Def$^{\underline{n}}$ of Annihilation & Creation Operators for the SHO via Dirac's method.

$\boxed{\text{APPENDIX}}$ QM SHO by Operator Techniques [Davydov, Secs. 26 & 32].

There is another way to do the QM SHO problem, by techniques originally suggested by Dirac (and in the style of Heisenberg). We present the method here, to/

- show that SHO quantization does not necessarily depend on wave mechanics;
- show there is a close connection between quantization & commutators ($[x,p]=i\hbar$);
- serve as an introduction to a (later) quantization of the radiation field.

The method is rather abstract, but nicely shows how the principles of QM work in a way (almost) independent of the mathematical representation used so far.

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1) We begin with the SHO Hamiltonian, and standard x-p commutator...

$$\rightarrow \mathcal{H} = \frac{1}{2m}p^2 + \frac{1}{2}m\omega^2 x^2 \text{ , and: } [x,p] = i\hbar. \tag{1}$$

In Schrödinger's method of wave mechanics, $[x,p]=i\hbar$ is satisfied by $x \rightarrow x_{op} = x$, $p \rightarrow p_{op} = -i\hbar\partial/\partial x$ (in configuration space), or $x_{op} = i\hbar\partial/\partial p$, $p_{op} = p$ (in momentum space). One then solves the eigenvalue eqtn $\mathcal{H}\psi_n = E_n\psi_n$ to find the eigenstates ψ_n & eigenenergies E_n of the system.

In Dirac's method, quantization is imposed solely by use of $[x,p]=i\hbar$, independent of how x, p, or ψ is represented. To begin Dirac's program, we define the following linear operators, a & $a^\dagger$, by...

$$\left[\begin{array}{l} a = \sqrt{\frac{m\omega}{2\hbar}}\,x + i\sqrt{(1/2m\hbar\omega)}\,p, \\ a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\,x - i\sqrt{(1/2m\hbar\omega)}\,p; \end{array}\right\} \text{w/ inverse: } \begin{array}{l} x = \sqrt{\frac{\hbar}{m\omega}}\,(a+a^\dagger)/\sqrt{2}, \\ p = -i\hbar\sqrt{\frac{m\omega}{\hbar}}\,(a-a^\dagger)/\sqrt{2}. \end{array} \tag{2}$$

a & $a^\dagger$ are called the "annihilation" & "creation" operators, resp., for reasons that will become apparent. In Schrödinger's terms, in configuration space, they are

$$\rightarrow a, a^\dagger = \frac{1}{\sqrt{2}}\left(\xi \pm \frac{\partial}{\partial \xi}\right), \text{ w/ } \xi = \alpha x \text{ \& } \alpha = \sqrt{m\omega/\hbar}. \tag{3}$$

Annihilation-Creation Operators: Version of $\mathcal{H}$ (SHO) & Eigenstates.

In the spirit of Dirac's calculation, the representation $a, a^\dagger = (\xi \pm \frac{\partial}{\partial \xi})/\sqrt{2}$ is of no particular consequence (it is not needed). Instead, we proceed by noting...

$$\rightarrow \underline{\underline{a a^\dagger}} = \left(\frac{m\omega}{2\hbar}\right) x^2 + \frac{1}{2m\hbar\omega} p^2 + i\sqrt{\frac{m\omega}{2\hbar}/2m\hbar\omega}\,(px - xp) \int \text{keeping track of order of mult}^{\underline{n}} \text{ of } x \,\&\, p\ldots$$

$$= \frac{1}{\hbar\omega}\left(\frac{1}{2m}p^2 + \frac{1}{2}m\omega^2\right) - \frac{i}{2\hbar}[x, p] = \underline{\underline{\frac{1}{\hbar\omega}\mathcal{H} + \frac{1}{2}}}. \qquad (4)$$

Similarly: $\underline{\underline{a^\dagger a = \frac{1}{\hbar\omega}\mathcal{H} - \frac{1}{2}}}$; so// $\underline{\mathcal{H} = \hbar\omega(a^\dagger a + \frac{1}{2})}$, used below. (5)

On the basis of Eqs. (4) & (5), we can conclude...

$$\left[\begin{array}{ll} \text{addition}: & a a^\dagger + a^\dagger a = (2/\hbar\omega)\mathcal{H}; \\ \text{subtraction}: & a a^\dagger - a^\dagger a = [a, a^\dagger] = 1. \end{array}\right. \qquad (6)$$

The SHO problem may thus be characterized by a Hamiltonian such that

$$\boxed{\mathcal{H} = \hbar\omega(a^\dagger a + \tfrac{1}{2}), \ \text{w/ commutation}: [a, a^\dagger] = 1.} \qquad (7)$$

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2) Now consider the product operator:

$\rightarrow \underline{\underline{\Lambda = a^\dagger a}}$ ... Λ is Hermitian, since: $\Lambda^\dagger = (a^\dagger a)^\dagger = a^\dagger a^{\dagger\dagger} = a^\dagger a = \Lambda$. (8)

Let us assume the existence of a complete, orthonormal set of eigenstates $|\lambda\rangle$, w//

$\rightarrow \underline{\underline{\Lambda|\lambda\rangle = \lambda|\lambda\rangle}}$, λ = eigenvalue of Λ w.r.t. eigenstates $|\lambda\rangle$. (9)

We assume only the existence of the eigenstates $|\lambda\rangle$ -- we are not necessarily interested in their detailed analytic behavior (as before, pp. Sol$^{\underline{n}}$s 15-19). Next, we look at the effect of operating w/ a on one of the eigenstates $|\lambda\rangle$...

$$a|\lambda\rangle = \underbrace{(a a^\dagger - a^\dagger a)}_{=1,\ \text{Eq.}(7)} a|\lambda\rangle = a\underbrace{(a^\dagger a)|\lambda\rangle}_{=\lambda|\lambda\rangle,(9)} - \underbrace{(a^\dagger a)}_{\Lambda,(8)} a|\lambda\rangle = (\lambda - \Lambda)a|\lambda\rangle$$

i.e.// $\underline{\underline{\Lambda(a|\lambda\rangle) = (\lambda - 1)(a|\lambda\rangle)}}$, so: $a|\lambda\rangle$ = eigenfcn of Λ w/ eigenvalue $(\lambda - 1)$

thus// $\underline{\underline{a|\lambda\rangle = A_\lambda|\lambda - 1\rangle}}$, with A_λ = cnst. (10)

More details on a & $a^\dagger$. Summary to date.

To fix the cnst A_λ in Eq. (10), multiply each ket by its companion bra...

$$\rightarrow |A_\lambda|^2 \underbrace{\langle\lambda-1|\lambda-1\rangle}_{=1,\text{ assumed norm}} = \langle a\lambda|a\lambda\rangle = \langle\lambda|\underbrace{a^\dagger a}_{\Lambda,\text{ itself}}|\lambda\rangle = \lambda\underbrace{\langle\lambda|\lambda\rangle}_{=1,\text{ by norm}}$$

So/ $|A_\lambda|^2 = \lambda$, or: $A_\lambda = \sqrt{\lambda}$, and: $\boxed{a|\lambda\rangle = \sqrt{\lambda}\,|\lambda-1\rangle}$. (11)

The adjoint operator $a^\dagger$ can be processed similarly (Note that $a^\dagger \neq a$, here):

$$a^\dagger|\lambda\rangle = a^\dagger\underbrace{(aa^\dagger - a^\dagger a)}_{1}|\lambda\rangle = (\Lambda-\lambda)a^\dagger|\lambda\rangle, \text{ or/ } \Lambda(a^\dagger|\lambda\rangle) = (\lambda+1)(a^\dagger|\lambda\rangle).$$

So/ $a^\dagger|\lambda\rangle = B_\lambda|\lambda+1\rangle$, w/ B_λ = cnst. Find B_λ by same process as in (11), viz...

$$\rightarrow |B_\lambda|^2\underbrace{\langle\lambda+1|\lambda+1\rangle}_{1,\text{ norm}} = \langle a^\dagger\lambda|a^\dagger\lambda\rangle = \langle\lambda|\underbrace{aa^\dagger}_{=a^\dagger a+1,\text{ Commutator}}|\lambda\rangle = (\lambda+1)\underbrace{\langle\lambda|\lambda\rangle}_{1,\text{ norm}}.$$

then/ $|B_\lambda|^2 = \lambda+1$, or $B_\lambda = \sqrt{\lambda+1}$, and: $\boxed{a^\dagger|\lambda\rangle = \sqrt{\lambda+1}\,|\lambda+1\rangle}$. (12)

Summarizing what we've learned so far about the operators a & $a^\dagger$...

1. SHO Hamiltonian: $\mathcal{H} = \hbar\omega(\Lambda+\frac{1}{2})$, w/ $\Lambda = a^\dagger a$ &/ $[a, a^\dagger] = 1$.
2. (Assumed) Eigenfcns $|\lambda\rangle$, such that: $\mathcal{H}|\lambda\rangle = (\lambda+\frac{1}{2})\hbar\omega|\lambda\rangle \Rightarrow$ Energies: $E_\lambda = (\lambda+\frac{1}{2})\hbar\omega$.
3. individual operations $\left\{\begin{array}{l} a|\lambda\rangle = \sqrt{\lambda}\,|\lambda-1\rangle \ldots \underline{a} \text{ is called step-down (lowering) or \underline{annihilation} operator;} \\ a^\dagger|\lambda\rangle = \sqrt{\lambda+1}\,|\lambda+1\rangle \ldots \underline{a^\dagger} \text{ is called step-up (raising) or \underline{creation} operator.} \end{array}\right.$ (13)

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3) **NOTE**: we have *not* yet achieved quantization -- to do that, we have to show $\lambda = 0, 1, 2, \ldots$ is a non-negative integer (for all we know in Eq. (13), λ could be continuous). But we *can* achieve quantization by the simple expedient of applying the annihilation operator a to state $|\lambda\rangle$ m times in succession...

$$\rightarrow a^m|\lambda\rangle = [\lambda(\lambda-1)(\lambda-2)\cdots(\lambda-m+1)]^{1/2}\,|\lambda-m\rangle. \qquad (14)$$

Do same trick in (14) as in (11) & (12): "square" both sides...

$$\rightarrow \lambda(\lambda-1)(\lambda-2)\cdots(\lambda-m+1)\langle\lambda-m|\lambda-m\rangle = \langle a^m\lambda|a^m\lambda\rangle \geqslant 0. \quad (15)$$

The "$\geqslant$" on the RHS results from the fact that $|\psi\rangle = a^m|\lambda\rangle$ does not lie outside the manifold of SHO states, so the modulus^2 $\langle\psi|\psi\rangle$ is not negative. Then, regarding λ as it appears in (15), we can say...

$\lambda(\lambda-1)(\lambda-2)\dots(\lambda-m+1) \geqslant 0$, for all λ and integers m;

$\Longrightarrow$ always true only if $\underline{\underline{\lambda = n, \text{ for some integer } n \geqslant 0}}$ (i.e. $n = 0, 1, 2, \dots$),

and// SHO state $|\lambda = n\rangle$ has corresponding energy: $\underline{\underline{E_n = (n+\frac{1}{2})\hbar\omega}}$. (16)

So we get the correct quantized energies (and discrete eigenfcns $|n\rangle$) without solving a diff^{l} eqtn. Per Schrödinger, the energy quantization results from applying boundary conditions to the solution of a diff^{l} eqtn; here, per Dirac, it just results from the commutator $[a, a^\dagger] = 1$ (which makes possible the relations a & $a^\dagger|\lambda\rangle \propto |\lambda \mp 1\rangle$ in (13)) plus the notion $\langle\psi|\psi\rangle \geqslant 0$ for any SHO state.

REMARKS On connections between SHO operators & wave mechanics.

1. To get explicit eigenfcns from Dirac's formalism, note first that with $n \geqslant 0$, there must be a ground state $|0\rangle$ for the system such that: $\underline{\underline{a^n|n\rangle = \sqrt{n!}\,|0\rangle}}$, and: $a|0\rangle = 0\cdot|-1\rangle = 0$. The "vacuum state" $|0\rangle$ has energy $E_0 = \frac{1}{2}\hbar\omega$, and -- if we use the coordinate representation of a in Eq.(3), we write...

$$\rightarrow a|0\rangle = \frac{1}{\sqrt{2}}\left(\xi + \frac{\partial}{\partial\xi}\right)|0\rangle = 0 \Rightarrow \underline{|0\rangle \propto e^{-\frac{1}{2}\xi^2}} \text{ (not yet normalized)}. \quad (17)$$

The excited states $|n\rangle$ follow from: $\sqrt{n!}\,|n\rangle = (a^\dagger)^n|0\rangle \propto \left(\frac{1}{\sqrt{2}}\right)^n\left(\xi - \frac{\partial}{\partial\xi}\right)^n|0\rangle$, etc.

2. Expectation values of x & p are simple. E.g. since $x = \sqrt{\hbar/2m\omega}\,(a + a^\dagger)$, by Eq.(2):

$$\rightarrow \sqrt{\frac{2m\omega}{\hbar}}\langle m|x|n\rangle = \langle m|a|n\rangle + \langle m|a^\dagger|n\rangle = \sqrt{n}\,\langle m|n-1\rangle + \sqrt{n+1}\,\langle m|n+1\rangle$$
$$= \underline{\sqrt{n}\,\delta_{m,n-1} + \sqrt{n+1}\,\delta_{m,n+1}} \leftarrow \text{same as Eq.(52), p. Sol}^{\text{ns}}\text{ 21.} \quad (18)$$

Similar results are possible for $p \propto (a - a^\dagger)$, and values $\langle m|p|n\rangle$.

3. Dirac used this formalism for the radiation field, where $\Delta E_n = \hbar\omega$. To be continued.