Iterative Born Approximations

9) The Born Approxn we have used [Eqs.(16)-(32)] is the lowest (leading) order approxn to the scattering problem... $A_B(\mathbf{q})$ of Eq.(16) is just first order in V. We close our presentation of scattering theory by remarking on higher-order approximations.

The integral equation, Eq.(8), for the scattering wavefcn ψ can be written

$$\rightarrow \psi(\mathbf{r}) = \phi(\mathbf{r}) + \int d\mathbf{r}_1\, K(\mathbf{r},\mathbf{r}_1) V(\mathbf{r}_1)\, \psi(\mathbf{r}_1)\,; \tag{33}$$

$$\text{w/ } K(\mathbf{r},\mathbf{r}_1) = -\frac{m}{2\pi\hbar^2}\left(\frac{e^{ik|\mathbf{r}-\mathbf{r}_1|}}{|\mathbf{r}-\mathbf{r}_1|}\right), \quad \int d\mathbf{r}_1 = \int_\infty d^3x_1 \;\; \int \begin{matrix}\text{volume integral over}\\ \text{entire space of } \mathbf{r}_1 \text{ cds.}\end{matrix}$$

In the Born Approxn (more properly the first Born Approxn), we have replaced $\psi(\mathbf{r}_1)$ on the RHS of (33) by the free-particle $\phi(\mathbf{r}_1)$ and have used

$$\rightarrow \psi^{(1)}(\mathbf{r}) = \phi(\mathbf{r}) + \int d\mathbf{r}_1\, K(\mathbf{r},\mathbf{r}_1) V(\mathbf{r}_1)\, \phi(\mathbf{r}_1)\,. \tag{34}$$

A better approximation to the actual $\psi(\mathbf{r})$ can be found by replacing $\psi(\mathbf{r}_1)$ on the RHS of (33) with $\psi^{(1)}(\mathbf{r}_1)$ rather than $\phi(\mathbf{r}_1)$. Then...

$$\psi^{(2)}(\mathbf{r}) = \phi(\mathbf{r}) + \int d\mathbf{r}_1\, K(\mathbf{r},\mathbf{r}_1) V(\mathbf{r}_1)\, \psi^{(1)}(\mathbf{r}_1)$$

or/

$$\psi^{(2)}(\mathbf{r}) = \phi(\mathbf{r}) + \int d\mathbf{r}_1\, K(\mathbf{r},\mathbf{r}_1) V(\mathbf{r}_1)\, \phi(\mathbf{r}_1) + $$
$$+ \int d\mathbf{r}_1 \int d\mathbf{r}_2\, K(\mathbf{r},\mathbf{r}_2) V(\mathbf{r}_2) K(\mathbf{r}_2,\mathbf{r}_1) V(\mathbf{r}_1)\, \phi(\mathbf{r}_1)\,. \tag{35}$$

$\psi^{(2)}$ is the second Born Approxn to ψ. It contains the first-Born $\psi^{(1)}$, correct to $O(V)$, as its first two terms, and also sports an $O(V^2)$ correction. One can continue the iteration in this manner, i.e. for $n=1,2,\ldots$

$$\rightarrow \psi^{(n)}(\mathbf{r}) = \phi(\mathbf{r}) + \int d\mathbf{r}_1\, K(\mathbf{r},\mathbf{r}_1) V(\mathbf{r}_1)\, \psi^{(n-1)}(\mathbf{r}_1)\,, \quad \text{w/ } \psi^{(0)}(\mathbf{r}) = \phi(\mathbf{r})\,. \tag{36}$$

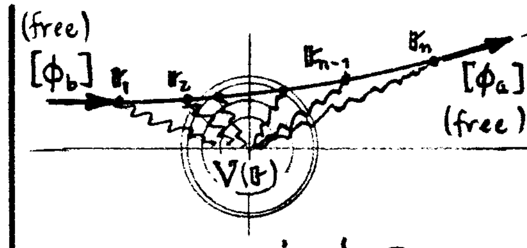
$\psi^{(n)}$ is the n^{th} order Born Approxn to ψ. It contains $\psi^{(n-1)}$ and also the

$\mathcal{O}(V^n)$ corrections. Specifically, for $n = 1, 2, 3, \ldots$ and $\psi^{(0)}(\mathbf{r}) = \phi(\mathbf{r}) \ldots$

$$\psi^{(n)}(\mathbf{r}) = \psi^{(n-1)}(\mathbf{r}) + \int d\mathbf{r}_1 \int d\mathbf{r}_2 \cdots \int d\mathbf{r}_n \, \mathcal{P}_n(\mathbf{r}; \mathbf{r}_n, \ldots \mathbf{r}_1)\, \phi(\mathbf{r}_1), \tag{37}$$

w/ $\mathcal{P}_n(\mathbf{r}; \mathbf{r}_n, \ldots, \mathbf{r}_1) = K(\mathbf{r}, \mathbf{r}_n) V(\mathbf{r}_n) K(\mathbf{r}_n, \mathbf{r}_{n-1}) V(\mathbf{r}_{n-1}) \cdots K(\mathbf{r}_2, \mathbf{r}_1) V(\mathbf{r}_1)$.

<u>$\mathcal{P}_n$ is an n-point "propagator"</u>... it takes the free particle wave $\phi(\mathbf{r}_1)$ at $\mathbf{r}_1$ into a scattering contact $V(\mathbf{r}_1)$, propagates it from $\mathbf{r}_1$ to $\mathbf{r}_2$ via $K(\mathbf{r}_2, \mathbf{r}_1)$ into a scattering contact $V(\mathbf{r}_2)$, thence $\mathbf{r}_2 \to \mathbf{r}_3$ via $K(\mathbf{r}_3, \mathbf{r}_2)$, etc... up to a scattering contact $V(\mathbf{r}_n)$. After the n scatterings $V(\mathbf{r}_1), V(\mathbf{r}_2), \ldots, V(\mathbf{r}_n)$, the wave is propagated from $\mathbf{r}_n$ to the observation point $\mathbf{r}$ via $K(\mathbf{r}, \mathbf{r}_n)$, and then contributes to $\psi^{(n)}(\mathbf{r})$. The integral in (37), $\int d\mathbf{r}_1 \cdots \int d\mathbf{r}_n \, \mathcal{P}_n \, \phi(\mathbf{r}_1)$, counts all the ways this n-fold scattering can happen, and hence the total contribution to $\psi^{(n)}(\mathbf{r})$ from $\mathcal{O}(V^n)$ processes.



propagation via $\mathcal{P}_n$

The above sketch shows the n-fold scattering at points $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_{n-1}, \mathbf{r}_n$ which have an obvious before → after time ordering. Of course the integral in Eq. (37) viz $\int d\mathbf{r}_1 \cdots \int d\mathbf{r}_n \, \mathcal{P}_n \, \phi(\mathbf{r}_1)$ has all possible "orderings" of $\mathbf{r}_1, \ldots, \mathbf{r}_n$.

What is remarkable about this method of solution is that we are solving for a $\psi(\mathbf{r})$ interacting with $V(\mathbf{r})$ by means of <u>free</u> particle ϕ's, and the simple Greens fcn K in Eq. (33). The process in Eq. (37) can be continued, so that

$$\psi(\mathbf{r}) = \phi(\mathbf{r}) + \int d\mathbf{r}_1 \, P(\mathbf{r}, \mathbf{r}_1)\, \phi(\mathbf{r}_1), \text{ is an \underline{exact} solution;} \tag{38}$$

w/ $P(\mathbf{r}, \mathbf{r}_1) = K(\mathbf{r}, \mathbf{r}_1) V(\mathbf{r}_1) + \sum_{n=2}^{\infty} \int d\mathbf{r}_n \int d\mathbf{r}_{n-1} \cdots \int d\mathbf{r}_2 \, \mathcal{P}_n(\mathbf{r}; \mathbf{r}_n, \ldots, \mathbf{r}_1)$.

The Born Approxn in Eq. (16) amounts to taking just the first $[\mathcal{O}(V)]$ term in the overall propagator $P(\mathbf{r}, \mathbf{r}_1)$. Feynman's QM "path integrals" originate with Eq. (38).