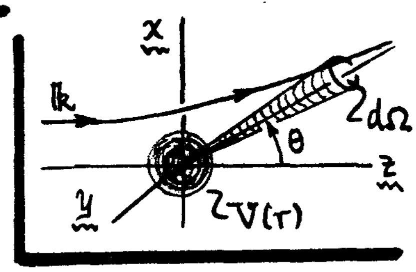
Total cross-section via partial waves. Optical Theorem.



This gives the differential scattering cross-section. The total scattering cross-section $\sigma(k)$ is obtained per...

$$\sigma(k) = \int_{4\pi} \left(\frac{d\sigma}{d\Omega}\right) d\Omega = 2\pi \int_0^\pi f_k^*(\theta) f_k(\theta) \sin\theta \, d\theta$$

$$= \frac{2\pi}{k^2} \sum_{l,\lambda=0}^{\infty} (2l+1)(2\lambda+1) e^{i(\delta_\lambda - \delta_l)} \sin\delta_\lambda \sin\delta_l \underbrace{\int_{-1}^{+1} P_l(\mu) P_\lambda(\mu) \, d\mu}_{=\left(\frac{2}{2l+1}\right)\delta_{l\lambda}},$$

or//

$$\boxed{\sigma(k) = \frac{4\pi}{k^2} \sum_{l=0}^{\infty} (2l+1) \sin^2\delta_l(k)}. \quad (18)$$

$\sigma(k)$ appears as a series of terms arranged in order of ascending values of ∡ momentum $l = 0, 1, 2, \ldots$, and can be written as...

$$\sigma(k) = \sum_{l=0}^{\infty} \sigma_l(k) \quad \text{w/} \quad \sigma_l(k) = \frac{4\pi}{k^2}(2l+1)\sin^2\delta_l(k) \quad (19)$$

called "l^{th} partial wave scattering cross-section"

$\sigma_l(k)$ represents the scattering by $V(r)$ of that part of the incident plane-wave which is (and remains) in ∡ momentum state l. Note that $\sigma_l(k)$ is weighted by the factor $(2l+1)$, which is the statistical weight of the permitted m states for that l $(-l \leq m \leq +l)$.

There is another way of getting at the total cross-section $\sigma(k)$. Evaluate the scattering amplitude $f_k(\theta)$ of Eq. (16) in the forward direction, i.e. $\theta = 0$. Then, since $P_l(\cos\theta)|_{\theta=0} = 1$, for all l, we can write...

$$f_k(0) = \frac{1}{k} \sum_{l=0}^{\infty} (2l+1)[e^{i\delta_l} \sin\delta_l], \text{ and: } f_k^*(0) = \frac{1}{k} \sum_{l=0}^{\infty} (2l+1)[e^{-i\delta_l} \sin\delta_l],$$

So//

$$\underbrace{f_k(0) - f_k^*(0)}_{2i\,\mathrm{Im}[f_k(0)]} = \frac{1}{k} \sum_{l=0}^{\infty} (2l+1) \underbrace{[e^{i\delta_l} - e^{-i\delta_l}]}_{2i \sin\delta_l} \sin\delta_l$$

or//

$$\mathrm{Im}[f_k(0)] = \frac{1}{k} \sum_{l=0}^{\infty} (2l+1) \sin^2\delta_l(k) \Rightarrow \boxed{\sigma(k) = \frac{4\pi}{k} \mathrm{Im}[f_k(0)]}. \quad (20)$$

This last result is called the "Optical Theorem" -- it holds very generally.

General features of the "phase shifts" $\delta_\ell(k)$.

4) For the scattering problem, the hypothesized "phase shifts" $\delta_\ell(k)$ neatly specify the scattering amplitude [$f_k(\theta)$ of Eq. (16)], the differential scattering cross-section [$\frac{d\sigma}{d\Omega}$ of Eq. (17)], and the total cross-section [$\sigma(k)$ of Eq. (18)]. It remains to show how to calculate the $\delta_\ell(k)$ from a given potential $V(r)$. Before that, however, we note a few general features of the $\delta_\ell(k)$.

1. The wave# k (for $r \to \infty$, either before or after the scattering) is a motion parameter, with $k = \sqrt{2mE/\hbar^2}$ for a free particle of kinetic energy E; the particle momentum is $p = \hbar k$. Both E & p are conserved in an elastic collision. Then, we expect:

$$\delta_\ell(k) \to 0, \text{ when } k \to 0, \tag{21}$$

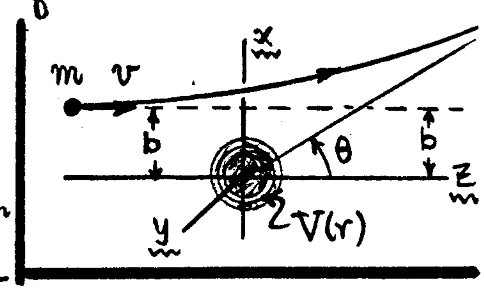
Since a stationary particle cannot be scattered by a "collision" with $V(r)$.

2. From Eq. (9), the $\delta_\ell(k)$ were introduced as a "distortion" in the radial wavefcns $v_{k\ell}$, vis-a-vis the free particle wavefcns, when $V(r) \neq 0$. From this, we expect:

$$\delta_\ell(k) \equiv 0, \text{ if } V(r) \equiv 0, \tag{22}$$

a remark made below Eq. (6). NOTE: Eq. (22) does not mean that $\delta_\ell(k) \to 0$ whenever $V(r) \to 0$ [as at $r \to \infty$]. If $V(r)$ is finite anywhere, it will generate $\delta_\ell(k) \neq 0$.

3. Finally, we can more directly interpret the ∡ momentum quantum # ℓ which occurs throughout this analysis (and in $\delta_\ell(k)$ itself). Classically, when a point particle of mass m and (initial) velocity v scatters from a central potential $V(r)$, both linear and ∡ momentum are conserved, and a convenient parameter describing the scattering encounter is the "impact parameter"



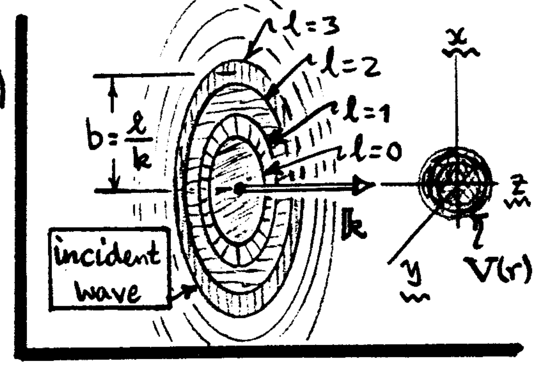
→ IMPACT PARAMETER: $$b = \frac{\text{∡ momentum}}{\text{linear momentum}} = \frac{mvb}{mv}. \tag{23}$$

In the QM case, the ∡ momentum is quantized: $mvb = \ell\hbar$, and so b becomes:

→ $$b = \ell\hbar/mv = \ell/(p/\hbar) = \ell/k, \text{ for QM scattering.} \tag{24}$$

Further general features of the phase shifts $\delta_\ell(k)$.

Eq. (24) suggests that for the QM case, for given k (i.e. given incident energy), those portions of the incident wavefcn at high ℓ-values are relatively far away from the scattering center, i.e. $b \propto \ell \to$ large. Then, since in our calculation we have assumed our incident wave is a planewave of ∞ transverse extent (all possible values $0 \leq b \to \infty$), the QM scattering will occur at all possible values of ℓ; the series e.g. for $f_k(\theta)$ in Eq. (16) really has an ∞ # terms.



4. We can also say that since $\delta_\ell(k)$ specifies scattering for that portion of the incident wave at ∡ mom^{m} ℓ, then it also describes scattering from that part of the wave @ minimum distance $b \sim \ell/k$ from the scattering center $V(r)$. The higher ℓ-states are scattered from positions farther away from $V(r)$. Now, if $V(r)$ is well-localized, so that $V(r)$ is negligible beyond some $r \sim a$, then ∡ momentum states with $\ell > ka$ will never participate in the scattering; for them, $\delta_\ell(k)=0$; they remain forever free. For potentials $V(r)$ that fall off less rapidly, we expect:

$$\delta_\ell(k) \to 0, \text{ when } \ell \to \infty. \tag{25}$$

5. In any case, $\delta_\ell(k)$ is called the "ℓ^{th} partial wave phase shift". The term "partial wave" refers to the (~fanciful) sketch above -- the incident wave is partitioned into an ∞ number of partial waves, each one specified by its ∡ mom^{m} $\ell\hbar$. One speaks of S-wave, P-wave, D-wave etc. scattering in referring to the calculation or measurement of $\delta_\ell(k)$ for $\ell=0$, $\ell=1$, $\ell=2$, etc.

Now we will actually calculate the $\delta_\ell(k)$ approximately -- in general, and for a specific case ("hard core" scattering). We will see that the features in Eqs. (21), (22) & (25) are borne out (Born out?) in detail.

General Formula for the Phase Shifts $\delta_\ell(k)$.

5) The phase shifts $\delta_\ell(k)$ characterize the distortions in the radial wavefcn $v_{k\ell}(r)$ caused by a scattering from $V(r)$, per Eq.(9); the δ_ℓ are <u>fixed</u> by $V(r)$. To establish the relation $\delta_\ell(k) \leftrightarrow V(r)$ directly, do the following:

1. Exact radial eqtn [w/ radial fcn $R(r) = \frac{1}{r} v(r)$] is, per Eq.(7), p. PW 3...

EQ. ① $$\left[\frac{d^2}{dr^2} + k^2 - \frac{\ell(\ell+1)}{r^2}\right] v_{k\ell}(r) = \left[\frac{2m}{\hbar^2} V(r)\right] v_{k\ell}(r), \quad \text{w/ } v_{k\ell}(0) = 0. \tag{26}$$

Radial eqtn for a free particle is, per Eq.(5), p. free 2...

EQ. ② $$\left[\frac{d^2}{dr^2} + k^2 - \frac{\ell(\ell+1)}{r^2}\right] u_{k\ell}(r) = 0, \quad \text{w/ } u_{k\ell}(0) = 0. \tag{27}$$

Multiply ① on the left by $u_{k\ell}$, ② on the left by $v_{k\ell}$, and <u>subtract</u> (à la Green):

$$\frac{d}{dr}\left[u_{k\ell}\frac{d}{dr}v_{k\ell} - v_{k\ell}\frac{d}{dr}u_{k\ell}\right] = \frac{2m}{\hbar^2} V(r)\, u_{k\ell}(r)\, v_{k\ell}(r) \tag{28}$$

... integrate via $\int_0^r dr$, imposing fact that $u_{k\ell}$ & $v_{k\ell}$ both $\equiv 0$ @ $r=0$...

$$u_{k\ell}(r)\frac{d}{dr}v_{k\ell}(r) - v_{k\ell}(r)\frac{d}{dr}u_{k\ell}(r) = \frac{2m}{\hbar^2}\int_0^r V(x)\, u_{k\ell}(x)\, v_{k\ell}(x)\, dx. \tag{29}$$

This eqtn is exact, and the free-particle fcns $u_{k\ell} \propto kr\, j_\ell(kr)$ are known.

2. In (29), let $r \to$ large, so we can use the asymptotic forms:

$$\left[\begin{array}{l} u_{k\ell}(r) \propto kr\, j_\ell(kr) \to \sin\left(kr - \frac{\ell}{2}\pi\right), \text{ for } kr \gg 1; \\ v_{k\ell}(r) \to \sin\left(kr - \frac{\ell}{2}\pi + \delta_\ell(k)\right), \text{ for } kr \gg 1 \end{array}\right. \tag{30}$$

a <u>definition</u> of the $\delta_\ell(k)$ per Eq.(9), p. PW 3.

We are of course trying to find the phase shifts $\delta_\ell(k)$. From (30), we can form the LHS of (29) in the asymptotic region. Then (29) yields

$$-k \sin\delta_\ell(k) = \frac{2m}{\hbar^2}\int_0^r V(x)\left[kx\, j_\ell(kx)\right] v_{k\ell}(x)\, dx, \quad \text{for } kr \gg 1, \tag{31}$$

where we have put $u_{k\ell}(x) = kx\, j_\ell(kx)$ in the integral RHS. (next page)