Φ 519 MidTerm Exam {in-class, 2hr. limit. — Sat. 19 Oct. 1991

This exam is open-book, open-notes, and is worth 120 points total. For each problem, put a box around your answer. Number your solution pages consecutively, write your name on page 1, and staple the pages together before handing them in.

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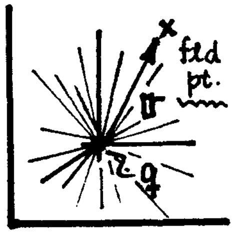
(1) [25pts]. A point charge q, situated at the origin, generates a non-Coulombic electric field: $\mathbb{E} = [q f(r)]\frac{\mathbf{r}}{r^3}$; the function $f(r)$ is such that $f(0)=1$, but $f(r)$ may vary with $r = |\mathbf{r}|$ at $r > 0$. (A) Find the electric flux $\Phi_E = \oint \mathbb{E} \cdot d\mathbb{S}$ passing through a sphere of radius R centered on q. Comment on how you could keep track of several q's inside the sphere by calculating the net Φ_E. (B) The first Maxwell equation is: $\nabla \cdot \mathbb{E} = 4\pi\rho$, where for a point charge q, one normally writes $\rho = q\delta(\mathbf{r})$. Find the form of the 1st Maxwell Eq. for the above $\mathbb{E}$.



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(2) [35pts.] Suppose you know a solution $\psi(\mathbf{r})$ to the PDE: $[\nabla^2 + K_0(\mathbf{r})]\psi(\mathbf{r}) = 0$. Let $K_0(\mathbf{r})$ be perturbed by a small amount $k(\mathbf{r})$, i.e. $K_0 \to K_0 + k$; then ψ is also perturbed: $\psi \to \tilde{\psi} = \psi + \lambda$, where $\lambda = \lambda(\mathbf{r})$ is a (small) correction function. Assume you can find a Green's function G satisfying: $[\nabla^2 + K_0(\mathbf{r})]G(\mathbf{r},\mathbf{r}') = -\delta(\mathbf{r}-\mathbf{r}')$. Using G, k & ψ, calculate the correction function $\lambda(\mathbf{r})$ to first order in the perturbation k. For simplicity, assume an ∞ domain, with ψ, etc. vanishing at ∞.

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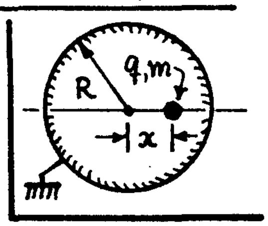
(3) [30pts.] A point charge q of mass m is held at distance $x < R$ from the center of a grounded conducting spherical shell of radius R, as shown. (A) Find the magnitude & direction of the force acting on q at x. (B) q is now released (from rest). Find a first integral of the motion, i.e. q's velocity.



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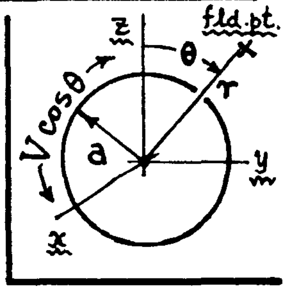
(4) [30pts.]. A non-conducting spherical shell of radius a has (by special arrangement) a surface potential held at $V\cos\theta$, w/ V = cnst, and θ the colatitude angle shown. Find the electrostatic potential $\phi(r,\theta,\varphi)$ everywhere in space. Do appropriate integrals.



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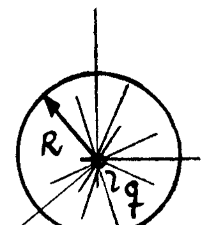
① [25pts]. Non-Coulombic electric field: $\mathbf{E} = [q\,f(r)]\frac{\mathbf{r}}{r^3}$.

(A) The R-sphere has $r = R = \text{cnst}$, and surface elements $d\mathbf{S} = \hat{r}\,dS$. So...

$$\rightarrow \Phi_E = \oint \mathbf{E}\cdot d\mathbf{S} = q\oint \left[f(r)\frac{\hat{r}}{r^2}\right]\Big|_{r=R}\cdot \hat{r}\,dS = q\,f(R)\frac{1}{R^2}\underbrace{\oint dS}_{4\pi R^2}$$



So $$\boxed{\Phi_E = 4\pi q\,f(R)} \quad (1)$$

If $f(R) \neq$ cnst, the flux generated by q depends on how far away you are from q when you measure it. If there are several q's randomly placed inside the R-sphere, then Φ_E(net) will depend not only on q(net) but also on where the individual q's are situated. So Φ_E(net) would no longer measure just q(net) as it does when $\mathbf{E}$ is Coulombic. In fact Φ_E is not a useful tool for non-Coulombic fields.

(B) Use identity $\nabla\cdot(\psi\mathbf{a}) = \psi\nabla\cdot\mathbf{a} + \mathbf{a}\cdot\nabla\psi$ (Jackson: inside front cover) to calculate:

$$\rightarrow \nabla\cdot\mathbf{E} = q\,\nabla\cdot\left[f(r)\frac{\mathbf{r}}{r^3}\right] = q\left[f(r)\underbrace{\nabla\cdot\left(\frac{\mathbf{r}}{r^3}\right)}_{①} + \left(\frac{\mathbf{r}}{r^3}\right)\cdot\underbrace{\nabla f(r)}_{②}\right]. \quad (2)$$

but/

① $= 4\pi\,\delta(\mathbf{r})$, by class notes on Helmholtz' Thm;

② $= \hat{r}\,(df/dr)$, for functions $f(r)$ which are indpt of $\measuredangle$'s; } (3)

So

$$\nabla\cdot\mathbf{E} = q\left[4\pi\,\cancel{f(r)}^{\,1,\text{ contributes only at } r=0,\text{ where } f(0)=1}\,\delta(\mathbf{r}) + \frac{1}{r^2}(df/dr)\right]$$

or

$$\boxed{\nabla\cdot\mathbf{E} = 4\pi q\left[\delta(\mathbf{r}) + \Delta(r)\right],\quad \Delta(r) = \frac{1}{4\pi r^2}(df/dr).} \quad (4)$$

The effective point charge distribution $\delta(\mathbf{r}) \rightarrow \delta(\mathbf{r}) + \Delta(r)$; q is "smeared out."

ϕ519 Exam #2 Solutions

Exam #2 MT2

② [30 pts]. Do first order perturbation theory on $[\nabla^2 + K_0(\mathbf{r})]\psi(\mathbf{r}) = 0$.

1) Know solution to $[\nabla^2 + K_0(\mathbf{r})]\psi(\mathbf{r}) = 0$, and wish to solve -- when $K_0 \to K_0 + k$:

$$\left.\begin{aligned} &[\nabla^2 + K_0(\mathbf{r})]\tilde{\psi}(\mathbf{r}) = -k(\mathbf{r})\,\tilde{\psi}(\mathbf{r}), \leftarrow \text{mult. on left by } G;\\ &[\nabla^2 + K_0(\mathbf{r})]\,G(\mathbf{r},\mathbf{r}') = -\delta(\mathbf{r}-\mathbf{r}'); \leftarrow \text{mult. on left by } \tilde{\psi}. \end{aligned}\right\} \quad (1)$$

w/

Interchange variables $\mathbf{r}$ & $\mathbf{r}'$, noting G is symmetric in $\mathbf{r}$ & $\mathbf{r}'$. Then, carry out indicated multiplications and subtract equations to get...

$$\left.\begin{aligned} \rightarrow \underbrace{G\nabla'^2\tilde{\psi} - \tilde{\psi}\nabla'^2 G}_{\nabla'\cdot[G\nabla'\tilde{\psi} - \tilde{\psi}\nabla' G],\ \text{Green's identity}} = -G(\mathbf{r},\mathbf{r}')\,k(\mathbf{r}')\,\tilde{\psi}(\mathbf{r}') + \delta(\mathbf{r}-\mathbf{r}')\,\tilde{\psi}(\mathbf{r}') \end{aligned}\right\} \quad (2)$$

2) Integrate over the domain D of definition $\int_D d^3x'$ and use Divergence Thm on the LHS of Eq. (2) to convert to a surface integral $\oint_S d^2x'$. Then...

$$\rightarrow \oint_S \left[G(\mathbf{r},\mathbf{r}')\frac{\partial\tilde{\psi}}{\partial n'} - \tilde{\psi}(\mathbf{r}')\frac{\partial G}{\partial n'}\right] d^2x' = -\int_D G(\mathbf{r},\mathbf{r}')\,k(\mathbf{r}')\,\tilde{\psi}(\mathbf{r}')\,d^3x' + \tilde{\psi}(\mathbf{r})$$

Let the domain $D \to \infty$, so surface S is at ∞ and $\oint_S [\ \ ]\,d^2x' \to 0$. Then have...

$$\rightarrow \tilde{\psi}(\mathbf{r}) = \int_\infty G(\mathbf{r},\mathbf{r}')\,k(\mathbf{r}')\,\tilde{\psi}(\mathbf{r}')\,d^3x' \quad (3)$$

3) Eq. (3), as it stands, is a particular integral of the PDE: $(\nabla^2 + K_0)\tilde{\psi} = -k\tilde{\psi}$, and we can add or subtract from $\tilde{\psi}$ on the LHS any fcn ψ which satisfies the homogeneous eqtn $(\nabla^2 + K_0)\psi = 0$. So put $\tilde{\psi} \to \tilde{\psi} - \psi = \lambda$ on LHS, to get...

$$\rightarrow \lambda(\mathbf{r}) = \int_\infty G(\mathbf{r},\mathbf{r}')\,k(\mathbf{r}')\,[\psi(\mathbf{r}') + \lambda(\mathbf{r}')]\,d^3x'. \quad (4)$$

Evidently λ is $O(k)$, and the contrib^n from λ on RHS is $O(k^2)$. To first order (lowest order) in the perturbation k, the correction fcn is therefore...

$$\boxed{\lambda(\mathbf{r}) \simeq \int_\infty G(\mathbf{r},\mathbf{r}')\,k(\mathbf{r}')\,\psi(\mathbf{r}')\,d^3x'} \quad (5)$$

Iteration provides a complete pertb^n theory on this problem.

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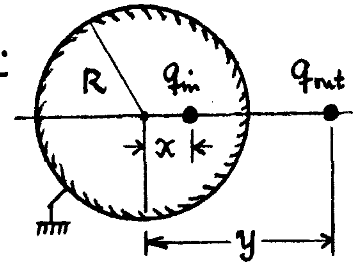
③ [30 pts.]. Force on point charge q inside grounded sphere.



(A) 1. Jackson solves the pt. charge–grounded sphere problem in his Sec.(2.2). He finds the sphere is at zero potential for an image pair (q_{in} at $x<R$) & (q_{out} at $y>R$) related by

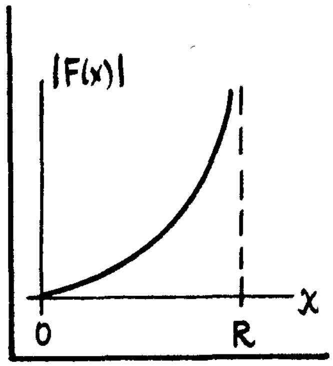
$$\rightarrow q_{in} = -(R/y)\, q_{out}, \quad x = R^2/y. \tag{1}$$

In our case, $q_{in}=q$ is the "real" charge, and $q_{out} = -(y/R)q$ is the image. We can eliminate y from the problem by using $y = R^2/x$.

2. The force on q (inside sphere) at distance x from the center is...

$$F = \frac{q_{in}\, q_{out}}{(y-x)^2} = -q^2 \frac{y}{R}\Big/(y-x)^2 \leftarrow \text{put in } y = R^2/x$$

so, $$\boxed{F(x) = -\frac{q^2}{R^2}\left(\frac{\xi}{(1-\xi^2)^2}\right)}, \quad \text{w/ } \xi = x/R,\ 0 \leq \xi < 1. \tag{2}$$



The (−) sign means that q_{in} & q_{out} attract each other; if $q_{in}=q$ is released from pt. x, it will accelerate toward the sphere surface.

(B) 3. When the force depends on distance x, it is appropriate to write the inertial part of Newton II as: $m\dot{v} = m\left(\frac{dx}{dt}\right)\frac{d}{dx}v = mv\frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{2}mv^2\right)$. Then, in this case: $m\dot{v} = |F(x)|$ [motion of (q,m) to right], with $x = R\xi$, gives...

$$\frac{1}{R}\frac{d}{d\xi}\left(\frac{1}{2}mv^2\right) = \frac{q^2}{R^2}\left(\frac{\xi}{(1-\xi^2)^2}\right) \leftarrow \text{integrate from } \xi = \frac{x_0}{R} = \xi_0,\ v=0 \text{ to } \xi, v\ldots \tag{3}$$

$$\frac{m}{2R}(v^2 - 0) = \frac{q^2}{R^2}\int_{\xi_0}^{\xi}\frac{\xi\, d\xi}{(1-\xi^2)^2} = \frac{q^2}{2R^2}\left(\frac{1}{1-\xi^2} - \frac{1}{1-\xi_0^2}\right),$$

or/ $$\boxed{v^2 = R^2\left(\frac{d\xi}{dt}\right)^2 = \frac{q^2}{mR}\left(\frac{1}{1-\xi^2} - \frac{1}{1-\xi_0^2}\right)}. \tag{4}$$

This is a first integral of q's motion; it gives q's velocity $v = v(\xi)$. The remaining integration will give $t = \text{fcn}(\xi)$.

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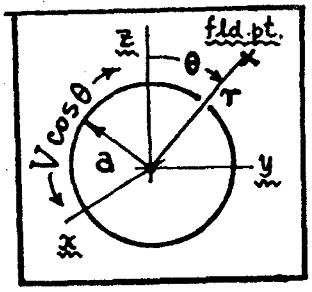

④ [30 pts]. Potential generated by a-sphere at $V\cos\theta$.

1) The problem has azimuthal symmetry (no φ-dependence), and the general solution is written down in Jackson's Eq. (3.33):

$$\phi(r,\theta) = \sum_{l=0}^{\infty} \left[A_l r^l + B_l r^{-(l+1)}\right] P_l(\cos\theta). \tag{1}$$

There is no charge at sphere center, $r=0$, so inside the sphere $(0 \leq r \leq a)$ all the $B_l \equiv 0$, and at most: $\phi(r \leq a, \theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta)$. Outside, ϕ cannot diverge as $r \to 0$, so all the $A_l \equiv 0$ (assuming $\phi \to$ cnst $= 0$ as $r \to 0$), and then: $\phi(r \geq a, \theta) = \sum_{l=0}^{\infty} B_l r^{-(l+1)} P_l(\cos\theta)$. All we need do is find the A_l (at $r \leq a$), and the B_l (at $r \geq a$).

2) At $r = a$, both the interior & exterior ϕ's must match the B.C. $V(\theta) = V\cos\theta$. Thus, by Jackson Eqs. (3.34)-(3.35) [and using orthogonality of the P_l's]:

$$A_l a^l = \left(\frac{2l+1}{2}\right)\int_0^{\pi} [V\underbrace{\cos\theta}_{\text{note: } \cos\theta = P_1(\cos\theta)}] P_l(\cos\theta)\sin\theta\, d\theta \tag{2}$$

$\leftarrow$ change variables to: $x = \cos\theta$,

by $A_l a^l = \left(\frac{2l+1}{2}\right) V \int_{-1}^{+1} P_1(x) P_l(x)\, dx = V\,\delta_{1l}$, by Jackson Eq. (3.21);

$$\text{i.e.,} \quad A_l = V/a \quad \text{for } l = 1; \text{ all other } A_l \equiv 0. \tag{3}$$

Similarly $B_l a^{-(l+1)} = V\,\delta_{1l}$ and $B_l = V a^2$ for $l = 1$ only. The potential everywhere in space is therefore...

$$\boxed{\phi(r,\theta) = \begin{cases} (r/a)\, V\cos\theta, & \text{for } 0 \leq r \leq a, \\ (a/r)^2\, V\cos\theta, & \text{for } a \leq r \to \infty. \end{cases}} \tag{4}$$

Note that the exterior solution goes as $1/r^2$, a dipole field characteristic of no <u>net</u> charge present. This happens because the sphere's average ptl $\langle V\cos\theta \rangle \equiv 0$.