An electron neutrino ν_e created in subatomic interactions can later be observed as a muon neutrino ν_μ, and vice versa, in a process known as *neutrino oscillation*. Such oscillations occur **not** as decays, but as a result of a mismatch between the neutrino *flavor eigenstates* (ν_e and ν_μ, with definite lepton numbers) and the *mass eigenstates* ν_1 and ν_2 (with definite masses m_1 and m_2, respectively). The eigenstates in these complete, orthonormal bases[1] are coupled via an arbitrary real unitary matrix:

$$U = \begin{pmatrix} \langle \nu_e | \, \nu_1 \rangle & \langle \nu_e | \, \nu_2 \rangle \\ \langle \nu_\mu | \, \nu_1 \rangle & \langle \nu_\mu | \, \nu_2 \rangle \end{pmatrix} \equiv \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

where the unknown *mixing angle* θ is real.

[1]We ignore here the tau neutrino ν_τ and any possible *sterile* neutrinos.

(a) Briefly discuss any physical constraints that require U to be unitary.

(b) Calculate the probability as a function of time t that a ν_e at time $t = 0$ with momentum p will transform into a ν_μ, expressing your answer in terms of θ and the energies $E_1 = \sqrt{p^2c^2 + m_1^2c^4}$ and $E_2 = \sqrt{p^2c^2 + m_2^2c^4}$ of the two mass eigenstates.

(c) Consider now the *ultrarelativistic limit*, *i.e.*, $pc \gg mc^2$ for both mass eigenstates. Re-express your answer from part (a) in terms of p, θ, $\Delta m^2 \equiv m_2^2 - m_1^2$, and distance traveled $L \simeq ct$.

(d) Make an accurate plot of probability as a function of p for given values of L, θ, and Δm^2, and describe an experiment or set of experiments that could be used to determine both θ and Δm^2.