Momentum as an operator in position space: $p_{op} = -i\hbar(\partial/\partial x)$.

or/ $$\langle p \rangle_0 = -i\hbar \int dx \underbrace{\left[\int dx' \delta(x'-x)\psi_0^*(x')\right]}_{\psi_0^*(x)} \frac{\partial}{\partial x}\psi_0(x)$$

$$\langle p \rangle_0 = \int dx\, \psi_0^*(x) \left\{-i\hbar \frac{\partial}{\partial x}\right\} \psi_0(x) = \langle -i\hbar(\partial/\partial x) \rangle. \tag{34}$$

Thus the momentum p, defined as $p = \hbar k$ w.r.t. the spectral fcn $\varphi(k)$ in Eq. (31B), takes the form of a linear operator $p = -i\hbar\partial/\partial x$ w.r.t. space wavefcn $\psi(x,0)$. Although we have considered the case $t=0$, generalization to $t \neq 0$ is simple, since t appears only as a parameter during x & k integrations...

1. for $t \neq 0$, use wavefcn: $\psi(x,t) = \frac{1}{\sqrt{2\pi}} \int \varphi(k) e^{i[kx - \omega(k)t]} dk$.

2. → Fourier inverse (in k) is: $\Phi(k,t) = \frac{1}{\sqrt{2\pi}} \int \psi(x,t) e^{-ikx} dx$,

i.e.// $$\Phi(k,t) = \frac{1}{2\pi} \int dx\, e^{-ikx} \int dk'\, \varphi(k') e^{ik'x - i\omega(k')t}$$

$$= \int dk' \varphi(k') e^{-i\omega(k')t} \cdot \underbrace{\left[\frac{1}{2\pi}\int dx\, e^{i(k'-k)x}\right]}_{=\delta(k'-k)} = \varphi(k) e^{-i\omega(k)t}.$$

3. Define momentum $\langle p \rangle$ @ $t \neq 0$ by...

$$\rightarrow \langle p \rangle_t = \int dk\, \Phi^*(k,t)\{\hbar k\}\Phi(k,t) = \int dk\, \varphi^*(k)\{\hbar k\}\varphi(k) = \langle p \rangle_0. \tag{35}$$

So, for a free particle, the mean momentum $\langle p \rangle$ does not change in time, i.e. $\langle p \rangle_t = \langle p \rangle_0 =$ cnst. This is reassuring, since a free particle -- by definition -- must have cnst momentum. Showing that the QM average momentum $\langle p \rangle =$ cnst here is as close as we can come to that classical fact.

In any case, the result in Eq. (34) is true as a general property of 1D (free particle) wavefcns. It implies the assignment of an operator to $\mathbf{p}$:

$$\boxed{\text{In an expectation value sense, momentum } \mathbf{p} = \hbar \mathbf{k} \text{ in } \mathbf{k}\text{-space is equivalent to the operator } \mathbf{p}_{op} = -i\hbar\nabla \text{ in } \mathbf{r} \text{ (position) space.}} \tag{36}$$

This justifies the remark at bottom of p. Sch. 9. We will now exploit expectation values.

SUMMARY : Properties & Uses of QM WaveFcns.

Properties of Wave Packets : Advantages & Disadvantages

- $\phi(x,t) = \int_{-\infty}^{+\infty} \varphi(k)\, e^{i[kx-\omega(k)]t}\, dk$ = wavepacket (wave group, localized in space to Δx).
- spectral fcn : $\varphi(k) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \phi(x,0)\, e^{-ikx}\, dx$; (width Δk), fixed by initial value of ϕ.

① widths Δx of ϕ & Δk of φ related via : $\boxed{\Delta k\, \Delta x \sim 1}$ (uncertainty relation).

② for photon ($m=0$), $\omega = kc$, and packet moves undistorted @ velocity $v = dx/dt = c$.

③ in a dispersive medium, $\omega = \omega(k)$, packet transport velocity is : $\boxed{v_g = \partial\omega/\partial k}$ (group velocity).

④ for free motion of mass m, $\omega = \hbar k^2/2m \longleftrightarrow v_g = \partial\omega/\partial k = p/m$ (particle velocity).

⑤ $v_g = \partial\omega/\partial k \longleftrightarrow \partial E/\partial p = v$, even works relativistically for motion of mass m.

⑥ for $m \neq 0$, packet disperses : width (as $t\to\infty$) : $\boxed{\delta x \simeq \alpha t/\delta x_0}$, δx_0 = initial width & $\alpha = \frac{\partial^2\omega}{\partial k^2}$.

⑦ packet intensity $|\phi(x,t)|^2$ cannot specify m's location at point x at time t, but $|\phi(x,t)|^2$ can specify probability that m arrives at pt. x at time t ($|\phi|^2$ disperses per uncertainty relⁿ).

Schrodinger's Wave Eqtn for the Packet Amplitude, or Wave Function

- differentiate ϕ (free particle wave packet), w/ $\omega = \hbar k^2/2m \Rightarrow \boxed{i\hbar\, \partial\phi/\partial t = -\frac{\hbar^2}{2m}\partial^2\phi/\partial x^2}$ SCHRÖDINGER'S WAVE EQTN.
- alternatively, let $\left\{\begin{array}{l} E \to E_{op} = i\hbar\,\partial/\partial t \\ p \to p_{op} = -i\hbar\,\partial/\partial x \end{array}\right\|$ then; $(E_{op})\phi = (p_{op}^2/2m)\phi \Rightarrow$ same wave eqtn.
- alternatively, use KG Eqtn for a massive photon : $\left[\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - (mc/\hbar)^2\right]\phi(\mathbf{r},t) = 0$.

Put : $\psi = \phi\, e^{i(mc^2/\hbar)t}$. Take NR limit of KG Eqtn $\Rightarrow$ get above Schrodinger's Eqtn.

① S. Eq. is not new... it describes a "photon" of mass $m>0$, moving @ $v \ll c$, and dispersing as it goes [per $\delta x \sim (\hbar/m\delta x_0)t$], even in free space. However, the QM interpretation of x & t cds, and $|\phi(x,t)|^2$ as a probability are new.

② Call $\psi(\mathbf{r},t) = \phi(\mathbf{r},t)\, e^{i(mc^2/\hbar)t}$ the "wave function": $i\hbar\,\partial\psi/\partial t = -\frac{\hbar^2}{2m}\nabla^2\psi$ {for free particle.

③ Define : $|\psi(\mathbf{r},t)|^2 d^3r$ = probability of finding m in volume d^3r at position $\mathbf{r}$ at time t. This probability is globally conserved : $(\partial/\partial t)\int_\infty |\psi|^2 d^3r = 0$ & $\int_\infty |\psi|^2 d^3r = 1$.

④ Probability also locally conserved : $\boxed{\frac{\partial\rho}{\partial t} + \nabla\cdot\mathbf{J} = 0}$ $\left\{\begin{array}{l} \rho = \psi^*\psi = |\psi|^2; \\ \mathbf{J} = \mathrm{Re}[\psi^*\{(\hbar/im)\nabla\}\psi]. \end{array}\right.$

⑤ If $\psi^*\psi\, d^3r$ = prob. distribution of m @ $\mathbf{r}$ & t, then $\varphi^*\varphi\, d^3k$ = prob. distribution for m with momentum $\mathbf{k}$. ψ & φ can both be normed : $\int_\infty |\psi|^2 d^3r = 1 = \int_\infty |\varphi|^2 d^3k$.

⑥ Maximum information available re the dynamical variable $f(\mathbf{r})$ is the mean value or "expectation value" : $\boxed{\langle f(t)\rangle = \int_\infty \psi^*(\mathbf{r},t)\{f(\mathbf{r})\}\psi(\mathbf{r},t)\, d^3r}$.

⑦ Momentum : $\langle\mathbf{p}\rangle = \int_\infty \varphi^*\{\hbar\mathbf{k}\}\varphi\, d^3k = \int_\infty \psi^*\{-i\hbar\nabla\}\psi\, d^3r$, i.e. $\mathbf{p} \to \boxed{\mathbf{p}_{op} = -i\hbar\nabla}$.