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1992 Φ520 MidTerm Exam (in class, 3hr.) Fri. 13 Mar. 1992

This exam is open-book, open-notes, and is worth 120 pts. total. For each problem, put a $\boxed{\text{box}}$ around your answer. Number your solution pages consecutively, write your name on page 1, and staple the pages together before handing them in.

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① [25 pts]. Consider the dispersion relations for the Re & Im parts of the dielectric constant $\epsilon(\omega)$, as written in Jackson's Eqs. (7.119), p. 311. Show that if the medium's phase velocity $v_p = \frac{\omega}{k} =$ cnst for all ω, then the medium does not attenuate an EM wave, i.e. "Any nondispersive medium must be nonabsorptive for EM waves". Is the converse ("Any dispersive medium must be absorptive for EM waves") also true?

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② [35 pts]. Combine Jackson's Eqs (7.49) & (7.50), p. 285, to form an equation for the macroscopic polarization $\mathbb{P} = -ne\mathbf{r}$, w/ n = # electrons/unit volume. Eqtn is: $\boxed{P_{tt} + 2\beta P_t + \omega_0^2 P = \omega_p^2 E(t)}$, for each component of $\mathbb{P}$ & $\mathbb{E}$, w/ $\omega_p^2 = ne^2/m$, and ω_0 the natural frequency of a single oscillator. The damping cnst 2β [here] = γ [Jk^n]. We want a particular integral for $P(t)$ when $E(t)$ is not monochromatic.

(A) Use Fourier transforms [$P(t) \to \tilde{P}(\omega) = \int_{-\infty}^{\infty} P(t) e^{-i\omega t} dt$, etc.] to show $\tilde{P}$ and $\tilde{E}$ are related by: $\tilde{P} = \omega_p^2 \tilde{E} / [(\omega_0^2 - \omega^2) + 2i\beta\omega]$.

(B) Invert the Fourier transform of part (A) to show: $P(t) = \int_0^\infty K(\tau) E(t-\tau) d\tau$, and obtain an integral expression for the "kernel" $K(\tau)$. This lower limit = 0. Why?

(C) Evaluate $K(\tau)$ by contour integration. Sketch $K(\tau)$ vs. τ for all τ.

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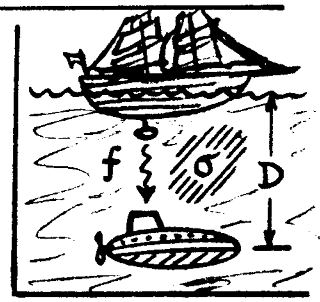
③ [25 pts]. A surface ship communicates with a submarine by sending EM waves at frequency f through the water. The sub remains submerged at depth D, and cannot detect the ship's signal if the power level falls below a nominal value S_m (let the ship broadcast at power level $B \gg S_m$). Assume that sea-water is a fairly good conductor [in fact: σ(sea-water) = 4.3 (ohm-m)$^{-1}$; note MKS units].



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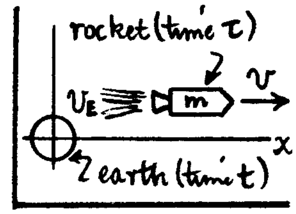
(A) Show that ship-to-sub communication is possible only if : $D\sqrt{f} \lesssim$ some #, and express "some #" in terms of σ, B, S_m and appropriate constants.

(B) For actual #'s, assume $B = 1000\, S_m$ (sub detects at 0.1% of ship broadcast level), take σ(seawater) given above, and fix $D = 100$m. What is the maximum frequency f which can be used for messages?



4 [35 pts]. A rocket of instantaneous rest mass $m(\tau)$ [τ = time on board rocket] -- which was initially at rest (at $\tau = 0$) on earth, with launch mass $m(0) = m_0$ -- is accelerated to relativistic speeds along a straight line by an engine burn scheme that produces acceleration $a(\tau)$ on board the rocket. $a(\tau)$ is not necessarily a constant, because the fuel burn rate $dm/d\tau$ and/or exhaust velocity v_E may be variable.

(A) Show that the rocket velocity is : $\beta(\tau) = \tanh\left\{\frac{1}{c}\int_0^\tau a(\tau')\,d\tau'\right\}$, in rocket time.

(B) Find the relation between earth time t and rocket time τ.

(C) Let $R(\tau) = m_0/m(\tau)$ be the "burn-ratio," and assume $v_E = v_E(\tau)$ by choice. Establish the relation : $\int_{R=1}^{R} v_E \, d\ln R = \int_0^\tau a(\tau')\,d\tau'$. What is the solution for $R(\tau)$ when v_E = cnst and a = cnst? What are the "best" choices for $v_E(\tau)$ and $a(\tau)$? Just comment on this last question; detailed solution is not needed.

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[25 pts]. Argue general dispersion/absorption features from $\epsilon(\omega)$ dispersion relations.

1) The dispersion relations for $\epsilon(\omega)$ in Jackson's Eqs. (7.119) are:

$$\rightarrow \mathrm{Re}\,\epsilon(\omega) = 1 + \frac{1}{\pi}\mathcal{P}\int_{-\infty}^{\infty}\frac{\mathrm{Im}\,\epsilon(x)}{x-\omega}dx\,,\quad \mathrm{Im}\,\epsilon(\omega) = \frac{1}{\pi}\mathcal{P}\int_{-\infty}^{\infty}\frac{[\mathrm{Re}\,\epsilon(x)-1]}{\omega-x}dx\,, \tag{1}$$

Where $\mathcal{P}$ denotes Cauchy principal value of the integral. Suppose the medium is non-dispersive, w/ phase (and group) velocity $v_p = \omega/k = c/n = \text{cnst}$. Here the index of refraction $n = \sqrt{\epsilon(\omega)}$ must be cnst, so ϵ is independent of ω. In particular, we must have $\mathrm{Re}\,\epsilon(\omega) = \text{cnst} = \epsilon_0$ for such a medium, and the 2nd of Eqs. (1) prescribes...

$$\mathrm{Im}\,\epsilon(\omega) = \frac{1}{\pi}(\epsilon_0-1)\,\mathcal{P}\int_{-\infty}^{\infty}\frac{dx}{\omega-x} = (-)\frac{1}{\pi}(\epsilon_0-1)\lim_{\substack{\delta\to 0\\ \Omega\to\infty}}\left\{\int_{-\Omega}^{\omega-\delta} + \int_{\omega+\delta}^{\Omega}\right\}d\ln(x-\omega)$$

$$= -\frac{1}{\pi}(\epsilon_0-1)\lim_{\substack{\delta\to 0\\ \Omega\to\infty}}\left\{\ln|x-\omega|\Big|_{x=-\Omega}^{x=\omega-\delta} + \ln|x-\omega|\Big|_{x=\omega+\delta}^{x=\Omega}\right\}$$

$$= -\frac{1}{\pi}(\epsilon_0-1)\lim_{\substack{\delta\to 0\\ \Omega\to\infty}}\left\{\ln\left(\frac{\delta}{\Omega+\omega}\right) + \ln\left(\frac{\Omega-\omega}{\delta}\right)\right\} = \frac{\epsilon_0-1}{\pi}\lim_{\Omega\to\infty}\ln\left(\frac{\Omega+\omega}{\Omega-\omega}\right) \equiv 0. \tag{2}$$

δ terms cancel

2) So, whenever $\mathrm{Re}\,\epsilon(\omega)$ is indpt of ω, $\mathrm{Im}\,\epsilon(\omega)$ must vanish. Since $\mathrm{Im}\,\epsilon(\omega)$ is a measure of the medium's absorption (dissipation) of the EM wave [ref Jkn Eq. (7.57), viz. $\mathrm{Im}\,\epsilon(\omega) \simeq 4\pi\sigma/\omega$, σ = conductivity $\Rightarrow$ Joule heating, etc], we can state:

$$\boxed{\text{Any non-dispersive medium is also non-absorptive for EM waves.}} \tag{3}$$

The contrapositive of this statement is also true: any dispersion of the medium implies some ω-dependence of $\mathrm{Re}\,\epsilon(\omega)$, and hence some nonzero functional dependence of $\mathrm{Im}\,\epsilon(\omega)$, per Eq. (1). In turn, $\mathrm{Im}\,\epsilon(\omega) \neq 0$ implies the medium is absorptive over some frequency range. Hence:

$$\boxed{\text{Any dispersive medium is also absorptive for EM waves, over some } \omega\text{-range.}} \tag{4}$$

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[35 pts]. Solve the SHO polarization model: $P_{tt} + 2\beta P_t + \omega_0^2 P = \omega_p^2 E(t)$.

(A) 1. If: $\tilde{F}(\omega) = \int_{-\infty}^{\infty} F(t)\, e^{-i\omega t}\, dt$, then: $\int_{-\infty}^{\infty} [\partial^n F(t)/\partial t^n]\, e^{-i\omega t}\, dt = (i\omega)^n \tilde{F}(\omega)$, for fcns $F(t)$ (and their derivatives) which vanish as $|t| \to \infty$. Then, by operating thru the P_{tt} eqtn with $\int_{-\infty}^{\infty} dt\, e^{-i\omega t} \times$, we easily get¶

$$\rightarrow (i\omega)^2 \tilde{P} + 2\beta\, i\omega \tilde{P} + \omega_0^2 \tilde{P} = \omega_p^2 \tilde{E} \Rightarrow \boxed{\tilde{P}(\omega) = \omega_p^2 \tilde{E}(\omega) / [(\omega_0^2 - \omega^2) + 2i\beta\omega]}. \quad (1)$$

NOTE Jk^{n} Eq.(7.50) is just a monochromatic version of Eq.(1), with $\tilde{E}(\omega) \to$ cnst.

(B) 2. The Fourier inverse of $\tilde{P}(\omega)$ of Eq.(1) is the desired particular integral. It is...

$$\rightarrow P(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{P}(\omega)\, e^{i\omega t}\, d\omega = \frac{\omega_p^2}{2\pi} \int_{-\infty}^{\infty} d\omega\, e^{i\omega t}\, \tilde{E}(\omega) / [(\omega_0^2 - \omega^2) + 2i\beta\omega]. \quad (2)$$

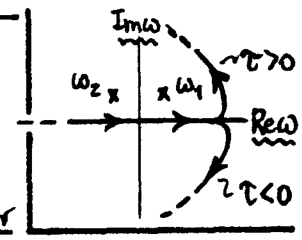
Put: $\tilde{E}(\omega) = \int_{-\infty}^{\infty} E(t')\, e^{-i\omega t'}\, dt'$ into the integral of Eq.(2) and rearrange terms

$$P(t) = \frac{\omega_p^2}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega\, e^{i\omega t}}{(\omega_0^2 - \omega^2) + 2i\beta\omega} \int_{-\infty}^{\infty} dt'\, e^{-i\omega t'} E(t') = \int_{-\infty}^{\infty} dt'\, E(t')\, K(t-t'),$$

w/ $\boxed{K(\tau) = \frac{\omega_p^2}{2\pi} \int_{-\infty}^{\infty} \frac{e^{i\omega\tau}\, d\omega}{(\omega_0^2 - \omega^2) + 2i\beta\omega}}$ (3) so $P(t) = \int_{\underbrace{-\infty}_{\to 0\ (\text{causality})}}^{\infty} d\tau\, E(t-\tau)\, K(\tau)$. (4)

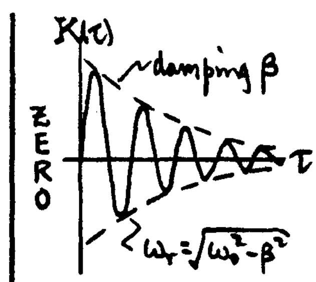
To respect causality, $P(t)$ cannot depend on values of $E(t')$ @ $t' > t$. In Eq.(4), this is accomplished by setting $\tau = 0$ at the lower limit, not $\tau = -\infty$.

(C) 3. The integrand for $K(\tau)$ has two simple poles in the upper half ω-plane: $\omega^2 - 2i\beta\omega - \omega_0^2 = 0 \Rightarrow \omega = \omega_{1,2} = \pm\omega_r + i\beta$, w/ $\omega_r = \sqrt{\omega_0^2 - \beta^2}$; ω_r is the damped SHO freq. When $\tau < 0$, contour is closed in the lower half-plane (why?); then $K(\tau < 0) \equiv 0$, which respects causality in Eq.(4). For $\tau > 0$, closure in upper half-plane, along with Residue Theorem and a bit of algebra yields:



$$\boxed{K(\tau) = \frac{\omega_p^2}{\omega_r} \theta(\tau)\, e^{-\beta\tau} \sin\omega_r\tau}, \quad \theta(\tau) = \text{unit step fcn}. \quad (5)$$

$K(\tau)$ vs. τ is sketched at right. It is a damped sinusoid, and represents the polarization response to a δ-fcn E-field excitation at $t = 0$.



¶ Any space dependence (on x) for either P or E just rides along as a spectator variable.

ϕ520 Midterm Solutions

(MT3)

③ [25 pts]. Analyze problems of rf communication underwater.

1. The ship's signal amplitudes fall off with distance x traveled through the water as

(A) $e^{-(x/\delta)}$, where δ is the "skin depth" of Jackson's Eq. (7.77), viz.

$$\rightarrow \delta = c/\sqrt{2\pi\mu\sigma\omega} \int \begin{matrix}\mu = 1 \text{ for seawater,}\\ \text{set } \omega = 2\pi f \ldots\end{matrix} \Rightarrow \delta = (c/2\pi\sqrt{\sigma})\frac{1}{\sqrt{f}}. \quad (1)$$

The signal power level (intensity) goes as $(e^{-x/\delta})^2$ and so if it was of strength B at $x=0$, it will be strength $B(e^{-D/\delta})^2$ at depth D. For sub to hear it:

$$\rightarrow B(e^{-D/\delta})^2 \geqslant S_m \Rightarrow e^{2D/\delta} \leqslant B/S_m, \text{ or } D \leqslant \frac{\delta}{2}\ln(B/S_m). \quad (2)$$

Put δ of Eq. (1) into Eq. (2) to get the required relation for ship→sub "talk"...

$$\boxed{D\sqrt{f} \leqslant (c/4\pi\sqrt{\sigma})\ln(B/S_m)}. \text{ The "Some \#" is the RHS of this inequality.} \quad (3)$$

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2. For actual #s, put σ(seawater) = 4.3 MKS units, as given, and -- from Jackson's

(B) Table 4, p. 820, note: $\underline{\sigma_{CGS} = 9\times10^9\,\sigma_{MKS}}$, so: σ(seawater) $= 3.87\times10^{10}$ Hz, CGS.

The coefficient on the RHS of Eq. (3) is then $(c/4\pi\sqrt{\sigma}) = 1.21\times10^4$ cm$/\sqrt{\text{Hz}}$.

If D is in meters, then Eq. (3) is numerically, in seawater...

$$\rightarrow D\sqrt{f} \leqslant 121\ln(B/S_m) \int \begin{matrix}D \text{ is in meters,}\\ f \text{ is in Hz.}\end{matrix} \quad (4)$$

If $D = 100$ m for sub's depth, and the signal strength ratio $B/S_m = 1000$, have

$$\rightarrow f \leqslant \left[\frac{121}{D}\ln(B/S_m)\right]^2 = [1.21\ln 10^3]^2 = 70\text{ Hz}. \quad (5)$$

Under these conditions, the sub "hears" frequencies only up to $\boxed{f \simeq 70\text{Hz, max.}}$

ϕ520 Midterm Solutions

④ [35 pts]. Analyse relativistic rocket for arbitrary acceln $a(\tau)$ & exhaust velocity v_E.

(A) 1. Ref. "Relativistic Rocket Trip" (class notes of 3/4/92). Divide both sides of Eq. (2)¶ by an incremental time $d\tau$ (rocket time), so...

$$\rightarrow \frac{dv}{d\tau} = (1-\beta^2)\frac{du}{d\tau}, \text{ or } \underbrace{\frac{d\beta}{d\tau} = (1-\beta^2)\frac{1}{c}a(\tau)} \quad \int a(\tau) = \frac{du}{d\tau}, \text{ acceln on rocket; } \beta = \frac{v}{c}, \text{ and } \frac{dv}{d\tau} = c\frac{d\beta}{d\tau}$$

$$\int \frac{d\beta}{1-\beta^2} = \frac{1}{c}\int a(\tau)\,d\tau, \text{ or } \boxed{\beta(\tau) = \tanh\left\{\frac{1}{c}\int_0^\tau a(\tau')\,d\tau'\right\}}, \text{ as required.} \quad (1)$$

(B) 2. Let $\phi(\tau) = \frac{1}{c}\int_0^\tau a(\tau')\,d\tau'$, so: $\beta(\tau) = \tanh\{\phi(\tau)\}$. Relation between t(earth) and τ(rocket) is found from Eq. (3) of "... Trip" notes, viz. $dt = d\tau/\sqrt{1-\beta^2}$. With $\beta = \tanh\phi$, have $1/\sqrt{1-\beta^2} = \cosh\phi$ (trig identity), so...

$$\rightarrow dt = [\cosh\phi(\tau)]\,d\tau \Rightarrow \boxed{t = \int_0^\tau [\cosh\phi(\tau')]\,d\tau'}, \text{ w/ } \phi(\tau) = \frac{1}{c}\int_0^\tau a(\tau')\,d\tau'. \quad (2)$$

(C) 3. If $R = m_0/m$ is the burn-ratio, then $dm = d(m_0/R) = -m_0\,dR/R^2$, and the relativistic rocket equation [Eq. (9) of "... Trip" notes] can be reorganized to...

$$\frac{v_E}{c}\left(\frac{dR}{R}\right) = \frac{d\beta}{1-\beta^2} = (\cosh^2\phi)\,d\tanh\phi = d\phi,$$

$$\text{so } \frac{1}{c}\int_{R=1}^{R} v_E\,d\ln R = \phi(\tau), \text{ or: } \boxed{\int_{R=1}^{R} v_E\,d\ln R = \int_0^\tau a(\tau')\,d\tau'}. \quad (3)$$

If $v_E = \text{cnst}$, this integrates to: $R(\tau) = \exp\left\{\frac{1}{v_E}\int_0^\tau a(\tau')\,d\tau'\right\}$. When also $a = \text{cnst} = A$, we get: $R(\tau) = \exp\left\{\frac{1}{c}A\tau/\epsilon\right\}$, w/ $\epsilon = v_E/c$, as in Eq. (11) of notes.
NOTE: both $v_E(\tau)$ and $a(\tau)$ are independently variable, resp. by adjusting the type and amount of fuel burned. We need an additional constraint on Eq. (3) to relate $v_E(\tau)$ & $a(\tau)$. Could be the reqt. that ratio of distance travelled to fuel burned, viz. $D(\tau)/R(\tau)$, be a maximum. Etc.

¶ Eq. (2) results from velocity addn formula: $(v+dv)_{\text{earth}} = [(v+du)/(1+\frac{v\,du}{c^2})]_{\text{rocket}}$. To 1st order ∞smal terms: $dv = [1-(v^2/c^2)]\,du$, as used in Eq. (1) above.