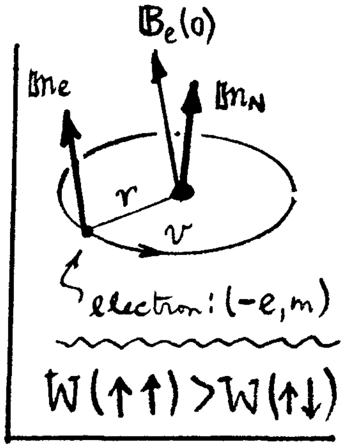


10) So long as we have the magnetic dipole interaction energy $W_{MD} = -\mathbb{m}\cdot\mathbb{B}$ in Eq. (33), let's touch on a famous problem in atomic physics -- the hyperfine structure interval in atomic hydrogen. This is a small energy splitting in the hydrogen energy levels due to the interaction of the magnetic dipole moments of the nucleus (proton $\mathbb{m}_N$) and the electron ($\mathbb{m}_e$). The classical Hamiltonian is...

$$\mathcal{H}_{hfs} = -\mathbb{m}_N\cdot\mathbb{B}_e(0),\quad \mathbb{B}_e(0) = \text{magnetic field generated at nucleus by electron}$$

w/
$$\mathbb{B}_e = \left(\frac{8\pi}{3}\right)\mathbb{m}_e\,\delta(\mathbb{r}) + \frac{1}{r^3}\left[3(\mathbb{m}_e\cdot\hat{n})\hat{n} - \mathbb{m}_e\right] + \frac{e}{mc}\left(\frac{\mathbb{L}_e}{r^3}\right)$$

↑ Fermi's coefficient for "contact term" — e orbit: $\mathbb{L}_e = m\,\mathbb{r}\times\mathbb{v}$

So/

$$\rightarrow \mathcal{H}_{hfs} = (-)\frac{8\pi}{3}(\mathbb{m}_N\cdot\mathbb{m}_e)\,\delta(\mathbb{r}) + \frac{1}{r^3}\left[(\mathbb{m}_N\cdot\mathbb{m}_e) - 3(\mathbb{m}_N\cdot\hat{n})(\mathbb{m}_e\cdot\hat{n}) - (e/mc)\,\mathbb{m}_N\cdot\mathbb{L}_e\right]. \tag{36}$$

QM energy in hydrogen state $\psi_n(\mathbb{r}) = |n\rangle$ is...

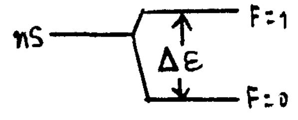
$$\rightarrow \mathcal{E}_{hfs} = \langle n|\mathcal{H}_{hfs}|n\rangle = (-)\frac{8\pi}{3}\langle\mathbb{m}_N\cdot\mathbb{m}_e\rangle\underbrace{|\psi_n(0)|^2}_{\text{due to }\delta(\mathbb{r})} + \langle n|\frac{1}{r^3}[\;]|n\rangle. \tag{37}$$

↑ dominant hfs term for S-states ($l=0$) ↑ hfs energy for $l\neq 0$

For a non-relativistic hydrogen atom...

$$|\psi_n(0)|^2 = \frac{1}{\pi}(1/na_0)^3$$ nS states only, $a_0 = \hbar^2/me^2$ (Bohr radius)

$$\langle\mathbb{m}_N\cdot\mathbb{m}_e\rangle_{nS} = m_N m_e\cdot\begin{cases}+1, & \text{for } \uparrow\uparrow\ (F=1,\text{ triplet})\\ -3, & \text{for } \uparrow\downarrow\ (F=0,\text{ singlet})\end{cases}$$



⋮

$$\rightarrow \Delta\mathcal{E}_{hfs}(nS) = \frac{1}{n^3}\cdot 1420.405\,751\,8\cdots(\pm 1/10^{12}),\ \text{MHz}. \tag{38}$$

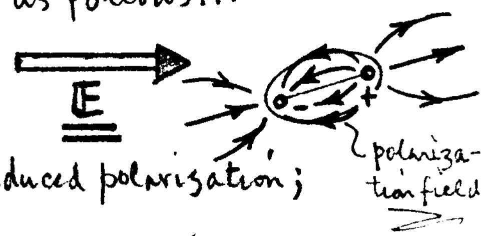
$[\lambda = 21.106\cdots\ \text{cm}]$

Certainly this term would have been hugely differently w/o the "contact interaction" due to $\delta(\mathbb{r})$ and w/o Fermi's $(8\pi/3)$ coefficient.

Magnetostatics (Cont'd)

11) For magnetic fields in matter, we expect some change in $\mathbb{B}$ due to possible <u>induced</u> magnetization. The effect is huge for ferromagnets, where a weak magnetizing field $\mathbb{B}_{applied}$ induces a very large $\mathbb{B}_{resultant}$ (think of the electromagnet on a crane). Discuss how we account for the induced fields by analogy with the $\mathbb{E}$-field case, as follows...

<u>$\mathbb{E}$-fields in materials</u> [Jk^{n} Sec. (4.3)]



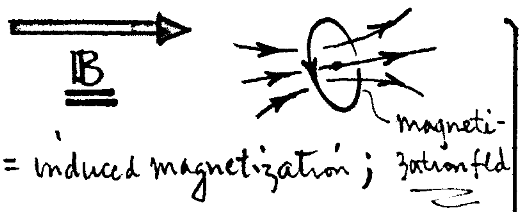
$\rho_{real} \to \rho_{eff} = \rho_{real} - \nabla\cdot\mathbb{P}$, w/ $\mathbb{P}$ = induced polarization;

and/ $\nabla\cdot\mathbb{E} = 4\pi\rho_{eff}$, or/ $\mathbb{E}\to\mathbb{D} = \mathbb{E} + 4\pi\mathbb{P}$, w/ $\underline{\nabla\cdot\mathbb{D} = 4\pi\rho_{real}}$ (39)

so/ Invent new field (and ϵ = dielectric cnst) $\Big\}$ $\underline{\mathbb{D} = \epsilon\,\mathbb{E}}$ $\int$ ϵ = cnst, or $\epsilon \to \underset{\sim}{\epsilon}$ (linear materials); or : $\epsilon \to \epsilon(\mathbb{E})$, for electrets.

$\to$ If $\mathbb{P} = \alpha\,\mathbb{E}$, α = "polarizability", then : $\epsilon = 1 + 4\pi\alpha$, etc.

<u>$\mathbb{B}$-fields in materials</u> [Jk^{n} Sec. (5.8)]



$\mathbb{J}_{real} \to \mathbb{J}_{eff} = \mathbb{J}_{real} + c\,\nabla\times\mathbb{M}$, w/ $\mathbb{M}$ = induced magnetization;

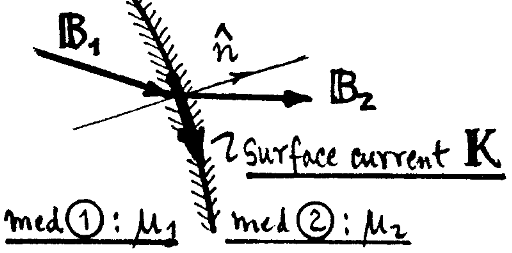
and/ $\nabla\times\mathbb{B} = \frac{4\pi}{c}\mathbb{J}_{eff}$, or/ $\mathbb{B}\to\mathbb{H} = \mathbb{B} - 4\pi\mathbb{M}$, w/ $\underline{\nabla\times\mathbb{H} = \frac{4\pi}{c}\mathbb{J}_{real}}$ (40)

so/ Invent new field (and μ = permeability) $\Big\}$ $\underline{\mathbb{H} = \frac{1}{\mu}\mathbb{B}}$ $\int$ $1/\mu$ = cnst, or $1/\mu \to \underset{\sim}{(1/\mu)}$ (linear materials); or : $1/\mu \to 1/\mu(\mathbb{B})$, ferromagnets.

$\to$ If $\mathbb{M} = \chi\,\mathbb{H}$, χ = "magnetic susceptibility", then $\mu = 1 + 4\pi\chi$, etc.

The add-on signs for $\mathbb{P}$ & $\mathbb{M}$ are opposite because of opposite <u>interior</u> fields.

The interface B.C. on $\mathbb{B}$ & $\mathbb{H}$ follow from amended Maxwell eqtns...



$\nabla\cdot\mathbb{B} = 0 \Rightarrow (\mathbb{B}_2 - \mathbb{B}_1)\cdot\hat{n} = 0$, B_{normal} is conserved;

$\nabla\times\mathbb{H} = \frac{4\pi}{c}\mathbb{J} \Rightarrow \hat{n}\times(\mathbb{H}_2 - \mathbb{H}_1) = \frac{4\pi}{c}\mathbb{K}$. (41)

Magnetostatics (cont'd)

12) In Jk[n] Sec. (5.9), there is a survey of how to solve magnetostatic problems. This is worth summarizing. The basic eqtns of magnetostatics are...*

$$\left[\begin{array}{l} ① \nabla\cdot\mathbb{B}=0, \\ ② \nabla\times\mathbb{H}=\frac{4\pi}{c}\mathbb{J}; \end{array}\right\| \text{ w/ } \mathbb{H}=\mathbb{B}-4\pi\mathbb{M} \;\Big\|\; \text{ \& } \begin{array}{l} \hat{n}\cdot(\mathbb{B}_2-\mathbb{B}_1)=0, \text{ at interface;} \\ \hat{n}\times(\mathbb{H}_2-\mathbb{H}_1)=\frac{4\pi}{c}\mathbb{K}_{surface}. \end{array} \tag{42}$$

There are some points in common with the electrostatics we already know.

A. Method of Vector Potential [$\mathbb{J}\neq 0$, $\mathbb{H}=$ fcn($\mathbb{B}$) given].

$$\nabla\cdot\mathbb{B}=0 \Rightarrow \underline{\mathbb{B}=\nabla\times\mathbb{A}}. \text{ If } \mathbb{H}=\mathbb{H}(\mathbb{B})\rightarrow\mathbb{H}(\mathbb{A}), \text{ then:}$$

$$\longrightarrow \nabla\times\mathbb{H}(\mathbb{A})=\frac{4\pi}{c}\mathbb{J} \Rightarrow \text{PDE for } \mathbb{A} \text{ with } \mathbb{J} \text{ as source term.} \tag{43}$$

e.g.// $\mathbb{H}=(1/\mu)\mathbb{B}$, with μ indpt. of $\mathbb{B}$ (linear medium)...

$$\longrightarrow \nabla\times\left(\frac{1}{\mu}\nabla\times\mathbb{A}\right)=\frac{4\pi}{c}\mathbb{J}. \tag{44}$$

If $\mu=$ cnst in the region of interest, then w/ Coulomb gauge ($\nabla\cdot\mathbb{A}=0$):

$$\left[\nabla^2\mathbb{A}=-\left(\frac{4\pi\mu}{c}\right)\mathbb{J} \Rightarrow \underbrace{\mathbb{A}(\mathbf{r})=\frac{\mu}{c}\int\frac{d^3x'}{|\mathbf{r}-\mathbf{r}'|}\mathbb{J}(\mathbf{r}')}_{\text{Helmholtz' solution}}+\left\{\begin{array}{c}\text{surface}\\ \text{terms}\end{array}\right\}.\right] \tag{45}$$

Then, with this $\mathbb{A}$, the solution is $\mathbb{B}=\nabla\times\mathbb{A}$.

B. Method of Scalar Potential [$\mathbb{J}=0$, $\mathbb{B}=$ fcn $\mathbb{H}$ given].

This works when $\mathbb{J}_{real}\equiv 0$ in the region of interest. Then have...

$$\nabla\times\mathbb{H}=0 \Rightarrow \underline{\mathbb{H}=-\nabla\phi_M}. \text{ If } \mathbb{B}=\mathbb{B}(\mathbb{H})\rightarrow\mathbb{B}(\phi_M), \text{ then:}$$

---

*Comp. with electrostatics...

$$\left[\begin{array}{l} ① \nabla\cdot\mathbb{D}=-4\pi\rho; \\ ② \nabla\times\mathbb{E}=0; \end{array}\right\| \text{ w/ } \mathbb{D}=\mathbb{E}+4\pi\mathbb{P} \;\Big\|\; \text{ \& } \begin{array}{l} \hat{n}\cdot(\mathbb{D}_2-\mathbb{D}_1)=4\pi\sigma_{surface}; \\ \hat{n}\times(\mathbb{E}_2-\mathbb{E}_1)=0; \text{ at interface.} \end{array}$$

Magnetostatics (cont'd)

$\rightarrow \nabla \cdot \mathbb{B}(\phi_M) = 0 \Rightarrow$ PDE for ϕ_M in a source-free region. (46)

e.g./ $\mathbb{B} = \mu\, \mathbb{H}$, with μ indpt of $\mathbb{B}$ (linear medium)...

$$\rightarrow \nabla \cdot (\mu \mathbb{H}) = -\nabla \cdot (\mu \nabla \phi_M) = 0. \tag{47}$$

If μ = cnst in the region of interest, then have...

$$\left[\!\left[\begin{array}{l} \nabla^2 \phi_M = 0, \text{ Laplace eqtn (plus B.C.)}. \\ \text{Solns: } \phi_M(r,\theta) = \sum_{l=0}^{\infty} \left[A_l r^l + B_l r^{-(l+1)}\right] P_l(\cos\theta) \Big\} \text{ azimuthal symmetry etc.} \end{array} \right]\!\right] \tag{48}$$

With such a ϕ_M, solution is: $\mathbb{B} = -\mu \nabla \phi_M$. This is analogous to $\mathbb{D} = -\epsilon \nabla \phi_E$ in electrostatics. The $\mathbb{J}$-free case in magnetostatics is as close as we come to the (ρ-free) electrostatic method of solution.

C. <u>Use of ϕ_M or $\mathbb{A}$</u> [$\mathbb{J} = 0$, $\mathbb{M}$ given].

(1) $\left.\begin{array}{l} \nabla \cdot \mathbb{B} = \nabla \cdot (\mathbb{H} + 4\pi \mathbb{M}) = 0, \\ \underline{\mathbb{H} = -\nabla \phi_M}, \text{ (since } \mathbb{J} = 0\text{)}; \end{array}\right\} \nabla^2 \phi_M = (-)4\pi \rho_M$, w/ $\rho_M = -\nabla \cdot \mathbb{M}$. (49)

(use ϕ_M) ← magnetization density

This is Poisson's eqtn (NOTE: ρ_M is <u>not</u> a monopole density, as in electrostatics). The solution on an ∞ domain is...

$$\phi_M(\mathbf{r}) = (-)\int_\infty \frac{d^3x'}{|\mathbf{r} - \mathbf{r}'|} \nabla' \cdot \mathbb{M}(\mathbf{r}')$$

use: $\frac{1}{R} \nabla' \cdot \mathbb{M} = \nabla' \cdot \left(\frac{\mathbb{M}}{R}\right) - \mathbb{M} \cdot \nabla'\left(\frac{1}{R}\right)$

convert to surface term: $\oint_\infty (\mathbb{M}/R) \cdot d\mathbb{S} \rightarrow 0$. put $\nabla' = (-)\nabla$

or/

$$\left[\!\left[\phi_M(\mathbf{r}) = \underbrace{(-)\nabla \cdot \left(\int_\infty \frac{d^3x'}{|\mathbf{r} - \mathbf{r}'|} \mathbb{M}(\mathbf{r}') \right)}_{\text{Helmholtz' solution (with one partial integration)}} \right]\!\right] \tag{50}$$

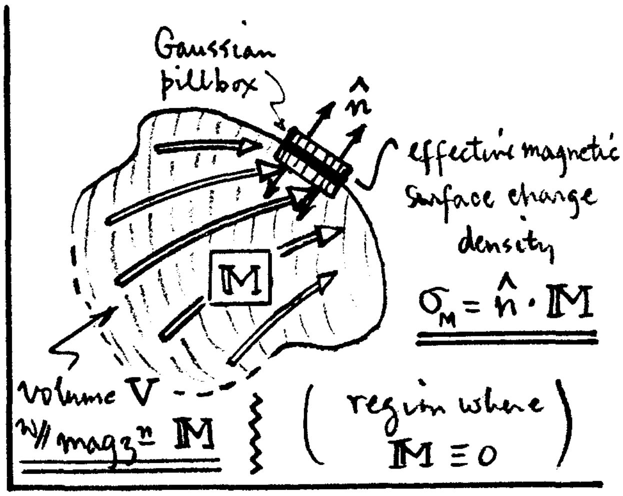
If $|\mathbf{r}\,(\text{observation distance})| \gg |\mathbf{r}'\,(\text{source size})|$, then: $\phi_M(\mathbf{r}) \simeq (-)\left[\nabla\left(\frac{1}{r}\right)\right] \cdot \int d^3x'\, \mathbb{M}(\mathbf{r}')$, or:

$\boxed{\phi_M(\mathbf{r}) \simeq (\mathbf{m} \cdot \mathbf{r})/r^3}$, w/ $\mathbf{m} = \int \mathbb{M}\, d^3x'$ = system mag. moment. See Prob. 36.

Magnetostatics (cont'd)

ASIDE

The assignment of a magnetization volume density $\underline{\rho_M = -\nabla\cdot\mathbb{M}}$ is done with some forethought... it helps in solving certain types of problems where the magnetization $\mathbb{M}$ is <u>localized</u>. At the surface of such a volume V, we can write...



magnetic charge in Gaussian pillox:

$$\rightarrow \int \rho_M\, dV = \oint(-\mathbb{M}\cdot\hat{n})\, dA$$

$$= +(\mathbb{M}\cdot\hat{n})\,\Delta A \quad \text{inner surface only}$$

so// effective magnetic surface charge density is: $\boxed{\sigma_M = \hat{n}\cdot\mathbb{M}}$. (51)

Then, when discontinuities in $\mathbb{M}$ are represented this way, the sol^n (50) is

$$\rightarrow \phi_M(\mathbf{r}) = -\int_V \frac{d^3x'}{|\mathbf{r}-\mathbf{r}'|}\nabla'\cdot\mathbb{M}(\mathbf{r}') + \oint_S \frac{dS'}{|\mathbf{r}-\mathbf{r}'|}\,\hat{n}'\cdot\mathbb{M}(\mathbf{r}') \tag{52}$$

In this form, the $\int_V$ cannot be partial-integrated as before, since $\mathbb{M}$ is not continuous. Eq. (52) is mainly used when $\mathbb{M}$ is uniform in V... then have $\nabla'\cdot\mathbb{M} \equiv 0$, and $\phi_M(\mathbf{r}) = \oint_S \frac{dS'}{R}(\hat{n}'\cdot\mathbb{M})$ is generated by the surface charge.

---

(2) (use $\mathbb{A}$)

$$\left.\begin{array}{l}\nabla\times\mathbb{H} = \nabla\times(\mathbb{B}-4\pi\mathbb{M}) = 0 \\ \underline{\mathbb{B} = \nabla\times\mathbb{A}} \quad (\text{and } \nabla\cdot\mathbb{A} = 0)\end{array}\right\} \nabla^2\mathbb{A} = -\frac{4\pi}{c}\mathbb{J}_M, \text{ w/ } \mathbb{J}_M = c\nabla\times\mathbb{M}. \tag{53}$$

magnetization current.

Again, Poisson's eqtn. Solution on an ∞ domain is...

$$\left[\!\left[\mathbb{A}(\mathbf{r}) = \int_\infty \frac{d^3x'}{|\mathbf{r}-\mathbf{r}'|}\nabla'\times\mathbb{M}(\mathbf{r}') = \left\{\begin{array}{c}\text{partial}\\ \text{integrate}\end{array}\right\} = \int_\infty \frac{\mathbb{M}(\mathbf{r}')\times\mathbb{R}}{R^3}d^3x'\right]\!\right] \quad \mathbb{R} = (\mathbf{r}-\mathbf{r}') \tag{54}$$

When $\mathbb{M}$ is localized in a volume V enclosed by surface S, it is possible to identify an effective surface current $\mathbb{K} = c\,\mathbb{M}\times\hat{n}$ (analogous to σ_M above), and

$$\rightarrow \mathbb{A}(\mathbf{r}) = \int_V \frac{d^3x'}{R}\nabla'\times\mathbb{M}(\mathbf{r}') + \frac{1}{c}\oint\frac{dS'}{R}\mathbb{K}(\mathbf{r}') \quad \text{Jk}^{\text{n}}\ \text{Eq. (5.103)} \tag{55}$$

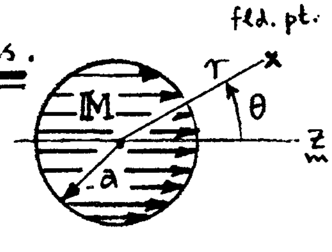
13) Finally, we shall give two examples of solutions to $\mathbb{B}$-$\mathbb{H}$ field problems...

I. Uniformly magnetized sphere [Jk^{n} Sec.(5.10)];

II. Permeable ($\mu > 1$) spherical shell in uniform $\mathbb{B}$ [Jk^{n} Sec.(5.12)].

Problem I is just a dipole problem; we can find Fermi's field $\mathbb{B}_m = \frac{8\pi}{3}\mathbb{M}$ inside the sphere. Problem II is important for concerns re magnetic shielding.

I. Sphere of radius a w/ magnetization $\mathbb{M}$ = cnst along z-axis.



(1) Let $\mathbb{M} = M_0\hat{z}$. $\nabla'\cdot\mathbb{M} \equiv 0$ inside (and outside), and the potential in Eq.(52) is generated by "surface charges"

$$\phi_M(\mathbf{r}) = \oint_S \frac{ds'}{R}\left[\hat{n}'\cdot\mathbb{M}(\mathbf{r}')\right] \quad \int R = |\mathbf{r}-\mathbf{r}'|, \text{ and: } \hat{n}'\cdot\mathbb{M}(\mathbf{r}') = M_0\cos\theta';$$

$$\xrightarrow{\text{So}} \phi_M(r,\theta) = M_0 a^2 \int_{4\pi} \frac{d\Omega'}{R}\cos\theta' \quad \int \text{azimuthal symmetry} \Rightarrow \text{no } \varphi\text{-dependence.} \tag{56}$$

Of course this formulation can be used since $\mathbb{J}_{real} \equiv 0$ in this problem. We are doing case C(1) on p. Mag 15, and the field will be $\mathbb{H} = -\nabla\phi_M$.

---

(2) The integral in (56) is easily evaluated,† with result

$$\longrightarrow \phi_M(r,\theta) = \frac{4\pi}{3}M_0 \cdot \begin{cases} r\cos\theta, & r<a \text{ (inside sphere)}; \\ (a^3/r^2)\cos\theta, & r>a \text{ (outside)}. \end{cases} \tag{57}$$

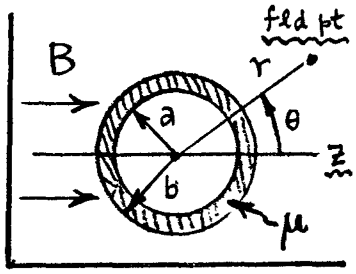
Inside: $\phi_M = \frac{4\pi}{3}M_0 z$, so: $\mathbb{H}_{in} = -\nabla\phi_M = -\frac{4\pi}{3}\mathbb{M}$. This immediately gives the Fermi result: $\mathbb{B}_{in} = \mathbb{H}_{in} + 4\pi\mathbb{M} = \underline{(8\pi/3)}\,\mathbb{M}$, as used in Jk^{n} Eq.(5.64).

Outside: $\phi_M = (m/r^2)\cos\theta$, with $m = (4\pi/3)a^3 M_0$ the sphere's dipole moment. Both $\mathbb{B}_{out}$ & $\mathbb{H}_{out} \equiv \mathbb{B}_{out}$ are just standard dipole fields. What else?

---

† By Jk^{n} Eq.(3.38), put $\frac{1}{R} = \frac{1}{r_>}\sum_{l=0}^{\infty}\left(\frac{r_<}{r_>}\right)^l P_l(\cos\gamma)$, w/ $\gamma = \measuredangle(\mathbf{r},\mathbf{r}')$. Put $\mathbf{r}$ on z-axis, so $\gamma = \theta'$, and $\phi_M(z,\theta=0) = (M_0 a^2/r_>)\sum_{l=0}^{\infty}(r_</r_>)^l \int_{4\pi} d\Omega' \cos\theta' P_l(\cos\theta')$. Integral is nonzero for $l=1$ only, and: $\phi_M(z,\theta=0) = M_0 a^2 (r_</r_>^2)\cdot\frac{4\pi}{3}$. Use trick in Jk^{n} Eq.(3.37) to get Eq.(57).

II. Spherical Shell of Permeability μ: Shielding of an External $\mathbb{B}$-field.



(1) Shell of magnetic material μ has radii a & b and is placed in cnst external field $\mathbb{B}$ along z-axis. Since $\mathbb{J}_{real}=0$, then $\mathbb{H}=-\nabla\phi_M$, and if $\mu=$ cnst $\nabla\cdot\mathbb{B}=0 \Rightarrow \nabla\cdot\mathbb{H}=0$. Then $\nabla^2\phi_M=0$, so we are looking at solutions to Laplace's eqtn of form: $\phi_M(r,\theta)=\sum_{l=0}^{\infty}\left(A_l r^l+\frac{B_l}{r^{l+1}}\right)P_l(\cos\theta)$ in regions $\begin{cases} r>b \text{ (outside)}, \\ b>r>a \text{ (in shell)}, \\ a>r \text{ (inside)}. \end{cases}$

That's not hard; what is complicated is imposing boundary conditions.

---

(2) Evidently, Laplace solutions in the three regions are...

$$\left.\begin{aligned}
&\underline{\underline{r>b}}:\ \phi_M(r,\theta)=-Br\cos\theta+\sum_{l=0}^{\infty}\frac{\alpha_l}{r^{l+1}}P_l(\cos\theta),\\
&\underline{\underline{b>r>a}}:\ \phi_M(r,\theta)=\sum_{l=0}^{\infty}\left[\beta_l r^l+\gamma_l r^{-(l+1)}\right]P_l(\cos\theta),\\
&\underline{\underline{a>r}}:\ \phi_M(r,\theta)=\sum_{l=0}^{\infty}\delta_l r^l P_l(\cos\theta).
\end{aligned}\right\}\quad \underline{(58)}$$

The B.C. are (note $\mathbb{B}\equiv\mathbb{H}$ for $r>b$ & $a>r$; $\mathbb{B}=\mu\mathbb{H}$ for $b>r>a$)...

$$\left.\begin{aligned}
&\underset{\text{(tangential }\mathbb{H})}{\underline{H_\theta \text{ cont}^{\underline{s}}}}:\ \text{①}\ \frac{\partial\phi_M}{\partial\theta}\Big|_{r=b+}=\frac{\partial\phi_M}{\partial\theta}\Big|_{r=b-},\quad \text{②}\ \frac{\partial\phi_M}{\partial\theta}\Big|_{r=a+}=\frac{\partial\phi_M}{\partial\theta}\Big|_{r=a-};\\
&\underset{\text{(normal }\mathbb{B})}{\underline{B_r \text{ cont}^{\underline{s}}}}\quad \text{③}\ \frac{\partial\phi_M}{\partial r}\Big|_{r=b+}=\mu\frac{\partial\phi_M}{\partial r}\Big|_{r=b-},\quad \text{④}\ \mu\frac{\partial\phi_M}{\partial r}\Big|_{r=a+}=\frac{\partial\phi_M}{\partial r}\Big|_{r=a-}.
\end{aligned}\right\}\quad \underline{(59)}$$

Eqs. (59) are 4 conditions on the 4 <u>sets</u> of coefficients $\{\alpha_l,\beta_l,\gamma_l,\delta_l\}$.

---

(3) Fortunately, all but the $l=1$ terms vanish in the various ϕ_M series. This is dictated by the form of the exterior potential $\phi_M\sim -Br\,P_1(\cos\theta)$ as $r\to\infty$, as we may see from B.C. ③ above:

$$\text{③}\Rightarrow -B\cos\theta-\sum_{l=0}^{\infty}\frac{(l+1)\alpha_l}{b^{l+2}}P_l(\cos\theta)=\mu\sum_{l=0}^{\infty}\left[l\beta_l b^{l-1}-\frac{(l+1)\gamma_l}{b^{l+2}}\right]P_l(\cos\theta),$$

(next page)

or/ $$\sum_{l=0}^{\infty}\frac{1}{b^3}\left[\left(\frac{l+1}{b^{l-1}}\right)\alpha_l+\mu l\, b^{l+2}\beta_l-\mu\left(\frac{l+1}{b^{l-1}}\right)\gamma_l\right]P_l(\cos\theta)=-B\,P_1(\cos\theta). \quad (60)$$

To satisfy this eqtn, the $[\;]\equiv 0$ for all l except $l=1$, and the easiest way to have $[\;]_{l\neq1}\equiv 0$ is to set $\{\alpha_l,\beta_l,\gamma_l\}_{l\neq 1}\equiv 0$. Then $\delta_{l\neq1}\equiv 0$ by B.C. ④. So only $\{\alpha_1,\beta_1,\gamma_1,\delta_1\}\neq 0$. Call them $\{\alpha,\beta,\gamma,\delta\}$ and write...

$$\left.\begin{aligned}&\underline{\underline{r>b}}:\ \phi_M=-Br\cos\theta+\frac{\alpha}{r^2}\cos\theta\ ;\quad \underline{\underline{a>r}}:\ \phi_M=\delta r\cos\theta;\\ &\underline{\underline{b>r>a}}:\ \phi_M=\left(\beta r+\frac{\gamma}{r^2}\right)\cos\theta.\end{aligned}\right\}(61)$$

---

(4) The B.C. of Eq. (59) are now imposed the ϕ_M's of Eq. (61). We still have 4 eqtns in 4 unknowns, so we get...

$$\begin{pmatrix}1&-b^3&-1&0\\0&a^3&1&-a^3\\2&\mu b^3&-2\mu&0\\0&\mu a^3&-2\mu&-a^3\end{pmatrix}\begin{pmatrix}\alpha\\\beta\\\gamma\\\delta\end{pmatrix}=\begin{pmatrix}b^3B\\0\\-b^3B\\0\end{pmatrix}. \quad (62)$$

This is ~ unpleasant, but the problem cannot be made simpler. The "interesting" coefficients are α & δ for $\phi_M(\text{out})$ & $\phi_M(\text{in})$ in Eq. (61).

The solution to Eq. (62) for α & δ, as quoted by Jk$^{\text{n}}$ in his Eq. (5.121), is...

$$\alpha=\left[\frac{\left(1+\frac{1}{2\mu}\right)\left(1-\frac{1}{\mu}\right)\left(1-\frac{a^3}{b^3}\right)}{\left(1+\frac{1}{2\mu}\right)\left(1+\frac{2}{\mu}\right)-\frac{a^3}{b^3}\left(1-\frac{1}{\mu}\right)^2}\right]b^3B\underset{\mu\gg1}{\simeq}\left[1-\frac{3}{2\mu}\left(\frac{2+\lambda}{1-\lambda}\right)\right]b^3B,\quad \underline{\underline{\lambda}}=\frac{a^3}{b^3}<1, \quad (63)$$

$$\delta=(-)\,\frac{9B}{2\mu}\Big/\left[\left(1+\frac{1}{2\mu}\right)\left(1+\frac{2}{\mu}\right)-\frac{a^3}{b^3}\left(1-\frac{1}{\mu}\right)^2\right]\underset{\mu\gg1}{\simeq}(-)\frac{9B}{2\mu(1-\lambda)}\left[1-\frac{1}{2\mu}\left(\frac{5+4\lambda}{1-\lambda}\right)\right].$$

For shielding purposes, μ can be ~ 10,000 (as for annealed iron), so the $\mu\gg1$ approxn is warranted. At high μ, the exterior field is characterized by: $\phi_M(\text{out})=-Bz\left(1-\frac{b^3}{r^3}\right)\longleftrightarrow$ uniform $\mathbb{B}$ + dipole correction (small for $r>$ few $\times\, b$). The shielding by the shell is measured by... (let $s=b-a=$ shell thickness)...

$$\longrightarrow B_{in}(r<a)/B_{out}(r\gg b)\simeq\frac{|\delta|}{B}\simeq\frac{9}{2\mu}\Big/\left[1-\left(\frac{a}{b}\right)^3\right]\simeq\frac{3}{2\mu}\left(\frac{b}{s}\right)\ll1 \quad (64)$$