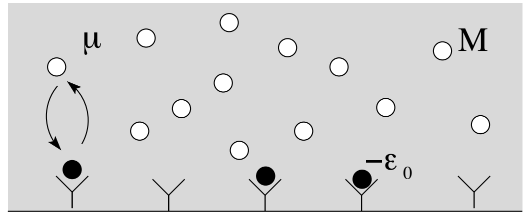
1)

A piece of space debris is a thin, uniform rod 6 meters long. It orbits the Earth in perfectly circular orbit of radius R.

a. Find the stable equilibrium orientation of the rod's axis relative to the radial direction. (Be sure to show your work).

b. What is frequency of small oscillations about this equilibrium?

2)



(a) Molecules of mass M form a part of some medium. These molecules can be adsorbed onto N_r receptors located at the interface with the medium. The binding energy of a receptor is $-\varepsilon_0$, chemical potential of molecules in the medium is μ. Find the average number of bound molecules, N_b/N_r (average occupation number of a receptor), in equilibrium at temperature T. Each receptor can accept only single molecule, and $1 \ll N_r \ll N$, where N is the number of molecules in the medium.

(b) Use this result to estimate the mass of antibiotic needed to be introduced into blood to occupy 99% of bacteria receptors, thus disrupting normal functioning of bacteria. Take typical mass of an antibiotic molecule to be $M \sim 400$ proton's masses ($m_p \sim 2 \cdot 10^{-27}\, kg$), binding energy $\varepsilon_0 \sim 1\ eV \approx 12000\ K$. Crudely assume that antibiotic forms ideal gas (in reality - *a solution*) in the blood volume $V \sim 10$ Liters, with chemical potential

$$\mu = -k_B T \ln\left[\frac{V}{N\lambda_T^3} Z_{int}(T)\right] \quad , \qquad \lambda_T = \frac{h}{\sqrt{2\pi M k_B T}} \approx 10^{-10} \sqrt{\frac{m_p}{M}} \sqrt{\frac{300\,\mathrm{K}}{T}}\ [\text{meters}]\,,$$

where $Z_{int}(T = 300\,K) \sim 10^4$ is the partition function of internal degrees of freedom of molecules (mostly rotations at room temperature). For the numerical estimate use $\ln 10 \approx 2.3 = 2(1 + 0.15)$.

You may find some of these formulas useful:
Stirling's formula for large A*,* $\ln A! \approx A \ln A - A$*; chemical potential is defined through* $dE = TdS - pdV + \mu dN$*, Helmholtz free energy is* $F = E - TS$*, grand thermodynamic potential is* $\Phi = F - \mu N$*.*

3)

In this problem you will find the two-slit diffraction pattern when a collimated beam of C_{60} fullerenes are incident on a pair of slits. The spherical C_{60} fullerenes are sometimes referred to as “buckyballs” since they resemble soccer balls with their hexagonal structure of carbon rings that combine to form the hollow, spherical shape with 60 carbon atoms. Assume that the C_{60} spheres of mass M are generated in an oven at temperature T. The spheres exiting the oven are then collimated with several collimating apertures. This horizontal collimated beam of C_{60} spherical molecules is then incident on a pair of slits, which are separated by a transverse distance d. The C_{60} molecules that are transmitted by the pair of slits are then detected a distance D downstream, in a detection plane that is perpendicular to the collimated beam. Consider each C_{60} molecule as a single quantum particle of mass M. You can also assume that the distance to the detection plane D is much larger than the separation d of the two slits (D>>d).

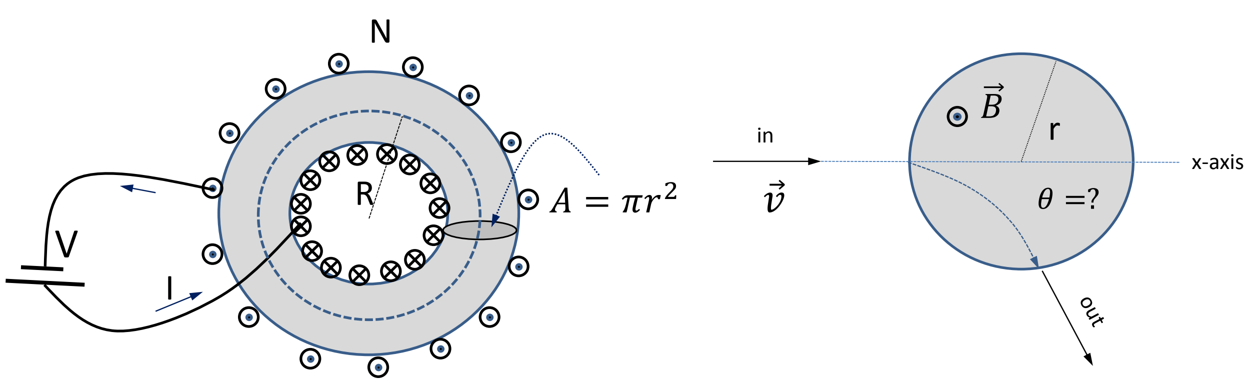
a) In the detection plane, find the location of the first maximum in the two-slit diffraction. Call this distance s and measure it from the center of the central lobe of the two-slit diffraction pattern.

b) Now use your answer in part a) to get a numerical value for the distance s to the first maximum if the temperature T of the oven is 1000K, the separation d of the two slits is 100nm, and the distance to the plane of the detector from the two slits is 1m. (Since you are not allowed to have calculators, make this estimate of the distance h to only one significant figure.)

4)

An electromagnet is made up of an iron ring of average radius R in the shape of a torus with a uniform circular cross-sectional area of $A = \pi r^2$ and a volume of $V = 2\pi RA$. Assume $R \gg r$. The toroidal iron yoke is uniformly wound with N turns of insulated wire of total electrical resistance R. The two ends of the wire are connected to a highly regulated d.c. power supply, creating a steady current I in the coil. The iron ring has a cut across it with zero initial separation, as shown in the figure on the left below. Assume that the permeability of iron is $\mu \gg \mu_o$ where μ_o is the permeability of air. For this problem assume that the magnetic field across the iron ring is uniform. Answer the following questions:

1. How much work is required to separate the opposite poles of the electromagnet across the cut to open up an air gap of separation $h \ll r$ in the iron ring while keeping the magnitude of the magnetic field B in the ring constant during this operation? Assume that the only force required to open up the gap is that of overcoming the attraction force between the opposite poles of the electromagnet. Ignore the mechanical force required to bend the iron ring. Assume also that there is no field leakage across the gap. Determine the force of attraction F_B between the poles of the electromagnet when a gap of separation h is established across the cut.
2. What would the force of attraction F_I between the poles of the electromagnet be if you were required to keep the current I in the coil constant instead of B? Is there a difference between F_I and F_B? Explain why?
3. An electron of initial velocity $\vec{v} = v\,\vec{i}$ is directed at the center of the gap, which has a uniform field $\vec{B}$. Moving parallel to the x-axis, it enters the gap region as shown in the figure below on the right. Determine the angle of deflection θ of the electron from the x-axis after the electron exits the gap region.



5)

(a) An electron (mass m, charge –e) in a hydrogen-like atom (with stationary nucleus of charge Ze) moves in a circular orbit of radius **r**. Treat this system classically, and assume the electron velocity $v \ll c$. Assume the electron radiates energy $\Delta E \ll |E|$, per orbit. Find the radiation power P in terms of **r** alone and constants. By equating P to the rate of loss of orbital energy, obtain a differential equation for the decrease of **r** due to radiation, as a function of r and constants. E is the total energy of the system and Z is the charge of the nucleus of the hydrogen-like atom.

[Hint: Radiation power is $P = (2e^2/3c^{3)}\ |\mathbf{a}|^2$, where **a** is acceleration.

(b) Consider a large synchrotron that maintains a beam of highly relativistic protons with charge e, rest energy $E_0 = M c^2$ in a circular orbit of radius ρ at total energy $E \gg E_0$. The machine supplies energy to the beam at constant rate (in lab) dU/dz (Mev/meter) per proton, where z is the distance the proton travels along its circular path. The machine also has magnets with high enough energy B-fields to contain the proton orbit for any “reasonable” E. Assume at first that the limit on E results from radiation losses. Find the limiting value for the Lorentz factor $\gamma = E/M c^2$ under these circumstances.

[Hint: Relativistic radiation power $P_{\text{rad}} = \frac{2}{3}\frac{e^2 c}{\rho^2}\ \beta^4\ \gamma^4$.]

6)

A fission chain reaction occurs as thermal neutrons diffuse though some material with diffusion coefficient D. Occasionally a neutron is absorbed by a nucleus where it is either lost or triggers a fission reaction releasing several new neutrons. The net effect is that each neutron produces new neutrons at a rate q (self-sustaining fission requires $q > 0$). The density of thermal neutrons, $N(\mathbf{x}, t)$, therefore evolves according to the equation

$$\frac{\partial N}{\partial t} = D\nabla^2 N + qN \ , \tag{1}$$

where q and D are constant properties of the material, say Plutonium. A piece of Plutonium is *super-critical* if the neutron density increases in time, $N \to \infty$ (a run-away chain reaction). It is *sub-critical* if neutrons decay away over time, $N \to 0$. The *critical mass*, or critical volume, of Plutonium is that exact amount which does neither of these things.

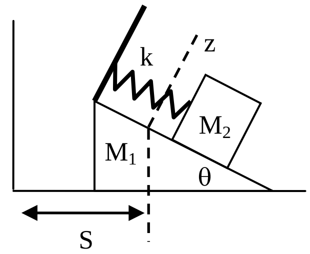
The simplest (but not most desirable) situation is when the Plutonium is surrounded by material transparent to neutrons, for example air. In that case $N = 0$ at the surface of the Plutonium piece. Consider this case and **find the critical volume for**

a. A solid cube of Plutonium.

b. A solid sphere of Plutonium.

7)

A block of mass M_2 slides on a frictionless wedge of mass M_1 and angle θ. The wedge in on a frictionless plane. Gravity is present. The block is attached to a spring with constant k to a post on the wedge. Let z be the generalized coordinate for the stretch of the spring. Let S be the generalized coordinate from the wall to the end of the unstretched spring on the wedge (shown by the dashed line).



a) Find the Lagrangian L for the system of the two masses. Take the potential energy V of mass M_2 to be zero when z=0.

b) Find the Lagrange equation for the generalized coordinate S.

c) Find the Lagrange equation for the generalized coordinate z.

d) Find the two normal mode frequencies for this system.

8)

Electrons on surface of liquid helium (weak dielectric) can form a monolayer. The attractive potential that electrons experience is due to the ‘image’ charge,

$$V(z) = -\frac{\alpha}{z},$$

where α is a positive constant, and z - height over the He surface. Electrons cannot penetrate into the liquid due to strong potential barrier at the surface, and one can neglect gravity. In the ground state, calculate the binding energy of electrons to He, and the average height of electrons above the surface. Sketch the wave function of electrons in the monolayer. Assume electrons do not interact with each other.

9)

In a 1 Dimensional arc discharge along the z-axis, a current pulse I(t) begins at t=0 and flows along a path of length ℓ. An observer at position (**r, θ**), with **r** >> ***ℓ***, detects the arc radiation (see Figure 1a). A model for the pulse I(t) is a capacitor C discharging through a series resister R and inductor L. C is switched at t=0, with initial voltage V_0 (see Figure 1b).

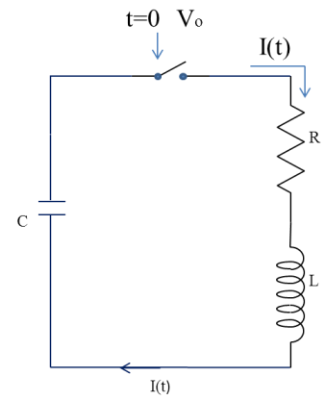


Fig. 1b

(a) For the overdamped case, show that the current is given by $I(t) = (V_0/L\Gamma)\ \exp(-\gamma t)\ \sinh(\Gamma t)$, where $\gamma = \frac{R}{2L}$ and $\Gamma = \sqrt{\gamma^2 - \left(\frac{1}{LC}\right)}$.

What is the condition for overdamping?

(b) Sketch I(t) vs t, and find the **approximate** expression for the pulse rise-time, duration, and peak amplitude.

(c) The total energy radiated per unit solid angle is:

$$\frac{dE}{d\Omega} = \int_{-\infty}^{+\infty} \sigma(\omega)\, d\omega,$$

$$\text{where } \sigma(\omega) = \frac{\sin^2\theta}{8\pi^2 c^3}\,\ell^2\omega^2 \left| \int_0^{\infty} I(t)\exp(-i\omega t)\, dt \right|^2$$

Calculate the arc frequency spectrum σ(ω) for this I(t) model. (Express your answer as a function of θ, ℓ, ω, V_0, C, L, R, and c.)

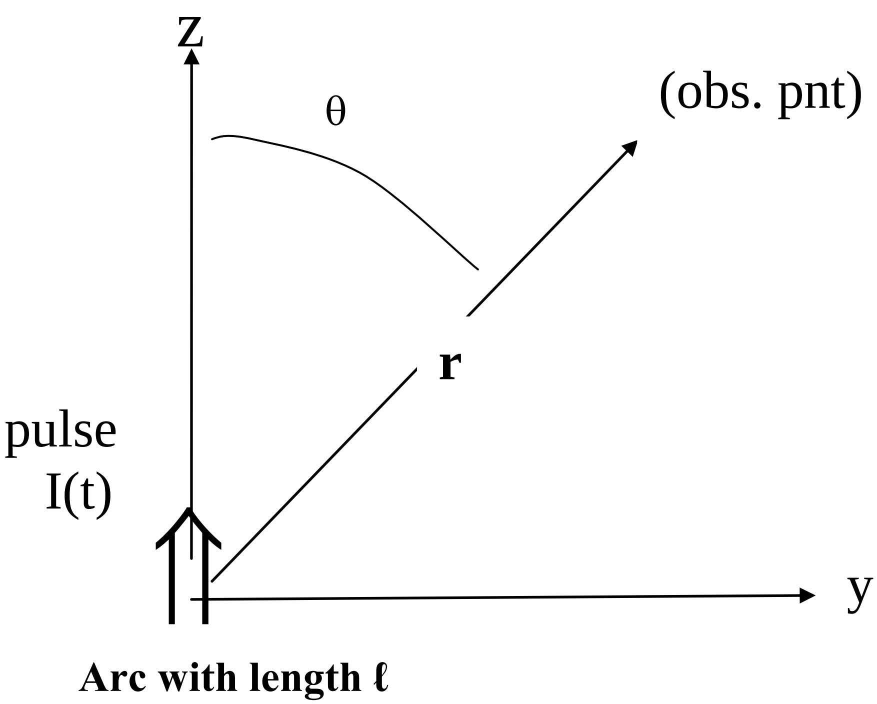


Fig.1a

10)

Answer the following questions:

a. It is well known that when a charged particle is accelerated such as in a circular motion it radiates electromagnetic waves called Larmor radiation and loses energy. Why is it then that the electron in its ground state in a hydrogen atom does not spiral down and fall into the nucleus?

b. The following observations have a common origin: The sky looks blue when you look away from the sun on earth but it looks black if you do the same in space outside the atmosphere. If you look directly at the sun on earth it looks yellow but in space it looks white. The sun looks red at sunset and sunrise. Can you explain these observations based on a common physical fact?

c. The solar electromagnetic radiation flux reaching earth is about $1369\ W/m^2$. About *70%* of the total loss in the solar mass is due to the electromagnetic radiation of the sun. Consider that light originating at the surface of the sun takes *8.3* minutes to arrive at the surface of the earth and that the sun has a radius of $7\times10^8 m$ and a mass of $2\times10^{30} kg$. Also assume that the sun will die when it runs out of its nuclear fuel, hydrogen, which constitutes *75%* of its total mass, and that only that hydrogen that is at the center (~10% of the total) is capable of producing thermonuclear energy. Furthermore, thermonuclear reactions consume 0.7% of the mass of the hydrogen. From these data determine the life expectancy of the sun in billions of years.

11)

Answer the following questions.

(a) Find the general solutions to:

$$y \, \frac{d^2 y}{d\,x^2} - \left(\frac{dy}{dx}\right)^2 - 6\,x\,y^2 = 0$$

(b) Generally, if $\frac{d^2 y}{d\,x^2} + f(x)\frac{dy}{dx} + g(x)\,y = 0$ had a particular solution $y = y_0\,(x)$, then a second solution could be manufactured as $y = y_0\,u$, where u is given by

$$u(x) = C\int .y_0^{-2} \exp\left[-\int f(x)dx\right] dx$$

with C = constant. Show that the two solutions y_0 and $(y_0\,u)$ are <u>linearly independent.</u>

[Hint: the definition of <u>linear independence</u> is that Det $\begin{matrix} y1 & y2 \\ \frac{dy1}{dx} & \frac{dy2}{dx} \end{matrix} \neq 0$.]

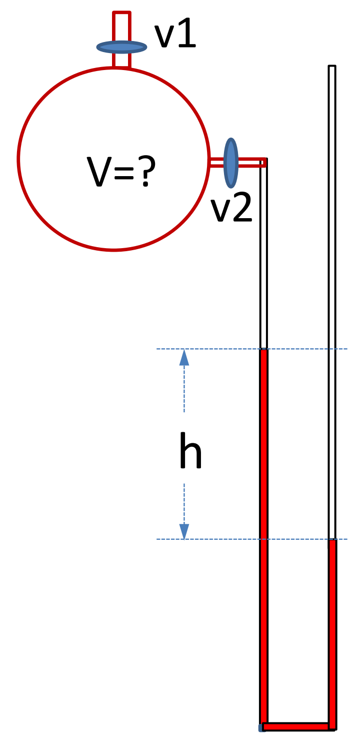
12)

The xy plane carries current with density $j_y(x) = j_0 \cos kx$. The $z < 0$ semi-space is a magnetic material with permeability μ, and $z > 0$ is vacuum. Find all components of the magnetic field everywhere in space. Check that your answer reduces to the known result for $k \to 0$ with vacuum on both sides of the plane.

13)

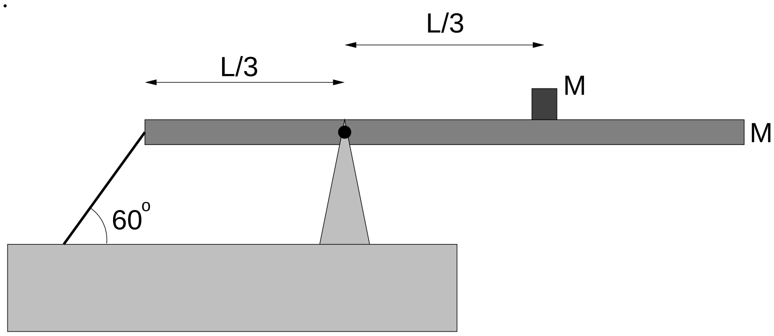
An experimentalist wants to determine the volume, *V*, of an irregularly shaped container confined between two valves, v1 and v2, as in the figure below. Valve v2 is connected to the left arm of a monometer with an inner diameter of $r=3$ *mm*, while the right arm of the monometer is open to the atmosphere. Initially volume *V* and the left arm of the monometer is pumped through the valves v1 and v2 with both valves open until the water column in the left arm rises and starts spilling into the container volume *V* through valve v2. At that moment valve v2 is closed and the container is first dried by pumping the spilled water back out and then filled with air until it reaches a pressure of $P_i = 1.2$ *atm*, at which time valve v1 is also closed. The temperature of the whole system is assumed to be in thermal equilibrium with the surrounding room temperature of 20^oC at all times. The experimentalist then opens valve v2, which causes the water column in the right arm of the monometer to rise by $h'=7$ *cm* while the height difference, h, between the water levels of the two arms of the monometer registers a value of $h=70$ *cm*, as shown in the figure below. From these data determine volume V.

Useful information: 1 atm=760 *torr*, $\rho_{H2O}=1$ g/cm^3, $\rho_{Hg}=13.6$ g/cm^3, 1 *torr*= 1 *mmHg*.



14)

A uniform wooden beam of mass M and length L pivots frictionlessly about an axis one-third of the distance from its end. The beam is tied down by a rope which makes an angle 60° from the horizontal (see figure). A small metal block with mass equal to the beam rests at a point one-third of the distance form the free end.



a. What is the force exerted by the pivot on the beam?

b. The rope is cut. At the instant following this cut what is the *normal force* of the block on the beam?

15)

Consider a harmonic oscillator with a mass m_e and oscillation frequency ω. Suppose that we put a perturbation in the middle of a harmonic oscillator potential. Let the perturbation be given by: $H'(x) = \varepsilon\left(\frac{\hbar}{m_e\omega} - x^2\right)$

where ε is a constant.

a) Find the first order correction to the energy E_n of the states.

Hint: $x = \sqrt{\frac{\hbar}{2m_e\omega}}\left(a_+ + a_-\right)$

b) Find the first order correction to the ground state wave function.

c) Make a sketch of the initial ground state wave function, the first order correction to the wave function and the resulting ground state wave function. Then explain why this new shape in the wave function makes sense from the perturbation. (Even though we are doing a perturbation here, make your sketch of the perturbation correction large enough so you can easily see the effect on the resulting ground state wave function.)