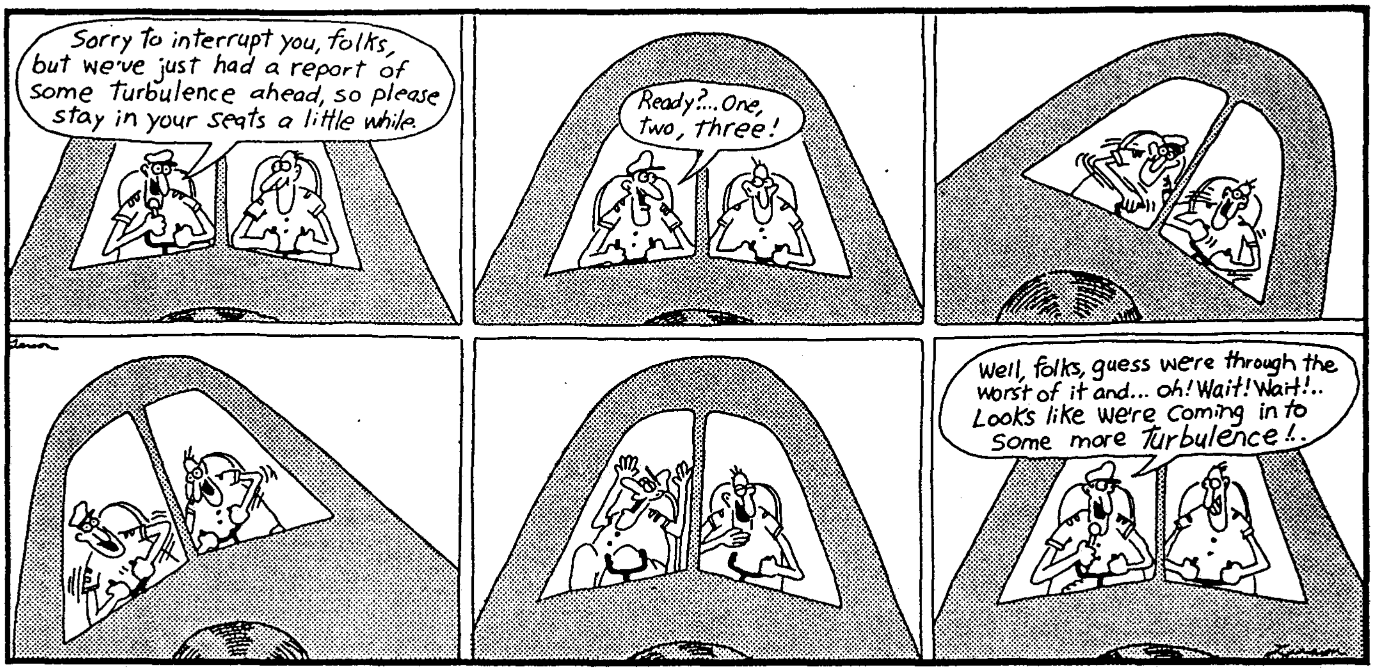
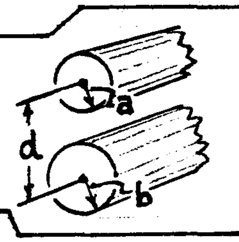
ϕ519 Problems {Set #④: Probs. 11–14. Assigned 10/14/88; due 10/21/88.

④



[redacted]

⑪ [Jackson Prob. (6.2)]. Calculate the self-inductance per unit length for an asymmetric transmission line. Show: $\bar{L} = \frac{1}{c^2}\left[1 + 2\ln(d^2/ab)\right]$.



⑫ [redacted]pts] In nonrelativistic QM, the motion of a particle (charge q, mass m) in an EM field w/ potentials $(\phi, \mathbb{A})$ is described by Schrödinger's eqtn: $\mathcal{H}\psi = i\hbar\frac{\partial\psi}{\partial t}$, with the "electromagnetic Hamiltonian" $\mathcal{H} = \frac{1}{2m}(\mathbb{p} - \frac{q}{c}\mathbb{A})^2 + q\phi$, and $\mathbb{p} \to (-)i\hbar\nabla$ as usual. Consider a gauge transform $\begin{cases}\mathbb{A} \to \mathbb{A}' = \mathbb{A} + \nabla g, \\ \phi \to \phi' = \phi - \frac{1}{c}(\partial g/\partial t).\end{cases}$ Insist that $\mathcal{H}\psi = i\hbar\frac{\partial\psi}{\partial t}$ remains form-invariant.

(A) Show that the required gauge transform for the wavefunction ψ implies a phase shift: $\psi \to \psi' = \psi\exp[i\beta(\mathbb{r},t)]$, and find β in terms of the gauge function g. The transform $(\phi, \mathbb{A}; \psi) \to (\phi', \mathbb{A}'; \psi')$ now ensures that Schrödinger's eqtn is gauge invariant.

(B) Do the problem backwards. First, demand that Schrodinger's theory be invariant under local phase shifts: $\psi \to \psi' = \psi e^{i\beta}$, β = same as part (A). Show that ψ & ψ' cannot both obey the free-particle eqtn: $\left(\frac{\mathbb{p}^2}{2m}\right)\psi = i\hbar\left(\frac{\partial\psi}{\partial t}\right)$. How must this eqtn be modified to be invariant under $\psi \to \psi' = \psi e^{i\beta}$? How are the modifications related to potentials $(\phi, \mathbb{A})$?

[redacted] This is the machinery of "gauge theories" in QM: a local invariance for ψ in some field is used to fix the nature of the field (potentials), and to fix possible couplings & wave eqtns.

Φ 519 Problems

(13) Consider Jackson's Sec. (6.6). Assume that: (1) you have solved Eq. (6.57) for the wave's Fourier transform $\tilde{\psi}$ a la Green, so: $\tilde{\psi}(\mathbf{r},\omega) = \int_\infty G_k^{(\pm)}(\mathbf{r},\mathbf{r}')\tilde{f}(\mathbf{r}',\omega)\,d\tau'$, over an ∞ domain, (2) you know $G_k^{(\pm)}(R) = \frac{1}{R}e^{\pm ikR}$, w/ $R = |\mathbf{r}-\mathbf{r}'|$, from Eq. (6.62). Now, instead of doing Jackson's Eqs. (6.63 → 6.69), obtain the wave amplitude $\psi(\mathbf{r},t)$ by inverting both of the Fourier transforms $\tilde{\psi}$ & $\tilde{f}$. Show that the same retarded/advanced Green's fcns $G^{(\pm)}(\mathbf{r},t;\mathbf{r}',t')$ result, per Jackson's Eq. (6.66). What happens to this approach in a dispersive medium, when $\frac{\partial\omega}{\partial k} \neq$ cnst?

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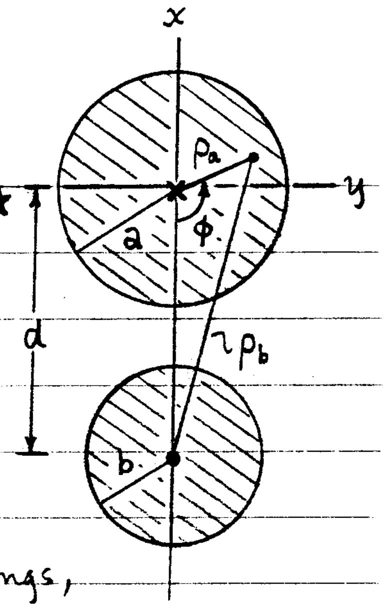
(14) [redacted]. The Green's fcn for a 1D "spherical wave" obeys: $\left(\frac{\partial^2}{\partial r^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)G = -4\pi\,\delta(r)\,\delta(t)$, where the radial distance $r \geq 0$, and the δ's are Dirac delta fcns. Carry out a Fourier transform $G(r,t) \to \tilde{G}(r,\omega)$, and solve the resulting ODE for $\tilde{G}$ [[redacted]: absorb the singularity at $r=0$ into $\frac{\partial\tilde{G}}{\partial r}$, and fix the multiplicative constant in $\tilde{G}$ by integrating the $\tilde{G}$ ODE over the small interval $0- \leq r \to 0+$]. Invert the transform, $\tilde{G} \to G$, and show: $\boxed{G(r,t) = \alpha\,\theta(ct-r)}$ where θ is the unit step fcn [$\theta(x) \equiv 0$, $x<0$; $\theta(x) \equiv 1$, $x>0$]. Find the constant α. Finally, write out a particular integral for a solution to the inhomogeneous 1D wave eqtn: $\left(\frac{\partial^2}{\partial r^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\psi(r,t) = -4\pi f(r,t)$.

ϕ 519 Prob. Solutions

[redacted]

[redacted] Calculate the self-inductance/length of a bifilar lead.



1. We use the result of prob^m ⑩ [Jackson's Prob^m (6.1)] in the form: ★

$$W_m = \frac{1}{2c}\int d^3r\, \mathbb{J}(\mathbf{r})\cdot \mathbb{A}(\mathbf{r}), \quad \mathbb{A}(\mathbf{r}) = \frac{1}{c}\int d^3r'\, \frac{\mathbb{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}. \quad (1)$$

10/12/88

$\mathbb{A}(\mathbf{r})$ is the vector potential produced in one wire by contributions both from the interior of that wire plus the (exterior) contribution from the other wire. There are two such pairings, and since $\mathbb{A}$ is always along the direction of $\mathbb{J}$, we can write...

$$2cW_m = \int_{\text{wire a}} d^3r\, J_a\left(A_a^{(int)} + A_b^{(ext)}\right) + \int_{\text{wire b}} d^3r\, J_b\left(A_b^{(int)} + A_a^{(ext)}\right). \quad (2)$$

2. In Eq. (2), the subscripts to wires a and b, and A will carry the algebraic sign of the J which produced it. As magnitudes, take $J_a = I/\pi a^2$ & $J_b = I/\pi b^2$, where I is the current, and then recall (by the definition of self-inductance L) that the magnetic energy $W_m = \frac{1}{2}LI^2$. To get $\bar{L}$, the self-inductance per unit length, we suppress integration along the z-axis.¶ Then: $\bar{L} = \frac{1}{cI^2}(2cW_m)\big|_{z=\text{cnst}}$, or from Eq. (2)...

$$\rightarrow \bar{L}\cdot \pi c I = \frac{1}{a^2}\int_{\text{wire a}} dS_a\left(A_a^{(int)} + A_b^{(ext)}\right) + \frac{1}{b^2}\int_{\text{wire b}} dS_b\left(A_b^{(int.)} + A_a^{(ext)}\right). \quad (3)$$

The integrations are now over the cross-sectional areas of the wires, where it is natural to use plane polar cds (ρ, ϕ) as indicated in the above diagram.

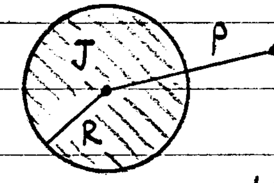
3. To get the A's in Eq. (3), we could try evaluating $\mathbb{A} = \int d^3r'\, \mathbb{J}/R$, but it is much easier to recognize that $\mathbb{A}$ & $\mathbb{J}$ are related by Poisson's eqtn:

¶ Assumption is that wire length >> a & b, so nothing changes along the z-axis.

★ The quoted form of $\mathbb{A}$ appears in Jackson's Eq. (5.32).

$$\nabla^2 \mathbb{A} = -(4\pi/c)\,\mathbb{J} \longrightarrow \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial A_z}{\partial \rho}\right) = -\frac{4\pi}{c} J_z \text{, in cylindrical symmetry:} \quad (4)$$

When J_z = cnst, as is the case here, this eqtn† can be integrated easily, w/o getting involved in ambiguous integrations over z. For a circular region of radius R, the solutions to Eq. (4) are easily found to be...



$$\begin{cases} 0 \leqslant \rho \leqslant R \ : \ A^{(int)} = \alpha - (\pi J/c)\,\rho^2, \quad \alpha = \text{cnst}; \\ \rho \geqslant R \ (\& \ J \equiv 0) : \ A^{(ext)} = \alpha - (\pi J/c)\,R^2 - \left(\frac{2\pi J}{c} R^2\right) \ln \frac{\rho}{R}. \end{cases} \quad (5)$$

Here we have dropped the subscript z, understanding $\mathbb{A}$ is along $\mathbb{J}$. The cnst α is ~ arbitrary (get same $\mathbb{B} = \nabla \times \mathbb{A}$ for any α), <u>except</u> for the fact that α must change sign when J does ... otherwise $\mathbb{A}$ would preferentially depend on some $\mathbb{J}$ direction, which is untidy & non-isotropic.

~~4. The A's in Eq. (3) may now be read off from the results of Eq. (5) ...~~

$$\longrightarrow A_a^{(int)} = \alpha - (I/a^2 c)\,\rho_a^2, \quad A_b^{(ext)} = -\alpha + (I/c) + (I/c) \ln \frac{\rho_b^2}{b^2}. \quad (6)$$

Since: $\rho_b^2 = \rho_a^2 + d^2 - 2\rho_a d \cos\phi$ (Law of Cosines), the 1st integral in (3) is... *

$$\frac{1}{a^2} \int_{\text{wire } a} dS_a \left(A_a^{(int)} + A_b^{(ext.)}\right) = \frac{I}{c a^2} \int_0^a \rho_a \, d\rho_a \int_0^{2\pi} d\phi \left[-\frac{\rho_a^2}{a^2} + 1 + \ln\left(\frac{\rho_a^2 + d^2 - 2\rho_a d \cos\phi}{b^2}\right)\right]$$

$$= \frac{2\pi I}{c a^2} \int_0^a \rho_a \, d\rho_a \left[1 - \frac{\rho_a^2}{a^2} + \ln(d^2/b^2)\right] = \frac{\pi I}{2c}\left[1 + 4 \ln(d/b)\right]. \quad (7)$$

The 2nd integral in (3) will be: $(\pi I/2c)\left[1 + 4\ln(d/a)\right]$, similarly. Adding these results, we get the desired inductance/length from (3), as...

$$\boxed{c^2 \bar{L} = 1 + 2 \ln(d^2/ab)}. \quad (8)$$

* Only tricky integration is $\int_0^{2\pi} d\phi \ln(\ ) = 2\pi \ln(1/b^2) + \int_0^{2\pi} d\phi \ln(\rho_a^2 - 2\rho_a d\cos\phi + d^2)$. Use Dwight #(865.73) to get: $\int_0^{2\pi} d\phi \ln(\ ) = 2\pi \ln(d^2/b^2)$, as used in Eq. (7).

† Eq. (4) ≡ Jackson's Eq. (5.31). It is also obvious from <u>form</u> of $\mathbb{A} = \int d^3r' \, \mathbb{J}/R$.

ϕ 519 Prob. Solutions

[redacted]

⑫ Establish gauge invariance of Schrödinger Eqtn, then do it backwards.

(10pts)

10/15/84

A. We take the hint that ψ transforms as: $\psi \to \psi' = \psi \exp[i\beta(\vec{x},t)]$, and note that we must have $\beta = 0$ when the gauge fcn $g = 0$. It is reasonable to set $\beta = \alpha g$, and look for a cnst α such that: $(\mathcal{H} - i\hbar\frac{\partial}{\partial t})\psi = 0 \to (\mathcal{H}' - i\hbar\frac{\partial}{\partial t})\psi' = 0$, is form-invariant under the gauge transform: $\mathbb{A} \to \mathbb{A}' = \mathbb{A} + \nabla g$, $\phi \to \phi' = \phi - \frac{1}{c}(\partial g/\partial t)$.
First note that the momentum operator for the gauge-transformed $\mathcal{H}$ acts as follows:

$$(\mathbb{P} - \tfrac{q}{c}\mathbb{A})'\psi' = (-i\hbar\nabla - \tfrac{q}{c}\mathbb{A} - \tfrac{q}{c}\nabla g)\psi e^{i\alpha g}$$

$$= e^{i\alpha g}(-i\hbar\nabla - \tfrac{q}{c}\mathbb{A})\psi + [(\hbar\alpha - \tfrac{q}{c})\nabla g]\psi e^{i\alpha g}. \qquad (1)$$

Evidently we can avoid large helpings of unpalatable arithmetic if we choose α such that the 2ND term here vanishes. Thus we try...

$$\underline{\underline{\alpha = q/\hbar c}} \qquad (2)$$

So// Eq (1) $\Rightarrow$ $\mathbb{P}'\psi' = e^{i\alpha g}\,\mathbb{P}\psi$, w/ $\mathbb{P} = -i\hbar\nabla - \frac{q}{c}\mathbb{A}$ (momentum operator). (3)

The hope is, of course, that this choice of α goes all the way through to the required gauge invariance of $\mathcal{H}\psi = i\hbar(\partial\psi/\partial t)$. Another application of $\mathbb{P}'$ yields...

$$(\mathbb{P}')^2\psi' = e^{i\alpha g}\,\mathbb{P}^2\psi \Rightarrow (\mathcal{H}' - q\phi')\psi' = e^{i\alpha g}(\mathcal{H} - q\phi)\psi;$$

$$\text{w// } \left.\begin{matrix}\phi' = \phi - \frac{1}{c}\dot{g} \\ \psi' = \psi e^{i\alpha g}\end{matrix}\right\} (\mathcal{H}' + \tfrac{q}{c}\dot{g})\psi' = e^{i\alpha g}(\mathcal{H}\psi). \qquad (4)$$

Now, note that the RHS of $\mathcal{H}\psi = i\hbar(\partial\psi/\partial t)$ requires we look at...

$$i\hbar\frac{\partial}{\partial t}\psi' = i\hbar\frac{\partial}{\partial t}(\psi e^{i\alpha g}) = e^{i\alpha g}(i\hbar\frac{\partial}{\partial t} - \hbar\alpha\dot{g})\psi. \qquad (5)$$

Subtract this equation: LHS−LHS = RHS−RHS from Eq. (4) to obtain...

ϕ 519 Prob. Solutions

$$\left(\mathcal{H}' - i\hbar\frac{\partial}{\partial t} + \frac{q}{c}\dot{g}\right)\psi' = e^{i\alpha g}\left(\mathcal{H} - i\hbar\frac{\partial}{\partial t} + \hbar\alpha\dot{g}\right)\psi. \quad (6)$$

cancel, if: $\hbar\alpha = q/c$

The same choice of α as in Eq. (2), $\hbar\alpha = \frac{q}{c}$, ensures that if: $\left(\mathcal{H} - i\hbar\frac{\partial}{\partial t}\right)\psi = 0$, then: $\left(\mathcal{H}' - i\hbar\frac{\partial}{\partial t}\right)\psi' = 0$. Thus the Schrödinger Eqtn is gauge invariant if:

$$\boxed{\psi \rightarrow \psi' = \psi \exp\left[i(q/\hbar c)\,g(\vec{r},t)\right]}, \quad g = \text{gauge transform fcn}. \quad (7)$$

(10 pts)

B. If $\psi \rightarrow \psi' = \psi e^{i\beta}$, $\beta = (q/\hbar c)g$, under a local phase shift β ["local" means the phase shift changes at each space-time point; it is not a global (uniform) phase shift everywhere in space-time], then it is clear that both ψ & ψ' cannot satisfy the free-particle Schrödinger eqtn: $-(\hbar^2/2m)\nabla^2\psi = i\hbar(\partial\psi/\partial t)$; one or the other of the ψ & ψ' eqtns will involve derivatives of β, ⊛ which cannot be arbitrarily set to zero. Demanding that: $\left[\nabla^2 + (2im/\hbar)\frac{\partial}{\partial t}\right]\psi = 0$, be form-invariant under $\psi \rightarrow \psi' = \psi e^{i\beta}$ requires new degrees of freedom for the operators ∇ & $\partial/\partial t$, say: $\nabla \rightarrow \nabla + \mathbb{K}$, $\partial/\partial t \rightarrow \partial/\partial t + \Omega$, where $\mathbb{K}$ & Ω are new vector & scalar fields constructed to make the $\psi \rightarrow \psi' = \psi e^{i\beta}$ invariance work.

Put $(\nabla + \mathbb{K})$ & $\left(\frac{\partial}{\partial t} + \Omega\right)$ into the free-particle Schrödinger eqtn. Then, when $\psi \rightarrow \psi' = \psi e^{i\beta}$, the phase invariance requires $\mathbb{K}$ & Ω to transform in a gauge-like way...

$$\rightarrow \mathbb{K} \rightarrow \mathbb{K}' = \mathbb{K} - i\nabla\beta, \quad \Omega \rightarrow \Omega' = \Omega - i\frac{\partial\beta}{\partial t}. \quad (8)$$

With $\beta = \alpha g$, the assignment: $\mathbb{K} = -i\alpha\mathbb{A}$, $\Omega = ic\alpha\phi$, brings us back to the standard potentials $\mathbb{A}$ & ϕ, and to the standard wave eqtn, viz.: $i\hbar\frac{\partial\psi}{\partial t} = \left[\frac{1}{2m}\left(-i\hbar\nabla - \frac{q}{c}\mathbb{A}\right)^2 + q\phi\right]\psi$. The procedure is unique so long as $\mathbb{K}$ & Ω are fields, not operators. In this way, the phase invariance dictates the dynamics, even existence of $\mathbb{A}$ & ϕ.

⊛ Specifically: $e^{-i\beta}\left(\frac{\hbar^2}{2m}\nabla^2 + i\hbar\frac{\partial}{\partial t}\right)\psi' = \left(\frac{\hbar^2}{2m} + i\hbar\frac{\partial}{\partial t}\right)\psi - \left(\hbar\beta\frac{\partial\beta}{\partial t}\right)\psi + \frac{i\hbar^2}{2m}\left\{\psi\left[\nabla^2\beta + i(\nabla\beta)^2\right] + 2(\nabla\psi)\cdot(\nabla\beta)\right\}$

The LHS, and 1st term RHS, cannot both be zero, for arbitrary β.

Alternate derivation of Jackson's Eqs. (6.63)-(6.69).

10/7/84

1) Assumptions (1) & (2) allow writing the solution for the wave's Fourier transform

$$\tilde{\psi}(\vec{x},\omega) = \int_\infty \frac{d\tau'}{R}\, e^{\pm ikR}\, \tilde{f}(\vec{r}',\omega). \qquad (1)$$

In real time, the wave is : $\psi(\vec{r},t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, \tilde{\psi}(\vec{r},\omega)\, e^{-i\omega t}$, and the source fcn transform : $\tilde{f}(\vec{r}',\omega) = \int_{-\infty}^{\infty} dt'\, f(\vec{r}',t')\, e^{i\omega t'}$. Notice that we have used t' as the integration variable, so as not to confuse it with the field point observation time t. Putting this together, we have...

$$2\pi\,\psi(\vec{r},t) = \int_{-\infty}^{\infty} d\omega \left\{ \int_\infty \frac{d\tau'}{R}\, e^{\pm ikR} \left[\int_{-\infty}^{\infty} dt'\, f(\vec{r}',t')\, e^{i\omega t'} \right] \right\} e^{-i\omega t}$$

$$= \int_\infty \frac{d\tau'}{R} \int_{-\infty}^{\infty} dt'\, f(\vec{r}',t') \left\{ \int_{-\infty}^{\infty} d\omega\, e^{i[\omega(t'-t) \pm kR]} \right\}. \qquad (2)$$

2) If we assume no dispersion : $k = \omega/c$, c = indpt. of ω, the last integral is

$$\{\,\} = \int_{-\infty}^{\infty} d\omega\, e^{i\omega[t' - (t \mp \frac{R}{c})]} = 2\pi\, \delta\left(t' - [t \mp \tfrac{R}{c}]\right), \qquad (3)$$

and then Eq. (2) yields...

$$\boxed{\psi(\vec{r},t) = \int_\infty d\tau' \int_{-\infty}^{\infty} dt'\, f(\vec{r}',t') \left\{ \frac{1}{R}\, \delta\left(t' - [t \mp \tfrac{R}{c}]\right) \right\}}. \qquad (4)$$

The { } is just Jackson's $G^{(\pm)}(\vec{r},t;\vec{r}',t')$ of his Eq. (6.66), and the solution for $\psi(\vec{r},t)$ is ≡ Jackson's un-numbered eqtn at bottom of his p. 225.

3) For a dispersive medium, let $k = \frac{\omega}{c} D(\omega)$. Then the integral in Eq. (3) becomes:
$\{\,\} = \int_{-\infty}^{\infty} d\omega\, e^{i\omega[(t'-t) \pm \frac{R}{c} D(\omega)]}$. This must be evaluated on a case-by-case basis, and the idea of retarded and advanced times $t' = t \mp \frac{R}{c}$ in Eq. (4) loses its meaning.

ϕ 519 Prob. Solutions

[redacted]

(14) Construct Green's Function for a 1D Spherical wave.

10/17/84

1) A Fourier transform: $G(r,t) \rightarrow \tilde{G}(r,\omega) = \int_{-\infty}^{\infty} G(r,t)\, e^{i\omega t}\, dt$, through the defining eqtn: $G_{rr} - \frac{1}{c^2} G_{tt} = -4\pi\, \delta(r)\, \delta(t)$, easily gives a harmonic oscillator eqtn & solution:

$$\tilde{G}_{rr} + (\omega/c)^2 \tilde{G} = -4\pi\,\delta(r) \Rightarrow \underline{\underline{\tilde{G}(r,\omega) = A \sin\left(\frac{\omega r}{c}\right)}} \quad \text{everywhere but at } r=0\text{; choose } \sin \text{ so } G\rightarrow 0 \text{ as } r\rightarrow 0+ \text{ or } 0-. \qquad (1)$$

2) To fix A, integrate the $\tilde{G}$ diff'l eqtn over $0- \leqslant r = \epsilon \rightarrow 0+$,* i.e. $\int_{0-}^{\epsilon} dr \times$ eqtn. Then:

$$(\partial\tilde{G}/\partial r)\Big|_{r=0-}^{r=\epsilon} + (\omega/c)^2 \int_{0-}^{\epsilon} \tilde{G}\, dr = -4\pi \int_{0-}^{\epsilon} \delta(r)\, dr = -4\pi. \qquad (2)$$

By choice, $G(r,t)$ & $\tilde{G}(r,\omega)$ are continuous (in fact $\equiv 0$) at $r=0$, so $\int_{0-}^{\epsilon} \tilde{G}\, dr \rightarrow 0$ as $\epsilon \rightarrow 0$. Since G must $\equiv 0$ for $r<0$, then also $(\partial\tilde{G}/\partial r) \equiv 0$ at $0-$, and Eq. (2) is...

$$(\partial\tilde{G}/\partial r)\big|_{r=\epsilon} = \frac{A\omega}{c}\cos(\omega\epsilon/c) = -4\pi \Rightarrow \underline{\underline{A = -4\pi c/\omega}}. \qquad (3)$$

3) So: $\tilde{G}(r,\omega) = -(4\pi c/\omega)\sin\left(\frac{\omega r}{c}\right)$, which has the Fourier inverse...

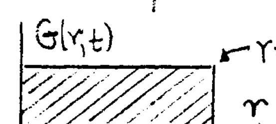
$$\longrightarrow G(r,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{G}(r,\omega)\, e^{-i\omega t}\, d\omega = (-)\, 2c \int_{-\infty}^{\infty} \frac{d\omega}{\omega}\, e^{-i\omega t} \sin\left(\frac{\omega r}{c}\right)$$

$$= (-)\, c \int_{-\infty}^{\infty} \frac{d\omega}{i\omega}\left[e^{-i(t-\frac{r}{c})\omega} - e^{-i(t+\frac{r}{c})\omega}\right], \qquad (4)$$

So, $$\frac{\partial G}{\partial t} = c\int_{-\infty}^{\infty} d\omega \left[e^{-i(t-\frac{r}{c})\omega} - e^{-i(t+\frac{r}{c})\omega}\right] = 2\pi c\left[\delta\left(t-\frac{r}{c}\right) - \delta\left(t+\frac{r}{c}\right)\right]. \qquad (5)$$

The 2ND δ-fcn does not contribute for $t=0+$ (since $t+\frac{r}{c}>0$). Then, as advertised

$$\boxed{G(r,t) = \alpha\, \theta\left(t-\frac{r}{c}\right), \quad \alpha = 2\pi c}\,, \quad \theta = \text{unit step fcn} \Rightarrow \qquad (6)$$



4) G of Eq. (6) is derived for a point source at $r=0$, $t=0$. If the source is at

* At any pathological point for a Green's fcn, $r=0$ in this case, one always considers the behavior of G as the singularity is approached from "above" & "below"; hence $0- \leqslant r \leqslant 0+$.

$t' \neq 0$, $r' \neq 0$, then we let $t \to (t-t')$, $r \to (r-r')$ in Eq. (6), and use...

$$\longrightarrow G(r,t;r',t') = 2\pi c\, \theta\left([t-t'] - \tfrac{1}{c}[r-r']\right), \quad t \geq t' \text{ understood.} \tag{7}$$

This fcn satisfies: $G_{rr} - \frac{1}{c^2}G_{tt} = -4\pi\,\delta(r-r')\,\delta(t-t')$. Note that by the symmetry of the δ-fcn: $\partial G/\partial t' = -\partial G/\partial t$, and $\partial G/\partial r' = -\partial G/\partial r$.

5) The utility of G for the 1D problem lies in the usual manufacture of a particular integral for the inhomogeneous problem, as follows...

<u>INHOM. EQTN</u>: $\psi_{r'r'} - \frac{1}{c^2}\psi_{t't'} = -4\pi f(r';t')$, $\psi = \psi(r';t')$ ← mult. on left by G

<u>GREEN'S EQTN</u>: $G_{r'r'} - \frac{1}{c^2}G_{t't'} = -4\pi\,\delta(r-r')\,\delta(t-t')$ ← mult. on left by ψ

subtract

$$\frac{\partial}{\partial r'}\left(G\psi_{r'} - \psi G_{r'}\right) - \frac{1}{c^2}\left(G\psi_{t'} - \psi G_{t'}\right) = -4\pi\left[Gf - \psi\,\delta(r-r')\,\delta(t-t')\right] \tag{8}$$

... integrate: $\int_{0-}^{a} dr' \int_{0}^{t+} dt'$, a = position of some bndy (with $r < a$); rearrange terms...

$$\psi(r,t) = \int_0^a dr' \int_0^{t+} dt'\, G f(r';t') + \frac{1}{4\pi}\int_0^{t+} dt' \left(G\psi_{r'} - \psi G_{r'}\right)\Big|_{r'=0}^{r'=a} - $$
$$- \frac{1}{4\pi c^2}\int_0^a dr' \left(G\psi_{t'} - \psi G_{t'}\right)\Big|_{t'=0}^{t'=t+} \tag{9}$$

In the last term, both G & $G_{t'}$ vanish at the upper limit (since $t - t+ < 0$). In the t' integrations, G & $G_{t'}$ are non-zero only for $(t-t') - \frac{1}{c}(r-r') \geq 0$, i.e. for $0 \leq t' \leq t_{ret.}$, where: $t_{ret.} = t - \frac{1}{c}(r-r')$. Thus we get, finally...

$$\boxed{\begin{aligned}\psi(r,t) &= \underbrace{\int_0^a dr' \int_0^{t_{ret.}} dt'\, G f(r';t')}_{\text{Particular integral}} + \underbrace{\frac{1}{4\pi}\int_0^{t_{ret.}} dt' \left(G\psi_{r'} - \psi G_{r'}\right)\Big|_{r'=0}^{r'=a}}_{\text{Boundary term}} + \\ &+ \underbrace{\frac{1}{4\pi c^2}\int_0^a dr' \left(G\psi_{t'} - \psi G_{t'}\right)\Big|_{t'=0}}_{\text{Propagation of initial }\psi}\end{aligned}} \tag{10}$$