## Maxwell Equations: Vector & Scalar Potentials $\mathbb{A}$ & $\phi$.

In the static (time-independent) case, and for linear media ($\mathbb{D}=\epsilon\mathbb{E}$, $\mathbb{B}=\mu\mathbb{H}$):

$$\left[\begin{array}{l} \nabla\cdot\mathbb{E}=\frac{4\pi}{\epsilon}\rho,\ \nabla\times\mathbb{E}=0 \Rightarrow \mathbb{E}=-\nabla\phi,\ \text{w/}\ \phi=\frac{1}{\epsilon}\int\frac{1}{R}\rho\, d^3x'; \\ \nabla\cdot\mathbb{B}=0,\ \nabla\times\mathbb{B}=\frac{4\pi\mu}{c}\mathbb{J} \Rightarrow \mathbb{B}=\nabla\times\mathbb{A},\ \text{w/}\ \mathbb{A}=\frac{\mu}{c}\int\frac{1}{R}\mathbb{J}\, d^3x'. \end{array}\right. \tag{1}$$

This would be all of E&M, if it were not for the t-dependent terms we have left out. To accommodate those terms, we must modify the roles of $\phi$ & $\mathbb{A}$ somewhat. The procedure goes as follows.

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**1)** For t-dept. case, we have the "<u>non-source</u>" Maxwell Equations...

① $\underline{\nabla\cdot\mathbb{B}=0}$ $\Rightarrow$ can use: $\boxed{\mathbb{B}=\nabla\times\mathbb{A}}$, with $\mathbb{A}=\mathbb{A}(\mathbb{r},t)$ now a fcn of $t$; (2)

② $\underline{\nabla\times\mathbb{E}+\frac{1}{c}\frac{\partial\mathbb{B}}{\partial t}=0}$, or: $\nabla\times\left(\mathbb{E}+\frac{1}{c}\frac{\partial\mathbb{A}}{\partial t}\right)=0$,

... so set: $\boxed{\mathbb{E}+\frac{1}{c}(\partial\mathbb{A}/\partial t)=-\nabla\phi}$, with $\phi=\phi(\mathbb{r},t)$ now a fcn of $t$. (3)

How $\mathbb{A}$ & $\phi$ depend on $t$ is dictated by the "<u>source</u>" Maxwell Equations...

| | | |
|---|---|---|
| ③ $\underline{\nabla\times\mathbb{H}=\frac{4\pi}{c}\mathbb{J}+\frac{1}{c}\frac{\partial\mathbb{D}}{\partial t}}$ | assume: $\mathbb{D}=\epsilon\mathbb{E}$ | $\nabla\times\mathbb{B}=\frac{4\pi\mu}{c}\mathbb{J}+\frac{\mu\epsilon}{c}(\partial\mathbb{E}/\partial t)$, |
| ④ $\underline{\nabla\cdot\mathbb{D}=4\pi\rho}$ | $\mathbb{B}=\mu\mathbb{H}$ | $\nabla\cdot\mathbb{E}=\frac{4\pi}{\epsilon}\rho$. (4) |

Put in: $\mathbb{B}=\nabla\times\mathbb{A}$, $\mathbb{E}=-\nabla\phi-\frac{1}{c}(\partial\mathbb{A}/\partial t)$, to Eqs. (4), use the identity $\nabla\times(\nabla\times\mathbb{A})=\nabla(\nabla\cdot\mathbb{A})-\nabla^2\mathbb{A}$, and rearrange terms to get...

$$\boxed{\begin{array}{l} \nabla^2\phi+\frac{1}{c}\frac{\partial}{\partial t}(\nabla\cdot\mathbb{A})=-\frac{4\pi}{\epsilon}\rho, \\ \nabla^2\mathbb{A}-(\mu\epsilon/c^2)\frac{\partial^2\mathbb{A}}{\partial t^2}-\nabla\left[(\nabla\cdot\mathbb{A})+\frac{\mu\epsilon}{c}\left(\frac{\partial\phi}{\partial t}\right)\right]=-\frac{4\pi\mu}{c}\mathbb{J}. \end{array}} \tag{5}$$

Notice how the choice of potentials $\mathbb{A}$ & $\phi$ in Eqs. (2) & (3) <u>automatically</u> satisfies Max. Eqs. ①&②, while (4) Max. Eqs. ③&④ are left to specify (4) potentials $(\phi,\mathbb{A})$.

**2)** The $(\phi, \mathbb{A})$ eqtns above [Eqs(5)] are 4 eqtns in 4 unknowns (i.e. $(\phi; A_x, A_y, A_z)$). The can be made simpler, even decoupled, by imposing an additional condition linking $\phi$ & $\mathbb{A}$. In particular, we can choose...

$$\underline{\underline{\nabla\cdot\mathbb{A} + \frac{\mu\epsilon}{c}\frac{\partial\phi}{\partial t} = 0}} \Rightarrow \boxed{\begin{aligned}\nabla^2\phi - \frac{\mu\epsilon}{c^2}\left(\frac{\partial^2\phi}{\partial t^2}\right) &= -(4\pi/\epsilon)\rho, \\ \nabla^2\mathbb{A} - \frac{\mu\epsilon}{c^2}\left(\frac{\partial^2\mathbb{A}}{\partial t^2}\right) &= -(4\pi\mu/c)\mathbb{J}.\end{aligned}} \tag{6}$$

Called a "gauge condition".

Marvelous, if true... now, by "just" solving these inhomogeneous wave eqtns for $\phi$ & $\mathbb{A}$, we can solve the most general form of Maxwell's Eqtns (in a linear medium) by calculating $\mathbb{B} = \nabla\times\mathbb{A}$, $\mathbb{E} = -\nabla\phi - \frac{1}{c}(\partial\mathbb{A}/\partial t)$.

Imposing the above "gauge condition" is possible because neither of the potentials $\phi$ & $\mathbb{A}$ are uniquely defined by the fields $\mathbb{E}$ & $\mathbb{B}$, i.e. more than one $(\phi, \mathbb{A})$ corresponds to a given $(\mathbb{E}, \mathbb{B})$. As follows...

if : $\phi \to \phi' = \phi - \frac{1}{c}(\partial g/\partial t)$, and : $\mathbb{A} \to \mathbb{A}' = \mathbb{A} + \nabla g$ {GAUGE TRANSFORM}, (7)

then : Same $\mathbb{B} = \nabla\times\mathbb{A}$ & $\mathbb{E} = -\nabla\phi - \frac{1}{c}\frac{\partial\mathbb{A}}{\partial t}$ result from $(\phi', \mathbb{A}')$ & $(\phi, \mathbb{A})$.

The gauge fcn $g(\mathbf{r}, t)$ is arbitrary, and it allows the following freedom...

If $(\phi, \mathbb{A})$ don't satisfy : $\nabla\cdot\mathbb{A} + \frac{\mu\epsilon}{c}\left(\frac{\partial\phi}{\partial t}\right) = 0$, then let $(\phi, \mathbb{A}) \to (\phi', \mathbb{A}')$, w/

$$\nabla\cdot\mathbb{A}' + \frac{\mu\epsilon}{c}\left(\frac{\partial\phi'}{\partial t}\right) = \underbrace{\left[\nabla\cdot\mathbb{A} + \frac{\mu\epsilon}{c}\left(\frac{\partial\phi}{\partial t}\right)\right]}_{①} + \underbrace{\left[\nabla^2 g - \frac{\mu\epsilon}{c^2}\left(\frac{\partial^2 g}{\partial t^2}\right)\right]}_{②} = 0. \tag{8}$$

"Gauge condition" is satisfied by choosing $g$ such that ② = −①.

The new potentials $(\phi', \mathbb{A}')$ satisfy the "gauge condition" in Eq.(6), they are solutions to the inhomogeneous wave eqtns in Eq.(6), and the fields $(\mathbb{E}, \mathbb{B})$ are unaffected by choice of $g$ in this way. It all works on the tacit assumption that only the fields have directly measurable effects; the potentials themselves are just spectators.

# Max. Eqs.: $\phi$ & $\mathbb{A}$

3) The gauge condition $\nabla\cdot\mathbb{A}+\frac{\mu\epsilon}{c}\left(\frac{\partial\phi}{\partial t}\right)=0$ is useful for deriving the wave eqtns in (6), but it is not a unique choice. Two particularized choices are in common use.

**LORENTZ GAUGE** ← used in SRT, where both $\phi$ & $\mathbb{A}$ relevant for particles.

$(\phi, \mathbb{A})$ are readily derived from Eqs. (6) which satisfy $\boxed{\nabla\cdot\mathbb{A}+\frac{\mu\epsilon}{c}\left(\frac{\partial\phi}{\partial t}\right)=0.}$

Now consider gauge transform $(\phi,\mathbb{A})\to(\phi',\mathbb{A}')$, via $\phi'=\phi-\frac{1}{c}\dot{g}$, $\mathbb{A}'=\mathbb{A}+\nabla g$.

Then $\xrightarrow{}$ $\nabla\cdot\mathbb{A}+\frac{\mu\epsilon}{c}\left(\frac{\partial\phi}{\partial t}\right)=0 \to \nabla\cdot\mathbb{A}'+\frac{\mu\epsilon}{c}\left(\frac{\partial\phi'}{\partial t}\right)=0$, only if $\left(\nabla^2-\frac{\mu\epsilon}{c^2}\frac{\partial^2}{\partial t^2}\right)g=0$. (9)

All potentials $(\phi,\mathbb{A})$, $(\phi',\mathbb{A}')$ related by a gauge transform, with $g$ restricted in this way, and with $\nabla\cdot\mathbb{A}+\frac{\mu\epsilon}{c}\left(\frac{\partial\phi}{\partial t}\right)=0$ for each pair, are said to belong to the "Lorentz Gauge". This is the most commonly used gauge condition.

**COULOMB (radiation) GAUGE** ← used in QED, where only $\mathbb{A}$ (photon) is important.

Impose $\boxed{\nabla\cdot\mathbb{A}=0}$ [instead of $\nabla\cdot\mathbb{A}+\frac{\mu\epsilon}{c}\left(\frac{\partial\phi}{\partial t}\right)=0$]. This condition is invariant under a gauge transform if we impose $\nabla^2 g=0$. In any case, from Eq. (5) above:

$$\begin{cases}\nabla^2\phi=-\frac{4\pi}{\epsilon}\rho \Rightarrow \phi(\mathbf{r},t)=\frac{1}{\epsilon}\int\frac{d^3x'}{R}\rho(\mathbf{r}',t), \text{ w/ } R=|\mathbf{r}-\mathbf{r}'| \text{ — instantaneous Coulomb pot'l.}\\[2ex] \nabla^2\mathbb{A}-\frac{\mu\epsilon}{c^2}\frac{\partial^2\mathbb{A}}{\partial t^2}=-\frac{4\pi\mu}{c}\mathbb{J}+\frac{\mu\epsilon}{c}\nabla(\partial\phi/\partial t).\end{cases} \tag{10}$$

If no charges are present, i.e. $\rho=0$ & $\mathbb{J}=0$, then $\begin{cases}\phi=0 \text{ — instead of homog. wave eqtn for } \phi\text{, as in Lorentz gauge;}\\ \left(\nabla^2-\frac{\mu\epsilon}{c^2}\partial^2/\partial t^2\right)\mathbb{A}=0.\end{cases}$

The general wave eqtn for $\mathbb{A}$, Eq. (10), can be simplified in this gauge ... see Jkⁿ Eqs. (6.47)-(6.52) and prob #43. First decompose $\mathbb{J}$ into transverse & longitudinal parts:

$$\to \mathbb{J}=\mathbb{J}_T+\mathbb{J}_L, \text{ w/ } \mathbb{J}_T=\frac{1}{4\pi}\nabla\times\left[\nabla\times\int\frac{d^3x'}{R}\mathbb{J}\right] \;\&\; \mathbb{J}_L=-\frac{1}{4\pi}\nabla\int\frac{d^3x'}{R}\nabla'\cdot\mathbb{J}. \tag{11}$$

Have $\nabla\cdot\mathbb{J}_T=0$ & $\nabla\times\mathbb{J}_L=0$. Now: $\phi=\frac{1}{\epsilon}\int\frac{d^3x'}{R}\rho$, and: $\frac{\partial\rho}{\partial t}+\nabla\cdot\mathbb{J}=0$, together imply that: $\nabla(\partial\phi/\partial t)=(4\pi/\epsilon)\mathbb{J}_L$. Use of this result in Eq. (10) yields a reduced eqtn:

$$\boxed{\left(\nabla^2-\frac{\mu\epsilon}{c^2}\partial^2/\partial t^2\right)\mathbb{A}=-(4\pi\mu/c)\mathbb{J}_T}. \tag{12}$$

# Maxwell Equations: Wave-like Solutions [Ref. Jackson Secs. (6.5) & (6.6)]

We have by now reduced Maxwell's Eqs. in a linear medium to the solution of two inhomogeneous wave equations...

$$\nabla\cdot\mathbf{E} = \left(\frac{4\pi}{\epsilon}\right)\rho,\quad \nabla\times\mathbf{E} = -\frac{1}{c}\partial\mathbf{B}/\partial t,\quad \nabla\cdot\mathbf{B} = 0,\quad \nabla\times\mathbf{B} = \frac{4\pi\mu}{c}\mathbf{J} + \frac{\mu\epsilon}{c}\frac{\partial\mathbf{E}}{\partial t};$$

use potentials...

$$\mathbf{B} = \nabla\times\mathbf{A},\quad \mathbf{E} = -\nabla\phi - \frac{1}{c}\frac{\partial\mathbf{A}}{\partial t};$$

... and Lorentz Gauge

$$\nabla\cdot\mathbf{A} + \frac{\mu\epsilon}{c}\frac{\partial\phi}{\partial t} = 0$$

$$\boxed{\begin{aligned}\nabla^2\phi - \frac{1}{v^2}\frac{\partial^2\phi}{\partial t^2} &= -\frac{4\pi}{\epsilon}\rho,\\ \nabla^2\mathbf{A} - \frac{1}{v^2}\frac{\partial^2\mathbf{A}}{\partial t^2} &= -\frac{4\pi\mu}{c}\mathbf{J};\end{aligned}}$$

Where: $v = c/\sqrt{\mu\epsilon}$. (1)

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1) To proceed, evidently we should look at solutions to the generic wave eqtn...

$$\boxed{\nabla^2\psi - \frac{1}{v^2}(\partial^2\psi/\partial t^2) = -4\pi f(\mathbf{r},t)} \quad (2)$$

$\psi = \psi(\mathbf{r},t)$ is the wave amplitude, $f(\mathbf{r},t)$ is a <u>known</u> source function.

**REMARK** <u>Plane wave solutions</u>.

The <u>homogeneous</u> eqtn is: $\left(\nabla^2 - \frac{1}{v^2}\frac{\partial^2}{\partial t^2}\right)\psi = 0$. Try plane wave: $\psi = Ae^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$.

So $\left(-k^2 + \frac{\omega^2}{v^2}\right)\psi = 0$ ... works if $\omega = \pm kv = \pm kc/\sqrt{\mu\epsilon}$ — relation: $\omega = \omega(k)$ is a "<u>dispersion relation</u>"

If $\mu$ & $\epsilon$ are indpt. of $\omega$, then $\begin{cases}\text{wave } \underline{\text{phase}} \text{ velocity}: \omega/k = \pm v\\ \text{wave } \underline{\text{group}} \text{ velocity}: \partial\omega/\partial k = \pm v\end{cases}$ both cnst (wave shows zero dispersion)

If $\mu$ & $\epsilon$ depend on freq. $\omega$, $v$(phase) & $v$(group) $\propto \omega$, and propagating wave distorts.

We first assume simplest case: $\omega = \text{cnst}\times k$, with no wave distortion/dispersion.

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2) With no dispersion [$v$(phase) & $v$(group) = cnst], Fourier analysis of Eq. (2) is useful:

transform ↓ inverse ↘

$$\tilde{\Psi}(\mathbf{r},\omega) = \int_{-\infty}^{\infty}\psi(\mathbf{r},t)\,e^{i\omega t}\,dt \leftrightarrow \psi(\mathbf{r},t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{\Psi}(\mathbf{r},\omega)\,e^{-i\omega t}\,d\omega,$$

$$\mathcal{F}(\mathbf{r},\omega) = \int_{-\infty}^{\infty}f(\mathbf{r},t)\,e^{i\omega t}\,dt \leftrightarrow f(\mathbf{r},t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathcal{F}(\mathbf{r},\omega)\,e^{-i\omega t}\,d\omega. \quad (3)$$

This formulation anticipates problems that are unbounded in time. Plug these in (2):

## ME Wave Solutions (cont'd)

$$\left(\nabla^2 - \frac{1}{v^2}\frac{\partial^2}{\partial t^2}\right)\psi = -4\pi f \Rightarrow \boxed{(\nabla^2 + k^2)\,\tilde{\psi}(\mathbf{r},\omega) = -4\pi \tilde{f}(\mathbf{r},\omega)}, \quad (4)$$

where $k = \frac{\omega}{v}$ is linear in $\omega$. The transformed wave eqtn in $\tilde{\psi}$ & $\tilde{f}$ is called an "inhomogeneous Helmholtz equation"; evidently it includes the Poisson eqtn [$\nabla^2\tilde{\psi} = -4\pi\tilde{f}$, for $k=0$] as a special case. What's important about the Fourier transform is that it has reduced the $t$-variation to a "spectator" variable $\omega$.

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3) PDE's of the Helmholtz (and allied) type can be solved by using Green's functions.

$$\begin{aligned} &\text{Want } \tilde{\psi} \text{ in}: \ (\nabla^2 + k^2)\,\tilde{\psi}(\mathbf{r},\omega) = -4\pi\tilde{f}(\mathbf{r},\omega), \quad ① \\ &\text{Define } G_k \text{ by}: \ (\nabla^2 + k^2)\,G_k(\mathbf{r},\mathbf{r}') = -4\pi\,\delta(\mathbf{r}-\mathbf{r}'). \ ② \end{aligned} \quad (5)$$

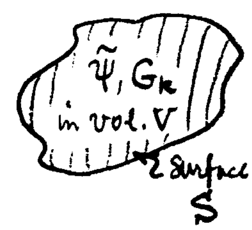
assume we can solve for $G_k(\mathbf{r},\mathbf{r}')$

Connect these equations by Green's identity...

$$\tilde{\psi}\,\nabla^2 G_k - G_k \nabla^2\tilde{\psi} = \nabla\cdot\left(\tilde{\psi}\nabla G_k - G_k\nabla\tilde{\psi}\right)$$

integrate $\int_V d^3x$, use Divergence Thm

$$\int_V \underbrace{\left(\tilde{\psi}\,\nabla^2 G_k - G_k\nabla^2\tilde{\psi}\right)}\,d^3x = \oint_S \left(\tilde{\psi}\nabla G_k - G_k\nabla\tilde{\psi}\right)\cdot d\mathbf{S}$$




integrand $= 4\pi\left[G_k\tilde{f} - \tilde{\psi}\,\delta(\mathbf{r}-\mathbf{r}')\right]$ obtained by forming qty $\tilde{\psi}\cdot$Eq②$-G_k\cdot$Eq①

So//

$$4\pi\left[\int_V G_k\tilde{f}\,d^3x - \tilde{\psi}(\mathbf{r}',\omega)\right] = -\oint_S\left(G_k\nabla\tilde{\psi} - \tilde{\psi}\nabla G_k\right)\cdot d\mathbf{S}. \quad (6)$$

Interchange labels $\mathbf{r}$ & $\mathbf{r}'$ [note that $G_k(\mathbf{r}',\mathbf{r}) = G_k(\mathbf{r},\mathbf{r}')$], rearrange terms to get

$$\boxed{\tilde{\psi}(\mathbf{r},\omega) = \int_V G_k(\mathbf{r},\mathbf{r}')\,\tilde{f}(\mathbf{r}',\omega)\,d^3x' + \frac{1}{4\pi}\oint_S\left(G_k\nabla'\tilde{\psi} - \tilde{\psi}\nabla' G_k\right)\cdot d\mathbf{S}'} \quad (7)$$

Except for the "spectator" variable $\omega$ (& $k=\omega/v$), this solution is the same as for Poisson's eqtn [see Jk$^{\underline{n}}$ Eq. (1.42)]. It gives a particular sol$^{\underline{n}}$ to $(\nabla^2+k^2)\tilde{\psi} = -4\pi\tilde{f}$, provided $G_k$ satisfies $(\nabla^2+k^2)G_k = -4\pi\,\delta(\mathbf{r}-\mathbf{r}')$. The $\int_V$ term accounts for $\tilde{\psi}$ generated in V by the source $\tilde{f}$; the $\oint_S$ term provides freedom for B.C. on surface.

## ME Wave Solutions (cont'd)

**REMARKS** on soln: $\tilde{\Psi} = \int_V G_k \tilde{f}\, d^3x' + \frac{1}{4\pi}\oint_S (G_k \nabla'\tilde{\Psi} - \tilde{\Psi}\nabla' G_k)\cdot d\mathbb{S}'$.

1. Any solⁿ $\tilde{\Psi}_0$ to homogeneous eqtn $(\nabla^2 + k^2)\tilde{\Psi}_0 = 0$ can be added to this $\tilde{\Psi}$.

2. The surface term can be adjusted to meet B.C. on surface $S$ enclosing volume V...

$$\left[\begin{array}{l}\left.\begin{array}{l}\text{Dirichlet}\\ \text{conditions}\end{array}\right\}\ \tilde{\Psi}\text{ given on } S \Rightarrow \left\{\begin{array}{l}\text{construct } G_k(\text{on } S)\equiv 0;\ \text{then:}\\ \text{surface term} \rightarrow (-)\frac{1}{4\pi}\oint_S(\tilde{\Psi}\nabla' G_k)\cdot d\mathbb{S};\end{array}\right. \qquad (8)\\ \\ \left.\begin{array}{l}\text{Neumann}\\ \text{conditions}\end{array}\right\}\ \nabla\tilde{\Psi}\text{ given on } S \Rightarrow \left\{\begin{array}{l}\text{construct } \nabla G_k(\text{on } S)\equiv 0;\ \text{then:}\\ \text{surface term} \rightarrow (+)\frac{1}{4\pi}\oint_S(G_k\nabla'\tilde{\Psi})\cdot d\mathbb{S}.\end{array}\right. \qquad (9)\end{array}\right.$$

Evidently the actual functional form of $G_k(\mathbf{r},\mathbf{r}')$ depends on the B.C. required.

3. Sometimes the region of interest is an $\infty$ domain, i.e. the surface $S$ is at $\infty$ -- where by definition $\tilde{\Psi}$ & $\nabla\tilde{\Psi}$ both vanish. Then our solution-to-date looks like:

$$\left[\begin{array}{l}\tilde{\Psi}(\mathbf{r},\omega) = \int_\infty G_{k\infty}(\mathbf{r},\mathbf{r}')\tilde{f}(\mathbf{r}',\omega)\,d^3x',\\ \text{with: } (\nabla^2 + k^2)G_{k\infty} = -4\pi\,\delta(\mathbf{r}-\mathbf{r}').\end{array}\right\Vert\ \begin{array}{l}\text{for } \infty \text{ domain, with}\\ G_{k\infty}\rightarrow 0 \text{ as } |\mathbf{r}-\mathbf{r}'|\rightarrow\infty.\end{array} \qquad (10)$$

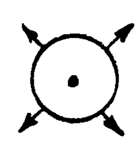
4. Jackson shows how to solve the $G_{k\infty}$ problem in his Eqs. (6.58)-(6.62). Results:

$$\boxed{G_{k\infty}(\mathbf{r},\mathbf{r}') = \frac{1}{R}e^{\pm ikR}},\ \text{w/ } R = |\mathbf{r}-\mathbf{r}'|,\ \text{on } \infty \text{ domain.} \qquad (11)$$

This can be verified by a plug-in, if you use $\nabla^2(1/R) = -4\pi\,\delta(\mathbf{r}-\mathbf{r}')$.

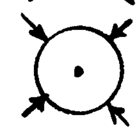
The results here have a geometrical interpretation...

(+) sign ⟷ <u>outgoing</u> spherical wave from point source at origin:



$\dfrac{e^{+ikR}}{R}$

(−) sign ⟷ <u>incoming</u> " " to " " " "



$\dfrac{e^{-ikR}}{R}$

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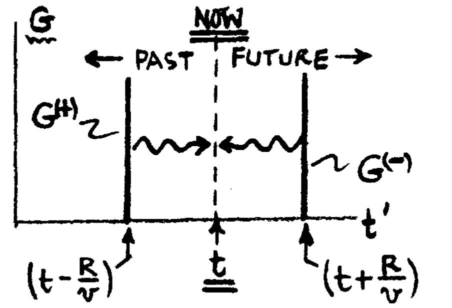
The wave amplitude $\Psi(\mathbf{r},t)$ [on an $\infty$ domain] can now be obtained by using $G_{k\infty}$ of Eq. (11) in $\tilde{\Psi}(\mathbf{r},\omega)$ of Eq. (10), and then inverting the Fourier transforms $\tilde{\Psi}$ and $\tilde{f}$. Jackson shows how this is done in his Eqs. (6.63)-(6.66).

## ME Wave Solutions (cont'd)

**4)** With the procedure just noted, the resulting solution to the inhomogeneous wave eqtn: $\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\psi = -4\pi f(\mathbf{r}, t)$, on an $\infty$ domain may be quoted as:

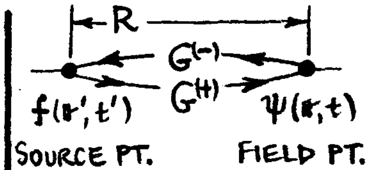
(12)
$$\boxed{\psi(\mathbf{r}, t) = \int_\infty d^3x' \int_{-\infty}^{\infty} dt'\, G^{(\pm)}(\mathbf{r}, t; \mathbf{r}', t')\, f(\mathbf{r}', t'),\quad \text{w/}\ G^{(\pm)} = \frac{1}{R}\,\delta\!\left(t' - \left[t \mp \frac{R}{v}\right]\right) \int R = |\mathbf{r} - \mathbf{r}'|,\ v = c/\sqrt{\mu\epsilon}.}$$




Signal (from $f$ at time $t'$) propagates to observation pt ($\psi$ at time $t$) from the past by $G^{(+)}$, or from future by $G^{(-)}$.

[this is the un-numbered eqtn on Jkn p. 225]. The $\delta$-fcn in $G^{(\pm)}$ is really present, and it has a Novel Feature: the signal from $f$ at time $t'$ can arrive to form the disturbance $\psi$ at time $t$, at: $\boxed{t = t' \pm (R/v)}$, i.e. $f \Rightarrow \psi$ from either the past or the future. There is no mathematical distinction between past & future (since wave eqtn is quadratic in $x$ & $t$).

Time-ordering aside, we at least have a rational result for the finite propagation velocity $v$ in the theory. The source & field points are (causally) connected only if: $R = v|t - t'|$.




Signal velocity $= v$

Details of the two solutions for $\psi$ go as follows...

① $G^{(+)}$ solution: $f$ signal at $t'$ reaches $\psi$ pt. at: $t = t' + \frac{R}{v} > t'$ {signal at $t$ is from past.

$\hookrightarrow$ $t' = t - \frac{R}{v} = t_{ret}$, is the "retarded time"; $G^{(+)}$ is called "retarded" Green's fcn;

and// $$\boxed{\psi(\mathbf{r}, t) = \Psi_{inc}(\mathbf{r}, t) + \int_\infty \frac{d^3x'}{R} f(\mathbf{r}', t_{ret})} \quad (13)$$
$\int$ $\Psi_{inc}$ = solution to homogeneous wave eqtn as $t \to (-)\infty$ [starts in past].

② $G^{(-)}$ solution: $f$ signal at $t'$ reaches $\psi$ pt. at: $t = t' - \frac{R}{v} < t'$ {signal at $t$ is from future.

$\hookrightarrow$ $t' = t + \frac{R}{v} = t_{adv.}$, is the "advanced time"; $G^{(-)}$ is called "advanced" Green's fcn;

and// $$\boxed{\psi(\mathbf{r}, t) = \Psi_{out}(\mathbf{r}, t) + \int_\infty \frac{d^3x'}{R} f(\mathbf{r}', t_{adv})} \quad (14)$$
$\int$ $\Psi_{out}$ = solution to homogeneous wave eqtn as $t \to (+)\infty$ [starts in future].

The choice of $G^{(\pm)}$ for the $\psi$ solution is dictated by whether we want the source integral to contribute $\sim$ zero at very early [$G^{(+)}$] or very late [$G^{(-)}$] times $t$.

# ME Wave Solutions (cont'd)

## 5) SUMMARY — A Complete Solution to Maxwell's Equations.

For a linear medium ($\mathbb{D} = \epsilon \mathbb{E}$, $\mathbb{B} = \mu \mathbb{H}$) ...  $\qquad$ L gauge: $\nabla \cdot \mathbb{A} + \frac{\mu\epsilon}{c}\left(\frac{\partial \phi}{\partial t}\right) = 0$

$$\left.\begin{array}{l} \nabla \cdot \begin{pmatrix} \mathbb{E} \\ \mathbb{B} \end{pmatrix} = \frac{4\pi}{\epsilon}\begin{pmatrix} \rho \\ 0 \end{pmatrix}, \\ \nabla \times \begin{pmatrix} \mathbb{E} \\ \mathbb{B} \end{pmatrix} = \frac{4\pi\mu}{c}\begin{pmatrix} 0 \\ \mathbb{J} \end{pmatrix} + \frac{1}{c}\begin{pmatrix} -\partial \mathbb{B}/\partial t \\ \mu\epsilon\, \partial \mathbb{E}/\partial t \end{pmatrix} \end{array}\right\} \left(\nabla^2 - \frac{1}{v^2}\frac{\partial^2}{\partial t^2}\right)\begin{pmatrix} \phi \\ \mathbb{A} \end{pmatrix} = -4\pi \begin{pmatrix} \rho/\epsilon \\ \frac{\mu}{c}\mathbb{J} \end{pmatrix},$$

where: $v = c/\sqrt{\mu\epsilon}$;

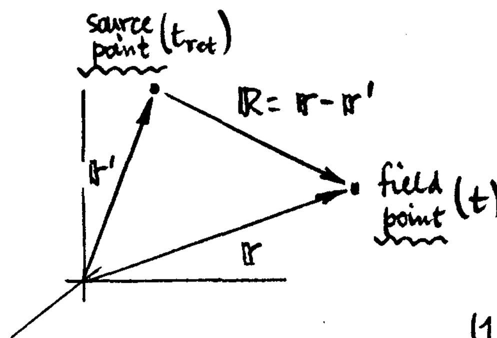
w/ $\mathbb{B} = \nabla \times \mathbb{A}$, $\mathbb{E} = -\nabla\phi - \frac{1}{c}(\partial \mathbb{A}/\partial t)$. $\qquad$ (15)

Solutions on an $\infty$ domain ...

$$\phi(\mathbf{r}, t) = \phi_0(\mathbf{r}, t) + \frac{1}{\epsilon}\int_{\text{sources}} \frac{d^3x'}{R}\, \rho(\mathbf{r}', t_{ret}),$$

$$\mathbb{A}(\mathbf{r}, t) = \mathbb{A}_0(\mathbf{r}, t) + \frac{\mu}{c}\int \frac{d^3x'}{R}\, \mathbb{J}(\mathbf{r}', t_{ret});$$

w/ $t_{ret} = t - \frac{1}{v}R(t_{ret})$, $R = |\mathbf{r} - \mathbf{r}'|$. $\qquad$ (16)




Here $\phi_0$ & $\mathbb{A}_0$ are solutions to the homogeneous eqtns: $\left(\nabla^2 - \frac{1}{v^2}\frac{\partial^2}{\partial t^2}\right)\begin{pmatrix} \phi_0 \\ \mathbb{A}_0 \end{pmatrix} = 0$. We have chosen the <u>retarded</u> solutions for $\phi$ & $\mathbb{A}$, per convention.

**REMARKS**

1. The only apparent effect of adding the time-derivative terms (in $\dot{\mathbb{B}}$ & $\dot{\mathbb{E}}$) to the static Maxwell Eqtns [see Eqs. (1) above] is that in the integrals for $\phi$ & $\mathbb{A}$ -- i.e. the integrals $\int \frac{d^3x'}{R}(\rho$ & $\mathbb{J})$ -- <u>the time $t \to t_{ret} = t - (R/v)$</u>. At first glance, this is an unexpectedly simple way to include the $\dot{\mathbb{B}}$ & $\dot{\mathbb{E}}$ terms... it just complicates the integrals a bit. <u>BUT</u>, the fact that the source pt - field pt distance $R$ now becomes an explicit function of the time difference $(t - t_{ret})$ will cause grief, later.

2. The solutions of Eq. (16) hold <u>only</u> on an $\infty$ domain (i.e. the only B.C. are that the fields $\mathbb{E}$ & $\mathbb{B}$ and potentials $\phi$ & $\mathbb{A}$ vanish at $\infty$). Problems which require B.C. on a finite domain are much more complicated, but are solvable w/ suitable surface terms.