Propagation of Plane EM Waves [Jkn Secs. 7.1–7.4]

1) A dramatic success of Maxwell's EM theory was the specification of light as an EM transverse wave which can propagate through empty space at a (calculated) velocity $c = 3\times10^{10}$ cm/sec. This characterization follows from the "wave equations" for the $\mathbb{E}$ & $\mathbb{B}$ fields, which follow directly from Maxwell's field equations. We shall now study wave solutions in Maxwell's theory in more detail.

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2) For a linear, homogeneous medium $\left\{ \begin{array}{l} \mathbb{D} = \epsilon\,\mathbb{E} \ \ (\text{permittivity}: \epsilon = 1+4\pi\alpha,\ \alpha = \text{polarizability}) \\ \mathbb{B} = \mu\,\mathbb{H} \ \ (\text{permeability}: \mu = 1+4\pi\chi,\ \chi = \text{susceptibility}) \end{array} \right\}$

w/ α & χ = cnsts (usu. frequency-dependent, however... consider monochromatic fields), the Maxwell Eqtns are...

$$\boxed{\begin{array}{ll} ① \ \nabla\cdot\mathbb{E} = 4\pi\rho/\epsilon, & ② \ \nabla\cdot\mathbb{B} = 0, \\ ③ \ \nabla\times\mathbb{E} = -\frac{1}{c}\frac{\partial\mathbb{B}}{\partial t}, & ④ \ \nabla\times\mathbb{B} = \frac{4\pi\mu}{c}\mathbb{J} + \frac{\mu\epsilon}{c}\frac{\partial\mathbb{E}}{\partial t}. \end{array}} \tag{1}$$

By separating the $\mathbb{E}$ & $\mathbb{B}$ variation (w/ by-now-familiar operations*), we straightforwardly generate wave equations for $\mathbb{E}$ & $\mathbb{B}$, viz.

$$\left(\nabla^2 - \frac{1}{v^2}\frac{\partial^2}{\partial t^2}\right)\mathbb{E} = 4\pi\,\nabla(\rho/\epsilon) + \left(\frac{4\pi\mu}{c^2}\right)\frac{\partial\mathbb{J}}{\partial t}, \tag{2a}$$

$$\left(\nabla^2 - \frac{1}{v^2}\frac{\partial^2}{\partial t^2}\right)\mathbb{B} = -\frac{4\pi\mu}{c}\nabla\times\mathbb{J}; \quad \text{w/ } v = c/\sqrt{\mu\epsilon} \leq c. \tag{2b}$$

We've assumed μ is indpt. of position $\mathbf{r}$. Assume the same for ϵ, and impose Ohm's Law: $\mathbb{J} = \sigma\mathbb{E}$, σ = medium's conductivity. Then write Eqs (2) as...

$$\boxed{\begin{array}{l} \nabla^2\mathbb{E} - \alpha\,\mathbb{E}_t - (1/v^2)\,\mathbb{E}_{tt} = (4\pi/\epsilon)\,\nabla\rho, \\ \nabla^2\mathbb{B} - \alpha\,\mathbb{B}_t - (1/v^2)\,\mathbb{B}_{tt} = 0. \end{array}} \quad \begin{array}{l} \alpha = 4\pi\mu\sigma/c^2 \text{ (attn. coeff.)}; \quad (3a) \\ \mathbb{E}_t = (\partial/\partial t)\,\mathbb{E}, \text{ etc.} \quad (3b) \end{array}$$

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* e.g. to get Eq. (2a), take $\nabla\times$ thru Eq. (1)③ $\Rightarrow \nabla(\nabla\cdot\mathbb{E}) - \nabla^2\mathbb{E} = -\frac{1}{c}\frac{\partial}{\partial t}(\nabla\times\mathbb{B})$. Now use Eq. (1)① for $\nabla\cdot\mathbb{E}$, and Eq. (1)④ for $\nabla\times\mathbb{B}$ here. Rearrange terms to get Eq. (2a).

Wave Solutions for Non-Conducting Media.

3) In a nonconducting medium, $\sigma = 0$ & $\alpha = 0$. If there are no free charges, $\rho = 0$, and -- according to Eqs. (3) -- all components of $\mathbb{E}$ & $\mathbb{B}$ obey the simple eqtn:

$$\left.\begin{matrix}\sigma = 0 \\ \rho = 0\end{matrix}\right\} \quad \boxed{\nabla^2 u - (1/v^2)\, u_{tt} = 0}\,, \quad u = \text{any comp}^{\underline{t}} \text{ of } \mathbb{E} \text{ or } \mathbb{B}. \qquad (4)$$

To look at "global" solutions to this PDE (prototype: hyperbolic), do as follows.

1. Go to 1D for simplicity: $u_{xx} - (1/v^2)\, u_{tt} = 0$.

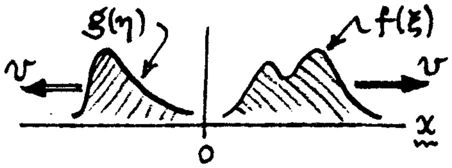
2. Define "normal coordinates": $\underline{\underline{\xi = x - vt}}$, $\underline{\underline{\eta = x + vt}}$ so/ $x = \frac{1}{2}(\eta + \xi)$, $t = \frac{1}{2v}(\eta - \xi)$.

and/
$$\left.\begin{matrix}\dfrac{\partial}{\partial x} = \dfrac{\partial}{\partial \eta} + \dfrac{\partial}{\partial \xi} \\[2ex] \dfrac{1}{v}\dfrac{\partial}{\partial t} = \dfrac{\partial}{\partial \eta} - \dfrac{\partial}{\partial \xi}\end{matrix}\right\} \quad \frac{\partial^2}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2}{\partial t^2} = \left(\frac{\partial}{\partial x} + \frac{1}{v}\frac{\partial}{\partial t}\right)\left(\frac{\partial}{\partial x} - \frac{1}{v}\frac{\partial}{\partial t}\right) = 4\left(\frac{\partial}{\partial \eta}\right)\left(\frac{\partial}{\partial \xi}\right);$$

thus// $u_{xx} - (1/v^2)\, u_{tt} = 0$, transforms to: $\boxed{\dfrac{\partial^2 u}{\partial \xi\, \partial \eta} = 0}$, for $u = u(\xi, \eta)$. (5)

3. The general solution to Eq. (5) is trivial, and can be written down by inspection:

D'Alembert Solution
$$\boxed{u = f(\xi) + g(\eta) = \underbrace{f(x - vt)}_{\text{rightward wave}} + \underbrace{g(x + vt)}_{\text{leftward wave}}} \qquad (6)$$



If the propagation velocity v does not depend on frequency ω for any of the ω's contained in the initial waveforms $f(x)$ & $g(x)$ [@ $t=0$], then these waveforms propagate without changing shape. However, if $v = c/\sqrt{\mu\epsilon} = v(\omega)$ [usu. because $\epsilon = \epsilon(\omega)$], then the waves are distorted as they propagate ... this effect is called "dispersion", and we shall study it later.

(recall SHO model for ϵ)

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4) A particular solution for u in Eq. (6) is often chosen to be in the form...

$$\left.\begin{matrix}f(x - vt) = A(k)\, e^{i(kx - \omega t)} \\ g(x + vt) = B(k)\, e^{i(kx + \omega t)}\end{matrix}\right\rangle \text{w/ } \underline{\underline{\omega = kv}}, \text{ so/ } \boxed{u_k(x,t) = A(k)\, e^{i(kx - \omega t)} + B(k)\, e^{i(kx + \omega t)}}$$

$\rightarrow$ in 3D: $u_k(\mathbf{r}, t) = A(\mathbf{k})\, e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)} + B(\mathbf{k})\, e^{i(\mathbf{k}\cdot\mathbf{r} + \omega t)}$. (7)

Planewave Solutions for Nonconducting Media.

Such solutions are called "plane-wave solutions", because the surfaces of constant phase ($\mathbf{k}\cdot\mathbf{r}$ = cnst, at a given time t) are planes. The utility of this formulation is... since $\nabla^2 u - (1/v^2)u_{tt} = 0$ is a linear eqtn, we can form the superposition:

$$\rightarrow u(\mathbf{r},t) = \int_\infty d^3k\,[A(\mathbf{k})e^{-i\omega t} + B(\mathbf{k})e^{+i\omega t}]\,e^{i\mathbf{k}\cdot\mathbf{r}}, \tag{8}$$

which is still a solution to the wave eqtn, and is also instantly recognizable as a Fourier representation of $u(\mathbf{r},t)$. So we have Fourier's machinery at our disposal.

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5) If $\mathbb{E}$ & $\mathbb{B}$ are represented by plane waves [per Eq.(7)], then certain vector relations must be obeyed for consistency with Maxwell's Eqtns. Suppose...

$$\rightarrow \mathbb{E}(\mathbf{r},t) = \mathcal{E}(\mathbf{k})\,e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}, \quad \mathbb{B}(\mathbf{r},t) = \mathcal{B}(\mathbf{k})\,e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} \tag{9}$$

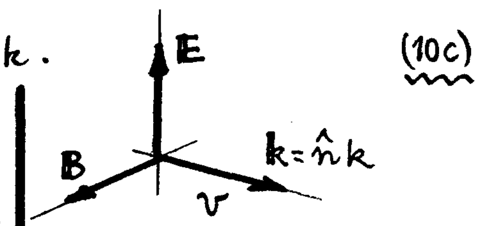
... $\mathcal{E}$ and $\mathcal{B}$ are constant vectors, independent of $\mathbf{r}$ & t ...

So/

$$\nabla\cdot\mathbb{E} = i(\mathbf{k}\cdot\mathcal{E})\,e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}, \text{ and/ } \nabla\cdot\mathbb{E} \overset{\text{no free charge}}{=} 0 \Rightarrow \underline{\underline{\mathbf{k}\cdot\mathcal{E} = 0}}; \tag{10a}$$

$$\nabla\cdot\mathbb{B} = i(\mathbf{k}\cdot\mathcal{B})\,e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}, \text{ and/ } \nabla\cdot\mathbb{B} \overset{\text{no MM's}}{=} 0 \Rightarrow \underline{\underline{\mathbf{k}\cdot\mathcal{B} = 0}}; \tag{10b}$$

$$\frac{\partial\mathbb{B}}{\partial t} = -c\,\nabla\times\mathbb{E} \Rightarrow \underline{\underline{\mathcal{B} = \sqrt{\mu\epsilon}\,(\hat{n}\times\mathcal{E})}}, \text{ w/ } \hat{n} = \mathbf{k}/k. \tag{10c}$$



TRANSVERSE PLANEWAVE

Eqs.(10) together ⇒ we have a transverse EM plane-wave, propagating in the direction of the "wavevector" $\mathbf{k}$, at velocity $v = c/\sqrt{\mu\epsilon}$, with $\mathbb{E}$, $\mathbb{B}$ & $\mathbf{k}$ forming an orthonormal triad, as shown. The energy transport for such a wave is:

Poynting Vector $\}$ $$\boxed{\mathbb{S} = \mathrm{Re}\left\{\frac{c}{4\pi}(\mathbb{E}\times\mathbb{H})\right\} = \frac{1}{2}\cdot\frac{c}{4\pi}\sqrt{\frac{\epsilon}{\mu}}\,|\mathcal{E}|^2\,\hat{n}}. \tag{11}$$

↳ Jackson Eq.(7.13)

↑ factor ½ for time-averaging.

The Re part here singles out $\sin^2$ & $\cos^2$ terms, whose averages over a cycle or more are $\langle\sin^2\rangle = \langle\cos^2\rangle = \frac{1}{2}$. Finally, the time-averaged wave energy density is

$$\rightarrow u = \mathrm{Re}\left\{\frac{1}{8\pi}\left(\epsilon\,\mathbb{E}^2 + \frac{1}{\mu}\mathbb{B}^2\right)\right\} = \frac{\epsilon}{8\pi}|\mathcal{E}|^2, \text{ and/ } \boxed{\mathbb{S} = u v \hat{n}}, \; v = \frac{c}{\sqrt{\mu\epsilon}}. \tag{12}$$