ϕ519 Problems $\left\{ \begin{array}{l} \text{Assigned} \\ 11/28/88. \end{array} \right.$ **9** & **10**

Set #9: Prs. 30, 31, 32; due 12/5/[redacted]
Set #10: Prs. 33, 34, 35; due 12/12/[redacted]

[redacted] The Maxwell field tensor $\underset{\sim}{F}$ and its dual $\underset{\sim}{\mathcal{F}}$ are given -- in their covariant and contravariant forms -- by Jackson's Eqs. (11.138) and (11.140). With the summation convention in effect (sum over repeated indices):

(A) Show that $F_{\alpha\beta}\mathcal{F}^{\alpha\beta}$ is a Lorentz invariant scalar.

(B) Evaluate $F_{\alpha\beta}\mathcal{F}^{\alpha\beta}$ explicitly (in terms of $\mathbb{E}$ & $\mathbb{B}$) to find the field invariant.

[redacted] [[redacted]] Jk$^{\underline{n}}$ Eq. (12.9): $L = -mc^2\sqrt{1-(u/c)^2} - q\phi + (q/c)\,\mathbf{u}\cdot\mathbb{A}$, is the EM Lagrangian.

4pts (A) Show that this L, plus the Lagrange eqtns-of-motion, give the Lorentz force law.

4pts (B) Jackson's Eq. (12.14): $\mathcal{H} = \left[(c\mathbb{P} - q\,\mathbb{A})^2 + (mc^2)^2\right]^{1/2} + q\phi$, is the EM Hamiltonian. Show that this $\mathcal{H}$, plus the Hamiltonian eqs-of-motion, also give the Lorentz force law.

4pts (C) Establish the nonrelativistic version of $\mathcal{H}$ of part (B), i.e. the lowest order $\mathcal{H}$ for $c\to\infty$. Only a <u>part</u> of $\mathcal{H}(\text{non rel.})$ is used in Schrödinger's eqtn: $\mathcal{H}\psi = i\hbar\dot{\psi}$. Why is this OK?

3pts (D) Again, in Schrödinger's eqtn, the <u>EM</u> momentum $\mathbb{P}\to(-)i\hbar\nabla$ is replaced by an operator, rather than the <u>kinetic</u> momentum $\mathbb{p} = \mathbb{P} - \frac{q}{c}\mathbb{A}$. Why is this procedure OK?

[redacted] [Jackson Prob. (12.2)]. (A) Show, from Hamilton's Principle, that Lagrangians differing only by a total time derivative of some fcn of the coordinates & time are "equivalent" -- i.e. they yield the <u>same</u> Euler-Lagrange eqtns-of-motion. (B) Show explicitly that a gauge transform: $A^\alpha \to A^\alpha + \partial^\alpha\Lambda$, for the potentials in the EM Lagrangian of Jk$^{\underline{n}}$ Eq. (12.9), merely generates an equivalent Lagrangian in the sense of part (A).

[redacted] [Jackson Prob. (12.13)]. An alternative Lagrange density for the EM field is given by:

$$\mathcal{L} = -\frac{1}{8\pi}(\partial_\alpha A_\beta)(\partial^\alpha A^\beta) - \frac{1}{c}J_\alpha A^\alpha.$$

(A) Derive the Lagrange eqtns-of-motion for this $\mathcal{L}$. What assumptions are needed to make them equivalent to Maxwell's eqtns? (B) Show explicitly (and with what assumptions) that the above $\mathcal{L}$ differs from Jk$^{\underline{n}}$ Eq. (12.85) by a 4-divergence. How does this added 4-divergence affect the eqtns-of-motion and the action integral?

ϕ 519 Problems

[redacted] Consider the scalar wavefcns $\psi(\mathbf{r},t)$ & $\psi^*(\mathbf{r},t)$ to be independent field variables, and -- for a particle of mass m in a potential $V=V(\mathbf{r},t)$ -- consider Lagrange density:

$$\boxed{\mathcal{L}=\frac{\hbar^2}{2m}(\nabla\psi)^*\cdot(\nabla\psi)+V\psi^*\psi-\frac{i\hbar}{2}\left[\psi^*\left(\frac{\partial\psi}{\partial t}\right)-\left(\frac{\partial\psi^*}{\partial t}\right)\psi\right]}.$$

(A) Show that the Lagrange eqtns-of-motion for this $\mathcal{L}$ generate Schrödinger's eqtn: $-(\hbar^2/2m)\nabla^2\psi+V\psi=i\hbar(\partial\psi/\partial t)$, and its complex conjugate.

(B) Neither the above $\mathcal{L}$, nor Schrödinger's eqtn, is covariant (why?). How would you proceed to make $\mathcal{L}$ and its eqtn-of-motion "manifestly covariant"?

[redacted] [[redacted] pts] [Jackson Prob. (12.14)]. Consider the Proca Eqtns for localized steady-state currents $\mathbb{J}$ which generate (only) a static magnetic moment. This model describes observable effects of a nonzero photon mass on the earth's magnetic field. [redacted] $\mathbb{J}$ is related to the magnetization $\mathbb{M}$ it generates by Jk$^{\text{II}}$ Eq (5.79): $\mathbb{J}=c\nabla\times\mathbb{M}$.

5pts (A) If $\mathbb{M}=\mathbf{m}f(\mathbf{r})$, with $\mathbf{m}$ a cnst vector, and $f(\mathbf{r})$ a scalar distribution fcn, show that the corresponding vector potential is:

$$\boxed{\mathbb{A}(\mathbf{r})=(-)\,\mathbf{m}\times\nabla\int\frac{d^3r'}{R}f(\mathbf{r}')\,e^{-\mu R}}\quad\int R=|\mathbf{r}-\mathbf{r}'|=\text{source - field separation};\ \mu=\frac{1}{\hbar}m_\gamma c=\text{photon wave number}.$$

5pts (B) If the source is a point-dipole at the origin, show that the earth's magnetic field is:

$$\boxed{\mathbb{B}(\mathbf{r})=\frac{e^{-\mu r}}{r^3}\left[3\hat{r}(\hat{r}\cdot\mathbf{m})-\mathbf{m}\right]\left(1+\mu r+\tfrac{1}{3}\mu^2r^2\right)-\tfrac{2}{3}\mu^2\,\mathbf{m}\,\frac{e^{-\mu r}}{r}}.$$

5pts (C) The result for $\mathbb{B}$(earth) shows that on the earth's surface, $r=R_e$, the magnetic field looks like the usual dipole field, plus an added constant field along $(-)\,\mathbf{m}$. Satellite magnetometer data imply that the added field -- if it exists -- is $<4\times10^{-3}$ times the dipole field at the magnetic equator. From this fact, estimate a lower limit on $1/\mu$ (in earth radii), and an upper limit on m_γ, the photon mass. Compare your results with discussion in Jackson's Sec. I.2, p. 6.

● Extract the field invariant $\mathbb{E}\cdot\mathbb{B}$ from Lorentz invariance of $F_{\alpha\beta}\mathcal{F}^{\alpha\beta}$.

A 1) Since $(F_{\alpha\beta})$ & $(\mathcal{F}^{\alpha\beta})$ are fully qualified (co- and contra-variant) tensors, they transform by the rules given in Jackson Eqs. (11.64) & (11.63). In the new (primed) frame:

11/24/88

$$\rightarrow F'_{\alpha\beta}\,\mathcal{F}'^{\alpha\beta} = \left[\left(\frac{\partial x^\gamma}{\partial x'^\alpha}\right)\left(\frac{\partial x^\delta}{\partial x'^\beta}\right)F_{\gamma\delta}\right]\left[\left(\frac{\partial x'^\alpha}{\partial x^\kappa}\right)\left(\frac{\partial x'^\beta}{\partial x^\lambda}\right)\mathcal{F}^{\kappa\lambda}\right] = F_{\gamma\delta}\,a^\gamma_\kappa\,a^\delta_\lambda\,\mathcal{F}^{\kappa\lambda}, \qquad (1)$$

where: $a^\gamma_\kappa = (\partial x^\gamma/\partial x'^\alpha)(\partial x'^\alpha/\partial x^\kappa)$, $a^\delta_\lambda = (\partial x^\delta/\partial x'^\beta)(\partial x'^\beta/\partial x^\lambda)$, by regrouping terms. With the summation convention: $a^\lambda_\kappa = \partial x^\gamma/\partial x^\kappa = \delta^\lambda_\kappa$, the Kronecker delta (because of the orthogonality of the x^γ cds). Similarly $a^\delta_\lambda = \delta^\delta_\lambda$. Then, on summing over the indices λ & κ, Eq. (1) yields...

$$\rightarrow F'_{\alpha\beta}\,\mathcal{F}'^{\alpha\beta} = F_{\gamma\delta}\,\delta^\gamma_\kappa\,\delta^\delta_\lambda\,\mathcal{F}^{\kappa\lambda} = F_{\gamma\delta}\,\delta^\gamma_\kappa\,\mathcal{F}^{\kappa\delta} = F_{\gamma\delta}\,\mathcal{F}^{\gamma\delta}. \qquad (2)$$

<u>So $F_{\alpha\beta}\mathcal{F}^{\alpha\beta}$ is a Lorentz invariant scalar</u>... it is $\equiv$ same in all Lorentz frames.

B 2) Since $\mathcal{F}$ is totally antisymmetric, $\mathcal{F}^{\alpha\beta} = -\mathcal{F}^{\beta\alpha}$, and the quantity of interest is:

$$F_{\alpha\beta}\mathcal{F}^{\alpha\beta} = -F_{\alpha\beta}\mathcal{F}^{\beta\alpha} = (-)\left[F\,\mathcal{F}^T\right]_{\alpha\alpha} = (-)\,\mathrm{Tr}\left[F\,\mathcal{F}^T\right]. \qquad (3)$$

"T" means transpose and "Tr" is trace ($\equiv$ sum of diagonal elements). We want:

$$\mathrm{Tr}\left[F\,\mathcal{F}^T\right] = \mathrm{Tr}\begin{bmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & -B_3 & B_2 \\ -E_2 & B_3 & 0 & -B_1 \\ -E_3 & -B_2 & B_1 & 0 \end{bmatrix}\begin{bmatrix} 0 & B_1 & B_2 & B_3 \\ -B_1 & 0 & -E_3 & E_2 \\ -B_2 & E_3 & 0 & -E_1 \\ -B_3 & -E_2 & E_1 & 0 \end{bmatrix} = \mathrm{Tr}\begin{bmatrix} -\mathbb{E}\cdot\mathbb{B} & & & \\ & -\mathbb{E}\cdot\mathbb{B} & & \\ & & -\mathbb{E}\cdot\mathbb{B} & \\ & & & -\mathbb{E}\cdot\mathbb{B} \end{bmatrix}$$

i.e.// $\mathrm{Tr}\left[F\,\mathcal{F}^T\right] = (-)4\,\mathbb{E}\cdot\mathbb{B}$, so// $\boxed{F_{\alpha\beta}\mathcal{F}^{\alpha\beta} = 4\,\mathbb{E}\cdot\mathbb{B}}$. (4)

<u>The required field invariant is $\mathbb{E}\cdot\mathbb{B} = E_k B_k$</u>. Note that we calculated only the diagonal elements of $[F\,\mathcal{F}^T]$; they were all we needed for Tr [ ].

[redacted]

● For q in $(\vec{A}, \phi)$, verify Lorentz force law from Lagrange & Ham$^{\underline{n}}$ eqtns.

4pts (a) For: $L = -mc^2\sqrt{1-(u/c)^2} - q\phi + \frac{q}{c}\,\vec{u}\cdot\vec{A}$, and for the space-like comps...

12/2/84

$$\frac{d}{dt}\left(\partial L/\partial\vec{u}\right) = \partial L/\partial\vec{x} \;\Rightarrow\; \frac{d}{dt}\Big(\overset{\text{particle momentum }\vec{p}}{\gamma m\vec{u}} + \frac{q}{c}\vec{A}\Big) = q\left[-\vec{\nabla}\phi + \frac{1}{c}\nabla(\vec{u}\cdot\vec{A})\right]$$

("gradients")

w/ $$\frac{d\vec{p}}{dt} = q\left[-\vec{\nabla}\phi + \frac{1}{c}\left\{\vec{\nabla}(\vec{u}\cdot\vec{A}) - \frac{d\vec{A}}{dt}\right\}\right]. \tag{1}$$

But: $\vec{\nabla}(\vec{u}\cdot\vec{A}) = (\vec{u}\cdot\vec{\nabla})\vec{A} + \cancelto{0}{(\vec{A}\cdot\vec{\nabla})\vec{u}} + \vec{u}\times(\vec{\nabla}\times\vec{A}) + \vec{A}\times\cancelto{0}{(\vec{\nabla}\times\vec{u})}$, and: $\frac{d\vec{A}}{dt} = \partial\vec{A}/\partial t + (\vec{u}\cdot\vec{\nabla})\vec{A}$. Putting these into Eq. (1), we find usual 3-vector law...

$$\frac{d\vec{p}}{dt} = q\left[-\left(\vec{\nabla}\phi + \frac{1}{c}\frac{\partial\vec{A}}{\partial t}\right) + \frac{1}{c}\vec{u}\times(\vec{\nabla}\times\vec{A})\right] = q\left(\vec{E} + \frac{1}{c}\vec{u}\times\vec{B}\right), \tag{2}$$

Since: $\vec{E} = -\left(\vec{\nabla}\phi + \frac{1}{c}\frac{\partial\vec{A}}{\partial t}\right)$, and $\vec{B} = \vec{\nabla}\times\vec{A}$. For the work-energy theorem, we know that with: $\mathcal{E} = \gamma mc^2$, $\mathcal{E}^2 = (c\vec{p})^2 + (mc^2)^2$, we must have...

$$\frac{d\mathcal{E}}{dt} = \left(\frac{c^2\vec{p}}{\mathcal{E}}\right)\cdot\frac{d\vec{p}}{dt} = \vec{u}\cdot\frac{d\vec{p}}{dt} = \vec{u}\cdot q\left(\vec{E} + \frac{1}{c}\vec{u}\times\vec{B}\right) = q\,\vec{u}\cdot\vec{E} \tag{3}$$

Eqs (2) & (3) together constitute the full Lorentz force law via the Lagrangian.

4pts (b) For $\mathcal{H} = [(c\vec{P} - q\vec{A})^2 + (mc^2)^2]^{\frac{1}{2}} + q\phi$, use the Ham$^{\underline{n}}$ eqtn...

$$\frac{d\vec{P}}{dt} = -\vec{\nabla}\mathcal{H} \;\Rightarrow\; \frac{d}{dt}\left(\vec{p} + \frac{q}{c}\vec{A}\right) = -q\vec{\nabla}\phi - \vec{\nabla}[\;\;]^{\frac{1}{2}}$$

w/ $$\frac{d\vec{p}}{dt} = q\left[-\vec{\nabla}\phi - \frac{1}{c}\frac{d\vec{A}}{dt}\right] - \frac{1}{2}\frac{c^2}{[\;\;]^{1/2}}\vec{\nabla}(\vec{\pi}\cdot\vec{\pi}),\quad \vec{\pi} = \vec{P} - \frac{q}{c}\vec{A}. \tag{4}$$

We have inserted the canonical momentum: $\vec{P} = \vec{p} + \frac{q}{c}\vec{A}$, on the LHS. If we remember: $\vec{u} = c^2\vec{\pi}/[\;\;]^{\frac{1}{2}}$, from class notes, then the 2^{ND} term RHS is...

(over)

$$\frac{1}{2}\frac{c^2}{[\;]^{1/2}}\vec{\nabla}(\vec{\Pi}\cdot\vec{\Pi}) = \frac{c^2}{[\;]^{1/2}}\{(\vec{\Pi}\cdot\vec{\nabla})\vec{\Pi} + \vec{\Pi}\times(\vec{\nabla}\times\vec{\Pi})\} =$$

$$= -\frac{q}{c}\left[(\vec{u}\cdot\vec{\nabla})\vec{A} + \vec{u}\times(\vec{\nabla}\times\vec{A})\right]. \tag{5}$$

Here we've used $\partial\vec{P}/\partial x_k \equiv 0$, since $\vec{P}$ & $\vec{x}$ are independent canonical cds. Inserting this result in Eq. (4), we get...

$$\frac{d\vec{p}}{dt} = q\left[-\vec{\nabla}\phi - \frac{1}{c}\underbrace{\left(\frac{d\vec{A}}{dt} - (\vec{u}\cdot\vec{\nabla})\vec{A}\right)}_{=\partial\vec{A}/\partial t} + \vec{u}\times(\vec{\nabla}\times\vec{A})\right] = q\left(\vec{E} + \frac{1}{c}\vec{u}\times\vec{B}\right). \tag{6}$$

This is the 3-vector part of the Lorentz law. The work-energy thm can be proven as in Eq. (3), and so we have verified the Lorentz law also via Ham^n eqtns.

4pts

(c) As $c\to\infty$, write the Ham^n as...

$$\mathcal{H} = q\phi + mc^2\left[1 + \left(\frac{c\vec{P} - q\vec{A}}{mc^2}\right)^2\right]^{\frac{1}{2}} \simeq q\phi + mc^2\left\{1 + \frac{1}{2}\left(\frac{c\vec{P}-q\vec{A}}{mc^2}\right)^2 + \cdots\right\}$$

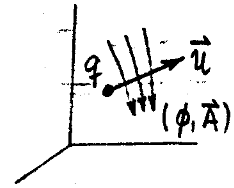
or// $$\mathcal{H} \simeq mc^2 + \left[q\phi + \frac{1}{2m}\left(\vec{P} - \frac{q}{c}\vec{A}\right)^2\right]. \tag{7}$$

The [ ] is the lowest order part of $\mathcal{H}$, which is used e.g. in the (non-relativistic) Schrödinger eqtn -- recall Prob 12. Only the [ ] need be used in the non-relativistic theory, because only energy level differences are measured, and -- without particle creation/annihilation -- the rest energy mc^2 subtracts out.

3pts

(d) The def^n of the Ham^n: $\mathcal{H} = \vec{P}\cdot\vec{u} - L$, involves the canonical momentum $\vec{P} = \partial L/\partial\vec{u}$, and it is this quantity which enters in the Poisson bracket formalism of classical mechanics. In the passage from CM → QM, have Poisson bracket: $\{Q_i, P_j\} \rightarrow \frac{1}{i\hbar}[Q_i, P_j] = \delta_{ij}$ (commutator bracket), so it is P_j which becomes an operator, not the kinematic p_j. See Schiff (3rd ed.), Sec. 24.

ϕ519 Prob. Solutions



12/3/84

● Examine effect of gauge transform$^{\underline{n}}$ on Lagrangian for q in $(\phi, \vec{A})$.

(a) For Lagrangians: L & $L' = L + \frac{d}{dt}\Gamma$, which differ by the total time derivative of some scalar fcn $\Gamma = \Gamma(\vec{x}, t)$, we have the actions...

$$A = \int_{t_1}^{t_2} L\,dt, \quad A' = \int_{t_1}^{t_2} L'\,dt = A + \Gamma\Big|_{t=t_1}^{t=t_2}.$$

A & A' differ only by the integrated term, which is fixed at the endpts t_1 & t_2. Any variational contribution $\delta\Gamma\Big|_{t_1}^{t_2}$ to Hamilton's principle will then be $\equiv 0$, since Γ must also be fixed at t_1 & t_2. So $\delta A = 0$ & $\delta A' = 0$ must lead to the same eqtns-of-motion, and L & L' are "gauge equivalent."

(b) For gauge transform: $A^\alpha \to A^\alpha + \partial^\alpha \Lambda$, work with Jackson's Eq. (12.8)...

$$L_{int} = -\frac{q}{\gamma c} U_\alpha A^\alpha \to L'_{int} = L_{int} - \frac{q}{\gamma c} U_\alpha \partial^\alpha \Lambda.$$

This is the only part of: $L_{total} = L_{free} + L_{int}$, for q in $A^\alpha = (\phi, \vec{A})$, which is changed by the gauge tranform. But with...

$$\left.\begin{aligned} U_\alpha &= \gamma(c, -\vec{u}) \leftarrow \text{Eq. (11.36)} \\ \partial^\alpha &= \left(\frac{\partial}{\partial ct}, -\vec{\nabla}\right) \leftarrow \text{Eq. (11.76)} \end{aligned}\right\} U_\alpha \partial^\alpha \Lambda = \gamma\left(\frac{\partial}{\partial t} + \vec{u}\cdot\vec{\nabla}\right)\Lambda,$$

So// $$L'_{int} = L_{int} - \frac{q}{c}\underbrace{\left[\left(\frac{\partial}{\partial t} + \vec{u}\cdot\vec{\nabla}\right)\Lambda\right]}_{d\Lambda/dt}$$

The [ ] is instantly recognized as the (convective) total derivative for $\Lambda = \Lambda(\vec{x}, t)$, so L'_{int} & L_{int} in fact differ only by a total t-derivative. By part (a), therefore, L'_{int} & L_{int} are "gauge equivalent."

Work out field eqtns for Lagrange density : $\mathcal{L} = -\frac{1}{8\pi}(\partial_\alpha A_\beta)(\partial^\alpha A^\beta) - \frac{1}{c} J_\alpha A^\alpha$.

12/4/84

(a) Make the covariant factor contravariant via : $x_\alpha = g_{\alpha\tau} x^\tau$ (Jackson's Eq. (11.72)). Then...

$$\mathcal{L} = -\frac{1}{8\pi} g_{\alpha\tau} g_{\beta\sigma} (\partial^\tau \partial A^\sigma)(\partial^\alpha A^\beta) - \frac{1}{c} J_\alpha A^\alpha \tag{1}$$

The field eqtns are : $\partial^\mu[\partial\mathcal{L}/\partial(\partial^\mu A^\nu)] = \partial\mathcal{L}/\partial A^\nu$. Trivially : $\frac{\partial\mathcal{L}}{\partial A^\nu} = -\frac{1}{c} J_\nu$.

On the LHS, we need to calculate...

Kronecker deltas

$$\frac{\partial\mathcal{L}}{\partial(\partial^\mu A^\nu)} = -\frac{1}{8\pi} g_{\alpha\tau} g_{\beta\sigma}\left[\delta^\tau_\mu \delta^\sigma_\nu (\partial^\alpha A^\beta) + (\partial^\tau A^\sigma)\delta^\alpha_\mu \delta^\beta_\nu\right]$$

interchange indices

... both terms the same $\Rightarrow \frac{\partial\mathcal{L}}{\partial(\partial^\mu A^\nu)} = -\frac{1}{4\pi} g_{\alpha\tau} g_{\beta\sigma} \delta^\tau_\mu \delta^\sigma_\nu (\partial^\alpha A^\beta) = -\frac{1}{4\pi} g_{\alpha\mu} g_{\beta\nu} (\partial^\alpha A^\beta)$

or// $$\frac{\partial\mathcal{L}}{\partial(\partial^\mu A^\nu)} = -\frac{1}{4\pi}(g_{\mu\alpha}\partial^\alpha)(g_{\nu\beta}A^\beta) = -\frac{1}{4\pi}(\partial_\mu A_\nu) \tag{2}$$

The field eqtns for $\mathcal{L}$ then yield the familiar Maxwell wave eqtn, Eq. (12.123)

$$\partial^\mu\left[\frac{\partial}{\partial(\partial^\mu \partial A^\nu)}\right] = \frac{\partial\mathcal{L}}{\partial A^\nu} \Rightarrow -\frac{1}{4\pi}\partial^\mu(\partial_\mu A_\nu) = -\frac{1}{c} J_\nu, \text{ or// } \boxed{\Box A_\nu = \frac{4\pi}{c} J_\nu} \tag{3}$$

This is equivalent to Maxwell's Eqtns, with the <u>assumption of the Lorentz gauge</u>.

(b) In Eq. (12.85) : $\mathcal{L}' = -\frac{1}{16\pi}(\partial_\alpha A_\beta - \partial_\beta A_\alpha)(\partial^\alpha A^\beta - \partial^\beta A^\alpha) - \frac{1}{c} J_\alpha A^\alpha$. Difference is:

$$\Delta\mathcal{L} = \mathcal{L} - \mathcal{L}' = -\frac{1}{8\pi}\left[(\partial_\alpha A_\beta)(\partial^\alpha A^\beta) - \frac{1}{2}(\partial_\alpha A_\beta - \partial_\beta A_\alpha)(\partial^\alpha A^\beta - \partial^\beta A^\alpha)\right]$$

i.e.// $$\Delta\mathcal{L} = -\frac{1}{8\pi}\left[(\partial_\alpha A_\beta)(\partial^\alpha A^\beta) - \frac{1}{2}\left\{(\partial_\alpha A_\beta)(\partial^\alpha A^\beta) - (\partial_\alpha A_\beta)(\partial^\beta A^\alpha) - (\partial_\beta A_\alpha)(\partial^\alpha A^\beta) + (\partial_\beta A_\alpha)(\partial^\beta A^\alpha)\right\}\right]$$

Σ these terms are $\equiv$; together they cancel first term in the [ ].

or// $$\Delta\mathcal{L} = -\frac{1}{8\pi}\cdot\frac{1}{2}\left[(\partial_\alpha A_\beta)(\partial^\beta A^\alpha) + (\partial_\beta A_\alpha)(\partial^\alpha A^\beta)\right] = -\frac{1}{8\pi}(\partial_\alpha A_\beta)(\partial^\beta A^\alpha). \tag{4}$$

This $\Delta\mathcal{L} = \partial_\alpha\left\{-\frac{1}{8\pi}(A_\beta \partial^\beta A^\alpha)\right\}$ <u>is</u> a 4-divergence (in Lorentz gauge). The action is unaffected, since : $\Delta A = \int d^4x\, \Delta\mathcal{L} \to 0$, if potls vanish at ∞. This $\Rightarrow$ field eqtns are identical.

● Rediscover Schrodinger's Eqtn via Lagrangian density formalism.

12/5/84

A. For: $\mathcal{L} = \frac{\hbar^2}{2m}\left(\frac{\partial\psi^*}{\partial x_k}\right)\left(\frac{\partial\psi}{\partial x_k}\right) + V\psi^*\psi - \frac{i\hbar}{2}\left[\psi^*\left(\frac{\partial\psi}{\partial t}\right) - \left(\frac{\partial\psi^*}{\partial t}\right)\psi\right]$, with $\xi = \psi, \psi^*$, there will be two eqtns of motion: $\partial^\mu[\partial\mathcal{L}/\partial(\partial^\mu\xi)] = \partial\mathcal{L}/\partial\xi$, i.e.

$$\frac{\partial}{\partial t}\left(\frac{\partial\mathcal{L}}{\partial\xi_t}\right) + \frac{\partial}{\partial x_k}\left(\frac{\partial\mathcal{L}}{\partial\xi_{x_k}}\right) = \frac{\partial\mathcal{L}}{\partial\xi}, \text{ for } \xi = \psi \text{ \& } \xi = \psi^*. \tag{1}$$

Take $\xi = \psi^*$ first, noting...

$$\partial\mathcal{L}/\partial\psi^* = V\psi - \frac{i\hbar}{2}\left(\frac{\partial\psi}{\partial t}\right), \quad \partial\mathcal{L}/\partial\psi_t^* = +\frac{i\hbar}{2}\psi$$

and// $$\partial\mathcal{L}/\partial\psi_{x_k}^* = \frac{\hbar^2}{2m}\left(\frac{\partial\psi}{\partial x_k}\right). \tag{2}$$

Putting this into Eq. (1), we easily get Schrodinger's wave eqtn...

$$\frac{\partial}{\partial t}\left(\frac{i\hbar}{2}\psi\right) + \frac{\partial}{\partial x_k}\left[\frac{\hbar^2}{2m}\left(\frac{\partial\psi}{\partial x_k}\right)\right] = V\psi - \frac{i\hbar}{2}\left(\frac{\partial\psi}{\partial t}\right),$$

or// $$\boxed{i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi + V\psi} \tag{3}$$

The Lagrange eqtn-of-motion [Eq (1)] for $\xi = \psi$ gives the complex-conjugate version of this, viz: $-i\hbar\dot{\psi}^* = [-(\hbar^2/2m)\nabla^2 + V]\psi^*$.

B. Above $\mathcal{L}$ and wave eqtn are manifestly non-covariant, since the space & time cds do not enter equivalently ($\mathcal{L}$ is linear in $\partial/\partial t$, but quadratic in $\partial/\partial x_k$). First step, in the quest for covariance, would be to replace the $\partial/\partial ?$ operators by the 4-vector del: $\partial_\mu = \partial/\partial x^\mu$, and to start out with: $\mathcal{L}_0 = \text{cnst}\times(\partial_\mu\phi)^2$, for a scalar wave field ϕ -- this is a Lorentz scalar, and quadratic in $\phi \Rightarrow$ linear wave eqtn. Then can add a mass term $\propto \mu^2\phi^2$, and some coupling of ϕ with a scalar source ρ, i.e. $\rho\phi$. So: $\mathcal{L} = \frac{1}{2}[(\partial_\mu\phi)^2 + \mu^2\phi^2] + \rho\phi$, is OK $\leftrightarrow (\Box + \mu^2)\phi = -\rho$ {Klein Gordon Eqtn

9 Dec. 1984

[redacted] Analyse changes in earth's magnetic field for m(photon) > 0, à la Proca.

5pts (a) The steady-state Proca Eq. is : $(\nabla^2 - \mu^2) A^\alpha = -\frac{4\pi}{c} J^\alpha$. If we momentarily set $\mu = ik$, then the Green's fcn for this eqtn is : $G_k = \frac{1}{R} e^{\pm ikR}$, by Jackson's Eq. (6.62), with : $R = |\mathbf{r} - \mathbf{r}'|$ the field pt - source pt separation. We evidently want the exponentially <u>damped</u> Green's fcn, i.e. $G_\mu = \frac{1}{R} e^{-\mu R}$, and the solution:

$$A^\alpha(\mathbf{r}) = \int \frac{d^3r'}{R} e^{-\mu R} \cdot \frac{1}{c} J^\alpha(\mathbf{r}'). \tag{1}$$

If $\phi = 0$, and $\mathbb{J} = c \nabla \times \mathbb{M}$, with $\mathbb{M} = \mathbb{m} f$ and $\mathbb{m}$ a cnst vector, then...

$$\mathbb{A}(\mathbf{r}) = -\mathbb{m} \times \int d^3r' \left(\frac{e^{-\mu R}}{R}\right) \nabla' f(\mathbf{r}'), \tag{2}$$

is the vector potential [we've used : $\nabla \times (\mathbb{m} f) = (\nabla f) \times \mathbb{m}$, for $\mathbb{m} = $ cnst].

Next, note that : $F(R) \nabla' f(\mathbf{r}') = \nabla'[F(R) f(\mathbf{r}')] - f(\mathbf{r}') \nabla' F(R)$, and $\nabla' F(R) = -\nabla F(R)$, for $F(R) = \frac{1}{R} e^{-\mu R}$, or in fact any fcn of R. Thus, with ∇ acting on the field pt $\mathbf{r}$ now, so: $-f(\mathbf{r}') \nabla' F(R) = \nabla[f(\mathbf{r}') F(R)]$

$$\rightarrow \mathbb{A}(\mathbf{r}) = -\mathbb{m} \times \nabla \int \frac{d^3r'}{R} e^{-\mu R} f(\mathbf{r}') - \mathbb{m} \times \underbrace{\int d^3r' \, \nabla' \left[\frac{e^{-\mu R}}{R} f(\mathbf{r}')\right]}_{\to 0} \tag{3}$$

The 2ND term integrates to the values of $(e^{-\mu R}/R) f(\mathbf{r}')$ at ∞; these vanish by assumption. The 1ST term gives the required vector potential.

5pts (b) For a point dipole at the origin, $f(\mathbf{r}') = \delta(\mathbf{r}')$, and Eq. (3) gives...

$$\mathbb{A}(r) = -\mathbb{m} \times \nabla\left(\frac{e^{-\mu r}}{r}\right) = -(\mathbb{m} \times \hat{r}) \frac{\partial}{\partial r}\left(\frac{e^{-\mu r}}{r}\right),$$

or// $\boxed{\mathbb{A}(r) = \mathbb{G}(r) \times \hat{r}}$, where $\begin{cases} \mathbb{G}(r) = \mathbb{m}\, g(r), \quad \hat{r} = \mathbf{r}/r, \\ \text{and// } g(r) = \frac{1}{r^2}(1 + \mu r) e^{-\mu r}. \end{cases}$ (4)

Now we need $\mathbb{B}(r) = \nabla \times \mathbb{A}(r)$. The usual vector identity prescribes...

[redacted]

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$$\mathbb{B}(r) = \overset{1}{\mathbb{G}(\nabla\cdot\hat{r})} - \overset{2}{(\mathbb{G}\cdot\nabla)\hat{r}} - \overset{3}{\hat{r}(\nabla\cdot\mathbb{G})} + \overset{4}{(\hat{r}\cdot\nabla)\mathbb{G}}. \tag{5}$$

We easily calculate: $(\nabla\cdot\hat{r}) = 2/r$, for the first term. For the other terms...

$$[2]_x = \left(G_x\frac{\partial}{\partial x} + G_y\frac{\partial}{\partial y} + G_z\frac{\partial}{\partial z}\right)\frac{x}{r} \quad ; \quad \frac{\partial}{\partial x}\left(\frac{x}{r}\right) = \frac{1}{r} - \frac{x^2}{r^3},\ \frac{\partial}{\partial y}\left(\frac{x}{r}\right) = -\frac{xy}{r^3}, \text{ etc.}$$

$$= \frac{G_x}{r} - \frac{x}{r^3}(\mathbb{r}\cdot\mathbb{G}) = \frac{1}{r}\left[\mathbb{G} - \hat{r}(\hat{r}\cdot\mathbb{G})\right]_x \tag{6}$$

So// $$1 - 2 = \frac{1}{r}\left[\mathbb{G} + \hat{r}(\hat{r}\cdot\mathbb{G})\right] = \frac{g}{r}\left[\mathbb{m} + \hat{r}(\hat{r}\cdot\mathbb{m})\right] \tag{7}$$

for 3: $$\nabla\cdot\mathbb{G} = \nabla\cdot(\mathbb{m}g) = \mathbb{m}\cdot\nabla g + g\overset{0}{\cancel{(\nabla\cdot\mathbb{m})}} = (\mathbb{m}\cdot\hat{r})\frac{\partial g}{\partial r} \tag{8}$$

and// $$[4]_x = \left(\frac{x}{r}\frac{\partial}{\partial x} + \frac{y}{r}\frac{\partial}{\partial y} + \frac{z}{r}\frac{\partial}{\partial z}\right)m_x g = m_x\left(\frac{\partial g}{\partial r}\right) = \left(\mathbb{m}\frac{\partial g}{\partial r}\right)_x$$

Put all this together to write the field of Eq. (5) as ...

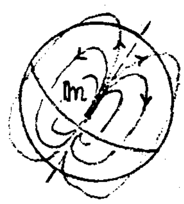
$$\mathbb{B}(r) = \left(\frac{g}{r} - \frac{\partial g}{r}\right)\hat{r}(\hat{r}\cdot\mathbb{m}) + \left(\frac{g}{r} + \frac{\partial g}{\partial r}\right)\mathbb{m}, \tag{9}$$

w// $g = \frac{1}{r^2}(1+\mu r)e^{-\mu r} \Rightarrow \partial g/\partial r = (-)\frac{1}{r^3}\left[2(1+\mu r) + \mu^2 r^2\right]$

Calculating the coefficients in g here, we find -- as required...

$$\boxed{\mathbb{B}(r) = \left[3\hat{r}(\hat{r}\cdot\mathbb{m}) - \mathbb{m}\right]\left(1 + \mu r + \tfrac{1}{3}\mu^2 r^2\right)\frac{e^{-\mu r}}{r^3} - \frac{2}{3}\mu^2\,\mathbb{m}\,\frac{e^{-\mu r}}{r}} \tag{10}$$

5pts (C) At the equator $(\hat{r}\cdot\mathbb{m} = 0)$: $B = -\frac{me^{-x}}{R_e^3}\left[\underbrace{(1 + x + \tfrac{1}{3}x^2)}_{\text{earth fld}} - \underbrace{\tfrac{2}{3}x^2}_{\text{added fld}}\right]$,



where $x = \mu R_e$, R_e = earth radius. The added/earth field ratio is

$$B_{added}/B_{earth} = \tfrac{2}{3}x^2/(1 + x + \tfrac{1}{3}x^2) < 4\times10^{-3} \Rightarrow x = \mu R_e < 0.080. \tag{11}$$

This gives the lower limit: $\mu^{-1} > R_e/0.080 = 12.5$ earth radii $= 79.6\times10^6$ m.
Since: $\mu = m_\gamma c/\hbar$, m_γ = photon mass, then: $\underline{m_\gamma < \frac{\hbar}{c}/7.96\times10^9\,\text{cm} = 4.4\times10^{-48}\,\text{gm}}$.
This agrees with (and supplies the argument for) Jackson's claim on his p. 6, re m_γ.