(26) With x & y the usual 2D cartesian coordinates, consider rotations in the xy plane, governed by the rotation matrix: $\underset{\sim}{R} = (R_{ik}) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$. Let: $\underset{\sim}{T} = \begin{pmatrix} -xy & -y^2 \\ x^2 & xy \end{pmatrix}$. Show that $\underset{\sim}{T}$ is an acceptable 2nd rank tensor, by virtue of transforming correctly under $\underset{\sim}{R}$, i.e. $T_{ij} \to T'_{ij} = R_{ik} R_{jl} T_{kl}$. Similarly, test: $\underset{\sim}{T}' = \begin{pmatrix} xy & y^2 \\ x^2 & -xy \end{pmatrix}$, as a tensor.

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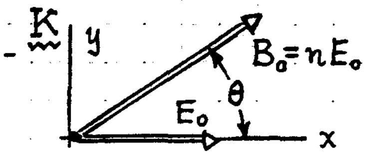
(27) [~ Jackson Prob. (11.13)]. In reference frame K, a uniform electric field E_0 is $\parallel$ x-axis, while a uniform magnetic field $B_0 = n E_0$ (n = some number > 0) lies in the xy plane. In K, $\mathbb{B}_0$ lies at a given $\measuredangle\,\theta$ relative to $\mathbb{E}_0$.



(A) Find the relative velocity β (magnitude & direction) of a moving frame K′ in which these fields appear to be parallel ($\theta' = 0$). Is β at all restricted by the size of n?

(B) Calculate the fields $\mathbb{E}'_0$ & $\mathbb{B}'_0$ in the K′ frame when $\theta \ll 1$, and when $\theta \to \frac{\pi}{2}$.

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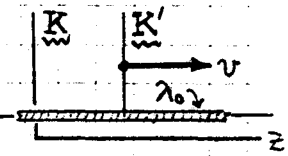
(28) [Jackson Prob. (11.11)]. A very long straight wire is at rest in inertial frame K′, where it has uniform charge/unit length λ_0. K′ (and the wire) move at velocity $\mathbb{v} \parallel$ wire down the z-axis of lab frame K. (A) Write down the $\mathbb{E}'$ & $\mathbb{B}'$ fields in cylindrical cds in K′. Using the Lorentz transforms for the fields, find $\mathbb{E}$ & $\mathbb{B}$ in the lab frame K. (B) What are the charge & current densities for the wire in K′? In K? (C) From the densities in K, calculate $\mathbb{E}$ & $\mathbb{B}$ in K directly. Compare with the $\mathbb{E}$ & $\mathbb{B}$ found in part (A). What do you conclude?



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(29) [[redacted] pts.]. For the free-space Maxwell Eqs. $\nabla \cdot \begin{Bmatrix} \mathbb{E} \\ \mathbb{B} \end{Bmatrix} = \begin{Bmatrix} 4\pi\rho \\ 0 \end{Bmatrix}$, $\nabla \times \begin{Bmatrix} \mathbb{E} \\ \mathbb{B} \end{Bmatrix} = \frac{1}{c}\begin{Bmatrix} (-)\dot{\mathbb{B}} \\ \dot{\mathbb{E}} + 4\pi\mathbb{J} \end{Bmatrix}$, let the fields & sources all be real, and define $\mathbb{M} = \mathbb{E} + i\,\mathbb{B}$. Condense the Maxwell Eqs. to just 2 eqs. involving $\nabla \cdot \mathbb{M}$ & $\nabla \times \mathbb{M}$. In turn, show that the Maxwell system ⇒ tensor divergence $\boxed{\partial_\alpha \mathcal{M}^{\alpha\beta} = (4\pi/c) J^\beta}$, w/ $\partial_\alpha = \left(\frac{\partial}{\partial x_0}, \nabla\right)$, $J^\beta = (c\rho, \mathbb{J})$, and:

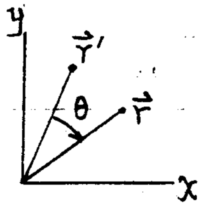
$$\mathcal{M}^{\alpha\beta} = \begin{pmatrix} 0 & -M_1 & -M_2 & -M_3 \\ M_1 & 0 & iM_3 & -iM_2 \\ M_2 & -iM_3 & 0 & iM_1 \\ M_3 & iM_2 & -iM_1 & 0 \end{pmatrix}.$$

Verify that the Maier field tensor $\mathcal{M}^{\alpha\beta}$ transforms properly for a Lorentz boost along the x_1-axis [[redacted]INT: see Jk^n Sec. (11.10)]. Finally, find the eigenvalues of the tensor $\mathcal{M}^{\alpha\beta}$.

[redacted]

11/17/84

(26) Test 2D matrix arrays for tensor character: $\underset{\sim}{T} = \begin{pmatrix} -xy & y^2 \\ x^2 & xy \end{pmatrix}$, $\underset{\sim}{T}' = \begin{pmatrix} xy & y^2 \\ x^2 & -xy \end{pmatrix}$.



The 2D rotation matrix is: $\underset{\sim}{R} = \begin{pmatrix} C & S \\ -S & C \end{pmatrix}$, where $\begin{cases} C = \cos\theta \\ S = \sin\theta \end{cases}$. Note that $\underset{\sim}{R}^{-1} = \underset{\sim}{R}^T = \begin{pmatrix} C & -S \\ S & C \end{pmatrix}$. For a proper tensor $\underset{\sim}{A}$, must have...

$$A_{ij} \to A'_{ij} = R_{ik} R_{jl} A_{kl} = R_{ik} A_{kl} R^T_{lj}, \text{ or: } \boxed{\underset{\sim}{A}' = \underset{\sim}{R}\,\underset{\sim}{A}\,\underset{\sim}{R}^T}$$

(a) Test: $\underset{\sim}{T} = \begin{pmatrix} -xy & -y^2 \\ x^2 & xy \end{pmatrix}$. Note that: $\begin{pmatrix} x' \\ y' \end{pmatrix} = \underset{\sim}{R} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} Cx + Sy \\ -Sx + Cy \end{pmatrix}$.

Check, $T' \overset{?}{=} \begin{pmatrix} C & S \\ -S & C \end{pmatrix} \begin{pmatrix} -xy & -y^2 \\ x^2 & xy \end{pmatrix} \begin{pmatrix} C & -S \\ S & C \end{pmatrix}$ ... do the matrix mult^n...

$$\begin{pmatrix} -x'y' & -y'^2 \\ x'^2 & x'y' \end{pmatrix} \overset{?}{=} \begin{pmatrix} (S^2 - C^2)xy + CS(x^2 - y^2) & 2CSxy - S^2x^2 - C^2y^2 \\ C^2x^2 + S^2y^2 + 2CSxy & -(S^2 - C^2)xy - CS(x^2 - y^2) \end{pmatrix}$$

But $\begin{matrix} x' = Cx + Sy \\ y' = -Sx + Cy \end{matrix} \Rightarrow -x'y' = (Cx + Sy)(Sx - Cy) = (S^2 - C^2)xy + CS(x^2 - y^2)$.

So T'_{11} transforms OK. Similarly: $T'_{21} = x'^2 = C^2x^2 + S^2y^2 + 2CSxy$, is OK.

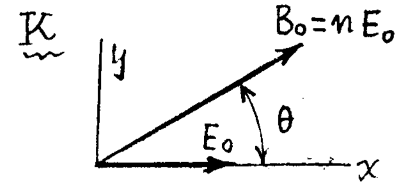
Finally: $T'_{12} = -T'_{21}$ Check out, and: $\underset{\sim}{T} = \begin{pmatrix} -xy & -y^2 \\ x^2 & xy \end{pmatrix}$, is an authentic tensor.

(b) Testing $\underset{\sim}{T} = \begin{pmatrix} xy & y^2 \\ x^2 & -xy \end{pmatrix} \to \underset{\sim}{T}' \overset{?}{=} \underset{\sim}{R}\,\underset{\sim}{T}\,\underset{\sim}{R}^T$, similarly, we calculate...

$$\begin{pmatrix} x'y' & y'^2 \\ x'^2 & -x'y' \end{pmatrix} \overset{?}{=} \begin{pmatrix} (C^2 - S^2)xy + CS(x^2 + y^2) & C^2y^2 - S^2x^2 - 2CSxy \\ C^2x^2 - S^2y^2 - 2CSxy & -(C^2 - S^2)xy - CS(x^2 + y^2) \end{pmatrix}$$

Re T'_{11}: $x'y' = (Cx + Sy)(Cy - Sx) = (C^2 - S^2)xy - CS(x^2 - y^2)$, so we do not have an identity as in part (a). The rest of the components of this $\underset{\sim}{T}'$ have similar problems, and $\underset{\sim}{T}' = \begin{pmatrix} xy & y^2 \\ x^2 & -xy \end{pmatrix}$ is not a qualified tensor. Note that the difference between tensor & non-tensor character is just a sign change in the 12 comp.

[redacted]



(27) Transform $\vec{E}$ & $\vec{B}$ fields into parallelism.

11/11/84

(a) Perusal of the field transf^n eqtns (11.148) $\Rightarrow$ $\vec{E}'$ & $\vec{B}'$ will be in the $x'y'$ plane if the Lorentz boost is along the z-axis: $\vec{\beta} = \beta\hat{z}$ [note: axes $(1,2,3) \leftrightarrow (z,x,y)$]. Then...

$$E_z' = E_z = 0,\quad E_x' = \gamma(E_x - \beta B_y) = \gamma E_0(1 - n\beta\sin\theta),\quad E_y' = \cdots = \gamma E_0 n\beta\cos\theta;$$

$$B_z' = B_z = 0,\quad B_x' = \gamma(B_x + \beta E_y) = \gamma E_0 n\beta\cos\theta,\quad B_y' = \cdots = \gamma E_0(n\sin\theta - \beta).$$

If the fields are to be parallel in K', then...

$$\frac{B_y'}{B_x'} = \frac{E_y'}{E_x'} \Rightarrow \frac{n\sin\theta - \beta}{n\beta\cos\theta} = \frac{n\beta\cos\theta}{1 - n\beta\sin\theta},\quad \text{or}/\!/\ (n\sin\theta)\beta^2 - (n^2+1)\beta + n\sin\theta = 0.$$

Solve this quadratic to get the req'd β. Choose $(-)\sqrt{\ }$ so $\beta \to 0$ as $\theta \to 0$...

$$\boxed{\beta = \frac{1}{2n\sin\theta}\left[(n^2+1) - \sqrt{(n^2+1)^2 - 4n^2\sin^2\theta}\,\right]}\ \Big/\ \text{Note: } n=1 \Rightarrow \beta = \tan(\theta/2).$$

For a possible β, the $\sqrt{\ }$ must be real, which $\Rightarrow (n^2+1) \geqslant 2n\sin\theta$. Subtract $2n$ from both sides to write this as: $(n-1)^2 > -2n(1-\sin\theta)$. Since this inequality is true for all n & θ, then $\left(\frac{n^2+1}{2n\sin\theta}\right) \geqslant 1$, and the $\sqrt{\ }$ is always real. Also, when $\tau = \left(\frac{n^2+1}{2n\sin\theta}\right) \geqslant 1$, it is easy to show $\beta = \tau - \sqrt{\tau^2 - 1}$, obeys $0 < \beta \leqslant 1$. Thus, <u>there is no restriction on β due to the size of n.</u>

(b) For the specific θ-values given, above formula for β allows calculating...

A. $\underline{\underline{\theta = \frac{\pi}{2} - \epsilon}},\ \epsilon \to 0 \Rightarrow \beta = \frac{1}{n} + O(\epsilon^2)$, and: $\gamma = n/\sqrt{n^2-1}$ (neglect $O(\epsilon^2)$)

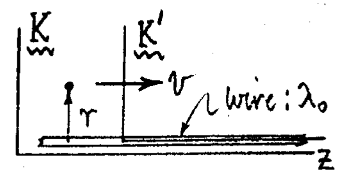
So// $$\underline{\underline{\vec{E}_0'}} \simeq \frac{nE_0}{\sqrt{n^2-1}}\Big[\underbrace{(1-\cos\epsilon)}_{0}\hat{x}' + \underbrace{(\sin\epsilon)}_{\epsilon}\hat{y}'\Big],\quad \underline{\underline{\vec{B}_0'}} \simeq \frac{n^2E_0}{\sqrt{n^2-1}}\Big[\underbrace{(\sin\epsilon)}_{\epsilon}\hat{x}' + \underbrace{\left(\cos\epsilon - \frac{1}{n^2}\right)}_{1-(1/n^2)}\hat{y}'\Big]$$

... neglect $O(\epsilon^2) \Rightarrow$

B. $\underline{\underline{\theta = \epsilon \ll 1}} \Rightarrow \beta \simeq n\epsilon/(n^2+1)$, to 1ST order in ϵ, and $\gamma \simeq 1$.

So// $$\underline{\underline{\vec{E}_0'}} \simeq E_0\left(\hat{x}' + \left(\frac{n^2\epsilon}{n^2+1}\right)\hat{y}'\right),\quad \underline{\underline{\vec{B}_0'}} \simeq nE_0\left(\hat{x}' + \left(\frac{n^2\epsilon}{n^2+1}\right)\hat{y}'\right),\ \text{to } 1^{ST} \text{ order in } \epsilon.$$

(28) Verify Lorentz transforms for $\vec{E}$ & $\vec{B}$ fields kinematically.



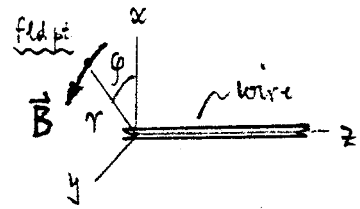
12/2/84

(a) In K', wire is at rest, so $\mathbb{B}' \equiv 0$, and: $\mathbb{E}' = \left(\frac{2\lambda_0}{r}\right)\hat{r}$, points radially outward (with: $r = \sqrt{x'^2 + y'^2}$, the radial distance) from the wire. These fields: $\mathbb{E}' = \frac{2\lambda_0}{r}(\cos\varphi, \sin\varphi, 0)$, $\mathbb{B}' = (0,0,0)$ transform into K via Jackson's Eqs. (11.148) [interchange primes & non-primes, set $\beta \to (-)\beta$], i.e. with $\gamma = 1/\sqrt{1-(v/c)^2}$:

$$\left.\begin{aligned} E_z &= 0, & B_z &= 0 \\ E_x &= \gamma(E_x' + \beta \cancelto{0}{B_y'}), & B_x &= \gamma(\cancelto{0}{B_x'} - \beta E_y') \\ E_y &= \gamma(E_y' - \beta \cancelto{0}{B_x'}), & B_y &= \gamma(\cancelto{0}{B_y'} + \beta E_x') \end{aligned}\right\} \Rightarrow \boxed{\mathbb{E} = \gamma \mathbb{E}', \; \mathbb{B} = \gamma\beta \frac{2\lambda_0}{r}(-\sin\varphi, \cos\varphi, 0)}$$

Note that since $\vec{r}$ is $\perp$ motion, it is the same in both K' & K: $r = \sqrt{x'^2+y'^2} = \sqrt{x^2+y^2}$.

$\vec{B}$ is tangent to circle of radius r in system K.



(b) In K', the charge density is λ_0 (by defn) while the current $\equiv 0$ (since wire at rest). In K: $\lambda_0 = \frac{\Delta\text{charge}}{\Delta\text{length}} \to \lambda = \frac{\Delta\text{charge}}{(\Delta\text{length})\sqrt{1-\beta^2}} = \gamma\lambda_0$, because of the length contraction between K' & K. Also, in K, this λ is moving at velocity v, so it constitutes an apparent current: $I = \lambda v = \gamma\lambda_0\beta c$. Altogether, in K...

Charge density: $\lambda = \gamma\lambda_0$, current: $I = (\gamma\beta\lambda_0)c$.

(c) If we did not know about relativity in K, we would write down directly:

$$\mathbb{E} = \left(\frac{2\lambda}{r}\right)\hat{r} \left\{\begin{matrix}\text{Gauss'} \\ \text{Law}\end{matrix}\right., \quad \mathbb{B} = \left(\frac{2I}{cr}\right)(-\sin\varphi, \cos\varphi, 0) \left\{\begin{matrix}\text{Biot-Savart} \\ \text{Law}^{\bullet}\end{matrix}\right.$$

With $\lambda = \gamma\lambda_0$, we see: $\mathbb{E}(\text{in K}) = \gamma\mathbb{E}'(\text{in K}')$, and with: $I/c = \gamma\beta\lambda_0$, we see $\mathbb{B}(\text{in K})$ is exactly what we've calculated here. Thus we have verified the field transf$^{\text{n}}$s of part (a) from purely kinematic considerations.

● Jackson's Eq. (5.6)

[redacted]

(29) [redacted] pts]. Write the Maxwell Eqs. as a tensor divergence on Maier's tensor $\mathcal{M}^{\alpha\beta}$.

1) For $\mathbb{M} = \mathbf{E} + i\,\mathbb{B}$, and $J_0 = c\rho$, Maxwell's divergence eqtns $\left\{ \begin{array}{l} \nabla\cdot\mathbb{E} = 4\pi\rho, \\ \nabla\cdot\mathbb{B} = 0, \end{array}\right.$ easily combine to give:

$$\longrightarrow \nabla\cdot\mathbb{M} = (4\pi/c)\,J_0\,. \qquad (1)$$

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In the same way (by simple addition) the curl eqtns $\left\{ \begin{array}{l} \nabla\times\mathbb{E} = -\frac{1}{c}(\partial\mathbb{B}/\partial t), \\ \nabla\times\mathbb{B} = +\frac{1}{c}\left(\frac{\partial\mathbb{E}}{\partial t}\right) + \frac{4\pi}{c}\mathbb{J}, \end{array}\right.$ yield:

$$\longrightarrow -\frac{\partial\mathbb{M}}{\partial x_0} - i\,\nabla\times\mathbb{M} = (4\pi/c)\,\mathbb{J}, \qquad (2)$$

where $x_0 = ct$ is the time coordinate.

2) In terms of components of the 4-position $\tilde{x} = (x_0, x_1, x_2, x_3)$, Eqs (1) & (2) are//

$$\left.\begin{array}{l}
\frac{\partial}{\partial x_0}(0) + \frac{\partial}{\partial x_1}(+M_1) + \frac{\partial}{\partial x_2}(+M_2) + \frac{\partial}{\partial x_3}(+M_3) = \frac{4\pi}{c}J_0; \\
\frac{\partial}{\partial x_0}(-M_1) + \frac{\partial}{\partial x_1}(0) + \frac{\partial}{\partial x_2}(-iM_3) + \frac{\partial}{\partial x_3}(iM_2) = \frac{4\pi}{c}J_1, \\
\frac{\partial}{\partial x_0}(-M_2) + \frac{\partial}{\partial x_1}(iM_3) + \frac{\partial}{\partial x_2}(0) + \frac{\partial}{\partial x_3}(-iM_1) = \frac{4\pi}{c}J_2, \\
\frac{\partial}{\partial x_0}(-M_3) + \frac{\partial}{\partial x_1}(-iM_2) + \frac{\partial}{\partial x_2}(iM_1) + \frac{\partial}{\partial x_3}(0) = \frac{4\pi}{c}J_3.
\end{array}\right\} \qquad (3)$$

Evidently this set of eqtns can be written in the form of a tensor divergence ...

$$\boxed{\partial_\alpha \mathcal{M}^{\alpha\beta} = (4\pi/c)\,J^\beta}\;, \text{ w/ } \partial_\alpha = \left(\frac{\partial}{\partial x_0}, \nabla\right),\; J^\beta = (c\rho, \mathbb{J}), \qquad (4)$$

$$\xrightarrow{\text{and//}} \mathcal{M} = (\mathcal{M}^{\alpha\beta}) = \begin{pmatrix} 0 & -M_1 & -M_2 & -M_3 \\ +M_1 & 0 & +iM_3 & -iM_2 \\ +M_2 & -iM_3 & 0 & +iM_1 \\ +M_3 & +iM_2 & -iM_1 & 0 \end{pmatrix}, \text{ the Maier tensor.} \qquad (5)$$

Eqs. (4) are ≡ Maxwell Eqtns, with $(\mathcal{M}^{\alpha\beta})$ a new form of the field tensor.

[redacted]

3) If $(\mathcal{M}^{\alpha\beta})$ is an acceptable field tensor, then under a Lorentz transform $\underset{\sim}{\Lambda}$, we must have $\underset{\sim}{\mathcal{M}} \rightarrow \underset{\sim}{\mathcal{M}}' = \underset{\sim}{\Lambda}\, \underset{\sim}{\mathcal{M}}\, \underset{\sim}{\Lambda}_T$ [Jk$^{\underline{n}}$ Eq. (11.147)]. For a Lorentz boost $\underset{\sim}{\Lambda} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ along the x_1 axis, this requires...

$$\begin{pmatrix} 0 & -M_1' & -M_2' & -M_3' \\ +M_1' & 0 & +iM_3' & -iM_2' \\ +M_2' & -iM_3' & 0 & +iM_1' \\ +M_3' & +iM_2' & -iM_1' & 0 \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -M_1 & -M_2 & -M_3 \\ +M_1 & 0 & +iM_3 & -iM_2 \\ +M_2 & -iM_3 & 0 & +iM_1 \\ +M_3 & +iM_2 & -iM_1 & 0 \end{pmatrix} \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

After some arithmetic, we find this relation is satisfied identically if ... (6)

$M_1' = M_1$,	or -- separating real &	$\boxed{E_1' = E_1, \quad B_1' = B_1}$
$M_2' = \gamma(M_2 + i\beta M_3)$,	imaginary parts of	$\boxed{E_2' = \gamma(E_2 - \beta B_3), \quad B_2' = \gamma(B_2 + \beta E_3)}$
$M_3' = \gamma(M_3 - i\beta M_2)$;	$M_k = E_k + i B_k$:	$\boxed{E_3' = \gamma(E_3 + \beta B_2), \quad B_3' = \gamma(B_3 - \beta E_2)}$

The boxed eqtns are precisely the Lorentz-transformed fields for an x_1-boost, according to Jk$^{\underline{n}}$ Eq. (11.148). So $(\mathcal{M}^{\alpha\beta})$ transforms properly, as a contravariant field tensor, at least for Lorentz boosts. (7)

4) The eigenvalues λ of $\underset{\sim}{\mathcal{M}}$ are found by imposing $\det(\underset{\sim}{\mathcal{M}} - \lambda \underset{\sim}{I}) = 0$, i.e.

$$\det \begin{pmatrix} -\lambda & -M_1 & -M_2 & -M_3 \\ +M_1 & -\lambda & +iM_3 & -iM_2 \\ +M_2 & -iM_3 & -\lambda & +iM_1 \\ +M_3 & +iM_2 & -iM_1 & -\lambda \end{pmatrix} = 0.$$

After some more arithmetic, this yields...

$$[\lambda^2 - (M_1^2 + M_2^2 + M_3^2)][\lambda^2 + (M_1^2 + M_2^2 + M_3^2)] = 0. \quad (8)$$

Since $(M_1^2 + M_2^2 + M_3^2) = (\mathbb{E}^2 - \mathbb{B}^2) + 2i\,\mathbb{E}\cdot\mathbb{B}$, then the eigenvalues of $\underset{\sim}{\mathcal{M}}$ are

$$\lambda = \pm[(\mathbb{E}^2 - \mathbb{B}^2) + 2i\,\mathbb{E}\cdot\mathbb{B}]^{1/2}, \quad \pm i[(\mathbb{E}^2 - \mathbb{B}^2) + 2i\,\mathbb{E}\cdot\mathbb{B}]^{1/2}. \quad (9)$$

Both quantities here, $(\mathbb{E}^2 - \mathbb{B}^2)$ & $\mathbb{E}\cdot\mathbb{B}$, are Lorentz invariants [cf Landau & Lifshitz "Cl. Theory of Fields" (2nd ed, 1962), ¶ 25]. So the λ's are also Lorentz invariant.