Modification of Bohr's Formula due to electron binding.

5) We can use the exact result in Eq. (14) [for the energy transfer $\Delta E(b)$ from (Q, M) to a bound electron $(-e, m)$] over the impact parameter range...

$$\rightarrow \underline{b_0 = Qe/\gamma m v^2} \leqslant b \rightarrow \infty \quad, \text{ or// } \xi_0 = \frac{Qe\omega_0}{\gamma^2 m v^3} \leqslant \xi \rightarrow \infty. \tag{17}$$

The lower limit b_0 is chosen to be consistent with the max. transfer ΔE_{max} in Eq. (4). We will adjust b_0 below, but for the moment we have the transfer:

$$\rightarrow \Delta E(\xi) = \frac{2Q^2e^2}{mv^2}\left(\frac{\omega_0}{\gamma v}\right)^2 \left\{ K_1^2(\xi) + \frac{1}{\gamma^2} K_0^2(\xi) \right\}, \text{ w// } \xi = (\omega_0/\gamma v)\, b. \tag{18}$$

Now we can consider Q's energy loss/unit distance when colliding with a <u>collection</u> of electrons at <u>different</u> bound frequencies $\omega_0 \rightarrow$ <u>set</u> of ω_k's. As follows...

$$\left[\begin{array}{l} N \text{ atoms/vol., } Z \text{ electrons per atom @ bound frequencies } \{\omega_k\}; \\ \text{the } e\text{'s have "oscillator strengths" } f_k \;\rfloor\; f_k \text{ measures relative contribution from } k^{th} \text{ electron, and } \sum_{k=1}^{Z} f_k = Z \text{, for norm.} \end{array}\right. \tag{19}$$

In analogy to Eq. (10), form...

$$\rightarrow \frac{dE}{dx} = N \sum_k f_k \int_{b_0}^{\infty} \Delta E(\xi) \cdot 2\pi b\, db$$

$$= 4\pi N \left(\frac{Q^2e^2}{mv^2}\right) \sum_k \int_{\xi_k}^{\infty} \left\{ K_1^2(\xi) + \frac{1}{\gamma^2} K_0^2(\xi) \right\} \xi\, d\xi, \quad \xi_k = \frac{Qe\omega_k}{\gamma^2 m v^3}. \tag{20}$$

The integration can be done, with the result a <u>modified</u> Bohr formula...

Jk$^{\underline{n}}$ Eq. (13.36)

$$\boxed{\frac{dE}{dx} = 4\pi N Z \left(\frac{Q^2e^2}{mv^2}\right) \left[\ln B_c - \tfrac{1}{2}(v^2/c^2) \right]} \tag{21}$$

$$\text{w// } \underline{B_c = 1.123\, \gamma^2 m v^3 / \langle\omega\rangle\, Qe}, \; \& \; \ln\langle\omega\rangle = \frac{1}{Z}\sum_k f_k \ln \omega_k.$$

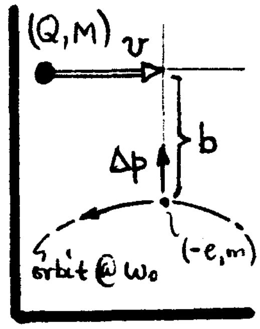
Compare with Eq. (10). This result is similar to the previous Bohr formula... but now we have an explicit $\theta(v/c)^2$ correction, and also a modification to the log argument: $B = \gamma^2 m v^3/\omega Qe \rightarrow B_c = 1.123\,(\omega/\langle\omega\rangle)\, B$. This is Bohr's work.

---

6) The weak point remaining in $\left(\frac{dE}{dx}\right)$ of Eq. (21) is the question of close encounters: $\underline{b \rightarrow b_0 = Qe/\gamma m v^2}$, classically. Here, there can be <u>QM corrections</u>, when the

Quantum considerations. Bethe's modification to Bohr's formula.

impact parameter $b \sim \Delta x \sim \hbar/\Delta p$... i.e. when $b \sim$ the position uncertainty Δx in the electron position associated with the collisional momentum Δp transferred to the electron during the $(Q,M) \rightarrow (-e,m)$ collision. Since $\Delta p \sim \gamma m v$ in the collision, we have a new QM lower limit : $b \sim \hbar/\gamma m v$.



Actually, QM questions can be asked also at large impact parameters. The (classical) energy transfer at $b \sim b_m = \gamma v/\omega_0$ is...

$$\Delta E(b_m) = \left(\frac{2Q^2e^2}{mv^2}\right)\frac{1}{b_m^2}$$ ← let $Q = ze$, $\alpha = e^2/\hbar c \simeq 1/137$ (fs cnst), and : $I = \frac{1}{2}\alpha^2 mc^2 = 13.6\text{ eV}$ (typical binding),

so/ $$\frac{\Delta E(b_m)}{\hbar\omega_0} = \frac{z^2}{\gamma^2}(\alpha c/v)^4\,[\hbar\omega_0/I]. \tag{22}$$

Evidently, when v(of M) >> $v \sim \alpha c$ (of m), then $\Delta E(b_m) << \hbar\omega_0$. QM might say that energy transfers this small are not probable; we should have $\Delta E(b) \sim \hbar\omega_0$. But, statistically, we can have $\langle\Delta E(b)\rangle << \hbar\omega_0$... this is the small average energy transfer in a large number of collisions: $\Delta E(b) \sim \hbar\omega_0$ is transferred in a few collisions at large b; for the rest $\Delta E(b) \sim 0$. So much for the QM hysteria at large b.

As for the case of (classically) small impact parameters $b \rightarrow b_0 = Qe/\gamma m v^2$, it is possible to reach the quantum limit $\hbar/\gamma m v$ first. By comparison...

for minimum impact parameter $$\left\{\begin{array}{l} b_0^{(QM)} = \hbar/\gamma m v \\ b_0^{(CM)} = Qe/\gamma m v^2 \end{array}\right\rangle \eta = \frac{b_0^{(CM)}}{b_0^{(QM)}} = \frac{Qe}{\hbar v} \tag{23}$$

Let $Q = ze$, $\alpha = \frac{e^2}{\hbar c}$. Then : $\eta = \frac{z}{(v/\alpha c)}$ ⟋ $b_0^{(CM)}$ dominates at $v < \alpha c$, $b_0^{(QM)}$ dominates at $v > \alpha c$.

Incorporating this refinement, we get Bethe's result [Jk$^{\text{n}}$ Eq. (13.44)]...

Jk Eq. (13.44)

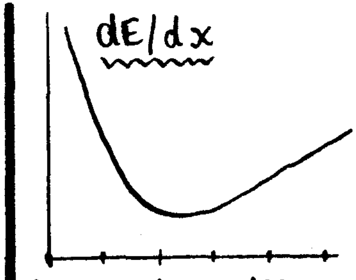
$$\boxed{\frac{dE}{dx} = 4\pi N Z\left(\frac{Q^2e^2}{mv^2}\right)\left\{\ln\left[\frac{2\gamma^2 mv^2}{\hbar\langle\omega\rangle}\cdot C(v)\right] - \frac{v^2}{c^2}\right\}}$$ ← NOTE ⟋ $C(v) = 1$, for $v > \alpha c$; $= \frac{1}{z}(v/\alpha c)$, $v < \alpha c$. (24)

Remarks on dE/dx. Notion of "range" for the stopping particle.

REMARKS on (dE/dx) to date.

1. $\frac{dE}{dx}\left[\begin{matrix}\text{Bohr}\\ \text{Eq.(21)}\end{matrix}\right]$ & $\frac{dE}{dx}\left[\begin{matrix}\text{Bethe}\\ \text{Eq.(24)}\end{matrix}\right]$ differ principally in the arguments of the ln...
$B_c(\text{Bohr}) = 1.123\,\gamma^2 m v^3/Qe\langle\omega\rangle \rightarrow B\,(\text{Bethe}) = 2\gamma^2 m v^2/\hbar\langle\omega\rangle$. This is the result of Bethe's introducing the multiplicative QM factor $\eta = Qe/\hbar v$.

2. The way dE/dx behaves as a fcn of Q's energy E is sketched at right. dE/dx goes through a broad minimum at $\gamma \sim 2$, i.e. when Q's K.E. $\simeq Mc^2$.



3. Since $\frac{dE}{dx} \propto v$, then a particle (Q,M) entering a solid at some v_{in} will see a changing loss rate as it slows down. We can get E as fcn of x = distance traveled by rewriting Bethe's formula of Eq. (24) as follows...

Let : $\epsilon = \frac{E}{Mc^2} = \gamma$ ⌠ Q's total energy in units of rest energy. So ; $\beta = \frac{v}{c} = \frac{1}{\epsilon}\sqrt{\epsilon^2 - 1}$.

Bethe's Eq.(24), viz : $\frac{dE}{dx} = 4\pi NZ\left(\frac{Qe^2}{mc^2}\right)\frac{1}{\beta^2}\left\{\ln\left[\frac{2mc^2}{\hbar\langle\omega\rangle}\gamma^2\beta^2\right] - \beta^2\right\}$,

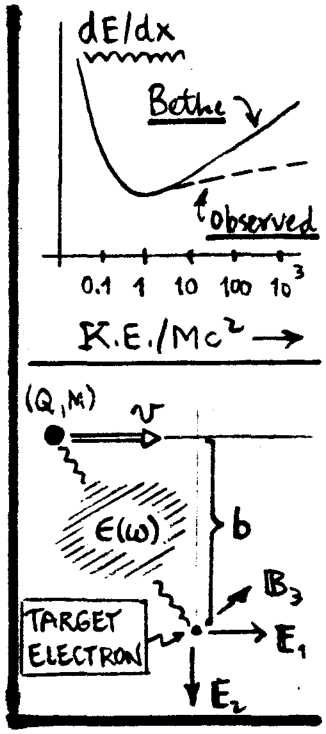
becomes //
$$\boxed{\frac{d\epsilon}{dx}\bigg|_{loss} = (-)\frac{1}{\lambda}\left\{\left(\frac{\epsilon^2}{\epsilon^2-1}\right)\ln\left[B(\epsilon^2-1)\right] - 1\right\}},\quad B = \frac{2mc^2}{\hbar\langle\omega\rangle} \gg 1 \qquad \underline{(25)}$$

w// $\lambda = \frac{M/m}{4\pi Z\zeta^2}\cdot\frac{a_0}{\alpha^4(Na_0^3)}$ for $Q = \zeta e$, $a_0 = \hbar^2/me^2 = 0.53\times10^{-8}$ cm, N = # atoms/vol., $\alpha = e^2/\hbar c \simeq 1/137$;

or// $$\underline{\underline{\lambda = 273\text{ cm}\times\left(\frac{M/M_p}{Z\zeta^2}\right)\frac{1}{Na_0^3}}},\quad M_p = \text{proton mass.} \qquad \underline{(26)}$$

λ gives a distance scale for the energy loss. One can now integrate Eq. (25) to get $x = x(\epsilon)$. The "range" for the stopping particle is the distance R traveled during an energy reduction from some input value $\epsilon = \epsilon_{in} > 1$ to $\epsilon \sim 1$. Details are left to a problem.

7) Bethe's modification to Bohr's formula for $\frac{dE}{dx}$ fitted lab data very well at low energies (K.E. < few × Mc^2), but was systematically high at higher energies. Fermi invented a correction which explained the discrepancy (1940); it is called the "density effect", and depends on the idea that the medium intervening between (Q, M) and a target electron is active rather than passive -- the medium has a dielectric cnst $\epsilon(\omega) \neq 1$ which modifies the interaction and energy transfer between (Q, M) and the target electron.



Fermi's calculation is fairly dense, but we shall outline it here--because it gives an easy way to explain the important phenomenon of Cerenkov radiation. Jackson gives details in his Secs. 13.4 & 13.5.

Fermi made the medium active by using the SHO model for $\epsilon(\omega)$, viz...

$$\to \epsilon(\omega) = 1 + (\omega_p^2/Z)\sum_{k=1}^{Z} f_k \Big/ \left[(\omega_k^2 - \omega^2) - i\omega\Gamma_k\right] \quad \Big\lceil \begin{array}{l} \mathrm{Re}\,\epsilon \sim n^2,\ n \leftrightarrow \text{refraction}; \\ \mathrm{Im}\,\epsilon \Rightarrow \text{absorption}; \end{array}$$

w/ $\underline{\underline{\omega_p}}^2 = 4\pi N Z e^2/m$, plasma frequency $\Big\lceil \begin{array}{l} N = \#\ \text{atoms/vol,} \\ Z = \#\ \text{electrons/atom} \end{array}$;

& k^{th} electron has oscillator strength f_k $\left(\begin{array}{c}\text{norm is:} \\ \sum_k f_k = Z\end{array}\right)$, natural freq. ω_k, damping cnst Γ_k. (27)

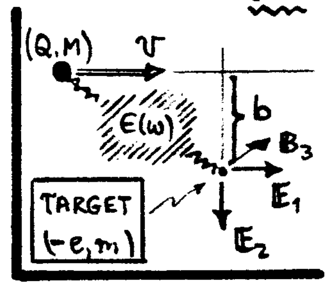
Here ω is the frequency of the (pulse-like) $\mathbb{E}$-field passing by the target electron; that electron will see the corrected field : $\mathbb{D}_\omega(\mathbf{r},t) = \epsilon(\omega)\,\mathbb{E}_\omega(\mathbf{r},t)$. Note that $\epsilon(\omega)$ is complex in general -- so it has an absorptive part as well as refractive. In Bohr's & Bethe's work, $\epsilon(\omega) \equiv 1$ implicitly, and (Q,M) interacted with each target electron individually. Here, in Fermi's picture, (Q,M) interacts collectively with the entire target. In that regard, the Fermi approach is much more realistic.

Summary of Fermi's density-effect calculation: Fermi's $\left(\frac{dE}{dx}\right)$.

Fermi's calculation is done with full Fourier transforms of the fields, like:

$$\rightarrow E(\mathbf{r},t) \rightarrow \tilde{E}(\mathbf{k},\omega) = (1/\sqrt{2\pi})^4 \int_{-\infty}^{\infty} dt \int_{\infty} d^3x \; E(\mathbf{r},t)\, e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega t)}. \qquad (28)$$

Details appear in Jk$^{\text{n}}$ Eqs. (13.53)-(13.68).



RESULTS for Fermi's density-effect calculation.

1. The $(Q,M)\rightarrow(-e,m)$ energy transfer at impact parameter b is

$$\rightarrow \Delta E(b) = \frac{1}{2\pi N}\,\mathrm{Re}\int_0^{\infty}\{-i\omega\epsilon(\omega)\}\,|\mathbb{E}(\omega,b)|^2\, d\omega, \qquad (29)$$

Jk$^{\text{n}}$ Eq. (13.65)

w/ $\mathbb{E}(\omega,b) = (1/\sqrt{2\pi})^3 \int_{\infty} d^3k\; \tilde{\mathbb{E}}(\mathbf{k},\omega)\, e^{ibk_2}$ — k_2 = comp. of $\mathbf{k}$ along $\mathbb{E}_2$ (transverse field) axis.

This form of $\Delta E(b)$ is a generalization of Eq. (12) above when $\epsilon\rightarrow\epsilon(\omega)$.

2. The amplitudes of the field transforms $\mathbb{E}(\omega,b)$ for the longitudinal & transverse fields $\mathbb{E}_1$ ($\parallel$ to $\mathbf{v}$) & $\mathbb{E}_2$ ($\perp$ to $\mathbf{v}$) are...

Jk$^{\text{n}}$ Eqs. (13.63) and (13.64)

$$\left[\begin{array}{ll} \text{Longitudinal field}: & E_1(\omega,b) = -\dfrac{iQ}{\omega\epsilon(\omega)}\sqrt{\dfrac{2}{\pi}}\,\lambda^2 K_0(\lambda b), \\ \text{transverse field}: & E_2(\omega,b) = \dfrac{Q}{v\,\epsilon(\omega)}\sqrt{\dfrac{2}{\pi}}\,\lambda K_1(\lambda b), \\ \text{transverse magnetic field}: & B_3(\omega,b) = \epsilon(\omega)\beta\, E_2(\omega,b); \end{array}\right\} \qquad (30)$$

Jk$^{\text{n}}$ Eq. (13.61) w/

$$\lambda = \frac{\omega}{v}\left[1-\beta^2\epsilon(\omega)\right]^{1/2}, \quad \epsilon(\omega) \text{ given in Eq. (27)}. \qquad (31)$$

NOTE: λ has the dimensions of a wave# (i.e. L^{-1}).

The magnetic field B_3 is "new" in that we have previously ignored it.

3. The fields of Eq. (30) are now put into the energy loss $\Delta E(b)$ of Eq. (29), and the loss for all collisions @ $b \geqslant a$ is found by: $\left(\frac{dE}{dx}\right)_{b\geqslant a} = 2\pi N\int_a^{\infty}\Delta E(b)\cdot b\,db$.

Result // Jk$^{\text{n}}$ Eq. (13.67)

$$\boxed{\left(\frac{dE}{dx}\right)_{b\geqslant a} = \frac{2Q^2 a}{\pi}\,\mathrm{Re}\int_0^{\infty}\left\{\frac{i|\lambda|^2}{\omega\epsilon(\omega)}\right\}\lambda K_1(\lambda^* a)\,K_0(\lambda a)\,d\omega}. \qquad (32)$$

This is Fermi's Stopping Power Formula. λ is defined in Eq. (31).

Remarks on Fermi's (dE/dx). Comparison with data as $v \to c$.

REMARKS on Fermi Stopping Power Formula, Eq.(32).

1. It is important to note that the parameter λ in Eq. (31) is complex in general:

$$\rightarrow \lambda = \frac{\omega}{v}\left[\left(1-\beta^2 \operatorname{Re}\epsilon(\omega)\right) - i\left(\beta^2 \operatorname{Im}\epsilon(\omega)\right)\right]^{1/2} \tag{33}$$

Suppose $\operatorname{Im}\epsilon(\omega)$ is negligible [i.e. ~no absorption of Q's radiation in the medium]. Set//

$$\operatorname{Re}\epsilon(\omega) = [n(\omega)]^2, \quad \text{refractive index}\Big\}\ n(\omega) = \frac{c}{v_p(\omega)} \ \Big\lceil\ v_p(\omega) = \text{phase velocity of EM waves at freq. } \omega.$$

...so//

$$\boxed{\lambda \simeq \frac{\omega}{v}\left[1-(v/v_p(\omega))^2\right]^{1/2}} \ \Big\lceil \ \begin{array}{l}\lambda \text{ is real for } v(Q,M) \leqslant v_p(\omega), \\ \lambda \text{ is imag. for } v(Q,M) > v_p(\omega).\end{array} \tag{34}$$

As we will see, the fact that $\lambda \to$ imag. when $v > v_p$ explains Cerenkov radn.

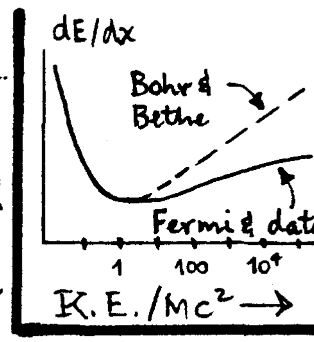
2. Fermi's formula can be shown to reduce to the (modified) Bohr formula [i.e. Eq.(21) above] when $\epsilon(\omega) \to 1$, so that $\lambda \to \omega/v\gamma$. At the other extreme, when $\epsilon(\omega) \neq 1$ is retained and $v \to c$...

indpt of structure of material.

$$\text{Fermi}: \ (dE/dx)_{b \gg a} \to Q^2 \frac{\omega_p^2}{c^2} \ln(\mathcal{N}c/a\omega_p) = \text{cnst},$$

$$\text{(modified) Bohr}: \ (dE/dx)_{b \gg a} \to Q^2 \frac{\omega_p^2}{c^2} \ln(\gamma\mathcal{N}c/a\langle\omega\rangle); \tag{35}$$

W// $\mathcal{N} = 1.123$, &// $\omega_p^2 = 4\pi N Z e^2/m$ (plasma freq.). Fermi wins...



3. Fermi's density correction evidently depends on there being an imaginary part to $\epsilon(\omega)$ and thus to λ in Eq. (33). Also, the correction is most pronounced as $v \to c$... only when $\beta \to 1$ does λ acquire a non-negligible imaginary part.

4. For future reference, we note that Fermi's Formula in Eq.(32) can be written:

$$\left(\frac{dE}{dx}\right)_{b \gg a} = \frac{Q^2}{c^2} \operatorname{Re}\int_0^\infty \left(i\omega\sqrt{\frac{\lambda^*}{\lambda}}\right)\left[\frac{1}{\beta^2\epsilon(\omega)} - 1\right]\left(\sqrt{\frac{2\lambda^* a}{\pi}}\,K_1(\lambda^* a)\right)\left(\sqrt{\frac{2\lambda a}{\pi}}\,K_0(\lambda a)\right)d\omega, \tag{36}$$

after rearranging. This form is helpful in analysing Cerenkov radn, next up.