Definitions of Terms: Convergence & Divergence

Notes on Infinite Series*

) NOMENCLATURE.

An infinite series is the sum of an ∞ number of terms:

$$\rightarrow S=\sum_{n=1}^{\infty} a_n = a_1+a_2+a_3+\dots \qquad (1)$$

The a_n depend on n, they may be of either sign (or even complex), and they may be variable; e.g. $a_n=(-x)^n$. Such sums occur in the Taylor series for some fcn $S(x)$, or in the power series solⁿ to an ODE, etc. Question-of-interest: is S finite or not? Answer as follows...

Define N^{th} "partial sum": $S_N = a_1+a_2+\dots+a_N=\sum_{n=1}^{N} a_n$;

then: $S=\lim_{N\to\infty} S_N = \sum_{n=1}^{N\to\infty} a_n$;

and // if this limit exists (i.e. is finite), $S=\sum_{n=1}^{\infty} a_n$ is a "convergent series";

if this limit does not exist (i.e. $S\to\infty$), S is a "divergent series". (2)

NOTE: S is convergent only if $\lim_{N\to\infty} S_N$ is definite and unique.

$\boxed{\text{Ex.}}$ Is: $S=\sum_{n=0}^{\infty}(-1)^n = \underset{\textcircled{1}}{1}-\underset{\textcircled{2}}{1}+\underset{\textcircled{3}}{1}-\underset{\textcircled{4}}{1}+\underset{\textcircled{5}}{1}-\underset{\textcircled{6}}{1}+\dots$ convergent or divergent?

Combine terms as: ①+② = 0, ③+④ = 0, etc. $\Rightarrow S=0$.

Isolate ①, and combine ②+③ = 0, ④+⑤ = 0, etc. $\Rightarrow S=+1$.

Isolate ②, and combine ①+④ = 0, ③+⑥ = 0, etc. $\Rightarrow S=-1$. (3)

Since addition is associative, the order in which the a_n are summed should not affect the value of a convergent $S=\sum_n^{\infty} a_n$. Here, however, by various associations, we see that we can produce any value desired for S (i.e. $-\infty \leq S \leq +\infty$); i.e. S is not unique. So, $S=\sum_n^{\infty}(-)^n$ is classified as divergent.

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* M&W, Chap. 2; Arfken, Chap. 5; Hassani, Chap. 7.

Geometric Series G(r). Preliminary Test.

Ex. Examine convergence of "Geometric Series": $G(r) = \sum_{n=1}^{\infty} r^n$.

Look at partial sums: $G_N = r + r^2 + \dots r^N = r(1-r^N)/(1-r)$.

(This closed form for G_N is available by simple algebra). As for G itself...

$$G(r) = \lim_{N\to\infty} G_N = \frac{r}{1-r} \lim_{N\to\infty}(1-r^N) = \begin{cases} \text{finite, when } |r|<1; \\ \text{infinite, when } |r|>1. \end{cases}$$

So//

$$\boxed{G(r) = \sum_{n=1}^{\infty} r^n \quad \begin{matrix} \text{converges to: } r/(1-r), \text{ when } |r|<1; \\ \text{diverges, when } |r| \geqslant 1. \end{matrix}} \tag{4}$$

The divergence at $|r|=1$ follows from: $G(r=\pm 1) = \pm 1 + 1 \pm 1 + 1 \pm 1 + \dots$, divergent.

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2) A DIVERGENCE TEST.

The most primitive test of lack of series convergence is the following:

Preliminary Test

> "Let $S = \sum_{n=1}^{\infty} a_n$. If $\lim_{n\to\infty} a_n \neq 0$, then S diverges. If $\lim_{n\to\infty} a_n = 0$, then S may either converge or diverge. Thus, S can converge only if $\lim_{n\to\infty} a_n = 0$."

The proof of this claim is the contrapositive of the following argument...

$$\text{If } S \text{ converges, then } \left\{ \begin{matrix} \lim_{N\to\infty} S_N = S \\ \lim_{N\to\infty} S_{N-1} = S \end{matrix} \right\rangle \text{ the } S_N\text{'s converge to a unique limit } S.$$

So//

$$\lim_{N\to\infty}(S_N - S_{N-1}) = \lim_{N\to\infty} a_N = S - S = 0, \tag{5}$$

As advertised, S is convergent only if $\lim_{n\to\infty} a_n = 0$. When this limit $\neq 0$, $\exists$ no unique limit for the partial sums S_N, and then S diverges.

The Preliminary Test is a weak test that establishes only divergence. But we can apply it to the following examples...

Preliminary Test Examples. Absolute Convergence.

Ex. Geometric Series: $G = \sum_{n=1}^{\infty} a_n$, w/ $a_n = r^n$. (6)

1. When $|r| \geq 1$, $\lim_{n\to\infty} a_n \neq 0$, so G diverges by Prelim$^{\underline{y}}$ Test.
2. When $|r| < 1$, $\lim_{n\to\infty} a_n = 0$, and G converges or diverges.

These conclusions are consistent with the result in Eq. (4). When $|r| < 1$, G happens to converge to $r/(1-r)$. The Prelim$^{\underline{y}}$ Test does not establish such convergence; the test only "allows" it to occur.

Ex. Harmonic Series: $H = \sum_{n=1}^{\infty} a_n$, w/ $a_n = \frac{1}{n}$. (7)

Since $\lim_{n\to\infty} a_n = 0$, then -- by Prelim$^{\underline{y}}$ Test -- H converges or diverges. In this case, H diverges, since we see it relates to the Taylor series...

$$\ln(1-x) = -\left(x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots\right) = (-)\sum_{n=1}^{\infty} \frac{x^n}{n},$$

so/ $$H = (-)\ln(1-x)\Big|_{x\to 1} = \lim_{x\to 1} \ln\left(\frac{1}{1-x}\right) = \ln(\infty) = \infty.$$

Ex. Alternating Harmonic Series: $\tilde{H} = \sum_{n=1}^{\infty} a_n$, w/ $a_n = \frac{(-)^{n+1}}{n}$. (8)

With $\lim_{n\to\infty} a_n = 0$, the Prelim$^{\underline{y}}$ Test $\Rightarrow$ $\tilde{H}$ converges or diverges. Here, insertion of the alternating sign factor $(-)^{n+1}$ turns things around, and $\tilde{H}$ converges, since

$$\tilde{H} = \sum_{n=1}^{\infty} (-)^{n+1}/n = \ln(1-x)\Big|_{x\to(-)1} = \ln 2 = 0.69315, \text{ finite.}$$

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3) ABSOLUTE CONVERGENCE.

The examples of Eqs. (7) & (8) suggest it is important to look at the signs of the terms in the series; depending on the signs, the series may diverge or converge. To look at signs, we claim the following...

Absolute Convergence defined. Comparison Test.

Defⁿ: The series $\sum_{n=1}^{\infty} a_n$ is "absolutely convergent" if $\sum_{n=1}^{\infty} |a_n|$ converges.

Lemma: If $\sum_{n=1}^{\infty} |a_n|$ converges, then so does $\sum_{n=1}^{\infty} a_n$. (9)

You will prove this Lemma in the homework; it is almost a matter of definition

NOTE: That $\sum_n |a_n|$ convergent $\Rightarrow \sum_n a_n$ convergent does not work in reverse, i.e. $\sum_n a_n$ convergent does not imply $\sum_n |a_n|$ is convergent. This is demonstrated by Eqs. (7) & (8): $\sum_n (-)^{n+1}/n$ converges, but $\sum_n |(-)^{n+1}/n| = \sum_n (1/n)$ does not.

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4) A CONVERGENCE TEST.

The first real convergence test we quote is the following:

Comparison Test

"Let $\sum_n u_n$ be a convergent series of positive terms. If, for the series $S = \sum_n a_n$, $|a_n| \leq u_n$ for all but a finite # of terms, then S is absolutely convergent, thus convergent. Conversely, if $\sum_n v_n$ is a divergent series of positive terms, and $|a_n| \geqslant v_n$ for all but a finite # of terms, then S is divergent." (10)

Here $\sum_n$ means $\sum_{n=1}^{\infty}$. We won't prove this test (it is "evident"), but will just use it, employing comparison series from the above examples. We will use...

Geometric Series: $G(r) = \sum_{n=1}^{\infty} r^n = r/(1-r)$, Convergent for $|r| < 1$; (11A)

Harmonic Series: $H = \sum_{n=1}^{\infty} (1/n)$, divergent. (11B)

Ex. Test: $S = \sum_{n=1}^{\infty} (1/n!)$, for convergence. (12)

Choose the convergent comparison series: $G(1/2) = \sum_{n=1}^{\infty} (1/2)^n = 1$, and note that $1/n! \leq 1/2^n$ for all $n > 3$. Then, Comparison Test $\Rightarrow$ S is convergent.

(We can find S explicitly here, since we know the Taylor series: $e^x = 1 + x/1! + x^2/2! + \cdots = \sum_{n=0}^{\infty} (x^n/n!)$. So: $S = (e^x - 1)|_{x=1} = 1.71828$.)

Riemann Zeta Fcn $\zeta(p)$. Cauchy Ratio Test.

$\boxed{\text{Ex.}}$ Test : $\zeta(p)=\sum_{n=1}^{\infty}(1/n)^p$, $0\leqslant p\leqslant 1$, for convergence. (13)

NOTE: $\zeta(p)$ is called the Riemann Zeta Fcn; it is so defined for all $p>0$.

Choose the divergent comparison series: $H=\sum_{n=1}^{\infty}(1/n)$, and observe that: $n^p\leqslant n$, or $(1/n)^p\geqslant(1/n)$, for all $n\geqslant 1$ & $0\leqslant p\leqslant 1$. So $\zeta(p)\geqslant H\rightarrow$ diverges

Later, we will show that although $\zeta(p)$ diverges for $0\leqslant p\leqslant 1$, it converges for $p>1$

) CAUCHY RATIO TEST.

This powerful test for series convergence/divergence can be stated as...:

Ratio Test

"Let $S=\sum_n a_n$. If $\lim_{n\to\infty}|a_{n+1}/a_n|<1$, S converges. If $\lim_{n\to\infty}|a_{n+1}/a_n|>1$, S diverges. If the (ratio) limit $=1$, S may converge or diverge." (14)

The proof goes as follows...

1. Write $\sum_{n=1}^{\infty}|a_n|=(|a_1|+|a_2|+\dots+|a_n|)+|a_n|\left\{\left|\frac{a_{n+1}}{a_n}\right|+\left|\frac{a_{n+2}}{a_n}\right|+\left|\frac{a_{n+3}}{a_n}\right|+\dots\right\}$.

2. Suppose: $|a_{n+1}/a_n|\leqslant r<1$, for sufficiently large n. Then, for terms in $\{\}$...,

$$\left|\frac{a_{n+2}}{a_n}\right|=\left|\frac{a_{n+1}}{a_n}\right|\cdot\left|\frac{a_{n+2}}{a_{n+1}}\right|\leqslant r^2;\quad \left|\frac{a_{n+3}}{a_n}\right|\leqslant r^3,\text{ similarly; etc.}$$

So// $$\sum_n|a_n|\leqslant\underbrace{(|a_1|+|a_2|+\dots+|a_n|)}_{\text{finite}}+|a_n|\underbrace{\{r+r^2+r^3+\dots\}}_{=G(r),\text{ convergent for }|r|<1.}$$

3. The last eqtn $\Rightarrow\sum_n|a_n|$ converges when $\lim_{n\to\infty}|a_{n+1}/a_n|\simeq r<1$; then $S=\sum_n a_n$ is (absolutely) convergent. When $\lim_{n\to\infty}|a_{n+1}/a_n|\simeq r>1$, $G(r)$ diverge and so does $\sum_n a_n$. When the (ratio) limit $=1$, the test is indeterminate.

Examples of use of the Ratio Test follow.

Examples for Ratio Test. Cauchy Integral Test.

Ex. Test: $S_k = \sum_{n=1}^{\infty}(n/k^n)$, $k = \text{cnst} > 0$, for convergence. (15)

Ratio: $|a_{n+1}/a_n| = \left(\frac{n+1}{k^{n+1}}\right)\Big/\left(\frac{n}{k^n}\right) = \frac{1}{k}\left(1+\frac{1}{n}\right)$;

so// $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{k}$, and// ratio limit is <1, for $k>1 \Rightarrow S_{k>1}$ converges;
>1, for $k<1 \Rightarrow S_{k<1}$ diverges.

Ex. Test: $\zeta(p) = \sum_{n=1}^{\infty}(1/n)^p$, $p>1$, for convergence (Riemann Zeta Fcn). (16)

Ratio: $|a_{n+1}/a_n| = n^p/(n+1)^p = \left[1/(1+\frac{1}{n})^p\right]$;

so// $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$, and// test is indeterminate $[\zeta(p>1)$ may $\begin{smallmatrix}\text{converge}\\ \text{diverge}\end{smallmatrix}]$.

We must wait for the next test to see what $\zeta(p>1)$ does.

6) CAUCHY INTEGRAL TEST.

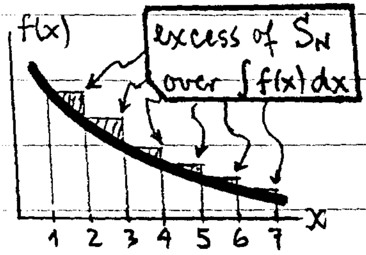
This test exploits the relation between an integral and a sum.

Integral Test

"Let $S = \sum_{n=1}^{\infty} a_n$ be a series of positive, non-increasing terms (i.e. $a_{n+1} \leq a_n$). Let $f(x)$ be a continuous, monotonically decreasing fcn w// $f(n) = a_n$. Then S converges if $\int_1^{\infty} f(x)\,dx$ is finite, and S diverges if the integral does." (17)

The proof proceeds by a kind of comparison test. Look at partial sums S_N...

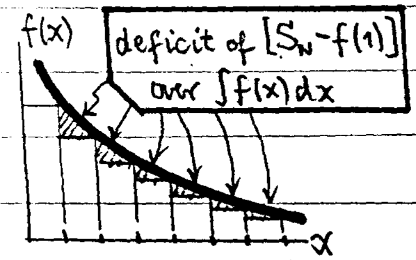
1.



$S_N = \sum_{n=1}^{N} a_n = \sum_{n=1}^{N} f(n)\,\Delta n$ } = area of unit width ($\Delta n=1$) rectangles from $x=1$ to N.

so// $S_N \geq \int_1^{N+1} f(x)\,dx$ } = area under $f(x)$ vs. x from $x=1$ to RH edge of N^{th} rectangle. (18A)

2.



By inspection: $\int_1^N f(x)\,dx \geq S_N - f(1)$.

so// $S_N \leq a_1 + \int_1^N f(x)\,dx$. (18B)

(next page)

Integral Test proof completed. Examples of use.

3. Eqs. (18A) & (18B) together establish lower & upper bounds for S_N...

$$\int_1^{N+1} f(x)\,dx \leqslant S_N \leqslant a_1 + \int_1^{N} f(x)\,dx$$

a_1 is (+)ve & finite (by assumption)

and// $N \to \infty \Rightarrow \underline{\int_1^\infty f(x)\,dx \leqslant S \leqslant a_1 + \int_1^\infty f(x)\,dx.}$ (19)

This shows that $S = \sum_{n=1}^\infty a_n$ exists, and is finite, depending on whether the integral exists. So, indeed, S converges or diverges with $\int_1^\infty f(x)\,dx$.

NOTE: By Eq.(19), $\int_1^\infty f(x)dx$ comes within a_1 of actually evaluating S.

$\boxed{\text{Ex.}}$ Test: $\zeta(p) = \sum_{n=1}^\infty (1/n)^p$, $p \geqslant 0$, for convergence (Riemann Zeta Fcn). (20)

Here $f(n) = 1/n^p$, so $f(x) = 1/x^p$, and we want the integral...

$$\int_1^\infty f(x)\,dx = \int_1^\infty dx/x^p = (-)\frac{1}{p-1}\left(1/x^{p-1}\right)\Big|_{x=1}^{x=\infty} = \frac{1}{p-1}\left[1 - \frac{1}{x^{p-1}}\right]\Big|_{x\to\infty}.$$

For $0 \leqslant p \leqslant 1$, evidently $\int_1^\infty f(x)dx$ diverges, and so does $\zeta(p)$ -- agreeing with the result in Eq.(13). For $p>1$, $\int_1^\infty f(x)dx$ converges, and so does $\zeta(p)$.

$\boxed{\text{ASIDE}}$ The convergence of $\zeta(p>1)$ gives a refinement to the Ratio Test.

$\zeta(p) = \sum_{n=1}^\infty a_n$, w// $a_n = (1/n)^p$, is convergent for $p>1$. ★

So// $|a_{n+1}/a_n| = \left(\frac{n}{n+1}\right)^p = \left(1+\frac{1}{n}\right)^{-p} \simeq 1 - (p/n)$, as $n \to \infty$. (21)

[By the Comparison Test [Eq.(10)], any series for which $|a_{n+1}/a_n| \leqslant 1-(p/n)$, for large n, and with $p>1$, must be convergent.

The Comparison, Ratio, and Integral Tests essentially exhaust the *simple* tests available for establishing series convergence. Several refinements of the first two tests do exist (see Arfken, Sec. 5.2), but they are rather specialized.

Specifically: $\zeta(2) = \sum_{n=1}^\infty (1/n^2) = \frac{\pi^2}{6}$, $\zeta(4) = \sum_{n=1}^\infty (1/n^4) = \frac{\pi^4}{90}$, etc. {Abramowitz & Stegun (Sec. 23.2)