de Broglie hypothesis ≃ Planck hypothesis read backwards.

De Broglie Relation & Wave Packets

1) From the "Squirrel cage" experiment (pp. Duality 1-3), we have reason to believe that we see quantization if and only if we are dealing with wave-particle duality. And we have just seen that all the dynamical variables of an electron bound in an H-atom are quantized. These two ideas together imply that the electron itself should show wave-particle duality.

If so, we need a relation which associates a wave-like property (e.g. a wavelength λ) with the well known particle-like properties of an electron (e.g. its momentum p)... just as the Planck relation $E=h\nu$ for a photon associated the particle-like energy E with the wave-like frequency ν. Just as for the photon, the relation for an electron must be an inspired guess. But not a wild guess -- it makes sense to be "inspired" by the photon, for which wave-particle duality is well established.

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2) Thus, just as Planck's hypothesis asserted...

(1900) [For (massless) photons, particle-like properties (E,p) are associated with the wave description (ν,λ) via: $E=h\nu$, $p=h/\lambda$. (18)

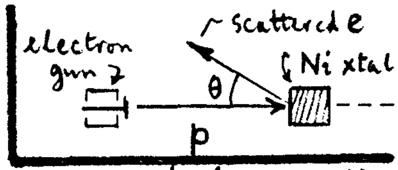
... de Broglie's hypothesis asserted...

(1923) [For all free (massive) particles, wave-like properties (ν,λ) are associated with the particle description (E,p) via: $\nu=E/h$, $\lambda=h/p$. (19)

Together, these hypotheses are pleasingly symmetric: waves & particles are completely interchangeable (possibly differing only by some "trivial" mass parameter), and de Broglie is just Planck read backwards. BUT, just as Planck's photon-as-particle needed experimental confirmation, now de Broglie's electron-as-wave required some real-world evidence.

Davisson-Germer Expt. How is a wave associated with a particle?

3) Experimental confirmation of de Broglie's hypothesis came soon, with the Davisson-Germer Expt. (1927). Here, electrons at momentum p are shot at a nickel crystal, and the intensity pattern of scattered electrons is recorded. The scattered electron distribution is not smooth, but shows well defined maxima at special angles θ which fit a diffraction formula: $n\lambda = d\sin\theta$, w/ $d \sim 1$ Å is characteristic of the xtal lattice spacing (and depends on the xtal orientation). When λ is extracted from the data, it is found to fit $\boxed{\lambda = h/p}$ over a large range of energies. So, in this expt, electrons really behave like waves.



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4) De Broglie's hypothesis associates waves with particles, but does not specify how the wave behaves in detail -- e.g. how is the wave affected by a force? We need a wave equation for the particle wave to say how it moves -- ultimately this will be Schrödinger's Equation. Here, we shall look into the question of how a wave can be associated with a particle to begin with.

For photons, we were lucky -- we knew both the wave eqtn (from classical EM: $[(\partial/\partial x)^2 - \frac{1}{c^2}(\partial/\partial t)^2]\phi = 0$) and the particle dynamics ($E^2 = p^2c^2 + \text{zero}$) from the outset -- "all" that needed doing was to connect the waves & particles via Planck's hypothesis ($E = h\nu$, $p = h/\lambda$). For electrons, on the other hand, we know the particle dynamics $[\mathbf{F} = -e(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B})]$, but we have scant information on the wave that is supposed to obey $\nu = E/h$, $\lambda = h/p$, per de Broglie.

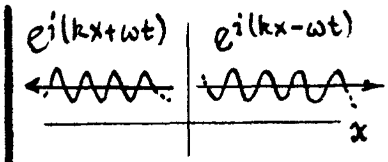
To gain some insight into how a particle's wave might come about, we again turn to what we know about light, and -- in particular -- how a photon (i.e. a localized concentration of light energy) can be constructed. All light in empty space obeys the EM wave equation... in 1D space...

Wave-packets for photons.

$$\left[(\partial/\partial x)^2 - \frac{1}{c^2}(\partial/\partial t)^2\right]\phi(x,t) = 0 \quad \int \begin{matrix} x = \text{space cd}, \ t = \text{time}; \ c = \text{light speed}; \\ \phi = \text{any comp}^{\text{nt}} \text{ of } \mathbb{E} \text{ or } \mathbb{B}. \end{matrix} \tag{20}$$

The amplitude ϕ specifies the wave intensity -- e.g. if $\phi \propto \mathbb{E}$, then $|\phi|^2 \propto |\mathbb{E}|^2$ is proportional to the wave's energy density. Elementary solutions to (20) are...

$$\rightarrow \phi(x,t) = \text{cnst} \times e^{i(kx \pm \omega t)}, \ \text{w/} \ k = \frac{\omega}{c} \quad \int \begin{matrix} (+) \Rightarrow \text{leftward} \\ (-) \Rightarrow \text{rightward} \end{matrix}$$



PLANE WAVES (21)

These are the so-called plane-wave solutions. They have a prominent feature that their intensity is everywhere uniform in both space & time, i.e. $|\phi(x,t)|^2 = |\text{cnst}|^2$. Since such solutions exhibit no localization at all of any dynamical property of light, then -- by themselves -- they cannot represent photons.

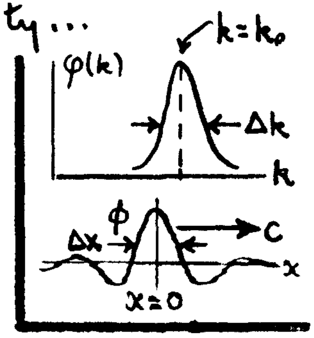
To get to a localized, photon-like, concentration of light, we can superimpose a a number of the above plane-waves, with amplitudes & phases chosen so that they destructively interfere in all but a finite region of space. A continuous superposition of such waves gives a Fourier-type integral...

$$\rightarrow \phi(x,t) = \int_{-\infty}^{\infty} \varphi(k)\, e^{i(kx-\omega t)}\, dk, \quad \omega = kc. \tag{22}$$

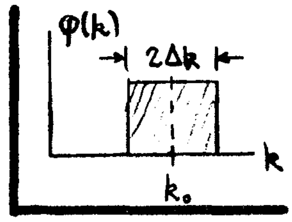
The amplitude fcn $\varphi(k)$ can be freely chosen (so long as it does not depend on x and/or t); in general, $\varphi(k)$ can be a complex fcn of k. The "dispersion relation" $\omega = kc$ ensures that ϕ in Eq. (22) actually obeys the wave eqtn of Eq. (20). The fact that ϕ[Eq. (22)] is a group of waves allows for such wave properties as interference/diffraction, common to both photons & electrons.

The superposition in Eq. (22) has the following general property...

[If $\varphi(k)$ is appreciable only in the range $k_0 \pm \Delta k$, then ϕ is localized in space to a size $\Delta x \sim 1/\Delta k$. Also, since $\Delta\omega = c\Delta k$, ϕ is localized in time to $\Delta t \sim 1/\Delta\omega$. So $\phi(x,t)$ is confined to a "wave-packet" of finite duration. (23)



An example of a photon wave-packet.



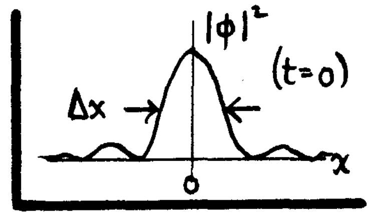
$\boxed{\text{EXAMPLE}}$ Wave-packet w/ amplitude: $\varphi(k) = \begin{cases} 1, & k \text{ in } k_0 \pm \Delta k; \\ 0, & \text{otherwise.} \end{cases}$

The packet in Eq. (22) is -- with $\omega = kc$ ...

$$\phi(x,t) = \int_{k_0-\Delta k}^{k_0+\Delta k} 1 \times e^{i(kx-\omega t)}\, dk = \int_{k_0-\Delta k}^{k_0+\Delta k} e^{i(x-ct)k}\, dk$$

or/ $$\phi(x,t) = \frac{2}{x-ct}\, e^{i(k_0x-\omega_0 t)} \sin(x\Delta k - t\Delta\omega) \quad \Big\{ \begin{matrix} \omega_0 = k_0 c, \\ \Delta\omega = c\Delta k. \end{matrix} \tag{24}$$

The space localization of ϕ is easily seen at a given time -- say $t=0$...

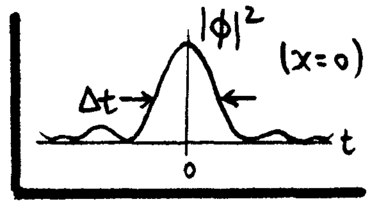


$\rightarrow |\phi(x,0)|^2 = \frac{4}{x^2} \sin^2(x\Delta k)$,

$\Rightarrow |\phi|^2$ appreciable only from $x=0$ to $\Delta x \sim 1/\Delta k$;

so/ $\boxed{\Delta k\, \Delta x \sim 1}$, for space localization of packet. (25)

Similarly, at a given position -- say $x=0$ -- one gets a time localization...



$\rightarrow |\phi(0,t)|^2 = \frac{4}{c^2t^2} \sin^2(t\,\Delta\omega)$,

$\Rightarrow |\phi|^2$ appreciable only from $t=0$ to $\Delta t \sim 1/\Delta\omega$;

so/ $\boxed{\Delta\omega\, \Delta t \sim 1}$, for time localization of packet. (26)

The packet reasonably portrays the photon's wave-particle duality -- it <u>is</u> wave-like, because it is composed of a (restricted) group of waves, and it <u>is</u> particle-like, in the sense of being localized in space & time. But it is <u>not</u> precisely a (monochromatic) wave, since it does not have a precise wavelength ($\lambda_0 = 2\pi/k_0$ is only nominal). And it is <u>not</u> precisely a (point) particle, because it is not precisely localized.

Note that in order to achieve this wave $\leftrightarrow$ particle "fuzziness", we must allow a <u>spread</u> ($\Delta\omega, \Delta k$) in the wave-like properties, and equally well a <u>spread</u> in the particle-like properties ($\Delta t, \Delta x$). These spreads are related by Eqs. (25) & (26).

5) Eqs. (25) & (26) express a very general property of wave-packets... in order to achieve localization in time & space to within Δt & Δx, we must admit spectral widths $\Delta\omega \sim 1/\Delta t$ & $\Delta k \sim 1/\Delta x$ in frequency & wavenumber, a priori. Conversely, if we specify that ω & k be known to within $\Delta\omega$ & Δk, the packet cannot be localized to better than $\Delta t \sim 1/\Delta\omega$ & $\Delta x \sim 1/\Delta k$. These relations therefore express our lack of precise knowledge of -- or uncertainty in -- whether the packet is "actually" a pure wave or pure particle. In view of wave-particle duality, this uncertainty is desirable -- we don't have to call the packet one or the other. Call the packet a "wavicle" if you wish.

If we decide to represent photons (& electrons, etc.) as "wavicles" -- as we will do soon -- we can get a preview of coming attractions by combining Eqs. (25) & (26) with the Einstein & de Broglie relations, as follows...

$$\Delta\omega\,\Delta t \sim 1, \text{ w/ } E = \hbar\omega \begin{pmatrix}\text{Einstein}\\ (\omega = 2\pi\nu)\end{pmatrix} \Rightarrow \boxed{\Delta E\,\Delta t \sim \hbar}$$
$$\Delta k\,\Delta x \sim 1, \text{ w/ } p = \hbar k \begin{pmatrix}\text{de Broglie}\\ (k = 2\pi/\lambda)\end{pmatrix} \Rightarrow \boxed{\Delta p\,\Delta x \sim \hbar}$$

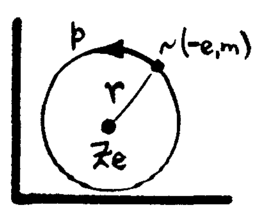
UNCERTAINTY RELATIONS (27)

This is just a translation of the wave-like uncertainties ($\Delta\omega$, Δk) into particle-like uncertainties (ΔE, Δp). In any case, to the extent that "wavicles" represent QM reality, these "uncertainty relations" must be a basic property of all matter. They are our first semi-quantitative statement of what is meant by wave-particle dualism.

To show that these "uncertainty relations" already contain substantial physics, use just Eqs (27) to quantize the H-atom in its lowest (ground) state...

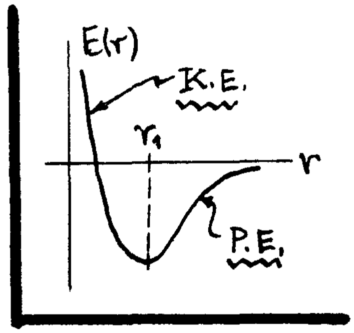
Assume a lowest stable orbit, of radius r.

Total energy: $E = \frac{1}{2m}p^2 - (Ze^2/r)$. (28)



Quantization of H-atom via Uncertainty Relations.

The very fact that the electron is <u>in</u> the orbit of radius r means that it is localized (about the nucleus) to a size $\underline{\Delta x \sim r}$. Such a localization means, by the second of Eqs. (27), that the electron must have momentum components at least of size $\underline{\Delta p \sim \hbar/\Delta x \sim \hbar/r}$. Now in Eq. (28), put the momentum $p \sim \Delta p \sim \hbar/r$. The orbit energy is then of order...



$$\rightarrow E \sim \frac{\hbar^2}{2mr^2} - \frac{Ze^2}{r}. \qquad (29)$$

The stable orbit occurs @ $dE/dr = 0$, i.e.

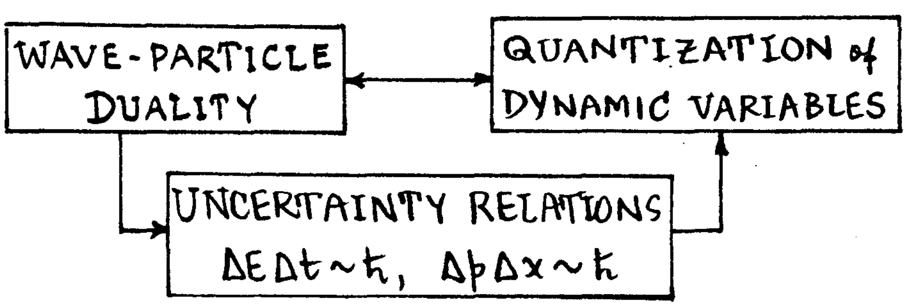
$$\left[\frac{dE}{dr} = -\frac{\hbar^2}{mr^3} + \frac{Ze^2}{r^2} = 0, \text{ so } \underline{\underline{r \sim \hbar^2/mZe^2 = r_1}}. \qquad (30)\right.$$

By balancing forces (classically), i.e. $\frac{mv^2}{r} = Ze^2/r^2$, we find we can write the orbit energy of Eq. (28) as...

$$\rightarrow E(r) = -Ze^2/2r, \text{ so for stable orbit : } \underline{\underline{E(r_1) \sim -\tfrac{1}{2}m(Ze^2/\hbar)^2}}. \qquad (31)$$

As it happens, the orbit radius r_1 and energy $E(r_1)$ agree exactly with the quantized values derived (via Correspondence Principle) in Eq. (16), p. Duality 7. The exact agreement is fortuitous -- we could have been off by factors of 2 in this estimate. But we got the right magnitude.

What we've shown here is that if we assume matter is characterized by a wave-particle duality, and then express that duality by the uncertainty relations [Eqs. (27)], we recover "well-known" quantization conditions. The uncertainty relations thus form a bridge between duality & quantization...



SUMMARY : QM Arguments to Date

Nature of Light

- light is wave-like (Young's diffraction expt., Newton's rings, etc.).
- light is particle-like (BB radⁿ, Photo-effect, Compton scattering).
- → ① light (photons) shows a wave-particle duality, w/ $\boxed{\underbrace{(\mathcal{E}, p)}_{\text{particle}} = \underbrace{\hbar(\omega, k)}_{\text{wave}}}$.

Squirrel-Cage Expt.

- if photon is a $\langle$ wave particle $\rangle$ at same time, then ∡ momentum L is quantized.
- assumed quantization of L, plus conservation of $L \Rightarrow$ electrons are wave-like.
- → ① WAVE-PARTICLE DUALITY $\Longleftrightarrow$ QUANTIZATION of DYNAMIC VARIABLES.
  - ② L appears to be quantized in units of $\hbar$.
  - ③ the electron behaves as if it had a wavelength: $\boxed{\lambda = \frac{h}{p}}$ (i.e. $k = p/\hbar$).

Correspondence Principle

- for large systems (large quantum #s), QM reduces to classical physics.
- → ① assume H-atom can emit a photon @ $\mathcal{E} = h\nu \Rightarrow \underline{\Delta L = \hbar}$; L is quantized.
  - ② L quantization in H-atom $\Rightarrow$ all its dynamical variables are quantized.

de Broglie's Hypothesis

- assume duality relations $\underline{(\mathcal{E}, p) = \hbar(\omega, k)}$ for massive particles as well as photons.
- that electrons do have wave-like properties shown by Davisson-Germer Expt.
- → ① Wave-particle duality is a universal characteristic of all matter.

Wave-Packets & Duality

- $\underline{\phi(x,t) = \int_{-\infty}^{\infty} \varphi(k)\, e^{i(kx-\omega t)}\, dk}$ is a packet of waves that can be localized in space (x) & time (t) to within tolerances Δx & Δt.
- to achieve a particle-like localization to within Δx & Δt, must allow a spread in the wave-like characteristics of size: $\Delta k \sim 1/\Delta x$, $\Delta\omega \sim 1/\Delta t$.
- → ① w/ deB hypothesis, the "spread" $\Rightarrow$ $\boxed{\Delta p\,\Delta x \sim \hbar,\ \Delta E\,\Delta t \sim \hbar}$, for all matter.
  - ② can again quantize H-atom, using just these "uncertainty relations."
  - ③ the uncertainty relations form a bridge between duality & quantⁿ

$$\boxed{\text{DUALITY}} \longleftrightarrow \boxed{\text{QUANTIZATION}}$$

$$\boxed{\Delta p\,\Delta x \sim \hbar,\ \Delta E\,\Delta t \sim \hbar}$$