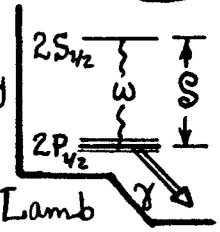
1992 ϕ507 Problems Assigned: 4/20. Due: 4/27/92. file $\mathcal{H}$13 (13)



(42) [20 pts]. The $2S_{1/2}$ level in hydrogen is metastable (lifetime $\tau_{2S} \sim \frac{1}{7}$ sec for decay $2S \to 1S$ by two photons). The nearby $2P_{1/2}$ level decays rapidly: the lifetime for $2P \to 1S + Ly\alpha$ (1216 Å) is $\tau_{2P} = 1.6\times10^{-9}$ sec. The levels are separated by the Lamb Shift $\mathcal{S}$ (in circular freq. $\mathcal{S} = 2\pi\times1058$ MHz) and can be coupled by an rf electric field via $V = [e\mathbf{r}\cdot\mathbf{E}(t)]\cos\omega t$, at freq. $\omega \simeq \mathcal{S}$. Since the next nearest level, $2P_{3/2}$, lies $\simeq 10^4$ MHz above $2S_{1/2}$, the $2S_{1/2}$ - $2P_{1/2}$ coupling is well-represented by a two-level problem, viz.

$$\boxed{i\dot{S} = \Omega^*(t)\, P e^{i\nu t}, \quad i\dot{P} = \Omega(t)\, S e^{-i\nu t} - \tfrac{1}{2} i\gamma P}$$

w/ $\nu = (\mathcal{S} - \omega)$, detuning frequency; $\gamma = 1/\tau_{2P}$, $2P\to1S$ decay rate.

$S(t)$ & $P(t)$ are the $2S_{1/2}$ & $2P_{1/2}$ amplitudes, and $\Omega(t) = \frac{1}{2\hbar}\langle\phi_{2P}|e\mathbf{r}\cdot\mathbf{E}(t)|\phi_{2S}\rangle$ is the envelope of the E-field pulse. The term in γ is added phenomenologically, so that -- when the coupling $\Omega \to 0$ -- $2P_{1/2}$ decays naturally, according to: $|P(t)|^2 = |P(0)|^2 e^{-\gamma t}$.

(A) A sample of $2S_{1/2}$ atoms experiences a weak rf pulse $\Omega = $ cnst, over $0 \leq t \leq T$, w/ $\tau_{2P} \ll T \ll \tau_{2S}$. Solve the above two-level problem to find the fraction $|S(t>T)|^2$ of $2S_{1/2}$ atoms remaining after the pulse. Sketch $|S(\text{after})|^2$ vs. ω. What is the <u>width</u> of this resonance?

(B) What fractional resolution in the linewidth [part (A)] is needed to measure $\mathcal{S}$ to 100 ppm?

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(43) [20 pts]. The <u>time-dependent</u> Schrödinger Eq. can be solved by Green's fcns. At $t<0$, start with a known stationary system: $\mathcal{H}u_n(\mathbf{r}) = \omega_n u_n(\mathbf{r})$, w/ $\mathcal{H} = -\frac{1}{2m}\nabla^2 + V(\mathbf{r})$ [units: $\hbar = 1$]. At $t \geq 0$, add coupling $W = W(\mathbf{r},t)$, so $\mathcal{H} \to \mathcal{H} + W$, and consider the time-dept Schrodinger Eq: $\underline{(\mathcal{H} - i\frac{\partial}{\partial t})\psi = -W(\mathbf{r},t)\psi}$. Now <u>define</u> K via: $\underline{(\mathcal{H} - i\frac{\partial}{\partial t})K = -i\,\delta(\mathbf{r}-\mathbf{r}_0)\,\delta(t-t_0)}$, for $t > t_0$, and $K \equiv 0$ for $t < t_0$. $K = K(\mathbf{r},t;\mathbf{r}_0,t_0)$ is the Green's fcn for the problem.

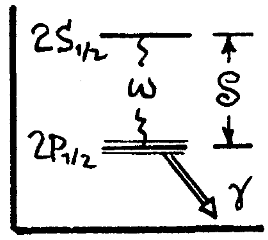
(A) Show that: $\psi(\mathbf{r},t) = \phi(\mathbf{r},t) - i\int_0^{t+} dt_0 \int_\infty d^3x_0\, K(\mathbf{r},t;\mathbf{r}_0,t_0)\, W(\mathbf{r}_0,t_0)\,\psi(\mathbf{r}_0,t_0)$, where $\phi(\mathbf{r},t) = \int_\infty d^3x_0\, K(\mathbf{r},t;\mathbf{r}_0,0)\,\psi(\mathbf{r}_0,0)$, and $t+ = \lim_{\epsilon\to0}(t+\epsilon)$.

(B) Verify that: $\underline{\underline{K(\mathbf{r},t;\mathbf{r}_0,t_0) = \theta(t-t_0)\sum_n u_n^*(\mathbf{r}_0)\, u_n(\mathbf{r})\, e^{-i\omega_n(t-t_0)}}}$, satisfies the equation which defines K. <u>NOTE</u>: the $\{u_n\}$ are assumed to be a complete set of eigenfcns.

(C) Specify the initial state of the system by: $\psi(\mathbf{r}_0,0) = \sum_k a_k u_k(\mathbf{r}_0)$, the $\{a_k\}$ = cnsts. With K of part (B), show that the first term in the solution for ψ in part (A) amounts to: $\underline{\phi(\mathbf{r},t) = \sum_n a_n u_n(\mathbf{r})\, e^{-i\omega_n t}}$. Clearly, $\phi(\mathbf{r},t)$ is the evolution of the <u>unperturbed</u> state $\psi(\mathbf{r},0)$.

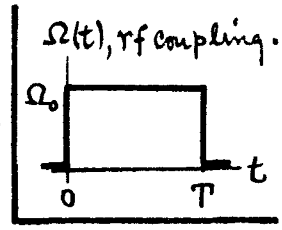
(D) Write down ψ of part (A) in the first Born Approxn. Discuss briefly how you would proceed to find ψ to terms higher order in W.

42 [20 pts]. $2S_{1/2}$ – $2P_{1/2}$ system: rf Lamb shift resonance for E-field pulse.



(A) 1. Except for the addition of the decay term in γ, this problem is the same as the two-level problem in prob. #39. Amplitude eqtns are:

$$\rightarrow i\dot{S} = \Omega^*(t)\,P\,e^{i\nu t},\quad i\dot{P} = \Omega(t)\,S\,e^{-i\nu t} - \tfrac{1}{2}i\gamma P, \qquad (1)$$



w/ $\underline{\nu} = S - \omega$ = detuning freq., and $\underline{\gamma} = 1/\tau_{2P}$ = $2P \rightarrow 1S$ decay rate. Initial conditions are: $S(t=0-)=1$, $P(t=0-)=0$, and the rf coupling is $\Omega(t) = \begin{cases} \Omega_0 = \text{cnst},\ 0 \leq t \leq T; \\ \text{zero, otherwise.}\end{cases}$ We separate Eqs (1) by taking $\frac{d}{dt}$ through the $\dot{S}$ eqtn, and using the $\dot{P}$ eqtn for substitutions. Then, when the coupling is on [i.e. $\Omega(t)=\Omega_0$, over $0 \leq t \leq T$], we find (ignoring transients @ $t=0$ & $t=T$)...

$$\underline{\underline{\ddot{S} - i\tilde{\nu}\dot{S} + |\Omega_0|^2 S = 0}},\ \text{w/}\ \underline{\underline{\tilde{\nu}}} = \nu + \tfrac{1}{2}i\gamma,\ \text{\&/ initial conditions:}\ \underline{S=1,\ \dot{S}=0 \ @\ t=0-}. \qquad (2)$$

The fact that $\tilde{\nu}$ is now a complex # changes the nature of the solution considerably. The fact that $\dot{S}(t=0-)=0$ follows from the first of Eqs.(1), with $P(t=0-)=0$.

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2. Solutions to (2) are $S(t) = e^{\alpha t}$, if α obeys the secular eqtn...

$$\rightarrow \alpha^2 - i\tilde{\nu}\alpha + |\Omega_0|^2 = 0 \Rightarrow \alpha_{1,2} = \frac{i\tilde{\nu}}{2}(1 \pm \tilde{Q}),\ \text{w/}\ \tilde{Q} = \left[1 + \left(\frac{2|\Omega_0|}{\tilde{\nu}}\right)^2\right]^{1/2}. \qquad (3)$$

The defn of $\tilde{Q}$ here parallels Q of prob. #39; except now $\tilde{Q}$ is complex. Then:

$$S(t) = Ae^{\alpha_1 t} + Be^{\alpha_2 t} \int \text{w/}\ \left.\begin{matrix} S(0) = A + B = 1, \\ \dot{S}(0) = \alpha_1 A + \alpha_2 B = 0 \end{matrix}\right\} \Rightarrow \begin{matrix} A = -\alpha_2/(\alpha_1 - \alpha_2), \\ B = \alpha_1/(\alpha_1 - \alpha_2). \end{matrix} \qquad (4)$$

$$\xrightarrow{\text{Soy}} S(t) = \frac{1}{\alpha_1 - \alpha_2}\left(\alpha_1 e^{\alpha_2 t} - \alpha_2 e^{\alpha_1 t}\right),\ \text{satisfies the initial conditions in (2).} \qquad (5)$$

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3. If the rf pulse amplitude Ω_0 is "weak", then we expand $\tilde{Q}$ of Eq.(3) as...

$$\left.\begin{matrix} \tilde{Q} \simeq 1 + 2(|\Omega_0|^2/\tilde{\nu}^2),\ \text{OK for:}\ (2|\Omega_0|)^2 \ll |\tilde{\nu}|^2 = \nu^2 + \tfrac{1}{4}\gamma^2; \\ \text{Soy}\ \alpha_1 = \frac{i\tilde{\nu}}{2}(1+Q) \simeq i\tilde{\nu}\left[1 + (|\Omega_0|^2/\tilde{\nu}^2)\right],\ \alpha_2 = \frac{i\tilde{\nu}}{2}(1-Q) \simeq -i|\Omega_0^2|/\tilde{\nu}. \end{matrix}\right\} \qquad (6)$$

The expansion is possible even at resonance ($\nu=0$) so long as $\underline{\underline{|\Omega_0| \ll \tfrac{1}{4}\gamma}}$. Clearly $|\alpha_1| \gg |\alpha_2|$, and so we write the metastable amplitude $S(t)$ in Eq.(5) as:

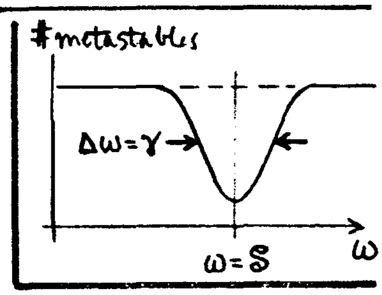
$$\rightarrow S(t) = \frac{\alpha_1 e^{\alpha_2 t}}{\alpha_1 - \alpha_2}\left[1 - \cancel{(\alpha_2/\alpha_1)}^{\,0}\, e^{(\alpha_1-\alpha_2)t}\right] \simeq \left(\frac{\alpha_1}{\alpha_1-\alpha_2}\right) e^{\alpha_2 t}. \tag{7}$$

The 2nd term in [ ] is negligible not just because $|\alpha_2/\alpha_1| \ll 1$, but also because (since $\alpha_1 - \alpha_2 \simeq i\tilde{\nu}$) it contains the factor $e^{i\tilde{\nu}t} = (e^{i\nu t})e^{-\frac{1}{2}\gamma t}$ which decays away for pulse durations $t \gg \tau_{2P}$, the 2P lifetime. Thus, at or near the end of the rf pulse $t \sim T$, the metastable amplitude is $\simeq$ last term in (7). Put in α_1 & α_2 of Eq. (6), and note the coefficient $\frac{\alpha_1}{\alpha_1-\alpha_2} \simeq 1 + (\alpha_2/\alpha_1) \simeq 1$. Then...

$$S(t) \simeq \exp[-i(|\Omega_0|^2/\tilde{\nu})t] = \exp\left[-\frac{|\Omega_0|^2 t}{\nu^2 + \frac{1}{4}\gamma^2}\left(i\nu + \frac{\gamma}{2}\right)\right]$$

So// $$\underline{\underline{|S(T)|^2 \simeq \exp\left\{-|\Omega_0|^2 \gamma T/(\nu^2 + \tfrac{1}{4}\gamma^2)\right\}}} \quad \int \text{for } |\Omega_0| \ll \tfrac{1}{4}\gamma, \text{ and at } T \gg \tau_{2P}. \tag{8}$$

The metastables decay in time (due to the Ω_0 coupling to the P-state), but the rate is very slow: $\Gamma = |\Omega_0|^2\gamma/(\nu^2 + \frac{1}{4}\gamma^2) \leqslant (4|\Omega_0|^2/\gamma^2)\gamma \ll \gamma$. If T is not too large, then enough metastable survive the rf pulse to be measurable.



4. With $\nu = (S - \omega)$, Eq. (8) predicts a resonant depletion of metables as the rf freq. $\omega \to$ Lamb shift S. If T is not too long, then Eq. (8) gives the depletion lineshape

$$\boxed{|S(T)|^2 \simeq 1 - \frac{|\Omega_0|^2 \gamma T}{(\omega - S)^2 + \frac{1}{4}\gamma^2}} \quad \int \text{good for... } |\Omega_0|^2 T \ll \tfrac{1}{4}\gamma. \tag{9}$$

(at any time after the pulse). The depletion curve is thus a Lorentzian, centered at $\omega = S$, and of width (FWHM) $\Delta\omega = \gamma$. This resonance line is broad, since

$$\rightarrow \gamma = 1/\tau_{2P} = 1/1.6\times10^{-9}\,\text{sec} \simeq \underline{\underline{2\pi \times 100\,\text{MHz}}}. \tag{10}$$

(B) 5. If $S = 2\pi \times 1058$ MHz is to be measured to 100 ppm, or $\Delta S \simeq 2\pi \times 0.1$ MHz, then we must locate the above line center to that accuracy. Hence we must work to a resolution: $\underline{\Delta\gamma/\gamma = 0.1\,\text{MHz}/100\,\text{MHz} = 0.1\%}$ of the linewidth. It can actually be done. See R.T. Robiscoe & B.L. Cosens, Phys. Rev. Letters 17, 69 (July 1966).

(43) [20 pts]. Solve time-dependent Schrödinger Equation via Green's functions.

(A) 1. Since K varies as $(t-t_0)$, then $\frac{\partial}{\partial t} = -\frac{\partial}{\partial t_0}$. Write the eqtn that defines ψ in terms of $(\mathbf{r}_0, t_0)$ variables, multiply the ψ eqtn on the left by K, the K eqtn on the left by K, and subtract...

$$\left.\begin{array}{l} K \times \left[\left(\mathcal{H}_0 - i\frac{\partial}{\partial t_0}\right)\psi(\mathbf{r}_0,t_0) = -W(\mathbf{r}_0,t_0)\psi(\mathbf{r}_0,t_0)\right] \\ \psi \times \left[\left(\mathcal{H} + i\frac{\partial}{\partial t_0}\right)K = -i\,\delta(\mathbf{r}-\mathbf{r}_0)\,\delta(t-t_0)\right] \end{array}\right\rangle \text{subtract}$$

$$\underbrace{(K\mathcal{H}_0\psi - \psi\mathcal{H}K)}_{1} - i\underbrace{\left(K\frac{\partial\psi}{\partial t_0} + \psi\frac{\partial K}{\partial t_0}\right)}_{2} = i\psi(\mathbf{r}_0,t_0)\,\delta(\mathbf{r}-\mathbf{r}_0)\,\delta(t-t_0) - K\,W(\mathbf{r}_0,t_0)\,\psi(\mathbf{r}_0,t_0). \quad (1)$$

2. Re term 1, the ∇^2 in $\mathcal{H}$ is equivalent to ∇_0^2 [since $\partial/\partial\mathbf{r} = (-)\partial/\partial\mathbf{r}_0$], and similarly, can replace $V(\mathbf{r})$ by $V(\mathbf{r}_0)$ [for the singularity at $\mathbf{r}=\mathbf{r}_0$]. So term 1 is: $(K\mathcal{H}_0\psi - \psi\mathcal{H}_0K) = \frac{1}{2m}(\psi\nabla_0^2K - K\nabla_0^2\psi) = \frac{1}{2m}\nabla_0\cdot(\psi\nabla_0K - K\nabla_0\psi)$.
Term 2 just $= \frac{\partial}{\partial t_0}(K\psi)$. Put these expressions in Eq.(1), and integrate through by $\int_0^{t+}dt_0\int_\infty d^3x_0$ (the range $0 \leq t_0 \leq t+$ is chosen to get all of $\delta(t-t_0)$). Then...

$$\rightarrow \frac{1}{2m}\int_0^{t+}dt_0 \underbrace{\int_\infty d^3x_0\,\nabla_0\cdot(\psi\nabla_0K - K\nabla_0\psi)}_{\Rightarrow \text{ Surface integral at } \infty\text{, which vanishes}} - i\int_\infty d^3x_0\int_0^{t+}dt_0\frac{\partial}{\partial t_0}(K\psi) = i\psi(\mathbf{r},t) - \int_0^{t+}dt_0\int_\infty d^3x_0\,KW\psi, \quad (2)$$

So

$$\rightarrow i\psi(\mathbf{r},t) = -i\int_\infty d^3x_0\,\overset{3}{(K\psi)}\Big|_{t_0=0}^{t_0=t+} + \int_0^{t+}dt_0\int_\infty d^3x_0\,KW\psi. \quad (3)$$

3. In term 3, $K\psi \equiv 0$ at the upper limit $t_0 = t+$, because $(t-t_0)<0$ there -- and by definition -- $K\equiv 0$ when $t<t_0$. The remaining integral is...

$$\rightarrow \int_\infty d^3x_0\,K(\mathbf{r},t;\mathbf{r}_0,0)\,\psi(\mathbf{r}_0,0) = \phi(\mathbf{r},t);\quad t>0; \quad (4)$$

and

$$\boxed{\psi(\mathbf{r},t) = \phi(\mathbf{r},t) - i\int_0^{t+}dt_0\int_\infty d^3x_0\,K(\mathbf{r},t;\mathbf{r}_0,t_0)\,W(\mathbf{r}_0,t_0)\,\psi(\mathbf{r}_0,t_0).} \quad (5)$$

As desired. The limit $t+ \rightarrow t$ can now taken, again since K vanishes @ $t_0 = t+$.

(B) 4. $\mathcal{H}$ operates only on $\mathbf{r}$-cds, and $\mathcal{H} u_n(\mathbf{r}) = \omega_n u_n(\mathbf{r})$ by def^n of the eigenfcns u_n. Then

$$\rightarrow \mathcal{H} K = \theta(t-t_0) \sum_n u_n^*(\mathbf{r}_0)\, u_n(\mathbf{r}) \cdot \omega_n\, e^{-i\omega_n(t-t_0)}. \quad (6)$$

$\partial K/\partial t$ has two terms. For the first, note $\frac{\partial}{\partial t}\theta(t-t_0) = \delta(t-t_0)$. Then...

$$\rightarrow i\frac{\partial K}{\partial t} = i\delta(t-t_0)\sum_n u_n^*(\mathbf{r}_0)\, u_n(\mathbf{r})\, e^{-i\omega_n(t-t_0)} + \underbrace{\theta(t-t_0)\sum_n u_n^*(\mathbf{r}_0)\, u_n(\mathbf{r}) \cdot \omega_n\, e^{-i\omega_n(t-t_0)}}_{\text{this} \equiv \mathcal{H}K,\ \text{per Eq. (6)}}, \quad (7)$$

$$\left[\left(\mathcal{H} - i\frac{\partial}{\partial t}\right) K = -i\,\delta(t-t_0)\sum_n u_n^*(\mathbf{r}_0)\, u_n(\mathbf{r})\, e^{-i\omega_n(t-t_0)}\right] \quad (8)$$

On the RHS of Eq. (8), the term contributes only at $t = t_0$, because of the δ-fcn. We then evaluate the exponential @ $t = t_0$, i.e. $e^{-i\omega_n(t-t_0)}\big|_{t=t_0} = 1$. Eq. (8) yields:

$$\left(\mathcal{H} - i\frac{\partial}{\partial t}\right) K = -i\,\delta(t-t_0)\sum_n u_n^*(\mathbf{r}_0)\, u_n(\mathbf{r}) = -i\,\delta(t-t_0)\,\delta(\mathbf{r}-\mathbf{r}_0). \quad (9)$$

The last step follows by closure on the complete set of eigenfcns $\{u_n(\mathbf{r})\}$. So we have shown that: $K(\mathbf{r},t;\mathbf{r}_0,t_0) = \theta(t-t_0)\sum_n u_n^*(\mathbf{r}_0)\, u_n(\mathbf{r})\, e^{-i\omega_n(t-t_0)}$ is the Green's fcn.

(C) 5. If the initial state $\psi(\mathbf{r}_0,0) = \sum_k a_k u_k(\mathbf{r}_0)$ is put in $\phi(\mathbf{r},t)$ of (4), together with K:

$$\phi(\mathbf{r},t) = \int_\infty d^3x_0 \left[\theta(t-0)\sum_n u_n^*(\mathbf{r}_0)\, u_n(\mathbf{r})\, e^{-i\omega_n(t-0)}\right]\left[\sum_k a_k u_k(\mathbf{r}_0)\right]$$

or

$$\rightarrow \phi(\mathbf{r},t) = \theta(t)\sum_{n,k} a_k\, u_n(\mathbf{r})\, e^{-i\omega_n t} \underbrace{\int_\infty d^3x_0\, u_n^*(\mathbf{r}_0)\, u_k(\mathbf{r}_0)}_{\delta_{nk},\ \text{by orthonormality}} = \theta(t)\sum_n a_n u_n(\mathbf{r})\, e^{-i\omega_n t}. \quad (10)$$

So, indeed, $\phi(\mathbf{r},t)$ is the evolution of $\psi(\mathbf{r},0)$ @ $t>0$ (nothing happens before then).

(D) 6. Let $\xi \leftrightarrow (\mathbf{r},t)$ stand for all 4 cds. The integral solution for ψ in Eq. (5) is -- symbolically: $\psi(\xi) = \phi(\xi) - i\int d\xi_0\, K(\xi,\xi_0)\, W(\xi_0)\, \psi(\xi_0)$. This is the same form as what we started with in the Born scattering analysis. The first Born approxn is to replace ψ in the integral by the unperturbed (incoming) wavefcn ϕ. Thus, to 1st order in W:

$$\boxed{\psi^{(1)}(\xi) = \phi(\xi) - i\int d\xi_0\, K(\xi,\xi_0)\, W(\xi_0)\, \phi(\xi_0)} \quad (11)$$

The higher order iterations follow the method in class notes, pp. ScT 15-16.