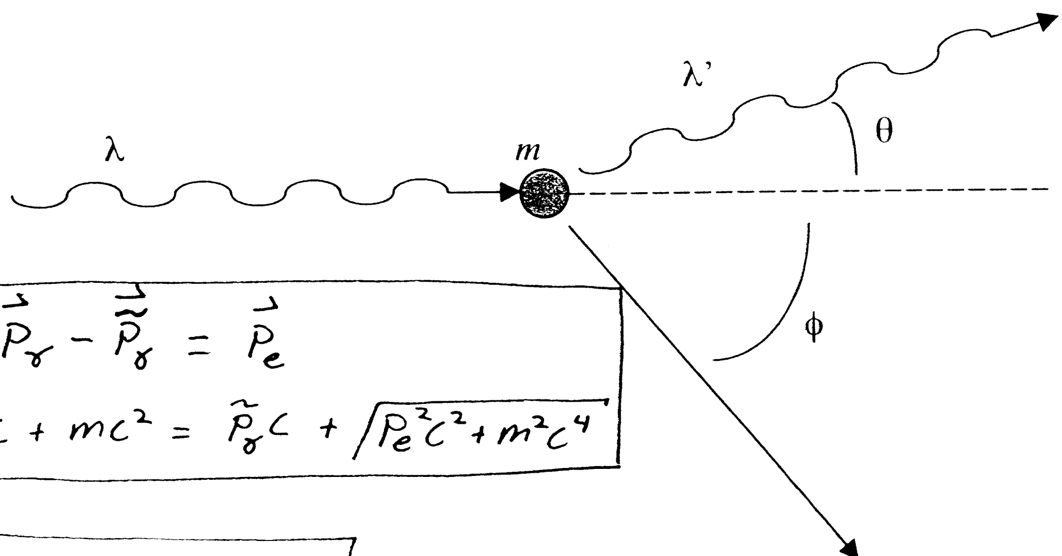
Problem #1

In 1927, Arthur H. Compton was awarded the Nobel Prize for the discovery of the Compton Effect. In the Compton Effect, a γ-ray photon of wavelength λ strikes a free, but initially stationary electron of mass m. The photon is scattered at an angle θ, and has a new wavelength λ'. The electron recoils at an angle ϕ (see figure below).

(a) Write the relativistic equations for momentum and energy conservation.

(b) Find an expression for the change $\lambda' - \lambda$ in the photon wavelength for the special case $\theta = \pi/2$.

(c) Qualitatively describe what would happen, if anything, to the scattered photon wavelength if the electron possessed internal degrees of freedom.

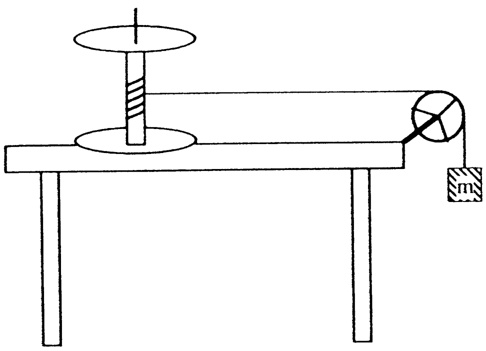


a) $$\boxed{\begin{aligned} \vec{P}_\gamma - \vec{\tilde{P}}_\gamma &= \vec{P}_e \\ P_\gamma c + mc^2 &= \tilde{P}_\gamma c + \sqrt{P_e^2 c^2 + m^2 c^4} \end{aligned}}$$

b) $$\boxed{\lambda' - \lambda = \frac{h}{mc}}$$

Problem #2

The following problem involves a device that can be used to measure the moment of inertia I_S of a spool based on the speed v of a mass. The spool is free to spin on a vertical axle fixed to a table; the spool's bearing is frictionless. A massless string is wrapped around the small radius r of the spool (see picture) in a single layer. The string passes over a pulley that spins on a frictionless bearing whose axle is horizontal to the table's surface. The round portion of the pulley has mass M, each of the three spokes has mass (1/3)M, and the radius is R. A mass m hangs freely from the end of the string, creating a tension that causes the pulley and spool to turn. The string does not slip on either the spool or the pulley, nor does it stretch. If m starts at rest, falls a distance y and reaches a speed v, find an expression for I_S in terms of the variables defined in the problem.



Answer

$$I_s = \left(\frac{2mgy}{v^2} - \frac{4}{3}M - m\right)r^2$$

Problem #3

A spherical shell of inner radius a and outer radius b has a permanent magnetization given by $\mathbf{M} = \mathrm{M}_o\,\hat{z}$. Answer the following questions. Your answers must be expressed in terms of the given parameters a, b, M_o, and θ, the angle from the z axis.

(a) Find the magnetic field $\mathbf{H}$ in all space: $r \leq a$, $a \leq r \leq b$, and $r \geq b$.

(b) Find the magnetic induction $\mathbf{B}$ in all space: $r \leq a$, $a \leq r \leq b$, and $r \geq b$.

a) $r < a \qquad \vec{H} = 0$

$a < r < b \qquad \vec{H} = -\frac{4\pi M_0}{3}\left\{\hat{r}\left(1 + \frac{2a^3}{r^3}\right)\cos\theta - \hat{\theta}\left(1 - \frac{a^3}{r^3}\right)\sin\theta\right\}$

$r > b \qquad \vec{H} = \frac{4\pi M_0}{3}\frac{(b^3 - a^3)}{r^3}\left\{2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right\}$

b) $r < a \qquad \vec{B} = 0$

$a < r < b \qquad \vec{B} = \frac{8\pi M_0}{3}\left\{\hat{r}\left(1 - \frac{a^3}{r^3}\right)\cos\theta - \hat{\theta}\left(1 + \frac{a^3}{2r^3}\right)\sin\theta\right\}$

$r > b \qquad \vec{H} = \frac{4\pi M_0}{3}\frac{(b^3 - a^3)}{r^3}\left\{2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right\}$

Problem #4

This problem investigates the variation of temperature and pressure as a function of altitude in the Earth's atmosphere, which is assumed to be an ideal gas. It is also assumed that the gravitational acceleration of g does not vary with altitude for the heights of interest (i.e. typical altitudes at which passenger planes fly). Answer the following questions, showing your work clearly:

a. Show that differential pressure dp and height z are related by $\frac{dp}{p} = -\frac{Mg}{RT(z)}dz$, where z is the altitude measured from sea level, $p = p(z)$ and $T(z)$ are the air pressure and temperature at altitude z, R is the ideal gas constant and M is the average molecular weight of air.

b. Suppose that the pressure decrease and cooling of the atmosphere are due to the adiabatic expansion of an ideal gas governed by $pV^{\gamma} = \text{const}$. Show that $\frac{dT}{dz} = (\frac{1}{\gamma} - 1)\frac{Mg}{R}$.

c. Show that the variation of pressure with altitude is given by

$$p(z) = p_o \exp\left(-\frac{1}{(1-\frac{1}{\gamma})}\ln\frac{\eta(z)}{\eta_o}\right),\text{ where } \frac{\eta(z)}{\eta_o} = 1 + (\frac{1}{\gamma} - 1)\frac{Mgz}{RT_o},\text{ where the}$$

subscript o indicates the values at $z = 0$.

Problem #5

Consider a quantum-mechanical system for which the properly normalized state $|\psi\rangle$ is expanded in terms of the discrete energy eigenstates $|n\rangle$ of the known time-independent *Hamiltonian* H:

$$|\psi\rangle = \sum_n |n\rangle \langle n \mid \psi\rangle \equiv \sum_n a_n |n\rangle \qquad \text{where} \quad H|n\rangle = E_n |n\rangle \,.$$

Assume that there are no continuum energy eigenstates. We construct the *density operator* $\rho \equiv |\psi\rangle\langle\psi|$ such that its matrix elements in the basis of energy eigenstates

$$\rho_{nm} = \langle n|\rho|m\rangle = \langle n \mid \psi\rangle \langle \psi \mid m\rangle = a_n\, a_m^{*}.$$

In the Schrödinger picture, the state $|\psi\rangle$ evolves in time, and we interpret the expansion coefficients as time-dependent complex numbers.

(a) Use the *Time-Dependent Schrödinger Equation* to determine the *equation of motion* of ρ, *i.e.*, find the rate of change of ρ. Compare and contrast your result with the equation of motion for dynamical observables, e.g., position, momentum, or energy, in the *Heisenberg picture*.

$$\boxed{i\hbar\,\frac{\partial\rho}{\partial t} = -[\rho, H]} \qquad \boxed{i\hbar\,\frac{dA_H}{dt} = [A_H, H]}$$

where the A_H means arbitrary operator A in the Heisenberg picture.

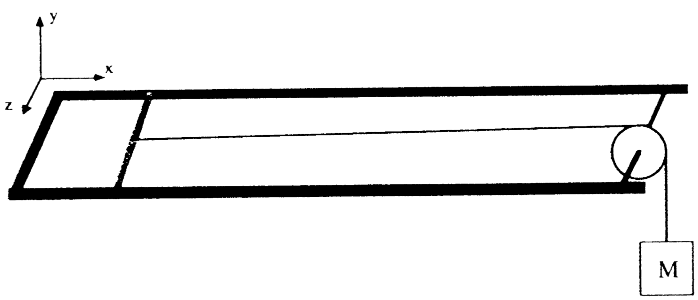
(b) What is the physical meaning of the *trace* of the density matrix (the sum of the diagonal elements)? Is it a time-dependent quantity? Explain.

$\boxed{\textit{The sum of the diagonal elements must be equal to one, always.}}$

Problem #6

A metal bar is formed into a long U-shape as shown in the diagram. The U is horizontal and thick enough that its electrical resistance is negligible; it lies in the x-z plane. Its two rails are a distance L apart; they point in the x-direction. A constant magnetic field B is directed in the y-direction, perpendicular through the plane in which the U lies. At the far end of the U, a frictionless pulley is mounted. A rod with resistance R and mass m is placed on the U. It can slide in a frictionless manner such that it remains perpendicular to the long axis of the U. Connected to the rod is a massless string from which a mass M hangs. At time t = 0, M is released which causes the rod to slide in the x-direction on the rails.

(a) Determine the speed of the rod as a function of time.
(b) From your result in (a), determine the terminal speed v_T of the rod. Assume that the rails are long enough to reach the speed v_T. Use v_T to calculate the energy dissipated in the resistance R at this speed and directly show that it is equivalent to the mechanical work done in moving the rod at constant speed v_T.



Answers

(a) $v(t) = \dfrac{MRg}{L^2B^2}\left(1 - e^{-\frac{L^2B^2}{R(M+m)}t}\right)$	(b) $v_{ter} = \dfrac{MRg}{L^2B^2}$

Problem #7

The partial differential equation

$$\nabla^2 u - \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2} = \alpha^2 u$$

describes the vibration of a stretched string embedded in a homogeneous elastic medium, for which u represents (transverse) displacement from equilibrium, v is the speed of waves on the string, and α is a measure of the medium's elasticity. Determine the characteristic frequencies of a such a string of length L.

Coincidentally, the variable substitutions $v \to c$ and $\alpha \to mc/\hbar$ produce the *Klein-Gordon equation*, which describes the relativistic quantum dynamics of free-particle spinless bosons of rest energy mc^2.

$$\boxed{\frac{v}{2\pi}\sqrt{\alpha^2 + \left(\frac{n\pi}{L}\right)^2}}$$

Problem #8

A spherical ball of mass M is attached to a massless string of length r (measured from the stand-off) which is attached to a massless spring with spring constant K (as shown in the diagram below). The entire apparatus rotates about the center post at a fixed angular velocity ω. At steady state while the apparatus is rotating at frequency ω, the length of the spring is l_l. Determine the following in reference to the mass M:

a) How many degrees of freedom are present?
b) What is the Lagrangian?
c) What are the equations of motion?
d) What are the steady state values (as a function of the parameters) for the variables describing the degrees of freedom?
e) What are the frequncies for small oscillations for the variables describing the degrees of freedom?

e) $a_{11}\delta\ddot{r} + C_{11}\delta r + C_{12}\delta\theta = 0$

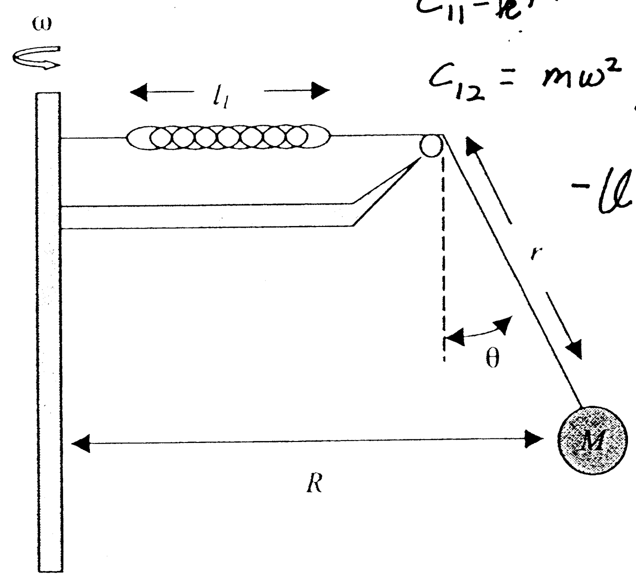
$C_{21}\delta r + a_{22}\delta\ddot{\theta} + C_{22}\delta\theta = 0$

where $r = r_0 + \delta r$ + $\theta = \theta_0 + \delta\theta$

$a_{11} = m$

$C_{11} = k + m\omega^2 \sin^2\theta_0 \left(\dfrac{3m(l_1 + r_0\sin\theta_0)^2 - I}{m(l_1 + r_0\sin\theta_0)^2 + I} \right)$

$C_{12} = m\omega^2 \left[\dfrac{4m(l_1 + r_0\sin\theta_0)^2 r_0 \sin\theta_0 \cos}{m(l_1 + r_0\sin\theta_0)^2 + I} - (l_1 + 2r_0\sin\theta_0)\cos\theta_0 \right] + mg\sin$



a) 2 r + θ

b) $L = T - V$; $T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{1}{2}\left[m(l_1 + r\sin\theta)^2 + I\right]\omega^2$

$V = \frac{1}{2}k(r - r_1) - mgr\cos\theta$

c) $\omega^2(l_1 + r_0\sin\theta_0)\cos\theta_0 = g\sin\theta_0$

$\omega^2(l_1 + r_0\sin\theta_0)\cos\theta_0 = -g\cos\theta_0 + \frac{k}{m}(r_0 - r_1)$

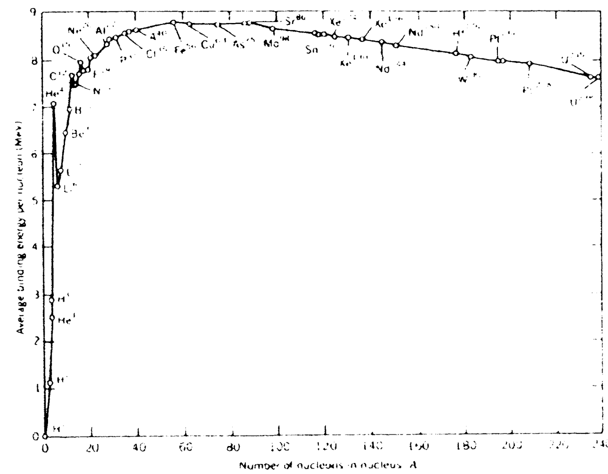
d) $\omega^2 = g\tan\theta_0 / \left(l_1 + r_1\sin\theta_0 + \frac{mg}{k}\tan\theta_0 \right)$

Problem #9

An atomic nucleus consists of Z protons and N neutrons, with mass number $A \equiv Z + N$. The *nuclear binding energy* B is the difference between the rest energies of the constituent protons and neutrons and the measured rest energy of the nucleus.

A plot of B/A as a function of A has clear trends.

This plot is given for information only and is not to be used in this problem.



B can be expressed semi-empirically as a sum of simple functions:

$$B = a_1\, A + a_2\, A^{2/3} + a_3\, \frac{Z(Z-1)}{A^{1/3}} + a_4\, \frac{(A-2Z)^2}{A} + \ldots$$

where the a_i are constants with units of energy. We use the convention here that $B > 0$, so that a larger value of B implies a more tightly bound nucleus.

No knowledge of nuclear physics beyond introductory physics is assumed in this problem.

Briefly explain your answers to all of the following questions:

(a) The dominant first term is referred to as the "volume term" because stable nuclei are roughly spherical with radii proportional to $A^{1/3}$. What is the sign of a_1? What can be concluded about the *strong interaction* from the observation that the dominant term is not proportional to $A(A-1)$?

The sign of a_1 must be positive. | *The strong interaction is therefore short-range.*

(b) What is the physical significance of the second term, which can be thought of as a correction to the first term, and what is the sign of a_2?

This term is the "surface term" and $a_2 < 0$.

(c) What is the physical significance of the third term, and what is the sign of a_3? Estimate the value of a_3 in eV to one significant figure.

The third term arises from the Coulomb repulsion of the protons, and $a_3 < 0$. | 0.7 MeV

(d) The fourth term arises in part from the *weak interaction*. If $a_4 < 0$, what relationship between Z and N is favored, i.e., provides tighter binding, by this term for a given A? For nuclei far from the favored relationship, how does this term compare to the previous three terms as A increases?

$Z = N$. As A increases, this term becomes less important than the other three.

Problem #10

Consider a quantum-mechanical particle of mass m subject to a one-dimensional potential.

(a) In coordinate space, the position operator $x_{\rm op}$ takes the form of multiplication by x, and the momentum operator $p_{\rm op}$ is represented as $-i\hbar\frac{\partial}{\partial x_{\rm op}}$. Prove that the *commutator*

$$[x_{\rm op}, p_{\rm op}] = i\hbar$$

independent of the state of the system. *You may not use part (b)!*

$\boxed{\textit{Answer given.}}$

(b) For an arbitrary scalar function f expressible as a power series in $p_{\rm op}$, prove that

$$[x_{\rm op}, f(p_{\rm op})] = i\hbar\frac{\partial f}{\partial p_{\rm op}}$$

independent of the state. Recall: $[A_{\rm op}, B_{\rm op}C_{\rm op}] = [A_{\rm op}, B_{\rm op}]\, C_{\rm op} + B_{\rm op}\, [A_{\rm op}, C_{\rm op}]$.

$\boxed{\textit{Answer given.}}$

(c) Now the particle is subject to the simple harmonic oscillator potential of characteristic frequency ω for which the normalized energy eigenstates $|n\rangle$ correspond to energy eigenvalues $E_n = \hbar\omega(n+\frac{1}{2})$ for $n = 0, 1, 2, \ldots$. Suppose at $t = 0$ the state vector is given by

$$|\psi(t=0)\rangle = e^{-ip_{\rm op}b/\hbar}\,|0\rangle$$

where b is a real number. Calculate the expectation value of position as a function of time for $t \geq 0$. You may find the following operator relations useful:

$$\begin{aligned} x_{\rm op} &= \tfrac{l_\circ}{\sqrt{2}}\left(a^\dagger_{\rm op} + a_{\rm op}\right) & p_{\rm op} &= \tfrac{i\hbar}{l_\circ\sqrt{2}}\left(a^\dagger_{\rm op} - a_{\rm op}\right) \\ a^\dagger_{\rm op}\,|n\rangle &= \sqrt{n+1}\,|n+1\rangle & a_{\rm op}\,|n\rangle &= \sqrt{n}\,|n-1\rangle \\ H_{\rm op} &= \hbar\omega\left(a^\dagger_{\rm op}a_{\rm op} + \tfrac{1}{2}\right) & \left[a_{\rm op}, a^\dagger_{\rm op}\right] &= 1 \end{aligned}$$

where $l_\circ = \sqrt{\hbar/m\omega}$ is the classical turning point corresponding to the ground-state energy. *Hint: Ehrenfest's Principle.*

$$\boxed{\langle x\rangle = b\cos(\omega t)}$$

Problem #11

A cross section of two very long coaxial conducting cylinders is shown below. The cylinders are both split into two halves on the same plane. The bottom half of the outer cylinder (radius ρ_2) and the top half of the inner cylinder (radius ρ_1) are grounded while the other half of each is kept at potential V_0. There is no leakage across the gaps between the upper and lower portions of each cylinder. Furthermore, the gaps are small enough so that you can neglect them.

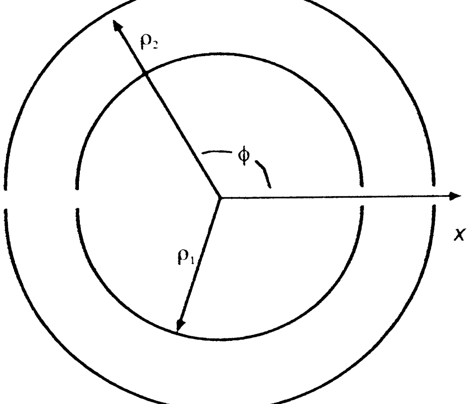
(a) Find the potential $\Phi(\rho,\phi)$ and electric field $\mathbf{E}(\rho,\phi)$ for $\rho \leq \rho_1$.

(b) Find the potential $\Phi(\rho,\phi)$ and electric field $\mathbf{E}(\rho,\phi)$ for $\rho_1 \leq \rho \leq \rho_2$.

(c) Find the potential $\Phi(\rho,\phi)$ and electric field $\mathbf{E}(\rho,\phi)$ for $\rho \geq \rho_2$.

a) $$\Phi(\rho,\phi) = \frac{V_0}{2} - \frac{2V_0}{\pi}\sum_{n\ \text{odd}} \left(\frac{\rho}{\rho_1}\right)^n \frac{\sin(n\phi)}{n}$$

$$\vec{E} = \hat{r}\,\frac{2V_0}{\pi}\sum_{n\ \text{odd}}\left(\frac{\rho^{n-1}}{\rho_1^n}\right)\sin(n\phi) + \hat{\phi}\,\frac{2V_0}{\pi}\sum_{n\ \text{odd}}\left(\frac{\rho^{n-1}}{\rho_1^n}\right)\cos(n\phi)$$



b) $$K = \frac{2V_0}{\pi}\,\frac{1}{\rho_1^n\rho_2^{-n} - \rho_2^n\rho_1^{-n}}$$

$$\Phi(\rho,\phi) = \frac{V_0}{2} - K\sum_{n\ \text{odd}}\left\{\frac{1}{n}\left[(\rho_2^{-n}+\rho_1^{-n})\rho^n - (\rho_2^n+\rho_1^n)\rho^{-n}\right]\sin(n\phi)\right\}$$

$$\vec{E} = \hat{r}K\sum_{n\ \text{odd}}\left[(\rho_2^{-n}+\rho_1^{-n})\rho^{n-1} + (\rho_2^n+\rho_1^n)\rho^{-n-1}\right]\sin(n\phi)$$
$$+\ \hat{\phi}K\sum_{n\ \text{odd}}\left[(\rho_2^{-n}+\rho_1^{-n})\rho^{n-1} - (\rho_2^n+\rho_1^n)\rho^{-n-1}\right]\cos(n\phi)$$

Problem 11 cont.

(c)

$$\Phi(\rho,\phi) = \frac{V_0}{2} + \frac{2V_0}{\pi}\sum_{n \text{ odd}} \left(\frac{\rho}{\rho_2}\right)^{-n} \frac{\sin(n\phi)}{n}$$

$$\vec{E} = \hat{r}\,\frac{2V_0}{\pi}\sum_{n \text{ odd}} \frac{\rho^{-n-1}}{\rho_2^{-n}} \sin(n\phi) + \hat{\phi}\,\frac{2V_0}{\pi}\sum_{n \text{ odd}} \frac{\rho^{-n-1}}{\rho_2^{-n}} \cos(n\phi)$$

Problem #12 (calculators are provided for this problem)

Part 1: What fraction of H_2 gas at sea level and $T = 300$ K has sufficient speed to escape from the Earth's gravitational field? You must show how to determine the escape velocity. (You may assume an ideal gas law behavior for H_2 gas.)

Part 2: Estimate how long it would take for a single H_2 molecule with this escape speed to leave the Earth's atmosphere. (To simplify this question, place the H_2 molecule in the upper atmosphere and model the atmosphere above the molecule as an inert gas that is present as an isothermal and homogeneous layer of thickness $d = 100$ km and at a temperature $T = 300$ K with a mean density of $n = 2.5 \times 10^{25}$ atom/m^3. Treat all collisions as elastic collisions.)

Possibly useful numbers:
$R_{earth} = 6.4 \times 10^3$ km
$M_{earth} = 6 \times 10^{24}$ kg
$M_{hydrogen\ molecule} = 3 \times 10^{-27}$ kg
$\pi = 3.14159$

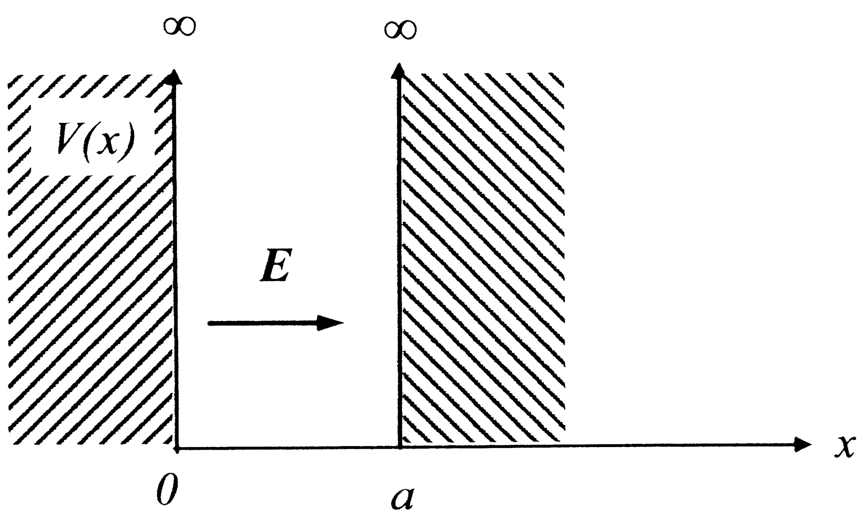
$\boxed{1.\ 6 \times 10^{-5}}$

$\boxed{2.\ 10^4 \text{ years}}$

Problem #13

Consider an electron that is confined in the one-dimensional infinite potential well shown in the figure below. Assume that a uniform electrical field of $\vec{E} = E_o\hat{x}$ is applied in the region where the electron is confined. Ignore electron spin and all other interactions of the electron with the environment, such as vacuum fluctuations. Assume that the maximum potential energy experienced by the electron in the applied field is much smaller than the ground state energy of the unperturbed system, and answer the following questions:

a. Determine the first-order corrections to the energy levels E_n of the unperturbed system. Are the separations between the levels changed?
b. By taking into account the perturbation of the ground state determine to a first-order approximation the probability that the electron will be found in the first excited state of the unperturbed system.
c. Assume that at $t=0$ the electric field is turned off. Ignoring states higher than the first excited state determine the time evolution of the system and comment on whether this is what would happen in reality and why.



a.

$$E_n = \varepsilon_n + \frac{2eE_o}{a}\int_0^a x\, Sin^2 k_n x\ dx = \varepsilon_n + \frac{2eE_o}{a}\left(\frac{a^2}{4} - \frac{1}{2}\int_0^a x\, Cos 2k_n x\ dx\right) = \varepsilon_n + \lambda, \text{ where } \lambda = \frac{eE_o a}{2}$$

b. $p = \dfrac{|\langle \phi_2 | \Phi_1 \rangle|^2}{|\langle \Phi_1 | \Phi_1 \rangle|^2} = \dfrac{\gamma^2}{(1+\gamma^2)} \approx \gamma^2$

c. $\varphi(x,t) = e^{-iH(t)/\hbar}\varphi(x,t=0) = \alpha\phi_1(x)e^{-i\omega_1 t} + \beta\phi_2(x)e^{-i\omega_2 t}$; where

$\omega_1 = \dfrac{\varepsilon_1}{\hbar}$ and $\omega_2 = \dfrac{\varepsilon_2}{\hbar}$, where the ε_n's are the unperturbed energies

Problem #14

Find the Green's function of the Helmholtz equation for the cube defined by $0 \le x, y, z \le L$ by solving the equation

$$\nabla^2 u + k^2 u = \delta(\mathbf{x} - \mathbf{x}'), \tag{1}$$

subject to the condition $u = 0$ on the surface of the cube.

for $0 \le z \le z'$

$$G = u = -\frac{\sin(P(L-z'))\sin(Pz)}{L^2 P \sin(PL)} \left\{ 4 \sum \sin\left(\frac{n\pi y}{L}\right) \sin\left(\frac{n\pi y'}{L}\right) \sin\left(\frac{m\pi x'}{L}\right) \sin\left(\frac{m\pi x}{L}\right) \right\}$$

for $z' < z < L$

$$G = u = -\frac{\sin(pz')\sin(p(L-z))}{L^2 p \sin(PL)} \left\{ 4 \sum \sin\left(\frac{n\pi y}{L}\right) \sin\left(\frac{n\pi y'}{L}\right) \sin\left(\frac{m\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) \right\}$$

Problem #15

Answer the following questions. The answers are short and simple.

a. Why is it that goggles provide clear vision underwater while without goggles objects are blurred? Hint: $n_{cornea} \approx 1.4$, $n_{water} \approx 1.3$ and $n_{air} \approx 1.0$.
b. Determine the wind velocity of a typical hurricane at a distance of 10 km from its center, assuming that the rate of pressure drop and the density of the air along the radial direction are constant and are typically equal to $dp/dr \sim 0.3\ N/m^3$ and $\rho \sim 1.3\ mg/cm^3$, respectively. Ignore the Coriolis force.
c. Assume that you are stripped of your clothes and placed inside a hollow metallic box of an arbitrary shape and are initially sitting on an insulating chair also placed inside the box. Suppose that the metallic box is connected to a 50,000-volt DC power supply by a copper wire pushed tightly through a hole drilled across the metallic box and you are asked to jump off the chair and grab the free end of the copper wire. Will your life be in danger? Explain.
d. A particle of mass m is in a stable circular orbit of radius r with a potential energy $V(r) = -\alpha r^{-\beta}$, where α and β are positive constants. Determine the relation between the angular momentum of particle and r.
e. Light of wavelength λ=600 nm is incident in air normally on a flat glass plate whose front surface is coated with a dielectric material of refractive index n_c=1.40. The refractive index of glass is n_g=1.30. What is the minimum thickness, τ, of the coating layer that will minimize the reflection of the incident light?

Answers

a. - -

b. ~173 km/h

c. nothing will happen to you

d. $L = \sqrt{\dfrac{\alpha\beta m}{r^{\beta-2}}}$

e. τ = 214 nm