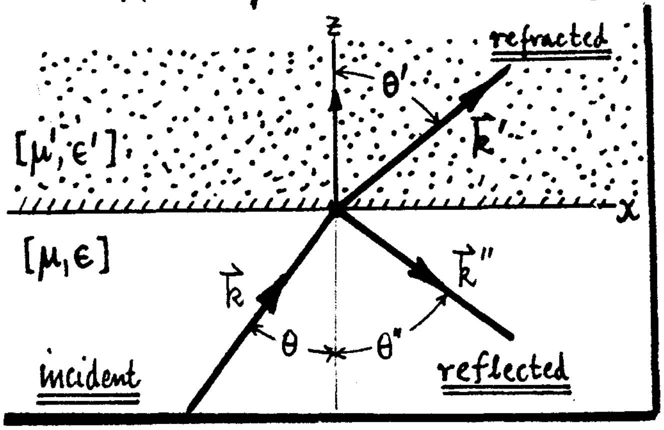
Reflection & Refraction at a Plane Boundary

Snell's Law & Fresnel Formulas

8) We <u>summarize</u> Jackson's account in Sec.(7.3) of the reflection and refraction of EM planewaves at a (plane) boundary between different dielectrics.



This exercise gives the basic laws of optics, viz. Snell's Law for <u>refraction</u>, plus the Fresnel Formulas for <u>reflection</u> & <u>transmission</u>.

Start by assuming you know almost nothing -- e.g. you <u>don't</u> even know the reflection ∡ θ'' & incident ∡ θ are equal. But <u>do</u> know:

$$|\mathbf{k}| = |\mathbf{k}''| = k = \frac{\omega}{c}\sqrt{\mu\epsilon}\ \left\{\begin{array}{l}\text{inc. \& refl.}\\ \text{waves are in}\\ \text{same medium}\end{array}\right. ;\quad |\mathbf{k}'| = k' = \frac{\omega}{c}\sqrt{\mu'\epsilon'} \neq k\ \left\{\begin{array}{l}\text{refracted wave}\\ \text{in new medium}\end{array}\right. ;$$

and,

$$\mathbb{E}(\text{incident}) = E_0\, e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)},\quad \mathbb{B}(\text{incident}) = \sqrt{\mu\epsilon}\ \hat{k}\times\mathbb{E}(\text{incident}). \tag{27}$$

There are similar expressions for $\mathbb{E}$ & $\mathbb{B}$ for the reflected & refracted waves, with appropriate changes $\mathbf{k}\to\mathbf{k}'$ & $(\mu,\epsilon)\to(\mu',\epsilon')$. The first important boundary condition is that at the plane interface, $z=0$ in above sketch, the <u>phases</u> of all the waves are the same, i.e. (at a fixed time -- say $t=0$):

$$⟨1⟩\ \left.\begin{array}{l}\text{at } z=0\\ (\& \ t=0)\end{array}\right\}\ \mathbf{k}\cdot\mathbf{r}\,[\text{incident}] = \mathbf{k}'\cdot\mathbf{r}\,[\text{refracted}] = \mathbf{k}''\cdot\mathbf{r}\,[\text{reflected}]. \tag{28A}$$

If this were not so, there would be discontinuities in $\mathbb{E}$ & $\mathbb{B}$ at the interface. This boundary condition implies:

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1. $\mathbf{k}$, $\mathbf{k}'$ & $\mathbf{k}''$ all lie in the same plane. (28B)
2. $|\mathbf{k}|\sin\theta = |\mathbf{k}''|\sin\theta''$, or: $\underline{\theta'' = \theta}$, i.e. ∡[reflection] = ∡[incidence].
3. $|\mathbf{k}|\sin\theta = |\mathbf{k}'|\sin\theta'$, or: $\underline{n'\sin\theta' = n\sin\theta}$, w/ $n=\sqrt{\mu\epsilon}$. **SNELL'S LAW.**

$n=\sqrt{\mu\epsilon}$ is called the "index of refraction". We've gotten a lot of mileage from ⟨1⟩.

Refraction & Reflection: Fresnel Formulas.

9) The second important boundary condition concerns the continuity of the fields. Let $\hat{z}$ be a unit vector normal to the plane interface. Then...

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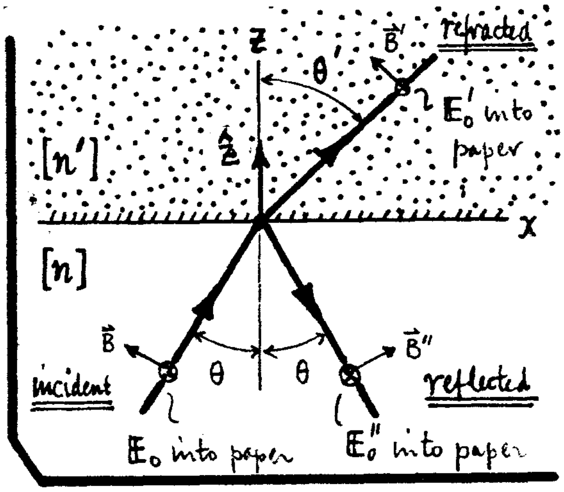
CONSERVED QTY	CONDITION IMPOSED
normal comp. of $\mathbb{D} = \epsilon\,\mathbb{E}$	$\epsilon(\mathbb{E}_0 + \mathbb{E}_0'') \cdot \hat{z} = (\epsilon' \mathbb{E}_0') \cdot \hat{z}$
normal comp. of $\mathbb{B}$	$(\mathbb{k} \times \mathbb{E}_0 + \mathbb{k}'' \times \mathbb{E}_0'') \cdot \hat{z} = (\mathbb{k}' \times \mathbb{E}_0') \cdot \hat{z}$
tangential comp. of $\mathbb{E}$	$(\mathbb{E}_0 + \mathbb{E}_0'') \times \hat{z} = \mathbb{E}_0' \times \hat{z}$
tangential comp. of $\mathbb{H} = \frac{1}{\mu}\mathbb{B}$	$\frac{1}{\mu}(\mathbb{k} \times \mathbb{E}_0 + \mathbb{k}'' \times \mathbb{E}_0'') \times \hat{z} = \frac{1}{\mu'}(\mathbb{k}' \times \mathbb{E}_0') \times \hat{z}$

(29)

These 8 eqtns, in the 9 unknowns (i.e. comps of $\mathbb{E}_0$, $\mathbb{E}_0'$ & $\mathbb{E}_0''$), allow calculation of the relative intensities of $\mathbb{E}_0$ [incident], $\mathbb{E}_0'$ [refracted), and $\mathbb{E}_0''$ [reflected]. The calculation is straightforward but algebraically cluttered.

For simplicity, consider two distinct polarizations...



[A] $\mathbb{E}$ is ⊥ plane of incidence:

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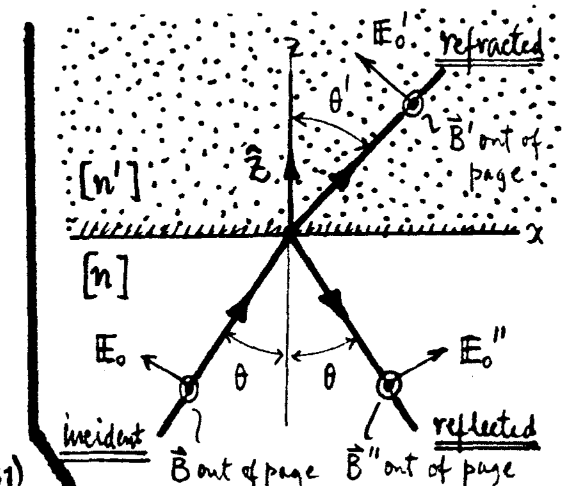
$$E_0'/E_0 = \frac{2n\cos\theta}{n\cos\theta + \frac{\mu}{\mu'}\sqrt{n'^2 - n^2\sin^2\theta}}$$

refracted

$$E_0''/E_0 = \frac{n\cos\theta - \frac{\mu}{\mu'}\sqrt{n'^2 - n^2\sin^2\theta}}{n\cos\theta + \frac{\mu}{\mu'}\sqrt{n'^2 - n^2\sin^2\theta}} \qquad (30)$$

reflected



[B] $\mathbb{E}$ is ∥ plane of incidence:

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$$E_0'/E_0 = \frac{2nn'\cos\theta}{\frac{\mu}{\mu'}n'^2\cos\theta + n\sqrt{n'^2 - n^2\sin^2\theta}}$$

refracted

$$E_0''/E_0 = \frac{\frac{\mu}{\mu'}n'^2\cos\theta - n\sqrt{n'^2 - n^2\sin^2\theta}}{\frac{\mu}{\mu'}n'^2\cos\theta + n\sqrt{n'^2 - n^2\sin^2\theta}} \qquad (31)$$

reflected

Details of Fresnel Formulas. Brewster's ∡.

Eqs. (30) & (31) are known as Fresnel Formulas for ⊥ & ∥ polarizations, resp. By combining $E_\perp$ & $E_\parallel$ results in appropriate ways, ratios E_0'/E_0 & E_0''/E_0 can be obtained for arbitrary input polarizations. NOTE: by use of Snell's Law: $\sqrt{n'^2 - n^2\sin^2\theta} = n'\cos\theta'$, can be used for the $\sqrt{\ }$ in (30) & (31).

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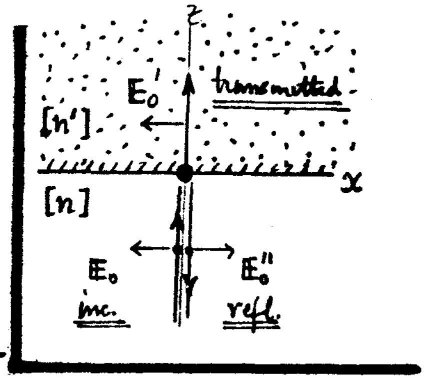
10) For normal incidence, $\theta = 0$, both above cases [A] & [B] reduce to:

$$\theta = 0 \Longrightarrow \left[\begin{array}{l} E_0'/E_0 = 2\Big/\left(1+\sqrt{\dfrac{\mu\epsilon'}{\mu'\epsilon}}\right) \rightarrow \dfrac{2n}{n'+n} \\[2ex] E_0''/E_0 = \dfrac{\sqrt{\mu\epsilon'/\mu'\epsilon}-1}{\sqrt{\mu\epsilon'/\mu'\epsilon}+1} \rightarrow \dfrac{n'-n}{n'+n} \end{array}\right. \text{ for } \mu' = \mu^{\star} \qquad (32)$$



NOTE: $E_0' + E_0'' = E_0$, so energy is conserved in the event.

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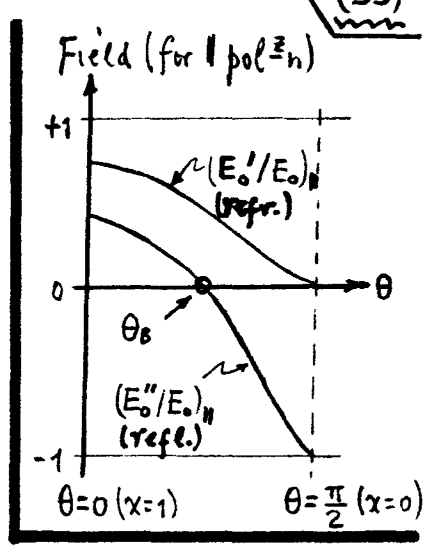
11) There is much detail contained in the Fresnel Formulas, Eqs. (30) & (31). Compactly:

Set: $\mu' = \mu$, $n'/n = r$, $\cos\theta = x$

POLARIZATION	(E_0'/E_0), for refracted ray	(E_0''/E_0), for reflected ray
$\mathbb{E}\perp$ plane of incidence	$2x\big/\left[x+\sqrt{(r^2-1)+x^2}\right]$	$\left(x-\sqrt{(r^2-1)+x^2}\right)\big/\left(x+\sqrt{(r^2-1)+x^2}\right)$
$\mathbb{E}\parallel$ plane of incidence	$2rx\big/\left[r^2x+\sqrt{(r^2-1)+x^2}\right]$	$\left(r^2x-\sqrt{(r^2-1)+x^2}\right)\big/\left(r^2x+\sqrt{(r^2-1)+x^2}\right)$

(33)

From these formulas, we can graph the fields vs. incident ∡ θ. An example is shown at right -- for ∥ polarization, and assumed ratio $r = n'/n > 1$. NOTE: the reflected wave amplitude E_0'' goes to zero at ∡ $\theta = \theta_B$ such that...



$$\theta = \theta_B \leftrightarrow E_0'' = 0 : \boxed{\theta_B = \tan^{-1}(n'/n)} \leftarrow \text{BREWSTER ANGLE} \qquad (34)$$

This ∡ of incidence is ~magic: if an incident wave of mixed polarization comes in at $\theta = \theta_B$, the reflected wave comes off w/ ⊥ polarization. This effect is used in lab to make polarized light.

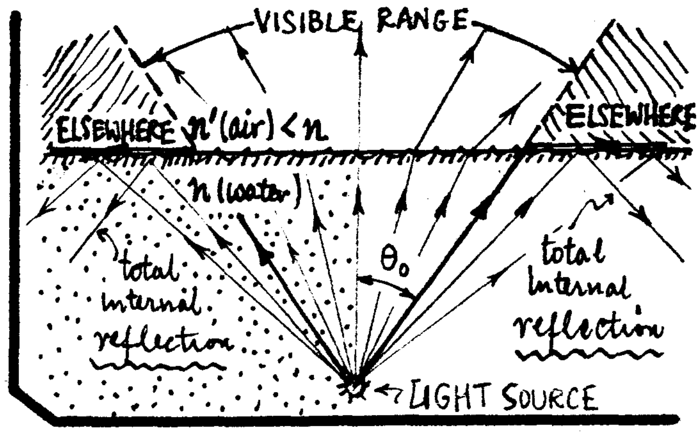
★ $\mu = 1 + 4\pi\chi$, w/ $\chi \sim$ few ppm $\begin{cases} (+) \text{ for paramagnetic matter,} \\ (-) \text{ for diamagnetic } \quad " \end{cases}$. $\chi \rightarrow$ large only for ferromagnets.

Total Internal Reflection

12) In the case where : $r = n'/n < 1$ (the wave is going from a denser medium n to a rarer medium n', e.g. from water to air), the $\sqrt{\ }$ appearing in Fresnel's Formulas [Eqs. (30) & (31)] goes as...

$$\sqrt{\ } = \underbrace{\sqrt{r^2 - \sin^2\theta}}_{\text{(use SNELL'S LAW)}} = r\cos\theta' \Longrightarrow \begin{cases} 0 \leq \theta \leq \theta_0 = \sin^{-1} r \longleftrightarrow 0 \leq \theta' \leq \frac{\pi}{2}; \\ \theta_0 < \theta \leq \frac{\pi}{2} \longleftrightarrow \theta' \text{ is } \underline{\text{imaginary}}. \end{cases} \quad (35)$$

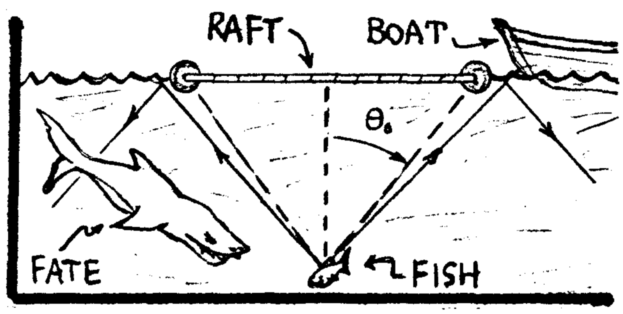
This mathematical oddity translates to the phenomenon of "total internal reflection", when a light ray propagates from a denser to a rarer medium. At incident ∡ $\theta < \theta_0$, the ray is transmitted, becoming more and more refracted as $\theta \to \theta_0$. When $\theta = \theta_0$, the transmitted ray travels along the interface. And for $\theta > \theta_0$, the ray is reflected and cannot be seen at all in the medium n'. What happens to the transmitted ray @ $\theta > \theta_0$ is:



$$E_0' \propto e^{ik'(x\sin\theta' + z\cos\theta')} = \underbrace{e^{-k'z\alpha}}\, e^{ik'x\sqrt{1+\alpha^2}}$$

$$\text{w/ } \alpha = \sqrt{(1/r^2)\sin^2\theta - 1} \quad \Big\| \; \alpha > 0 \text{ when } \theta > \theta_0 \; \Big\| \quad \uparrow \text{ the refracted wave is } \underline{\text{attenuated}} \text{ in } n'. \quad (36)$$

A clever fish can make use of this effect to hide from a fisherman's boat at (or above) a critical depth. Because of total internal reflection, the fisherman will never see the fish. But fate may intervene...



The Fresnel Formulas make clear the central importance of the index of refraction: $n = \sqrt{\mu\epsilon}$, in determining how an EM wave propagates in a medium. So far, we've treated n as a cnst, but now we will consider n to depend on frequency ω, thru the dielectric cnst ϵ, i.e. $\boxed{n(\omega) = \sqrt{\epsilon(\omega)}}$, for $\mu = 1$.