From this, we conclude that the $\pm$ exchange symmetry of Ψ is a "constant of the motion", which must be included in any listing of quantum numbers that specify the overall state Ψ

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4) The even or odd exchange symmetry of the wavefcn Ψ for QM systems of n identical particles would just be a mathematical curiosity [and a source of degeneracy, as for the reflection symmetry in Eq. (1)] were it not for the following Law of Nature regarding two distinct kinds of identical particles.

I. Identical particles with integral spin $S = 0, \hbar, 2\hbar, \ldots$ [e.g. photons w/ spin $\hbar$, ground state He atoms (spin 0), π-mesons (spin 0), etc] can only be described by overall wavefcns Ψ which show even (i.e. +1, or "symmetric") exchange symmetry. Such particles can only obey Bose-Einstein statistics; they are "bosons".

II. Identical particles with half-integral spin $S = \frac{1}{2}\hbar, \frac{3}{2}\hbar, \ldots$ [e.g. electrons, protons, neutrons (all spin $\frac{1}{2}\hbar$)] can only be described by overall wavefcns Ψ which show odd (i.e. (−1), or "antisymmetric") exchange symmetry. Such particles can only obey Fermi-Dirac statistics; they are called "fermions".

More succinctly...

$$\left.\begin{aligned} &\Psi(1\ldots l\ldots k\ldots n;\ \text{BOSONS}) = (+)\,\Psi(1\ldots k\ldots l\ldots n;\ \text{BOSONS});\\ &\Psi(1\ldots l\ldots k\ldots n;\ \text{FERMIONS}) = (-)\,\Psi(1\ldots k\ldots l\ldots n;\ \text{FERMIONS}). \end{aligned}\right\} \quad (8)$$

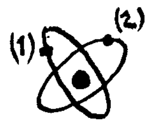
This is known as the "symmetrization postulate". In the context of nonrelativistic QM, it must be accepted as a Revealed Truth. In (relativistic) quantum field theory, it is possible to prove the spin-statistics connection [i.e. integral spin $\leftrightarrow$ Bose-Einstein, $\frac{1}{2}$ integral spin $\leftrightarrow$ Fermi-Dirac]. Beyond that, it is an empirical fact that mixed symmetries don't occur for Ψ(BOSONS) or Ψ(FERMIONS).

Application of Symmetrization Postulate.

EXAMPLE Use of Symmetrization Postulate.

Consider two weakly interacting particles in an external potential V. In lowest order (ignoring inter-particle interaction) system Ham$^{\underline{n}}$ is:

$$\rightarrow \mathcal{H}(1,2) = \mathcal{H}(1) + \mathcal{H}(2), \quad \text{w/} \quad \mathcal{H}(k) = \frac{1}{2m}\mathbb{P}_k^2 + V(k), \quad k = 1 \,\&\, 2. \tag{9}$$

... e.g. He atom 

, w/ V(k) = Coulomb potential for k$^{\underline{th}}$ electron.

Suppose one particle is in state α, and the other is in state $\beta \neq \alpha$:

$$\begin{bmatrix} \mathcal{H}(1)\,\phi_\alpha(1) = E_\alpha\,\phi_\alpha(1), \\ \mathcal{H}(2)\,\phi_\beta(2) = E_\beta\,\phi_\beta(2); \end{bmatrix} \quad \underline{\underline{\text{OR}}} \quad \begin{matrix} \mathcal{H}(1)\,\phi_\beta(1) = E_\beta\,\phi_\beta(1), \\ \mathcal{H}(2)\,\phi_\alpha(2) = E\,\phi_\alpha(2). \end{matrix} \tag{10}$$

Now $\Psi_A(1,2) = \phi_\alpha(1)\,\phi_\beta(2)$ is an eigenfcn for the system, because: $\mathcal{H}(1,2)\,\Psi_A = (E_\alpha + E_\beta)\Psi_A$, but this Ψ_A does <u>not</u> have any particular exchange symmetry. The same remarks apply to $\Psi_B(1,2) = \phi_\beta(1)\,\phi_\alpha(2)$. However, we can <u>construct</u> an appropriately symmetrized eigenstate by taking the linear combinations $\Psi_A \pm \Psi_B$, i.e.

$$\rightarrow \Psi(1,2) = \mathcal{N}\left[\phi_\alpha(1)\,\phi_\beta(2) \pm \phi_\beta(1)\,\phi_\alpha(2)\right], \quad \mathcal{N} = \text{norm cnst}. \tag{11}$$

This "symmetrized eigenstate" still has energy eigenvalue $(E_\alpha + E_\beta)$, but now shows the required exchange symmetry: $\underline{\Psi(2,1) = \pm\Psi(1,2)}$. Ψ consists of an equal admixture of [particle #1 in state α, #2 in β] and [particle #1 in state β, #2 in α], so it is equally likely to find either particle in either state -- in this way Ψ in Eq (11) treats the particles as being identical and indistinguishable. In detail...

Calculate probability of finding particle #1 in state α, #2 in β.

Norm$^{\underline{3}}$n for Ψ: $\int |\Psi(1,2)|^2\, dx = 1 \Rightarrow \mathcal{N} = 1/\sqrt{2}$, in Eq. (11).

(next page)

Required probability is:

$$P(\#1 \text{ in } \alpha, \#2 \text{ in } \beta) = |\langle \phi_\alpha(1)\phi_\beta(2) | \Psi(1,2)\rangle|^2$$

$$= \frac{1}{2}\Big|\underbrace{\langle \phi_\alpha(1)\phi_\beta(2) | \phi_\alpha(1)\phi_\beta(2)\rangle}_{1} \pm \underbrace{\langle \phi_\alpha(1)\phi_\beta(2) | \phi_\beta(1)\phi_\alpha(2)\rangle}_{0}\Big|^2 = \frac{1}{2}. \quad (12)$$

Similarly: $P(\#1 \text{ in } \beta, \#2 \text{ in } \alpha) = \frac{1}{2}$. So #1 is equally likely to be found in α or β while #2 is in β or α, so long as we use Ψ of Eq. (11).

Finally, and most importantly, in using Ψ of Eq. (11) the symmetrization postulate requires ... when the individual states are α and $\beta \neq \alpha$ ...

for BOSONS: $\Psi(2,1) = +\Psi(1,2)$, so:

$$\Psi_B(1,2) = \frac{1}{\sqrt{2}}\left[\phi_\alpha(1)\phi_\beta(2) + \phi_\beta(1)\phi_\alpha(2)\right]; \quad (13A)$$

for FERMIONS: $\Psi(2,1) = (-)\Psi(1,2)$, so:

$$\Psi_F(1,2) = \frac{1}{\sqrt{2}}\left[\phi_\alpha(1)\phi_\beta(2) - \phi_\beta(1)\phi_\alpha(2)\right]. \quad (13B)$$

These are distinctly different states, as one may see by letting $\beta = \alpha$, i.e. putting both particles in the <u>same</u> individual state. Immediately...

> When $\beta = \alpha$, the fermion wavefcn $\Psi_F(1,2) \equiv 0$, and no two identical fermions can occupy a given individual QM state. (14)

This is a statement of the <u>Pauli Exclusion Principle</u> for fermions. No such exclusion applies to bosons... when $\beta = \alpha$, take $\Psi_B(1,2) = \phi_\alpha(1)\phi_\alpha(2)$... which has the correct exchange symmetry... so an <u>arbitrary</u> number of bosons can occupy the state α, with: $\Psi_B(1,2,\ldots,n) = \phi_\alpha(1)\phi_\alpha(2)\cdots\phi_\alpha(n)$.

NOTE Fact that electrons are fermions ⇒ shell structure for multi-e atoms (K shell, L shell, etc.). What would happen if electrons were <u>bosons</u>?