ϕ507 Final Exam (3hr., in-class) Ths. 12 May '94 1

This exam is open-book, open notes, and is worth 300 pts. total. For each of the 6 problems, $\boxed{\text{box}}$ the answer on your solution sheet. Number your solution pages in order, put your name on p. 1, and staple the pages together before handing in.

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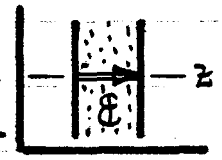
(1) [50 pts]. A sample of ground state H atoms is placed in a ||-plate capacitor. A time-dependent & spatially uniform electric field is applied as: $\mathbb{E}(t) = 0$, for $t<0$; $\mathbb{E}(t) = \mathcal{E}\, e^{-t/\tau}$, for $t>0$. $\tau = \text{cnst} > 0$, and $\mathcal{E} = \text{cnst}$.



(A) Use 1st order time-dep$^{\underline{t}}$ perturbation theory to estimate the probability of finding excited states n [i.e. transitions: g(ground) $\rightarrow$ n(excited)] in the sample @ $t \gg \tau$.

(B) Finesse the relevant matrix element $\langle n | \overset{\text{Stark}}{\text{coupling}} | g \rangle$ by setting it equal to a numerical coefficient $\mathcal{N}$ × suitable scale factors. What is the limit of the $g \rightarrow n$ probability of part (A) when τ is "large", say $\tau \sim 10^{-9}$ sec?

(C) If $|\mathcal{E}| \sim 10^{6}$ volts/cm, about how big is the probability calculated in part (B)?

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(2) [40 pts]. For a QM ∡ momentum $\mathbb{L}$, make the following assumptions:

1. Space is isotropic, i.e. the x, y and z axes are all equivalent.
2. The possible values of any one component of $\mathbb{L}$ are $m\hbar$, where m ranges over the $2l+1$ values $-l, -l+1, \ldots, 0, \ldots, +l$ [note: l is an integer].
3. All m-values occur with equal a priori probability.

From these assumptions, show that the average value of $\mathbb{L}^2$ is $\langle \mathbb{L}^2 \rangle = l(l+1)\hbar^2$.

HINT: 1 $\Rightarrow \langle \mathbb{L}^2 \rangle = 3\langle L_z^2 \rangle = 3\langle m^2 \rangle \hbar^2$. Find the avg. $\langle m^2 \rangle$ using 2 & 3.

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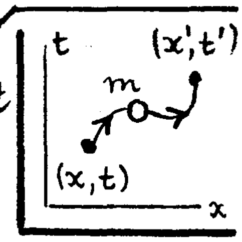
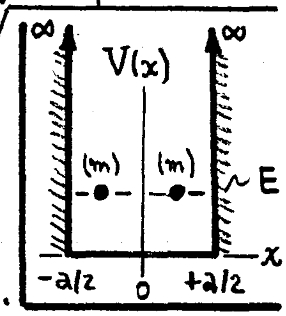
(3) [50 pts]. The wavefunction describing the motion of a free particle that starts out at (x, t) and moves to (x', t') can depend only on the differences between initial and final coordinates. Consequently, the free particle propagator G_0 is at most a function of $(x'-x)$ & $(t'-t)$. In 1D, a full Fourier representation of G_0 must then assume the form...



$$G_0(x'-x, t'-t) \doteq \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} d\omega\, g(k,\omega)\, e^{ik(x'-x)}\, e^{-i\omega(t'-t)}.$$

(next page)

$g(k,\omega)$ is called the free particle propagator in momentum space. Derive an expression for g, by using the fact that G_0 obeys the point-source Schrödinger Eqtn, viz: $[i\hbar(\partial/\partial t') - \frac{1}{2m}p'^2]\,G_0 = \hbar\,\delta(x'-x)\,\delta(t'-t)$.



④ [50pts]. Two identical spin $\frac{1}{2}$ fermions (each of mass m) move in 1D in a QM "box" of length a as shown. The box has infinitely high walls at $x = \pm\frac{a}{2}$. For parts (A) & (B), assume the particles do not interact.

(A) Find the ground-state energy (i.e. lowest permitted energy) when the particles are in a spin triplet configuration. Call this energy E_T.

(B) Find the ground-state energy (lowest permitted energy) when the particles are in a spin singlet configuration. Call this energy E_S.

(C) Now let the particles interact via a strong, attractive, short-range (delta-fcn) potential: $U(x_1,x_2) = -\lambda\,\delta(x_1-x_2)$, w/ λ = (+)ve cnst, and x_1 & x_2 the particle positions. Use 1st order perturbation theory to discuss what happens to the E_T & E_S energies.

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⑤ [50pts]. Dirac's wave equation for a free zero-mass, spin $\frac{1}{2}$ particle (a neutrino) can be written as: $c(\sigma\cdot\mathbb{p})\psi = i\hbar\,\partial\psi/\partial t$, w/ σ = (Pauli matrices) and $\mathbb{p}$ = linear momentum operator.

(A) What angular momentum is conserved for the motion of this particle?

(B) Show that the spin of this particle in a positive energy state is parallel to its momentum, while the spin in a negative energy state is antiparallel to $\mathbb{p}$.

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⑥ [60pts]. The Klein-Gordon Eqtn is: $\left[\frac{\partial^2}{\partial x_\mu^2} - k_0^2\right]\psi = 0$, w/ $k_0 = \frac{mc}{\hbar}$, for a free particle of mass m. The wavefcn $\psi = \psi(x_\mu)$ depends on all four space-time coordinates x_μ. We want to show the covariance of the KG Eq. under a Lorentz Transform: $x_\mu \to x'_\mu = \Lambda_{\mu\nu}x_\nu$. It is sufficient to consider an infinitesimal transform: $\Lambda_{\mu\nu} = \delta_{\mu\nu} + \epsilon_{\mu\nu}$, w/ $\delta_{\mu\nu}$ = Kronecker delta, $\epsilon_{\nu\mu} = (-)\epsilon_{\mu\nu}$ [antisymmetric] & $O(\epsilon^2)$ neglected. Under this $\Lambda_{\mu\nu}$, show that the KG Eq. transforms covariantly to the primed frame as: $\left[\frac{\partial^2}{\partial x_\mu'^2} - k_0^2\right]\psi' = 0$, where: $\psi' = \psi(x'_\mu)$. HINT: expand $\psi(x'_\mu) = \psi(x_\mu + \epsilon_{\mu\nu}x_\nu)$ in a Taylor series. The key is to show that: $[\partial^2/\partial x_\mu^2 - k_0^2]\,\epsilon_{\mu\nu}x_\nu(\partial\psi/\partial x_\mu) = 0$.

Φ507 Final Exam Solutions (1994)

(1) [50 pts.]. Estimate H-atom excitations by a pulsed electric field $\mathbb{E} = \mathcal{E}\, e^{-t/\tau}$.

(A) 1. By 1st order t-dept. perturbation theory, the amplitude for $g \to n$ is [Class notes p. tD5, Eq. (13)]:

$$\rightarrow a(t) = -\frac{i}{\hbar}\int_{t_0}^{t} V_{ng}(t')\, e^{i\omega_{ng}t'}\, dt', \quad \text{w/}\ V_{ng}(t') = \langle \underset{\text{excited}}{n} | V(x,t') | \overset{\text{ground}}{g} \rangle. \tag{1}$$

Here, V is a Stark coupling $e\,\mathbb{E}\cdot\mathbf{r}$, and so -- with the given $\mathbb{E} = \mathcal{E} e^{-t/\tau}$ @ $t>0$:

$$a(\infty) = -\frac{i}{\hbar}\langle n | e\,\mathcal{E}\cdot\mathbf{r} | g\rangle \int_0^\infty e^{-t/\tau} e^{i\omega_{ng}t}\, dt = -\frac{i}{\hbar}\langle n | e\,\mathcal{E}\cdot\mathbf{r} | g\rangle \frac{\tau}{1 - i\omega_{ng}\tau}$$

and//

$$\boxed{|a(\infty)|^2 = \frac{\tau^2/\hbar^2}{1+\omega_{ng}^2\tau^2}\, |\langle n | e\,\mathcal{E}\cdot\mathbf{r} | g\rangle|^2}. \tag{2}$$

$|a(\infty)|^2$ is the probability for g(ground) $\to$ n(excited) @ $t \gg \tau$.

(B) 2. The matrix element in Eq. (2) is $\langle n | e\,\mathcal{E}\cdot\mathbf{r} | g\rangle = e\,\mathcal{E}\cdot\langle n|\mathbf{r}|g\rangle$, and we put this equal to $e\mathcal{E}\cdot\mathcal{N}a_0$, where $a_0 = \hbar^2/me^2$ is the Bohr radius. The numerical coefficient $\mathcal{N}$ contains geometry of $\mathcal{E}$ as well as the "strength" of the dipole matrix element $\langle n|\mathbf{r}|g\rangle$. The transition probability of Eq. (2) is

$$\rightarrow |a(\infty)|^2 = \mathcal{N}^2\left[\frac{(\omega_{ng}\tau)^2}{1+(\omega_{ng}\tau)^2}\right](e\mathcal{E}a_0/\hbar\omega_{ng})^2 \rightarrow \underline{\underline{\mathcal{N}^2(e\mathcal{E}a_0/E_{ng})^2}}. \tag{3}$$

$E_{ng} = (E_n - E_g)$ is the $g \to n$ transition energy, and the expression on the far RHS of Eq. (3) is valid when $\omega_{ng}\tau \gg 1$. Since the first possible transition is $g(1S) \to n(2P)$, with $\omega_{ng} = (10.2\,\text{eV})/\hbar = 2\pi\times 2.5\times 10^{15}\,\text{Hz}$, then certainly this expression is valid for "large" $\tau \sim 10^{-9}$ sec. At this point, the $g \to n$ transition probability is actually independent of τ.

(C) 3. The $g \to n$ probability just calculated in part (B) can be written as:

$$\rightarrow |a(\infty)|^2 = \mathcal{N}^2(\mathcal{E}/\mathcal{E}_{ng})^2, \quad \text{w/}\ \mathcal{E}_{ng} = E_{ng}/ea_0 \text{ (an electric field)}. \tag{4}$$

For the first possible transition: $g(1S) \to n(2P)$, $E_{ng} = 10.2\,\text{eV}$, and the scale field: $\mathcal{E}_{ng} \simeq 10\,\text{volts}/a_0 = 2\times 10^9$ volts/cm. If $\mathcal{E} \sim 10^6$ volts/cm, then the transition probability is: $\boxed{|a(\infty)|^2 \sim \frac{1}{4}\mathcal{N}^2\times 10^{-6}}$, certainly < 1 ppm.

② [40pts]. Show $\langle \mathbb{L}^2 \rangle = l(l+1)\hbar^2$ from $\langle L_k \rangle = m\hbar$ and isotropy of space.

1. Following the hint, the assumption that all three space axes are equivalent plus the assumption that for any one of them, say the z-axis, $\langle L_z \rangle = m\hbar$, allows us to write¶

$$\rightarrow \langle \mathbb{L}^2 \rangle = 3 \cdot \langle L_z^2 \rangle = 3\hbar^2 \langle m^2 \rangle. \tag{1}$$

So we need to find $\langle m^2 \rangle$, i.e. the average value of m^2 among the m-states.

2. m assumes the integer values $-l, -l+1, \ldots, +l$, altogether $(2l+1)$ in number, and with equal a priori probability. The chance of seeing a particular m-value is thus $1/(2l+1)$, and the average m^2 is...

$$\rightarrow \langle m^2 \rangle = \frac{1}{2l+1} \sum_{\mu=-l}^{\mu=+l} \mu^2 = \frac{2}{2l+1} \sum_{\mu=1}^{\mu=l} \mu^2. \tag{2}$$

From tables [e.g. Gradshteyn & Ryzhik #(0.121.2)]: $\sum_1^l \mu^2 = \frac{1}{6} l(l+1)(2l+1)$,

$$\xrightarrow{\text{so}} \langle m^2 \rangle = \frac{2}{2l+1} \cdot \frac{1}{6} l(l+1)(2l+1) = \frac{1}{3} l(l+1). \tag{3}$$

3. By use of Eq. (3) in Eq. (1), we have the desired result...

$$\boxed{\langle \mathbb{L}^2 \rangle = l(l+1)\hbar^2}. \quad \textbf{QED}$$

NOTE This works fine when l = integer. Can you make it work for a QM ∡ momentum $\mathbb{J}$ where j = half-integer? For $j = l + \frac{1}{2}$ and $m = -j, -j+1, \ldots, +j$, you have to show: $\langle m^2 \rangle = \frac{2}{2j+1} \sum_{\mu=1/2}^{\mu=l+1/2} \mu^2 = \frac{1}{3} j(j+1)$. This turns out to be true, after a bit of algebra. So the average $\langle \mathbb{J}^2 \rangle = j(j+1)\hbar^2$ is a QM Truth.

¶ For an eigenstate of L_z, note that $\langle L_z \rangle^2 = \langle L_z^2 \rangle$, because its uncertainty $= 0$. For an eigenstate of operator Q in general: $(\Delta Q)^2 = \langle Q^2 \rangle - \langle Q \rangle^2 \equiv 0$.

③ [50 pts]. Derivation of free particle propagator $g(k,\omega)$ in momentum space.

1. G_0 obeys: $\left(i\hbar\frac{\partial}{\partial t'} - \frac{1}{2m}p'^2\right)G_0 = \hbar\,\delta(x'-x)\,\delta(t'-t)$, w/ $p' = (\hbar/i)\,\partial/\partial x'$. On the LHS of this equation, carry out the differentiations on the Fourier integral for G_0, i.e. (all integrals are $\int_{-\infty}^{\infty}$)...

$$\xrightarrow{\text{LHS}} \left(i\hbar\frac{\partial}{\partial t'} - \frac{1}{2m}p'^2\right)G_0 = \frac{\hbar}{(2\pi)^2}\int dk\int d\omega\, g(k,\omega)\left[i\frac{\partial}{\partial t'} + \frac{\hbar}{2m}\frac{\partial^2}{\partial x'^2}\right]\cdot$$
$$\cdot\, e^{ik(x'-x)}\,e^{-i\omega(t'-t)},$$
$$= \frac{\hbar}{(2\pi)^2}\int dk\int d\omega\left\{g(k,\omega)\left[\omega - \frac{\hbar k^2}{2m}\right]\right\}e^{ik(x'-x)}e^{-i\omega(t'-t)}. \quad (1)$$

2. Now, on the RHS of the PDE for G_0, put in the standard Fourier integral rep$^{\underline{n}}$ of the delta fcns (i.e. $\delta(\kappa) = \frac{1}{2\pi}\int d\xi\, e^{\pm i\kappa\xi}$), so that

$$\xrightarrow{\text{RHS}} \hbar\,\delta(x'-x)\,\delta(t'-t) = \frac{\hbar}{(2\pi)^2}\int dk\int d\omega\,\{1\}\,e^{ik(x'-x)}\,e^{-i\omega(t'-t)}, \quad (2)$$

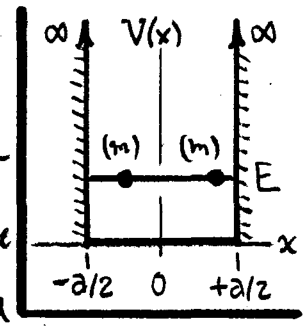
3. By equating the LHS & RHS expressions in Eqs. (1) & (2), we see that the PDE which defines G_0 requires that its Fourier rep$^{\underline{n}}$ satisfies...

$$\rightarrow \int dk\int d\omega\left\{g(k,\omega)\left[\omega - \frac{\hbar k^2}{2m}\right]\right\}e^{ik(x'-x)-i\omega(t'-t)} =$$
$$= \int dk\int d\omega\,\{1\}\,e^{ik(x'-x)-i\omega(t'-t)}. \quad (3)$$

This is an identity only if the momentum-space amplitude is

$$\boxed{g(k,\omega) = 1\Big/\left[\omega - \frac{\hbar k^2}{2m}\right]}. \quad (4)$$

4. $g(k,\omega)$ in Eq. (4) evidently shows a pole (big resonance!) at $\omega = \hbar k^2/2m$ on the real axis -- this is at free particle energy: $E = \hbar\omega = \frac{1}{2m}(\hbar k)^2$. The pole is handled by letting $\omega \to \omega + i\epsilon$. Then $\epsilon \to 0+$ gives the retarded propagator $G_0^{(r)}$, w/ $G_0^{(r)} \equiv 0$ for $t' < t$, while $\epsilon \to 0-$ gives the advanced propagator $G_0^{(a)}$, w/ $G_0^{(a)} \equiv 0$ for $t' > t$. The particle remains happy in both cases.



(4) [50 pts]. Analyse case of two identical fermions in a box.

The particle-in-a-box problem is solved everywhere, e.g. in Davydov, Sec. 25. For a single particle of mass m, the eigenenergies are $E_n = n^2 E_1$, w/ $E_1 = \pi^2\hbar^2/2ma^2$, and $n = 1, 2, 3, \ldots$ The normalized eigenfcns are: $\phi_n(x) = \sqrt{\frac{2}{a}}\cos(n\pi x/a)$, n=odd; $\phi_n(x) = \sqrt{\frac{2}{a}}\sin(n\pi x/a)$, n=even.

(A) & (B) 1) For a spin triplet ($\uparrow\uparrow$), the Exclusion Principle forbids both fermions being in the same space state, and symmetrization requires the overall system wavefcn $u(x_1, x_2)$ be odd under exchange, $x_1 \leftrightarrow x_2$. So: $u_T(x_1,x_2) = \frac{1}{\sqrt{2}}[\phi_m(x_1)\phi_n(x_2) - \phi_n(x_1)\phi_m(x_2)]$ with $m \neq n$. For a spin singlet ($\uparrow\downarrow$): $u_S(x_1,x_2) = \frac{1}{\sqrt{2}}[\phi_m(x_1)\phi_n(x_2) + \phi_n(x_1)\phi_m(x_2)]$, and $m = n$ is allowed. The energies in these states are -- where H_1 & H_2 are the Hamiltonians for particles 1 & 2 that give the above eigenenergies E_n --

$$\rightarrow E_{T,S} = \langle u_{T,S}|H_1 + H_2|u_{T,S}\rangle = (E_m + E_n) \mp (E_m + E_n)\langle\phi_m(x_1)\phi_n(x_2)|\phi_n(x_1)\phi_m(x_2)\rangle. \quad (1)$$

The upper (−) sign on the RHS is for triplets; the lower (+) is for singlets. In doing this calculation, we have assumed the $\phi_n(x)$ are orthonormal.

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2) For T states, $m \neq n$ in Eq. (1), and the overlap term vanishes. Lowest energy is:

$$\rightarrow \text{triplet ground state energy: } E_T = E_1 + E_2 = 5E_1, \text{ w/ } E_1 = \frac{\pi^2\hbar^2}{2ma^2}. \quad (2)$$

For S states, can have $m = n = 1$ in Eq. (1). The overlap integral is $\equiv 1$, and we get:

$$\rightarrow \text{Singlet ground state energy: } E_S = (E_1 + E_1) + (E_1 + E_1) = 4E_1, \text{ w/ } E_1 = \frac{\pi^2\hbar^2}{2ma^2}. \quad (3)$$

The singlet is lower in energy, as it is for the He ground state.

C) 3) Using the symmetry of the δ-fcn, and by the sort of calculation as in Eq. (1)...

$$\rightarrow V_{T,S} = -\lambda\langle u_{T,S}|V|u_{T,S}\rangle = -2\lambda\left\{\int dx\,|\phi_m(x)|^2|\phi_n(x)|^2 \mp \int dx\,|\phi_m(x)|^2|\phi_n(x)|^2\right\} \quad (4)$$

The perturbation V_T on the triplet state E_T vanishes. The singlet state E_S is reduced by an amount: $V_S = -4\lambda\int_{-a/2}^{+a/2} dx\,|\phi_1(x)|^4 = -6\lambda/a$.

ϕ507 Final Solutions (cont'd)

⑤ [50 pts]. Analyse Dirac Eqtn for a (free) neutrino.

(A) 1. When the particle is free & massless, the Dirac Hamiltonian is $\mathcal{H} = c(\boldsymbol{\sigma}\cdot\mathbf{p})$.
For the angular momentum dynamics, look at the commutators ... for spin ...

$$\rightarrow [\mathcal{H}, \boldsymbol{\sigma}]_j = c[\sigma_i p_i, \sigma_j] = c p_i [\sigma_i, \sigma_j] \leftarrow \text{use } [\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k{}^{\star}$$

$$= 2ic\epsilon_{ijk} p_i \sigma_k = -2ic\epsilon_{jik} p_i \sigma_k = -2ic(\mathbf{p}\times\boldsymbol{\sigma})_j,$$

so/ $\underline{[\mathcal{H}, \frac{\hbar}{2}\boldsymbol{\sigma}] = +i\hbar c(\boldsymbol{\sigma}\times\mathbf{p})}$, spin is not a constant of the motion; (1)

... and for orbital ∡ momentum $\mathbf{L} = \mathbf{r}\times\mathbf{p}$ ...

$$\rightarrow [\mathcal{H}, \mathbf{L}]_k = c[\sigma_i p_i, \epsilon_{kij} x_i p_j] = c\epsilon_{kij}\sigma_i [p_i, x_i p_j]$$

$$= -i\hbar c\epsilon_{kij}\sigma_i \left\{\frac{\partial}{\partial x_i}(x_i p_j) - (x_i p_j)\frac{\partial}{\partial x_i}\right\} = -i\hbar c\epsilon_{kij}\sigma_i p_j$$

so/ $\underline{[\mathcal{H}, \mathbf{L}] = -i\hbar c(\boldsymbol{\sigma}\times\mathbf{p})}$, orbital ∡ momentum is not constant. (2)

Neither $\mathbf{L}$ nor spin $\mathbf{S} = \frac{\hbar}{2}\boldsymbol{\sigma}$ is separately cnst, but the <u>total</u> ∡ momentum $\boxed{\mathbf{J} = \mathbf{L} + \mathbf{S}}$ <u>is</u> a cnst of the motion for a (free) neutrino.

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2. In an eigenstate of energy E, have $i\hbar\,\partial\psi/\partial t = E\psi$, so Dirac's Eqtn is:

$$\rightarrow E\psi = c(\boldsymbol{\sigma}\cdot\mathbf{p})\psi = (cp\cos\phi)\psi, \quad \phi = \measuredangle(\boldsymbol{\sigma}, \mathbf{p}). \tag{3}$$

ϕ is the fixed ∡ between the particle's spin $\frac{\hbar}{2}\boldsymbol{\sigma}$ and its momentum $\mathbf{p}$; this definition is permissible since for fixed E the momentum $\mathbf{p}$ is a cnst of the motion. In fact, $E^2 = c^2p^2$ for this massless particle, so its energy can be either $E = +cp$ or $E = -cp$. Then Eq. (3) requires ...

$$\left[\begin{array}{l} (+)\text{energy}: E = +cp \Rightarrow \cos\phi = +1 \;\&\; \phi = 0^\circ, \text{ so: spin } \boldsymbol{\sigma} \text{ is } \parallel \mathbf{p}; \\ (-)\text{energy}: E = -cp \Rightarrow \cos\phi = -1 \;\&\; \phi = 180^\circ, \text{ so: spin } \boldsymbol{\sigma} \text{ is anti-}\parallel \mathbf{p}. \end{array}\right. \tag{4}$$

★ Davydov, Eq. (59.15). $\epsilon_{ijk} = \begin{cases} +1, \text{ when } ijk = \widehat{123} \\ -1, \text{ when } ijk = \widehat{132} \end{cases}$, and $\epsilon_{ijk} \equiv 0$, otherwise.

⑥ [60pts]. Establish covariance of the free-particle Klein-Gordon Eqtn.

1. The original KG Eq. is: $K\psi = 0$, w/ $K = \partial^2/\partial x_\lambda^2 - k_0^2$. Under an ∞smal Lorentz transform: $\Lambda_{\mu\nu} = \delta_{\mu\nu} + \epsilon_{\mu\nu}$, coordinates transform as $x_\mu \to x'_\mu = \Lambda_{\mu\nu} x_\nu$, or $x'_\mu = x_\mu + \epsilon_{\mu\nu} x_\nu$ [$\epsilon_{\nu\mu} = -\epsilon_{\mu\nu}$ & $\mathcal{O}(\epsilon^2)$ negligible]. Then the transformed ψ is:

$$\rightarrow \psi' = \psi(x'_\mu) = \psi(x_\mu) + \epsilon_{\mu\nu} x_\nu (\partial\psi/\partial x_\mu), \text{ to } \mathcal{O}(\epsilon). \tag{1}$$

This is the lowest-order Taylor series for $\psi(x'_\mu)$.

2. The operator K is a Lorentz scalar (since k_0^2 is, and the D'Alembertian $\partial^2/\partial x_\mu^2$ is a Lorentz invariant), so K transforms as $K \to K' \equiv K$. Then we have:

$$\rightarrow K'\psi' = K\left(\psi + \epsilon_{\mu\nu} x_\nu \frac{\partial\psi}{\partial x_\mu}\right) = K\psi + K\left[\epsilon_{\mu\nu} x_\nu \frac{\partial\psi}{\partial x_\mu}\right]. \tag{2}$$

To establish covariance, we must show that if $K\psi = 0$, then $K'\psi' = 0$ also. From Eq. (2), this is true when...

$$\rightarrow K\psi = 0 \leftrightarrow K'\psi' = 0, \text{ iff}: K[\epsilon_{\mu\nu} x_\nu (\partial\psi/\partial x_\mu)] = 0. \tag{3}$$

3. Calculating derivatives relevant to Eq. (3), we find...

use: $\dfrac{\partial^2\psi}{\partial x_\lambda^2} = k_0^2\psi$ from KG Eqtn.

$$\frac{\partial}{\partial x_\lambda}\left(x_\nu \frac{\partial\psi}{\partial x_\mu}\right) = \delta_{\nu\lambda}\frac{\partial\psi}{\partial x_\mu} + x_\nu \frac{\partial^2\psi}{\partial x_\lambda \partial x_\mu},$$

$$\frac{\partial^2}{\partial x_\lambda^2}\left(x_\nu \frac{\partial\psi}{\partial x_\mu}\right) = \delta_{\nu\lambda}\frac{\partial^2\psi}{\partial x_\lambda \partial x_\mu} + \delta_{\nu\lambda}\frac{\partial^2\psi}{\partial x_\lambda \partial x_\mu} + x_\nu \frac{\partial}{\partial x_\mu}\left(\frac{\partial^2\psi}{\partial x_\lambda^2}\right),$$

$$= 2(\partial^2\psi/\partial x_\nu \partial x_\mu) + k_0^2 x_\nu (\partial\psi/\partial x_\mu). \tag{4}$$

Eq. (4) can be rewritten, with $K = \partial^2/\partial x_\lambda^2 - k_0^2$, and with $\epsilon_{\mu\nu}$ indpt of x_μ...

$$\rightarrow K\left(x_\nu \frac{\partial\psi}{\partial x_\mu}\right) = 2(\partial^2\psi/\partial x_\mu \partial x_\nu), \text{ so } K\left[\epsilon_{\mu\nu} x_\nu \frac{\partial\psi}{\partial x_\mu}\right] = 2\epsilon_{\mu\nu}\frac{\partial^2\psi}{\partial x_\mu \partial x_\nu}. \tag{5}$$

4. Now, per Eq. (3), we need to show that in Eq (5): $K[\;] = 0$. Write Eq. (5) as...

$$\rightarrow K[\;] = \epsilon_{\mu\nu}\frac{\partial^2\psi}{\partial x_\mu \partial x_\nu} + \epsilon_{\nu\mu}\frac{\partial^2\psi}{\partial x_\nu \partial x_\mu} = \epsilon_{\mu\nu}\left[\partial^2\psi/\partial x_\mu \partial x_\nu - \partial^2\psi/\partial x_\nu \partial x_\mu\right] = 0. \tag{6}$$

(1st step: interchange dummy indices ν & μ; 2nd step: use: $\epsilon_{\nu\mu} = -\epsilon_{\mu\nu}$; 3rd step: use $\partial_\nu\partial_\mu = \partial_\mu\partial_\nu$ w.r.t. ψ). So, per (3), have $K[\;] = 0$, and the KG Eq. is covariant.