(28) [15 pts]. Consider Schrödinger's Eqtn in 1D for a delta-fcn potential: $V(x) = -A\,\delta(x)$, A = +ve cnst. For such a singular potential, Ψ is continuous but Ψ' is not.

(A) By directly integrating the wave eqtn, show that the discontinuity in Ψ' at the origin is measured by: $\Psi'(0+) - \Psi'(0-) = -(2mA/\hbar^2)\Psi(0)$. Here, $0\pm$ means $\lim_{\epsilon\to 0}(0\pm\epsilon)$, i.e. $x=0$ approached from the right or left, resp.

(B) Using part (A), show that there is just one bound state in $V(x) = -A\,\delta(x)$. Find the energy of this bound state. Think about the eigenfunction.

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(29) The confluent hypergeometric ODE is: $z\frac{d^2F}{dz^2} + (c-z)\frac{dF}{dz} - aF = 0$, w/ c & a = cnsts.

(A) By direct substitution, show that a solution to this ODE can be written as a Confluent hypergeometric series: $F(a,c;z) = \frac{\Gamma(c)}{\Gamma(a)}\sum_{k=0}^{\infty}[\Gamma(a+k)/\Gamma(c+k)\Gamma(k+1)]z^k$.

(B) When $a \to (-)n$, w/ $n = 0, 1, 2, \ldots$, show that the series in part (A) reduces to a polynomial of degree n, viz. $F(-n,c;z) = \sum_{k=0}^{n}\frac{\Gamma(c)}{\Gamma(c+k)}\binom{n}{k}[-z]^k$, $\binom{n}{k}$ = binomial coefficient.

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(30) [15 pts] With $H_k(x)$ the Hermite polynomials ($k = 0, 1, 2, \ldots$):

(A) Consider the orthogonality integral: $I_{mn} = \int_{-\infty}^{\infty} e^{-x^2} H_m(x) H_n(x)\,dx$. Let $m \leq n$ to fix ideas. By suitable partial integrations, and using the properties: $H_k(x) = (-)^k e^{x^2}(d/dx)^k e^{-x^2}$, $(d/dx)H_k(x) = 2k\,H_{k-1}(x)$, Show that: $I_{mn} = (2^n n!\sqrt{\pi})\,\delta_{mn}$.

(B) By using the Hermite differential eqtn: $H_k''(x) - 2x H_k'(x) + 2k\,H_k(x) = 0$, and the relation: $H_k'(x) = 2k\,H_{k-1}(x)$, establish a recurrence formula for Hermite polynomials, viz: $H_{k+1}(x) - 2x H_k(x) + 2k\,H_{k-1}(x) = 0$.

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(31) For a 1D simple harmonic oscillator, use the position-momentum uncertainty relation to show that the lowest total energy in the system cannot be less than $\frac{1}{2}\hbar\omega$. So $E_n = (n+\frac{1}{2})\hbar\omega$ is consistent w/ Heisenberg, even when $n=0$.

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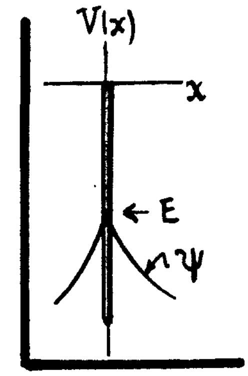
(28) [15 pts]. Find the bound state in an attractive delta-fcn well: $V(x) = -A\delta(x)$.



(A) 1. If there is a bound state, at energy $E<0$, then Schrödinger's Eqtn is...

$$\rightarrow \frac{d}{dx}\psi'(x) + \frac{2m}{\hbar^2}\left[E + A\delta(x)\right]\psi(x) = 0. \qquad (1)$$

Operate through this eqtn by $\lim_{\epsilon\to 0}\int_{-\epsilon}^{\epsilon} dx$. Then...

$$\rightarrow \lim_{\epsilon\to 0}\left\{\left[\psi'(0+\epsilon) - \psi'(0-\epsilon)\right] + \frac{2m}{\hbar^2}E\int_{-\epsilon}^{+\epsilon}\psi(x)\,dx + \frac{2m}{\hbar^2}A\,\psi(0) = 0\right\}. \qquad (2)$$

Since $\psi(x)$ is continuous near $x=0$, then $\int_{-\epsilon}^{+\epsilon}\psi(x)\,dx \simeq 2\epsilon\,\psi(0)$, by the mean value theorem. Then $\lim_{\epsilon\to 0}\int_{-\epsilon}^{+\epsilon}\psi(x)\,dx = 0$, and Eq. (2) reads -- as desired:

$$\boxed{\psi'(0+) - \psi'(0-) = -(2mA/\hbar^2)\,\psi(0)}. \qquad (3)$$

This condition does not depend on E; it is valid even if there is no bound state.

(B) 2. For a bound state set $E = -\hbar^2\kappa^2/2m$, with κ to be found. Eq. (1) is then:

$$\rightarrow \frac{d^2\psi}{dx^2} - \kappa^2\psi(x) = 0, \text{ for } x \neq 0 \Rightarrow \psi(x) = \begin{cases} F\,e^{+\kappa x}, & \text{for } x<0; \\ G\,e^{-\kappa x}, & \text{for } x>0 \end{cases} \qquad (4)$$

Continuity in ψ at $x=0$ requires $G=F$. The $\psi'(0)$ condition in Eq. (3) then requires (after cancelling the cnsts F & $G=F$):

$$\rightarrow -\kappa F - \kappa F = -(2mA/\hbar^2)\,F \Rightarrow \underline{\underline{\kappa = mA/\hbar^2}} \qquad (5)$$

This is satisfied for only one value of κ... the continuity (& discontinuity) conditions thus permit only one bound state, whose energy is...

$$\left[\!\left[\, E = -\frac{\hbar^2}{2m}\kappa^2\Big|_{\kappa \text{ in Eq.(5)}}, \text{ i.e. } \boxed{E = -mA^2/2\hbar^2} \right. \checkmark \text{ bound state energy.} \qquad (6)$$

3. The eigenfcn for this bound state is $\psi(x) = \text{cnst} \times e^{-\kappa|x|}$, from Eq. (4). This ψ has a cusp at $x=0$; that is the only part of ψ inside the well. ψ is normalizable, and extends for a distance $\Delta x \sim \frac{1}{\kappa} = \hbar^2/mA$ outside the well.

(29) Some aspects of confluent hypergeometric eqtns: series & polynomial solutions.

1. Write $\Gamma(k+1)=k!$ for $k=0,1,2,\ldots$. Note that...

(A) $\rightarrow \Gamma(a+k)/\Gamma(a) = a(a+1)(a+2)\cdots(a+k-1) = (a)_k \int$ Pochhammer symbol (see NBS Hndbk, p.256) (1)

So// $\underline{F(a,c;z) = \sum_{k=0}^{\infty} [(a)_k/(c)_k] \frac{z^k}{k!}}$, as alternate form of confl. hyper. series. (2)

Differentiate F once, and use fact that $(a)_{k+1} = a(a+1)_k^{\star}\ldots$

$$\left.\begin{aligned} \frac{dF}{dz} &= \sum_{k=1}^{\infty}\left[\frac{(a)_k}{(c)_k}\right]\frac{z^{k-1}}{(k-1)!} = \sum_{k=0}^{\infty}\left[\frac{(a)_{k+1}}{(c)_{k+1}}\right]\frac{z^k}{k!} = \frac{a}{c}F(a+1,c+1;z);\\ \frac{d^2F}{dz^2} &= \frac{a(a+1)}{c(c+1)}F(a+2,c+2;z), \text{ similarly.}\end{aligned}\right\} \quad (3)$$

2. Plug the results of (3) into the ODE, i.e. $\underline{zF''+(c-z)F'-aF=0}$, to obtain:

$$\frac{a(a+1)}{c(c+1)}zF(a+2,c+2;z) + \frac{a}{c}(c-z)F(a+1,c+1;z) - aF(a,c;z) = 0$$

$$\text{or}// \rightarrow \sum_{k=0}^{\infty}\frac{z^{k+1}}{k!}\left[\frac{(a)_{k+2}}{(c)_{k+2}} - \frac{(a)_{k+1}}{(c)_{k+1}}\right] + a\sum_{k=0}^{\infty}\frac{z^k}{k!}\left[\frac{(a+1)_k}{(c+1)_k} - \frac{(a)_k}{(c)_k}\right] = 0. \quad (4)$$

We've used the rule: $(a)_{k+n} = (a)_n(a+n)_k^{\star}$. We must show Eq.(4) is an identity, i.e. that the LHS vanishes. Then $F(a,c;z)$ of (2) will in fact be a solution to the ODE. Since $(a)_0=1$ (see Eq.(1)), then the $k=0$ term in the 2nd sum LHS in (4) vanishes, and that sum is $= a\sum_{k=0}^{\infty}\frac{z^{k+1}}{(k+1)!}\left[\frac{(a+1)_{k+1}}{(c+1)_{k+1}} - \frac{(a)_{k+1}}{(c)_{k+1}}\right]$. In combination with the 1st sum LHS in Eq.(4), we obtain:

$$\rightarrow \sum_{k=0}^{\infty}\frac{z^{k+1}}{k!}\left\{\frac{(a)_{k+2}}{(c)_{k+2}}\left[1+\frac{c}{k+1}\right] - \frac{(a)_{k+1}}{(c)_{k+1}}\left[1+\frac{a}{k+1}\right]\right\} = 0. \quad (5)$$

3. For an identity, we need to show that in (5) the $\{\,\}\equiv 0$, i.e. $(c+k+1)\frac{(a)_{k+2}}{(c)_{k+2}} \equiv (a+k+1)\frac{(a)_{k+1}}{(c)_{k+1}}$. But this is clearly true, since on the LHS: $(c+k+1)/(c)_{k+2} = 1/(c)_{k+1}$, and on the RHS: $(a+k+1)(a)_{k+1} = (a)_{k+2}$. Thus, in Eq.(5), the $\{\,\}\equiv 0$, and we have shown: $\underline{F=\sum_{k=0}^{\infty}[(a)_k/(c)_k]\frac{z^k}{k!}, \text{ satisfies: } zF''+(c-z)F'-aF=0}$. **QED**

$\star$ Can be extended to: $(a)_{k+n} = a(a+1)\cdots(a+n-1)(a+n)_k = (a)_n(a+n)_k$.

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4. Now we want to see what happens to $F(a,c;z)$ when $a \to (-)n$, w/ $n=0,1,2,\ldots$

It is clear from the definition of the Pochhammer symbol in Eq.(1), viz.

(B)

$$\rightarrow (a)_k = \Gamma(a+k)/\Gamma(a) = a(a+1)(a+2)\cdots(a+k-1), \tag{6}$$

that when $a=(-)n$, the last nonzero factor in $(a)_k$ will occur for $k=n$; when $k=n+1$, the last factor RHS in (6) vanishes. So $(-n)_k \equiv 0$ for all $k>n$, and the confluent hypergeometric series in Eq.(2) clearly reduces to a polynomial of degree n, viz.

$$\rightarrow \underline{F(-n,c;z) = \sum_{k=0}^{n} [(-n)_k/(c)_k]\frac{z^k}{k!}}, \tag{7}$$

$$\text{w/ } (-n)_k = (-)n(-n+1)(-n+2)\cdots(-n+k-1). \tag{8}$$

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5. $(-n)_k$ in Eq.(8) has k factors on the RHS. If we extract $(-)^k$, then...

$$(-n)_k = (-)^k n(n-1)(n-2)\cdots(n-(k-1)) = (-)^k \frac{n!}{(n-k)!}$$

or/

$$\rightarrow (-n)_k = (-)^k k!\binom{n}{k}, \quad \text{w/ } \binom{n}{k} = n!/k!(n-k)! \leftarrow \text{Binomial Coefficient} \tag{9}$$

Put this result into Eq.(7), and restore $(c)_k = \Gamma(c+k)/\Gamma(c)$. Then...

$$\rightarrow F(-n,c;z) = \sum_{k=0}^{n}\left[(-)^k k!\binom{n}{k}\Big/\frac{\Gamma(c+k)}{\Gamma(c)}\right]\frac{z^k}{k!}$$

$$\text{or/ } \boxed{F(-n,c;z) = \sum_{k=0}^{n}\left[\frac{\Gamma(c)}{\Gamma(c+k)}\binom{n}{k}\right](-z)^k}. \quad \underline{\text{QED}} \tag{10}$$

This is the required form of $F(a,c;z)$ when $a \to (-)n$.

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(30) [15 pts]. Hermite polynomials: orthogonality & recurrence relations.

(A)

1. For $I(m,n) = \int_{-\infty}^{\infty} e^{-x^2} H_m(x) H_n(x)\,dx$, insert Rodrigues' form for H_n, and integrate by parts once...

$$\rightarrow I(m,n) = (-)^n \int_{-\infty}^{\infty} H_m \left(\frac{d}{dx}\right)^n e^{-x^2} dx = (-)^n \left\{ \cancel{H_m \left(\frac{d}{dx}\right)^{n-1} e^{-x^2} \Big|_{-\infty}^{\infty}}^{\,0} - \int_{-\infty}^{\infty} \overbrace{H_m'}^{\text{set} = 2m H_{m-1}} \left(\frac{d}{dx}\right)^{n-1} e^{-x^2} dx \right\}$$

$$= 2m \cdot (-)^{n-1} \int_{-\infty}^{\infty} H_{m-1} \left(\frac{d}{dx}\right)^{n-1} e^{-x^2} dx = 2m\, I(m-1, n-1). \quad (1)$$

After $k \leq m$ such partial integrations, this recursion relation predicts that $I(m,n) = 2^k \cdot m(m-1)\cdots(m-(k-1))\, I(m-k, n-k)$. So, if $k = m$ and $n \geq m$, then:

$$\rightarrow I(m,n) = 2^m m!\, I(0, n-m) = 2^m m! (-)^{n-m} \int_{-\infty}^{\infty} (d/dx)^{n-m} e^{-x^2} dx. \quad (2)$$

We've used the fact that $H_0(x) = 1$.

2. If $m = n$ in Eq. (2), the integral is: $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$ (tabulated), and so:

$$\rightarrow I(n,n) = 2^n n! \sqrt{\pi}. \quad (3)$$

On the other hand, if $n - m = p > 0$, the integral in Eq (2) vanishes, since $\int_{-\infty}^{\infty} (d/dx)^p e^{-x^2} dx = (d/dx)^{p-1} e^{-x^2} \Big|_{-\infty}^{\infty} = 0$. So we get, as desired...

$$\boxed{I(m,n) = (2^n n! \sqrt{\pi})\, \delta_{mn}}. \quad \textbf{QED} \quad (4)$$

(B)

3. The Hermite differential eqtn is: $2k H_k - 2x H_k' + \frac{d}{dx} H_k' = 0$. By directly substituting $H_k' = 2k H_{k-1}$ and cancelling factors $2k$ (assume $k > 0$):

$$\rightarrow H_k - 2x H_{k-1} + H_{k-1}' = 0. \quad (5)$$

Put $k = n+1$, and use $H_n' = 2n H_{n-1}$ for the 3rd term LHS in (5)...

$$\boxed{H_{n+1} - 2x H_n + 2n H_{n-1} = 0}, \quad n \geq 0. \quad \textbf{QED} \quad (6)$$

As required.

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(31) Minimum SHO energy via uncertainty relations.

1. The mean (total) SHO energy -- W/ x = position & p = momentum -- is given by

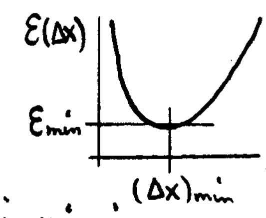
$$\rightarrow \bar{E} = \frac{1}{2}\,\overline{p^2}/m + \frac{1}{2} m\omega^2\, \overline{x^2}, \tag{1}$$

where m = mass of the oscillating particle, and ω is its oscillation frequency. As the oscillator sinks lower in energy, $\overline{p^2}$ and $\overline{x^2}$ are replaced by the QM fluctuations $(\Delta p)^2$ and $(\Delta x)^2$ allowed by the uncertainty relations.

$$\text{So} \rightarrow \bar{E} = \frac{1}{2m}(\Delta p)^2 + \frac{1}{2} m\omega^2 (\Delta x)^2. \tag{2}$$

But Δp & Δx are related by $\Delta p\, \Delta x \geqslant \frac{1}{2}\hbar$ (NOTES: p. Prop. 20, Eq. (13)). Then, since $\Delta p \geqslant \frac{1}{2}\hbar/\Delta x$, Eq. (2) yields...

$$\bar{E} \geqslant \frac{\hbar^2}{8m}\frac{1}{(\Delta x)^2} + \frac{1}{2} m\omega^2 (\Delta x)^2 = \mathcal{E}(\Delta x). \tag{3}$$



The SHO mean energy $\bar{E}$ will be minimum when $\mathcal{E}(\Delta x)$ is minimum.

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2. Finding the minimum in $\mathcal{E}(\Delta x)$...

$$\frac{\partial \mathcal{E}}{\partial(\Delta x)} = -\frac{\hbar^2}{4m}\frac{1}{(\Delta x)^3} + m\omega^2(\Delta x) = 0 \Rightarrow \underline{\underline{(\Delta x)^2 = \hbar/2m\omega}} \text{ @ min.} \tag{4}$$

$$\text{So} \rightarrow \mathcal{E}_{min} = \frac{\hbar^2}{8m}\left(\frac{2m\omega}{\hbar}\right) + \frac{1}{2} m\omega^2\left(\frac{\hbar}{2m\omega}\right) = \underbrace{\frac{\hbar\omega}{4}}_{\text{kinetic}} + \underbrace{\frac{\hbar\omega}{4}}_{\text{potential}} = \frac{\hbar\omega}{2}. \tag{5}$$

Note that the kinetic & potential energy contributions to $\mathcal{E}_{min}$ are equal (this is the result of the Virial Theorem). The absolute minimum for the SHO total energy -- consistent with the uncertainty relations -- is then

$$\boxed{\bar{E}_{min} \geqslant \mathcal{E}_{min} = \frac{1}{2}\hbar\omega}. \tag{6}$$

As advertised.