Notes on the WKB Method

Basic Theory

for ϕ 566

Aut. 1997

WKB : Examples of use.

The WKB Method

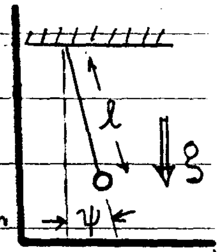
ref. M&W, pp. 27-37 ★
Davydov "QM", Ch III

1) The WKB method is a way of obtaining approximate solutions to 2nd order ODE's of the form of a generalized SHO (simple harmonic oscillator) eqtn:

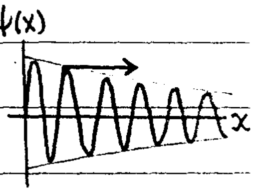
$$\boxed{\frac{d^2\psi}{dx^2} + k^2(x)\,\psi = 0} \quad \begin{cases} \psi = \psi(x) \text{ is an "amplitude" of some sort,} \\ k = k(x) \text{ is a variable "spring cnst".} \end{cases} \qquad (1)$$

The method works to the extent that $k(x)$ varies "slowly" with x [we will define "slowly" below]; it works best when $k(x) \to$ cnst. In any case, eqtns of this type arise in many examples in physics, e.g....

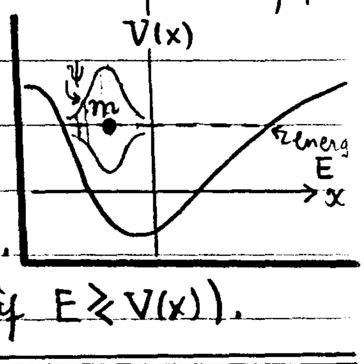
A. If $x = t$ (time), and ψ = displacement of a pendulum, then Eq. (1) is the pendulum's eqtn-of-motion, with $k(x) \leftrightarrow \omega(t)$ the natural frequency. $\omega = \omega(t)$ can depend on time if the length of the pendulum changes with t [$\omega = \sqrt{g/l}$ = fcn(t)] (e.g. m on a rubber band)



B. If x = position, and ψ = field amplitude of an EM wave, then Eq. (1) is the space-dependent part of the wave eqtn for the propagation. $k(x)$ is the "wavenumber", related to wavelength λ by $k = 2\pi/\lambda$. $k = k(x)$ if the wave propagates through a medium whose index of refraction n is changing ($\lambda = c/n\nu \Rightarrow k = 2\pi\frac{n\nu}{c}$, $n = n(x)$).



C. The 1D Schrödinger Eqtn of QM is of form of Eq. (1), with x = position, ψ = system "wavefcn", and $\hbar k(x)$ the system momentum. For a particle of mass m & energy E moving in an external potential $V(x)$, have: $\left[\!\left[k(x) = \left(\frac{2m}{\hbar^2}[E - V(x)]\right)^{1/2} \right]\!\right]$ Here $\hbar$ = Planck's cnst. Clearly $k = k(x)$ wherever $V(x) \neq$ cnst. Note that in this problem, k can be real or imaginary (if $E \gtrless V(x)$).



WKB ↔ Wentzel, Kramers, Brillouin... physicists who "popularized" the method in the early days of QM. Method actually invented by Jefferies (British mathⁿ) at ~1920.

D. Finally, from a math standpoint alone, we easily see that <u>any</u> 2nd order homogeneous ODE of the form...

$$\rightarrow \quad y'' + f(x)\,y' + g(x)\,y = 0, \quad \text{for } y = y(x); \tag{2}$$

can be cast into the WKB form with the substitution:

$$y(x) = \psi(x)\exp\left[-\frac{1}{2}\int^x f(\xi)\,d\xi\right]$$

Why doesn't integral have a lower limit? What difference?

$$\Rightarrow \boxed{\psi'' + k^2(x)\,\psi = 0, \quad \text{w/ } k(x) = \pm\sqrt{g(x) - \frac{1}{2}\left[f'(x) + \frac{1}{2}f^2(x)\right]}}. \tag{3}$$

So a WKB solution to this problem approximates a very general 2nd order (homogeneous) ODE... provided $k(x)$ is "slowly varying" with x.

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2) <u>A clue as to how to proceed</u> to solve the WKB eqtn [Eq. (1) above] is found by looking at the solutions when k actually <u>is</u> cnst, say $k = k_0$. Then...

$$\left[\!\left[\begin{array}{l} k = k_0 = \text{cnst} \Rightarrow \text{WKB eqtn: } \psi'' + k_0^2\,\psi = 0; \\ \ldots \text{ solutions are: } \psi(x) \propto e^{\pm i k_0 x} = \exp\left(\pm i \int^x k_0\,d\xi\right). \end{array}\right.\tag{4}$$

This suggests that if $k \rightarrow k(x)$ varies slowly with x, $\psi(x)$ will resemble:

$$\longrightarrow \psi(x) = e^{iS(x)}, \quad S(x) \simeq \pm\int^x k(\xi)\,d\xi \quad (\text{when } k(\xi) \simeq \text{cnst}). \tag{5}$$

To get a better fix on the "phase" $S(x)$, we change dept. variables by the substitution: $\psi(x) = e^{iS(x)}$. This gives an <u>exact</u> (nonlinear) eqtn for $S(x)$, viz

$$\left[\!\left[\begin{array}{l} \psi(x) = e^{iS(x)} \text{ into } \psi'' + k^2(x)\,\psi = 0; \\ \Rightarrow \boxed{(dS/dx)^2 = k^2(x) + i\,(d^2S/dx^2)} \end{array}\right.\tag{6}$$

This eqtn <u>cannot</u> be solved for S when $k(x)$ is an arbitrary fcn. But...

... if, in this eqtn, $k \sim k_0 = \text{cnst}$, then $S(x) \sim \pm k_0 x$, $dS/dx \sim \pm k_0$, and $S'' \sim 0$. This suggests that when $k(x)$ is "slowly varying", the effect in the eqtn for $S(x)$ will be that S'' is "small"; more specifically: $\underline{|S''| \ll |k^2(x)|}$.

3) Elaborate on the last idea, that k "slowly varying" $\Rightarrow |S''| \ll |k|^2$...

$|S''| \ll |k|^2 \Rightarrow$ Eq. (6) is : $\underline{(dS/dx)^2 \simeq k^2(x)}$

... solutions : $S(x) \simeq \pm \int^x k(\xi)\, d\xi$ ∫ in accord with Eq. (5) above. (7)

Now plug this (approximate) solution back into the "slowly varying" condition to find a condition on k for the Whole Approach to be valid...

$|S''| \ll |k|^2$, with : $S(x) \simeq \pm \int^x k(\xi)\, d\xi \Rightarrow |S''| = \left|\frac{dk}{dx}\right|$.

SLOWLY VARYING $\Rightarrow \left|\frac{dk}{dx}\right| \ll |k|^2$, or $\boxed{\boxed{\left|\frac{1}{k}\left(\frac{dk}{dx}\right)\right| \ll |k|}}$. (8)

This says that for a "slowly varying" fcn $k(x)$, the fractional change in the fcn, dk/k, per interval dx, should be small compared to the fcn k itself in that interval. OK... that's intuitive for a weak variation in $k(x)$. NOTE : condition of Eq. (8) fails whenever $|k| \to 0$ but $|dk/dx| \neq 0$, so the WKB method has big problems when $|k| \to 0$... e.g. it doesn't work.

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4) Now we assume the "slowly-varying" condition of Eq. (8), and seek to improve the approximate solution $S(x) \simeq \pm \int^x k(\xi)\, d\xi$ of Eq. (7) by iteration. Have.

$(S')^2 = k^2 + i S''$ ← exact, Eq. (6)

approx. soln : $S(x) \simeq S_0(x) = \pm \int^x k(\xi)\, d\xi$ ← approx., Eq. (7)

... for small term on RHS of exact eqtn, put : $S'' \simeq S_0'' = \pm \left(\frac{dk}{dx}\right)$...

So // $(S')^2 \simeq k^2 + i S_0'' = k^2 \left[1 \pm i \frac{1}{k^2}\left(\frac{dk}{dx}\right)\right]$,

or // $$dS/dx \simeq \pm k \left[1 \pm i \underbrace{\frac{1}{k^2}\left(\frac{dk}{dx}\right)}_{\text{"small" by Eq. (8)}}\right]^{1/2} \underset{\text{Binomial Expansion}}{\simeq} \pm k + \frac{i}{2} \underbrace{\frac{1}{k}\left(\frac{dk}{dx}\right)}_{= \frac{d}{dx} \ln k}. \quad (9)$$

WKB (cont'd) The WKB solution. Remarks on accuracy.

This last eqtn is easily integrated to give an improved solution for $S(x)$, viz:

$$\rightarrow S(x) \simeq S_0(x) + S_1(x) \int^{w/} \left.\begin{array}{l} S_0(x) = \pm \int^x k(\xi)\,d\xi \leftarrow \text{soln of Eq. (7)} \\ S_1(x) = \frac{i}{2} \ln k(x) + \text{cnst} \leftarrow \text{new correction} \end{array}\right\} \quad (10)$$

In Eq. (6), the solution proposed for $\psi'' + k^2\psi = 0$ was $\psi = e^{iS}$, so we form

$$\rightarrow \psi = e^{iS} \simeq e^{i(S_0+S_1)} = \exp\left[\pm i \int^x k(\xi)\,d\xi\right] \times \underbrace{\exp\left[-\tfrac{1}{2}\ln k(x) + \text{cnst}\right]}_{=\,\text{cnst}'/\sqrt{k(x)}} \ldots$$

... and we can state...

KB
UTION

$$\boxed{\begin{array}{l} \psi(x) = \left(\dfrac{\text{cnst}}{\sqrt{k(x)}}\right) \exp\left[\pm i \int^x k(\xi)\,d\xi\right], \text{ is an approximate solution to:} \\ (d^2\psi/dx^2) + k^2(x)\,\psi = 0, \ \text{provided: } \left|\dfrac{1}{k^2}\left(\dfrac{dk}{dx}\right)\right| \ll 1. \end{array}} \quad (11)$$

This form of ψ is called the WKB solution to the problem $\psi'' + k^2\psi = 0$.

REMARKS on WKB solution, Eq. (11).

A. The WKB solution for ψ in Eq. (11) is approximate in that it doesn't quite solve $\psi'' + k^2\psi = 0$; in fact...

$$\left[\begin{array}{l} \psi(x) = \left(\dfrac{\text{cnst}}{\sqrt{k(x)}}\right) \exp\left[\pm i \int^x k(\xi)\,d\xi\right], \text{ obeys: } \underline{\psi'' + k^2(1-\epsilon)\,\psi = 0}, \\ \text{where: } \underline{\underline{\epsilon(x)}} = \dfrac{3}{4}\left[\dfrac{1}{k^2}\left(\dfrac{dk}{dx}\right)\right]^2 - \dfrac{1}{2k^3}\left(\dfrac{d^2k}{dx^2}\right). \end{array}\right. \quad (12)$$

The WKB version of ψ is "good" only if $|\epsilon(x)| \ll 1$. The 1st term of $\epsilon(x)$ is small (by assumption) because of the "slowly-varying" condition of Eq. (8). The 2nd term of $\epsilon(x)$, involving k'', will usually be small if k' is small. More precisely, note that: $\frac{d}{dx}(k'/k^2) = \frac{1}{k^2}k'' - \frac{2}{k^3}(k')^2$, and rewrite...

$$\rightarrow \epsilon(x) = (-)\left[f + \frac{1}{k}\left(\frac{d}{dx}\right)\right] f, \quad w/\ f(x) = \frac{1}{2k^2}\left(\frac{dk}{dx}\right) = \frac{1}{k}\frac{d}{dx}\left(\ln\sqrt{k}\right). \quad (13)$$

For $|\epsilon(x)| \ll 1$, we need both (k'/k^2) and its derivative $\frac{d}{dx}(k'/k^2) \rightarrow$ small.

REMARKS (cont'd)

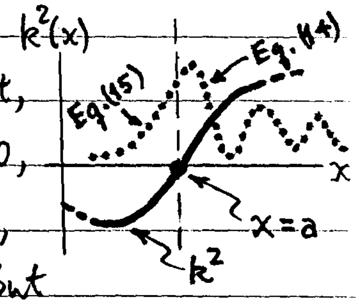
B. To provide the required two independent solutions to $\psi'' + k^2\psi = 0$, choose the + & − exponents in Eq. (11), and -- with A & B = integration consts -- form

$$\longrightarrow \psi(x) = \frac{1}{\sqrt{k(x)}}\left[A \exp\left(+i\int k(x)\,dx\right) + B \exp\left(-i\int k(x)\,dx\right)\right],\quad k^2 > 0. \quad (14)$$

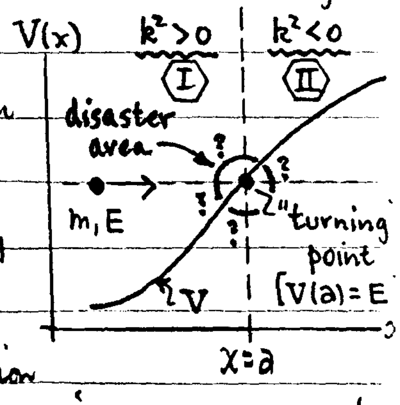
This general WKB solution is evidently oscillatory when k is real, i.e. when $k^2 > 0$. In some problems, however, it may be that $k^2 < 0$ over all or part of the range of x. If, say, $k^2(x) = (-)K^2(x)$, then the appropriate square root is $k = \pm iK$, and the above oscillatory solution becomes exponential:

$$\longrightarrow \psi(x) = \frac{1}{\sqrt{K(x)}}\left[C \exp\left(+\int K(x)\,dx\right) + D \exp\left(-\int K(x)\,dx\right)\right],\quad k^2 = (-)K^2 < 0. \quad (15)$$

This is the general WKB solution to $\psi'' - K^2\psi = 0$, with C & D = arb^{y} consts. In both cases, WKB is "good" if: $|k'/k^2| \ll 1$ (for (14)), $|K'/K^2| \ll 1$ (for (15)).



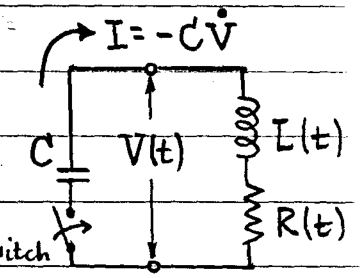
C. If $k^2(x)$ is a fcn which passes through zero at some point, say $x = a$ as shown at right, then: (1) @ $x < a$, when $k^2 < 0$, use the exponential solution of Eq. (15) above, (2) @ $x > a$, when $k^2 > 0$, use the oscillatory solution of Eq. (14). But in the neighborhood $a - \delta \leq x \leq a + \delta$, $\delta \to 0$, we run into Big Trouble... because $|k(x)| \to 0$ @ $x = a$, both types of WKB solns -- which $\propto \frac{1}{\sqrt{k(x)}}$ -- diverge at $x = a$. To boot, the "slowly-varying" condition $|k'/k^2| \ll 1$ is no good.



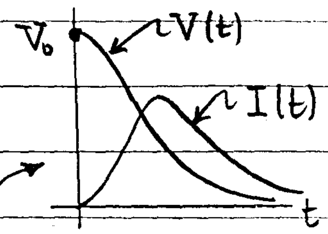
This annoyance occurs frequently in QM, where (recall from p. WKB 1): $\hbar k(x) = \sqrt{2m[E - V(x)]}$. When m approaches $x = a$, where the potential $V(a) = E$, then $k(x) \to 0$, and any WKB solution to this problem breaks down. While Eq. (14) holds in region I ($x < a$ & $k^2 > 0$), and Eq. (15) is OK in region II ($x > a$ & $k^2 < 0$), we have no WKB soln near $x = a$. The disaster area $x \sim a$ is called a "turning point", since a classical m would reverse its motion there.

WKB (cont'd) Arc discharge circuit problem.

5) We shall deal with WKB "turning point" problems in detail, but later. Here we wish to discuss a method for finding out by how much ψ(WKB) actually differs from the ψ which satisfies: $\psi'' + k^2(x)\psi = 0$. Rather than imposing inequalities like: $|k'/k^2| \ll 1$, for WKB validity, we shall estimate the correction: $\Delta\psi = \psi(\text{actual}) - \psi(\text{WKB})$... which (of course) depends on how rapidly k varies. Anyway, a knowledge of the size of $|\Delta\psi/\psi|$ is the "bottom line" mathematics here... if $|\Delta\psi/\psi| \to 1$, WKB is ~useless.



To fix ideas, we shall consider a physical example -- an ODE which describes the discharge of a capacitor C through an external circuit consisting of an inductor L & resistor R. The switch is closed at time $t=0$, when C is charged up to voltage V_0. If C = just a passive cnst, then a current $I = (-)C\dot{V}$ proceeds to flow in the circuit, and the voltage $V(t)$ across C diminishes; the expected behavior of V & I goes as...



Here's the twist... while we fix C = cnst, we let L & R = fcns of time (unlike the usual textbook examples). $L = L(t)$ & $R = R(t)$ would arise, for example, if the "external circuit" were a plasma, and we were trying to model the discharge of a highly electrified region (C) through an arc (L & R). We can choose $L(t)$ & $R(t)$ at will, and -- like Zeus -- we can manufacture our own lightning bolts. Thunder comes later in the course

When L & R = fcns of t, the circuit eqtn for $V = V(t)$ is...

$$V = RI + \frac{d}{dt}(LI), \quad I = -C\dot{V}$$

$$\Rightarrow \boxed{\ddot{V} + 2\Gamma(t)\dot{V} + \omega^2(t)V = 0}$$

$\Gamma(t) = \frac{1}{2}\left(\frac{R}{L} + \frac{\dot{L}}{L}\right)$, t-dept damping;

$\omega^2(t) = 1/LC$, t-dept resonant freq;

initial conditions $\}$ $V(0) = V_0$, $\dot{V}(0) = 0$ $[I(0) = 0]$.

We will now WKB this eqtn.

WKB (cont'd) Circuit problem: WKB form.

6) Convert Eq. (16) to standard WKB form by substitution...

$\rightarrow V(t) = v(t) \exp\left[-\int_0^t \Gamma(\tau)d\tau\right] \Rightarrow \boxed{\ddot{v} + \Omega^2(t) v = 0}$, w/ $\Omega = \sqrt{\omega^2 - (\Gamma^2 + \dot{\Gamma})}$. (17)

REMARKS on Eq. (17).

1. $V(t)$ will decay ~ exponentially with time t (which is reasonable) if the decay rate $\Gamma(t)$ is not too weird [need: $\Gamma = \frac{1}{2}\left(\frac{R}{L} + \frac{\dot{L}}{L}\right) > 0$, on avg., for $0 \leq t \to \infty$.
2. The WKB frequency Ω can be real or imaginary depending on the relative size of ω^2 & Γ^2. Basically, if $\dot{L}/L$ is "small", then: (A) Ω is real when $\omega^2 > \Gamma^2$, or $4L/CR^2 > 1$ (conventionally, such a CLR cct is "under-damped"), (B) Ω is imaginary when $\omega^2 < \Gamma^2$, or $4L/CR^2 < 1$ (the cct is "overdamped"). The WKB solns are: v(case A) ~ oscillatory, v(case B) ~ exponential.
3. A WKB solution for $v(t)$ in Eq. (17) will be "good" if Ω is "slowly-varying"...

$$|\dot{\Omega}/\Omega^2| = \left|\frac{1}{\Omega^3}\left[\omega\dot{\omega} - (\Gamma\dot{\Gamma} + \tfrac{1}{2}\ddot{\Gamma})\right]\right| << 1, \text{ w/ } \omega^2 \,\&\, \Gamma \text{ of Eq. (16)};$$

i.e. $$|\dot{\Omega}/\Omega^2| = \left|\frac{1}{2\Omega^3}\left[\omega^2\frac{\dot{L}}{L} + \Gamma\frac{d}{dt}\left(\frac{R}{L} + \frac{\dot{L}}{L}\right) + \frac{1}{2}\frac{d^2}{dt^2}\left(\frac{R}{L} + \frac{\dot{L}}{L}\right)\right]\right| << 1. \quad (18)$$

This condition is so complicated as to be ~ useless [although it does ⇒ that significant changes in $(\dot{L}/L)$ & $(\dot{R}/R)$ should occur on a time scale long compared to the natural scale $|\Omega|^{-1}$]. The point is: the simple imposition of "slowly-varying" $(|\dot{\Omega}/\Omega^2| << 1)$ does not always provide a transparent idea of how well the WKB method will work.

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7) A better way of assessing the accuracy of the WKB solution proceeds by comparing the WKB fcns with the actual solution. With Eq. (17) as a typical WKB problem, proceed as follows...

$\rightarrow$ Change indpt variable: $t \rightarrow s = \int_0^t \Omega(\tau)d\tau$, so/ $\frac{d}{dt} = \Omega\frac{d}{ds}$,

and/ $\ddot{v} + \Omega^2 v = 0$ ... becomes ... $v'' + (\Omega'/\Omega)v' + v = 0$. (19)

$\uparrow$ dot $\leftrightarrow \frac{d}{dt}$ $\qquad\qquad \uparrow$ prime $\leftrightarrow \frac{d}{ds}$

WKB (cont'd) Circuit problem: WKB form.

→ Change <u>dept.</u> variable: $v(s) = u(s)/\sqrt{\Omega(s)}$, $u(s)$ to be found,

So// v'' eqtn [Eq. (19)] becomes: $\boxed{u'' + [1 + b(s)]\,u = 0}$, (20)

where: $\underline{b(s)} = \frac{1}{4}\left(\frac{\Omega'}{\Omega}\right)^2 - \frac{1}{2}\left(\frac{\Omega''}{\Omega}\right)$, $s = \int_0^t \Omega(\tau)\,d\tau$.

If $b(s) \to$ small [note that $\Omega'/\Omega = \dot{\Omega}/\Omega^2$ is the old WKB small parameter... $|\dot{\Omega}/\Omega^2| \ll 1$ is the "slowly-varying" condition], then the u'' eqtn collapses to the triviality: $u'' + u \simeq 0$, and we have got a pretty good soln. In any case, we are now working with the system...

$$\boxed{\begin{array}{l} \text{Soln to: } \ddot{V} + 2\Gamma(t)\dot{V} + \omega^2(t)V = 0, \text{ is...} \\ V(t) = \dfrac{u(s)}{\sqrt{\Omega(s)}}\, e^{-\int_0^t \Gamma(\tau)\,d\tau}, \quad \text{w/ } \Omega = \sqrt{\omega^2 - (\Gamma^2 + \dot{\Gamma})}, \\ \qquad\qquad s = \int_0^t \Omega(\tau)\,d\tau; \\ \text{w/ } u(s) \text{ a soln to Eq. (20): } u'' + [1 + b(s)]\,u = 0. \end{array}}$$ (21)

NOTE: this eqtn is <u>exact</u>.

8) The neat thing about this formulation is that when $b(s) \equiv 0$, the solution to the u'' problem produce the usual WKB forms. Write Eq. (20) as...

→ $[\![u'' + u = -b(s)u]\!]$ ... when eqtn is homogeneous [$b(s) \to 0$], solns...

are// $u_H(s) = e^{\pm is} = e^{\pm i\int_0^t \Omega(\tau)\,d\tau}$, ← HOMOGENEOUS SOLNS

and// $v(t) = u_H/\sqrt{\Omega} = \frac{1}{\sqrt{\Omega(t)}}\, e^{\pm i\int_0^t \Omega(\tau)\,d\tau}$ } Standard <u>WKB forms</u>. (22)

So if $b(s) \neq 0$, the RHS contribution to the u'' eqtn will measure just how far $u_H(s)$ = WKB soln differs from the actual value of u. This rests on the idea that the u'' eqtn here can be solved iteratively in powers of the supposedly small parameter $b(s)$.

WKB (cont'd) Circuit problem. WKB soln as an integral eqtn.

9) To be more precise, recall a result from the treatment of "osculating parameters":

if// $p(s)\,u'' + q(s)\,u' + r(s)\,u = f(s)$, and// $u_{1,2}(s)$ = solns to homog^{s} eqtn,

then// $$u(s) = u_2(s)\int^s \frac{f(\sigma)}{p(\sigma)W}\,u_1(\sigma)\,d\sigma - u_1(s)\int^s \frac{f(\sigma)}{p(\sigma)W}\,u_2(\sigma)\,d\sigma$$ is a particular integral ★

Here: $W = u_1u_2' - u_1'u_2$ is the Wronskian. Apply to u'' eqtn in Eq. (22)... $\underbrace{}_{u''+u=-bu}$ (23)

$p(s)=1,\ q(s)=0,\ r(s)=1;\ f(s) = -b(s)\,u(s)$;

homogeneous solutions are: $u_{1,2}(s) = e^{\pm is}$ (solns to $u''+u=0$);

So// $W = e^{is}(-ie^{-is}) - (ie^{is})e^{-is} = -2i$, and// particular integral is...

$$u(s) = e^{-is}\int^s \frac{[+b(\sigma)u(\sigma)]}{(+2i)}\,e^{i\sigma}\,d\sigma - e^{is}\int^s \frac{[+b(\sigma)u(\sigma)]}{(+2i)}\,e^{-i\sigma}\,d\sigma$$

$$\rightarrow u_p(s) = \int_0^s u(\sigma)\,b(\sigma)\,\sin(\sigma - s)\,d\sigma \quad (24)$$ is a particular integral for eqtn: $u''+u=-b(s)u$.

The lower limit $s=0$ here is chosen for convenience; it makes no difference in the overall solution. We now have a <u>full</u> solution to Eq. (20)..

$\rightarrow u'' + [1+b(s)]\,u = 0$, has $\begin{cases}\text{homog. solns } u_{1,2}(s) = e^{\pm is},\\ \text{particular integral } u_p(s) \text{ of Eq. (24)};\end{cases}$

So// $$\boxed{u(s) = \underbrace{\left(Ae^{+is} + Be^{-is}\right)}_{\text{homog}^{s}\text{ soln} \equiv u(\text{WKB})\ldots} + \underbrace{\int_0^s u(\sigma)\,b(\sigma)\,\sin(\sigma-s)\,d\sigma}_{\text{Correction term} \propto \text{size of } b(s).}} \quad (25)$$

fcn u still exact

All this is still exact (we've made no "smallness" approxns). It appears we have an exact solution for $u(s)$. <u>But</u> this is a <u>integral eqtn</u> for u, since u (the unknown fcn) appears under the integral RHS. However, iteration is "easy".

★ Verify against solution to Arfken prob. # (8.6.25), p. 479.

10) Define: $\underline{w(s)} = Ae^{+is} + Be^{-is}$, $\underline{s} = \int_0^t \Omega(\tau)\,d\tau$ ... this is WKB soln†. So

$$\text{Eq. (25)}: \left[\!\left[\underset{\text{exact soln}}{u(s)} = \underset{\text{WKB approxn}}{w(s)} + \int_0^s \underset{\text{exact soln}}{u(\sigma)}\, \underset{\text{correction factor}}{b(\sigma)}\, \underset{\text{kernel fcn}}{\sin(\sigma - s)}\, d\sigma\right]\!\right], \quad \underline{\underline{b(s)}} = \left(\frac{\Omega'}{2\Omega}\right)^2 - \left(\frac{\Omega''}{2\Omega}\right). \tag{26}$$

This is a Volterra Integral Eqtn of the 2nd Kind for $u(s)$. Solvable by iteration procedure when $b(s) \to$ small...

$K^0)$] zeroth approxn } $u_0(s) = w(s)$ ... this is WKB ... applies strictly only when $b(s) \to 0$;

$K^1)$] first approxn } $u_1(s) = u_0(s) + \int_0^s u_0(\sigma)\, K(\sigma, s)\, d\sigma$ ... w/ $\underline{\underline{K(\sigma,s)}} = b(\sigma)\sin(\sigma - s)$;

$K^2)$] second approxn } $u_2(s) = u_1(s) + \int_0^s u_1(\sigma)\, K(\sigma, s)\, d\sigma$

⋮ etc ...

$$\boxed{u_{n+1}(s) = u_n(s) + \int_0^s u_n(\sigma)\, K(\sigma, s)\, d\sigma}, \quad n = 0, 1, 2, \ldots \tag{27}$$

terms of a Neumann series

Thus we can iterate WKB to arbitrary accuracy, in principle. There is of course the question of whether the iterative series [basically in powers of $b(s)$] converges. What counts here is the first iteration...

1st iteration: $u(s) = w(s) + \int_0^s w(\sigma)\, b(\sigma) \sin(\sigma - s)\, d\sigma$. ∫ everything on RHS is calculable

So// fractional error }
$$\boxed{\Delta(s) = \frac{u(s) - w(s)}{w(s)} = \frac{1}{w(s)} \int_0^s w(\sigma)\, b(\sigma) \sin(\sigma - s)\, d\sigma} \tag{28}$$

$\Delta(s)$ is evidently the fractional error in u(ACTUAL) vs. u(WKB), in first approxn (this was promised on p. 7). Also $\Delta(s)$ is the effective expansion parameter in the iterative expansion of Eq. (27). This claim is not precise, but ... roughly speaking ... the expansion works, and WKB is ~ good, when $\underline{|\Delta(s)| \lesssim 1}$.

Q// What happens when $\Omega(t) \to i|\Omega(t)|$?

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† i.e. in Eq. (21): $V(t) = [w(s)/\sqrt{\Omega(t)}]\, e^{-\int_0^t \Gamma(\tau)\, d\tau}$, is WKB approxn to problem.