4) Eq. (17) is a "<u>Confluent</u> <u>hypergeometric</u> <u>equation</u>." The standard form is gotten by one more (trivial) change of variables...

$z = 2\kappa\rho$, $b = 2(\ell+1)$, $a = (\ell+1) - \frac{Z}{\kappa}$, so Eq. (17) becomes:

$$\left[\begin{array}{l} z\frac{d^2f}{dz^2} + (b-z)\frac{df}{dz} - af = 0 \leftarrow \text{standard form of Confl. HyperG. Eq.} \\ \text{w/ solution: } f(z) = {}_1F_1(a;b;z) \leftarrow {}_1F_1 \text{ is a Confl. HyperG. Fcn.} \end{array}\right. \tag{18}$$

The solution for the Coulomb boundstate radial wavefcn is then...

$$\boxed{\begin{array}{l} R(\rho) = \text{cnst} \times \rho^{\ell+1} e^{-\kappa\rho}\, {}_1F_1\left(\ell+1-\frac{Z}{\kappa};\, 2\ell+2;\, 2\kappa\rho\right), \\ \text{where: } \rho = r/a_0,\ \kappa = \sqrt{2|E|/E_0}\ ;\ E = -|E| = \begin{smallmatrix}\text{orbit}\\ \text{energy}\end{smallmatrix}. \end{array}} \tag{19}$$

) A quantization condition on κ, and hence the energy E, results from the fact that: ${}_1F_1(a;b;2\kappa\rho) \sim e^{2\kappa\rho}$, as $\rho\to\infty$, unless $a = -N$, w/ $N = 0,1,2,\ldots$. Hence, to keep $R(\rho\to\infty)$ finite, we need to impose...

$$\rightarrow \ell+1-\frac{Z}{\kappa} = -N \Rightarrow \kappa = Z/(N+\ell+1)\ ,\ N = 0,1,2,\ldots \int \underline{\text{radial quantum \#}}$$

$$\text{so } E = -\tfrac{1}{2}E_0\kappa^2, \text{ or } \boxed{E_n = -\tfrac{1}{2}(Z\alpha)^2 mc^2/n^2} \int \begin{array}{l} n = N+\ell+1 = 1,2,\ldots \int \underline{\text{principal quantum\#}} \\ \text{and } 0 \leqslant \ell \leqslant n-1. \end{array} \tag{20}$$

The E_n are <u>Bohr's</u> <u>quantum</u> <u>energies</u> for the hydrogenlike atom. The condition on ℓ arises from $N = n-(\ell+1) \geqslant 0 \Rightarrow \ell \leqslant (n-1)$.

Before going on, we review some facts about the Confluent HyperGeometric Eqtn.

ASIDE <u>Confluent Hypergeometric Lore</u>

A commonly occurring ODE in math. physics (cf. Eq (18) above) is...

$$\left[z\frac{d^2f}{dz^2} + (b-z)\frac{df}{dz} - af = 0 \int \begin{array}{l} \text{on } 0 \leqslant |z| \leqslant \infty, \text{ typically;} \\ a \,\&\, b \text{ are parametric constants.} \end{array}\right. \tag{21a}$$

ASIDE Confluent Eqn. (cont'd)

(21a) is called the "confluent hypergeometric equation". It is the prototype 2nd order ODE with one regular & one irregular singularity (@ $z=0$ & $z=\infty$, resp.), and it is related to the "hypergeometric equation" by a scale change of variables.

Standard power series treatment gives a solution which is regular at $z=0$:

$$f(z) = {}_1F_1(a;b;z) = 1 + \frac{a}{b}\cdot\frac{z}{1!} + \frac{a(a+1)}{b(b+1)}\cdot\frac{z^2}{2!} + \frac{a(a+1)(a+2)}{b(b+1)(b+2)}\cdot\frac{z^3}{3!} + \cdots,$$

or $${}_1F_1(a;b;z) = \sum_{k=0}^{\infty}\frac{(a)_k}{(b)_k}\cdot\frac{z^k}{k!}, \text{ where } (a)_k \text{ is the Pochhammer symbol:}$$

$$(a)_k = a(a+1)(a+2)\cdots(a+k-1) = \frac{\Gamma(a+k)}{\Gamma(a)},\ (a)_0 = 1. \qquad (21b)$$

The ODE in (21a), and solutions like ${}_1F_1$ of (21b) are discussed in many places.* We shall drop the indices† and just list some properties of $F(a;b;z)$.

1. $F(a;b;z)$ is not defined [i.e. blows up] when $b=0$, or $b=(-)$integer.

---

2. When $b \neq 1,2,3,\ldots$ a second, linearly independent solution to the ODE is

$$\rightarrow g(z) = z^{1-b}\,F(a-b+1;\,2-b;\,z). \qquad (21c)$$

This solution generally diverges as $z\to 0$, in contrast to the solution in (21b).

---

† The indices in ${}_mF_n$ simply count the number of Pochhammer symbols in the numerator & denominator (resp.) of the series solution in Eq. (21b). Recall the series solution to the hypergeometric ODE: $x(1-x)f'' + [\gamma - (1+\alpha+\beta)x]f' - \alpha\beta f = 0$, was: $f(x) = {}_2F_1(\alpha,\beta;\gamma;x) = \sum_{k=0}^{\infty}[(\alpha)_k(\beta)_k/(\gamma)_k]\cdot\frac{x^k}{k!}$. Solutions ${}_mF_n$ can be generalized (see G&R, Sec. 9.26).

* Gradshteyn & Ryzhik "Tables..." (1980), Sec. 9.21. There ${}_1F_1(a;b;z)$ is denoted $\Phi(\alpha,\gamma;z)$.
Abramowitz & Stegun "Handbook..." (NBS, 1965), Ch. 13. ${}_1F_1(a;b;z)$ is denoted $M(a,b,z)$.
Mathews & Walker "Math. Methods..." (1970), Sec. 7-4. ${}_1F_1(a;b;z)$ is denoted ${}_1F_1(a;c;z)$.
Davydov "QM" (1965 ed.), App. D. There ${}_1F_1(a;b;z)$ is denoted by $F(a,c;z)$.

ASIDE Confluent Lore (cont'd)

3. When $b=a$, Eq. (21b) $\Rightarrow F(a;a;z)=e^z$, a series which converges for all a & z. Many other elementary fcns can be represented similarly, e.g.* (21d)

a	b	z	$f(x)=F(a;b;z)$	REMARKS
α	α	x	e^x	exponential; α = arbitrary
1	2	$-2ix$	$(e^{-ix}/x)\sin x$	trigonometric fcns
1	2	$2x$	$(e^x/x)\sinh x$	hyperbolic fcns
$\nu+\frac{1}{2}$	$2\nu+1$	$2ix$	$\Gamma(1+\nu)\,e^{ix}\,(x/2)^{-\nu}\,J_\nu(x)$	ordinary Bessel fcns
$\nu+\frac{1}{2}$	$2\nu+1$	$2x$	$\Gamma(1+\nu)\,e^x\,(x/2)^{-\nu}\,I_\nu(x)$	modified Bessel fcns
$n+1$	$2n+2$	$2ix$	$\Gamma(\frac{3}{2}+n)\,e^{ix}\,(x/2)^{-(n+\frac{1}{2})}\,J_{n+\frac{1}{2}}(x)$	spherical Bessel fcns
$-n$	$\alpha+1$	x	$[n!/(\alpha+1)_n]\,L_n^{(\alpha)}(x)$	Laguerre polynomials
$-n$	$3/2$	x^2	$(-)^n\,[n!/(2n)!]\,H_{2n}(x)$	Hermite polynomials
α	$\alpha+1$	$-x$	$\alpha x^{-\alpha}\gamma(\alpha,x)$, w/ $\gamma(\alpha,x)=\int_0^x e^{-t}t^{\alpha-1}dt$	Incomplete gamma fcn
$1/2$	$3/2$	$-x^2$	$(\sqrt{\pi}/2x)\,\mathrm{erf}\,x$, w/ $\mathrm{erf}\,x=(2/\sqrt{\pi})\int_0^x e^{-t^2}dt$	Error function
$1/2$	$3/2$	$i\frac{\pi x^2}{2}$	$\frac{1}{x}[C(x)+iS(x)]$, w/ $\left\{\begin{matrix}C(x)\\S(x)\end{matrix}\right\}=\int_0^x\left\{\begin{matrix}\cos\\\sin\end{matrix}\right\}\left(\frac{\pi t^2}{2}\right)dt$	Fresnel integrals.

The table is not complete; many other entries can be made. Evidently, any general statement about $F(a;b;z)$ is valuable insofar as applying to all these fcns.

4. For example, recursion relations can be derived from the series for F...

$$(b-a)F(a-1;b;z)+(2a-b+z)F(a;b;z)=aF(a+1;b;z) \leftarrow \text{NBS\#}(13.4.1),$$

$$(a-b+1)F(a;b;z)+(b-1)F(a;b-1;z)=aF(a+1;b;z) \leftarrow \text{NBS\#}(13.4.3),$$

... etc. There 31 more such relations in Sec. 13.4 of NBS Handbook. (21e)

And a general differentiation formula can be derived...

$$\boxed{\frac{d^n}{dz^n}F(a;b;z)=\left(\frac{\Gamma(b)\Gamma(a+n)}{\Gamma(a)\Gamma(b+n)}\right)F(a+n;b+n;z)} \leftarrow \text{NBS}^{*}(13.4.9). \quad (21f)$$

* from TABLE 13.6 of "NBS Handbook of Math. Fcns" Abramovitz & Stegun

ASIDE Confluent Lore (cont'd)

5. A useful integral representation of $F(a;b;z)$ can be obtained as follows. Put $(a)_k = \Gamma(a+k)/\Gamma(a)$, etc. into the series of Eq. (21b), so...

$$\rightarrow F(a;b;z) = \frac{\Gamma(b)}{\Gamma(a)}\sum_{k=0}^{\infty}\left[\frac{\Gamma(a+k)}{\Gamma(b+k)}\right]\frac{z^k}{k!} = \frac{\Gamma(b)}{\Gamma(a)}\sum_{k=0}^{\infty}\left[\frac{1}{\Gamma(b-a)}B(a+k,b-a)\right]\frac{z^k}{k!}. \quad (21g)$$

In the [ ] we have put the "beta fcn" B, defined by [Mathews & Walker, Sec. 3-4]:

$$\rightarrow B(r,s) = \Gamma(r)\Gamma(s)/\Gamma(r+s) = \int_0^1 t^{r-1}(1-t)^{s-1}\,dt; \quad (21h)$$

this follows from the integral defⁿ of the gamma fcn: $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,dt$. So:

$$\rightarrow F(a;b;z) = \frac{\Gamma(b)}{\Gamma(a)\Gamma(b-a)}\sum_{k=0}^{\infty}\frac{z^k}{k!}\int_0^1 t^{a+k-1}(1-t)^{b-a-1}\,dt$$

combine: $\sum_{k=0}^{\infty}\frac{1}{k!}(zt)^k = e^{zt}$

$$\boxed{F(a;b;z) = \frac{\Gamma(b)}{\Gamma(a)\Gamma(b-a)}\int_0^1 e^{zt}\,t^{a-1}(1-t)^{b-a-1}\,dt,} \quad (21i)$$

where: $\mathrm{Re}\,b > \mathrm{Re}\,a > 0$, for convergence. From this, it is easy to derive the:

[[**KUMMER TRANSFORM**: $F(a;b;z) = e^z F(b-a;b;-z)$]] (21j)

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6. That $F(\alpha;\alpha;z) = e^z$ suggests $F(a;b;z) \sim e^z$ as $|z|\to\infty$. In fact...

$$\rightarrow F(a;b;z) \simeq \frac{\Gamma(b)}{\Gamma(a)}e^z z^{a-b}\left[1+\theta\left(\tfrac{1}{|z|}\right)\right], \quad |z|\to\infty \;\&\; \mathrm{Re}\,z>0;$$

NBS # (13.5.1)

$$\simeq [\Gamma(b)/\Gamma(b-a)](-z)^{-a}\left[1+\theta\left(\tfrac{1}{|z|}\right)\right], \quad |z|\to\infty \;\&\; \mathrm{Re}\,z<0. \quad (21k)$$

The point here is that for the hydrogen atom, whose radial wavefcns go as Eq. (19): $R(\rho) \propto \rho^{l+1}e^{-\kappa\rho}F(l+1-\frac{Z}{\kappa};\,2l+2;\,2\kappa\rho)$, $R(\rho)$ will naturally diverge as $\rho\to\infty$ [as $R(\rho)\sim e^{+\kappa\rho}\rho^{-Z/\kappa}$] unless the parameter $a=-N$, with $N=0,1,2,\ldots$. When $a=-N$, the $\Gamma(a)=\Gamma(-N)=\infty$ factor in Eq. (21k) knocks out the exponentially divergent term. We have already used the condition $a = l+1-(Z/\kappa) = -N$ to get the hydrogen Bohr energies in Eq. (20).

ASIDE Confluent Lore (cont'd)

7. When $a=-N$, $N=0,1,2,\ldots$, $F(a;b;z)$ reduces to a polynomial of degree N, i.e.

$$F(-N;b;z) = 1 - \frac{N}{b}z + \frac{N(N-1)}{b(b+1)}\frac{z^2}{2!} + \cdots + (-)^N \frac{\Gamma(b)}{\Gamma(b+N)} z^N$$

$$= \frac{\Gamma(b)}{\Gamma(b+N)} z^{1-b}\left\{e^z \frac{d^N}{dz^N}\left(z^{b-1+N} e^{-z}\right)\right\}. \qquad (21\ell)$$

These fcns are called Laguerre polynomials, and there are many different normalizations in the literature. Davydov defines...★

LAGUERRE POLYNOMIAL $\Big\}$ $$L_N^b(z) = z^{-b} e^z \frac{d^N}{dz^N}\left(z^{b+N} e^{-z}\right) = \frac{\Gamma(b+1+N)}{\Gamma(b+1)} F(-N;b+1;z). \qquad (21m)$$

We shall stick with Davydov's definition.

At this point, we halt the saga of confluent hypergeometric functions, although much more can be said and done with them -- e.g. see the listings in footnote ★ on p. H7. We have ~ enough information to handle the F's as they appear in the H-atom radial wavefcns $R(\rho)$ of Eq. (19) above.

You will have an opportunity to verify some of the above "Confluent Lore" in problem assignments. Then, with a sincere act-of-faith, you can rely on tabulated results -- per NBS Handbook, Gradshteyn & Ryzhik, etc.

Next, we apply some of the "Confluent Lore" to the H-atom wavefcns $R(\rho)$.

Nonrel. H-Atom

H 10

ERRATUM

Observation of Infinite Number of Sputter-Induced Transitions of Chaotic States in Supercritical Mixtures of Superbanana Orbits of Dynamical Pseudo Goldstone Bosons in Anharmonic Periodic Fields near the $n=3$ Threshold around $Z_1=9$ and $Z_2=26$ in $CeAl_2$ in Stellarator Geometries

As a result of a transposition of manuscript pages, the originally printed version of this Letter was badly garbled. The resulting confusion was such that it could not be rectified by an Erratum of the usual form, and therefore the Letter is reproduced below in its entirety.