Magnetostatics (cont'd)

3. In MKS units, choose $k = \mu_0/4\pi$ rather than $k = 1/c$ (CGS), so the MKS formulas for B are gotten by replacing $1/c$ by $\mu_0/4\pi$, e.g.:
$B\left(\substack{\text{long}\\\text{wire}}\right) = \frac{1}{c}(2I/r)$ [CGS] $\rightarrow \frac{\mu_0}{4\pi}(2I/r) = \mu_0 I/2\pi r$ [MKS].

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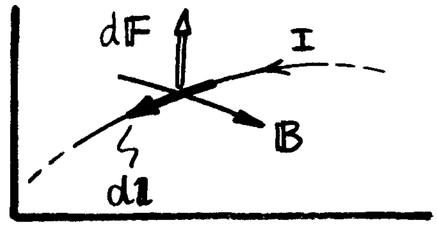
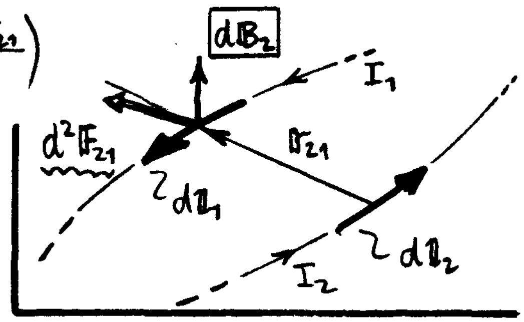
3) Eq. (11) [~ Ampere's Law] shows how a source $I\,d\mathbb{l}$ generates a magnetic field $d\mathbb{B}$, but we still need to know how $I\,d\mathbb{l}$ couples to an already existing field $\mathbb{B}$. Answer is...



$$\boxed{d\mathbb{F} = \frac{1}{c} I\,(d\mathbb{l} \times \mathbb{B})} \quad [\text{Lorentz' Law}]. \tag{13}$$

If I is due to the motion of a single charge Δq, then $I\,d\mathbb{l} = (\Delta q)\mathbb{v}$, and: $\Delta\mathbb{F} = (\Delta q/c)\,\mathbb{v}\times\mathbb{B}$, which is Lorentz' Law.

Now, consider the magnetic interaction between two elemental sources $I_1\,d\mathbb{l}_1$ & $I_2\,d\mathbb{l}_2$...



$$d^2\mathbb{F}_{2\,on\,1} = \frac{1}{c} I_1 (d\mathbb{l}_1 \times d\mathbb{B}_2), \qquad d\mathbb{B}_2 = \frac{I_2}{c}\left(\frac{d\mathbb{l}_2 \times \mathbb{r}_{21}}{r_{21}^3}\right)$$

i.e.// $$d^2\mathbb{F}_{21} = \frac{I_1 I_2}{c^2}\left[\frac{d\mathbb{l}_1 \times (d\mathbb{l}_2 \times \mathbb{r}_{21})}{r_{21}^3}\right]$$

$$\xrightarrow{\text{or}//} d^2\mathbb{F}_{21} = \frac{I_1 I_2}{c^2 r_{21}^2}\left[(d\mathbb{l}_1\cdot\hat{r}_{21})\,d\mathbb{l}_2 - (d\mathbb{l}_1\cdot d\mathbb{l}_2)\,\hat{r}_{21}\right]. \tag{14}$$

This is the force by $I_2\,d\mathbb{l}_2$ on $I_1\,d\mathbb{l}_1$. Reversing the roles...

$$d^2\mathbb{F}_{12} = \frac{I_2 I_1}{c^2 r_{12}^2}\left[(d\mathbb{l}_2\cdot\hat{r}_{12})\,d\mathbb{l}_1 - (d\mathbb{l}_2\cdot d\mathbb{l}_1)\,\hat{r}_{12}\right] \quad \hat{r}_{12} = (-)\,\hat{r}_{21},\; r_{12} = r_{21}$$

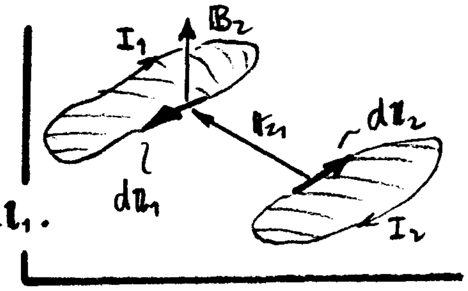
So// $$\left[\!\left[\, d^2\mathbb{F}_{12} + d^2\mathbb{F}_{21} = \frac{I_1 I_2}{c^2 r_{21}^2}\left[(d\mathbb{l}_1\cdot\hat{r}_{21})\,d\mathbb{l}_2 - (d\mathbb{l}_2\cdot\hat{r}_{21})\,d\mathbb{l}_1\right] \neq 0.\right]\!\right] \tag{15}$$

A seeming Disaster... by Newton III, should have: $d^2\mathbb{F}_{12} + d^2\mathbb{F}_{21} \equiv 0$. What has been left out is that both $I_k\,d\mathbb{l}_k$ are parts of current loops.

Magnetostatics (cont'd)

1) What must be done to recover from this disaster is to include the loops...

$$d\mathbb{F}_{21} = \frac{1}{c} I_1 (d\mathbb{l}_1 \times \mathbb{B}_2), \qquad (16)$$



w/ $\mathbb{B}_2 = \frac{I_2}{c} \oint_{\text{loop 2}} \frac{d\mathbb{l}_2 \times \hat{r}_{21}}{r_{21}^2}$ $\int$ entire field due to loop #2 at site of $I_1\, d\mathbb{l}_1$.

Entire force by loop #2 on loop #1 $\Big\}$ $\mathbb{F}_{2\,on\,1} = \oint_{\text{loop 1}} d\mathbb{F}_{21} = \frac{I_1 I_2}{c^2} \oint_{\#1} \oint_{\#2} \frac{1}{r_{21}^2}\, d\mathbb{l}_1 \times (d\mathbb{l}_2 \times \hat{r}_{21})$

or// $$\mathbb{F}_{2\,on\,1} = \frac{I_1 I_2}{c^2} \oint_{\#1} \oint_{\#2} \frac{1}{r_{21}^2} \Big[\underbrace{(d\mathbb{l}_1 \cdot \hat{r}_{21})\, d\mathbb{l}_2}_{\text{Contributes zero on integration}} - (d\mathbb{l}_1 \cdot d\mathbb{l}_2)\, \hat{r}_{21}\Big]. \qquad (17)$$

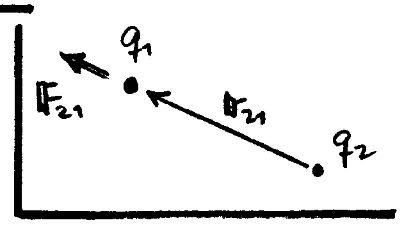
The indicated term contributes zero because it is $\propto \oint_{\#2} d\mathbb{l}_2 \oint_{\#1} d\mathbb{l}_1 \cdot \frac{\mathbb{r}_{21}}{r_{21}^3}$, and the $\oint_{\#1} \propto \oint_{\text{loop 1}} d\mathbb{r} \cdot \frac{\mathbb{r}}{r^3} = \frac{1}{2} \oint_{\#1} \frac{dr}{r^2} = -\frac{1}{2} \oint_{\#1} d(1/r) \equiv 0$. The total force law is then...

$$\boxed{\mathbb{F}_{2\,on\,1} = (-)\, \frac{I_1 I_2}{c^2} \oint_{\#1} \oint_{\#2} \frac{1}{r_{21}^2} (d\mathbb{l}_1 \cdot d\mathbb{l}_2)\, \hat{r}_{21}} \quad [\text{Jk}^{\text{n}}\ \text{Eq. (5.10)}]. \qquad (18)$$

Now, clearly, $\mathbb{F}_{1\,on\,2} = (-)\, \mathbb{F}_{2\,on\,1}$, and Newton III is obeyed, as must be.

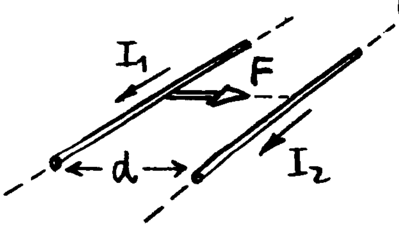
Contrast $\mathbb{F}_{2\,on\,1}$ (magnetic) with $\mathbb{F}_{2\,on\,1}$ (electric)...

$\mathbb{F}_{2\,on\,1} = q_1 \mathbb{E}_2$, $\mathbb{E}_2 = q_2 \hat{r}_{21} / r_{21}^2$



so// $$\left[\mathbb{F}_{2\,on\,1}(\text{electric}) = q_1 q_2 \frac{\hat{r}_{21}}{r_{21}^2}\right] \qquad (19)$$

The sources of $\mathbb{B}$, refusing to be monopoles, really complicate the force law. About the only simple application of Eq. (18) is to two $\parallel$ long wires...



force/unit length on I_1 by I_2... $\Big\}$ $$\boxed{\frac{dF}{dl} = 2 I_1 I_2 / c^2 d} \qquad (20)$$

$\mathbb{F}$ is attractive if I_1 & I_2 flow $\parallel$, repulsive if I_1 & I_2 anti-$\parallel$.