1992 $\phi$ 507 Problems  Assigned: 1/27. Due 2/3/92  HW2 ② file

⑤ [15pts]. This problem concerns details of H-atom radial wavefcns in Davydov ¶ 38.

(A) When $a=-N$, w/ $N=0,1,2,\ldots$, show that the confluent hypergeometric fcn $F(a;b;x)$ reduces to the polynomial: $F(-N;b;x)=\sum_{k=0}^{N}\frac{\Gamma(b)}{\Gamma(k+b)}\binom{N}{k}(-x)^k$, w/ $\binom{N}{k}=\frac{N!}{k!(N-k)!}$ the binomial coefficient. Using this result, find an explicit form for the full H-atom radial wavefcn $f_{n\ell}(\rho)=\frac{1}{\rho}R_{n\ell}(\rho)$ for the 3s state. Compare with Davydov Table 8.

(B) H-atom states $|n\ell\rangle$ with maximum allowed ∡ momentum $\ell=n-1$ are called "raster" states. Find the general form of the full radial wavefcn $f_{n\ell}(\rho)$ when $\ell=n-1$.

(C) Calculate expectation values of powers of $\rho$, viz. $\langle\rho^\lambda\rangle$, in the states $|n,\ell=n-1\rangle$ you found in part (B). For $\lambda=-3$, specifically, compare with Davydov's Eq. 38.17e.

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⑥ A QM ∡ momentum $\hat{\mathbf{J}}$ has eigenfcns $|jm\rangle$. Consider the ladder operators $\hat{J}_\pm=\hat{J}_x\pm i\hat{J}_y$.

(A) Show that $\hat{J}_\pm|jm\rangle$ is an eigenfcn of $\hat{\mathbf{J}}^2$, with $j$-value unchanged.

(B) Show that $\hat{J}_\pm|jm\rangle$ is an eigenfcn of $\hat{J}_z$, corresponding to eigenvalues $m\pm1$.

(C) Using the $\hat{J}_\pm$, find the most general matrix elements of $\hat{J}_x$ & $\hat{J}_y$ -- i.e. evaluate $\langle\alpha' j' m'|\hat{J}_{x,y}|\alpha j m\rangle$, with pertinent selection rules for the quantum #s $\alpha\alpha'$; $jj'$, $mm'$.

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⑦ Consider the Pauli matrices $\vec{\sigma}=(\sigma_x,\sigma_y,\sigma_z)$ for spin $\frac{1}{2}$; they obey the commutation rule: $[\sigma_\alpha,\sigma_\beta]=2i\sigma_\gamma$, w/ $\alpha\beta\gamma$ = cyclic permutation of $xyz$ [Sakurai, Sec 3.2].

(A) Prove the anti-commutation rule: $\{\sigma_\alpha,\sigma_\beta\}=\sigma_\alpha\sigma_\beta+\sigma_\beta\sigma_\alpha=2\delta_{\alpha\beta}$.

(B) If $\vec{A}$ & $\vec{B}$ are any two vector operators that commute with $\vec{\sigma}$, use $[\sigma_\alpha,\sigma_\beta]$ and $\{\sigma_\alpha,\sigma_\beta\}$ to prove the Dirac identity: $\boxed{(\vec{\sigma}\cdot\vec{A})(\vec{\sigma}\cdot\vec{B})=\vec{A}\cdot\vec{B}+i\vec{\sigma}\cdot(\vec{A}\times\vec{B})}$.

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⑧ If vector operators $\vec{A}$ & $\vec{B}$ are both $\vec{T}$-vectors w.r.t. a QM ∡ momentum operator $\vec{J}$, show that: $[\vec{J},\vec{A}\cdot\vec{B}]=0$. Why does this establish $\vec{A}\cdot\vec{B}$ as a "true scalar"?

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⑨ [5pts]. Given: noncommuting operators $\hat{P}$ & $\hat{Q}$ and a set of basis fcns $\{u_k(x)\}$. If $P_{ij}=\int dx\, u_i^*(x)\hat{P}u_j(x)$, verify the matrix eqtn: $(PQ)_{k\ell}=\sum_m P_{km}Q_{m\ell}$, directly. What assumption(s) must be made about the set $\{u_k(x)\}$?

## φ507 Solutions

⑤ [15 pts]. Explore details of H-atom radial wavefcns in Davydov ¶38.

(A) $F(a;b;x) = \sum_{k=0}^{\infty} [(a)_k/(b)_k] \frac{x^k}{k!}$, and with $(a)_k = \Gamma(k+a)/\Gamma(a)$, we can write:

$$\rightarrow F(-N;b;x) = \sum_{k=0}^{\infty} \frac{\Gamma(b)}{\Gamma(k+b)} \left[\frac{\Gamma(k-N)}{\Gamma(-N)}\right] \frac{x^k}{k!}, \quad N=0,1,2,\ldots \tag{1}$$

We must deal with the [ ]. $\Gamma(z)$ diverges at $z=(-)N$, and so the $[\,] \equiv 0$ for all $k>N$. This means the series terminates at $k=N$. For $0 \leq k \leq N$, the [ ] is of indeterminate form $\pm\infty/\infty$. But from the reflection formula for Γ-fcns [see NBS Handbook # (6.1.17)]: $\Gamma(z)\Gamma(1-z) = \pi \csc \pi z$, it is easy to show...

$$\rightarrow n!\,\Gamma(-n+\epsilon) \simeq (-)^n/\epsilon, \text{ leading order in } \epsilon \text{ as } \epsilon \to 0, \text{ w/ } n=0,1,2,\ldots \tag{2}$$

Thus, except for sign, $(-)^n$, $\Gamma(z)$ diverges in the same way (namely as $\frac{1}{\epsilon}$) when $z \to -n$, any (−)ve integer. If $m$ & $n$ are integers: $m!\,\Gamma(-m)/n!\,\Gamma(-n) = (-)^{m-n}$, is finite. Applying this result to Eq. (1)...

$$\left[\frac{\Gamma(k-N)}{\Gamma(-N)}\right] = (-)^k \frac{N!}{(N-k)!} \Rightarrow \boxed{F(-N;b;x) = \sum_{k=0}^{N} \frac{\Gamma(b)}{\Gamma(k+b)} \binom{N}{k} (-x)^k}. \tag{3}$$

$\binom{N}{k} = N!/k!(N-k)!$ is the binomial coefficient. NOTE: this result can be gotten also by using Leibniz' formula to differentiate the definition of the Laguerre polynomial $L_N^{b-1}(x)$ associated with $F(-N;b;x)$ [Davydov, Math. App. D, Eqs. (D7) & (D8)].

In his Eq. (38.16), Davydov writes the full H-atom radial wavefcn as...

$$\rightarrow f_{n\ell}(\rho) = N_{n\ell}\, x^\ell F(-N;b;x)\, e^{-x/2}, \text{ w/ } x = \frac{2Z\rho}{n}, N = n-\ell-1, b = 2\ell+2;$$

$$\text{where: } N_{n\ell} = \frac{(2Z/n)^{3/2}}{(2\ell+1)!} \sqrt{(n+\ell)!/2n\,N!}, \text{ norm factor.} \tag{4}$$

For the 3s state, $n=3$ & $\ell=0$, so $N=2$ & $b=2$. Then $N_{30} = \frac{1}{\sqrt{2}}(2Z/3)^{\frac{3}{2}}$, and $x = \frac{2}{3}Z\rho$. Also [by Eq. (3)]: $F(-N;b;x) = F(-2;2;x) = 1 - x + \frac{1}{6}x^2$. Then have:

$$\boxed{f_{30}(\rho) = N_{30}\, F(-2;2;x)\, e^{-x/2} = \frac{2Z^{3/2}}{3\sqrt{3}} \left[1 - \frac{2}{3}Z\rho + \frac{2}{27}(Z\rho)^2\right] e^{-\frac{1}{3}Z\rho}}. \tag{5}$$

This agrees with the 3s entry in Davydov's Table 8 when $Z=1$ (hydrogen).

(B) For the "raster" states $|n, l=n-1\rangle$, the radial quantum # $N = n-(l+1) = 0$, and -- by Eq. (3) -- have $F(0; b; x) \equiv 1$. By Eq. (4), the radial wave fcns reduce to

$$\underline{\underline{f_{n,n-1}(\rho) = N_{n,n-1}\, x^{n-1} e^{-x/2}}}, \quad \text{w/}\ x = \frac{2Z\rho}{n},\ N_{n,n-1} = \left(\frac{2Z}{n}\right)^{\frac{3}{2}} \sqrt{\frac{1}{(2n)!}}. \qquad (6)$$

(C) The expectation value of $\rho^\lambda$ in the raster state of Eq. (6) is...

$$\rightarrow \langle\rho^\lambda\rangle = \int_0^\infty f^*_{n,n-1}(\rho)\,[\rho^\lambda]\, f_{n,n-1}(\rho)\cdot \rho^2 d\rho = \int_0^\infty d\rho\, \rho^{\lambda+2}\, [f_{n,n-1}(\rho)]^2. \qquad (7)$$

The $\measuredangle$ lar integration $\int_{4\pi} d\Omega\, |Y_{lm}(\theta,\varphi)|^2 = 1$ has been done. With $f_{n,n-1}(\rho)$ the real radial wavefcn of Eq. (6)...

$$\langle\rho^\lambda\rangle = \frac{1}{(2n)!}\left(\frac{2Z}{n}\right)^{\frac{3}{2}} \int_0^\infty d\rho\, \rho^{\lambda+2}\, x^{2n-2} e^{-x}, \quad x = (2Z/n)\rho$$

$$= \frac{1}{(2n)!}\left(\frac{n}{2Z}\right)^{\lambda} \int_0^\infty dx\, x^{2n+\lambda} e^{-x} \leftarrow \text{tabulated integral [e.g. Dwight \#(860.07)]}$$

So, $$\boxed{\langle\rho^\lambda\rangle = \left(\frac{n}{2Z}\right)^{\lambda} \frac{\Gamma(2n+\lambda+1)}{(2n)!}}; \quad \langle\rho^\lambda\rangle = \left(\frac{n}{2Z}\right)^{\lambda} \frac{(2n+\lambda)!}{(2n)!}, \text{ if } \lambda = \text{integer}. \qquad (8)$$

For $\lambda = -3$, Eq. (8) gives...

$$\left\langle\frac{1}{\rho^3}\right\rangle = \left(\frac{n}{2Z}\right)^{-3} \frac{(2n-3)!}{(2n)!} = \left(\frac{2Z}{n}\right)^3 \Big/ 2n(2n-1)(2n-2)$$

Davydov Eq. (38.17e)

$$\text{w/} \boxed{\left\langle\frac{1}{\rho^3}\right\rangle = \left(\frac{Z}{n}\right)^3 \Big/ n(n-\tfrac{1}{2})(n-1)} \stackrel{!}{=} \left(\frac{Z}{n}\right)^3 \Big/ (l+1)(l+\tfrac{1}{2})\, l, \ \text{w/}\ l = n-1. \qquad (9)$$

Our result for $\langle 1/\rho^3\rangle$ agrees with Davydov's Eq. (38.17e) in the case we are considering, viz. $l = n-1$. Similarly, Eq. (8) gives, with $l = n-1$ properly inserted

$$\left.\begin{aligned}
\langle 1/\rho^2\rangle &= \left(\frac{Z}{n}\right)^2 \Big/ n(n-\tfrac{1}{2}) = \frac{Z^2}{n^3}\Big/(l+\tfrac{1}{2}) \leftarrow \text{Davydov Eq. (38.17d)}\\
\langle 1/\rho\rangle &= Z/n^2 \ \ (\text{no } l\text{-dependence}) \leftarrow \ \text{" \ " (38.17c)}\\
\langle \rho\rangle &= \frac{n}{Z}(n+\tfrac{1}{2}) = \left.\frac{3n^2 - l(l+1)}{2Z}\right|_{l=n-1} \leftarrow \ \text{" \ " (38.17a).}
\end{aligned}\right\} \qquad (10)$$

etc.

 $\boxed{\hbar = 1, \text{ throughout}}$ 

⑥ Elementary operations with ∡ momentum ladder operators $\hat{J}_\pm$ [Sakurai, Sec. 3.5].

(A) $\hat{\mathbb{J}}^2$ commutes with every one of the components $\hat{J}_k$ of $\hat{\mathbb{J}}$, i.e. $[\hat{\mathbb{J}}^2, \hat{J}_k] = 0$, where $k = x, y, z$ [Sakurai, Eq. (3.5.2)]. So, obviously $[\hat{\mathbb{J}}^2, \hat{J}_\pm] = 0$. Now consider $\psi = |jm\rangle$ an eigenfcn of $\hat{\mathbb{J}}^2$, i.e. $\hat{\mathbb{J}}^2 \psi = j(j+1)\psi$. Let $\phi_\pm = \hat{J}_\pm \psi$, and look at

$$\rightarrow \underline{\underline{\hat{\mathbb{J}}^2 \phi_\pm}} = \hat{\mathbb{J}}^2 \hat{J}_\pm \psi = \hat{J}_\pm \hat{\mathbb{J}}^2 \psi = j(j+1) \hat{J}_\pm \psi = \underline{\underline{j(j+1)\phi_\pm}}. \quad (1)$$

So, as required, $\phi_\pm = \hat{J}_\pm |jm\rangle$ is an eigenfcn of $\hat{\mathbb{J}}^2$ with $j$ unchanged.

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(B) $\psi = |jm\rangle$ is an eigenfcn of $\hat{J}_z$, i.e. $\hat{J}_z \psi = m\psi$. Now consider $\hat{J}_z \phi_\pm$, where (as above) $\phi_\pm = \hat{J}_\pm \psi$. By adding & subtracting $\hat{J}_\pm \hat{J}_z$, we can write...

$$\rightarrow \hat{J}_z \phi_\pm = \hat{J}_z \hat{J}_\pm \psi = \hat{J}_\pm \hat{J}_z \psi + [\hat{J}_z, \hat{J}_\pm]\psi. \quad (2)$$

The first term RHS is just $m\phi_\pm$. As for the second term RHS, calculate

$$\left[\!\left[ [\hat{J}_z, \hat{J}_\pm] = \pm \hat{J}_\pm \leftarrow \text{Sakurai Eq. (3.5.6b) [use of } [\hat{J}_\alpha, \hat{J}_\beta] = i\hat{J}_\gamma, \text{ etc].} \right.\right. \quad (3)$$

so// Eq. (2) $\Rightarrow$ $\underline{\underline{\hat{J}_z \phi_\pm = (m \pm 1)\phi_\pm}}$ (4)

As required, $\phi_\pm = \hat{J}_\pm |jm\rangle$ is an eigenfcn of $\hat{J}_z$ with eigenvalue $m \pm 1$.

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(C) We can express: $\hat{J}_x = \frac{1}{2}(\hat{J}_+ + \hat{J}_-)$, and; $\hat{J}_y = \frac{1}{2i}(\hat{J}_+ - \hat{J}_-)$, and we "know"...

$$\left[\!\left[ \hat{J}_\pm |\alpha j m\rangle = \sqrt{(j \mp m)(j \pm m + 1)}\, |\alpha j\, m \pm 1\rangle \leftarrow \text{Sakurai Eqs (3.5.39 \& 40).} \right.\right. \quad (5)$$

The quantum #s $j$ (total ∡ momentum) and $\alpha$ (all other quantum #s) remain unchanged, and the matrix element $\langle \alpha' j' m' | \hat{J}_x | \alpha j m\rangle = \delta_{\alpha\alpha'} \delta_{jj'} \langle \alpha j m' | \hat{J}_x | \alpha j m\rangle$. When we insert $\hat{J}_x = \frac{1}{2}(\hat{J}_+ + \hat{J}_-)$, this M.E. vanishes except when $m' = m \pm 1$. So...

$$\langle \alpha' j' m' | \hat{J}_x | \alpha j m \rangle = \frac{1}{2} \delta_{\alpha\alpha'} \delta_{jj'} \begin{cases} \sqrt{(j-m)(j+m+1)}, & \text{when } m' = m+1; \\ \sqrt{(j+m)(j-m+1)}, & \text{when } m' = m-1. \end{cases} \quad (6a)$$

$\boxed{\text{selection: } \Delta\alpha = 0, \Delta j = 0, \Delta m = \pm 1}$

$$\langle \alpha' j' m' | \hat{J}_y | \alpha j m \rangle = \frac{1}{2i} \delta_{\alpha\alpha'} \delta_{jj'} \begin{cases} \sqrt{(j-m)(j+m+1)}, & \text{when } m' = m+1; \\ (-)\sqrt{(j+m)(j-m+1)}, & \text{when } m' = m-1. \end{cases} \quad (6b)$$

⑦ Carry out manipulations with the Pauli matrices $\vec{\sigma}$ for spin 1/2.

(A) From the explicit representation: $(\sigma_x, \sigma_y, \sigma_z) = \left(\begin{pmatrix}0&1\\1&0\end{pmatrix}, \begin{pmatrix}0&-i\\i&0\end{pmatrix}, \begin{pmatrix}1&0\\0&-1\end{pmatrix}\right)$ -- see Sakurai, Eq. (3.2.32) -- we verify directly that: $\sigma_x^2 = \begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix} = \begin{pmatrix}1&0\\0&1\end{pmatrix} = 1$. Also, $\sigma_y^2 = 1$ & $\sigma_z^2 = 1$, similarly. Thus $\sigma_\alpha^2 = 1$ for each of $\alpha = x, y, z$.

For $\alpha \neq \beta$, again look at explicit forms...

$$\rightarrow \sigma_x\sigma_y + \sigma_y\sigma_x = \begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}0&-i\\i&0\end{pmatrix} + \begin{pmatrix}0&-i\\i&0\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix} = \begin{pmatrix}i&0\\0&-i\end{pmatrix} + \begin{pmatrix}-i&0\\0&i\end{pmatrix} = 0;$$

$$\sigma_y\sigma_z + \sigma_z\sigma_y = \begin{pmatrix}0&-i\\i&0\end{pmatrix}\begin{pmatrix}1&0\\0&-1\end{pmatrix} + \begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}0&-i\\i&0\end{pmatrix} = \begin{pmatrix}0&i\\i&0\end{pmatrix} + \begin{pmatrix}0&-i\\-i&0\end{pmatrix} = 0;$$

$$\sigma_z\sigma_x + \sigma_x\sigma_z = \begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix} + \begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}1&0\\0&-1\end{pmatrix} = \begin{pmatrix}0&1\\-1&0\end{pmatrix} + \begin{pmatrix}0&-1\\1&0\end{pmatrix} = 0. \quad (1)$$

So: $\sigma_\alpha\sigma_\beta + \sigma_\beta\sigma_\alpha = 0$, when $\alpha \neq \beta$. Combined with $\sigma_\alpha^2 = 1$, we get

$$\boxed{\{\sigma_\alpha, \sigma_\beta\} = \sigma_\alpha\sigma_\beta + \sigma_\beta\sigma_\alpha = 2\delta_{\alpha\beta}}, \quad (2)$$

which is the desired anticommutation rule.

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(B) If we just add the equations $\{\sigma_\alpha, \sigma_\beta\} = 2\delta_{\alpha\beta}$ and $[\sigma_\alpha, \sigma_\beta] = 2i\epsilon_{\alpha\beta\gamma}\sigma_\gamma$, then

$$\rightarrow \sigma_\alpha\sigma_\beta = \delta_{\alpha\beta} + i\,\epsilon_{\alpha\beta\gamma}\sigma_\gamma \quad \text{w/ } \epsilon_{\alpha\beta\gamma} = \begin{cases}\pm 1, & \text{for } \alpha\beta\gamma = \binom{\text{even}}{\text{odd}} \text{ perm}^{\text{n}} \text{ of } xyz;\\ 0, & \text{otherwise.}\end{cases} \quad (3)$$

Then the Dirac identity is easy...

$\vec{A}$ & $\vec{B}$ commute with $\vec{\sigma}$

$$\underline{\underline{(\vec{\sigma}\cdot\vec{A})(\vec{\sigma}\cdot\vec{B})}} = \sum_{\alpha,\beta}(\sigma_\alpha A_\alpha)(\sigma_\beta B_\beta) = \sum_{\alpha,\beta}(\sigma_\alpha\sigma_\beta)(A_\alpha B_\beta)$$

$$= \sum_{\alpha,\beta}(\delta_{\alpha\beta} + i\,\epsilon_{\alpha\beta\gamma}\sigma_\gamma)A_\alpha B_\beta$$

$$= \sum_\alpha A_\alpha B_\alpha + i\sum_{\alpha,\beta}\sigma_\gamma\,(\underbrace{\epsilon_{\alpha\beta\gamma}A_\alpha B_\beta}_{(\vec{A}\times\vec{B})_\gamma})$$

$$= \underline{\underline{\vec{A}\cdot\vec{B} + i\,\vec{\sigma}\cdot(\vec{A}\times\vec{B})}}. \quad \underline{\underline{\text{QED}}} \quad (4)$$

$\boxed{\text{REMARKS}}$

1. Sakurai proves the Dirac identity in his Eq. (3.2.40).

2. For $\vec{\sigma}$'s for spin 1 & spin 3/2, see Schiff "QM" (3rd ed., 1968), Sec. 27. NOTE: the relation $\{\sigma_\alpha, \sigma_\beta\} = 2\delta_{\alpha\beta}$ is obeyed only for spin 1/2.

⑧ Show that $[\vec{J}, \vec{A}\cdot\vec{B}]=0$, for $\vec{A}$ & $\vec{B}$ as $\vec{T}$-vectors w.r.t. $\vec{J}$.

Consider the $\alpha$ component of the commutator. It can be written as...

$$\rightarrow [\vec{J}, \vec{A}\cdot\vec{B}]_\alpha = \sum_\beta [J_\alpha, A_\beta B_\beta] = \sum_\beta \{A_\beta [J_\alpha, B_\beta] + [J_\alpha, A_\beta] B_\beta\}. \quad (1)$$

We have used the commutator identity: $[P, QR] = Q[P,R] + [P,Q]R$. Since $\vec{A}$ & $\vec{B}$ are both $\vec{T}$-vectors w.r.t. $\vec{J}$, then in Eq. (1) we can set...

$$\left[\begin{array}{l} [J_\alpha, B_\beta] = i\sum_\gamma \epsilon_{\alpha\beta\gamma} B_\gamma, \\ [J_\alpha, A_\beta] = i\sum_\gamma \epsilon_{\alpha\beta\gamma} A_\gamma. \end{array}\right. \quad (2)$$

Here $\hbar=1$, and $\epsilon_{\alpha\beta\gamma}$ = Levi-Civita density.* The sum over $\gamma$ is redundant (but useful).

$$\overset{\text{Soy}}{\rightarrow} [\vec{J}, \vec{A}\cdot\vec{B}]_\alpha = i\sum_{\beta,\gamma} \epsilon_{\alpha\beta\gamma}\{A_\beta B_\gamma + A_\gamma B_\beta\}$$

β & γ interchanged here

$$= i\left\{\sum_{\beta,\gamma} \epsilon_{\alpha\beta\gamma} A_\beta B_\gamma - \sum_{\gamma,\beta} \epsilon_{\alpha\gamma\beta} A_\gamma B_\beta\right\}. \quad (3)$$

Each term on RHS of Eq. (3) is equivalent to $(\vec{A}\times\vec{B})_\alpha$. So, as required...

$$\boxed{[\vec{J}, \vec{A}\cdot\vec{B}] = i\{(\vec{A}\times\vec{B}) - (\vec{A}\times\vec{B})\} = 0.} \quad (4)$$

This result is independent of whether $\vec{A}$ & $\vec{B}$ commute with each other.

Under the $\infty$smal rotation operator (rotation by ∡ $\delta\varphi$ about axis $\hat{n}$), viz. $R(\delta\varphi) = 1 - i\,\delta\varphi(\hat{n}\cdot\vec{J})$, [Sakurai Eq. (3.1.15)], a scalar $S$ transforms as

$$\left[\begin{array}{l} S \rightarrow S' = R^{-1} S R = S + i\,\delta\varphi[\hat{n}\cdot\vec{J}, S], \text{ to } 1^{st} \text{ order in } \delta\varphi; \\ \text{soy} \quad \delta S = S' - S = i\,\delta\varphi[\hat{n}\cdot\vec{J}, S]. \end{array}\right. \quad (5)$$

If $S$ is a "true scalar", it will be unaffected by such a rotation, i.e. $\delta S = 0$. This requires $[\hat{n}\cdot\vec{J}, S] = 0$... or that $S$ commute with each component of $\vec{J}$, i.e. $[\vec{J}, S] = 0$. Then $S = \vec{A}\cdot\vec{B}$ is a "true scalar" by virtue of Eq. (4).

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* $\epsilon_{\alpha\beta\gamma} = \pm 1$ if $\alpha\beta\gamma = \left\{\begin{smallmatrix}\text{even}\\ \text{odd}\end{smallmatrix}\right\}$ permutation of 123. Otherwise $\epsilon_{\alpha\beta\gamma} \equiv 0$.

⑨ [5pts]. Prove : $(PQ)_{k\ell} = \sum_m P_{km} Q_{m\ell}$ w.r.t. basis $\{u_k(x)\}$.

The RHS of the identity is

$$\rightarrow \sum_m P_{km} Q_{m\ell} = \sum_m \int dx\, u_k^*(x)\, \hat{P}\, u_m(x) \int dx'\, u_m^*(x')\, \hat{Q}\, u_\ell(x')$$

$$= \int dx\, u_k^*(x)\, \hat{P} \int dx' \Big[ \sum_m u_m(x)\, u_m^*(x') \Big] \hat{Q}\, u_\ell(x'). \qquad (1)$$

But the "basis" $\{u_k(x)\}$ is by assumption a complete set of fcns on the (common) domain of $\hat{P}$ & $\hat{Q}$. Such a complete set obey the closure relation:

$$\rightarrow \sum_m u_m(x)\, u_m^*(x') = \delta(x-x'), \text{ Dirac delta fcn.} \qquad (2)$$

The [ ] in Eq. (1) can be replaced by the δ-fcn, and we have -- as desired

$$\Big[\!\Big[ \sum_m P_{km} Q_{m\ell} = \int dx\, u_k^*(x)\, \hat{P} \int dx'\, \delta(x-x')\, \hat{Q}\, u_\ell(x')$$

$$= \int dx\, u_k^*(x)\, \hat{P}\hat{Q}\, u_\ell(x) = (PQ)_{k\ell}. \Big]\!\Big] \quad \underline{\underline{\text{QED}}} \qquad (3)$$

The ordering of $\hat{P}$ & $\hat{Q}$ has been respected, so the proof holds whether or not $\hat{P}$ & $\hat{Q}$ commute. One need only assume the $\{u_k(x)\}$ are a complete set.

In the language of bras and kets, the proof goes as...

$$\rightarrow \sum_m P_{km} Q_{m\ell} = \sum_m \langle k|\hat{P} \underbrace{|m\rangle\langle m|}_{\sum_m |m\rangle\langle m| = \hat{I},\ \text{identity matrix, for complete set}} \hat{Q}|\ell\rangle = \langle k|\hat{P}\hat{Q}|\ell\rangle = (PQ)_{k\ell}. \qquad (4)$$

More efficient, but more abstract.