Φ519 Problems★ NOTE: problems are graded @ 10 pts each, unless noted otherwise.

① Apparently, charge equality between electrons and protons is exact. If it were not, one could claim that the expansion of the universe might be due to matter carrying a net electric charge [Lyttleton & Bondi, Proc. Roy. Soc. A252, 313 (1959)]. Consider a spherically symmetric universe (centered on a "genesis point") containing un-ionized hydrogen atoms at density n $\left(\frac{\text{atoms}}{\text{unit vol.}}\right)$. Assume the proton & electron charges are slightly different, viz.: $|e_p/e_e| = 1+\beta$, w/ $|\beta| \ll 1$, but $\beta \neq 0$.

(A) Find the minimum value β_m of β for which this universe begins expanding.

(B) Assume n remains constant due to "continuous creation of matter" (deus ex machina). For $\beta > \beta_m$, show that the repulsive force on an atom is proportional to r, its distance from the genesis point. As consequences of this fact, show: (1) the atom's radial expansion velocity $v_r = (\text{cnst}) \times r$, (2) this universe expands exponentially in time.

(C) Show that $v_r = r/T$, where T is the time required for expansion by factor e. If $T \sim 10^{10}$ yr. ($\sim$ age of universe), and the observed average density $n \sim 6\ \frac{\text{atoms}}{\text{m}^3}$, find the size of β needed to "explain" the expansion of the universe.

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② If $\psi = \psi(x_i)$ & $\mathbb{A} = \mathbb{A}(x_i)$ are resp. scalar & vector fields in 3D space [$(x_i) = (x_1, x_2, x_3)$ are the 3 rectangular space coordinates], and $\nabla = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right)$ is the 3D gradient operator, prove the following identities:

(A) $\nabla \times (\nabla \psi) = 0$, curl grad $= 0$; (C) $\nabla \times (\nabla \times \mathbb{A}) = \nabla(\nabla \cdot \mathbb{A}) - \nabla^2 \mathbb{A}$,

(B) $\nabla \cdot (\nabla \times \mathbb{A}) = 0$, div curl $= 0$; (D) $\nabla(1/r) = -\mathbb{r}/r^3$, for $\mathbb{r} = (x_1, x_2, x_3)$.

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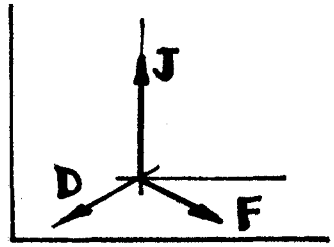
③ Suppose $\mathbb{F}$ is an unknown vector, about which we do know:

[redacted] $\mathbb{D} \cdot \mathbb{F} = \rho$, $\mathbb{D} \times \mathbb{F} = \mathbb{J}$, where $\mathbb{D}$, ρ & $\mathbb{J}$ are all known.

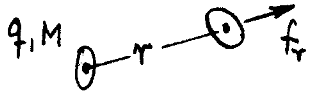


Solve for $\mathbb{F}$ in terms of $\mathbb{D}$, ρ & $\mathbb{J}$. Your solution should be a vector equation for $\mathbb{F}$ which does not involve components or direction cosines. Comment on analogies to solution of the system: $\nabla \cdot \mathbb{F} = \rho$, $\nabla \times \mathbb{F} = \mathbb{J}$.

★ Assigned: 8/30/91. Due: 9/6/91.

1 Explore how e-p charge imbalance might explain expansion of universe.

A. The excess charge $q = \beta e$ on each atom (assumed to be mass M) can -- at atom-atom separation r -- cause a net repulsion...

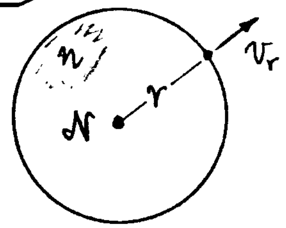


$$\left.\right\} f_r = \frac{k(\beta e)^2}{r^2} - G\frac{M^2}{r^2} = \frac{ke^2}{r^2}(\beta^2 - \beta_m^2), \quad \beta_m = \sqrt{\frac{GM^2}{ke^2}},$$

where: $k = 1/4\pi\epsilon_0 = 9\times10^{9}$ MKS, $e = 1.6\times10^{-19}$ C

grav. const $\}$ $G = 6.67\times10^{-11}$ MKS, $M = 1.67\times10^{-27}$ kgm $\Big\}$ $\underline{\beta_m = 0.90\times10^{-18}}$.

$\beta \geq \beta_m$ ensures $f_r \geq 0$, i.e. actual repulsion, so this $\Rightarrow$ the universe expands.



B. f_r is an inverse-square law, so Gauss' Law applies, and an atom at radial distance r experiences a repulsive force as though all the atoms in the r-sphere were concentrated at its center; the atoms outside the r-sphere don't count. This $\Rightarrow$ a net force F_r on atom at r...

$$N = \begin{matrix}\#\text{ atoms in}\\ r\text{-sphere}\end{matrix} = \left(\frac{4\pi}{3}r^3\right)n \Rightarrow \boxed{F_r = Nf_r = Kr}, \quad \underline{K} = \frac{4\pi n}{3}ke^2(\beta^2 - \beta_m^2).$$

The eqtn of motion for the radial velocity $v_r = dr/dt$ is...

$$M\frac{dv_r}{dt} = F_r, \quad \text{or}// \quad Mv_r\frac{dv_r}{dr} = Kr \Rightarrow \underline{v_r = \Omega r}, \text{ where: } \Omega = \sqrt{\frac{K}{M}}.$$

So $v_r \propto r$ as advertised. Since: $dr/dt = \Omega r$, another trivial integration yields

$\underline{r(t) = r(0)\exp(\Omega t)}$, and: Δt req'd for ex expansion is: $T = \frac{1}{\Omega} = \sqrt{\frac{M}{K}}$.

C. For a given T, need $K = M/T^2$. Numerically, for $T = 10^{10}$ yrs, this yields...

$(\beta^2 - \beta_m^2)n = 1.74\times10^{-35}$, or: $\beta^2 = \beta_m^2 + 2.90\times10^{-36}$, when $n = 6$ atoms/m^3.

Then: $\underline{\beta = 1.92\times10^{-18}} \simeq 2\beta_m$ charge imbalance is "all" that is required.

② Prove various identities involving $\psi(x_i)$, $\mathbb{A}(x_i)$ & $\nabla = (\partial/\partial x_i)$.

(A) Let indices ijk = cyclic permutation of 123 (ijk = 123, 231, or 312). Then:

$$\rightarrow [\nabla\times(\nabla\psi)]_k = \frac{\partial}{\partial x_i}\left(\frac{\partial\psi}{\partial x_j}\right) - \frac{\partial}{\partial x_j}\left(\frac{\partial\psi}{\partial x_i}\right) = \left[\frac{\partial^2}{\partial x_i\partial x_j} - \frac{\partial^2}{\partial x_j\partial x_i}\right]\psi = 0$$

since order of differentiation is immaterial.

Since each comp$^{\underline{t}}$ $[\;]_k = 0$, then $\nabla\times(\nabla\times\psi) = 0$, as required. **NOTE**: if we consider ∇ just to be a vector $\mathbb{D}$, this identity is: $\mathbb{D}\times(\mathbb{D}\psi) = 0$, which is "evident" by the fact that the vector product of $\mathbb{D}$ with itself vanishes.

(B) If $\nabla \leftrightarrow \mathbb{D}$, then: $\nabla\cdot(\nabla\times\mathbb{A}) \leftrightarrow \mathbb{D}\cdot(\mathbb{D}\times\mathbb{A})$. The latter expression $\equiv 0$ since $(\mathbb{D}\times\mathbb{A})$ is $\perp$ $\mathbb{D}$. So we've "proven" that $\nabla\cdot(\nabla\times\mathbb{A}) = 0$, as required.

Such proofs should be checked by looking at comp$^{\underline{t}}$s:

$$\rightarrow \nabla\cdot(\nabla\times\mathbb{A}) = \sum_k \frac{\partial}{\partial x_k}\left(\frac{\partial A_j}{\partial x_i} - \frac{\partial A_i}{\partial x_j}\right), \quad ijk = \overset{\frown}{123}$$

cancel

$$= \frac{\partial}{\partial x_3}\left(\frac{\partial A_2}{\partial x_1} - \cancel{\frac{\partial A_1}{\partial x_2}}\right) + \frac{\partial}{\partial x_2}\left(\cancel{\frac{\partial A_1}{\partial x_3}} - \cancel{\frac{\partial A_3}{\partial x_1}}\right) + \frac{\partial}{\partial x_1}\left(\cancel{\frac{\partial A_3}{\partial x_2}} - \cancel{\frac{\partial A_2}{\partial x_3}}\right) = 0$$

cancel — cancel

(C) The standard triple product: $\mathbb{A}\times(\mathbb{B}\times\mathbb{C}) = \mathbb{B}(\mathbb{A}\cdot\mathbb{C}) - \mathbb{C}(\mathbb{A}\cdot\mathbb{B})$ translates [when $\mathbb{A}=\mathbb{B}=\mathbb{D}$, $\mathbb{C}=\mathbb{A}$] to: $\mathbb{D}\times(\mathbb{D}\times\mathbb{A}) = \mathbb{D}(\mathbb{D}\cdot\mathbb{A}) - \mathbb{A}(\mathbb{D}\cdot\mathbb{D})$. In turn, this can be written: $\mathbb{D}\times(\mathbb{D}\times\mathbb{A}) = \mathbb{D}(\mathbb{D}\cdot\mathbb{A}) - D^2\mathbb{A}$. If we put $\mathbb{D}\leftrightarrow\nabla$...

$$\rightarrow \nabla\times(\nabla\times\mathbb{A}) = \nabla(\nabla\cdot\mathbb{A}) - \nabla^2\mathbb{A}, \text{ as required.}$$

This "proof" should be checked by comp$^{\underline{t}}$s. Do it. It works.

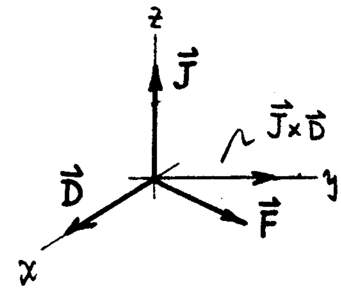
(D) If $\mathbb{r} = (x_1, x_2, x_3)$ is the position vector in 3D space: $r = (x_1^2 + x_2^2 + x_3^2)^{1/2}$.

Sol $$\left[\nabla\left(\frac{1}{r}\right)\right]_k = \frac{\partial}{\partial x_k}\left(\frac{1}{r}\right) = -\frac{1}{r^2}\left(\frac{\partial r}{\partial x_k}\right) = -\frac{1}{r^2}\,\frac{1}{2r}\,\underbrace{\frac{\partial}{\partial x_k}(x_1^2 + x_2^2 + x_3^2)}_{\text{only } x_k \text{ contributes to this}}$$

$$= -\frac{1}{r^2}\,\frac{1}{2r}\cdot 2x_k = -\frac{x_k}{r^3}.$$

Evidently, for the full vector: $\nabla(1/r) = -\mathbb{r}/r^3$, as advertised.

③ Solve the system : $\vec{D}\cdot\vec{F}=\rho$, $\vec{D}\times\vec{F}=\vec{J}$, for unknown $\vec{F}$ ($\vec{D}$, ρ & $\vec{J}$ known).



1) Label the axes xyz as shown : x-axis along $\vec{D}$, z-axis along $\vec{J}$, which must be $\perp$ the xy plane that contains both $\vec{D}$ & $\vec{F}$ (this is because $\vec{D}\times\vec{F}=\vec{J}$). Notice that $\vec{J}\times\vec{D}$ lies along the y-axis. Then $\vec{F}$, lying in the xy plane, must be a linear combination of the form...

$$\vec{F}=\alpha\vec{D}+\beta\,\vec{J}\times\vec{D}\,,\quad \alpha\ \&\ \beta = \text{coefficients to be found.}$$

2) But : $\rho=\vec{D}\cdot\vec{F}=\alpha D^2+\beta\underbrace{\vec{D}\cdot(\vec{J}\times\vec{D})}_{0}=\alpha D^2$, so : $\alpha=\rho/D^2$.

And : $\vec{J}=\vec{D}\times\vec{F}=\alpha\underbrace{\vec{D}\times\vec{D}}_{0}+\beta\,\vec{D}\times(\vec{J}\times\vec{D})=\beta\big[\vec{J}D^2-\vec{D}\underbrace{(\vec{D}\cdot\vec{J})}_{0\ \text{(obvious)}}\big]$,

So, $\beta=1/D^2$, and overall the desired vector $\vec{F}$ is...

$$\boxed{\vec{F}=\frac{1}{D^2}\left[\vec{D}\rho-\vec{D}\times\vec{J}\right]}\quad \text{where : } \rho=\vec{D}\cdot\vec{F},\ \ \vec{J}=\vec{D}\times\vec{F}$$

3) If we replace $\vec{D}$ by the symbol $\vec{\nabla}$, have : $\vec{F}=(1/\nabla^2)[\vec{\nabla}\rho-\vec{\nabla}\times\vec{J}]$. Then, if ∇^2 is a differential operator, $1/\nabla^2$ (the inverse operator) must be some kind of integral operator (in fact it is). This suggests that when $\vec{F}$ varies throughout space, the solution of the system : $\vec{\nabla}\cdot\vec{F}=\rho$, $\vec{\nabla}\times\vec{F}=\vec{J}$, will look like ...

$$\vec{F}\sim\vec{\nabla}\int\rho\cdot\text{something}\cdot d\tau-\vec{\nabla}\times\int\vec{J}\cdot\text{something}\cdot d\tau\,.$$

In fact this turns out to be the case, as we know from Helmholtz' Theorem.