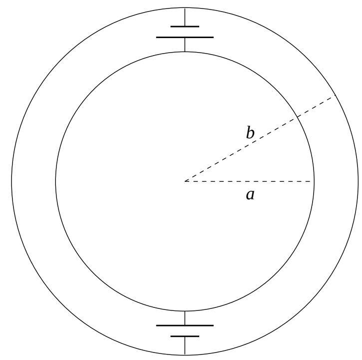
DEPARTMENT OF PHYSICS

2010 COMPREHENSIVE EXAM

23-25 August 2010

1. Two perfectly conducting, coincentric spherical shells of radius a and $b > a$ are connected via two batteries, one that is fully charged, and the other that is in need of recharging. The battery being charged is connected between the south poles of the inner and outer sphercial shells, while the battery doing the charging is connected between the north poles. The positive terminals of each battery are connected to the inner sphere, V is the potential difference between the spheres, and I is the current flowing in the circuit.

(a) Compute the Electric field in the gap between the spheres.
(b) Compute the Magnetic field in the gap between the spheres.
(c) Show that the Poynting flux flowing through any plane with $\theta = \mathrm{constant}$ is equal to the power flow, $P = IV$, from the charged to the un-charged battery.

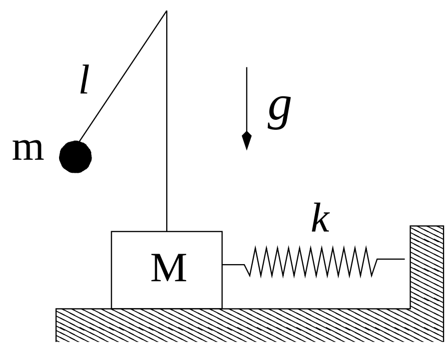


2. An oscillating system consists of a block of mass M, on a frictionless table, connected to a wall by a massless spring with constant k; and a pendulum (length l, mass m) attached to a pole on top of the block. The pole and the pendulum rod are rigid and massless, the block and pendulum only perform left-right motions, the block cannot rise above the table.

(a) Write the Lagrangian for this system. Here consider arbitrary large oscillations.

(b) For small oscillations find the Lagrangian, equations of motion, and the normal modes.

(c) Give physical interpretation of the limiting cases: $k = 0$, $k \to \infty$, $l \to 0$.



3. A particle of spin ½ is at rest in a uniform magnetic field $\vec{B} = B_0\hat{k}$. In the z basis, the Hamiltonian in matrix form is $H = -\frac{\gamma B_0 \hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ with eigenstates $\chi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\chi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, where γ is the gyromagnetic ratio for the particle.

a) Find the energies E_+ and E_- of the two eigenstates.

b) Now assume the particle is initially (at t=0) in a superposition state $\chi(0) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Find the time dependent superposition state $\chi(t)$.

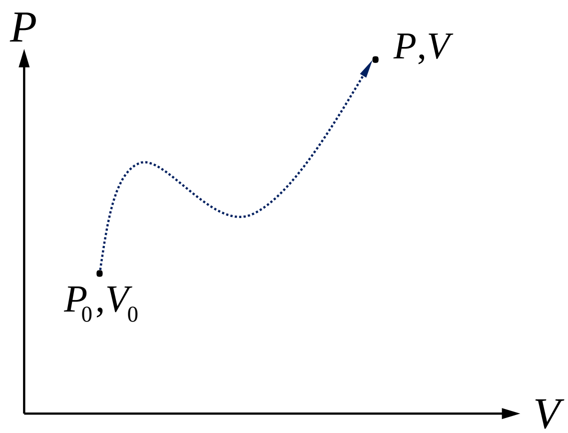
c) Now find the expectation value $\langle S_x \rangle$. Let $\gamma B_0 = \omega$ to simplify the notation.

d) If you measure S_x, the component of the spin angular momentum along the +x direction, at time t, what is the probability that you will get $+\frac{\hbar}{2}$?

e) Plot $\langle S_x \rangle$ from part c) over the region $0 < \omega t < 2\pi$. Then just below this plot, using the same ωt scale, plot the probability from part d).

Comment on the values of the two plots at $\omega t = \frac{\pi}{2}$. Are they consistent with what you expect for this time in the precession?

4. The object of this problem is to determine the change in the entropy, ΔS, of an n mole ideal gas when it was taken from an initial state of P_0, V_0 to a final state of P, V along an arbitrary reversible path on a P vs. V diagram as shown in the figure below. Assume that the molar heat capacities at constant pressure and constant volume of the ideal gas are C_p and C_v and that they are constant. Recall that $\gamma = C_p / C_v$ and that for a reversible process $dS = \delta Q / T$. Determine ΔS in terms of P_0, V_0, P, V, n, C_p and C_v.

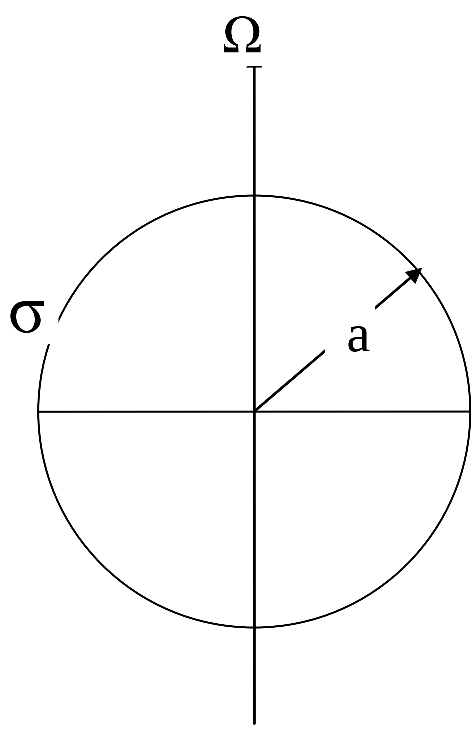


5. A spherical shell of radius **a** has an uniform surface density **σ**. The shell rotates with angular velocity Ω about a diameter (see Figure).

(a) Write the scalar potential both inside and outside the shell as a sum of Legendre polynomials.

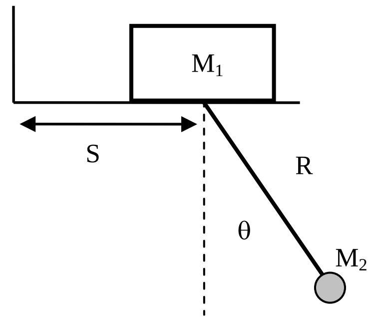
(b) Apply the appropriate boundary conditions to find the coefficients.

(c) Find the magnetic field **B** and sketch it both inside and outside the shell.



Figure

6. A pendulum is made from a small ball of mass M_2 hanging on a massless rope of length R. The other end of the rope is connected to a block of mass M_1. The block is accelerated at a constant acceleration a to the right such that the distance of the block from the origin is given by $S = \frac{1}{2}at^2$



a) Find the Lagrangian L for this system in terms of the generalized coordinates S and θ. Take the potential energy V of mass M_2 to be zero when $\theta=0$.
b) Find Lagrange's equation for the generalized coordinate θ .
c) Use the equation found in b) to find the equilibrium angle θ_o that the pendulum will have during the acceleration of M_1.
d) Now during the acceleration, let the pendulum have a small oscillation α in angle about the equilibrium, such that $\alpha = \theta - \theta_o$. Use the equation that you found in b) to find this oscillation frequency.

7. A model for amorphous medium, glass, consists of a large collection of two-level systems (that arise due to complicated potential landscape and tunneling between different local minima). Each two-level system with energy levels ε_1 and ε_2 is characterized by the gap between them $\Delta = \varepsilon_2 - \varepsilon_1$, and the probability density $P(\Delta)$ (number of two-level systems with gaps in the interval $[\Delta, \Delta + d\Delta]$).

(a) Find the free energy, entropy and heat capacity of a single two-level system.

(b) Find the temperature dependence of heat capacity of glass at low temperatures. Use the fact that $P(\Delta)$ is almost constant at energies below some Λ, i.e. $P(\Delta < \Lambda) = P_0$ and we are interested in $T \ll \Lambda$; you don't have to evaluate the overall constant.

8. A driven, damped, harmonic oscillator is described the the ordinary differential equation

$$\ddot{x} + 4\dot{x} + 8x = \sin(4t)$$

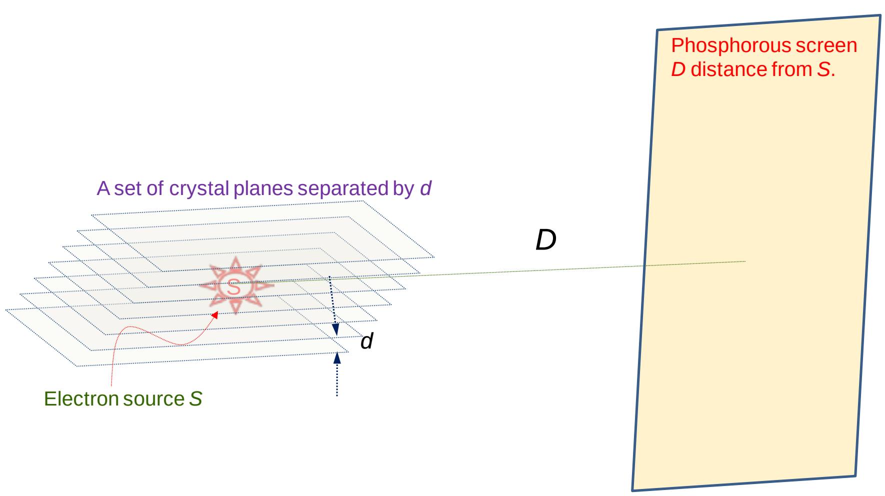
and has initial conditions $x(0) = 0$ and $\dot{x}(0) = 4/5$. Using a Laplace transform, or otherwise, solve for the subsequent motion of the system.

9. Figure below shows a point source S of monoenergetic electrons at $E=20$-keV energy emitting such electrons uniformly in all directions. The source is located in the interior of a crystal lattice whose physical size is small, so that electrons can escape the crystal. Normally a crystal has large number of crystal planes oriented in different directions. This problem focuses only *one set* of such planes oriented as shown in the figure below. A small number of this crystal planes spaced by a distance d, is drawn in the figure. For simplicity assume that these crystal planes are perpendicular to the plane of a phosphorous screen located at $D >> d$ distance from source S. Furthermore assume that these high-energy electrons diffract from these crystal planes at specific incident angles given by Bragg's diffraction condition, $2dSin\theta = n\lambda$. For simplicity assume that only the diffracted electrons produce a visible pattern on the phosphorous screen, that all the other electrons lose their energy in the crystal and do not make it to the phosphorous screen. Answer the following questions:

a. Using your knowledge of typical crystal lattice spacing show that $\lambda << d$, where λ is the wavelength of the diffracted electron.

b. Determine the diffraction pattern of these electrons on the phosphorous screen. Consider only the first-order diffraction and only the set of crystal planes that is described in the text.

c. Determine lattice spacing d in terms of D, E , m_e (the mass of the electron) and separation δ between the two diffraction lines observed on the phosphorous screen. Give a numerical answer assuming $D=20$ cm and $\delta \simeq 0.5$ cm.



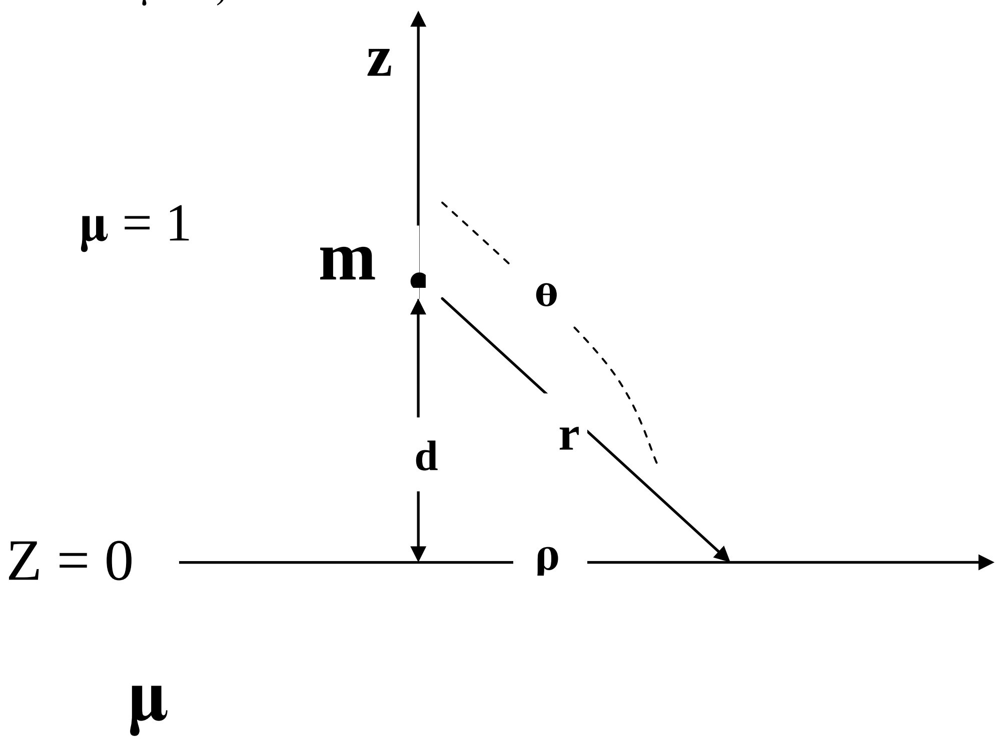
Hint: Because $\lambda << d$ you can assume that source S (even though it is inside the crystal) is far from the crystal planes. Do not concern yourself with proving the Bragg's condition but using it.

10. Magnetic dipole **m** is located distance **d** on the **z** axis above an infinite **z** = 0 plane. Beneath the plane (**z** < 0) the material consists of a paramagnetic substance with magnetic permeability **μ,** while **μ =** 1 above the plane (see Figure). Using *the image method*, answer the following questions.

(a) Find the magnetic field **B** just above ($\mathbf{B}^+$) and just below ($\mathbf{B}^-$) the **z** = 0 plane. Give $\mathbf{B}^+$ and $\mathbf{B}^-$ as a function of **r** (distance from the dipole to a point at **ρ** from the origin on the **z** = 0 plane) and **θ** (polar angle away from the **z** axis to point **ρ** (see Figure).

(b) Applying the boundary conditions at the **z** = 0 plane, express all image magnetic dipoles in terms of **m** and **μ.**

(c) What happens when **μ** becomes infinity at **z = 0** (i.e., what is **B** just above and below the **z** = 0 plane when **μ** = ∞)?



Figure

11. Assume a mass *m* is attached to the end of a massless spring of spring constant k and that the mass and spring system is spinning initially with angular velocity ω_0 in a circle of radius r_0 on a horizontal frictionless plane. Now assume that during its circular motion the mass *m* is given a radial impulse which sets the spring to oscillate along the radial direction with an amplitude of $|u| << r_0$. Answer the following questions:

1. Write down Newton's equation of motion for the mass and identify any physical quantity that remains conserved after the impulse.

2. Determine the frequency, ν, of the oscillations of mass *m* along the radial direction in terms of spring constant *k*, mass *m* and initial angular rotation ω_0.

Hint: For $\alpha \ll 1,\ (1+\alpha)^k \approx 1 + k\alpha$

12. A number of high energy cosmic ray experiments have recently reported the detection of the "Greisen-Zatsepin-Kuzmin cut-off" in the cosmic ray spectrum. This effect was first predicted in the 1960's, when it was realized that the high energy protons p that make up the cosmic rays would undergo inelastic scattering off the cosmic microwave photons γ via $\gamma + p \to \pi^0 + p$ (it is also possible to produce a neutron and a π^+, but we will ignore that here). Your task will be to estimate the proton energy threshold for this reaction, following the steps outlined below.

a) The cosmic microwave background (CMB) photons have a temperature of 2.72^o K and a black body spectrum. What is the energy (in eV or Joules) of a typical CMB photon?

b) Before embarking on the calculation, describe in words the physical picture of the collision at threshold - what can you say about the directions of the incident particles and the motion of the pion and proton following the collision?

c) What is the proton energy threshold for π^0 production? (The protons have rest mass $0.94\ \mathrm{GeV}/c^2$, and the pions have rest mass $0.135\ \mathrm{GeV}/c^2$).

d) The cross section for pion production near threshold is very low $\sigma \sim 10^{-35}\ \mathrm{m}^2$, and the actual GZK cut-off is dominated by the so-called Delta resonance $\gamma + p \to \Delta^+ \to \pi^0 + p$, whereby a Delta particle of rest mass $1.23\ \mathrm{GeV}/c^2$ is first produced before it quickly decays into a proton and a pion. What is the proton energy threshold for this reaction?

e) Compare the energy you computed in part c) to a well hit tennis ball (you have to estimate the mass and speed of the tennis ball).

f) What is the fractional change in energy of the proton at the Delta threshold (in other words $(E_{\rm in} - E_{\rm out})/E_{\rm in}$)?

g) The cross section at the Delta resonance is $\sigma = 6 \times 10^{-31}\ \mathrm{m}^2$ and the number density of CMB photons is currently $413\,\mathrm{cm}^{-3}$. What is the mean free path for a cosmic ray proton near the Delta threshold? Factoring in what you found in part e), what is the approximate stopping distance, in lightyears, for these high energy protons?

13. Find the potential ϕ for all points inside a region defined by $-\infty<z<\infty$, $0<x<x_o$, $-y_o<y<y_o$. The potential on the four boundaries is given by $\phi(x,+y_o,z)= \phi(x,-y_o,z)=0$ and $\phi(0,y,z)= \phi(x_o,y,z)=+\phi_o$.

14. Infinite dielectric slab of thickness **d** is located with the bottom surface at the **z** = 0 plane (see Figure). The slab's dielectric constant is **ϵ.** The slab is placed in an external electric field with:

$$\mathbf{E}_2 = A\,\mathbf{z} + B\,\mathbf{x}$$

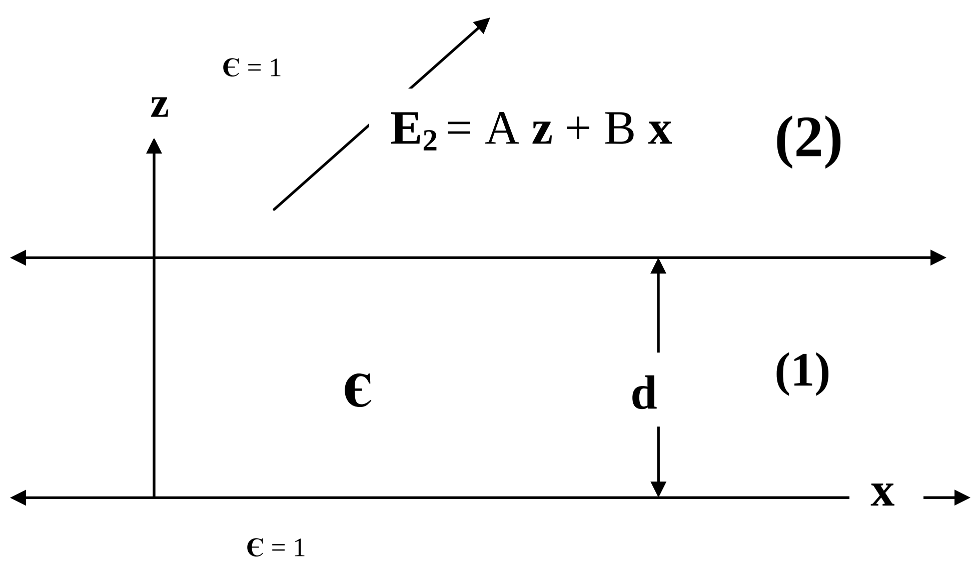
where **ϵ,** A and B are constants. (**z** and **x** are *unit* vectors in the z and x direction.) (See Figure.)

(a) Find the electric field **E** within the slab (1) (in terms of the given constants **ϵ,** A and B).

(b)Find the polarization **P** and volume polarization density $\boldsymbol{\rho_p}$ within the slab (in terms of the given constants **ϵ,** A and B).

(c) Find the surface polarization density $\boldsymbol{\sigma_p}$ on the top surface and bottom surface of the slab (in terms of the given constants **ϵ,** A and B).

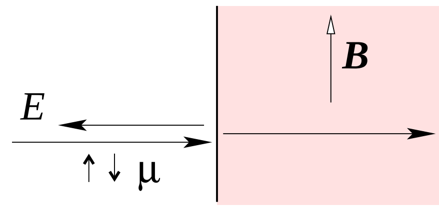
Figure



15. A unpolarized beam of neutral particles with spin $\hbar/2$ is normally incident on a semi-infinite region with uniform magnetic field $\mathbf{B}$. The particles in the beam have energy E and magnetic moments $\pm\mu$, corresponding to spins up/down.

(a) Justify that the (one-dimensional) Hamiltonian describing this system is

$$\mathcal{H} = -\frac{\hbar^2}{2m}\boldsymbol{\nabla}_x^2 - \mu\mathbf{B}\cdot\boldsymbol{\sigma}\,\theta(x)$$



where x is the direction perpendicular to the barrier, $\boldsymbol{\sigma}$ are the Pauli matrices, and $\theta(x)$ is the Heaviside step-function.

(b) Find the spin polarization (difference between densities of spin-up and spin-down particles) of the reflected beam. Schematically plot the dependence of polarization on E (for this you need to analyse the general formula in several typical limits: small and large E, and slightly above $E^* = \mu B$). Explain the sign of the polarization, and why above a certain energy there is a significant reduction in the value of the polarization.