ϕ519 Problems   Assigned 10/4/91. Due 10/11/91.

(18) [15pts]. Many second order ODE's can be written in the form of a "hypergeometric differential equation": $x(1-x)y'' + [\gamma - (1+\alpha+\beta)x]y' - \alpha\beta y = 0$, where α, β & γ are constants. Use a power series solution: $y(x) = x^k \sum_{\lambda=0}^{\infty} a_\lambda x^\lambda$, to show:

(A) The indicial equation is: $k(k-1) + k\gamma = 0$, so that $k=0$, or $k = 1-\gamma$.

(B) For $k=0$, the recursion relation is: $a_{\lambda+1} = [(\alpha+\lambda)(\beta+\lambda)/(1+\lambda)(\gamma+\lambda)]\,a_\lambda$.

(C) Iteration of the $a_\lambda \to a_{\lambda+1}$ relation in (B) produces the hypergeometric series (for $k=0$):

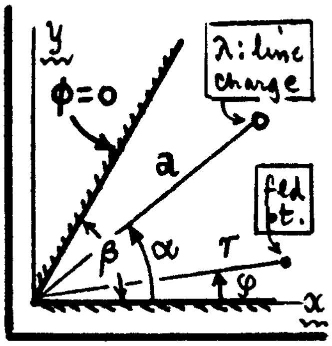
$$\rightarrow y(x) = a_0\left[1 + \frac{\alpha\beta}{\gamma}\cdot\frac{x}{1!} + \frac{\alpha(\alpha+1)\beta(\beta+1)}{\gamma(\gamma+1)}\cdot\frac{x^2}{2!} + \cdots + \frac{\alpha(\alpha+1)\cdots(\alpha+n-1)\beta(\beta+1)\cdots(\beta+n-1)}{\gamma(\gamma+1)\cdots(\gamma+n-1)}\cdot\frac{x^n}{n!} + \cdots\right]$$

The series in [ ] here is usually denoted by ${}_2F_1(\alpha, \beta; \gamma; x)$, a hypergeometric function.

(D) For the other k-value, the second solution is: $y(x) = a_0 x^{1-\gamma}{}_2F_1(\alpha', \beta'; \gamma'; x)$. Find the new indices $(\alpha', \beta', \gamma')$ in terms of the original set (α, β, γ).

---

(19) Here is a variant of the 2D "wedge" problem that Jackson does in his Sec. (2.11). Two conducting planes intersect at $\measuredangle\beta$; the planes are grounded (potential $\phi \equiv 0$). At a point with polar cds (a, α) inside the wedge, there is a line charge ∥ z-axis which carries a uniform charge per unit length λ. Fld pt. is at (r, φ).



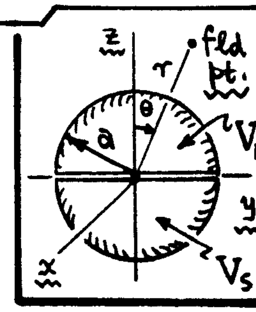
(A) Write expressions for the potentials $\phi_1(r<a, \varphi)$ & $\phi_2(r>a, \varphi)$, valid over $0 \leq \varphi \leq \beta$, such that $\phi_1 \to 0$ as $r \to 0$, and $\phi_2 \to 0$ as $r \to \infty$. Note that $\phi_2 \equiv \phi_1$ at $r = a$.

(B) There is a source discontinuity @ $(r=a, \varphi=\alpha)$. Account for this by a singular surface charge density on the cylinder $r=a$, and -- from the resulting field singularity -- determine the unknown coefficients in the expressions for ϕ_1 & ϕ_2 in part (A).

(C) Find the charge density σ on the plates ($\varphi = 0$ & $\varphi = \beta$) "in close," at $r<a$. Comment.

---

(20) Another variant [on Jackson's hemispheres, his Eq. (3.36)]. Suppose a conducting sphere of radius a is split into hemispheres, resp. held at potentials V_N and V_S. Find the potential $\phi(r, \theta)$ outside the sphere.



ϕ 519 Solutions

Hyper Geometric Equation

18 [15 pts.]. Solve: $x(1-x)y''+[\gamma-(1+\alpha+\beta)x]y'-\alpha\beta y=0$, by power series.

1) Put in $y=x^k\sum_{\lambda=0}^{\infty}a_\lambda x^\lambda$, and corresponding y' & y''. Collect like powers of x to get...

$$\longrightarrow a_0[k(k-1)+\gamma k]x^{k-1}+\sum_{\lambda=0}^{\infty}\{a_{\lambda+1}[(k+\lambda+1)(k+\lambda+\gamma)]-a_\lambda[(k+\lambda)(k+\lambda+\alpha+\beta)+ +\alpha\beta]\}x^{k+\lambda}=0. \quad (1)$$

This $\Rightarrow$ all information needed to specify a series solution.

A 2) The indicial eqtn, specifying desired k-values, results from the first term in Eq. (1)-- if $a_0\neq 0$ (and for all x) we must have the [ ] $\equiv 0$, which gives...

$$[k(k-1)+\gamma k]\equiv 0 \Rightarrow \underline{k=0,\ \text{or}\ k=1-\gamma}. \quad (2)$$

The cnst $a_0\neq 0$, and these two k-values, will generate 2 separate series expansions; this provides the 2 degrees of freedom which are necessary to solve the ODE.

B 3) For the 1st series solution, set $k=0$, and impose the { } in Eq. (1) is $\equiv 0$. This gives the recursion relation (for 1st solution)...

$$a_{\lambda+1}=\left(\underbrace{[\lambda(\lambda+\alpha+\beta)+\alpha\beta]}_{=(\alpha+\lambda)(\beta+\lambda)}/(\lambda+1)(\lambda+\gamma)\right)a_\lambda$$

Soy

$$a_{\lambda+1}=[(\alpha+\lambda)(\beta+\lambda)/(\lambda+1)(\lambda+\gamma)]a_\lambda,\quad \lambda=0,1,2,\ldots \quad (3)$$

C 4) Eq. (3) can be iterated "by hand", as...

$$a_1=\frac{\alpha\beta}{1\cdot\gamma}a_0,\quad a_2=\frac{(\alpha+1)(\beta+1)}{2\cdot(\gamma+1)}a_1=\frac{\alpha(\alpha+1)\beta(\beta+1)}{2\cdot1\cdot\gamma(\gamma+1)}a_0,\quad a_3=\text{etc.}$$

and// $$a_n=\frac{\alpha(\alpha+1)\cdots(\alpha+n-1)\,\beta(\beta+1)\cdots(\beta+n-1)}{n!\,\gamma(\gamma+1)\cdots(\gamma+n-1)}a_0,\ \text{by induction.} \quad (4)$$

The desired hypergeometric series -- for $k=0$ -- is thus

$$\boxed{y_1(x)=a_0\left\{1+\sum_{n=1}^{\infty}\frac{[\alpha(\alpha+1)\cdots(\alpha+n-1)][\beta(\beta+1)\cdots(\beta+n-1)]}{[\gamma(\gamma+1)\cdots(\gamma+n-1)]}\cdot\frac{x^n}{n!}\right\}} \quad (5)$$

(next page)

ϕ 519 Solutions

(18) (cont'd). **ASIDE** The multiplicative combinations in [ ]'s in Eq. (5) occur so often in the mathematics of special fcns that they are given a symbol, per...

POCHHAMMER'S SYMBOL * $\Big\}$ $(z)_n \equiv z(z+1)(z+2)\cdots(z+n-1) = \frac{\Gamma(z+n)}{\Gamma(z)}$ $\int$ n = (+)ve integer, z = anything. **(6)**

Also: $(z)_0 \equiv 1$; then $(z)_1 = z$, $(z)_2 = z(z+1)$, etc. The Γ's are Gamma functions.†

In these terms, the hypergeometric solution of Eq. (5) is written...

$$\longrightarrow y_1(x) = a_0\, F(\alpha,\beta;\gamma;x)\ ;\ \boxed{F(\alpha,\beta;\gamma;x) = \sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n}{(\gamma)_n}\cdot\frac{x^n}{n!}}. \tag{7}$$

This is standard notation for the hypergeometric fcn F; see NBS Handbk, Ch. 13.

D 5) For a 2nd series solution $y_2(x)$, set $k=1-\gamma$, after factoring the numerator of the general recursion relation obtained from setting { } = 0 in Eq. (1)...

$$a_{\lambda+1} = \left[\frac{(\alpha+k+\lambda)(\beta+k+\lambda)}{(\gamma+k+\lambda)(1+k+\lambda)}\right] a_\lambda \underset{k=1-\gamma}{=} \frac{[(\alpha+1-\gamma)+\lambda][(\beta+1-\gamma)+\lambda]}{(\lambda+1)[(2-\gamma)+\lambda]}\, a_\lambda$$

or// $$a_{\lambda+1} = [(\alpha'+\lambda)(\beta'+\lambda)/(\lambda+1)(\gamma'+\lambda)]\, a_\lambda \int^{\text{w/}} \boxed{\begin{aligned}\alpha' &= \alpha+1-\gamma,\\ \beta' &= \beta+1-\gamma,\\ \gamma' &= 2-\gamma.\end{aligned}} \tag{8}$$

In this form, the $k=1-\gamma$ recursion relation is the same as in Eq. (3), except (α',β',γ') replace (α,β,γ). The 2nd solution proceeds as above, with the result...

$$\boxed{y_2(x) = a_0\, x^{1-\gamma}\, F(\alpha',\beta';\gamma';x)} \int \begin{matrix}\alpha',\beta',\gamma' \text{ in Eq. (8)},\\ F \text{ series in Eq. (7)}.\end{matrix} \tag{9}$$

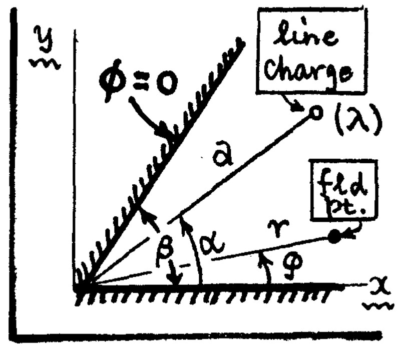
It turns out that y_1 (Eq. (5)) & y_2 (Eq. (9)) are linearly indpt if $\gamma>0$ & $\gamma \neq$ integer.

We will study the hypergeometric series $F(\alpha,\beta;\gamma;x)$ in detail later. For various choices of α, β & γ, the F's represent <u>many</u> different elementary fcns (e.g. e^x, $\sin x$ & $\cos x$, $\sinh x$ & $\cosh x$ (and inverses), Legendre fcns, elliptic fcns, etc.)

* NBS Handbk Eq. (6.1.22). † NBS Handbk, Ch. 6.

ϕ519 Solutions

(19) Potential in a 2D conducting (grounded) wedge with line-charge present.



(A) The line charge constitutes a surface charge density $\sigma(\varphi)$ on the cylinder $r=a$ which is singular at $\varphi=\alpha$. To accommodate this singularity, we shall write potentials valid for $0 \leq r < a$, and $a < r \leq \infty$, and later impose matching at $r=a$. After Jackson Eqs. (2.71) & (2.72), solutions are:

$$\left.\begin{array}{l} \phi_1(r<a,\varphi) = \sum_{n=1}^{\infty} A_n \left(\frac{r}{a}\right)^{\nu_n} \sin\nu_n\varphi \quad, \ \phi = 0 \text{ @ } r=0 \\ \phi_2(r>a,\varphi) = \sum_{n=1}^{\infty} B_n \left(\frac{a}{r}\right)^{\nu_n} \sin\nu_n\varphi \ , \ \phi\to 0 \text{ as } r\to\infty \end{array}\right\} \text{ w/ } \underline{\underline{\nu_n}} = \frac{n\pi}{\beta}. \quad (1)$$

ν_n is chosen to fit the B.C. that $\phi=0$ @ $\varphi=0$ & $\varphi=\beta$. The coefficients $B_n = A_n$ to make ϕ continuous at $r=a$. So the problem is solved if we find the A_n.

(B) The "surface charge" at (a,α) can be written as: $\sigma(\varphi) = \lambda\,\delta(\varphi-\alpha)$, where λ is the charge per unit length, and dimensions are kept right if the $\measuredangle$ lar delta fcn is defined so that: $a\int_0^\beta f(\varphi)\,\delta(\varphi-\alpha)\,d\varphi = f(\alpha)$. The singularity at (a,α) is then incorporated in the field at $r=a$ by...

$$\rightarrow \sigma(\varphi) = \lambda\,\delta(\varphi-\alpha) = -\frac{1}{4\pi}\left[\frac{\partial\phi_2}{\partial r} - \frac{\partial\phi_1}{\partial r}\right]\bigg|_{r=a} = \frac{1}{2\pi a}\sum_{n=1}^{\infty} A_n \nu_n \sin\nu_n\varphi. \quad (2)$$

project out A_m by $\int_0^\beta d\varphi \cdot \sin\nu_m\varphi \cdots$ $\Big\}$

$$\frac{\lambda}{a}\sin\nu_m\alpha = \frac{1}{2\pi a}\sum_{n=1}^{\infty} A_n\nu_n \underbrace{\int_0^\beta \sin\nu_m\varphi\,\sin\nu_n\varphi\,d\varphi}_{\text{this integral } = \frac{\beta}{2}\delta_{mn}}$$

So// $\underline{\underline{A_m = (4\lambda/m)\sin\nu_m\alpha}}$, $\quad (3)$

and//

$$\boxed{\phi(r,\varphi) = 4\lambda\sum_{n=1}^{\infty}\left[\frac{1}{n}\sin\left(\frac{n\pi\alpha}{\beta}\right)\right]\left\{\begin{array}{l}(r/a)^{n\pi/\beta}, \text{ if } r\leq a \\ (a/r)^{n\pi/\beta}, \text{ if } r \geq a\end{array}\right\}\sin\left(\frac{n\pi\varphi}{\beta}\right)}. \quad (4)$$

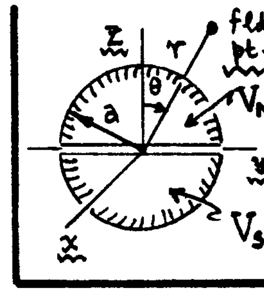
(C) Charge density on the plates "in close" is $\sigma(r<a) = \mp\frac{1}{4\pi r}\left(\frac{\partial\phi}{\partial\varphi}\right)\Big|_{\varphi=0,\beta}$, or...

$$\sigma(r<a) = \mp\frac{\lambda}{\beta a}\sum_{n=1}^{\infty}\left[\sin\left(\frac{n\pi\alpha}{\beta}\right)\right]\left[\left(\frac{r}{a}\right)^{\frac{n\pi}{\beta}-1}\right]\left\{\begin{array}{l}1, \text{ if } \varphi=0 \\ (-)^n, \text{ if } \varphi=\beta\end{array}\right\} \ \begin{array}{l}(-) \text{ for } \varphi=0 \text{ plate,} \\ (+) \ \text{"} \ \varphi=\beta \ \text{"}.\end{array} \quad (5)$$

Because of the presence of the line charge, the σ's are generally different (except $\alpha=\beta/2$).

ϕ 519 Solutions

S25



⑳ Exterior potential for hemispheres held at V_N & V_s, resp.

1. As we showed in class (Xerox notes p. II BV4), for a problem with spherical and azimuthal symmetry, the exterior solution is:

$$\left[\begin{array}{l} \phi(r \geq a, \theta) = \sum_{l=0}^{\infty} v_l \left(\frac{a}{r}\right)^{l+1} P_l(\cos\theta), \quad r \geq a, \\ \text{w/}\quad v_l = \left(\frac{2l+1}{2}\right) \int_{-1}^{+1} V(\theta) P_l(x)\,dx, \quad x = \cos\theta \begin{cases} 0 \leq \theta \leq \pi/2 \leftrightarrow +1 \geq x \geq 0, \\ \frac{\pi}{2} \leq \theta \leq \pi \leftrightarrow 0 \geq x \geq -1. \end{cases} \end{array}\right] \quad (1)$$

So all we have to do here for a solution is to evaluate the coefficients v_l.

2. Since $V(\theta) = V_N$ over $0 \leq \theta \leq \frac{\pi}{2} \leftrightarrow 1 \geq x \geq 0$ and $V(\theta) = V_s$ over $0 \geq x \geq -1$:

$$\rightarrow \left(\frac{2}{2l+1}\right) v_l = V_N \int_0^1 P_l(x)\,dx + V_s \int_{-1}^{0} P_l(x)\,dx. \quad (2)$$

In the second integral here, change variables $x \to (-)x$, and use the fact that $P_l(-x) = (-)^l P_l(x)$. Then get...

$$\rightarrow \left(\frac{2}{2l+1}\right) v_l = \left[V_N + (-)^l V_s\right] \int_0^1 P_l(x)\,dx, \quad (3)$$

and the problem amounts to evaluating $\int_0^1 P_l(x)\,dx$ for any $l = 0, 1, 2, 3, \ldots$

3. Evidently $\int_0^1 P_l(x)\,dx = 1$ for $l = 0$ (since $P_0(x) = 1$). For $l = 2n$ = even ($n = 1, 2, 3, \ldots$) use $P_l = \frac{1}{2l+1}(P_{l+1}' - P_{l-1}')$ [Jkⁿ Eq. (3.28)] to find: $\int_0^1 P_l(x)\,dx = \frac{1}{2l+1}\left[P_{l+1}(x) - P_{l-1}(x)\right]\Big|_{x=0}^{x=1} \equiv 0$, since all $P_\lambda(x=1) = 1$, and $P_\lambda(x=0) \equiv 0$ for λ odd. What's left is $l = 2n-1$ = odd ($n = 1, 2, 3, \ldots$), for which Jackson gives

$$\rightarrow \int_0^1 P_l(x)\,dx = (-)^{n-1} \frac{(2n-3)!!}{2^n n!}, \quad \text{for } l = 2n-1 \ (n = 1, 2, 3, \ldots), \quad (4)$$

... in his Eq. (3.26). The coefficients v_l in Eq. (1) above are therefore...

$$\left[\begin{array}{l} v_0 = \frac{1}{2}(V_N + V_s), \\ v_l = \frac{1}{2}(V_N - V_s) \cdot (4n-1)(-)^{n-1} \frac{(2n-3)!!}{2^n n!} \end{array}\right] \quad (5)$$

Insertion of these v_l into Eq. (1) gives the desired soln, which agrees with Jkⁿ Eq. (3.36) when $V_s = (-)V_N$.