1992 Φ507 Problems   Assigned: 2/14. Due: 2/24/92.   ℋ5 ⑤ file

⑰ [15 pts]. For a particle (q,m) in an EM field specified by a 4-potential $(A_\mu)=(\vec{A}, i\phi)$, the Klein-Gordon wave equation and continuity equation are [$^{w/}$ $(x_\mu)=(\mathbf{r}, ict)$]...

$$\left[\!\!\left[\begin{array}{l} \left[\left(\frac{\partial}{\partial x_\mu}-\frac{iq}{\hbar c}A_\mu\right)^2-k_0^2\right]\psi=0, \quad {}^{w/}\ k_0=mc/\hbar\,; \\ \partial S_\mu/\partial x_\mu=0, \quad {}^{w/}\ S_\mu=\frac{\hbar}{2im}\left[\psi^*\left(\frac{\partial}{\partial x_\mu}-\frac{iq}{\hbar c}\right)\psi-\mathrm{c.c.}\right]. \end{array}\right]\!\!\right]$$

Consider the gauge transformation: $A_\mu\to A'_\mu=A_\mu+\partial\eta/\partial x_\mu$, η = arbitrary fcn. Given that $\psi\to\psi'=\psi\exp[i(q/\hbar c)\eta]$ under this transform, show that: (A) S_μ is gauge invariant, and: (B) the KG Eqtn itself is gauge covariant (form-invariant).

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⑱ [15 pts]. Consider a particle of mass m in a 3D attractive spherical potential well of depth V and radius a. Using the Klein-Gordon Eqtn for S-states only (set the orbital ∡ momentum $l=0$), find the minimum well depth V_{KG} which just barely binds the particle. State your answer in terms of the well-known result from the Schrödinger Eqtn, viz: $V_S=\pi^2\hbar^2/8ma^2$. Interpret the difference between V_{KG} and V_S.

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⑲ [15 pts]. A Schrödinger-type form for the free-particle Klein-Gordon Eqtn can be manufactured as follows. Define a fcn ξ via: $(mc^2)\xi=i\hbar\,\partial\psi/\partial t$. Then the KG Eqtn is: $\frac{1}{m}[\vec{p}^{\,2}+(mc)^2]\psi=i\hbar\,\partial\xi/\partial t$. Next, define a two-component wavefunction by: $\Psi=\begin{pmatrix}\psi_+\\ \psi_-\end{pmatrix}=\frac{1}{2}\begin{pmatrix}\psi+\xi\\ \psi-\xi\end{pmatrix}$. In these terms, show the KG Eqtn can be written as:

$$\boxed{i\hbar\,\partial\Psi/\partial t=\mathcal{H}\Psi, \quad {}^{w/}\ \mathcal{H}=\begin{pmatrix}1&0\\0&-1\end{pmatrix}mc^2+\begin{pmatrix}1&1\\-1&-1\end{pmatrix}\frac{\vec{p}^{\,2}}{2m}}.$$

Notice that this "Hamiltonian" $\mathcal{H}$ is not Hermitian. For nonrelativistic particles ($\vec{p}^{\,2}/2m \ll mc^2$), evidently ψ_+ is the solution for positive energy states $E\simeq+mc^2$, while ψ_- is the solution for negative energy states $E\simeq(-)mc^2$. Now show that the KG "probability density": $\rho=-(\hbar/mc^2)\,\mathrm{Im}[\psi^*(\partial\psi/\partial t)]$, class notes p. fs 16, can be written as a charge density: $\tilde{\rho}=q\rho=q\{|\psi_+|^2-|\psi_-|^2\}$. Then (+)ve energy solutions (ψ_+ dominant) have $\tilde{\rho}\doteq+q$, while (−)ve energy solutions (ψ_- dominant) have $\tilde{\rho}\doteq(-)q$. We will see that the Dirac Eqtn has similar features.

(17) [15 pts]. KG Eqtn under a gauge transform: invariance of S_μ, covariance of KG Eq.

1. The KG current for a mass m & charge q in an external EM potential A_μ has components

(A) $$\rightarrow S_\mu = (\hbar/2im)\left[\psi^*\left(\frac{\partial}{\partial x_\mu} - \frac{iq}{\hbar c}A_\mu\right)\psi - \text{C.C.}\right], \tag{1}$$

as cited. Under a gauge transform by an arbitrary scalar fcn $\eta = \eta(x_\mu)$

$$\rightarrow A_\mu \rightarrow A'_\mu = A_\mu + (\partial\eta/\partial x_\mu), \quad \psi \rightarrow \psi' = \psi e^{i(q/\hbar c)\eta}, \tag{2}$$

we note that $\partial\psi/\partial x_\mu$ transforms as...

$$\frac{\partial}{\partial x_\mu}\psi \rightarrow \frac{\partial}{\partial x_\mu}\psi' = \frac{\partial}{\partial x_\mu}\psi e^{i(q/\hbar c)\eta} = e^{i(q/\hbar c)\eta}\left[\frac{\partial}{\partial x_\mu} + \frac{iq}{\hbar c}\left(\frac{\partial\eta}{\partial x_\mu}\right)\right]\psi \tag{3}$$

$$\xrightarrow{\text{So}} \psi^*\left(\frac{\partial}{\partial x_\mu} - \frac{iq}{\hbar c}A_\mu\right)\psi \rightarrow \psi'^*\left(\frac{\partial}{\partial x_\mu} - \frac{iq}{\hbar c}A'_\mu\right)\psi' = \checkmark \text{ assume } \eta \text{ is a real fcn}$$

$$= \underbrace{\left(\psi^* e^{-i(q/\hbar c)\eta}\right)\, e^{i(q/\hbar c)\eta}}_{\text{product} \equiv 1}\left[\frac{\partial}{\partial x_\mu} + \cancel{\frac{iq}{\hbar c}\left(\frac{\partial\eta}{\partial x_\mu}\right)} - \frac{iq}{\hbar c}\left(A_\mu + \cancel{\frac{\partial\eta}{\partial x_\mu}}\right)\right]\psi$$

cancel

$$= \psi^*\left(\frac{\partial}{\partial x_\mu} - \frac{iq}{\hbar c}A_\mu\right)\psi \longleftarrow \text{this quantity is } \underline{\text{invariant}} \text{ under gauge transf}^{\text{n}}. \tag{4}$$

From this, it is evident that <u>S_μ of Eq. (1) is gauge invariant</u>, since the C.C. transforms the same way: $S_\mu \rightarrow S'_\mu \equiv S_\mu$, under the gauge transform of Eq. (2).

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2. To establish gauge covariance for the KG Eqtn in A_μ, i.e. for

(B) $$\rightarrow \left[\left(\frac{\partial}{\partial x_\mu} - \frac{iq}{\hbar c}A_\mu\right)^2 - k_0^2\right]\psi = 0, \tag{5}$$

we need to show that the same <u>form</u> of the eqtn holds when $(A_\mu, \psi) \rightarrow (A'_\mu, \psi')$ via Eq. (2). That is, we must show Eq. (5) also holds with A_μ replaced by A'_μ and ψ replaced by ψ', w/o any additional terms.

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3. We already know from Eq. (3) that...

$$\left(\frac{\partial}{\partial x_\mu} - \frac{iq}{\hbar c}A'_\mu\right)\psi' = e^{i(q/\hbar c)\eta}\left(\frac{\partial}{\partial x_\mu} - \frac{iq}{\hbar c}A_\mu\right)\psi. \tag{6}$$

Applying the LHS operator a second time, we find...

$$\left(\frac{\partial}{\partial x_\mu} - \frac{iq}{\hbar c}A'_\mu\right)^2\psi' = \left[\frac{\partial}{\partial x_\mu} - \frac{iq}{\hbar c}\left(A_\mu + \frac{\partial\eta}{\partial x_\mu}\right)\right]\left\{e^{i(q/\hbar c)\eta}\left(\frac{\partial}{\partial x_\mu} - \frac{iq}{\hbar c}A_\mu\right)\psi\right\}. \quad (7)$$

... the $\frac{\partial}{\partial x_\mu}$ here operates on each of the three factors in the { }, so we get...

$$\left(\frac{\partial}{\partial x_\mu} - \frac{iq}{\hbar c}A'_\mu\right)^2\psi' = e^{i(q/\hbar c)\eta}\cancel{\left[\left(\frac{iq}{\hbar c}\right)\frac{\partial\eta}{\partial x_\mu}\right]\left(\frac{\partial}{\partial x_\mu} - \frac{iq}{\hbar c}A_\mu\right)}\psi + \quad (8)$$

term ①

$$+\ e^{i(q/\hbar c)\eta}\frac{\partial}{\partial x_\mu}\left(\frac{\partial}{\partial x_\mu} - \frac{iq}{\hbar c}A_\mu\right)\psi - e^{i(q/\hbar c)\eta}\frac{iq}{\hbar c}\left(A_\mu + \cancel{\frac{\partial\eta}{\partial x_\mu}}\right)\left(\frac{\partial}{\partial x_\mu} - \frac{iq}{\hbar c}A_\mu\right)\psi$$

② ③ ④

There is no need to perform the differentiation in term ②. Note that term ① is cancelled altogether by term ④. Then, by combining terms ② & ③, we have...

$$\left(\frac{\partial}{\partial x_\mu} - \frac{iq}{\hbar c}A'_\mu\right)^2\psi' = e^{i(q/\hbar c)\eta}\left[\left(\frac{\partial}{\partial x_\mu} - \frac{iq}{\hbar c}A_\mu\right)\left(\frac{\partial}{\partial x_\mu} - \frac{iq}{\hbar c}A_\mu\right)\psi\right] = e^{i(q/\hbar c)\eta}\left(\frac{\partial}{\partial x_\mu} - \frac{iq}{\hbar c}A_\mu\right)^2\psi. \quad (9)$$

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4. On the basis of Eq. (9), we can say the gauge-transformed LHS of Eq. (5) will look like... [note: $k_0^2\psi' = e^{i(q/\hbar c)\eta}k_0^2\psi$]...

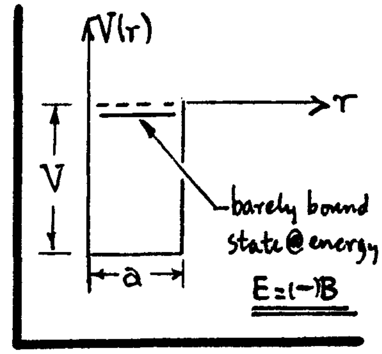
$$\left[\left(\frac{\partial}{\partial x_\mu} - \frac{iq}{\hbar c}A'_\mu\right)^2 - k_0^2\right]\psi' = e^{i(q/\hbar c)\eta}\underbrace{\left[\left(\frac{\partial}{\partial x_\mu} - \frac{iq}{\hbar c}A_\mu\right)^2 - k_0^2\right]\psi}_{0} = 0. \quad (10)$$

The indicated [ ]$\psi = 0$ because of (5), i.e. that is the original (untransformed) KG Eqtn. So we have shown gauge covariance of the KG Eqtn, i.e. if it is true that $[(\partial/\partial x_\mu - i(q/\hbar c)A_\mu)^2 - k_0^2]\psi = 0$, then also $[(\partial/\partial x_\mu - i(q/\hbar c)A'_\mu)^2 - k_0^2]\psi' = 0$.

–QED–

(18) [15 pts]. Particle in a 3D spherical well $[V, a]$: barely bound in a KG S-state.

1. The spherical potential well is...



$$V(r) = \begin{cases} -V, \text{ cnst, for } 0 \leq r \leq a; \\ 0, \text{ for } r > a. \end{cases} \quad (1)$$

... and we seek a barely bound S-state ($l=0$) @ energy $E = (-)B \to 0$ near the top of the well. If the radial wavefcn is written as $\psi(r) = \frac{1}{r} R(r)$, then the radial wave eqtns are (for $l=0$)...

$$\left[\begin{array}{ll} \underline{\text{SCHRÖDINGER}}: & \frac{d^2R}{dr^2} + \frac{2m}{\hbar^2}[E - V(r)]\, R = 0, \\ \underline{\text{KLEIN-GORDON}}: & \frac{d^2R}{dr^2} + \frac{2m}{\hbar^2}[E - V(r)]\left\{1 + \left(\frac{E - V(r)}{2mc^2}\right)\right\} R = 0. \end{array}\right\} \quad (2)$$

Here E is the conventional eigen-energy of the problem (i.e. E = total energy $- mc^2$). The KG Eqtn has the characteristic relativistic correction $(E-V)/2mc^2$ in the $\{\ \}$, and in the limit $c \to \infty$, the KG Eqtn reduces to the Schrödinger form.

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2. We are looking for a bound state w/ $E = (-)B$, and in Eqs (2): $V(r) = -V$, @ $r \leq a$. Then the eqtns are of the form...

$\underline{0 \leq r \leq a}$: $\underline{\frac{d^2R}{dr^2} + \alpha^2 R = 0}$, w/ $\alpha = \alpha_S = \left[\frac{2m}{\hbar^2}(V - B)\right]^{1/2}$, for S. Eq.; (3)

$$\alpha = \alpha_K = \left[\frac{2m}{\hbar^2}(V - B)\left\{1 + \frac{V - B}{2mc^2}\right\}\right]^{1/2}, \text{ for KG Eq.}$$

$\underline{r \geq a}$ $\underline{\frac{d^2R}{dr^2} - \beta^2 R = 0}$, w/ $\beta = \beta_S = [2mB/\hbar^2]^{1/2}$, for S. Eq.; (4)

$$\beta = \beta_K = \left[\frac{2mB}{\hbar^2}\left\{1 - \frac{B}{2mc^2}\right\}\right]^{1/2}, \text{ for KG Eq.}$$

In both cases, the acceptable solutions (w/ ψ finite @ $r=0$ and $\psi \to 0$ as $r \to \infty$) are

$\rightarrow$ $0 \leq r \leq a$: $R(r) = A \sin \alpha r$; $\quad r \geq a$: $R(r) = C e^{-\beta r}$. (5)

A & C are cnsts in Eq. (5). We impose the condition that R & dR/dr be continuous at r = a ... this gives a quantization condition on the energies B, as...

$$\rightarrow \lim_{r\to a-}\left[\frac{1}{R}\left(\frac{dR}{dr}\right)\right] = \lim_{r\to a+}\left[\frac{1}{R}\left(\frac{dR}{dr}\right)\right] \Rightarrow \boxed{\alpha \operatorname{ctn} \alpha a = (-)\beta}. \qquad (6)$$

Solutions to this (transcendental) equation -- for the α & β assignments in Eqs. (3) & (4) -- will specify discrete values of the binding B in terms of well parameters.

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3. For a barely bound particle, $B \to 0$, and in Eq. (6) $\beta \to 0$ for both the S. Eq. and KG Eq. [ref. Eq. (4)]. So we want...

$$\left[\begin{matrix}\text{barely bound}\\ \text{state: } B\to 0\end{matrix}\right\} \;\beta \to 0 \Rightarrow \operatorname{ctn}\alpha a \to 0, \text{ and: } \underline{\underline{\alpha a \to \pi/2}}. \qquad (7)$$

With the α-values in Eq. (2), this condition gives the minimum well depth V which will just barely bind the particle in an S-state. For the two cases...

$$\underline{\underline{\text{S. Eq.}}}: \;\alpha_s a = \left[\frac{2m}{\hbar^2}(V_s - \cancelto{0}{B})\right]^{1/2} a = \frac{\pi}{2} \Rightarrow \boxed{V_s = \pi^2\hbar^2/8ma^2}. \qquad (8)$$

$$\underline{\underline{\text{KG Eq.}}}: \;\alpha_K a = \left[\frac{2m}{\hbar^2}(V_K - \cancelto{0}{B})\left\{1 + \frac{V_K - \cancelto{0}{B}}{2mc^2}\right\}\right]^{1/2} = \frac{\pi}{2} \Rightarrow V_K\left\{1 + \frac{V_K}{2mc^2}\right\} = V_s$$

$$\text{or}/\!/ \;\boxed{V_K = mc^2\left\{\left[1 + \frac{2V_s}{mc^2}\right]^{1/2} - 1\right\} \overset{V_s \ll mc^2}{\simeq} V_s\left[1 - \tfrac{1}{2}(V_s/mc^2)\right]}.\;¶ \qquad (9)$$

We see that the min. KG binding $V_K < V_s$ in the nonrelativistic limit. In fact $V_K < V_s$ at all finite values of mc^2. In effect, relativistic effects reduce the Schrödinger value $V_s = \pi^2\hbar^2/8ma^2$. A way to think about this is that V_s scales as $1/m$, so heavier particles are easier to bind (for a given size a). And, we know that a relativistic particle shows: $\vec{p} = m\vec{v} \to \left(m/\sqrt{1-(v/c)^2}\right)\vec{v}$ for its momentum... in effect its mass increases: $m \to m/\sqrt{1-(v/c)^2}$. So, expect V_s decreases due to relativistic corrections, and thus $V_K < V_s$.

¶ The nonrelativistic approxn is reasonable since $V_s \sim mc^2$ only if the well size $a \sim \hbar/mc$, the particle's Compton wavelength. Wells this small rarely occur in nature.

ϕ507 Solutions

⑲ [15pts]. KG Eqtn: two-component formulation; identification of charge density.

1. The free-particle KG Eqtn can be written as

$$\rightarrow c^2\left[\vec{p}^{\,2}+(mc)^2\right]\psi=\left(i\hbar\frac{\partial}{\partial t}\right)^2\psi, \text{ w/ } \vec{p}=-i\hbar\vec{\nabla}. \qquad (1)$$

If we define a new fcn ξ by: $(mc^2)\xi=i\hbar\,\partial\psi/\partial t$, then Eq. (1) becomes

$$\rightarrow \frac{1}{m}\left[\vec{p}^{\,2}+(mc)^2\right]\psi=i\hbar\frac{\partial\xi}{\partial t}. \qquad (2)$$

We have therefore split the 2nd order KG Eqtn into two 1st order pieces, viz.

$$\rightarrow i\hbar\frac{\partial\psi}{\partial t}=(mc^2)\xi, \text{ and } i\hbar\frac{\partial\xi}{\partial t}=(mc^2)\psi+(\vec{p}^{\,2}/m)\psi. \qquad (3)$$

These can already be written in a Hamiltonian form, as...

$$\left.\begin{array}{l} \text{If: } \Phi=\begin{pmatrix}\psi\\ \xi\end{pmatrix}, \text{ then: } i\hbar\,\partial\Phi/\partial t=K\Phi, \\ \text{where: } K=\begin{pmatrix}0&1\\1&0\end{pmatrix}mc^2+\begin{pmatrix}0&0\\1&0\end{pmatrix}\frac{\vec{p}^{\,2}}{m}. \end{array}\right\} \qquad (4)$$

K is not Hermitian, since $K^\dagger=\begin{pmatrix}0&1\\1&0\end{pmatrix}mc^2+\begin{pmatrix}0&1\\0&0\end{pmatrix}\frac{\vec{p}^{\,2}}{m}\neq K$.

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2. We transform Φ in Eq. (4) to a new wavefcn Ψ, as defined by...

$$\left.\begin{array}{l} \Psi=\begin{pmatrix}\psi_+\\ \psi_-\end{pmatrix}=\frac{1}{2}\begin{pmatrix}\psi+\xi\\ \psi-\xi\end{pmatrix}=\frac{1}{2}\begin{pmatrix}1&1\\1&-1\end{pmatrix}\begin{pmatrix}\psi\\ \xi\end{pmatrix}; \\ \text{i.e. } \Psi=U\Phi, \text{ with: } U=\frac{1}{2}\begin{pmatrix}1&1\\1&-1\end{pmatrix}. \end{array}\right\} \qquad (5)$$

It is easy to show that the inverse of U is

$$U^{-1}=\begin{pmatrix}1&1\\1&-1\end{pmatrix}, \text{ with: } U^{-1}U=UU^{-1}=\begin{pmatrix}1&0\\0&1\end{pmatrix}. \qquad (6)$$

We see that $U^{-1}=2U^\dagger$, so U is quasi-unitary. In any case, with U in hand, we can easily transform Eq. (4) to the Ψ representation.

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3. Multiply through Eq. (4) on the left by U. Then...

$$\left[\begin{array}{l} i\hbar \frac{\partial}{\partial t} U\Phi = UK\Phi = (UKU^{-1})U\Phi, \\ \text{or } \underline{\underline{i\hbar\,\partial\Psi/\partial t = \mathcal{H}\Psi}}, \text{ w/ } \Psi = U\Phi \text{ \& } \mathcal{H} = UKU^{-1}. \end{array}\right. \tag{7}$$

This is the desired Hamiltonian form for $\Psi = \binom{\psi_+}{\psi_-}$, with $\mathcal{H}$ given by

$$\mathcal{H} = \frac{1}{2}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\left\{\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}mc^2 + \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}\frac{\vec{p}^{\,2}}{m}\right\}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}$$

or $$\boxed{\mathcal{H} = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}mc^2 + \begin{pmatrix}1 & 1\\ -1 & -1\end{pmatrix}\frac{\vec{p}^{\,2}}{2m}}, \text{ as required.} \tag{8}$$

Evidently $\mathcal{H}$ is not Hermitian, since $\mathcal{H}^\dagger = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}mc^2 + \begin{pmatrix}1 & -1\\ 1 & -1\end{pmatrix}\frac{\vec{p}^{\,2}}{2m} \neq \mathcal{H}$. When the Newtonian K.E. $\vec{p}^{\,2}/2m \ll mc^2$, however, the eigenvalues of $\mathcal{H}$ are the (+)ve and (−)ve energy solutions $\pm mc^2$, with associated eigenfcns $\psi_\pm$.¶

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4. The main utility in this formulation of the KG Eqtn is to provide a rational interpretation of the probability density $\rho = -(\hbar/mc^2)\,\mathrm{Im}[\psi^*(\partial\psi/\partial t)]$. Multiply by the charge q of m (q could be (±)ve, or zero), so we are looking at...

$$\rightarrow \tilde{\rho} = -(q\hbar/mc^2)\,\mathrm{Im}[\psi^*(\partial\psi/\partial t)]. \tag{9}$$

Our transform has been to: $\psi_\pm = \frac{1}{2}(\psi \pm \xi)$, so: $\psi = \psi_+ + \psi_-$, and: $\xi = \psi_+ - \psi_- = (i\hbar/mc^2)\,\partial\psi/\partial t$. The charge density in Eq. (9) is therefore...

$$\tilde{\rho} = -\left(\frac{q\hbar}{mc^2}\right)\mathrm{Im}\left[(\psi_+ + \psi_-)^*\,\frac{mc^2}{i\hbar}(\psi_+ - \psi_-)\right] = q\,\mathrm{Im}\left[i(\psi_+^* + \psi_-^*)(\psi_+ - \psi_-)\right]$$

$$= q\,\mathrm{Im}\Big[i\,\underbrace{(|\psi_+|^2 - |\psi_-|^2)}_{1} - i\,\underbrace{(\psi_+^*\psi_- - \psi_+\psi_-^*)}_{2}\Big]$$

term 1 is pure real and contributes; term 2 is pure imaginary and does not contribute

i.e. $$\boxed{\tilde{\rho} = q\,(|\psi_+|^2 - |\psi_-|^2)}, \text{ as required.} \tag{10}$$

The (+)ve energy solutions (ψ_+ dominant) show $\tilde{\rho} = +q$; the (−)ve energy solutions have $\tilde{\rho} = (-)q$. These are interpreted in Dirac theory as particle–antiparticle modes.

¶ If $\vec{p}$ is not an operator, eigenvalues of $\mathcal{H}$ of Eq. are $\pm\sqrt{(mc^2)^2 + (\vec{p}c)^2}$.