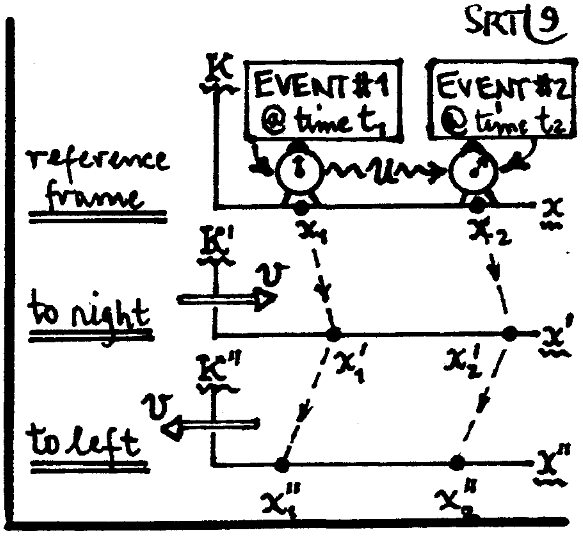
SRT Introd^n Link: Causality ↔ c, as limit.



5) We have just claimed that "causality" is linked with c as a limiting signal velocity. The following example shows why. Consider events #1 & #2 that occur in reference frame K at times t_1 & t_2. The events are observed also by passing frames K' (to right @ v) & K" (to left @ v). The observed times between events are...

$$\left[\begin{array}{l} \text{for } K: (t_2 - t_1) = \Delta t; \\ \text{for } K': (t_2' - t_1') = \Delta t' = \gamma\left[\Delta t - \frac{v}{c^2}(x_2 - x_1)\right]; \\ \text{for } K'': (t_2'' - t_1'') = \Delta t'' = \gamma\left[\Delta t + \frac{v}{c^2}(x_2 - x_1)\right]. \end{array}\right. \quad \left(\begin{array}{c} \gamma = \frac{1}{\sqrt{1 - v^2/c^2}} \\ \text{for both } K' \\ \text{and } K'' \end{array}\right) \qquad (23)$$

signs

Suppose the events are simultaneous for K, i.e. $\Delta t = 0$. Other observers see:

$$\rightarrow \Delta t = 0 \text{ in } K \Rightarrow \begin{cases} \Delta t' = (-)\frac{\gamma v}{c^2}\Delta x < 0, \text{ in } K' \int \text{event \#2 occurred } \underline{\text{before}} \text{ \#1}; \\ \Delta t'' = (+)\frac{\gamma v}{c^2}\Delta x > 0, \text{ in } K'' \int \text{event \#2 occurred } \underline{\text{after}} \text{ \#1}. \end{cases} \qquad (24)$$

Here $\Delta x = x_2 - x_1$ is the spatial separation between the event locations in K. Note that neither K' nor K" agree on simultaneity in K, and -- because their motions v are oppositely directed -- they don't even agree on the sequence of the events.

Now, suppose K claims event #1 caused #2. This makes sense to K", since he sees #2 after #1. But in K', we see #2 before #1, and the time-ordering for cause → effect is violated. Clearly, to preserve causality for all three observers [i.e. that they all agree event #1 (cause) precedes event #2 (effect)], we must look more carefully at the sequencing #1 ⇒ #2.

(next page)

Causality. Def$^{\underline{n}}$ of Proper Time

If K claims #1 <u>causes</u> #2, he must send a <u>signal</u> -- at some velocity u -- from position x_1 to x_2 in order to preserve the cause-effect time ordering, so...

$$\xrightarrow{\text{Say}} \Delta t = t_2(\text{effect}) - t_1(\text{cause}) = \frac{x_2 - x_1}{u} > 0, \text{ in K } \int \begin{matrix} u = \text{signal velocity} \\ \text{for K(cause} \rightarrow \text{effect)}. \end{matrix} \quad (25)$$

u(signal) is so far unknown. In K″, $\Delta t'' > 0$ (per Eq.(23)), so cause → effect is preserved. Now, with $\Delta t > 0$, K′ sees a new temporal ordering...

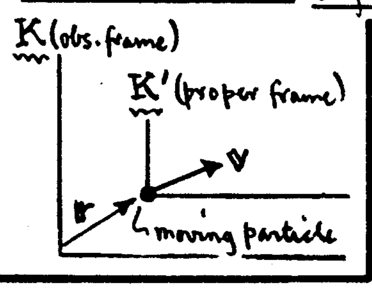
$$\Delta t' = \gamma\left[\Delta t - \frac{v}{c^2}\Delta x\right] = \gamma\left(\frac{c}{u} - \frac{v}{c}\right)\frac{\Delta x}{c},$$

$$\text{Say} \left.\begin{matrix} \text{cause} \rightarrow \text{effect ordering} \\ \text{restored in K}' \text{ if } \Delta t' > 0 \end{matrix}\right\} \Rightarrow \boxed{\left(\frac{c}{u} - \frac{v}{c}\right) > 0}. \quad (26)$$

The signal velocity u (in K) must satisfy this inequality in order that both K′ & K″ agree on the cause → effect time-ordering. But $v/c < 1$ for all physical observers¶, and so: <u>$c/u > v/c$, for all $v/c < 1 \Rightarrow c/u \geq 1$</u>. Thus we claim...

> To preserve causality in all inertial frames (in relative motion @ $v < c$), no information-carrying signal can be transmitted at speed $u > c$. Moreover, c = limit speed for all signals that can causally link events. (27)

AFTERTHOUGHT <u>Def$^{\underline{n}}$ of particle "proper time" via invariant spacetime interval</u>:



$$(ds)^2 = \underbrace{(c\,dt)^2 - (\mathbf{v}\,dt)^2}_{\text{movement in K}} = \underbrace{(c\,d\tau)^2 - (0)^2}_{\text{no movement in K}'} \quad \left(\begin{matrix} \text{K}' \text{ is attached to} \\ \text{particle; it is the} \\ \text{particle "rest-frame"} \end{matrix}\right)$$

$$\Rightarrow \left.\begin{matrix} \text{"proper time" in} \\ \text{particle rest frame} \end{matrix}\right\} \boxed{d\tau = dt\sqrt{1-\beta^2}}, \quad \beta = |\mathbf{v}|/c. \quad (28)$$

While the observer time dt changes with the relative motion, the proper time $d\tau$ is a Lorentz invariant -- it cannot change size when viewed from any inertial frame. It is also the <u>least</u> time, since: $t_2 - t_1 = \int_{t_1}^{t_2} dt = \int_{\tau_1}^{\tau_2} d\tau / \sqrt{1-\beta^2(\tau)} > (\tau_2 - \tau_1)$.

¶ An upcoming problem... no physical frame can be accelerated to $v = c$.