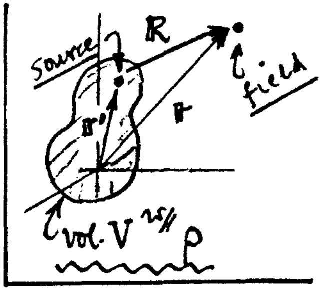
4. There is a feature of ϕ which we glossed over in Helmholtz' Thm, Viz...

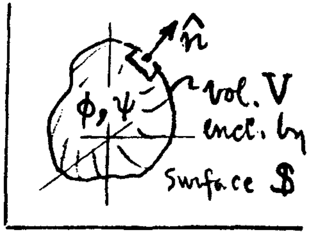
→ Solution: $\phi(\mathbf{r}) = \int_V \frac{\rho(\mathbf{r}')}{R}\, d^3x_i'$; $\int$ is complete only if the volume V contains all charges ρ. (8)



$\rho \to 0$ (always) at ∞, but what if we choose to integrate over a finite V such that some charges lie outside V?

What happens is that ϕ picks up an additional term (for finite V). To see how this happens, we use Green's Theorem [Jackson Eq. (1.35)]:

→ ϕ & ψ = any "smooth" (cont^s & twice differentiable) scalar fields in V enclosed by S. Then...



$$\int_V (\phi \nabla^2 \psi - \psi \nabla^2 \phi)\, dV = \oint_S \left(\phi \frac{\partial \psi}{\partial n} - \psi \frac{\partial \phi}{\partial n}\right) dS,$$

where: $\partial\psi/\partial n = \hat{n}\cdot\nabla\psi$, normal derivative on S. (9)

Proof is via Divergence Thm. We are free to choose any "smooth" ϕ & ψ.

Electrostatic ϕ (cont'd)

5. In Green's identity, we choose ...

$$\begin{cases} \phi = \text{desired solution to Poisson Eq.}: \ \nabla^2\phi = -4\pi\rho\,; & (10) \\ \psi = 1/R = \text{pt. source potential } (R = |\mathbf{r}-\mathbf{r}'|): \ \nabla^2\psi = -4\pi\,\delta(\mathbf{r}-\mathbf{r}')\,. \end{cases}$$

Plug into Green's Thm to get Jackson Eq. (1.36):

$$\boxed{\phi(\mathbf{r}) = \int_V \frac{\rho(\mathbf{r}')}{R}\,d^3x' + \frac{1}{4\pi}\oint_S \left[\frac{1}{R}\frac{\partial\phi}{\partial n'} - \phi\frac{\partial}{\partial n'}\left(\frac{1}{R}\right)\right] dS'}^{\dagger}\,. \qquad (11)$$

1^{ST} term RHS is recognized from previous Helmholtz Thm. The $\oint_S$ = new toy... it vanishes as $S\to\infty$ (which was tacit assumption for Helmholtz), but for S finite, it represents potential generated by charges on distant surfaces.

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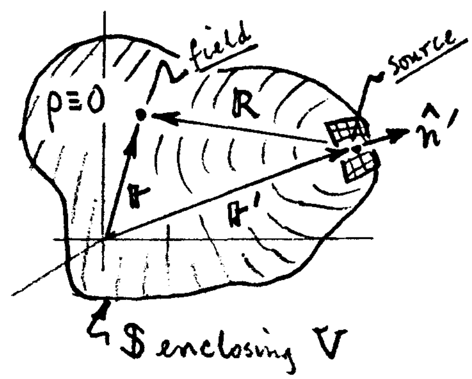
6. Suppose now we have the situation:



$\rho \equiv 0$ in vol. V, fld pt. $\mathbf{r}$ <u>inside</u> V,

and ϕ and/or $\dfrac{\partial\phi}{\partial n}$ known on S enclosing V;

$$\xrightarrow{\text{say}} \phi(\mathbf{r}) = \frac{1}{4\pi}\oint_S \left[\frac{1}{R}\left(\frac{\partial\phi}{\partial n'}\right) - \phi\frac{\partial}{\partial n'}\left(\frac{1}{R}\right)\right] dS'. \qquad (12)$$

This is ~ remarkable: in a charge-free region, ϕ is specified everywhere inside the region by values of ϕ and/or $(\partial\phi/\partial n')$ on any surrounding surface.

<u>CAVEAT</u> There is such a thing as too much information. Per Jk^{n} Sec. (1.9):

If, on S: ϕ is given, but <u>not</u> $\dfrac{\partial\phi}{\partial n'}$ indptly, then ϕ(inside) is unique [<u>Dirichlet</u>];

$\dfrac{\partial\phi}{\partial n'}$ is given, but <u>not</u> ϕ indptly, then ϕ(inside) is unique up to a constant [<u>Neumann</u>];

[Cauchy] → both ϕ & $\dfrac{\partial\phi}{\partial n'}$ are assumed given & indpt, ϕ(inside) generally doesn't exist.

This is because ϕ & $\mathbf{E} = -\nabla\phi$ are <u>not</u> indpt

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$\dagger$ $\mathbf{r}$ must lie <u>inside</u> S, otherwise LHS of Eq. (11) is $\equiv 0$.

7. In using Green's Thm, Eq. (9), we chose ϕ to be our electrostatic potential of interest, so $\nabla^2\phi = -4\pi\rho$, and we chose $\psi = 1/R$. More general & useful forms for ψ are possible. For example...

let// $\psi = \underbrace{1/R + F(\mathbf{r}, \mathbf{r}')}_{\text{call this fcn } G(\mathbf{r},\mathbf{r}')}$ with: $R = |\mathbf{r}-\mathbf{r}'|$, and F = an arbitrary scalar fcn w/ $\nabla^2 F = 0$;

so// $$\nabla^2\psi = \nabla^2 G = \nabla^2(1/R) + \cancelto{0}{\nabla^2 F} = -4\pi\,\delta(\mathbf{r}-\mathbf{r}'). \quad (13)$$

This $\psi = G$, a ϕ such that $\nabla^2\phi = -4\pi\rho$, and Green's Thm $\Rightarrow$

$$\boxed{\phi(\mathbf{r}) = \int_V \rho(\mathbf{r}')G(\mathbf{r},\mathbf{r}')\,d^3x' + \frac{1}{4\pi}\oint_S \left[G(\mathbf{r},\mathbf{r}')\frac{\partial\phi}{\partial n'} - \phi(\mathbf{r}')\frac{\partial G}{\partial n'}\right] dS'} \quad (14)$$

The G here is called a "Green's Function". It is a pt-source fcn in that $\nabla^2 G = -4\pi\delta(\mathbf{r}-\mathbf{r}')$ is singular. The degree of freedom inherent in choice of F can be used to make either the term in $(\partial\phi/\partial n')$ or $\phi(\mathbf{r}')$ in the surface integral vanish. We shall have use for the formulation in Eq. (14) later.

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8. A last (for now) use for ϕ is in calculating the electrostatic energy of an assembly of charges. We have seen in Eq. (6) that ϕ is related to work, so...

$dW = \phi(\mathbf{r})\,dq$ = work done on moving dq from ∞ $[\phi(\infty)=0]$ to position $\mathbf{r}$

... let $dq = \rho(\mathbf{r})\,d^3x$ be part of distrib$^{\underline{n}}$ $\rho(\mathbf{r})$, so $dW = \phi\rho\,d^3x$, and...

self energy

energy of assembly of distribution $\rho(\mathbf{r})$ $\Bigg\}$ $$\left[\!\left[W_E = \frac{1}{2}\int_{\substack{\text{all space}\\(\text{i.e. }\infty)}} \phi(\mathbf{r})\rho(\mathbf{r})\,d^3x \right]\!\right] \quad (15)$$ the factor $\frac{1}{2}$ corrects for counting $dq_1 \rightleftarrows dq_2$ twice.

Poisson... $\rho = -\frac{1}{4\pi}\nabla^2\phi$ $$\Rightarrow W_E = (-)\frac{1}{8\pi}\int_\infty \phi\left(\sum_i \frac{\partial}{\partial x_i}(\nabla\phi)_i\right)d^3x = +\frac{1}{8\pi}\int_\infty |\nabla\phi|^2 d^3x. \quad (16)$$

this step by partial integration and claim $\phi\nabla\phi \to 0$ at ∞.

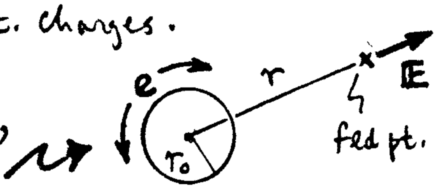
... and $\mathbf{E} = -\nabla\phi$ $\Bigg\}$ $$\boxed{W_E = \int_\infty (\mathbf{E}^2/8\pi)\,d^3x} \quad (17)$$ Remarkable! The energy resides in the field created while assembling ρ.

9. This calculation permits us to identify an important quantity, viz.

energy per unit vol. residing in field $\mathbb{E}$ $\Big\}$ $\boxed{u_E = \mathbb{E}^2/8\pi}$ (ergs/cm^3, in cgs). (18)

u_E exists by virtue of $\mathbb{E}$ itself, w/o detailed nature or structure of the sources. However, there is a basic problem with pt. charges.

... let $\mathbb{E}$ be generated by a "point charge" of size e, with e spread uniformly over sph. shell of rad. r_0...



$$E = \begin{cases} e/r^2, & r > r_0 \\ 0, & r < r_0 \end{cases} \Rightarrow W_E = \int_{r_0}^{\infty} \frac{1}{8\pi}\left(\frac{e}{r^2}\right)^2 \cdot 4\pi r^2 dr = \frac{e^2}{2r_0}. \quad (19)$$

This is the "self-energy" of the charge; it diverges as $r_0 \to 0$, and poses a fundamental difficulty with a theory of "actual" point e's. More, later.

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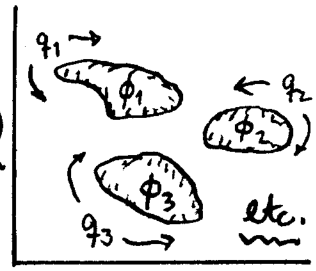
10. There is a useful formulation for the field energy when the sources can be localized on surfaces (like conducting spheres, etc.).



$\Bigg[$ System of n conductors: charges q_i, potentials ϕ_i $\Bigg\}$ $\phi_i = \sum_{j=1}^{n} p_{ij}\, q_j$, $i = 1 \to n$. (20) — geometry

ϕ_i is linear in the q_j (superposition!). The p_{ij} depend on the geometry of the conductors (dimensions, separations). Invert the series:

$$\to q_j = \sum_{k=1}^{n} C_{jk}\,\phi_k, \quad j = 1 \to n \quad (21)$$

$C_{j=k}$ are coeff^s of capacitance

$C_{j \neq k}$ " " " inductance

The C_{jk} are purely geometrical (and generally hard to calculate). But once they are determined, the system's (field) energy can be written:

$$\left[\!\left[W_E = \frac{1}{2}\sum_j q_j \phi_j = \frac{1}{2}\sum_{j,k} C_{jk}\,\phi_j \phi_k \right]\!\right]. \quad (22)$$

corrects for counting $j \leftrightarrow k$ twice.