SRT Math: Length of 4-Vectors. Minkowski space. {ref. Jackson, Sec. 11.7.

8) We now go back to pay more attention to "why" our 4-vectors $\tilde{A} = (A_0, \mathbb{A})$ have the peculiar $(-)$ sign in their "length": $\tilde{A}^2 = A_0^2 - \mathbb{A}^2$. It is not really a question of why, but an assertion that because ... the $(-)$ sign is there because, by definition, a 4-vector $\tilde{A}$ has an invariant "length" of the same form as the spacetime interval $d\tilde{x}$, viz. $(d\tilde{x})^2 = (dx_0)^2 - (d\mathbf{x})^2$; see the definition in Eq.(2), p. SRT 12. For the prototype 4-vector position $(x_0, \mathbf{x})$, it is $(x_0^2 - \mathbf{x}^2)$ which is the Lorentz invariant, not $(x_0^2 + \mathbf{x}^2)$.

For a 4-vector $\tilde{A} = (A_0, A_1, A_2, A_3)$, the "length" $\tilde{A}\cdot\tilde{A} = A_0^2 - A_1^2 - A_2^2 - A_3^2$ does resemble the invariant length of a 3-vector $\mathbb{V}$ under rotations $\underset{\sim}{R}$ ...

$$\rightarrow \mathbb{V} = (V_1, V_2, V_3),\ ^{w/}\ \mathbb{V}\cdot\mathbb{V} = V_1^2 + V_2^2 + V_3^2 \Big\} \begin{array}{l}\text{an invariant under rotation:}\\ \mathbb{V} \rightarrow \mathbb{V}' = \underset{\sim}{R}\,\mathbb{V}\end{array} \tag{19}$$

but -- while $\mathbb{V}\cdot\mathbb{V}$ has terms all of the same sign, corresponding to the Euclidean space in which it is defined -- $\tilde{A}\cdot\tilde{A}$ has different signs for its TIMELIKE term A_0^2 and SPACELIKE terms A_k^2. That's OK... the space in which 4-vectors are defined is just non-Euclidean. It is called Minkowski space.

We can accommodate the $(-)$ sign in Minkowski space as follows. We redefine the usual scalar product between vectors with four components, viz...

Let: $\tilde{A} = (A_0, A_1, A_2, A_3)$ & $\tilde{B} = (B_0, B_1, B_2, B_3)$ be 4-comp$^{\underline{t}}$ vectors. (20)

for normal vectors } $\tilde{A}\cdot\tilde{B} = \underbrace{[A_0, A_1, A_2, A_3]}_{\text{row vector}} \underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{\underset{\sim}{e} = \underset{\sim}{I}\text{ (identity), Euclidean metric tensor;}} \underbrace{\begin{pmatrix} B_0 \\ B_1 \\ B_2 \\ B_3 \end{pmatrix}}_{\text{column vector}} = A_0B_0 + A_1B_1 + A_2B_2 + A_3B_3.$ (20a)

This product is invariant under ordinary spatial rotations $\underset{\sim}{R}$.

for four vectors } $\tilde{A}\cdot\tilde{B} = [A_0, A_1, A_2, A_3] \underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}}_{\underset{\sim}{g} = \text{Minkowski metric tensor; } \underset{\sim}{g}^2 = \underset{\sim}{I}.} \begin{pmatrix} B_0 \\ B_1 \\ B_2 \\ B_3 \end{pmatrix} = A_0B_0 - A_1B_1 - A_2B_2 - A_3B_3.$ (20b)

This product is invariant under Lorentz Transformations $\underset{\sim}{\Lambda}$.

SRT Math: Def$^{\underline{n}}$ of $\underset{\sim}{\Lambda}$ as a Minkowski rotation.

: If in general we define: $\tilde{A}\cdot\tilde{B} = [\tilde{A}]_{row}\, \underset{\sim}{g}\, (\tilde{B})_{column}$, we can make the scalar product turn out to be anything we deem useful by adjusting the metric tensor $\underset{\sim}{g}$ ... $\underset{\sim}{g}$ just characterizes the space we are working in. With choice of metric tensor, Lorentz matrices $\underset{\sim}{\Lambda}$ amount to rotations in Minkowski space just as rotation matrices $\underset{\sim}{R}$ produce coordinate rot$^{\underline{n}}$s in ordinary space.

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9) With these notions, we define Lorentz Transformations (matrices) $\underset{\sim}{\Lambda}$ in general as a <u>group</u> of linear transformations which preserves lengths in Minkowski space:

$$\rightarrow\ \tilde{A} \rightarrow \tilde{A}' = \underset{\sim}{\Lambda}\tilde{A}, \quad \text{such that:}\ \underline{\underline{\tilde{A}'\cdot \underset{\sim}{g}\, A' \equiv \tilde{A}\cdot\underset{\sim}{g}\,\tilde{A}}}$$

4-vector length invariance under Lorentz Transf $\underset{\sim}{\Lambda}$

$$\text{w/}\quad \underset{\sim}{g} = \begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix}, \text{ for Minkowski space.} \qquad \underset{\sim}{(21)}$$

$\rightarrow$ Put $\tilde{A}' = \underset{\sim}{\Lambda}\tilde{A}$ into the length-invariance condition...¶

$$\underbrace{(\underset{\sim}{\Lambda}\tilde{A})}_{\text{row}}\cdot \underset{\sim}{g}\,\overbrace{(\underset{\sim}{\Lambda}\tilde{A})}^{\text{column}} = \tilde{A}\cdot\big(\underbrace{\underset{\sim}{\Lambda}_T\,\underset{\sim}{g}\,\underset{\sim}{\Lambda}}_{\text{must be equal}}\big)\tilde{A} \equiv \tilde{A}\cdot\underset{\sim}{g}\,\tilde{A},$$

"T" means "transpose" (interchange rows & columns)

So, $\boxed{\underset{\sim}{\Lambda}_T\,\underset{\sim}{g}\,\underset{\sim}{\Lambda} \equiv \underset{\sim}{g}}$, <u>defines</u> LT's $\underset{\sim}{\Lambda}$. $\underset{\sim}{(22)}$

Eq. (22) is a necessary and sufficient condition on $\underset{\sim}{\Lambda}$, considered as a rotation in Minkowski space ($\underset{\sim}{g}$), to preserve the length $\tilde{A}^2 = \tilde{A}\cdot\underset{\sim}{g}\,\tilde{A}$. By taking the determinant of both sides, we find

$$\rightarrow\ \det(\underset{\sim}{g}) = \det(\underset{\sim}{\Lambda}_T\,\underset{\sim}{g}\,\underset{\sim}{\Lambda}) = [\det(\underset{\sim}{g})][\det(\underset{\sim}{\Lambda})]^2 \Rightarrow \underline{\underline{\det(\underset{\sim}{\Lambda}) = \pm 1}}. \qquad \underset{\sim}{(23)}$$

$\det(\underset{\sim}{\Lambda}) = -1$ corresponds to $\underset{\sim}{\Lambda}$'s that include space inversions; such $\underset{\sim}{\Lambda}$'s are called <u>improper</u> LT's, and we will skip them. We consider <u>proper</u> LT's, w/ $\det(\underset{\sim}{\Lambda}) = +1$.

¶ If $\tilde{A}' = \underset{\sim}{\Lambda}\tilde{A}$ is a column vector, then its row vector counterpart is $\tilde{A}'_T = \tilde{A}\underset{\sim}{\Lambda}_T$.

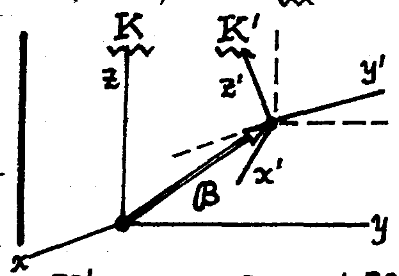
SRT Math: Construction of most general Λ.

10) All Lorentz Transforms Λ allowed in SRT theory can be deduced from

$$\underline{\underline{\Lambda_T\, g\, \Lambda = g}} \;\left\{\begin{array}{l}\text{preserves length}\\ \tilde{A}\cdot g\,\tilde{A}\end{array}\right., \quad \text{and}: \quad \underline{\underline{\det\Lambda = +1}} \;\left\{\begin{array}{l}\text{no space or time}\\ \text{reversals allowed.}\end{array}\right. \tag{24}$$

Jackson does this in his Sec. 11.7, pp. 536-541. We shall not repeat the details, but we note the following...

1. Because: $\tilde{A}' = \Lambda\tilde{A}$ $\left(\begin{array}{c}\text{column}\\ \text{vector}\end{array}\right)$, and its transpose: $\tilde{A}'_T = \tilde{A}\Lambda_T$ $\left(\begin{array}{c}\text{row}\\ \text{vector}\end{array}\right)$, must be the same transformation between relatively moving frames K & K', then Λ is a symmetric matrix: $\Lambda_T = \Lambda$. This condition, plus the defining Eq. (22): $\Lambda g \Lambda = g$, reduces the # free parameters for the 4×4 matrix Λ from 16 to 6. The remaining 6 free parameters in Λ are just enough to specify the 3 components of the relative K-K' velocity β, and 3 ∡s giving the relative orientation of the (x', y', z') & (x, y, z) axes.



K′ moves at β w.r.t K, and K′ is rotated w.r.t. K

2. One writes the unknown $\Lambda = \exp(L) = \sum_{n=0}^{\infty}\frac{1}{n!}L^n$, and finds for L...

$$\left.\begin{array}{l} L \text{ is real, and traceless}: \; \mathrm{Tr}\,L = \sum_{\mu=0}^{3} L_{\mu\mu} = 0; \\[6pt] \Lambda g \Lambda = g \;\Rightarrow\; (gL)_T = -(gL) \,\Big/\, gL \text{ is anti-symmetric} \end{array}\right\} \text{so}/\!/ \; L = \begin{pmatrix} 0 & L_{01} & L_{02} & L_{03} \\ L_{01} & 0 & L_{12} & L_{13} \\ L_{02} & -L_{12} & 0 & L_{23} \\ L_{03} & -L_{13} & -L_{23} & 0 \end{pmatrix} \tag{25}$$

L has the prescribed 6 free parameters. The L_{0k} are for a "boost"; the L_{jk} for rotations.

3. The group of 4×4 matrices L with 6 free parameters can be written [Jkn Eq. (11.91)]

$$\underline{L = -\omega\cdot\mathbb{S} - \zeta\cdot\mathbb{K}} \quad \Big/\!/ \;\; \begin{array}{l}\mathbb{S} = (S_1, S_2, S_3), \text{ are 3 basis matrices for 3 parameter rot}^{\underline{n}}\text{s};\\ \mathbb{K} = (K_1, K_2, K_3), \text{ are 3 basis matrices for 3 parameter boosts.}\end{array}$$

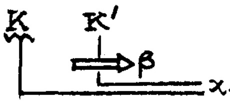
This spans the space of L's. Free parameters are now 3-vectors ω (rot$^{\underline{n}}$) & ζ (boost). (26)

$$\text{so}/\!/ \quad \boxed{\Lambda = e^{-\underset{\substack{\uparrow\\ \text{rotation}\\ \text{between}\\ K \,\&\, K'}}{\omega\cdot\mathbb{S}} - \underset{\substack{\uparrow\\ \text{Lorentz}\\ \text{boost}\\ K\to K'}}{\zeta\cdot\mathbb{K}}}} \tag{27}$$

The $K \to K'$ Lorentz transf$^{\underline{n}}$ Λ constructed this way has the most general features allowed and required by the theory.

11) Jackson cites several examples using: $\underset{\sim}{\Lambda}(\omega, \zeta) = e^{-\omega\cdot\mathbb{S} - \zeta\cdot\mathbb{K}}$, in his Sec. 11.7. The simplest (and most practical) is: $\omega \equiv 0$ ($\Rightarrow$ Lorentz frames K & K' have ∥ axes), and $\zeta = \zeta\,\hat{e}_1$ (motion is along one space axis x_1). Above formalism yields ...

$$\rightarrow \underset{\sim}{L} = -\zeta\,\underset{\sim}{K}_1 = \left(\begin{array}{c|ccc} 0 & -\zeta & 0 & 0 \\ \hline -\zeta & & & \\ 0 & & \underbrace{0}_{\text{no rotations}} & \\ 0 & & & \end{array}\right), \text{ and: } \underset{\sim}{\Lambda} = \begin{pmatrix} \cosh\zeta & -\sinh\zeta & 0 & 0 \\ -\sinh\zeta & \cosh\zeta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (28)$$

This $\underset{\sim}{\Lambda}$ is for our prototype Lorentz transformation (), w/ $\cosh\zeta = \gamma = 1/\sqrt{1-\beta^2}$. The transf^n is called a "boost" because K is boosted to K' by motion alone.

The boost parameter ζ is related generally to the β of the transform by...

$$\boxed{\zeta = \hat{\beta}\tanh^{-1}\beta} \;\; \text{w/ } \hat{\beta} = \boldsymbol{\beta}/\beta = \text{unit vector along motion } \mathbf{v} = \boldsymbol{\beta}c. \quad (29)$$

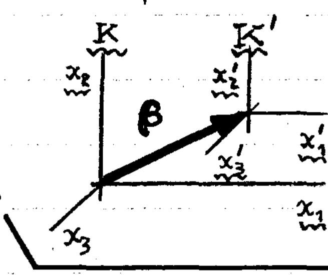
So, along β, have: $\tanh\zeta = \beta \Rightarrow \cosh\zeta = 1/\sqrt{1-\tanh^2\zeta} = \gamma$, as quoted above. Also,

$$\underset{\sim}{L} = -(\hat{\beta}\cdot\mathbb{K})\tanh^{-1}\beta, \text{ for most general pure Lorentz boost } \left(\substack{\text{zero} \\ \text{rot}^{\underline{n}}}\right), \quad (30)$$

thus,

$$\boxed{\underset{\sim}{\Lambda}(\beta) = \begin{pmatrix} \gamma & -\gamma\beta_1 & -\gamma\beta_2 & -\gamma\beta_3 \\ -\gamma\beta_1 & 1+\tau\beta_1^2 & \tau\beta_1\beta_2 & \tau\beta_1\beta_3 \\ -\gamma\beta_2 & \tau\beta_1\beta_2 & 1+\tau\beta_2^2 & \tau\beta_2\beta_3 \\ -\gamma\beta_3 & \tau\beta_1\beta_3 & \tau\beta_2\beta_3 & 1+\tau\beta_3^2 \end{pmatrix}}$$

w/ $\tau = \frac{1}{\beta^2}(\gamma - 1)$. For K→K':



K & K' axes ∥. $\boldsymbol{\beta} = (\beta_1, \beta_2, \beta_3)$ is in arbitrary direction.

This is Jackson's Eq. (11.98). $\underset{\sim}{\Lambda}$ has a pleasing symmetry (e.g. all 3 space cds 1, 2, 3 enter equivalently), and the matrix equation: $\tilde{x}' = \underset{\sim}{\Lambda}(\beta)\,\tilde{x}$ [w/ $\tilde{x} = (x_0, \mathbf{x})$ the position 4-vector] is certainly more elegant than the transf^n eqtns given in Jk^n Eq. (11.19), viz

$$\left.\begin{array}{l} \text{K} \rightarrow \text{K}', \text{ by } \beta \\ \text{i.e. } \tilde{x}' = \underset{\sim}{\Lambda}(\beta)\,\tilde{x} \end{array}\right\} \Rightarrow \begin{array}{l} x_0' = \gamma(x_0 - \boldsymbol{\beta}\cdot\mathbf{x}), \\ \mathbf{x}' = \gamma(\mathbf{x} - \boldsymbol{\beta}x_0) - (\gamma - 1)\left[\mathbf{x} - \frac{1}{\beta^2}(\boldsymbol{\beta}\cdot\mathbf{x})\boldsymbol{\beta}\right]. \end{array} \quad (31)$$