Simp. Rad (cont'd) Radiation Zone: Dipole Radiation.

3) Since the terms in $\tilde{\mathbb{A}}$ of Eq. (8) go as $(d/\lambda)^m \ll 1$, the dominant term is for $m=0$. Then the vector potential is:

$$\longrightarrow \tilde{\mathbb{A}}_0(\mathbb{r},\omega) = \left(\frac{e^{ikr}}{cr}\right)\int d^3x'\,\tilde{\mathbb{J}}(\mathbb{r}',\omega), \quad m=0 \text{ term only.} \left\{\begin{array}{l}\text{for DIPOLE}\\ \text{RADIATION.}\end{array}\right. \qquad (9)$$

Actually, this $\tilde{\mathbb{A}}_0$ is the leading term in $\tilde{\mathbb{A}}$ for sourcesize $d \to 0$, no matter what the relative size of λ and r. So it holds in <u>all 3 zones</u> as $d \to 0$.

Now we transform the integral in Eq. (9) to a venerable quantity called the "electric dipole moment" (EDM) of the system. This is done by two tricks:

(1) $\int \tilde{\mathbb{J}}\,d^3x' = -\int \mathbb{r}'(\nabla'\cdot\tilde{\mathbb{J}})\,d^3x'$, by partial integration (in 1D, this is just $\int x(\partial J/\partial x)\,dx = xJ - \int J\,dx$, with the integrated term $\to 0$ at source bndys);

(2) Continuity Eqtn: $\nabla\cdot\mathbb{J} = -(\partial\rho/\partial t) \Rightarrow \nabla'\cdot\tilde{\mathbb{J}} = i\omega\tilde{\rho}$, in terms of amplitudes.

Then the integral in Eq. (9) above: $\int d^3x'\,\tilde{\mathbb{J}} = -i\omega\int d^3x'\,\mathbb{r}'\tilde{\rho}$, and so...

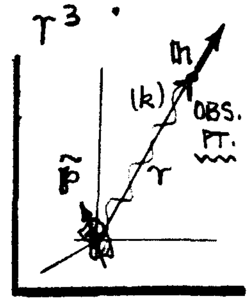
$$\boxed{\tilde{\mathbb{A}}_0(\mathbb{r},\omega) = (-)ik\left(\frac{e^{ikr}}{r}\right)\tilde{\mathbb{p}}, \quad \tilde{\mathbb{p}} = \int d^3x'\,\mathbb{r}'\tilde{\rho}(\mathbb{r}',\omega) = \text{system } \widetilde{\text{EDM}}.} \qquad (10)$$

This is the leading term approx^n to the (nonrel^c) radiation problem for <u>harmonic</u> sources $\mathbb{J}(\mathbb{r},t), \rho(\mathbb{r},t) = [\tilde{\mathbb{J}}(\mathbb{r},\omega), \tilde{\rho}(\mathbb{r},\omega)]e^{-i\omega t}$, w/ $k=\omega/c$. Requires only $d\to 0$.

4) The fields which follow from $\tilde{\mathbb{A}}_0$ of Eq. (10) are... (by arithmetic)...

$$\left[\begin{array}{l}\tilde{\mathbb{B}}_0 = \nabla\times\tilde{\mathbb{A}}_0 = k^2(\mathbb{n}\times\tilde{\mathbb{p}})\dfrac{e^{ikr}}{r}\left(1-\dfrac{1}{ikr}\right), \quad \text{w/ } k = 2\pi/\lambda = \omega/c;\\[2ex] \tilde{\mathbb{E}}_0 = \dfrac{i}{k}\nabla\times\tilde{\mathbb{B}}_0 = k^2(\mathbb{n}\times\tilde{\mathbb{p}})\times\mathbb{n}\dfrac{e^{ikr}}{r} + [3\mathbb{n}(\mathbb{n}\cdot\tilde{\mathbb{p}})-\tilde{\mathbb{p}}](1-ikr)\dfrac{e^{ikr}}{r^3}.\end{array}\right. \qquad (11)$$

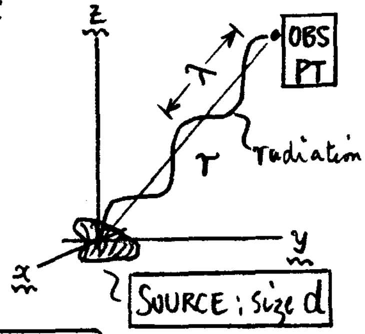
REMARKS



1. $\tilde{\mathbb{B}}_0$ & $\tilde{\mathbb{E}}_0$ are called "dipole rad^n fields", since they are $\propto$ EDM $\tilde{\mathbb{p}}$.
2. Both $\tilde{\mathbb{B}}_0$ & $\tilde{\mathbb{E}}_0$ have components $\propto 1/r$, so they are true rad^n fields.
3. $\tilde{\mathbb{B}}_0$ is transverse to the propagation direction $\mathbb{n}$, but in general $\tilde{\mathbb{E}}_0$ is <u>not</u> transverse to $\mathbb{n}$. But the leading term (in $1/r$) of $\tilde{\mathbb{E}}_0$ <u>is</u> transverse to $\mathbb{n}$.
4. If the radiating system has non-vanishing EDM $\tilde{\mathbb{p}}$, Eqs. (11) are the dominant fields.

Summary of Simple Radiation Theory

$\mathbb{A}(\mathbf{r},t)$ & $\mathbb{J}(\mathbf{r},t) \rightarrow$ Fourier components $\tilde{\mathbb{A}}(\mathbf{r},\omega)$ & $\tilde{\mathbb{J}}(\mathbf{r},\omega)$.★



So// $\tilde{\mathbb{A}}(\mathbf{r},\omega) = \frac{1}{c}\int_{\text{source}} d^3x' \left(\frac{e^{ikR}}{R}\right)\tilde{\mathbb{J}}(\mathbf{r},\omega)$ {wave eq. solⁿ a la Fourier (exact)

and// $\tilde{\mathbb{B}} = \nabla\times\tilde{\mathbb{A}}$, $\tilde{\mathbb{E}} = \frac{i}{k}\nabla\times\tilde{\mathbb{B}}$ {outside source

---

approxⁿ ① $R = |\mathbf{r}(\text{to obs.}) - \mathbf{r}'(\text{source})| \gg d$, observer is "far away" from source.

So// $R \simeq r - \mathbf{n}\cdot\mathbf{r}'$, $\mathbf{n}$ = unit vector from <u>origin</u> to observer ($\mathbf{n}$ is <u>fixed</u> in $\int d^3x'$)

approxⁿ ② $d \ll \lambda$, for many "interesting" problems {equiv. to $v \ll c$ (nonrelativistic q-motion) ignores high freq. (good up to $\nu \sim c/d$).

So// $e^{ikR} \overset{①}{\simeq} e^{ikr}\, e^{-ik\mathbf{n}\cdot\mathbf{r}'}$ ← this phase is $\sim \frac{d}{\lambda} \ll 1$.

Then for the "radiation zone" $\boxed{d \ll \lambda \ll r}$, above soln for $\tilde{\mathbb{A}}$ is

$$\tilde{\mathbb{A}}(\mathbf{r},\omega) \simeq \left(\frac{e^{ikr}}{cr}\right)\sum_{m=0}^{\infty}\frac{(-ik)^m}{m!}\underbrace{\int_{\text{source}} d^3x'\,\tilde{\mathbb{J}}(\mathbf{r}',\omega)[\mathbf{n}\cdot\mathbf{r}']^m}_{m^{\text{th}}\text{ moment of }\tilde{\mathbb{J}}}$$

note simple r-dependence (spherical wave at obs. pt.)

m^{th} term in the series {gives 2^{m+1}-pole radiation ($m=0 \Rightarrow$ dipole, $m=1 \Rightarrow$ quadrupole, etc) is of relative strength $(d/\lambda)^m$ [declines rapidly]

<u>Dipole Radiation</u>: take $m=0$ term in above expansion. Then...

$\tilde{\mathbb{A}}_0(\mathbf{r},\omega) = -ik\left(\frac{e^{ikr}}{r}\right)\tilde{\mathbb{p}}$, $\tilde{\mathbb{p}} = \int_{\text{source}} d^3x'\,[\mathbf{r}'\tilde{\rho}(\mathbf{r}',\omega)] \equiv$ system's EDM

So// $\tilde{\mathbb{B}}_0 = \nabla\times\tilde{\mathbb{A}}_0 = k^2(\mathbf{n}\times\tilde{\mathbb{p}})\left(\frac{e^{ikr}}{\underline{\underline{r}}}\right)\left(1-\cancel{\frac{1}{ikr}}\right)$,

The terms in $1/r$ are the ones which transport energy.

$\tilde{\mathbb{E}}_0 = \frac{i}{k}\nabla\times\tilde{\mathbb{B}}_0 = k^2(\mathbf{n}\times\tilde{\mathbb{p}})\times\mathbf{n}\left(\frac{e^{ikr}}{\underline{\underline{r}}}\right) + [3\mathbf{n}(\mathbf{n}\cdot\tilde{\mathbb{p}})-\mathbb{p}](1-ikr)\cancel{\left(\frac{e^{ikr}}{r^3}\right)}$.

★ Convention: $F(\mathbf{r},t) = \int_{-\infty}^{\infty}\tilde{F}(\mathbf{r},\omega)\,e^{-i\omega t}\,d\omega$, $\tilde{F}(\mathbf{r},\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty}F(\mathbf{r},t)\,e^{i\omega t}\,dt$.

Simp. Rad$^{\underline{n}}$ (cont'd) Lowest-order Approximation: $\lambda \gg d \to 0$.

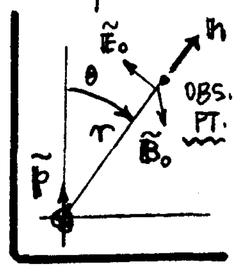
4) In the static zone ($r \ll \lambda \Rightarrow kr \to 0$) the fields of Eq. (11) [for $d \to 0$] reduce to:

$$\rightarrow \tilde{\mathbb{B}}_0 \simeq ikr(\mathbb{n}\times\tilde{\mathbb{p}})\frac{1}{r^3}, \quad \tilde{\mathbb{E}}_0 = [3\mathbb{n}(\mathbb{n}\cdot\tilde{\mathbb{p}}) - \tilde{\mathbb{p}}]\frac{1}{r^3}, \text{ as } kr \to 0. \tag{12}$$

$\tilde{\mathbb{E}}_0$ is just the familiar electric dipole field, while $\tilde{\mathbb{B}}_0$ is very small (down by the factor $kr \to 0$). These fields do not propagate -- they just "circulate", according to the factor $e^{-i\omega t}$, when appended thereonto.

In the radiation zone ($r \gg \lambda \Rightarrow kr \to \infty$) the dipole fields have leading terms:

$$\rightarrow \tilde{\mathbb{B}}_0 = k^2(\mathbb{n}\times\tilde{\mathbb{p}})\frac{e^{ikr}}{r}, \quad \tilde{\mathbb{E}}_0 = \tilde{\mathbb{B}}_0 \times \mathbb{n}. \tag{13}$$



Now we have "clean" radiation fields... they both fall off as $1/r$, and both are $\perp\ \mathbb{n}$, so we have a transverse wave.

5) One can now process the dipole rad$^{\underline{n}}$ fields of Eq. (13) [in the far zone] by the vast panoply of EM wave theory. A key quantity is the Poynting vector (energy flow per unit time & area), which is defined for fields that $\propto e^{\pm i\omega t}$ as: $\mathbb{S} = \frac{1}{2}\cdot\frac{c}{4\pi}\mathrm{Re}(\mathbb{E}\times\mathbb{B}^*)$.† This is combined in the usual way to form [Jk$^{\underline{n}}$ Eq. (9.22)]

radiated power per unit solid ∡ $\Big\}$ $$\boxed{\frac{d\overline{P}}{d\Omega} = r^2\,\mathbb{n}\cdot\mathbb{S} = \frac{ck^4}{8\pi}|\mathbb{n}\times(\mathbb{n}\times\tilde{\mathbb{p}})|^2.} \; ¶ \tag{14}$$

The vector product here is what is left of the $[\mathbb{n}\times(\mathbb{n}\times\beta)]$ term in the general radiation formula of Jk$^{\underline{n}}$ Eq. (14.67)... only here we already have the answer (no integral to do), and need only worry about ∡s. If $\tilde{\mathbb{p}}$ is on the z-axis (as above)

$$\left[\!\left[\ \begin{matrix}\text{dipole rad}^{\underline{n}}\\ \text{∡ distrib}^{\underline{n}}\end{matrix}\right\}\ \frac{d\overline{P}}{d\Omega} = \frac{ck^4}{8\pi}|\tilde{\mathbb{p}}|^2\sin^2\theta \;;\quad \left.\begin{matrix}\text{total radiated}\\ \text{power}\end{matrix}\right\}\ \overline{P} = \int\left(\frac{d\overline{P}}{d\Omega}\right)d\Omega = \frac{ck^4}{3}|\tilde{\mathbb{p}}|^2. \tag{15}$$

In this way, in leading order (and for sources $\propto e^{\pm i\omega t}$), the whole radiation problem is reduced to calculating the system EDM $\tilde{\mathbb{p}}$.

Q. What if $\tilde{\mathbb{p}} \to 0$?
A. Punt. Or see Jk$^{\underline{n}}$ Sec. 9.3

¶ NOTE: Because of the $k^4 = (2\pi/\lambda)^4$ factor, short λ's radiate much better than long λ's.

† The extra $\frac{1}{2}$ out in front is due to a time-average of the harmonic terms $\sin^2$ & $\cos^2\omega t$.

Time-dependence of Dipole Radiation Fields.

7) We restore the harmonic time dependence to the dipole radiation fields of Eq. (13): $(\tilde{\mathbb{B}}_0, \tilde{\mathbb{E}}_0) \rightarrow (\mathbb{B}, \mathbb{E}) = (\tilde{\mathbb{B}}_0, \tilde{\mathbb{E}}_0) e^{-i\omega t}$. Then we can write...

$$\mathbb{B}(\mathbb{r},t) = \frac{1}{r}\, \mathbb{n} \times \left[\frac{\omega^2}{c^2} \int d^3x'\, \mathbb{r}'\, \underbrace{\tilde{\rho}(\mathbb{r}',\omega)\, e^{-i\omega(t-\frac{r}{c})}}_{=\rho(\mathbb{r}',t'),\ t'=t-\frac{r}{c}\ (\text{ret. time})}\right] = \frac{1}{c^2 r}\, \mathbb{n} \times \Big[\underbrace{(-)\frac{\partial^2}{\partial t'^2}}_{\text{gives } \omega^2} \mathbb{p}(t')\Big]$$

Where // $\mathbb{p}(t') = \int d^3x'\, \mathbb{r}'\, \rho(\mathbb{r}',t')$ = system EDM (not $\tilde{\text{EDM}}$).

So //
$$\boxed{\mathbb{B}(\mathbb{r},t) = \frac{1}{c^2 r}\left[\ddot{\mathbb{p}}(t') \times \mathbb{n}\right], \quad \mathbb{E}(\mathbb{r},t) = \mathbb{B}(\mathbb{r},t) \times \mathbb{n}.} \tag{16}$$

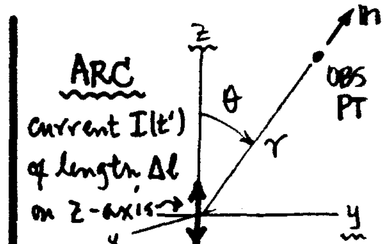
The "$\cdot$" denotes $\frac{\partial}{\partial t'}$. These are the leading-term expressions for the radiation flds which you will see in many texts [e.g. Landau & Lifshitz "Classical Theory of Fields" (Addison-Wesley, 1965), Sec. 67]. They no longer are tied to motion @ $e^{\pm i\omega t}$; the time dependence is now specified by behavior of $\ddot{\mathbb{p}}$.

In Eq. (16), $\mathbb{E}$ & $\mathbb{B}$ form a transverse wave ($\mathbb{B}$, $\mathbb{E}$, $\mathbb{n}$) and it follows...

POYNTING VECTOR } $\mathbb{S} = \frac{c}{4\pi}(\mathbb{E} \times \mathbb{B}) = \frac{c}{4\pi} B^2 \mathbb{n} = \frac{1}{4\pi c^3 r^2} |\mathbb{n} \times \ddot{\mathbb{p}}(t')|^2\, \mathbb{n}$;

POWER per SOLID ∡ } $\frac{dP}{d\Omega} = r^2\, \mathbb{n} \cdot \mathbb{S} = \frac{1}{4\pi c^3} |\mathbb{n} \times \ddot{\mathbb{p}}(t')|^2 = \frac{\sin^2\theta}{4\pi c^3} |\ddot{\mathbb{p}}(t')|^2$. (17)

θ is the ∡ between $\ddot{\mathbb{p}}$ & the obs$^{\underline{n}}$n direction $\mathbb{n}$.



8) Small application of Eq. (17). Consider an arc, which is a pulse of current $I(t')$ of length Δl along z-axis. The charge transport/unit time is I, so we have a changing dipole moment $\dot{\mathbb{p}} = I \Delta l$ in general. Then the magnitude of the Poynting vector is

$$\rightarrow S(r,t) = \left(\frac{\sin^2\theta}{4\pi r^2}\right) \frac{1}{c^3}\left[\dot{I}(t')\, \Delta l\right]^2. \tag{18}$$

This is used for arc radiation analysis in Eq. (43) of Robiscoe & Sui, J. Appl. Phys. **64** 4364 (Nov. 1988).