QM 507 Final Exam 2 June 1972

This is a take-home exam, which is due in my office no later than 5 PM, Wednesday 7 June.

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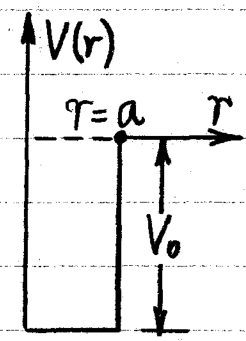
50 pts. ① Do problem (84) from the QM problem sheets.

25 pts. ② A neutral tritium atom (isotope H^3), in its ground state, undergoes spontaneous beta decay, emitting an electron of energy on the order of 10 KeV. The system remaining is an He^3 positive ion. Neglecting nuclear recoil, calculate the probability that this ion is left in a state with principal quantum number $n=2$ (i.e. a 2S or 2P state). Hint: write the wave functions for the initial and final States of the system, then use the sudden approximation.

25 pts. ③ It is desired to accurately calculate the S-wave ($\ell=0$) phase shift and cross section for the low energy scattering of a particle of mass m and energy E from a 3D rectangular potential well



$$V(r) = \begin{cases} 0, & \text{for } r > a, \\ -V_0, & \text{for } r < a. \end{cases}$$

a) Show that acceptable solutions to the radial equation (with wave fcn $\psi_k(r,\vartheta) \propto \frac{1}{r} v_{k0}(r)\, P_0(\cos\vartheta)$) in the two regions are

$$v_{k0}(r) = A \sin(kr + \delta_0), \quad k^2 = 2mE/\hbar^2, \text{ for } r > a$$

$$v_{k0}(r) = B \sin Kr, \quad K^2 = \frac{2mV_0}{\hbar^2} + k^2, \text{ for } r < a$$

where A & B are normalization constants, and $\delta_0 = \delta_0(k)$ may be interpreted as the S-wave phase shift.

b) By imposing continuity in v_{k0} and v_{k0}' at $r = a$, explicitly solve for $\tan\delta_0$ in terms of ka and the parameter $\beta = Ka \operatorname{ctn} Ka$. Show that as $k \to 0$, the result is

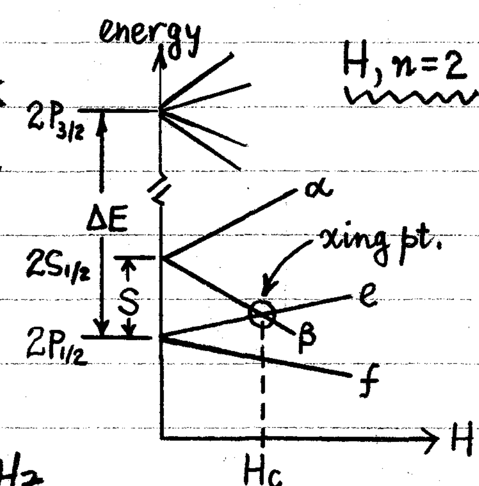
$$\tan\delta_0 \underset{k\to 0}{\simeq} (ka/\beta) - \tan ka.$$

c) Show that in the limit of part (b), the total S-wave scattering cross section is given by

$$\sigma_0(k) \underset{k\to 0}{\simeq} 4\pi a^2 \left(\frac{\tan ka}{ka} - \frac{\tan Ka}{Ka} \right)^2.$$

d) Physically interpret the fact that for $k = 0$, $\sigma_0(k)$ of part (c) blows up whenever $Ka = (2n+1)\frac{\pi}{2}$ for $n = 0, 1, 2, \ldots$

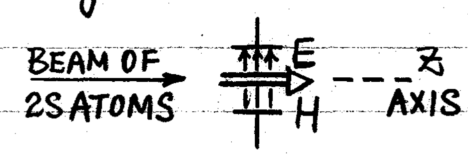
100 pts. **4** The $n=2$ state of atomic hydrogen consists of levels $2P_{3/2}$, $2S_{1/2}$ and $2P_{1/2}$ as shown. Neglecting hyperfine structure, the 2P levels are split by the fine structure interaction, $\Delta E = 10{,}969$ MHz. The degeneracy between the $2S_{1/2}$ and $2P_{1/2}$ is lifted by the Lamb shift, $S = 1058$ MHz (which is a quantum electrodynamic effect). The 2S levels



are "metastable", in that the lifetime for a decay $2S \rightarrow 1S$ (which can only take place by a double photon emission) is very long, namely $\sim 1/10$ sec. By contrast, the $2P \rightarrow 1S$ decay takes place very rapidly (by dipole radiation), with a lifetime $\tau \sim 10^{-9}$ sec. Finally, in an external magnetic field H, the levels split as shown in the diagram; it is traditional to refer to the $2S_{1/2}$ levels as α and β, and the $2P_{1/2}$ levels as e and f, as indicated.

a) Using a linear theory of the Zeeman effect, calculate the field H_c (in gauss) at which the levels β and e cross one another.

b) Suppose a beam of metastables enters a region where there is maintained a magnetic field H along the z-axis (beam axis), and a weak electric field E which is $\perp$ that axis,



as shown. The 2S and 2P levels are then coupled via a Stark matrix element $V_{PS} = \langle \phi_{2P} | e\vec{E}\cdot\vec{x} | \phi_{2S} \rangle$. Neglect $2S_{1/2}$ coupling to the $2P_{3/2}$ states, which are "far away". Of the remaining possible $2S_{1/2} - 2P_{1/2}$ couplings, show that for the chosen geometry, $V = 0$ for αe and βf coupling, while $|V| = N e a_0 E$ for αf and βe coupling. Calculate the numerical factor N. What relative orientation of $\vec{E}$ and $\vec{H}$ would give αe and βf coupling? What is the selection rule operating here?

c) As a function of the magnetic field H, the βe level separation may be written $E_\beta - E_e = \hbar\omega = hS - g\mu_0 H$, where g is adjusted so that $\omega = 0$ at $H = H_c$. Near H_c, levels β and e are close together so the Stark coupling is relatively much stronger for βe than for αf. To the extent that αf coupling can be ignored, the βe coupling becomes a two level problem. Assume the state superposition: $\Psi = a_\beta \phi_\beta e^{-\frac{i}{\hbar}E_\beta t} + a_e \phi_e e^{-\frac{i}{\hbar}E_e t}$, and use the Schrodinger equation to derive the amplitude equations

$$i\hbar \dot{a}_\beta = V^* a_e e^{+i\omega t}, \quad i\hbar \dot{a}_e = V a_\beta e^{-i\omega t} - \frac{1}{2} i\hbar \gamma a_e$$

where a_β & a_e are respectively the time-dependent S & P amplitudes, $V = \langle \phi_e | e\vec{E}\cdot\vec{x} | \phi_\beta \rangle$, and $\gamma = 1/\tau \sim 10^9/\text{sec}$ is the spontaneous decay rate of state e. The terms in V follow from the S. eqtn, while the term in γ in the 2ND eqtn is added phenomenologically, such that for $V=0$, the P state amplitude decays as $|a_e|^2 = e^{-\gamma t}$, representing the $2P \to 1S$ spontaneous decay. To relate to the experiment of part (b), solve these eqtns with the boundary conditions that $a_\beta = 1$, $a_e = 0$ at $t=0$, which is the entry time of the 2S atom into the field region. Suppose the coupling V is "weak", i.e. $|V| \ll \frac{1}{2}\hbar\gamma$ -- which is the natural width of the P level. By looking at the time dependence of $|a_\beta|^2$, show that the atom develops an effective decay rate $\Gamma \simeq C E^2 \gamma$, by virtue of the coupling via E. Calculate the proportionality factor C, and show that Γ, plotted vs. H (at cnst E) shows a Lorentzian resonance at the xing pt. field H_c. What is the halfwidth of this resonance (in gauss)?

d) For part (b), assume the initial intensity of the 2S beam is B_0, and it enters the field region from the left at velocity v. Suppose E & H are constant over a length l. Calculate the 2S intensity B to the right of the E-H region. Plot B vs. H, and show that it has a resonant decrease at $H = H_c$. For $v = 10^6$ cm/sec and $l = 1$ cm, what E-field (in volts/cm) destroys 50% of the 2S beam. For an H_c measured in this way, what is the Lamb shift S? Hint: See Phys. Rev. __138__, A22 ('65).