Maxwell Equations: Vector & Scalar Potentials $\mathbb{A}$ & ϕ.

In the static (time-independent) case, and for linear media ($\mathbb{D}=\epsilon\mathbb{E}$, $\mathbb{B}=\mu\mathbb{H}$):

$$\begin{cases} \nabla\cdot\mathbb{E}=\frac{4\pi}{\epsilon}\rho,\ \nabla\times\mathbb{E}=0 \Rightarrow \mathbb{E}=-\nabla\phi,\ \text{w/}\ \phi=\frac{1}{\epsilon}\int\frac{1}{R}\rho\, d^3x'; \\ \nabla\cdot\mathbb{B}=0,\ \nabla\times\mathbb{B}=\frac{4\pi\mu}{c}\mathbb{J} \Rightarrow \mathbb{B}=\nabla\times\mathbb{A},\ \text{w/}\ \mathbb{A}=\frac{\mu}{c}\int\frac{1}{R}\mathbb{J}\, d^3x'. \end{cases} \quad (1)$$

This would be all of E&M, if it were not for the t-dependent terms we have left out. To accommodate those terms, we must modify the roles of ϕ & $\mathbb{A}$ somewhat. The procedure goes as follows.

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1) For t-dept. case, we have the "non-source" Maxwell Equations...

① $\nabla\cdot\mathbb{B}=0$ $\Rightarrow$ can use: $\boxed{\mathbb{B}=\nabla\times\mathbb{A}}$, with $\mathbb{A}=\mathbb{A}(\mathbf{r},t)$ now a fcn of t; (2)

② $\nabla\times\mathbb{E}+\frac{1}{c}\frac{\partial\mathbb{B}}{\partial t}=0$, or: $\nabla\times\left(\mathbb{E}+\frac{1}{c}\frac{\partial\mathbb{A}}{\partial t}\right)=0$,

... so set: $\boxed{\mathbb{E}+\frac{1}{c}(\partial\mathbb{A}/\partial t)=-\nabla\phi}$, with $\phi=\phi(\mathbf{r},t)$ now a fcn of t. (3)

How $\mathbb{A}$ & ϕ depend on t is dictated by the "source" Maxwell Equations...

③ $\nabla\times\mathbb{H}=\frac{4\pi}{c}\mathbb{J}+\frac{1}{c}\frac{\partial\mathbb{D}}{\partial t}$	assume: $\mathbb{D}=\epsilon\mathbb{E}$	$\nabla\times\mathbb{B}=\frac{4\pi\mu}{c}\mathbb{J}+\frac{\mu\epsilon}{c}(\partial\mathbb{E}/\partial t)$,
④ $\nabla\cdot\mathbb{D}=4\pi\rho$	$\mathbb{B}=\mu\mathbb{H}$	$\nabla\cdot\mathbb{E}=\frac{4\pi}{\epsilon}\rho$. (4)

Put in: $\mathbb{B}=\nabla\times\mathbb{A}$, $\mathbb{E}=-\nabla\phi-\frac{1}{c}(\partial\mathbb{A}/\partial t)$, to Eqs. (4), use the identity $\nabla\times(\nabla\times\mathbb{A})=\nabla(\nabla\cdot\mathbb{A})-\nabla^2\mathbb{A}$, and rearrange terms to get...

$$\boxed{\begin{aligned} &\nabla^2\phi+\frac{1}{c}\frac{\partial}{\partial t}(\nabla\cdot\mathbb{A})=-\frac{4\pi}{\epsilon}\rho, \\ &\nabla^2\mathbb{A}-(\mu\epsilon/c^2)\frac{\partial^2\mathbb{A}}{\partial t^2}-\nabla\left[(\nabla\cdot\mathbb{A})+\frac{\mu\epsilon}{c}\left(\frac{\partial\phi}{\partial t}\right)\right]=-\frac{4\pi\mu}{c}\mathbb{J}. \end{aligned}} \quad (5)$$

Notice how the choice of potentials $\mathbb{A}$ & ϕ in Eqs. (2) & (3) automatically satisfies Max. Eqs. ①&②, while (4) Max. Eqs. ③&④ are left to specify (4) potentials $(\phi, \mathbb{A})$.

2) The $(\phi, \mathbb{A})$ eqtns above [Eqs(5)] are 4 eqtns in 4 unknowns (i.e. $(\phi; A_x, A_y, A_z)$). The can be made simpler, even decoupled, by imposing an additional condition linking ϕ & $\mathbb{A}$. In particular, we can choose...

$$\underline{\underline{\nabla\cdot\mathbb{A} + \frac{\mu\epsilon}{c}\frac{\partial\phi}{\partial t} = 0}} \Rightarrow \boxed{\begin{aligned}\nabla^2\phi - \frac{\mu\epsilon}{c^2}\left(\frac{\partial^2\phi}{\partial t^2}\right) &= -(4\pi/\epsilon)\rho, \\ \nabla^2\mathbb{A} - \frac{\mu\epsilon}{c^2}\left(\frac{\partial^2\mathbb{A}}{\partial t^2}\right) &= -(4\pi\mu/c)\mathbb{J}.\end{aligned}} \qquad (6)$$

called a "gauge condition".

Marvelous, if true... now, by "just" solving these inhomogeneous wave eqtns for ϕ & $\mathbb{A}$, we can solve the most general from of Maxwell's Eqtns (in a linear medium) by calculating $\mathbb{B} = \nabla\times\mathbb{A}$, $\mathbb{E} = -\nabla\phi - \frac{1}{c}(\partial\mathbb{A}/\partial t)$.

Imposing the above "gauge condition" is possible because neither of the potentials ϕ & $\mathbb{A}$ are uniquely defined by the fields $\mathbb{E}$ & $\mathbb{B}$, i.e. more than one $(\phi, \mathbb{A})$ corresponds to a given $(\mathbb{E}, \mathbb{B})$. As follows...

if : $\phi \to \phi' = \phi - \frac{1}{c}(\partial g/\partial t)$, and : $\mathbb{A} \to \mathbb{A}' = \mathbb{A} + \nabla g$ $\left\{\text{GAUGE TRANSFORM}\right.$, (7)

then : Same $\mathbb{B} = \nabla\times\mathbb{A}$ & $\mathbb{E} = -\nabla\phi - \frac{1}{c}\frac{\partial\mathbb{A}}{\partial t}$ result from $(\phi', \mathbb{A}')$ & $(\phi, \mathbb{A})$.

The gauge fcn $g(\mathbb{r}, t)$ is arbitrary, and it allows the following freedom...

If $(\phi, \mathbb{A})$ don't satisfy : $\nabla\cdot\mathbb{A} + \frac{\mu\epsilon}{c}\left(\frac{\partial\phi}{\partial t}\right) = 0$, then let $(\phi, \mathbb{A}) \to (\phi', \mathbb{A}')$, w/

$$\nabla\cdot\mathbb{A}' + \frac{\mu\epsilon}{c}\left(\frac{\partial\phi'}{\partial t}\right) = \underbrace{\left[\nabla\cdot\mathbb{A} + \frac{\mu\epsilon}{c}\left(\frac{\partial\phi}{\partial t}\right)\right]}_{①} + \underbrace{\left[\nabla^2 g - \frac{\mu\epsilon}{c^2}\left(\frac{\partial^2 g}{\partial t^2}\right)\right]}_{②} = 0. \qquad (8)$$

"Gauge condition" is satisfied by choosing g such that ② = −①.

The new potentials $(\phi', \mathbb{A}')$ satisfy the "gauge condition" in Eq.(6), they are solutions to the inhomogeneous wave eqtns in Eq.(6), and the fields $(\mathbb{E}, \mathbb{B})$ are unaffected by choice of g in this way. It all works on the tacit assumption that only the fields have directly measurable effects; the potentials themselves are just spectators.

Max. Eqs. : ϕ & $\mathbb{A}$

3) The gauge condition $\nabla\cdot\mathbb{A}+\frac{\mu\epsilon}{c}\left(\frac{\partial\phi}{\partial t}\right)=0$ is useful for deriving the wave eqtns in (6), but it is not a unique choice. Two particularized choices are in common use.

LORENTZ GAUGE ← used in SRT, where both ϕ & $\mathbb{A}$ relevant for particles.

$(\phi,\mathbb{A})$ are readily derived from Eqs. (6) which satisfy $\boxed{\nabla\cdot\mathbb{A}+\frac{\mu\epsilon}{c}\left(\frac{\partial\phi}{\partial t}\right)=0.}$

Now consider gauge transform $(\phi,\mathbb{A})\rightarrow(\phi',\mathbb{A}')$, via $\phi'=\phi-\frac{1}{c}\dot{g}$, $\mathbb{A}'=\mathbb{A}+\nabla g$.

$$\xrightarrow{\text{Then}}\nabla\cdot\mathbb{A}+\frac{\mu\epsilon}{c}\left(\frac{\partial\phi}{\partial t}\right)=0\rightarrow\nabla\cdot\mathbb{A}'+\frac{\mu\epsilon}{c}\left(\frac{\partial\phi'}{\partial t}\right)=0,\ \text{only if}\ \left(\nabla^2-\frac{\mu\epsilon}{c^2}\frac{\partial^2}{\partial t^2}\right)g=0. \quad (9)$$

All potentials $(\phi,\mathbb{A})$, $(\phi',\mathbb{A}')$ related by a gauge transform, with g restricted in this way, and with $\nabla\cdot\mathbb{A}+\frac{\mu\epsilon}{c}\left(\frac{\partial\phi}{\partial t}\right)=0$ for each pair, are said to belong to the "Lorentz Gauge". This is the most commonly used gauge condition.

COULOMB (radiation) GAUGE ← used in QED, where only $\mathbb{A}$(photon) is important.

Impose $\boxed{\nabla\cdot\mathbb{A}=0}$ [instead of $\nabla\cdot\mathbb{A}+\frac{\mu\epsilon}{c}\left(\frac{\partial\phi}{\partial t}\right)=0$]. This condition is invariant under a gauge transform if we impose $\underline{\nabla^2 g=0}$. In any case, from Eq. (5) above:

$$\nabla^2\phi=-\frac{4\pi}{\epsilon}\rho\Rightarrow\phi(\mathbf{r},t)=\frac{1}{\epsilon}\int\frac{d^3x'}{R}\rho(\mathbf{r}',t),\ \text{w/}\ R=|\mathbf{r}-\mathbf{r}'|\ \ \text{instantaneous Coulomb pot'l.}$$

$$\nabla^2\mathbb{A}-\frac{\mu\epsilon}{c^2}\frac{\partial^2\mathbb{A}}{\partial t^2}=-\frac{4\pi\mu}{c}\mathbb{J}+\frac{\mu\epsilon}{c}\nabla(\partial\phi/\partial t). \quad (10)$$

If no charges are present, i.e. $\rho=0$ & $\mathbb{J}=0$, then $\begin{cases}\phi=0 & \text{instead of homog}^{s}\text{ wave eqtn for }\phi\text{, as in Lorentz gauge;}\\ \left(\nabla^2-\frac{\mu\epsilon}{c^2}\partial^2/\partial t^2\right)\mathbb{A}=0.\end{cases}$

The general wave eqtn for $\mathbb{A}$, Eq. (10), can be simplified in this gauge... see Jk^{n} Eqs. (6.47)-(6.52) and prob #40. First decompose $\mathbb{J}$ into transverse & longitudinal parts:

$$\rightarrow \mathbb{J}=\mathbb{J}_T+\mathbb{J}_L,\ \text{w/}\ \mathbb{J}_T=\frac{1}{4\pi}\nabla\times\left[\nabla\times\int\frac{d^3x'}{R}\mathbb{J}\right]\ \&\ \mathbb{J}_L=-\frac{1}{4\pi}\nabla\int\frac{d^3x'}{R}\nabla'\cdot\mathbb{J}. \quad (11)$$

Have $\nabla\cdot\mathbb{J}_T=0$ & $\nabla\times\mathbb{J}_L=0$. Now: $\phi=\frac{1}{\epsilon}\int\frac{d^3x'}{R}\rho$, and: $\frac{\partial\rho}{\partial t}+\nabla\cdot\mathbb{J}=0$, together imply that: $\nabla(\partial\phi/\partial t)=(4\pi/\epsilon)\mathbb{J}_L$. Use of this result in Eq. (10) yields a reduced eqtn:

$$\boxed{\left(\nabla^2-\frac{\mu\epsilon}{c^2}\partial^2/\partial t^2\right)\mathbb{A}=-(4\pi\mu/c)\mathbb{J}_T}. \quad (12)$$