Lorentz Force Law. Invariance of charge q.

The Covariance of ElectroDynamics [ref. Jkⁿ Sec. 11.9].

With the 4-vector and Lorentz transfⁿ formalism in hand, it is relatively easy to show the form-invariance -- or "covariance" -- of Maxwell's Eqtns, and thus all of electrodynamics, under the Lorentz transfⁿ. This means that any two observers in inertial frames -- no matter how fast they may be moving relative to one another -- will write down <u>exactly</u> the same laws of E & M.*

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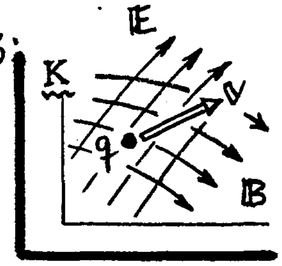
1) We work first on the Lorentz force law [per Jkⁿ Sec. 11.9], viz.



$$\rightarrow \frac{d\mathbf{p}}{dt} = q\left(\mathbf{E} + \frac{\mathbf{v}}{c}\times\mathbf{B}\right), \text{ for charge } q \text{ @ } \mathbf{v} \text{ in ref. frame } K. \quad (1)$$

This is a 3-vector force law; we want a 4-vector law.

<u>FIRST</u>; <u>we assert that charge q is a relativistic (Lorentz) invariant</u>. Evidence:

$$\left[\!\left[\begin{array}{l}\text{electron in } n^{th} \text{ orbit} \\ \text{about nucleus of at.\# } Z\end{array}\right\} \; \beta = \frac{v}{c} \simeq \frac{Z\alpha}{n}, \; \text{w/ } \alpha = \frac{e^2}{\hbar c} = \frac{1}{137}. \quad (2)$$

This β is exact for a 1e atom, and is ~ OK for low n in high Z atoms. Calculate;

$$\left[\!\left[\begin{array}{l}\text{for hydrogen: } \begin{array}{l}Z=1\\ n=1\end{array}\Big\} \; \beta_H = \frac{1}{137} = 0.0073, \text{ and: } \gamma_H = 1/\sqrt{1-\beta_H^2} = 1 + \underline{27\,\text{ppm}} \\ \text{for cesium: } \begin{array}{l}Z=55\\ n=1\end{array}\Big\} \; \beta_{Cs} = \frac{55}{137} = 0.4015, \text{ and: } \gamma_{Cs} = 1/\sqrt{1-\beta_{Cs}^2} = 1 + \underline{91{,}900\,\text{ppm}}\end{array}\right. \quad (3)$$

If the charge e transformed as $e \rightarrow \gamma e$, we would expect charge imbalances of order $\Delta e/e = \Delta\gamma \sim 9.2\%$ to develop between the e-p couplings in H & Cs. They <u>don't</u>... both atoms are electrically neutral within $|\Delta e/e| \sim 10^{-20}$. And so to that accuracy, q is a <u>Lorentz scalar</u>, just as is the particle mass m.

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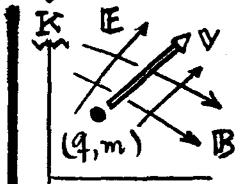
* Curiously, when the "ancients" (Lorentz, Fitzgerald, Poincaré) tried to explain the null results of the Michelson-Morley experiment, they discovered that Maxwell's Eqtns were Lorentz covariant (~10 yrs before Einstein's SRT).

4-vector form of Lorentz force law.

SECOND: put the particle (q,m) proper time $d\tau = dt/\gamma_v$ into Lorentz law, so:

$$\rightarrow \frac{d\mathbb{p}}{d\tau} = \frac{q}{c}\left[(\gamma_v c)\mathbb{E} + (\gamma_v \mathbb{v})\times\mathbb{B}\right], \quad (4)$$

now referred to q's rest frame.



$$\gamma_v = \frac{1}{\sqrt{1-(v/c)^2}}$$

$$d\tau = dt/\gamma_v$$

... recall 4-velocity: $\tilde{v} = \gamma_v(c, \mathbb{v}) = (u_0, \mathbb{u})$ $\begin{cases} u_0 = \gamma_v c, \\ \mathbb{u} = \gamma_v \mathbb{v} \end{cases}$ ...

so// $\underline{\underline{\dfrac{d\mathbb{p}}{d\tau} = \dfrac{q}{c}(u_0\mathbb{E} + \mathbb{u}\times\mathbb{B})}}$ NOTE: $\tilde{p} = m(u_0, \mathbb{u})$ is the relativistic 4-momentum of m. $\quad (5)$

The LHS of Eq. (5) is by now the spacelike part of a 4-vector, namely the Minkowski force $\tilde{K} = m\,\frac{d\tilde{u}}{d\tau}$ [recall prob<u>m</u> 64]. This 4-vector force is...

$$\rightarrow \tilde{K} = \frac{d\tilde{p}}{d\tau} = \left(K_0, \frac{d\mathbb{p}}{d\tau}\right), \quad \text{w/ } K_0 = dp_0/d\tau = \frac{1}{c}\frac{d}{d\tau}\underbrace{(\gamma_v mc^2)}_{\text{total energy } \mathcal{E}}$$

$$\text{so// } K_0 = \frac{\gamma_v}{c}\frac{d\mathcal{E}}{dt} = \frac{\gamma_v}{c}\overset{\text{as seen in K.}}{\mathbb{F}}\cdot\mathbb{v} = \frac{\gamma_v}{c}\,q\left(\mathbb{E} + \cancelto{0\ldots}{\frac{\mathbb{v}}{c}\times\mathbb{B}}\right)\cdot\mathbb{v},$$

i.e.// $\underline{K_0 = \dfrac{q}{c}\,\mathbb{E}\cdot\mathbb{u}}$, w/ $\mathbb{u} = \gamma_v\mathbb{v}$. $\quad (6)$

The relation $K_0 = \frac{dp_0}{d\tau} = \frac{q}{c}\mathbb{E}\cdot\mathbb{u}$ is the relativistic version of the work-energy theorem for the EM field. The 4-vector form of the Lorentz force law can now be written as...

$$\boxed{\frac{d\tilde{p}}{d\tau} = \frac{q}{c}\left(\mathbb{u}\cdot\mathbb{E},\ u_0\mathbb{E} + \mathbb{u}\times\mathbb{B}\right)} \quad u_0 = \gamma_v c,\ \mathbb{u} = \gamma_v\mathbb{v}, \text{ and: } \gamma_v = 1/\sqrt{1-(v/c)^2}. \quad (7)$$

The LHS of Eq. (7) is manifestly a 4-vector. With q as an invariant, it must be true that the ( ) on the RHS is also a 4-vector. Then, the Lorentz transf<u>n</u> properties of the ( ) on the RHS can be used to establish how the $\mathbb{E}$ & $\mathbb{B}$ fields must transform. We could do this, but won't.

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2) Instead of unravelling the $\mathbb{E}$ & $\mathbb{B}$ transforms from the Lorentz law, Eq. (7), we shall find those transforms by looking at how the potentials ϕ & $\mathbb{A}$ behave (recall: $\mathbb{E} = -\nabla\phi - \frac{1}{c}(\partial\mathbb{A}/\partial t)$, $\mathbb{B} = \nabla\times\mathbb{A}$). This alternative is pursued in order to show how the potentials themselves form a 4-vector $\underline{\underline{\tilde{A} = (\phi, \mathbb{A})}}$,

4-vector character of $\tilde{J} = (c\rho, \mathbb{J})$ and $\tilde{A} = (\phi, \mathbb{A})$.

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and also how the sources ρ & $\mathbb{J}$ = charge & current density form another 4-vector $\underline{\tilde{J} = (c\rho, \mathbb{J})}$. In fact, that $\tilde{J}$ is a 4-vector is demanded by charge conservation -- a universal requirement -- as we can see from...

CONTINUITY EQUATION } $\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbb{J} = 0$, or/ $\frac{\partial}{\partial x^0}(c\rho) + \frac{\partial}{\partial x^k} J^k = 0$, ← sum over $k = 1, 2, 3$

or/ $\boxed{\partial_\alpha J^\alpha = 0}$ ⌠ $\partial_\alpha = (\partial/\partial x^0, \nabla)$ ← covariant del [Jk$^{\underline{n}}$ Eq. (11.76)], $J^\alpha = (c\rho, \mathbb{J})$ ← 4-current source [Jk$^{\underline{n}}$ Eq. (11.128)]. (8)

Since ∂_α is a 4-vector, the only way that $\partial_\alpha J^\alpha = 0$ in all Lorentz frames (i.e. that the 0, signifying charge conservation, always appears RHS), is for J^α to be a 4-vector. Then $\partial_\alpha J^\alpha$ is a 4-vector scalar product, the same in all Lorentz frames, and $\partial_\alpha J^\alpha = 0$ is a Lorentz-invariant statement.

As for the 4-vector character of $\tilde{A} = (\phi, \mathbb{A})$, recall the defining eqtns...

Lorentz gauge : $\frac{1}{c}\frac{\partial \phi}{\partial t} + \nabla \cdot \mathbb{A} = 0 \Rightarrow \boxed{\partial_\alpha A^\alpha = 0}$, w/ $A^\alpha = (\phi, \mathbb{A})$. (9)

The Lorentz gauge choice is available to all inertial observers if A^α is a 4-vector. A^α __is__ a 4-vector because it obeys the wave eqtn...

$$\underbrace{\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)}_{\partial_\alpha \partial^\alpha = \Box,\ \text{D'Alembertian [Jk}^{\underline{n}}\text{(11.78)]}}(\phi, \mathbb{A}) = \frac{4\pi}{c}(c\rho, \mathbb{J}), \quad \text{or/} \quad \boxed{\Box \tilde{A} = (4\pi/c)\tilde{J}}$$ 4-VECTOR WAVE EQTN (10)

Now $\Box = \partial_\alpha \partial^\alpha$ is a Lorentz scalar, so the wave eqtn reads: $\left(\begin{smallmatrix}\text{Lorentz}\\ \text{scalar}\end{smallmatrix}\right)\tilde{A} = \left(\begin{smallmatrix}\text{Lorentz}\\ \text{scalar}\end{smallmatrix}\right)\tilde{J}$, so $\tilde{A}$ has the same Lorentz transf$^{\underline{n}}$ character as $\tilde{J}$ -- $\tilde{A}$ is a 4-vector if $\tilde{J}$ is.

φ519 Notes, p. ME 18 w/ Jackson, Sec. 6.4

3) We shall now write the fields $\mathbb{E}$ & $\mathbb{B}$ in terms of the 4-potential $A^\alpha = (\phi, \mathbb{A})$. In so doing, we find it convenient to use the __contravariant__ del operator $\partial^\alpha = (\partial/\partial x_0, -\nabla)$, [Jk$^{\underline{n}}$ Eq. (11.76)]. With $A^0 = \phi$, and with the coordi-

Field components via the "field tensor": $F^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha$.

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nate assignment $(x^0, x^1, x^2, x^3) = (ct, x, y, z)$, we calculate...

contravariant del

$$\left[\begin{aligned} E_x &= -\frac{\partial}{\partial x}\phi - \frac{1}{c}\frac{\partial}{\partial t}A_x = -\frac{\partial}{\partial x^1}A^0 - \frac{\partial}{\partial x^0}A^1 = -(\partial^0 A^1 - \partial^1 A^0), \\ B_x &= \frac{\partial}{\partial y}A_z - \frac{\partial}{\partial z}A_y = \frac{\partial}{\partial x^2}A^3 - \frac{\partial}{\partial x^3}A^2 = -(\partial^2 A^3 - \partial^3 A^2). \end{aligned}\right. \qquad (11)$$

Evidently, the $\mathbb{E}$ & $\mathbb{B}$ fields are components of a 4×4 second rank contravariant <u>tensor</u> $F^{\alpha\beta}$. $F^{\alpha\beta}$ is called the <u>EM field tensor</u>, and is defined:

$$\boxed{F^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha}, \text{ w/ } F^{\alpha\alpha} \equiv 0,\ F^{01} = -E_x,\ F^{23} = -B_x, \text{ etc.}$$

i.e.

① $$F^{\alpha\beta} = \left[\begin{array}{c|ccc} 0 & -E_x & -E_y & -E_z \\ \hline E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{array}\right] \qquad (12)$$

$F^{\alpha\beta}$ is 4×4, 2nd rank, and totally antisymmetric. The $\mathbb{E}$ fld enters along the "boost" part of $F^{\alpha\beta}$ (1st row & column); $\mathbb{B}$ is in the "rotation" part.

Several other versions of $F^{\alpha\beta}$ are used...

② <u>covariant</u> $\underset{\sim}{F}$: $F_{\alpha\beta} = g_{\alpha\lambda} F^{\lambda\epsilon} g_{\epsilon\beta}$,

i.e.

$$F_{\alpha\beta} = \left[\begin{array}{c|ccc} 0 & E_x & E_y & E_z \\ \hline -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{array}\right] \qquad (13)$$

NOTE: $F_{\alpha\beta} = F^{\alpha\beta}\big|_{\mathbb{E}\to(-)\mathbb{E}}$ ✓ space inversion

③ <u>dual</u> of $\underset{\sim}{F}$: $\mathcal{F}^{\alpha\beta} = \frac{1}{2}\epsilon^{\alpha\beta\gamma\epsilon} F_{\gamma\epsilon}$, ★

i.e.

$$\mathcal{F}^{\alpha\beta} = \left[\begin{array}{c|ccc} 0 & -B_x & -B_y & -B_z \\ \hline B_x & 0 & E_z & -E_y \\ B_y & -E_z & 0 & E_x \\ B_z & E_y & -E_x & 0 \end{array}\right] \qquad (14)$$

NOTE: $\mathcal{F}^{\alpha\beta} = F^{\alpha\beta}\big|_{\mathbb{E}\to\mathbb{B},\ \mathbb{B}\to(-)\mathbb{E}}$ ✓ duality transf$^{\underline{n}}$

So far, these various "field tensors" are just bookkeeping procedures, to keep track of how to calculate field components from the 4-potential A^α (e.g. in Eq. (12): $F^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha = -E_x$ when $\alpha,\beta = 0,1$, etc.). But now we will show: <u>(1)</u> the Maxwell Eqtns can be written in a very elegant form in terms of $F^{\alpha\beta}$, <u>(2)</u> $F^{\alpha\beta}$ is actually a <u>Lorentz 4-tensor</u> (per Eq. (39a), p. SRT 21).

★ $\epsilon^{\alpha\beta\gamma\epsilon}$ = Levi-Civita pseudo-tensor of 4 indices. It is totally antisymmetric, with: $\epsilon^{\alpha\beta\gamma\epsilon} = \pm1$, when $\alpha\beta\gamma\epsilon = {}^{\text{even}}_{\text{odd}}$ permutation of 0123; $\epsilon^{\alpha\beta\gamma\epsilon} \equiv 0$, when two indices same.

Maxwell's source-dependent eqtns in form: $\partial_\alpha F^{\alpha\beta} = (4\pi/c) J^\beta$.

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4) Recall the operation of tensor "divergence"...*

$$\rightarrow (\mathrm{div}\, \underset{\sim}{F})^\beta = \frac{\partial}{\partial x^\alpha} F^{\alpha\beta} = \partial_\alpha F^{\alpha\beta} = \partial_0 F^{0\beta} + \partial_1 F^{1\beta} + \partial_2 F^{2\beta} + \partial_3 F^{3\beta}. \quad (15)$$

This produces a 4-vector out of a (2nd rank) 4-tensor. Apply this operation to the field tensor $F^{\alpha\beta}$ of Eq. (12) above...

$$\begin{cases} \partial_\alpha F^{\alpha 0} = \partial_0 F^{00} + \frac{\partial}{\partial x} E_x + \frac{\partial}{\partial y} E_y + \frac{\partial}{\partial z} E_z = \nabla \cdot \mathbb{E} = \frac{4\pi}{c}(c\rho); \\ \partial_\alpha F^{\alpha 1} = \left(-\frac{1}{c}\frac{\partial \mathbb{E}}{\partial t} + \nabla \times \mathbb{B}\right)_x = \frac{4\pi}{c} J_x, \quad \text{etc.} \end{cases} \quad (16)$$

Evidently the inhomogeneous (source-dept.) Maxwell Eqtns can be written

$$\text{(A)} \ \boxed{\partial_\alpha F^{\alpha\beta} = \frac{4\pi}{c} J^\beta} \ \Big/ \ \text{w/ } F^{\alpha\beta} \text{ in Eq. (12),} \ J^\beta = (c\rho, \mathbb{J}) \ \Big| \Rightarrow \ \begin{aligned} &\nabla \cdot \mathbb{E} = 4\pi\rho, \\ &\nabla \times \mathbb{B} - \frac{1}{c}\frac{\partial \mathbb{E}}{\partial t} = \frac{4\pi}{c}\mathbb{J}. \end{aligned} \quad (17)$$

This compact operation on $F^{\alpha\beta}$ thus produces <u>Gauss' Law</u> and the <u>Ampere-Maxwell Law</u>. We note further the transformation properties of $F^{\alpha\beta}$ according to (17): we have a known 4-vector J^β on the RHS, and a known 4-vector ∂_α on the LHS... what's in between, $F^{\alpha\beta}$, can only transform as a (2nd rank) Lorentz 4-tensor. $F^{\alpha\beta}$'s transform is:

$$F^{\alpha\beta} \to F'^{\alpha\beta} = \left(\frac{\partial x'^\alpha}{\partial x^\gamma}\right)\left(\frac{\partial' x^\beta}{\partial x^\epsilon}\right) F^{\gamma\epsilon} = \Lambda^\alpha_\gamma F^{\gamma\epsilon} \Lambda^\beta_\epsilon \quad \Big/ \ \underset{\sim}{\Lambda} \text{ is Lorentz transform: } K \to K' \quad (18)$$

(per def^{n} in Eq. (39a), p. SRT 21 of class notes). <u>So the EM field tensor $F^{\alpha\beta}$ is, in fact, a contravariant Lorentz 4-tensor</u>. This means that the Maxwell Eqtns in (A) above, viz. $\partial_\alpha F^{\alpha\beta} = (4\pi/c) J^\beta$, are "Lorentz covariant"... any two inertial observers, exchanging information via Lorentz transforms, will write down <u>exactly</u> the same equations.

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* ϕ 519 Notes, p. ME 18, w/ Jk^{n} Sec. 6.8, Eq. (6.119) [on Poynting's Thm].

The soure-independent eqtns: $\partial_\alpha \mathcal{F}^{\alpha\beta}=0$. Manifest covariance.

5) Maxwell's homogeneous (source-indpt) eqtns, $\nabla\cdot\mathbb{B}=0$ & $\nabla\times\mathbb{E}=-\frac{1}{c}\frac{\partial\mathbb{B}}{\partial t}$, are actually ensured by the way we have chosen the potentials $(\phi,\mathbb{A})$. $\mathbb{E}=-\nabla\phi-\frac{1}{c}(\partial\mathbb{A}/\partial t)$ and $\mathbb{B}=\nabla\times\mathbb{A}$ immediately give $\nabla\cdot\mathbb{B}=0$ and $\nabla\times\mathbb{E}=-\frac{1}{c}(\partial\mathbb{B}/\partial t)$. So the very <u>form</u> of $F^{\alpha\beta}=\partial^\alpha A^\beta-\partial^\beta A^\alpha$, with $A^\alpha=(\phi,\mathbb{A})$ implicitly $\Rightarrow$ the source-free eqtns.

But also we can write the source-free eqtns in terms of the dual $\mathcal{F}^{\alpha\beta}$:

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Ⓑ $\boxed{\partial_\alpha \mathcal{F}^{\alpha\beta}=0}$ $\int^{\text{w/}}\mathcal{F}^{\alpha\beta}$ in Eq. (14) $\Rightarrow$ $\nabla\cdot\mathbb{B}=0$, $\nabla\times\mathbb{E}+\frac{1}{c}\frac{\partial\mathbb{B}}{\partial t}=0$. (19)

Recall $F^{\alpha\beta}\to\mathcal{F}^{\alpha\beta}$ under $\mathbb{E}\to\mathbb{B}$, $\mathbb{B}\to(-)\mathbb{E}$; then clearly (19) implies the same operations as (17) on the fields on the LHS, and the RHS $\equiv 0$ because there is no magnetic monopole 4-current.† While $\mathcal{F}^{\alpha\beta}$ clearly exposes the nature of the source-free Maxwell Eqtns, they can also be recovered from $F^{\alpha\beta}$ directly, but in a somewhat clumsy form.¶

Because $\mathcal{F}^{\alpha\beta}=\frac{1}{2}\epsilon^{\alpha\beta\gamma\epsilon}F_{\gamma\epsilon}$, and $F_{\gamma\epsilon}$ is a Lorentz 4-tensor, then <u>$\mathcal{F}^{\alpha\beta}$ is also a Lorentz 4-tensor</u>, and Ⓑ above, viz. $\partial_\alpha\mathcal{F}^{\alpha\beta}=0$, is a Lorentz covariant statement... all inertial observers agree on these Maxwell Eqs.

We now have a manifestly covariant statement of Maxwell's Eqtns, by Ⓐ & Ⓑ above: $\boxed{\partial_\alpha F^{\alpha\beta}=(4\pi/c)J^\beta,\ \partial_\alpha\mathcal{F}^{\alpha\beta}=0}$. These elegant equations imply:

1. Maxwell's E&M can be cast in 4-vector, 4-tensor form, obeying the Lorentz transform and thus SRT. So E&M <u>is</u> universal, and $c\equiv$ cnst everywhere.
2. The Lorentz transform $F^{\alpha\beta}\to F'^{\alpha\beta}$ of Eq. (18) will tell us how the <u>fields</u> transform. $\mathbb{E}$ & $\mathbb{B}$ do <u>not</u> transform as vectors, but as elements of a 4-tensor.

¶ Jk^D Eq. (11.143): $\partial^\alpha F^{\beta\gamma}+\partial^\beta F^{\gamma\alpha}+\partial^\gamma F^{\alpha\beta}=0$ $\left\{\begin{array}{l}(\alpha,\beta,\gamma)=\text{any three of }(0,1,2,3),\\ \text{and }\alpha\neq\beta\neq\gamma.\end{array}\right.$

† If there <u>were</u> a MM 4-current $\tilde{J}_{MM}$, (19) would read: $\partial_\alpha\mathcal{F}^{\alpha\beta}=\frac{4\pi}{c}J^\beta_{MM}$.

Lorentz transf$^{\underline{n}}$ for the $\mathbb{E}$ & $\mathbb{B}$ field components.

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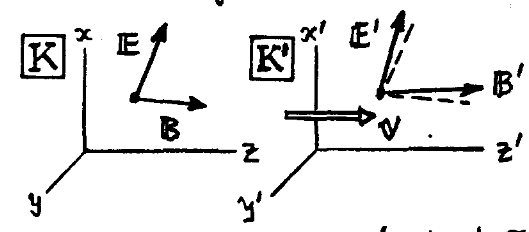
6) In Eq. (18), we have that the EM field tensor $F^{\alpha\beta}$ transforms as a 4-tensor:

$$\left[\begin{array}{l} F^{\alpha\beta}(\text{in } K) \rightarrow F'^{\alpha\beta}(\text{in } K') = \Lambda^{\alpha}_{\gamma} F^{\gamma\epsilon} \Lambda^{\beta}_{\epsilon} = (\underset{\sim}{\Lambda}\, \underset{\sim}{F}\, \underset{\sim}{\Lambda}_T)^{\alpha\beta}, \\ \text{w/ } \Lambda^{\kappa}_{\lambda} = (\partial x'^{\kappa}/\partial x^{\lambda}) \text{ is the Lorentz transf}^{\underline{n}} \text{ for } K \rightarrow K'. \end{array}\right\} \quad (20)$$

By doing the actual matrix mult$^{\underline{n}}$, one finds [Jk$^{\underline{n}}$ Eq. (11.148)]...

$$\boxed{\begin{array}{l|l} E'_z = E_z & B'_z = B_z \\ E'_x = \gamma(E_x - \beta B_y) & B'_x = \gamma(B_x + \beta E_y) \\ E'_y = \gamma(E_y + \beta B_x) & B'_y = \gamma(B_y - \beta E_x) \end{array}} \quad (21)$$



$(\mathbb{E},\mathbb{B})$ in K seen as $(\mathbb{E}',\mathbb{B}')$ in K'.
$\beta = v/c$, $\gamma = 1/\sqrt{1-\beta^2}$, as usual.

for a velocity boost along the z-axis. These laws of transf$^{\underline{n}}$ for $(\mathbb{E},\mathbb{B})$ ensure that Maxwell's Eqtns (e.g. $\nabla\cdot\mathbb{E} = 4\pi\rho$) are manifestly covariant, i.e. assume exactly the same form in K & K'. Note that the fields are completely <u>mixed together</u> by the relative motion of K & K'... in that sense, the fields do <u>not</u> show unique & independent existences. <u>One should not speak of $\mathbb{E}$ & $\mathbb{B}$ separately, but rather just the "EM field" embodied in the tensor $F^{\alpha\beta}$.</u>

In case the K-K' relative velocity $\boldsymbol{\beta} = \mathbf{v}/c$ is not $\parallel$ some common K-K' axis (but the K & K' axes are still $\parallel$ each other), the $(\mathbb{E},\mathbb{B}) \rightarrow (\mathbb{E}',\mathbb{B}')$ transform in Eq. (21) is generalized to [Jk$^{\underline{n}}$ Eq. (11.149)]...

$$\boxed{\begin{array}{l} \mathbb{E}' = \gamma(\mathbb{E} + \boldsymbol{\beta}\times\mathbb{B}) - \dfrac{\gamma^2}{\gamma+1}(\boldsymbol{\beta}\cdot\mathbb{E})\boldsymbol{\beta}, \\ \mathbb{B}' = \gamma(\mathbb{B} - \boldsymbol{\beta}\times\mathbb{E}) - \dfrac{\gamma^2}{\gamma+1}(\boldsymbol{\beta}\cdot\mathbb{B})\boldsymbol{\beta}. \end{array}} \quad (22)$$

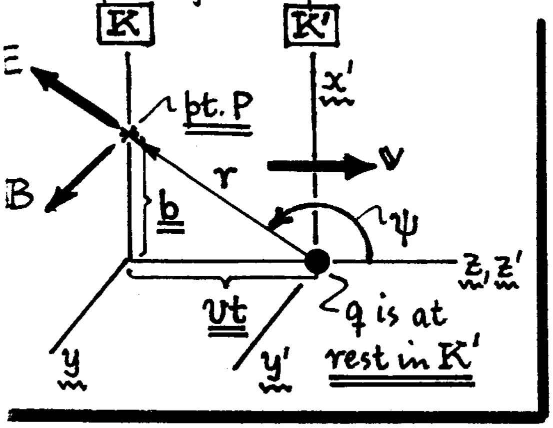
<u>NOTE</u>: the first terms RHS look like Lorentz force laws as $\beta\rightarrow 0$, e.g. $\mathbb{E}' \simeq (\mathbb{E} + \frac{\mathbf{v}}{c}\times\mathbb{B})$, as $\beta\rightarrow 0$.

Note also: $(\mathbb{E}=0,\mathbb{B}) \rightarrow \left(\mathbb{E}' = \gamma\boldsymbol{\beta}\times\mathbb{B},\ \mathbb{B}' = \gamma\left[\mathbb{B} - \frac{\gamma\beta}{\gamma+1}(\boldsymbol{\beta}\cdot\mathbb{B})\right]\right)$. $\mathbb{E}'$ is called a "motional electric field", generated by <u>movement</u> thru $\mathbb{B}$. Also $(\mathbb{E},0)\rightarrow(\mathbb{E}',\mathbb{B}'\neq 0)$.

Eqs. (11)–(22) Completes the program for establishing the covariance of Maxwell Eqs.

Explicit $\mathbb{E}$ & $\mathbb{B}$ transformation.

7) As an example of field transforms, consider the fields for a single q, moving uniformly. We follow the treatment in Jackson's Sec. 11.10, pp. 552-556.



q, fixed at origin of K', passes K origin at velocity $\mathbb{v}$, when $t'=t=0$. Impact parameter (closest approach) $= b$.

$$\left[\begin{array}{l}\text{fields at pt. P}\\ \text{(in K' cds)}\end{array}\right\} \quad E_z' = -\frac{qvt'}{r'^3},\ E_x' = \frac{qb}{r'^3},\ E_y' \equiv 0;$$

and $\mathbb{B}' \equiv 0$, w/ $r' = \sqrt{b^2+(vt')^2}$ = dist. $q \leftrightarrow P$. (23)

Can transform K' → K time via: $t' = \gamma\left[t - \frac{v}{c^2}z\right] = \gamma t$, since z-cd ≡ 0 for pt. P. Rewrite the above E_i' in terms of K-time t. Then transform $(\mathbb{E}', \mathbb{B}') \to (\mathbb{E}, \mathbb{B})$ by the Lorentz transformation prescribed for the fields. The result is...

$$\left[\begin{array}{l}\text{fields at pt. P}\\ \text{(in K cds)}\end{array}\right\} \quad \boxed{\begin{array}{l} E_z = (-)\gamma q v t/r^3,\ E_x = \gamma q b/r^3,\ E_y \equiv 0;\\ B_z \equiv 0,\ B_x = 0,\ B_y = \beta E_x,\ \text{w/}\ r = \sqrt{b^2+(\gamma v t)^2}. \end{array}} \tag{24}$$

The radial position $\mathbb{r}'(\text{in K'}) = (b, 0, -vt') \to \mathbb{r}(\text{in K}) = (b, 0, -\gamma v t)$.

REMARKS on fields of a moving charge.

(this is the Biot-Savart law)

1. We now have a motional magnetic field: $\mathbb{B} = \gamma\frac{q}{c}\left(\frac{vb}{r^3}\right)\hat{y} = \gamma\frac{q}{c}\left(\frac{\mathbb{v}\times\mathbb{r}}{r^3}\right)$, in frame K. $\mathbb{B} \neq 0$ so long as $\beta \neq 0$, and becomes as large as $|\mathbb{E}|$ when $\beta \to 1$.

2. Write $\mathbb{E}$(in K) in terms of q's "present position": $\mathbb{R} = (b, 0, vt)$. Then...

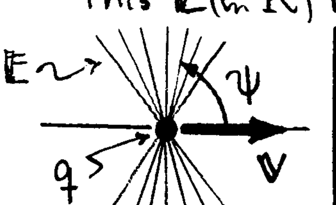
$$\mathbb{E} = \left(\frac{\gamma q}{r^3}\right)\mathbb{R} \int \begin{array}{l} \text{but}/\!/\ r^2 = b^2 + \gamma^2(vt)^2 = R^2\left[1+(\gamma^2-1)\left(\frac{vt}{R}\right)^2\right];\\ \text{let}/\!/\ \frac{vt}{R} = (-)\cos\psi,\ \gamma^2-1 = \gamma^2\beta^2 \Rightarrow r^2 = \gamma^2R^2(1-\beta^2\sin^2\psi); \end{array} \quad \Big\| \ \begin{array}{l}\psi = \measuredangle(\mathbb{v}, \mathbb{r})\\ \text{as above.}\end{array}$$

so//

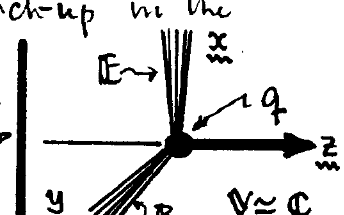
$$\boxed{\mathbb{E} = (q\mathbb{R}/R^3)\cdot\left[(1-\beta^2)/(1-\beta^2\sin^2\psi)^{3/2}\right]} \tag{25}$$

← equivalent to Jk$^{\text{n}}$ Eq. (11.154).

This $\mathbb{E}$(in K) is radial, per Coulomb law, but the field lines "bunch-up" in the



equatorial plane ($\psi \sim 90°$); $\mathbb{B}$ behaves similarly. In the limit $v \to c$, K sees only an $\mathbb{E}$-$\mathbb{B}$ comb$^{\text{n}}$ which is ~ transverse to $\mathbb{v}$. Kinda like a photon?



Extensions of the Covariant Formulation of E & M.

AFTERTHOUGHTS on the covariant formulation of E & M.

The 4-vector $\leftrightarrow$ 4-tensor formulation of E & M is an elegant, compact, and useful way to write down the principal equations of the theory, particularly w.r.t. to making certain that Lorentz covariance is respected -- this requirement is an absolute "must" for all meaningful E & M equations. The fields $\mathbb{E}$ & $\mathbb{B}$ now disappear from the theory and are replaced by the field tensor $F^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha$. Maxwells Eqtns are: $\partial_\alpha F^{\alpha\beta} = \frac{4\pi}{c} J^\beta$, and: $\partial_\alpha \mathcal{F}^{\alpha\beta} = 0$ [Eqs (17) & (19) above], the wave eqtn [Eq. (10) above] is: $(\partial_\alpha \partial^\alpha) A^\beta = (4\pi/c) J^\beta$, etc. Other covariant statements are:

1. Lorentz force law [Eq. (7) above] becomes Jkn Eq. (11.144):

$$\boxed{m\dot{u}^\alpha = \frac{q}{c} F^{\alpha\beta} u_\beta} \quad \Big/ \; u^\alpha = \gamma_v(c, \mathbb{V}) \text{ is the 4-velocity,} \quad \text{(26)}$$

$\cdot \Rightarrow$ differentiation $d/d\tau$ (propertime).

2. Poynting's Theorems [ϕ 519 Notes: pp. ME2-ME9], involving energy & momentum balances between particle (sources) & fields, can be done covariantly -- see Jkn Secs. 12.10 & 17.5. We will look at this later; there are some surprises [e.g. Poynting's original form of the field energy density $(E^2+B^2)/8\pi$ has to be corrected].

3. The covariant formulation of E & M can be extended, but with some difficulty, to material media. There $\mathbb{E} \rightarrow \mathbb{D} = \epsilon \mathbb{E}$, $\mathbb{B} \rightarrow \mathbb{H} = \frac{1}{\mu}\mathbb{B}$, and the field tensor: $F^{\alpha\beta}(\mathbb{E}, \mathbb{B}) \rightarrow G^{\alpha\beta}(\mathbb{D}, \mathbb{H})$. Maxwell's Eqtns. in a medium have the same general form, viz. $\partial_\alpha G^{\alpha\beta} = (4\pi/c) J^\beta$, $\partial_\alpha \mathcal{F}^{\alpha\beta} = 0$, but care is needed in how one defines polarization $\mathbb{P}$, magnetization $\mathbb{M}$, etc.

In any case, in much of what follows, we will be using covariant notation -- particularly in analysing the motion of relativistic particles.

ϕ520 : Maxwell Eqtns $\longleftrightarrow$ Covariance Summary.

$\boxed{\text{SUM}^{\underline{N}}\ \text{CON}^{\underline{V}}\text{N in effect}}$

1) Charge conservation : $\underline{\underline{\partial_\alpha J^\alpha = 0}}$, w/ $J^\alpha = (c\rho, \mathbb{J})$ = 4-vector source density.

2) Lorentz gauge : $\underline{\underline{\partial_\alpha A^\alpha = 0}}$, w/ $A^\alpha = (\phi, \mathbb{A})$ = 4-vector potential.

Wave equation : $\underline{\underline{(\partial_\alpha \partial^\alpha) A^\beta = (4\pi/c) J^\beta}}$, $\partial_\alpha \partial^\alpha = \Box \begin{cases} \text{D'Alembertian} \\ \text{(Lorentz invariant)} \end{cases}$

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3) Maxwell field tensor...

$$\underline{\underline{F^{\alpha\beta}}} = \partial^\alpha A^\beta - \partial^\beta A^\alpha = \left[\begin{array}{c|ccc} 0 & -E_1 & -E_2 & -E_3 \\ \hline E_1 & 0 & -B_3 & B_2 \\ E_2 & B_3 & 0 & -B_1 \\ E_3 & -B_2 & B_1 & 0 \end{array}\right], \text{ for } \begin{cases} \mathbb{E} = (E_1, E_2, E_3) \\ \mathbb{B} = (B_1, B_2, B_3) \end{cases}$$

$\underline{\text{[a Lorentz 4-tensor]}}$

... dual tensor...

$$\underline{\underline{\mathcal{F}^{\alpha\beta}}} = \begin{bmatrix} 0 & -B_1 & -B_2 & -B_3 \\ B_1 & 0 & E_3 & -E_2 \\ B_2 & -E_3 & 0 & E_1 \\ B_3 & E_2 & -E_1 & 0 \end{bmatrix} = F^{\alpha}\begin{pmatrix} \mathbb{E} \to +\mathbb{B} \\ \mathbb{B} \to -\mathbb{E} \end{pmatrix} \text{ duality transform}$$

$\underline{\text{[a Lorentz 4-tensor]}}$

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4) Maxwell Equations...

$$\underline{\underline{\partial_\alpha F^{\alpha\beta} = (4\pi/c) J^\beta_{elec.}}} \Rightarrow \begin{cases} \nabla \cdot \mathbb{E} = 4\pi\rho, & \text{(GAUSS' LAW)} \\ \nabla \times \mathbb{B} - \frac{1}{c}(\partial \mathbb{E}/\partial t) = \frac{4\pi}{c}\mathbb{J}; & \text{(AMPERE MAXWELL)} \end{cases}$$

$$\underline{\underline{\partial_\alpha \mathcal{F}^{\alpha\beta} = (4\pi/c) J^\beta_{magc}}} = 0 \Rightarrow \begin{cases} \nabla \cdot \mathbb{B} = 0, & \text{(DIRAC LAW)} \\ \nabla \times \mathbb{E} + \frac{1}{c}(\partial \mathbb{B}/\partial t) = 0 & \text{(FARADAY LAW)} \end{cases}$$

These laws are "manifestly covariant" ← (<u>Same</u> <u>form</u> in all inertial frames).

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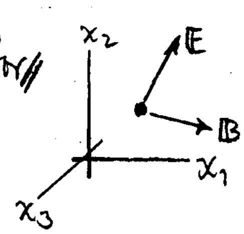
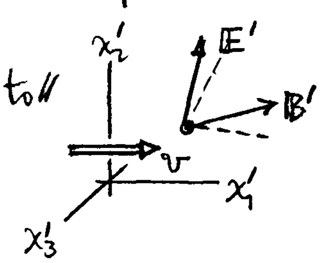
5) From Lorentz transf$^{\underline{n}}$ on field 4-tensor $F^{\alpha\beta}$, get field transf$^{\underline{n}}$s...

$E_1' = E_1$	$B_1' = B_1$
$E_2' = \gamma(E_2 - \beta B_3)$	$B_2' = \gamma(B_2 + \beta E_3)$
$E_3' = \gamma(E_3 + \beta B_2)$	$B_3' = \gamma(B_3 - \gamma E_2)$

for//



to//



for velocity boost @ $\beta = v/c$ (w/ $\gamma = 1/\sqrt{1-\beta^2}$) along the 1-axis. The components of $\mathbb{E}$ & $\mathbb{B}$ transform as components of a 2nd rank <u>tensor</u> (i.e. $F^{\alpha\beta}$).