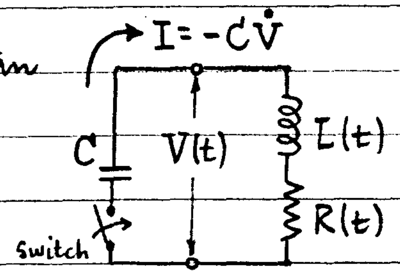
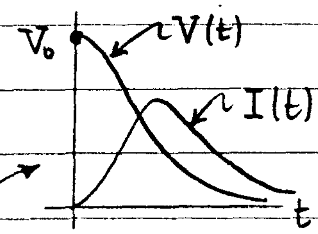
WKB (cont'd) Arc discharge circuit problem.

5) We shall deal with WKB "turning point" problems in detail, but later. Here we wish to discuss a method for finding out by how much ψ(WKB) actually differs from the ψ which satisfies: $\psi'' + k^2(x)\,\psi = 0$. Rather than imposing inequalities like: $|k'/k^2| \ll 1$, for WKB validity, we shall estimate the correction: $\Delta\psi = \psi(\text{actual}) - \psi(\text{WKB})$... which (of course) depends on how rapidly k varies. Anyway, a knowledge of the size of $|\Delta\psi/\psi|$ is the "bottom line" mathematics here ... if $|\Delta\psi/\psi| \to 1$, WKB is ~useless.

To fix ideas, we shall consider a physical example -- an ODE which describes the discharge of a capacitor C through an external circuit consisting of an inductor L & resistor R. The switch is closed at time $t = 0$, when C is charged up to voltage V_0. If C = just a passive cnst, then a current $I = (-)C\dot{V}$ proceeds to flow in the circuit, and the voltage $V(t)$ across C diminishes; the expected behavior of V & I goes as...





Here's the twist... while we fix C = cnst, we let L & R = fcns of time (unlike the usual textbook examples). $L = L(t)$ & $R = R(t)$ would arise, for example, if the "external circuit" were a plasma, and we were trying to model the discharge of a highly electrified region (C) through an arc (L & R). We can choose $L(t)$ & $R(t)$ at will, and -- like Zeus -- we can manufacture our own lightning bolts. Thunder comes later in the course.

When L & R = fcns of t, the circuit eqtn for $V = V(t)$ is...

$$V = RI + \frac{d}{dt}(LI), \quad I = -C\dot{V}$$

$$\Rightarrow \boxed{\ddot{V} + 2\Gamma(t)\,\dot{V} + \omega^2(t)\,V = 0} \qquad \Gamma(t) = \frac{1}{2}\left(\frac{R}{L} + \frac{\dot{L}}{L}\right),\ \text{t-dept damping}; \quad \omega^2(t) = 1/LC,\ \text{t-dept resonant freq}; \tag{16}$$

initial conditions $\}$ $V(0) = V_0$, $\dot{V}(0) = 0$ [$I(0) = 0$].

We will now WKB this eqtn.

WKB (cont'd) Circuit problem: WKB form.

6) Convert Eq. (16) to standard WKB form by substitution...

$$\longrightarrow V(t) = v(t)\exp\left[-\int_0^t \Gamma(\tau)d\tau\right] \Rightarrow \boxed{\ddot{v} + \Omega^2(t)v = 0}, \text{ w/ } \Omega = \sqrt{\omega^2 - (\Gamma^2 + \dot{\Gamma})}. \quad (17)$$

REMARKS on Eq. (17).

1. $V(t)$ will decay ~ exponentially with time t (which is reasonable) if the decay rate $\Gamma(t)$ is not too weird [need: $\Gamma = \frac{1}{2}\left(\frac{R}{L} + \frac{\dot{L}}{L}\right) > 0$, on avg., for $0 < t \to \infty$].
2. The WKB frequency Ω can be real or imaginary depending on the relative size of ω^2 & Γ^2. Basically, if $\dot{L}/L$ is "small", then: (A) Ω is real when $\omega^2 > \Gamma^2$, or $4L/CR^2 > 1$ (conventionally, such a CLR cct is "underdamped"), (B) Ω is imaginary when $\omega^2 < \Gamma^2$, or $4L/CR^2 < 1$ (the cct is "overdamped"). The WKB solns are: v(case A) ~ oscillatory, v(case B) ~ exponential.
3. A WKB solution for $v(t)$ in Eq. (17) will be "good" if Ω is "slowly-varying"...

$$\left|\dot{\Omega}/\Omega^2\right| = \left|\frac{1}{\Omega^3}\left[\omega\dot{\omega} - (\Gamma\dot{\Gamma} + \tfrac{1}{2}\ddot{\Gamma})\right]\right| \ll 1, \text{ w/ } \omega^2 \text{ \& } \Gamma \text{ of Eq. (16)};$$

i.e./ $$\left|\dot{\Omega}/\Omega^2\right| = \left|\frac{1}{2\Omega^3}\left[\omega^2\frac{\dot{L}}{L} + \Gamma\frac{d}{dt}\left(\frac{R}{L} + \frac{\dot{L}}{L}\right) + \frac{1}{2}\frac{d^2}{dt^2}\left(\frac{R}{L} + \frac{\dot{L}}{L}\right)\right]\right| \ll 1. \quad (18)$$

This condition is so complicated as to be ~ useless [although it does ⇒ that significant changes in $(\dot{L}/L)$ & $(\dot{R}/R)$ should occur on a time scale long compared to the natural scale $|\Omega|^{-1}$]. The point is: the simple imposition of "slowly-varying" ($|\dot{\Omega}/\Omega^2| \ll 1$) does not always provide a transparent idea of how well the WKB method will work.

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7) A better way of assessing the accuracy of the WKB solution proceeds by comparing the WKB fcns with the actual solution. With Eq. (17) as a typical WKB problem, proceed as follows...

$\longrightarrow$ Change indpt variable: $t \to s = \int_0^t \Omega(\tau)d\tau$, so/ $\frac{d}{dt} = \Omega\frac{d}{ds}$,

and/ $$\ddot{v} + \Omega^2 v = 0 \;\ldots\text{ becomes }\ldots\; v'' + (\Omega'/\Omega)v' + v = 0. \quad (19)$$

$\uparrow$ dot $\leftrightarrow \frac{d}{dt}$ $\qquad\qquad$ $\uparrow$ prime $\leftrightarrow \frac{d}{ds}$

WKB (cont'd) Circuit problem: WKB form.

→ Change **dept.** variable: $v(s) = u(s)/\sqrt{\Omega(s)}$, $u(s)$ to be found,

So// v'' eqtn [Eq. (19)] becomes: $\boxed{u'' + [1 + b(s)]\,u = 0}$, (20)

where: $\underline{b(s)} = \frac{1}{4}\left(\frac{\Omega'}{\Omega}\right)^2 - \frac{1}{2}\left(\frac{\Omega''}{\Omega}\right)$, $s = \int_0^t \Omega(\tau)\,d\tau$.

If $b(s) \to$ small [note that $\Omega'/\Omega = \dot{\Omega}/\Omega^2$ is the old WKB small parameter... $|\dot{\Omega}/\Omega^2| \ll 1$ is the "slowly-varying" condition], then the u'' eqtn collapses to the triviality: $u'' + u \simeq 0$, and we have got a pretty good soln. In any case, we are now working with the system...

$$\boxed{\begin{array}{l}
\text{Soln to: } \ddot{V} + 2\Gamma(t)\dot{V} + \omega^2(t)V = 0, \text{ is...} \\[1em]
V(t) = \dfrac{u(s)}{\sqrt{\Omega(s)}}\, e^{-\int_0^t \Gamma(\tau)\,d\tau}, \quad \text{w/ } \Omega = \sqrt{\omega^2 - (\Gamma^2 + \dot{\Gamma})}, \\[1em]
\qquad\qquad\qquad\qquad\qquad\qquad s = \int_0^t \Omega(\tau)\,d\tau; \\[1em]
\text{w/ } u(s) \text{ a soln to Eq. (20): } u'' + [1 + b(s)]\,u = 0.
\end{array}}$$

NOTE: this eqtn is **exact**. (21)

8) The neat thing about this formulation is that when $b(s) \equiv 0$, the solutions to the u'' problem produce the usual WKB forms. Write Eq. (20) as...

→ $[\![\, u'' + u = -b(s)\,u \,]\!]$... when eqtn is homogeneous [$b(s) \to 0$], solns...

are// $u_H(s) = e^{\pm is} = e^{\pm i\int_0^t \Omega(\tau)\,d\tau}$, ← HOMOGENEOUS SOLNS

and// $v(t) = u_H/\sqrt{\Omega} = \frac{1}{\sqrt{\Omega(t)}}\, e^{\pm i\int_0^t \Omega(\tau)\,d\tau}$ } standard **WKB forms**. (22)

So if $b(s) \neq 0$, the RHS contribution to the u'' eqtn will measure just how far $u_H(s)$ = WKB soln differs from the actual value of u. This rests on the idea that the u'' eqtn here can be solved iteratively in powers of the supposedly small parameter $b(s)$.

WKB (cont'd) Circuit problem. WKB soln as an integral eqtn.

9) To be more precise, recall a result from the treatment of "osculating parameters":

if// $p(s)\,u'' + q(s)\,u' + r(s)\,u = f(s)$, and// $u_{1,2}(s)$ = solns to homog^s eqtn,

then// $$u(s) = u_2(s)\int^s \frac{f(\sigma)}{p(\sigma)W}\,u_1(\sigma)\,d\sigma - u_1(s)\int^s \frac{f(\sigma)}{p(\sigma)W}\,u_2(\sigma)\,d\sigma$$ $\int$ is a particular integral ★

(23)

Here: $W = u_1 u_2' - u_1' u_2$ is the Wronskian. Apply to u'' eqtn in Eq. (22)... $\underbrace{}_{u''+u=-bu}$

$p(s) = 1,\ q(s) = 0,\ r(s) = 1\ ;\ f(s) = -b(s)\,u(s)\ ;$

homogeneous solutions are: $u_{1,2}(s) = e^{\pm is}$ (solns to $u'' + u = 0$);

So// $W = e^{is}(-ie^{-is}) - (ie^{is})e^{-is} = -2i$, and// particular integral is...

$$u(s) = e^{-is}\int^s \frac{[+b(\sigma)u(\sigma)]}{(+2i)}\,e^{i\sigma}\,d\sigma - e^{is}\int^s \frac{[+b(\sigma)u(\sigma)]}{(+2i)}\,e^{-i\sigma}\,d\sigma$$

$$\rightarrow u_p(s) = \int_0^s u(\sigma)\,b(\sigma)\sin(\sigma - s)\,d\sigma$$ $\int$ is a particular integral for eqtn: $u'' + u = -b(s)\,u$. (24)

The lower limit $s=0$ here is chosen for convenience; it makes no difference in the overall solution. We now have a full solution to Eq. (20)...

$\rightarrow u'' + [1 + b(s)]\,u = 0$, has $\begin{cases}\text{homog. solns } u_{1,2}(s) = e^{\pm is},\\ \text{particular integral } u_p(s) \text{ of Eq. (24)};\end{cases}$

So// $$\boxed{u(s) = \underbrace{\left(Ae^{+is} + Be^{-is}\right)}_{\text{homog}^s \text{ soln} \equiv u(\text{WKB})\ldots} + \underbrace{\int_0^s u(\sigma)\,b(\sigma)\sin(\sigma - s)\,d\sigma}_{\text{Correction term} \propto \text{size of } b(s).}}$$ (25)

fcn u still exact

All this is still exact (we've made no "smallness" approxns). It appears we have an exact solution for $u(s)$. But this is a integral eqtn for u, since u (the unknown fcn) appears under the integral RHS. However, iteration is "easy".

★ Verify against solution to Arfken prob. # (8.6.25), p. 479.

WKB (cont'd) WKB soln ≡ zeroth order term of a Neumann series.

10) Define : $\underline{w(s)} = Ae^{+is} + Be^{-is}$, $\underline{s} = \int_0^t \Omega(\tau)\,d\tau$ ... this is WKB soln†. So

$$\text{Eq.(25)}: \left[\!\left[\, \underset{\text{exact soln}}{u(s)} = \underset{\text{WKB approxn}}{w(s)} + \int_0^s \underset{\text{exact soln}}{u(\sigma)}\, \underset{\text{correction factor}}{b(\sigma)}\, \underset{\text{kernel fcn}}{\sin(\sigma - s)}\, d\sigma \right]\!\right], \quad \underline{\underline{b(s)}} = \left(\frac{\Omega'}{2\Omega}\right)^2 - \left(\frac{\Omega''}{2\Omega}\right). \quad (26)$$

This is a Volterra Integral Eqtn of the 2nd Kind for $u(s)$. Solvable by iteration procedure when $b(s) \to$ small...

$[\mathcal{O}(K^0)]$ zeroth approxn } $u_0(s) = w(s)$ ... this is WKB... applies strictly only when $b(s) \to 0$;

$[\mathcal{O}(K^1)]$ first approxn } $u_1(s) = u_0(s) + \int_0^s u_0(\sigma) K(\sigma, s)\, d\sigma$ ... w/ $\underline{\underline{K(\sigma,s)}} = b(\sigma)\sin(\sigma - s)$;

$[\mathcal{O}(K^2)]$ second approxn } $u_2(s) = u_1(s) + \int_0^s u_1(\sigma) K(\sigma, s)\, d\sigma$

2 terms of a Neumann series

etc...
$$\boxed{u_{n+1}(s) = u_n(s) + \int_0^s u_n(\sigma)\, K(\sigma, s)\, d\sigma}, \quad n = 0, 1, 2, \ldots \quad (27)$$

Thus we can iterate WKB to arbitrary accuracy, in principle. There is of course the question of whether the iterative series [basically in powers of $b(s)$] converges. What counts here is the first iteration...

1st iteration : $u(s) = w(s) + \int_0^s w(\sigma)\, b(\sigma) \sin(\sigma - s)\, d\sigma$. everything on RHS is calculable

So// fractional error }
$$\boxed{\Delta(s) = \frac{u(s) - w(s)}{w(s)} = \frac{1}{w(s)} \int_0^s w(\sigma)\, b(\sigma) \sin(\sigma - s)\, d\sigma} \quad (28)$$

$\Delta(s)$ is evidently the fractional error in u(ACTUAL) vs. u(WKB), in first approxn (this was promised on p.7). Also $\Delta(s)$ is the effective expansion parameter in the iterative expansion of Eq.(27). This claim is not precise, but... roughly speaking... the expansion works, and WKB is ~ good, when $\underline{\underline{|\Delta(s)| < 1}}$.

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† i.e. in Eq.(21): $V(t) = \left[w(s)/\sqrt{\Omega(t)}\right] e^{-\int_0^t \Gamma(\tau) d\tau}$, is WKB approxn to problem.

Now can go back and "get" the circuit.
Q// What happens when $\Omega(t) \to i|\Omega(t)|$?