given; that is, the pattern is the same for all z.]

b) Now consider a small particle moving across the light interference fringes. Describe in some detail the scattered light signal that will be picked up as the particle (smaller in diameter than the spatial scale of the interference pattern) passes through the interference pattern.

c) Show that the apparatus can be used to measure an aspect or aspects of the particle's velocity. Determine precisely what aspect or aspects can be measured, being sure to also point out any ambiguity in the resulting measurement.

d) If one of the light beams were polarized in the plane of the diagram instead of perpendicular to it, would the capabilities be expanded? Explain.

Comp Question #2 - 2006

let $k \cos\frac{\theta}{2} = k_x$ and $k \sin\frac{\theta}{2} = k_y$

2(a) $$E = E_0\left[e^{i(\vec{k}_1\cdot\vec{r}-\omega t)} + e^{i(\vec{k}_2\cdot\vec{r}-\omega t)}\right]$$

$$= E_0\left[e^{i(k_x x + k_y y - \omega t)} + e^{i(k_x x - k_y y - \omega t)}\right]$$

$$= E_0 e^{i(k_x x - \omega t)}\underbrace{\left[e^{i k_y y} + e^{-i k_y y}\right]}_{2\cos k_y y}$$

$$\boxed{I = \frac{E^* E}{2} = 4\left(\frac{E_0^2}{2}\right)\cos^2 k_y y}$$ where $k_y = k \sin\frac{\theta}{2}$

Note: Note that this is a "standing wave".

To find spatial period of fringes, $k \sin\theta \; y = \pi$ ← period of $\cos^2$ function

$$y = \frac{\pi}{\frac{2\pi}{\lambda}\sin\frac{\theta}{2}}$$

$$\boxed{y = \frac{\lambda}{2\sin\frac{\theta}{2}}}$$

(This page is a H.W. Solution for Phys. 426.)

2. b & c - see attached sheets.

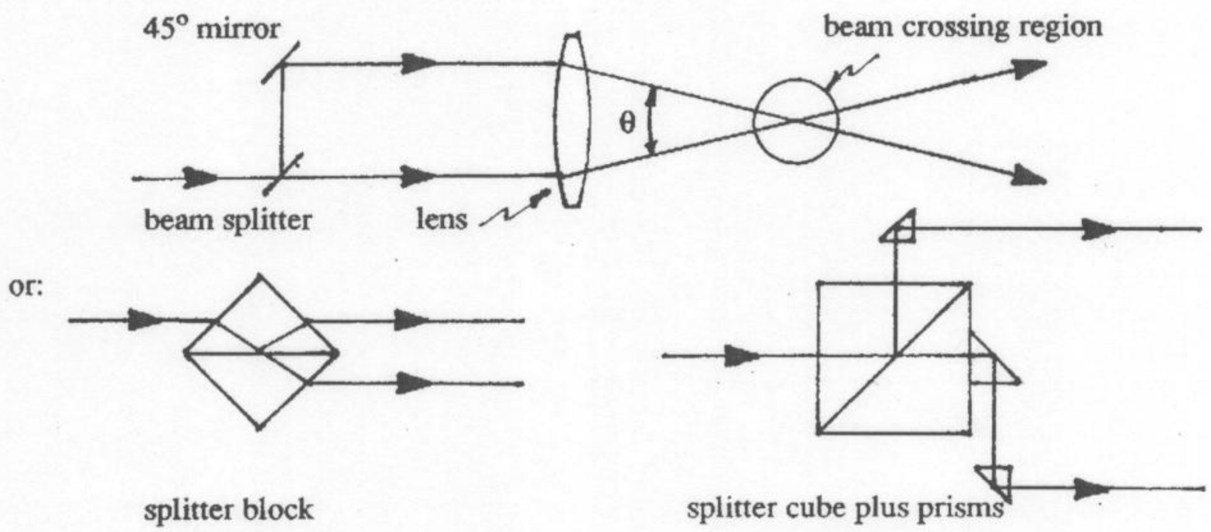
2. (d) Perpendicularly polarized waves cannot interfere.

Laser Doppler Velocimetry/Laser Doppler Anemometry - LDV or LDA

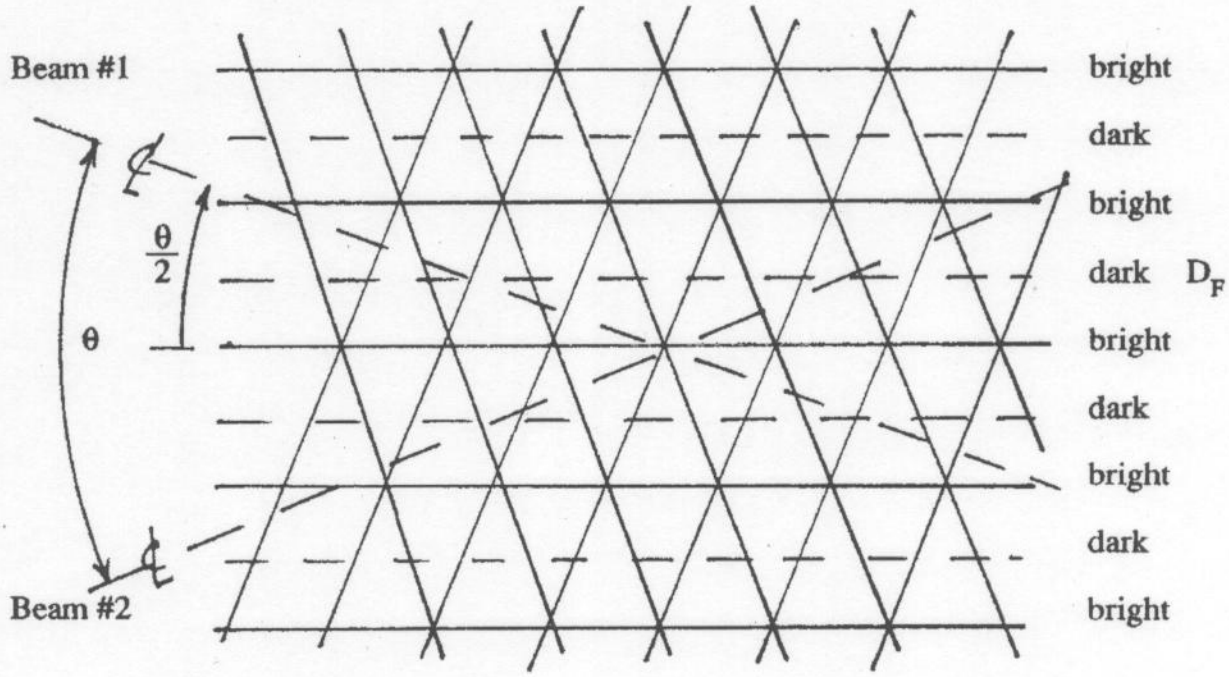
As a simple and very concrete illustration of widely used light scattering applications, we consider the use of scattering for *velocity measurement in a fluid* (gas or liquid). Several methods of mathematical analysis exist, and each can play a role in developing understanding of this measurement technique and related processes that allow us, for example, to design *acousto-optic light modulators and beam deflectors*, devices which can manipulate light signals at the fast nanosecond time scale. We consider first the conceptually simplest *fringe interpretation of LDV*, and then we introduce the more general concepts of light scattering and *light-beating spectroscopy* (related to the beats you hear when you tune your guitar or other musical instrument.

Basic LDV Apparatus and Theory

Consider two coherent crossed light beams, which could be derived from a single laser beam in the following way:



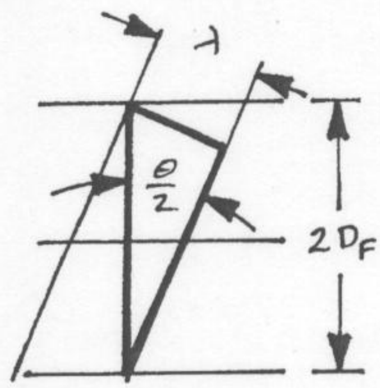
In the *beam crossing region*, the two laser beams overlap and interference occurs. In typical LDV setups, one wants a lens to focus both beams to small waists in the beam crossing region as shown in the diagram. The spatial overlap of two tightly focused beams defines a very small *measurement volume* in that case. Our analysis will show that a particle gives a different light scattering signal when it passes through this measurement volume than that given at other positions in the fluid. We will pick up that signal with a photomultiplier detector and use the detected electrical signal to measure the particle's velocity **v**. The small measurement volume provides one of the major advantages of this measurement technique over other techniques like hot wire anemometry — the laser technique has wonderful spatial resolution. This resolution is very important in cases as different as wind tunnel measurements around supersonic aircraft and blood flow measurements in the human body. We can also measure the velocities of individual particles in the fluid and establish statistics that tell us about turbulence (velocity fluctuations), and turbulence is a very important property of flows in many technological applications. Creation of an ideal fuel-air mixture in combustion is one example where turbulence is desired and where characterization of it is important for engineers improving engine efficiency and emissions.



Within the measurement volume defined by the overlap of the two beams, the Gaussian laser beams can be considered "plane waves" to a reasonable approximation since we are near their beam waists. That simplifies the analysis significantly. Of course more complicated model calculations have been carried out to investigate the accuracy of this simplified picture, but from here on, we assume that plane waves cross at the angle θ as shown above and that their interference sets up a pattern of equally-spaced interference fringes. The interference pattern in the measurement volume is shown on the top of the next page.

Constructive interference takes place where the *wavefronts* from the two waves cross (where both waves have the same phase). Destructive interference takes place at points halfway between (where the waves are π out of phase). The result is pattern of alternating regions of brightness and darkness, namely an *interference fringe pattern*. We will call the separation of the bright fringes D_F.

Derivation of D_F using a segment of the figure above, as reproduced and labeled here:



For the triangle whose hypotenuse is vertical,

$$\lambda = 2D_F \sin(\theta/2)$$

Solving for D_F gives the equation:

$$D_F = \frac{\lambda}{2\sin(\theta/2)}.$$

Now consider a small particle moving across the light interference fringes at a right angle to the fringes; we will call the particle a *scattering center*. In some cases the scattering center is the basic constituent of the fluid and in others it might be a *seed particle* that travels along with the fluid. The scattering center will alternately cross bright and dark areas of the interference pattern, alternately being exposed to strong light fields and scattering strongly and then not being exposed at all and thus not scattering. The result is an alternating periodic signal of light 'flashes' emanating from the particle as it moves through the fringe pattern. If the scattering center is moving perpendicular to the fringes, the time T_{signal} between the between bright regions is:

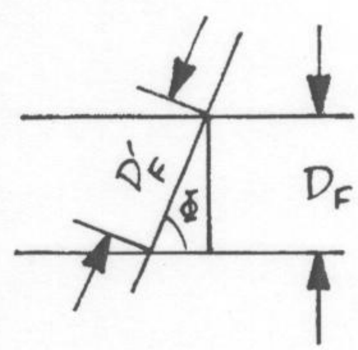
$T_{signal} = D_F/v_{particle}$.

The frequency of the observed flashing light signal for the perpendicular case considered above is:

$\nu_{signal} = 1/T_{signal} = v/D_F$.

The light signal thus becomes strong and weak at a rate *directly proportional to the particle's speed* v; moreover, the constant of proportionality that sets the speed calibration is easily determined using simple geometrical factors and the light wavelength.

Suppose that the particle crosses the fringes at an angle $\Phi \neq 90^\circ$. The new distance D'_F between bright regions for this path becomes longer than D_F.



Since $D_F = D'_F \sin\Phi$,

$$D'_F = \frac{D_F}{\sin\Phi}$$

and $$\nu_{signal} = \frac{v}{D'_F} = \frac{v}{D_F/\sin\Phi} = \frac{v\sin\Phi}{D_f} = \frac{v_{perpendicular}}{D_f}$$

We note that v sin Φ is the perpendicular component of the velocity vector. This simple LDV system thus measures the *velocity component perpendicular to the fringes* for any particle traversing the fringe region where the beams overlap.

As noted earlier, since the beams may be focused to a small size in the beam overlap region, the measurement volume may be quite small, allowing one to measure velocity as a function of position with high spatial resolution.

By contrast with classic methods like hot-wire anemometry, this LDV method has the further advantage that it involves no disturbance of the system being monitored, except perhaps for a seeding of the fluid with small scattering centers (depending on the nature of the fluid). In systems at normal temperatures, micron-sized polystyrene spheres are commonly used to seed liquids. Note though that we have so far implicitly assumed that the light scattering particles move at the same speed as the fluid. This assumption must be critically examined in each individual case.

PROBLEM #3

The ground state ($1s$) of a single electron bound to a nucleus with charge Ze is given by

$$\psi(r,\theta,\phi) = \frac{1}{\sqrt{\pi}}\left(\frac{Z}{a_0}\right)^{3/2} \exp(-Zr/a_0)$$

where $a_0 = 5.3 \times 10^{-9}$ cm is the Bohr radius.

a. Find the expectation of the potential energy

$$V = -\frac{Ze^2}{r}$$

in this state, for an arbitrary charge.

An atom of tritium ($Z = 1$) is in its ground state when it nucleus undergoes a β decay, changing the nucleus to ^{3}He ($Z = 2$). Consider this as an instantaneous change to the nucleus at $t = 0$ **which leaves the electron wave function unchanged.**

b. Write down the values of the electron's potential energy expectation both **before and after** the nuclear decay? Be sure you have eliminated all appearances of Z.

c. What is the probability of finding the electron in its ground state (i.e. of $Z = 2$) at some $t > 0$?

The following integral might prove useful

$$\int_0^\infty u^n e^{-\alpha u}\, du = \frac{n!}{\alpha^{n+1}} \qquad \text{(where } 0! = 1\text{)}$$

Problem #3

a. Denote by ψ_Z, the ground state for a nucleus charge Z. To calculate the expected potential energy we must find

$$\langle\psi_Z|\frac{1}{r}|\psi_Z\rangle = 4\pi\int_0^\infty \frac{1}{r}\psi_Z^2(r)\,r^2\,dr \;=\; 4\left(\frac{Z}{a_0}\right)^3 \underbrace{\int_0^\infty e^{-2Zr/a_0}\,r\,dr}_{1!\,(a_0/2Z)^2} = \frac{Z}{a_0}$$

In terms of this the expectation of the potential energy is

$$\langle V\rangle \;=\; -\,Ze^2\langle\psi_Z|\frac{1}{r}|\psi_Z\rangle \;=\; -\,\frac{Z^2e^2}{a_0} \tag{3-1}$$

(b) Before the nuclear decay the electron is in the groud state with $Z = 1$ so

$$\langle V\rangle_{\mathrm{i}} \;=\; -\,\frac{e^2}{a_0} \tag{3-2}$$

Afterward the electron is still in state $|\psi_1\rangle$ but the nuclear charge is now $Z = 2$. This makes

$$\langle V\rangle_f \;=\; -\,2e^2\langle\psi_1|\frac{1}{r}|\psi_1\rangle \;=\; -\,\frac{2e^2}{a_0} \tag{3-3}$$

so the expected potential energy has *dropped* by e^2/a_0 as a result of the decay.

(c) To calculate the probability of finding the electron is state $|\psi_2\rangle$ we must evaluate the inner product

$$\langle\psi_1|\psi_2\rangle = 4\pi\int_0^\infty \psi_1(r)\psi_2(r)\,r^2\,dr = 2^{3/2}4a_0^{-3}\underbrace{\int_0^\infty e^{-3r/a_0}\,r^2\,dr}_{2!\,(a_0/3)^3} = \frac{2^{3/2}8}{3^3}$$

The probability is the square magnitude of this

$$P \;=\; |\langle\psi_1|\psi_2\rangle|^2 \;=\; \frac{8^3}{9^3} \;=\; 0.70 \tag{3-4}$$

there is about a 70% chance of finding the electron in its ground state, where its potential energy would be $-4e^2/a_0$.

PROBLEM #4

A circular polarizing filter, a disk of radius 0.1 m, absorbs 100% of the left-hand circularly polarized component of incoming light and transmits 100% of the right-hand circularly polarized component. Sunlight is perpendicularly incident on one side of the polarizer, with the usual intensity of $I = 1.4\,\mathrm{kW/m^2}$. Estimate (a) **the force** and (b) **the torque** the light extert on the filter.

Problem #4

A circular polarizing filter, a disk of radius 0.1 m, absorbs 100% of the left-hand circularly polarized component of incoming light and transmits 100% of the right-hand circularly polarized component. Sunlight is perpendicularly incident on one side of the polarizer, with the usual intensity of $I = 1.4\,\mathrm{kW/m^2}$. Estimate (a) **the force** and (b) **the torque** the light extert on the filter.

Solution

(a) The momentum flux carried in the beam of sunlight is

$$\mathcal{P} \;=\; I/c \;=\; 4.7 \times 10^{-6}\,\mathrm{kg\,m^{-1}\,s^{-2}}$$

The incoming light is *unpolarized*, and therefore consists of equal parts right and left circular polarization. The filter absorbs half of this (the LHC component). The force on an area $A = \pi(0.1\mathrm{m})^2 = 0.0314\,\mathrm{m}^2$ is

$$F = 0.5\,\mathcal{P}\,A = 0.5 I A/c \;=\; 7.3 \times 10^{-8}\,\mathrm{N} \tag{4-1}$$

(b) For the purpose of estimation we will assume that sunlight consists of photons whose wavelength is $\lambda = 500$ nm. This means each photon has an energy

$$E_\gamma = hc/\lambda = 4 \times 10^{-19}\,\mathrm{J}$$

The power incident on the filter is

$$P_{\mathrm{fil}} = I\,A = 44.0\,\mathrm{W}$$

So the number of LHC *photons* incident on the filter is approximately

$$\phi \;=\; \frac{1}{2}\frac{P_{\mathrm{fil}}}{E_\gamma} \;=\; \frac{IA\lambda}{2hc} \;=\; 5 \times 10^{19}\,\mathrm{photons/sec}$$

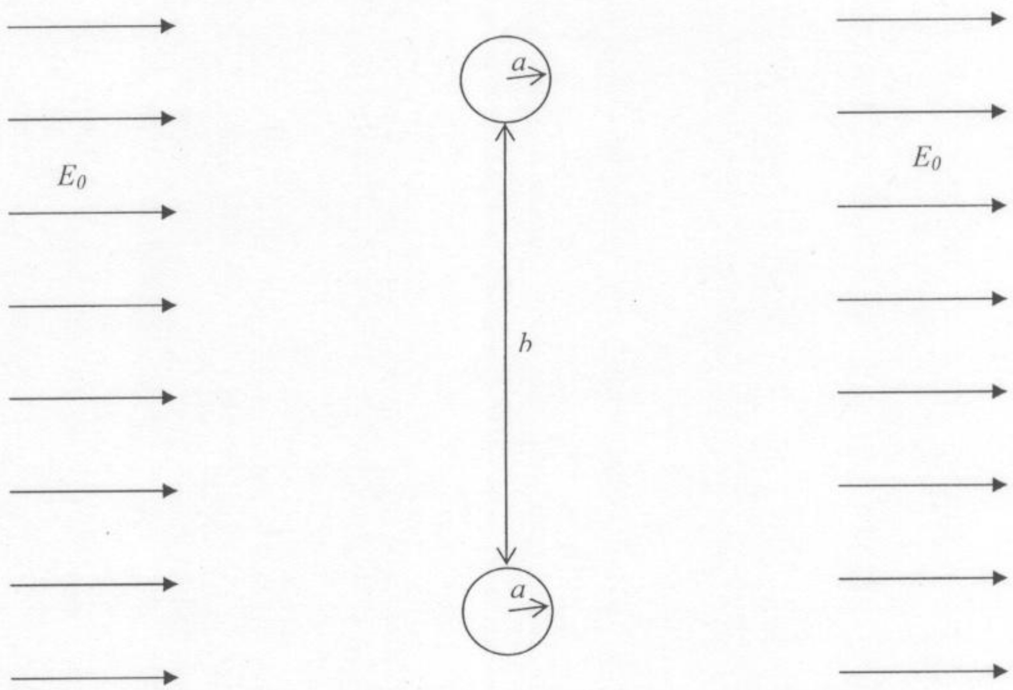
Each of these photons carries an angular momentum $\hbar$ which is absorbed by the filter. This exerts a torque

$$N \;=\; \hbar\phi \;=\; \frac{IA\lambda}{4\pi c} \;=\; F\lambda/2\pi \;=\; 6 \times 10^{-15}\,\mathrm{N\,m} \tag{4-2}$$

Note that the torque and the force are in the same direction so they are not directly related.

Problem #5

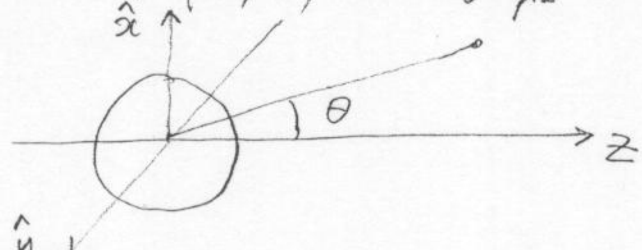
Two conducting grounded spheres of radius a are positioned at distance b apart from each other. External electric field of constant strength E_0 is applied perpendicular to the line connecting the two spheres. Find the force acting between the two spheres with the assumption that the distance between the spheres is large compared to the radius of the sphere, $b >> a$.



Problem # 5

In an external uniform field a conducting sphere gives rise to a dipole potential (Jackson 2.5)

$$\Phi(r,\theta) = E_0 \frac{a^3}{r^2}\cos\theta = \frac{(\vec{d}\cdot\vec{r})}{r^3}\ ; \ \vec{d} = E_0 a^3 \cdot \hat{z}$$



Field created by the 1^{st} dipole at the location of the 2^{nd} dipole is

$$\vec{E}(b,0,0) = \Big[-\vec{\nabla}\Phi(x,y,z)\Big]_{\vec{x}=\{b,0,0\}} =$$

$$-E_0 a^3 \left\{ \frac{3zx}{(x^2+y^2+z^2)^{5/2}},\ \frac{3zy}{(x^2+y^2+z^2)^{5/2}},\ \frac{x^2+y^2-2z^2}{(x^2+y^2+z^2)^{5/2}} \right\}_{\vec{x}=\{b,0,0\}}$$

$$= -E_0 a^3 \left\{ 0,\ 0,\ \frac{1}{b^3} \right\}$$

Energy of the 2^{nd} dipole: $\quad W = \vec{d}\cdot\vec{E}$

$$= -E_0 a^3 \cdot \frac{1}{b^3} \cdot E_0 a^3 = -\frac{E_0^2 a^6}{b^3}$$

Force acting on the 2^{nd} dipole

$$\vec{F} = -\frac{dW}{db}\hat{x} = \underline{\underline{\frac{3E_0^2 a^6}{b^4}\hat{x}}}$$

PROBLEM #6

A cylinder with a piston contains a monoatomic ideal gas in thermodynamic equilibrium with state variables P, V, T, U, and S. The cylinder is surrounded by a heat reservoir at the same temperature T. The walls and piston can be either a perfect thermal conductor or a perfect thermal insulator. The piston is moved to produce a small volume change $+\Delta V$ or $-\Delta V$. The piston moves either "slow" or "fast" where slow is a rate much less than the mean molecular speed at temperature T and fast is much greater than the mean molecular speed. For each of the processes given below, determine whether the changes (after equilibrium is re-established) in the other quantities are positive (+), negative (-), or zero (0).
[You may enter your answers on the table below, but briefly describe your reasoning for your answers.]

CONDITIONS	ΔT	ΔU	ΔS	ΔP
$+\Delta V$ (piston out), "slow" piston, conducting walls				
$+\Delta V$ (piston out), "slow" piston, insulating walls				
$+\Delta V$ (piston out), "fast" piston, insulating walls				
$+\Delta V$ (piston out), "fast" piston, conducting walls				
$-\Delta V$ (piston in), "fast" piston, conducting walls				

SOLUTION Problem # 6

	ΔT	ΔU	ΔS	ΔP
$+\Delta V$ slow conducting	0	0	+	-
$+\Delta V$ slow insulating	-	-	0	-
$+\Delta V$ fast insulating	0	0	+	-
$+\Delta V$ fast conducting	0	0	+	-
$-\Delta V$ fast conducting	0	0	-	+

The thermodynamic fundamental equation is $\Delta U = \Delta Q - \Delta W_{\text{by system}}$

Conducting walls in contact with a heat reservoir is an isothermal situation. $\Delta T = 0$.

Insulating walls means no heat transfer. $\Delta Q = 0$.

Fast moving wall means free expansion and no work can be done. $\Delta W = 0$.

For an ideal gas, $PV = nRT$ and the internal energy is only a function of the temperature.

$$\Delta U = C\,\Delta T, \text{ with } C > 0.$$

CASE 1: Isothermal expansion. This is an irreversible process and heat flows into the gas, so $\Delta S > 0$. An isothermal process has $P\,V$ = Constant and therefore $\Delta P = -\Delta V$ and $\Delta U = 0$ and $\Delta T = 0$.

CASE 2: Adiabatic expansion. This is a reversible process (done slowly) so $\Delta S = 0$. Adiabatic process means that $P\,V^{\gamma}$ = constant and therefore $\Delta P = -\Delta V$. The work done comes from the internal energy in the gas so $\Delta U < 0$ and $\Delta T < 0$.

CASE 3: Adiabatic free expansion. This is an irreversible process (done fast) so $\Delta S > 0$. Adiabatic process means that $P\,V^{\gamma}$ = constant and therefore $\Delta P = -\Delta V$ and with no work done, $\Delta U = 0$ and $\Delta T = 0$.

CASE 4: Isothermal free expansion. This is an irreversible process and heat flows into the gas, so $\Delta S > 0$. An isothermal process has $P\,V$ = Constant and therefore $\Delta P = -\Delta V$ and $\Delta U = 0$ and $\Delta T = 0$.

CASE 5: Isothermal free compression. This is an irreversible process and heat flows out of the gas, so $\Delta S < 0$ ($\Delta S > 0$ for the gas plus heat reservoir). An isothermal process has $P\,V$ = Constant and therefore $\Delta P = -\Delta V$ and $\Delta U = 0$ and $\Delta T = 0$.

Problem #7

A point mass m moves along the curve C_1 defined as the intersection between the surfaces S_1 and S_2 (see figure)

$$S_1:\ x^2+y^2+z^2-1=0,$$
$$S_2:\ 2x^2+y^2-1=0.$$

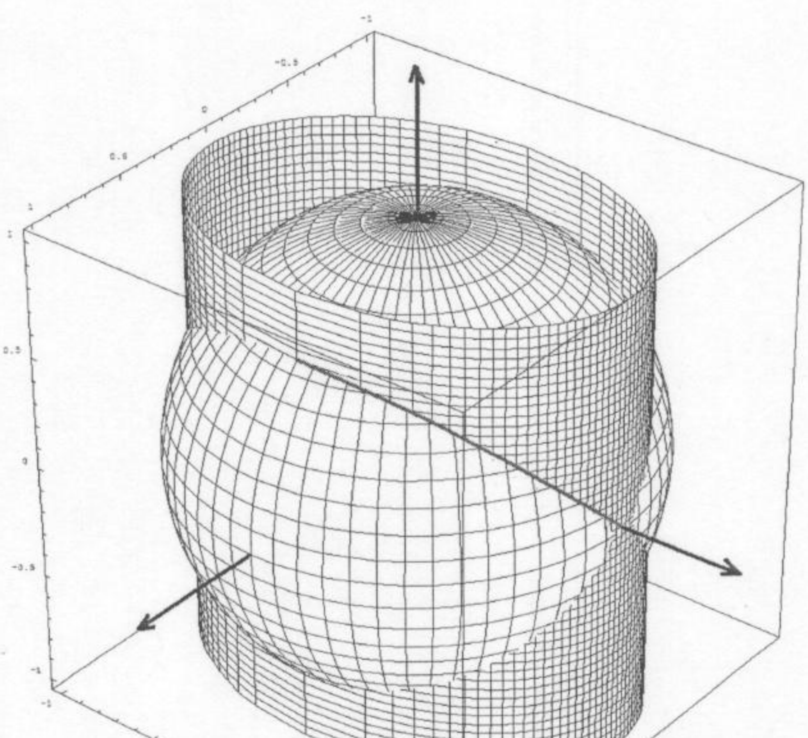
(a) Find a suitable parametric representation of C_1 of the form, $\{x(t), y(t), z(t)\}$, which is valid in the region $\{0<x, 0<y, 0<z\}$.

(b) Calculate the length of the segment of C_1 between the points $P_2=\{0,1,0\}$ and $P_2=\{\frac{1}{\sqrt{2}},0,\frac{1}{\sqrt{2}}\}$ in the region $\{0<x, 0<y, 0<z\}$.

(c) Find the maximum value of an external potential,

$$\Phi(x,y,z)=x^2+y+z^2.$$

on the curve C_1 between the points the P_1 and P_2.



Problem # 7

(a) Parametrization

$$\left.\begin{array}{l} x^2+y^2+z^2-1=0 \\ 2x^2+y^2 \qquad -1=0 \end{array}\right\} \Rightarrow -x^2+z^2=0 \Rightarrow x^2=z^2$$

for $x>0, z>0 \Rightarrow x=z$

$$\hookrightarrow y^2 = 1-2x^2 \Rightarrow y=\sqrt{1-2x^2}$$

$$x(t)=t \;;\; y(t)=\sqrt{1-2t^2}\;;\; z(t)=t \qquad 0<t<\frac{1}{\sqrt{2}}$$

(b) $$\int_0^{1/\sqrt{2}} \left[\left(\frac{dx(t)}{dt}\right)^2+\left(\frac{dy(t)}{dt}\right)^2+\left(\frac{dz(t)}{dt}\right)^2\right]^{1/2} dt$$

$$= \int_0^{1/\sqrt{2}} \left[1+\left(\frac{-2t}{\sqrt{1-2t^2}}\right)^2+1\right]^{1/2} dt = \int_0^{1/\sqrt{2}} \sqrt{2+\frac{4t^2}{1-2t^2}}\, dt = \frac{\pi}{2}$$

(c) $$\frac{d}{dt}\Phi(x(t), y(t), z(t)) = \frac{d}{dt}\left[t^2 + \sqrt{1-2t^2} + t^2\right]$$

$$= \frac{d}{dt}\left[2t^2 + \sqrt{1-2t^2}\right] = 4t - \frac{2t}{\sqrt{1-2t^2}} = 0$$

$$\Rightarrow \sqrt{1-2t^2} = \frac{1}{2}$$

$$\Rightarrow t = \frac{\sqrt{3}}{2\sqrt{2}} \quad \Rightarrow \Phi_{max} = 2\cdot\frac{3}{8} + \sqrt{1-\frac{3}{4}} = \frac{3}{4} + \frac{1}{2} = \underline{\underline{\frac{5}{4}}}$$

Alternative method: Lagrange multipliers

$$\left.\begin{array}{l} 2x - 2\lambda x - 4\mu x = 0 \\ 1 - 2\lambda y - 2\mu y = 0 \\ 2z - 2\lambda z = 0 \\ x^2 + y^2 + z^2 - 1 = 0 \\ 2x^2 + y^2 - 1 = 0 \end{array}\right\}$$

$\Rightarrow 1 - 2y - 2\mu y = 0 \Rightarrow \mu = \frac{1-2y}{2y}$

$\Rightarrow \lambda = 1$

$\Big\} \Rightarrow -x^2 + z^2 = 0 \qquad x^2 = z^2$

in region $x>0, y>0, z>0 \qquad x = z$

$$\cancel{2x} - \cancel{2x} - 4x\left(\frac{1-2y}{2y}\right) = 0 \quad \Rightarrow \quad y = \frac{1}{2}$$

$$z = x = \sqrt{\frac{1}{2}(1-y^2)} = \frac{\sqrt{3}}{2\sqrt{2}}$$

$$\Phi_{max} = \frac{3}{8} + \frac{1}{2} + \frac{3}{8} = \underline{\underline{\frac{5}{4}}}$$

PROBLEM #8

A scanning tunning microscope (STM) tip, modeled as a perfectly conducting cone $\theta \leq \alpha$ in spherical coordinates, is in contact with the surface of a perfectly conducting object occupying the half-space $\theta \geq \pi/2$. The STM tip is held at potential $\Phi = V$, while $\Phi = 0$ within the object. The infinitesimal point of contact (the origin) has resistance R.

A. Find the magnetic field, $\mathbf{B}(\mathbf{r})$, for $\alpha \leq \theta \leq \pi/2$.

B. Find the electric field, $\mathbf{E}(\mathbf{r})$, for $\alpha \leq \theta \leq \pi/2$.

C. Calculate (or deduce by physical insight) the value of

$$\oint_D \mathbf{S} \cdot \mathbf{da},$$

where $\mathbf{S}$ is the Poynting flux, and the domain D is the sphere $r = b$. Interpret your result physically.

Problem 8. A scanning tunning microscope (STM) tip, modeled as a perfectly conducting cone $\theta \leq \alpha$ in spherical coordinates, is in contact with the surface of a perfectly conducting object occupying the half-space $\theta \geq \pi/2$. The STM tip is held at potential $\Phi = V$, while $\Phi = 0$ within the object. The infinitesimal point of contact (the origin) has resistance R.

A. Find the magnetic field, $\mathbf{B}(\mathbf{r})$, for $\alpha \leq \theta \leq \pi/2$.

The current is $I = V/R$. Taking advantage of the axial symmetry,

$$\oint \mathbf{B} \cdot \mathbf{dl} = \mu_0 I_{enc}$$

$$\mathbf{B} = -\frac{\mu_0 V \hat{\phi}}{2\pi R \rho},$$

where ρ is the cylindrical radial coordinate.

B. Find the electric field, $\mathbf{E}(\mathbf{r})$, for $\alpha \leq \theta \leq \pi/2$.

The problem is azimuthally symmmetric, so $\nabla^2 \Phi(r, \theta) = 0$. Another symmetry: if $r \to kr$, the boundary conditions do not change. So the equipotentials must be radial, and Φ is a function of θ only. Consequently,

$$0 = \frac{d}{d\theta}\left(\sin\theta \frac{d\Phi}{d\theta}\right).$$

Solving by direct integration and applying the boundary conditions,

$$\Phi(\theta) = C \ln[\tan(\theta/2)] + D = V \frac{\ln[\tan(\theta/2)]}{\ln[\tan(\alpha/2)]}.$$

$$\mathbf{E} = -\nabla\Phi = \frac{-V\hat{\theta}}{2r\ \ln[\tan(\alpha/2)]}\left[\cot(\theta/2) + \tan(\theta/2)\right].$$

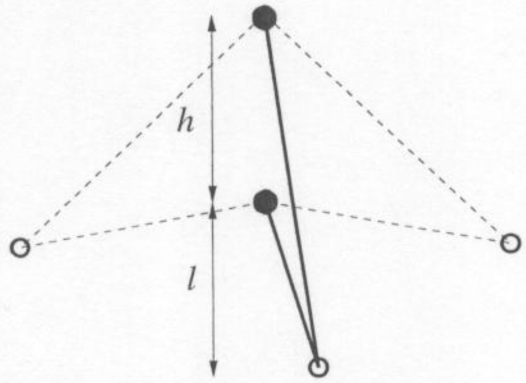
C. Calculate (or deduce by physical insight) the value of

$$\oint_D \mathbf{S} \cdot \mathbf{da},$$

where $\mathbf{S}$ is the Poynting flux, and the domain D is the sphere $r = b$. Interpret your result physically.

Because of Poynting's theorem, $\oint_S \mathbf{S} \cdot \mathbf{da} = -V^2/R$ for *any* surface S enclosing the origin. In other words, the fields must be consistent with the dissipation of power in the resistor placed at the origin.

PROBLEM #9



A small bead slides frictionlessly on a massless string. The ends of the string are fastened to two points separated vertically by a distance h. In equilibrium, the bead hangs a distance ℓ below the bottom point. The length of the string is therefore $L = 2\ell + h$.

(a) Writing the location of the bead x, y with the origin at the equilibrium express the effect of the string as a constraint function $C(x, y) = 0$.

(b) Using a Lagrange multiplier and the constraint function from a., find the equation of motion for **small displacements** from equilibrium.

(c) What is the frequency of small oscillation about the equilibrium?

Problem #9

A small bead slides frictionlessly on a massless string. The ends of the string are fastened to two points separated vertically by a distance h. In equilibrium, the bead hangs a distance ℓ below the bottom point. The length of the string is therefore $L = 2\ell + h$.

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(c) What is the frequency of small oscillation about the equilibrium?

Solution

(a) The string is fastened to points $\mathbf{x}_1 = (0, \ell)$ and $\mathbf{x}_2 = (0, \ell + h)$. The distances to each of the points from a point (x, y) is

$$d_1 = \sqrt{x^2 + (y-\ell)^2} \quad , \quad d_2 = \sqrt{x^2 + (y - \ell - h)^2} \ .$$

The constraint is to require that $d_1 + d_2 = 2\ell + h$, for which the function

$$C(x,y) = \sqrt{x^2 + (y-\ell)^2} + \sqrt{x^2 + (y-\ell-h)^2} - 2\ell - h \tag{9-1}$$

must vanish.

(b) The Lagrangian, including the constraint with undetermined multiplier λ, is

$$L(x, y, \dot{x}, \dot{y}) = \tfrac{1}{2}m\dot{x}^2 + \tfrac{1}{2}m\dot{y}^2 - mgy + \lambda C(x,y)$$

The Euler-Lagrange equations for each component are

$$m\ddot{x} = \lambda\frac{\partial C}{\partial x} \quad , \quad m\ddot{y} = -mg + \lambda\frac{\partial C}{\partial y}$$

The derivatives of C, expanded to lowest order in the small parameters, x and y are

$$\frac{\partial C}{\partial x} = \frac{x}{\sqrt{x^2 + (y-\ell)^2}} + \frac{x}{\sqrt{x^2 + (y-\ell-h)^2}} \simeq \frac{x}{\ell} + \frac{x}{\ell + h}$$

$$\frac{\partial C}{\partial y} = \frac{y-\ell}{\sqrt{x^2 + (y-\ell)^2}} + \frac{y-\ell-h}{\sqrt{x^2 + (y-\ell-h)^2}} \simeq \frac{y-\ell}{|y-\ell|} + \frac{y-\ell-h}{|y-\ell-h|} = -2$$

The y equation then becomes $m\ddot{y} = -mg - 2\lambda$, which means that to lowest order $\lambda = -mg/2$ so that $\ddot{y} = 0$ at equilibrium. The equations of motion for x is therefore

$$m\ddot{x} = -\tfrac{1}{2}mg\left(\frac{1}{\ell} + \frac{1}{\ell + h}\right)x \tag{9-2}$$

(c) Proposing an oscilliatory solution, $x(t) = Ae^{-i\omega t}$, to eq. (9-2) gives a frequency

$$\omega = \sqrt{\frac{g}{2}\left(\frac{1}{\ell} + \frac{1}{\ell + h}\right)} = \sqrt{\frac{g}{2}}\sqrt{\frac{2\ell + h}{\ell(\ell + h)}} \tag{9-3}$$

The bead moves along an ellipse and its equilibrium point is at the major axis, where the radius of curvature is (apparently) $\ell(\ell + h)/(\ell + h/2)$.

9a

Alternative Solution

It is also possible to use polar coordinates — although with more difficulty.

(a) Begin with coordinates (r, ϕ) and (r', ϕ') about the bottom and top points respectively. The length of the string can be used to place one constraint immediately:

$$r + r' = L = 2\ell + h \quad . \tag{9-4}$$

Equating the horizontal extents of each piece provides the second constraint

$$r \sin\phi = r' \sin\phi' \quad .$$

Since both angles are small we can use this to eliminate ϕ' as

$$\phi' \;=\; \frac{r}{r'}\phi \;=\; \frac{r}{L-r}\phi \quad . \tag{9-5}$$

The final constraint comes from the difference in vertical extents of the string pieces

$$r' \cos\phi' - r\cos\phi \;=\; h$$

Making the small-angle approximation and using (9-4) and (9-5) gives a constraint function relating r and ϕ

$$C(r,\phi) = 2(\ell - r) + \tfrac{1}{2} r \left(1 - \frac{r}{L-r}\right)\phi^2 \;=\; 0 \tag{9-6}$$

(b) Now we use the standard polar-coordinate expression for kinetic energy to arrive that the Lagrangian

$$L(r,\phi,\dot{r},\dot{\phi}) \;=\; \tfrac{1}{2}m\dot{r}^2 + \tfrac{1}{2}mr^2\dot{\phi}^2 + mgr\cos\phi + \lambda C(r,\phi)$$

including the constraint function with an undetermined multiplier λ. The Euler-Lagrange equations are

$$m\ddot{r} \;=\; mr\dot{\phi}^2 + mg\cos\phi \;+\; \lambda\frac{\partial C}{\partial r} \tag{9-7}$$

$$mr^2\ddot{\phi} + 2r\dot{r}\dot{\phi} \;=\; -mgr\sin\phi \;+\; \lambda\frac{\partial C}{\partial \phi} \tag{9-8}$$

Evaluating the derivitaives of C to lowest non-vanishing order of ϕ and $r - \ell$ gives

$$\frac{\partial C}{\partial r} \;=\; -2 \ , \quad \frac{\partial C}{\partial \phi} \;=\; \frac{\ell h}{\ell + h}\phi$$

To lowest order in the small coordinates (9-7) becomes $m\ddot{r} = mg - 2\lambda$, whose solution requires $\lambda = -mg/2$ (exactly as in the Cartesian case). Placing this into the exapansion of (9-8) gives

$$m\ell^2\ddot{\phi} = -mg\ell\phi - \tfrac{1}{2}mg\frac{\ell h}{\ell+h}\phi \;=\; -\,mg\frac{\ell(\ell + h/2)}{\ell+h} \tag{9-9}$$

(c) Proposing an oscillatory solution $\phi(t) = Ae^{-i\omega t}$ yields the frequency

$$\omega = \sqrt{\frac{g}{\ell}}\sqrt{\frac{\ell + h/2}{\ell + h}} \tag{9-10}$$

9b

Problem #10

Consider a one-dimensional system consisting of two particles of mass m. The coordinates of the particles are x_1 and x_2. There is attractive force,

$$|F| = |k(x_1 - x_2)|,$$

acting between the particles (*k>0* is constant).

(a) Write down Schrödinger equation that describes relative motion of the particles.

(b) Suppose that at some time *t* the state of the system is described by the wave function,

$$\psi_1 = e^{i\frac{P(x_1+x_2)}{2\hbar}} e^{-\sqrt{\frac{mk}{2}}\frac{(x_1-x_2)^2}{2\hbar}},$$

where *P* is constant. Find the expectation value of energy associated with the relative motion of the particles.

(c) Show that the expectation value of the absolute value of momentum, $|p|$, associated with the relative motion of the particles in the state ψ_1 is,

$$\langle |p| \rangle = \sqrt{\frac{\hbar}{\pi}\sqrt{\frac{mk}{2}}}$$

(d) If one would measure the absolute value of momentum, $|p|$, then what is the probability of obtaining the value,

$$0 \le |p| < \sqrt{\hbar\sqrt{\frac{mk}{2}}} ?$$

Hint: In (d) use momentum representation.

Problem # 10

-1-

$$F = -k(\underbrace{x_1 - x_2}_{r}) = -\frac{\partial}{\partial r}\left(\underbrace{\frac{1}{2} k r^2}_{u(r)}\right)$$

$$\boxed{\begin{aligned} \mu &= \frac{1}{2} m \\ \alpha &= \frac{P(x_1+x_2)}{2\hbar} \\ \beta &= \sqrt{\frac{mk}{2\hbar^2}} \end{aligned}}$$

$$\hat{H}_{rel} = -\frac{\hbar^2}{2\mu}\frac{\partial^2}{\partial r^2} + \frac{1}{2} k r^2$$

$$\Psi = e^{i\frac{P(x_1+x_2)}{2\hbar}} e^{-\frac{1}{2}\sqrt{\frac{mk}{2\hbar^2}}\, r^2} = e^{i\alpha}\, \overbrace{e^{-\frac{\beta}{2} r^2}}^{\psi(r)}$$

$$\int_{-\infty}^{\infty} |\psi|^2 dr = \int_{-\infty}^{\infty} e^{-\beta r^2} dr = \sqrt{\frac{\pi}{\beta}} = \frac{\sqrt{\pi}}{\left(\frac{mk}{2\hbar^2}\right)^{1/4}} = \frac{1}{c_1^2}$$ normalization factor

$$E = \langle c_1\psi(r) | \hat{H}_{rel} | c_1\psi(r)\rangle = c_1^2 \int_{-\infty}^{\infty} e^{-\frac{\beta}{2} r^2}\left(-\frac{\hbar^2}{2\mu}\frac{\partial^2}{\partial r^2} + \frac{1}{2} k r^2\right) e^{-\frac{\beta}{2} r^2} dr$$

$$= c_1^2 \int e^{-\frac{\beta}{2} r^2}\left[-\frac{\hbar^2}{2\mu}\frac{\partial}{\partial r}\left(-\beta r e^{-\frac{\beta}{2} r^2}\right) + \frac{1}{2} k r^2 e^{-\frac{\beta}{2} r^2}\right] dr$$

$$= c_1^2 \int e^{-\frac{\beta}{2} r^2}\left[-\frac{\hbar^2}{2\mu}\left(-\beta e^{-\frac{\beta}{2} r^2} + \beta^2 r^2 e^{-\frac{\beta}{2} r^2}\right) + \frac{1}{2} k r^2 e^{-\frac{\beta}{2} r^2}\right] dr$$

$$= c_1^2 \int e^{-\beta r^2} \underbrace{\left[-\frac{\hbar^2}{2\mu}\left(-\beta + \beta^2 r^2\right) + \frac{1}{2} k r^2\right]}_{} dr$$

$$= \frac{\hbar^2}{2\mu}\sqrt{\frac{mk}{2\hbar^2}} - \cancel{\frac{\hbar^2}{2\cdot\frac{m}{2}}\cdot\frac{mk}{2\hbar^2}\cdot r^2} + \cancel{\frac{1}{2} k r^2} = \hbar\sqrt{\frac{k}{2m}}$$

$$E = \hbar\sqrt{\frac{k}{2m}} \cdot c_1^2 \int_{-\infty}^{\infty} e^{-\beta r^2} dr = \underline{\underline{\hbar\sqrt{\frac{k}{2m}}}}$$

(c) $\langle|\hat{p}|\rangle = \int_{-\infty}^{\infty} \left| c_1 e^{-\frac{\beta}{2}r^2} \left(-i\hbar \frac{\partial}{\partial r}\right) c_1 e^{-\frac{\beta}{2}r^2} \right| dr$

$= c_1^2 \hbar \int_{-\infty}^{\infty} \left| e^{-\frac{\beta}{2}r^2} (-\beta r) e^{-\frac{\beta}{2}r^2} \right| dr = c_1^2 \hbar\, 2 \int_0^{\infty} \beta r e^{-\beta r^2} dr$

$z = \beta r^2$
$dz = 2\beta r\, dr$

$= c_1^2 \hbar \int_0^{\infty} e^{-z} dz = \hbar \left(\frac{mk}{2\hbar^2}\right)^{1/4} \cdot \frac{1}{\sqrt{\pi}} = \underline{\underline{\frac{\sqrt{\hbar}}{\sqrt{\pi}} \left(\frac{mk}{2}\right)^{1/4}}}$

(d) Wave function in momentum representation

$\psi(p) = \int_0^{\infty} e^{i\frac{1}{\hbar} p\cdot r} \psi(r)\, dr$ (~Fourier transformation)

$= \int_{-\infty}^{\infty} e^{i\frac{1}{\hbar} p\cdot r} e^{-\frac{\beta}{2}r^2} dr = \sqrt{\frac{2}{\beta}} \int_{-\infty}^{\infty} e^{iwy} e^{-y^2} dy$

$= \sqrt{\frac{2\pi}{\beta}}\, e^{-\frac{w^2}{4}} \Rightarrow$ normalized wave function $(0 \le p < \infty)$ $\psi(p) = \frac{\sqrt{2}}{(\pi\beta\hbar^2)^{1/4}} e^{-\frac{p^2}{2\beta\hbar^2}}$

$$\boxed{\begin{aligned} y^2 &= \tfrac{\beta}{2} r^2 \\ y &= \sqrt{\tfrac{\beta}{2}}\, r \\ wy &= \tfrac{1}{\hbar} p\cdot r \\ w\sqrt{\tfrac{\beta}{2}} &= \tfrac{1}{\hbar} p \\ w &= \sqrt{\tfrac{2}{\beta}} \cdot \tfrac{1}{\hbar} p \end{aligned}}$$

Probability $0 < p < p_0 = \sqrt{\hbar} \sqrt{\frac{mk}{2}}$

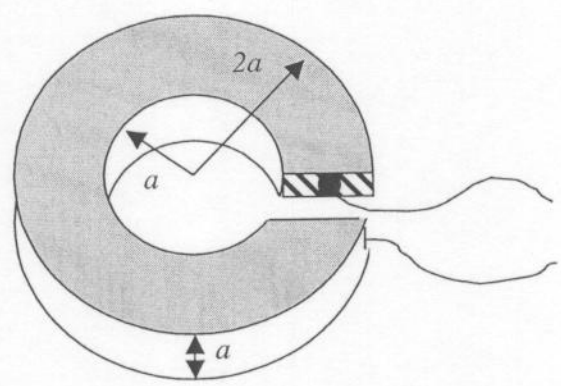
$P = \int_0^{p_0} |\psi(p)|^2 dp = \frac{2}{(\pi\beta\hbar^2)^{1/2}} \int_0^{p_0} e^{-\frac{1}{\beta\hbar^2}\cdot p^2} dp$

$= \frac{2\, \hbar\sqrt{\beta}}{\sqrt{\pi\beta}\cdot\hbar} \int_0^{q_0} e^{-q^2} dq = \underbrace{\frac{2}{\sqrt{\pi}} \int_0^1 e^{-q^2} dq}_{Erf[1] \approx 0.84} \approx \underline{\underline{0.84}}$

$$\boxed{\begin{aligned} q^2 &= \tfrac{1}{\beta\hbar^2} p^2 \\ q &= \tfrac{1}{\sqrt{\beta}} \tfrac{1}{\hbar} p \\ dp &= \hbar\sqrt{\beta} \cdot dq \\ q_0 &= \tfrac{1}{\sqrt{\beta}} \tfrac{1}{\hbar} \sqrt{\hbar} \left(\tfrac{mk}{2}\right)^{1/4} \\ &= \left(\tfrac{2}{mk}\right)^{1/4} \left(\tfrac{mk}{2}\right)^{1/4} = 1 \end{aligned}}$$

PROBLEM #11

I want to find the total resistance of a washer that has a very small slit splitting the washer into a C-shape (see figure below). The washer is made of a dielectric of resistivity ρ and has an inner radius of $\boldsymbol{a}$, thickness $\boldsymbol{a}$, and an outer radius of $\boldsymbol{2a}$ (resulting in a square cross sectional area for the washer of $\boldsymbol{a}^2$). Assume that the wires and solder (used to connect the wires to the washer) have negligible resistance, and that the solder covers the entire exposed face of the cut washer (cross hatched area).



Problem #11 SOLUTION:

Let me begin by finding the resistance of a bar of cross sectional area a^2 and length l with resistivity ρ. In this case R is proportional to the length and inversely proportional to the area, or

$$R = \frac{l\rho}{a^2}$$

For comparison with the washer problem, the length of the bar is between $2\pi a$ and $2\pi(2a) = 4\pi a$. I estimate that the resistance of the washer must be

$$\frac{2\pi\rho}{a} > R_{washer} > \frac{4\pi\rho}{a}$$

For the actual problem: From the cylindrical symmetry of the washer, there is no radial dependence to the potential. For a potential difference of the faces of the split washer of V, we can sum the infinitesimal current rings to find the total current and therefore the total resistance.

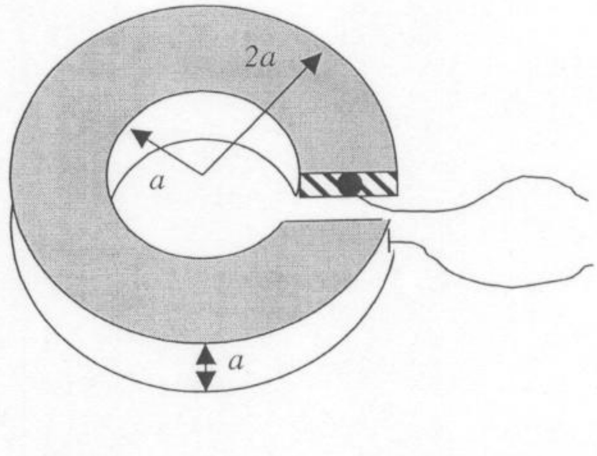
$$dI = \frac{V \times d(Area)}{Length \times \rho} = \frac{Vadr}{2\pi r\rho}$$

where r is the radius of the current ring. Integrating gives

$$I = \int_a^{2a} dI = \int_a^{2a} \frac{Vadr}{2\pi r\rho} = \frac{a\ln 2}{2\pi\rho} V$$

or

$$R_{washer} = \frac{V}{I} = \frac{2\pi\rho}{a\ln 2} \approx 2.9\frac{\pi\rho}{a}$$



PROBLEM #12

Two p electrons (with $\ell = 1$) occupy unfilled shells in an atom. The energy levels associated with a particular pair of angular momentum values S and L are called a *Term* in the language of atomic spectroscopy.

a) Determine the allowed values of total spin angular momentum S and total orbital angular momentum L if these electrons are *inequivalent* (that is, they have different *n* values), and thus list the allowed *terms* for this system of two inequivalent electrons.

b) Determine the allowed values of total spin angular momentum S and total orbital angular momentum L if these electrons are *equivalent* (that is, they have the same *n* value), and thus list the allowed *terms* for this system of two equivalent electrons.

c) The angular momentum coupling scheme where the atomic quantum numbers are S, M_S (the projection of S on the z axis), L, and M_L (the projection of L on the z axis) is called Russell-Saunders coupling. Show by direct computation that J^2 defined by

$$\mathbf{J} = \mathbf{L} + \mathbf{S}$$

commutes with the operator

$$\mathbf{L} \cdot \mathbf{S},$$

and hence with the so-called spin-orbit Hamiltonian, which in this coupling scheme is written as

$$\xi(r)\, \mathbf{L} \cdot \mathbf{S}$$

d) The operator J^2 also commutes with the kinetic energy operator and central potentials You are not required to show this, but if you have trouble completing part (e) you could do so to recover some of the credit for that important part of the problem. What is the significant consequence in terms of "good quantum numbers" of the observation that J^2 commutes with these three leading terms in the Hamiltonian of a multi-electron atom?

e) Consider the energy level structure associated with the terms you found in part (b). Use first order perturbation theory to describe the effects of the spin-orbit interaction on that term structure assuming implicitly that the spin orbit energies are much less than the separations of the terms. Sketch a level diagram and label the resultant levels with appropriate quantum numbers. If you use any specialized spectroscopic notation (not required), you are asked to define the notation. In this level of approximation, the radial integral of $\xi(r)$ gives rise to a new spin orbit interaction parameter usually denoted as ξ.

page 1/3

#12 - Comp 2006

(a) For inequivalent electrons the exclusion principle places no restrictions on the S & L values. By addition of angular momenta,
$\left\{ \begin{array}{l} L = 0, 1, 2 \\ S = 0, 1 \end{array} \right\}$ ^1D, ^3D, ^1P, ^3P, ^1S, ^3S are allowed

(b) For equivalent electrons, each has to have different angular momentum quantum numbers. The easiest approach tabulates all possible combinations and assigns M_S & M_L values and then identifies the corresponding S & L values

M_L \ M_S	−1	0	1
−2	none	$(-1^+ \; -1^-)$	none
−1	$(-1^- \; 0^-)$	$(-1^+ \; 0^-)$ $(-1^- \; 0^+)$	$(-1^+ \; 0^+)$
0	$(1^- \; -1^-)$	$(1^+ \; -1^-)$ $(-1^+ \; 1^-)$ $(0^+ \; 0^-)$	$(1^+ \; -1^+)$
+1	$(1^- \; 0^-)$	$(1^+ \; 0^-)$ $(1^- \; 0^+)$	$(1^+ \; 0^+)$
+2		$(1^+ \; 1^-)$	

identify ^1D, ^3P, ^1S

(c) algebra - see textbooks

(d) A complete set of commuting operators for the Hamiltonian determines the good quantum numbers S, L, J, M_J

(e) $\left(\vec{J}\right)^2 = \left(\vec{L}+\vec{S}\right)^2 = \vec{L}^2 + \vec{S}^2 + 2\,\vec{L}\cdot\vec{S}$

So $\vec{L}\cdot\vec{S} = \frac{1}{2}\left(\vec{J}^2 - \vec{L}^2 - \vec{S}^2\right)$

For 1st order perturbation theory we need only diagonal elements

$$\langle {}^1S_0 | \vec{L}\cdot\vec{S} | {}^1S_0 \rangle = \frac{1}{2}(0 - 0 - 0) = 0$$

$$\langle {}^3P_2 | \vec{L}\cdot\vec{S} | {}^3P_2 \rangle = \frac{1}{2}\left(2(3) - 1(2) - 1(2)\right) = \underline{1}$$

$$\langle {}^3P_1 | \vec{L}\cdot\vec{S} | {}^3P_1 \rangle = \frac{1}{2}\left(1(2) - 1(2) - 1(2)\right) = -1$$

$$\langle {}^3P_0 | \vec{L}\cdot\vec{S} | {}^3P_0 \rangle = \frac{1}{2}\left(0 - 1(2) - 1(2)\right) = -2$$

$$\langle {}^1D_2 | \vec{L}\cdot\vec{S} | {}^1D_2 \rangle = \frac{1}{2}\left(2(3) - 2(3) - 0\right) = 0$$

So, to first order ${}^1S_0$ & ${}^1D_2$ are <u>not</u> shifted

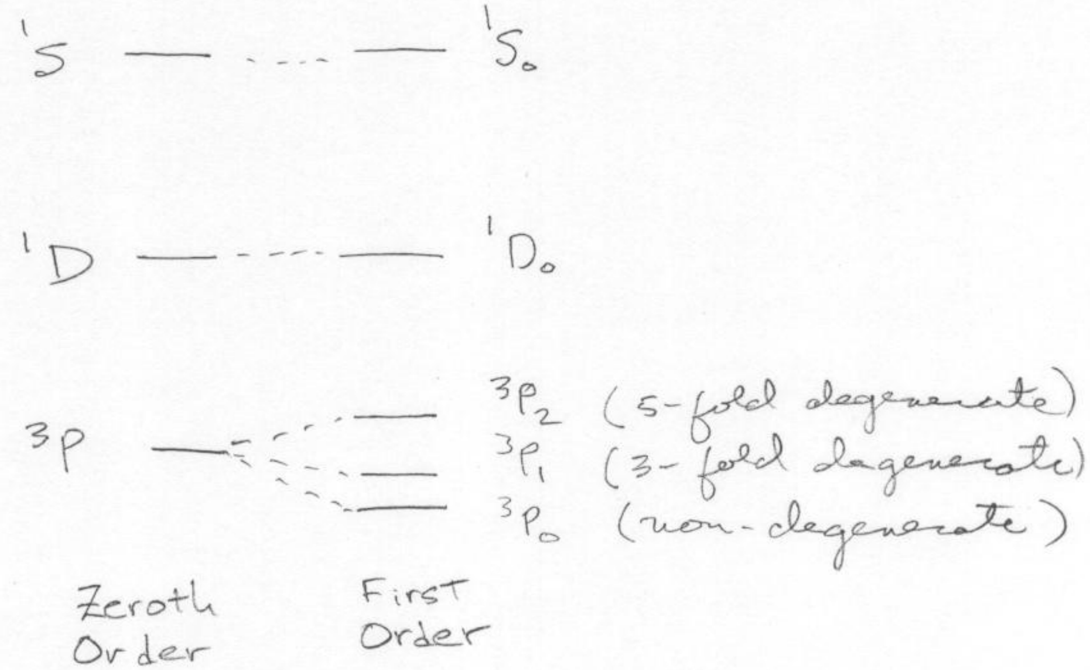
${}^3P_2$ has a 1st order shift of $(1)\lambda$

${}^3P_1$ has a 1st order shift of $(-1)\lambda$

${}^3P_0$ has a 1st order shift of $(-2)\lambda$

As a check of the algebra, we note that the center of gravity is not changed $+5\lambda - 3\lambda - 2\lambda = 0$

#12 page 3/3


^1S
$^1S_0$
^1D
$^1D_0$
^3P
$^3P_2$ (5-fold degenerate)
$^3P_1$ (3-fold degenerate)
$^3P_0$ (non-degenerate)
Zeroth Order
First Order

PROBLEM #13

A glass test tube (mass M, density ρ_t) is inverted and submerged in Flathead Lake at depth z. An air bubble (N molecules, mean molecular mass m) is trapped in the test tube.

A. If the pressure at the surface of the lake is P_{atm} and the water density is ρ_w, then what is the pressure at a depth z?

B. Find the net force $\mathbf{F}$ on the test tube as a function of depth z. For simplicity, assume that the bubble temperature T is constant.

C. At some depth $z = d$, the test tube is neutrally buoyant ($\mathbf{F} = 0$). If we place the test tube at this depth, will it remain there? Why or why not? Again, assume that T is constant.

D. Suppose that the bubble behaved adiabatically. Would this change the answer to the previous part?

Problem 13. A glass test tube (mass M, density ρ_t) is inverted and submerged in Flathead Lake at depth z. An air bubble (N molecules, mean molecular mass m) is trapped in the test tube.

A. If the pressure at the surface of the lake is P_{atm} and the water density is ρ_w, then what is the pressure at a depth z?

Pressure increases with depth, $dP = \rho_w\, g\, dz$, with boundary condition $P(0) = P_{atm}$, so that $P(z) = P_{atm} + \rho_w g z$. Note that the z-axis points downward.

B. Find the net force $\mathbf{F}$ on the test tube as a function of depth z. For simplicity, assume that the bubble temperature T is constant.

In addition to the weights (terms 1 & 3 below), there is the buoyant force equal to the weight of displaced water (terms 2 & 4).

$$\mathbf{F} = \hat{z}\left[Mg - \frac{\rho_w}{\rho_t}Mg + Nmg - \rho_w V g\right].$$

V is the volume of the bubble. Using the ideal gas law,

$$V = \frac{NkT}{P} = \frac{NkT}{P_{atm} + \rho_w g z}.$$

The force is then

$$\mathbf{F} = g\hat{z}\left[M\left(1 - \frac{\rho_w}{\rho_t}\right) + N\left(m - \frac{\rho_w kT}{P_{atm} + \rho_w g z}\right)\right].$$

C. At some depth $z = d$, the test tube is neutrally buoyant ($\mathbf{F} = 0$). If we place the test tube at this depth, will it remain there? Why or why not? Again, assume that T is constant.

$$\text{No, because } \frac{dF}{dz} > 0.$$

This is true for all z, so any equilibrium will be unstable.

D. Suppose that the bubble behaved adiabatically. Would this change the answer to the previous part?

No. If the bubble behaves adiabatically, then PV^γ is constant. Volume still decreases as pressure increases, and $\frac{dF}{dz} > 0$.

PROBLEM #14

Historically, the *Planck distribution function* is one of the stepping stones that led to the discovery of *quantum mechanics*. This problem is a modern visitation of this topic. Consider a photon gas in a volume V in thermal equilibrium with the walls of the volume kept at temperature T. Answer the following questions:

a. The *Planck distribution function* is given by $<n>=\frac{1}{e^{\beta\hbar\omega}-1}$, where $\beta = 1/kT$, explain precisely and briefly what $<n>$ means?
b. For a single photon of *mode* $\hbar\omega$ determine its thermal average energy using the standard methods of statistical physics. From this result prove that the *Planck distribution function* is given by the relationship shown in (a) above.
c. Note that $<n>$ is independent of volume V, and yet the total average radiation energy, $<E>$, being an extensive quantity, is proportional to volume V and given by the *Stefan-Boltzmann law* of radiation, $<E>=V\alpha T^4$, where α is a constant. Explain by using first principles in a simple model why the larger of the two volumes at the same temperature would have more radiation energy even though the average energy per mode is the same for both volumes?

Solution to PROBLEM #14

a. A single photon of *mode* $\hbar\omega$ is a quantum oscillator with energy levels given by the well-known relation $e_n = (n+1/2)\,\hbar\omega$, where $n = 0, 1, 2, 3, \ldots \infty$ is the instantaneous photon number in *mode* $\hbar\omega$, and $\hbar\omega/2$ is the so-called zero point (or vacuum fluctuation) energy. When this photon is in thermal equilibrium with the walls of a volume V at temperature T all n's are accessible with a certain probability as described below. The $<n>$ is simply the thermal average number of such photons in *mode* $\hbar\omega$ at thermal equilibrium with a system at temperature T.

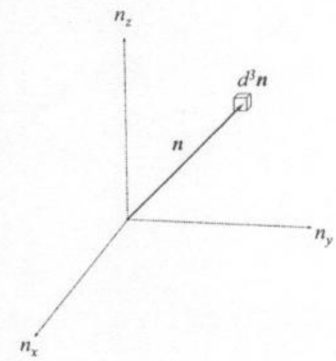
b. The thermal average energy $<e_n> = <n\hbar\omega> + \hbar\omega/2 = <\varepsilon_n> + \hbar\omega/2$ is given by the standard relation: $<e_n> = \sum_{n=0}^{\infty} \varepsilon_n P_n + \hbar\omega/2$, where P_n is the probability of finding the quantum oscillator (photon) in state n of energy ε_n, and $\hbar\omega/2$, the zero point energy, has no statistical significance though has great fundamental importance in quantum physics of photons. As you know, in statistical physics P_n is proportional to the *Boltzmann factor*, $e^{-\beta\varepsilon_n}$, and is given by the relation $P_n = e^{-\beta\varepsilon_n}/Z$, where $Z = \sum_{n=0}^{\infty} e^{-\beta\varepsilon_n}$, a normalization factor, is known as the *partition function* for the quantum oscillator. The average energy above the zero point energy can then easily be calculated by noting that

$$<\varepsilon_n> = \sum_{n=0}^{\infty} \varepsilon_n P_n = \frac{\sum_{n=0}^{\infty} \varepsilon_n e^{-\beta\varepsilon_n}}{\sum_{n=0}^{\infty} e^{-\beta\varepsilon_n}} = \frac{-\partial Z/\partial\beta}{Z} = -\partial \ln Z/\partial\beta$$ and that Z is a geometric

series and is given by $Z = 1/(1-e^{-\beta\hbar\omega})$. This immediately yields $<\varepsilon_n> = -\partial \ln Z/\partial\beta = \hbar\omega e^{-\beta\hbar\omega}/(1-e^{-\beta\hbar\omega}) = \hbar\omega/(e^{\beta\hbar\omega}-1)$. Taking into account the fact that $<\varepsilon_n> = <n\hbar\omega> = <n>\hbar\omega$ this gives the desired relation for the *Planck distribution function*: $<n> = 1/(e^{\beta\hbar\omega}-1)$

c. The short answer to the question is that even though the thermal average energy per *mode* is independent of volume V the number of modes that fall between $\hbar\omega$ and $\hbar(\omega + d\omega)$ depends on the volume and is proportional to the volume as demonstrated below: A photon field of *mode* ω inside a cavity of volume $V=L_xL_yL_z$, where L_x, L_y, L_z are the dimensions of volume V, can be represented by a traveling wave of $\boldsymbol{E(r,t)}=E_o\, e^{i(\mathbf{k}.\mathbf{r}-\omega t)}\boldsymbol{e}$. In the latter relation $\boldsymbol{e}$ is the polarization unit vector lying in a plane perpendicular to $\mathbf{k}=k_x\vec{i}+k_y\vec{j}+k_z\vec{k}$, which is the wave vector associated with the photon. The values of $k=\sqrt{k_x^2+k_y^2+k_z^2}$ and ω are not independent and are related to each other via dispersion relation $\omega = ck$, which can be traced to the Maxwell's wave equation for radiation. For a given k (hence for a given ω) there are two independent polarization vectors in a the plane perpendicular to $\boldsymbol{k}$.

The wave vector $\boldsymbol{k}$ is quantized because $\boldsymbol{E}$ must satisfy the periodic boundary conditions. That is, $\boldsymbol{E}(r, t)=\boldsymbol{E}(\mathbf{r}+\boldsymbol{R}, t)$, where $\mathbf{r}=x\vec{i}+y\vec{j}+z\vec{k}$ and $\boldsymbol{R}=L_x\vec{i}+L_y\vec{j}+L_z\vec{k}$. This implies that $e^{i\mathbf{k.r}}=e^{i(\mathbf{k.r+k.R})}$, and for this to happen $k_xL_x=2\pi n_x$, $k_yL_y=2\pi n_y$ and $k_zL_z=2\pi n_z$, where n_x, n_y, and n_z are independent integers in the range $0, \pm 1, \pm 2, \pm 3, \cdots, \pm\infty$. Therefore, to count the number of modes in volume V it is convenient to work in the *number space* defined by n_x, n_y, and n_z as shown in the figure below. If we consider a small volume, $d^3\boldsymbol{n}$, in the number space (see figure), the number of modes within this volume for a given polarization is given by $d^3\boldsymbol{n}=dn_xdn_ydn_z=\frac{L_xL_yL_z}{(2\pi)^3}dk_xdk_ydk_z=\frac{V}{(2\pi)^3}d^3\boldsymbol{k}$, and hence the number of modes in $d^3\boldsymbol{n}$ is directly proportional to the volume V.



(Up to this point is what is required for part (c); the rest is extra knowledge.)

This last relation needs to be multiplied by two in order to take into account the two independent polarizations for a given $\boldsymbol{k}$. Furthermore, the dispersion relation $\omega=ck$ has spherical symmetry; hence it is best to work in spherical coordinates rather than Cartesian coordinates. That is, $d^3\boldsymbol{k}=4\pi k^2dk=4\pi\frac{\omega^2}{c^2}\frac{d\omega}{c}=\frac{4\pi\omega^2}{c^3}d\omega$. Inserting this relation into the equation for $d^3\boldsymbol{n}$ above and taking into account the two polarizations for a given $\boldsymbol{k}$ we then determine the number of states that lie between $\hbar\omega$ and $\hbar(\omega+d\omega)$, which is given by $d^3\boldsymbol{n}=g(\omega)d\omega=\frac{V}{\pi^2c^3}\omega^2\,d\omega$, where $g(\omega)=\frac{V}{\pi^2c^3}\omega^2$ is known as the *density of states* for the photon modes. We can then determine the total average thermal energy, $<E>=\int<\varepsilon_n>d^3\boldsymbol{n}$, above the zero point (or vacuum fluctuation) energy. It is given by the integration:

$<E>=\int<\varepsilon_n>d^3\boldsymbol{n}$

$=\int_0^\infty\frac{\hbar\omega}{(e^{\beta\hbar\omega}-1)}g(\omega)d\omega=\frac{V}{\beta^4\pi^2\hbar^3c^3}\int_0^\infty\frac{x^3}{e^x-1}dx=\frac{V\pi^2k^4}{15\hbar^3c^3}T^4=V\alpha T^4$, where

$\int_0^\infty\frac{x^3}{e^x-1}dx=\frac{\pi^4}{15}$ is used.

PROBLEM #15

An arbitrary voltage $V(t)$ is applied to a series RC circuit.

A. Write the differential equation for charge $q(t)$ on the capacitor.

B. Find the Green's function (response to $V(t) = \delta(t - t')$).

C. Find $q(t)$ for

$$V(t) = \begin{cases} 0, & t < 0; \\ V_0 \exp\left(+\frac{t}{RC}\right), & 0 < t < RC; \\ 0, & t > RC. \end{cases}$$

Problem 15. An arbitrary voltage $V(t)$ is applied to a series RC circuit.

A. Write the differential equation for charge $q(t)$ on the capacitor.

Straightforward application of Kirchoff's loop rule leads to

$$R\dot{q} + \frac{q}{C} = V(t).$$

B. Find the Green's function (response to $V(t) = \delta(t - t')$).

We seek $G(t,t')$ such that $R\dot{G} + G/C = \delta(t - t')$. For $t \neq t'$, we use the homogeneous solution $A\exp(-t/RC)$, with A an arbitrary constant. Fitting the boundary conditions implied by the delta function,

$$G(t,t') = \begin{cases} 0, & t < t', \\ Ae^{-t/RC}, & t > t'. \end{cases}$$

To obtain A, we integrate the ODE over an infinitesimal domain:

$$\int_{t'-\epsilon}^{t'+\epsilon} \left(R\dot{G} + \frac{G}{C} \right) dt = \int_{t'-\epsilon}^{t'+\epsilon} \delta(t - t') dt = 1.$$

$$RG(t' + \epsilon, t') - RG(t' - \epsilon, t') = RAe^{-t'/RC} = 1$$

$$G(t,t') = \begin{cases} 0, & t < t'; \\ \frac{1}{R}e^{-(t-t')/RC}, & t > t'. \end{cases}$$

C. Find $q(t)$ for

$$V(t) = \begin{cases} 0, & t < 0; \\ V_0 \exp\left(+\frac{t}{RC}\right), & 0 < t < RC; \\ 0, & t > RC. \end{cases}$$

Paying scrupulous attention to the piecewise-continuous definitions of V and G,

$$q(t) = \int_{-\infty}^{\infty} V(t')\, G(t,t')\, dt' = \int_{-\infty}^{t} V(t')\, G(t,t')\, dt'$$

$$= \begin{cases} 0, & t < 0; \\ \frac{V_0 C}{2} \left(e^{t/RC} - e^{-t/RC} \right), & 0 < t < RC; \\ \frac{V_0 C}{2} e^{-t/RC}(e^2 - 1), & t > RC. \end{cases}$$