Q → e Collisional Energy Transfer

Charged Particle Collisions: Energy Loss [Jkⁿ Ch. 13].

1) We consider the rate of loss of mechanical energy by a heavy, charged particle moving at high velocities through matter -- where it undergoes numerous "small" collisions with the ambient electrons and nuclei. The subject is important to:

A. Cosmic-ray ϕ: can identify high-E particles by tracks left in emulsions;

B. High-energy ϕ: specify collision products by bubble-chamber tracks & Cerenkov radⁿ;

C. Solid-State ϕ / Bioϕ: analyse radiation damage in materials & organisms.

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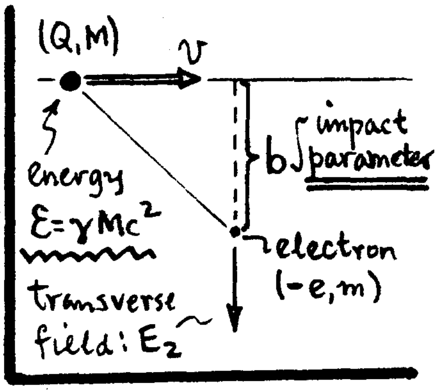
2) The calculation begins with the simple picture at right... The following assumptions are made:



a) Major energy transfer from Q is to the electrons; the Q→e collisions cause negligible deviations in Q's path.

b) Collisions of Q with nuclei ⇒ path deviations, but -- on average -- are rare, and cause ~ negligible energy loss.

c) Q's energy loss is mechanical (transfers K.E. to electrons); radiation is ignored.

d) Q's velocity v >> orbital velocity of the e's; the e's end up w/ nonrelativistic energy.

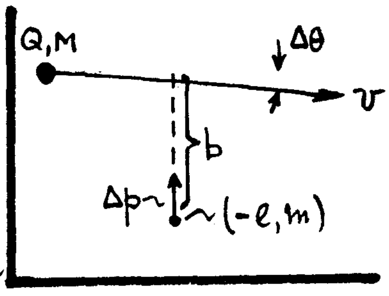
Now calculate the transverse momentum (impulse) delivered to e by passing Q...

$$\rightarrow \Delta p = \int_{-\infty}^{\infty} eE_2(t)\,dt\,, \quad \text{w/}\ E_2(t) = \gamma Qb/(b^2+\gamma^2v^2t^2)^{\frac{3}{2}}$$ ∫ Jkⁿ Eq. (11.152); t = time in e's frame

$$\text{or/}\ \Delta p = \frac{eQb}{v^3}\int_{-\infty}^{\infty} d\tau/[(b/v)^2+\tau^2]^{\frac{3}{2}} = 2Qe/vb. \quad (1)$$ ∫ v (of Q) is assumed unchanged by collision.

So/ Energy transferred to electron: $\boxed{\Delta E(b) = \frac{(\Delta p)^2}{2m} = \left(\frac{2Q^2e^2}{mv^2}\right)\frac{1}{b^2}}$ ∫ Jkⁿ Eq. (13.2) (2)

REMARKS on Q → e collision.



1. ∡ deflection of M: $\Delta\theta \simeq \frac{\Delta p}{p} \simeq \frac{2Qe}{pvb}$, w/ $p = \gamma Mv$;

$$\text{or/}\ \left[\Delta\theta \simeq \frac{2}{\beta^2}\left(\frac{Qe/b}{\mathcal{E}}\right) \sim 2\left(\frac{\text{P.E. of Q at b}}{\text{total energy of M}}\right) \ll 1. \quad (3)\right.$$ ∫ ∡ lar deflection ~ negligible

Remarks on the Q→e collision. Lower & upper limits on b.

REMARKS on Q→e collision (cont'd)

2. Q→e energy transfer } $\Delta E(b) = \left(\frac{2Q^2e^2}{mv^2}\right)\frac{1}{b^2}$ ∫ depends on Q & v, but not on M; impact parameter $b \to \left(\begin{smallmatrix}\text{small}\\ \text{large}\end{smallmatrix}\right) \Rightarrow \Delta E(b) \to \left(\begin{smallmatrix}\text{large}\\ \text{small}\end{smallmatrix}\right)$.

3. $b \to 0 \Rightarrow \Delta E(b) \to \infty$, nonsense. For head-on collisions ($b\to 0$), must fix up b...

LOWER LIMIT

Maxm energy transfer for Q-e collision : $\Delta E_{max} \simeq 2\gamma^2 mv^2$ (for $M \gg m$), ← Jk[n] Eq. (13.5).

So $\Delta E(b) \leqslant \Delta E_{max} \Rightarrow \underline{b \geqslant b_0 = Qe/\gamma m v^2}$. ★ (4)

For $b < b_0$, we must modify $\Delta E(b)$ of Eq. (2).

4. A modification to $\Delta E(b)$, which respects $\Delta E(b) \leqslant \Delta E_{max}$ as $b\to 0$ is :

$$\boxed{\Delta E(b) = \frac{2Q^2e^2}{mv^2}\Big/(b^2+b_0^2)}, \quad b_0 = Qe/\gamma m v^2. \tag{5}$$

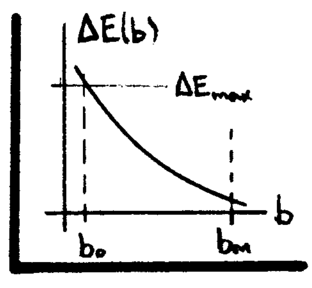
∫ Jk[n] Eq. (13.7)

5. What about an upper limit on b? There is one, by following argument.

U. LIMIT

The assumption that $v(\text{of } Q) \gg v(\text{of } e\text{'s})$ essentially treats the e's as being free. But in fact they are bound. Now when $b\to$ large, the Q-e collision time $\Delta t \sim b/\gamma v$ can exceed the electron orbital period, so the e's will orbit their nuclei as though bound (certainly not free). For an "effective" collision, impose :

$$\rightarrow \Delta t\left(\begin{smallmatrix}\text{collision}\\ \text{time}\end{smallmatrix}\right) \sim b/\gamma v < \frac{1}{\omega}\left(\begin{smallmatrix}\text{e-orbital}\\ \text{period}\end{smallmatrix}\right) \Rightarrow b < b_m \simeq \frac{\gamma v}{\omega}. \tag{6}$$



6. By comparing Eqs. (4) for b_0 (lower limit) and (6) for b_m (upper limit), we see easily that $b_0 \ll b_m$ (this is true so long as $\beta(\text{of } Q) > \alpha \simeq 1/137$). And, with these limits in mind, we expect that the loss

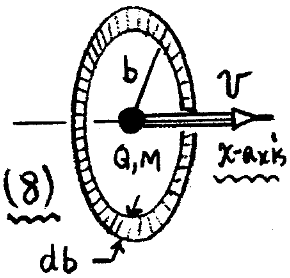
$$\rightarrow \Delta E(b) \simeq (2Q^2e^2/mv^2)\frac{1}{b^2}, \text{ holds over } b_0 \lesssim b \lesssim b_m. \tag{7}$$

★ During collision, avg. electron velocity is $v_e \sim \Delta p/2m$, and e moves $d = v_e \Delta t$, w/ collision time in e-frame $\Delta t \sim b/\gamma v$. Then: $d \sim (\Delta p/2m)\Delta t \sim Qe/\gamma m v^2 = b_0$.

3) Now $\Delta E(b)$ of Eq. (7) is good (?) for the collision of Q with a single electron. Look for an integrated result for Q colliding w/ many electrons...

[Solid: N atoms/vol., Z electrons/atom

$\Rightarrow$ # electrons in vol. of length δx, w/ impact parameters in b to $b+db$ $\Big\}$ $\delta n = NZ \cdot 2\pi b\, db\, \delta x$. (8)



So Q's energy loss to the δn e's : $\delta \mathcal{E} = [\Delta E(b)]\,\delta n$, or...

$$\rightarrow \frac{\delta E}{\delta x} = 2\pi NZ[\Delta E(b)]\,b\,db = 2\pi NZ\left[\left(\frac{2Q^2e^2}{mv^2}\right)\frac{1}{b^2}\right] b\,db \quad \text{valid over } b_0 \leq b \leq b_m. \tag{9}$$

Integrate $(\delta E/\delta x\, db)$ over b to get Q's total energy loss/unit length...

Jk$^{\underline{n}}$ Eq. (13.13)

$$\boxed{\frac{dE}{dx} = \int_{b=b_0}^{b=b_m} (\delta E/\delta x\, db)\, db = 4\pi NZ\left(\frac{Q^2e^2}{mv^2}\right)\ln B\,,\quad B = \frac{b_m}{b_0} \simeq \frac{\gamma^2 m v^3}{\omega Q e} \;*} \tag{10}$$

This result for dE/dx, which measures the rate at which the material can Stop Q by absorbing its K.E., is known as Bohr's Stopping Power Formula. Much of the rest of Jackson's Chap. 13 concerns various ~small corrections to this formula -- particularly adjustments to the impact parameter limits b_0 & b_m.

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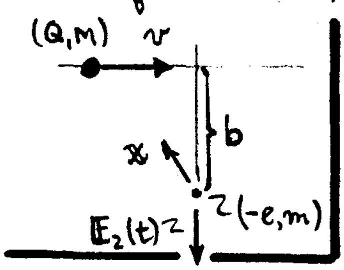
4) The first adjustment to $\left(\frac{dE}{dx}\right)_{\text{BOHR}}$ concerns the accuracy of the upper limit $b_m \sim \gamma v/\omega$. In Sec. 13.2, he does the "distant passage" problem (as suggested in Eq. (6) above) more carefully... with about the same result for b_m. To wit:



(Q,M) passes by at v and generates (mainly) a transverse field $E_2(t)$... but now electron is explicitly bound in a damped SHO...

$$\rightarrow \ddot{\mathbf{x}} + \Gamma\dot{\mathbf{x}} + \omega_0^2\mathbf{x} = -\frac{e}{m}\mathbb{E}(t). \tag{11}$$

$\mathbb{E}(t)$ can now include both $\mathbb{E}_2$ (transverse) & $\mathbb{E}_1$ (longitudinal). Details appear in Jk$^{\underline{n}}$ Eqs. (13.15)-(13.33). The principal results are...

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* Worth noting: m = electron mass, ω = electron orbit freq., w/ γ & $v \leftrightarrow M$'s motion.

Verify that upper limit $b_m \simeq \gamma v/\omega_0$ is reasonable.

RESULTS "Distant passage": (Q,M) collision w/ bound $(-e,m)$.

1. For an e bound in a (nonrelativistic) SHO as in Eq. (11), the energy transfer to the electron from any electric field impulse $\mathbb{E}(t)$ is...

Jk^n Eq. (13.26)

$$\boxed{\Delta E = (e^2/2m)\left|\int_{-\infty}^{\infty} \mathbb{E}(t)\, e^{i\omega_0 t}\, dt\right|^2}, \quad \omega_0 = e\text{'s SHO frequency}. \tag{12}$$

This is a general result, and holds even when $\mathbb{E}(t)$ is a (photon) radiation field. [†]

2. In (12), we separately evaluate ΔE for each component of $\mathbb{E}$(passage), viz.

$$\left.\begin{array}{l} \perp\ \mathbb{E}(t) = \mathbb{E}_2(t),\ \text{transverse, w/ } E_2(t) = \gamma Q b/(b^2+\gamma^2 v^2 t^2)^{3/2}; \\ \text{and } \parallel\ \mathbb{E}(t) = \mathbb{E}_1(t),\ \text{long}^{\text{d}}\text{nal, w/ } E_1(t) = -\gamma Q v t/(b^2+\gamma^2 v^2 t^2)^{3/2}. \end{array}\right\} \tag{13}$$

Since $\mathbb{E}_1 \perp \mathbb{E}_2$, the integrals in (12) add as the sum of squares, with result:

Jk^n Eq. (..31)

$$\boxed{\Delta E(b) = \left[\left(\frac{2Q^2e^2}{mv^2}\right)\frac{1}{b^2}\right]\left\{\xi^2 K_1^2(\xi) + \frac{1}{\gamma^2}\xi^2 K_0^2(\xi)\right\}}, \text{ w/ } \underline{\underline{\xi}} = \left(\frac{b}{\gamma v}\right)\omega_0; \tag{14}$$

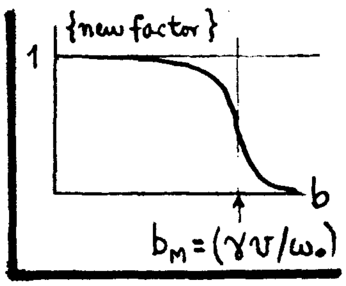
previous result [Eq. (2)]; from transverse field $\mathbb{E}_2$; from long$^{\text{d}}$nal field $\mathbb{E}_1$; collision time Δt of Eq. (6)

w/ modified Bessel fcn[¶]:

$$\left.\begin{array}{l} K_\nu(z) \simeq \sqrt{\dfrac{\pi}{2z}}\, e^{-z}\left[1 + (4\nu^2-1)\dfrac{1}{z} + \dots\right], \text{ as } z\to\infty; \\ \text{\&/ } K_\nu(z) \simeq \dfrac{1}{2}\Gamma(\nu)\left[\dfrac{2}{z}\right]^\nu, \text{ as } z\to 0\ (\nu\neq 0);\quad K_0(z) \simeq \ln\left[\dfrac{2}{z}\right] - 0.5772\dots,\ z\to 0. \end{array}\right\} \tag{15}$$

3. Eq. (14) $\Rightarrow$ $\Delta E(b)$ = [previous result] $\times$ {new factor}, where the { } accurately for the electron binding. The correction factor behaves asymptotically as...

$$\{\text{new factor}\} \sim \begin{cases} 1 - \mathcal{O}(\xi^2), & \text{as } \xi = \omega_0 \Delta t \to 0; \\ (1+\gamma^{-2})\frac{\pi}{2}\,\xi\, e^{-2\xi}, & \text{as } \xi \to \infty. \end{cases} \tag{16} \Rightarrow$$



The previous estimate of a cutoff in $\Delta E(b)$ @ $b \simeq b_M = \frac{\gamma v}{\omega_0}$ (i.e. $\xi \simeq 1$) in Eq. (6) was pretty good; there is little more at $b > b_m$.

¶ See, e.g., NBS Handbook (Abramovitz & Stegun), Chap. 9, Sec. 9.6.

† These fields were calculated in Jk^n Sec. (11.10). See Jk^n Eq. (11.152), p. 554.

Modification of Bohr's Formula due to electron binding.

5) We can use the exact result in Eq. (14) [for the energy transfer $\Delta E(b)$ from (Q, M) to a bound electron $(-e, m)$] over the impact parameter range...

$$\rightarrow \underline{b_0 = Qe/\gamma m v^2} \leq b \to \infty \quad , \quad ^{or}\!/\!/ \quad \xi_0 = \frac{Qe\omega_0}{\gamma^2 m v^3} \leq \xi \to \infty . \qquad (17)$$

The lower limit b_0 is chosen to be consistent with the max. transfer ΔE_{max} in Eq. (4). We will adjust b_0 below, but for the moment we have the transfer:

$$\rightarrow \Delta E(\xi) = \frac{2Q^2e^2}{mv^2}\left(\frac{\omega_0}{\gamma v}\right)^2 \left\{ K_1^2(\xi) + \frac{1}{\gamma^2} K_0^2(\xi) \right\} , \quad ^{w}\!/\!/ \quad \xi = (\omega_0/\gamma v)\, b . \qquad (18)$$

Now we can consider Q's energy loss/unit distance when colliding with a collection of electrons at different bound frequencies $\omega_0 \to$ set of ω_k's. As follows...

$$\left[\begin{array}{l} N \text{ atoms/vol.}, \; Z \text{ electrons per atom @ bound frequencies } \{\omega_k\}; \\ \text{the } e\text{'s have "oscillator strengths" } f_k \int \begin{array}{l} f_k \text{ measures relative contribution from} \\ k^{th} \text{ electron, and } \sum_{k=1}^{Z} f_k = Z, \text{ for norm.} \end{array} \end{array}\right. \qquad (19)$$

In analogy to Eq. (10), form...

$$\rightarrow \frac{dE}{dx} = N \sum_k f_k \int_{b_0}^{\infty} \Delta E(\xi) \cdot 2\pi b\, db$$

$$= 4\pi N \left(\frac{Q^2e^2}{mv^2}\right) \sum_k \int_{\xi_k}^{\infty} \left\{ K_1^2(\xi) + \frac{1}{\gamma^2} K_0^2(\xi) \right\} \xi\, d\xi , \quad \xi_k = \frac{Qe\omega_k}{\gamma^2 m v^3} . \qquad (20)$$

The integration can be done, with the result a modified Bohr formula...

Jk$^{\underline{n}}$ Eq. (13.36)

$$\boxed{\frac{dE}{dx} = 4\pi N Z \left(\frac{Q^2e^2}{mv^2}\right) \left[\ln B_c - \tfrac{1}{2}(v^2/c^2) \right]} \qquad (21)$$

$$^{w}\!/\!/ \quad \underline{B_c = 1.123\, \gamma^2 m v^3 / \langle\omega\rangle\, Qe} \quad , \quad \& \quad \ln\langle\omega\rangle = \frac{1}{Z}\sum_k f_k \ln \omega_k .$$

Compare with Eq. (10). This result is similar to the previous Bohr formula... but now we have an explicit $\Theta(v/c)^2$ correction, and also a modification to the log argument: $B = \gamma^2 m v^3/\omega\, Qe \to B_c = 1.123\, (\omega/\langle\omega\rangle)\, B$. This is Bohr's work.

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6) The weak point remaining in $\left(\frac{dE}{dx}\right)$ of Eq. (21) is the question of close encounters: $\underline{b \to b_0 = Qe/\gamma m v^2}$, classically. Here, there can be QM corrections, when the