Electrostatic B.V. Problems II

Es gibt hier die Elementen von Kap. 3 aus Jackson.

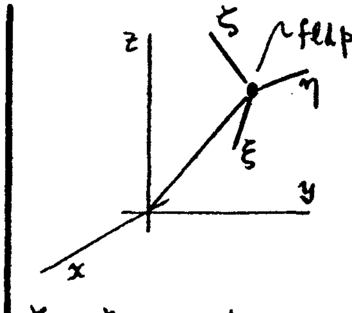
1) We have solved Laplace Eq. $\nabla^2\phi = 0$ (for the potential in a charge-free region) for several problems in rectangular symmetry. Now we shall solve the same eqtn, $\underline{\nabla^2\phi = 0}$, <u>for problems with spherical and cylindrical symmetry.</u>

The procedure, generally speaking, is the same as before -- we pick a coordinate system which matches the <u>symmetry</u> of the problem (i.e. conforms as much as possible to the given <u>shapes</u> of the conductors, charge distributions, etc.), then separate $\nabla^2\phi = 0$ into 3 ODE's with known solutions, i.e.,...

in cd. system (ξ,η,ζ) $\Big\}$ $\phi = U(\alpha,\xi)V(\beta,\eta)W(\gamma,\zeta)$,

and

$$\frac{1}{\phi}\nabla^2\phi = 0 \Rightarrow \text{ODE's}\begin{cases} \mathcal{D}_\xi U(\alpha,\xi) = 0, \\ \mathcal{D}_\eta V(\beta,\eta) = 0, \\ \mathcal{D}_\zeta W(\gamma,\zeta) = 0. \end{cases} \text{separation.} \quad (1)$$



ξ,η,ζ cds orthogonal

Separation $\Rightarrow$ $\gamma = f(\alpha,\beta)$; i.e. the sep^n cnsts not independent. By superposition, the general soln for ϕ can be formed as

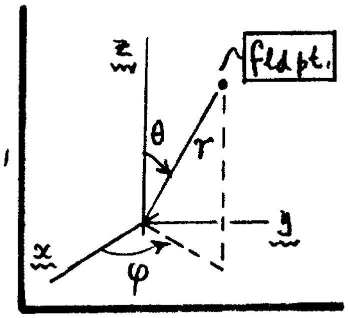
$$\longrightarrow \phi(\xi,\eta,\zeta) = \sum_{\alpha,\beta} C_{\alpha\beta}\, U(\alpha,\xi)V(\beta,\eta)W(\gamma,\zeta),\ \gamma = f(\alpha,\beta). \quad (2)$$

The cnsts α & β, and coefficients $C_{\alpha\beta}$ are used to fix B.C. $\left(\substack{\text{boundary}\\ \text{conditions}}\right)$.

The specific cd systems of interest here are **(1)** spherical polar (r,θ,φ), **(2)** cylindrical polar (ρ,φ,z). From the expressions for $\nabla^2_{sph.}$ & $\nabla^2_{cyl.}$ (inside back cover of text), you can expect the separation of $\nabla^2\phi = 0$ in these curvilinear cd systems will be <u>much</u> different than in rect^r cds.

2) $\nabla^2\phi = 0$ in spherical polar cds (r,θ,φ) ; Jackson Secs. (3.1)–(3.3).

1. $\phi(r,\theta,\varphi) = \frac{1}{r}U(r)P(\theta)Q(\varphi)$ into $\frac{1}{\phi}\nabla^2_{sph}\phi = 0$ separates into 3 ODE's. There are two indpt separation cnsts, conventionally written as m^2 & $l(l+1)$. The Q-eqtn is:



$$\rightarrow \frac{1}{Q}\left(\frac{d^2Q}{d\varphi^2}\right) = -m^2 \Rightarrow \boxed{Q(\varphi) \propto e^{\pm im\varphi}}, \text{ or } \begin{cases} \sin m\varphi \\ \cos m\varphi \end{cases}. \quad (3)$$

NOTE: if $Q(\varphi)$ is to be single-valued for an azimuthal rotation, i.e. $\varphi = 0 \rightarrow 2\pi$, then m must be integral ; $m = 0, \pm 1, \pm 2, \ldots$ This "quantization" of m is independent of any specific B.C. on ϕ.

The U-eqtn is also simple...

$$\rightarrow \frac{d^2U}{dr^2} - \left(\frac{l(l+1)}{r^2}\right)U = 0 \Rightarrow \boxed{\frac{1}{r}U(r) = Ar^l + Br^{-(l+1)}}. \quad (4)$$

with l still free. So far, so good. But the P-eqtn is not simple...

$$\rightarrow \frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dP}{d\theta}\right) + \left[l(l+1) - \frac{m^2}{\sin^2\theta}\right]P = 0 \Rightarrow P(\theta) = \text{what?} \quad (5)$$

So we have generated a non-trivial ODE by the separation in spherical cds.

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2. Eq. (5) is called Legendre's ODE. Change variables to $x = \cos\theta$; then

$$\rightarrow \frac{d}{dx}\left[(1-x^2)\frac{dP}{dx}\right] + \left[l(l+1) - \frac{m^2}{1-x^2}\right]P = 0; \quad (6)$$

So// Eqtn is Sturm-Liouville type, with $\begin{cases} p(x) = 1-x^2,\ q(x) = -\frac{m^2}{1-x^2}, \\ w(x) = 1, \text{ and } \lambda = l(l+1). \end{cases}$

From the general results of S-L theory, we immediately know that Legendre's Eqtn (on $|x| = |\cos\theta| \leq 1$) will generate an ∞ set of orthogonal solutions which can be labeled by the eigenvalues l & m, e.g. $P(l,m;x)$.

3. If the $\nabla^2\phi = 0$ problem has rotational symmetry (about z-axis), then there is no φ-dependence, and we can set $m=0$ in Eq. (6). Thus we consider:

$$\longrightarrow [(1-x^2)P']' + \ell(\ell+1)P = 0. \tag{7}$$

A power series soln can be constructed $\left[P(x) = x^\alpha \sum_{n=0}^{\infty} a_n x^n, \text{etc.}\right]$ which quickly shows: $P(x)$ converges on $x^2<1$ for any ℓ, BUT $P(x)$ diverges at $x^2=1$ unless ℓ = non-negative integer. So we choose $\ell = 0, 1, 2, \ldots$ for $P(x)$ convergent on the entire interval.† Then, can show following:

a) for $\ell = 0, 1, 2, \ldots$, the solutions $P_\ell(x)$ are the Legendre polynomials:

$$P_\ell(x) = \frac{1}{2^\ell \ell!}\left[\left(\frac{d}{dx}\right)^\ell (x^2-1)^\ell\right] \Rightarrow P_0(x)=1,\ P_1(x)=x,\ P_2(x)=\tfrac{1}{2}(3x^2-1),$$

etc [see Jkn Eq. (3.15)]; (7A)

NOTE: norm chosen here so that all $P_\ell(1)=1$ (at $\theta=0$).

b) the $P_\ell(x)$ are orthogonal on $x^2 \leq 1$:

$$\int_{-1}^{+1} P_\ell(x) P_{\ell'}(x)\, dx = \left(\frac{2}{2\ell+1}\right)\delta_{\ell\ell'}; \tag{7B}$$

c) the $\{P_\ell(x)\}$ are a complete set on $x^2 \leq 1$:

$$\text{if}/\!/\ f(x) = \sum_{\ell=0}^{\infty} A_\ell P_\ell(x), \quad \text{then}/\!/\ A_\ell = \frac{2\ell+1}{2}\int_{-1}^{+1} f(x) P_\ell(x)\, dx; \tag{7C}$$

d) Parity $\left(\begin{array}{c}\text{reflection thru origin:}\\ \theta \to \theta+\pi \Rightarrow x \to (-)x\end{array}\right)$: $P_\ell(-x) = (-)^\ell P_\ell(x)$. (7D)

e) Recurrence: $P'_{\ell+1} - P'_{\ell-1} = (2\ell+1)P_\ell$, etc. See Jkn Eq. (3.29). (7E)

Other properties of the $P_\ell(x)$ are listed in M. Abramowitz & I. A. Stegun "Handbook of Mathematical Functions" (NBS Series • 55), Chap. 8. There you will also find the other branch of Legendre fcns, $Q_\ell(x)$, which diverge at $x^2=1$.

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† By now, both the m & ℓ quantization are due to functional demands, not B.C.