QM Continuity Eqtn for Schrödinger Eqtn.

ASIDE A "continuity equation" for the QM probability $|\psi|^2$.

Notice than in analysing $\dot{\mathcal{P}}$ in Eqs (16), we have proven the identity...

$$\rightarrow \frac{\partial}{\partial t}|\psi|^2 = \left(\frac{i\hbar}{2m}\right)\nabla\cdot\left[\psi^*(\nabla\psi)-(\nabla\psi^*)\psi\right] \qquad \text{ref. Eqs. (16A) \& (16C) (under the integrals).} \tag{18}$$

This follows directly from Schrödinger's Eqtn. We *define* the quantities:

$$\text{PROBABILITY DENSITY: } \rho = \psi^*\psi, \text{ for } \psi = \psi(\mathbf{r},t);$$
$$\text{PROBABILITY CURRENT DENSITY: } \mathbf{J} = \frac{\hbar}{2im}\left[\psi^*(\nabla\psi)-(\nabla\psi^*)\psi\right]. \tag{19}$$

So// Eq. (18) $\Rightarrow$ $\boxed{\frac{\partial\rho}{\partial t}+\nabla\cdot\mathbf{J}=0}$ QM Continuity Eqtn, which guarantees probability conservation. (20)

This equation has exactly the same form, and virtually the same meaning, as the equation in EM that governs charge conservation, viz. [Jackson Eq. (5.2)]:

if// $\left.\begin{matrix}\rho = \text{charge density} \\ \mathbf{J} = \text{current density}\end{matrix}\right\}$ $\frac{\partial\rho}{\partial t}+\nabla\cdot\mathbf{J}=0$ EM Continuity Eqtn, which guarantees charge conservation. (21)

The QM Continuity Eqtn is the essential working ingredient for the global probability conservation shown in Eq. (16F). But it is also more... here, in Eq. (20), we have a microscopic balance between the density ρ and its current $\mathbf{J}$... ρ does not increase in any region of space without $\mathbf{J}$ flowing in to supply it.

To continue with the EM analogy, we know we can write: $\mathbf{J}_{EM} = \rho_{EM}\mathbf{v}$, for a flux of charge density ρ_{EM} flowing at velocity $\mathbf{v}$ through a surface. In the QM case:

$$\mathbf{J}_{QM} = \mathrm{Re}\left[\psi^*\left(\frac{\hbar}{im}\nabla\right)\psi\right], \text{ from Eq. (19) above.} \tag{22}$$

The density $\rho_{QM} = \psi^*\psi$ does appear on the RHS (as in $\mathbf{J}_{EM} = \rho_{EM}\mathbf{v}$), but it split by the *operator* $\mathbf{v}_{op} = (\hbar/im)\nabla$, which has the dimensions of a velocity. The companion momentum operator is: $\boxed{\mathbf{p}_{op} = m\mathbf{v}_{op} = (\hbar/i)\nabla}$. This harks back to Eq. (4), p. Sch 2, where we also tried $\mathbf{p}$ as an operator. More to come...