DEPARTMENT OF PHYSICS

1996 COMPREHENSIVE EXAM

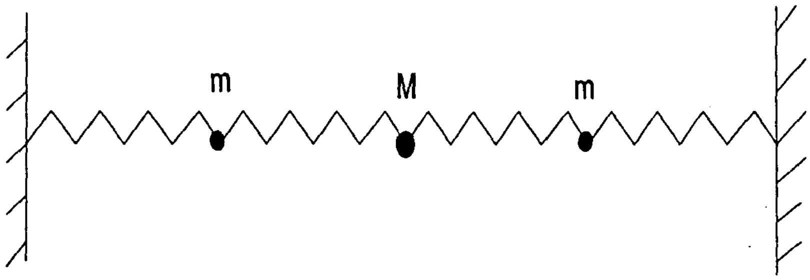
Answer each of the following questions, all of which carry equal weight. Begin your answer to each question on a new sheet of paper; solutions to different questions must not appear on the same sheet. Each sheet of paper must be labeled with your name and the problem number in the upper right hand corner. When more than one sheet is submitted for a problem, be sure the pages are ordered properly.

PHYSICAL CONSTANTS

Quantity	Symbol	Value
acceleration due to gravity	g	9.8 m s^{-2}
gravitational constant	G	$6.67 \times 10^{-11} \text{ N m}^2 \text{ kg}^2$
permittivity of vacuum	ε_o	$8.85 \times 10^{-12} \text{ C}^2 \text{ N}^{-1} \text{ m}^{-2}$
permeability of vacuum	μ_o	$4\pi \times 10^{-7} \text{ N A}^{-2}$
speed of light in vacuum (or air)	c	$3.00 \times 10^{-8} \text{ m s}^{-1}$
elementary charge	e	$1.602 \times 10^{-19} \text{ C}$
mass of electron	m_e	$9.11 \times 10^{-31} \text{ kg}$
mass of proton	m_p	$1.673 \times 10^{-27} \text{ kg}$
Planck constant	h	$6.63 \times 10^{-34} \text{ J s}$
Avogadro constant	N_A	$6.02 \times 10^{23} \text{ mol}^{-1}$
Boltzmann constant	k	$1.38 \times 10^{-23} \text{ J K}^{-1}$
molar gas constant	R	$8.31 \text{ J mol}^{-1} \text{ K}^{-1}$
standard atmospheric pressure		$1.013 \times 10^{5} \text{ Pa}$

1996 COMPREHENSIVE EXAM

1. Three point masses move on a straight, frictionless wire under the influence of four identical, ideal springs as shown below. The spring constant is k and $M >> m$.



Derive the normal mode frequencies, to lowest order in m/M.

JH2 Solution

$$T = \frac{m}{2}\left(\dot{x}_1^2 + \gamma \dot{x}_2^2 + \dot{x}_3^2\right), \text{ where } \gamma = M/m \gg 1$$

$$V = \frac{k}{2}\left(x_1^2 + (x_1 - x_2)^2 + (x_2 - x_3)^2 + x_3^2\right)$$

$$= k\left(x_1^2 - x_1 x_2 + x_2^2 - x_2 x_3 + x_3^2\right)$$

To determine the normal mode frequencies, diagonalize the matrix

$$\underline{V} - \omega^2 \underline{m} = \begin{pmatrix} 2k - m\omega^2 & -k & 0 \\ -k & 2k - M\omega^2 & -k \\ 0 & -k & 2k - m\omega^2 \end{pmatrix}$$

(or set up Lagrange's equations in x_1, x_2, x_3)

This leads to the secular equation

$$0 = \begin{vmatrix} 2a - x & -a & 0 \\ -a & 2a - \gamma x & -a \\ 0 & -a & 2a - x \end{vmatrix} \quad \begin{pmatrix} a \equiv k/m \\ x \equiv \omega^2 \end{pmatrix}$$

$$= (2a - x)^2 (2a - \gamma x) - 2a^2 (2a - x)$$

$$= (2a - x)\left(\gamma x^2 - 2(\gamma + 1) a x + 2a^2\right)$$

Soln's

$$x_1 = 2a \qquad \boxed{\omega_1 = \sqrt{\frac{2k}{m}}}$$

(M fixed)

$$x_{2,3} = \frac{(\gamma+1)a}{\gamma} \pm \sqrt{\left(\frac{(\gamma+1)a}{\gamma}\right)^2 - \frac{8a^2}{\gamma}}$$

$$= a\left\{1 + \gamma^{-1} \pm \sqrt{1 - 6\gamma^{-1}}\right\}, \quad \gamma = \frac{M}{m} \gg 1$$

Then $x_{2,3} \approx a\left(1+\gamma^{-1} \pm (1-3\gamma^{-1})\right)$

$$x_2 \approx 2a(1-\gamma^{-1}) \approx 2a \quad \text{to lowest order}$$

$$x_3 \approx a(0+4\gamma^{-1}) = \frac{4a}{\gamma} \quad \text{to lowest order}$$

$$= \frac{4k}{M}$$

Altogether,

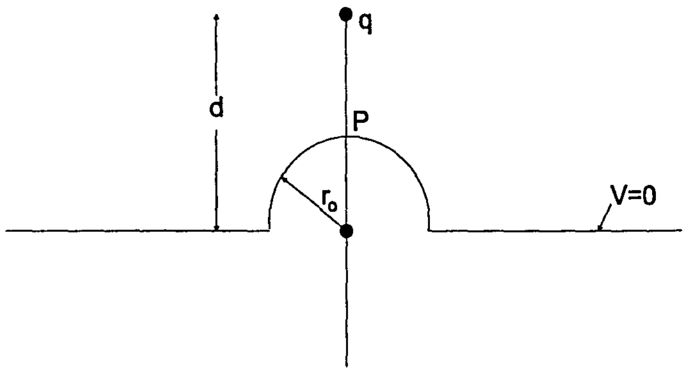
$$\omega_1 = \sqrt{\frac{2k}{m}}$$

$$\omega_2 = \sqrt{\frac{2k}{m}} \quad \text{to lowest order}$$

$$\omega_3 = \sqrt{\frac{4k}{M}} \quad \text{to lowest order}$$

1996 COMPREHENSIVE EXAM

2. A thick conductor at potential $V = 0$ has the shape of an infinite plane except for a hemispherical bulge of radius r_o. A charge q is placed above the center of the bulge a distance $d > r_o$ from the plane as shown in the figure below. Find the charge density accumulated at point P on the top of the hemispherical bulge.



Guidelines for grading:

1. Mention of image charge: 1
2. Solving plane portion: 1
3. Solving spherical portion: 3
   Attempting Laplace eq. in spherical coordinates (3/10)
4. Setting up $\vec{E}.\hat{n} = 4\pi\sigma$: 1
5. Setting up correct superposition: 2
6. Carry out math: 2

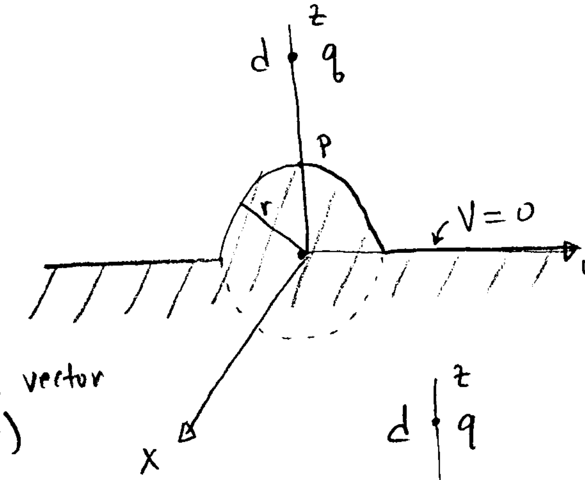
\+ 10

Total:

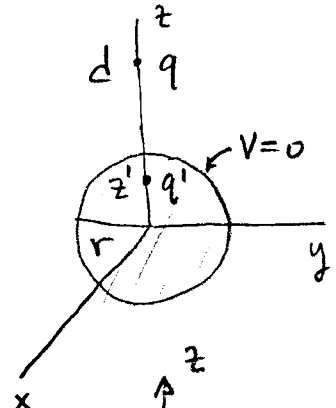
E & M

RA

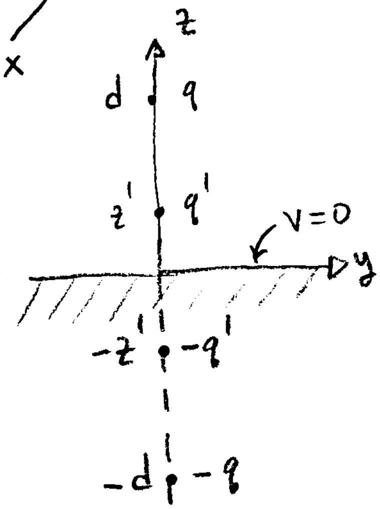
2. **Hemispherical bulge in a plane:** A thick conductor at potential $V = 0$ has the shape of an infinite plane except for an hemispherical bulge of radius r_o. A charge q is placed above the center of the bulge, a distance $d > r_o$ from the plane as shown in the figure below. Find the charge density accumulated at point P on the top of the hemispherical bulge.



Choose the coordinate system as shown. Gauss' law relates the charge density at P to the normal component of $\vec{E}$ via: $\vec{E}\cdot\hat{n} = 4\pi\sigma$, where $\hat{n} = \hat{k}$, $\hat{}$ is the unit normal vector at point P. (Here CGS units are used)



Problem can be solved by image charge method. First consider only the sphere of radius r with the charge q at $z=d$: An image charge $q' = -\frac{r}{d}q$ located at $z' = \frac{r^2}{d}$ will satisfy $V=0$ on the sphere. We must also maintain $V=0$ on a plane passing through the xy-plane while maintaining $V=0$ on the sphere. This is easily achieved with two more image charges at $z=-z'$ and $z=-d$ with corresponding image charges $-q'$ and $-q$, respectively. The superposition of the fields by these 4 charges not only satisfy the the boundary conditions but solve the Poisson's equation $\nabla^2 V = -4\pi\rho$, hence the uniqueness theorem guarentees that this is the only solution.



$$\Rightarrow \sigma = \frac{\vec{E}\cdot\hat{k}}{4\pi} = \frac{1}{4\pi}\left(\frac{-q}{(d-r)^2} + \frac{-\frac{r}{d}q}{(r-\frac{r^2}{d})^2} + \frac{\frac{r}{d}q}{(r+\frac{r^2}{d})^2} - \frac{q}{(d+r)^2}\right)$$

$$\sigma = -\frac{q}{4\pi r}\left(\frac{(r+d)^3 + (r-d)^3}{(d^2-r^2)^2}\right) = \boxed{-\frac{q}{2\pi}\frac{r^2+3d^2}{(d^2-r^2)^2}}$$

3.

a) A transmission diffraction grating may be treated, at a simple level, as an array of N equally-spaced, long, narrow slits a small distance d apart. Determine the condition for constructive interference in the *Fraunhofer limit* (plane wave incident on the grating) for light of wavelength λ incident at angle α relative to the grating normal and diffracting at angle β (both angles are in a plane perpendicular to the grating and to the slits). Illustrate your calculation with a sketch labeled to show α and β and the grating spacing d.

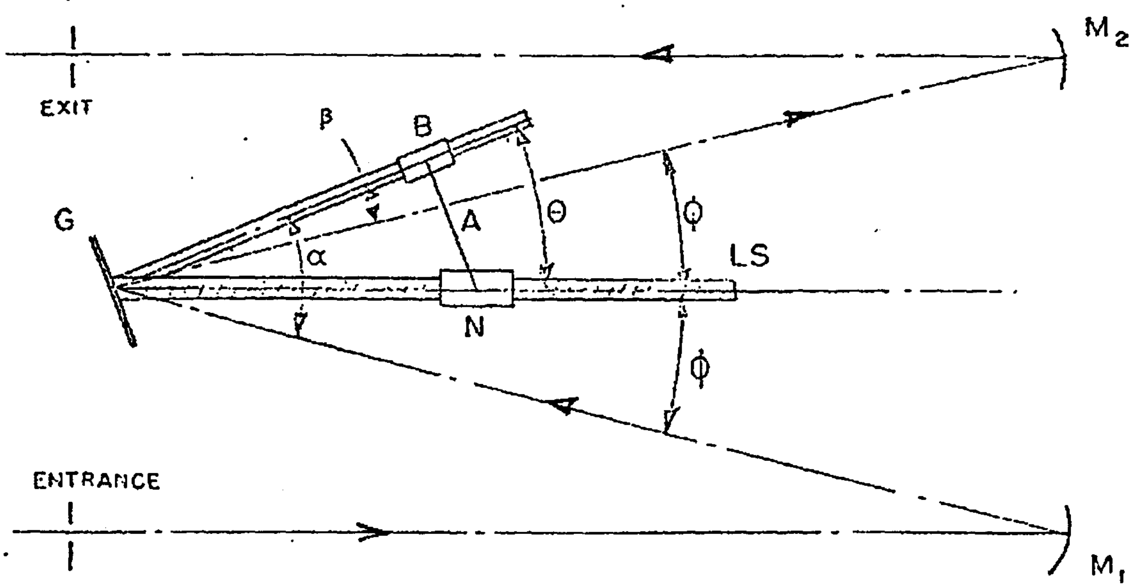
b) Derive for finite N the irradiance (interference) pattern expected in the *Fraunhofer limit* for N slits of width a and spacing d. Sketch a reasonably labeled graph of the expected irradiance pattern for rather large but finite N.

c) Apply the general grating formula to the "monochromator" described in the diagram below (note that the grating works by reflection instead of transmission in this case), and thus derive the grating formula applicable to the actual geometry and angles of this system:

$$m\lambda = d\left(2\sin\theta\cos\phi\right).$$

Note: The arm BG is normal to the surface of the grating.

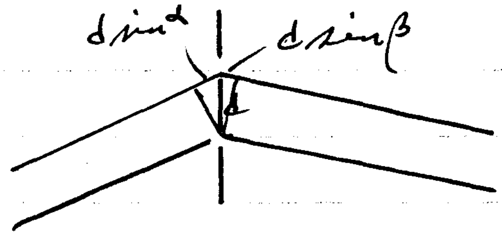
ϕ = 6° 15' and cos ϕ = 0.9942
θ = grating rotation measured from zero, its position at the direct image



G = Grating
A = Arm
N = Nut
LS = Leadscrew
B = Slide
M_1 & M_2 = Mirrors

Optics Problem - Cone 1996

(a)



sign depends on "sense" of β

path difference $= d\sin\alpha + d\sin\beta$
$= m\lambda$

(b) path difference above converts into phase difference δ

$$\delta = k\left[y(\sin\alpha+\sin\beta)\right]$$ where y is distance along grating

$$E = E_0 \int_0^a e^{i\delta}\,dy + \int_d^{d+a} e^{i\delta}\,dy + \int_{2d}^{2d+a} e^{i\delta}\,dy$$
$$+ \dots \int_{(N-1)d}^{(N-1)d+a} e^{i\delta}\,dy$$ N slits

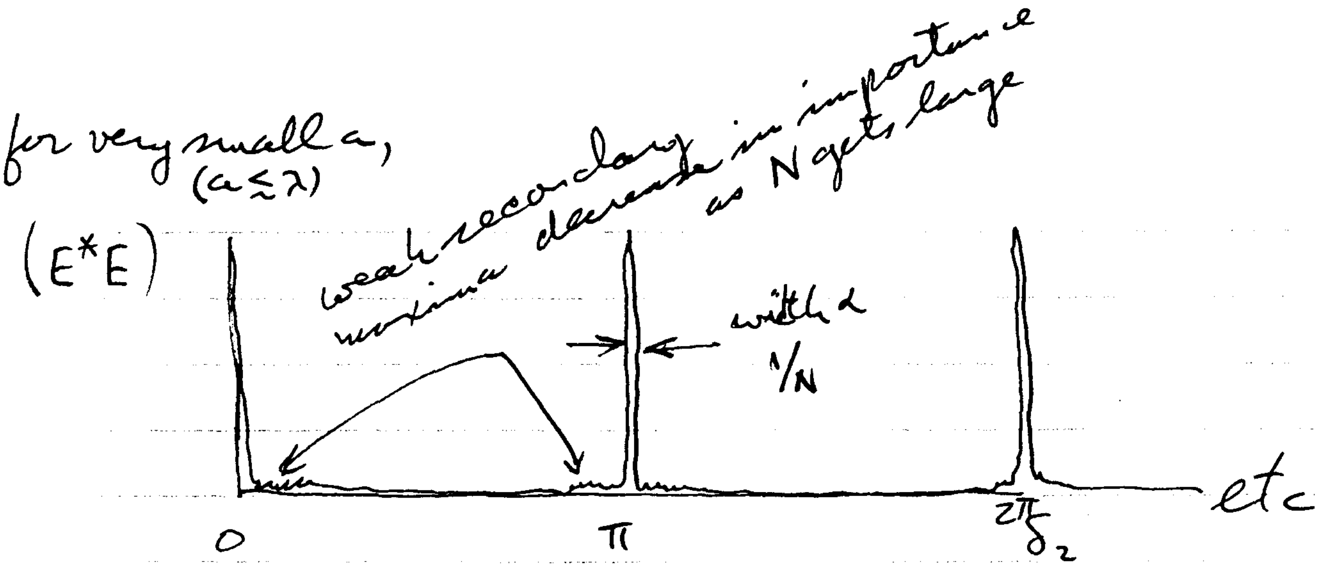
$$= E_0 \frac{e^{i\cancel{k}\delta_1}-1}{ik(\cancel{a}\sin\alpha+\sin\beta)}\left[1 + e^{ikd(\sin\alpha+\sin\beta)} + \dots e^{i(N-1)kd(\sin\alpha+\sin\beta)}\right]$$

$$= E_0 \frac{e^{i\cancel{k}\delta_1}-1}{ik(\sin\alpha+\sin\beta)} \; \frac{1-e^{iN\cancel{k}\delta_2}}{1-e^{i\cancel{k}\delta_2}}$$

$$\begin{cases} \delta_1 \equiv ka(\sin\alpha+\sin\beta) \\ \delta_2 \equiv kd(\sin\alpha+\sin\beta) \end{cases}$$

$$\boxed{E^*E = |E_0|^2 a^2 \left(\frac{\sin(\delta_1/2)}{(\delta_1/2)}\right)^2 \left(\frac{\sin(N\cdot\delta_2/2)}{\sin(\delta_2/2)}\right)^2}$$

for very small a, $(a \lesssim \lambda)$



as a becomes larger a "single slit diffraction pattern" envelope modulates the relative intensities of the peaks shown above.

(c) $m\lambda = d(\sin\alpha + \sin\beta) = d\left\{2\sin\left[\tfrac{1}{2}(\alpha+\beta)\right]\cos\left[\tfrac{1}{2}(\alpha-\beta)\right]\right.$

$$\begin{cases} \alpha - \beta = 2\phi \\ \alpha = 2\phi + \beta, \quad \text{so } \alpha + \beta = 2\phi + 2\beta = 2\theta \end{cases}$$

Hence $\boxed{m\lambda = d\,(2\sin\theta\cos\phi)}$

4. The parity operator $\hat{P}$ is defined such that
$$\hat{P}F(\mathbf{r}) = F(-\mathbf{r})$$
for any operator or function F.

a) Prove that for spherically symmetric potentials, $\hat{P}$ commutes with the Hamiltonian, i.e., $[\hat{P},\hat{H}] = 0$. What, therefore, may we conclude about the geometry of the energy eigenfunctions? Note that in spherical polar coordinates the Hamiltonian may be written:
$$\hat{H} = \frac{\hbar^2}{2\mu}\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r} + \frac{\hat{L}^2}{2\mu r^2} + V(r)$$
where the orbital angular momentum operator $\hat{L}$ is given by:
$$\hat{L}^2 = -\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right].$$

b) Now, consider perturbation of the energy levels of a hydrogen atom placed in a uniform electric field $\mathbf{E}$ (Stark effect). The interaction Hamiltonian may be written:
$$\hat{H}^1 = -\mathbf{E}\cdot\hat{\mathbf{d}}$$
where $\hat{\mathbf{d}}$ is the dipole moment operator. Using the result of (a), give a clear physical explanation as to why the $|100\rangle$ state shows its lowest order perturbation in second order whereas the $|2lm\rangle$ states show first order perturbations.

c) Using first order degenerate perturbation theory, calculate the energy shifts of the $n = 2$ states of hydrogen in a uniform electric field $\mathbf{E} = E\hat{\mathbf{z}}$. To do so, calculate the matrix $\langle 2lm|\hat{H}^1|2l'm'\rangle$ (symmetry arguments will save you time) and find its eigenvalues.

You may wish to use the hydrogen wave functions:

$$U_{nlm} = R_{nl}(r)Y_{lm}(\theta\phi)$$

where

$$R_{20} = \frac{1}{(2a_o)^{\frac{3}{2}}}\left(2 - \frac{r}{a_o}\right)e^{-r/2a_o},$$

$$R_{21} = \frac{1}{(2a_o)^{\frac{3}{2}}}\frac{r}{a_o\sqrt{3}}e^{-r/2a_o},$$

$$Y_{00} = \frac{1}{\sqrt{4\pi}},$$

and

$$Y_{10} = \sqrt{\frac{3}{4\pi}}\cos\theta,$$

$$Y_{1\pm1} = \mp\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{\pm i\phi}.$$

Solution

a) The action of the parity operator may be written in spherical coordinates: $\hat{P}F(r,\theta,\phi) = F(r,\pi-\theta,\pi+\phi)$. So,

$$\begin{aligned}
\hat{P}\hat{L}^2\Psi &= -\hbar^2\left[\frac{1}{\sin(\pi-\theta)}\left(-\frac{\partial}{\partial\theta}\right)\sin(\pi-\theta)\left(-\frac{\partial}{\partial\theta}\right)+\frac{1}{\sin^2(\pi-\theta)}\frac{\partial^2}{\partial\phi^2}\right]\Psi(r,\pi-\theta,\pi+\phi)\\
&= -\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]\hat{P}\Psi\\
&= \hat{L}^2\hat{P}\Psi
\end{aligned}$$

therefore $[\hat{P},\hat{L}^2]=0$. Clearly, $[\hat{P},F(r)]=0$ for any function of r including the spherically symmetric potential and the radial part of the Laplacian operator, therefore $[\hat{P},\hat{H}]=0$. We can conclude that there exist simultaneous eigenfunctions of the energy and parity operators.

b) The $|100>$ state of hydrogen has even parity and therefore has no dipole moment. There is no energy perturbation to first order, but there is a perturbation of the wave function caused by the external electric field. The energy perturbation in second order results from the dipole moment of the wave function induced in first order. The $|21m>$ states have odd parity and thereore have a dipole moment which couples to the field causing an energy shift in first order. $|200>$ participates in this shift even though it is symmetric because it is degenerate with $|21m>$ and mixes with these states under the perturbation.

c) We require the matrix elements $< 2lm|\hat{H}^1|2l'm' >= -eE < 2lm|z|2l'm' >$. Since z has ǒdd parity, and Y_{lm} have parity -1^l, we conclude that these matrix elements are 0 unless $l+l'$ is odd. In addition, the azimuthal integration kills the matrix elements unless $m-m'=0$. The only nonzero matrix elements are $eE < 210|z|200 >$ and its complex conjugate. Since $z = r\cos\theta = \sqrt{\frac{4\pi}{3}}rY_{10}$ we find:

$$\begin{aligned}
-eE < 210|z|200 > &= \frac{-eE}{\sqrt{3}}\int R_{21}R_{20}r^3\,dr\\
&= \frac{-eE}{24a_o^4}\int\left(2r^4-\frac{r^5}{a_0}\right)e^{-r/a_o}\,dr\\
&= \frac{-eE}{24a_o^4}\left(48a_0^5-120a_o^5\right)\\
&= 3a_oeE
\end{aligned}$$

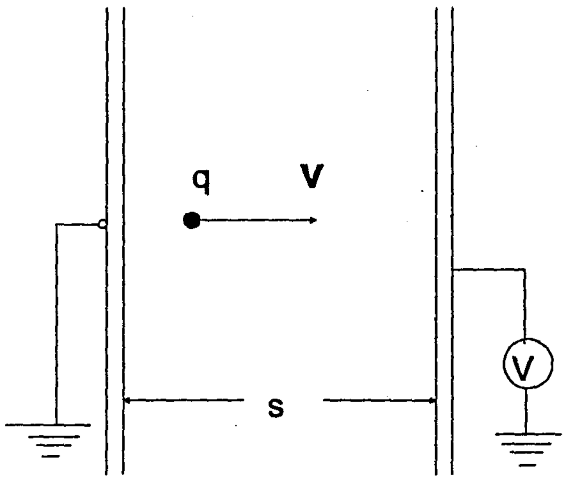
Diagonalizing this 4×4 matrix with only 2 nonzero elements is trivial giving energy shifts $\delta = \pm3a_oeE$.

5. A capacitor (with an electron source at one plate) is placed in a vacuum. The capacitor plate separation s can be adjusted from 0.1 cm to a few cm. To accelerate an electron across the gap, you have available a dc voltage supply with a maximum output of $V_m = 100$ kV. You also have some fast electronics, capable of resolving electron pulse arrival times to ~ 0.1 ns.

   a) If the electron starts from the left-hand plate at time $t = 0$ with velocity $v = 0$, find relativistic forms for $v(t)$ and the distance traveled at time t. [Give your answers in terms of the lab time t, the voltage V, the plate separation s, the electron mass m, the electronic charge q, and the velocity of light c.]

Hint: The relativistic equation of motion is: $d(\gamma mv)/dt = qE =$ constant.

   b) By adjusting V and s, find the maximum value of $\beta = v/c$ possible for this experiment. Find the numerical value for β_{max} for electrons. Is it relativistic?



Comp: Relativity

~22 min

Answer:

(a) In transit between the plates, the charge has the relativistic equation of motion $d(\gamma m v)/dt = qE = \text{const}$ ① ~11 min.

$\gamma = 1/\sqrt{1-\beta^2}$; $\beta = v/c$

∘ The field $E = V/S$ ③.

∘ ① $\Rightarrow d(\beta/\sqrt{1-\beta^2})/dt = qE/(mc) \equiv \Omega$ ②

Define Ω by ②.

The soln is for release from the left-hand plate at time $t=0$ when $v=0$. So the velocity v & distance D traveled at t are:

Solving ②, $\beta = \Omega t\sqrt{1-\beta^2} \rightarrow \beta^2 = (\Omega t)^2(1-\beta^2)$

$\beta^2[1+(\Omega t)^2] = (\Omega t)^2 \rightarrow \beta(t) = \Omega t/\sqrt{1+(\Omega t)^2} = v/c$

$$\boxed{\begin{array}{l} \rightarrow v = c\Omega t/\sqrt{1+(\Omega t)^2} \quad ④ \\ \text{where, from ② ③} \Rightarrow \Omega \equiv qE/mc = qV/(mcS) \quad ④' \end{array}}$$ Ans. for v.

3

∘ $dx/dt = v = c\beta \Rightarrow D(t) = \int_0^D dx = \int_0^t c\beta(t')dt'$ ⑥

⑥ $\Rightarrow D(t) = \int_0^t c[\Omega t'/\sqrt{1+(\Omega t')^2}]dt' = (c/\Omega)(\sqrt{1+(\Omega t)^2} - 1]$ ⑦

So, $\boxed{D(t) = \frac{c}{\Omega}\left[\sqrt{1+(\Omega t)^2} - 1\right]}$ ⑦ Ans for D with Ω as defined in ④'.

~11 min (b) With $\Omega t = \beta/\sqrt{1-\beta^2}$ from ②, D of eqn ⑦ can be rewritten as

$D(\beta) = (c/\Omega)[\sqrt{1+(\Omega t)^2} - 1] = (c/\Omega)[\sqrt{1+\beta^2/(1-\beta^2)} - 1]$

$= (c/\Omega)[\sqrt{1-\beta^2+\beta^2/(1-\beta^2)} - 1] = (c/\Omega)\left(\frac{1}{\sqrt{1-\beta^2}} - 1\right) = \frac{c}{\Omega}(\gamma - 1)$ ⑧

where $c/\Omega = Smc^2/qV$. ⑨ from ⑤ & ③.

3 Maximum β is when $D_m(\beta_m) = S = (c/\Omega)(\gamma_m - 1) = \frac{Smc^2}{qV}(\gamma_m - 1)$ ⑩

$\rightarrow$ So $\gamma_m - 1 = qV/mc^2 \rightarrow \gamma_m = 1 + qV/(mc^2) = \frac{1}{\sqrt{1-\beta_m}}$ ⑪

$\rightarrow \beta_m = \sqrt{1 - 1/\gamma_m^2}$ ⑫ with γ_m given by ⑪.

$\therefore$ $\boxed{\beta_{max} = \sqrt{1 - \frac{1}{\gamma_m^2}} \text{ when } \gamma_m = 1 + \frac{qV}{mc^2}}$ Ans

∘ For electrons, $mc^2 = 511$ keV & $V_M = 100$ kV. So

$qV_M = eV_M = 100$ KeV. $\Rightarrow qV/mc^2 = 100/511 = 0.1957$.

$\rightarrow \gamma_m = 1.1957 \rightarrow \beta_m = \sqrt{1-(1/\gamma_m^2)} = \boxed{0.548}$ Ans.

So $\boxed{\text{mildly Relativistic!}}$ Ans

6. Assume that you are studying motion on a large, horizontal merry-go-round that is rotating at a constant angular velocity $\omega = \omega\hat{\mathbf{z}}$, where $\omega = 1$ rad/s. There is no friction between a hockey puck (mass = m) and the surface of the merry-go-round. At $t = 0$, the fixed and the rotating coordinate systems coincide and the puck is released from rest in the rotating system at the coordinates (r_o, 0, 0), where $r_o = 1$ m. We are interested in the position of the puck at the time $t_f = \pi/10$ s.

a) The general equation for motion in a rotating coordinate system is

$$\mathbf{F}_{eff} = m\mathbf{a}_r = \mathbf{F} - m\ddot{\mathbf{R}}_f - m\dot{\omega}\times\mathbf{r}_r - m\omega\times(\omega\times\mathbf{r}_r) - 2m\omega\times\mathbf{v}_r$$

where **R** is the position of the origin of the rotating system measured in the fixed system, **r** and **v** are the position and velocity of the puck, and the subscripts f and r refer to the fixed and rotating coordinate systems, respectively.

Show that the equations of motion for the puck in the rotating system are given by

$$\ddot{x} = \omega^2 x + 2\omega\dot{y}$$
$$\ddot{y} = \omega^2 y - 2\omega\dot{x}$$

[*Note:* Parts b and c of this problem are independent of each other and can be worked in either order.]

b) Under the assumption that the puck stays close to the x-axis in the moving system, find an approximate equation for the puck's position along the x-axis and its location at time $t = t_f$.

Use this result to find the puck's approximate deflection in the y-direction.

c) Let's now look at the same motion from the laboratory (non-rotating) frame. Describe the motion in words.

Find the coordinates of the puck at time $t = t_f$ measured relative to the laboratory axes.

Transform these coordinates to the rotating frame.

d) Are your answers to parts b and c consistent?

Solution

a) From the description of the problem $\vec{F}=0$, $\vec{R}=0$, and $\dot{\vec{\omega}}=0$

$$\Rightarrow \quad \vec{a}_r = -\vec{\omega}\times(\vec{\omega}\times\vec{r}_r) - 2\vec{\omega}\times\vec{v}_r$$

$$\vec{\omega} = \omega\hat{z}_r \qquad \vec{v}_r = \dot{x}\hat{x}_r + \dot{y}\hat{y}_r \qquad \vec{r}_r = x\hat{x}_r + y\hat{y}_r$$

$$\vec{\omega}\times\vec{v}_r = \omega\dot{x}\hat{y}_r - \omega\dot{y}\hat{x}_r$$

$$\vec{\omega}\times\vec{r}_r = \omega x\hat{y}_r - \omega y\hat{x}_r$$

$$\vec{\omega}\times(\vec{\omega}\times\vec{r}_r) = -\omega^2 x\,\hat{x}_r - \omega^2 y\,\hat{y}_r$$

$$\Rightarrow \quad \ddot{x} = \omega^2 x + 2\omega\dot{y} \qquad (1)$$

$$\ddot{y} = \omega^2 y - 2\omega\dot{x} \qquad (2)$$

b) In equation (1), ignore the term $2\omega\dot{y}$ as the puck stays close to the x-axis

$$\Rightarrow \ddot{x} = \omega^2 x$$

$$x = Ae^{\omega t} + Be^{-\omega t}$$

$$\dot{x} = \omega Ae^{\omega t} - \omega Be^{-\omega t}$$

initial conditions

$$r_0 = A + B$$

$$0 = A - B$$

$$\Rightarrow A = B = \tfrac{1}{2}r_0$$

$$x = \tfrac{1}{2}r_0\left[e^{\omega t} + e^{-\omega t}\right] = r_0\cosh\omega t$$

$$x\left(t = \tfrac{\pi}{10}\,\mathrm{s}\right) = 1.0498\ \mathrm{m}$$

In equation (2), ignore the term $\omega^2 y$

$$\Rightarrow \ddot{y} = -2\omega\dot{x}$$

$$= -r_0\,\omega^2\left(e^{\omega t} - e^{-\omega t}\right)$$

$$\dot{y} = -r_0\,\omega\left(e^{\omega t} + e^{-\omega t}\right) + C$$

b cont)

Since $\dot{y}(t=0)=0$, $c = 2r_0\omega$

and $\dot{y} = -r_0\omega\,(e^{\omega t} + e^{-\omega t} - 2)$

$$y = -r_0\,(e^{\omega t} - e^{-\omega t} - 2\omega t) + c' \quad \text{w/ } c'=0$$

$$y = 2r_0\,(\omega t - \sinh \omega t)$$

$$y(t = \pi/10\,s) = -0.0104\ m$$

c) In the lab frame, the initial velocity is $\omega r_0 \hat{y}_f$ which remains constant since there is no net force.

$$x_f = r_0 = 1\,m$$

$$y_f = \omega r_0 t_f = r_0\theta = \tfrac{\pi}{10}\ m$$

Now transform to the rotating frame

$$\begin{pmatrix} x_m \\ y_m \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} r_0 \\ r_0\theta \end{pmatrix}$$

$$x_m = r_0\,(\cos\theta + \theta\sin\theta) = 1.0499\ m$$

$$y_m = \underbrace{r_0\,(-\sin\theta + \theta\cos\theta)}_{\text{note that these are exact expressions}} = -0.0102\ m$$

d) yes

7. A linearly polarized, transverse electromagnetic wave

$$\mathbf{E} = \mathbf{E}_o e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}$$
$$\mathbf{B} = \mathbf{B}_o e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}$$

propagates in a uniform, non-magnetic conductor with dielectric constant ε and conductivity σ. Assume that ε, σ, and ω are positive, real constants and that $\frac{4\pi\sigma}{\varepsilon\omega} >> 1$. Determine the real and imaginary parts of the wavevector k, and interpret your answer in physical terms.

JH3. Solution

Begin with Maxwell's equations for this system, using $\vec{J} = \sigma \vec{E}$ and $\rho = 0$:

$$\nabla \cdot E = 0 \qquad \nabla \cdot B = 0$$

$$\nabla \times E + \frac{1}{c}\frac{\partial B}{\partial t} = 0 \qquad \nabla \times B - \frac{\epsilon}{c}\frac{\partial E}{\partial t} - \frac{4\pi\sigma}{c} E = 0$$

Using $\nabla \cdot E = ik \cdot E$ and $\nabla \times E = ik \times E$

$\nabla \cdot B = ik \cdot B$ $\qquad \nabla \times B = ik \times B$

We have

$$ik \times E - i\frac{\omega}{c} B = 0 \longrightarrow B = \frac{c}{\omega} k \times E$$

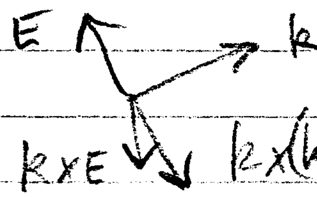
$$ik \times B + i\frac{\epsilon\omega}{c} E - \frac{4\pi\sigma}{c} E = 0$$

$$k \times B + \frac{\epsilon\omega}{c} E + i\frac{4\pi\sigma}{c} E = 0$$

Also, $E, B \perp k$

$$\frac{c}{\omega} k \times (k \times E) + \frac{\epsilon\omega}{c} E + i\frac{4\pi\sigma}{c} E = 0$$

Draw the vectors:



$k \times (k \times E) = -k^2 E$

Then

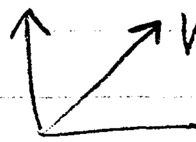
$$\frac{c}{\omega}\left[-k^2 + \frac{\epsilon\omega^2}{c^2} + i\frac{4\pi\sigma\omega}{c^2}\right] E = 0$$

$$\left[k^2 - \left(\frac{\epsilon\omega^2}{c^2} + i\frac{4\pi\sigma\omega}{c^2}\right)\right] E = 0$$

For a nontrivial solution $E \neq 0$,

$$\underline{k^2 = \frac{\omega^2}{c^2}\left(\epsilon + i\frac{4\pi\sigma}{\omega}\right)}$$

Given that $\frac{4\pi\sigma}{\epsilon\omega} \gg 1$ "Good conductor"

i  $\sqrt{i} = \frac{1+i}{\sqrt{2}}$

$$k = \frac{\omega}{c}\sqrt{i\,\frac{4\pi\sigma}{\omega}}$$

$$= \frac{\omega}{c}\sqrt{\frac{4\pi\sigma}{\omega}}\,\frac{1+i}{\sqrt{2}}$$

$$= \boxed{\sqrt{\frac{2\pi\sigma\omega}{c^2}}\,(1+i) = k_1 + i k_2}$$

Interpretation

$$k_1^{-1} = \frac{c}{\sqrt{2\pi\sigma\omega}} = \frac{\lambda}{2\pi} = \text{wavelength}/2\pi$$

$$k_2^{-1} = \frac{c}{\sqrt{2\pi\sigma\omega}} = \text{penetration depth at a free surface}$$

8. Two spin ½ particles are coupled by an interaction J in such a manner that the Hamiltonian can be written $H = -J\, \mathbf{S}_1 \cdot \mathbf{S}_2$.

   a) Calculate the energy levels for this dimer (two-spin) system. [There are several ways to do this, but an easy one is to define a vector $\mathbf{S} = \mathbf{S}_1 + \mathbf{S}_2$ and take the dot product of **S** with itself and change to the **S** representation.]

   b) Write the partition function Z.

   c) Assume a magnetic field **H** is applied and that each spin has a magnetic dipole moment $\mu\mathbf{S}$. Write the new partition function.

   d) The magnetization (per unit volume) can be written as $M = \frac{kT}{V}\frac{\partial}{\partial H}\ell n Z$. Calculate the magnetization.

Statistical Mehanics J. Drumheller

Spin Dimer

Two spin ½ particles are coupled by an interaction J, in such a manner that the Hamiltonian ~~is~~ can be written $\mathcal{H} = -J\vec{S}_1 \cdot \vec{S}_2$.

a. Calculate the energy levels for this dimer system. [There are several ways to this but an easy one is to define a vector $\vec{S} = \vec{S}_1 + \vec{S}_2$ ~~(and take it's own dot product)~~ (and take the dot product of S with itself) and change to the S representation.]

b. Write the partition function Z.

c. Assume a magnetic field H is applied and write the new partition function. ~~[illegible]~~

d. Magnetization (per unit volume) ~~is~~ (can be written as) ~~given by~~ $M = \frac{kT}{V}\frac{\partial}{\partial H}\ln Z$.

Calculate the magnetization.

Stat. Mech. sol'n Drumheller

Spin Dimer

$$\mathcal{H} = -J\,\vec{S}_1\cdot\vec{S}_2$$

Following the hint: $\vec{S} = \vec{S}_1 + \vec{S}_2$

$$\vec{S}_1\cdot\vec{S}_2 = \frac{1}{2}\left(S^2 - S_1^2 - S_2^2\right)$$

$$\therefore\quad E = -J\frac{1}{2}\Big(S(S+1) - S_1(S_1+1) - S_2(S_2+1)\Big)$$

a./
$$\begin{cases} E_{S=1} = -\frac{J}{2}\left(1(2) - \frac{1}{2}\left(\frac{3}{2}\right) - \frac{1}{2}\left(\frac{3}{2}\right)\right) = -\frac{J}{2}\left(2 - \frac{3}{2}\right) = -\frac{J}{4} & \text{(triply degenerate)} \\ E_{S=0} = -\frac{J}{2}\left(0 - \frac{3}{2}\right) = \frac{3J}{4} \end{cases}$$

2

b./
$$Z = 3\exp\frac{J}{4kT} + \exp\left(-\frac{3J}{4kT}\right) = e^{j/4}\left(3 + e^{-j}\right)$$

3

Since there are three states; one each for $M_S = 1, 0, -1$.

$$j = \frac{J}{kT}$$

c./ Add'l Hamiltonian term is $\mathcal{H}' = -\mu\vec{H}\cdot\vec{S}$

$$E_{\substack{S=1\\M_S=1}} = -\frac{J}{4} - \mu H \qquad E_{\substack{S=1\\M_S=0}} = -\frac{J}{4} \qquad E_{\substack{S=1\\M_S=-1}} = -\frac{J}{4} + \mu H$$

3

$$E_{S=0} = -\frac{3J}{4}$$

$$Z = \exp\left(\frac{J}{4kT} + \frac{\mu H}{kT}\right) + \exp\left(\frac{J}{4kT}\right) + \exp\left(\frac{J}{4kT} - \frac{\mu H}{kT}\right) + \exp\left(-\frac{3J}{4kT}\right)$$

$$= e^{j/4}\left(e^{h} + e^{-h} + 1 + e^{-j}\right) \qquad j = \frac{J}{kT}\quad h = \frac{\mu H}{kT}$$

d./
$$M = \frac{kT}{V}\frac{\partial}{\partial H}\ln Z = \frac{kT}{V}\left(\frac{kT}{\mu}\right)^{-1}\frac{\partial}{\partial h}\ln Z = \frac{\mu}{VZ}\left(e^{h} - e^{-h}\right)e^{j/4}$$

2

i/
$$= \frac{\mu}{V}\,\frac{e^{h} - e^{-h}}{e^{h} + e^{-h} + 1 + e^{-j}}$$

9. A hollow cube has conducting walls defined by six planes $x = 0$, $y = 0$, $z = 0$, and $x = a$, $y = a$, $z = a$. The walls $z = 0$ and $z = a$ are held at a constant potential V. The other four sides are at zero potential.

   Find the potential $\phi(x, y, z)$ at any point inside the cube.

~17 min.

MP (ST)

Answer:

(a) $\nabla^2\Phi = 0 \Rightarrow \frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial z^2} = 0$ ①

$\Phi = X(x)Y(y)Z(z)$ ②

Then; $X(x) \propto \sin\left(\frac{n\pi x}{a}\right)$; $Y(y) \propto \sin\left(\frac{m\pi y}{a}\right)$;
$Z(z) \propto e^{\pm\frac{\pi}{a}\sqrt{m^2+n^2}\,z}$ ③

& $\Phi = \sum_{nm} \left(A_1 e^{\frac{\pi}{a}\sqrt{m^2+n^2}\,z} + A_2 e^{-\frac{\pi}{a}\sqrt{m^2+n^2}\,z}\right) \sin\frac{n\pi x}{a} \sin\frac{m\pi y}{a}$ ④

From BC (boundary condn) at $z=0$, $\Phi = V \rightarrow$

$V = \sum_{nm} (A_1 + A_2) \sin\frac{n\pi x}{a} \sin\frac{m\pi y}{a}$ ⑤

BC at $z = a \Rightarrow \Phi = V \rightarrow$

$A_1 e^{\sqrt{m^2+n^2}\,\pi} + A_2 e^{-\sqrt{m^2+n^2}\,\pi} = \frac{4}{a^2} V \frac{a}{n\pi}\frac{a}{m\pi}\left[(-1)^n - 1\right]\left[(-1)^m - 1\right]$ ⑥

From ⑤ & ⑥, if n or m even $\rightarrow A_1 = A_2 = 0$.

So, with n & m all odd, we get

$$A_1 = \frac{16 V e^{-\sqrt{m^2+n^2}\,\pi}}{nm\pi^2\left[1 + e^{-\sqrt{m^2+n^2}\,\pi}\right]}; \quad \& \quad A_2 = \frac{16V}{nm\pi^2\left[1 + e^{-\sqrt{m^2+n^2}\,\pi}\right]} \quad ⑦$$

⑦ → ④ & get

$$\Phi = \sum_{n,m\ \text{odd}} \left(\frac{16V}{nm\pi^2\left[1 + e^{-\sqrt{m^2+n^2}\,\pi}\right]}\right)\left[e^{-\sqrt{m^2+n^2}\,\pi} e^{\sqrt{n^2+m^2}\,\frac{\pi}{a} z} + e^{-\sqrt{m^2+n^2}\,\frac{\pi}{a} z}\right] \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi y}{a}\right) \quad ⑧$$

Ans.

10. A particle of mass m interacts in one dimension with an attractive delta potential

$V(x) = -V_0\delta(x)$.

a) Prove that there is only one bound state, and determine the eigenfunction and eigenvalue of the bound state.

b) Determine the uncertainties in x and in p_x when the particle is in the bound state. Note that $\Delta x = \sqrt{<x^2> - <x>^2}$ and $\Delta p_x = \sqrt{<p_x^2> - <p_x>^2}$.

Grading policy:

(a) $\psi^{\pm}$ & $E = -\frac{\hbar^2}{2m}k^2$ (1)

$\psi^- = \psi^+$ @ $x=0$ (1)

$\frac{d\psi^+}{dx}\Big|_0 - \frac{d\psi^-}{dx}\Big|_0 \Rightarrow k = \frac{V_0 m}{\hbar^2}$ (2)

$A = \sqrt{k}$ (1)

(b) $\langle x \rangle = \langle p_x \rangle = 0$ (1)

$\Delta x = \frac{1}{\sqrt{2}}\frac{1}{k} = \frac{1}{\sqrt{2}}\frac{\hbar^2}{V_0 m}$ (2)

$\Delta p_x = \hbar k = \frac{V_0 m}{\hbar}$ (2)

$\llcorner \; \Delta x \Delta p_x \sim \hbar$

4. **Attractive delta potential:** A particle of mass m interacts in one dimension with an attractive delta potential

$$V(x) = -V_o\delta(x)$$

(a) Prove that there is only one bound state, and determine the eigenfunction and eigenvalue of the bound state.

(b) Determine the uncertainties in x and in p_x when the particle is in the bound state. Note that $\Delta x = \sqrt{<x^2> - <x>^2}$, and $\Delta p_x = \sqrt{<p_x^2> - <p_x>^2}$.

(a) $$H = \frac{p_x^2}{2m} - V_o\,\delta(x), \quad \Rightarrow \quad -\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} - V_o\,\delta(x)\,\psi(x) = E\,\psi(x)$$

Search solutions in two regions:

$x<0$: $$-\frac{\hbar^2}{2m}\frac{d^2\psi^-}{dx^2} = E\psi^- \Rightarrow \psi^- = A e^{kx}, \quad E(k) = -\frac{\hbar^2}{2m}k^2$$

similarly, for $x>0$ $$\Rightarrow \psi^+ = B e^{-kx}, \quad E = -\frac{\hbar^2}{2m}k^2$$

where A, B & k need to be determined

Continuity across $x=0$ yields $\psi^- = \psi^+ \Rightarrow A=B$

Integrating Schrödinger equation across $x=0$ yields:

$$\lim_{\epsilon\to 0} -\frac{\hbar^2}{2m}\left.\frac{d\psi}{dx}\right|_{-\epsilon}^{+\epsilon} - V_o\,\psi(0) = E\,\psi(0)\,2\epsilon = 0$$

$$\Rightarrow -\frac{\hbar^2}{2m}\left(\left.\frac{d\psi^+}{dx}\right|_{x=0} - \left.\frac{d\psi^-}{dx}\right|_{x=0}\right) = \frac{\hbar^2}{2m}2kA = V_o A \Rightarrow \boxed{k = \frac{V_o m}{\hbar^2}}$$

A is determined from normalization:

$$\int_{-\infty}^{+\infty}|\psi|^2dx = 1 \Rightarrow \int_{-\infty}^{0}|\psi^-|^2dx + \int_0^{\infty}|\psi^+|^2dx$$

$$|A|^2\left(\left.\frac{1}{2k}e^{2kx}\right|_{-\infty}^{0} - \left.\frac{1}{2k}e^{-2kx}\right|_0^{\infty}\right) = 1 \Rightarrow \boxed{A = \sqrt{k}}$$

(over)

Results:

$$\boxed{\begin{array}{l} \Psi = \begin{cases} \sqrt{k}\, e^{kx} & ; \; x<0 \\ \sqrt{k}\, e^{-kx} & ; \; x>0 \end{cases} \text{, and} \\[2ex] E = -\dfrac{\hbar^2 k^2}{2m} = -\dfrac{V_0^{\,2}}{2}\left(\dfrac{m}{\hbar^2}\right) \end{array}}$$

Since $k = \dfrac{V_0 m}{\hbar^2}$ is the only value for the bound state the eigenfunction & eigenvalue given above is the only solution.

(b) From symmetry $\langle x \rangle = \langle P_x \rangle = 0$

$$\langle x^2 \rangle = \langle \Psi | x^2 | \Psi \rangle = \int_{-\infty}^{+\infty} x^2 |\Psi|^2 dx = 2k \int_0^{\infty} x^2 e^{-2kx} dx$$

Use partial integration: $\int_a^b u\,dw = uw\Big|_a^b - \int_a^b w\,du$

$$\langle x^2 \rangle = -2k \int_0^{\infty} (-) \frac{2x}{2k} e^{-2kx} dx = -2 \frac{1}{-2k} \int_0^{\infty} e^{-2kx} dx = \frac{1}{2k^2}$$

$$\Rightarrow \boxed{\Delta x = \frac{1}{\sqrt{2}} \frac{1}{k}}$$

$\langle P_x^2 \rangle = 2m \langle H + V_0 \delta(x) \rangle$ (From schrödinger equation)

$$\langle P_x^2 \rangle = 2m\left(\langle H \rangle + V_0 \langle \delta(x) \rangle\right) = 2m\left(E + V_0 \underbrace{\int_{-\infty}^{+\infty} |\Psi|^2 \delta(x)\, dx}_{k}\right)$$

$$\langle P_x^2 \rangle = 2m\left(-\frac{\hbar^2 k^2}{2m} + \frac{\hbar^2 k^2}{m}\right) = \hbar^2 k^2$$

$$\Rightarrow \boxed{\Delta P_x = \hbar k}$$

Notice that $\boxed{\Delta x \, \Delta P_x = \dfrac{\hbar}{\sqrt{2}}}$ Consistent with the principle of uncertainty.

— 0 —

11. We wish to solve the differential equation

$$L\Psi(x) + \Phi(x) = 0 \tag{1}$$

$-1 \le x \le 1$, where L is the self adjoint differential operator,

$$L = \frac{d}{dx}\left[\left(1-x^2\right)\frac{d}{dx}\right]$$

$\Phi(x)$ is the 'source' function,

$$\Phi = 1 - 3x^2$$

and $\Psi(x)$ is nonsingular at $x = \pm 1$

10 a) First, find the solutions to the eigenfunction equation

$$LU_n(x) + \lambda_n U_n(x) = 0$$

What are the orthonormal functions U_n and eigenvalues λ_n? (You need not give a rigorous solution.)

5 b) Write $\Phi(x)$ as a sum over the U_n's

10 c) Now, solve equation 1 for $\Psi(x)$ using either an orthogonal expansion or the Green function method. Note that the Green function can be written:

$$G(x,\eta) = \sum_n \frac{U_n(x)U_n(\eta)}{\lambda_n}$$

Solutions

D. Caditz

11.

a) Gradshteyn & Ryzhik 8.820;

$$L\, P_n + n(n+1)\, P_n = 0 \qquad n = 0, 1, 2, \ldots$$

P_n are the Legendre Polynomials

note $\displaystyle\int_{-1}^{1} P_n(x) P_m(x)\, dx = \frac{2}{2n+1} \delta_{mn}$ (GR 7.221.1)

so, $$\boxed{U_n(x) = \sqrt{\frac{2n+1}{2}}\, P_n(x)}$$

b) $\phi(x) = 1 - 3x^2$

$= -2 P_2$ (GR 8.912)

$$\boxed{\phi(x) = -2\sqrt{\frac{2}{5}}\, U_2}$$

c) orthogonal expansion method:

$$\psi = \sum_n a_n U_n \qquad \phi = \sum_n b_n U_n$$

$$L\psi + \phi = 0 \quad \Rightarrow \quad \sum_n (-\lambda_n a_n + b_n) U_n = 0$$

$$\Rightarrow \quad a_n = \frac{b_n}{\lambda_n}$$

$$a_2 = \frac{1}{6}\left(-\frac{2\sqrt{2}}{\sqrt{5}}\right)$$

$$a_2 = -\frac{\sqrt{2}}{3\sqrt{5}}$$

$$\psi = -\frac{\sqrt{2}}{3\sqrt{5}} U_2$$

$$= -\frac{\sqrt{2}}{3\sqrt{5}} \cdot \frac{\sqrt{5}}{\sqrt{2}} \cdot \frac{1}{2}(3x^2-1)$$

$$\boxed{\psi(x) = \frac{1}{6}(1-3x^2)}$$

c) Green function method:

$$\psi(x) = \int_{-1}^{1} G(x,\eta)\, \phi(\eta)\, d\eta$$

$$= \sum_n \frac{U_n(x)}{\lambda_n} \int_{-1}^{1} U_n(\eta) \left(\frac{-2\sqrt{2}}{\sqrt{5}}\right) U_2(\eta)\, d\eta$$

$$= -\frac{2\sqrt{2}}{\sqrt{5}} \sum_n \frac{U_n(x)}{\lambda_n} \delta_{n2}$$

$$= -\frac{2\sqrt{2}}{\sqrt{5}} \frac{1}{6} U_2$$

$$\psi(x) = -\frac{\sqrt{2}}{3\sqrt{5}} U_2$$

$$\boxed{\psi(x) = \frac{1}{6}\left(1 - 3x^2\right)}$$

12. The chemical potential of an ideal gas with no internal degrees of freedom is given by

$$\mu = kT \ln(\rho/\rho_Q),$$

where $\rho = N/V$ is the number density and $\rho_Q = (MkT/2\pi\hbar^2)^{3/2}$ is the quantum concentration.

a) Determine the Helmholtz free energy F of the gas. Assume that the constant of integration is zero.

b) Determine the entropy S of the gas.

c) Prove that for an adiabatic expansion $T_i^{3/2} V_i = T_f^{3/2} V_f$.

Hint: $dF = -SdT - pdV + \mu dN$ and $\int_0^N \ln x \, dx = N \ln N - N$

Grading

(a)

$$\mu = \left(\frac{\partial F}{\partial N}\right)_{T,V} \quad \text{(1)}$$

$$F = \int_0^N \mu(N')\, dN' \quad \text{(2)}$$

$$F = kTN\left(\ln\frac{\rho}{\rho_Q} - 1\right) \quad \text{(1)}$$

(4)

(b)

$$S = -\left(\frac{\partial F}{\partial T}\right)_{V,N} \quad \text{(1)}$$

$$\ln \rho_Q = \frac{3}{2}\ln T + \cdots \quad \text{(1)}$$

$$S = kN\left(\ln\frac{\rho_Q}{\rho} + \frac{5}{2}\right. \quad \text{(1)}$$

(3)

(c)

$$S_i = S_f \quad \text{(1)}$$

$$\left(\frac{\rho_Q}{\rho}\right)_i = \left(\frac{\rho_Q}{\rho}\right)_f \quad \text{(1)}$$

$$T_i^{3/2} V_i = T_f^{3/2} V_f \quad \text{(1)}$$

(3)

(10)

July 3, 1996

Questions‡

1. **Ideal gas:** The chemical potential of an ideal gas with no internal degrees of freedom is given by $\mu = kT\ln(\rho/\rho_Q)$ where $\rho = N/V$ is the number density and $\rho_Q = (MkT/2\pi\hbar^2)^{3/2}$ is the quantum concentration.

(a) Determine the Helmholtz free energy, F, of the gas. Assume that the constant of integration is zero in determining F.

(b) Determine the entropy, S, of the gas.

(c) Prove that for an adiabatic expansion $T_i^{3/2}V_i = T_f^{3/2}V_f$.

(Hint: $dF = -SdT - pdV + \mu dN$ and $\int_0^N \ln x dx = N\ln N - N$)

(a) From the hint:

$$\mu = \left(\frac{\partial F}{\partial N}\right)_{T,V} \Rightarrow F = \int_0^N \mu(N')dN' + C(T,V) \text{, where } C(T,V) = 0$$

{ The reason for this is F is extensive hence $C(T,V) = V\beta(T)$. From the hint $p = -\left(\frac{\partial F}{\partial V}\right)_{T,N} = \underbrace{\frac{NkT}{V} - \beta(T)}_{\text{(see below)}^*} \Rightarrow \beta(T) = 0$ }

$$\Rightarrow F = kT\underbrace{\int_0^N \ln N' dN'}_{\text{From the hint: } kTN\ln N - kTN} - kTN\ln V - kTN\ln\rho_Q$$

$$F = kTN\left(\ln \underbrace{\frac{N}{V}}_{\rho} - \ln\rho_Q - 1\right) \Rightarrow \boxed{F = kTN\left(\ln\left(\frac{\rho}{\rho_Q}\right) - 1\right)}$$

(*) Note that $p = -\left(\frac{\partial F}{\partial V}\right)_{T,N} = +\frac{kTN}{V}$, which is the ideal gas law.

‡ *Please be clear and to the point.*

Ideal gas cont'd

(b) From the hint

$$S = -\left(\frac{\partial F}{\partial T}\right)_{V,N}$$

$$S = -kN\left(\ln\left(\frac{\rho}{\rho_Q}\right)-1\right) + kTN\,\frac{\partial}{\partial T}\ln\rho_Q$$

note that $\ln\rho_Q = \frac{3}{2}\ln T + \cdots$ (terms independent of T)

$$\frac{\partial}{\partial T}\ln\rho_Q = \frac{3}{2}\frac{1}{T}$$

$$\Rightarrow \quad S = kN\left(\ln\left(\frac{\rho_Q}{\rho}\right) + 1 + \frac{3}{2}\right)$$

$$\boxed{S = kN\left(\ln\left(\frac{\rho_Q}{\rho}\right) + \frac{5}{2}\right)}$$

(Sackur-Tedrode equation)

(c) For an adiabatic process $S_i = S_f$

Assuming N is not changing, which is the case here,

this means $\left(\frac{\rho_Q}{\rho}\right)_i = \left(\frac{\rho_Q}{\rho}\right)_f$, or using

$$\rho_Q \sim T^{3/2}, \quad \rho = \frac{N}{V} \quad \Rightarrow \quad \boxed{T_i^{3/2} V_i = T_f^{3/2} V_f}$$

— o —

13. The Hamiltonian for a one-dimensional simple harmonic oscillator of natural frequency ω can be written as

$$H = (a^\dagger a + \tfrac{1}{2})\, \hbar\omega,$$

where $a^\dagger a$ has eigenvalues 0, 1, 2, ⋯, ∞ and $aa^\dagger - a^\dagger a = 1$.

a) Derive relations for p and x in terms of a and $a^\dagger$.

b) Derive the matrix elements of a and $a^\dagger$ in the representation where $a^\dagger a$ is diagonal.

c) Given an initial state

$$|\varphi\rangle = \frac{1}{\sqrt{2}}(|1\rangle + |2\rangle),$$

where $|1\rangle$ and $|2\rangle$ are the first and second excited states of the oscillator, find the expectation value $\langle H\rangle$ of the Hamiltonian as a function of the time.

d) For the initial state given in (c), find $\cancel{\langle x|}$ as a function of the time.

OOPS, $\langle x \rangle$

JH1. The Hamiltonian ~~H~~ for a one-dimensional simple harmonic oscillator of natural frequency ω can be written as

$$H = \left(a^{\dagger}a + \tfrac{1}{2}\right)\hbar\omega$$

where $a^{\dagger}a$ has eigenvalues $0, 1, 2, \cdots, \infty$, and $a a^{\dagger} - a^{\dagger} a = 1$.

(a) Derive relations for p and x in terms ~~as functions~~ of a and $a^{\dagger}$.

(b) Derive the matrix elements of a and $a^{\dagger}$ in the representation where $a^{\dagger}a$ is diagonal.

(c) Given an initial state

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|1\rangle + |2\rangle\right),$$

where $|1\rangle$ and $|2\rangle$ are the first and second excited states of the oscillator, find the expectation value $\langle H \rangle$ of the Hamiltonian as a function of the time.

(d) For the initial state given in (c), find $\langle x \rangle$ as a function of the time

JH1 Solution

(a) Factor $\frac{p^2}{2m} + \frac{m\omega^2}{2}x^2 = \frac{m\omega^2}{2}\left(x^2 + \frac{p^2}{m^2\omega^2}\right)$

$$= \frac{m\omega^2}{2}\left(x - \frac{ip}{m\omega}\right)\left(x + \frac{ip}{m\omega}\right) + \underbrace{\frac{i\omega}{2}(px - xp)}_{\frac{i\omega}{2}(-i\hbar) = \frac{\hbar\omega}{2}}$$

[Require] $= \hbar\omega a^\dagger a + \frac{\hbar\omega}{2}$

Thus $\sqrt{\frac{m\omega^2}{2}}\left(x - \frac{ip}{m\omega}\right) = \sqrt{\hbar\omega}\, a^\dagger$
$\sqrt{\frac{m\omega^2}{2}}\left(x + \frac{ip}{m\omega}\right) = \sqrt{\hbar\omega}\, a$ } mutually conjugate, and $[a, a^\dagger] = 1$

or $\boxed{x = \sqrt{\frac{\hbar}{2m\omega}}\,(a^\dagger + a) \; ; \; p = i\sqrt{\frac{m\omega\hbar}{2}}\,(a^\dagger - a)}$

(b) Given $\langle n | a^\dagger a | n \rangle = n$

$$\langle n | a^\dagger | n-1 \rangle\langle n-1 | a | n \rangle = n$$

But $\langle n | a^\dagger | n-1 \rangle = \langle n-1 | a | n \rangle^*$; def'n of adjoint
Choosing the m.e.'s to be real and positive,

$$\langle n | a^\dagger | n-1 \rangle = \langle n-1 | a | n \rangle = \sqrt{n}$$

or
$$\boxed{\begin{aligned} \langle n-1 | a | n \rangle &= \sqrt{n} \\ \langle n+1 | a^\dagger | n \rangle &= \sqrt{n+1} \end{aligned}}$$

(c) Given an initial state $|\psi\rangle = \frac{1}{\sqrt{2}}\left(|1\rangle + |2\rangle\right)$,

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}}\left[|1\rangle e^{-i\omega_1 t} + |2\rangle e^{-i\omega_2 t}\right]$$

$$\langle H\rangle = \frac{1}{2}\Big[\langle 1|H|1\rangle + \langle 2|H|2\rangle$$

$$+ \underbrace{\cancel{\langle 1|H|2\rangle}}_{0} e^{i(\omega_1-\omega_2)t} + \underbrace{\cancel{\langle 2|H|1\rangle}}_{0} e^{i(\omega_2-\omega_1)t}\Big]$$

$$= \frac{1}{2}\left(\frac{3}{2}\hbar\omega + \frac{5}{2}\hbar\omega\right)$$

$$\boxed{\langle H\rangle = 2\hbar\omega}$$ constant in time

(d) $$\langle x\rangle = \frac{1}{2}\Big[\overbrace{\cancel{\langle 1|x|1\rangle}}^{0} + \overbrace{\cancel{\langle 2|x|2\rangle}}^{0}$$

$$+ \langle 1|x|2\rangle e^{i\omega_{12}t} + c.c.\Big]$$

$$\langle 1|x|2\rangle = \sqrt{\frac{\hbar}{2m\omega}}\langle 1|a^\dagger + a|2\rangle$$

$$= \sqrt{\frac{\hbar}{2m\omega}}\langle 1|a|2\rangle = \sqrt{\frac{\hbar}{2m\omega}}\sqrt{2}$$

$$= \sqrt{\frac{\hbar}{m\omega}}$$

$$\langle x\rangle = \frac{1}{2}\left[\sqrt{\frac{\hbar}{m\omega}}e^{i\omega_{12}t} + \sqrt{\frac{\hbar}{m\omega}}e^{-i\omega_{12}t}\right]$$

$$\boxed{\langle x\rangle = \sqrt{\frac{\hbar}{m\omega}}\cos\omega t}$$ $\omega = \omega_{12} = \omega_1 - \omega_2$

14.

a) Discuss briefly the experimental evidence for assigning to the electron an intrinsic spin of $\frac{1}{2}\hbar$.

b) Explain qualitatively the origin of fine structure in atomic spectra.

c) The Sodium D lines are produced by 3p → 3s transitions of the valence electron. In zero applied magnetic field there are two lines at 589.0 nm and at 589.6 nm. List the quantum numbers for each state, label them in standard atomic notation $^{2S+1}L_J$, and use vertical arrows to show the transitions between the states in a labeled energy level diagram. (The information in part d below may be helpful as you answer this part of the question.)

d) When a magnetic field of 1 Tesla (10,000 G) is applied, the line at 589.0 nm is split into six components with wavelengths 589.0 ± 0.005, ± 0.016, and ± 0.027 nm; the line at 589.6 nm is split into four components at 589.6 ± 0.011, and ± 0.022 nm.

Obtain an expression for the splitting of a single electron energy level of specified *s, l, j* in a weak magnetic field (the Lande g_j factor will be useful) and hence show that these data are consistent with an electron spin of $\frac{1}{2}\hbar$ and an electron g_s factor of approximately 2. The Bohr magneton $\mu_B = \beta = -9.27 \times 10^{24}$ J/T = - 0.4669 cm^{-1}/T.

Sketch a diagram showing the energy levels and their relevant quantum numbers in an applied field and showing the allowed transitions giving rise to the spectrum quoted above. Briefly explain why these transitions are allowed.

a) Stern-Gerlach or $\langle$Rabi$\rangle$ atomic beam experiments | some detail expected
optical spectroscopy (see attached) |

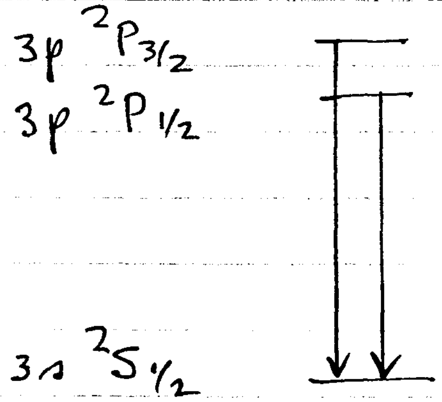
b) magnetic interaction of electron's "intrinsic" magnetic moment μ_B with the field arising from its "orbital" motion

current loop => field proportional to $\vec{l}$
magnetic moment $\propto \vec{s}$

energy $\propto \vec{l} \cdot \vec{s}$ $\left\{ \begin{array}{l} \text{"SPIN-ORBIT"} \\ \text{Interaction} \end{array} \right.$

c) 3p: $l=1$ $s=\frac{1}{2}$ $\left\{ \begin{array}{ll} j = 3/2 & \text{4 } m_j \text{ components} \\ j = 1/2 & \text{2 } m_j \text{ components} \end{array} \right.$

3s: $l=0$ $s=\frac{1}{2}$ $\left\{ j = 1/2 \right.$ 2 m_j components



d) $\mathcal{H}_{zeeman} = -\vec{\mu}\cdot\vec{B}$ $\qquad \vec{\mu} \equiv \mu_B\,(\vec{l} + g_s\vec{s})$

$\approx \mu_B\,(\vec{l} + 2\vec{s})$

put field in z dir

$$\mathcal{H}_z = -\mu_B(l_z + 2s_z)B_z = -\mu_B\, g_J\, J_z\, B_z$$

Lande g_J factor

calc g_J

$$\vec{l} + 2\vec{s} \doteq g_J\vec{j} \qquad \text{dot } j \text{ with both sides}$$

$$l^2 + \vec{s}\cdot\vec{l} + 2\vec{s}\cdot\vec{l} + 2s^2 = g_J j^2 \qquad \text{operators}$$

but

$$j^2 = l^2 + s^2 + 2\vec{l}\cdot\vec{s} \quad \text{so} \quad \vec{l}\cdot\vec{s} = \tfrac{1}{2}\left[j(j+1) - l(l+1) - s(s+1)\right]$$

and

$$g_J\left[j(j+1)\right] = l(l+1) + 2s(s+1) + 3\cdot\tfrac{1}{2}\left[j(j+1) - l(l+1) - s(s+1)\right]$$

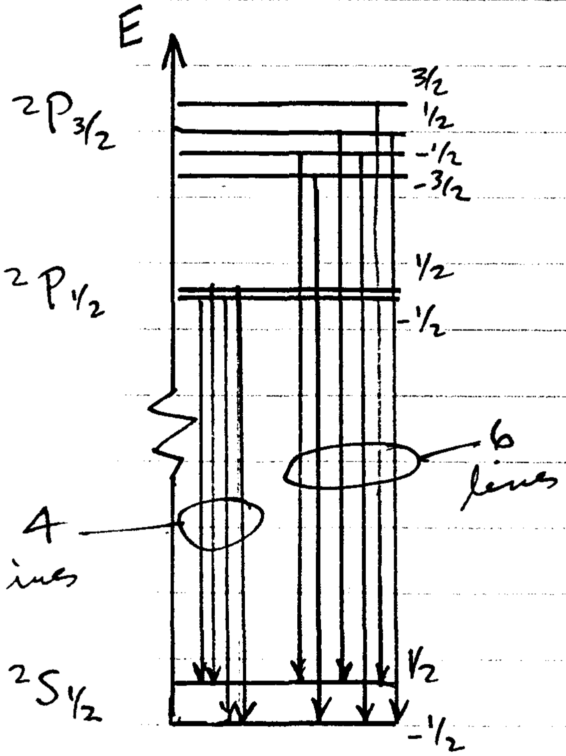
$$\boxed{g_J = 1 + \frac{j(j+1) - l(l+1) + s(s+1)}{2j(j+1)}}$$

${}^3P_{3/2}$ $\qquad g_J = 4/3$

${}^2P_{1/2}$ $\qquad g_J = 2/3$

${}^2S_{1/2}$ $\qquad g_J = 2$

state	m_j	$-\overset{-0.47}{\mu_B}\, g_J\, m_j\, \overset{1.0}{B}$	
$^2P_{3/2}$	$3/2$	0.94	cm^{-1}
	$1/2$	0.31	
	$-1/2$	-0.31	
	$-3/2$	-0.94	
$^2P_{1/2}$	$1/2$	0.16	cm^{-1}
	$-1/2$	-0.16	
$^2S_{1/2}$	$1/2$	0.47	
	$-1/2$	-0.47	



$\Delta m_j = \pm 1, 0$ (using both polarization)

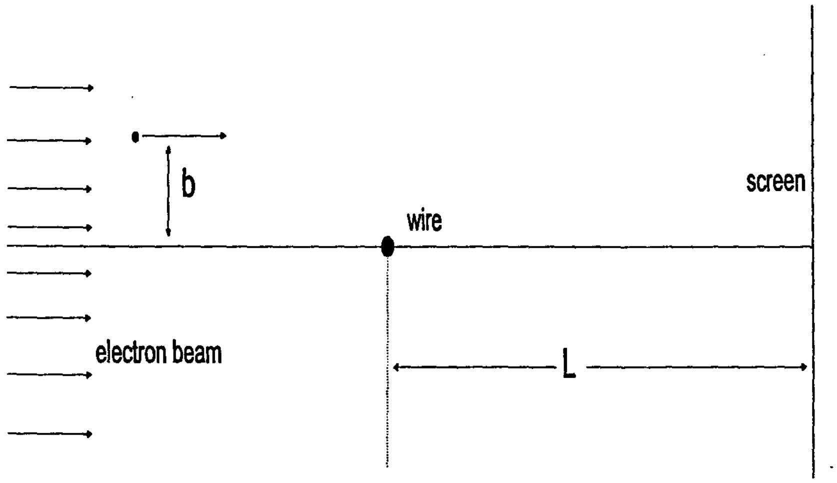
6 line group	$\pm 0.78\ cm^{-1}$	⇒ pattern OK
1:3:5	± 0.47	
	± 0.16	

4 line group	$\pm 0.63\ cm^{-1}$	⇒ pattern OK
1:2	± 0.31	

$$d\lambda = \frac{\lambda^2}{c}\, d\nu = (589.6 \times 10^{-9})^2 \, (0.31) \left(\frac{10^2\, cm}{m}\right)$$

$$\underline{d\lambda = 0.011\ nm} \quad \text{check}$$

15. An accelerating voltage V_o produces a uniform, parallel beam of electrons. The electrons pass a long, thin positively charged wire at right angles to the beam as shown in the figure. The impact parameter b is the distance an electron would pass the wire if the wire were uncharged. The electrons then proceed to a distant screen ($L >> b$). The beam initially extends to distances $\pm b_{max}$ with respect to the wire. Both the width of the beam and the length of the wire may be considered infinite in the direction perpendicular to the page. Neglect the mutual repulsion of the electrons in the beam.



Numerical data:

radius of the wire = $r_o = 10^{-6}$ m
maximum value of $b = b_{max} = 10^{-4}$ m
charge per unit length of wire = $\lambda = 4.4 \times 10^{-11}$ C/m
accelerating voltage = $V_o = 2 \times 10^4$ V
distance from wire to screen = $L = 0.3$ m

a) Calculate the electric field **E** produced by the wire.

b) Let θ_f denote the very small angle of deflection between the initial velocity of the electron and the velocity when the electron strikes the screen. Use classical physics to show that θ_f is independent of the impact parameter b and obtain its value.

Note: If you cannot complete part b, assume $\theta_f = 10^{-4}$ and continue with the problem.

c) Calculate and sketch the pattern of impacts (that is, the intensity distribution) on the screen that classical physics predicts.

d) Calculate and sketch the pattern of impacts on the screen that quantum physics predicts.

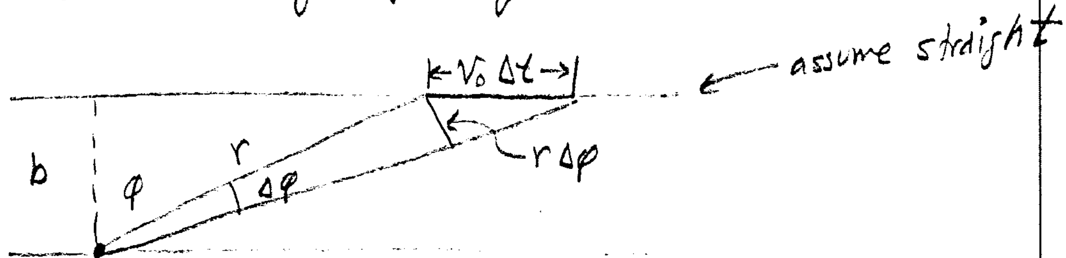
Solution

a) Use Gauss's law $\oint \vec{E} \cdot d\vec{A} = \frac{q_{enc}}{\varepsilon_0}$

$$2\pi r E = \frac{\lambda}{\varepsilon_0} \qquad (r \geq r_0)$$

$$\vec{E} = \frac{\lambda}{2\pi r \varepsilon_0}\hat{r} \qquad E = 0 \quad (r < r_0)$$

b) Find $\Delta p_\perp$ assuming θ_f very small



$$F_\perp = eE\cos\varphi = \frac{e\lambda}{2\pi\varepsilon_0 r}\cos\varphi$$

$$v_0 \Delta t \cos\varphi = r\Delta\varphi \Rightarrow \Delta t = \frac{r\Delta\varphi}{v_0\cos\varphi}$$

$$F_\perp \Delta t = \frac{e\lambda}{2\pi\varepsilon_0 r}\cos\varphi\,\frac{r\Delta\varphi}{v_0\cos\varphi} = \frac{e\lambda}{2\pi\varepsilon_0 v_0}\Delta\varphi$$

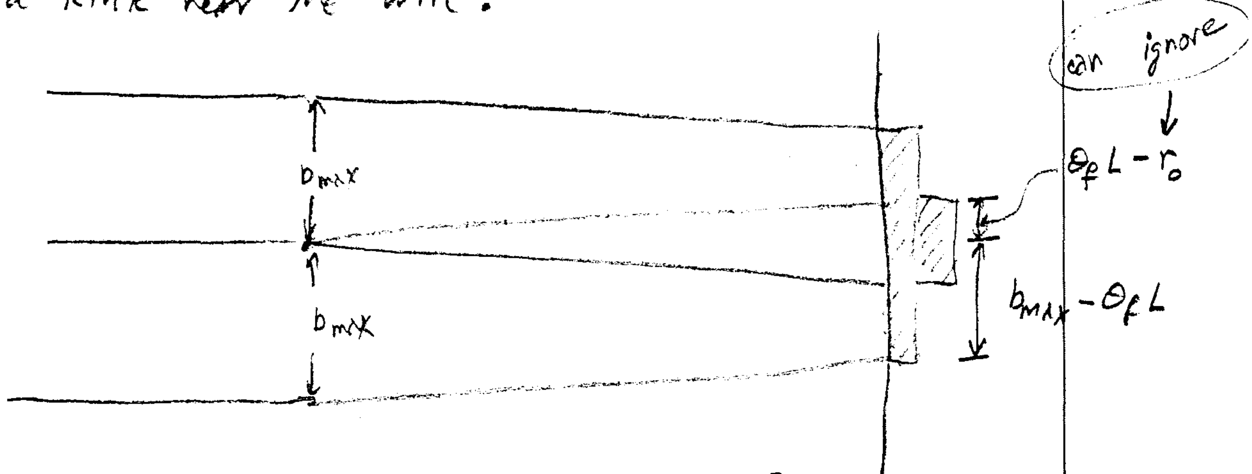
$$\int_{-\pi/2}^{\pi/2} F_\perp \Delta t = \frac{e\lambda}{2\varepsilon_0 v_0} = \Delta p_\perp \Rightarrow \text{independent of } b$$

$$\Delta p_\perp = p\sin\theta_f \cong m v_0 \theta_f$$

$$\theta_f = \frac{\Delta p_\perp}{m v_0} = \frac{e\lambda}{2\varepsilon_0 m v_0^2} = \frac{\lambda}{4\varepsilon_0 V_0} = 6.21 \times 10^{-5}$$

Note θ_f is very small in agreement w/ our assumption

c) Because most of the bending occurs near the wire, we approximate the trajectory by two straight lines with a kink near the wire.



transverse displacement $= \theta_f L = 1.86 \times 10^{-5}\,m \approx 19\, r_0$

$\Rightarrow \theta_f L - r_0 \approx 18\, r_0 \approx 1.8 \times 10^{-5}\,m$

$b_{max} - \theta_f L \approx 8.2 \times 10^{-5}\,m$

d) QM says electrons have a de Broglie wavelength

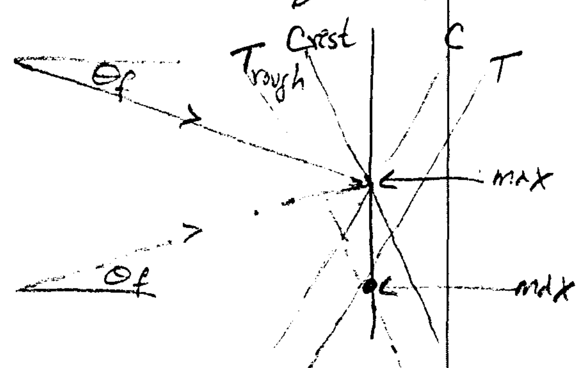
$$\lambda = \frac{h}{m v_0} = \frac{h}{\sqrt{2 m e V_0}} = 8.68 \times 10^{-12}\,m$$

To the right of the wire, we have the interference of two plane waves at a fixed relative angle of $2\theta_f$

$$\Delta y = \frac{\lambda/2}{\sin\theta_f} \approx \frac{\lambda/2}{\theta_f}$$

$$= \frac{\frac{1}{2} \times 8.68 \times 10^{-12}\,m}{6.21 \times 10^{-5}}$$

$$= 7.00 \times 10^{-8}\,m$$



$\Rightarrow$ Roughly 500 interference maxima will occur in overlap region (independent of b and b_{max}).