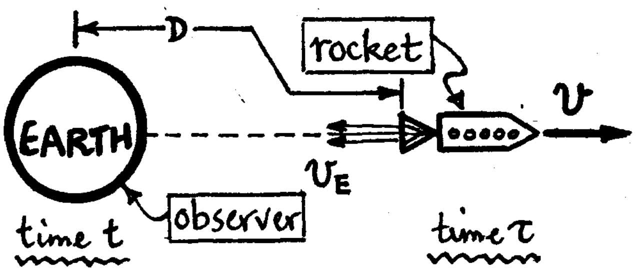
At or near $t=0$ in a Star-Trek adventure.

A Relativistic Rocket Trip



on-board the rocket, have:

acceleration : $A = \text{cnst}$,
exhaust velocity : $v_E = \text{cnst}$.

The rocket accelerates so that passengers on-board feel a <u>constant</u> acceleration A, and -- since the rocket engines perform reliably -- the engineers on-board the rocket record a <u>constant</u> fuel exhaust velocity v_E. If t is the rocket trip time as measured by an earth observer, and τ the corresponding time on the rocket, we want to find the distance D traveled by the rocket in a given time t or τ, the fuel fraction remaining at that time, etc. How far can we go?

---

1) Define the usual velocity parameter β; with c = light velocity:

$\rightarrow \beta = v/c$, v = rocket velocity w.r.t. earth. (1)

Earth & rocket observers agree on values of β, but they measure β according to different clock times t(EARTH) & τ(ROCKET). These times are related by:

$\rightarrow dt = d\tau/\sqrt{1-\beta^2}$, instantaneously (when $\beta(t) = \beta(\tau)$). (2)

To connect the t(EARTH) & τ(ROCKET) motions, note that an increment in v of dv (EARTH) is related to the corresponding increment du(ROCKET) by...

$\rightarrow dv = (1-\beta^2)\, du$. (3)

This result follows from the relativistic velocity-addition formula. By combining Eqs. (2) & (3), we can write...

$$\boxed{dv/dt = (1-\beta^2)^{3/2}\, du/d\tau}. \quad (4)$$

Here, $du/d\tau$ is the rocket acceleration felt by its occupants, while dv/dt is the apparent acceleration as logged by an earth observer. We want

Relations between earth & rocket times & distances.

$du/d\tau = A = \text{cnst}$ for the rocket occupants, so Eq. (4) reads...

$$d\beta/dt = \frac{A}{c}(1-\beta^2)^{3/2} \longrightarrow \underline{\beta(t) = (At/c)/[1+(At/c)^2]^{1/2}}. \quad (5)$$

Note that we can get to $\beta \to 1$, but only at $t \to \infty$ (when everyone will have forgotten this impropriety). Anyway, using Eq. (5) in Eq. (2), we can integrate to relate earth & rocket times t & τ, and then get β in terms of τ, as...

$$\boxed{t = \frac{c}{A}\sinh(A\tau/c)}, \quad \boxed{\beta(\tau) = \tanh(A\tau/c)}. \quad (6)\ (7)$$

---

2) The distance D traveled in earth time t is found by integrating Eq. (5)...

$$\rightarrow D = \int_0^t c\,\beta(t')\,dt' = \frac{c^2}{A}\left[\sqrt{1+(At/c)^2} - 1\right] \longrightarrow ct, \text{ as } t \to \text{large}. \quad (8A)$$

D grows linearly with earth time t. The situation is quite different for rocket time τ. By plugging Eq. (6), i.e. $At/c = \sinh(A\tau/c)$, into (8A)...

$$\underline{\underline{D(\tau) = (c^2/A)[\cosh(A\tau/c) - 1]}} \longrightarrow (c^2/2A)\,e^{A\tau/c}, \text{ as } \tau \to \text{large}. \quad (8B)$$

We see that D grows exponentially with time τ on-board the rocket.

---

3) So far, our results don't depend on the rocket fuel exhaust velocity v_E. To bring in v_E, let m be the instantaneous rocket rest mass (m decreases with time), and by momentum conservation write the "rocket equation"

$$\rightarrow mc\,d\beta + (1-\beta^2)\,v_E\,dm = 0. \quad (9)$$

$v_E = \text{cnst}$ (on-board the rocket), and we can solve (9) for β as a fcn of m. The result can be quoted in terms of two convenient ratios, R & ϵ, as

$$\left[\begin{array}{lcl} \text{BURN-RATIO} & : & R = m_0/m \\ \text{EXHAUST PARAMETER} & : & \epsilon = v_E/c \end{array}\right\} \quad \boxed{\beta = (R^{2\epsilon} - 1)/(R^{2\epsilon} + 1) < 1}. \quad (10)$$

m_0 is the initial rocket mass (as it leaves earth), so R measures the mass of fuel that must be burned to deliver a unit (payload) mass at journey's end. ϵ is just the rocket fuel exhaust velocity in units of light speed c.

4) $\boxed{\text{REMARKS}}$ on: $\beta = (R^{2\epsilon}-1)/(R^{2\epsilon}+1)$, $R = m_0/m$ & $\epsilon = v_E/c$. (10)

1. For chemical rockets, ϵ is very small... $\epsilon \sim 10^{-4}$ is typical. Then (10) ⇒

$\rightarrow \beta \simeq \epsilon \ln R$, or $v \simeq v_E \ln(m_0/m)$, for a chemical rocket. (11)

Such a rocket never gets close to light speed for any reasonable burn ratio. If $\epsilon \sim 10^{-4}$ & $R = 10^N$ (i.e., 10^N tons of fuel burned for each ton of payload), then: $\beta \simeq 2.3 \times 10^{-4} N$. So even if $R = 10^{15}$, only get to $\beta \simeq 0.0035$. NOTE: 10^{15} tons of liquid fuel occupies a volume of $\sim 2 \times 10^5$ cubic miles.

2. For fixed burn ratio $R > 1$, the rocket velocity β increases as the exhaust velocity ϵ increases. The fastest -- and farthest -- trip possible thus employs a "photon rocket", with $\epsilon = 1$. Here, by some engineering miracle, the spent fuel is ejected from the rocket engine at light speed, $v_E = c$.

3. For fixed ϵ (i.e. fixed engine design), β increases as R increases -- so you can get going fast if you are willing to burn enough fuel. But how much? A relevant form for the burn ratio R follows from setting β[Eq.(7)] = β[Eq.(10)]:

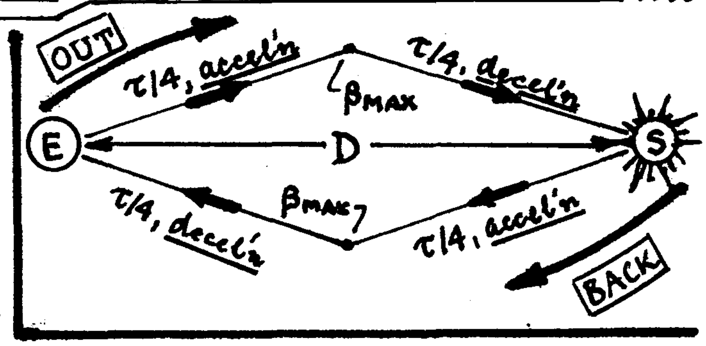
$$\boxed{R(\tau) = m_0/m(\tau) = \exp(A\tau/\epsilon c)}\text{, for cnst on-board accel}^{\text{n}}\ A. \quad (12)$$

R increases exponentially fast with rocket trip-time τ (and thus with distance traveled); $R \to$ large is a dominant factor in limiting our rocket trip.

4. The physical parameters most relevant to planning our rocket trip are:
   - burn-ratio R... how big a rocket can we build?
   - exhaust parameter ϵ... can we design a photon rocket, w/ $\epsilon \to 1$?
   - on-board trip-time τ... for passenger comfort: $\tau <$ human lifetime.

To get the trip distance D in terms of R, ϵ and τ, use Eq.(12) to eliminate A, via $A\tau/c = \epsilon \ln R$, and plug this into Eq.(8B) to get...

$$\boxed{D = [c\tau / \tfrac{1}{2}\epsilon \ln R]\,\sinh^2(\tfrac{1}{2}\epsilon \ln R)} \tag{13}$$



5) Consider a "symmetric" round trip from earth E to a star S lying at distance D. The trip takes total rocket time τ, with equal times $\tau/4$ spent on each leg of acceleration & deceleration (at rate A = cnst), per sketch. The maximum rocket velocity β_{MAX} is reached at the trip midpoints, and the same distance $D/2$ is traveled on each leg. β_{max} is given by Eq.(7) evaluated at $\frac{\tau}{4}$, namely: $\beta_{MAX} = \tanh(A\tau/4c)$, and the burn-ratio $R = m_0/m$ ($^{w/}m_0$ = initial rocket mass, $^{\&/}m$ = mass that returns to earth) is given by Eq.(12): $R(\tau) = \exp(A\tau/\epsilon c)$. Eqs.(6)&(8) are modified slightly to give the elapsed earth time t and $E \leftrightarrow S$ distance D...

$$t = (4c/A)\sinh(A\tau/4c), \quad D = (2c^2/A)[\cosh(A\tau/4c) - 1]. \tag{14}$$

We are free to choose the on-board acceleration A -- for passenger comfort, we design the engines to give $A = 10\ \mathrm{m/sec}^2 \simeq 1\,g$ (one earth gravity). Also, we can pretend to have invented a photon rocket, with exhaust velocity $\epsilon = 1$, so as to take the longest possible trip in a given time τ. Other relevant numbers are...

- 1 earth year = 3.1536×10^7 sec,
- 1 l.y. (light year) = $c \times 1$ year = 9.4543×10^{12} km, (15)
- mass of earth: $M_e = 5.983 \times 10^{24}$ kgm = 6.595×10^{21} tons.

Then it turns out that the critical limiting parameter for the trip is the burn-ratio R. If we want the payload mass $m \simeq 100$ tons (to return to earth) and also want the initial mass $m_0 < M_e$, we don't have enough fuel to travel for more than $\tau \simeq 43$ yr. Representative numbers appear below...

Various round trip parameters for a photon rocket.

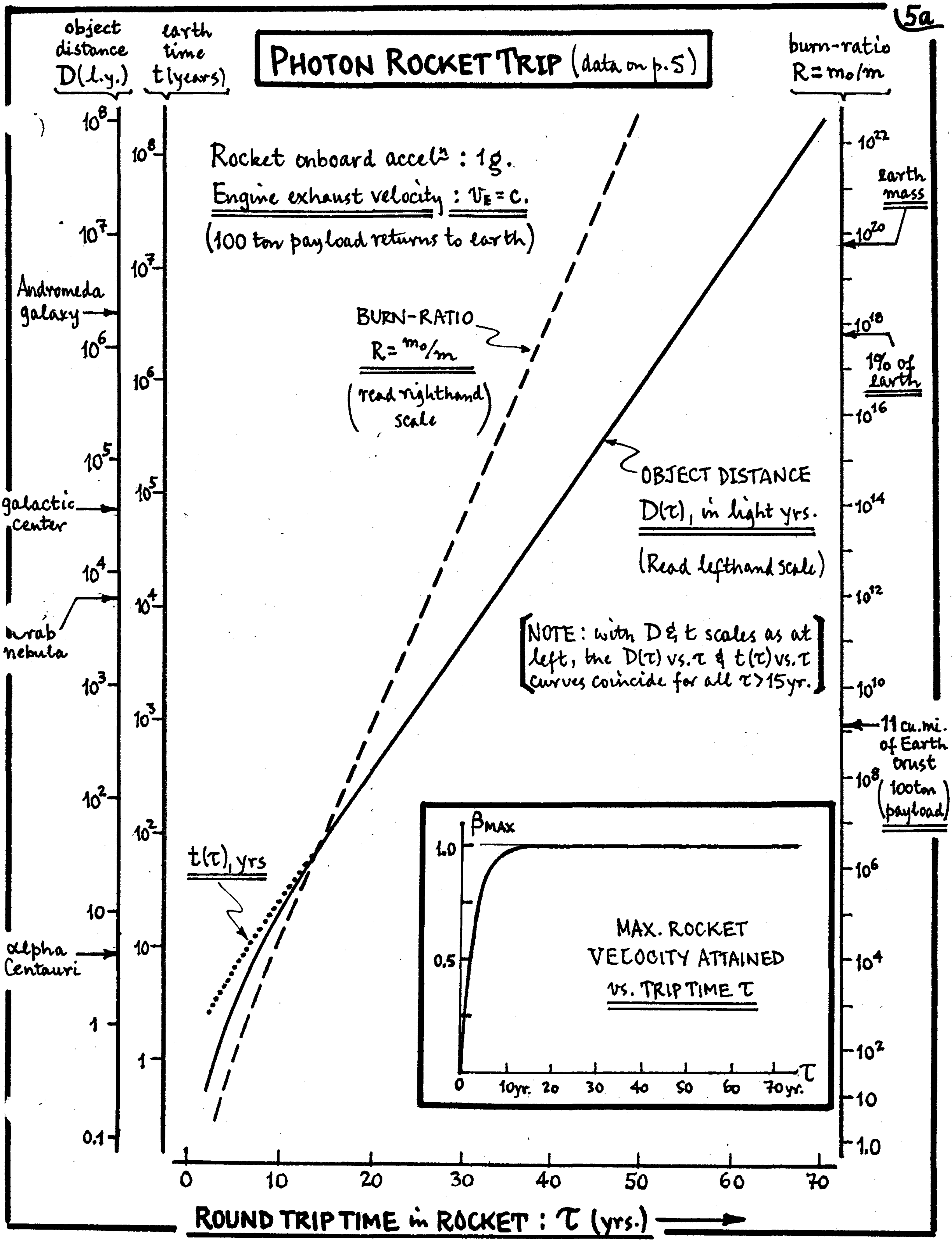
PHOTON ROCKET: $\epsilon = 1$ @ accel'n $A = 10\ m/sec^2 \simeq 1g$

rocket time τ, yr.	earth time t, yr	trip distance D, l.y.	max.vel./c β_{MAX}	burn ratio $R = m_0/m$	REMARKS
1	1.0016	0.066	0.2569	2.86	$D \simeq 50 \times$ solar system diam.
3	3.32	0.623	0.6575	23.4	
5	6.57	1.893	0.8653	192	
7	11.7	4.236	0.9508	1569	Get to α-Centauri.
10	26.2	11.34	0.9896	36,750	Visit ~15 nearby stars.
15	98.0	47.13	0.9992	7.05×10^6	
20	365	180.5	$1 - 5.4 \times 10^{-5}$	1.35×10^9	for $m = 100$ ton payload, need $m_0 \sim 11$ cu.mi. of earth's crust.
30	5050	2523	$1 - 2.8 \times 10^{-7}$	4.97×10^{13}	Crab Nebula @ $\tau = 33$ yr.
40	69,930	34,960	$1 - 1.5 \times 10^{-9}$	1.83×10^{18}	Approaching galactic center.
50	9.68×10^5	4.84×10^5	$1 - 1.0 \times 10^{-11}$	6.71×10^{22}	Andromeda Galaxy @ $\tau = 55$ yr.
60	1.34×10^7	6.70×10^6	$\simeq 1$	2.47×10^{27}	$m_0 \sim 100$ solar masses
70	1.86×10^8	9.28×10^7	$\simeq 1$	9.06×10^{31}	$m_0 \sim 3.7 \times 10^6$ solar masses.

These data are plotted on a graph, next page.

REMARKS on above photon rocket round-trip.

1. For a $\tau = 1$ yr round-trip, we only get out to see the Oort cloud. Worth it?
2. First "interesting" trip is $\tau = 7$ yrs; we get to $D \simeq$ location of α-Centauri, our nearest star. For $m = 100$ tons payload, we need an $m_0 = 157{,}000$ ton rocket.
3. A $\tau = 10$ yr. trip might be practical: for a 100 ton payload, you only need a 3.7 million-ton rocket. You get to $D \simeq 11.3$ l.y.; there are ~15 stars to inspect.
4. At $\tau \simeq 33$ yr, you visit the Crab Nebula (@ $D \simeq 6000$ l.y.). Expenses are: $t(\text{earth}) = 11{,}110$ yr (What do you come home to?), $m_0 \simeq 10^{17}$ tons $\simeq 8.1 \times 10^6$ cu.mi. of earth's crust.
5. $\tau = 40$ yr. $\Rightarrow$ visit to galactic center, but $m_0 \simeq 2.8\% \times M_e$ (or $2.25 \times M_{moon}$). Extra-galactic trips ($>10^5$ l.y.) require burning up planets, stars, galaxies.
6. If you lived $\tau = 82$ yr., you could get to $D \simeq 2.2 \times 10^9$ l.y., burning $m_0 \simeq 10^{12}$ solar masses ($\sim 10^4$ galaxies). But $t(\text{earth}) \simeq 4.4 \times 10^9$ yr $\Rightarrow$ sun burns out before you return.


PHOTON ROCKET TRIP (data on p.5)
object distance D(l.y.)
earth time t (years)
burn-ratio R = m₀/m
Rocket onboard accelⁿ : 1g.
Engine exhaust velocity : v_E = c.
(100 ton payload returns to earth)
BURN-RATIO R = m₀/m (read righthand scale)
OBJECT DISTANCE D(τ), in light yrs. (Read lefthand scale)
NOTE: with D & t scales as at left, the D(τ) vs. τ & t(τ) vs. τ curves coincide for all τ > 15 yr.
t(τ), yrs
Andromeda galaxy
galactic center
crab nebula
alpha Centauri
earth mass
1% of earth
11 cu.mi. of Earth crust (100 ton payload)
10^8
10^7
10^6
10^5
10^4
10^3
10^2
10
1
0.1
10^{22}
10^{20}
10^{18}
10^{16}
10^{14}
10^{12}
10^{10}
10^8
10^6
10^4
10^2
10
1.0
β_{MAX}
1.0
0.5
MAX. ROCKET VELOCITY ATTAINED vs. TRIP TIME τ
0
10yr.
20
30
40
50
60
70yr.
τ
0
10
20
30
40
50
60
70
ROUND TRIP TIME in ROCKET : τ (yrs.)

Problems with a rocket with exhaust velocity $\epsilon < 1$.

6) What happens if we are not clever enough to design a photon rocket that can burn up galaxies as fuel? Answer: the burn-ratio R quickly gets out of hand, for a given trip. The counterpart of Eq. (13), for the round trip sketched on p. RR4, with D = trip distance & τ = trip time, is

$$\rightarrow D = [4c\tau/\epsilon \ln R] \sinh^2\left(\tfrac{1}{8}\epsilon \ln R\right). \tag{16}$$

With D & τ fixed for the trip, we must have $\epsilon \ln R$ = cnst. For example, the trip to α-Centauri (table on p. RR5) prescribes D = 4.24 l.y. in τ = 7 yr, with: $\epsilon \ln R = 1 \times \ln 1569 = 7.3582$. Then...

$$\left[\begin{array}{l}\text{to } \alpha\text{-Centauri:} \\ D = 4.24 \text{ l.y.}, \tau = 7 \text{yr}\end{array}\right\} \; R = 10^{3.1956/\epsilon}. \tag{17}$$

For $\epsilon = 1$ (photon rocket), we get the previous result: R = 1569, or a rocket that starts at mass m_0 = 157,000 tons for an m = 100 ton payload. But if $\epsilon = 0.5$, $R \rightarrow 2.46 \times 10^6 \Rightarrow m_0$ = 246 million tons.

ϵ	$R = m_0/m$	m_0, tons
1.0	1569	157,000
0.8	9877	$\sim 10^6$
0.6	2.12×10^5	21×10^6
0.4	9.76×10^7	10×10^9
0.2	9.52×10^{15}	$\sim 10^{18}$
0.1612	6.6×10^{19}	~ M(earth)
0.1261	2.2×10^{25}	~ M(sun)

The table shows how much R must increase to compensate for a reduced exhaust velocity ϵ... this increase in R must occur in order to maintain the on-board acceleration at $A \simeq 1g$ = cnst. Evidently, trips to α-Centauri are not practical at ϵ-values < 0.2. And ϵ-values > 0.2 are not possible for chemical rockets.

If we are limited by a manageable rocket mass m_0, and by engines with $\epsilon < 0.1$, then -- to get anyplace "interesting" (e.g. α-Centauri @ D = 4.3 l.y.) -- we must relax our restrictions on trip-time τ and onboard acceleration A. In particular, we have to anticipate that -- practically -- $\epsilon \ln R$ is quite small.

"Practical" rocket trips: to α-Centauri; in one lifetime.

7) If, indeed, the rocket design parameter $\epsilon \ln R$ $\left\{\begin{matrix}\epsilon = \text{exhaust velocity} \\ R = \text{burn ratio}\end{matrix}\right\}$ is "small", then in Eq.(16), we can approximate $\sinh(\frac{1}{8}\epsilon \ln R) \simeq \frac{1}{8}\epsilon \ln R$, and thus find:

$$\boxed{\tau \simeq (16/\epsilon \ln R)\frac{D}{c}}, \text{ for } \epsilon \ln R < 1. \tag{18}$$

This gives the rocket round-trip time τ in terms of the objective distance D, and the design parameter $\epsilon \ln R$. Suppose $D \simeq 4.2$ l.y. (for α-Centauri), $\epsilon \simeq 2\times10^{-4}$ (a really "hot" chemical rocket), and $R \simeq 10^5$ (with a 100 ton payload, the rocket starts off at $m_0 \simeq 10$ million tons). Then $\epsilon \ln R \simeq 2.3\times10^{-3}$, and for this trip, Eq.(18) yields: $\underline{\tau = 29{,}200 \text{ years}}$. The Centaurians can rest easy.

One more numerical example: how far out could we go (and return) in one human lifetime with a "practical" relativistic rocket... say $\tau = 80$yr, and $R = 10^5$ (10 million tons mass at start). Eq.(16) gives...

$$\left.\begin{matrix}\tau = 80\text{yr} \\ R = 10^5\end{matrix}\right\} \underline{\underline{D_{MAX} = \frac{27.8}{\epsilon}\sinh^2(1.439\epsilon)}}, \text{ in l.y.} \tag{19}$$

Assuming we are not stuck with chemical rockets, look at the ϵ-variation:

exhaust velocity ϵ	1.0 (photon rocket)	0.7	0.4	0.1	10^{-2}	10^{-3}	10^{-4}
max. distance D_{MAX}, l.y.	110	55.9	25.7	5.80	0.576	0.0576	0.0058
on-board acceln: A/g	1/7.2	1/10.2	1/17.9	1/71.6	1/716	1/7160	1/71,600
max. velocity: β_{MAX}	0.994	0.965	0.818	0.280	0.029	$\sim3\times10^{-3}$	$\sim3\times10^{-4}$
earth time: t, yrs	246	146	98.9	81.1	80yrs + 4 days	80yrs + 1 hour	80yrs + 35 sec.

(20)

Clearly, we are bound to a fairly small neighborhood of our own galaxy, even with a photon rocket. What we really need is hyperdrive, spacewarp, and -- mainly -- a repeal of the laws of relativity. Pace, Captain Kirk.