Frequency Dependence of EM Wave Propagation.

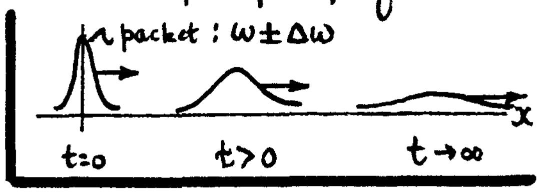
$$\boxed{c \to v = c/\sqrt{\mu\epsilon};\quad n = c/v = \sqrt{\mu\epsilon}.}$$

Dispersion & Attenuation for EM Waves in Dissipative Media [Jackson: Secs 7.5 & 7.7].

1) As we have noted for the Fresnel Formulas, the index of refraction, namely

$$\boxed{n = \sqrt{\mu\epsilon} = \sqrt{\epsilon(\omega)}} \left\{ \begin{matrix}\text{non-permeable}\\ \text{medium, } \mu = 1\end{matrix} \right. \quad (1)$$

is of central importance in describing how an EM wave propagates in a material. In actual materials, the atoms/molecules can resonate at many different frequencies under stimulation by EM waves; at these resonant frequencies, where the atoms absorb the EM wave energy, the character of the wave propagation must change radically. We expect $n = n(\omega)$ becomes a fcn of frequency, and should show absorption features at specific frequencies.



Also, the propagation velocity $v = c/n \to v(\omega)$, and so a "wavepacket" containing some spread of frequencies $\omega \pm \Delta\omega$ will spread out, or disperse, as it propagates, as sketched. Finally, we have only briefly noted the effects of attenuation due to the medium, through its conductivity σ. We now want to account in more detail for,

$$\epsilon \to \epsilon(\omega)\ [\text{ABSORPTION}],\quad v \to v(\omega)\ [\text{DISPERSION}],\quad \sigma \to \sigma(\omega)\ [\text{ATTENUATION}]. \quad (2)$$

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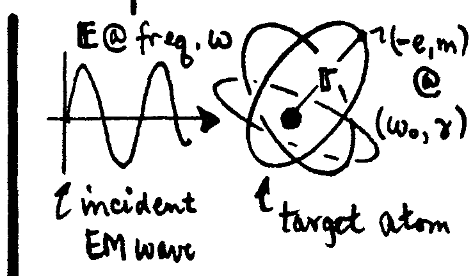
2) The place to begin is to construct a model for the dielectric cnst $\epsilon = \epsilon(\omega)$. This will tell us how the medium is polarized by the EM wave field $\mathbb{E}(\mathbf{x}, t)$ during its passage; this polarization is the basic interaction between the wave $\mathbb{E}$ and the medium's electrons $(-e, m)$. Simplest is a damped SHO model:

$$\to m\,[\ddot{\mathbf{r}} + \gamma\dot{\mathbf{r}} + \omega_0^2 \mathbf{r}] = -e\,\mathbb{E}(\mathbf{x}, t). \quad (3)$$



Here the electron's coupling to $\mathbb{E}$ is characterized by the two cnsts ω_0 & γ... ω_0 is its bound orbital frequency in its parent atom, and γ is a damping cnst represent-

SHO model for the dielectric constant $\epsilon(\omega)$.

ing an inhibition of the electron motion due to interaction with neighboring e's. Now if $\mathbb{E} = \mathbb{E}_0 e^{-i\omega t}$ is monochromatic at freq. ω, then $\mathbb{r} = \mathbb{r}_0 e^{-i\omega t}$ (after transients) and Eq. (3) yields -- upon division by $e^{-i\omega t}$...

$$\rightarrow m\mathbb{r}_0\left[-\omega_0^2 + i\gamma\omega + \omega_0^2\right] = -e\mathbb{E}_0 \underset{\text{MOMENT}}{\overset{\text{DIPOLE}}{\Longrightarrow}} \mathbb{p} = -e\mathbb{r}_0 = \frac{(e^2/m)\,\mathbb{E}_0}{(\omega_0^2-\omega^2) - i\gamma\omega},$$

i.e., $\underline{\mathbb{p} = \alpha\,\mathbb{E}_0}$, w/ $\boxed{\alpha = \frac{e^2}{m}\Big/\left[(\omega_0^2-\omega^2) - i\gamma\omega\right]}$ polarizability for one e in orbit with (ω_0, γ). (4)

Generalize this α(one e) result to α for Z electrons per atom and N atoms per unit volume by simple addition (assumes all e's act independently)...

$$\alpha \longrightarrow N\frac{e^2}{m}\sum_{j=1}^{Z} f_j\Big/\left[(\omega_j^2-\omega^2) - i\gamma_j\omega\right] \quad (\omega_j, \gamma_j) \text{ characterize orbit of } j^{\text{th}} \text{ electron.} \tag{5}$$

The f_j are called "oscillator strengths": they are numbers, w/ norm $\sum_j f_j = Z$, that specify the relative polarization contribution of the j^{th} electron (e's that are strongly bound have ~ small f_j's, while weakly bound e's have ~ large f_j's).

The dielectric cnst for a medium with the polarizability in Eq.(5) is then:

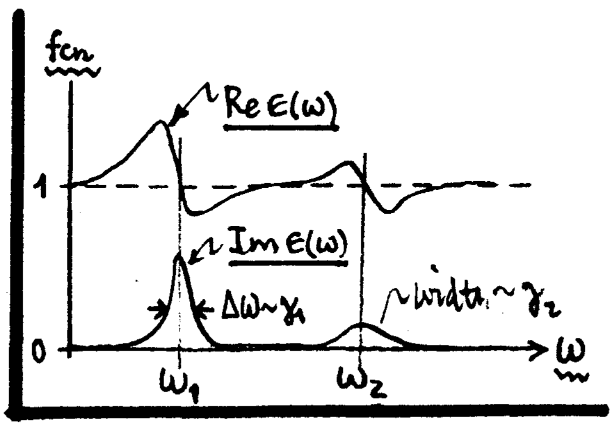
$$\boxed{\epsilon(\omega) = 1 + 4\pi\alpha = 1 + \omega_p^2\sum_{j=1}^{Z} g_j\Big/\left[(\omega_j^2-\omega^2) - i\gamma_j\omega\right]} \quad \text{Jackson Eq. (7.51)}$$

w/ $\omega_p^2 = 4\pi NZe^2/m$ {called the "plasma freq."}, $g_j = f_j/Z$ {$g_j \leq 1$ is a "fractional oscillator strength"} (6)

$$\underline{\mathrm{Re}\,\epsilon(\omega)} = 1 + \omega_p^2\sum_j g_j\left[\frac{\omega_j^2-\omega^2}{\mathcal{D}_j(\omega)}\right], \quad \underline{\mathrm{Im}\,\epsilon(\omega)} = \omega_p^2\sum_j g_j\left[\frac{\gamma_j\,\omega}{\mathcal{D}_j(\omega)}\right];$$

w/ $\mathcal{D}_j(\omega) = (\omega_j^2-\omega^2)^2 + (\gamma_j\omega)^2$. (7)

The frequency dependence of this model is sketched at right. Both $\mathrm{Re}\,\epsilon(\omega)$ & $\mathrm{Im}\,\epsilon(\omega)$ show rapid variation with ω near each of the atomic resonant frequencies ω_j.



Wave Propagation for a frequency dependent $\epsilon(\omega)$.

3) The functional form of $\epsilon(\omega)$ connects to actual wave propagation as follows. Recall that the wave number k for propagation in the medium is...

$$\rightarrow k = \frac{\omega}{v} = \frac{\omega}{c}\sqrt{\mu\epsilon} \quad \ldots\ \epsilon\text{, and so } k\text{, is now } \underline{\text{complex}};\ \text{set } \underline{\mu = 1} \left(\begin{matrix}\text{nonpermeable}\\ \text{medium}\end{matrix}\right). \tag{8}$$

... put: $k = \beta + \frac{1}{2}i\alpha$, and define: $\underline{k_0 = \frac{\omega}{c}}$ ✓ free space wave # ...

$$\text{So}/\!/\ \underline{\underline{k^2 = \left(\beta^2 - \frac{\alpha^2}{4}\right) + i\alpha\beta = k_0^2\,\epsilon(\omega)}}\ \checkmark \quad \text{with: } \beta = \mathrm{Re}\,k,\ \alpha = 2\,\mathrm{Im}\,k. \tag{9}$$

Eq. (9) can be solved for α & β separately, with the following results...

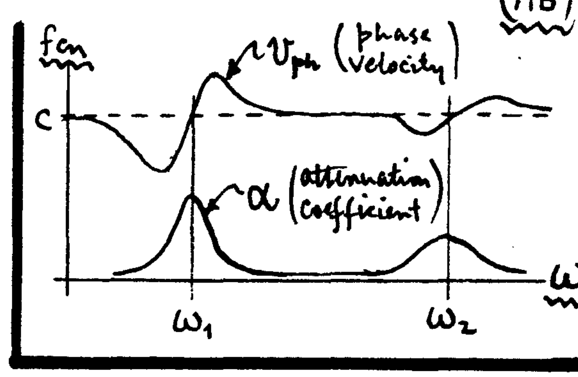
$$\left[\!\!\left[\begin{matrix} \left.\begin{matrix}\text{effective}\\ \text{wave \#}\end{matrix}\right\}\ \underline{\beta = k_0\sqrt{\epsilon_R}\,U(\epsilon)}, & \left.\begin{matrix}\text{attenuation}\\ \text{coefficient}\end{matrix}\right\}\ \underline{\alpha = \frac{\beta}{[U(\epsilon)]^2}(\epsilon_I/\epsilon_R)}; \\ \text{w}/\!/\ \epsilon_R = \mathrm{Re}\,\epsilon(\omega),\ \epsilon_I = \mathrm{Im}\,\epsilon(\omega), & \text{and}/\!/\ U(\epsilon) = \left[\frac{1}{2}\left(1 + \sqrt{1 + (\epsilon_I/\epsilon_R)^2}\right)\right]. \end{matrix}\right]\!\!\right] \tag{10}$$

The parameter $\alpha = 2\,\mathrm{Im}\,k$ is clearly connected with <u>attenuation</u> of the wave, as

$$\left.\begin{matrix}\text{wave}\\ \text{field}\end{matrix}\right\}\ \mathbb{E} \propto e^{ikz} = \left(e^{-\frac{1}{2}\alpha z}\right)e^{i\beta z} \Rightarrow \left.\begin{matrix}\text{wave}\\ \text{intensity}\end{matrix}\right\}\ \boxed{|\mathbb{E}|^2 \propto e^{-\alpha z}}. \tag{11A}$$

The physics of $\beta = \mathrm{Re}\,k$ is less apparent, but we can define a <u>phase velocity</u>:

$$\rightarrow v_{ph} = \omega/\beta = c/U(\epsilon)\sqrt{\epsilon_R} = v_{ph}(\omega). \tag{11B}$$



The frequency dependence of the attenuation coefficient $\alpha(\omega)$ & wave phase velocity $v_{ph}(\omega)$ is sketched at right. α & v_{ph} roughly repeat the behavior of $\mathrm{Im}\,\epsilon$ & $1/\mathrm{Re}\,\epsilon$ per the sketch last page -- they show rapid variation near the atomic resonance freqs. ω_j [<u>NOTE</u>: $U(\epsilon) \simeq 1$ except near the resonances]. Near the resonances, the wave is both <u>absorbed</u> & <u>dispersed</u>.

<u>**CONCLUSION**</u>: when $\epsilon \rightarrow \epsilon(\omega)$, the propagation problem is seriously altered.

from J. D. Jackson "Classical ElectroDynamics" (Wiley, 2nd ed., 1975).

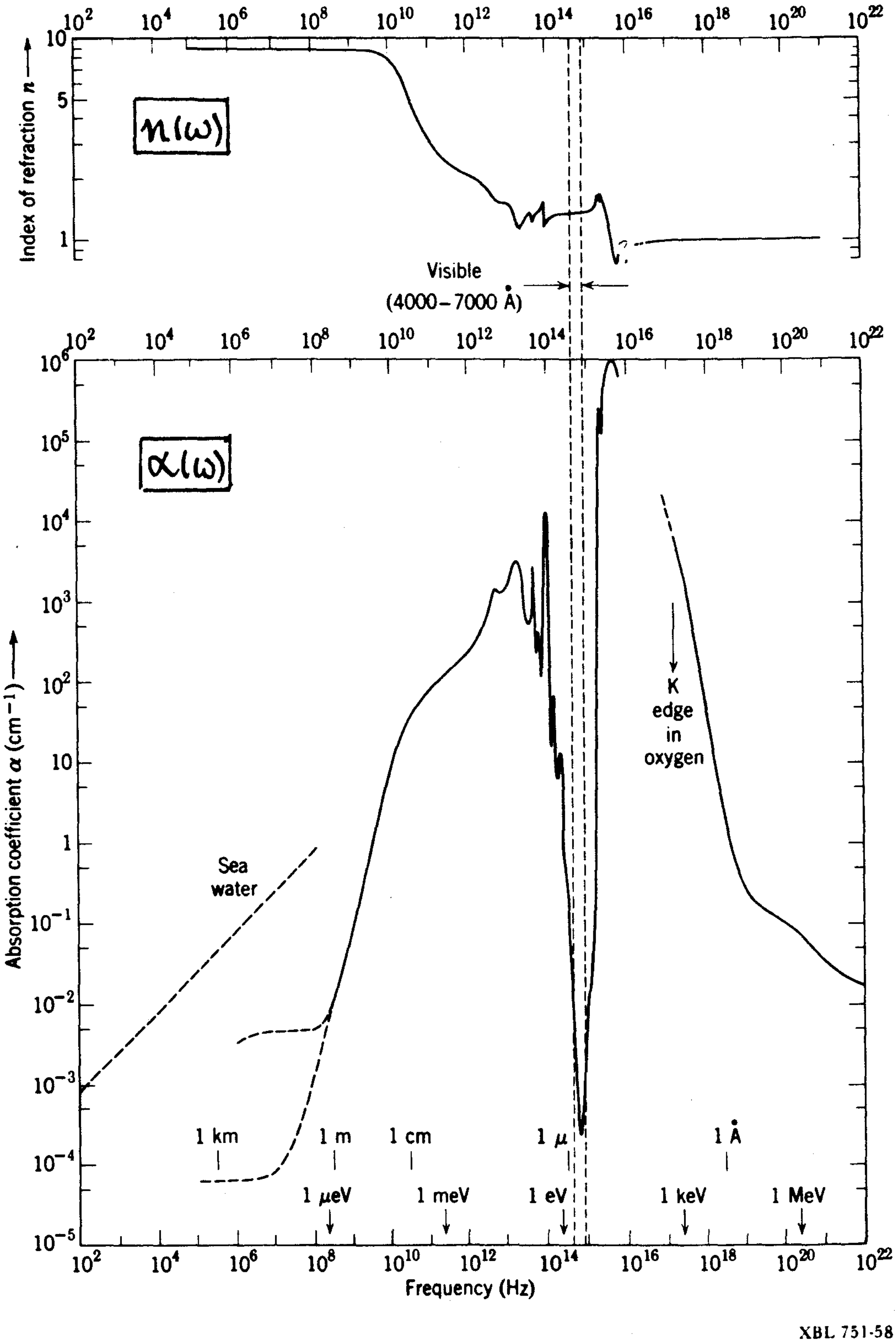


Fig. 7.9 The index of refraction (top) and absorption coefficient (bottom) for liquid water as a function of linear frequency. Also shown as abscissas are an energy scale (arrows) and a wavelength scale (vertical lines). The visible region of the frequency spectrum is indicated by the vertical dashed lines. The absorption coefficient for sea water is indicated by the dashed diagonal line at the left. Note that the scales are logarithmic in both directions.