Physics 505 Hour Exam 13 Nov. 1970

① 10 points

Consider the product operator $C = AB$. Suppose both A and B are Hermitian operators. What condition must be imposed on A and B to insure that C is also Hermitian?

✓ ② 15 points

The phase velocity of an electromagnetic wave in a waveguide is

$$v_p = c/\sqrt{1-(\omega_0/\omega)^2}$$

where c is the velocity of light, ω is the angular frequency, and ω_0 is a characteristic cutoff frequency. What is the group velocity for such a wave?

③ 20 points

For a one-dimensional system described by the Hamiltonian $H = (p^2/2m) + V(x)$, obtain an expression for the time rate of change of kinetic energy $d\langle p^2/2m\rangle/dt$. What relation does this have to the classical work-energy theorem?

✓ ④ 25 points

The total energy of a one-dimensional harmonic oscillator (mass m, natural frequency ω) can be written as $E = (p^2/2m) + \frac{1}{2}m\omega^2x^2$, where p is the momentum and x the position. Use the uncertainty relations to estimate the minimum energy of the oscillator.

⑤ 30 points

A particle is in a state described by the wave function

$$\psi(x) = A(a^2 - x^2),\ A = \text{cnst, for } -a \leq x \leq +a;\ \psi(x) = 0, \text{ for } |x| > a.$$

What is the probability that a measurement of the particle's position will yield a value between $-a/2$ and $+a/2$?

Solutions to Phys. 505 Hour Exam

13 Nov 70

1 Given $C = AB$, $A = A^\dagger$ & $B = B^\dagger$.

Note $C^\dagger = (AB)^\dagger = B^\dagger A^\dagger = BA$.

If $C^\dagger = C$, then $BA = AB$, or $[A,B] = 0$.

2 $v_p = c/\sqrt{1-(\omega_0/\omega)^2} = \omega/k$ for phase velocity

~~Solve for $\omega = \omega(k)$...~~

$$\omega\sqrt{1-\left(\frac{\omega_0}{\omega}\right)^2} = \sqrt{\omega^2-\omega_0^2} = kc \Rightarrow \omega^2 = k^2c^2 + \omega_0^2$$

$v_g = \partial\omega/\partial k$. So form

$$\omega\frac{\partial\omega}{\partial k} = kc^2 \Rightarrow v_g = c^2\frac{k}{\omega} = c^2/v_p$$

$$\therefore v_g = c\sqrt{1-\left(\frac{\omega_0}{\omega}\right)^2}.\ \text{Note } v_g v_p = c^2 \ \left\{\begin{array}{l}\text{std result for}\\ \text{waveguides}\end{array}\right.$$

3

\# 4 on ϕ 506 Mid-Term (Oct'93)

$$\frac{d}{dt}\left\langle\frac{p^2}{2m}\right\rangle = \frac{i}{\hbar}\left\langle\left[H,\frac{p^2}{2m}\right]\right\rangle = \frac{i}{2m\hbar}\left\langle[V,p^2]\right\rangle$$

But $[V,p^2] = p[V,p] + [V,p]p$

And $[V,p] = -i\hbar\left[V,\frac{\partial}{\partial x}\right] = +i\hbar\frac{\partial V}{\partial x}$

$$\therefore [V,p^2] = i\hbar\left\{p\frac{\partial V}{\partial x} + \frac{\partial V}{\partial x}p\right\}$$

$$\Rightarrow \frac{d}{dt}\left\langle\frac{p^2}{2m}\right\rangle = -\frac{1}{2m}\left\langle p\frac{\partial V}{\partial x} + \frac{\partial V}{\partial x}p\right\rangle \ \left\{\begin{array}{l}\text{Set } F = -\partial V/\partial x = \text{force}\\ v = p/m = \text{velocity}\end{array}\right.$$

$$\therefore \frac{d}{dt}\langle \underbrace{T}_{\text{K.E.}}\rangle = \frac{1}{2}\langle vF + Fv\rangle \text{ is desired QM expression.}$$

The classical work-energy thm is $\frac{dT}{dt} = \vec{F}\cdot\vec{v} = Fv$, for 1D.

We see we must replace Fv QMly with $\frac{1}{2}(Fv + vF)$.

4 $E = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2$ for 1D SHO

Unc. relations $\Rightarrow$ $\Delta x \Delta p \geq \frac{1}{2}\hbar$.

For given Δx, get $\Delta p \simeq \frac{1}{2}\hbar/\Delta x$ at least

$\therefore E \simeq \frac{1}{4}\frac{\hbar^2}{(\Delta x)^2}\frac{1}{2m} + \frac{1}{2}m\omega^2(\Delta x)^2$ at least

Minimize E w.r.t. Δx ...

$$\frac{\partial E}{\partial \Delta x} = 0 \Rightarrow -\frac{\hbar^2}{8m}\frac{2}{(\Delta x)^3} + m\omega^2 \Delta x = 0$$

$$\Rightarrow \frac{\hbar^2}{4m}\frac{1}{(\Delta x)^3} = m\omega^2 \Delta x \quad \text{or} \quad (\Delta x)^2 = \frac{\hbar}{2m\omega}$$

$$\therefore E_{min} \simeq \frac{\hbar^2}{8m}\frac{2m\omega}{\hbar} + \frac{1}{2}m\omega^2\frac{\hbar}{2m\omega} = \frac{1}{2}\hbar\omega$$

5 on
ϕ 506 MidTerm
(Oct. '93)

5 Desired prob is: $P = \int_{-a/2}^{+a/2} |\psi(x)|^2 dx \Big/ \int_{-a}^{+a} |\psi(x)|^2 dx$

$$\int_{-a}^{+a} |\psi(x)|^2 dx = A^2 \int_{-a}^{+a} (a^2 - x^2)^2 dx = A^2 \int_{-a}^{+a} (a^4 - 2a^2x^2 + x^4)\, dx$$

$$= A^2\left(2a^5 - \frac{4}{3}a^5 + \frac{2}{5}a^5\right) = A^2\left(\frac{16}{15}\right)a^5$$

$$\int_{-a/2}^{+a/2} |\psi(x)|^2 dx = A^2\left(a^5 - \frac{4}{3}a^5\left(\frac{1}{2}\right)^3 + \frac{2}{5}a^5\left(\frac{1}{2}\right)^5\right) = A^2\left(\frac{5}{6} + \frac{1}{80}\right)a^5$$

$$\therefore P = \frac{\frac{5}{6} + \frac{1}{80}}{16/15} = \frac{\cancel{15}^{5}}{16} \times \frac{5}{\cancel{6}_{2}} + \frac{\cancel{15}^{3}}{16}\frac{1}{\cancel{80}_{16}} = \frac{8\times 25}{8\times 32} + \frac{3}{256} = \frac{203}{256} = 0.792$$