Scattering from a screened Coulomb potential.

$\boxed{\text{EXAMPLE}}$ Cross-section for screened Coulomb potential: $V(r) = \frac{Q_1 Q_2}{r} e^{-r/r_0}$.

1. According to Eq. (31), the relevant Fourier amplitude is...

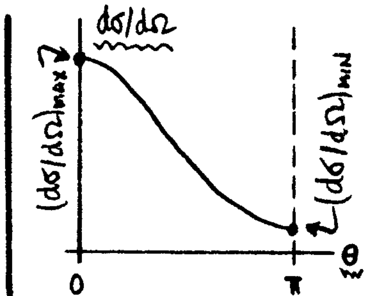
$$\tilde{V}(q) = \frac{4\pi}{q} Q_1 Q_2 \int_0^\infty e^{-r/r_0} \sin qr \, dr = \frac{4\pi Q_1 Q_2}{q^2 + (1/r_0^2)}$$

put in: $q = 2k\sin\frac{\theta}{2}$, $k^2 = 2mE/\hbar^2$;

so $\tilde{V}(q) = \frac{\pi\hbar^2}{2mE} Q_1 Q_2 / [\sin^2\frac{\theta}{2} + (1/\rho^2)]$, $\rho = 2kr_0$. (32a)

2. E is the incident energy of m (in CM cds). The differential cross-section is...

$$\rightarrow \frac{d\sigma}{d\Omega} = \left[\frac{m}{2\pi\hbar^2}|\tilde{V}(q)|\right]^2 = \left(\frac{Q_1 Q_2}{4E}\right)^2 \Big/ \left[\sin^2\frac{\theta}{2} + \frac{1}{\rho^2}\right]^2. \quad (32b)$$



$$\left(\frac{d\sigma}{d\Omega}\right)_{MAX} = \left(\frac{2mr_0^2}{\hbar^2} Q_1 Q_2\right)^2$$

$$\left(\frac{d\sigma}{d\Omega}\right)_{MIN} = \frac{(d\sigma/d\Omega)_{MAX}}{(1+\rho^2)^2}$$

This cross-section is finite for all scattering ∡s θ so long as the screening length r_0 is finite; it behaves as sketched at right. If $r_0 \to \infty$, so that we have a pure Coulomb scattering, then $\rho \to \infty$, and (32b) gives...

$$\rightarrow \frac{d\sigma}{d\Omega} = \left(\frac{Q_1 Q_2}{4E}\right)^2 \Big/ \sin^4(\theta/2), \text{ for } r_0 \to \infty \left(\begin{smallmatrix}\text{Coulomb}\\ \text{scattering}\end{smallmatrix}\right). \quad (32c)$$

This is recognized as the Rutherford cross-section★, and is exactly the same as the result calculated by classical means (for scattering of Q_1 by Q_2 interacting via a Coulomb force $Q_1 Q_2/r^2$).

3. The **total** scattering cross-section is: $\sigma = \int_{4\pi} (d\sigma/d\Omega)\, d\Omega$, integrated over all 4π solid ∡. For spherical symmetry: $d\Omega = 2\pi \sin\theta\, d\theta = 8\pi \sin\frac{\theta}{2}\, d\sin\frac{\theta}{2}$, so for the screened Coulomb case in (32b) [for all scattering, $0 \leq \theta \leq \pi$]:

$$\rightarrow \sigma = \left(\frac{Q_1 Q_2}{4E}\right)^2 \cdot 8\pi \int_0^1 x\,dx \Big/ \left(x^2 + \frac{1}{\rho^2}\right)^2 = \frac{\pi}{4}\left(\frac{Q_1 Q_2}{E}\right)^2 \frac{\rho^4}{1+\rho^2} = \frac{4\pi r_0^2}{1+\rho^2}\left(Q_1 Q_2 \frac{2mr_0}{\hbar^2}\right)^2. \quad (32d)$$

σ diverges if the screening length $r_0 \to \infty$. But, for an electron-ion scattering ($|Q_1| = e$, $Q_2 = Ze$) screened at $r_0 \sim a = \hbar^2/Zme^2$, get: $\sigma \simeq 16\pi a^2 / [1 + 8(\frac{E}{Ze^2/a})]$. At high E, have: $\sigma \simeq 2\pi a^2 [(Ze^2/a)/E]$. The E-falloff is observed in the lab.

★ See e.g. Fetter & Walecka "Theoretical Mechanics" (McGraw-Hill, 1980), Eq. (5.28).