DEPARTMENT OF PHYSICS

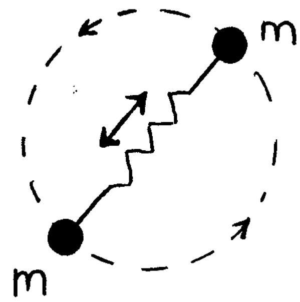
PH.D. COMPREHENSIVE EXAMINATION

MONDAY, SEPTEMBER 18, 1989, 9 A.M. - 12 NOON

Answer each of the following questions, all of which carry equal weight. Begin your answer to each question on a new sheet of paper; solutions to different questions must not appear on the same sheet. Each sheet of paper must be labeled with your name and the problem number in the upper right hand corner. When more than one sheet is submitted for a problem, be sure the pages are ordered properly. [Problems 1-4].

1989

1. A classical model for estimating the vibrational properties of a rotating, diatomic molecule can be constructed as follows. Two masses, each of mass m, are attached to the ends of a massless spring, with spring constant K, and unstretched length l_0. We set the system in motion with total angular momentum L. Without changing L, we give the system a symmetric kick (impulse) so that the separation between masses, l, oscillates about some average value. Assume that the oscillations are small, and that gravity does not play a role in this problem (atoms).



a) Reduce this two-body problem to a one-body problem and write down the two single-body equations which must be solved to determine the motion for this system.

b) Determine the vibration frequency for the molecule.

1

Final Exam

1. Diatomic Molecule

1 No external forces acting $\Rightarrow$ $\boxed{M_{Tot}\ddot{\bar{R}}_{cm} = \vec{F}^{ext} = 0}$ 1st E

so center-mass moves with uniform velocity

2 $2^{\underline{nd}}$ Equation is $\mu\ddot{\bar{r}} = \vec{F}_1^{int}$

For this problem, have $\mu\ddot{\bar{l}} = -k(l-l_0)\hat{r}$

For plane polar coordinates

given on board $\Rightarrow$ $\ddot{\bar{l}} = (\ddot{l} - l\dot{\theta}^2)\hat{r} + (l\ddot{\theta} + 2\dot{l}\dot{\theta})\hat{\theta}$

No θ component to $\vec{F}_1^{int}$ so

2 $\boxed{\mu\ddot{l} - \mu l\dot{\theta}^2 = -k(l-l_0)}$ $2^{\underline{nd}}$ Eqn to solve

arb

See that $\hat{\theta}$ component gives conservation of angular momentum as usual, since

$$|\vec{L}| = |\mu\,\bar{r}\times\bar{v}| = \mu l(l\dot{\theta}) = \mu l^2\dot{\theta}$$

and since

$$l\ddot{\theta} + 2\dot{l}\dot{\theta} = 0$$

$$\Rightarrow \frac{d}{dt}(l^2\dot{\theta}) = 0 = \frac{d}{dt}(\mu l^2\dot{\theta}) = \frac{dL}{dt}$$

1 $\therefore L = \text{const}$

2

(b) Solve $\mu \ddot{l} = -k(l-l_0) + \mu l \dot{\theta}^2$ but $L = \mu l^2 \dot{\theta}$

$$\mu \ddot{l} = -k(l-l_0) + \frac{L^2}{\mu l^3}$$

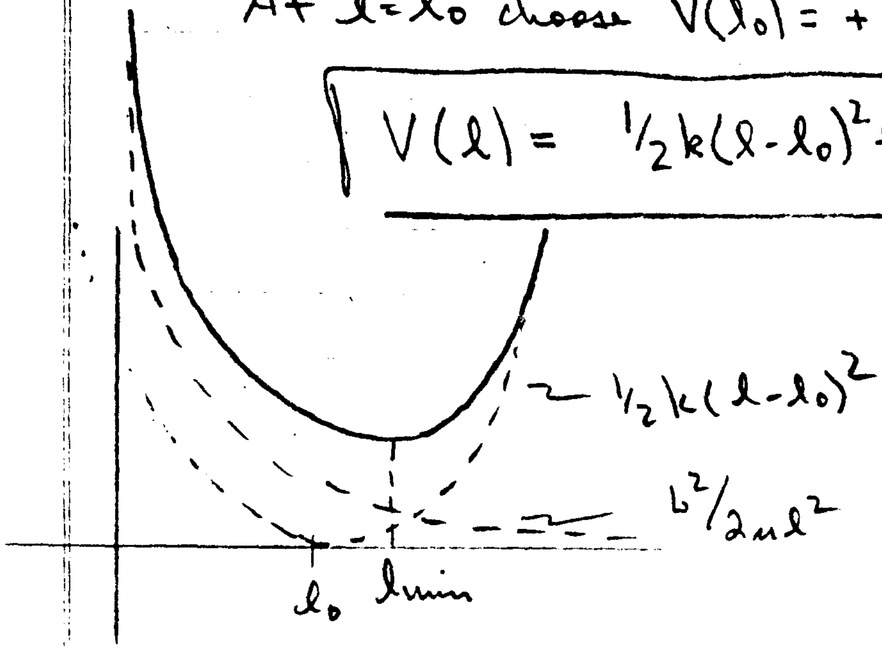
↑ effective force

Using an effective potential

$$V(l) - V(l_0) = -\int_{l_0}^{l} F \cdot dl = \frac{1}{2}k(l-l_0)^2 + \frac{L^2}{2\mu l^2} - \frac{L^2}{2\mu l_0^2}$$

At $l = l_0$ choose $V(l_0) = + L^2/2\mu l_0^2$ for convenience

$$\boxed{V(l) = \tfrac{1}{2}k(l-l_0)^2 + L^2/2\mu l^2}$$



Have min val in $V(l)$ at $\frac{dV}{dl} = 0 = k(l-l_0) - \frac{L^2}{\mu l^3}$

Solve for l_m: $l_m^3(l_m - l_0) = L^2/\mu K$

To get frequency: $\left.\frac{d^2V}{dl^2}\right|_{l_m}$ = effective spring constant

$$= K + \frac{3L^2}{\mu l_m^4}$$

4

So $\boxed{\omega^2 = \frac{1}{\mu}\left[K + \frac{3L^2}{\mu l_m^4}\right]} =$

③

If instead we linearize the differential eqn:

Let $x = l - l_0$ so $\ddot{x} = \ddot{l}$

$$\mu \ddot{x} + kx - \frac{L^2}{\mu (l_0 + x)^3} = 0$$

Expand last term for small x: (given expans on board

$$\mu \ddot{x} + kx - \frac{L^2}{\mu l_0^3}\left(1 - 3^{x}/_{l_0}\right) = 0$$

$$\ddot{x} + \frac{1}{\mu}\underbrace{\left(k + \frac{3L^2}{\mu l_0^4}\right)}_{\text{frequency term}} x = \frac{L^2}{\mu^2 l_0^3}$$

$$\boxed{\omega = \left[{}^{k}/_{\mu} + \frac{3L^2}{\mu^2 l_0^4} \right]^{1/2}}$$ Frequency of vibration

Note: difference in 2nd approach. Linearization has effectively set $l_m = l_0$ in exact approach using an effective potential.

2. Consider the spherical wave solution of Maxwell's equations shown below.

a) Calculate $\vec{H}$ and $\vec{E}$.

b) From the limiting values of $\vec{H}$ for small r show that this field would result from an oscillating magnetic dipole and find the magnitude of the magnetic moment.

c) Show that $\vec{E}$ and $\vec{H}$ are transverse fields at large r.

$$H_r = \frac{u}{r}$$

$$H_\theta = \frac{1}{2r}\frac{\partial}{\partial\theta}\frac{\partial}{\partial r}(ru)$$

$$E_\phi = \frac{jk}{2r}\sqrt{\frac{\mu_0}{\varepsilon_0}}\;\frac{\partial(ru)}{\partial\theta}$$

$$H_\phi = E_r = E_\theta = 0$$

$$u = c\cos\theta\left[\frac{-1}{kr} + \frac{j}{(kr)^2}\right]e^{j(\omega t - kr)}$$

$$k = \sqrt{\mu_0\varepsilon_0}\,\omega\,;\ c = \text{constant}$$

Sol.

$$\frac{\partial}{\partial r}(ru) = \frac{\partial}{\partial r} C\cos\theta\left[\frac{-1}{k} + \frac{j}{k^2 r}\right] e^{j(\omega t - kr)}$$

$$= C\cos\theta\left[\frac{-1}{k} + \frac{j}{k^2 r}\right](-jk)e^{j(\omega t - kr)} - \frac{jC\cos\theta}{k^2 r^2} e^{j(\omega t - kr)}$$

$$\frac{\partial}{\partial\theta}\frac{\partial}{\partial r}(ru) = -C\sin\theta\, e^{j(\omega t - kr)}\left[j + \frac{1}{kr} - \frac{j}{(kr)^2}\right]$$

$$\Rightarrow H_\theta = \frac{C\sin\theta}{2r}\left[\frac{j}{(kr)^2} + \frac{-1}{kr} - j\right] e^{j(\omega t - kr)}$$

$$\frac{\partial ru}{\partial\theta} = -C\sin\theta\, r\left[\frac{-1}{kr} + \frac{j}{(kr)^2}\right] e^{j(\omega t - kr)}$$

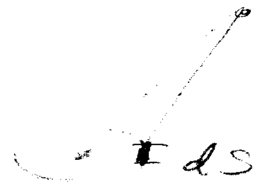
$$\Rightarrow E_\phi = \frac{jk}{2}\sqrt{\frac{\mu}{\epsilon}}\, C\sin\theta\left[\frac{1}{kr} - \frac{j}{(kr)^2}\right] e^{j(\omega t - kr)}$$

$$\rightarrow H_r = \frac{C\cos\theta}{r}\left[\frac{-1}{kr} + \frac{j}{(kr)^2}\right] e^{j(\omega t - kr)}$$

Find H at small r

$$H_r \doteq j\frac{C\cos\theta}{k^2 r^3} e^{j\omega t}\,;\quad H_\theta \doteq j\frac{C\sin\theta}{2r^3 k^2} e^{j\omega t}$$

Look at dipole



$$\Phi = \frac{\mu_0 i}{4\pi}\int_A \frac{dA\cdot\vec{r}}{r^3}$$

$dA\cdot r = dA\, r\cos\theta$

$$= \frac{\mu_0 i}{4\pi r^2}\int_A dA\cos\theta$$

$$= \frac{\mu_0 i A}{4\pi r^2}\cos\theta \quad , \; iA = m \text{ mag. moment}$$

$$\vec{B} = -\nabla\Phi \;,\; \vec{B} = \mu_0\vec{H} \;,\; \vec{H} = -\frac{\nabla\Phi}{\mu_0}$$

$$H_r = \frac{-1}{\mu_0}\frac{\partial\Phi}{\partial r} \;,\; H_\theta = \frac{-1}{\mu_0 r}\frac{\partial\Phi}{\partial\theta}$$

$$\therefore\; H_r = \frac{-1}{\mu_0}\left(\frac{-2\mu_0 m\cos\theta}{4\pi r^3}\right) = \frac{m\cos\theta}{2\pi r^3}$$

$$H_\theta = \frac{-1}{\mu_0 r}\left(\frac{-\sin\theta\,\mu_0 m}{4\pi r^2}\right) = \frac{m\sin\theta}{4\pi r^3}$$

So the field above is dipolar

Comparing

$$\frac{m}{2\pi} = j\frac{c}{k^2}e^{j\omega t}$$

$$m = j\frac{2\pi c}{k^2}e^{j\omega t} \;;\; c = \frac{mk^2}{2\pi j}e^{j\omega t}$$

so $m(t) = |m|e^{j\omega t} = \frac{2\pi c}{k^2}e^{j\omega t}$

e) at large r

$$H_r \rightarrow 0$$

$$H_\theta = \frac{c \sin\theta}{2r}(-j)\, e^{j(\omega t - kr)}$$

$$E_\phi = \sqrt{\frac{\mu}{\epsilon}}\, \frac{c \sin\theta}{2r}(-j)\, e^{j(\omega t - kr)}$$

$$H_\phi = E_r = E_\theta = 0$$

which is a transverse wave.

3. A particle of mass m in a one-dimensional harmonic oscillator potential $m\omega_0^2 x^2/2$ is prepared at time t = 0 in an eigenstate $|\psi_A\rangle$ of the lowering operator $\hat{a}$:

$\hat{a}\,|\,\psi_A\rangle = A\,|\,\psi_A\rangle$ where A is a complex number.

a) Find the expectation values of $\hat{x}$ and the Hamiltonian $\hat{H}$ in state $|\psi_A\rangle$. Recall that

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega_0}}(\hat{a} + \hat{a}^{+})$$

and
$$\hat{H} = \hbar\omega_0\left(\hat{a}^{+}\hat{a} + \frac{1}{2}\right).$$

b) Find the likelihood that a measurement of the energy will give a value $\hbar\omega_0\left(n + \frac{1}{2}\right)$ (n an integer) in the state $|\psi_A\rangle$.

c) Show that two such states $|\psi_A\rangle$ and $|\psi_{A'}\rangle$ are not orthogonal even if $A \neq A'$, where $|\psi_{A'}\rangle$ is defined by $\hat{a}\,|\,\psi_{A'}\rangle = A'\,|\,\psi_{A'}\rangle$.

A particle of mass m in a one-dimensional harmonic oscillator potential $\frac{1}{2} m \omega_0^2 x^2$ is prepared at time ~~in an~~ $t=0$ in an eigenstate $|\psi_A\rangle$ of the lowering operator $\hat{a}$:

$$\hat{a}|\psi_A\rangle = A|\psi_A\rangle \qquad \text{where } A \text{ is a complex number.}$$

(a) Find the expectation values of $\hat{x}$ and the Hamiltonian $\hat{H}$ in state $|\psi_A\rangle$. Recall that $\hat{x} = \sqrt{\frac{\hbar}{2m\omega_0}}\,(\hat{a}+\hat{a}^\dagger)$ and $\hat{H} = \hbar\omega_0\,(\hat{a}^\dagger\hat{a} + \frac{1}{2})$.

(b) Find the likelihood that a measurement of the energy will give a value $\hbar\omega_0(n+\frac{1}{2})$ (n an integer) in the state $|\psi_A\rangle$.

(c) Show that two such states $|\psi_A\rangle$ and $|\psi_{A'}\rangle$ are not orthogonal even if $A \neq A'$

~~(d) Find $\langle\hat{x}\rangle$ at arbitrary time $t>0$.~~

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soln (a)

$$\langle\hat{x}\rangle = \sqrt{\frac{\hbar}{2m\omega_0}}\,\langle\psi_A|\hat{a}+\hat{a}^\dagger|\psi_A\rangle$$

$$= \sqrt{\frac{\hbar^2}{2m\omega_0}}\;A + A^* \;=\; \underline{2\sqrt{\frac{\hbar}{2m\omega_0}}\;Re(A)}$$

$$\langle\hat{H}\rangle = \frac{\hbar\omega_0}{2} + \hbar\omega_0\underbrace{\langle\psi_A|\hat{a}^\dagger\hat{a}|\psi\rangle}_{A^*A} = \underline{\hbar\omega_0\left(|A|^2+\frac{1}{2}\right)}$$

(b) expand $|\psi_A\rangle$ in energy eigenfunctions $|n\rangle$: $|\psi_A\rangle = \sum_{n=0}^{\infty} a_n |n\rangle$

$$\hat{a}|\psi_A\rangle = A|\psi_A\rangle = \sum_{n=1}^{\infty} a_n \sqrt{n}\, |n-1\rangle$$

$$= \sum_{0}^{\infty} a_{n+1} \sqrt{n+1}\, |n\rangle$$

So $A\, a_n = a_{n+1}\sqrt{n+1}$ or $a_{n+1} = \frac{A}{\sqrt{n+1}} a_n = \ldots$

$$= \frac{A^{n+1}}{\sqrt{(n+1)!}} a_1$$

ie, $a_n = \frac{A^n}{\sqrt{n!}} a_1$

Normalize: $\sum_n |a_n|^2 = 1 = \sum \frac{|A|^{2n}}{n!} |a_1|^2 = |a_1|^2 e^{|A|^2}$

So take $a_1 = e^{-|A|^2/2}$

Then $a_n = e^{-|A|^2/2} \frac{A^n}{\sqrt{n!}}$

and probability (energy $= \hbar\omega_0(n+\frac{1}{2})$) $= |a_n|^2 = e^{-|A|^2} \frac{|A|^{2n}}{n!}$

(c) $\langle\psi_{A'}|\psi_A\rangle = \sum_n e^{-\frac{|A|^2}{2} - \frac{|A'|^2}{2}} \frac{(A'^* A)^n}{n!} = e^{-\frac{|A|^2}{2} - \frac{|A'|^2}{2} + A'^* A}$

This is not zero, since

$$|\langle\psi_{A'}|\psi_A\rangle|^2 = e^{-|A|^2 - |A'|^2 + A'^* A + A' A^*}$$

$$= e^{-|A-A'|^2}$$

4. The hemispherical electron energy analyzer consists of two concentric metal spheres S_1 and S_2 with radii r_1 and r_2, along with a plate P with two slits to let the electrons in and out. A vertical cross section of the device is shown in the picture below. All three elements are at different potentials. S_1 is grounded, $V_1 = 0$. The potential on S_2 is V_2, and the potential on P is V_P, which is set to be the average potential between S_1 and S_2.

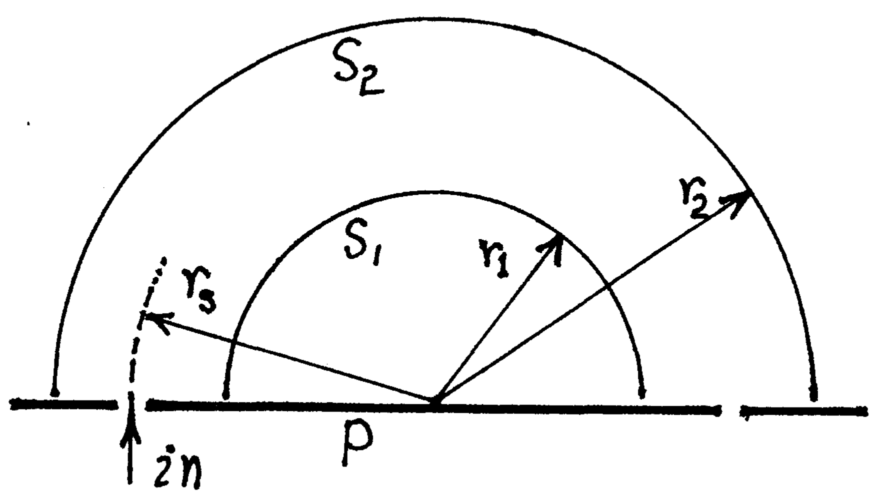
a) Neglecting distortion of the potential near P, write an expression for the potential distribution between the hemispheres and calculate the force on an electron placed between the hemispheres.

b) At what position r_s between S_1 and S_2 should the slits be put so they will be at the mean potential of the hemispheres?

c) Electrons are injected into the space between the hemispheres and normal to the plate P. By analogy with planetary motion, what are the orbits of the electrons that pass through the device and strike at or near the exit slit. Discuss this topic in the light of your knowledge of planetary motion and explain how this device works as an energy analyzer.

d) The potential V_2 on S_2 is adjusted so that an electron of kinetic energy E passes through the analyzer and out the exit slit.

How must V_2 be adjusted to pass a proton of the same energy? The proton has the opposite charge to the electron and has 1836 times as much mass.



(2a)

Answer to part 1

By inspection it must be a $\frac{1}{r}$ potential, but to be so, the potential at infinity is taken to be a constant value other than zero.

Therefore we have

$$V = a + \frac{b}{r}$$, where a is the potential at infinity.

Calculate the parameters a and b

@ $r1$: $a + \frac{b}{r1} = V1$

@ $r2$ $a + \frac{b}{r2} = V2$

Solve for a and b: $a = \frac{r_2 V_2}{r_2 - r_1}$

$b = \frac{-r_1 r_2 V_2}{r_2 - r_1}$

potential $= V = \frac{r_2 V_2}{(r_2 - r_1)} \left(1 - \frac{r_1}{r}\right)$

Force $= \frac{dV}{dr} = \frac{r_1 r_2 V_2}{r_2 - r_1} \left(\frac{1}{r^2}\right)$ inverse square force law, the same as gravity.

2b

Answer to part 2)

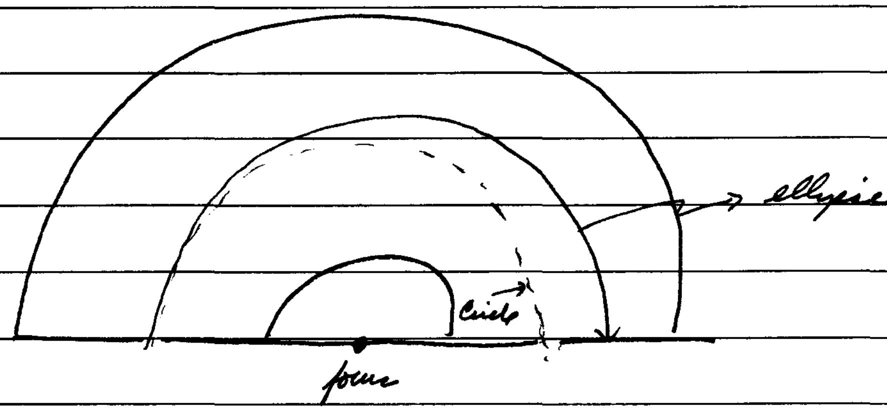
mean potential $= \frac{V_2}{2} = \frac{r_2 V_2}{(r_2 - r_1)}\left(1 - \frac{r_1}{r_s}\right)$

Solve for the slit radius $\boxed{r_s = \frac{2 r_1 r_2}{r_2 + r_1}}$

2c

circles & ellipses, by Kepler's laws for motion in an inverse-square force field.

angular momentum about the center of the sphere, one of the foci of the ellipse, is conserved, since the force is radial the torque
$L = \vec{F} \times \vec{r} = 0$, since $\vec{F}$ and $\vec{r}$ are collinear.



Answer to part 4

It is only necessary to change the polarity of V2. In an electrostatic energy analyzer, the operation ~~of~~ is independent of the mass of the analyzed particle.

Proof.

As the particle passes through, the electrostatic force on it is ~~equal to~~ balanced by the centrifugal force.

The electrostatic force is eE where E is the electric field.

The centrifugal force is ~~$\frac{mv^2}{r}$, where~~

$C = \frac{mv^2}{r}$, where v is the velocity.

But the kinetic energy $K = \frac{1}{2}mv^2$

Therefore the centrifugal force is

$C = \frac{2K}{r}$, and hence the two forces that operate depend explicitly only on the charge and kinetic ~~operate as independent of the mass~~.

energy of the particle, and the mass drops out.

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5. Consider a particle of mass m described by the time independent Schrödinger equation with the spherically symmetric potential $V(r) = V_0 r^2/(a^2 - r^2)$. Use the calculus of variations approach to approximating eigenvalues (Rayleigh-Ritz variational principle) to estimate the energy of the ground state. Choose any trial function for $\psi(r)$ that you wish; keep in mind, however, simplicity and boundary conditions. (Potentials of this form have been used in bag models of hadrons).

Bag Model Solution

Hiscock

Rayleigh-Ritz tells us that the energy is approximately given by

$$E \simeq \frac{\int \psi H \psi}{\int \psi \psi} \quad \text{where } \psi \text{ is our trial function.}$$

here, $H = -\frac{\hbar^2}{2m}\nabla^2 + \frac{V_0 r^2}{a^2 - r^2}$

I will take as my trial function $\psi = 1 - \left(\frac{r}{a}\right)^2$. This is simple and obeys the boundary condition $\psi(r=a) = 0$. It is independent of θ and ϕ, as befits the ground state wave function.

In spherical coordinates $\nabla^2\psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \psi}{\partial r}\right)$

$$\frac{\partial \psi}{\partial r} = -2r/a^2 \; ; \; r^2 \frac{\partial \psi}{\partial r} = -2r^3/a^2 \; ; \; \frac{\partial}{\partial r}\left(r^2 \frac{\partial \psi}{\partial r}\right) = -6r^2/a^2$$

$$\nabla^2 \psi = -6/a^2 \; ; \qquad V(r)\psi = \frac{V_0 r^2}{a^2 - r^2}\left[1 - \left(\frac{r}{a}\right)^2\right] = V_0 \frac{r^2}{a^2}$$

$$H\psi = \frac{\hbar^2}{2m}\left(\frac{6}{a^2}\right) + \frac{V_0 r^2}{a^2} = \frac{3\hbar^2}{ma^2} + \frac{V_0 r^2}{a^2}$$

$$\psi H \psi = \frac{3\hbar^2}{ma^2}\left[1 - \left(\frac{r}{a}\right)^2\right] + \frac{V_0 r^2}{a^2}\left[1 - \left(\frac{r}{a}\right)^2\right]$$

$$\int \psi H \psi \, dV = 4\pi \int_0^a \left\{ \frac{3\hbar^2}{ma^2}\left(r^2 - \frac{r^4}{a^2}\right) + \frac{V_0}{a^2}\left(r^4 - \frac{r^6}{a^2}\right) \right\} dr$$

$$= 4\pi \left\{ \frac{3\hbar^2}{ma^2}\left(\frac{a^3}{3} - \frac{a^3}{5}\right) + \frac{V_0}{a^2}\left(\frac{a^5}{5} - \frac{a^7}{7}\right) \right\}$$

$$= 4\pi \left\{ \frac{2\hbar^2 a}{m} + \frac{2V_0 a^3}{35} \right\}$$

next,
$$\int \psi\psi \, dV = 4\pi \int_0^a \left[r^2 - 2 \cdot \frac{r^4}{a^2} + \frac{r^6}{a^4} \right] dr$$

$$= 4\pi \left\{ \frac{1}{3} - \frac{2}{5} + \frac{1}{7} \right\} a^3 = 4\pi \left(\frac{8}{105} \right) a^3$$

finally,
$$\boxed{E \simeq \frac{2 \left[\frac{\hbar^2 a}{m} + \frac{V_0 a^3}{35} \right]}{\left(\frac{8}{105} \right) a^3} = \frac{105 \hbar^2}{4 m a^2} + \frac{3 V_0}{4}}$$

6. A conducting sphere of radius a carries charge q. The dielectric constant outside the sphere varies with radial distance from the center of the sphere according to

$$\varepsilon = 1 + b/r \qquad r > a$$

a) Find the potential in the region outside the sphere; $r > a$.

b) What will the polarization surface charge density be on the dielectric surface at $r = a$?

E+M #2 #6 D. Smith

A conducting sphere of radius a carries charge q. The dielectric constant outside the sphere varies with radial distance from the center of the sphere according to

$$\epsilon = 1 + b/r$$

6 (a) Find the potential in the region outside the sphere; $r > b$

4 (b) What will the polarization surface charge density on the dielectric surface at $r = a$ be?

— " —

Electrostatics problem; use $\vec{\nabla} \times \vec{E} = 0$

$\vec{\nabla} \cdot \vec{D} = 4\pi\rho$ with $\vec{D} = \epsilon \vec{E}$

For sphere of radius r outside the conductor, use Gauss' theorem:

$$\int_S (\vec{\nabla} \cdot \vec{D})\, da = 4\pi q \text{ , free charge}$$

$$D\, 4\pi r^2 = 4\pi q$$

$$\vec{D} = q/r^2 \, \hat{r}$$

Then

$$\vec{E} = \vec{D}/\epsilon = q/\epsilon r^2 \, \hat{r} = -\vec{\nabla}\Phi$$

Let $\Phi = 0$ at $r = \infty$, so

Integral tables used?

$$\Phi(r) = -\int_\infty^r \vec{E} \cdot d\vec{r} = -\int_\infty^r \frac{q\, dr}{r^2(1+b/r)} = -\int_\infty^r \frac{q\, dr}{r(b+r)}$$

$$= -q\left(-\tfrac{1}{b} \ln \frac{r}{r+b}\right)_\infty^r$$

$$\boxed{\Phi(r) = q/b \ln \frac{b+r}{r} = -q/b \ln\left(\frac{r}{r+b}\right)} \qquad r > b$$

(b) Polarization surface charge density, $\sigma = \vec{P} \cdot \hat{n}$ ← surface normal

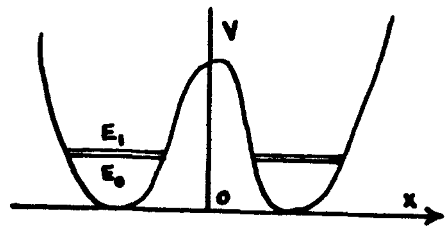
$$\vec{D} = \vec{E} + 4\pi \vec{P} = \epsilon \vec{E}$$

$$\vec{P} = \frac{(\epsilon - 1)}{4\pi} \vec{E} = \frac{b}{r} \frac{1}{4\pi} \frac{q}{\epsilon r^2} \hat{r}$$

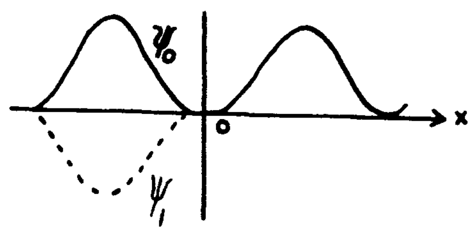
Normal to dielectric surface at r is $\hat{n} = -\hat{r}$ so

$$\vec{\sigma} = \vec{P} \cdot \hat{n}\Big|_a = -\frac{b}{4\pi a^3} \frac{q}{(1 + b/a)} = \boxed{-\frac{q b}{4\pi a^2 (a+b)}}$$

7. A particle moves in one dimension in a potential V(x) which is an even function of x and has the form of two potential wells separated by a barrier. If the barrier is high, the ground state and the first excited state have nearly equal energies, E_0 and E_1.



The corresponding wave functions, $\psi_0(x)$ and $\psi_1(x)$, are even and odd functions, respectively, and look roughly as shown: The existence of second and higher excited states can be disregarded throughout.



a) At time t = 0 the particle is almost certainly in the right-hand well. After how long a time (expressed in terms of E_0 and E_1) is it almost certainly in the left-hand well, and what happens thereafter?

b) The particle has charge q, and a uniform electrostatic field of strength F is applied along the x-axis. In the limit of very small F, the energies of the new stationary states can be represented as $E_0 + a_0F + b_0F^2$ and $E_1 + a_1F + b_1F^2$, where the *a*'s and the *b*'s are independent of *F*. Say as much as you can about the values of the *a*'s and the *b*'s and any relations between them.

c) How is the polarizability of the ground state connected with these constants?

$\left(\frac{1}{\sqrt{2}}\right)$

5a) At $t = 0$ the wave function must be $(\psi_0 + \psi_1)$ to make the probability of finding the particle in the left-hand well essentially zero. Then the time-dependent wave function is

$$\psi(x, t) = \psi_0(x)\, e^{-iE_0t/\hbar} + \psi_1(x)\, e^{-iE_1t/\hbar}$$

The particle is in the left-hand well when this function has become proportional to $\psi_0 - \psi_1$, i.e. when

$$e^{-iE_1t/\hbar} = -e^{-iE_0t/\hbar} \quad \text{or} \quad (E_1 - E_0)t/\hbar = \pi,\ 3\pi,\ 5\pi,\ \ldots\ldots \quad .$$

Define $T = \dfrac{2\pi\hbar}{E_1-E_0}$. The particle tunnels into the left-hand well in a time $T/2$. At time T is has tunneled back to the right-hand well, and thereafter it repeats the process with period T.

b) The perturbing Hamiltonian is $H' = -qFx$ and its matrix element between unperturbed stationary states ψ_m and ψ_n will be denoted by H'_{mn}. Since H' is an odd function and ψ_0 has definite parity, the ground-state energy has no term linear in F:

$$a_oF = H'_{oo} = -qFx_{oo} = 0$$

Hence $a_o = 0$ and similarly $a_1 = 0$.

By second-order perturbation theory, $\displaystyle b_oF^2 = \sum_{n>0} \frac{|H'_{0n}|^2}{E_o-E_n} \cong q^2F^2\,\frac{|x_{01}|^2}{E_0-E_1}$,

where we neglect second and higher excited states. Thus

$$b_0 = \frac{-q^2\,|x_{01}|^2}{E_1 - E_0} < 0 \quad \text{and similarly} \quad b_1 \cong -b_o > 0.$$

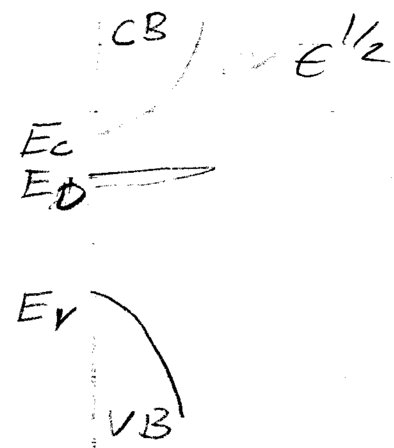
c) A system with polarizability α, if placed in an electric field F, acquires an induced moment αF and an energy $-1/2\ \alpha\ F^2$. Hence the ground state has a positive polarizability $-2b_0$ and the first excited state has a negative polarizability $-2b_1 \cong 2b_0$.

8. Consider the free electron model of a semiconductor. Find the expression for the position of the Fermi level E_F for a semiconductor whose only dopant is a single donor level of density N_d. Consider the case where the donor binding energy is much greater than four times the thermal energy and the fundamental gap is very much larger than the donor binding energy. (A donor is a localized state which is neutral when occupied by an electron and has a charge of positive e when empty.)

Sol

a) A donor is a localized impurity which is neutral when occupied by an electron and has a positive charge when empty.

b) electric charge neutrality ... holds



$$p - n + N_D^+ - N_A^- = 0$$

since

$$E_c - E_d \ll E_c - E_V$$

p is neg & crystal does not have acceptors

$$\rightarrow \quad n = N_D^+$$

$$N_D^+ = N_d(1 - F_{E_D}) = N_D\left(1 - \frac{1}{e^{\frac{E_D - E_F}{kT}} + 1}\right)$$

$$= N_D\left(\frac{1}{1 + e^{\frac{E_F - E_D}{kT}}}\right)$$

$$n = \int_{E_C}^{\infty} G(E)F(E)\,dE \; ; \; G(E) = \underbrace{C}_{\frac{4\pi(2m)^{3/2}}{h^3}}(E - E_c)^{1/2}$$

since the ... binding energy is larger than kT $\quad F_E = \frac{1}{e^{\frac{E - E_F}{kT}} + 1} \doteq e^{-(E-E_F)/kT}$

the Maxwell Boltzmann approx

$\swarrow E' + E_c$ $\quad E' = E - E_c$

$$n = C\int_{E_c}^{\infty} (E - E_c)^{1/2} e^{E_F - E/kT} \, dE$$

$$= C\int_{0}^{\infty} E'^{1/2} e^{\frac{E_F - E_c}{kT}} e^{-E'/kT} \, dE$$

$$= C e^{\frac{E_F - E_c}{kT}} \frac{1}{2} \sqrt{\pi} (kT)^{3/2}$$ USE TABLE

since $\int_0^{\infty} e^{-ax} \sqrt{x} = 1/2 \frac{\sqrt{\pi}}{a^{3/2}}$

if one knows $C \rightarrow n = \frac{2(2\pi m kT)^{3/2}}{h^3} e^{\frac{E_F - E_c}{kT}}$

write

$$n = N_C e^{-\frac{E_c - E_F}{kT}}$$

combine

$$n = N_D \left(\frac{1}{1 + e^{(E_F - E_D)/kT}} \right) \doteq N_D e^{-\left(\frac{E_F - E_D}{kT}\right)}$$

$\Rightarrow$ since $n \ll N_D$

$$n = N_C e^{-\left(\frac{E_c - E_F}{kT}\right)} = N_D e^{\left(\frac{E_D - E_F}{kT}\right)}$$

$$e^{\frac{2E_F - E_D - E_c}{kT}} = \frac{N_D}{N_c}$$

$$\frac{2E_F - E_D - E_c}{kT} = \ln \frac{N_D}{N_c}$$

$$\boxed{E_F = \frac{E_d + E_c}{2} + \frac{kT}{2} \ln \frac{N_d}{N_c}}$$

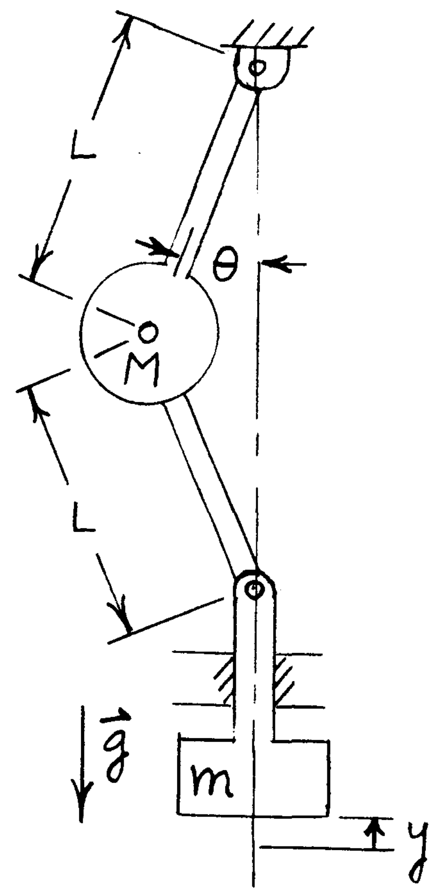
DEPARTMENT OF PHYSICS

PH.D. COMPREHENSIVE EXAMINATION

MONDAY, SEPTEMBER 19, 1989, 9 A.M. - 12 NOON

Answer each of the following questions, all of which carry equal weight. Begin your answer to each question on a new sheet of paper; solutions to different questions must not appear on the same sheet. Each sheet of paper must be labeled with your name and the problem number in the upper right hand corner. When more than one sheet is submitted for a problem, be sure the pages are ordered properly. [Problems 9-12].

9. A pendulum of point mass M and length L is connected by a massless rod of length L to a rod holding mass m and constrained to move vertically by a hole lined up with the pendulum pivot.



a) Given $y=0$ when $\theta=0$, find exact expressions for $y(\theta)$ and $\dot{y}(\theta,\dot{\theta})$.

b) Write the exact Lagrangian $L(\theta,\dot{\theta})$ for this system.

c) Now write $L(\theta,\dot{\theta})$ in the small-θ approximation.

d) Using the small-θ approximation for $L(\theta,\dot{\theta})$, write the Lagrange equation for this system.

e) Making suitable approximations, solve the Lagrange equation to find the frequency ω of small oscillations.

f) Show that this system acts as a mechanical frequency doubler, for small oscillations.

a) $y = 2L(1-\cos\theta)$ $\qquad \dot{y} = 2L\dot{\theta}\sin\theta$

b) $$L = T - V = \frac{1}{2}ML^2\dot{\theta}^2 + \frac{1}{2}m(4L^2\dot{\theta}^2\sin^2\theta) - MgL(1-\cos\theta) - 2mgL(1-\cos\theta)$$

$$L = \frac{1}{2}ML^2\dot{\theta}^2 + 2mL^2\dot{\theta}^2\sin^2\theta - (M+2m)gL(1-\cos\theta)$$

c) $$L \simeq \frac{1}{2}ML^2\dot{\theta}^2 + 2mL^2\theta^2\dot{\theta}^2 - \frac{1}{2}(M+2m)gL\theta^2$$

d) $$\frac{d}{dt}\frac{\partial L}{\partial\dot{\theta}} - \frac{\partial L}{\partial\theta} = ML^2\ddot{\theta} + 4mL^2\theta^2\ddot{\theta} + 8mL^2\theta\dot{\theta}^2 - 4mL^2\theta\dot{\theta}^2 + (M+2m)gL\theta$$
$$= ML^2\ddot{\theta} + 4mL^2\theta^2\ddot{\theta} + 4mL^2\theta\dot{\theta}^2 + (M+2m)gL\theta = 0$$

e) For small oscillations, the above $\theta^2\ddot{\theta}$ and $\theta\dot{\theta}^2$ terms can be neglected. Then

$$\ddot{\theta} = -\frac{M+2m}{M}\frac{g}{L}\theta$$ which has solution

$$\theta = \theta_0\cos\left[\sqrt{\frac{M+2m}{M}\frac{g}{L}}\,t\right]$$ so $$\omega = \sqrt{\frac{M+2m}{M}\frac{g}{L}}$$

f) For $\theta = \theta_0\cos\omega t$, $$y = 2L(1-\cos\theta) \simeq L\theta^2 = L\theta_0^2\cos^2\omega t = \frac{1}{2}L\theta_0^2(1+\cos 2\omega t)$$

so the motion of m is at double the frequency of the motion of M.

10. A plasma has n electrons per unit volume of charge -e and mass m.

a) Given that $\vec{J} = \sigma\vec{E}$, find σ for an applied field $\vec{E} = \vec{E}_0 e^{i\omega t}$ for this plasma.

b) There is a plasma cutoff frequency ω_P which limits the frequency range over which electromagnetic waves can propagate in a plasma. Is this the upper limit, or the lower limit?

c) Find ω_P for the above plasma.

a) $\vec{J} = nq\mu\vec{E} = nq\vec{v}$

Solve equation of motion to find $\vec{v}$:

$$m\ddot{\vec{x}} = q\vec{E} \qquad \ddot{\vec{x}} = \frac{q}{m}\vec{E}_0 e^{j\omega t} \qquad \dot{\vec{x}} = \frac{q}{m}E_0\frac{e^{j\omega t}}{j\omega}$$

$$\vec{v} = -\frac{e}{m}\vec{E}_0\left(-\frac{j}{\omega}\right)e^{j\omega t}$$

so $\vec{J} = \frac{-jne^2}{m\omega}\vec{E}_0 e^{j\omega t} = \sigma\vec{E}_0 e^{j\omega t}$

Thus $\boxed{\sigma = -\frac{jne^2}{m\omega}}$

b) ω_p is lower limit; at high frequency, inertia limits the effect of the plasma on E-M wave propagation.

c) Consider E-M wave propagating along z, with $\vec{E}$ along x. Look for wave solution to Maxwell's equations:

$$\vec{E} = E_0\hat{i}\,e^{j(\omega t - kz)} \qquad \vec{B} = B_0\hat{j}\,e^{j(\omega t - kz)}$$

$$\vec{\nabla}\times\vec{E} = \hat{j}\frac{\partial E_x}{\partial z} = -\hat{j}\,jkE_0 e^{j(\omega t - kz)} = -\frac{\partial\vec{B}}{\partial t} = -\hat{j}\,j\omega B_0 e^{j(\omega t - kz)}$$

so we have $kE_0 = \omega B_0$

$$\vec{\nabla}\times\vec{B} = -\hat{i}\frac{\partial B_y}{\partial z} = \hat{i}\,jkB_0 e^{j(\omega t - kz)} = \mu_0\vec{J} + \frac{1}{c^2}\frac{\partial\vec{E}}{\partial t}$$

$$= -\frac{\hat{i}\,jne^2\mu_0 E_0 e^{j(\omega t - kz)}}{m\omega} + \frac{\hat{i}\,j\omega E_0 e^{j(\omega t - kz)}}{c^2}$$

so $kB_0 = \left(\frac{\omega}{c^2} - \frac{ne^2\mu_0}{m\omega}\right)E_0$

Combine the two relations between E_0 and B_0:

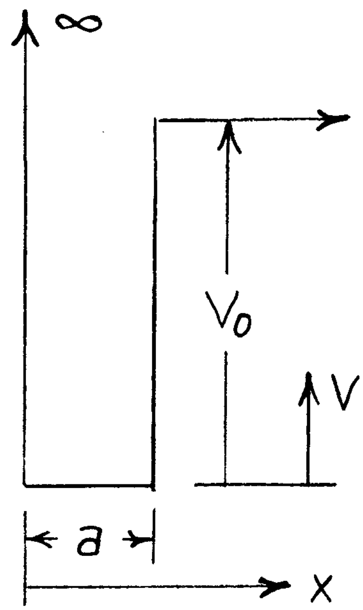
$$\frac{k^2}{\omega} E_0 = \left(\frac{\omega}{c^2} - \frac{n e^2 \mu_0}{m \omega} \right) E_0$$

A wave solution exists only if both coefficients are positive, so

$$\omega_p^2 = \frac{n e^2 \mu_0 c^2}{m} = \frac{n e^2 \mu_0}{m \epsilon_0 \mu_0} = \boxed{\frac{n e^2}{m \epsilon_0} = \omega_p^2}$$

11. a) For a particle of mass m in this 1-d well, find the energy E_0 of the ground state if $V_0 \to \infty$.

b) Now let V_0 be finite but with $V_0 \gg E_0$. Find an approximate value for the new ground state energy E_1.



a) $-\frac{\hbar^2}{2m}\frac{\partial^2 \psi}{\partial x^2} = -\frac{\hbar^2}{2m}\frac{\partial^2 \psi_0 \sin\frac{\pi x}{a}}{\partial x^2}$

$= \frac{\pi^2 \hbar^2 \psi_0 \sin\frac{\pi x}{a}}{2ma^2} = E_0 \psi_0 \sin\frac{\pi x}{a}$

so $\boxed{E_0 = \frac{\pi^2 \hbar^2}{2ma^2}}$

b) Inside well, $\psi_1 = \psi_{10} \sin kx$

Outside well, for $x \geq a$, $\psi_2 = \psi_{20} e^{-cx}$

From Schrödinger equation,

$+\frac{\hbar^2 k^2}{2m} = E_1$ and $\frac{\hbar^2 c^2}{2m} = V_0 - E_1$

Match amplitudes at boundary:

$\psi_{10} \sin ka \simeq \psi_{10}(\pi - ka) = \psi_{20} e^{-ca}$

Match slopes at boundary:

$k\psi_{10} \cos ka \simeq -k\psi_{10} \simeq -\frac{\pi}{a}\psi_{10} = -c\psi_{20} e^{-ca}$

so $-k\psi_{10} = -c\psi_{10}(\pi - ka)$ and $c = \frac{k}{\pi - ka}$

$\frac{\hbar^2 c^2}{2m} = \frac{\hbar^2}{2m}\frac{k^2}{(\pi - ka)^2} = V_0 - \frac{\hbar^2 k^2}{2m}$ so $k^2\left(1 + \frac{1}{(\pi - ka)^2}\right) = \frac{2mV_0}{\hbar^2}$,

or $k^2 \simeq (\pi - ka)^2 \frac{2mV_0}{\hbar^2}$, or $(\pi - ka)^2 \simeq \frac{\pi^2}{a^2}\frac{\hbar^2}{2mV_0}$

$k \simeq \frac{\pi}{a} - \frac{\pi\hbar}{a^2\sqrt{2mV_0}}$ $\boxed{E_1 \simeq \frac{\hbar^2}{2m}\frac{\pi^2}{a^2}\left(1 - \frac{\hbar}{a}\sqrt{\frac{2}{mV_0}}\right)}$

12. The Σ^{*0} hyperon can decay into (among other things) (a) $\Sigma^+ + \pi^-$, (b) $\Sigma^0 + \pi^0$, and (c) $\Sigma^- + \pi^+$. (Note: $\Sigma^* \neq \Sigma$). Each of these particles carries a quantum number known as isospin (invented by Heisenberg in 1932), <u>which obeys exactly the same mathematical rules as ordinary angular momentum in quantum mechanics</u>. The isospin values of the particles described above are: (in the form $|j,m\rangle$):

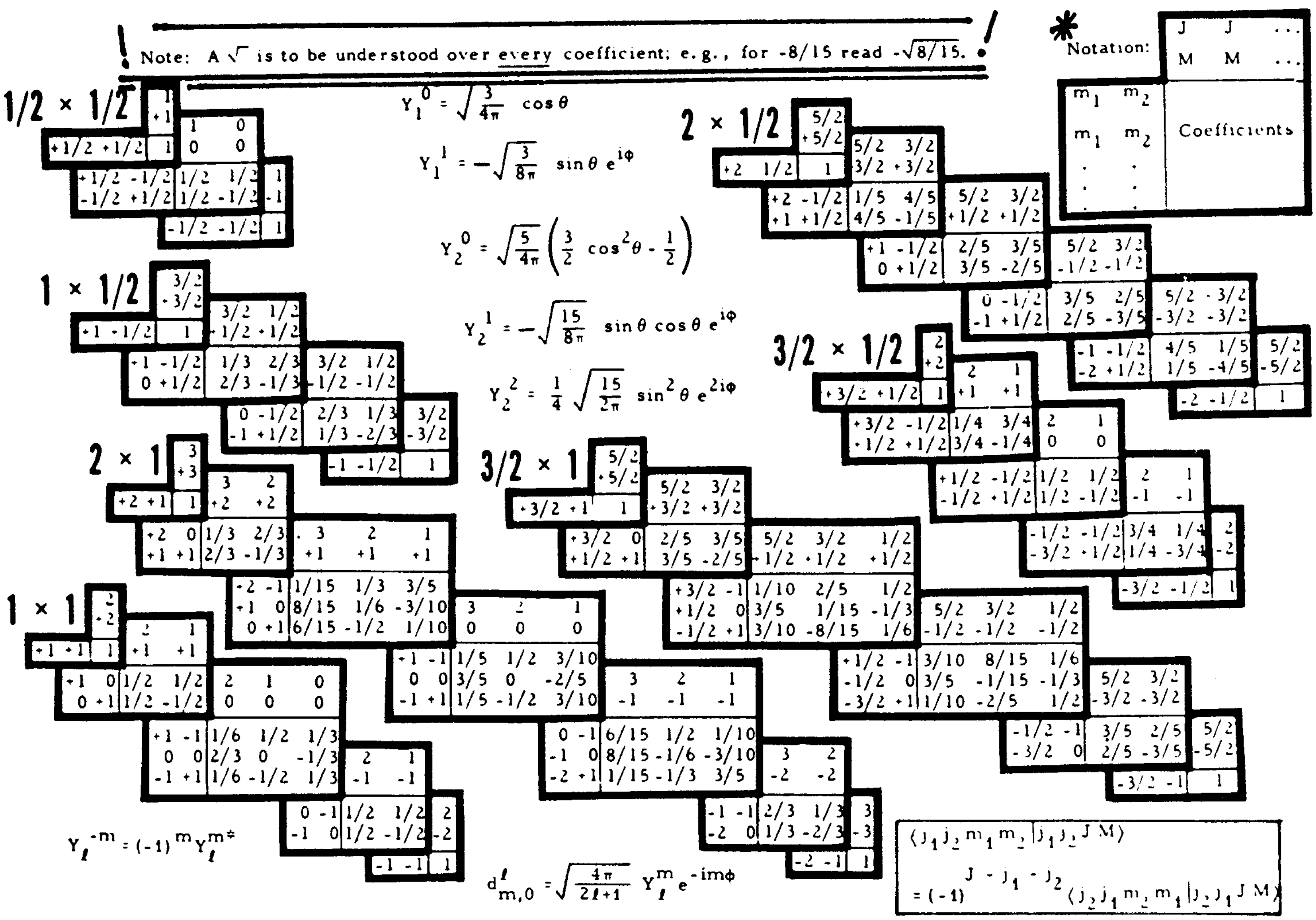
$$\Sigma^{*0} : |1,0\rangle$$

$$\Sigma^+ : |1,1\rangle \qquad \Sigma^0 : |1,0\rangle \qquad \Sigma^- : |1,-1\rangle$$

$$\pi^+ : |1,1\rangle \qquad \pi^0 : |1,0\rangle \qquad \pi^- : |1,-1\rangle$$

Use the rules for combining quantum mechanical angular momenta to relate the decay rates for the three processes, (a), (b), (c). If you observed 100 disintegrations of the form $\Sigma^{*0} \rightarrow \Sigma + \pi$, how many would you expect to see of each of the three types (a), (b), (c)?

CLEBSCH-GORDAN COEFFICIENTS, SPHERICAL HARMONICS, & *d* FUNCTIONS



?/11/89 Isospin in Σ^{*0} decay - solution Hiscock

decay (a) $\Sigma^{*0} \to \Sigma^{+} + \pi^{-}$

$$|1,0\rangle \to |1,1\rangle + |1,-1\rangle$$

$$|1,1\rangle + |1,-1\rangle = \frac{1}{\sqrt{6}}|2,0\rangle + \frac{1}{\sqrt{2}}|1,0\rangle + \frac{1}{\sqrt{3}}|0,0\rangle$$

decay (b): $\Sigma^{*0} \to \Sigma^{0} + \pi^{0}$

$$|1,0\rangle \to |1,0\rangle + |1,0\rangle$$

$$|1,0\rangle + |1,0\rangle = \sqrt{\tfrac{2}{3}}|2,0\rangle - \frac{1}{\sqrt{3}}|0,0\rangle$$

decay (c): $\Sigma^{*0} \to \Sigma^{-} + \pi^{+}$

$$|1,0\rangle \to |1,-1\rangle + |1,1\rangle$$

$$|1,-1\rangle + |1,1\rangle = \frac{1}{\sqrt{6}}|2,0\rangle + \frac{1}{\sqrt{2}}|1,0\rangle + \frac{1}{\sqrt{3}}|0,0\rangle$$

Since the initial state is isospin 1, only those final states with total isospin = 1 will contribute. Note that the final state for decay (b) contains <u>no</u> $|1 0\rangle$ component.

The ratio of the amplitudes $\mathcal{M}$ for the three decays are then:

$$\mathcal{M}_a : \mathcal{M}_b : \mathcal{M}_c = 1/\sqrt{2} : 0 : 1/\sqrt{2}$$

The decay rate goes like the amplitude squared; here, this implies that <u>out of 100</u> $\Sigma^{*0} \to \Sigma + \pi$ decays, we will see:

50	$\Sigma^{*0} \to \Sigma^{+} + \pi^{-}$
0	$\Sigma^{*0} \to \Sigma^{0} + \pi^{0}$
50	$\Sigma^{*0} \to \Sigma^{-} + \pi^{+}$

DEPARTMENT OF PHYSICS

PH.D. COMPREHENSIVE EXAMINATION

MONDAY, SEPTEMBER 19, 1989, 2 P.M. - 5 P.M.

Answer each of the following questions, all of which carry equal weight. Begin your answer to each question on a new sheet of paper; solutions to different questions must not appear on the same sheet. Each sheet of paper must be labeled with your name and the problem number in the upper right hand corner. When more than one sheet is submitted for a problem, be sure the pages are ordered properly. [Problems 13-16].

13. Find approximate expressions for the roots of

$$x^3 - x^2 + \varepsilon = 0 \quad , \quad \varepsilon \ll 1$$

to linear order in ε.

Math

Find ~~the roots of~~ approximate expressions for the roots of Math

$$x^3 - x^2 + \epsilon = 0 \quad , \quad \epsilon \ll 1$$

to linear order in ϵ.

When $\epsilon = 0$ the roots are $x = 1$ and 0. For $\epsilon \neq 0$ the root at $x=0$ gives rise to two ~~solutions~~ unequal roots. Note that if we try $x = a_1 \epsilon$ to find them, we get

$$a_1^3 \epsilon^3 - a_1^2 \epsilon^2 + \epsilon = 0$$

which ~~contains a~~ gets us nowhere. Trying instead $x = a_1 \epsilon^n$, we first look for the value of n needed to balance the eqn in lowest order:

$$a_1^3 \epsilon^{3n} - a_1^2 \epsilon^{2n} + \epsilon = 0$$

This forces $n = 1/2$ and $a_1 = \pm 1$. So now try a series in $\epsilon^{1/2}$:

$$x = \pm \epsilon^{1/2} + a_2 \epsilon$$

for $a_1 = +1$:

$$\epsilon^{3/2} + 3a_2 \epsilon^2 \ldots - \cancel{\epsilon} - 2a_2 \epsilon^{3/2} \ldots + \cancel{\epsilon} = 0$$

so $a_2 = \frac{1}{2}$ & $\boxed{x = \epsilon^{1/2} + \frac{1}{2}\epsilon}$

for $a_1 = -1$

$$-\epsilon^{3/2} + 3a_2 \epsilon^2 \ldots - \cancel{\epsilon} + 2a_2 \epsilon^{3/2} \ldots + \cancel{\epsilon} = 0$$

and $a_2 = 1/2$ again

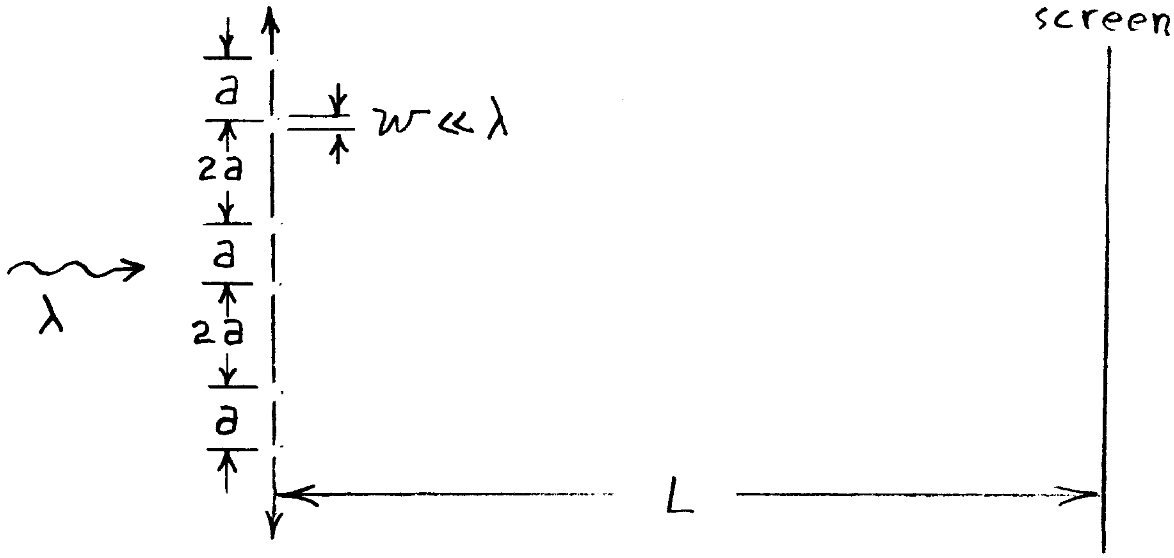
so $\boxed{x = -\epsilon^{1/2} + \frac{1}{2}\epsilon}$

Finally, for the root that in zeroth order is just 1, we try $x = 1 + a_1 \epsilon \ldots$;

$$\cancel{1} + 3a_1 \epsilon \ldots - \cancel{1} - 2a_1 \epsilon \ldots + \epsilon = 0$$

$a_1 = -$ and $\boxed{x = 1 - \epsilon}$

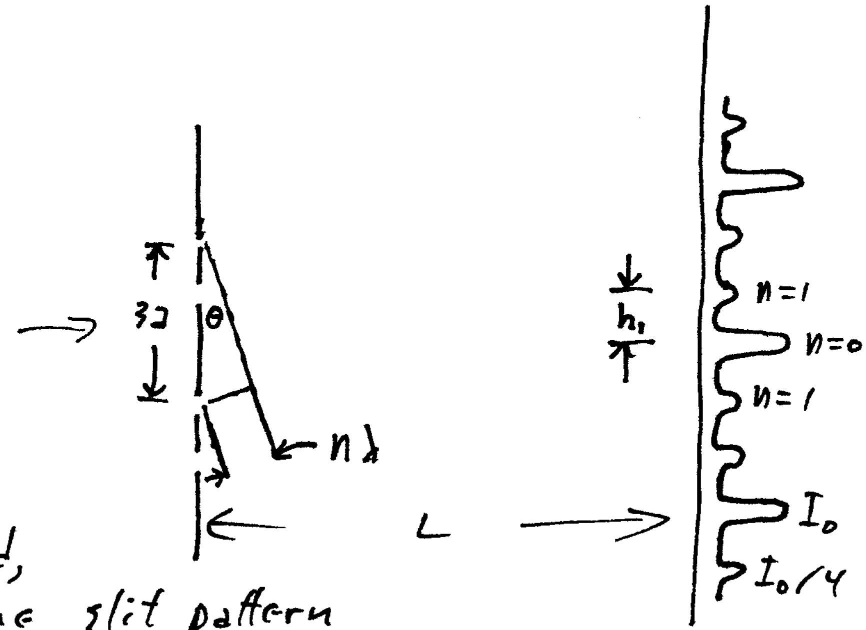
14.



A grating consists of slits of width w with alternate spacings a and 2a as shown. It is illuminated normally by light of wavelength λ. Describe the spacing and relative intensities of the diffraction bands on a screen a distance L behind the grating, if the center band has intensity I_0.

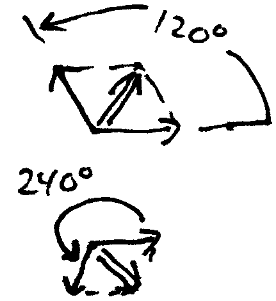
To get constructive interference, the condition

$$\sin\theta_n = \frac{n\lambda}{3a}$$



must be satisfied, because $3a$ is the slit pattern repeat distance.

For $n = 1, 4, 7, \cdots$ the phase angle of light from adjacent slits is 120°, so the resultant amplitude is the same as from a single slit.



For $n = 2, 5, 8, \cdots$ the phase angle is 240° and again the amplitude is that for one slit.

For $n = 0, 3, 6, 9, \cdots$ the phase angle is 0°, the amplitude is twice that for other n, so the intensity is 4 times that for other n.

The fringe position $h_n = \pm L \tan\theta_n$. For small θ_n this becomes

$$h_n \simeq \pm L\theta_n \simeq \frac{n\lambda L}{3a}.$$

15. Consider a system of N identical fermions enclosed in volume V at temperature T. A single fermion has a density of states $g(E)$ given by $g(E) = \alpha V$ where α is a constant with dimension $(\text{energy})^{-1}(\text{volume})^{-1}$.

a) Find the chemical potential μ as a function of T and V.

b) Find the variation of the ground state energy with the volume: $(\partial U/\partial V)_{T=0} = ?$

c) Calculate the isothermal compressibility $\kappa_T = -(1/V)(\partial V/\partial P)_T$ for this system.

<u>Stat / Thermo</u>

Consider a system of N identical fermions, enclosed in volume V at temperature T. A single fermion has a density of states $g(E)$ given by $g(E) = \alpha V$ where α is a constant with dimension $(\text{energy})^{-1}(\text{volume})^{-1}$. The fermions do not interact, and you may ignore their spin

(a) Find the chemical potential μ as a function of T and V.

(b) Find the variation of the ground state energy with the volume: $\left(\frac{\partial U}{\partial V}\right)_{T=0} = ?$

(c) Calculate the isothermal compressibility $\kappa_T = -\frac{1}{V}\left(\frac{\partial V}{\partial p}\right)_T$ for this system.

<u>Solution</u>

(a) The particle # N is fixed, so

$$N = \int_0^\infty g(E)\, dE \frac{1}{e^{\beta(E-\mu)}+1} = \frac{\alpha V}{\beta} \int_0^\infty dx \underbrace{\frac{e^{\beta\mu} e^{-x}}{1+e^{\beta\mu} e^{-x}}}$$

$$-\ln\left(1+e^{\beta\mu} e^{-x}\right)\Big]_0^\infty$$

$$= \ln\left(1+e^{\beta\mu}\right)$$

so $\underline{\mu = \frac{1}{\beta} \ln\left(e^{\frac{\beta N}{\alpha V}} - 1\right)}$

(b) $U = \int_0^\infty g(E)\,dE\; E \;\frac{1}{e^{\beta(E-\mu)}+1}$

At $T=0$, μ takes on its low-T limit, the Fermi energy ϵ_F $\quad \mu \to \lim_{\beta\to\infty} \frac{1}{\beta}\ln\left(e^{\frac{\beta N}{\alpha V}}-1\right) = \frac{N}{\alpha V} = \epsilon_F$

& the function $\frac{1}{e^{\beta(E-\mu)}+1}$ becomes a step fcn at $E=\epsilon_F$

$$U(T=0) = \alpha V \int_0^{\epsilon_F} dE\; E = \alpha V \frac{\epsilon_F^2}{2}$$

$$= \alpha V \frac{N^2}{2(\alpha V)^2} = \frac{N^2}{2\alpha V}$$

then $\left(\frac{\partial U}{\partial V}\right)_{T=0} = -\frac{N^2}{2\alpha V^2}$

(c) Using the grand canonical partition function $\mathcal{Q}$

$$\frac{pV}{kT} = \ln \mathcal{Q} = \int g(E)\,dE\; \ln\left(1+e^{\beta(\mu-E)}\right)$$

or $p = \alpha kT \int_0^\infty dE \;\ln\left(1+e^{\beta(\mu-E)}\right)$

then $\left(\frac{\partial p}{\partial V}\right)_T = \alpha kT \int dE \;\frac{e^{\beta(\mu-E)}}{1+e^{\beta(\mu-E)}}\left[\beta\left(\frac{\partial \mu}{\partial V}\right)_T\right]$

$$= \frac{1}{V}\left(\frac{\partial \mu}{\partial V}\right)_T \alpha V \int dE\; \frac{e^{\beta(\mu-E)}}{1+e^{\beta(\mu-E)}} = \frac{N}{V}\left(\frac{\partial \mu}{\partial V}\right)_T$$

or $-\frac{1}{V}\left(\frac{\partial V}{\partial p}\right)_T = -\frac{1}{N}\frac{1}{\left(\frac{\partial \mu}{\partial V}\right)_T} = \frac{\alpha V^2}{N^2}\left(1-e^{-\beta N/\alpha V}\right)$

16. A particle of mass m traveling at speed v approaches an identical particle at rest.

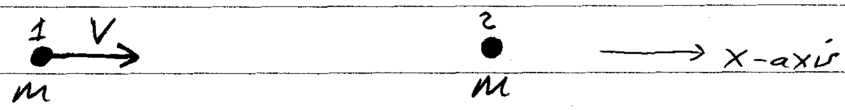
a) What is the speed of each particle in the center-of-mass (CM) frame? (Warning: it is not $v/2$; that value is merely a low-velocity approximation to the correct answer).

b) Compute the kinetic energy of the particle with speed v as a function of the kinetic energy in the CM frame.

c) Suppose the particles are electrons, and their kinetic energy in the CM frame is 50 GeV. What would the kinetic energy of one of the electrons be in the frame where the other is at rest? (These numbers describe the Stanford Linear Collider, which is currently studying the physics of the Z boson).

d) Based on your results, comment on whether future accelerators should be of the fixed target (one particle at rest in the lab frame) or colliding beam design.

9/1/89 Relativistic center of mass - solution Hiscock



In the CM frame, the particles will possess equal and opposite three-velocities of magnitude V_{CM}. Their energies will be equal, as will their momenta (in magnitude). In the initial frame, the particle's four momenta are: (setting $c=1$, of course)

$$p_1^\mu = (E_1, p_1, 0, 0)$$
$$p_2^\mu = (m, 0, 0, 0)$$

and the total 4-momentum is: $P^\mu = p_1^\mu + p_2^\mu$

$$P^\mu = (E_1 + m, p_1, 0, 0)$$

In the CM frame,

$$p_{1CM}^\mu = (E_{CM}, p_{CM}, 0, 0)$$
$$p_{2CM}^\mu = (E_{CM}, -p_{CM}, 0, 0)$$

and P^μ total in the CM frame is:

$$P_{CM}^\mu = (2E_{CM}, 0, 0, 0)$$

Now, while $P_{CM}^\mu \neq P^\mu$ (components of a vector depend on frame) $|P_{CM}^\mu|^2 = |P^\mu|^2$ [scalars are frame-independent], so,

$$4E_{CM}^2 = (E_1 + m)^2 - p_1^2 = E_1^2 + 2E_1 m + m^2 - p_1^2$$

and, since $E_1^2 - p_1^2 = m^2$,

$$4E_{CM}^2 = 2E_1 m + 2m^2 \rightarrow \boxed{2E_{CM}^2 = E_1 m + m^2}$$

now:

$$E_1 = \gamma m = \frac{m}{\sqrt{1 - V^2}} \qquad E_{CM} = \gamma_{CM} m = \frac{m}{\sqrt{1 - V_{CM}^2}}$$

so ...

(a)

$$\frac{2}{1-V_{cM}^2} = \frac{1+\sqrt{1-V^2}}{\sqrt{1-V^2}} \rightarrow \frac{2\sqrt{1-V^2}}{1+\sqrt{1-V^2}} = 1 - V_{cM}^2$$

$$\boxed{V_{cM}^2 = \frac{1-\sqrt{1-V^2}}{1+\sqrt{1-V^2}}} \text{ or } \boxed{V_{cM} = \left[\frac{1-\sqrt{1-V^2}}{1+\sqrt{1-V^2}}\right]^{1/2} \overset{\text{or}}{=} \frac{V}{1+\sqrt{1-V^2}} \overset{\text{or}}{=} \frac{1-\sqrt{1-V^2}}{V}}$$

note that as $V \to 0$, this may be approximated by

$$V_{cM} = V/2 + \mathcal{O}(V/c)^2 \cdot V$$

(b) $E_1 = T_1 + m \quad E_{cM} = T_{cM} + m$, so

$$2(T_{cM}+m)^2 = T_1 m + 2m^2 \qquad \boxed{T_1 = (2T_{cM}^2 + 4T_{cM}m)/m}$$

(c) $T_{cM} = 50$ GeV $\quad m \simeq 5\times10^{-4}$ GeV

then

$$T_1 = \frac{[5\times10^3 + 10^{-1}]}{5\times10^{-4}} \text{ GeV} \simeq \boxed{10^7 \text{ GeV} = T_1 \ !}$$

(d) It is obviously better (cheaper) to build colliders rather than fixed-target accelerators once ultra-relativistic energies are desired.

Taking the SLC as an example, if building a 50 GeV fixed-target accelerator cost n dollars, building a 50 on 50 collider would cost (at most - it was really accomplished for much less) $2n$ dollars; for comparison upgrading to a 10^7 GeV fixed target accelerator would cost $\simeq 10^5 n$ dollars. Colliders are better, because $10^5 \gg 2$ [the ratio of energies gets even worse as the cM energy is increased further]