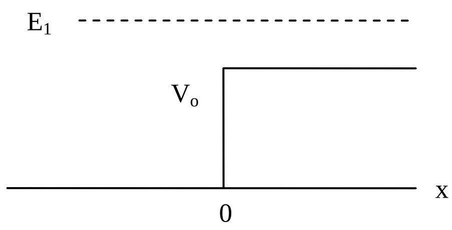
1) Solution

Consider the 1-D scattering of a quantum particle of mass m and at energy E_1 by a step potential $V_o > 0$ as shown in the figure. Assume that the potential energy to the left of the step potential (x<0) is zero. Also assume that $E>V_o$ for this 1-D scattering process.



The incident wave function, traveling from left to right, is given by:

$$\Psi_1(x,t) = Ae^{+ik_1x-i\omega_1t}$$

where the wave vector k_1 and the angular frequency ω_1 of this incident traveling wave are both positive and real.

a) For this incident quantum wave, use Schrödinger's equation to find B/A, the ratio of the reflection amplitude B to the incident amplitude A.

b) Then use Schrödinger's equation to find the transmission amplitude C/A, the ratio of the transmission amplitude C to the incident amplitude A.

c) Finally, find the reflection coefficient R and the transmission coefficient T and verify that R and T will indeed sum to unity.

a) First we find the wave vector k_1 , by plugging the wave function into Schrödinger's time independent equation to relate k to the energy E. Or you can just note that the wave vector k_1 of the incident wave can be related to the deBroglie wavelength λ which can be related to the momentum p of the particle which can be related to the kinetic energy, which is just the energy E_1 in this region since V=0 for x<0.

$$k_1 = \frac{2\pi}{\lambda} = \frac{2\pi p}{h} = \frac{\sqrt{2mE_1}}{\hbar}$$

$$k_1 = \frac{\sqrt{2mE_1}}{\hbar}$$

The frequency of the incident wave is just related to the usual phase factor that depends on the total energy of the particle. Thus we find that the frequency ω_1 is related to the total energy E_1 by:

$$\omega_1 = \frac{E_1}{\hbar}$$

For the wave vector k_2 of the reflected wave, we note that it must be of the same magnitude as k_1 since x<0 also, but the propagation is in the opposite direction so that the sign of k changes:

$$k_2 = -\frac{\sqrt{2mE_1}}{\hbar}$$

The frequency of the reflected wave is the same since the total energy E_1 is not changing so that:

$$\omega_2 = \frac{E_1}{\hbar}$$

For the wave vector k_3 of the transmitted wave, we note that the kinetic energy in this region (x>0) is now E_1-V_o so that:

$$k_3 = \frac{\sqrt{2m\left(E_1 - V_o\right)}}{\hbar}$$

And again, the frequency of the transmitted wave is the same since the total energy E_1 is not changing so that:

$$\omega_3 = \frac{E_1}{\hbar}$$

Now to find B/A, we need to match the total wave function on each side of the step potential. Note that the time dependence of all three waves are the same. Thus we get:

$$\left[Ae^{ik_1x} + Be^{-ik_1x}\right]_{x=0} = \left[Ce^{ik_3x}\right]_{x=0}$$
$$\rightarrow A + B = C$$

To get a second equation, we also match the slope of the total wave function on each side of the step potential.

$$\left[ik_1Ae^{ik_1x} - ik_1Be^{-ik_1x}\right]_{x=0} = \left[ik_3Ce^{ik_3x}\right]_{x=0}$$
$$\rightarrow \frac{k_1}{k_3}\left(A - B\right) = C$$

Equating these two equation gives:

$$A + B = \frac{k_1}{k_3}\left(A - B\right) \rightarrow A\left(1 - \frac{k_1}{k_3}\right) = -B\left(1 + \frac{k_1}{k_3}\right)$$

$$\boxed{\frac{B}{A} = -\frac{\left(1 - \frac{k_1}{k_3}\right)}{\left(1 + \frac{k_1}{k_3}\right)}}$$

b) Now to find C/A, we can use either of the two equations we found at the step potential edge at x=0 to get:

$$A + B = C \rightarrow \frac{C}{A} = 1 + \frac{B}{A} = 1 - \frac{\left(1 - \frac{k_1}{k_3}\right)}{\left(1 + \frac{k_1}{k_3}\right)} = \frac{\left(1 + \frac{k_1}{k_3}\right) - \left(1 - \frac{k_1}{k_3}\right)}{\left(1 + \frac{k_1}{k_3}\right)} = \frac{2k_1}{\left(k_3 + k_1\right)}$$

$$\boxed{\frac{C}{A} = \frac{2k_1}{\left(k_3 + k_1\right)}}$$

c) Finally, to get the reflection coefficient R, we simply need to find:

$$R = \left|\frac{B}{A}\right|^2 = \left| -\frac{\left(1 - \frac{k_1}{k_3}\right)}{\left(1 + \frac{k_1}{k_3}\right)} \right|^2$$

$$\boxed{R = \left|\frac{(k_3 - k_1)}{(k_3 + k_1)}\right|^2}$$

To find the transmission coefficient T, we not only need the modulus squared of C/A but we also must multiply by the ratio of the speed of the transmitted wave v_t to the speed of the incident wave v_i to obtain:

$$T = \left|\frac{C}{A}\right|^2 \left(\frac{v_t}{v_i}\right) = \left|\frac{2k_1}{(k_3 + k_1)}\right|^2 \left(\frac{\sqrt{2m(E_1 - V_o)}}{\sqrt{2mE_1}}\right) = \frac{4k_1^2}{(k_3 + k_1)^2}\left(\frac{k_3}{k_1}\right)$$

$$\boxed{T = \frac{4k_1k_3}{(k_3 + k_1)^2}}$$

Now we can check to see that probability is conserved by adding the reflection coefficient R to the transmission coefficient T to get:

$$R + T = \frac{(k_3 - k_1)^2}{(k_3 + k_1)^2} + \frac{4k_1k_3}{(k_3 + k_1)^2} = \frac{(k_3 - k_1)^2 + 4k_1k_3}{(k_3 + k_1)^2} = \frac{k_3^2 - 2k_3k_1 + k_1^2 + 4k_1k_3}{(k_3 + k_1)^2}$$

$$= \frac{k_3^2 + 2k_3k_1 + k_1^2}{(k_3 + k_1)^2} = \frac{(k_3 + k_1)^2}{(k_3 + k_1)^2}$$

$$\boxed{R + T = 1}$$

Thus the probability is conserved as we expect.

end

2. SOLUTION

Setting all rates to ν yields a system

$$\frac{d}{dt}\begin{bmatrix} f_0 \\ f_1 \\ f_2 \end{bmatrix} = \underbrace{\begin{bmatrix} -\nu & \nu & 0 \\ \nu & -2\nu & \nu \\ 0 & \nu & -\nu \end{bmatrix}}_{\underline{\underline{M}}} \cdot \begin{bmatrix} f_0 \\ f_1 \\ f_2 \end{bmatrix} \tag{4}$$

The eigenvalues λ solve the equation

$$\det(\underline{\underline{M}} - \lambda\,\underline{\underline{I}}) \;=\; -\,(\lambda+2\nu)(\lambda+\nu)^2 + 2\nu^2(\lambda+\nu) \;=\; -\,\lambda(\lambda+\nu)(\lambda+3\nu) \;=\; 0 \quad .$$

Eigenvalues and eigenvectors are therefore

$$\lambda_1 = 0\ ,\ \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad , \quad \lambda_2 = -\nu\ ,\ \vec{v}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \quad , \quad \lambda_3 = -3\nu\ ,\ \vec{v}_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

We choose to leave them un-normalized since they will be multiplied by coefficients we must yet find. The general solution is

$$\begin{bmatrix} f_0(t) \\ f_1(t) \\ f_2(t) \end{bmatrix} = a_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + a_2 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} e^{-\nu t} + a_3 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} e^{-3\nu t} \quad . \tag{5}$$

where a_i are constants. These constants are set by the initial condition

$$\begin{bmatrix} f_0(0) \\ f_1(0) \\ f_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = a_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + a_2 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + a_3 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \quad . \tag{6}$$

This system can be solved to yield $a_1 = 1/3$, $a_2 = 1/2$ and $a_3 = 1/6$. The doubly-ionized fraction therefore evolves as

$$\boxed{f_2(t) \;=\; \frac{1}{3} \;-\; \frac{1}{2}e^{-\nu t} \;+\; \frac{1}{6}e^{-3\nu t} \quad . \tag{7}}$$

Check:

$$\begin{aligned}
f_2(0) &= \frac{1}{3} - \frac{1}{2} + \frac{1}{6} = 0 \\
\dot{f}_2(0) &= 0 + \frac{\nu}{2} - \frac{3\nu}{6} = 0 \\
&\quad \text{matches } \dot{f}_2(0) \;=\; 0\times f_0(0) \;+\; \nu f_1(0) \;-\; \nu f_2(0) \\
\ddot{f}_2(0) &= 0 - \frac{\nu^2}{2} + \frac{9\nu^2}{6} = \nu^2 \\
&\quad \text{matches } \ddot{f}_2(0) \;=\; 0\times \dot{f}_0(0) \;+\; \nu \dot{f}_1(0) \;-\; \nu \dot{f}_2(0) \\
&= 0 + \nu \underbrace{\Big[\nu f_0(0) - 2\nu f_1(0) + \nu f_2(0)\Big]}_{\dot{f}_1(0)} - \nu\times 0 \;=\; \nu^2 f_0(0)
\end{aligned}$$

3) Solution

a. Consider an isolated electron as the center plate of an infinitely large spherical capacitor with a capacitance $\lim_{R\to\infty} C = \frac{4\pi\varepsilon_o}{(1/r - 1/R)} = 4\pi\varepsilon_o r$ .We also know that the total energy stored in a capacitor in the electric field lines is given by $U = Q^2/2C = e^2/2C$, where we take *Q*=-*e*. The total energy stored in a capacitor by the charge of the electron cannot exceed its rest mass, or $U < m_e c^2$. This puts a lower limit on the classical electron radius: $U = e^2/2C = e^2/8\pi\varepsilon_o r < m_e c^2$. This implies $r > e^2/(8\pi\varepsilon_o m_e c^2)$. Inserting numbers gives a lower limit for *r*: $r > (1.6\times10^{-19})^2/(8\pi\times8.9\times10^{-12}\times9.1\times10^{-31}\times(3\times10^8)^2) \approx 1.4\times10^{-15}$ m .

b. The photon originating from the center of the sun does a random walk as it scatters elastically on the electrons. The mean free path between two scatterings is given by $\ell = 1/(n\sigma_T)$, where *n* is the number density of electrons. The number density of electrons, assuming most of the mass of the star is due to ionized hydrogen atoms, can be obtained from $n = \rho/(m_e + m_p) = 1.5\times10^5/(9.1\times10^{-31} + 1.7\times10^{-27}) \approx 9\times10^{31}$ m^{-3}. This yields a mean free path of: $\ell = 1/(n\sigma_T) = 1/(9\times10^{31}\times7\times10^{-29}) = 1.6\times10^{-4}$ m . The average time per step is given by $\tau = \ell/c = 1.6\times10^{-4}/3\times10^8 = 5.3\times10^{-13}$ s . The number of steps is given by $N = \langle x^2 \rangle/\ell^2 = (10^8/1.6\times10^{-4})^2 \approx 4\times10^{23}$. The time it takes for the light originating from the center is $t = N\tau = 4\times10^{23}\times5.3\times10^{-13} \approx 2\times10^{11}$ s $\approx 7,000$ years. In reality it would take light longer than 7000 years to come out of the core of the star. These calculations consider only elastic scattering of light by electrons. They do not take into account, for example, inelastic scattering of light by electrons (Compton Scattering) or inelastic scattering by protons. These scattering processes will delay the photons from exiting the core of the sun by many more thousands of years.

4) Solution

The Saros Cycle. The orbital plane of the moon around Earth is tilted by 5.15 deg with respect to the orbital plane of the Earth around the Sun (the so-called *ecliptic*) as schematically depicted below (not to scale!). If you view the Earth-Moon system as a spinning top, the Sun exhibits tidal forces on that system that try to force the lunar orbit into the ecliptic.

Make a *leading-order* estimate of the time-averaged torque and the precession period of the moon-Earth system due to the Sun's influence. Give the precession period in years.

Side note: the relative position of Earth, Sun and moon governs the occurrence of both solar and lunar eclipses. The precession period that you will calculate corresponds to a naturally occurring cycle in the occurrence of solar and lunar eclipses, the so-called *Saros Cycle* that was already observationally known to the ancient Babylonians!

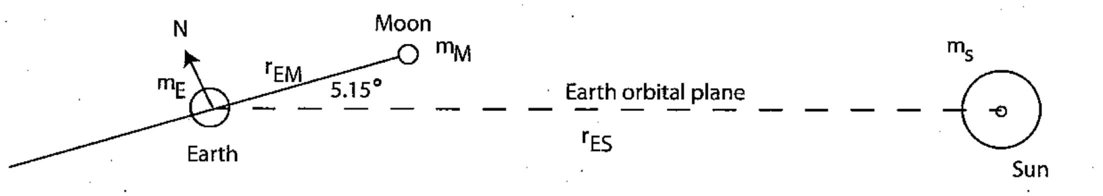


FIG. 1: Sketch of the setup of the problem.

In good approximation, the center of mass of the Earth-Moon system describes a circular path around the Sun. The gravitational attraction is on average balanced by the centrifugal force of the Earth-Moon system.

$$\frac{(m_M + m_E)}{r_{ES}} v_E^2 = G\frac{(m_M + m_E)}{r_{ES}^2} m_S \,, \tag{1}$$

and

$$\frac{m_M}{r_{EM}} v_M^2 = G\frac{m_M m_E}{r_{EM}^2} \,. \tag{2}$$

where Eq. (1) *refers to the motion of the combined Earth-Moon system around the Sun and Eq.* (**??**) *refers to the motion of the moon around the Earth. The distance of the moon to the sun varies between full moon and new moon phases by* $\pm r_{EM} \cos 5.15° \approx \pm r_{EM}$ *and thus the moon experiences a tidal force*

$$F = G m_M m_S \left[-\frac{1}{r_{ES}^2} + \frac{1}{(r_{ES} \pm r_{EM})^2} \right] \,. \tag{3}$$

Since $r_{EM} \ll r_{ES}$*, the square bracket simples to* $\approx 2r_{EM}/r_{ES}^3$ *and thus*

$$F = \pm G m_M m_S \frac{2r_{EM}}{r_{ES}^3} = \pm \frac{m_M v_E^2}{r_{ES}^2} 2r_{EM} \,, \tag{4}$$

where in the last step we have made use of Eq. (1)*. We now approximate using that the mass of the moon is on average on a circle around the Earth. The torque* $\tau = F \sin 5.15° r_{EM}$ *during new and full moon tries to tilt the lunar orbital plane into the ecliptic. If the moon is at a distance of angle* ϕ *away from the new moon position,* F *is reduced by* $\cos\phi$ *since the distance of the moon to the ecliptic is reduced by that factor. In addition, the angle is reduced by that factor* $\cos\phi$ *leading to a reduction of* $\cos^2\phi$ *in the torque. Thus, the average torque is reduced by* $< \cos^2\phi >= 1/2$*. (In addition, the lunar orbit during the course of the year is not always tilted against the Sun, which reduces the torque by another factor of 2, but that's not super relevant here). Thus, the effective torque is*

$$\tau_{eff} = \frac{m_M}{r_{ES}^2} v_E^2 \frac{r_{EM}^2}{2} \sin 5.15° \tag{5}$$

and the angular momentum of the moon is

$$L = r_{EM} m_M v_M \,. \tag{6}$$

The effective torque is equal to $\tau_{eff} = \omega_p L \sin 5.15°$ *and thus the precession of the lunar orbital plane is*

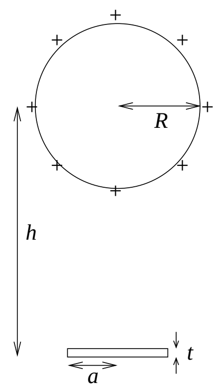
$$\omega_p = \frac{m_M}{r_{ES}^2} v_E^2 \frac{r_{EM}^2}{2} \sin 5.15° \frac{1}{r_{EM} m_M v_M \sin 5.15°} = \frac{1}{2} \frac{r_{EM} v_E^2}{r_{ES}^2 v_M} = \frac{\omega_{ES}^2}{2\omega_M} \,, \tag{7}$$

where we have assumed in the last steps that the angle is not zero between Earth and Moon. Thus, the precession period T_p *is given by*

$$T_p = \frac{2\pi}{\omega_p} = \frac{2T_{ES}^2}{T_M} = 2 \times 365 \times 365/30 = 24 \text{ years} \,. \tag{8}$$

The exact calculation shows that the precession period for the Saros cycle is actually more like 18.5 years. The Saros cycle dominates the series of lunar and solar eclipses.

5. A small, circular, copper disk, of mass density ρ, radius a, and thickness $t \ll a$, lies on a flat, non-conducting table. A glass sphere of radius R and uniformly distributed positive charge Q is suspended with its center a distance $h > R$ above the disk. Assume that $h \gg a$ and $h \gg t$.



(a) Explain why we can take the electric field due to the sphere as approximately uniform over the volume of the disk.

The field due to the sphere is directed radially away $\hat{r}$ *from the center of the sphere, and its magnitude goes as one over distance squared from the center of the sphere* r. *Because* $h \gg t$, *the sphere's field will not vary much within the height of the disk:* $\hat{r} \pm t\hat{z} \simeq \hat{r}$ *and* $r \pm t \simeq r$. *A similar argument can be made for the horizontal displacement of the field point within the disk:* $\hat{r} \pm a\hat{x} \simeq \hat{r}$ *and* $\sqrt{r^2 + a^2} \simeq r$.

(b) Assuming that the field is uniform, determine the total charge induced on the upper surface of the disk due to the presence of the sphere. Why must its sign be negative?

Because we are dealing with a conductor, the field within the disk must be zero. To accomplish this, the charge separation within the disk must create a field upward which just cancels the downward field due to the sphere, so electrons will migrate to the upper surface, leaving a net positive charge on the lower surface. If the field due to the sphere is uniform, the field due to the charge separation must be uniform, too. Ignoring edge effects, a uniform field between two parallel surfaces is achieved by equal and opposite, uniform charge distributions on those surfaces. Then, we have a kind of parallel-plate capacitor with a field simply proportional to the charge density on the positive plate.

$$\frac{Q}{4\pi\epsilon_\circ h^2} = \frac{q/\pi a^2}{\epsilon_\circ} \qquad \Rightarrow \qquad q_{\rm top} = \boxed{-\frac{Qa^2}{4h^2}}$$

I have inserted a negative sign for the negative charge on the top surface.

(c) Consider the interaction between the upper surface charges from part (b) and those on the sphere. Determine the value of h such that their electrostatic attraction equals the magnitude of the force of gravity on the disk.

$$\begin{aligned} F_g &= mg = \rho\pi a^2 tg \\ F_e &= \frac{Q}{4\pi\epsilon_\circ h^2}\frac{Qa^2}{4h^2} \\ h &= \boxed{\left(\frac{Q^2}{16\pi^2\epsilon_\circ\rho tg}\right)^{1/4}} \end{aligned}$$

(d) In the laboratory, you find that the disk begins to rise from the table at a value of h that is much smaller than you predicted in your analysis in part (c). Explain qualitatively how your analysis can be modified to give a better description of your observations.

We have neglected the repulsion of the positive charges (actually a deficit of negative charges) on the bottom of the disk! That repulsion will reduce the electrostatic attraction of the disk to the sphere, so that we will need to get the sphere closer to the disk to exert a sufficient force to lift the disk. If we simply use the approximation of a uniform field from the sphere, however, as we did in part (b), we will find that there is zero net electrostatic attraction between the disk and the sphere.

To better describe the observations, we will need to account for the non-uniformity of the field due to the sphere. The field will be stronger, and more dependent on lateral displacements, on the top of the disk than on the bottom. Together, this will cause a more concentrated charge density on the top than on the bottom. This will, in turn, cause the attraction of the charges on the top surface to be larger than the repulsion of the positive charges on the bottom surface. This is a very messy problem to solve quantitatively because we can no longer resort to the nice parallel-plate capacitor model.

6) Solution

1. The partition function is given by $Z = \sum_s e^{-E_s/kT} = \sum_{j=0}^{\infty}(2j+1)e^{-j(j+1)\varepsilon/kT} = \sum_{j=0}^{\infty}(2j+1)e^{-j(j+1)\varepsilon\beta}$ where *s* stands for all possible states associated with the rotation of the molecule and E_s is the energy associated with specific state *s*. In writing *Z* above we took into account that the states of a rotating molecule are uniquely determined by the spherical harmonics $Y_j^m(\theta,\phi)$ and each energy level associated with j is degenerate and has *2j+1* states associated with it.
2. a. For CO $\varepsilon/k \approx 2.8$ K; hence, $kT >> \varepsilon$ is well justified. We can then easily determine Z by simply integrating: $Z = \sum_{j=0}^{\infty}(2j+1)e^{-j(j+1)\varepsilon\beta} = \int_0^{\infty}(2j+1)e^{-j(j+1)\varepsilon\beta}dj = 1/\beta\varepsilon = kT/\varepsilon$

   b. For H_2: $\varepsilon/k \approx 66$ K; hence, at low temperatures $kT << \varepsilon$ is justified since the J=0 nuclear spin state is a symmetric function with respect to switching the two identical hydrogen nuclei around. This means only those $Y_j^m(\theta,\phi)$ with even *j*'s (which are symmetric with respect to switching the two identical hydrogen atoms around) are allowed. Because hydrogen atoms are identical, *Z* has to be divided by *2!=2*. This yields:

   $Z = (1/2)\sum_{j=0}^{\infty}(2j+1)e^{-j(j+1)\varepsilon\beta} \approx (1/2)(1+5e^{-6\beta\varepsilon})$ where we take only the *j=0* and *j=2* terms.

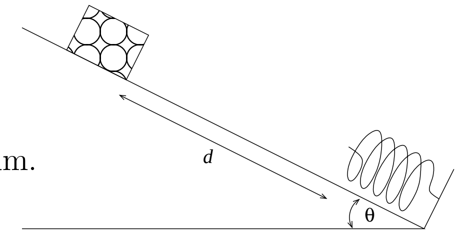
3. a. For CO: $\bar{E} = -\frac{\partial \ln Z}{\partial\beta} = \frac{\partial \ln\varepsilon}{\partial\beta} + \frac{\partial\ln\beta}{\partial\beta} = 1/\beta = kT$ (This is expected: each degree of rotation gets $kT/2$ energy. The specific heat, then, is given by $C_V = \frac{\partial\bar{E}}{\partial T} = k$.

   b. For H_2: Here we assumed $\beta\varepsilon = \varepsilon/kT >> 1$. This immediately yields

   $\bar{E} = -\frac{\partial\ln Z}{\partial\beta} = -\frac{\partial\ln(1+5e^{-6\beta\varepsilon})}{\partial\beta} = \frac{30\varepsilon e^{-6\beta\varepsilon}}{1+5e^{-6\beta\varepsilon}} \approx 30\varepsilon e^{-6\beta\varepsilon}$, where we use $5e^{-6\beta\varepsilon} << 1$

   $C_V = \frac{\partial\bar{E}}{\partial T} = \frac{\partial}{\partial T}30\varepsilon e^{-6\beta\varepsilon} = 180k(\beta\varepsilon)^2 e^{-6\beta\varepsilon}$

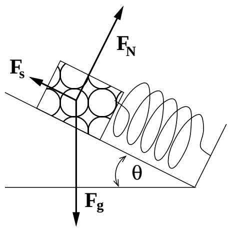
7. A block of mass M is released from rest on a frictionless surface inclined at angle θ from horizontal, a distance d from the end of massless spring of spring constant κ, which is in equilibrium.



(a) Derive an expression for the compression of the spring at the instant that the block has its maximum speed in two independent ways:

- using Newton's Laws of Motion accompanied by the relevant free-body diagram, and

  Let $\vec{F}_g$ be the force of gravity, $\vec{F}_N$ the normal force, and $\vec{F}_s$ the restoring force of the spring. Let h be the compression of the spring. In the direction to the left and up along the incline, the net force gives:



$$ma = \kappa h - Mg \sin\theta$$

  Maximum speed occurs when the acceleration is zero.

$$h = \boxed{\frac{Mg\sin\theta}{\kappa}}$$

- using conservation of energy.

  Decrease in gravitational potential energy equals increase in elastic potential energy and kinetic energy.

$$\begin{aligned} \frac{1}{2}\kappa h^2 + \frac{1}{2}mv^2 &= Mg|\Delta y| = Mg\,(d+h)\sin\theta \\ v &= \sqrt{2g(d+h)\sin\theta - \frac{\kappa h^2}{M}} \end{aligned}$$

  The speed will be maximized when $dv/dh = 0$.

$$\frac{dv}{dh} = \frac{\frac{1}{2}\left(2g\sin\theta - \frac{2\kappa h}{M}\right)}{\sqrt{2g(d+h)\sin\theta - \frac{\kappa h^2}{M}}}$$

  $dv/dh = 0$ when the numerator is zero. Solving for h, we again arrive at

$$h = \boxed{\frac{Mg\sin\theta}{\kappa}}$$

Discuss the salient features of your two results. Using the values $M = 10$ kg, $d = 4.0$ m, $\kappa = 250$ N/m, and $\theta = 30°$, calculate the value of the compression to one significant figure.

The two results agree, as they must, because we are simply using two different techniques to describe the same problem. Also, note that the slide distance d

does not appear in the expression, because the initial kinematic conditions are irrelevant when balancing the forces.

$$h = \frac{(10\ \text{kg})(10\ \text{m/s}^2)\left(\frac{1}{2}\right)}{250\ \text{N/m}} = \boxed{20\ \text{cm}}$$

(b) Derive an expression for the maximum compression of the spring.

I'll use conservation of energy here. Maximum compression occurs when the block's speed is zero, so the decrease in gravitational potential energy is just the increase in elastic potential energy.

$$\frac{1}{2}\kappa h^2 = Mg|\Delta y| = Mg\,(d+h)\sin\theta$$

$$h = \boxed{\frac{Mg\sin\theta \pm \sqrt{(Mg\sin\theta)^2 + 2\kappa Mgd\sin\theta}}{\kappa}}$$

Choosing the positive root in the numerator gives the compression when the block is traveling downward. Choosing the negative root will give the extension of the spring when the block is pushed back up the incline.

(c) Now, instead, the surface is *not* frictionless, and the block slides with a coefficient of kinetic friction μ. Do you expect the spring compressions from parts (a) and (b) to increase, decrease, or remain the same? Rederive each of your expressions to include the effects of friction, and show them to be consistent with your qualitative expectations.

I expect that friction will remove energy from the block-spring system, so that the maximum compression will decrease. I also expect that the force of friction acting with the elastic force to retard the block will reduce the compression at maximum speed. So I expect both compressions to be reduced. First, I'll redo part (a) using Newton, with an additional force of friction added parallel to the restoring force. In the direction normal to the incline, the normal force will be $Mg\cos\theta$.

$$ma = \kappa h + \mu Mg\cos\theta - Mg\sin\theta$$

Solving for h with $a = 0$ again gives:

$$\boxed{h = \frac{Mg(\sin\theta - \mu\cos\theta)}{\kappa}}$$

Now, because $\mu > 0$ and $\cos\theta > 0$, the numerator will be smaller than it had been in part (a), so the compression will decrease.

For part (b), I'll use conservation of energy again, but now with the frictional force acting over the distance $d + h$.

$$\frac{1}{2}\kappa h^2 = Mg|\Delta y| - \mu Mg\cos\theta(d+h) = Mg\,(d+h)\,(\sin\theta - \mu\cos\theta)$$

$$h = \boxed{\frac{Mg(\sin\theta - \mu\cos\theta) + \sqrt{[Mg(\sin\theta - \mu\cos\theta)]^2 + 4\frac{1}{2}\kappa Mgd(\sin\theta - \mu\cos\theta)}}{2\frac{1}{2}\kappa}}$$

Here, we must choose the positive root because we have only included friction on the way down the incline. Now, we can see that each of the sine functions has been reduced by $\mu \cos\theta$*, which is positive, so the numerator is smaller than in part (b), so the compression will decrease.*

8) Solution

Asymptotic Expansions. Find the first term in the asymptotic expansion of the following integral (i.e. the behavior of the integral in the limit $x \gg 1$):

$$I = \frac{1}{\pi} \int_0^{\pi} \left(t^4 + 2t^6\right)^{1/2} e^{x \cos t} \cos nt \, dt \,, \tag{1}$$

where n is a constant. Show *all* your work.

The basic idea in asymptotic expansions involving exponentials is that only the region near where the maximum of $x \cos t$ lies contributes in the limit $x \gg 1$. This occurs at $t = 0$. So the integral can be written

$$I \sim \frac{1}{\pi} \int_0^{\epsilon} \left(t^4 + 2t^6\right)^{1/2} e^{x \cos t} \cos nt \, dt \,, \tag{2}$$

where ϵ is small, and the final answer must not depend on it. Next, Taylor expand everything around $t = 0$, keeping only enough terms so that the answer doesn't depend on ϵ.

$$I \sim \frac{1}{\pi} \int_0^{\epsilon} t^2 e^{x(1-t^2/2)} \, dt = \frac{e^x}{\pi} \int_0^{\epsilon} t^2 e^{-xt^2/2} \, dt. \tag{3}$$

Note that we needed two terms in the exponential, since if we approximated $e^{x \cos t} \sim e^x$, then the integral would have depended explicitly on ϵ. With the second term included, the integral has a term that, when $x \gg 1$, does not depend upon ϵ. For the asymptotic limit, we can perform the integral using any upper limit, so replace ϵ with ∞.

$$I \sim \frac{e^x}{\pi} \int_0^{\infty} t^2 e^{-xt^2/2} \, dt \,. \tag{4}$$

Changing variables to $t = \sqrt{2/x}\, u^{1/2}$, or $dt = u^{-1/2}/\sqrt{2x}\, du$, the integral becomes

$$I \sim \frac{e^x \sqrt{2}}{\pi x^{3/2}} \int_0^{\infty} u^{1/2} e^{-u} du = \frac{e^x \sqrt{2}}{\pi x^{3/2}} \Gamma(3/2) \,. \tag{5}$$

Finally, using $\Gamma(3/2) = 1/2\, \Gamma(1/2) = \sqrt{\pi}/2$, we have

$$I \sim \frac{e^x}{\sqrt{2\pi x^3}} \,. \tag{6}$$

9. SOLUTION

a. One *could* the compute field from a current on loop 1 and integrate this over a surface bounded by loop 2 — but that would be **hard** (see below). You get the **exact** same answer if current I_2 flowing on the outer loop creates a field which you integrate over the very the small inner loop — **much easier!** Since $a_1 \ll a_2$ you only need the field at the origin

$$\mathbf{B}_0 \;=\; \frac{\mu_0 I_2}{4\pi}\oint \frac{a_2\, d\phi\, \hat{\mathbf{z}}}{a_2^2} \;=\; \frac{\mu_0 I_2}{2a_2}\,\hat{\mathbf{z}} \quad , \tag{2}$$

taking I_2 to be counter-clockwise. The flux through the small inner loop is therefore

$$\Phi_1 \;=\; \int \mathbf{B}\cdot d\mathbf{a} \;\simeq\; \mathbf{B}_0\cdot\hat{\mathbf{n}}_1\,\pi a_1^2 \;=\; \frac{\mu_0\pi a_1^2}{2a_2}\cos\theta\, I_2 \quad , \tag{3}$$

since $\mathbf{B}$ is approximately constant, $=\mathbf{B}_0$, over the inner loop and $\hat{\mathbf{n}}_1\cdot\hat{\mathbf{z}} = \cos\theta$. The mutual inductance is

$$\boxed{M_{12} \;=\; \frac{\Phi_1}{I_2} \;=\; \frac{\mu_0\pi a_1^2}{2a_2}\cos\theta \quad .} \tag{4}$$

b. The flux through the outer loop is

$$\Phi_2 \;=\; M_{21}I_1 \;=\; M_{12}I_1 \;=\; \frac{\mu_0\pi a_1^2}{2a_2}\cos\theta\, I_1 \quad , \tag{5}$$

after using the fact that $M_{21} = M_{12}$. The EMF is therefore

$$\mathcal{E} \;=\; -\frac{d\Phi_2}{dt} \;=\; -\frac{d}{dt}\left(\frac{\mu_0\pi a_1^2}{2a_2}\cos\theta\, I_1\right) \;=\; \frac{\mu_0\pi a_1^2}{2a_2}\sin\theta\, I_1\omega \quad . \tag{6}$$

Neglecting self inductance this must equal $I_2\,R_2$, leading to the current

$$\boxed{I_2 \;=\; \frac{\mu_0\pi a_1^2\omega}{2a_2 R_2}\sin\theta\, I_1 \quad .} \tag{7}$$

c. Placing this expression into eq. (2) yields the magnetic field at the origin

$$\mathbf{B}_0 \;=\; \frac{\mu_0 I_2}{2a_2}\,\hat{\mathbf{z}} \;=\; \frac{\mu_0^2\pi a_1^2\omega}{4a_2^2 R_2}\sin\theta\, I_1\,\hat{\mathbf{z}} \quad . \tag{8}$$

The dipole moment of the inner loop is

$$\mathbf{m}_1 \;=\; A_1\, I_1\,\hat{\mathbf{n}}_1 \;=\; \pi a_1^2 I_1\,(\sin\theta\,\hat{\mathbf{x}} + \cos\theta\,\hat{\mathbf{z}}) \quad , \tag{9}$$

where $\hat{\mathbf{n}}_1$ rotates with angular velocity $\boldsymbol{\omega} = +\omega\hat{\mathbf{y}}$. The torque on the dipole is

$$\boxed{\mathbf{N} \;=\; \mathbf{m}_1\times\mathbf{B}_0 \;=\; -\frac{\mu_0^2\pi^2 a_1^4\omega}{4a_2^2 R_2}\sin^2\theta\, I_1^2\,\hat{\mathbf{y}} \quad .} \tag{10}$$

d. The torque exerted by the loop **on the driver** is $-\mathbf{N}$ so the power delivered is

$$P_\omega \;=\; -\mathbf{N}\cdot\boldsymbol{\omega} \;=\; -\mathbf{N}\cdot(\omega\hat{\mathbf{y}}) \;=\; \frac{\mu_0^2\pi^2 a_1^4\omega^2}{4a_2^2 R_2}\sin^2\theta\, I_1^2 \quad , \tag{11}$$

where $\boldsymbol{\omega} = +\omega\hat{\mathbf{y}}$ is the constant angular velocity. The powered dissipated by the resistor is

$$P_R \;=\; R_2 I_2^2 \;=\; \frac{\mu_0^2 \pi^2 a_1^4 \omega^2}{4a_2^2 R_2} \sin^2\theta\, I_1^2 \quad . \tag{12}$$

Since $P_R/P_\omega = 1$, **100% of the power delivered to the axle is dissipated resistively.** By dropping the self-inductance of the outer loop, we neglect any magnetic energy it might store. By working with lumped circuit formulation, i.e. inductances and resistances, we have neglected radiation.

Part a.) the hard way

For radii $r \gg a_1$ the inner loop may be treated as a point dipole with dipole moment

$$\mathbf{m}_1 \;=\; A_1\, I_1\, \hat{\mathbf{n}}_1 \;=\; \pi a_1^2 I_1\,(\,\sin\theta\,\hat{\mathbf{x}} + \cos\theta\,\hat{\mathbf{z}}) \tag{13}$$

We can think of this as two different time-varying dipoles parallel to $\hat{\mathbf{x}}$ and $\hat{\mathbf{z}}$. The $\hat{\mathbf{x}}$ dipole will contribute **no flux** to loop 2 and can be forgotten. A point dipole with moment $\mathbf{m}_1 = m_{1,z}\hat{\mathbf{z}}$ makes a field in the x–y plane

$$\mathbf{B}(x,y,0) \;=\; -\frac{\mu_0 m_{1,z}}{4\pi r^3}\,\hat{\mathbf{z}} \;=\; -\frac{\mu_0 a_1^2 I_1}{4r^3}\cos\theta\,\hat{\mathbf{z}} \quad . \tag{14}$$

We must now compute the flux through loop 2 by integrating $\mathbf{B}(\mathbf{r})$ over some surface bounded by $r = a_2$. The natural choice is a simple disk **EXCEPT** if we choose that surface we cannot use expression (14) which is only valid for $r \gg a_1$. Instead we choose a surface which covers the entire x–y plane **outside** $r = a_2$, and closes over a hemisphere at infinite distance. (Think of deforming the disk by blowing a soap bubble — a **huge** soap bubble.) The integral over the hemispheric portion will vanish. The surface element over the x–y plane will be $d\mathbf{a} = -r\,d\phi\,dr\,\hat{\mathbf{z}}$ directed **downward**. This makes the enclosed flux

$$\begin{aligned}
\Phi_2 \;&=\; \int \mathbf{B}\cdot d\mathbf{a} \;=\; \int_0^{2\pi} d\phi \int_{a_2}^{\infty} \mathbf{B}(r)\cdot(-\hat{\mathbf{z}})r\,dr \\
&=\; 2\pi\frac{\mu_0 a_1^2 I_1}{4}\cos\theta \underbrace{\int_{a_2}^{\infty}\frac{r\,dr}{r^3}}_{a_2^{-1}} \;=\; \frac{\mu_0 \pi a_1^2 I_1}{2a_2}\cos\theta \quad .
\end{aligned} \tag{15}$$

Naturally this matches expression (5), found the **easy** way. We now compute

$$M_{21} \;=\; \frac{\Phi_2}{I_1} \;=\; \frac{\mu_0 \pi a_1^2}{2a_2}\cos\theta \quad , \tag{16}$$

showing that $M_{21} = M_{12}$ from eq. (4).

10) Solution

A superposition quantum state can be written as:

$$\left|\Psi(t)\right\rangle = \sum_n c_n \left|n\right\rangle e^{-iE_n t/\hbar}$$

where the $\left|n\right\rangle$ are the eigenstates of the time independent Hamiltonian with eigenvalues of the energy E_n and coefficients c_n that satisfy the normalization requirement:

$$\sum_n \left|c_n\right|^2 = 1$$

Now consider the interaction of the eigenstates $\left|n\right\rangle$ with a time dependent perturbation given by

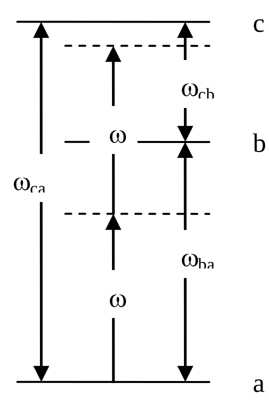
$$H'(t) = \begin{cases} V\cos(\omega t), & t \geq 0 \\ 0, & t < 0 \end{cases},$$

where V is the spatial part of the perturbation that leads to coupling between the states. This time dependent perturbation leads to a coupling between the coefficients c_n that can be shown via Schrödinger's equation to be given by:

$$c_n(t) = -\frac{i}{\hbar}\int_0^t H'_{nm}(t')e^{i\omega_{nm}t'}c_m(t')dt' + c_n(0)$$

where $H'_{ba}(t) = \left\langle b\right|H'(t)\left|a\right\rangle$ **and the frequency is related to the energies by** $\omega_{ij} = \dfrac{E_i - E_j}{\hbar}$.

Now let this perturbation interact with a three level system as shown in the figure. We assume that only these three states are accessible. Assume the initial conditions on the coefficients for the superposition state are:
$c_a(0) = 1,\ c_b(0) = 0$ and $c_c(0) = 0$.
Also assume that state c has the same symmetry as state a so that V_{ac}=0 but that both V_{ab} and V_{bc} are nonzero. Here the notation is $V_{ba} = \left\langle b\right|V\left|a\right\rangle$.



a) Use first order time dependent perturbation theory to find the time dependent coefficient for the first excited state b $c_b^{(1)}(t)$. Simplify your answer by dropping the non-resonant term (some books call this the the Rotating Wave Approximation).

b) Now use your simplified result from a) and also second order time dependent perturbation theory to find the time dependent coefficient for the second excited state $c_c^{(2)}(t)$. Again, use the Rotating Wave Approximation to drop the non-resonant terms.

c) Now let $\left|\omega_{ca} - 2\omega\right| \ll \left|\omega_{cb} - \omega\right|$. Using this approximation, find an expression for the value of the product $\left|V_{cb}\right|\left|V_{ba}\right|$ needed to get the maximum probability $\left|c_c^{(2)}\right|^2$ will be equal to 0.01.

d) For the case in c), find an expression for the time t_{max} to reach the maximum probability $\left|c_c^{(2)}\right|^2 = 0.01$ for the first time.

a) To find the first order perturbation for the coefficient for the first excited state we use the integral equation with the zeroth order values for $c_a(t')$ and $c_b(0)$:

$$c_b^{(1)}(t) = -\frac{i}{\hbar}\int_0^t H'_{ba}(t')e^{i\omega_{ba}t'}c_a^{(0)}(t')dt' + c_b^{(0)}(0)$$

$$= -\frac{i}{\hbar}\int_0^t V_{ba}\cos(\omega t')e^{i\omega_{ba}t'}(1)dt' + 0$$

$$= -\frac{iV_{ba}}{\hbar}\int_0^t\left[\frac{e^{i\omega t'}+e^{-i\omega t'}}{2}\right]e^{i\omega_{ba}t'}dt'$$

$$= -\frac{iV_{ba}}{2\hbar}\int_0^t\left[e^{i(\omega_{ba}+\omega)t'}+e^{i(\omega_{ba}-\omega)t'}\right]dt'$$

$$= -\frac{iV_{ba}}{2\hbar}\left\{\int_0^t e^{i(\omega_{ba}+\omega)t'}dt' + \int_0^t e^{i(\omega_{ba}-\omega)t'}dt'\right\}$$

$$= -\frac{iV_{ba}}{2\hbar}\left\{\left[\frac{e^{i(\omega_{ba}+\omega)t'}}{i(\omega_{ba}+\omega)}\right]_0^t + \left[\frac{e^{i(\omega_{ba}-\omega)t'}}{i(\omega_{ba}-\omega)}\right]_0^t\right\}$$

$$= -\frac{V_{ba}}{2\hbar}\left\{\left[\frac{e^{i(\omega_{ba}+\omega)t}-1}{(\omega_{ba}+\omega)}\right] + \left[\frac{e^{i(\omega_{ba}-\omega)t}-1}{(\omega_{ba}-\omega)}\right]\right\} \quad .$$

Now we make the rotating wave approximation and drop the first term getting:

$$c_b^{(1)}(t) = -\frac{V_{ba}}{2\hbar}\left[\frac{e^{i(\omega_{ba}-\omega)t}-1}{(\omega_{ba}-\omega)}\right]$$

b) Now we go to second order to find the coefficient for the second excited state we again use the integral solution but now put in the first order result for $c_b(t')$ and the starting value for $c_c(0)$:

$$c_c^{(2)}(t) = -\frac{i}{\hbar}\int_0^t H'_{cb}(t')e^{i\omega_{cb}t'}c_b^{(1)}(t')dt' + c_c^{(1)}(0)$$

$$= -\frac{i}{\hbar}\int_0^t V_{cb}\cos(\omega t')e^{i\omega_{cb}t'}\left\{-\frac{V_{ba}}{2\hbar}\left[\frac{e^{i(\omega_{ba}-\omega)t'}-1}{(\omega_{ba}-\omega)}\right]\right\}dt' + 0$$

$$= \frac{iV_{cb}}{\hbar}\frac{V_{ba}}{2\hbar(\omega_{ba}-\omega)}\int_0^t\left[\frac{e^{i\omega t'}+e^{-i\omega t'}}{2}\right]\left[e^{i(\omega_{ba}-\omega)t'}-1\right]e^{i\omega_{cb}t'}dt'$$

$$= \frac{iV_{cb}}{2\hbar}\frac{V_{ba}}{2\hbar(\omega_{ba}-\omega)}\int_0^t\left[e^{i(\omega_{cb}+\omega)t'}+e^{i(\omega_{cb}-\omega)t'}\right]\left[e^{i(\omega_{ba}-\omega)t'}-1\right]dt'$$

$$= \frac{iV_{cb}}{2\hbar}\frac{V_{ba}}{2\hbar(\omega_{ba}-\omega)}\left\{\int_0^t e^{i(\omega_{cb}+\omega+\omega_{ba}-\omega)t'}dt' - \int_0^t e^{i(\omega_{cb}+\omega)t'}dt' + \int_0^t e^{i(\omega_{cb}-\omega+\omega_{ba}-\omega)t'}dt' - \int_0^t e^{i(\omega_{cb}-\omega)t'}dt'\right\}$$

$$= \frac{iV_{cb}}{2\hbar}\frac{V_{ba}}{2\hbar(\omega_{ba}-\omega)}\left\{\left[\frac{e^{i(\omega_{cb}+\omega_{ba})t'}}{i(\omega_{cb}+\omega_{ba})}\right]_0^t - \left[\frac{e^{i(\omega_{cb}+\omega)t'}}{i(\omega_{cb}+\omega)}\right]_0^t + \left[\frac{e^{i(\omega_{ca}-2\omega)t'}}{i(\omega_{ca}-2\omega)}\right]_0^t - \left[\frac{e^{i(\omega_{cb}-\omega)t'}}{i(\omega_{cb}-\omega)}\right]_0^t\right\}$$

$$= \frac{V_{cb}}{2\hbar}\frac{V_{ba}}{2\hbar(\omega_{ba}-\omega)}\left\{\left[\frac{e^{i(\omega_{cb}+\omega_{ba})t}-1}{(\omega_{cb}+\omega_{ba})}\right] - \left[\frac{e^{i(\omega_{cb}+\omega)t}-1}{(\omega_{cb}+\omega)}\right] + \left[\frac{e^{i(\omega_{ca}-2\omega)t}-1}{(\omega_{ca}-2\omega)}\right] - \left[\frac{e^{i(\omega_{cb}-\omega)t}-1}{(\omega_{cb}-\omega)}\right]\right\}$$

Now we look for non-resonant terms and drop the first two terms. Thus we get:

$$\boxed{c_c^{(2)}(t) = \frac{V_{cb}}{2\hbar}\frac{V_{ba}}{2\hbar(\omega_{ba}-\omega)}\left\{\left[\frac{e^{i(\omega_{ca}-2\omega)t}-1}{(\omega_{ca}-2\omega)}\right] - \left[\frac{e^{i(\omega_{cb}-\omega)t}-1}{(\omega_{cb}-\omega)}\right]\right\}}$$

c) Now we use the fact that $|\omega_{ca}-2\omega| \ll |\omega_{cb}-\omega|$ to drop the last term to get:

$$c_c^{(2)}(t) = \frac{V_{cb}}{2\hbar}\frac{V_{ba}}{2\hbar(\omega_{ba}-\omega)}\left[\frac{e^{i(\omega_{ca}-2\omega)t}-1}{(\omega_{ca}-2\omega)}\right]$$

$$= \frac{iV_{cb}}{\hbar}\frac{V_{ba}e^{i(\omega_{ca}-2\omega)t/2}}{2\hbar(\omega_{ba}-\omega)(\omega_{ca}-2\omega)}\left[\frac{e^{i(\omega_{ca}-2\omega)t/2}-e^{-i(\omega_{ca}-2\omega)t/2}}{2i}\right]$$

$$= \frac{iV_{cb}}{\hbar}\frac{V_{ba}e^{i(\omega_{ca}-2\omega)t/2}}{2\hbar(\omega_{ba}-\omega)(\omega_{ca}-2\omega)}\sin\left[(\omega_{ca}-2\omega)t/2\right]$$

Now to get the probability to excite the atom into the level c we only need to take the modulus square of $c_c^{(2)}(t)$ to get:

$$P_c(t) = \left|c_c^{(2)}(t)\right|^2 = \left|\frac{iV_{cb}}{\hbar}\frac{V_{ba}e^{i(\omega_{ca}-2\omega)t/2}}{2\hbar(\omega_{ba}-\omega)(\omega_{ca}-2\omega)}\sin\left[(\omega_{ca}-2\omega)t/2\right]\right|^2$$

$$= \frac{|V_{cb}|^2|V_{ba}|^2}{4\hbar^4(\omega_{ba}-\omega)^2(\omega_{ca}-2\omega)^2}\sin^2\left[(\omega_{ca}-2\omega)t/2\right]$$

Now this will be a maximum in time when the sine=1, so that we can solve for $|V_{cb}||V_{ba}|$ to get:

$$\left|V_{cb}\right|\left|V_{ba}\right| = 2\hbar^2 \left|\omega_{ba} - \omega\right|\left|\omega_{ca} - 2\omega\right|\sqrt{P_c(t)}$$

$$= 2\hbar^2 \left|\omega_{ba} - \omega\right|\left|\omega_{ca} - 2\omega\right|\sqrt{0.01}$$

Thus we get:

$$\boxed{\left|V_{cb}\right|\left|V_{ba}\right| = 0.2\hbar^2 \left|\omega_{ba} - \omega\right|\left|\omega_{ca} - 2\omega\right|}$$

d) Finally we find the time needed to first reach this maximum. This will happen when:

$$\sin\left[\left(\omega_{ca} - 2\omega\right)t_{max} / 2\right] = 1$$

$$\rightarrow \left(\omega_{ca} - 2\omega\right)t_{max} / 2 = \frac{\pi}{2}$$

$$\boxed{t_{max} = \frac{\pi}{\left(\omega_{ca} - 2\omega\right)}}$$

end

11. One of the fundamental characteristics of the non-relativistic superfluid state is that the system behaves as if a non-zero fraction of the atoms is described by the same time-independent wave function $\psi(\vec{r})$ in non-relativistic quantum mechanics. Here, we explore some of the surprising properties of this macroscopic quantum fluid.

   (a) Explain why the general form of this wave function can be written $\psi(\vec{r}) = |\psi(\vec{r})|e^{iS(\vec{r})}$, where S is a purely real, scalar function.

   Any complex number can be written in terms of a real modulus and phase, i.e., $|\psi|$ and S, so this is true of any wave function.

   (b) Explain why both ψ and $\vec{\nabla}\psi$ are required to be continuous, single-valued functions within a confining cylindrical volume. From these conditions, determine the mathematical constraints on S.

   The wave function is required to be continuous and single-valued so that the probability density can be uniquely characterized with no holes. The gradient of the wave function is required to be continuous in regions where the potential is bounded and continuous so that the integrated Schrödinger Equation is valid. Because e^{iS} must be single-valued,

$$S(\rho, \phi + 2\pi k, z) = S(\rho, \phi, z) + 2\pi n$$

   where k and n are both integers. S must be continuous so that e^{iS} is continuous. The gradient of S, which will appear in the gradient of ψ, must also be continuous and single-valued.

   (c) Recall that the *probability current density* for a particle of mass m and charge q in a real vector potential $\vec{A}$ is defined

$$\vec{j} \equiv \frac{1}{2m}\left[\psi^*\left(-i\hbar\vec{\nabla} - \frac{q}{c}\vec{A}\right)\psi + \psi\left(+i\hbar\vec{\nabla} - \frac{q}{c}\vec{A}\right)\psi^*\right]$$

   in Gaussian units. We can further introduce the velocity field $\vec{v}$ such that $\vec{j} = |\psi|^2\vec{v}$. Determine a manifestly real expression for $\vec{v}$ in terms of the modulus and phase of ψ. For the case that the magnetic field is zero, show that $\vec{\nabla}\times\vec{v} = 0$, i.e., that the velocity field has zero *vorticity*, except at special singular points.

$$\begin{aligned}
\vec{j} &= \frac{1}{2m}\left[|\psi|e^{-iS}\left(-i\hbar\vec{\nabla} - \frac{q}{c}\vec{A}\right)|\psi|e^{iS} + |\psi|e^{iS}\left(+i\hbar\vec{\nabla} - \frac{q}{c}\vec{A}\right)|\psi|e^{-iS}\right] \\
&= \frac{1}{2m}\left[-i\hbar|\psi|\left(\vec{\nabla}|\psi| + i|\psi|\vec{\nabla}S\right) - \frac{q}{c}\vec{A}|\psi|^2 + i\hbar|\psi|\left(\vec{\nabla}|\psi| - i|\psi|\vec{\nabla}S\right) - \frac{q}{c}\vec{A}|\psi|^2\right] \\
&= \frac{1}{2m}\left[2\hbar|\psi|^2\vec{\nabla}S - \frac{2q}{c}\vec{A}|\psi|^2\right] = |\psi|^2\left(\frac{\hbar}{m}\vec{\nabla}S - \frac{q}{mc}\vec{A}\right) \\
\vec{v} &= \boxed{\frac{\hbar}{m}\vec{\nabla}S - \frac{q}{mc}\vec{A}} \qquad \textit{Independent of } |\psi|!
\end{aligned}$$

   The curl of $\vec{v}$ is the sum of the curl of the gradient of a scalar field ($\vec{\nabla}\times\vec{\nabla}S = 0$) and the curl of the vector potential ($\vec{\nabla}\times\vec{A} = \vec{B} = 0$).

(d) The *circulation* of any vector field is defined by its closed-path integral. For a superfluid, the circulation is maintained with zero dissipation. Evaluate the circulation of your general expression for $\vec{v}$ about an arbitrary point. Show that it is quantized in units of $2\pi\hbar/m$ for the case of an electrically neutral superfluid, *e.g.*, atomic helium in a terrestrial lab or neutrons in a neutron star. This results in the appearance of persistent vortices when the system is put in rotation.

Electrically neutral particles means $q = 0$, so the term with the vector potential becomes irrelevant. I'll choose a path that makes k loops.

$$\oint \vec{v} \cdot d\vec{\ell} \;=\; \frac{\hbar}{m} \oint \vec{\nabla} S \cdot d\vec{\ell} = \frac{\hbar}{m} \left[S(\rho, \phi + 2\pi k, z) - S(\rho, \phi, z) \right] = \frac{\hbar}{m} 2\pi n = n \left(\frac{2\pi\hbar}{m} \right)$$

The Fundamental Theorem of Integral Calculus to the rescue!

(e) If, instead, the superfluid is *charged*, show that an additional term containing the *magnetic flux* appears in the circulation. The magnetic flux quantum is important in describing *superconductivity*.

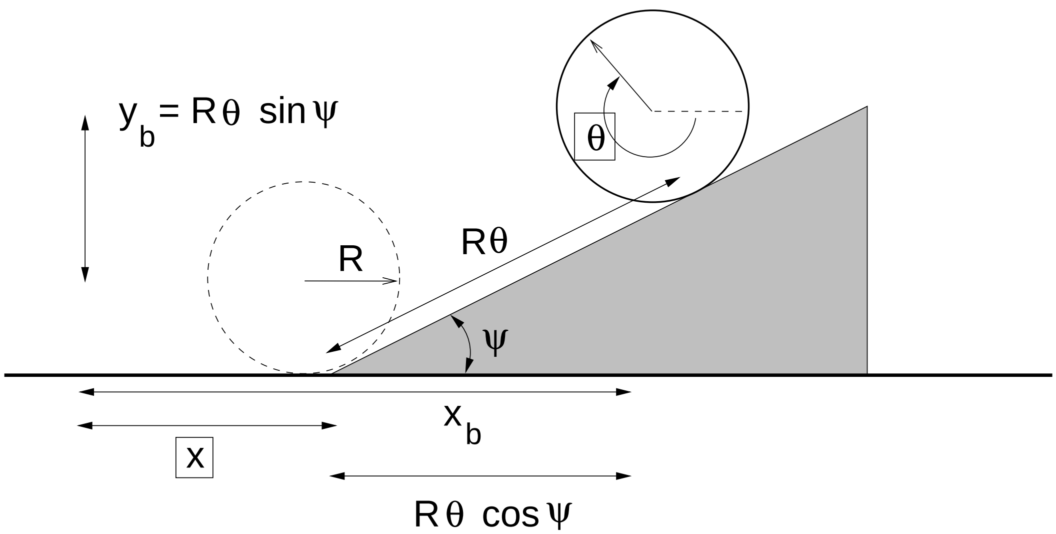
The term we've neglected in $\vec{v}$ is proportional to the vector potential.

$$-\frac{q}{mc} \oint \vec{A} \cdot d\vec{\ell} = -\frac{q}{mc} \int \vec{\nabla} \times \vec{A} \cdot d\vec{\sigma} = -\frac{q}{mc} \int \vec{B} \cdot d\vec{\sigma}$$

12. SOLUTION

a. Two generalized coordinates (2 degree-of-freedom system):

- θ — angle of the ball's orientation (clockwise). $\theta = 0$ corresponds to its orientation when it is at the bottom of the slope. So its distance up the slope is $R\theta$.
- x — horizontal displacement of the wedge on the table.



The position and velocity of the of the ball's center is

$$\mathbf{x}_b \;=\; (x + R\theta\cos\psi)\,\hat{\mathbf{x}} \;+\; R\theta\sin\psi\,\hat{\mathbf{y}} \quad , \quad \mathbf{v}_b \;=\; (\dot{x} + R\cos\psi\,\dot{\theta})\,\hat{\mathbf{x}} \;+\; R\sin\psi\,\dot{\theta}\,\hat{\mathbf{y}} \quad . \tag{1}$$

The ball's kinetic energy is thus

$$\begin{aligned} T_b \;&=\; \tfrac{1}{2}\,m_2|\mathbf{v}_b|^2 + \tfrac{1}{2}\,I\dot{\theta}^2 = \tfrac{1}{2}\,m_2(\dot{x} + R\cos\psi\,\dot{\theta})^2 + \tfrac{1}{2}\,m_2R^2\sin^2\psi\,\dot{\theta}^2 + \tfrac{1}{5}\,m_2R^2\dot{\theta}^2 \\ &=\; \tfrac{1}{2}\,m_2\dot{x}^2 \;+\; m_2R\cos\psi\,\dot{x}\dot{\theta} \;+\; \tfrac{7}{10}\,m_2R^2\dot{\theta}^2 \end{aligned} \tag{2}$$

The potential energy of the ball and kinetic energy of the wedge are

$$V_b \;=\; m_2gy_b \;=\; m_2gR\sin\psi\,\theta \quad , \quad T_w \;=\; \tfrac{1}{2}\,m_1\,\dot{x}^2 \quad . \tag{3}$$

The Lagrangian is therefore

$$\boxed{L(x,\dot{x}\,\theta,\dot{\theta}) \;=\; \tfrac{1}{2}\,(m_1 + m_2)\,\dot{x}^2 \;+\; m_2R\cos\psi\,\dot{x}\,\dot{\theta} \;+\; \tfrac{7}{10}\,m_2R^2\dot{\theta}^2 \;-\; m_2gR\sin\psi\,\theta \quad . \qquad (4)}$$

b. The Euler-Lagrange equations from this are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) \;=\; (m_1 + m_2)\,\ddot{x} \;+\; m_2R\cos\psi\,\ddot{\theta} \;=\; \frac{\partial L}{\partial x} \;=\; 0 \tag{5}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) \;=\; m_2R\cos\psi\,\ddot{x} \;+\; \tfrac{7}{5}\,m_2R^2\,\ddot{\theta} \;=\; \frac{\partial L}{\partial \theta} \;=\; -\,m_2gR\sin\psi \tag{6}$$

Equation (5) can be solved for $\ddot{\theta}$,

$$\ddot{\theta} \;=\; -\,\frac{m_1 + m_2}{m_2R\cos\psi}\,\ddot{x} \quad ,$$

and substituted into eq. (6)

$$m_2 R\cos\psi\,\ddot{x} - \tfrac{7}{5}\,m_2 R^2\,\frac{m_1+m_2}{m_2 R\cos\psi}\,\ddot{x} \;=\; R\frac{m_2\cos^2\psi - \frac{7}{5}(m_1+m_2)}{\cos\psi}\ddot{x} \;=\; -\,m_2 g R\sin\psi \quad .$$

This can be solved to yield the acceleration of the wedge

$$\boxed{\ddot{x} \;=\; \frac{5m_2\sin\psi\;\cos\psi}{7(m_1+m_2)-5m_2\cos^2\psi}\,g \;=\; \frac{5\sin\psi\;\cos\psi}{7(m_1/m_2)+2+5\sin^2\psi}\,g \quad . \qquad (7)}$$

The final expression makes clear that the wedge accelerates to the right — as expected.

13) Solution

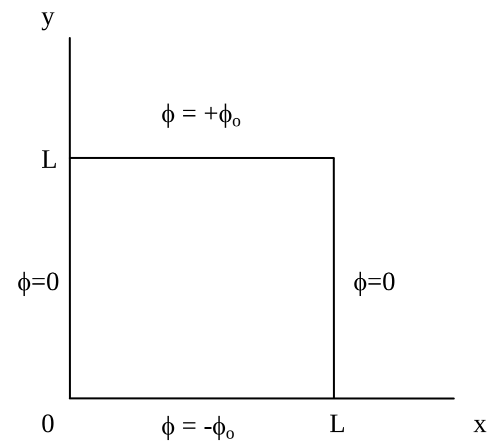
1. Take $\frac{dp}{dT} = \frac{L}{T\Delta v}$, where $\Delta v = v_g - v_l$, v_g is the volume of one mole of water vapor and v_l is the volume of one mole of liquid water. The volume of one mole of water vapor is much greater than that of liquid water (v_g / v_l ~800); hence, $\Delta v = v_g - v_l \approx v_g$. Using the ideal gas law for water vapor we can rewrite v_g as: $\Delta v \approx v_g = V/n = RT/p$. We insert this into dp/dT :
   $\frac{dp}{dT} = \frac{L}{T\Delta v} = \frac{Lp}{RT^2}$, which can easily be solved for *p*:
   $\int \frac{dp}{p} = \int \frac{L}{R} \frac{dT}{T^2} \quad \Rightarrow \quad \ln p / p_o = -\frac{L}{RT} \quad \Rightarrow \quad p = p_o e^{-L/RT}$ where p_o is an integration constant. To determine p_o we use the boiling point of water at ambient pressure:
   $p_o / p \approx e^{(40{,}000/8.31 \times 373)} \approx e^{13} \approx 4 \times 10^5$, where p = 1 atm. This yields $p_o = 4 \times 10^{10}\ N/m^2$.
2. As implied in $p(T) = p_o \exp(-L/RT)$ the boiling point of water is very much a function of pressure. For example, the boiling point of water in Bozeman (at ~5000 ft) is about 95 ^oC. The boiling point of water would be much higher under high pressure. At a depth of h below the surface *p* vs. *T* can be written as: $p = p_s + \rho g h = p_o e^{-L/R(T_s + \alpha h)}$ where p_s is the atmospheric pressure on the surface at the ground level and *g* is the acceleration due to gravity. This equation can be solved numerically (we did not expect you to solve it). The solution is $h \approx 200$ m, temperature $T = T_s + \alpha h$ ~ $220\ ^oC$ (assuming T_s ~ 20 ^oC) and $p = p_s + \rho g h \approx 20$ atm.

When boiling starts at a depth of 200 m, the steam bubbles rise and push the water out of the flue, the pressure decreases to ~1 atm and the superheated water at ~220 ^oC starts to boil violently, producing as much steam as is necessary to reduce its temperature to below 100 ^oC. The cycle continues after the reservior cools off to below the boiling point. After the eruption, underground springs in the subsurface region refill the cavity and the flue to the ground level within about half an hour. The heat transfer from the surrounding rocks to the cooled reservoir takes another hour to bring the reservoir water to the boiling point, and the process repeats itself.

14) Solution

Find the potential ϕ at all points <u>inside</u> a region given by $0<x<L$ and $0<y<L$, and $-\infty<z<\infty$. The potentials for the four boundary planes are given by:

$$\phi(0,y,z)=0$$
$$\phi(L,y,z)=0$$
$$\phi(x,0,z)=-\phi_o$$
$$\phi(x,L,z)=+\phi_o$$



This is a boundary value problem that we can solve using the general form for the solution which is:

$$\phi(x,y,z)=\sum_{\alpha,\beta,\gamma}\left(a_1e^{\alpha x}+a_2e^{-\alpha x}\right)\left(b_1e^{\beta y}+b_2e^{-\beta y}\right)\left(c_1e^{\gamma z}+c_2e^{-\gamma z}\right).$$

with the restriction that

$$\alpha^2+\beta^2+\gamma^2=0.$$

Since ϕ is z independent, set γ=0. In this case our restriction becomes:

$$\alpha^2+\beta^2=0.$$

Also let $c_1+c_2=1$ since we still have the a's and b's to match the boundary conditions.
Now since ϕ=0 at x=0 and x=L, we expect our solution to have a sin(x) type of term. Thus to get the sin(x) form, we let β be real so that α is pure imaginary with $\alpha=i\beta$ since in this problem $\alpha^2+\beta^2=0$.
Also to get the sin(x) form, we let $a_2=-a_1$. Also let $a_1=(1/2i)$ since we still have the b's to get the overall constants.
With all of these changes our solution form now becomes:

$$\phi(x,y)=\sum_{\beta}\left(\frac{e^{+i\beta x}+e^{-i\beta x}}{2i}\right)\left(b_1e^{\beta y}+b_2e^{-\beta y}\right)$$
$$=\sum_{\beta}\sin(\beta x)\left(b_1e^{\beta y}+b_2e^{-\beta y}\right)$$

Note that this form satisfies ϕ=0 at x=0. Now to satisfy ϕ=0 at x=L, we need sin(βx)=0, so that we have a restriction on β which is:

$$\boxed{\beta=\frac{n\pi}{L}}$$

Now we need to find b_1 and b_2. Note that by symmetry of the boundary conditions, ϕ(y=L/2)=0, so that we need:

$$b_1e^{\beta L/2}+b_2e^{-\beta L/2}=0$$
$$\rightarrow b_2=-b_1e^{\beta L/2}e^{\beta L/2}=-b_1e^{\beta L}$$

so that our solution becomes:

$$\begin{aligned}\phi(x,y) &= \sum_{\beta} b_1 \sin(\beta x)\left(e^{\beta y} - e^{-\beta y}e^{\beta L}\right)\\ &= \sum_{\beta} b_1 \sin(\beta x) e^{\beta L/2}\left(e^{\beta y}e^{-\beta L/2} - e^{-\beta y}e^{\beta L/2}\right)\\ &= \sum_{\beta} b_1 \sin(\beta x) e^{\beta L/2}\left(e^{\beta(y-L/2)} - e^{-\beta(y-L/2)}\right)\\ &= \sum_{\beta} b_1 \sin(\beta x) e^{\beta L/2} 2\sinh\left[\beta(y-L/2)\right]\\ &= \sum_{\beta} B\sin(\beta x)\sinh\left[\beta(y-L/2)\right] \qquad \text{with } B = 2b_1 e^{\beta L/2}\end{aligned}$$

Now if we put our value of $\beta = \frac{n\pi}{L}$ into this equation, our solution becomes:

$$\phi(x,y) = \sum_n B_n \sin\left(\frac{n\pi x}{L}\right)\sinh\left[\frac{n\pi(y-L/2)}{L}\right]$$

where we now sum over the integer n.

All that is left is to find the values for the coefficients B_n. To do this, we use the boundary value at y=L to get:

$$\begin{aligned}\phi(x,y=L) = +\phi_o &= \sum_n B_n \sin\left(\frac{n\pi x}{L}\right)\sinh\left[\frac{n\pi(L-L/2)}{L}\right]\\ &= \sum_n B_n \sin\left(\frac{n\pi x}{L}\right)\sinh\left(\frac{n\pi}{2}\right)\end{aligned}$$

Now to find the coefficients B_n, we multiply both sides of this equation by sin(mπx/L) and integrate over x to get:

$$\begin{aligned}\int_0^L \phi_o \sin\left(\frac{m\pi x}{L}\right)dx &= \sum_n B_n \sinh\left(\frac{n\pi}{2}\right)\int_0^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx\\ &= \sum_n B_n \sinh\left(\frac{n\pi}{2}\right)\left(\frac{L}{2}\delta_{nm}\right)\\ &= B_m \sinh\left(\frac{m\pi}{2}\right)\left(\frac{L}{2}\right)\end{aligned}$$

Using this we can solve for B_n:

$$\begin{aligned}B_n &= \frac{2}{L\sinh\left(\frac{m\pi}{2}\right)}\int_0^L \phi_o \sin\left(\frac{n\pi x}{L}\right)dx = \frac{2\phi_o}{L\sinh\left(\frac{n\pi}{2}\right)}\left[-\left(\frac{L}{n\pi}\right)\cos\left(\frac{n\pi x}{L}\right)\right]_0^L\\ &= \frac{2\phi_o}{L\sinh\left(\frac{n\pi}{2}\right)}\left(\frac{L}{n\pi}\right)\left[1-\cos(n\pi)\right] = \frac{2\phi_o}{(n\pi)\sinh\left(\frac{n\pi}{2}\right)}\left[1-\cos(n\pi)\right]\end{aligned}$$

Thus we find that:

$$B_n = \begin{cases} \dfrac{4\phi_o}{(n\pi)\sinh\left(\frac{n\pi}{2}\right)} & \text{for } n \text{ odd}\\ 0 & \text{for } n \text{ even}\end{cases}$$

Thus our final answer for the potential ϕ, given the boundary conditions, is:

$$\boxed{\phi(x,y)=\frac{4\phi_o}{\pi}\sum_{n\ \mathrm{odd}}\frac{1}{n}\sin\left(\frac{n\pi x}{L}\right)\frac{\sinh\left[\frac{n\pi(y-L/2)}{L}\right]}{\sinh\left(\frac{n\pi}{2}\right)}}$$

end

15) Solution

Null Rays around a Black Hole. The trajectory of a null ray (light) in the geometry of a Schwarzschild black hole, i.e. one with $SO(3)$ symmetry, can be written as $r(\phi)$, where $u \equiv GM/(c^2 r(\phi))$ satisfies the equation

$$\frac{d^2u}{d\phi^2} + u = 3u^2 \,, \tag{1}$$

with $r(\phi)$ is the distance from the black hole to the photon ray and M the black hole mass. Denote by b the distance of closest approach, i.e. the periholium, and consider a non-grazing collision so that $GM/(c^2 b) \ll 1$. Given this:

(i) Draw a diagram!

 (ia) Imagine first that the black hole mass is tiny, and draw the trajectory of the "unperturbed" ray, i.e. a ray that travels unaffected by the presence of the black hole. Label this trajectory r_0 or u_0.

 (ib) Imagine now that the black hole mass is not negligible, so that the light ray is deflected. Draw the trajectory for the "perturbed" path, so that both the perturbed and unperturbed rays are *initially* parallel to each other. Denote this trajectory by r_1 or u_1.

 (ic) Denote the angle between the unperturbed and the perturbed path $\delta\phi$. Indicate this angle in your diagram, as well as the locations of $r = \infty$ at $t = -\infty$ and $t = +\infty$, the distance b and the angle $\delta\phi$. Why are there no additional coordinates needed to solve this problem?

(ii) Solve for the unperturbed trajectory of light, i.e. for the leading-order in M/b solution? Interpret this solution physically, i.e. explain it in words, and compare it to your expectation from part (i).

(iii) Now solve for the perturbed trajectory to next-order in $GM/(c^2 b)$. What is the angular deflection, $\delta\phi$ of a photon from its unperturbed path as it passes the spherical body? You may assume that $\delta\phi$ is small and solve for it to leading order in $GM/(c^2 b)$.

First, since this is spherical body with SO(3) symmetry, you can pick an orbit on any plane you want, just like when studying orbits around a Newtonian $1/r$ potential. Be smart, and pick an orbit on the equatorial plane: $\theta = \pi/2$. The problem says nothing about loss of energy due to gravitational radiation, so you are free to ignore such dissipative effects. In the absence of these, the orbit is fully conservative and you could work within a Lagrangian formulation if you wanted. Note, however, that a Lagrangian is not needed! The only generalized coordinate is the angle ϕ, the azimuthal angle subtended as the body swings by the black hole from one tangent to the perturbed trajectory to the next as $r \to \infty$.

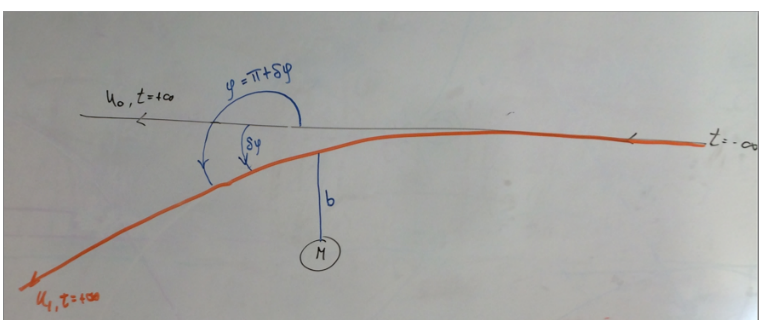


FIG. 1: Schematic diagram of the process described in the problem. This is one way to set up the problem; there are many others, but this is the simplest one.

Henceforth, in these solutions, I will use geometric units in which $G = 1 = c$. In the limit $M/b \ll 1$, the second term in Eq. (1) can be neglected, since it is higher-order in M/b. That is, the largest u can be is at $r = b$, but there $u(r = b) = M/b \ll 1$, and thus $u^2 \ll u$. Thus, the trajectory satisfies a simple harmonic oscillator equation. The leading-order solution is just

$$u_0 = A_0 \sin\phi + B_0 \cos\phi \,. \tag{2}$$

Using that at $t = -\infty$, $r = \infty$, and then $u_0 = 0$ let's say at $\phi = 0$, you find $B_0 = 0$. Using that $r(\pi/2) = b$, then $u_0(\pi/2) = M/b$ and $A_0 = M/b$. Notice that this is just a straight line up a distance b from the black hole, to leading order in M/b, as you should have guessed when you drew the diagram in part (i).

Now write $u = u_0 + \epsilon u_1 + \mathcal{O}(\epsilon^2)$, where u_0 is the unperturbed path, u_1 is a perturbation to u_0 and $\epsilon \ll 1$ is an order-counting parameter. Equation (1) then becomes

$$\frac{d^2 u_1}{d\phi^2} + u_1 = 3u_0^2 = 3\left(\frac{M}{b}\right)^2 \sin^2\phi = \frac{3}{2}\left(\frac{M}{b}\right)^2 (1 - \cos 2\phi)\ , \tag{3}$$

where in the last line we have done a spherical harmonic decomposition. The solution to this differential equation is the linear superposition of a homogeneous and a particular solution:

$$u_1 = u_{1,h} + u_{1,p}\,, \tag{4}$$

where the homogenous solution is functionally equal to u_0, namely

$$u_{1,h} = A_1 \sin\phi + B_1 \cos\phi\,, \tag{5}$$

while the particular solution can be solved by inspection to find

$$u_{1,p} = \frac{1}{2}\left(\frac{M}{b}\right)^2 (3 + \cos 2\phi) \tag{6}$$

The integration constants can again be found by imposing the initial conditions: $u(0) = 0$ and $u(\pi/2) = b$, which lead to $A_1 = -(M/b)^2$ and $B_1 = -2(M/b)^2$. You see then that the second-order solution modifies the integration constants slightly so that the full, perturbed solution reaches b at $\phi = \pi/2$ and $r = \infty$ when $\phi = 0$.

Now you can find the total deflection angle by calculating the angle ϕ_d at which $r(\phi_d) = \infty$, or alternatively $u(\phi_d) = 0$. Since the unperturbed path reaches $u_0 = 0$ at $\phi_d = \pi$, we linearize about that value: $\phi_d = \pi + \delta\phi$, where $\delta\phi \ll 1$, and in fact, $\delta\phi$ must be of $\mathcal{O}(M/b)$. One then finds

$$0 = \left(\frac{M}{b} - \frac{M^2}{b^2}\right)\sin\phi - 2\frac{M^2}{b^2}\cos\phi + \frac{1}{2}\frac{M^2}{b^2}(3 + \cos 2\phi)\ , \tag{7}$$

$$\mathcal{O}(\delta\phi^2) = -\left(\frac{M}{b} - \frac{M^2}{b^2}\right)\delta\phi + 2\frac{M^2}{b^2} + 2\frac{M^2}{b^2} \tag{8}$$

$$\delta\phi = \frac{4M}{b} + \mathcal{O}\left(\frac{M^2}{b^2}\right)\ . \tag{9}$$

You can't prove this here, but if you were using Newtonian gravity, then the differential equation for the trajectory of the photon would be slightly different and you would get the answer wrong by a factor of 2. The deflection of light was one of the first experimental verifications (a post-diction, really) of Einstein's theory of General Relativity, but of course, the deflecting body in that case was the Sun and Eddington had to wait for a Solar eclipse to measure it.