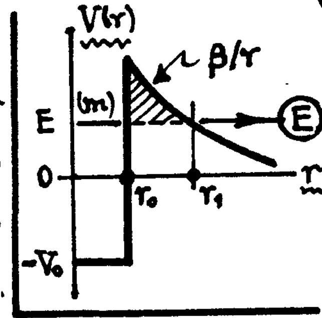
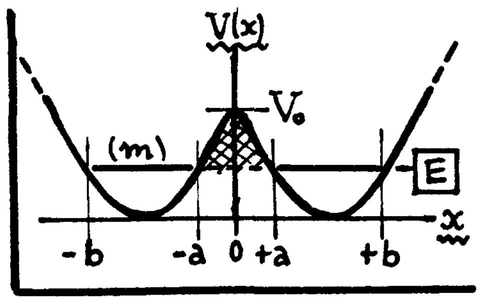


(28) [20 pts]. A particle of mass m and total energy $E>0$ is initially bound in a nuclear potential well of depth V_0 and radial size r_0. m tunnels thru the Coulomb barrier β/r, emerging at r_1 with zero ∡ momentum.

(A) Per WKB, calculate the probability $T(E)$ that the tunneling occurs. Show that for high barriers ($E \ll \beta/r_0$): $T(E) \simeq \exp\{-\frac{\pi\beta}{\hbar}\sqrt{2m/E}\}$, independent of r_0.

(B) Consider deuterium fusion: ${}_1H^2 + {}_1H^2 \rightarrow {}_2He^3 + n$ (3.2 MeV), by collisions between ${}_1H^2$ nuclei. Calculate the tunneling factor for ${}_1H^2 \rightarrow {}_1H^2$ penetration at room temperature (300°K).

(C) Consider ${}_1H^2$ gas at STP, w/ density n & thermal speed $\bar{v}$. The probability/unit time of ordinary collisions is: $\Gamma_0 = n\sigma_A\bar{v}$, w/ σ_A = atomic collision cross-section. The fusion rate is: $\Gamma_f = n\sigma_D\bar{v}\,T(\bar{v})$, w/ $\sigma_D = {}_1H^2$ nuclear cross-section. Approximate σ_A & σ_D as geometrical, and estimate Γ_f/Γ_0. Is "cold fusion" plausible?



(29) [30 pts]. A symmetric potential $V(x)$ consists of two wells separated by a barrier of height V_0 as shown. A particle of mass m and energy $E<V_0$ is initially placed in one well. m can tunnel thru the barrier ($-a \leq x \leq a$), coupling the wells.

(A) Use the WKB method to show that the condition determining the system eigenenergies is:

$$\boxed{\operatorname{ctn}\phi = \pm\frac{1}{2}e^{-\theta}} \quad \text{w/} \quad \begin{aligned}\phi &= \int_a^b k(x)\,dx, \quad k(x) = \sqrt{(2m/\hbar^2)[E-V(x)]};\\ \theta &= \int_a^b \kappa(x)\,dx, \quad \kappa(x) = \sqrt{(2m/\hbar^2)[V(x)-E]}.\end{aligned}$$

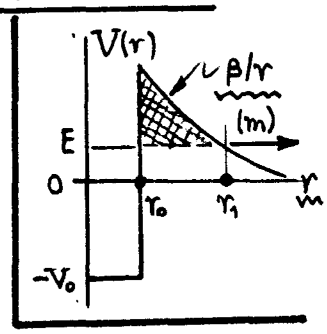
Please use this notation.

HINT: establish this condition by starting out with $\psi_1 = (A/\sqrt{\kappa})\,e^{-\int_x^{-b}\kappa\,dx'}$ in the region $x<-b$, and connecting $\psi_1 \rightarrow \psi_2 \rightarrow \psi_3 \rightarrow \psi_4 \rightarrow \psi_5$ in $x>b$. Make sure ψ_5 doesn't diverge.

(B) For $V_0 \gg E$, $\theta \rightarrow$ "large", and the condition of part (A) is: $\phi \simeq (n+\frac{1}{2})\pi \pm \frac{1}{2}e^{-\theta}$. Let $E_n^{(0)}$ be the nth energy level of either well alone (w/o barrier). Show that the presence of a penetrable barrier perturbs $E_n^{(0)}$ by an amount which is approximated to lowest order by: $\Delta E_n = \pm(\hbar\omega_n/2\pi)\exp\{-\int_{-a}^{+a}\sqrt{(2m/\hbar^2)[V(x)-E_n^{(0)}]}\,dx\}$. Here ω is the classical natural frequency of motion in the well, defined by: natural period $= \frac{2\pi}{\omega} = 2\int_a^b dx/[p(x)/m]$.

(C) Suppose the well is: $V(x) = \frac{1}{2}m\omega^2(|x|-x_0)^2$ [double SHO well]. Calculate the splitting ΔE_0 (in the $n=0$ ground state) explicitly in terms of ω & $V_0 = \frac{1}{2}m\omega^2 x_0^2$.

(3B) [20 pts]. Penetration of a Coulomb barrier (via WKB). Will "cold fusion" work?



(A) 1) The centrally symmetric problem reduces to a 1D motion along the radial direction r, and if the tunneling particle (m, E) has zero ∡ momentum, there is no centrifugal barrier term -- the potential in the tunneling region is just β/r. We can therefore use the transmission coefficient T of Eq.(11), p WKB 23 of classnotes directly:

$$\rightarrow T = \exp\left\{-\frac{2}{\hbar}J(E)\right\}, \text{ w/ } J(E) = \int_{r_0}^{r_1}\sqrt{2m[(\beta/r)-E]}\,dr. \qquad (1)$$

2) The initial barrier contact point is r_0 = nuclear radius, and the exit point r_1 is such that $\beta/r_1 = E$, i.e. $r_1 = \beta/E$. By a simple change of variables...

$$\rightarrow u = \frac{\beta}{Er} \Rightarrow J(E) = \beta\sqrt{\frac{2m}{E}}\int_1^{u_0}\frac{du}{u^2}\sqrt{u-1}, \text{ w/ } u_0 = \beta/Er_0. \qquad (2)$$

Integrals of this form are tabulated, and the result for J(E) is...

$$\rightarrow J(E) = \beta\sqrt{\frac{2m}{E}}\left[\tan^{-1}\sqrt{u_0-1} - \frac{1}{u_0}\sqrt{u_0-1}\right], \quad \underline{\underline{u_0 = \frac{\beta}{r_0}/E}}. \qquad (3)$$

Note that u_0 = ratio of initial barrier height to particle energy. In the limits...

$$\begin{cases} E\to 0+,\ u_0\to\infty:\ J(E) \simeq \frac{\pi}{2}\beta\sqrt{\frac{2m}{E}}\left[1-\frac{4}{\pi}(1/\sqrt{u_0})\right]; \\ E\to\frac{\beta}{r_0}-,\ u\to 1+:\ J(E) \simeq \beta\sqrt{2m/E}\,(u_0-1)^{3/2}/u_0. \end{cases} \qquad (4)$$

For high barriers, $\beta/r_0 \gg E$, $u_0\to$ large, and the tunneling probability is

$$\boxed{T(E) \simeq \exp\left(-\frac{\pi\beta}{\hbar}\sqrt{2m/E}\right)}. \qquad (5)$$

(B) 3) If the emergent particle is not relativistic (~always true), then in Eq.(5): $E = \frac{1}{2}mv_{out}^2$, and: $T(v_{out}) \simeq \exp(-2\pi\beta/\hbar v_{out})$, where v_{out} is the velocity of m outside the barrier. Furthermore, $\beta = e^2\times$(some factor f), so...

fs cnst: $\alpha = e^2/\hbar c \simeq 1/137$.

$$\rightarrow T(v_{out}) \simeq \exp\left(-2\pi f \frac{e^2}{\hbar c}\frac{c}{v_{out}}\right) = \exp[-2\pi f \alpha (c/v_{out})]. \quad (6)$$

For ${}_1H^2$ at room temperature (300°K), the K.E. is (1/38.7) eV, so

$$\frac{v_{out}}{c} = \sqrt{\frac{2E_{out}}{mc^2}} = \sqrt{\frac{2\times(1/38.7)}{2\times 932\times 10^6}} = 1/1.9\times 10^5. \quad (7)$$

(we've take $m = 2$ a.m.u. for ${}_1H^2$). With $f=1$ in Eq. (6) for a barrier penetration of ${}_1H^2$ by ${}_1H^2$ (both charged at $+e$), we find the tunneling factor in Eq. (6): $\underline{\underline{T(v_{out}) = e^{-871.4} = 3.6\times 10^{-379}}}$. Which is a mite small.

(C) 4) For ${}_1H^2$ gas at STP, $n = 2.7\times 10^{19}/cm^3$ (Loschmidt #), and $\bar{v} = c/190 = 1.58\times 10^8$ cm/sec. But these numbers drop out when we take the ratio of the collision rates...

$$\left.\begin{array}{l}\Gamma(\text{fusion}) = n\sigma_D \bar{v}\, T(\bar{v}) \\ \Gamma(\text{atomic}) = n\sigma_A \bar{v}\end{array}\right\} \quad \frac{\Gamma(\text{fusion})}{\Gamma(\text{atomic})} = \left(\frac{\sigma_D}{\sigma_A}\right) T(\bar{v}). \quad (8)$$

So it doesn't much matter whether we work with liquid or gaseous ${}_1H^2$. The geometrical cross-sections are: $\sigma_D \sim \pi\times(2\times 10^{-13}\,cm)^2$ *, $\sigma_A = \pi\times(0.53\times 10^{-8}\,cm)^2$, so: $\sigma_D/\sigma_A \sim 1.42\times 10^{-9}$, and the relative fusion reaction rate is

$$\rightarrow \Gamma(\text{fusion})/\Gamma(\text{atomic}) \sim 1.42\times 10^{-9}\, T(\bar{v}). \quad (9)$$

For room temp, $T(\bar{v}) = 3.6\times 10^{-379}$, as calculated in part (B), so then this ratio is: $\underline{\underline{\Gamma(\text{fusion})/\Gamma(\text{atomic}) \sim 5\times 10^{-388}}}$. At room temp, fusions occur "spontaneously" ~ one time per 2×10^{387} collisions. Does not appear too promising.

To make the fusion work, you have to heat the ${}_1H^2$ gas, to increase $T(E)$. At a temp $\sim 300\times 10^6$ °K, $E = 26$ keV, and $T(E) \simeq 1.64\times 10^{-4}$. Then $\Gamma(\text{fusion})/\Gamma(\text{atomic}) \sim 2\times 10^{-13}$, which begins to approach the realm of the possible.

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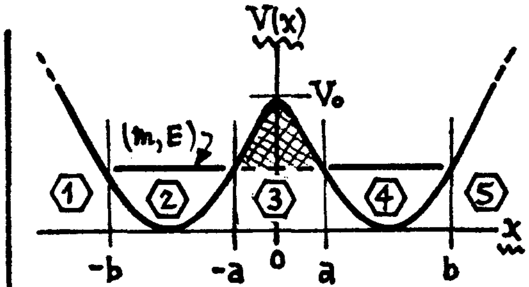
* Ref. A. Arya "Fund$^{\underline{s}}$ of Nuclear ϕ" (Allyn-Bacon 1966), p. 123: $r \simeq (1.35\times 10^{-13}\,cm)\times A^{1/3}$.

19 [30 pts]. Double-well analysis via WKB method.



$k(x) = \sqrt{\frac{2m}{\hbar^2}[E - V(x)]}$, in ② & ④;

$\kappa(x) = \sqrt{\frac{2m}{\hbar^2}[V(x) - E]}$, in ①, ③, ⑤.

1) Per hint, start with WKB fcn in region ① $(x < (-)b)$:

(A) $\rightarrow \psi_1 = \frac{A}{\sqrt{\kappa}} e^{-\int_x^{-b} \kappa(x')dx'}$, for $x < -b$. (1)

By the connection formulas [Eqs. (53) & (54), p. 18 of WKB Notes], $\psi_1 \rightarrow \psi_2 = \frac{2A}{\sqrt{k}} \sin\left(\int_{-b}^{x} k(x')dx' + \frac{\pi}{4}\right)$ in region ②. Refer the integral in ψ_2 to the RH edge $x = -a$ (via $\int_{-b}^{x} = \int_{-b}^{-a} - \int_{x}^{-a}$; this picks up a phase: $\underline{\underline{\phi = \int_{-b}^{-a} k(x)dx = \int_a^b k(x)dx}}$). So:

$$\rightarrow \psi_2 = \frac{2A}{\sqrt{k}} \left\{ (\cos\phi) \cos\left(\int_x^{-a} k\,dx' + \frac{\pi}{4}\right) + (\sin\phi) \sin\left(\int_x^{-a} k\,dx' + \frac{\pi}{4}\right) \right\}. \quad (2)$$

2) When $\psi_2 \rightarrow \psi_3$ in region ③, the $\cos(\ ) \rightarrow e^{+\int_{-a}^{x} \kappa dx'}$ by the connection formulas, while $\sin(\ ) \rightarrow \frac{1}{2} e^{-\int_{-a}^{x} \kappa dx'}$. Refer the new integrals to the RH edge of ③; this generates another "phase": $\underline{\underline{\theta = \int_{-a}^{+a} \kappa(x)dx}}$. Result is:

$$\rightarrow \psi_3 = \frac{2A}{\sqrt{\kappa}} (e^{\theta}\cos\phi) e^{-\int_x^a \kappa dx'} + \frac{A}{\sqrt{\kappa}} (e^{-\theta}\sin\phi) e^{+\int_x^a \kappa dx'}. \quad (3)$$

Continuing (literally), the $e^- \rightarrow 2\sin\left(\int_a^x k\,dx' + \frac{\pi}{4}\right)$ in going from ③ to ④, while the $e^+ \rightarrow \cos\left(\int_a^x k\,dx' + \frac{\pi}{4}\right)$. Again, shift reference points in the integrals, via $\int_a^x k\,dx' = \int_a^b k\,dx' - \int_x^b k\,dx'$. We again pick up: $\phi = \int_a^b k\,dx$, as phase. Then:

$$\psi_4 = \frac{4A}{\sqrt{k}} (e^{\theta}\cos\phi) \cos\left[\phi - \left(\int_x^b k\,dx' + \frac{\pi}{4}\right)\right] - \frac{A}{\sqrt{k}} (e^{-\theta}\sin\phi) \sin\left[\phi - \left(\int_x^b k\,dx' + \frac{\pi}{4}\right)\right]$$

or, $\rightarrow \psi_4 = \frac{A}{\sqrt{k}} \Big\{ [4e^{\theta}\cos^2\phi - e^{-\theta}\sin^2\phi] \cos\left(\int_x^b k\,dx' + \frac{\pi}{4}\right) +$

$$+ \left[(4e^{\theta} + e^{-\theta}) \sin\phi\cos\phi\right] \sin\left(\int_x^b k\,dx' + \frac{\pi}{4}\right) \Big\}. \quad (4)$$

3) Finally, continue $\psi_4 \rightarrow \psi_5$. In Eq. (4), the $\cos(\ ) \rightarrow e^{+\int_b^x \kappa dx'}$, and the $\sin(\ ) \rightarrow 2e^{-\int_b^x \kappa dx'}$. This specifies the WKB wavefcn in region ⑤ as...

$$\rightarrow \psi_5 = \frac{A}{\sqrt{K}} \underbrace{[4e^{\theta}\cos^2\phi - e^{-\theta}\sin^2\phi]}_{C} e^{+\int_b^x K dx'} + \frac{2A}{\sqrt{K}}[(4e^{\theta}+e^{-\theta})\sin\phi\cos\phi]\, e^{-\int_b^x K dx'}. \quad (5)$$

Now ψ_5 is in the classically inaccessible region ⑤, so it must <u>decrease</u> exponentially for $x > b$. This requires that the coefficient $C \equiv 0$, so-- as required...

$$\left[\begin{array}{l} C \equiv 0 \Rightarrow 4e^{\theta}\cos^2\phi = e^{-\theta}\sin^2\phi, \text{ or } \boxed{\mathrm{ctn}\,\phi = \pm\frac{1}{2}e^{-\theta}}, \\ \text{where}: \underline{\phi = \int_a^b k(x)\,dx}, \ \underline{\theta = \int_{-a}^{+a} K(x)\,dx}. \end{array}\right\} \quad (6)$$

(B) 4) For $\theta \to$ "large", $e^{-\theta} \to$ small, and the quantum condition of Eq. (6) is (approx'ly):

$$\left[\!\left[\phi = \int_a^b k(x)\,dx \simeq (n+\tfrac{1}{2})\pi \pm \tfrac{1}{2}e^{-\theta}\right]\!\right]. \quad (7)$$

Now if $E_n^{(0)}$ are the energy levels of either well separately, then

$$\rightarrow \int_a^b k_n^{(0)}(x)\,dx = (n+\tfrac{1}{2})\pi, \text{ w/ } k_n^{(0)}(x) = \sqrt{(2m/\hbar^2)[E_n^{(0)} - V(x)]}, \quad (8)$$

by the Bohr-Sommerfeld rule. The term in $e^{-\theta}$ in Eq. (7) perturbs the energies: $E_n^{(0)} \to E_n = E_n^{(0)} + \Delta E_n$; so also $k_n^{(0)}(x) \to k_n(x) = \sqrt{(2m/\hbar^2)[E_n - V(x)]}$. Then for small ΔE_n, $k_n(x)$ can be expanded as

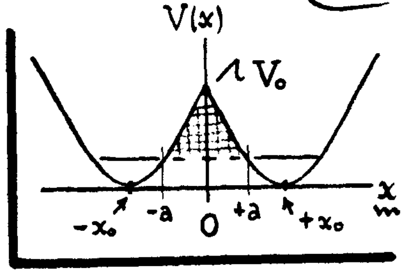
$$\rightarrow k_n(x) = \left(\frac{2m}{\hbar^2}[E_n^{(0)} + \Delta E_n - V(x)]\right)^{\frac{1}{2}} \simeq k_n^{(0)}(x) + \frac{m}{\hbar}\Delta E_n \Big/ \sqrt{2m[E_n^{(0)} - V(x)]}. \quad (9)$$

Identify: $\int_a^b k_n(x)\,dx = (n+\frac{1}{2})\pi \pm \frac{1}{2}e^{-\theta}$, by Eq. (7). Then, with (8), (9) yields

$$\rightarrow \frac{m}{\hbar}\Delta E_n \int_a^b dx / p_n^{(0)}(x) \simeq \pm \tfrac{1}{2}e^{-\theta}, \text{ w/ } p_n^{(0)}(x) = \sqrt{2m[E_n^{(0)} - V(x)]}. \quad (10)$$

On the LHS here: $m\int_a^b dx/p_n^{(0)}(x) = \frac{1}{2}(2\pi/\omega_n)$, w/ ω_n the natural frequency in the (unperturbed) state. So Eq. (10) gives the energy splitting due to tunneling:

$$\boxed{\Delta E_n \simeq \pm(\hbar\omega_n/2\pi)\exp\left[(-)\int_{-a}^{a} K(x)\,dx\right]}, \quad K(x) = \sqrt{(2m/\hbar^2)[V(x) - E_n^{(0)}]}. \quad (11)$$



5) We calculate the total splitting in the $n=0$ ground state,

(C) where the (unperturbed) energy is $E_0^{(0)} = \frac{1}{2}\hbar\omega$, and the natural frequency is ω. According to Eq. (11), this is:

$$\rightarrow \Delta E_0 = (\hbar\omega/\pi)\exp[-J], \text{ w/ } J = \int_{-a}^{a}\sqrt{(2m/\hbar^2)[V(x)-E_0^{(0)}]}\,dx. \qquad (12)$$

Put in: $V(x) = \frac{1}{2}m\omega^2(|x|-x_0)^2$, which is symmetric about $x=0$. Then...

$$\rightarrow J = 2\int_0^a \left\{\frac{2m}{\hbar^2}\left[\tfrac{1}{2}m\omega^2(x-x_0)^2 - E_0^{(0)}\right]\right\}^{1/2}dx$$

$$= 2\sqrt{2mE_0^{(0)}/\hbar^2}\int_0^a\left\{\frac{1}{2}\frac{m\omega^2}{E_0^{(0)}}(x-x_0)^2 - 1\right\}^{1/2}dx. \qquad (13)$$

Let $\xi^2 = (m\omega^2/2E_0^{(0)})(x_0-x)^2$, so: $dx = (-)\sqrt{2E_0^{(0)}/m\omega^2}\,d\xi$. The integral is...

$$\rightarrow J = 2\sqrt{\frac{2mE_0^{(0)}}{\hbar^2}\cdot\frac{2E_0^{(0)}}{m\omega^2}}\int_{x=a}^{x=0}\{\xi^2-1\}^{1/2}d\xi. \qquad (14)$$

Out in front here, the $\sqrt{} = 2E_0^{(0)}/\hbar\omega = 1$. The integral limit $x=0 \Rightarrow \xi = \xi_0 = \sqrt{m\omega^2/2E_0^{(0)}}\,x_0 = \sqrt{V_0/E_0^{(0)}}$, where $V_0 = V(0)$ is the barrier height. At the other limit $x=a$ (such that $V(a) = E_0^{(0)}$ is a turning point), we have $\xi = 1$. Thus...

$$J = 2\int_1^{\xi_0}\sqrt{\xi^2-1}\,d\xi, \text{ w/ } \underline{\underline{\xi_0}} = \sqrt{V_0/E_0^{(0)}} = \sqrt{m\omega/\hbar}\,x_0 \gg 1;$$

$$\xrightarrow{\text{or}} J = \xi_0\sqrt{\xi_0^2-1} - \ln\left(\xi_0+\sqrt{\xi_0^2-1}\right) \simeq \xi_0^2 - \ln 2\xi_0, \text{ for } \xi_0 \gg 1. \qquad (15)$$

$\xi_0 \gg 1$ because by WKB conditions, the particle energy $E_0^{(0)}$ must lie well below the barrier height. Put J of (15) into Eq. (12) to obtain the total splitting...

$$\boxed{\Delta E_0 = (\hbar\omega/\pi)\cdot 2\xi_0 e^{-\xi_0^2} = \frac{2\hbar\omega}{\pi}\sqrt{2V_0/\hbar\omega}\,e^{-(2V_0/\hbar\omega)}}, \qquad (16)$$

good for $V_0 \gg \frac{1}{2}\hbar\omega$. Considered as a fcn of $(2V_0/\hbar\omega)$, ΔE_0 actually goes thru a maxm @ $(2V_0/\hbar\omega) = \frac{1}{2}$. This is too small to qualify for the present approxn.