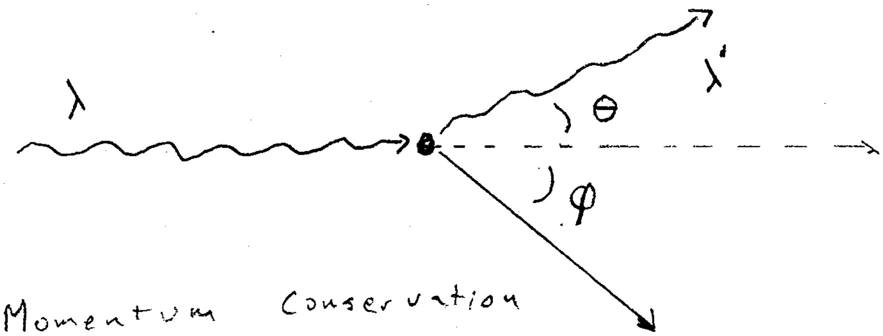
Problem #1:




a) Momentum Conservation

$$\vec{P}_\gamma = \vec{\tilde{P}}_\gamma + \vec{P}_e$$

or $$\boxed{\vec{P}_\gamma - \vec{\tilde{P}}_\gamma = \vec{P}_e} \qquad (1)$$

Energy Conservation

$$\mathcal{E}_\gamma + \mathcal{E}_e = \tilde{\mathcal{E}}_\gamma + \tilde{\mathcal{E}}_e$$

$$\mathcal{E}_\gamma = P_\gamma c \qquad \tilde{\mathcal{E}}_\gamma = \tilde{P}_\gamma c$$

$$\mathcal{E}_e = mc^2 \qquad \tilde{\mathcal{E}}_e = \sqrt{P_e^2 c^2 + m^2 c^4}$$

or $$\boxed{P_\gamma c + mc^2 = \tilde{P}_\gamma c + \sqrt{P_e^2 c^2 + m^2 c^4}} \qquad (2)$$

b) Eq. #2 can be rewritten as

$$P_e^2 = (P_\gamma - \tilde{P}_\gamma)^2 + 2mc\,(P_\gamma - \tilde{P}_\gamma)$$

and eq. #1 can be written as

$$P_e^2 = (\vec{P}_\gamma - \vec{\tilde{P}}_\gamma)^2 = P_\gamma^2 + \tilde{P}_\gamma^2 - 2P_\gamma \tilde{P}_\gamma \cos\theta$$

or $$P_\gamma \tilde{P}_\gamma (1 - \cos\theta) = mc\,(P_\gamma - \tilde{P}_\gamma)$$

For the special case $\theta = \frac{\pi}{2}$

$$P_\gamma \tilde{P}_\gamma = mc\,(P_\gamma - \tilde{P}_\gamma)$$

or $$1 = mc\left(\frac{1}{\tilde{P}_\gamma} - \frac{1}{P_\gamma}\right)$$

$$\boxed{\lambda' - \lambda = \frac{h}{mc}}$$

Since $$P = \frac{h}{\lambda}$$

c) If the electron has internal degrees of freedom than energy can be lost to those internal excitations.

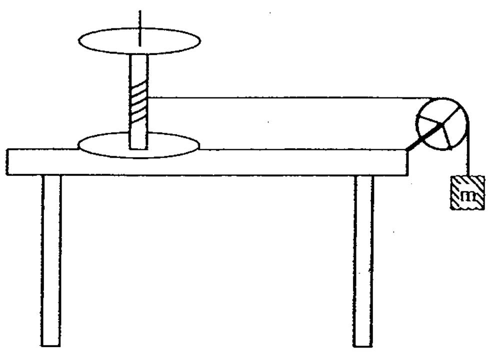
In that case, the wavelength shift calculated above is the <u>lower limit</u> and instead we should write

$$\lambda' - \lambda \geq \frac{h}{mc}$$

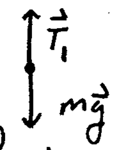
In the experiment we would see wavelengths greater (energies lower) than this cut-off value.

*Problem #2*

The following problem involves a device that can be used to measure the moment of inertia $I_S$ of a spool based on the speed v of a mass. The spool is free to spin on a vertical axle fixed to a table; the spool's bearing is frictionless. A massless string is wrapped around the small radius r of the spool (see picture) in a single layer. The string passes over a pulley that spins on a frictionless bearing whose axle is horizontal to the table's surface. The round portion of the pulley has mass M, each of the three spokes has mass (1/3)M, and the radius is R. A mass m hangs freely from the end of the string, creating a tension that causes the pulley and spool to turn. The string does not slip on either the spool or the pulley, nor does it stretch. If m starts at rest, falls a distance y and reaches a speed v, find an expression for $I_S$ in terms of the variables defined in the problem.

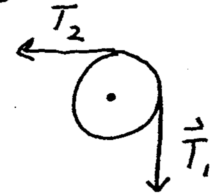



Free-body diagram for m $\longrightarrow$




$T_1 - mg = -ma \Rightarrow \boxed{T_1 = mg - ma}$ (1)

Free-body diagram for pulley $\longrightarrow$




$$T_2 R - T_1 R = -I_p \alpha_p$$
$$= -I_p \frac{a}{R}$$

* need $I_p$

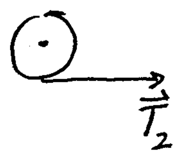
$$I_p = MR^2 + 3\left(\tfrac{1}{3}\left(\tfrac{1}{3}M\right)R^2\right)$$
$$= \tfrac{4}{3}MR^2$$

Using $I_p$ and pulley equation...

$$T_2 R - T_1 R = -\tfrac{4}{3}MR^2 \frac{a}{R}$$

$$\boxed{T_2 - T_1 = -\tfrac{4}{3}Ma}$$ (2)

For the spool...




$$\boxed{T_2 r = I_s \frac{a}{r}}$$ (3)

Put equations $1 \rightarrow 3$ together noting $T_2 = I_s \frac{a}{r^2}$ from (3)

$$I_s \frac{a}{r^2} - m(g-a) = -\tfrac{4}{3}Ma \Rightarrow a\left(\tfrac{4}{3}M + m + \frac{I_s}{r^2}\right) = mg$$

and
$$\boxed{a = \frac{mg}{\frac{4}{3}M + m + \frac{I_s}{r^2}}}$$

now, use the kinematic equations to relate $v$ to a, i.e. $v_f^2 - v_0^2 = 2a\Delta y$

so
$$v^2 = \frac{2mgy}{\frac{4}{3}M + m + \frac{I_s}{r^2}}$$

solve for $I_s = \frac{2mgy}{v^2}r^2 - \left(\frac{4}{3}M + m\right)r^2$

or

$$I_s = \left(\frac{2mgy}{v^2} - \frac{4}{3}M - m\right)r^2$$

## Problem #3

(a) $\bar{H} = -\bar{\nabla}\Phi_m;\quad \nabla^2\Phi_m = -4\pi\rho_m$ (1)

$\rho_m = -\bar{\nabla}\cdot\bar{M};\quad \sigma_m = \hat{n}\cdot\bar{M}$ (2)

since $\bar{M} = M_0\hat{z};\quad M_0 = \text{const.}$ (3)

$\rightarrow$ (2) $\quad \rho_m = -\bar{\nabla}\cdot\bar{M} = 0$ (4a)

$$\sigma_m = \hat{n}\cdot\bar{M} = \begin{cases} M\cos\theta & r = b \\ -M\cos\theta & r = a \end{cases} \quad (4b)$$

So $$\Phi_m = M_0\left\{\int\frac{b^2\cos\theta'\sin\theta'\,d\theta'\,d\phi}{|\bar{x}-\bar{x}'_b|} - \int\frac{a^2\cos\theta'\sin\theta'\,d\theta'\,d\phi'}{|\bar{x}-\bar{x}'_a|}\right\} \quad (5)$$

for $\bar{x}$ on $\hat{z}$ axis,

$$\Phi_m = 2\pi M_0\left\{\frac{b^2 r_<^{\ell}}{r_>^{\ell+1}} - \underbrace{\frac{a^2 r_<^{\ell}}{r_>^{\ell+1}}}_{(x,\,a)}\right\}\underbrace{\int P_\ell(\cos\theta')\sin\theta'\cos\theta'\,d\theta'}_{\frac{2}{3}\delta_{\ell 1}} \quad (6)$$

$r_<, r_> \in (x, b)$

So ~~$r<a$ $\Phi$~~ $$\Phi_m(\bar{x}) = \frac{4\pi M_0}{3}\left\{b^2\frac{r_<}{r_>^2} - a^2\frac{r_<}{r_>^2}\right\}\cos\theta \quad (7)$$

So,

$$\boxed{\begin{array}{ll} r<a & \Phi_m = 0 \\ a<r<b & \Phi_m = \dfrac{4\pi M_0}{3}\left\{r - \dfrac{a^3}{r^2}\right\}\cos\theta \\ b<r & \Phi_m = \dfrac{4\pi M_0}{3}\left(\dfrac{b^3-a^3}{r^2}\right)\cos\theta \end{array}} \quad (8)$$

Now $H = -\bar{\nabla}\Phi_m$ so (8) $\rightarrow$ (9)

$r<a: \bar{H}=0$ ans

$a<r<b: \bar{H} = -\frac{4\pi M_0}{3}\left\{\hat{r}\left(1+\frac{2a^3}{r^3}\right)\cos\theta - \hat{\theta}\left(1-\frac{a^3}{r^3}\right)\sin\theta\right\}$

$= \bar{H} = \frac{4\pi M_0}{3}\left\{-\hat{r}\left(1+\frac{2a^3}{r^3}\right)\cos\theta + \hat{\theta}\left(1-\frac{a^3}{r^3}\right)\sin\theta\right\}$ ans.

$b<r: \bar{H} = -\frac{4\pi M_0}{3}(b^3-a^3)\left\{-\frac{2}{r^3}\cos\theta\,\hat{r} - \frac{1}{r^3}\sin\theta\,\hat{\theta}\right\}$

$= \bar{H} = \frac{4\pi M_0}{3}\frac{(b^3-a^3)}{r^3}\left\{2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right\}$ // ans.

(b) $\bar{B} = \bar{H} + 4\pi\bar{M} = \bar{H} + 4\pi M(\overbrace{\cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}}^{\hat{z}})$ (9)'

(8) + (9)'

$r<a \quad \bar{B}=0$ ans.

$a<r<b$:

$\bar{B} = \frac{4\pi M_0}{3}\left\{\hat{r}\left(2-\frac{2a^3}{r^3}\right)\cos\theta - \hat{\theta}\left(2+\frac{a^3}{r^3}\right)\sin\theta\right\}$

$= \bar{B} = \frac{8\pi M_0}{3}\left\{\hat{r}\left(1-\frac{a^3}{r^3}\right)\cos\theta - \hat{\theta}\left(1+\frac{a^3}{2r^3}\right)\sin\theta\right\}$ // ans

$b<r: \bar{H} = -\frac{4\pi M_0}{3}(b^3-a^3)\left\{-\frac{2}{r^3}\cos\theta\,\hat{r} - \frac{1}{r^3}\sin\theta\,\hat{\theta}\right\}$

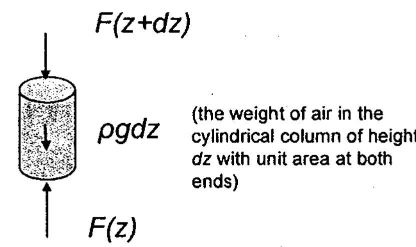
$= \bar{H} = \frac{4\pi M_0}{3}\frac{(b^3-a^3)}{r^3}\left\{2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right\}$ // ans

$\cancel{\bar{B}=\bar{H}}$

#4

Solution:

a. The mechanical equilibrium of a cylindrical column of air (of unit area) at altitude z, as shown in the figure on the right, can be represented as $p(z) = \rho g\, dz + p(z+dz)$, where the force per unit area by definition is the pressure, and ρ is the density of the air in the column and is defined by $\rho = \frac{nM}{V}$. Using the ideal gas law, $pV = nRT$, immediately yields




$\rho = \frac{p}{RT}M$. Using the definition $dp = p(z+dz) - p(z) = -\rho g dz$ and inserting ρ from the above equation into the last equation immediately yields the desired result: $\frac{dp}{p} = -\frac{Mg}{RT}dz$.

b. Eliminating *V* from the ideal gas law, $pV = nRT$, using the adiabatic relation $pV^{\gamma} = \text{const.}$ immediately yields $pT^{\frac{\gamma}{1-\gamma}} = \text{constant}$. Differentiating the latter relation yields $dp\, T^{\frac{\gamma}{1-\gamma}} + \frac{\gamma}{1-\gamma} T^{(\frac{\gamma}{1-\gamma}-1)} p\, dT = 0$. This last result can be reduced to $\frac{dT}{T} = \frac{\gamma - 1}{\gamma}\frac{dp}{p}$. Combining the result of (a) above with this last relation yields the desired relation: $\frac{dT}{dz} = (\frac{1}{\gamma} - 1)\frac{Mg}{R}$.

c. Solution of the simple differential equations in (b) yields the variation of temperature with altitude: $T(z) = T_o + (\frac{1}{\gamma} - 1)\frac{Mg}{R}z$. By inserting *T(z)* into the last equation in (a) we reduce it to $\frac{dp}{p} = -\frac{Mg}{RT_o + (\frac{1-\gamma}{\gamma})Mg\, z}dz$. Changing the variable to $\eta = RT_o + (\frac{1-\gamma}{\gamma})Mg\, z$, and using $dz = \frac{d\eta}{(\frac{1}{\gamma} - 1)\, Mg}$, the last differential equation can be reduced to $\frac{dp}{p} = -\frac{1}{(\frac{1}{\gamma} - 1)}\frac{d\eta}{\eta}$. The solution of this simple equation is $\ln\frac{p(z)}{p_o} = -\frac{1}{(1 - \frac{1}{\gamma})}\ln\frac{\eta}{\eta_o}$, where $\frac{\eta(z)}{\eta_o} = 1 + (\frac{1}{\gamma} - 1)\frac{Mgz}{RT_o}$.

#4 cont.

Some numerical values (this part was not in the exam): Let us see what $T$ and $p$ are at a typical crusing altitude of *35,000* ft: We can obtain the numerical values for $T(z)$ and $p(z)$ by inserting at $z=0$ (sea level) $T_o=300$ K, $M=0.029$ kg/mol, $R=8.31$ J/K mol, $g=9.8$ m/s$^2$ and $z=35,000$ ft * *0.3048* m/ft=*10,668* m. We obtain

$$T = 300 + (\frac{1}{1.4} - 1)\frac{0.029 \times 9.8 \times 35,000 \times 0.3048}{8.31}\ ;\ 196\ K = -77^{\circ}C\ ;\ \text{similarly,}$$

$$\ln\frac{p}{p_o} = \frac{1}{(1-\frac{1}{1.4})}\ln(1 + (\frac{1}{1.4} - 1)\frac{0.029 \times 9.8 \times 10,668}{8.31 \times 300}) = -1.49,\ \text{or}$$

$$p = 0.22 p_o = 0.22\ \text{atm.}$$

*Problem #5*

Consider a quantum-mechanical system for which the properly normalized state $|\psi\rangle$ is expanded in terms of the discrete energy eigenstates $|n\rangle$ of the known time-independent *Hamiltonian* $H$:

$$|\psi\rangle = \sum_n |n\rangle \langle n \mid \psi\rangle \equiv \sum_n a_n |n\rangle \qquad \text{where} \;\; H|n\rangle = E_n |n\rangle .$$

Assume that there are no continuum energy eigenstates. We construct the *density operator* $\rho \equiv |\psi\rangle\langle\psi|$ such that its matrix elements in the basis of energy eigenstates

$$\rho_{nm} = \langle n |\rho| m\rangle = \langle n \mid \psi\rangle \langle \psi \mid m\rangle = a_n a_m^* .$$

In the Schrödinger picture, the state $|\psi\rangle$ evolves in time, and we interpret the expansion coefficients as time-dependent complex numbers.

(a) Use the *Time-Dependent Schrödinger Equation* to determine the *equation of motion* of $\rho$, *i.e.*, find the rate of change of $\rho$. Compare and contrast your result with the equation of motion for dynamical observables, *e.g.*, position, momentum, or energy, in the *Heisenberg picture.*

$$\begin{aligned}
\frac{\partial \rho}{\partial t} &= \frac{\partial |\psi\rangle}{\partial t} \langle\psi| + |\psi\rangle \frac{\partial \langle\psi|}{\partial t} \\
H|\psi\rangle &= i\hbar \frac{\partial |\psi\rangle}{\partial t} \qquad \text{Hermitian adjoint} \;\; \langle\psi| H = -i\hbar \frac{\partial \langle\psi|}{\partial t} \\
\frac{\partial \rho}{\partial t} &= \frac{1}{i\hbar} H|\psi\rangle \langle\psi| + |\psi\rangle \frac{1}{-i\hbar} \langle\psi| H = \frac{1}{i\hbar}\left(H|\psi\rangle\langle\psi| - |\psi\rangle\langle\psi| H\right) \\
& \boxed{i\hbar \frac{\partial \rho}{\partial t} = -[\rho, H]}
\end{aligned}$$

*The dynamical observables for which the operators in the Schrödinger picture are not intrinsically time-dependent, acquire a time-dependence in the Heisenberg picture according to*

$$i\hbar \frac{dA_H}{dt} = [A_H, H] ,$$

where the $H$ subscript means Heisenberg picture. The two results – for a state-based operator like $\rho$ in the Schrödinger picture, and for a dynamical operator in the Heisenberg picture – differ by a minus sign.

(b) What is the physical meaning of the *trace* of the density matrix (the sum of the diagonal elements)? Is it a time-dependent quantity? Explain.

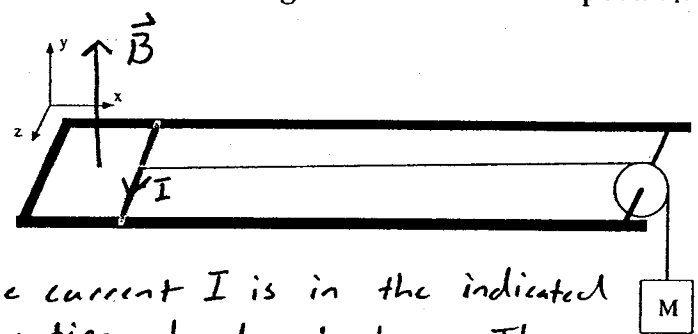
*The diagonal elements* $\rho_{nn} = |a_n|^2$ *give the probability to find the system in the state* $|n\rangle$. *The sum of these numbers for a properly normalized state must be equal to one, independent of time.*

## Problem #6

A metal bar is formed into a long U-shape as shown in the diagram. The U is horizontal and thick enough that its electrical resistance is negligible; it lies in the x-z plane. Its two rails are a distance L apart; they point in the x-direction. A constant magnetic field B is directed in the y-direction, perpendicular through the plane in which the U lies. At the far end of the U, a frictionless pulley is mounted. A rod with resistance R and mass m is placed on the U. It can slide in a frictionless manner such that it remains perpendicular to the long axis of the U. Connected to the rod is a massless string from which a mass M hangs. At time t = 0, M is released which causes the rod to slide in the x-direction on the rails.

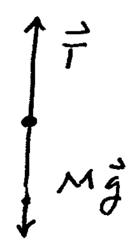
(a) Determine the speed of the rod as a function of time.

(b) From your result in (a), determine the terminal speed $v_T$ of the rod. Assume that the rails are long enough to reach the speed $v_T$. Use $v_T$ to calculate the energy dissipated in the resistance R at this speed and directly show that it is equivalent to the mechanical work done in moving the rod at constant speed $v_T$.




Free-body diagram for the mass

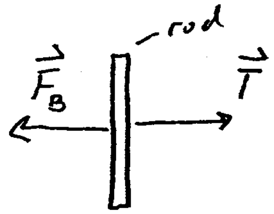
$$\boxed{T - Mg = -Ma} \quad ①$$




acceleration for rod is same as mass.

The current $I$ is in the indicated direction, by Lenz's law. The emf generated by the changing flux is $\mathcal{E} = -LBv$, $v$ is the speed of the rod.

Because of the $\vec{F} = I\vec{L} \times \vec{B}$ force, there is a force to the left on rod.




$$\Rightarrow \quad T - ILB = m\frac{dv}{dt} \qquad \text{use ① to find } T = Mg - M\frac{dv}{dt}$$

so $$Mg - ILB = (m+M)\frac{dv}{dt} \Rightarrow \frac{dv}{dt} = \frac{Mg}{M+m} - \frac{ILB}{M+m}$$

note $$I = \frac{\mathcal{E}}{R} = \frac{LBv}{R} \quad \therefore \quad \frac{dv}{dt} = \frac{Mg}{M+m} - \frac{L^2B^2v}{R(M+m)}$$

$$\frac{dv}{\frac{L^2B^2}{R(M+m)}v - \frac{Mg}{M+m}} = -dt \quad \Rightarrow \quad \frac{dv}{v - \frac{MRg}{L^2B^2}} = \frac{-L^2B^2}{R(M+m)}dt$$

and $$\int_0^v \frac{dv'}{v' - \frac{MRg}{L^2B^2}} = -\int_0^t \frac{L^2B^2}{R(M+m)}dt'$$

#6 cont.

$$\ln\left(v - \frac{MRg}{L^2B^2}\right) - \ln\left(-\frac{MRg}{L^2B^2}\right) = \frac{-L^2B^2}{R(M+m)}t$$

$$\ln\left(1 - \frac{L^2B^2}{MRg}v\right) = \frac{-L^2B^2}{R(M+m)}t$$

$$1 - \frac{L^2B^2}{MRg}v = e^{-\frac{L^2B^2}{R(M+m)}t}$$

$$v = v(t) = \frac{MRg}{L^2B^2}\left(1 - e^{-\frac{L^2B^2}{R(M+m)}t}\right)$$

b)

$$v_T = \frac{MRg}{L^2B^2} \qquad P = I^2R \qquad I = \frac{LBv_T}{R}$$

$$P = \frac{L^2B^2}{R}v_T^2$$

$$P = \frac{dW}{dt} = \frac{d}{dt}(Fx) = F\frac{dx}{dt} = Fv$$

F constant @ $v_T$

so $P = Fv_T = ILBv_T = \left(\frac{LBv_T}{R}\right)LBv_T$

$$= \frac{L^2B^2}{R}v_T^2$$

*Problem #7*

The partial differential equation

$$\nabla^2 u - \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2} = \alpha^2 u$$

describes the vibration of a stretched string embedded in a homogeneous elastic medium, for which $u$ represents (transverse) displacement from equilibrium, $v$ is the speed of waves on the string, and $\alpha$ is a measure of the medium's elasticity. Determine the characteristic frequencies of a such a string of length $L$.

Coincidentally, the variable substitutions $v \to c$ and $\alpha \to mc/\hbar$ produce the *Klein-Gordon equation*, which describes the relativistic quantum dynamics of free-particle spinless bosons of rest energy $mc^2$.

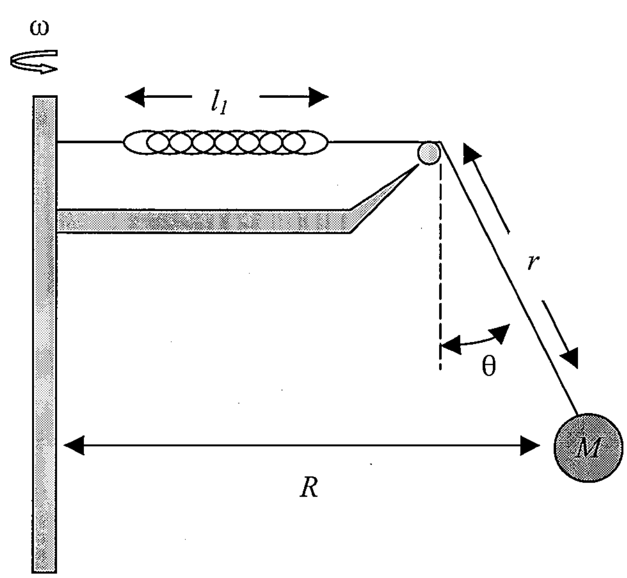
*Assume that the string is fixed at both ends so that the displacement from equilibrium is zero $u(0,t) = 0 = u(L,t)$, and that displacement can be written as a product of a function of $x$ and a function of $t$: $u(x,t) = f(x)g(t)$.*

$$\begin{aligned}
gf'' - \frac{1}{v^2} f\ddot{g} &= \alpha^2 fg \\
\frac{f''}{f} - \alpha^2 &= \frac{1}{v^2}\frac{\ddot{g}}{g} \equiv -\beta^2 \\
f'' &= -\left(\beta^2 - \alpha^2\right) f \equiv -k^2 f \\
f(x) &= A\cos(kx) + B\sin(kx) \\
f(0) &= A = 0 \qquad \Rightarrow f(x) = B\sin(kx) \\
f(L) &= B\sin(kL) = 0 \qquad \Rightarrow kL = n\pi \qquad n = 1, 2, 3, \ldots \\
\ddot{g} &= -v^2\beta^2 g \equiv -\omega^2 g \\
f_n &= \frac{\omega_n}{2\pi} = \frac{v\beta}{2\pi} = \frac{v\sqrt{\alpha^2 + k^2}}{2\pi} = \boxed{\tfrac{v}{2\pi}\sqrt{\alpha^2 + \left(\tfrac{n\pi}{L}\right)^2}}
\end{aligned}$$

Problem #8

A spherical ball of mass $M$ is attached to a massless string of length $r$ (measured from the stand-off) which is attached to a massless spring with spring constant $K$ (as shown in the diagram below). The entire apparatus rotates about the center post at a fixed angular velocity ω. At steady state while the apparatus is rotating at frequency ω, the length of the spring is $l_1$. Determine the following in reference to the mass $M$:

a) How many degrees of freedom are present?
b) What is the Lagrangian?
c) What are the equations of motion?
d) What are the steady state values (as a function of the parameters) for the variables describing the degrees of freedom?
e) What are the frequncies for small oscillations for the variables describing the degrees of freedom?



Note: I expected no one to solve part e), just wanted to see how it was started

e) To find the frequencies for small oscillations you can use many methods. The most direct is the Routhian function

$$R = L - \Sigma P_i Q_i$$

leading to

$$\frac{d}{dt}\left(\frac{\partial R}{\partial \dot{q}_i}\right) - \frac{\partial R}{\partial q_i} = 0$$

But free of any ignorable co-ordinates

$$\rightarrow \quad R = \tfrac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - \tfrac{1}{2}c^2\left[m(\ell_1 + r\sin\theta)^2 + I\right]^{-1}$$
$$+ mgr\cos\theta - \tfrac{1}{2}k(r - r_1)^2$$

where $c = P_\psi = \frac{\partial L}{\partial \dot{\psi}} = \left[m(\ell_1 + r\sin\theta)^2 + I\right]\omega = \text{constant}$

equations of motion become

$$m\ddot{r} - mr\dot{\theta}^2 - mc^2\left[m(\ell_1 + r\sin\theta)^2 + I\right]^{-2}(\ell_1 + r\sin\theta)\sin\theta$$
$$- mg\cos\theta + k(r - r_1) = 0$$

And

$$mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta} - mc^2\left[m(\ell_1 + r\sin\theta)^2 + I\right]^{-2}(\ell_1 + r\sin\theta)\,r\cos\theta$$
$$+ mgr\sin\theta = 0$$

Writing $r = r_0 + \delta r \qquad \theta = \theta_0 + \delta\theta$

After much math

$$a_{11}\,\delta\ddot{r} + c_{11}\,\delta r + c_{12}\,\delta\theta = 0$$
$$c_{21}\,\delta r + a_{22}\,\delta\ddot{\theta} + c_{22}\,\delta\theta = 0$$

where $a_{11} = m$, $c_{11} = m\,\omega^2\sin^2\theta_0\left(\dfrac{3m(\ell_1 + r_0\sin\theta_0)^2 - I}{m(\ell_1 + r_0\sin\theta_0)^2 + I}\right) + k$

and

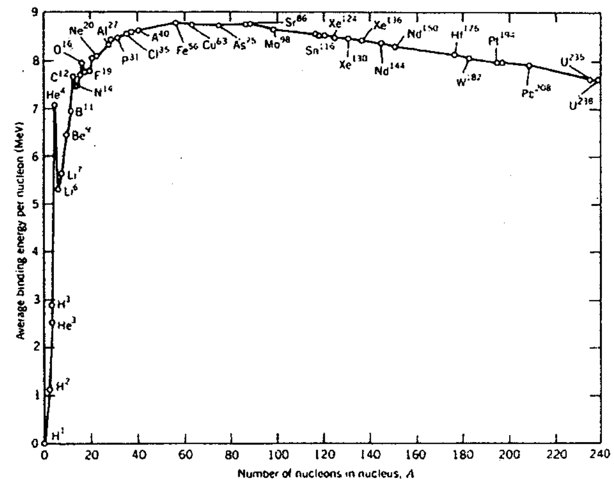
$$c_{12} = m\omega^2\left[\frac{4m(\ell_1 + r_0\sin\theta_0)^2\, r_0\sin\theta_0\cos\theta_0}{m(\ell_1 + r_0\sin\theta_0)^2 + I}\right.$$
$$\left. - (\ell_1 + 2r_0\sin\theta_0)\cos\theta_0\right] + mg\sin\theta_0$$

*Problem #9*

An atomic nucleus consists of $Z$ protons and $N$ neutrons, with mass number $A \equiv Z + N$. The *nuclear binding energy* $B$ is the difference between the rest energies of the constituent protons and neutrons and the measured rest energy of the nucleus.

A plot of $B/A$ as a function of $A$ has clear trends.

**This plot is given for information only and is not to be used in this problem.**




$B$ can be expressed semi-empirically as a sum of simple functions:

$$B = a_1\, A + a_2\, A^{2/3} + a_3\, \frac{Z(Z-1)}{A^{1/3}} + a_4\, \frac{(A-2Z)^2}{A} + \ldots$$

where the $a_i$ are constants with units of energy. We use the convention here that $B > 0$, so that a larger value of $B$ implies a more tightly bound nucleus.

*No knowledge of nuclear physics beyond introductory physics is assumed in this problem.*

**Briefly explain your answers to all of the following questions:**

(a) The dominant first term is referred to as the "volume term" because stable nuclei are roughly spherical with radii proportional to $A^{1/3}$. What is the sign of $a_1$? What can be concluded about the *strong interaction* from the observation that the dominant term is not proportional to $A(A-1)$?

*The sign of $a_1$ must be positive if the first term is the dominant term, because it is the attractive strong interaction that provides the binding of nuclei.*

*A factor of $A(A-1)$ would arise by coupling each nucleon to all other nucleons in the sphere. A linear dependence on $A$ means each nucleon does not feel the attraction to all others, just its nearest neighbors. The strong interaction is therefore short-range. Each nucleon has about the same number of neighbors; each nucleon thus contributes roughly the same amount to the binding energy.*

(b) What is the physical significance of the second term, which can be thought of as a correction to the first term, and what is the sign of $a_2$?

*This term is the "surface term". The nuclei at the surface have fewer neighbors than those within the volume, so they are less tightly bound. Because the first term gives full weight to the surface nucleons, we must subtract from $B$ a term proportional to the surface area, so $a_2 < 0$.*

(c) What is the physical significance of the third term, and what is the sign of $a_3$? Estimate the value of $a_3$ in eV to one significant figure.

#9 cont.

*The third term arises from the Coulomb repulsion of the protons. The electromagnetic interaction makes the nucleus less tightly bound, so $a_3 < 0$.*

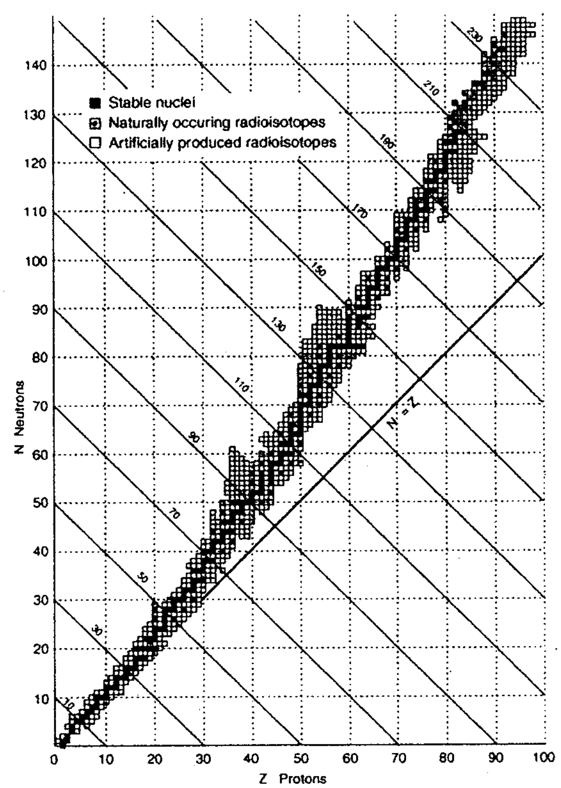
*For a uniformly charged sphere of radius $R = R_\circ A^{1/3}$ with $R_\circ \simeq 1.2$ fm:*

$$\begin{aligned} E_C &= \frac{3}{5}\frac{e^2}{4\pi\epsilon_\circ}\frac{Z(Z-1)}{R} = \frac{3}{5}\frac{1}{4\pi\epsilon_\circ}\frac{e^2}{R_\circ}\frac{Z(Z-1)}{A^{1/3}} \\ &= \left(6\times10^{-1}\right)\left(9.0\times10^{9}\mathrm{J}\cdot\mathrm{m/C^2}\right)\frac{(1.6\times10^{-19}\ \mathrm{C})^2}{1.2\times10^{-15}\ \mathrm{m}}\times\left(\frac{1\ \mathrm{eV}}{1.6\times10^{-19}\ \mathrm{J}}\right)\frac{Z(Z-1)}{A^{1/3}} \\ &= (6\times9\times1.6/1.2)\times10^4\frac{Z(Z-1)}{A^{1/3}} = 0.72\ \mathrm{MeV}\frac{Z(Z-1)}{A^{1/3}} \end{aligned}$$

(d) The fourth term arises in part from the *weak interaction*. If $a_4 < 0$, what relationship between $Z$ and $N$ is favored, *i.e.*, provides tighter binding, by this term for a given $A$? For nuclei far from the favored relationship, how does this term compare to the previous three terms as $A$ increases?

*Because $a_4 < 0$, a value of zero for the numerator in this term will give the largest $B$, so $A = 2Z$ or $Z = N$.*

*For a given value of $Z - N$, the fourth term decreases faster with increasing $A$ than the third term, and the first two terms increase with increasing $A$. Therefore, as $A$ increases, this term becomes less important than the other three. The stable, heavy elements, therefore, have a neutron excess so that the strong interaction can counteract the Coulomb repulsion without paying much of a price for this fourth term.*



*Problem #10*

Consider a quantum-mechanical particle of mass $m$ subject to a one-dimensional potential.

(a) In coordinate space, the position operator $x_{\rm op}$ takes the form of multiplication by $x$, and the momentum operator $p_{\rm op}$ is represented as $-i\hbar\frac{\partial}{\partial x_{\rm op}}$. Prove that the *commutator*

$$[x_{\rm op}, p_{\rm op}] = i\hbar$$

independent of the state of the system. *You may not use part (b)!*

*We proceed in coordinate space.*

$$\begin{aligned}
[x_{\rm op}, p_{\rm op}]\,\psi(x,t) &= (x_{\rm op}p_{\rm op} - p_{\rm op}x_{\rm op})\,\psi(x,t) = -i\hbar\, x\,\frac{\partial\psi}{\partial x} + i\hbar\,\frac{\partial}{\partial x}\,(x\psi) \\
&= -i\hbar x\,\frac{\partial\psi}{\partial x} + i\hbar\left(\psi + x\,\frac{\partial\psi}{\partial x}\right) = i\hbar\,\psi(x,t) \qquad \textit{q.e.d.}
\end{aligned}$$

(b) For an arbitrary scalar function $f$ expressible as a power series in $p_{\rm op}$, prove that

$$[x_{\rm op}, f(p_{\rm op})] = i\hbar\frac{\partial f}{\partial p_{\rm op}}$$

independent of the state. Recall: $[A_{\rm op}, B_{\rm op}C_{\rm op}] = [A_{\rm op}, B_{\rm op}]\,C_{\rm op} + B_{\rm op}\,[A_{\rm op}, C_{\rm op}]$.

$$\begin{aligned}
[x_{\rm op}, f(p_{\rm op})] &= \left[x_{\rm op}, \sum_n a_n p_{\rm op}^n\right] = \sum_n a_n \left[x_{\rm op}, p_{\rm op}^n\right] \\
&= a_0\left[x_{\rm op}, p_{\rm op}^0\right] + a_1\left[x_{\rm op}, p_{\rm op}^1\right] + a_2\left[x_{\rm op}, p_{\rm op}^2\right] + a_3\left[x_{\rm op}, p_{\rm op}^3\right] + \ldots \\
\left[x_{\rm op}, p_{\rm op}^2\right] &= [x_{\rm op}, p_{\rm op}]\,p_{\rm op} + p_{\rm op}\,[x_{\rm op}, p_{\rm op}] = i\hbar p_{\rm op} + p_{\rm op} i\hbar = 2i\hbar p_{\rm op} \\
\left[x_{\rm op}, p_{\rm op}^3\right] &= \left[x_{\rm op}, p_{\rm op}^2\right] p_{\rm op} + p_{\rm op}^2\,[x_{\rm op}, p_{\rm op}] = 2i\hbar p_{\rm op} p_{\rm op} + p_{\rm op}^2 i\hbar = 3i\hbar p_{\rm op}^2 \\
[x_{\rm op}, f(p_{\rm op})] &= i\hbar a_1 + 2i\hbar a_2 p_{\rm op} + 3i\hbar a_3 p_{\rm op}^2 + \ldots = i\hbar\sum_n n a_n p_{\rm op}^{n-1} = i\hbar\frac{\partial f}{\partial p_{\rm op}} \qquad \textit{q.e.d.}
\end{aligned}$$

(c) Now the particle is subject to the simple harmonic oscillator potential of characteristic frequency $\omega$ for which the normalized energy eigenstates $|n\rangle$ correspond to energy eigenvalues $E_n = \hbar\omega(n + \frac{1}{2})$ for $n = 0, 1, 2, \ldots$. Suppose at $t = 0$ the state vector is given by

$$|\psi(t=0)\rangle = e^{-ip_{\rm op}b/\hbar}\,|0\rangle$$

where $b$ is a real number. Calculate the expectation value of position as a function of time for $t \geq 0$. You may find the following operator relations useful:

$$\begin{aligned}
x_{\rm op} &= \tfrac{l_\circ}{\sqrt{2}}\left(a_{\rm op}^\dagger + a_{\rm op}\right) & p_{\rm op} &= \tfrac{i\hbar}{l_\circ\sqrt{2}}\left(a_{\rm op}^\dagger - a_{\rm op}\right) \\
a_{\rm op}^\dagger\,|n\rangle &= \sqrt{n+1}\,|n+1\rangle & a_{\rm op}\,|n\rangle &= \sqrt{n}\,|n-1\rangle \\
H_{\rm op} &= \hbar\omega\left(a_{\rm op}^\dagger a_{\rm op} + \tfrac{1}{2}\right) & \left[a_{\rm op}, a_{\rm op}^\dagger\right] &= 1
\end{aligned}$$

where $l_\circ = \sqrt{\hbar/m\omega}$ is the classical turning point corresponding to the ground-state energy. *Hint: Ehrenfest's Principle.*

# #10 cont.

*Ehrenfest's Principle states that expectation values obey classical equations of motion.*

$$\frac{d^2 \langle x \rangle}{dt^2} = -\omega^2 \langle x \rangle \quad \text{and} \quad \frac{d\langle x \rangle}{dt} = \frac{\langle p \rangle}{m} \qquad \Rightarrow \langle x \rangle = A\cos(\omega t) + B\sin(\omega t)$$

*Now, we need two initial conditions to set the values of $A$ and $B$: the expectation values of position and momentum. For these calculations, we'll need to use commutators.*

$$\begin{aligned}
\langle p \rangle_0 &= \langle \psi(0) | p_{\rm op} | \psi(0) \rangle = \left\langle 0 \left| e^{ip_{\rm op}b/\hbar}\, p_{\rm op}\, e^{-ip_{\rm op}b/\hbar} \right| 0 \right\rangle = \left\langle 0 \left| e^{ip_{\rm op}b/\hbar}\, e^{-ip_{\rm op}b/\hbar}\, p_{\rm op} \right| 0 \right\rangle \\
&= \langle 0 | p_{\rm op} | 0 \rangle = \frac{i\hbar}{l_\circ \sqrt{2}} \left\langle 0 \left| \left( a^\dagger_{\rm op} - a_{\rm op} \right) \right| 0 \right\rangle = 0
\end{aligned}$$

*... where we have used (a) an operator commutes with functions of itself, (b) the ground state is the lowest possible state, and (c) non-degenerate energy eigenstates are orthogonal. This result sets the constant $B = 0$:*

$$\begin{aligned}
\frac{d\langle x \rangle}{dt} &= -A\omega \sin(\omega t) + B\omega \cos(\omega t) \\
\left. \frac{d\langle x \rangle}{dt} \right|_{t=0} &= B\omega = \left. \frac{\langle p \rangle}{m} \right|_{t=0} = 0
\end{aligned}$$

*Now, we have only to fix the constant $A$.*

$$\begin{aligned}
\langle x \rangle_0 &= \langle \psi(0) | x_{\rm op} | \psi(0) \rangle = \left\langle 0 \left| e^{ip_{\rm op}b/\hbar}\, x_{\rm op}\, e^{-ip_{\rm op}b/\hbar} \right| 0 \right\rangle \\
&= \left\langle 0 \left| e^{ip_{\rm op}b/\hbar} \left\{ \left[ x_{\rm op}, e^{-ip_{\rm op}b/\hbar} \right] + e^{-ip_{\rm op}b/\hbar} x_{\rm op} \right\} \right| 0 \right\rangle \\
&= \left\langle 0 \left| e^{ip_{\rm op}b/\hbar} \left\{ i\hbar \left( -\frac{ib}{\hbar} \right) e^{-ip_{\rm op}b/\hbar} + e^{-ip_{\rm op}b/\hbar} x_{\rm op} \right\} \right| 0 \right\rangle = \langle 0 | (b + x_{\rm op}) | 0 \rangle \\
&= b + \frac{l_\circ}{\sqrt{2}} \left\langle 0 \left| \left( a^\dagger_{\rm op} + a_{\rm op} \right) \right| 0 \right\rangle = b \\
\langle x \rangle_0 &= A = b \\
& \boxed{\langle x \rangle = b\cos(\omega t)}
\end{aligned}$$

Problem #11

~~10 min.~~ general z independent solution for full azimuthal range $0<\phi<2\pi$.

$$\Phi(\rho,\phi)=a_0+b_0\ln\rho+\sum_n(a_n\rho^n+b_n\rho^{-n})[A\cos(n\phi)+B_n\sin(n\phi)] \quad (1)$$

Boundary conditions determine all a's, b's, A's & B's.

(a) $\rho<\rho_1$:

~10 min. all $\rho^{-n}$ & $\ln\rho$ terms blow up as $\rho\to 0$.
Therefore all b's = 0.

So $$\Phi(\rho,\phi)=a_0+\sum_n a_n\rho^n[A_n\cos(n\phi)+B_n\sin(n\phi)] \quad (2)$$

$$a_0=\frac{1}{2\pi}\int_0^{2\pi}\Phi(\rho,\phi)\,d\phi=\frac{V_0}{2\pi}\int_\pi^{2\pi}d\phi=\frac{V_0}{2} \quad (3)$$

$$a_n\rho_1^n A_n=\frac{1}{\pi}\int_0^{2\pi}\Phi(\rho_1,\phi)\cos(n\phi)\,d\phi$$
$$=\frac{V_0}{\pi}\int_\pi^{2\pi}\cos(n\phi)\,d\phi=\frac{V_0}{\pi n}\sin(n\phi)\Big|_\pi^{2\pi}=0 \quad (4)$$

$$a_n\rho_1^n B_n=\frac{1}{\pi}\int_0^{2\pi}\Phi(\rho,\phi)\sin(n\phi)\,d\phi=\frac{V_0}{\pi}\int_\pi^{2\pi}\sin(n\phi)\,d\phi$$
$$=\frac{V_0}{n\pi}\cos(n\phi)\Big|_{2\pi}^{\pi}=\begin{cases}-\dfrac{2V_0}{n\pi} & n\text{ odd}\\ =0 & n\text{ even}\end{cases} \quad (5)$$

$\therefore$ (2) (3) (4) (5) $\to$

Ans: $$\boxed{\Phi(\rho,\phi)=\frac{V_0}{2}-\frac{2V_0}{\pi}\sum_{n\,odd}\left(\frac{\rho}{\rho_1}\right)^n\frac{\sin(n\phi)}{n}} \quad (\rho<\rho_1) \quad (6)$$

note: $\Phi(0)=\frac{V_0}{2}$ = average of $\Phi$ over surface.

$$\boxed{\begin{aligned}\bar{E}=-\nabla\Phi&=\hat{r}\frac{2V_0}{\pi}\sum_{n\,odd}\left(\frac{\rho^{n-1}}{\rho_1^n}\right)\sin(n\phi)\\&+\hat{\phi}\frac{2V_0}{\pi}\sum_{n\,odd}\left(\frac{\rho^{n-1}}{\rho_1^n}\right)\cos(n\phi)\end{aligned}} \quad (7)$$

$(\rho<\rho_1)$

10 min **(b) $\rho_1 < \rho < \rho_2$** similarly,

$$a_0 + b_0 \ln \rho_1 = \frac{1}{2\pi}\int_0^{2\pi} \Phi(\rho_1, \phi)\, d\phi = \frac{V_0}{2\pi}\int_\pi^{2\pi} d\phi = \frac{V_0}{2} \quad (8)$$

$$a_0 + b_0 \ln \rho_2 = \int_0^{2\pi} \Phi(\rho_2, \phi)\, d\phi = \frac{V_0}{2\pi}\int_0^{\pi} d\phi = \frac{V_0}{2} \quad (9)$$

(8) & (9) $\rightarrow$

$$a_0 = \frac{V_0}{2}; \quad b_0 = 0 \quad (10)$$

$$(a_n \rho_1^n + b_n \rho_1^{-n}) A_n = \frac{V_0}{\pi}\int_\pi^{2\pi} \cos(n\phi)\, d\phi = 0 \rightarrow A_n = 0 \quad (11)$$

$$\begin{cases} (a_n \rho_1^n + b_n \rho_1^{-n}) B_n = \frac{V_0}{\pi}\int_\pi^{2\pi} \sin(n\phi)\, d\phi = -\frac{2V_0}{n\pi} & n \text{ odd} \quad (12) \\ (a_n \rho_2^n + b_n \rho_2^{-n}) B_n = \frac{V_0}{\pi}\int_0^{\pi} \sin(n\phi)\, d\phi = \frac{2V_0}{n\pi} & n \text{ odd} \quad (13) \end{cases}$$

(12) (13) $\rightarrow$

$$\left.\begin{cases} a_n B_n = -\frac{1}{n} K (\rho_2^{-n} + \rho_1^{-n}) & (14a) \\ b_n B_n = \frac{1}{n} K (\rho_2^{n} + \rho_1^{n}) & (14b) \end{cases}\right\} n \text{ odd.}$$

where

$$\boxed{K \equiv \frac{2V_0}{\pi} \frac{1}{\rho_1^n \rho_2^{-n} - \rho_2^n \rho_1^{-n}}} \quad (15)$$

$n$ odd.

So, ans.

$$\boxed{\Phi(\rho, \phi) = \frac{V_0}{2} - K \sum_{n\,\text{odd}} \left\{ \frac{1}{n}\left[ (\rho_2^{-n} + \rho_1^{-n})\rho^n - (\rho_2^n + \rho_1^n)\rho^{-n} \right] \times \sin(n\phi) \right\} \quad (\rho_1 < \rho < \rho_2)}$$

Ans.

$$\boxed{\begin{aligned} \bar{E} = \bar{\nabla}\Phi = \hat{\rho} K \sum_{n\,\text{odd}} \left[ (\rho_2^{-n} + \rho_1^{-n})\rho^{n-1} + (\rho_2^n + \rho_1^n)\rho^{-n-1} \right] \sin(n\phi) \\ + \hat{\phi} K \sum_{n\,\text{odd}} \left[ (\rho_2^{-n} + \rho_1^{-n})\rho^{n-1} - (\rho_2^n + \rho_1^n)\rho^{-n-1} \right] \cos(n\phi) \\ (\rho_1 < \rho < \rho_2) \end{aligned}}$$

(c) $\rho > \rho_2$:

$b_0$ and all $a$'s must be $=0$

for $\Phi_m$ to remain finite as $\rho \to \infty$

$$\Phi_m(\rho,\phi) = a_0 + \sum_n b_n \rho^{-n}\left[A_n \cos(n\phi) + B_n \sin(n\phi)\right]$$

$$\rightarrow a_0 = \frac{1}{2\pi}\int_0^{2\pi} \Phi_m(\rho_2,\phi)\,d\phi = \frac{V_0}{2\pi}\int_0^{\pi} d\phi = \frac{V_0}{2}$$

$$b_n \rho_2^{-n} A_n = \frac{1}{\pi}\int_0^{2\pi} \Phi_m(\rho_2,\phi)\cos(n\phi)\,d\phi$$

$$= \frac{V_0}{\pi}\int_0^{\pi} \cos(n\phi)\,d\phi = \frac{V_0}{n\pi} \sin(n\phi)\Big|_0^{\pi} = 0$$

$$b_n \rho_2^{n} B_n = \frac{1}{n}\int_0^{2\pi} \Phi_m(\rho_2,\phi)\sin(n\phi)\,d\theta$$

$$= \frac{V_0}{\pi}\int_0^{\pi} \sin(n\phi)\,d\phi = \frac{V_0}{n\pi}\cos(n\phi)\Big|_{\pi}^{0} = \frac{2V_0}{n\pi}$$

$n$ odd

So

$$\boxed{\Phi_m(\rho,\phi) = \frac{V_0}{2} + \frac{2V_0}{\pi}\sum_{n\,\text{odd}}\left(\frac{\rho}{\rho_2}\right)^{-n}\frac{\sin(n\phi)}{n} \quad \rho > \rho_2}$$

Ans

$$\boxed{\begin{aligned}\bar{E} = -\nabla\Phi_m &= \hat{r}\,\frac{2V_0}{\pi}\sum_{n\,\text{odd}}\frac{\rho^{-n-1}}{\rho_2^{-n}}\sin(n\phi) \\ &+ \hat{\phi}\,\frac{2V_0}{\pi}\sum_{n\,\text{odd}}\frac{\rho^{-n-1}}{\rho_2^{-n}}\cos(n\phi) \qquad \rho > \rho_2\end{aligned}}$$

Ans

Problem #12

a) The Maxwell velocity distribution is

$$F(v)\,dv = 4\pi \left(\frac{m}{2\pi kT}\right)^{3/2} e^{-\frac{mv^2}{2kT}} v^2\,dv$$

The Earth's escape velocity is determined

$$\frac{1}{2} m v_e^2 = \frac{G m M_e}{R_e} \qquad \left(K.E. = V(R)\right) \quad \left[V(\infty)=0\right]$$

$$\rightarrow V_e = \left(\frac{2GM_e}{R_e}\right)^{1/2}$$

The Fraction that escape have $V \geq V_e$

$$\rightarrow \boxed{F = 4\pi \left(\frac{m}{2\pi kT}\right)^{3/2} \int_{V_e}^{\infty} v^2 e^{-\frac{mv^2}{2kT}}\,dv}$$

Setting $x^2 = \frac{mv^2}{2kT}$ then $\alpha = V_e / \sqrt{\frac{2kT}{m}}$

Equivalent

$$\boxed{f = \frac{4}{\sqrt{\pi}} \int_{\alpha}^{\infty} x^2 e^{-x^2}\,dx}$$

For $T = 300\ K$, $M_e = 6 \times 10^{24}\ kg$ $R_e = 6.4 \times 10^3\ km$

$$f = \frac{2}{\sqrt{\pi}} \left[\alpha e^{-\alpha^2} + \frac{2}{\sqrt{\pi}} \int_{\alpha}^{\infty} e^{-x^2}\,dx\right]$$

$$= 1.4 \times 10^{-5} + 1.13 \int_{3.55}^{\infty} e^{-x^2}\,dx = \underline{\underline{6 \times 10^{-5}}}$$

Problem #12 (cont.)

b) The average time for an $H_2$ molecule to escape is the time it takes $H_2$ to diffuse elastically through the atmosphere.

The mean-free-path between scattering events is the inverse of the density × cross sectional area of $H_2$

$$\lambda = \left(\frac{1}{nA}\right) = \frac{1}{(2.5\times10^{25})(1\times10^{-10})^2} = 4\times10^{-6}\ m \qquad A = (1\ \text{Å})^2, \quad n = 2.5\times10^{25}\ \text{atom}/m^3$$

Time between collisions is

$$\tau = \frac{\lambda}{V_e} = 5\times10^{-10}\ \text{sec}$$

After $N$ collisions the mean-square of the diffusion displacement is

$$\langle x^2 \rangle = N\lambda^2$$

The total time is $(d = 100\ km)$

$$t = N\tau = \frac{\tau d^2}{\lambda^2}$$

$$= \frac{5\times10^{-10}\ \text{sec} \times (10^5\ m)^2}{(4\times10^{-6}\ m)^2} \cong 3\times10^{11}\ \text{sec} \doteq \underline{\underline{10^4\ \text{yrs}!}}$$

#13

Solution:

General: Normalized unperturbed states $\phi_n$ and corresponding eigenvalues $\varepsilon_n$ are given by $\phi_n = \sqrt{\frac{2}{a}}\, Sin k_n x$, where $k_n = \frac{n\pi}{a}$, and $\varepsilon_n = \frac{\hbar^2 k_n^{\,2}}{2m}$, where n=1, 2, 3... Note that negative $n$'s do not give different states: they only introduce a phase shift in the unperturbed states. Hence they do not introduce independent degenerate eigenstates which are orthogonal to those corresponding to $n > 0$, so we only consider the positive $n$'s. The first-order corrections to the $\varepsilon_n$'s and the $\phi_n$'s are given by the non-degenerate stationary perturbation approach as $E_n = \varepsilon_n + <\phi_n | V | \phi_n>$ and $\Phi_n = \phi_n + \sum_{m \neq n} \frac{<\phi_m|V|\phi_n>}{\varepsilon_n - \varepsilon_m} \phi_m$, respectively, where $V$ is the perturbation introduced on the electron of charge $-e$ by the uniform electric field $\vec{E} = E_o \hat{x}$ acting inside the infinite well and is given by

$V(x) = -(-e)\int_0^x \vec{E}.d\vec{x} = eE_o x$, where $0 \le x \le a$. Now we can solve the problem:

(a) From the above, $E_n = \varepsilon_n + <\phi_n | V | \phi_n> = \varepsilon_n + \int_0^a \frac{2}{a} Sin k_n x\, (eE_o x)\, Sin k_n x\, dx$, which can easily be evaluated to be

$$E_n = \varepsilon_n + \frac{2eE_o}{a}\int_0^a x\, Sin^2 k_n x\ \ dx = \varepsilon_n + \frac{2eE_o}{a}\left(\frac{a^2}{4} - \frac{1}{2}\int_0^a x\, Cos 2k_n x\ \ dx\right) = \varepsilon_n + \lambda, \text{ where } \lambda = \frac{eE_o a}{2}$$

which suggests that all levels are, to a first approximation, rigidly shifted by the same amount, $\lambda = \frac{eE_o a}{2}$, and hence that the separations between the levels are not altered. It is, of course, assumed that $\lambda << \varepsilon_1$.

(b) From the general review above

$$\Phi_1 = \phi_1 + \sum_{m \neq n} \frac{<\phi_m|V|\phi_1>}{\varepsilon_1 - \varepsilon_m}\phi_m \ ;\ \phi_1 + \frac{<\phi_2|V|\phi_1>}{\varepsilon_1 - \varepsilon_2}\phi_2 = \phi_1 - \frac{2m}{3\hbar^2 k_1^{\,2}} <\phi_2 | V | \phi_1> \phi_2, \text{ where}$$

$$<\phi_2 | V | \phi_1> = \frac{2}{a}\int_0^a Sin k_2 x\, (eE_o x)\, Sin k_1 x\, dx = \frac{4eE_o}{a}\int_0^a x\, Sin^2 k_1 x\, Cos k_1 x\, dx = -\frac{4eE_o}{a}\frac{4}{9k_1^{\,2}}$$

This yields $\Phi_1 = \phi_1 + \gamma\phi_2$, where $\gamma = \frac{2m}{3\hbar^2 k_1^{\,2}}\frac{4eE_o}{a}\frac{4}{9k_1^{\,2}} = \frac{32}{27\pi^2}\frac{\lambda}{\varepsilon_1}$, where it is clear that $\gamma << 1$. Therefore, to a first-order approximation the probability $p$ that the electron will be found in the first excited state of the unperturbed system is given by $p = \frac{|<\phi_2 | \Phi_1>|^2}{|<\Phi_1 | \Phi_1>|^2} = \frac{\gamma^2}{(1+\gamma^2)} \approx \gamma^2$.

(c) Once the electric field is lifted, the electron will be in the state of $\Phi_1(x, t=0) = \phi_1(x) + \gamma\phi_2(x)$. In the absence of any other interactions of the electron with the environment, the time evolution of the normalized version of $\Phi_1(x, t=0)$, which is given by $\varphi(x, t=0) = \alpha\phi_1(x) + \beta\phi_2(x)$, where

#13 cont.

$\alpha = \frac{1}{\sqrt{1+\gamma^2}}$ and $\beta = \frac{\gamma}{\sqrt{1+\gamma^2}}$, will be determined by the time evolution operator: $\varphi(x,t) = e^{-iH(t)/\text{h}}\varphi(x,t=0) = \alpha\phi_1(x)e^{-i\omega_1 t} + \beta\phi_2(x)e^{-i\omega_2 t}$, where $\omega_1$ and $\omega_2$ are given by

$\omega_1 = \frac{\varepsilon_1}{\text{h}}$ and $\omega_2 = \frac{\varepsilon_2}{\text{h}}$, where the $\varepsilon_n$'s are the unperturbed energies associated with the $\phi_n$'s. The energy of the system is given by the expectation value of the Hamiltonian, $\varepsilon = <\varphi(x,t)|H|\varphi(x,t)> = \alpha^2\varepsilon_1 + \beta^2\varepsilon_2$, and is independent of time. This energy is slightly larger than $\varepsilon_1$ and is given by $\varepsilon ; \varepsilon_1 + \left(\frac{32}{27\pi^2}\right)^2\left(\frac{3\lambda}{\varepsilon_1}\right)\lambda$. In reality, however, the system will not stay in this excited state. It will decay to the ground state by interacting with the vacuum fluctuations and end up dissipating its excess energy and settling down at ground state $\phi_1$ with energy $\varepsilon_1$.

## Problem #14

$$\nabla^2 u + K^2 u = \delta(\bar{x} - \bar{x}') \quad (1)$$

subject to the boundary condition that $u = 0$ on the surface of the cube $0 \le x, y, z \le L$.

First, factor the $\delta$-function.

$$\delta^3(\bar{x} - \bar{x}') = \delta(x - x')\,\delta(y - y')\,\delta(z - z') \quad (2)$$

Expand $\delta(y - y')$ and $\delta(x - x')$ into eigenfunctions

$$\int_0^L \delta(y - y') = \sum_n A_n \sin\left(\frac{n\pi y}{L}\right)$$

$$\rightarrow \int_0^L \delta(y - y') \sin\frac{n\pi y}{L}\,dy = \sin\left(\frac{n\pi y'}{L}\right) = \frac{L}{2} A_n$$

$$\therefore \delta(y - y') = \frac{2}{L} \sum_n \sin\left(\frac{n\pi y}{L}\right) \sin\left(\frac{n\pi y'}{L}\right) \quad (3)$$

Similarly,

$$\delta(x - x') = \frac{2}{L} \sum_m \sin\left(\frac{m\pi x}{L}\right) \sin\left(\frac{m\pi x'}{L}\right) \quad (4)$$

now we look at the inhomogeneous equation in the z' direction. Let

$$u = \sin\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi y}{L}\right) f(z) \quad (5)$$

(5) → (1) & get

$$\left.\begin{aligned} &\frac{\partial^2 f}{\partial z^2} - \left(\frac{n\pi}{L}\right)^2 f - \left(\frac{m\pi}{L}\right)^2 f + K^2 f \\ &= \frac{4}{L^2} \sin\left(\frac{n\pi y'}{L}\right) \sin\left(\frac{m\pi x'}{L}\right) \delta(z - z') \end{aligned}\right\} \quad (6)$$

For $z \neq z'$ this is homogenous, with solutions

$$0 \le z \le z' \quad f(z) = A_{mn} \sin pz \quad (7)$$

$$z' < z < L \quad f(z) = B_{mn} \sin p(L - z) \quad (8)$$

where $p^2 = K^2 - \frac{\pi^2}{L^2}(n^2 + m^2) \quad (9)$

The matching conditions are that $f(z)$ is continuous at $z=z'$, and $f'(z)$ has a step-fn discontinuity there. We get

(7) & (8) →

(10) $A_{mn} \sin pz' = B_{mn} \sin p(L-z')$ at $z=z'$

(11) $A_{mn}\, p \cos pz' = -B_{mn}\, p \cos p(L-z')$
$$-\underbrace{\frac{4}{L^2} \sin\left(\frac{n\pi y'}{L}\right) \sin\left(\frac{m\pi x'}{L}\right)}_{\delta(x-x')} \text{ at } z=z'$$

by $u'(0 \le z \le z') = u'(z' \le z \le L)$ at $z=z'$ $+\delta(\vec{x}-\vec{x}')$

Solving (10) & (11) simultaneously, we get

$$A_{mn} = \frac{-4 \sin\left(\frac{n\pi y'}{L}\right) \sin\left(\frac{m\pi x'}{L}\right) \sin p(L-z')}{L^2 p \sin pL}$$

& $$B_{mn} = \frac{-4 \sin\left(\frac{n\pi y'}{L}\right) \sin\left(\frac{m\pi x'}{L}\right) \sin pz'}{L^2 p \sin pL}$$

The Green's fn is thus:

Ans

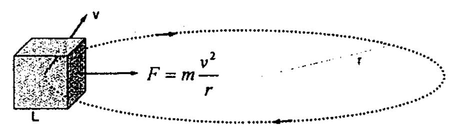
$0 \le z \le z'$:
$$G \equiv u = \frac{-4 \sum \sin\left(\frac{n\pi y}{L}\right) \sin\left(\frac{n\pi y'}{L}\right) \sin\left(\frac{m\pi x}{L}\right) \sin\left(\frac{m\pi x'}{L}\right)}{L^2 p \sin pL}$$
$$\times \sin p(L-z') \sin pz.$$

$z' < z \le L$:
$$G \equiv u = \frac{-4 \sum \sin\left(\frac{n\pi y}{L}\right) \sin\left(\frac{n\pi y'}{L}\right) \sin\left(\frac{m\pi x'}{L}\right) \sin\left(\frac{m\pi x}{L}\right)}{L^2 p \sin pL}$$
$$\times \sin pz' \sin p(L-z)$$

#15

a. The eye is composed of two lenses. The first one is the cornea, the outer lens, and the other, known as the eye lens, is located behind the cornea, inside the eye. To focus objects on the retina, the majority (~75%) of the refraction (bending) of light takes place at the cornea, the outside surface of the eye. The eye lens simply fine-tunes the focusing in order to bring the image onto the retina. Recall Snell's Law, $n_1 Sin\theta_1 = n_2 Sin\theta_2 \Rightarrow Sin\theta_2 = \frac{n_1 Sin\theta_1}{n_2}$, which simply means that the angle of refraction ($\theta_2$) by the cornea in water is less than that in air for a given angle of incidence $\theta_1$. This means the eye cannot focus the object on the retina: the object is focused behind the retina. When you wear a goggle with an air space between your eye and the water, the refraction of your eye works as it does in air: hence you see normally, as though you were in air.

b. Consider a cubic volume ($L^3$) of air circulating (with velocity v) around the center of the eye of the hurricane at distance r from the center. An air volume of mass $m = \rho L^3$ is held around a circle of radius r by the centripetal force originating from the pressure difference between the inner and outer surfaces of volume $L^3$. Centripetal force




$F = \delta p\, L^2 = (dp/dr)\, \delta r\, L^2 = (dp/dr)\, L\, L^2 = (dp/dr)\, L^3$. This immediately yields $F = (dp/dr)\, L^3 = m\frac{v^2}{r} = m = \rho L^3 \frac{v^2}{r} \Rightarrow v = \sqrt{\frac{(dp/dr)r}{\rho}}$. Substituting the numbers yields $v = \sqrt{\frac{0.3\times 10{,}000}{1.3\times 10^{-6}/10^{-6}}} = 48\ m/s \approx 173\ km/hr$, which is typical of hurricane speeds.

c. Nothing will happen to you. All the charges will be distributed outside the metallic box, the electric field inside the box will be zero everywhere, and the electric potential will be constant everywhere. Therefore, even though you will be at 50,000 V above the ground, since there is no potential difference inside the box no currents will flow through your body and you will not feel a thing unless you stick your hand outside the box and touch somewhere that is grounded: in that case you will be in serious trouble.

d. The centripetal force is given by $F = -\frac{\partial V}{\partial r} = -\alpha\beta r^{-\beta-1}$. The negative sign in $F$ indicates that the force is pointing inwards towards the center as it should be. Using $F = m\frac{v^2}{r}$ we can find the velocity in terms of other parameters, $v = \sqrt{\frac{\alpha\beta}{mr^{\beta}}}$. Using the last equation in the definition of angular momentum, $L = mvr$, we obtain $L = \sqrt{\frac{\alpha\beta m}{r^{\beta-2}}}$.

e. Reflected light will suffer a $\pi$ phase shift (or a half-wavelength phase shift) at the air/thin-film interface (because $n_{air} < n_c$), but it will not suffer any phase shift at the thin-film/glass interface because $n_c > n_g$. Taking this phase shift into account in

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calculating the optical path difference between the two reflected rays (one at the air/thin-film and the other at the thin-film/ glass interface) we require that the optical path difference be an integer multiple of the wavelength: this way the two rays will be out of phase because of the half-wavelength phase shift at the air/thin-film interface. This condition will give the minimum reflection. This yields $2n\tau = m\lambda \Rightarrow \tau = \dfrac{m\lambda}{2n}$. By setting $m$=*1* for minimum thickness $\tau$ we obtain $\tau = \dfrac{m\lambda}{2n} = \dfrac{1\times 600}{2\times 1.4}$; $214\ nm$.