Φ 519 MidTerm Exam (in class, 2 hr. limit) $\left\{ \begin{array}{l} \text{HI} = 83 \\ \text{LO} = 30 \\ \text{AVG} = 57 \\ \quad (67\%) \end{array} \right.$ Ths. 10 Nov. 1988

This exam is open-book, open-notes, and is worth 85 points total. For each problem, put your answer in a $\boxed{\text{box}}$ on your solution sheets. $\boxed{\text{Number}}$ your solution pages, put your $\boxed{\text{name}}$ on page 1, and $\boxed{\text{staple}}$ the pages together before handing them in.

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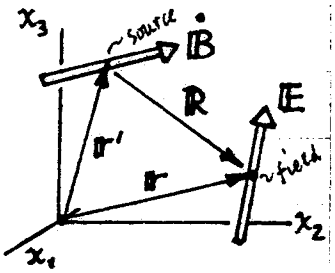
(1) [redacted]]. Consider a region of space where the magnetic field is changing at a rate $\dot{\mathbb{B}}(x'_\mu)$ [the dot $\Rightarrow \frac{\partial}{\partial t}$; $x'_\mu \Rightarrow$ space & time cds]. Show that the induced electric field is $\mathbb{E}(x_\mu) = \frac{1}{4\pi c}\int \frac{d\tau'}{R^3}\left[\mathbb{R}\times\dot{\mathbb{B}}(x'_\mu)\right]$. Here, $\mathbb{R} = \mathbb{r} - \mathbb{r}'$, per the figure.



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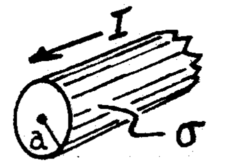
(2) [redacted]]. A long straight wire of radius a and uniform conductivity σ carries a steady current I. Find the magnitude and direction of $\mathbb{S}$, the Poynting vector, at the surface of the wire. Integrate the normal component of $\mathbb{S}$ over the surface of the wire for a segment of length l, and compare your result with the Joule heat produced in that segment. What is the origin of the energy flow represented by $\mathbb{S}$?



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(3) [redacted]]. For any vacuum electromagnetic field $(\mathbb{E}, \mathbb{B})$, verify the conservation law:

$$\boxed{\nabla\cdot\mathbb{P} + \frac{\partial\delta}{\partial t} = 0} \int^{\text{w/}} \begin{array}{l} \mathbb{P} = \mathbb{E}\times\dot{\mathbb{E}} + \mathbb{B}\times\dot{\mathbb{B}}, \\ \delta = \mathbb{E}\cdot(\nabla\times\mathbb{E}) + \mathbb{B}\cdot(\nabla\times\mathbb{B}). \end{array}$$

The dots $\Rightarrow \frac{\partial}{\partial t}$. Discuss this "continuity equation" for a linearly polarized light wave, where -- during propagation -- the $\mathbb{E}$ & $\mathbb{B}$ maintain fixed directions in space.

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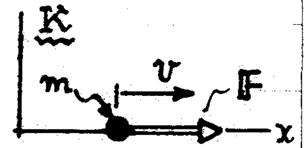
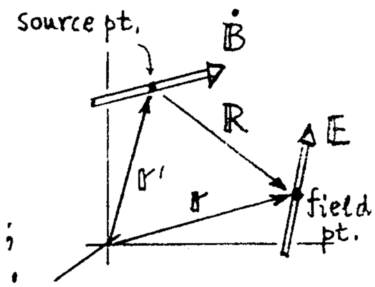
(4) [redacted] pts.]. A particle of (constant) mass m, initially at rest in reference frame $\underset{\sim}{K}$, is accelerated to relativistic speeds along a straight line by a force $\mathbb{F}(\tau)$. $F(\tau)$ acts in m's rest frame, and $F(\tau)$ is given as a function of m's proper time τ.



(A) Show that m's speed relative to $\underset{\sim}{K}$ is : $\beta(\tau) = \tanh\left[(1/mc)\int_0^\tau F(\tau')\,d\tau'\right]$.

(B) Find the relation between the elapsed time τ in m's frame, and the corresponding elapsed time t in the reference frame $\underset{\sim}{K}$ (while $\mathbb{F}$ is acting).

[redacted]



① [redacted]pts]. Show how $\partial\mathbb{B}/\partial t$ induces an $\mathbb{E}$-field.

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1) Faraday's Law prescribes: $\nabla\times\mathbb{E} = -\frac{1}{c}\dot{\mathbb{B}}$, where $\dot{\mathbb{B}} = \frac{\partial\mathbb{B}}{\partial t}$; the RHS of this eqtn is an effective current density $\mathbb{J} = (-)\frac{1}{c}\dot{\mathbb{B}}$.

2) From the Vector Calculus Theorem proved in class, we know that the solution to $\nabla\times\mathbb{E} = \mathbb{J}$ can be written as: $\mathbb{E} = \nabla\times\left[\frac{1}{4\pi}\int\frac{d\tau'}{R}\mathbb{J}\right]$, where $\mathbb{R} = |\mathbf{r}-\mathbf{r}'|$ is the distance between field pt $\mathbf{r}$ and source pt. $\mathbf{r}'$. Here the $-\nabla\phi$ (scalar potential) part of $\mathbb{E}$ vanishes, because there is no charge density present. Putting $\mathbb{J} = (-)\frac{1}{c}\dot{\mathbb{B}}$ in the solution for $\mathbb{E}$, we have...

$$\rightarrow \mathbb{E}(x_\mu) = (-)\frac{1}{4\pi c}\nabla\times\int\frac{d\tau'}{R}\dot{\mathbb{B}}(x'_\mu). \qquad (1)$$

3) The ∇ in Eq. (1) operates on the space components of the field pt. (x_i), not the source pt. cds (x'_i). When ∇ is moved inside the integral, we must find

$$\nabla\times\left[\frac{1}{R}\dot{\mathbb{B}}(x'_\mu)\right] = \underbrace{\nabla\left(\frac{1}{R}\right)}_{(-)\mathbb{R}/R^3,\ \text{well-known identity}}\times\dot{\mathbb{B}}(x'_\mu) + \frac{1}{R}\underbrace{\nabla\times\dot{\mathbb{B}}(x'_\mu)}_{0,\ \text{because}\ \nabla=\nabla_{x_i}}. \qquad (2)$$

Putting this result into Eq. (1), we find -- as required...

$$\boxed{\mathbb{E}(x_\mu) = +\frac{1}{4\pi c}\int\frac{d\tau'}{R^3}\left[\mathbb{R}\times\dot{\mathbb{B}}(x'_\mu)\right]}. \qquad (3)$$

[redacted]

② [[redacted]pts]. Find Poynting vector & energy flux at surface of current-carrying wire.

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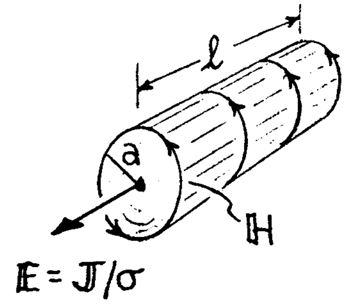
1) The magnetic field at the surface of the wire is [see Jk^{n} Eq. (5.6), or just use Ampere's Law]...



$$H = 2I/ca, \qquad (1)$$

in magnitude; the direction of $\mathbb{H}$ is along concentric circles (obeying a RH rule w.r.t. $\mathbb{I}$) around the wire. Further, since the current is DC, the current density $J = I/\pi a^2$ is uniform over the wire cross-section, $\mathbb{J}$ is along $\mathbb{I}$, and so is $\mathbb{E} = \mathbb{J}/\sigma$ (Ohm's Law). The fields are as shown above.

2) Over a length l of the wire: $E = V/l$, where V is the voltage drop in that segment. Then the Poynting vector is [Jk^{n} Eq. (6.109)]

$$\mathbb{S} = \frac{c}{4\pi}(\mathbb{E}\times\mathbb{H}) = -\frac{c}{4\pi}(EH)\,\hat{r}, \quad \hat{r} = \underline{\text{outward}} \text{ unit normal on surface;}$$

So// $\mathbb{S}$ points radially <u>inward</u> on surface, and is of magnitude $\Big\}$ $\boxed{S = \frac{c}{4\pi}\cdot\frac{V}{l}\cdot\frac{2I}{ca} = \frac{IV}{2\pi a l}}$. (2)

3) Integrating (the normal component of) $\mathbb{S}$ over the wire surface of radius a & length l, we find the energy/time carried <u>into</u> the wire segment

$$\longrightarrow \mathcal{P} = \int \mathbb{S}\cdot d\mathbb{A} = S\cdot 2\pi a l = IV, \ \underline{\text{inward}} \text{ energy flux.} \qquad (3)$$

This is exactly the rate of Joule heating occurring in the segment, and -- per Jk^{n} Eq. (6.108) -- this relation expresses conservation of energy (fields + mechanical) for this system. Conventionally, a source of "emf" is thought to produce the Joule heating... here the "emf" is replaced in that role by the fields it creates.

[redacted]

③ [redacted]]. For EM fields in a vacuum, show: $\nabla\cdot\mathbb{P}+\frac{\partial\delta}{\partial t}=0\ \begin{cases}\mathbb{P}=\mathbb{E}\times\dot{\mathbb{E}}+\mathbb{B}\times\dot{\mathbb{B}},\\ \delta=\mathbb{E}\cdot(\nabla\times\mathbb{E})+\mathbb{B}\cdot(\nabla\times\mathbb{B})\end{cases}$

1) Maxwell's Eqtns for the electric & magnetic fields $\mathbb{E}$ & $\mathbb{B}$ in free space (charge density ρ & current $\mathbf{J}$ both $\equiv 0$) are...

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$$\left.\begin{array}{l}\nabla\cdot\mathbb{E}=0,\quad \nabla\times\mathbb{E}=-\frac{1}{c}\dot{\mathbb{B}},\\ \nabla\cdot\mathbb{B}=0,\quad \nabla\times\mathbb{B}=+\frac{1}{c}\dot{\mathbb{E}};\end{array}\right\}\text{the dot}\Rightarrow \partial/\partial t. \tag{1}$$

2) We can then form the quantity (using the curl relations)...

$$[\mathbb{E}\cdot(\nabla\times\mathbb{E})+\mathbb{B}\cdot(\nabla\times\mathbb{B})]=\frac{1}{c}[-\mathbb{E}\cdot\dot{\mathbb{B}}+\dot{\mathbb{E}}\cdot\mathbb{B}],$$

$$\xrightarrow{\text{so}}\ \frac{\partial}{\partial t}[\ ]=(-)\frac{1}{c}(\mathbb{E}\cdot\ddot{\mathbb{B}}-\ddot{\mathbb{E}}\cdot\mathbb{B}). \tag{2}$$

The other quantity in the required identity is...*

$$\nabla\cdot(\mathbb{E}\times\dot{\mathbb{E}}+\mathbb{B}\times\dot{\mathbb{B}})=\cancel{\dot{\mathbb{E}}\cdot\underbrace{(\nabla\times\mathbb{E})}_{=-\frac{1}{c}\dot{\mathbb{B}}}}-\mathbb{E}\cdot\overbrace{(\nabla\times\dot{\mathbb{E}})}^{=-\frac{1}{c}\ddot{\mathbb{B}}}+ \tag{3}$$

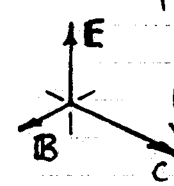
$$\cancel{+\dot{\mathbb{B}}\cdot\underbrace{(\nabla\times\mathbb{B})}_{=+\frac{1}{c}\dot{\mathbb{E}}}}-\mathbb{B}\cdot\overbrace{(\nabla\times\dot{\mathbb{B}})}^{+\frac{1}{c}\ddot{\mathbb{E}}}.$$

1st & 3rd terms RHS cancel, so...

$$\rightarrow\nabla\cdot(\mathbb{E}\times\dot{\mathbb{E}}+\mathbb{B}\times\dot{\mathbb{B}})=+\frac{1}{c}(\mathbb{E}\cdot\ddot{\mathbb{B}}-\ddot{\mathbb{E}}\cdot\mathbb{B}). \tag{4}$$

3) Comparison of Eqs. (2) & (4) shows indeed we have the required "continuity eqtn" //

$$\boxed{\nabla\cdot\mathbb{P}+\frac{\partial\delta}{\partial t}=0}\ \text{w/}\ \begin{cases}\mathbb{P}=\mathbb{E}\times\dot{\mathbb{E}}+\mathbb{B}\times\dot{\mathbb{B}},\\ \delta=\mathbb{E}\cdot(\nabla\times\mathbb{E})+\mathbb{B}\cdot(\nabla\times\mathbb{B}).\end{cases} \tag{5}$$



For a linearly polarized wave, where $\mathbb{E}$ & $\dot{\mathbb{E}}$ are collinear, as are $\mathbb{B}$ & $\dot{\mathbb{B}}$, evidently $\mathbb{P}\equiv 0$. Then, for such a wave: $\delta=\frac{1}{c}(\dot{\mathbb{E}}\cdot\mathbb{B}-\mathbb{E}\cdot\dot{\mathbb{B}})=$ cnst. In fact, the cnst is $\equiv 0$ in this case, since $\mathbb{E}\perp\mathbb{B}$ (& $\dot{\mathbb{B}}$) and $\mathbb{B}\perp\mathbb{E}$ (& $\dot{\mathbb{E}}$). The situation is <u>not</u> trivial, however, for a circularly polarized wave.

* Use the vector formula: $\nabla\cdot(\mathbb{P}\times\mathbb{Q})=\mathbb{Q}\cdot(\nabla\times\mathbb{P})-\mathbb{P}\cdot(\nabla\times\mathbb{Q})$. See Jackson, front cover.

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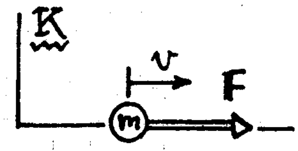
[redacted]



④ [redacted]. Relativistic acceleration of m by a proper force F.

(A) 1) As in the relativistic rocket rocket problem, a velocity increment dv' in m's "rest frame" (i.e. a frame instantaneously at rest w.r.t. m) transforms to an increment dv in K as: $dv = (1-\beta^2)\, dv'$, w/ $\beta = v/c$.✱ Dividing by an increment $d\tau$ of m's proper time, we have...

$$\frac{dv}{d\tau} = c\,\frac{d\beta}{d\tau} = (1-\beta^2)\,\frac{dv'}{d\tau}. \qquad (1)$$

2) But $dv'/d\tau$ is m's proper acceleration, so: $dv'/d\tau = \frac{1}{m}F(\tau)$, where $F(\tau)$ is the given proper force. Then Eq. (1) prescribes...

$$\frac{d\beta}{d\tau} = (1-\beta^2)\,\frac{F(\tau)}{mc} \Rightarrow \underbrace{\int \frac{d\beta}{1-\beta^2}}_{\tanh^{-1}\beta,\ \text{from tables}} = \underbrace{\frac{1}{mc}\int F(\tau)\, d\tau}_{\text{call this } f(\tau)}. \qquad (2)$$

Since $\beta(0) = 0$, then...

$$\boxed{\beta(\tau) = \tanh[f(\tau)]}, \quad f(\tau) = \frac{1}{mc}\int_0^\tau F(\tau')\, d\tau'. \qquad (3)$$

(B) 3) The proper time τ (m's frame) and reference time t (in K) are related incrementally by: $dt = d\tau/\sqrt{1-\beta^2(\tau)}$. With $\beta(\tau)$ given in Eq. (3), and with the hyperbolic identity: $1-\tanh^2 = \mathrm{sech}^2 = 1/\cosh^2$, we have

$$dt = \frac{1}{\sqrt{1-\beta^2}}\, d\tau = \cosh[f(\tau)]\, d\tau \Rightarrow \boxed{t = \int_0^\tau \cosh[f(\tau')]\, d\tau'} \qquad (4)$$

Since $\cosh[f]\, d\tau = [\dot f]^{-1} d \sinh[f]$ (w/ $\dot f = df/d\tau$), Eq. (4) can be partial-integrated to give an expression whose first term is ~ previous rocket result...

$$\longrightarrow t = \frac{1}{\dot f}\sinh f + \int \sinh f\, [\ddot f/(\dot f)^2]\, d\tau \qquad (5)$$

✱ This results from the velocity addition formula: $(v+dv)_{\text{in } K} = (v+dv')_{\text{in } m}/(1+\frac{v\,dv'}{c^2})$. To terms 1st order in the infinitesimals: $dv = (1-\beta^2)\,dv'$, as quoted above.