ϕ520 Problems

② [~Jk$^{\underline{n}}$ #7.11] A solution to the EM waveqtn in 1D is $u(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} A(k)\, e^{i(kx-\omega t)}\, dk$, with dispersion relation $\omega = \omega(k)$ given for a specific medium. Suppose that at $t=0$, the waveform is ~ monochromatic, with wave # k_0 & envelope $f(x)$, viz. $u(x,0) = f(x)\, e^{ik_0 x}$. For each $f(x)$ below: find the spectral intensity $|A(k)|^2$, sketch $|u(x,0)|^2$ vs. x & $|A(k)|^2$ vs. k, and -- by "reasonably" defining the widths Δx of $|u|^2$ and Δk of $|A|^2$ -- find the product $\Delta x \Delta k$. What is the minimum possible value of $\Delta x \Delta k$? The envelopes are: (A) $f(x) = \mathcal{N} e^{-\frac{1}{2}\alpha|x|}$, (B) $f(x) = \mathcal{N} e^{-\frac{1}{4}\alpha^2 x^2}$, (C) $f(x) = \begin{cases} \mathcal{N}, & \text{for } |x| < a; \\ 0, & \text{otherwise.} \end{cases}$

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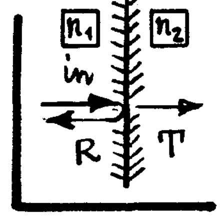
③ An EM plane wave is incident normally on the (flat) interface between materials with refractive indices n_1 and $n_2 > n_1$. (A) Find the reflection & transmission coefficients R & T, and show that $R + T = 1$. NOTE: R & T are ratios of intensities, not amplitudes. (B) If $n_1 \simeq 1$ (air) and $n_2 \simeq 1.5$ (glass), what are R & T? What happens when $n_2 \to \infty$, and what physics might this be?



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④ [20 pts]. Consider Jk$^{\underline{n}}$ Eq. (7.49) in 1D, for a field $E = E(t)$ that is an arbitrary fcn of t, but independent of x. Write: $\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = a(t)$, w/ damping cnst $\gamma = 2\beta$, and $a(t) = -\frac{e}{m}E(t)$ the acceleration. We want a particular integral for $x = x(t)$.

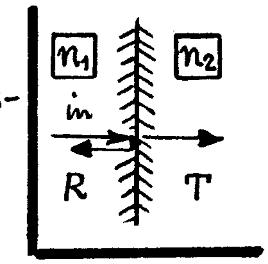
(A) Using Fourier transforms $[x(t) \leftrightarrow \tilde{x}(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-i\omega t}\, dt]$, show: $\tilde{x} = \tilde{a}/(\omega_0^2 + 2i\beta\omega - \omega^2)$. How is the Fourier inverse $x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{x}(\omega)\, e^{i\omega t}\, d\omega$ related to Jk$^{\underline{n}}$ Eq. (7.50)?

(B) Put the integral for $\tilde{a}(\omega)$ into the $x(t)$ integral of part (A), and -- by regrouping things -- show: $x(t) = \int_T^{\infty} a(t-\tau)\, K(\tau)\, d\tau$. Specify the "kernel" $K(\tau)$ as an integral over ω for now (will evaluate $K(\tau)$ in part (C)). Re the lower limit T for $x(t)$: what is T formally? what must T be to respect causality?

(C) Evaluate $K(\tau)$ explicitly, for $\beta < \omega_0$. Show: $K(\tau) = \frac{1}{\omega_r} e^{-\beta\tau} \sin\omega_r\tau$, for $\tau \geq 0$, and find the frequency ω_r in terms of ω_0 & β. HINT: contour integration, anyone? Sketch $K(\tau)$ vs. τ. How does $K(\tau)$ relate to a Green's fcn for this problem?

③ Reflection & transmission coefficients R & T at normal incidence.

(A) For normal incidence, the Fresnel formulas reduce to simple forms for the relative field strengths of the reflected and transmitted waves (CLASS NOTES, Eq. (32), p. Waves 9), viz...



$$\left[\!\left[\frac{E(\text{refl.})}{E(\text{in})} = \frac{n_2 - n_1}{n_2 + n_1}, \quad \frac{E(\text{trans.})}{E(\text{in})} = \frac{2n_1}{n_2 + n_1}. \right.\right. \qquad (1)$$

For purposes of calculating the coefficients R & T, which describe transport of energy, it is important to note that the Poynting vector $\mathbb{S}$ depends on n, the local index of refraction... from CLASS NOTES, Eq. (11), p. Waves 3...

$$\rightarrow \mathbb{S} = (c/8\pi)\, n\, |\mathbb{E}|^2\, \hat{k}, \quad \text{plane wave propagating in direction } \hat{k}. \qquad (2)$$

With this in mind, we easily get...

$$R = \frac{S(\text{refl})}{S(\text{in})}, \text{ both in medium 1} \Rightarrow \boxed{R = \left(\frac{n_2 - n_1}{n_2 + n_1}\right)^2}. \qquad (3)$$

$$T = \frac{S(\text{trans}) \sim \text{medium 2}}{S(\text{in}) \sim \text{medium 1}} \Rightarrow \boxed{T = \frac{n_2}{n_1}\left(\frac{2n_1}{n_1 + n_2}\right)^2 = \frac{4 n_2 n_1}{(n_2 + n_1)^2}}. \qquad (4)$$

It is then a simple algebraic identity that: $R + T = 1$.

(B) If $n_1 \simeq 1$ (air) and $n_2 \simeq 1.5$ (glass), numerical values are...

$$\left[\!\left[\text{air} \rightarrow \text{glass}: \; R = \left(\frac{0.5}{2.5}\right)^2 = 0.04, \quad T = 1 - R = 0.96. \right.\right. \qquad (5)$$

Thus, glass transmits (visible) light @ 96% efficiency. Surprised?

When $n_2 \rightarrow \infty$, i.e. $n_1/n_2 \ll 1$, have...

$$\rightarrow R \simeq 1 - 4(n_1/n_2), \quad T \simeq 4(n_1/n_2). \qquad (6)$$

A material with large n_2 becomes an (almost) perfect reflector. This situation might (ought to) characterize a metallic surface.