DEPARTMENT OF PHYSICS

PH. D. COMPREHENSIVE EXAMINATION

SEPTEMBER 17-18, 1985

DEPARTMENT OF PHYSICS

PH.D. COMPREHENSIVE EXAMINATION

TUESDAY, SEPTEMBER 17, 1985, 9-12 AM

Answer each of the following questions, all of which carry equal weight. Begin your answer to each question on a new sheet of paper; solutions to different questions must not appear on the same sheet. Each sheet of paper must be labeled with your name and the problem number, in the upper right-hand corner. When more than one sheet is submitted for a problem, be sure the pages are ordered properly.

1. Assume that a particle of mass m moves on the frictionless inner surface of a paraboloid of revolution $x^2 + y^2 = az$, in the presence of a uniform gravitational field acting in the negative z-direction.

   a. Obtain the equations of motion.

   b. Calculate the frequency of circular motion if z is restricted to a constant height h.

(Mechanics) Drumheller

~~Return to J. Drumheller by Monday, Sept. 16~~

①

1. Assume that a particle of mass m moves on the frictionless inner surface of a paraboloid of revolution $x^2 + y^2 = az$, in the presence of a uniform gravitational field acting in the negative z-direction.

a/ Obtain the equations of motion.

b/ ~~[illegible]~~ Calculate the frequency of circular rotation if z is restricted to a constant height h.

~~Should we ask for a discussion of the stability of the circular orbit?~~

a/ Convert to cylindrical coordinates $\left(\begin{array}{l} x = \rho\cos\phi \\ y = \rho\sin\phi \\ z = z \end{array}\right)$

where $T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 \cancel{\phi} + \dot{z}^2)$

and $V = +mgz$ and write Lagrangian

$$L = \frac{1}{2}m(\dot{x}^2+\dot{y}^2+\dot{z}^2) - mgz = \frac{1}{2}m(\dot{\rho}^2 + \rho^2\dot{\phi}^2 + \dot{z}^2) - mgz$$

Also $x^2+y^2 = \rho^2$.

The eqn. for the paraboloid is $x^2+y^2 = az$ therefore eqn. of Constraint is

$$\rho^2 = az \quad \text{or} \quad 2\rho\, d\rho = a\, dz$$

Eq. for constraint is $A_\rho d\rho + A_\phi d\phi + A_z dz = 0$

only one eqn. of constraint

$$\therefore A_\rho = 2\rho \quad A_\phi = 0 \quad A_z = -a$$

Then Lagrange's Eqns with multiplier becomes

$$\frac{d}{dt}\frac{\partial L}{\partial\dot{\rho}} - \frac{\partial L}{\partial\rho} = \lambda\, 2\rho \qquad \frac{d}{dt}\frac{\partial L}{\partial\dot{\phi}} - \frac{\partial L}{\partial\phi} = 0 \qquad \frac{d}{dt}\frac{\partial L}{\partial\dot{z}} - \frac{\partial L}{\partial z} = -\lambda a$$

$$\frac{d}{dt}\frac{\partial L}{\partial\dot{\rho}} = m\ddot{\rho} \qquad \frac{\partial L}{\partial\rho} = m\rho\dot{\phi}^2$$

$$\frac{d}{dt}\frac{\partial L}{\partial\dot{\phi}} = m\rho^2\ddot{\phi} \qquad \frac{\partial L}{\partial\phi} = 0$$

$$\frac{d}{dt}\frac{\partial L}{\partial\dot{z}} = m\ddot{z} \qquad \frac{\partial L}{\partial z} = -mg$$

$$m\ddot{\rho} - m\rho\dot{\phi}^2 = 2\rho\lambda$$

$$m\rho^2\ddot{\phi} = 0 \quad \text{or} \quad m\rho^2\dot{\phi} = \text{const.}$$

$$m\ddot{z} + mg = -a\lambda$$

$$2\rho\dot{\rho} = a\dot{z}$$

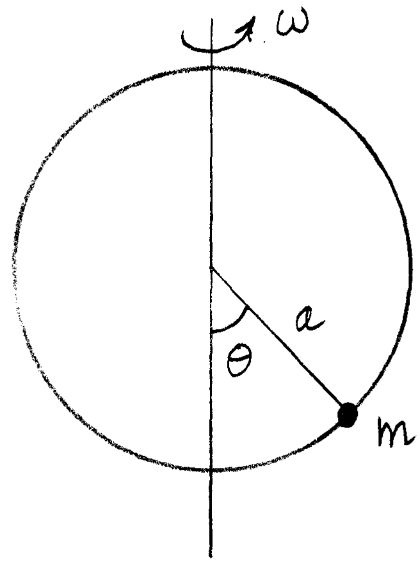
from which ρ, ϕ, z, λ can be found.

b/ ~~[illegible]~~ $\rho = \sqrt{az} \Rightarrow \sqrt{ah}$ and $mg = -a\lambda$

$$-\cancel{m}\cancel{\sqrt{ah}}\,\omega^2 = 2\cancel{\phi}\left(-\frac{mg}{a}\right)\cancel{\sqrt{ah}}$$

$$\therefore \quad \omega = \sqrt{\frac{2g}{a}}$$

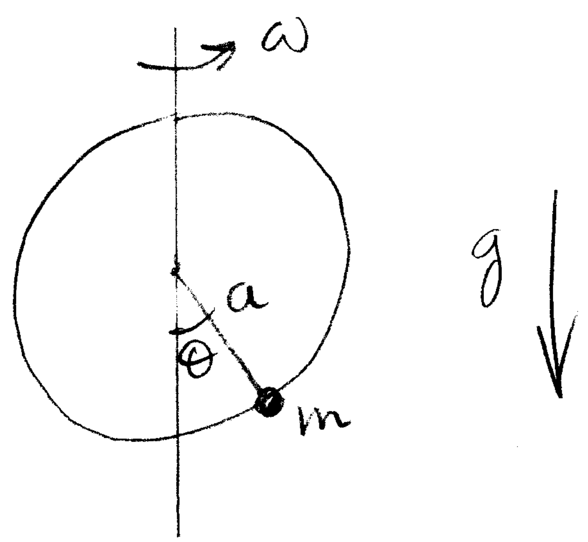
2. A point particle of mass m is constrained to move on a frictionless circular ring of radius a. The ring rotates with constant angular frequency ω about its diameter, which is vertical as shown. Gravity acts downward. Determine the Lagrangian, the canonical momentum, and the Hamiltonian for this system. How large must ω be in order that the particle have a position of stable equilibrium at $\theta > 0$? <u>Prove</u> the stability.



(2)

Classical Mechanics J. Hermanson

2. A point particle of mass m is constrained to move on a frictionless circular ring of radius a. The ring rotates with constant angular frequency ω about its diameter, which is vertical as shown. Gravity acts downward. Determine the Lagrangian, the canonical momentum, and the Hamiltonian for this system. How large must ω be in order that the particle have a position of stable equilibrium at $\theta > 0$?



$$\underline{\text{Sol}^{n}}: L = T - V = \frac{m}{2}\left(a^2\dot{\theta}^2 + a^2\sin^2\theta\,\omega^2\right) \underset{(-mga)}{+ mga\cos\theta}$$

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = \underline{ma^2\dot{\theta}}$$

$$H = p_\theta \dot{\theta} - L$$

$$= \frac{ma^2\dot{\theta}^2}{2} - \frac{1}{2}ma^2\sin^2\theta\,\omega^2 - mga\cos\theta$$

$$= \underline{\frac{p_\theta^2}{2ma^2} - \frac{1}{2}ma^2\sin^2\theta\,\omega^2 - mga\cos\theta}$$

$$= \frac{p_\theta^2}{2ma^2} + V_{eff}(\theta)$$

stable equilibrium:

$$\dot{p}_\theta = -\frac{\partial H}{\partial \theta} = ma^2\omega^2\sin\theta\cos\theta - mga\sin\theta = 0$$

$$ma\sin\theta\left(a\omega^2\cos\theta - g\right) = 0$$

$$\cos\theta = \frac{g}{a\omega^2}$$

For $\theta > 0$ require $\cos\theta < 1$

or $\boxed{\omega > \sqrt{g/a}}$

stable?

$$V_{eff}(\mu) = -\frac{1}{2}ma^2\omega^2(1-\mu^2) - mga\mu; \quad \mu = \cos\theta$$

$$V'_{eff}(\mu) = ma^2\omega^2\mu - mga \; (= 0 \text{ when } \mu = \tfrac{g}{a\omega^2} \checkmark)$$

$$V''_{eff}(\mu) = ma^2\omega^2 > 0 \quad \checkmark$$

3. Absorption cross section of a celestial body

Imagine a planet moving through a cloud of dust with relative velocity V. If the planet has a radius R it will surely sweep up all the dust inside a cylindrical column of area πR^2. However, because of the attractive gravitational force, it will actually sweep out more - all the dust in a column with area σ. The area σ can be considered the gravitational absorption cross section of the planet.

a. Calculate σ in terms of R, V, M(the mass of the planet), G, etc.

b. If the sun is surrounded by a cloud of stationary dust, how much larger is the Earth's cross section than its geometric cross section (i.e., find $\sigma/\pi R^2$)?

Hint: Use conservation laws.

$G = 6.67 \times 10^{-8}\ cm^3/g \cdot sec^2$
$M_{Earth} = 6x10^{27}\ g$
$R_{Earth} = 6x10^{8}\ cm$
Mean Earth - Sun distance = $1.5x10^{13}$ cm
1 year $\simeq 3.16x10^{7}$ sec

(3) Classical Mechanics Hiscock

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$R_{Earth} = 6 \times 10^{8}\ cm$

Mean Earth-Sun distance $= 1.5 \times 10^{13}$ cm

1 year $\simeq 3.16 \times 10^{7}$ sec

Solution

(a) $\sigma = \pi h^2$ $\quad h =$ impact parameter corresponding to a distance of closest approach $= R$

$$\frac{1}{2} m v^2 = \frac{1}{2} m v'^2 - \frac{GMm}{R} \qquad \text{conservation of energy} \quad \left(v' = \text{nearest approach velocity} \right)$$

$$m v h = m v' R \qquad \text{conservation of angular momentum}$$

$$\Rightarrow h = (v'/v) R$$

$$\to \quad 1 = \left(\frac{v'}{v}\right)^2 - \frac{2GM}{Rv^2} \quad \Rightarrow \left(\frac{v'}{v}\right)^2 = 1 + \frac{2GM}{Rv^2}$$

$$\sigma = \pi h^2 = \pi \left(\frac{v'}{v}\right)^2 R^2 = \boxed{\pi R^2 \left[1 + \frac{2GM}{Rv^2}\right] = \sigma}$$

(b) $$\frac{\sigma}{\pi R^2} = \left[1 + \frac{2GM}{Rv^2}\right]$$

Let $r =$ mean earth-sun distance

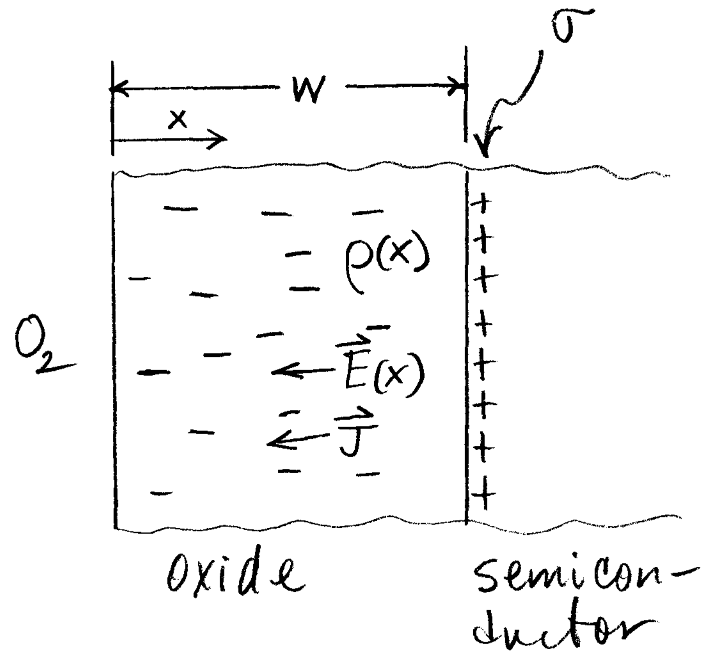
$$v = \frac{2\pi r}{T} \qquad T = 1\,\text{year}$$

$$v = \frac{2\pi (1.5 \times 10^{13}\,\text{cm})}{3.16 \times 10^7\,\text{sec}} = 3 \times 10^6 \,\frac{\text{cm}}{\text{sec}}$$

$$\boxed{\frac{\sigma}{\pi R^2} = 1 + \frac{2 \cdot (6.67 \times 10^{-8})(6 \times 10^{27})}{(6 \times 10^8)\,(3 \times 10^6)^2} = 1.15}$$

So the absorption cross section of the Earth in its orbit is some 15% larger than the geometric cross section

4. In a model for oxidation of semiconductors, an oxygen molecule settles on the oxide surface and ionizes, forming an O_2^- molecular ion and a hole. The hole tunnels to the semiconductor-oxide interface, contributing to a charge density σ which sets up an electric field $\vec{E}(x)$ in the oxide, where x is measured as shown. The O_2^- ions drift to the right in this field and cause a space-charge density $\rho(x)$ which modifies the field, giving it its x-dependence. These ions have charge $-e$ and mobility μ. Their drift sets up a current density $\vec{J}$ independent of x.



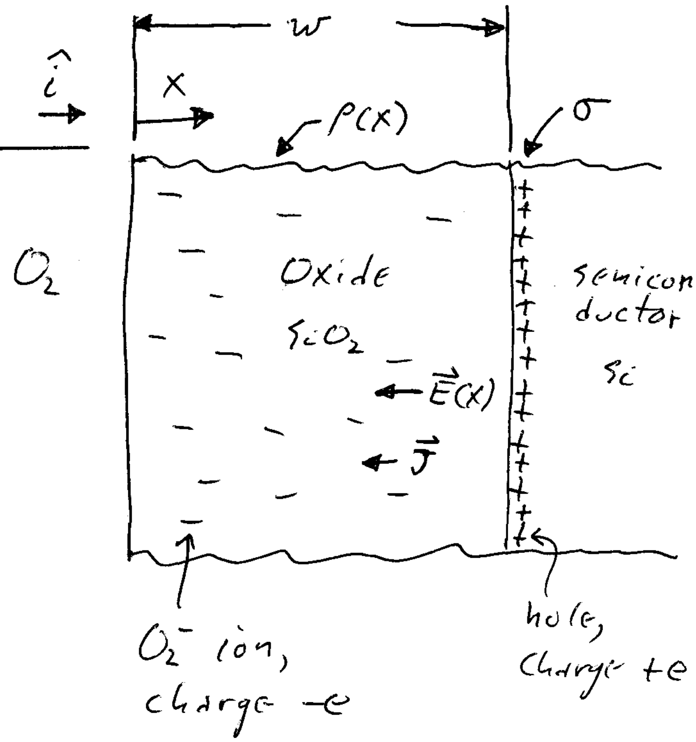
Given σ and given overall charge neutrality so that $\int_0^W \rho\, dx = -\sigma$, find $\rho(x)$, $\vec{E}(x)$, and $\vec{J}$. Let dielectric permittivity ε (= $\varepsilon_0 \varepsilon_r$) be given.

(4)

E & M

Problem

4. In a model for oxidation of semiconductors, an oxygen molecule settles on the oxide surface and ionizes, forming an O_2^- molecular ion and a hole. The hole tunnels to the semiconductor-oxide interface, contributing to a charge density σ which sets up an electric field $\vec{E}(x)$ in the oxide. The O_2^- ions drift to the right in this field, and cause a space charge density $\rho(x)$ which modifies the field, giving it its x-dependence. These ions have charge $-e$ and mobility μ. Their drift sets up a current density $\vec{J}$ independent of x.



Given σ and given overall charge neutrality so that $\int_0^w \rho\, dx = -\sigma$, find $\rho(x)$, $\vec{E}(x)$, and $\vec{J}$.
Let dielectric permittivity ϵ $(= \epsilon_0 \epsilon_r)$ be given.

E&M

Solution

$\vec{J} = n(x)\vec{E}(x)\mu e$ $\quad$ n = ion concentration

$\rho(x) = -en(x) = -\vec{J}/\mu\vec{E}(x)$

Gauss' Law: $\oint \vec{E}\cdot d\vec{A} = EA = \frac{1}{\epsilon}\int \rho\, d\tau = \frac{A}{\epsilon}\int_0^x \rho\, dx = -\frac{JA}{\epsilon\mu}\int_0^x \frac{dx}{E}$

Solve for E by differentiating & then solving diff. eq.

$\frac{\partial E}{\partial x} = -\frac{J}{\epsilon\mu E}$ $\quad$ so $\quad \int E\, dE = -\frac{J}{\epsilon\mu}\int dx$

or $\frac{E^2}{2} = -\frac{Jx}{\epsilon\mu}$ $\qquad E = -\sqrt{\frac{2Jx}{\epsilon\mu}}$

$\rho = -\sqrt{\frac{J\epsilon}{2\mu x}}$ $\qquad \int \rho\, dx = -2\sqrt{\frac{J\epsilon x}{2\mu}} = -\sqrt{\frac{2J\epsilon x}{\mu}} \Rightarrow -\sigma$ for $x = w$

so $\boxed{\vec{J} = -\hat{i}\,\frac{\mu\sigma^2}{2\epsilon w}}$

$\rho(x) = -\sqrt{\frac{\epsilon}{2\mu x}\,\frac{\mu\sigma^2}{2\epsilon w}}$ $\qquad \boxed{\rho(x) = -\frac{\sigma}{2}\sqrt{\frac{1}{wx}}}$

$\vec{E} = -\hat{i}\sqrt{\frac{2x}{\epsilon\mu}\,\frac{\mu\sigma^2}{2\epsilon w}} = \boxed{\vec{E}(x) = -\hat{i}\,\frac{\sigma}{\epsilon}\sqrt{\frac{x}{w}}}$

DEPARTMENT OF PHYSICS

PH.D. COMPREHENSIVE EXAMINATION

TUESDAY, SEPTEMBER 17, 1985, 2-5 PM

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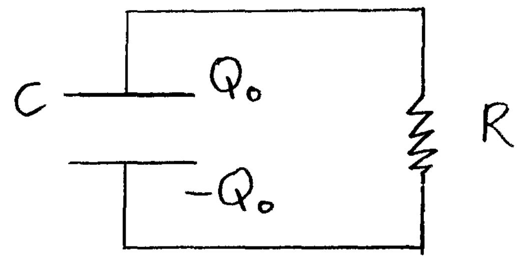
5. A capacitor with capacitance C and initial charge Q_0 at $t = 0$ discharges through a resistor with resistance R in the circuit shown. The resistor is a long, uniform circular cylinder of length L and radius a. For $t > 0$

a. Determine the rate at which energy is absorbed in the resistor, in terms of Q_0, R and C .

b. Find the Poynting vector at the resistor's surface, and evaluate the rate at which energy flows into the resistor from the electromagnetic field outside.



(5)

E & M   J. Hermanson

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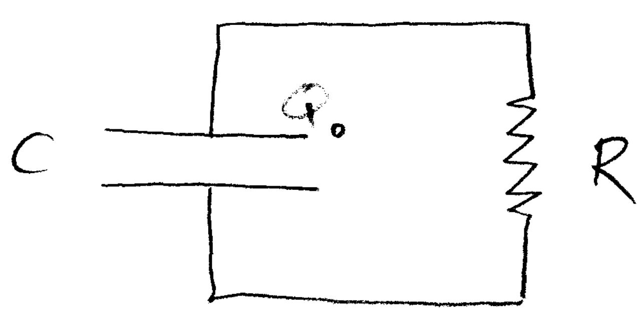
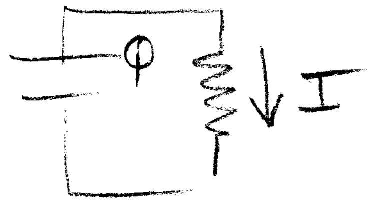




Sol'n a) $\frac{Q}{C} = IR$ [Here $I = -\frac{dQ}{dt}$]

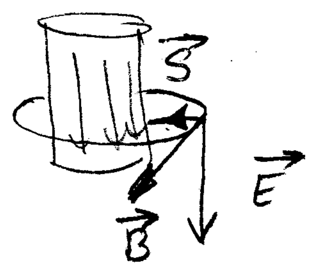
$$\frac{dQ}{dt} + \frac{1}{RC} Q = 0$$

$$Q = Q_0 e^{-t/RC}$$

$$I = -\frac{dQ}{dt} = \frac{Q_0}{RC} e^{-t/RC}$$

$$\boxed{\text{Power} = I^2 R = \frac{Q_0^2}{RC^2} e^{-2t/RC}}$$

b) $\vec{S} = \frac{c}{4\pi} \vec{E} \times \vec{B}$



$$E = \frac{V}{l} = \frac{Q}{Cl}$$

Ampere's L: $B \cdot 2\pi a = \frac{4\pi}{c} I$

$$B = \frac{2I}{ca} = \frac{2Q}{cRCa}$$

$$S = \frac{\cancel{c}}{4\pi} \cdot \frac{Q}{Cl} \cdot \frac{2Q}{\cancel{c}RCa} = \frac{Q^2}{2\pi RC^2 al}$$

$$\boxed{S = \frac{Q_0^2}{2\pi RC^2 al} e^{-2t/RC}}$$

$$\boxed{\text{Power} = S \cdot 2\pi al = \frac{Q_0^2}{RC^2} e^{-2t/RC}}$$ as in a) !

6. Consider transverse electromagnetic waves propagating through the earth's ionosphere, in a direction <u>parallel</u> to a line of the earth's magnetic field $\vec{B}$. For such waves, the dispersion relation can be shown to be

$$(\nu \pm f)(\nu^2 - \kappa^2) = \nu$$

Here, if ω_p is the plasma frequency for the ionosphere, and ω and k are the wave's frequency and wavenumber, then $\nu = \omega/\omega_p$, $\kappa = kc/\omega_p$, and $f = \Omega/\omega_p$, where $\Omega = eB/mc$ is the gyrofrequency. The $\pm$ refer to the motion of left and right circularly polarized waves, respectively. Assume B and the ionospheric particle density n are constant.

a. Look for high-frequency solutions, $\nu \gg 1$, to the dispersion relation. Show that $k \simeq (\omega/c) \pm \Delta k$, and specify Δk by known quantities.

b. Show that a nonzero Δk (from part (a)) implies a <u>rotation</u> of the plane of polarization of a linearly polarized wave. Calculate the rate of rotation, i.e. the derivative d(polarization angle)/d(distance traveled).

c. Show there is a low-frequency ($\nu \ll 1$) solution to the dispersion relation. Find the approximate relation between ω and k, and calculate the group velocity for these waves, in terms of the wave frequency.

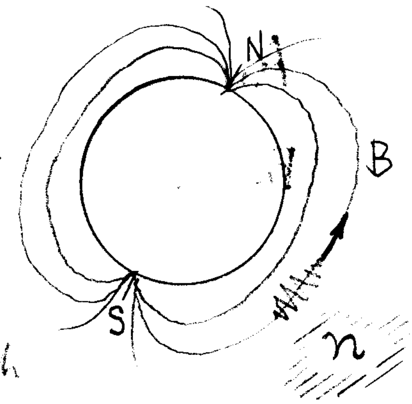
17 June 85

MSU Ph.D. Exam : Sept. 1985

(from Robisson)

E&M : Plasma Physics

6



9
6. Consider transverse electromagnetic waves propagating through the earth's ionosphere, in a direction parallel to a line of the earth's magnetic field B. For such waves, the dispersion relation can be shown to be

l.c. Greek "nu"

$$(\nu \pm \overset{f}{\cancel{\beta}})(\nu^2 - \kappa^2) = \nu .$$

Here, if ω_p is the plasma frequency for the ionosphere, and ω and k are the wave's frequency and wavenumber, then : $\nu = \omega/\omega_p$, $\kappa = kc/\omega_p$, and $\overset{f}{\cancel{\beta}} = \Omega/\omega_p$, where $\Omega = eB/mc$ is the gyrofrequency. The $\pm$ refer to the motion of left and right circularly polarized waves, resp. Assume B and the ionospheric particle density n are constant.

~~A. How is the plasma frequency ω_p related to the particle density n?~~

(a) ~~B~~. Look for high-frequency solutions, $\nu \gg 1$, to the dispersion relation. Show that : $k \simeq (\omega/c) \pm \Delta k$, and specify Δk by known quantities.

(b) ~~C~~. Show that a nonzero Δk (from part (a)) implies a rotation of the plane of polarization of a linearly polarized wave. Calculate the rate of rotation, i.e. d(polarization angle)/d(distance travelled).

(c) ~~D~~. Show there is a low-frequency ($\nu \ll 1$) solution to the dispersion relation. Find the approximate relation between ω and k, and calculate the group velocity for these waves, in terms of the wave frequency.

Solution (over)...

$$\nu = \frac{\omega}{\omega_p}, \quad \kappa = \frac{kc}{\omega_p}, \quad g \equiv \frac{\Omega}{\omega_p}$$

A. $\omega_p^2 = 4\pi n e^2/m_e$, by definition. This is just informational.

(a) ~~B~~. Solve dispⁿ relation for κ. Easily get: $\kappa = \nu\left[1 - \frac{1}{\nu^2(1 \pm g/\nu)}\right]^{1/2}$.

With $\nu \gg 1$, and $\frac{g}{\nu} = \Omega/\omega$ small, then: $\kappa \simeq \nu\left[1 - \frac{1}{2\nu^2}\left(1 \mp \frac{g}{\nu}\right)\right]$, or...

$$\kappa \simeq \nu\left[1 - \cancel{\tfrac{1}{2\nu^2}}^{\text{ignore}}\right] \pm \frac{1}{2\nu}\left(\frac{g}{\nu}\right) \Rightarrow k \simeq (\omega/c) \pm \Delta k, \quad \boxed{\Delta k = \frac{\Omega}{2c}\left(\frac{\omega_p}{\omega}\right)^2}.$$

(b) ~~C~~. A wave with $k = (\omega/c) \pm \Delta k$ [the $\pm$ for L & R circular polarization] will propagate according to...

$$e^{i(kx - \omega t)} = e^{i\omega\left(\frac{x}{c} - t\right)} e^{\pm i(\Delta k)x}.$$

Both the L & R components are initially present in a linearly polarized wave. After travelling distance x, the wave looks like

$$E(x,t) = \frac{E_0}{2} e^{i\omega\left(\frac{x}{c} - t\right)}\left[e^{+i(\Delta k)x} + e^{-i(\Delta k)x}\right]$$

$$= \left[E_0 \cos(\Delta k)x\right] e^{i\omega\left(\frac{x}{c} - t\right)}.$$

Evidently E <u>rotated</u> through $\measuredangle\ \phi = (\Delta k)x \Rightarrow \boxed{\frac{d\phi}{dx} = \Delta k = \frac{\Omega}{2c}\left(\frac{\omega_p}{\omega}\right)^2}.$

(c) ~~D~~. For $\nu \ll 1$, dispⁿ relation $\Rightarrow \kappa^2 = \cancel{\nu^2}^{\text{ignore}} - \frac{\nu}{\nu \pm g} \simeq (-)\frac{(\nu/g)}{(\nu/g) \pm 1}$. Choose $(-)$ sign [$\Rightarrow$ R wave], and ignore $(\nu/g) \ll 1$ in denom. Then get...

$$\kappa^2 \simeq \nu/g, \quad \text{or} \quad \boxed{\omega \simeq \Omega\left(\frac{kc}{\omega_p}\right)^2 \ll \Omega.}$$

Group velocity: $\left[\!\left[v_g = \frac{\partial\omega}{\partial k} = 2c\,\frac{\Omega kc}{\omega_p^2} = c \cdot \frac{2\Omega}{\omega_p}\sqrt{\frac{\omega}{\Omega}} \right]\!\right]$ These are "whistler" waves.

7. A particle of mass m moves in one dimension in a box of width a with infinitely hard walls. The particle is in its quantum-mechanical ground state. Suddenly, the box is expanded symmetrically to width 2a. Determine the probability that the particle will be found in the ground state of the expanded box. Estimate the time scale for the expansion if it is to be considered ''sudden''.

(7)

Quantum Mechanics I J. Hermanson

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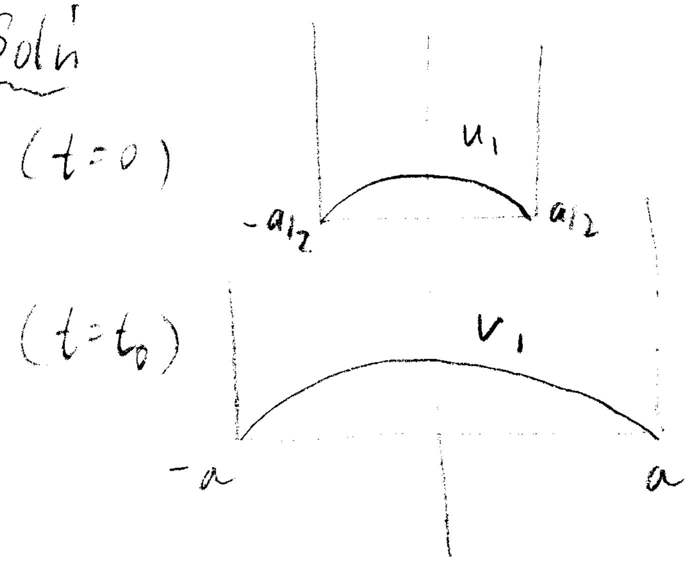
Sol'n



$u_1 = \sqrt{\frac{2}{a}} \cos kx \quad , \; k = \frac{\pi}{a}$

$v_1 = \sqrt{\frac{1}{a}} \cos \frac{1}{2} kx \quad (a \to 2a)$

$\text{Prob} \cong |\langle v_1 | u_1 \rangle|^2$

in the sudden approx.

$$\langle \psi_1 | k_1 \rangle = \int_{-a/2}^{a/2} dx \sqrt{\frac{1}{a}} \cos \tfrac{1}{2} kx \sqrt{\frac{2}{a}} \cos kx$$

$$= \frac{\sqrt{2}}{ka} \int_{-\pi/2}^{\pi/2} dy \cos \tfrac{1}{2} y \cos y \,, \quad y = kx$$

$$= \frac{\sqrt{2}}{ka} \cdot 2 \left(\frac{\sin \frac{1}{2} y}{2(\frac{1}{2})} + \frac{\sin \frac{3}{2} y}{2(\frac{3}{2})} \right) \Bigg|_0^{\pi/2}$$

$$= \frac{\cancel{\sqrt{2}}}{ka} \, 2 \left(\frac{1}{\cancel{\sqrt{2}}} + \frac{1}{3\cancel{\sqrt{2}}} \right)$$

$$= \frac{8}{3ka} = \frac{8}{3\pi} = 0.85$$

$$\boxed{\text{Prob} = (0.85)^2 = 0.72}$$

The expansion is "sudden" if its time scale t_0 satisfies

$$t_0 \ll \frac{1}{\Delta\omega} = \frac{\hbar}{\Delta E}$$

where ΔE is a characteristic energy diff

$$\Delta E \sim \frac{\hbar^2}{ma^2}$$

$$\boxed{t_0 \ll \frac{ma^2}{\hbar}}$$

8. Evaluate the expectation values of L^2, S^2 and J^2 for a spin-1/2 particle in the spinor state

$$[\Psi] = \frac{1}{\sqrt{2}} \begin{bmatrix} Y_1^1 \\ (Y_1{}^o - Y_o{}^o)/\sqrt{2} \end{bmatrix} R(r)$$

where the Y_ℓ^m are spherical harmonics and R satisfies

$$\int_o^\infty r^2 R^2 \, dr = 1.$$

(Hint: Introduce raising and lowering operators when evaluating $\langle J^2 \rangle$)

(8)

QM 2. J. Hermanson

8. Evaluate the expectation values of L^2, S^2 and J^2 for a spin-½ particle in the spinor state

$$[\psi] = \frac{1}{\sqrt{2}} \begin{pmatrix} Y_1^1 \\ \frac{Y_1^0 - Y_0^0}{\sqrt{2}} \end{pmatrix} R(r) \, ,$$

where the Y_ℓ^m are spherical harmonics and R satisfies

$$\int_0^\infty r^2 R^2 dr = 1 \, .$$

[Hint: Introduce raising and lowering operators when evaluating $\langle J^2 \rangle$]

Sol'n: $\langle L^2 \rangle = \int [\psi^+] L^2 [\psi] d^3r$

$$= \frac{1}{2} \int_0^\infty r^2 R^2 dr \int \left(Y_1^{1*}, \frac{Y_1^{0*} - Y_0^{0*}}{\sqrt{2}} \right) \begin{pmatrix} 2\hbar^2 Y_1^1 \\ \frac{2\hbar^2 Y_1^0 - 0 Y_0^0}{\sqrt{2}} \end{pmatrix} d\Omega$$

$$= \frac{\hbar^2}{2} \int \left\{ 2|Y_1^1|^2 + \left(|Y_1^0|^2 - Y_0^{0*} Y_1^0 \right) \right\} d\Omega$$

$$= \frac{\hbar^2}{2} (2 + 1 - 0) = \boxed{\frac{3\hbar^2}{2}}$$

$$\langle S^2 \rangle = \frac{\hbar}{2} \left(\frac{3\hbar}{2} \right) = \boxed{\frac{3\hbar^2}{4}} \quad \text{for spin-}1/2$$

$$\langle J^2 \rangle = \langle L^2 \rangle + \langle S^2 \rangle + \langle 2L\cdot S \rangle$$

$$2L\cdot S = L_+ S_- + L_- S_+ + 2L_z S_z$$

$$= \hbar \begin{pmatrix} L_z & L_- \\ L_+ & -L_z \end{pmatrix}$$

$$\langle 2L\cdot S \rangle = \frac{\hbar}{2} \int \left(Y_1^{1*}, \frac{Y_1^{0*} - Y_0^{0*}}{\sqrt{2}} \right) \begin{pmatrix} L_z Y_1^1 + L_- \left(\frac{Y_1^0 - Y_0^0}{\sqrt{2}} \right) \\ L_+ Y_1^1 - L_z \left(\frac{Y_1^0 - Y_0^0}{\sqrt{2}} \right) \end{pmatrix}$$

But $L_- Y_0^0 = L_+ Y_1^1 = L_z Y_1^0 = L_z Y_0^0 = 0$

So $\langle 2L\cdot S \rangle = \frac{\hbar}{2} \int \left(Y_1^{1*} L_z Y_1^1 + \cancel{\frac{1}{\sqrt{2}} Y_1^{1*} L_- Y_1^0} \right) d\Omega$

0, since $L_- Y_1^0 \sim Y_1^{-1}$

$$= \frac{\hbar^2}{2}$$

$$\langle J^2 \rangle = \frac{3\hbar^2}{2} + \frac{3\hbar^2}{4} + \frac{\hbar^2}{2} = \boxed{\frac{11\hbar^2}{4}}$$

DEPARTMENT OF PHYSICS

PH.D. COMPREHENSIVE EXAMINATION

WEDNESDAY, SEPTEMBER 18, 1985, 9-12 AM

Answer each of the following questions, all of which carry equal weight. Begin your answer to each question on a new sheet of paper; solutions to different questions must <u>not</u> appear on the same sheet. Each sheet of paper must be labeled with your name and the problem number, in the upper right-hand corner. When more than one sheet is submitted for a problem, be sure the pages are ordered properly.

9. Light can be scattered by vibrational or rotational levels of a molecule. Considering rotational scattering only, it can be of two kinds: no change in rotational quantum number (Rayleigh), or a change in quantum number by two units of angular momentum (rotational Raman).

Consider a molecule with total spin of $S = 3/2$ and total orbital angular momentum of $L = 1$, with strong spin-orbit coupling, and assume that this molecule can be treated as a rigid rotator. What are the feasible energy shifts that could be observed in Raman scattering? Which value of total angular momentum gives the smallest shift?

Drumheller

(Quantum Mechanics) (9)

Rigid rotator and Raman scattering.

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9. ~~When light is incident on a system of molecules~~ it can be scattered by vibrational or rotational level ~~in~~ of ~~the~~ a molecule. ~~[illegible]~~ Considering rotational scattering only, it can be of two kinds: no change in rotational quantum number (Rayleigh) or a change in quantum number by two units of angular momentum (~~Ram~~ rotational Raman).

Consider a molecule with total spin of 3/2 and total orbital angular momentum of 1 with strong ~~strong~~ spin-orbit coupling and assume that this rigid molecule can be treated as a rigid rotator. What are the possible energy shifts that could be observed in Raman scattering? Which value of ~~an~~ total angular momentum gives ~~has~~ the smallest shift?

Sol'n.

$S = \frac{3}{2} \quad L = 1 \qquad J = 5/2,\ 3/2,\ 1/2$

For the rigid rotator the energy levels come from the Hamiltonian in which all radial pts are fixed:

$$\hat{L}^2 \psi = \hbar^2 \ell(\ell+1)\psi$$

$$\therefore \quad \hat{H}\psi = E\psi = \frac{\hbar^2 \ell(\ell+1)}{2I}\psi$$

recall $\frac{1}{2} I\omega^2 = \frac{I^2\omega^2}{2mr^2} = \frac{\hbar^2 \ell(\ell+1)}{2mr^2} = E$

Since $I\omega$ = ang mom.

or, for our case

$$\boxed{E_J = \frac{\hbar^2 J(J+1)}{2I}}$$

Selection Rules: $\Delta J = 0$ (Rayleigh)
$= 2$ (Raman)

$$\Delta E = E_{J+2} - E_J = \frac{\hbar^2}{2I}\left[(J+2)(J+3) - J(J+1)\right]$$

$$= \frac{\hbar^2}{2I}\left(\cancel{J^2} + 5J + 6 - \cancel{J^2} - J\right)$$

$$= \frac{\hbar^2}{2I}(4J+6) = \frac{\hbar^2}{I}(2J+3)$$

$J = \frac{5}{2}$: $\Delta E_{5/2} = \frac{\hbar^2}{I}\left(2\left(\frac{5}{2}\right)+3\right) = 8\frac{\hbar^2}{I}$

$J = \frac{3}{2}$: $\Delta E_{3/2} = \frac{\hbar^2}{I} \cdots = 6\frac{\hbar^2}{I}$

$J = \frac{1}{2}$: $\Delta E_{1/2} = \cdots\cdots = 4\frac{\hbar^2}{I}$ ← gives smallest shift.

No ΔM selection rules given. Could interpret this to mean that the system must remain within the possible J's or $J = 5/2 \leftrightarrow J = 1/2$ is only transition.

10. Define an operator

$$A(\lambda) = e^{i\lambda H} A e^{-i\lambda H}$$

where H is the (Hermitian) Hamiltonian of a system and λ is a real number. It is given that

$$[H\ [H, A]] = \omega_o^2\ A$$

where ω_o^2 is a real, positive number.

a. Show that

$$A(\lambda) = A \cos \omega_o \lambda + \frac{i}{\omega_o} [H, A] \sin \omega_o \lambda$$

b. Assume that H is the Hamiltonian of a particle in a three-dimensional isotropic harmonic oscillator with spring constant C. Show that the Heisenberg representations of the operators $\vec{r}$ and $\vec{p}$ are periodic functions of time.

11. Let u(x) and v(x) be linearly independent, real functions defined on the interval $a \leq x \leq b$, which vanish at the endpoints. Consider the general differential operator

$$D = f(x)\ [d^2/dx^2] + g(x)\ [d/dx] + h(x),$$

where f, g, and h are also real functions defined on the same interval. D is called a ''self-adjoint'' operator if and only if

$$\int_a^b u(x)\ \{D\ v(x)\}\ dx = \int_a^b v(x)\ \{D\ u(x)\}\ dx$$

a. If D is self-adjoint, how must f, g, and h be interrelated?

b. If the relations of part a are <u>not</u> obeyed, find a function m(x) such that D' = m(x)D <u>is</u> a self-adjoint operator.

MSU Ph.D. Exam: Sept. 1985 17 June 85

(from Robiscoe)

Math ϕ: Self-Adjoint Operator (11)

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11. Let $u(x)$ and $v(x)$ be linearly independent, real functions defined on $a \leq x \leq b$, which vanish at the endpoints. Consider the general differential operator

$$D = f(x)\,[d^2/dx^2] + g(x)\,[d/dx] + h(x),$$

where f, g, and h are also real functions defined on the same interval. D is called a "self-adjoint" operator if and only if

$$\int_a^b u(x)\,\{D v(x)\}\,dx = \int_a^b v(x)\,\{D u(x)\}\,dx.$$

A. If D is self-adjoint, how must f, g, and h be interrelated?

B. If the relations of part A are not obeyed, find a function $m(x)$ such that: $D' = m(x)D$, is a self-adjoint operator.

Solution...

A. Let $\int = \int_a^b$. The self-adjoint condition requires (direct plug-in)

$$\int u f v''\,dx + \int u g v'\,dx + \underbrace{\int u h v\,dx}_{\text{cancel}} = \int v f u''\,dx + \int v g u'\,dx + \int v h u\,dx,$$

or//

$$\int u f\,dv' + \int u g\,dv = \int v f\,du' + \int v g\,du.$$

$h(x)$ drops out here, so there are no special conditions on it. Now partial integrate each of the integrals in the last expression...

$$u f v'\Big|_a^b - \int v'\,d(uf) + u g v\Big|_a^b - \int v\,d(ug) = v f u'\Big|_a^b - \int u'\,d(vf) +$$

... reorganize ... cancel

$$+ v g u\Big|_a^b - \int u\,d(vg),$$

$$\int v' u\,df + \int v g\,du - \int u' v\,df - \int u g\,dv = (uv' - vu')f\Big|_a^b = 0.$$

Several integrals have cancelled. The RHS = 0 because u & v vanish at the endpoints. The terms on the LHS can be regrouped to read

$$\int_a^b (uv' - vu')\left[\frac{df}{dx} - g\right]dx = 0.$$

The first factor in the Wronskian of u & v; this is nonzero by assumption of linear independence. The only way this equation can be satisfied identically is for f & g to be related by.... ∃

$\boxed{g(x) = df/dx}$ ← necessary (& sufficient) for D ∈ self-adjoint

B. If D is such that $g \neq df/dx$, look at $D \rightarrow D' = mD$. Then $f \rightarrow mf$ and $g \rightarrow mg$, and we need to satisfy

$$mg = \frac{d}{dx}(mf) \Rightarrow \frac{m'}{m} = \frac{g - f'}{f}$$

Integrating, we find the necessary multiplier $m(x)$

$$\boxed{m(x) = \exp\left\{\int_a^x [(g - f')/f]\,dx'\right\}} = \frac{f(a)}{f(x)}\exp\left\{\int_a^x (g/f)\,dx'\right\}$$

for which $D' = m(x)D$ is always self-adjoint.

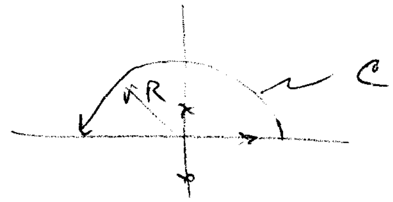
12. Evaluate by contour integration: $I = \int_{0}^{\infty} \frac{dx}{(1 + x^2)^2}$

Math Phys (12) Ford

12. Evaluate by contour integration: ~~using contour~~ $I = \int_0^\infty \frac{dx}{(1+x^2)^2}$

Soln

Let $\mathcal{I} = \oint_C \frac{dz}{(1+z^2)^2}$



and $f(z) = \frac{1}{(1+z^2)^2} = \frac{1}{(z+i)^2}\frac{1}{(z-i)^2}$

along $z = Re^{i\theta}$

$$f(z)\,dz = \frac{1}{(1+R^2e^{2i\theta})^2}\, iR\,d\theta\, e^{i\theta}$$

$$\left|\frac{iRe^{i\theta}}{(1+R^2e^{2i\theta})^2}\right| \leq \frac{R}{(1-R^2)^2} \overset{R\to\infty}{\sim} \frac{1}{R^3}$$

By Jordan's lemma the contribution along the arc vanishes

$$\therefore\ I = \frac{1}{2}\lim_{R\to\infty}\mathcal{I} \quad \text{(by symmetry)}$$

$$\mathrm{Res}\,[f(z), z=i] = \frac{1}{(n-1)!}\frac{d^{n-1}}{dz^{n-1}}\left[(z-i)^n f(z)\right]\Big|_{n=2}$$

$$= \cancel{4}\,\frac{-2}{(z+i)^3}\Big|_{z=i} = \frac{+2}{8i} = \frac{-i}{4}$$

$$\mathcal{I} = 2\pi i\left(\frac{-i}{4}\right) = \frac{\pi}{2} \Rightarrow \boxed{I = \frac{\pi}{4}}$$

DEPARTMENT OF PHYSICS

PH.D. COMPREHENSIVE EXAMINATION

WEDNESDAY, SEPTEMBER 18, 1985, 2-5 PM

Answer each of the following questions, all of which carry equal weight. Begin your answer to each question on a new sheet of paper; solutions to different questions must not appear on the same sheet. Each sheet of paper must be labeled with your name and the problem number, in the upper right-hand corner. When more than one sheet is submitted for a problem, be sure the pages are ordered properly.

13. In a certain thermodynamic cycle 2 moles of an ideal gas ($\gamma = 1.4$) starts at $0^{o}C$ and 1 atmosphere. The gas is heated at constant volume to $T_2 = 150^{o}C$, then expands adiabatically until the pressure is again 1 atmosphere. Finally, it is compressed at constant pressure back to its original state.

a. What is the temperature T_3 after the adiabatic expansion?

b. How much heat enters or leaves the system in each segment of the cycle?

c. What is the efficiency of this cycle and how does it compare to a Carnot cycle?

[R (universal gas constant) = 8.314 J/mol·K = 0.082 L·atm/mol·K and 1 L·atm = 101.3 J]

(Thermodynamics) Drumheller

13.

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13. In a ~~[illegible]~~ certain thermodynamic cycle 2 mol. of an ideal gas ($\gamma = 1.4$) starts at $0°C$ and 1 atmosphere. The gas is heated at ~~constant~~ constant volume to $T_2 = 150°C$, then expands adiabatically until the pressure is again 1 atmosphere. Finally it is compressed at constant pressure back to its original state.

a.) What is the temperature T_3 after the adiabatic expansion?

b.) How much heat enters or leaves the system in each segment of the cycle?

c.) What is the efficiency of this cycle and how does it compare to a Carnot cycle?

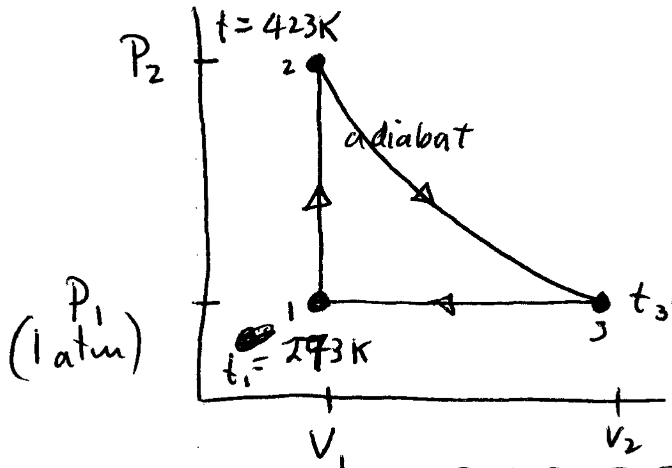
R (universal gas constant) = 8.314 J/mol-K = 0.082 L·atm/mol.K

or 101.3 J = 1 L-atm.

Too simple? (Wouldn't be for <u>me</u>!)

Sol'n

Known:



$\gamma = 1.4 \quad \therefore \quad \dfrac{C_p}{C_v} = 1.4 = \dfrac{7}{5} = \dfrac{7/2\, nR}{5/2\, nR}$

$\left.\begin{array}{l} C_p = 7/2\, nR \\ C_v = 5/2\, nR \end{array}\right\}$ ideal gas.

$n = 2$

a.) $t_3 = ?$ $\quad$ (adiabat) $PV^{\gamma} = \text{const}$ $\quad$ but also $\quad$ (ideal gas) $PV = nRT$ or $V = \dfrac{nRT}{P}$

$\therefore P\left(\dfrac{nRT}{P}\right)^{\gamma} = \text{const}$

So that $\quad P_2^{1-\gamma} T_2^{\gamma} = P_3^{1-\gamma} T_3^{\gamma}$

$$\boxed{T_3 = \left(\frac{P_2}{P_3}\right)^{\frac{1-\gamma}{\gamma}} T_2}$$

need P_2 : $\quad P_1 V_1 = nRT_1 \qquad P_2 V_2 = nRT_2 \qquad$ but $V_1 = V_2$

$\therefore \boxed{\dfrac{T_1}{P_1} = \dfrac{T_2}{P_2}} \quad$ or $\quad P_2 = P_1 \dfrac{T_2}{T_1} = (1\text{ atm}) \dfrac{423}{293} = \underline{1.55\text{ atm}}$

Finally $\quad T_3 = \left(\dfrac{1.55}{1}\right)^{\frac{1-1.4}{1.4}} 423 = 1.55^{-0.286}\, 423 = \underline{\underline{374\text{ K}}}$

b.) 1^{st} segment; Isovolumic $\qquad \Delta Q = C_v \Delta T = \frac{5}{2} 2 \left(8.314 \frac{J}{K}\right) 150\text{ K}$

$= \underline{6.235\text{ kJ}}$

2^{nd} Segment : adiabatic $\qquad \underline{\Delta Q = 0}$

3^{rd} Segment : isobaric $\qquad \Delta Q = C_p \Delta T = \frac{7}{2} 2 \left(8.314 \frac{J}{K}\right) 101\text{ K}$

$= \underline{-5.878\text{ kJ}}$

c.) $\varepsilon = 1 - \dfrac{Q_c}{Q_h} = 1 - \dfrac{5.878}{6.235} = \underline{5.7\ \%}$

A Carnot cycle would be $\varepsilon = 1 - \dfrac{T_c}{T_h} = 1 - \dfrac{273}{423} \approx 35.5\%$

14. The Debye Model

The Debye model of solids considers each independent mode of vibration as an independent harmonic oscillator. The harmonic oscillator can have any energy e_n given by

$$e_n = (n+1/2)\hbar\omega, \quad n = 0,1,2,3,\ldots$$

Each mode is a lattice-displacement wave of the form

$$u_i = u^o{}_i \exp\{i(\vec{k}\cdot\vec{r}-\omega t)\}$$

a. Write the partition function for a system containing only one mode. From its partition function compute the occupation number for phonons in this mode.

b. Find an expression for the internal energy of the full three-dimensional solid.

c. Compute the specific heat in the low-temperature and high-temperature limits. Interpret these results.

15. Calculate the threshold energy of a cosmic ray proton for it to undergo the reaction

$$\gamma + p \longrightarrow p + \pi^o \quad ,$$

where γ represents a photon of temperature 3 K (the cosmic microwave background). Assume the collision is head-on; take the photon energy to be $kT \cong 2.5 \times 10^{-10}$ MeV; $m_p = 940$ MeV; $m_\pi = 140$ MeV. This reaction provides an upper limit to the energies of cosmic ray protons.

15. Calculate the threshold energy of a cosmic ray proton for it to undergo the reaction

$$\gamma + p \rightarrow p + \pi^0$$

where γ represents a photon of temperature 3 K (the cosmic microwave background). Assume the collision is head-on; take the photon energy to be $\sim kT = 2.5\times10^{-10}$ MeV; $m_p = 940$ MeV; $m_\pi = 140$ MeV. This reaction provides an upper limit to the energies of cosmic ray protons.

Solution: Conservation of four-momentum: $\overset{\Delta}{P} = (E, \vec{p})$

$$\overset{\Delta}{P}_\gamma + \overset{\Delta}{P}_p = \overset{\Delta}{P}_p{}' + \overset{\Delta}{P}_\pi$$

so $$|\overset{\Delta}{P}_\gamma + \overset{\Delta}{P}_p|^2 = |\overset{\Delta}{P}_p{}' + \overset{\Delta}{P}_\pi|^2$$

in C.M. frame at threshold, $\overset{\Delta}{P}_p{}' + \overset{\Delta}{P}_\pi = (m_p + m_\pi, 0)$

so

$$|\overset{\Delta}{P}_\gamma|^2 + 2\overset{\Delta}{P}_\gamma \cdot \overset{\Delta}{P}_p + |\overset{\Delta}{P}_p|^2 = -(m_p + m_\pi)^2$$

$$2\overset{\Delta}{P}_\gamma \cdot \overset{\Delta}{P}_p - m_p^2 = -(m_p + m_\pi)^2$$

$$\overset{\Delta}{P}_\gamma \cdot \overset{\Delta}{P}_p = -E_\gamma E_p + \vec{p}_\gamma \cdot \vec{p}_p \rightarrow$$

head-on collision $\Rightarrow \vec{p}_\gamma \cdot \vec{p}_p = -p_\gamma p_p$

$|\overset{\Delta}{P}_\gamma|^2 = 0 \Rightarrow p_\gamma = E_\gamma$

$p_p = (E_p^2 - m_p^2)^{1/2}$

so, again

$$-2E_\gamma\left[E_p + (E_p^2 - m_p^2)^{1/2}\right] - m_p^2 = -(m_p + m_\pi)^2$$

$$\left[E_p + (E_p^2 - m_p^2)^{1/2}\right] = \frac{2m_p m_\pi + m_\pi^2}{2E_\gamma} = \frac{2\cdot 940\cdot 140 + 140^2}{2\cdot(2.5\times10^{-10})}\text{ MeV} = \underline{6\times10^{14}\text{ MeV}}$$

Since, clearly, $E_p \gg m_p$, we may replace $(E_p^2 - m_p^2)^{1/2}$ with E_p, to find

$$\boxed{E_p \approx 3\times10^{14}\text{ MeV}}$$

16. About 5 billion years from now, our sun ($M_\odot = 2\text{x}10^{33}$ g, $R_\odot = 7\text{x}10^{10}$ cm) will collapse to form a white dwarf star ($R_{WD} \simeq 5\text{x}10^{8}$ cm) supported by the degeneracy of its electrons. Assume that no mass loss occurs (so $M_{WD} = M_\odot$); that the initial temperature of the sun just before collapse is absolute zero; and that the sun is composed entirely of ^{4}He ($m_4 \simeq 7\text{x}10^{-24}$ g).

a. Assuming that all the energy of collapse is converted into thermal motion of the ^{4}He ions, which act like a monatomic ideal gas ($C_V = 3/2$ k), what is the approximate temperature of the white dwarf star shortly after it is formed? (Don't worry about factors of order 2, etc.)

b. Assuming that the white dwarf radiates like an ideal black body, find its temperature as a function of time, in terms of R_{WD}, σ, k, etc. How many years does it take the white dwarf to cool to a ''black'' dwarf ($T \approx 1000$ K)?

1 yr = $3.16\text{x}10^{7}$ sec
k = $1.38\text{x}10^{-16}$ erg/K
G = $6.67\text{x}10^{-8}$ cm^3/g$\cdot$ sec^2
σ = $5.67\text{x}10^{-5}$ g/sec$^3\cdot$K^4

16. About 5 billion years from now, our Sun ($M_\odot \approx 2\times10^{33}$ g, $R_\odot = 7\times10^{10}$ cm) will collapse to form a white dwarf star ($R_{WD} \approx 5\times10^{8}$ cm) supported by the degeneracy of its electrons. Assume that no mass loss occurs (so $M_{WD} = M_\odot$); that the initial temperature of the Sun just before collapse is absolute zero; and that the Sun is composed entirely of ^{4}He ($m_{^4He} \approx 7\times10^{-24}$ g)

(a) Assuming that all the energy of collapse is converted into thermal motion of the ^{4}He ions, which act like a monatomic ideal gas ($c_v = 3/2\, k$), what is the approximate temperature of the white dwarf star shortly after it is formed? (Don't worry about factors of order 2, etc.).

(b) Assuming that the white dwarf radiates like an ideal black body, find its temperature as a function of time, in terms of R_{WD}, σ, k, etc. How many years does it take the white dwarf to cool to a "black" dwarf ($T \approx 1000$K)?

$1\text{ yr} = 3.16\times10^{7}$ sec

$k = 1.38\times10^{-16}$ erg / K

$G = 6.67\times10^{-8}$ cm^3 g^{-1} sec^{-2}

$\sigma = 5.67\times10^{-5}$ g sec^{-3} K^{-4}

Solution

(a) gain in energy by an "average" He ion = Δu

$$\langle \Delta U \rangle \simeq \frac{1}{2} G M_\odot M_{He} \left[\frac{1}{R_{WD}} - \frac{1}{R_\odot} \right]$$

$$\Delta T = \frac{\langle \Delta U \rangle}{c_v} = \frac{1}{3} \frac{G M_\odot M_{He}}{k} \left[\frac{1}{R_{WD}} - \frac{1}{R_\odot} \right]$$

$$= \frac{1}{3} \frac{(6.67\times10^{-8})(2\times10^{33})(7\times10^{-24})}{1.38\times10^{-16}} \left[\frac{1}{5\times10^{8}} - \frac{1}{7\times10^{10}} \right] \quad K$$

$$\boxed{T \simeq 4\times10^{9}\, K}$$

(b) $L = \sigma A T^4 \qquad A = 4\pi R_{WD}^2$

$$L = -\frac{dU}{dt} \;;\; U = \frac{3}{2} N k T = \frac{3}{2} \frac{M_\odot}{M_{He}} kT \;;\; \frac{dU}{dt} = \frac{3}{2} N k \frac{dT}{dt}$$

$$-\frac{3}{2} N k \frac{dT}{dt} = \sigma A T^4 \qquad \frac{dT}{T^4} = -\frac{2\sigma A}{3Nk} dt$$

$$+\frac{1}{\cancel{3}} T^{-3} = +\frac{2}{\cancel{3}} \frac{\sigma A}{Nk} (t - t_0)$$

$$\boxed{\begin{aligned} T(t) &= \left[\frac{2\sigma A}{Nk} (t - t_0) \right]^{-\frac{1}{3}} \\ T(t) &= \left[\frac{8\pi\sigma R_{WD}^2 M_{He}}{M_\odot k} (t - t_0) \right]^{-\frac{1}{3}} \end{aligned}}$$

time to cool to 10^3 K: Let $T(t_1) = T_1 = 4\times10^9$ K

$T(t_2) = T_2 = 10^3$ K

$$t_1 - t_0 = \frac{Nk}{2\sigma A} \frac{1}{T_1^3} \;;\; t_2 - t_0 = \frac{Nk}{2\sigma A} \frac{1}{T_2^3} \;;\; t_2 - t_0 - (t_1 - t_0) = t_2 - t_1$$

$$t_2 - t_1 = \frac{Nk}{2\sigma A} \left(\frac{1}{T_2^3} - \frac{1}{T_1^3} \right) = \frac{M_\odot k}{8\pi\sigma R_{WD}^2 M_{He}} \left(\frac{1}{T_2^3} - \frac{1}{T_1^3} \right)$$

$$= \frac{(2\times10^{33})(1.38\times10^{-16})}{8\pi(5.67\times10^{-5})(5\times10^{8})^2(7\times10^{-24})} \left\{ \frac{1}{10^9} - \underbrace{\frac{1}{(4\times10^9)^3}}_{\text{ignore}} \right\}$$

$$\boxed{\Delta t = 1.1\times10^{17}\ \text{sec} = 3.48\times10^{9}\ \text{years}}$$