DEPARTMENT OF PHYSICS
M.S./PH.D. QUALIFYING/COMPREHENSIVE EXAMINATION
JUNE 22, 24, 26, 1992

Answer each of the following questions, all of which carry equal weight. Begin your answer to each question on a new sheet of paper; solutions to different questions must not appear on the same sheet. Each sheet of paper must be labeled with your name and the problem number in the upper right hand corner. When more than one sheet is submitted for a problem, be sure the pages are ordered properly.

1. An idealized governor, sketched below, has frictionless bearings and is rigidly supported at A and B. The collar C is frictionless and massless. The connecting rods, of length a, are also massless, and the whole assembly is constrained to rotate at constant angular velocity Ω about the vertical axis AB. For this system:

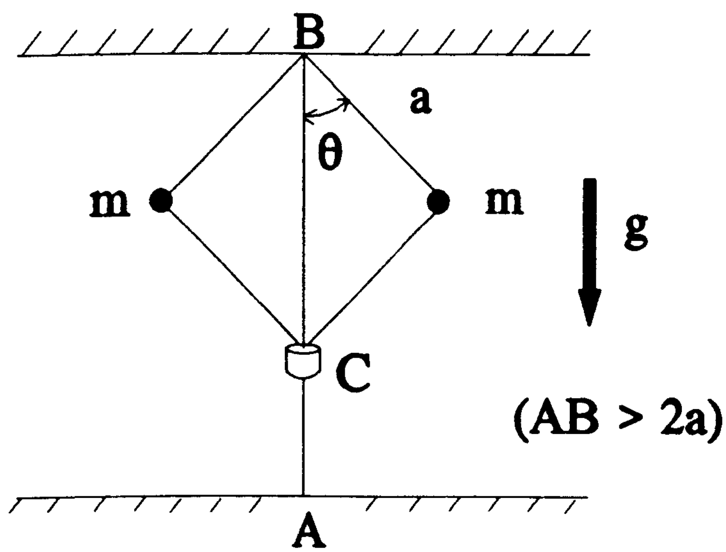
   a) Write the Hamiltonian H in terms of a generalized coordinate and its conjugate momentum.

   b) Is H conserved? Is the energy conserved?

   c) Write Hamilton's equations of motion.

   d) If $\Omega^2 > g/a$, at what angle $\theta_o > 0$ does the governor have a solution θ = constant?

   e) Determine the frequency for small oscillations about θ_o. Assume $\Omega^2 > g/a$.



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C(1) Soln (1)

$$L = T - V = ma^2(\dot{\theta}^2 + \sin^2\theta \cdot \Omega^2) + 2mga\cos\theta$$

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = 2ma^2\dot{\theta}$$

(a) $$H = p_\theta \dot{\theta} - L = ma^2(\dot{\theta}^2 - \sin^2\theta \cdot \Omega^2) - 2mga\cos\theta$$

$$\boxed{H = \frac{p_\theta^2}{4ma^2} - ma^2\left(\sin^2\theta \cdot \Omega^2 + \frac{2g}{a}\cos\theta\right)}$$

(b) $\frac{\partial H}{\partial t} = 0 \Rightarrow$ H is conserved

but $E = T + V \neq H$ is not conserved

(c)
$$\begin{cases} \dot{p}_\theta = -\frac{\partial H}{\partial \theta} = 2ma^2\left[\sin\theta\cos\theta \cdot \Omega^2 - \frac{g}{a}\sin\theta\right] \\ \dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{2ma^2} \end{cases}$$

(d) equilibrium requires $\dot{p}_\theta = 0$ (or $\dot{\theta} = 0$)*

From (c) this happens if

$$\cos\theta_0 \cdot \Omega^2 - \frac{g}{a} = 0$$

or $$\boxed{\theta_0 = \cos^{-1}\left(\frac{g}{a\Omega^2}\right)}$$

* the torque changes sign about $\theta = \theta_0$ so the governor will oscillate about θ_0.

(e) Rewrite (c) as

$$2ma^2 \ddot{\theta} = 2ma^2 \left[\sin\theta \cos\theta \cdot \Omega^2 - \frac{g}{a} \sin\theta \right]$$

$$\ddot{\theta} = \Omega^2 \sin\theta \, (\cos\theta - \cos\theta_0)$$

Let $\theta = \theta_0 + \Delta$

To first order in Δ,

$$LHS = \Omega^2 \sin\theta_0 \left(\cos(\theta_0 + \Delta) - \cos\theta_0 \right)$$

but $\cos(\theta_0 + \Delta) = \cos\theta_0 \cos\Delta - \sin\theta_0 \sin\Delta$

$$\simeq \cos\theta_0 - \Delta \sin\theta_0$$

$$LHS = \Omega^2 \sin\theta_0 \, (-\Delta \sin\theta_0)$$

Thus $\ddot{\Delta} + \Omega^2 \sin^2\theta_0 \cdot \Delta = 0$

$$\boxed{\text{frequency} = \Omega \sin\theta_0 = \Omega \sqrt{1 - \left(\frac{g}{a\Omega^2}\right)^2}}$$

2. An infinite slab of material is oriented perpendicular to the x axis, and extends from $x = 0$ to $x = L$. Within the slab the temperature $T(x,t)$ obeys the equation $\frac{\partial T}{\partial t} = k\frac{\partial^2 T}{\partial x^2}$ where k is constant. Initially the slab is at temperature T_i. Starting at $t = 0$, the two faces are maintained at temperature T_f where $T_f < T_i$.

a) Derive an expression for $T(x,t)$ valid at all times $t > 0$. Give an approximate form for the long time behavior of T at the center of the slab.

b) Now suppose the material itself evolves heat through some activated process, so that T obeys

$$\frac{\partial T}{\partial t} = k\frac{\partial^2 T}{\partial x^2} + \gamma(T-T_f)$$

where γ is also a constant. Again derive an expression for $T(x,t)$ valid for all $t > 0$. What is the minimum value of γ such that the slab <u>never</u> cools to T_f?

c) Finally, suppose the constant k itself increases with temperature, so that $k = aT$ where a is constant. Then

$$\frac{\partial T}{\partial t} = aT\frac{\partial^2 T}{\partial x^2}$$

Discuss the dependence of T upon time, if the initial temperature distribution is parabolic:

$$T(x,0) = T_f + \theta(x/L - x^2/L^2), \text{ where } \theta \text{ is constant.}$$

2\. Solve $\dfrac{\partial T}{\partial t} - \kappa \dfrac{\partial^2 T}{\partial x^2} = 0$

subject to $T(x,0) = T_i$

$T(0,t) = T(L,t) = T_f \qquad t > 0$

Use separation of variables

$$T(x,t) = f(x)\, g(t)$$

$$g(t) \sim e^{-\lambda t}$$

$$f(x) \sim \sin kx,\ \cos kx \quad \text{where}$$

$$\lambda = \kappa k^2$$

Boundary conditions force us to use sine, and also force $k = \dfrac{n\pi}{L}$ $\quad n = 1, 2, 3, \ldots$. So

$$T(x,t) = T_f + \sum_{n=1}^{\infty} A_n e^{-\lambda_n t} \sin \frac{n\pi x}{L}$$

where $\lambda_n = \dfrac{n^2\pi^2\kappa}{L^2}$

At $t=0$:

$$T_i - T_f = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{L}$$

$$A_n = \frac{2}{L}(T_i - T_f) \int_0^L \sin \frac{n\pi x}{L}\, dx$$

$$= \frac{4}{n\pi}(T_i - T_f) \quad \text{for } n \text{ odd}$$

$$= 0 \quad \text{" " even.}$$

So $$T(x,t) = T_f + \sum_{n\ \text{odd}} \frac{4(T_i - T_f)}{n\pi} e^{-\frac{n^2\pi^2\kappa}{L^2} t} \sin \frac{n\pi x}{L}$$

$$T\left(\frac{L}{2}, t\right) = T_f + \sum_{n\ \text{odd}} \frac{4(T_i - T_f)}{n\pi} e^{-\frac{n^2\pi^2\kappa}{L^2} t} (-1)^{\frac{n-1}{2}}$$

$$\sim T_f + \frac{4}{\pi}(T_i - T_f)\, e^{-\frac{\pi^2\kappa t}{L^2}} \quad \text{for large } t$$

2. (b) Can still solve by sep. of variables in the same way, but now

$$-\lambda = -k^2\kappa + \gamma$$

or $$\lambda = k^2\kappa - \gamma$$

As before this becomes

$$\lambda_n = \frac{n^2\pi^2\kappa}{L^2} - \gamma$$

As long as all the λ_n's are positive, the temp. will decay to T_f. – in particular as long as λ_1 is positive

So the limiting value of γ is

$$\gamma = \frac{\pi^2\kappa}{L^2}$$

(c) Separation of variables still works!

$$T = f(x)g(t)$$

$$\frac{dg}{dt} f = \kappa f g^2 \frac{d^2f}{dx^2}$$

$$\frac{1}{g^2}\frac{dg}{dt} = \kappa \frac{d^2f}{dx^2}$$

$$g = \frac{1}{1+\lambda t}$$

$$f = -\frac{\lambda}{\kappa}\frac{x^2}{2} + C_1 x + C_2$$ which is also parabolic.

Matching to $T(x,0)$: $\frac{\lambda}{2\kappa} = \Delta/L^2$ so $\lambda = \frac{2\Delta\kappa}{L^2}$

$C_1 = \Delta/L$

$C_2 = T_f$

For long times the decay is not exponential but $1/t$:

$$T \sim 1/\lambda t$$

3. a) Using the Bohr model, find the energies (in eV) of the first three levels in hydrogen (n=1,2,3).

   b) With the spin orbit coupling included, list the possible states with n=1,2, or 3 and the values for the quantum numbers n, ℓ, j, and m_j for each state. Also write down the spectroscopic notation for each allowed state; *e.g.* $^2P_{1/2}$.

   c) Give the selection rules for the allowed electric dipole transitions among the states in b). Give the allowed transitions between n=1 and n=2.

   d) Using a model that you think will be reasonable, obtain an order of magnitude estimate for the spin orbit splitting in eV between the $^2P_{3/2}$ and the $^2P_{1/2}$ states in the n=2 level.

   Useful constants:
   $m_e = 9.109 \times 10^{-31}$ kg
   $h = 6.626 \times 10^{-34}$ joule
   $e = 1.602 \times 10^{-19}$ coul
   $\mu_o = 4\pi \times 10^{-7}$ nt/(amp)^2
   $k = 1/4\pi\epsilon_o = 8.988 \times 10^9$ nt-m^2/coul^2

(Q) 3

Atomic Physics

a) $E_n = -13.6\,eV/n^2$ $\rightarrow$ $E_1 = -13.6\,eV$

$E_2 = -3.4\,eV$

$E_3 = -1.51\,eV$

If you don't recall the above formula, you can quickly derive it from Bohr theory,

$$E = PE + KE = -\frac{ke^2}{r} + \frac{1}{2}mv^2 \qquad a = \frac{v^2}{r} = \frac{F}{m} = \frac{1}{m}\frac{ke^2}{r^2}$$

$$\rightarrow E = -\frac{1}{2}ke^2/r \qquad \rightarrow \frac{1}{2}mv^2 = \frac{1}{2}ke^2/r$$

$$L = n\hbar = mvr \rightarrow v^2 = \frac{n^2\hbar^2}{m^2r^2} = \frac{ke^2}{mr}$$

$$\rightarrow \frac{1}{r} = \frac{mke^2}{n^2\hbar^2}$$

$$\rightarrow E = -\frac{1}{2}\frac{mk^2e^4}{\hbar^2n^2} = -\frac{13.6}{n^2}\ \checkmark$$

b)

n	l	j	m_j	
1	0	1/2	±1/2	$^2S_{1/2}$
2	0	1/2	±1/2	$^2S_{1/2}$
2	1	1/2	±1/2	$^2P_{1/2}$
2	1	3/2	±3/2, ±1/2	$^2P_{3/2}$
3	0	1/2	±1/2	$^2S_{1/2}$
3	1	1/2	±1/2	$^2P_{1/2}$
3	1	3/2	±3/2, ±1/2	$^2P_{3/2}$
3	2	3/2	±3/2, ±1/2	$^2D_{3/2}$
3	2	5/2	±5/2, ±3/2	$^2D_{5/2}$

c) The selection rules for the electric dipole transitions are

$\Delta j = 0, \pm 1$ (not $j=0$ to $j=0$)

$\Delta l = \pm 1$

$\Delta m_j = 0, \pm 1$

Thus transitions between $n=1$ and $n=2$ are

$(n=1)\ ^2S_{1/2,1/2} \leftrightarrow (n=2)\ ^2P_{1/2,1/2}$

$(n=1)\ ^2S_{1/2,-1/2} \leftrightarrow (n=2)\ ^2P_{1/2,-1/2}$

$(n=1)\ ^2S_{1/2,1/2} \leftrightarrow (n=2)\ ^2P_{3/2,1/2}$

$(n=1)\ ^2S_{1/2,-1/2} \leftrightarrow (n=2)\ ^2P_{3/2,-1/2}$

$(n=1)\ ^2S_{1/2,1/2} \leftrightarrow (n=2)\ ^2P_{3/2,3/2}$

$(n=1)\ ^2S_{1/2,-1/2} \leftrightarrow (n=2)\ ^2P_{3/2,-3/2}$

(4) 3

Atomic Physics (cont.)

d) Spin-orbit splitting

spin orbit interaction can be viewed as coming from interaction of electron magnetic moment and the magnetic field seen by orbiting electron.

Thus for $^2P_{3/2}$ state $\vec{l}$ and $\vec{s}$ are aligned $\rightarrow U = -\vec{\mu}\cdot\vec{B} = +\mu B$

$^2P_{1/2}$ state $\vec{l}$ and $\vec{s}$ are anti aligned $\rightarrow U = -\mu B$

Thus $\left.\begin{matrix} \_\_\_ \ ^2P_{3/2} \\ \_\_\_ \ ^2P_{1/2} \end{matrix}\right\} \Delta E = 2\mu B$

Estimate μ for electron by model of spinning charged loop

$$\mu = IA = \frac{e}{T}\pi r^2 = \frac{e}{2}\underbrace{\frac{2\pi r}{T}}_{v}\frac{mr}{m} = \frac{e}{2}\frac{vmr}{m} = \frac{eL}{2m}$$

for the electron $L = \frac{1}{2}\hbar \rightarrow \mu = \frac{e\hbar}{4m}$ (Actually $\mu = \frac{e\hbar}{2m}$)

To estimate B, use proton in circular orbit around electron.

$$B = \frac{\mu_0}{4\pi}\int\frac{I\,ds}{r^2} = \frac{\mu_0}{4\pi}\frac{qv}{r^2} = \frac{\mu_0}{4\pi}\frac{ev}{r^2}$$

From a) we know $\frac{1}{r^2} = \left(\frac{mke^2}{\hbar^2 n^2}\right)^2$ and $v = \frac{ke^2}{\hbar n}$

$$\rightarrow B = \frac{\mu_0}{4\pi}e\left(\frac{ke^2}{\hbar n}\right)\left(\frac{mke^2}{\hbar^2 n^2}\right)^2$$

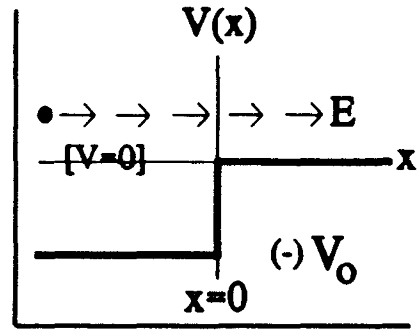
$$\Delta E = 2\mu B = 2\left(\frac{e\hbar}{2m}\right)\left(\frac{\mu_0 e}{4\pi}\right)\left(\frac{ke^2}{\hbar n}\right)\left(\frac{m^2k^2e^4}{\hbar^4 n^4}\right) = \frac{\mu_0 m k^3 e^8}{4\pi\hbar^4 n^5}$$

$$= \frac{(4\pi\times10^{-7}\,N/A^2)(0.91\times10^{-30}\,kg)(9\times10^9\,Nm^2/C^2)(1.6\times10^{-19}\,C)^8}{4\pi\,(1.06\times10^{-34}\,J\text{-}s)^4\,(2)^5}$$

$$= 6.8\times10^{-24}\,J$$

$$\underline{\Delta E = 4.3\times10^{-5}\,eV}$$

4. In the emission of electrons from metals, it is possible that an electron with enough energy to leave the metal is reflected at the metal surface. Consider a one-dimensional model with a potential: $V(x) = -V_o$, for $x < 0$ (inside metal), and $V(x) = 0$, for $x > 0$ (outside). Let an electron at energy $E > 0$ be incident from the left ($x < 0$) on the metal surface at $x = 0$.



a) Find the reflection coefficient R(E) for the electron at the surface. (Note: R(E) is the probability that the electron, incident from $x < 0$, will not be found at $x > 0$).

b) R(E) of part (a) can be written as a function of the ratio $\epsilon = E/V_o$. Find that value $\epsilon_{1/2}$ such that when $\epsilon < \epsilon_{1/2}$, $R(E) > 1/2$. Thus, electrons at energies less than $\epsilon_{1/2} V_o$ have better than a 50% chance of being reflected at the surface.

c) In a real metal, the discontinuity in V(x) at the surface is not sharp; V(x) changes smoothly over a region of size on the order of the interatomic spacings in the metal. If V(x) is smoothed out this way in this neighborhood of $x = 0$, do you expect the reflection coefficient R(E) of part (a) to increase or decrease at a given energy? Discuss your reasoning for how R(E) changes.

June 1996 Exam

9 QM-2 : Key

SOLUTION : Reflections on an electron emitted from a metal.

(A) After encountering the discontinuity in V at $x=0$, the electron wavefunction (which is a plane wave wherever V = const.) inside the metal will consist of a rightward and leftward traveling component

$$\rightarrow \psi_i(x) = A e^{ik_i x} + B e^{-ik_i x}, \text{ @ } x<0, \text{ w/ } k_i = \sqrt{\frac{2m}{\hbar^2}(E+V_0)}. \quad (1)$$

The latter represents the reflected wave, and the reflection coefficient will be $R = |B|^2/|A|^2$. The transmitted (emitted) wave will be

$$\rightarrow \psi_o(x) = C e^{ik_o x}, \text{ @ } x>0, \text{ w/ } k_o = \sqrt{\frac{2m}{\hbar^2}E}. \quad (2)$$

ψ and $d\psi/dx$ must be continuous at the surface, $x=0$, so...

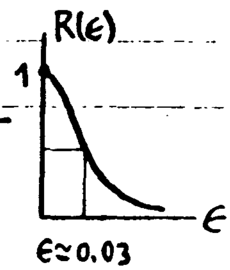
$$\left[\begin{array}{l} \psi_i(0) = \psi_o(0) \Rightarrow A+B=C \\ \psi_i'(0) = \psi_o'(0) \Rightarrow k_i(A-B) = k_o C \end{array}\right. \Rightarrow \frac{B}{A} = \frac{k_i - k_o}{k_i + k_o}. \quad (3)$$

The required reflection coefficient is...

$$R(E) = \left|\frac{B}{A}\right|^2 = \left(\frac{k_i - k_o}{k_i + k_o}\right)^2 = \left(\frac{\sqrt{E+V_0} - \sqrt{E}}{\sqrt{E+V_0} + \sqrt{E}}\right)^2 = \frac{V_0^2}{(\sqrt{E+V_0} + \sqrt{E})^4}$$

$$\text{or } \boxed{R(\epsilon) = 1/(\sqrt{1+\epsilon} + \sqrt{\epsilon})^4}, \text{ w/ } \epsilon = E/V_0. \quad (4)$$

R(ε) decreases rapidly with increasing energy, as sketched...



(B) If $R(\epsilon) > 1/2$, then by Eq. (4), we must have:

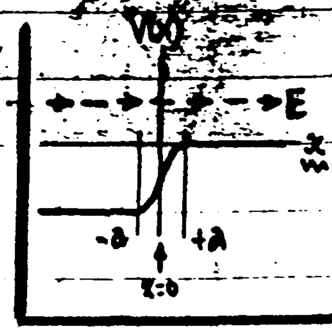
$$\rightarrow (\sqrt{1+\epsilon} + \sqrt{\epsilon})^4 < 2, \text{ or } \sqrt{1+\epsilon} < 2^{1/4} - \sqrt{\epsilon} \quad (5)$$

Square both sides of Eq. (5) and solve for ϵ...

(b) $\frac{4}{1+\cancel{\epsilon}} < \sqrt{2} - 2\cdot 2^{1/4}\sqrt{\epsilon} + \cancel{\epsilon} \Rightarrow \sqrt{\epsilon} < (\sqrt{2}-1)/2^{5/4}$

i.e. $\boxed{\epsilon < \epsilon_{1/2} = \frac{1}{2^{5/2}}(\sqrt{2}-1)^2 = 0.0303}$ (6)

~~Indeed R(E) of part (a) falls off rapidly with energy...~~ $R > \frac{1}{2}$ only for $E < 0.0303\, V_0$.



(C) If $V(x)$ is rounded off ~~over~~ $-a < x < a$ as shown (a is an interatomic spacing), then the "inside" and "outside" wavenumbers k_i and k_o of Eqs. (1) & (2) are changed as $x \to 0$: $k_i \propto \sqrt{E+V(x)}$ for $x<0$ is evidently decreased, while $k_o \propto \sqrt{E+V(x)}$ for $x>0$ is increased. Thus the difference $(k_i - k_o)$ is made smaller, and in Eq. (4), the reflection coefficient $R(E) \propto (k_i - k_o)^2$ should decrease. In other words, the reflection from a "soft" wall (with repulsive force $-(dV/dx)$ always finite) should be less than the reflection from a "hard" wall (where $-dV/dx \to$ infinite at some point).

Semi-quantitatively (and with the cnst $2m/\hbar^2$ set $= 1$), near $x = 0$...

$$\to k_i \to \bar{k}_i = \frac{1}{a}\int_{-a}^{0}\Big[(E+V_0) - \underbrace{(a+x)}_{u}\Big(\frac{dV}{dx}\Big)_0\Big]^{1/2} dx = \frac{\sqrt{E+V_0}}{a}\int_0^a \sqrt{1-Qu}\, du,\quad Q = \frac{(dV/dx)_0}{E+V_0}$$

$$\Rightarrow \bar{k}_i = \sqrt{E+V_0}\,\frac{2}{3Qa}\Big[1-(1-Qa)^{3/2}\Big] \simeq \sqrt{E+V_0}\Big[1 - \frac{a}{4}\frac{(dV/dx)_0}{E+V_0}\Big]; \quad (7)$$

$$\to k_o \to \bar{k}_o = \frac{1}{a}\int_0^a \Big[E + (a-x)\Big(\frac{dV}{dx}\Big)_0\Big]^{1/2} dx \simeq \sqrt{E}\Big[1 + \frac{a}{4}\frac{(dV/dx)_0}{E}\Big]; \quad (8)$$

1st order in a

so $\bar{R} = \left(\frac{\bar{k}_i - \bar{k}_o}{\bar{k}_i + \bar{k}_o}\right)^2 \simeq R(1-\delta)$, w/ $R = \left(\frac{\sqrt{E+V_0} - \sqrt{E}}{\sqrt{E+V_0} + \sqrt{E}}\right)^2$, per Eq. (4),

and: $\delta = \frac{a}{2}\frac{(dV/dx)_0}{\sqrt{(E+V_0)E}}\left(\frac{1}{\sqrt{R}} + \sqrt{R}\right)$, to first order in a.

Indeed the step fcn R is reduced to $\bar{R}$ when the "smoothing" range $a > 0$.

5. A pion travelling at speed v decays into a muon and a muon antineutrino,

$$\pi^- \rightarrow \mu^- + \bar{\nu}_\mu$$

In a particular decay, the neutrino emerges at 90° to the original pion direction of motion in the laboratory frame, as shown in the diagram below.

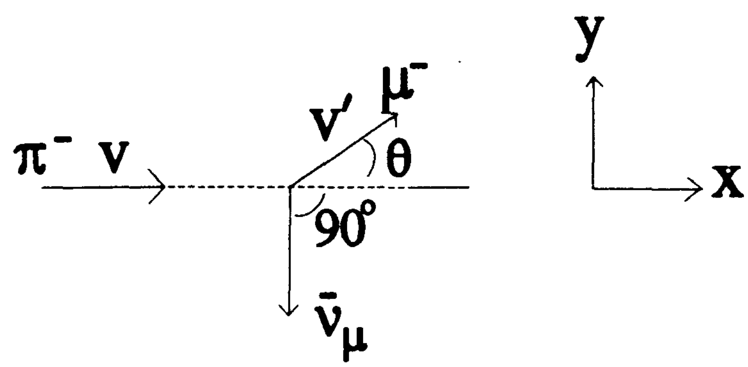
a) Find E_ν, the energy of the emitted antineutrino.

b) Find θ, the angle between the pion's direction of motion and the muon's direction of motion.

Note: Answers to a) and b) should be in terms of

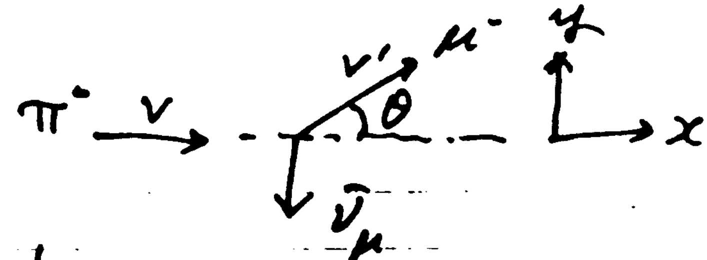
$$m_\pi,\ m_\mu,\ v \text{ and } \gamma = \frac{1}{\sqrt{1-v^2}}$$

i.e., v in units of the velocity of light c.



5 REL: Key

$\pi^- \to \mu^- + \bar{\nu}_\mu$



(a). $\gamma = \dfrac{1}{\sqrt{1-v^2}}$, $\gamma' = \dfrac{1}{\sqrt{1-v'^2}}$

4 momenta.

$P^*_\pi = (\gamma m_\pi, \gamma m_\pi v, 0, 0)$.

$P^*_\mu = (\gamma' m_\mu, \gamma' m_\mu v' \cos\theta, \gamma' m_\mu v' \sin\theta, 0)$.

$P^*_{\bar\nu} = (E_\nu, 0, -E_\nu, 0)$.

Conserve 4-momentum:

$\gamma m_\pi = \gamma' m_\mu + E_\nu$ $\quad p^t \quad (1)$

$\gamma m_\pi v = \gamma' m_\mu v' \cos\theta$ $\quad p^x \quad (2)$

$0 = \gamma' m_\mu v' \sin\theta - E_\nu$ $\quad p^y \quad (3)$

Rewrite (3).

$\gamma' m_\mu v' \sin\theta = E_\nu$ $\quad (4)$. θ to find

$\tan\theta = \dfrac{E_\nu}{\gamma m_\pi v}$

$((2))^2 + ((4))^2 \to$

$(\gamma' m_\mu v')^2(\cos^2\theta + \sin^2\theta) = E_\nu^2 + (\gamma m_\pi v)^2$

$(\gamma' m_\mu v')^2 = E_\nu^2 + (\gamma m_\pi v)^2 \qquad (5)$

move E_ν to LHS of (1) & square:

$(\gamma m_\pi)^2 - 2\gamma m_\pi E_\nu + E_\nu^2 = (\gamma' m_\mu)^2 \quad (6)$

Solve (5) for $(\gamma' m_\mu)^2$:

$(5) \to m_\mu^2 \frac{v'^2}{1-v'^2} = E_\nu^2 + (\gamma m_\pi v)^2.$

add m_μ^2 to both side:

$m_\mu^2 \left(1 + \frac{v'^2}{1-v'^2}\right) = E_\nu^2 + (\gamma m_\pi v)^2.$

add m_μ^2 to both side:

$m_\mu^2 \left(1 + \frac{v'^2}{1-v'^2}\right) = E_\nu^2 + (\gamma m_\pi v)^2 + m_\mu^2.$

$m_\mu^2 \left(\frac{1}{1-v'^2}\right) = E_\nu^2 + (\gamma m_\pi v)^2 + m_\mu^2.$

$\therefore (m_\mu \gamma')^2 = E_\nu^2 + (\gamma m_\pi v)^2 + m_\mu^2.$

Substitute this into RHS of (6)

$\to (\gamma m_\pi)^2 - 2\gamma m_\pi E_\nu + E_\nu^2 = E_\nu^2 + (\gamma m_\pi v)^2 + m_\mu^2.$

solve for E_ν. $\to (\gamma m_\pi)^2 (1 - v^2) - m_\mu^2 = 2\gamma m_\pi E_\nu$

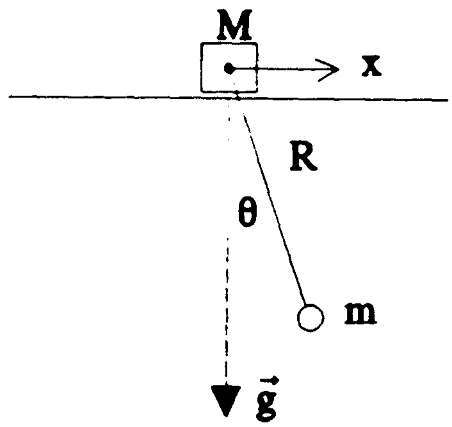
$\therefore$ Ans. $\boxed{E_\nu = \frac{m_\pi^2 - m_\mu^2}{2\gamma m_\pi}}$

(b). Substitute the above E_ν into the eqn for $\tan\theta$ & get

$\tan\theta = \frac{m_\pi^2 - m_\mu^2}{2\gamma^2 v\, m_\pi^2} = \boxed{\left(1 - \frac{m_\mu^2}{m_\pi^2}\right) \Big/ (2\gamma^2 v)}$

Ans.

6. A point mass m is connected by a massless string of length R to a mass M which can slide on a frictionless horizontal surface, as shown.



   a) Write the Lagrangian L for this system.

   b) Write the Lagrange equations for this system.

   c) Solve these equations to find the frequency of small oscillations.

6 (2)

HS

CM2 : Key

Classical Mechanics Solution

a) $L = T - V$ Start with cartesian coordinates and transform to generalized coordinates

$$T = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m\left[(\dot{x} + R\dot{\theta}\cos\theta)^2 + R^2\dot{\theta}^2\sin^2\theta\right]$$

$$V = -mgR\cos\theta$$

$$L = \frac{1}{2}(M+m)\dot{x}^2 + mR\dot{x}\dot{\theta}\cos\theta + \frac{1}{2}mR^2\dot{\theta}^2 + mgR\cos\theta$$

b) $$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = \frac{d}{dt}\left[(M+m)\dot{x} + mR\dot{\theta}\cos\theta\right]$$

$$= (M+m)\ddot{x} + mR\ddot{\theta}\cos\theta - mR\dot{\theta}^2\sin\theta = 0$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = \frac{d}{dt}\left[mR\dot{x}\cos\theta + mR^2\dot{\theta}\right] + mR\dot{x}\dot{\theta}\sin\theta + mgR\sin\theta = 0$$

$$mR\left\{\ddot{x}\cos\theta - \dot{x}\dot{\theta}\sin\theta + R\ddot{\theta} + (\dot{x}\dot{\theta} + g)\sin\theta\right\} = 0$$

c) For small oscillations, $\cos\theta \to 1$, $\sin\theta \to \theta$ and we ignore 3rd-order terms such as $\dot{x}\dot{\theta}\theta$. The Lagrange equations become

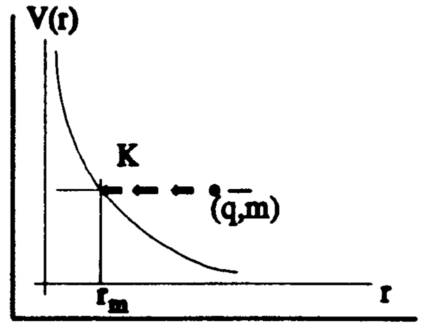
$$(M+m)\ddot{x} + mR\ddot{\theta} = 0 \qquad \ddot{x} + R\ddot{\theta} + g\theta = 0.$$

Combining, we get

$$-(M+m)R\ddot{\theta} - (M+m)g\theta + mR\ddot{\theta} = 0, \text{ so } \ddot{\theta} = -\frac{M+m}{M}\frac{g}{R}\theta$$

$$\theta = \theta_0 e^{i\sqrt{\frac{M+m}{M}\frac{g}{R}}\,t} \quad \text{so} \quad \omega = \sqrt{\frac{M+m}{M}\frac{g}{R}}$$

7. A nonrelativistic particle of charge q, mass m, and kinetic energy K makes a head-on collision with a fixed central force field. The interaction is repulsive, and is specified by a potential $V(r)$, with $V(r)$ increasing as the separation r decreases, and $V(r) > K$ when $r < r_m$. Thus r_m is the "distance of closest approach" of m to the force center during the collision.



a) Show that the total energy radiated by q during the collision is

$$\Delta W = \frac{4}{3c}(q/mc)^2\sqrt{m/2}\int_{r_m}^{\infty}\left(\frac{dV}{dr}\right)^2[K - V(r)]^{-1/2}\,dr\,.$$

Discuss any assumptions needed to arrive at this result.

b) Let $V(r) = V_0 \exp(-r/a)$, with $V_0 > K$. Evaluate the radiated energy ΔW of part (a) for a collision of (q,m) with this field.

c) Implicit in the result of part (a) is the assumption that the radiative loss ΔW is negligible compared to the incident kinetic energy K. Show (numerically) that this assumption is justified for $V(r)$ of part b), if the particle (q,m) is an electron and the range a is of atomic dimensions: $a \sim \hbar^2/me^2 = 0.53 \times 10^{-8}$ cm (Bohr radius).

June 1992 Exam (8) Key

SOLUTION: Radiation during a (nonrelativistic) scattering event.

(A) Total energy radiated is: $\Delta W = \int_{-\infty}^{\infty} P\,dt$, w/ $P = (2q^2/3c^3)|dv/dt|^2$ the Larmor radiation rate. But the acceleration $dv/dt = \frac{1}{m}(dp/dt)$, and since the collision is head-on (along r-coordinate only), then by Newton II: $|dp/dt| = |(-)dV/dr|$. Hence: $P = (2q^2/m^2c^3)|dV/dr|^2$, and

$$\longrightarrow \Delta W = (2q^2/3m^2c^3)\int_{t=-\infty}^{t=+\infty}[dV/dr]^2\,dt. \qquad (1)$$

Assume the radiation loss ΔW is small compared to the incident energy K. Then mechanical energy is conserved, so that the particle velocity at any r, i.e. $v = dr/dt$ is such that...

$$\frac{1}{2}mv^2 + V(r) = K \Rightarrow v = \frac{dr}{dt} = \sqrt{\frac{2}{m}}\,[K - V(r)]^{1/2}$$

$$\xrightarrow{\text{so}} dt = \sqrt{\frac{m}{2}}\,dr/[K - V(r)]^{1/2}. \qquad (2)$$

Use this to convert Eq. (1) to an integration over r, noting that the collision is symmetric in time about the closest approach: $-\infty \leq t \to 0 \Rightarrow \infty \geq r \geq r_m$, and: $0 \leq t \leq \infty \Rightarrow r_m \leq r \leq \infty$. Then, as required:

$$\Delta W = \frac{2q^2}{3m^2c^3}\cdot 2\int_{r_m}^{\infty}[dV/dr]^2\sqrt{\frac{m}{2}}\,dr/[K - V(r)]^{1/2},$$

$$\text{so} \quad \boxed{\Delta W = \frac{4}{3c}(q/mc)^2\sqrt{\frac{m}{2}}\int_{r_m}^{\infty}\left(\frac{dV}{dr}\right)^2[K - V(r)]^{-1/2}\,dr.} \qquad (3)$$

The closest approach distance is defined by $V(r_m) = K$.

(B) For: $V(r) = V_0 e^{-r/a}$, have: $dV/dr = -\frac{1}{a}V(r)$. The integral in (3) is:

$$J(K) = \frac{1}{a^2}\int_{r_m}^{\infty}\left\{[V(r)]^2/\sqrt{K - V(r)}\right\}dr. \qquad (4)$$

June 1992 Exam

Define a new variable $z = K - V(r)$, so that...

$r = r_m \Rightarrow z = 0$, $r = \infty \Rightarrow z = K$;

$dz = -(dV/dr)\,dr = \frac{1}{a}V(r)\,dr$, so// $dr = [a/V(r)]\,dz$;

and// $$J(K) = \frac{1}{a^2}\int_{z=0}^{z=K}\{[V(r)]^2/\sqrt{z}\}\frac{a}{V(r)}\,dz = \frac{1}{a}\int_0^K \frac{dz}{\sqrt{z}}\{K-z\}$$

$$= \frac{1}{a}\left\{K\int_0^K \frac{dz}{\sqrt{z}} - \int_0^K \sqrt{z}\,dz\right\}$$

or// $$J(K) = \frac{1}{a}\left\{2K\sqrt{z}\Big|_0^K - \frac{2}{3}z^{3/2}\Big|_0^K\right\} = \frac{4}{3a}K^{3/2}. \qquad (5)$$

With this result, the radiation loss of Eq. (3) is...

$$\Delta W = \frac{4}{3c}(q/mc)^2\sqrt{\frac{m}{2}}\cdot\frac{4}{3a}K^{3/2}$$

or// $\boxed{\Delta W = \frac{8}{9c}\left(\frac{r_0}{a}\right)\sqrt{\frac{2}{m}}\,K^{3/2}}$, $r_0 = q^2/mc^2 =$ classical charge radius of (q/m). (6)

(C) If (q,m) is an electron, and $a \sim \hbar^2/me^2$ (Bohr radius) is of atomic dimensions, then in (6): $r_0/a \sim (e^2/mc^2)/(\hbar^2/me^2) = \alpha^2$, where $\alpha = e^2/\hbar c \simeq 1/137$ is the fine structure constant. And if $K = \frac{1}{2}mv^2$ (at ∞ separation), then $\sqrt{2/m}\,K^{1/2} = v$. Consequently, the ratio...

$$\Delta W/K = \frac{8}{9c}(r_0/a)\sqrt{\frac{2}{m}}\,K^{1/2} \sim \frac{8}{9}\alpha^2\frac{v}{c}$$

i.e.// $\boxed{\Delta W/K \sim (4.74\times 10^{-5})\,\frac{v}{c}}$. (7)

Since $v < c$ (in fact $v \lll c$ for this nonrelativistic calculation), then certainly: $\Delta W/K < 50$ ppm. So, indeed ΔW is negligible w.r.t. K, as assumed in Eq. (2) above.

8. a) A converging lens has a focal length of 10 cm and is placed 15 cm from an object that is 2 cm in height. Find the size and location of the image. Is the image real or virtual?

   b) Place the object 5 cm from the lens and find the size and location of the image and discuss whether the image is real or virtual.

   c) Use a converging lens with focal length f_o as a magnifying lens and derive an expression for the magnifying power of the lens. Use this expression to estimate the maximum magnifying power of a 10 cm lens when it is placed in front of the eye. The magnifying power is defined as how much larger the angular size of the image appears to the eye when viewed with the lens.

   d) Assuming that the lens in the human eye is diffraction limited, estimate the minimum size of an object that the unaided eye can focus on and resolve. Estimate the thickness of this paper and see if you can resolve this thickness when viewing the paper's edge from the side.

JC

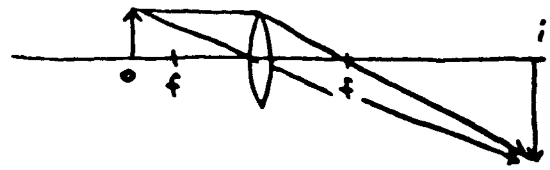
8

Optics : Key

a) Use $\frac{1}{d_o} + \frac{1}{d_i} = \frac{1}{f}$

$\rightarrow \frac{1}{d_i} = \frac{1}{f} - \frac{1}{d_o} = \frac{1}{10} - \frac{1}{15} = \frac{1}{30}$

$\rightarrow \underline{d_i = 30\text{ cm}}$

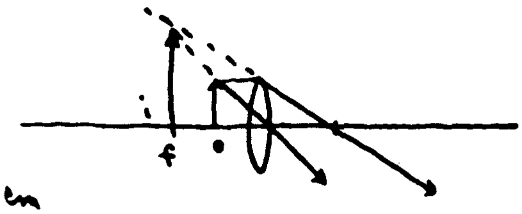


$h_i = -\frac{h_o d_i}{d_o} = -\frac{(2\text{cm})(30)}{15} = \underline{-4\text{ cm}}$ real image

b) $\frac{1}{d_i} = \frac{1}{f} - \frac{1}{d_o} = \frac{1}{10} - \frac{1}{5} = -\frac{1}{10}$

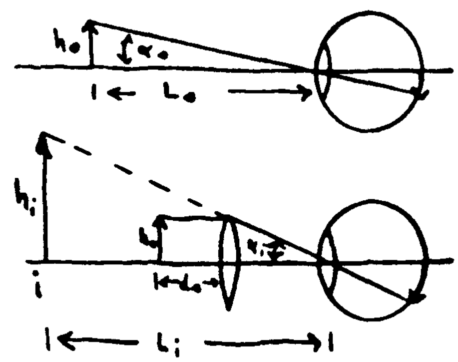
$\rightarrow \underline{d_i = -10\text{ cm}}$

$h_i = -\frac{h_o d_i}{d_o} = \frac{-(2\text{ cm})(-10\text{ cm})}{5\text{ cm}} = \underline{+4\text{ cm}}$



c) Magnifying Power $MP = \frac{\alpha_i}{\alpha_o}$

$\alpha_o = \frac{h_o}{L_o} \qquad \alpha_i = \frac{h_i}{L_i}$



$\rightarrow MP = \frac{h_i}{h_o}\,\frac{L_o}{L_i}$

$h_i = -\frac{h_o d_i}{d_o} \rightarrow \frac{h_i}{h_o} = -\frac{d_i}{d_o}$

$\rightarrow MP = \left(-\frac{d_i}{d_o}\right)\left(\frac{L_o}{L_i}\right)$

$\frac{1}{d_i} + \frac{1}{d_o} = \frac{1}{f} \rightarrow \frac{1}{d_o} = \frac{1}{f} - \frac{1}{d_i} \rightarrow \frac{d_i}{d_o} = -\left(1 - \frac{d_i}{f}\right)$

$\rightarrow \boxed{MP = \left(1 - \frac{d_i}{f}\right)\frac{L_o}{L_i}}$

Now assume L_o is the min. dist. that the eye can focus.
To get the max MP, set $L_i = L_o$

$\rightarrow MP = \left(1 + \frac{|d_i|}{f}\right)$ since $d_i < 0$

To further maximize MP set $d_i = L_o$ which puts lens close to eye.

$\rightarrow \underline{MP = \left(1 + \frac{L_o}{f}\right)}$

Now for the human eye L_o is a minimum of $L_o \approx 25\text{ cm}$

Thus $MP = \left(1 + \frac{25\text{ cm}}{10\text{ cm}}\right) = \underline{3.5}$ for $f = 10\text{cm}$ and $L_o \approx 25\text{ cm}$

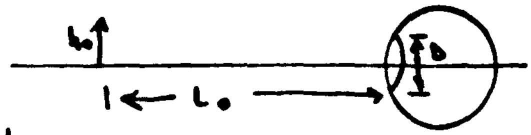
0

Optics (cont) Key

JC

d) The resolution of the eye is determined by D, the size of the aperture and L_0, the minimum focus distance.

Assume $L_0 \approx 25$ cm
$D \approx 4$ mm



Diffraction limit for a circular aperture is $\theta = \frac{1.22\lambda}{D}$

For $\lambda = 0.5\,\mu m \rightarrow \theta = \frac{1.22(0.5\times10^{-6} m)}{4\times10^{-3} m} = 1.5\times10^{-4}$ rad

$\rightarrow h_{min} = \theta L_0 = (1.5\times10^{-4}\text{ rad})(25\text{ cm}) = \underline{37\,\mu m}$

To estimate the thickness of the paper, compare the thickness of seven sheets with a pencil lead (0.7 mm = 700μ). This indicates that the paper is ~100μ thick which the normal eye can resolve.

9. The Hamiltonian of a rigid rotator in a magnetic field perpendicular to the x-axis is of the form

$$H = AL^2 + BL_z + CL_y .$$

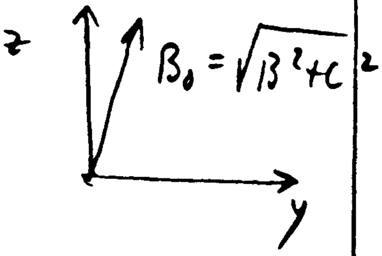
a) Obtain the exact energy eigenvalues of H.

b) Assuming B >> C, use second-order perturbation theory to get approximate eigenvalues.

c) Compare the eigenvalues obtained in a) and b).

9.

(a) Given $H = AL^2 + BL_z + CL_y$



In the rotated frame
(aligned with B_0)

$$H = AL^2 + B_0 L_z \quad ; \quad \psi \sim Y_\ell^m$$

$$\boxed{E_{\ell m} = A\,\ell(\ell+1)\hbar^2 + m\hbar B_0}$$ exact

(b) Using 2nd order perturbation theory in the original reference frame:

$$H = H_0 + V \quad \begin{cases} H_0 = AL^2 + BL_z \\ V = CL_y = \dfrac{C}{2i}(L_+ - L_-) \end{cases}$$

Unperturbed eigenvalues:

$$E^0_{\ell m} = A\,\ell(\ell+1)\hbar^2 + m\hbar B$$

$$|\psi^0_{\ell m}\rangle = |\ell m\rangle$$

First-order m.e.'s:

$$\langle \ell m | L_+ - L_- | \ell m\rangle = 0$$ no 1st-order shift

Second-order m.e.'s:

$$\langle \ell m \pm 1 | L_+ - L_- | \ell m\rangle \neq 0$$ 2nd order shift

energy shifts:

$$\Delta E_{\ell m} = \frac{C^2}{4}\left\{ \frac{|\langle \ell m-1|L_-|\ell m\rangle|^2}{m\hbar B - (m-1)\hbar B} + \frac{|\langle \ell m+1|L_+|\ell m\rangle|^2}{m\hbar B - (m+1)\hbar B} \right\}$$

$$= \frac{C^2}{4}\left\{ \frac{(\ell+m)(\ell-m+1)\hbar^2}{\hbar B} + \frac{(\ell-m)(\ell+m+1)\hbar^2}{-\hbar B} \right\}$$

$$\boxed{\Delta E_{\ell m} = \frac{C^2}{2B}\, m\hbar}$$

(c) 9 Comparison to 2nd order in C:

(a) exact: $E_{lm+1} - E_{lm} = \hbar B_0$

(b) perturbation theory:

$$E_{lm+1} - E_{lm} = \hbar B + \frac{C^2}{2B}\hbar$$

$$= \hbar B\left(1 + \frac{C^2}{2B^2}\right)$$

This agrees with (a) to second order:

$$\hbar B_0 = \hbar\sqrt{B^2 + C^2} = \hbar B\left(1 + \frac{C^2}{2B^2} + \cdots\right)$$

10. A gas of N particles obeys the van der Waals equation of state

$$P = \frac{Nk_BT}{V-Nb} - \frac{N^2a}{V^2}$$

where P, V and T are the pressure, volume and temperature, k_B is Boltzmann's constant, and a, b are two constants, characterizing the system.

a) Using dimensional analysis, construct from the constants a, b and k_B a characteristic pressure, volume and temperature. Apart from numerical factors, these will be the values of P, V and T at the critical point. For the remainder of this problem, assume that the temperature is above the critical temperature, so the system is stable.

b) Suppose further that the heat capacity at constant volume is given by

$$C_v = \frac{3}{2} Nk_B \frac{T}{T+\tau}$$

where τ is a constant with the dimension of temperature. Using this relation and the equation of state above, find the entropy S(V,T) of the system. Assume that S is known to have the value S_o in a reference state V_o, T_o.

c) Find the internal energy U(V,T) assuming it has the value U_o at V_o, T_o.

(a) m - mass, l - length, t - time, T - temp.

$$[a] = \frac{ml}{t^2 \cdot l^2} \cdot l^6 = \frac{ml^5}{t^2}, \quad [b] = l^3, \quad [k_B] = \frac{ml^2}{t^2 \cdot T}$$

So $$\left[\frac{a}{bk_B}\right] = T \qquad T_c \propto \frac{a}{bk_B}$$

$$[b] = l^3 \qquad V_c \propto b$$

$$\left[\frac{a}{b^2}\right] = \frac{ml}{t^2 \cdot l^2} \qquad P_c \propto \frac{a}{b^2}$$

(b) $$dS = \left(\frac{\partial S}{\partial T}\right)_V dT + \left(\frac{\partial S}{\partial V}\right)_T dV \qquad \left(\frac{\partial S}{\partial T}\right)_V = \frac{C_V}{T} = \frac{3}{2}\frac{Nk}{T+\tau}$$

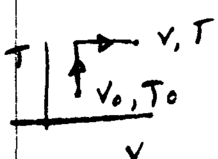
$$\to \left(\frac{\partial S}{\partial V}\right)_T = \left(\frac{\partial P}{\partial T}\right)_V = \frac{Nk}{V-Nb}$$

T

AG

P

$$S(V,T) - S(V_0,T_0) = \int_{T_0}^{T} \frac{3}{2}Nk \frac{dT'}{T'+\tau} + \int_{V_0}^{V} \frac{Nk}{V'-Nb}\, dV'$$



$$S(V,T) = \frac{3}{2} Nk \ln \frac{T+\tau}{T_0+\tau} + Nk \ln \frac{V-Nb}{V_0-Nb} + S_0$$

(c) $dU = \left(\frac{\partial U}{\partial T}\right)_V dT + \left(\frac{\partial U}{\partial V}\right)_T dV$

$$\left(\frac{\partial U}{\partial T}\right)_V = C_V = \frac{3}{2}\frac{NkT}{T+\tau}$$

$$\left(\frac{\partial U}{\partial V}\right)_T = \left(\frac{\partial U}{\partial V}\right)_S + \left(\frac{\partial U}{\partial S}\right)_V \left(\frac{\partial S}{\partial V}\right)_T = -P + T\left(\frac{\partial P}{\partial T}\right)_V$$

$$= -\frac{NkT}{V-Nb} + \frac{N^2a}{V^2} + \frac{NkT}{V-Nb} = \frac{N^2a}{V^2}$$

$$U(V,T) - U(V_0,T_0) = +\int_{V_0}^{V}\frac{N^2a}{V'^2}dV', \quad + \frac{3}{2}Nk\int_{T_0}^{T}\frac{T'}{T'+\tau}dT'$$

$$= -N^2a\left(\frac{1}{V}-\frac{1}{V_0}\right) + \frac{3}{2}Nk\left\{(T-T_0) - \tau\ln\frac{T+\tau}{T_0+\tau}\right\}$$

$$U(V,T) = N^2a\left(\frac{1}{V_0}-\frac{1}{V}\right) + \frac{3}{2}Nk(T-T_0) - \frac{3}{2}Nk\ln\frac{T+\tau}{T_0+\tau} + U_0$$

11. Consider functions $f_n(x)$ defined by the following three relationships,

$$f_o(x) = \sum_{n=0}^{\infty} \frac{x^n}{(n!)^2}$$

$$(n+1)f_{n+1} = x\, f_n - f_{n+2}$$

$$\frac{df_n}{dx} = f_{n-1}\,.$$

Find a generating function $G(x,t)$ such that

$$G(x,t) = \sum_{n=-\infty}^{\infty} f_n(x)t^n\,.$$

92, 11.

ϕ566 Solutions

$\boxed{\begin{array}{l} \underline{\text{(A)}}\ (n+1) f_{n+1} = x f_n - f_{n+2} \\ \underline{\text{(B)}}\ f_n' = f_{n-1},\ \underline{\text{(C)}}\ f_0(x) = \sum_{k=0}^{\infty} x^k/(k!)^2. \end{array}}$

(37) [15 pts]. Find generating fcn $G(x,t)$ for fcns $\{f_n(x)\}$ that are defined via

1) Multiply the recurrence relation (A) by t^n and sum over all n... $(-\infty \leq n \leq +\infty)$...

A. $$\sum_n (n+1) f_{n+1} t^n = x \sum_n f_n t^n - \sum_n f_{n+2} t^n \leftarrow \text{put } G = \sum_n f_n t^n \text{ (by def}^{\underline{n}}\text{)}$$

$$\Rightarrow \partial G/\partial t = xG - \frac{1}{t^2} G, \quad \text{a//} \quad \frac{\partial G}{G} = \left(x - \frac{1}{t^2}\right)\partial t$$

... integrating: $$\underline{\underline{G(x,t) = g(x) \exp\left(xt + \frac{1}{t}\right)}}$$ $\int$ $g(x)$ is an arbitrary multiplicative fcn of x. (1)

2) $g(x)$ in Eq. (1) needs to be fixed so as to be consistent with relation (C). First note...

$$\frac{\partial G}{\partial x} = \sum_n [f_n'] t^n \overset{\text{by relation (B)}}{=} \sum_n [f_{n-1}] t^n = tG$$

$$\updownarrow$$

$$[g' e^{(xt + \frac{1}{t})} + tG] = tG, \quad \text{n//} \quad g' = 0 \Rightarrow \underline{\underline{g(x) = \text{cnst} = g_0}}. \qquad (2)$$

3) To get the cnst g_0, compare G with $f_0(x)$ as follows...

$$G(x,t) = g_0 \exp\left(xt + \frac{1}{t}\right) = \sum_n [f_n(x)] t^n$$

$$\text{a//} \quad g_0 \underbrace{(e^{xt})(e^{1/t})} = f_0(x) + \sum_{n \neq 0} [f_n(x)] t^n$$

$$= \left(\sum_{m=0}^{\infty} \frac{(xt)^m}{m!}\right)\left(\sum_{n=0}^{\infty} \frac{1}{n!} \frac{1}{t^n}\right) = \sum_{m,n} \left(\frac{x^m}{m!\,n!}\right) t^{m-n} \qquad (3)$$

Break last sum into terms with $m=n$ & $m \neq n$. Put in $f_0(x)$ from relation (C). So

$$g_0 \left\{ \sum_{n=0}^{\infty} \frac{x^n}{(n!)^2} + \sum_{m \neq n} \left(\frac{x^m}{m!\,n!}\right) t^{m-n} \right\} = \sum_{k=0}^{\infty} \frac{x^k}{(k!)^2} + \sum_{n \neq 0} [f_n(x)] t^n$$

compare

This is true for all t (e.g. $t=0$) only if $g_0 = 1$. So desired generating fcn is

$$\boxed{G(x,t) = e^{(xt + \frac{1}{t})} = \sum_n [f_n(x)] t^n}$$ $\int$ such that $\underline{A}$ $(n+1) f_{n+1} = x f_n - f_{n+2}$, $\underline{B}$ $f_n' = f_{n-1}$, $\underline{C}$ $f_0(x) = \sum_{k=0}^{\infty} x^k/(k!)^2$. (4)

(over)

12. a) A particle of mass m is trapped in a spherical shell, defined by

$$V(r) = \begin{cases} 0 \ , & a < r < b \\ \infty \ , & \text{otherwise} \ . \end{cases}$$

i) What is the degeneracy of the nth excited state of the system?

ii) Calculate the ground state energy and wavefunction (including normalization).

b) Suppose the shell is of finite depth:

$$V(r) = \begin{cases} -V_o \ , & a < r < b \\ 0 \ , & \text{otherwise} \ . \end{cases}$$

Calculate the elastic scattering cross-section $\frac{d\sigma}{d\Omega}$ of a beam of particles (mass m, momentum $\vec{p}$) from this shell in the Born approximation. What is $\frac{d\sigma}{d\Omega}$ in the forward direction?

(a) The potential has spherical symmetry, so the angular dependence of the wavefcns is given by the usual spherical harmonics, and those of the same l will be degenerate. So the l^{th} energy will be $2l+1$ - fold degenerate.

Ground state: $l=0$, no angular dependence:

$$-\frac{\hbar^2}{2m}\left(\frac{1}{r^2}\frac{d}{dr}r^2\frac{d}{dr}\right)\psi = E\psi \quad \text{inside the shell}$$

$$\psi'' + \frac{2}{r}\psi' = -\frac{2mE}{\hbar^2}\psi$$

let $\psi = \frac{u(r)}{r} \Rightarrow u'' = -k^2 u \qquad k = \sqrt{\frac{2mE}{\hbar^2}}$

$$u = A\sin kr + B\cos kr$$

$$u(a) = 0 = A\sin ka + B\cos ka$$

$$u(b) = 0 = A\sin kb + B\cos kb$$

$$\Rightarrow \frac{\sin ka}{\sin kb} = \frac{\cos ka}{\cos kb}$$

$$\sin k(b-a) = 0 \Rightarrow k = \frac{\pi}{b-a}$$

$\frac{B}{A} = -\tan ka$ $\qquad E = \frac{\hbar^2\pi^2}{2m(b-a)^2}$

$u(r) = A(\sin kr - \tan ka\cos kr)$ ground state energy

Norm

$$1 = 4\pi|A|^2\int_a^b r^2dr\,\frac{(\sin kr - \tan ka\cos kr)^2}{r^2} = \frac{4\pi|A|^2}{\cos^2 ka}\frac{(b-a)}{2k}$$

$$|A| = \frac{\cos ka}{2(b-a)}$$

(b) Scattering

$$\frac{d\sigma}{d\Omega} = |f(\theta)|^2 \quad \text{where}$$

$$f(\theta) = -\frac{m}{2\pi\hbar^2}\int V(\vec{r})\, e^{i\vec{r}\cdot\vec{K}}\, d\vec{r}$$

where $\vec{K} = \vec{k}' - \vec{k}$ $\qquad \vec{k} = \vec{P}/\hbar$

and θ is the scattering angle between $\vec{k}'$ and $\vec{k}$.

$V(\vec{r})$ is spherically symmetric so

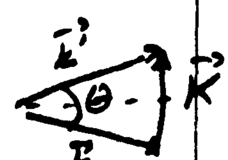
$$\int V(\vec{r})\, e^{i\vec{r}\cdot\vec{K}}\, d^3r = \frac{4\pi}{K}\int_0^\infty r\,dr\, V(r)\sin Kr$$

$$= -V_0\frac{4\pi}{K}\int_a^b r\,dr\,\sin Kr$$

$$= +\frac{4\pi V_0}{K^3}\left[Kb\cos Kb - Ka\cos Ka - \sin Kb + \sin Ka\right]$$

For elastic scattering, $|\vec{k}| = |\vec{k}'| = k$.

$$K = 2k\sin\frac{\theta}{2}$$



So
$$\frac{d\sigma}{d\Omega} = \left[-\frac{2mV_0}{\hbar^2K^3}\left(Kb\cos Kb - Ka\cos Ka - \sin Kb + \sin Ka\right)\right]^2$$

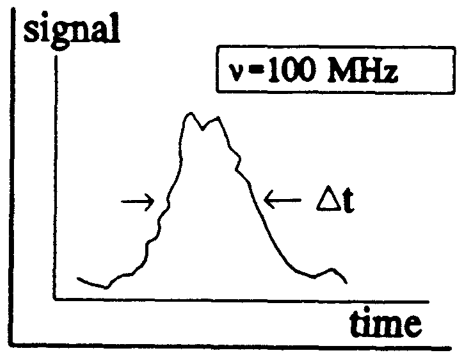
In the small-K limit, the expression in brackets becomes

$$(\;\;) \cong Kb\left(1-\frac{K^2b^2}{2}\right) - Ka\left(1-\frac{K^2a^2}{2}\right) - Kb\left(1-\frac{K^2b^2}{6}\right) + Ka\left(1-K^2a^2/6\right)$$

$$\cong -\frac{K^3}{3}(b^3-a^3)$$

So
$$\frac{d\sigma}{d\Omega}(\theta=0) = \frac{4}{9}\frac{m^2V_0^2}{\hbar^4}(b^3-a^3)^2$$

13. Signals from a pulsar in the Crab Nebula, about 6500 light years distant from earth, can be detected at radio frequencies: $\nu = 100$ MHz. The signals consist of a train of pulses, repeating regularly at 30 ms intervals, and each pulse has a width in time of $\Delta t = 2$ ms.



Assume the pulse width Δt is due to a velocity dispersion of the pulse in transit, and that this dispersion is due to a finite photon mass m. From the given data, establish an upper limit on m.

Finally, numerically, quote your limit on the photon mass m as a ratio, m/m_e, where m_e is the electron mass. It helps to know:

$m_e = 9.1 \times 10^{-28}$ gm,
$c = 3.0 \times 10^{10}$ cm/sec,
$h = 6.6 \times 10^{-27}$ erg-sec.

(B) June 1992 Exam (40) RR

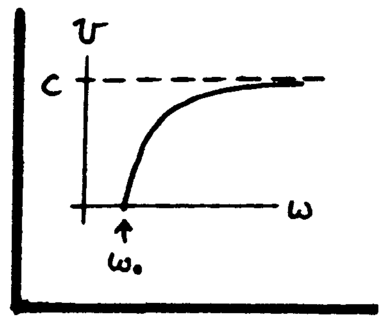
~~EM72~~ : Key [illegible]

SOLUTION : Photon mass limit from pulsar data.

1. If the photon has mass m, then it obeys a dispersion relation:

$$\rightarrow \omega = \sqrt{k^2c^2 + \omega_0^2} \quad \left\{ \begin{array}{l} \omega = 2\pi\nu = \text{frequency},\ k = \text{wave \#},\ c = \text{light speed}, \\ \text{and: } \omega_0 = mc^2/\hbar,\ \hbar = \text{Planck cnst.} \end{array} \right. \tag{1}$$

The photon group velocity is then...



$$\rightarrow v = \frac{\partial\omega}{\partial k} = c \cdot \frac{kc}{\sqrt{k^2c^2+\omega_0^2}} = c\sqrt{1-(\omega_0/\omega)^2}\ ; \tag{2}$$

so, $v \simeq c\left[1 - \frac{1}{2}(\omega_0/\omega)^2\right]$, for $\omega_0 \ll \omega$;

and, $$\frac{\partial v}{\partial\omega} \simeq c\,\frac{\omega_0^2}{\omega^3}, \text{ for } \omega_0 \ll \omega. \tag{3}$$

Signals at frequencies in a range $\Delta\omega$ about ω thus show a velocity dispersion Δv of size:

$$\Delta v \simeq (\partial v/\partial\omega)\,\Delta\omega, \quad \text{i.e.,} \quad \frac{\Delta v}{c} \simeq \left(\frac{\omega_0}{\omega}\right)^2 \frac{\Delta\omega}{\omega}. \tag{4}$$

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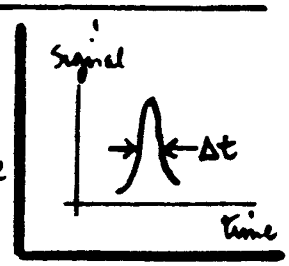
2. If the signal pulse is spread out in time by Δt by the velocity dispersion just calculated, then -- since the pulse has been in transit for time D/c, where D is the distance to the source -- we can write



$$\rightarrow \frac{D}{c}\Delta v \leq c\Delta t, \quad \text{or,} \quad \frac{\Delta v}{c} \leq \Delta t/(D/c),$$

so, Eq.(4) $\Rightarrow$ $$\boxed{\left(\frac{\nu_0}{\nu}\right)^2 \frac{\Delta\nu}{\nu} \leq \left(\frac{\Delta t}{D/c}\right)}. \tag{5}$$

We have converted to linear freq. $\nu = \frac{\omega}{2\pi}$. Now $\nu_0 = mc^2/h$.

13 EM-2 : Key General

3. The frequency spread in Eq. (5) is $\Delta\nu \simeq 1/\Delta t$ (per Fourier Thm), and so the upper limit on the photon mass term is

$$\boxed{\nu_0 = mc^2/h \lesssim (\nu \Delta t)\sqrt{\nu/(D/c)}} \quad (6)$$

If $\nu = 100$ MHz, $\Delta t = 2$ ms, and $D = c \times 6500$ years†

$$\rightarrow \nu_0 = \frac{mc^2}{h} \lesssim 10^8 \times 2\times10^{-3}\sqrt{\frac{10^8}{6500\times3.156\times10^7}} = \underline{\underline{4.42\times10^3 \text{ Hz}}}. \quad (7)$$

---

4. In (7): $\nu_0 = \frac{m}{m_e}(m_e c^2/h)$, and for the electron...

$$m_e c^2/h = c/(h/m_e c) = \frac{3\times10^{10}\text{ cm/sec}}{2.43\times10^{-10}\text{ cm}} = 1.23\times10^{20}\text{ Hz}. \quad (8)$$

So Eq. (7) reads...

$$1.23\times10^{20}\text{ Hz} \times \frac{m}{m_e} \lesssim 4.42\times10^3\text{ Hz}.$$

$$\text{or}\quad \boxed{m/m_e \lesssim 3.6\times10^{-17}},\ \text{at } \nu = 100\text{ MHz}. \quad (9)$$

This limit is ~4 orders of magnitude <u>less</u> sensitive than that established from geophysical data. To compete with the geophysical data, the pulse measurements here would have to be pushed down to frequencies $\nu \simeq 200$ kHz. Earth's ionosphere prevents measurements below $\nu \sim 10$ MHz, and so at best $m/m_e < 10^{-18}$ from pulsars.

† 1 year = 3.156×10^7 sec.

MP-1: Key.

$|x| < a/2$; $|y| < a/2$ and $|z| < a/2$.

at $t=0$, $T = \delta(\bar{x})$. find $T = T(t)$ at $(a/4, 0, 0)$.

Boundary $T=0$ at all times

(a). For small t, use the Green's function for an infinite solid, and suppose δ-function sources with alternating signs at center of each cube in an infinite lattice. The temperature will be

$$T = \Sigma \left(\frac{1}{4\pi K t}\right)^{3/2} e^{-r^2/4Kt}$$

where r is the distance from source to observation point. For small t, only the nearest two contribute, namely the one at the origin, for which $r = a/4$ and the adjacent one for which $a = 3a/4$.

$$\therefore \boxed{T \approx \left(\frac{1}{4\pi K t}\right)^{3/2} \left[e^{-a^2/64Kt} - e^{-9a^2/64Kt}\right]}$$

for $t \to 0$. Ans

(b). For large t, the Green's function approach is not useful. Instead we use separation of variables. T obeys $\nabla^2 T = \frac{1}{K}\frac{\partial T}{\partial t}$.

The normal mode solutions are $T \sim e^{-\lambda t}$.

$$\therefore T \sim \cos\left(\frac{n\pi x}{a}\right)\cos\left(\frac{m\pi y}{a}\right)\cos\left(\frac{\ell\pi z}{a}\right)e^{-\lambda t},$$

where $-\frac{\pi^2}{a}(n^2+m^2+\ell^2) + \frac{\lambda}{K} = 0$.

The ones which survive best for large t is the ones with the smallest λ, i.e. for $(m, n, \ell) = (1,1,1), (1,1,2), (1,2,1)$, and $(2,1,1)$.

MP-1. Key

Initially $T = \delta^3(\vec{x})$

$$= \Sigma C_{lmn} \cos\left(\frac{n\pi x}{a}\right)\cos\left(\frac{m\pi y}{a}\right)\cos\left(\frac{l\pi z}{a}\right).$$

Multiply by $\cos\left(\frac{n'\pi x}{a}\right)\cos\left(\frac{m'\pi y}{a}\right)\cos\left(\frac{l'\pi z}{a}\right)$, and integrate over the cube. On the left hand side we get just 1. On the right hand side, we get $C_{l'm'l'}(a^3/8)$.

$\therefore\ 1 = C_{l'm'l'}\, a^3/8 \rightarrow \therefore\ \underline{C_{l'm'n'} = 8/a^3}.$

Now at $(a/4, 0, 0)$

$$\cos\left(\frac{\pi x}{a}\right)\cos\left(\frac{\pi y}{a}\right)\cos\left(\frac{\pi z}{a}\right) = \cos\frac{\pi}{4} = \frac{1}{2}\sqrt{2}.$$

$$\cos\left(\frac{2\pi x}{a}\right)\cos\left(\frac{\pi y}{a}\right)\cos\left(\frac{\pi z}{a}\right) = \cos\frac{\pi}{2} = 0$$

$$\cos\left(\frac{\pi x}{a}\right)\cos\left(\frac{2\pi y}{a}\right)\cos\left(\frac{\pi z}{a}\right) = \frac{1}{2}\sqrt{2}$$

$$\cos\left(\frac{\pi x}{a}\right)\cos\left(\frac{\pi y}{a}\right)\cos\left(\frac{2\pi z}{a}\right) = \frac{1}{2}\sqrt{2}.$$

Hence, at large t, at $(a/4, 0, 0)$, T is

$$\boxed{T \approx \frac{4\sqrt{2}}{a^3}\, e^{-3K\pi^2 t/a^2} + \frac{8\sqrt{2}}{a^3}\, e^{-6K\pi^2 t/a^2}}$$

Ans

14 E M

J C

a) Write Maxwell's equations in term of **E** and **B** when the medium is a homogeneous, nonmagnetic, nonconducting dielectric with a free charge density of ρ_f and a free current density of $\mathbf{J}_f$. You may use gaussian units or MKSA.

b) Now assume there is an interface between region 1 and region 2. Using Maxwell's equations from part a), write down the boundary conditions on **E** and **B** at the interface.

c) Now, in the absence of free charges and free currents, consider a plane wave incident at an angle θ_i on a boundary at $z = 0$ that divides two media as described in part a). Region 1 is characterized by a permittivity ϵ_1 and region 2 is characterized by a permittivity ϵ_2. Use the boundary conditions to derive a relation between the incident angle θ_i and the transmitted angle θ_t. Find an expression for the critical angle θ_c, beyond which all intensity will be totally reflected. Assume $\epsilon_1 > \epsilon_2$.

d) Now assume that the E field of the incident wave is polarized in the plane of ~~incident~~ ce and for $\theta_i > \theta_c$, derive the ratio of the transmitted electric field to the incident electric field at the boundary.

incidence

e) If the real part of the incident wave is given by

$$\vec{E}_i = \vec{E}_{io}\cos(\vec{k}_i \cdot \vec{r} - \omega t)$$

derive an expression for the real part of the evanescent wave in region 2.

14 (8) EM: Key

EM

JC

a) Maxwell's Equations (in MKSA) for homogeneous, nonmagnetic, nonconducting dielectric

$$\nabla\cdot\vec{E} = \frac{\rho_f}{\epsilon}$$

$$\nabla\cdot\vec{B} = 0$$

$$\nabla\times\vec{E} = -\frac{\partial}{\partial t}\vec{B}$$

$$\nabla\times\vec{B} = \mu_0\vec{J}_f + \mu_0\epsilon\frac{\partial}{\partial t}\vec{E}$$

b) $\epsilon_2 E_{2n} - \epsilon_1 E_{1n} = \sigma_f$ σ_f is free surface charge density

$B_{2n} = B_{1n}$

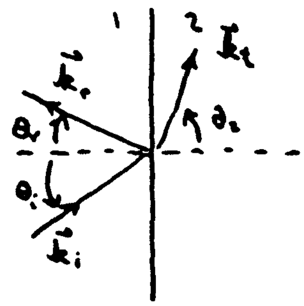
$E_{2t} = E_{1t}$

$B_{2t} - B_{1t} = \mu_0 K_f$ K_f is free surface current density

c) let $\vec{E}_i = \vec{E}_{i0}\, e^{i(\vec{k}_i\cdot\vec{r} - \omega_i t)}$

$\vec{E}_r = \vec{E}_{r0}\, e^{i(\vec{k}_r\cdot\vec{r} - \omega_r t)}$

$\vec{E}_t = \vec{E}_{t0}\, e^{i(\vec{k}_t\cdot\vec{r} - \omega_t t)}$



to match boundary conditions for all time need $\omega_i = \omega_r = \omega_t = \omega$

to match boundary condition for all points on boundary need $k_i \sin\theta_i = k_r \sin\theta_r = k_t \sin\theta_t$

$\rightarrow \theta_i = \theta_r$ since $k_i = k_r$

and $\rightarrow \boxed{\sqrt{\epsilon_1}\sin\theta_i = \sqrt{\epsilon_2}\sin\theta_t}$ since $k = \frac{\omega}{c}\sqrt{\frac{\epsilon}{\epsilon_0}} = \frac{\omega}{c}n$

this is snell's law

14 EM Key

EM (cont)

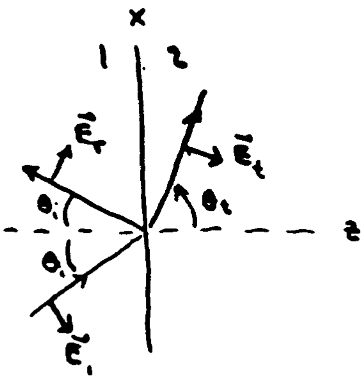
JC

c) (cont) To find critical angle, let $\theta_t = \pi/2$

$$\rightarrow \sqrt{\epsilon_1}\sin\theta_i = \sqrt{\epsilon_2}\sin(\pi/2) = \sqrt{\epsilon_2}$$

$$\boxed{\sin\theta_c = \sqrt{\epsilon_2/\epsilon_1}}$$ thus need $\epsilon_1 > \epsilon_2$ for θ_c to exist.

d) $\vec{E}_i$ in the plane of incidence



Boundary condition $E_{1t} = E_{2t}$

$$\rightarrow E_{i0}\cos\theta_i - E_{r0}\cos\theta_i = E_{t0}\cos\theta_t \quad (*)$$

Now use $\nabla\times\vec{E} = -\frac{\partial}{\partial t}\vec{B}$

$$\vec{k}\times\vec{E} = -(-i\omega)\vec{B} = i\omega\vec{B}$$

$$\vec{B} = \frac{\vec{k}\times\vec{E}}{i\omega} = \frac{\omega}{c}\sqrt{\epsilon}\,\frac{E}{i\omega}(-\hat{y}) = i\frac{\sqrt{\epsilon}\,E}{c}\hat{y}$$

Boundary condition $B_{1t} = B_{2t}$

$$\rightarrow \sqrt{\epsilon_1}E_{i0} + \sqrt{\epsilon_1}E_{r0} = \sqrt{\epsilon_2}E_{t0} \quad (**)$$

Need to solve for $\frac{E_{r0}}{E_{i0}}$ and $\frac{E_{t0}}{E_{i0}}$

$$\left(\frac{E_{t0}}{E_{i0}}\right)\cos\theta_t + \left(\frac{E_{r0}}{E_{i0}}\right)\cos\theta_i = \cos\theta_i \qquad \text{from}(*)$$

$$\sqrt{\frac{\epsilon_2}{\epsilon_1}}\left(\frac{E_{t0}}{E_{i0}}\right) - \left(\frac{E_{r0}}{E_{i0}}\right) = 1 \qquad \text{from}(**)$$

$$\rightarrow \left(\frac{E_{t0}}{E_{i0}}\right) = \frac{-\cos\theta_i - \cos\theta_i}{-\cos\theta_t - \sqrt{\frac{\epsilon_2}{\epsilon_1}}\cos\theta_i} = \frac{2\cos\theta_i}{\sqrt{\frac{\epsilon_2}{\epsilon_1}}\cos\theta_i + \left(1 - \frac{\epsilon_1}{\epsilon_2}\sin^2\theta_i\right)^{1/2}}$$

Now for $\theta_i > \theta_c$ $\quad \frac{\epsilon_1}{\epsilon_2}\sin^2\theta_i > 1$

so write

$$\boxed{\left(\frac{E_{t0}}{E_{i0}}\right) = \frac{2\cos\theta_i}{\sqrt{\frac{\epsilon_2}{\epsilon_1}}\cos\theta_i + i\left(\frac{\epsilon_1}{\epsilon_2}\sin^2\theta_i - 1\right)^{1/2}}}$$

14 E11-Key

EM (cont)

e) use $\vec{E}_i = \vec{E}_{i0}\, e^{i(\vec{k}_i\cdot\vec{r} - wt)}$ for complex form of incident wave

and $\vec{E}_t = \vec{E}_{t0}\, e^{i(\vec{k}_t\cdot\vec{r} - wt)}$ for transmitted wave in complex form

write in rectangular coordinates

$$\vec{E}_t = \vec{E}_{t0}\, e^{ik_t\cos\theta_t z}\, e^{ik_t\sin\theta_t x}\, e^{-iwt}$$

But $k_t\sin\theta_t = k_i\sin\theta_i$ snell's law

and $k_t\cos\theta_t = \frac{\omega}{c}\sqrt{\frac{\epsilon_2}{\epsilon_0}}\; i\left(\frac{\epsilon_1}{\epsilon_2}\sin^2\theta_i - 1\right)^{1/2} \equiv i\,K$

Thus $\vec{E}_t = \vec{E}_{t0}\, e^{-Kz}\, e^{i(k_i\sin\theta_i x - wt)}$

Since wave propagates in x direction $\vec{E}_{t0} = \hat{z}\, E_{t0}$

Thus $\vec{E}_t = \hat{z}\, E_{t0}\, e^{-Kz}\, e^{i(k_i\sin\theta_i x - wt)}$

But to get real part we need to get E_{t0}

$$\frac{E_{t0}}{E_{i0}} = \frac{2\cos\theta_i}{\sqrt{\frac{\epsilon_2}{\epsilon_1}}\cos\theta_i + i\,K} = \left|\frac{E_{t0}}{E_{i0}}\right| e^{i\phi}$$

$$\phi = \tan^{-1}\left(\frac{-K}{\sqrt{\frac{\epsilon_2}{\epsilon_1}}\cos\theta_i}\right)$$

$$\rightarrow \vec{E}_t = \hat{z}\, E_{i0}\left|\frac{E_{t0}}{E_{i0}}\right| e^{-Kz}\, e^{i(k_i\sin\theta_i x - wt + \phi)}$$

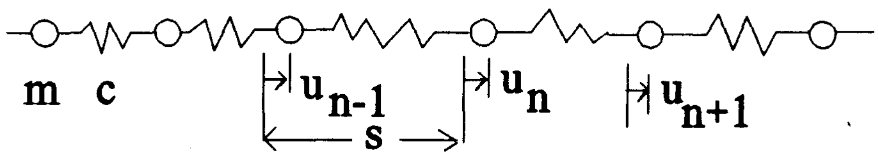
$$\left|\frac{E_{t0}}{E_{i0}}\right| = \frac{2\cos\theta_i}{\left[\frac{\epsilon_2}{\epsilon_1}\cos^2\theta_i + K^2\right]^{1/2}}$$

Finally real part is

$$\boxed{\vec{E}_t = \hat{z}\, E_{i0}\left|\frac{E_{t0}}{E_{i0}}\right| e^{-Kz}\cos(k_i\sin\theta_i x - wt + \phi)}$$

Evanescent wave propagates in x direction while it decays in z direction.

15. Consider an infinite chain of atoms of mass m and equilibrium spacing s connected by springs of spring constant c, as shown below.



a) Write the equation of motion for the nth atom, in terms of the displacements from equilibrium u_n, u_{n-1}, and u_{n+1} of that atom and its neighbors.

b) Assume a wave solution for displacements in terms of equilibrium positions (n-1)s, ns, (n+1)s,

$$u_n = u_o \exp\,[i\,(kns - \omega t)],\ u_{n\pm1} = u_n \exp\,(\pm iks).$$

Plug this solution into the equation of motion and obtain the dispersion relation $\omega(k)$.

c) Find expressions for the phase velocity v_p (k) and the group velocity v_g (k) for these waves (which when quantized are called phonons).

15

Solid State Solution

a) From Newton's 2nd Law, $F_n=c(u_{n+1}+u_{n-1}-2u_n)=md^2u_n/dt^2$

b) Plugging the solution into the equation of motion and factoring out u_n, we get

$$c[\exp(iKs)+\exp(-iKs)-2]=c[2\cos(Ks)-2]=c[2-4\sin^2(Ks/2)-2]=-m\omega^2,$$

so $\omega=2(c/m)^{1/2}\sin(Ks/2)$.

c) The phase velocity $v_p(K)$ is the velocity of a point of constant phase, which without loss of generality can be the point where the phase $Kns-\omega t$ is zero. For this condition, the position $ns=\omega t/K$, and its time rate of change or phase velocity is $v_p(K)=\omega/K$.

The group velocity $v_g(K)$ is the velocity of a point for which the phases remain equal for two waves of slightly different wavevector, such as K and $K+dK$. These waves will have frequencies ω and $\omega+d\omega$. Setting the phases equal for these two waves yields

$$Kns-\omega t=(K+dK)ns-(\omega+d\omega)t, \text{ so } ns=(d\omega/dK)t \text{ and}$$

$$v_g(K)=dns/dt=d\omega/dK=s(c/m)^{1/2}\cos(Ks/2).$$