#1 : JH

DEPARTMENT OF PHYSICS
PH.D. QUALIFYING EXAMINATION
MONDAY, AUGUST 19, 1991

Answer each of the following questions, all of which carry equal weight. Begin your answer to each question on a new sheet of paper; solutions to different questions must not appear on the same sheet. Each sheet of paper must be labeled with your name and the problem number in the upper right hand corner. When more than one sheet is submitted for a problem, be sure the pages are ordered properly. [Problems 1-4]

1. A particle of mass m and charge q, constrained to move along the x-axis, is bound by a linear restoring force with spring constant k. The system is subjected to an electric pulse of duration τ; the perturbation is described by the Hamiltonian

$$H'(t) = \begin{cases} -qEx\ , & 0 < t < \tau \\ 0\ , & \text{otherwise} \end{cases}$$

where E is a constant.

(a) Determine the first-order probability that the particle will be found in its first excited state at $t > \tau$ if it was in its ground state at $t < 0$.

(b) Under what circumstances is the first-order result accurate?

Sol'n

$$H = H_0 + H' \left\{ \begin{array}{l} H_0 = (a^\dagger a + \tfrac{1}{2})\hbar\omega \\ H' = -q\mathcal{E}x \,, \quad 0<t<\tau \end{array} \right.$$

Here $x = \frac{\Delta}{\sqrt{2}}(a+a^\dagger)$; $\Delta = \sqrt{\frac{\hbar}{m\omega}}$; $\omega = \sqrt{\frac{k}{m}}$

After the pulse, to 1st order,

$$P = |a_1|^2 = \frac{1}{\hbar^2}\left|\int_0^\tau dt\, H'_{10}\, e^{i\omega t}\right|^2$$

where $H'_{10} = -q\mathcal{E}\langle 1|x|0\rangle$

$$= -q\mathcal{E}\frac{\Delta}{\sqrt{2}}\underbrace{\langle 1|a+a^\dagger|0\rangle}_{1}$$

$$= -q\mathcal{E}\frac{\Delta}{\sqrt{2}}$$

7 pts.

$$P = \frac{1}{\hbar^2}\left(\frac{q\mathcal{E}\Delta}{\sqrt{2}}\right)^2\left|\int_0^\tau dt\, e^{i\omega t}\right|^2$$

$$= \left(\frac{q\mathcal{E}\Delta}{\sqrt{2}\,\hbar}\right)^2\left|\frac{e^{i\omega\tau}-1}{i\omega}\right|^2$$

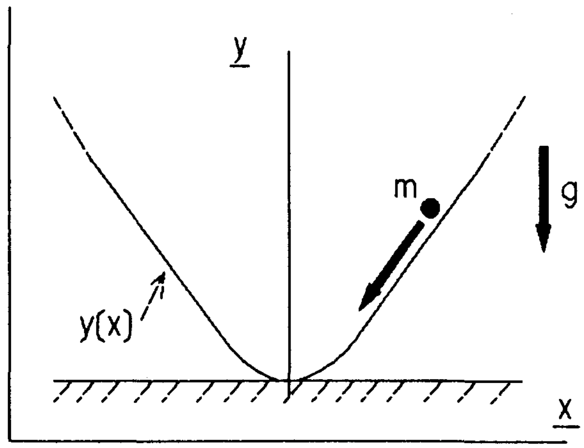
$$= \left(\frac{q\mathcal{E}\Delta}{\sqrt{2}\,\hbar}\right)^2\left|\frac{e^{i\omega\tau/2}-e^{-i\omega\tau/2}}{i\omega}\right|^2$$

$$P = 2\left(\frac{q\mathcal{E}\Delta}{\hbar\omega_0}\right)^2\sin^2(\omega_0\tau/2) = \frac{2q^2\mathcal{E}^2}{m\hbar\omega_0^3}\sin^2(\omega_0\tau/2)$$

(b) The result is valid provided $q\mathcal{E}\Delta \ll \hbar\omega_0$ 3 pts.

#2 : DR

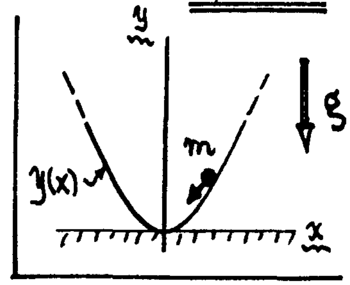
2. A particle of mass m executes oscillations on a frictionless wire described by the curve y = y(x) under gravity g (see sketch). The oscillation amplitude is <u>not</u> necessarily small. Determine the functional form of y(x) such that m's oscillation frequency is <u>independent</u> of its oscillation amplitude.



M: Isochronous Oscillations l.c. g

7/29/91

"A particle of mass m executes oscillations on a curve $y = y(x)$ under gravity g (see sketch). The oscillation amplitude is not necessarily small. Determine the functional form of $y(x)$ such that m's oscillation frequency is independent of its oscillation amplitude."



Solution The motion must be that of a SHO, where--if s = arc length along the curve -- the K.E. & P.E. are: $K = \frac{1}{2} m\dot{s}^2$ & $U = \frac{1}{2} k s^2$ [k = spring cnst & s measured from origin], and the osc. freq. is: $\omega = \sqrt{k/m}$, indpt of s. Any departure from $U \propto s^2$ brings amplitude dependence into ω.

Along the curve: $U = \frac{1}{2} k s^2 = mgy$, so: $s^2 = (2g/\omega^2)\,y$. Then, since...

$$\left.\begin{array}{l} ds^2 = dx^2 + dy^2 \Rightarrow (dx/dy)^2 = (ds/dy)^2 - 1, \\ \text{and } s = (2g/\omega^2)^{\frac{1}{2}}\sqrt{y} \Rightarrow \left(\frac{ds}{dy}\right)^2 = (g/2\omega^2)\frac{1}{y}, \end{array}\right\} \quad \frac{dx}{dy} = \left[\left(\frac{g}{2\omega^2}\right)\frac{1}{y} - 1\right]^{1/2}.$$

Integrate this last equation, assuming curve passes through origin...

$$\rightarrow x = \int_0^y \left[(a/z) - 1\right]^{1/2} dz, \text{ w/ } \underline{\underline{a}} = \frac{g}{2\omega^2}. \longleftarrow \text{substitute: } z = \frac{a}{2}(1-\cos\phi'), \; dz = \frac{a}{2}\sin\phi'\,d\phi', \text{ etc.}$$

so// $x = \frac{a}{2}\int_0^\phi \left(\frac{1+\cos\phi'}{1-\cos\phi'}\right)^{\frac{1}{2}} \sin\phi'\,d\phi'$, w/ $\boxed{y = \frac{a}{2}(1-\cos\phi)}$

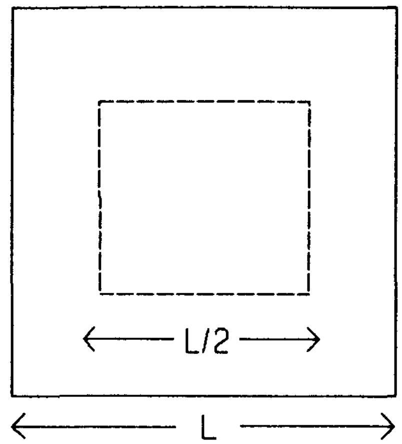
or// $x = 2a\int_0^\phi \cos^2\left(\frac{\phi'}{2}\right) d\left(\frac{\phi'}{2}\right)$, by trig identities.

and// $\boxed{x = \frac{a}{2}(\phi + \sin\phi)}$, with $a = g/2\omega^2$.

The boxed eqtns, with $0 \leqslant \phi$, are the parametric eqtns for a cycloid, and $y = y(x)$ is the desired curve, where -- for any excursion of size $y \leqslant a$ by m -- the oscillation freq. $\omega = \sqrt{k/m}$ is cnst. k is fixed by the desired steepness of $y(x)$.

#3 : ST

3. A square membrane with fixed sides of length L is at rest. The central square section, with sides of length L/2 that are parallel to the fixed sides, is struck a blow such that the struck area moves initially with uniform speed v_o upward. Find the amplitude of the lowest mode of vibration of the membrane (for given v_o).



Comp. Exam 1991 Tsuruta ③

Key

$\boxed{\text{Prm 3}}$

Initial condn at $t=0$ $u(x,y,0)=0$

$$\frac{\partial u}{\partial t} \equiv V(x,y,0) \begin{cases} = V_0 & \text{for } \frac{L}{4} \le x \le \frac{3L}{4} \\ = 0 & \text{elsewhere} \end{cases}$$

Boundary condns: $u(x,y,t)=0$ for $\begin{cases} 0 \le x \le L \\ 0 \le y \le L \end{cases}$

A general solution which satisfies the above conditions is

$$u(x,y,t) = \sum_{nm} a_{nm} \sin\frac{n\pi x}{L} \sin\frac{m\pi y}{L} \sin\omega_{nm} t.$$

where $\omega_{nm}^2 = (n^2+m^2)\left(\frac{\pi c}{L}\right)^2$ ①

by using separation of variables

a_{nm} are obtained from the above initial conditions on $V = \partial u/\partial t = V_0$ at $t=0$.

$$\Rightarrow \omega_{nm} a_{nm} = \left(\frac{2}{L}\right)^2 \int_0^L \int_0^L V(x,y,0) \sin\frac{n\pi x}{L} \sin\frac{m\pi y}{L}\, dx\, dy$$

$$\rightarrow \text{get } \omega_{nm} a_{nm} = \left(\frac{16 V_0}{\pi^2 nm}\right) \cos\frac{n\pi}{4} \cos\frac{m\pi}{4}.$$

The lowest mode is for $n=m=1$

So the amplitude is

$$\boxed{a_{11} = \frac{8V_0}{\omega_{11}\pi^2} = \frac{8V_0 L}{\sqrt{2}\pi^3 c}} \quad \text{Ans.}$$

since $\omega_{11} = \frac{\sqrt{2}\pi c}{L}$ from eqn ①.

4. (a) What is the complete ground electron configuration of the neutral sodium atom?

#4 : RC

(b) A particular transition of the neutral sodium atom is denoted by 3p $^2P_{3/2}$ - 3d $^2D_{5/2}$. Explain the meaning and significance of each part of the notation used.

(c) Describe all other states that may be associated with the "3d" configuration.

(d) Define the parity quantum number and discuss its relevance to the states of part (b).

(e) Comment briefly on the changes in electronic structure that take place when neutral sodium forms the crystal sodium chloride.

Atomic Physics

a) $(1s)^2(2s)^2(2p)^3\ 3s$ for the ground state [$n=3$ clearly not filled; Li occupies 2^{nd} row $\therefore$ must be in 3^{rd} row]

b) valence electron promoted to $n=3$ $l=1$ (3p)
$l=1$ couples to $s=\frac{1}{2}$, giving total angular momentum $j=3/2$ super 2 => $S=\frac{1}{2}$; P => $l=1$, sub $3/2$ is the j value. $({}^2P_{3/2})$

the valence electron promoted to $n=3$ $l=2$ (3d)
$l=2$ couples to $s=\frac{1}{2}$, giving total angular momentum $j=5/2$ $({}^2D_{5/2})$

c) $l=2$ may couple with $s=1/2$ to give $j=3/2$ also

d) parity is the quantum # associated with inversion and is assigned the value +1 (even) or (−1) odd. For single electrons, it is given by $(-1)^l$

e) The valence electron is removed leaving a sodium ion with closed shells (stable). The bonding is <u>ionic</u>.

#5 : JH

DEPARTMENT OF PHYSICS
PH.D. QUALIFYING EXAMINATION
TUESDAY, AUGUST 20,1991

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5. A particle is in the spinor state

$$\psi = \frac{1}{\sqrt{2}}\begin{pmatrix} Y_1^1 \\ \dfrac{Y_1^0 - Y_0^0}{\sqrt{2}} \end{pmatrix} R(r)$$

where $\int_0^\infty |R|^2 r^2 dr = 1$, and the Y_ℓ^m are spherical harmonics. Evaluate $\langle L^2 \rangle$ and $\langle J^2 \rangle$ for this state.

Soln

$$\langle L^2\rangle = \int \psi^{+} L^2 \psi \, d^3r$$

$$= \frac{1}{2}\int_0^\infty r^2 |R|^2 dr \int \left(Y_1^{1*}, \ \frac{Y_1^{0*} - Y_0^{0*}}{\sqrt{2}} \right) \begin{pmatrix} 2\hbar^2 Y_1^1 \\ \frac{2\hbar^2 Y_1^0}{\sqrt{2}} \end{pmatrix} d\Omega$$

, since $L^2 Y_0^0 = 0$.

$$\langle L^2\rangle = \frac{\hbar^2}{2}\int \left\{ 2|Y_1^1|^2 + |Y_1^0|^2 - \cancel{Y_0^{0*} Y_1^0} \right\} d\Omega$$

$$= \frac{\hbar^2}{2}(2+1)$$

$$\boxed{\langle L^2\rangle = \frac{3}{2}\hbar^2}$$

5 pts

$$\langle J^2\rangle = \langle (\underset{\sim}{L} + \underset{\sim}{S})^2\rangle = \langle L^2\rangle + \langle S^2\rangle + 2\langle \underset{\sim}{L}\cdot\underset{\sim}{S}\rangle$$

$$= \frac{3}{2}\hbar^2 + \frac{3}{4}\hbar^2 + 2\langle \underset{\sim}{L}\cdot\underset{\sim}{S}\rangle$$

Now $2L\cdot S = L_+ S_- + L_- S_+ + 2L_z S_z$

$$= \hbar \begin{pmatrix} L_z & L_- \\ L_+ & -L_z \end{pmatrix}$$

$$2\langle \underset{\sim}{L}\cdot\underset{\sim}{S}\rangle = \frac{\hbar}{2}\int \left(Y_1^{1*}, \frac{Y_1^{0*} - Y_0^{0*}}{\sqrt{2}} \right) \begin{pmatrix} L_z Y_1^1 + \frac{1}{\sqrt{2}} L_- Y_1^0 - \frac{1}{\sqrt{2}} L_- Y_0^0 \\ L_+ Y_1^1 - \frac{1}{\sqrt{2}} L_z Y_1^0 + \frac{1}{\sqrt{2}} L_z Y_0^0 \end{pmatrix} d\Omega$$

where $L_- Y_0^0 = L_+ Y_1^1 = L_z Y_1^0 = L_z Y_0^0 = 0$

$$2\langle \underset{\sim}{L}\cdot\underset{\sim}{S}\rangle = \frac{\hbar}{2}\int \left(Y_1^{1*} L_z Y_1^1 + \cancel{\frac{1}{\sqrt{2}} Y_1^{1*} L_- Y_1^0} \right) d\Omega$$

$$= \frac{\hbar}{2}\cdot\hbar = \frac{1}{2}\hbar^2$$

$$\boxed{\langle J^2\rangle = \frac{3}{2}\hbar^2 + \frac{3}{4}\hbar^2 + \frac{1}{2}\hbar^2 = \frac{11}{4}\hbar^2}$$

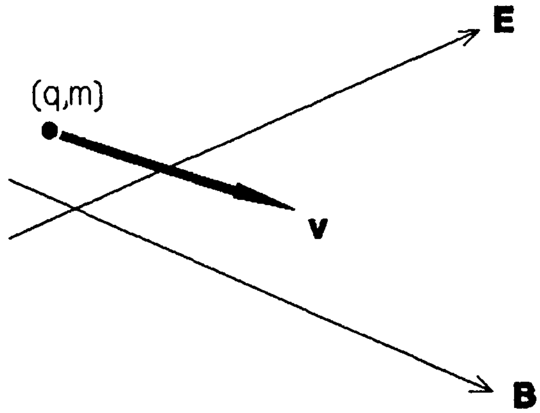
5 pts

6. A particle of mass m and charge q moves at velocity $\mathbf{v}$ in an electromagnetic field which is specified by the scalar and vector potentials ϕ and $\mathbf{A}$. It is claimed that the Lagrangian for this motion is

#6: DR

$$L = -mc^2[1-(v/c)^2]^{1/2} - q\phi + (q/c)\mathbf{v}\cdot\mathbf{A}.$$

Show that for this L the Lagrange equations of motion generate the Lorentz force law. What form does the work-energy theorem take for this system?

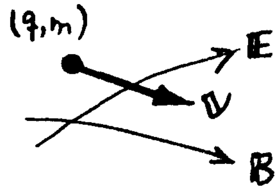


for MSU Exam of Aug. 1991

1: Field Lagrangian

7/30/91

"A particle of mass m and charge q moves at velocity $\mathbf{v}$ in an electromagnetic field which is specified by the scalar and vector potentials ϕ and $\mathbf{A}$. It is claimed that the Lagrangian for this motion is



$$L = -mc^2[1-(v/c)^2]^{1/2} - q\phi + (q/c)\,\mathbf{v}\cdot\mathbf{A}.$$

Show that for this L the Lagrange equations of motion generate the Lorentz force law. What form does the work-energy theorem take for this system?

Solution The Lagrange equations of motion are...

$$\frac{d}{dt}(\partial L/\partial \mathbf{v}) = \underbrace{(\partial L/\partial \mathbf{r})}_{\text{gradients}} \Rightarrow \frac{d}{dt}(\underbrace{\gamma m\mathbf{v}}_{\text{particle momentum } \mathbf{p},\ w/\ \gamma = 1/\sqrt{1-(v/c)^2}} + \frac{q}{c}\mathbf{A}) = q[-\nabla\phi + \frac{1}{c}\nabla(\mathbf{v}\cdot\mathbf{A})], \quad \nabla = \partial/\partial\mathbf{r}$$

(

$$\xrightarrow{\text{so}} d\mathbf{p}/dt = q[-\nabla\phi + \frac{1}{c}\{\nabla(\mathbf{v}\cdot\mathbf{A}) - \frac{d\mathbf{A}}{dt}\}]. \qquad (1)$$

But: $\nabla(\mathbf{v}\cdot\mathbf{A}) = (\mathbf{v}\cdot\nabla)\mathbf{A} + \cancelto{0}{(\mathbf{A}\cdot\nabla)\mathbf{v}} + \mathbf{v}\times(\nabla\times\mathbf{A}) + \mathbf{A}\times\cancelto{0}{(\nabla\times\mathbf{v})}$, and: $d\mathbf{A}/dt = \partial\mathbf{A}/\partial t + (\mathbf{v}\cdot\nabla)\mathbf{A}$. Put these results into Eq. (1) to find...

$$\boxed{\frac{d\mathbf{p}}{dt} = q\Big[-\underbrace{\Big(\nabla\phi + \frac{1}{c}\Big(\frac{\partial\mathbf{A}}{\partial t}\Big)\Big)}_{\mathbf{E} \text{ in terms of } \phi\ \&\ \mathbf{A}} + \frac{1}{c}\mathbf{v}\times\underbrace{(\nabla\times\mathbf{A})}_{\mathbf{B}}\Big] = q\Big[\mathbf{E} + \frac{\mathbf{v}}{c}\times\mathbf{B}\Big].} \quad (2)$$

Eq. (2) is the desired Lorentz force law, so we have a QED.

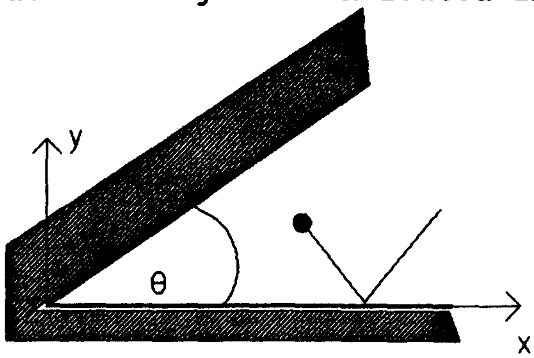
For work-energy thm, it is always true [for energy $\mathcal{E} = \gamma mc^2$, and $\mathcal{E}^2 = (c\mathbf{p})^2 + (mc^2)^2$] that: $d\mathcal{E}/dt = (c^2\mathbf{p}/\mathcal{E})\cdot\frac{d\mathbf{p}}{dt} = \mathbf{v}\cdot(d\mathbf{p}/dt)$. Here

$$\longrightarrow \frac{d\mathcal{E}}{dt} = \mathbf{v}\cdot\frac{d\mathbf{p}}{dt} = \mathbf{v}\cdot q[\mathbf{E} + \frac{\mathbf{v}}{c}\times\mathbf{B}] = q\mathbf{v}\cdot\mathbf{E} + \cancelto{0}{\frac{q}{c}\mathbf{v}\cdot(\mathbf{v}\times\mathbf{B})}. \qquad (3)$$

++ T. L L

7. (a) Consider a particle of mass m moving in a 2-dimensional plane subject to the central potential $V(r) = -C/r^3$. Start the particle at a distance R from the origin at a point on the x-axis with the initial velocity vector $\mathbf{V} = (V_x, V_y)$. Assume that $V_x < 0$ while $V_y \neq 0$.
   (i) Under what circumstances will the particle escape to infinity?
   (ii) Under what circumstances will the particle reach the origin?

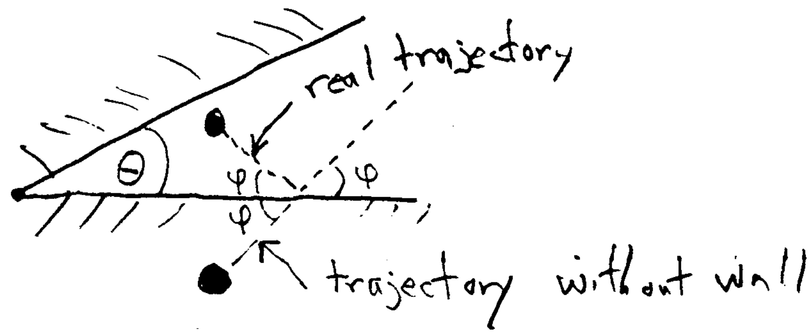
   (b) Now consider a particle of mass m moving in 2 dimensions between two perfectly reflecting walls which intersect at an angle θ at the origin as indicated in the figure below.



Assume that when the particle is reflected, its speed is unchanged and its angle of incidence equals its angle of reflection. The particle is attracted to the origin by the same potential as in part (a): $V(r) = -C/r^3$. And as above, start the particle at a distance R from the origin at a point on the x-axis with the initial velocity vector $\mathbf{V} = (V_x, V_y)$. Assume that $V_x < 0$ while $V_y \neq 0$. Now, give the answers to (i) and (ii) above for this situation as well.

Solution:

Each time the particle bounces off a wall its trajectory becomes the mirror image of what it would have been had the wall not been there:



Therefore the real trajectory could be obtained by finding the trajectory in the space without walls, and then "folding up" the space into a wedge of angle θ. In any case the answers to questions a), b), and c) are the same whether the space is folded or not. Thus, henceforth, we ignore the walls.

The energy of the particle is:

$$E = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - \frac{c}{r^3}$$

Its angular momentum is

$$J = mr^2\dot{\theta}$$

(1)

Since we are dealing with a spherical potential, the angular momentum is conserved. The energy is also conserved since the potential is time independent. The initial values of these quantities are:

$$J = mRV^y$$
$$E = \frac{1}{2}m\left[(V^x)^2 + (V^y)^2\right] - \frac{C}{R^3}$$

Using the angular momentum to solve for $\dot{\Theta}$ the Energy can be expressed as:

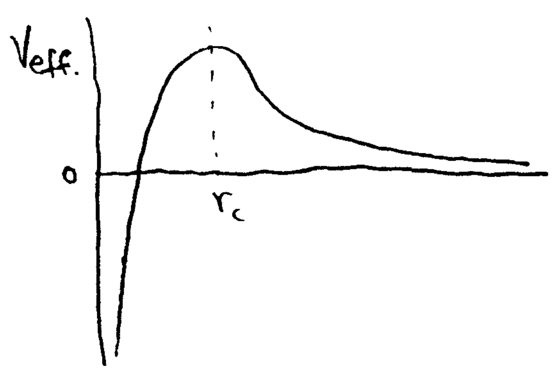
$$E = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}m\left(\frac{J}{mr}\right)^2 - \frac{C}{r^3}$$

$$= \frac{1}{2}m\dot{r}^2 + V_{eff}(r)$$

Where the effective potential is defined by:

$$V_{eff}(r) = \frac{1}{r^2}\left\{\frac{J^2}{2m} - \frac{C}{r}\right\}$$

Since $J \neq 0$ $V_{eff} > 0$ for very large r, while $V_{eff} < 0$ for very small r. Thus V_{eff} looks something like:



3

V_{eff} has a maximum at $r = r_c$, where:

$$0 = \left.\frac{dV_{eff}}{dr}\right|_{r=r_c} = -\frac{J^2}{r_c^3 m} + \frac{3C}{r_c^4}$$

$$\Rightarrow \quad r_c = \frac{3Cm}{J^2} = \frac{3C}{mR^2(v_y)^2}$$

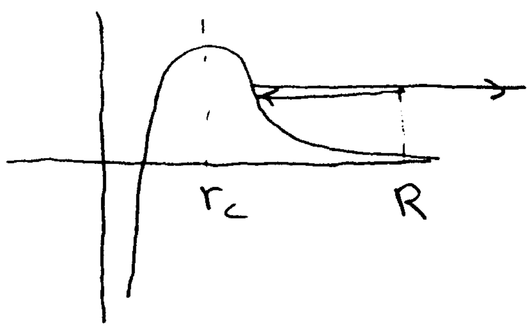
The value of the effective energy at r_c is

$$V_{eff}(r_c) = \frac{1}{r_c^2}\left\{\frac{J^2}{2m} - \frac{C}{r_c}\right\}$$

$$= \frac{J^2}{6mr_c^2} = \frac{m^2R^2(v_y)^2}{6m}\,\frac{m^2R^4(v_y)^4}{9C^2}$$

$$= \frac{m^3R^6(v_y)^6}{54\,C^2}$$

a) The particle can only reach infinity if it gets reflected away by the centrifugal barrier.



Thus we must have $R > r_c$ and $E < V_{eff}(r_c)$

The first condition is equivalent to

$$\boxed{(v_y)^2 > \frac{3C}{mR^3}}$$

While the second condition is

$$\frac{1}{2}m\left[(V^x)^2 + (V^y)^2\right] - \frac{C}{R^3} < \frac{m^3R^6}{54C^2}(V^y)^2$$

That is $$\boxed{(V^x)^2 < \frac{2C}{mR^3} + \left[\frac{m^2R^6}{27C^2} - 1\right](V^y)^2}$$

b) The particle will reach $r=0$ if either of these conditions is violated i.e. if

$$r_c > R \quad \text{or if} \quad E > V_{eff}(r_c).$$

c) In this potential there are no other choices when $V^x \neq 0$. The particle must either fall in to $r=0$ or escape to $r=\infty$.

8. The free energy F and thermodynamic potential G are defined as #8 : ST

$$F = U - TS, \qquad G = F + PV,$$

where U, T, S, P and V are the internal energy, temperature, entropy, pressure and specific volume, respectively.

Using the first law of thermodynamics, derive the following relationships.

2 (a) $S = -\left(\frac{\partial F}{\partial T}\right)_V$ and $V = \left(\frac{\partial G}{\partial P}\right)_T$

3 (b) $\left(\frac{\partial S}{\partial P}\right)_T = -\left(\frac{\partial V}{\partial T}\right)_P$

5 ~~4~~ (c) $\left(\frac{\partial U}{\partial V}\right)_T = T\left(\frac{\partial P}{\partial T}\right)_V - P$

Prm 8

(a) $dQ = dU + PdV$ & $dS = dQ/T$

$\therefore\ TdS = dU + PdV \rightarrow \boxed{dU = TdS - PdV}$ ①

given $F = U - TS \rightarrow dF = dU - TdS - SdT$ ②

① → ② $dF = \cancel{TdS} - PdV - \cancel{TdS} - SdT$

$dF = -PdV - SdT$ ③

$\therefore\ \left.\frac{\partial F}{\partial T}\right)_V = -S \rightarrow \boxed{S = -\left(\frac{\partial F}{\partial T}\right)_V}$ QED.

given $G = F + PV \rightarrow dG = dF + PdV + VdP$ ④

③ → ④ $dG = -\cancel{PdV} - SdT + \cancel{PdV} + VdP$

$\therefore\ dG = -SdT + VdP$ ⑤

$\therefore\ \left.\frac{\partial G}{\partial P}\right)_T = V \rightarrow \boxed{V = \left(\frac{\partial G}{\partial P}\right)_T}$ QED.

(b)

from eqn ⑤ in (a),

$S = -\left.\frac{\partial G}{\partial T}\right)_P$ & $V = \left.\frac{\partial G}{\partial P}\right)_T$

$\therefore\ -\left.\frac{\partial S}{\partial P}\right)_T = \frac{\partial^2 G}{\partial P \partial T}$

& $\left.\frac{\partial V}{\partial T}\right)_P = \frac{\partial^2 G}{\partial T \partial P}$

but $\frac{\partial^2 G}{\partial P \partial T} = \frac{\partial^2 G}{\partial T \partial P}$ $\therefore$ $\left.\frac{\partial S}{\partial P}\right)_T = \left.\frac{\partial V}{\partial T}\right)_P$

Prm 8 (cont'd)

(c) ① → $T\,dS = dU + P\,dV \rightarrow dS = \frac{1}{T}(dU + P\,dV)$ ⑤

$U = U(V,T) \rightarrow dU = \left(\frac{\partial U}{\partial V}\right)_T dV + \left(\frac{\partial U}{\partial T}\right)_V dT$ ⑥

⑥ → ⑤

$$dS = \frac{1}{T}\left[\left(\frac{\partial U}{\partial V}\right)_T dV + \left(\frac{\partial U}{\partial T}\right)_V dT + P\,dV\right]$$

$$= \underbrace{\frac{1}{T}\left[\left(\frac{\partial U}{\partial V}\right)_T + P\right]}_{ⓐ} dV + \underbrace{\left(\frac{\partial U}{\partial T}\right)_V}_{ⓒ} dT$$

$S = S(V,T)$

$$\therefore dS = \underbrace{\left(\frac{\partial S}{\partial V}\right)_T}_{ⓑ} dV + \underbrace{\left(\frac{\partial S}{\partial T}\right)_V}_{ⓓ} dT$$

$\therefore$ ⓐ = ⓑ → $\frac{\partial}{\partial T}\left(\frac{\partial S}{\partial V}\right)_T = \frac{\partial}{\partial T}\left\{\frac{1}{T}\left[\left(\frac{\partial U}{\partial V}\right)_T + P\right]\right\}$

ⓒ = ⓓ → $\frac{\partial}{\partial V}\left(\frac{\partial S}{\partial T}\right)_V = \frac{\partial}{\partial V}\left[\frac{1}{T}\left(\frac{\partial U}{\partial T}\right)_V\right]$

but $\frac{\partial}{\partial T}\left(\frac{\partial S}{\partial V}\right)_T = \frac{\partial}{\partial V}\left(\frac{\partial S}{\partial T}\right)_V$

$$\therefore \frac{\partial}{\partial T}\left\{\frac{1}{T}\left[\left(\frac{\partial U}{\partial V}\right)_T + P\right]\right\} = \frac{\partial}{\partial V}\left[\frac{1}{T}\left(\frac{\partial U}{\partial T}\right)_V\right]$$

$$\therefore -\frac{1}{T^2}\left[\left(\frac{\partial U}{\partial V}\right)_T + P\right] + \frac{1}{T}\left(\frac{\partial P}{\partial T}\right)_V = 0$$

$$\therefore -\frac{1}{T}\left[\left(\frac{\partial U}{\partial V}\right)_T + P\right] + \left(\frac{\partial P}{\partial T}\right)_V = 0$$

$$\left(\frac{\partial U}{\partial V}\right)_T + P = T\left(\frac{\partial P}{\partial T}\right)_V$$

$$\therefore \boxed{\left(\frac{\partial U}{\partial V}\right)_T = T\left(\frac{\partial P}{\partial T}\right)_V - P}$$ QED.

#9: DR

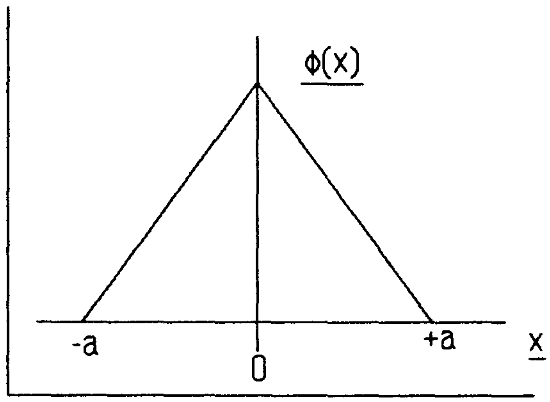
DEPARTMENT OF PHYSICS
PH.D. QUALIFYING EXAMINATION
WEDNESDAY, AUGUST 21, 1991

Answer each of the following questions, all of which carry equal weight. Begin your answer to each question on a new sheet of paper; solutions to different questions must not appear on the same sheet. Each sheet of paper must be labeled with your name and the problem number in the upper right hand corner. When more than one sheet is submitted for a problem, be sure the pages are ordered properly. [Problems 9-12]

9. Approximate the ground state of the quantum mechanical simple harmonic oscillator by using the trial wavefunction

$$\phi(x) = \begin{cases} N[1-(|x|/a)], & \text{for } -a \le x \le a; \\ 0, & \text{for } |x| > a. \end{cases}$$

N is a normalization constant, and a is an adjustable parameter. Using this wavefunction, make a best estimate of the SHO ground state energy and compare your estimate to the known value.

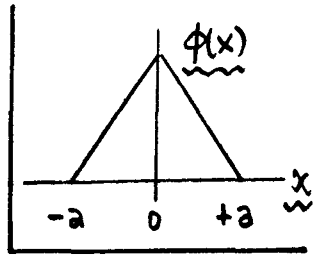


for MSU Exam of Aug. 1991

M: Estimate of Energy Eigenvalue 7/29/91

" Approximate the ground state of the QM simple harmonic oscillator by using the trial wavefunction

$$\phi(x) = \begin{cases} N[1-(|x|/a)], & \text{for } -a \leq x \leq a; \\ 0 & \text{, for } |x| > a. \end{cases}$$



N is a normalization constant, and a is an adjustable parameter. Find the total SHO energy for this ϕ, optimize that energy w.r.t. a, and compare your result with the known ground state energy."

Solution Normalize ϕ...

$$\rightarrow \langle\phi|\phi\rangle = N^2 \int_{-a}^{+a} \left(1-\frac{|x|}{a}\right)^2 dx = \cdots = \frac{2}{3} a N^2 = 1 \Rightarrow \underline{N} = \sqrt{3/2a}.$$

The SHO has P.E. $= \frac{1}{2} m\omega^2 x^2$, ω = natural freq. Total energy is...

$$\rightarrow E = \langle\phi|H|\phi\rangle = N^2 \int_{-a}^{+a} \left(1-\frac{|x|}{a}\right)\left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2} m\omega^2 x^2\right]\left(1-\frac{|x|}{a}\right) dx$$

Now $\frac{d^2}{dx^2}|x| = 2\delta(x)$, since $\int_{0-}^{0+} \frac{d}{dx}\left(\frac{d}{dx}|x|\right) dx = \left(\frac{d}{dx}|x|\right)_{x=0-}^{x=0+} = 2$. So...

$$\rightarrow E = \frac{3}{2a}\left[\int_{-a}^{+a}\left(1-\frac{|x|}{a}\right)\frac{\hbar^2}{ma}\delta(x)\,dx + \frac{1}{2} m\omega^2 \int_{-a}^{+a} x^2\left(1-\frac{|x|}{a}\right)^2 dx\right]$$

or

$$E = \frac{3}{2a}\left[(\hbar^2/ma) + m\omega^2 a^3 \int_0^1 u^2(1-u)^2 du\right] = \frac{3}{2}\left(\frac{\hbar^2}{ma^2}\right) + \frac{1}{20} m\omega^2 a^2$$

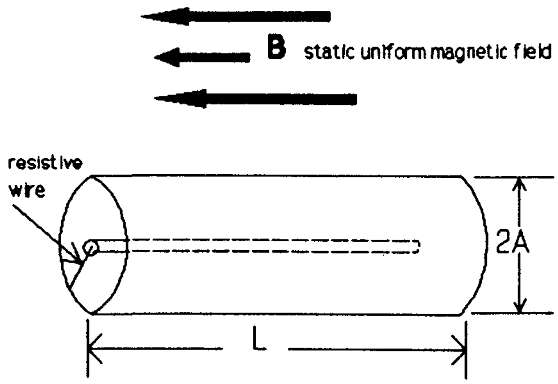
optimize E w.r.t. a $\Big\}$ $\partial E/\partial a = 0 \Rightarrow \underline{a^2} = \sqrt{30}\,(\hbar/m\omega)$.

As a result : $\boxed{E = \sqrt{\frac{6}{5}} \times E_0}$, w/ $E_0 = \frac{\hbar\omega}{2}$ is known energy.

The calculated $E(\phi) = \sqrt{1.2}\, E_0 = 1.095\, E_0$, i.e. about 10% above the actual ground state energy.

#10 : LL

10. Consider a capacitor which consists of a long thin metal cylinder of length L and inner radius A (with A << L) and a coaxial solid metal cylinder of radius a << A. Assume that the capacitor is charged initially with charges Q and -Q on the inner and outer cylinders respectively. Assume further that the capacitor is placed in a static uniform magnetic field which is parallel to the axis of the capacitor.



(a) Compute the total angular momentum about the axis of the cylinder contained in the electromagnetic field associated with this capacitor.

(b) Assume that the capacitor can rotate freely about its axis, and that it is initially non-rotating. At time t = 0, let a piece of resistive wire (of length A and total resistance R) be attached rigidly between the outer cylinder (of mass M) and the inner cylinder (of mass m << M) of the capacitor. Use the Lorentz force law to compute the angular momentum of the capacitor as a function of time.

(c) Show that the result of part (b) is consistent with part (a) and conservation of angular momentum.

(1)

Solution

a) Use Gauss' law to deduce that the radial component of the electric field is

$$2\pi r L \, E^r = 4\pi Q \qquad \Rightarrow \quad E^r = \frac{2Q}{rL}$$

The momentum density contained in the electromagnetic field is

$$\vec{g} = \frac{1}{4\pi c}\vec{E}\times\vec{B} = \frac{1}{4\pi c}\frac{2QB}{rL}\,\hat{r}\times\hat{z}$$

Thus, the angular-momentum density about the capacitor's axis is

$$\vec{j} = \vec{r}\times\vec{g} = \frac{QB}{2\pi cL}\,\hat{r}\times(\hat{r}\times\hat{z}) = -\frac{QB}{2\pi cL}\,\hat{z}$$

Integrating this angular-momentum density over the interior of the capacitor we find the total angular momentum to be:

$$\boxed{J^z = -\frac{QBA^2}{2c}}$$

b) At time $t=0$ a current begins to flow from the inner to the outer cylinder. The magnitude of this current is determined by Ohm's law.

$$I = \frac{V}{R}$$

②

Where V is the potential difference between the inner and outer cylinders. This potential difference can also be expressed in terms of the capacitance as:

$$V = \frac{Q}{C}$$

Thus: $$I = -\frac{dQ}{dt} = \frac{Q}{RC} \Rightarrow Q(t) = Q_0 e^{-t/RC}$$

So $$I(t) = \frac{Q_0}{RC} e^{-t/RC}$$

This current will in turn cause a torque to be applied to the outer cylinder due to the Lorentz force. The torque per unit length applied to the resistive wire then is

$$d\vec{\tau} = \vec{r} \times \vec{F} = \vec{r} \times \frac{1}{c}\left(\vec{I} \times \vec{B}\right) = \frac{Br}{c} I \, \hat{r} \times (\hat{r} \times \hat{z})$$
$$= -\frac{Br}{c} \frac{Q_0}{RC} e^{-t/RC} \hat{z}$$

Integrating this along the wire, we find the total torque applied to the cylinder to be:

$$\vec{\tau} = \int d\vec{\tau} = -\frac{Bf^2}{2c} \frac{Q_0}{RC} e^{-t/RC}$$

This torque causes the angular momentum of the cylinder to change via Newton's law:

$$\frac{d\vec{J}}{dt} = \vec{\tau}$$

10c

(3)

So: $$\frac{dJ^z}{dt} = -\frac{BA^2}{2c}\frac{Q_0}{RC}e^{-t/RC} = \frac{Q_0BA^2}{2c}\frac{d}{dt}e^{-t/RC}$$

Thus: $$\boxed{J^z = \frac{Q_0BA^2}{2c}\left[e^{-t/RC} - 1\right]}$$

c) At any instant of time the angular momentum contained in the electromagnetic field is (from a))

$$(J^z)_{field} = -\frac{QBA^2}{2c} = -\frac{Q_0BA^2}{2c}e^{-t/RC}$$

While from b) the angular momentum of the cylinder is:

$$(J^z)_{mechanical} = \frac{Q_0BA^2}{2c}\left[e^{-t/RC} - 1\right]$$

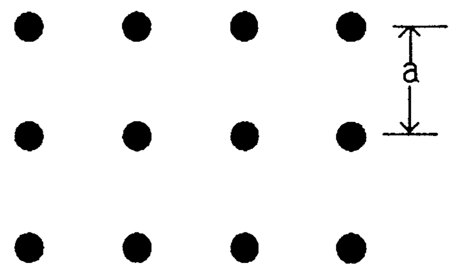
The total angular momentum thus is

$$(J^z)_{total} = (J^z)_{field} + (J^z)_{mechanical} = -\frac{Q_0BA^2}{2c}$$

a constant!

11. A single atomic plane of Cu atoms is suspended in free space; the atoms are arranged in a square net as shown below.

#11 : JH



Suppose a beam of electrons of energy E is incident normal to the Cu monolayer.

(a) With what momenta will elastically scattered electrons emerge from the monolayer?

(b) Describe the pattern produced on a fluorescent screen parallel to the Cu layer.

(a) Bravais Lattice of the Plane

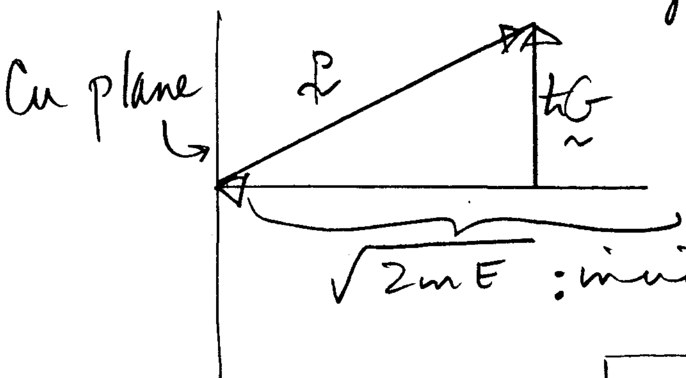
$$\underset{\sim}{R} = a\,(i, j) \qquad 2D;\ i, j = \text{integer}$$

Reciprocal Lattice (allowed transverse wavevectors)

$$\underset{\sim}{G} = \frac{2\pi}{a}\,(m, n) \qquad 2D;\ m, n = \text{integer}$$

check: $\underset{\sim}{G} \cdot \underset{\sim}{R} = 2\pi(mi + nj) = \text{multiple of } 2\pi$ ✓

momentum and energy conservation



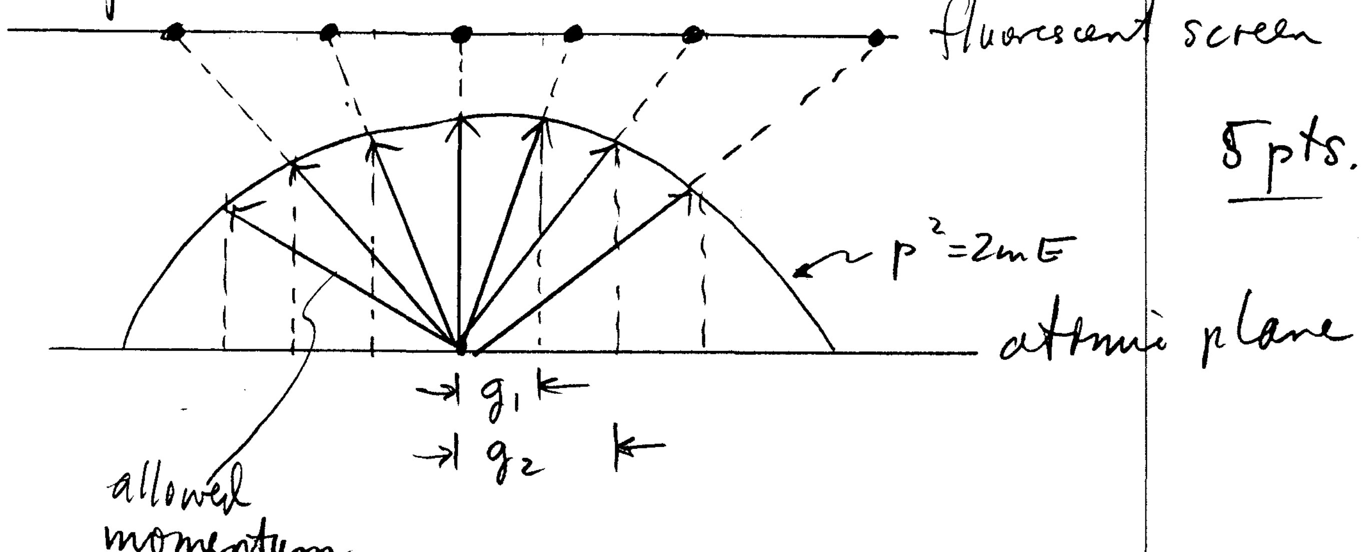
p = final momentum
ħG = transv. momentum

$$\frac{p_z^2}{2m} + \frac{\hbar^2 G^2}{2m} = E \Rightarrow \boxed{\begin{aligned} p_z &= \sqrt{2mE - \hbar^2 G^2} \\ p_{\parallel} &= \hbar \underset{\sim}{G} \end{aligned}}$$

where $\underset{\sim}{G} = \frac{2\pi}{a}(m, n)$

5 pts

(b) Spot pattern



5 pts.

#12 : LL

12. Consider the partial differential equation

$$\nabla^2\nabla^2 \phi = 0, \tag{1}$$

where ∇^2 is the three dimensional Laplace operator:

$$\nabla^2\phi \equiv \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} .$$

(a) Find the most general solution to eq. (1) in the region between two concentric spheres, of radii R_1 and R_2 respectively, which satisfies the boundary conditions:

$$\phi(R_1) = \phi_1,$$

$$\phi(R_2) = \phi_2,$$

$$\left(\frac{\partial\phi}{\partial r}\right)_{r = R_1} = \left(\frac{\partial\phi}{\partial r}\right)_{r = R_2} = 0,$$

where ϕ_1 and ϕ_2 are constants and r is the spherical radial coordinate.

(b) Prove that the solution to eq. (1) is uniquely specified in a closed region if the values of ϕ and $\nabla\phi$ are given on the boundary of that region.

Solution:

a) Look for a spherical solution: $\varphi = \varphi(r)$

Then $\nabla^2\varphi = \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\varphi}{dr}\right)$, so

$$\nabla^2\nabla^2\varphi = \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}\left[\nabla^2\varphi\right]\right) = 0$$

Now integrate:

$$\Rightarrow \frac{d}{dr}\left(r^2\frac{d}{dr}\left[\nabla^2\varphi\right]\right) = 0 \Rightarrow C_1 = r^2\frac{d}{dr}\left[\nabla^2\varphi\right]$$

$$\Rightarrow \frac{d}{dr}\left[\nabla^2\varphi\right] - \frac{C_1}{r^2} = 0 \Rightarrow C_2 = \nabla^2\varphi + \frac{C_1}{r}$$

And again:

$$\nabla^2\varphi = \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\varphi}{dr}\right) = C_2 - \frac{C_1}{r}$$

$$\Rightarrow \frac{d}{dr}\left(r^2\frac{d\varphi}{dr}\right) + C_1 r - C_2 r^2 = 0 \Rightarrow C_3 = r^2\frac{d\varphi}{dr} + \tfrac{1}{2}C_1 r^2 - \tfrac{1}{3}C_2 r^3$$

$$\Rightarrow \frac{d\varphi}{dr} + \tfrac{1}{2}C_1 - \tfrac{1}{3}C_2 r - \frac{C_3}{r^2} = 0$$

$$\Rightarrow \boxed{\varphi(r) = \tfrac{1}{2}C_1 r - \tfrac{1}{6}C_2 r^2 + \frac{C_3}{r} + C_4}$$

This is the most general spherically symmetric solution where C_1, C_2, C_3 and C_4 are arbitrary constants.

(2)

We can introduce new constants $\bar{C}_1, \bar{C}_2, \bar{C}_3, \bar{C}_4$ such that

$$\varphi(r) = \bar{C}_1(r-R_1) + \bar{C}_2(r-R_1)^2 + \bar{C}_3\left(\frac{1}{r}-\frac{1}{R_1}\right) + \bar{C}_4$$

Then $\varphi(R_1) = \bar{C}_4 = \varphi_1$

$$\varphi(R_2) = \bar{C}_1(R_2-R_1) + \bar{C}_2(R_2-R_1)^2 + \bar{C}_3\left(\frac{1}{R_2}-\frac{1}{R_1}\right) + \varphi_1$$

$$\frac{\partial\varphi}{\partial r} = \bar{C}_1 + 2\bar{C}_2(r-R_1) - \frac{\bar{C}_3}{r^2}$$

$$\frac{\partial\varphi}{\partial r}(R_1) = \bar{C}_1 - \frac{\bar{C}_3}{R_1^2} = 0 \quad \Rightarrow \quad \bar{C}_1 = \frac{\bar{C}_3}{R_1^2}$$

$$\frac{\partial\varphi}{\partial r}(R_2) = \bar{C}_1 + 2\bar{C}_2(R_2-R_1) - \frac{\bar{C}_3}{R_2^2} = 0$$

$$2\bar{C}_2(R_2-R_1) = \bar{C}_3\left(\frac{1}{R_2^2}-\frac{1}{R_1^2}\right)$$

$$\bar{C}_2 = -\bar{C}_3\,\frac{R_1+R_2}{2R_1^2R_2^2}$$

$$\Rightarrow \varphi(R_2) = \frac{\bar{C}_3}{R_1^2}(R_2-R_1) - \bar{C}_3\,\frac{R_1+R_2}{2R_1^2R_2^2}(R_2-R_1)^2 + \bar{C}_3\,\frac{R_1-R_2}{R_1R_2} + \varphi_1 = \varphi_2$$

$$\frac{\bar{C}_3}{R_1^2R_2^2}(R_2-R_1)\left\{R_2^2 - \frac{1}{2}(R_2^2-R_1^2) - R_1R_2\right\} + \varphi_1 = \varphi_2$$

$$\frac{\bar{C}_3}{2R_1^2R_2^2}(R_2-R_1)^3 + \varphi_1 = \varphi_2$$

$$\Rightarrow \quad \bar{C}_3 = 2\frac{\varphi_2-\varphi_1}{(R_2-R_1)^3}R_1^2R_2^2$$

Thus, the solution is:

$$\boxed{\varphi(r) = 2\frac{\varphi_2-\varphi_1}{(R_2-R_1)^3}\left\{R_2^2(r-R_1) - \frac{1}{2}(R_1+R_2)(r-R_1)^2 + R_1^2R_2^2\left(\frac{1}{r}-\frac{1}{R_1}\right)\right\} + \varphi_1}$$

b) Consider two solutions φ_1 and φ_2 of the equation:

$$\nabla^2\nabla^2\varphi_1 = \nabla^2\nabla^2\varphi_2 = 0$$

in some closed region S with boundary ∂S.

Assume that $\varphi_1 = \varphi_2$ on ∂S and that $\vec{\nabla}\varphi_1 = \nabla\varphi_2$ on ∂S

Then $\varphi = \varphi_1 - \varphi_2$ satisfies $\nabla^2\nabla^2\varphi = 0$ in S.

$\varphi = \vec{\nabla}\varphi = 0$ on ∂S.

④

Now integrate $\varphi \nabla^2 \nabla^2 \varphi$ over S. Integrate by parts twice:

$$0 = \int_S \varphi \nabla^2 \nabla^2 \varphi \, dV$$

$$= \int_{\partial S} \cancelto{0}{\varphi} \vec{n} \cdot \vec{\nabla} \nabla^2 \varphi \, ds - \int_S \sum_i \frac{\partial \varphi}{\partial x_i} \frac{\partial}{\partial x_i} \nabla^2 \varphi$$

$$= - \sum_{i,j} \int_{\partial S} \cancelto{0}{\frac{\partial \varphi}{\partial x_i}} n_j \frac{\partial^2 \varphi}{\partial x_i \partial x_j} ds + \sum_{i,j} \int_S \left(\frac{\partial^2 \varphi}{\partial x^i \partial x^j} \right)^2 dV$$

The boundary integrals vanish because $\varphi = \vec{\nabla} \varphi = 0$ on the boundary.

Thus $$0 = \sum_{i,j} \int_S \left(\frac{\partial^2 \varphi}{\partial x^i \partial x^j} \right)^2 dV$$

Since the integrand is positive definite it must vanish.

$$0 = \frac{\partial^2 \varphi}{\partial x^i \partial x^j}$$

$$\Rightarrow \quad \frac{\partial \varphi}{\partial x^i} = a_i \doteq \text{constant}.$$

$$\Rightarrow \quad \varphi = \sum_i a_i x^i \quad \text{is a linear functi[on]}$$

$$= a_x x + a_y y + a_z z.$$

⑤ De

But, φ vanishes on a closed surface

$$\Rightarrow \quad a_x = a_y = a_z = 0$$

$$\Rightarrow \quad \varphi = 0$$

$$\Rightarrow \quad \varphi_1 = \varphi_2$$

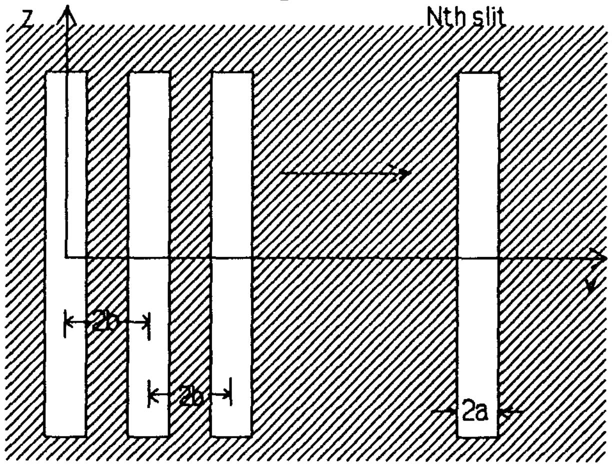
Thus the solution $\nabla^2\nabla^2\varphi = 0$ is uniquely determined by the boundary values of φ and $\nabla\varphi$.

#13 : ST

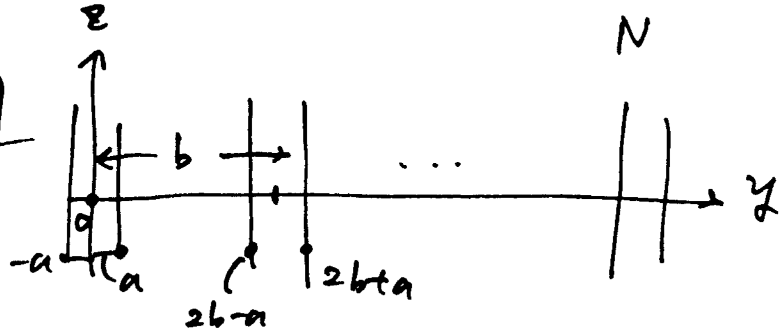
DEPARTMENT OF PHYSICS
PH.D. QUALIFYING EXAMINATION
THURSDAY, AUGUST 22, 1991

Answer each of the following questions, all of which carry equal weight. Begin your answer to each question on a new sheet of paper; solutions to different questions must not appear on the same sheet. Each sheet of paper must be labeled with your name and the problem number in the upper right hand corner. When more than one sheet is submitted for a problem, be sure the pages are ordered properly. [Problems 13-16]

13. There are N slits, very long in the z direction, uniformly spaced along the y axis. The width of each slit is 2a and the intervals between adjacent slits are 2b. For normally incident electromagnetic waves of length λ, find the distribution of intensities of the diffracted waves at infinity, in a plane parallel to the y-z plane.



Prm 13



$\lambda = \frac{2\pi}{k}$.

$$\psi = \psi_0 \int_{y=0} e^{ik_y y}\, dy, \quad \text{where } k_y = k \sin\theta$$

$$\psi = \psi_0 \left[\int_{-a}^{a} + \int_{2b-a}^{2b+a} + \cdots + \int_{2(N-1)b-a}^{2(N-1)b+a} \right] e^{ik_y y}\, dy.$$

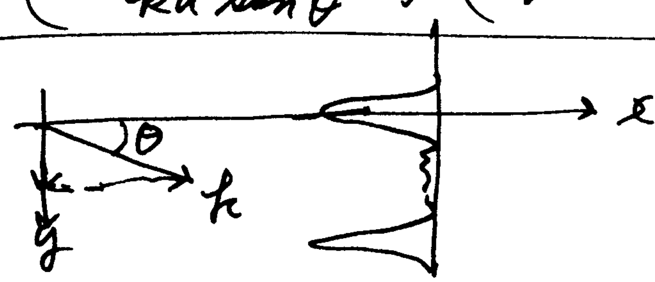
$$= \psi_0 \frac{1}{ik_y} \left(e^{+ik_y a} - e^{-ik_y a} \right) \sum_{n=0}^{N-1} e^{2nbk_y}$$

$$\psi = 2a\psi_0 \left(\frac{\sin k_y a}{k_y a} \right) \frac{1 - e^{i2Nbk_y}}{1 - e^{i2bk_y}}$$

$$I = |\psi|^2 = |2a\psi_0|^2 \left(\frac{\sin k_y a}{k_y a} \right)^2 \left(\frac{\sin Nbk_y}{\sin bk_y} \right)^2$$

$\therefore$ The intensity distribution is

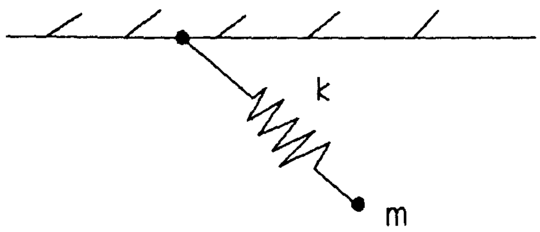
$$\boxed{I = |2a\psi_0|^2 \left(\frac{\sin(ka\sin\theta)}{ka\sin\theta} \right)^2 \left(\frac{\sin(Nbk\sin\theta)}{\sin(bk\sin\theta)} \right)^2}$$



$k_y = k \sin\theta$

#14: JH

14. Consider the "elastic pendulum" consisting of a mass m and a spring with constant k, as shown below. The relaxed length of the pendulum, in the absence of gravity, is L_o. The motion is confined to a vertical plane, and the (massless) pendulum arm remains in a linear configuration; the bearing is frictionless and damping is negligible. Find the frequencies of small oscillations about the equilibrium configuration.



coords (L, θ)

$$V = -mgL\cos\theta + \tfrac{1}{2}k(L-L_0)^2$$

$$T = \tfrac{1}{2}m\left[\dot{L}^2 + L^2\dot{\theta}^2\right]$$

Equil: $\theta_e = 0$; $k(L_e - L_0) = mg \Rightarrow L_e = L_0 + mg/k$

Let $L = L_e + x$

then $V = -mg(L_e + x)\cos\theta + \tfrac{1}{2}k(\cancel{L_e} + x - \cancel{L_e} + mg/k)^2$

$= -mgL_e\cos\theta - mgx\cos\theta + \tfrac{1}{2}k(x + mg/k)^2$

$T = \tfrac{1}{2}m[\dot{x}^2 + (L_e + x)^2\dot{\theta}^2]$

(θ) $\frac{d}{dt}\, m(L_e + x)^2\dot{\theta} + mg(L_e + x)\sin\theta = 0$

small $\theta, x \Rightarrow$ $mL_e^2\ddot{\theta} + mgL_e\theta = 0$

$$\ddot{\theta} + \frac{g}{L_e}\theta = 0$$

$$\omega_1 = \sqrt{g/L_e} = \sqrt{g/(L_0 + mg/k)}$$

(x) $m\ddot{x} - m(L_e + x)\dot{\theta}^2 - mg\cos\theta + k\left(x + \frac{mg}{k}\right) = 0$

small $\theta, x \Rightarrow$

$$m\ddot{x} - \cancel{mg} + kx + \cancel{mg} = 0$$

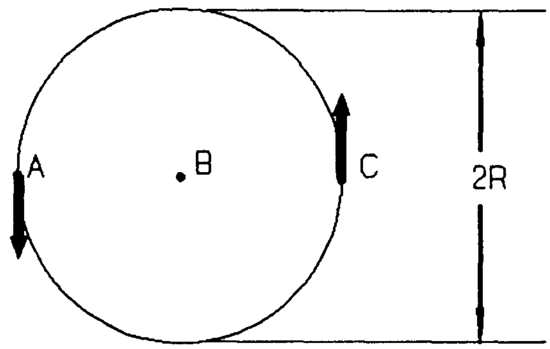
$$\omega_2 = \sqrt{k/m}$$

For small oscillations, the θ and x modes are not coupled.

15. Consider three observers A, B, and C mounted rigidly on a rotating platform as shown in the figure. Observer B is located at the center of the platform, while A and C are located diametrically opposite one another. The platform rotates with angular velocity Ω as measured in an inertial frame at rest with the center of the platform. In this same inertial frame the spatial distance between observers A and C is 2R. You may not assume that $R\Omega \ll c$, where c is the speed of light.

#15:

LL



Assume that observer A transmits a radio signal with frequency ω_o as measured in A's frame of reference. Observers B and C each have radio receivers with them on the platform. At what frequencies must B and C tune their respective receivers so that they can tune in to A's broadcasts?

Solution:

The 3-speeds of observer's A and C are

$$V = R\Omega$$

thus the components of their 4-velocities as a function of the inertial time t are:

$$\vec{u}_A = \gamma\left(1, \frac{V}{c}\sin\Omega t, -\frac{V}{c}\cos\Omega t, 0\right)$$
$$\vec{u}_B = (1, 0, 0, 0)$$
$$\vec{u}_C = \gamma\left(1, -\frac{V}{c}\sin\Omega t, \frac{V}{c}\cos\Omega t, 0\right)$$

where $\gamma = \left(1 - \frac{V^2}{c^2}\right)^{-1/2}$.

Radio signals are propagated along path's whose tangents $\vec{k}$ which are null vectors: $\vec{k}\cdot\vec{k} = 0$. The inner product of $\vec{k}$ with the 4-velocity of an observer gives the frequency measured by that observer. Thus, for example

$$\omega_0 = -\vec{k}\cdot\vec{u}_A$$
$$\omega_B = -\vec{k}\cdot\vec{u}_B \quad , \text{ etc.}$$

Now, let A emit a signal at time $t = 0$. It propagates along the x-axis to arrive at B. Thus, the components of $\vec{k}$ have the form:

2

$$\vec{k} = (k^t, k^x, 0, 0)$$

$\vec{k}$ is null, so $\vec{k}\cdot\vec{k} = 0 = -(k^t)^2 + (k^x)^2$ which implies $k^x = k^t$. Now taking the inner product:

$$\omega_0 = -\vec{k}\cdot\vec{u}_A = \gamma k^t$$

(remember, this is at the instant $t=0$). Thus, we have:

$$k^t = \frac{\omega_0}{\gamma}$$

Now compute the inner product to find ω_B:

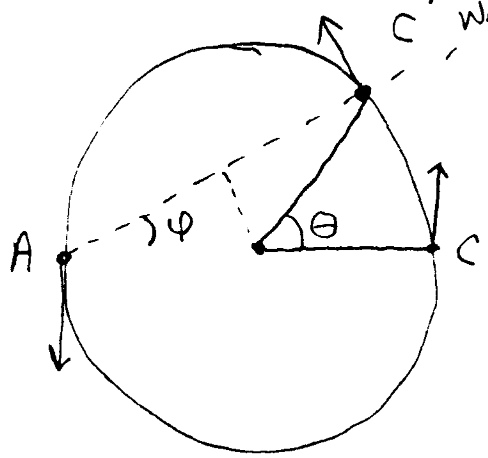
$$\omega_B = -\vec{k}\cdot\vec{u}_B = k^t = \frac{\omega_0}{\gamma}$$

Thus, the frequency that observer B sees is:

$$\boxed{\omega_B = \omega_0 \left(1 - \frac{R^2\Omega^2}{c^2}\right)^{1/2}}$$

The situation is a little more complicated for observer C. Let A emit a signal at $t=0$ as before. Now, however, the signal cannot move directly along the x-axis for C will have moved before it arrives. Thus, the signal that reaches C must be sent at an angle φ such that the time needed for the photon to reach C is the same as the time needed for C itself to arrive:

(3)



The distance between A and C' is

$$D = 2R\cos\varphi$$

Thus the time needed for the signal to reach C' is

$$t' = \frac{D}{c} = \frac{2R}{c}\cos\varphi$$

During the time t' observer C rotates on angle

$$\theta = \Omega t' = \frac{2R\Omega}{c}\cos\varphi$$

In order for C and the signal to arrive at the same place at the same time we must have:

$$\theta = 2\varphi = \frac{2R\Omega}{c}\cos\varphi$$

Thus $\boxed{\varphi = \frac{R\Omega}{c}\cos\varphi = \frac{V}{c}\cos\varphi}$

Now, the signal moves along the path:

$$\vec{K} = k^t \left(1, \cos\varphi, \sin\varphi, 0\right)$$

Thus:
$$\omega_0 = -\vec{u}_A \cdot \vec{K}$$
$$= \gamma k^t \left(1 + \frac{V}{c}\sin\varphi\right)$$

where the inner product is taken at the instant the signal is emitted, $t=0$. So:

$$k^t = \frac{\omega_0}{\gamma\left(1+\frac{V}{c}\sin\varphi\right)}$$

Finally, to get ω_c we compute

$$\omega_c = -\vec{u}_c \cdot \vec{K}$$
$$= -k^t\gamma\left(1, -\frac{V}{c}\sin\Theta, \frac{V}{c}\cos\Theta, 0\right)\cdot\left(1, \cos\varphi, \sin\varphi, 0\right)$$
$$= k^t\gamma\left(1 + \frac{V}{c}\left[\sin\Theta\cos\varphi - \cos\Theta\sin\varphi\right]\right)$$

So
$$\sin\Theta\cos\varphi - \cos\Theta\sin\varphi = \sin\left(\Theta - \varphi\right)$$

$$\omega_c = \frac{\omega_0}{1+\frac{V}{c}\sin\varphi}\left\{1 + \frac{V}{c}\sin\left[\frac{2R\Omega}{c}\cos\varphi - \varphi\right]\right\}$$

$$= \omega_0 \frac{1+\frac{V}{c}\sin\varphi}{1+\frac{V}{c}\sin\varphi} = \omega_0$$

Thus $\boxed{\omega_c = \omega_0}$ Observer C observes the <u>same</u> frequency as that emitted by A!

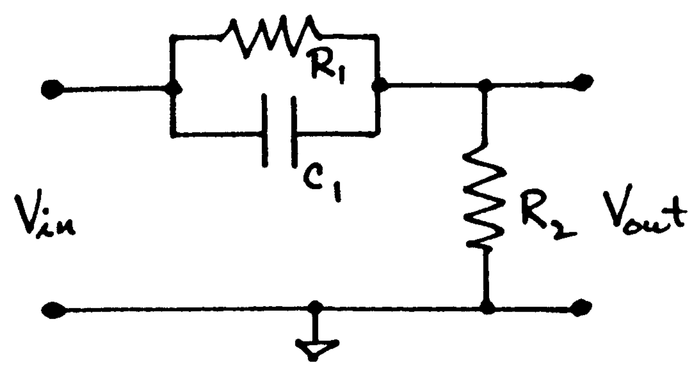
#16 · KC

16. (a) Consider the RC circuit below, with the input on the left and the output on the right. Answer each of the questions.

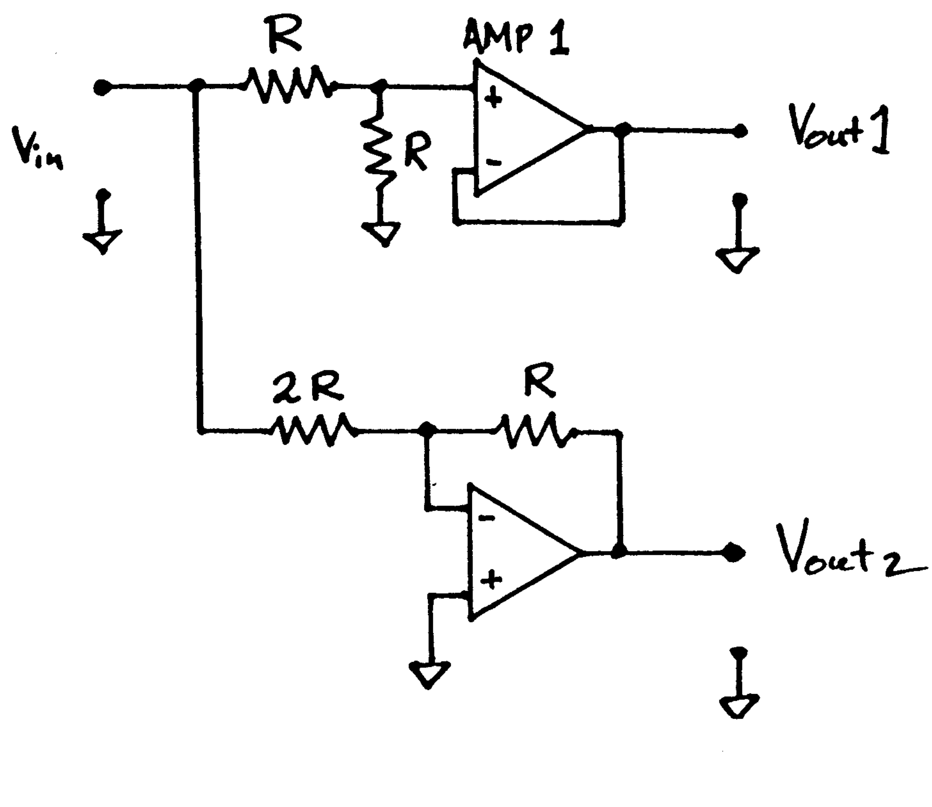
(1) What is the *transfer function* H = V_{out}/V_{in} for a DC signal V_{in}?

(2) What is the *transfer function* $H(j\omega) = V_{out}/V_{in}$ for an input signal of the form $e^{j\omega t}$ (where as usual in problems of this sort, $j = \sqrt{-1}$)?

(3) Plot the log of the absolute value of the *transfer function* $H(j\omega)$ versus log ω, pointing out interesting features of your graph and describing a type of signal-processing function that the circuit can perform.



(b) For the *operational amplifier* circuit shown below, find V_{out1} and V_{out2} in terms of V_{in} and the parameter R. It is not sufficient to simply write the answer down; to receive any credit, you must provide some justification for your answer.



Instrumentation Problem

a) ① Voltage divider - capacitor may be neglected in this case

$$H = \frac{V_o}{V_i} = \frac{R_2}{R_1+R_2}$$

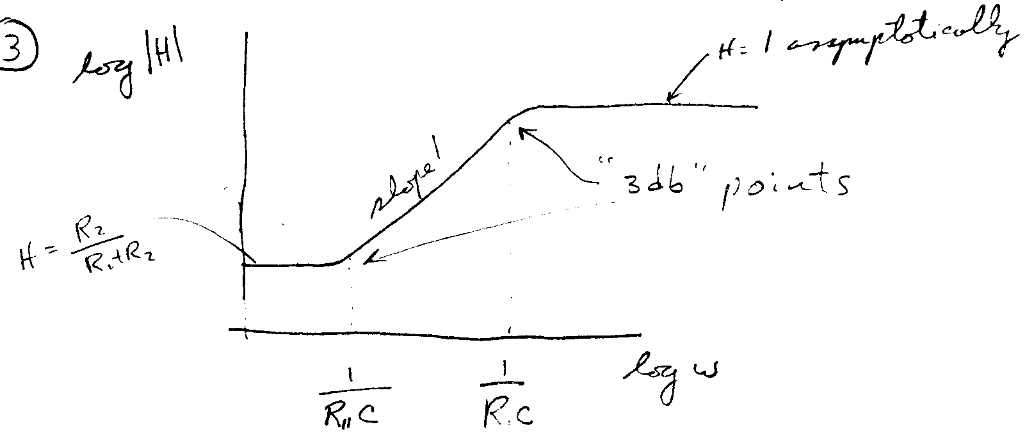
② $$\frac{1}{Z_1} = \frac{1}{R_1} + j\omega C = \frac{1+j\omega R_1 C}{R_1}$$

$$Z_2 = R_2$$

$$H = \frac{Z_2}{Z_1+Z_2} = \frac{R_2}{R_1/(1+j\omega R_1 C) + R_2} = \frac{R_2(1+j\omega R_1 C)}{R_1 + R_2(1+j\omega R_1 C)}$$

$$\boxed{H = \frac{R_2(1+j\omega R_1 C)}{(R_1+R_2) + j\omega R_1 R_2 C}}$$

③

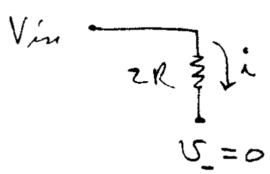


↳ where $R_{\parallel} = \frac{R_1 R_2}{R_1+R_2}$

b) Use "virtual equality of inputs" applicable to negative feedback systems

Amp 1
$$\left.\begin{aligned} v_+ &= \frac{R}{R+R} v_{in} = \frac{v_{in}}{2} \\ v_- &= v_{out1} \end{aligned}\right\}$$ set $v_+ = v_-$ & find $\boxed{v_{out1} = \frac{v_{in}}{2}}$

<u>Amp 2</u> $v_+ \equiv 0$ (grounded) $\Rightarrow v_- = 0$



$$i = \frac{v_{in}}{2R}$$

current i must flow through feedback resistor R

Hence, $v_{out_2} = v_- - iR = 0 - iR = -\frac{v_{in}}{2}$.

$$\boxed{v_{out} = -\frac{v_{in}}{2}}$$