QM Operators for Translation & Rotation.

Angular Momentum : Summary

$\boxed{\text{refs. Sakurai, Secs. 3.1 \& 3.5; Davydov, ¶ 40.}}$

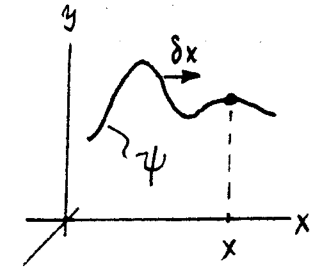
1) Conserved Qtys $\left\{\begin{array}{l}\text{A. linear mom. } \vec{P} \leftrightarrow \text{translational isotropy of space,} \\ \text{B. angular mom. } \vec{J} \leftrightarrow \text{rotational " " ",} \\ \text{C. energy .... } E \leftrightarrow \text{isotropy of time.}\end{array}\right.$

All accounted by Nöther's Theorem in classical mechanics.

(1) In QM : $\vec{P} = (\hbar/i)\vec{\nabla}$. Apply to ψ...

$$\rightarrow P_x\psi = (\hbar/i)\frac{\partial\psi}{\partial x} = \lim_{\delta x\to 0}\frac{\hbar}{i}\left[\frac{\psi(x)-\psi(x-\delta x)}{\delta x}\right]$$

$$= \lim_{\delta x \to 0} i\hbar\left[\frac{T(\delta x)-1}{\delta x}\right]\psi(x). \qquad (1)$$



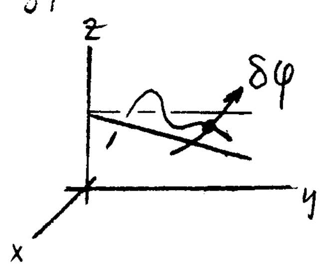
(ψ shifts, not cd. sys.)

$T(\delta x)$ is ∞smal translation operator : $\boxed{T(\delta x) = 1 - \frac{i}{\hbar}(\delta x) P_x.}$

For finite translations, can show : $T(\vec{r}) = \exp\left[-\frac{i}{\hbar}(\vec{r}\cdot\vec{p})\right]$. (2)

(2) For QM $\vec{J}$, define rotn operator $R(\delta\varphi)$ by analogy...

$$\boxed{R(\delta\varphi) = 1 - \frac{i}{\hbar}(\delta\varphi) J_z}, \qquad (3)$$



where $\left\{\begin{array}{l} J_z = \text{QM } \measuredangle \text{ mom. operator on } z\text{-axis,} \\ [R(\delta\varphi)]\psi \Rightarrow \psi \text{ rotated by } \delta\varphi \text{ around } z. \end{array}\right.$

For finite rotations... $\varphi = n\,\delta\varphi = \text{cnst}$ ($n\to\infty$ & $\delta\varphi\to 0$)

$$R(\varphi) = [R(\delta\varphi)]^n \to \lim_{n\to\infty}\left[1 - \frac{1}{n}\left(\frac{i}{\hbar}\varphi J_z\right)\right]^n = e^{-\frac{i}{\hbar}\varphi J_z},$$

and // $\boxed{R(\varphi) = e^{-\frac{i}{\hbar}\varphi(\hat{n}\cdot\vec{J})}}$ $\left\{\begin{array}{l}\text{rotn by } \varphi \text{ about axis } \hat{n} \\ \text{with : } \vec{J} = (J_x, J_y, J_z).\end{array}\right.$ (4)

$\vec{J}$ Summary Commutation Relations, Etc.

2) Have no explicit form for $\vec{J}$, in same way as $\vec{P} = -i\hbar\vec{\nabla}$. However, rules can be developed--from $R(\varphi)$-- which are adequate to handle $\vec{J}$ prob^ms.

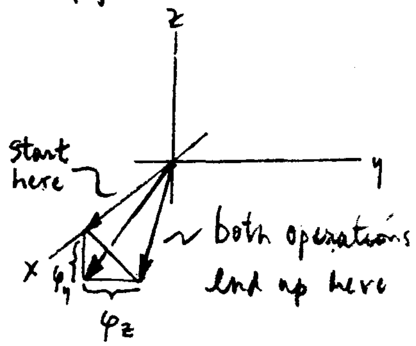
N.B. Finite rot^ns do not commute in gen'l ; $R(\varphi_x)R(\varphi_y) \neq R(\varphi_y)R(\varphi_x)$.

[Illustrate with book...]

However, these operations are related if φ_x & φ_y ~ "small". In fact...

$$\underline{R(\varphi_x)\,R(\varphi_y) = R(\varphi_z)\,R(\varphi_y)\,R(\varphi_x)}, \qquad (5)$$

↳ to order φ^2, with $\varphi_z = \varphi_x\varphi_y$.



Plug in: $R(\varphi_x) = \exp(-\frac{i}{\hbar}\varphi_x J_x)$, etc. Preserve order among the J_α. Calculate to $O(\varphi^2)$.

Get // $\boxed{[J_x, J_y] = i\hbar J_z}$, or // $\vec{J}\times\vec{J} = i\hbar\vec{J}$. (6)

This constraint on the operator form of $\vec{J}$ ⇒ sufficient to realize an explicit rep^n for any given problem.

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3) Properties which follow from the commutation relation above are...

(a) $\vec{J}$ is Hermitian ;

(b) $[\vec{J}^2, J_z] = 0$ ⇒ simultaneous eigenfcns of $\vec{J}^2$ & J_z ;

(c) $\vec{J}^2$ has eigenvalues $j(j+1)$, $j = \left\{\begin{matrix}\text{integral, or}\\ \text{half-integral}\end{matrix}\right\}$ only ;

(d) J_z has eigenvalues m, w/ $m = j, j-1, \ldots, -j$ ($2j+1$ values). (7)

Here we've set $\hbar = 1$. A useful relation to remember is...

$J_\pm = J_x \pm i J_y$ ← ladder operators ; $|jm\rangle$ = eigenfcn of $\vec{J}^2$ & J_z ⇒

(e) $\boxed{J_\pm |jm\rangle = [\,j(j+1) - m(m\pm1)\,]^{\frac{1}{2}} |j\; m\pm1\rangle}$ (8)

Clebsch-Gordan Transformation

refs. Sakurai, Sec. 3.7; Davydov, § 41.

1) Add two <u>independent</u> ∡ momenta $\vec{J}_1$ & $\vec{J}_2$ to form a resultant $\vec{J}$...

[[$\vec{J} = \vec{J}_1 + \vec{J}_2$ ✓ $\vec{J}_1$ & $\vec{J}_2$ each have eigenfcns $|j_1 m_1\rangle$ & $|j_2 m_2\rangle$ (resp^{t}), and they commute: $[J_{1\alpha}, J_{2\beta}] \equiv 0$ for <u>all</u> α & β.

Show easily that: $\vec{J} \times \vec{J} = i\vec{J}$, so $\vec{J}$ <u>is</u> an ∡ momentum. What are eigenfcns of $\vec{J}$? Try "uncoupled" rep^{n} for $\vec{J}$ eigenfcns by direct products:

$$\rightarrow |j_1 m_1\rangle \cdot |j_2 m_2\rangle = |j_1 j_2 m_1 m_2\rangle. \qquad (1)$$

These <u>are</u> eigenfcns of $J_z = (J_{1z} + J_{2z})$, as...

$$\rightarrow J_z |j_1 j_2 m_1 m_2\rangle = (m_1 + m_2)|j_1 j_2 m_1 m_2\rangle \leftarrow J_z \text{ diagonal}, \qquad (2)$$

... but they are <u>not</u> eigenfcns of $\vec{J}^2 = (\vec{J}_1 + \vec{J}_2)^2$, as...

$$\rightarrow \vec{J}^2 |j_1 j_2 m_1 m_2\rangle = [\, j_1(j_1+1) + j_2(j_2+1) + 2\vec{J}_1 \cdot \vec{J}_2 \,]|j_1 j_2 m_1 m_2\rangle; \qquad (3)$$

<u>mixes</u> states with $m_1 \pm 1$ & $m_2 \mp 1 \leftarrow \vec{J}^2$ <u>not</u> diag^{l}.

Can show: $\vec{J}_1 \cdot \vec{J}_2 = J_{1z} J_{2z} + \frac{1}{2}(J_1^{(+)} J_2^{(-)} + J_1^{(-)} J_2^{(+)})$,

We need to look for some linear comb^{n} of the $|j_1 j_2 m_1 m_2\rangle$ which renders $\vec{J}^2$ = diagonal. The transformation to such a linear comb^{n} ≡ Clebsch-Gordan transf^{n}. Hereafter, <u>abbreviate this by CGTr.</u>

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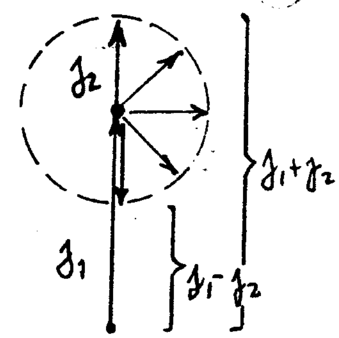
2) To see what to work with, recall that for a composite $\vec{J} = \vec{J}_1 + \vec{J}_2$ ...

[[Allowed j-values: $j = (j_1 + j_2), (j_1 + j_2) - 1, \ldots, |j_1 - j_2|$. $\qquad (4)$

The # j-values allowed here is $2j_1+1$ (if $j_1 \leq j_2$), or $2j_2+1$ (if $j_2 \leq j_1$). If $j_1 > j_2$, as shown at right, the j's add like vectors, but j_2 can assume only $(2j_2+1)$ discrete orientations w.r.t. j_1. Anyway, since each j-value requires $(2j+1)$ discrete m-values, the # eigenstates of $\vec{J}^2$ & J_z needed for our CGTr is...



$$\sum_{j_1-j_2}^{j_1+j_2} (2j+1) = (2j_1+1)(2j_2+1). \qquad (5)$$

Fortunately (and obviously), this is exactly the # of uncoupled states $|j_1 j_2 m_1 m_2\rangle$ available [from Eq. (1)]. We can look for a one-to-one mapping of these states onto the new ones, and write...

$$|j_1 j_2 j m\rangle = \sum_{\mu_1 \mu_2} C(j_1 j_2 \mu_1 \mu_2 j m)\, |j_1 j_2 \mu_1 \mu_2\rangle \qquad (6)$$

② ③ ①

① = uncoupled state, eigenfcn of $\vec{J}_1^2$, $\vec{J}_2^2$, J_{1z} & J_{2z} (via $j_1 j_2 \mu_1 \mu_2$);

② = coupled state, eigenfcn of $\vec{J}_1^2$, $\vec{J}_2^2$, $\vec{J}^2$ & J_z (via $j_1 j_2 j m$);

③ = Clebsch-Gordan coefficient, j_1 & j_2 are fixed.

N.B. Impose $C \equiv 0$, unless $\mu_1 + \mu_2 = m$, so that $|j_1 j_2 j m\rangle$ = eigenfcn of J_z.

Since j_1 & j_2 are fixed, drop them from the notation, and write...

$$\longrightarrow |jm\rangle = \sum_{\mu_1 \mu_2} C(\mu_1 \mu_2 j m)\, |\mu_1 \mu_2\rangle \Big|_{\mu_1+\mu_2=m},$$

so $$C(m_1 m_2 j m) = \langle m_1 m_2 | j m\rangle \Big|_{m_1+m_2=m}. \qquad (7)$$

This last relation follows from the orthogonality of the states $|\mu_1 \mu_2\rangle$.

3) Now generate explicit C's via ladder operators. Note first...

$$\left[\!\left[\left.\begin{matrix}\text{only possible eigenfcn} \\ \text{with } j=j_1+j_2,\ m=j\end{matrix}\right\} |j=j_1+j_2, m=j\rangle = 1 \cdot |m_1=j_1, m_2=j_2\rangle; \right.\right. \quad \text{(8)}$$

this $= \underline{C(j_1 j_2, j_1+j_2, j_1+j_2)}$.

Operate on this with step-down operator: $J^- = J_1^- + J_2^-$ ★ ...

$$\rightarrow \text{LHS} = J^-|j=j_1+j_2, m=j\rangle = \underbrace{[j(j+1)-j(j-1)]^{\frac{1}{2}}}_{=\sqrt{2j}\ =\ \sqrt{2(j_1+j_2)}}|j=j_1+j_2, m=j-1\rangle,$$

$$\rightarrow \text{RHS} = (J_1^- + J_2^-)|m_1=j_1, m_2=j_2\rangle = \sqrt{2j_1}\,|m_1=j_1-1, m_2=j_2\rangle +$$

$$+\sqrt{2j_2}\,|m_1=j_1, m_2=j_2-1\rangle,$$

and// LHS = RHS ⇒

$C(j_1, j_2-1, j_1+j_2, j_1+j_2-1)$ $\quad$ $C(j_1-1, j_2, j_1+j_2, j_1+j_2-1)$

$$\left[\!\left[|j=j_1+j_2, m=j-1\rangle = \sqrt{\frac{j_2}{j_1+j_2}}\,|m_1=j_1, m_2=j_2-1\rangle + \sqrt{\frac{j_1}{j_1+j_2}}\,|m_1=j_1-1, m_2=j_2\rangle. \right.\right. \quad \text{(9)}$$

Applying J^- again gives...

$C(j_1, j_2-2, j_1+j_2, j_1+j_2-2)$, etc.

$$\left[\!\left[|j=j_1+j_2, m=j-2\rangle = \sqrt{\frac{j_2(2j_2-1)}{\mathcal{D}}}\,|m_1=j_1, m_2=j_2-2\rangle + \right.\right.$$

$$+\sqrt{4j_1j_2/\mathcal{D}}\,|m_1=j_1-1, m_2=j_2-1\rangle + \sqrt{\frac{j_1(2j_1-1)}{\mathcal{D}}}\,|m_1=j_1-2, m_2=j_2\rangle, \quad \text{(10)}$$

with: $\underline{\underline{\mathcal{D}}} = (j_1+j_2)(2j_1+2j_2-1)$.

★ In problem ◯, you will show that $J^-|jm\rangle$ is an eigenfcn of $\vec{J}^2$, with the same j-value, and it is a eigenfcn of J_z, with eigenvalue $m-1$.

And so forth. By successive applications of J^-, you can step down the entire ladder of m-values for $j=j_1+j_2$. This is ~ simple for j_1 & j_2 = given (and "small"). It is evidently tedious for the general case.

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4) To get the eigenfcns for the next lowest j-value, $j \doteq (j_1+j_2)-1$, note that at the maxm m-value, can only have the combination...

$$\rightarrow |j=j_1+j_2-1, m=j\rangle = a\,|m_1=j_1, m_2=j_2-1\rangle + b\,|m_1=j_1-1, m_2=j_2\rangle; \qquad (11)$$

The cnsts a & b are fixed by imposing this eigenfcn is orthonormal to the one with the same m-value in the $j=j_1+j_2$ manifold, viz Eq.(9)...

$$|j=j_1+j_2-1, m=j\rangle \perp |j=j_1+j_2, m=j-1\rangle \Rightarrow \qquad (\ )$$

$$a\sqrt{j_2/(j_1+j_2)} + b\sqrt{j_1/(j_1+j_2)} = 0 \quad \Bigg\| \quad a = +\sqrt{j_1/(j_1+j_2)}$$

$$\text{and (norm)}: \ a^2+b^2 = 1 \ \ldots\ldots \quad \Bigg\| \quad b = (-)\sqrt{j_2/(j_1+j_2)} \qquad (12)$$

so $\Downarrow$

$$|j=j_1+j_2-1, m=j\rangle = \sqrt{j_1/(j_1+j_2)}\,|m_1=j_1, m_2=j_2-1\rangle - \sqrt{j_2/(j_1+j_2)}\,|m_1=j_1-1, m_2=j_2\rangle.$$

Now apply J_- to get (j_1+j_2-1) manifold. E.g. with $\underline{\underline{\Delta'}} = (j_1+j_2)(j_1+j_2-1)$...

$$\left[\ \begin{aligned} |j=j_1+j_2-1, m=j-1\rangle &= \sqrt{j_1(2j_2-1)/\Delta'}\,|m_1=j_1, m_2=j_2-2\rangle + \\ &+ \frac{j_1-j_2}{\sqrt{\Delta'}}\,|m_1=j_1-1, m_2=j_2-1\rangle - \sqrt{j_2(2j_1-1)/\Delta'}\,|m_1=j_1-2, m_2=j_2\rangle. \end{aligned} \right. \qquad (13)$$

These expressions agree with Schiff (3rd ed.) pp. 216-217. Tables of the C-coefficients appear in various references -- e.g. Condon & Shortley; see pp. 76-77 for the cases of j_1 = arbitrary, $j_2 = \frac{1}{2}, 1, \frac{3}{2}, 2$.