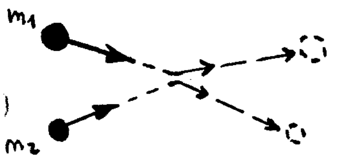
Space-Time & Charge Symmetries of EM Quantities.

1) The question of how any field theory behaves under the following operations:

OPERATOR	NAME	EFFECT of OPERATOR × SYSTEM
C	charge conjugation	all charges reverse sign: $q \to (-)q$.*
P	parity (space inversion)	coordinate vector is mirror-imaged: $\mathbf{r} \to (-)\mathbf{r}$.
T	time reversal	time t runs "backward": $t \to (-)t$.

has become relevant in the past 35 years, since it has been noticed that all acceptable theories in physics are <u>invariant</u> under one or more (or a combination) of these operations. What this means can be illustrated simply for T...

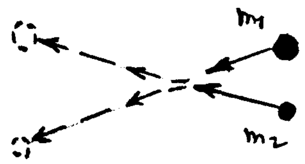
SYSTEM ①: collision



use: $\dfrac{d\mathbf{p}}{dt} = -\nabla U$

$$\underbrace{T(\text{sys.}①)}_{\text{physically possible}} = \underbrace{\text{sys.}①'}_{\text{equally possible}}$$

SYSTEM ①': collision



use: $\dfrac{d\mathbf{p}'}{dt'} = -\nabla' U'$

① & T① = ①' are both equally possible, and both are described by the <u>same</u> physics (Newton II).

One says that <u>Newton II is T-invariant</u>, i.e. $T \times \left[\dfrac{d\mathbf{p}}{dt} = -\nabla U\right]$ does not change the physics, or generate some new and unobserved event. "Invariance" here does <u>not</u> mean the system does not change [obviously ①' = T① ≠ ①; if ① runs "forward" in time, then ①' is running "backwards"]. What invariance <u>does</u> mean is that both ① & ①' are physically (and equivalently) realizable, and both can be described by the <u>same</u> theory. The theory cannot pick out a preferred time-direction.

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2) In olden times (≲1956), it was thought that all theories would show C, P, & T invariances separately. Actually, this was not thought out very carefully; it seemed obvious and not very important. Then, in ~1956, and at the suggestion

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* More generally: C×(SYSTEM) changes particles to anti-particles and vice-versa.

of T.D. Lee & C.N. Yang, an experiment was performed by C.S. Wu at Columbia University on the β-decay of Co^{60}. She found...★

$$\left[\!\left[\; P(\underbrace{\beta\text{-decay}}_{\text{observable}}) = \underbrace{(\beta\text{-decay})'}_{\text{not observable}} \;\right\} \right.$$

P is not conserved in β-decay. So theory of weak interactions (governing β-decay) is not P-invariant.

However, Wu noted that PC was conserved, i.e. $P \times C \times (\beta\text{-decay}) = (\beta\text{-decay})''$ was a physically observable outcome. So, weak interaction is PC-invariant, not P-invariant.

This unexpected result led to a critical examination of just what invariances were obeyed in which theories. What is believed these days is the "CPT Theorem" [W. Pauli, G. Lüders, J. Schwinger, ca. 1954]. This says, symbolically...

$$\left[\!\left[\; CPT \times (\underbrace{\text{system}①}_{\text{use theory A}}) = \underbrace{\text{system}①'}_{\text{use theory A}'} \;\right\|\right.$$

If ① is physically observable, then so is ①'. And theory $A \rightarrow A'$ is form-invariant under CPT.

CPT invariance is thought to be a very firm requirement -- were it to be violated, then causality and Lorentz covariance would also fail.

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3) With this background, we examine the C, P, & T transform properties of Maxwell's field theory. This is important because EM interactions are the dominant feature of atoms & molecules, and are non-negligible even in nuclei. So we are looking at the C, P & T characteristics of ~ all of (low energy) laboratory matter.

The working tools of any respectable field theory are scalars, vectors & tensors. So we begin by recalling how these objects are defined w.r.t. transformations.

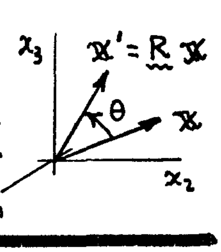
★ Specifically, Wu showed the Co^{60} β-decay rate contained a term $\propto \mathbf{S}\cdot\mathbf{p}$, where $\mathbf{S}$ is the nuclear spin and $\mathbf{p}$ is the emergent β momentum. Now $P\mathbf{S} = +\mathbf{S}$ (axial vector), while $P\mathbf{p} = (-)\mathbf{p}$ (polar vector), so $P[\mathbf{S}\cdot\mathbf{p}] = (-)[\mathbf{S}\cdot\mathbf{p}]$; see Eq (9) below. Thus $\mathbf{S}\cdot\mathbf{p}$ is a pseudoscalar, and the Co^{60} decay process is not P-invariant.

Definitions of scalars, vectors, matrices, tensors.

Scalars, vectors & tensors in 3D space are defined in terms of their properties under rotation, as represented by a rotation matrix $\underset{\sim}{R}$:

$\mathbf{x} \to \mathbf{x}' = \underset{\sim}{R}\,\mathbf{x}$, $\underset{\sim}{R} = (R_{ik})$ an $n \times n$ matrix in nD space.

e.g.// in 2D: $\underset{\sim}{R} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$, $\theta = \measuredangle(\mathbf{x}, \mathbf{x}')$ √ $\mathbf{x}$ is rotated, not the cds. (1)



NOTE: Length $\mathbf{x}^2$ invariant $\Rightarrow$ $\underline{R_{ij} R_{ik} = \delta_{jk} = R_{ji} R_{ki}}$, $\underline{R^{-1}_{ik} = R_{ki}}$. (using summation convention.)

Any rotation matrix $\underset{\sim}{R}$ preserving the length $\mathbf{x}^2$ is thus orthogonal.

also// $\det \underset{\sim}{R} = +1$, for "proper" rotations; $\det \underset{\sim}{R} = (-)1 \Rightarrow$ "improper" {reflection; need P. (2)

(a) Then, for a vector $\mathbb{A} = (\cdots A_i \cdots)$; under rotation:

if// $\underline{\underline{A_i \to A_i' = R_{ik} A_k}}$, then// $\mathbb{A}$ is a **vector** {e.g. position $\mathbb{r}$, … momentum $\mathbb{p} = m\mathbb{v}$, … current density $\mathbb{J} = nq\mathbb{v}$. (3)

Note, for gradient operator $\nabla = (\cdots \partial/\partial x_i \cdots)$:

$$\frac{\partial}{\partial x_i} \to \frac{\partial}{\partial x_i'} = \left(\frac{\partial x_k}{\partial x_i'}\right)\frac{\partial}{\partial x_k} \quad \ldots \text{ but } \; x_k = R^{-1}_{kj} x_j' = R_{jk} x_j' \Rightarrow \frac{\partial x_k}{\partial x_i'} = R_{ik}$$

$$= R_{ik}\frac{\partial}{\partial x_k}, \text{ so } \nabla \text{ is a \underline{vector} operator.} \tag{4}$$

(b) For a scalar field $\phi = \phi(x_i)$, under rotation:

if// $\underline{\underline{\phi(x_i) \to \phi'(x_i') \equiv \phi(x_i)}}$, then// ϕ is a **scalar** {e.g. $\phi = \mathbb{A}\cdot\mathbb{B}$ (if $\mathbb{A}$ & $\mathbb{B}$ are polar), … $\phi = \nabla\cdot\mathbb{E}$ (if $\mathbb{E}$ is polar); [for "polar", see Eq.(9)]. (5)

(c) For matrices $\underset{\sim}{M} = (M_{ij})$ = 2D array, under rotation:

if// $\underline{\underline{M_{ij} \to M_{ij}' = R_{ik} R_{jl} M_{kl}}}$, then// $\underset{\sim}{M}$ is a **matrix** {e.g. $\underset{\sim}{M} = \underset{\sim}{R}$, a rot^{n}; … $\underset{\sim}{M}$ = Max. stress tensor. (6)

(d) For a tensor $\underset{\sim}{T} = (T_{\alpha\beta\gamma\cdots\epsilon})$ = nD array (called a "tensor of rank n"):

if// $\boxed{T_{\alpha\beta\gamma\ldots\epsilon} \to T'_{\alpha\beta\gamma\ldots\epsilon} = (R_{\alpha\kappa} R_{\beta\lambda} R_{\gamma\mu} \cdots R_{\epsilon\nu})\, T_{\kappa\lambda\mu\ldots\nu}}$ then// $\underset{\sim}{T}$ is a **tensor**. (7)

Classification of ϕ, $\mathbb{A}$, ... under parity operation.

NOTE: All these quantities can be classified as tensors by defining: scalar $\phi \equiv$ tensor of rank 0 (carries 0 indices and rotates by R^0), vector $\mathbb{A} \equiv$ tensor of rank 1 (comps carry 1 index, and rotate by R^1), matrix $\underset{\sim}{M}$ = tensor of rank 2 (2 indices, rotate by R^2), etc. Then Eq. (7) gives the required rotational transform for the general case.

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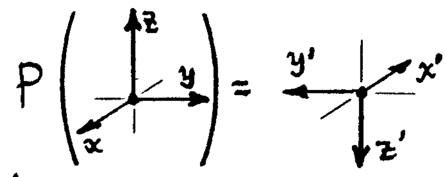
4) For the tensors in Eqs. (3)-(7), behavior under spatial rotations $\underset{\sim}{R}$ is not the whole story. We need a classification under the other space-like operation, namely spatial inversions P, i.e. the parity operation.



P is not attained by rotations. A 180° rot$^{\underline{n}}$ about x-axis sends $y \to y'$, $z \to z'$, but not $x \to x'$.

P, or mirror-imaging, sends $\mathbb{r} \to \mathbb{r}' = (-)\mathbb{r}$; this operation cannot be achieved by ordinary (continuous) rotations (by 180°, or anything else -- see sketch). The P operation is discontinuous; it can be represented as:

$$x_i \to x_i' = (-)\delta_{ik}\, x_k, \quad \text{i.e.}/\!/ \quad P = \begin{pmatrix} -1 & & 0 \\ & -1 & \\ 0 & & \ddots \end{pmatrix}. \tag{8}$$

NOTE: in 3D, $\det P = \det\begin{pmatrix} -1 & & 0 \\ & -1 & \\ 0 & & -1 \end{pmatrix} = (-)1$; this is sign of "improper rot$^{\underline{n}}$".

Now, behavior under P gives a bifurcation for each of tensors in Eqs. (3)-(7). E.g.

$$\text{(a)}' \quad \text{if}/\!/ \quad P\mathbb{A} = \begin{cases} (-)\mathbb{A}, \text{ then } \mathbb{A} \text{ is a "polar (true) vector" ... e.g. } \mathbb{r}, \mathbb{p}, \mathbf{J}, \mathbf{E}, \text{... are polar;} \\ (+)\mathbb{A}, \text{ then } \mathbb{A} \text{ is an "axial (pseudo) vector" ... any } \mathbb{G}\times\mathbb{H},\ \mathbb{G}\ \&\ \mathbb{H} \text{ polar.} \end{cases} \tag{9}$$

$$\text{(b)}' \quad \text{if}/\!/ \quad P\phi = \begin{cases} (+)\phi, \text{ then } \phi \text{ is a "true scalar" ... e.g. } \phi = \nabla\cdot\mathbf{E}, \text{ w/ } \mathbf{E} \text{ polar;} \\ (-)\phi, \text{ then } \phi \text{ is a "pseudoscalar" ... } \phi = \mathbb{A}\cdot(\mathbb{B}\times\mathbb{C}), \text{ all polar.} \end{cases} \tag{10}$$

⋮

$$\text{(d)}' \quad \text{For } n^{\underline{th}} \text{ rank tensor } \underset{\sim}{T}: \quad P\underset{\sim}{T} = \begin{cases} (-)^n\, \underset{\sim}{T} \Rightarrow \underset{\sim}{T} \text{ is a "true tensor"} \\ (-)^{n+1}\, \underset{\sim}{T} \Rightarrow \underset{\sim}{T} \text{ is a "pseudotensor."} \end{cases} \tag{11}$$

Armed with these definitions, we can now classify the various elements of Maxwell's field theory. Each element can be signed with (C, P, T) numbers, according to how it behaves under these transforms. E.g. $\rho\left(\begin{smallmatrix}\text{charge}\\ \text{density}\end{smallmatrix}\right)$ has $(C,P,T) = (-1, +1, +1)$. See next page.

TRANSFORMATIONS of EM DESCRIPTORS under P (parity), T (time-reversal), & C (charge-conjugation).†

QUANTITY	NAME	transform under P	T	C	overall CPT	REMARKS
$\mathbf{r}=(x,y,z)$	position	− (polar)	+	+	−	$\mathbf{r}$ is the prototype polar vector.
$\mathbf{v}=d\mathbf{r}/dt$	velocity	− (polar)	−	+	+	$\mathbf{v}$ is T-odd; $\mathbf{a}=\frac{d\mathbf{v}}{dt}$ is T-even.
$\mathbf{L}=m\,\mathbf{r}\times\mathbf{v}$	angular momentum	+ (axial)	−	+	−	$\mathbf{L}$ is the prototype axial vector.
$\rho,\ \phi=\int\frac{\rho}{R}\,dV$	scalar density & potential	+	+	−	−	ρ is Lorentz invariant {P & T signs are by convention.
$\mathbf{J},\ \mathbf{A}=\int\frac{\mathbf{J}}{R}\,dV$	vector density & potential	− (polar)	−	−	−	$\mathbf{J}=ne\mathbf{v}$ is evidently polar.
$\mathbf{E};\ \mathbf{D}=\epsilon\mathbf{E},\ \mathbf{P}=\frac{1}{4\pi}(\mathbf{D}-\mathbf{E})$	electric field vectors	− (polar)	+	−	+	$\mathbf{E}=-\nabla\phi-\frac{1}{c}(\partial\mathbf{A}/\partial t)$ is polar.
$\mathbf{H};\ \mathbf{B}=\mu\mathbf{H},\ \mathbf{M}=\frac{1}{4\pi}(\mathbf{B}-\mathbf{H})$	magnetic field vectors	+ (axial)	−	−	+	$\mathbf{B}=\nabla\times\mathbf{A}$ is evidently axial.
$\mathbf{F}=\rho\mathbf{E}+\frac{1}{c}\mathbf{J}\times\mathbf{B}$	Lorentz force / unit volume	− (polar)	+	+	−	… follows from $\mathbf{E}$ & $\mathbf{B}$, ρ & $\mathbf{J}$ transforms. Both elec. & mag. terms transform same way.
$u=\frac{1}{8\pi}(\mathbf{E}\cdot\mathbf{D}+\mathbf{H}\cdot\mathbf{B})$	EM field energy density	+	+	+	+	… follows from $\mathbf{E}$ & $\mathbf{B}$ transforms.
$\mathbf{S}=\frac{c}{4\pi}(\mathbf{E}\times\mathbf{H}),\ \mathbf{g}=\frac{\mu\epsilon}{c^2}\mathbf{S}$	Poynting (transport) vectors	− (polar)	−	+	+	… ditto.
$T_{ik}=\frac{1}{4\pi}(E_iD_k+H_iB_k)-u\,\delta_{ik}$	Maxwell stress tensor	+	+	+	+	… ditto.

† Augments Table (6.1), p. 249 of J. D. Jackson "Classical ElectroDynamics" (Wiley, 2nd ed., 1975).