DEPARTMENT OF PHYSICS

2002 COMPREHENSIVE EXAM

Answer each of the following questions, all of which carry equal weight. Begin your answer to each question on a new sheet of paper; solutions to different questions must not appear on the same sheet. Each sheet of paper must be labeled with your name and the problem number in the upper right hand corner. When more than one sheet is submitted for a problem, be sure the pages are ordered properly.

PHYSICAL CONSTANTS

Quantity	Symbol	Value
acceleration due to gravity	g	9.8 m s^{-2}
gravitational constant	G	$6.67 \times 10^{-11} \text{ N m}^2 \text{ kg}^{-2}$
permittivity of vacuum	ε_o	$8.85 \times 10^{-12} \text{ C}^2 \text{ N}^{-1} \text{ m}^{-2}$
permeability of vacuum	μ_o	$4\pi \times 10^{-7} \text{ N A}^{-2}$
speed of light in vacuum	c	$3.00 \times 10^{8} \text{ m s}^{-1}$
elementary charge	e	$1.602 \times 10^{-19} \text{ C}$
mass of electron	m_e	$9.11 \times 10^{-31} \text{ kg}$
mass of proton	m_p	$1.673 \times 10^{-27} \text{ kg}$
Planck constant	h	$6.63 \times 10^{-34} \text{ J s}$
Avogadro constant	N_A	$6.02 \times 10^{23} \text{ mol}^{-1}$
Boltzmann constant	k	$1.38 \times 10^{-23} \text{ J K}^{-1}$
molar gas constant	R	$8.31 \text{ J mol}^{-1} \text{ K}^{-1}$
standard atmospheric pressure		$1.013 \times 10^{5} \text{ Pa}$

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As a model of an elementary particle, consider a charge q with a spherically symmetric charge density $\rho(r) \sim \exp(-r/a)$, where r is the radial distance from the center of the charge and $a \geq 0$ is a scale length.

(a) Normalize $\rho(r)$ so that $\int \rho \, dV = q$.

(b) Find the electric field $E(R)$ at radial distance R. Obtain asymptotic forms for $E(R)$ when $R << a$ and $R >> a$.

(c) Sketch $E(R)$ *vs.* R over the interval $(0, \infty)$. At what approximate R-value is $E(R)$ maximum?

e & m / Tsuruta

$\rho(r) \propto e^{-r/a}$ Key

$\rightarrow \rho(r) = A e^{-r/a}$ ①

(a). $\int^{\infty} \rho \, dV = q.$ ②

② $\rightarrow$ $q = \int^{\infty} A e^{-r/a} dV = A \int_0^{2\pi} d\varphi \int_0^{\pi} \sin\theta \, d\theta \int_0^{\infty} e^{-r/a} r^2 dr$

$= 4\pi A \, 2a^3$

$\rightarrow$ $\boxed{A = q/(8\pi a^3)}$ ans.

& $\boxed{\rho(r) = \frac{q}{8\pi a^3} e^{-r/a}}$ ans. ③

(b). (i) Use Gauss Law. $\int_S \bar{E} \cdot d\bar{a} = 4\pi Q$ ④

where

$Q = \int_0^{2\pi} d\varphi \int_0^{\pi} \sin\theta \, d\theta \int_0^R \rho(r) r^2 dr$

$= 4\pi \frac{q}{8\pi a^3} \int_0^R e^{-r/a} r^2 dr = \frac{q}{2a^3} \int_0^R e^{-r/a} r^2 dr$

$= q\left[1 - e^{-R/a} - \left(\frac{R}{a}\right) e^{-R/a} - \frac{1}{2}\left(\frac{R}{a}\right)^2 e^{-R/a} \right]$ ⑤

⑤ $\rightarrow$ ④ & get

$4\pi R^2 E(R) = 4\pi Q$

$\rightarrow$ $E(R) = \frac{q}{R^2}\left[1 - e^{-R/a} - \left(\frac{R}{a}\right) e^{-R/a} - \frac{1}{2}\left(\frac{R}{a}\right)^2 e^{-R/a} \right]$ ⑥ // ans.

(ii) $R \gg a$ ⑥ $\rightarrow$

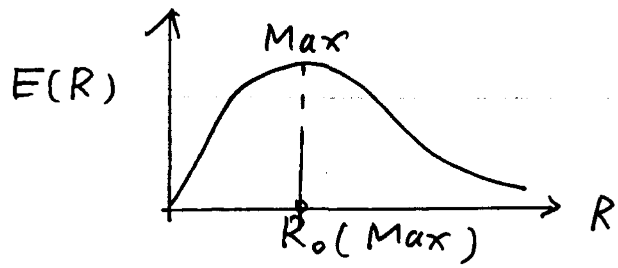
$E(R) \rightarrow \boxed{\frac{q}{R^2}}$ // ans. ⑦

(iii) $R \ll a$ ⑥ $\rightarrow$

$E(R) = \frac{q}{R^2}\left[1 - \left(1 - \frac{R}{a} + \frac{1}{2}\left(\frac{R}{a}\right)^2 - \frac{1}{3.2}\left(\frac{R}{a}\right)^3 + \cdots\right) - \frac{R}{a}\left(1 - \frac{R}{a} + \frac{1}{2}\left(\frac{R}{a}\right)^2 + \cdots \right) \right]$

$\approx \frac{1}{6} \frac{q}{R^2}\left(\frac{R}{a}\right)^3 = \boxed{\frac{1}{6} \frac{R}{a^3} q}$ ⑧ // ans

(c)



get $R_0 = R(\max)$ from

$$\left.\frac{dE(R)}{dR}\right|_{R=R_0} = 0 \qquad (9)$$

(6) → (9) & get.

$$0 = \left|\frac{q}{R^2}\left[\frac{1}{a}e^{-R/a} - \frac{1}{a}e^{-R/a} + \frac{R}{a^2}e^{-R/a} - \frac{R}{a^2}e^{-R/a} + \frac{1}{2}\frac{R^2}{a^3}e^{-\frac{R}{a}}\right]\right.$$
$$\left. - \frac{q}{R^3}\left[1 - e^{-R/a} - \frac{R}{a}e^{-R/a} - \frac{1}{2}\left(\frac{R}{a}\right)^2 e^{-R/a}\right]\right|_{R=R_0}, \qquad (10)$$

Let $R_0/a = x$. Then (10) →

$$x^3 - 4\left(e^x - 1 - x - \tfrac{1}{2}x^2\right) = 0$$

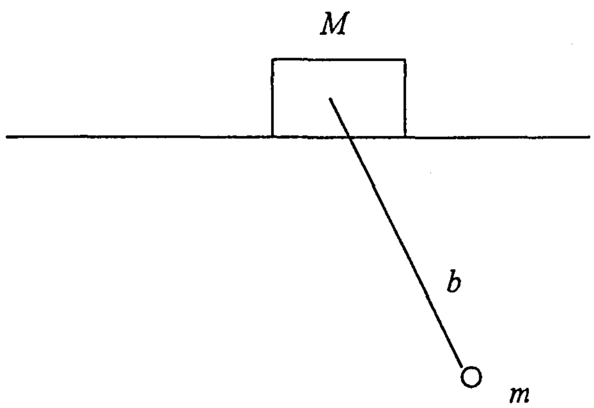
$$\to x^3 + 2x^2 + 4x + 4 = 4e^x$$

$$\approx 4\left(1 + x + x^2/2 + x^3/6 + x^4/24 + x^5/125 + \cdots\right)$$

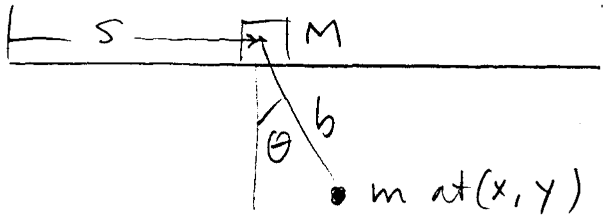
So $x \approx \frac{2}{3}$ &

$$\boxed{R_0 = \frac{2}{3}a}$$ Ans.

A mass M moves horizontally along a smooth, straight rail. A pendulum hung from M consists of a massless rigid rod of length b and a point mass m at its end. The pendulum swings without friction in a vertical plane that contains the rail. Find the eigenfrequencies of the system and describe the normal modes for small oscillations.



A mass M moves horizontally along a smooth, straight track. A pendulum is hung from M with a weightless rod of length b and point mass m at its end. The pendulum swings without friction in a vertical plane that contains the rail. Find the eigenfrequencies of the system and describe the normal modes for small oscillations.



s and θ are good generalized coords

For small oscillations,

$$x = s + b\sin\theta \simeq s + b\theta \; ; \quad \dot{x} = \dot{s} + b\dot{\theta}$$

$$y = b(1-\cos\theta) \simeq \tfrac{1}{2} b\theta^2 \; ; \quad \dot{y} = b\theta\dot{\theta} \simeq 0$$

$$T = \tfrac{1}{2} M\dot{s}^2 + \tfrac{1}{2} m(\dot{x}^2 + \dot{y}^2) \simeq \tfrac{1}{2} M\dot{s}^2 + \tfrac{1}{2} m(\dot{s} + b\dot{\theta})^2$$

$$V = mgy = \tfrac{1}{2} mgb\theta^2$$

$$L = \tfrac{1}{2} M\dot{s}^2 + \tfrac{1}{2} m(\dot{s}^2 + b^2\dot{\theta}^2 + 2b\dot{s}\dot{\theta}) - \tfrac{1}{2} mgb\theta^2$$

EOMs

$$(s) \quad (M+m)\ddot{s} + mb\ddot{\theta} = 0$$

$$(\theta) \quad mb^2\ddot{\theta} + mb\ddot{s} + mgb\theta = 0$$

$$\begin{pmatrix} -\omega^2(M+m) & -\omega^2 mb \\ -\omega^2 mb & -\omega^2 mb^2 + mgb \end{pmatrix} \begin{pmatrix} s \\ \theta \end{pmatrix} = 0$$

<u>Eigenfrequencies</u> from secular equation:

$$\omega = 0, \; \sqrt{\frac{M+m}{M} \cdot \frac{g}{b}}$$

<u>Normal modes</u> from algebraic equations above

$\underline{\omega = 0 \Rightarrow \theta = 0 \text{ uniform translation}}$

$$\underline{\omega = \sqrt{\frac{M+m}{M} \cdot \frac{g}{b}}} \Rightarrow \frac{M+m}{M} \cdot \frac{g}{b} \cdot s + \frac{m}{M}\theta = 0$$

or $m(s + b\theta) + Ms = 0$

or $mx + Ms = 0$

<u>pendulum swings about the CM</u>

3

Answer all of the following.

(a) When the H_2 molecule is in its ground state, the two electrons are in a spin singlet state -- that is, a state with total electron spin 0. Discuss the physical origin of this fact, and in particular how it relates to the molecular bonding.

(b) In its ground state, the D_2 molecule carries no orbital angular momentum (i.e., the wavefunction for the two deuterons has l=0). What value(s) for the total spin of the two-nucleus system is/are permitted for this state? Explain your answer.

(c) Carbon has two 2p electrons in an unfilled shell. What combinations of total s (total spin quantum number) and total l (total orbital angular momentum quantum number) are allowed for these two electrons? How many linearly independent angular momentum states do these represent in all? What values of j (total angular momentum quantum number) are allowed for carbon?

#3

Problem 3 – Solution

(a) The spin singlet state is exchange-antisymmetric, so for the total 2-electron state to be exchange-antisymmetric, their spatial state must be symmetric. This gives an enhanced probability that the two electrons will lie close together, between the protons. This screens the proton-proton Coulomb interaction, and binds the atoms together – a covalent bond.

(b) The $l = 0$ orbital angular momentum state for the two deuterons is an exchange-symmetric spatial state. Since the deuterons are bosons, their total spin state must therefore be exchange-symmetric as well. They are spin-1 particles, and so their total spin could in principle be 2, 1 or 0. But the total spin state $S_{tot} = 1$ is exchange-antisymmetric and so is not allowed. The states $S_{tot} = 2$ and $S_{tot} = 0$ are the allowed symmetric ones.

(c) Each $l = 1$ electron could in principle have one of three orbital angular momentum azimuthal quantum numbers ($m_l = 1, 0$ or -1) and one of two spin azimuthal quantum numbers ($m_s = ½$ or $-½$), for a total of six possibilities of (m_l, m_s). But the two electrons are indistinguishable and can't have the same set of quantum numbers, so the number of possibilities for the two is not $6^2 = 36$, but rather $(6\cdot 5/2) = 15$. So there are 15 linearly independent angular momentum states for the 2-electron system. In terms of total l and total s: Total l could be 2, 1 or 0 for the two p-electrons, and total s could be 1 or 0. But $s = 1$ is exchange-symmetric, so the orbital angular momentum state would have to be antisymmetric, forcing $l = 1$. Similarly, $s = 0$ is antisymmetric, forcing $l = 2$ or 0.

Answer all of the following.

(a) A spherically symmetric distribution of gas contains mass $M(r)$ inside radius r. For a body of very small mass orbiting the center **through the gas**, in a circular orbit, find the orbital velocity v_φ as a function of r and $M(r)$.

(b) Stars in spiral galaxies (including our own) orbit the galactic center with a "flat rotation profile" meaning $v_\varphi = v_0 = \text{constant}$. What is the mass density $\rho(r)$ implied by this? Express your answer in terms of v_0 and universal constants.

(c) Assume a galaxy has the same density distribution $\rho(r)$ found in part (b) **plus** a central point mass m_0 (perhaps a black hole). For which radii will the orbital velocity differ by more that 10% from v_0?

#4

(4)

Solution

(a) The acceleration of the circular orbit must match the gravitational acceleration due to the *enclosed mass*

$$\frac{v_\phi^2}{r} = \frac{GM(r)}{r^2} \quad . \tag{1}$$

Solving for v_ϕ gives the expression

$$v_\phi = \sqrt{\frac{GM(r)}{r}} \quad . \tag{2}$$

(b) If the azimuthal velocity $v_\phi = v_0$ in eq. (1) we find that the enclosed mass is

$$M(r) = \frac{v_0^2 r}{G} \quad .$$

This must equal the integral of the mass density $\rho(r)$ within the spherical shell

$$M(r) = 4\pi \int_0^r \rho(s)\, s^2\, ds \quad .$$

Differentiating this expression we find

$$\rho(r) = \frac{M'(r)}{4\pi r^2} = \frac{v_0^2}{4\pi G r^2} \quad . \tag{3}$$

(This cannot apply all the way to the galactic center since the mass density cannot really diverge.)

(c) With the addition of the point mass we find

$$M(r) = \frac{v_0^2 r}{G} + m_0 \quad .$$

Replacing this in expression (1) gives

$$v_\phi^2 = \frac{GM(r)}{r} = v_0^2 + \frac{Gm_0}{r} \quad . \tag{4}$$

Clearly $v_\phi > v_0$ so the dividing line will be where $v_\phi = 1.1\, v_0$ or $v_\phi^2 = 1.21\, v_0^2$. Using this in (4) and solving for r gives the radius

$$r_* = \frac{Gm_0}{0.21 v_0^2} \quad . \tag{5}$$

At all radii $r < r_*$ the orbital velocity will differ from v_0 by **more** than 10%.

5

Helicity may be defined for a spin-1/2 particle as the operator

$$A = \vec{S} \cdot \vec{P}$$

where $\vec{S}$ and $\vec{P}$ are the spin matrix and momentum operator, respectively.

(a) Find the eigenvalues of A.

(b) Find the eigenvectors of A.

(c) Interpret the eigenvectors as explicitly as you can.

Hint: $\vec{S}$ and $\vec{P}$ commute.

(5)

Given $A = \vec{S} \cdot \vec{P}$; $S_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $S_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$
$S_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

(a) Eigenvalue problem

Let $\psi = \chi \varphi_{\vec{p}}$; $\varphi_{\vec{p}} \sim e^{i\vec{p}\cdot\vec{x}/\hbar}$

Then $A\psi = \lambda \psi$

becomes $\vec{S}\cdot\vec{p}\, \chi\cancel{\psi} = \lambda \chi \cancel{\psi}$; cancel the ψ

Now $\vec{S}\cdot\vec{p} = \frac{\hbar}{2}\begin{pmatrix} p_z & p_x - ip_y \\ p_x + ip_y & -p_z \end{pmatrix}$ = hermitian

The eigenvalues are found from the secular eq'n

$$\begin{vmatrix} (\frac{\hbar}{2}p_z - \lambda) & \frac{\hbar}{2}(p_x - ip_y) \\ \frac{\hbar}{2}(p_x + ip_y) & (-\frac{\hbar}{2}p_z - \lambda) \end{vmatrix} = \lambda^2 - \frac{\hbar^2}{4}\left(p_x^2 + p_y^2 + p_z^2\right) = 0$$

$$\Rightarrow \lambda = \pm \frac{\hbar}{2}\sqrt{p_x^2 + p_y^2 + p_z^2}$$

$$\boxed{\lambda = \pm \frac{\hbar}{2} p}$$

(b) The eigenvectors $\chi \varphi_{\vec{p}}$ follow from

$$\vec{S}\cdot\vec{p}\, \chi_\pm = \pm \frac{\hbar}{2} p\, \chi_\pm$$

where $\vec{S}\cdot\vec{p}$ is the 2×2 matrix given above

I find $\boxed{\chi_\pm = \frac{1}{\sqrt{2p(p+p_z)}} \begin{pmatrix} \pm p + p_z \\ p_x + i p_y \end{pmatrix}}$

(c) To interpret χ_+ (for example), rewrite

$$\chi_+ = \begin{pmatrix} \sqrt{\frac{p+p_z}{2p}} \\ \frac{p_x + i p_y}{\sqrt{2p(p+p_z)}} \end{pmatrix};$$

Below, let (θ, ϕ) be the direction of $\vec{p}$

But $\sqrt{\frac{p+p_z}{2p}} = \sqrt{\frac{p(1+\cos\theta)}{2p}} = \sqrt{\frac{1+\cos\theta}{2}} = \cos\frac{\theta}{2}$

$$\frac{p_x + i p_y}{\sqrt{2p(p+p_z)}} = \frac{\sqrt{p_x^2+p_y^2}\,(\cos\phi + i\sin\phi)}{\sqrt{2p(p+p_z)}}$$

$$= \sqrt{\frac{p^2 - p_z^2}{2p(p+p_z)}}\, e^{i\phi}$$

$$= \sqrt{\frac{p-p_z}{2p}}\, e^{i\phi} = \sqrt{\tfrac{1}{2}(1-\cos\theta)}\, e^{i\phi}$$

$$= \sin\frac{\theta}{2} \cdot e^{i\phi}$$

$$\chi_+ = \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \cdot e^{i\phi} \end{pmatrix} = \overbrace{e^{i\frac{\phi}{2}}}^{\text{ignore}} \begin{pmatrix} \cos\frac{\theta}{2}\, e^{-i\frac{\phi}{2}} \\ \sin\frac{\theta}{2}\, e^{i\frac{\phi}{2}} \end{pmatrix}$$

Thus, χ_+ describes a spin aligned along $\vec{p}$

Similarly χ_- describes a spin antialigned wrt $\vec{p}$

A rotating iron magnet that is a good conductor can be used to produce an EMF. Consider a uniformly magnetized perfectly conducting sphere of radius R that rotates about its magnetization axis (chosen as the z-axis) with angular velocity ω, where $\omega R << c$. Given that the sphere produces a magnetic field

$$\vec{B}(\vec{x}) = \frac{2\mu_0}{3}\vec{M} \quad \text{for } |\vec{x}| \leq R$$

and

$$\vec{B}(\vec{x}) = \frac{\mu_0 R^3}{3}\left(\frac{3\hat{x}(\hat{x}.\vec{M}) - \vec{M}}{|\vec{x}|^3}\right) \quad \text{for } |\vec{x}| \geq R,$$

where $\vec{M} = M\hat{z}$ is the magnetization,

(a) Find the electric field generated inside the sphere.

(b) Find the EMF between the north pole and the equator.

6 #6

(a) Since the charges are free to move, they will have to establish an equilibrium configuration where the Lorentz force is zero

$\Rightarrow \quad \underset{\sim}{F} = q(\underset{\sim}{E} + \underset{\sim}{v} \times \underset{\sim}{B}) = 0 \quad \Rightarrow \quad \underset{\sim}{E} = -\underset{\sim}{v} \times \underset{\sim}{B}$

Using cylindrical coordinates we have $\underset{\sim}{v} = \omega \rho \hat{\phi}$, $\underset{\sim}{B} = \frac{2\mu_0}{3} M \hat{z}$

Convert to cartesian coordinates $\Rightarrow \rho = \sqrt{x^2+y^2}$, $\hat{\phi} = -\sin\phi\,\hat{\imath} + \cos\phi\,\hat{\jmath}$

$$\Rightarrow \quad \underset{\sim}{E} = -\underset{\sim}{v} \times \underset{\sim}{B} = -\omega\rho(\sin\phi\,\hat{\jmath} + \cos\phi\,\hat{\imath})\frac{2\mu_0}{3} M$$

$$\Rightarrow \quad \boxed{\underset{\sim}{E} = -\frac{2\mu_0 M}{3}\omega\rho\,\hat{\rho}}$$

(b) Method I

$$\mathcal{E} = \int_{pole}^{equator} \underset{\sim}{E} \cdot d\underset{\sim}{\ell}$$

$$d\underset{\sim}{\ell} = R\,d\theta\,\hat{\theta} = R\,d\theta(\cos\theta\cos\phi\,\hat{\imath} + \cos\theta\sin\phi\,\hat{\jmath} - \sin\theta\,\hat{k})$$

$$\rho = R\sin\theta$$

$$\Rightarrow \quad \mathcal{E} = \int_{\pi/2}^{0} -\omega R^2 \frac{2\mu_0 M}{3} \sin\theta\cos\theta\,d\theta = \frac{1}{3}\mu_0 M\omega R^2$$

Method II

$$\underset{\sim}{E} = -\underset{\sim}{\nabla}\Phi \quad \Rightarrow \quad \Phi = \tfrac{1}{3}\mu_0 M\omega\rho^2$$

$$\Rightarrow \quad \mathcal{E} = \Phi(\text{equator}) - \Phi(\text{pole})$$

$$= \tfrac{1}{3}\mu_0 M\omega R^2 - 0$$

$$\Rightarrow \quad \boxed{\mathcal{E} = \tfrac{1}{3}\mu_0 M\omega R^2}$$

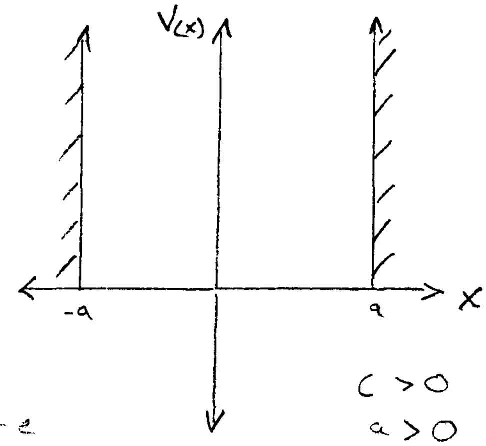
A particle of mass m is confined to a box in one dimension, $-a < x < a$, and the box has walls of infinite potential. An attractive delta function potential is at the center of the box at $x = 0$ given by $V(x) = -C\,a\,\delta(x)$, where $a > 0$ and $C > 0$.

(a) Find an implicit equation for the ground state energy of this system.

(b) Find the value of C such that the ground state energy is zero. Find and sketch the corresponding wavefunction.

(c) If the system is in the ground state and the strength of the potential is changed suddenly from C to C', what is the probability that the particle will be found in the ground state of the new potential?

Quantum Mechanics
Delta Function in a Box.



a) The potential (shown →) is given by

$$V(x) = \infty \qquad x \leq -a \quad x \geq a$$

$$= -Ca\,\delta(x) \qquad -a < x < a$$

$C > 0$
$a > 0$

ψ^{odd} The odd parity states are unaffected by the delta function $-Ca\delta(x)$

$$\rightarrow \psi_n^{odd}(x) = \frac{1}{\sqrt{a}} \sin\left(\frac{n\pi x}{a}\right)$$

$$E_n^{odd} = \frac{(\hbar\pi n)^2}{2ma^2} \quad (>0)$$

ψ^{even} For $\psi_n^{even}(x)$ breakdown potential.

For a delta function without a box the eigenstates are $\psi = Ae^{-\alpha|x|}$ (exponentials). With the box the eigenstates must vanish at $x = \pm a$. The states which do this are

$$\psi_n(x) = A_n e^{-\alpha_n|x|} + B_n e^{+\alpha_n|x|}$$

B.C. $\psi_n(x{=}a) = \psi_n(x{=}-a) = 0 = A_n e^{-\alpha_n a} + B e^{+\alpha_n a}$

$$\rightarrow B_n = -A_n e^{-2\alpha_n a}$$

$$\therefore \psi_n(x) = A_n\left(e^{-\alpha_n|x|} - e^{+\alpha_n a - 2\alpha_n a}\right)$$

$$= -2A_n e^{-\alpha a}\left(\underline{e^{+\alpha_n(a-|x|)} - e^{-\alpha_n(a-|x|)}}\right)$$

$$= A_n' \sinh \alpha_n(a - |x|)$$

The energy is simply given by

$$E_n = -\frac{\hbar^2}{2m}\alpha_n^2$$

To determine α_n we recall for a delta Fn.

$$-Ca\,\psi_n(0) = -\frac{\hbar^2}{2m}\left[\frac{\partial\psi_n}{\partial x}\Big|_{0^+} - \frac{\partial\psi_n}{\partial x}\Big|_{0^-}\right]$$

and note that

$$\frac{d\psi_n}{\partial x}\Big|_{0^\pm} = \mp\alpha_n A\cosh(\alpha_n a)$$

$$\rightarrow -CaA\sinh(\alpha_n a) = -\frac{\hbar^2}{2m}\left(2\alpha_n A\cosh(\alpha_n a)\right)$$

or $$\boxed{\frac{C}{\hbar^2/ma^2} = \alpha_n a\coth(\alpha_n a)}$$

Which we can solve graphically for all α_n.

b) Lowest eigenvalue $E_n \Rightarrow 0$ implies $\alpha_n \rightarrow 0$.

$$\lim_{\alpha_n\Rightarrow 0}\left(\alpha_n a\coth(\alpha_n a)\right) = 1 \quad\rightarrow\quad \boxed{C = \frac{\hbar^2}{ma^2}}$$

c) If the system is in the first odd parity state changing the strength of the delta function does not change the occupancy of the state

$$\psi^{odd}(x=0) = 0 \quad \text{and} \quad \langle\psi^{odd}|\psi'^{\,odd}\rangle = 1 \quad \text{or}$$

$$\rightarrow \boxed{P = 1}$$

In the rest frame of the Sun a monochromatic radio wave propagates in the $\vec{k}$ direction with amplitude (as measured at some fixed location) $A(t) = A_0 \cos \Phi(t)$ and frequency $f = (1/2\pi)\,(d\Phi(t)/dt)$.

(a) Assuming that the Earth orbits the Sun on a circular path with radius 1 AU, find the frequency and amplitude of the wave as measured on Earth using Earth-bound clocks. Neglect gravitational redshifts and the Earth's rotation on its own axis. Your answer should depend on the source's sky location (θ,φ).

(b) Estimate the angular resolution that could be achieved with one day of observations (i.e., how well could we measure θ and φ?) with a telescope that can measure phase shifts of one radian for radio waves with frequencies of 1 GHz. You can make a rough estimate even if you couldn't answer part (a)

(a) Earth's location $\vec{x} \rightarrow (ct,\ R\cos\omega t,\ R\sin\omega t,\ 0)$ where

$R = 1\ \text{AU}$ and $\omega = 2\pi/\text{year}$.

Earth's 4-velocity $\vec{u}_E = \dfrac{d\vec{x}}{d\tau} = \gamma\,(c,\ -R\omega\sin\omega t,\ R\omega\cos\omega t,\ 0)$

$\gamma = \dfrac{1}{\sqrt{1-v^2/c^2}}$, $v = \omega R$ = constant, $\dfrac{dt}{d\tau} = \gamma \Rightarrow t = \gamma\tau$

Here t is time measured in Sun's rest frame and τ is time measured on Earth.

4-momentum of wave is $\vec{k} = -k\,(\sin\theta\cos\phi,\ \sin\theta\sin\phi,\ \cos\theta)$

Using the invariant $E_{Earth} = hf_E = -\vec{k}\cdot\vec{u}_E$ and $k = hf$

$$\Rightarrow \quad f_E = \gamma f\left(1 - \frac{v}{c}\sin\theta\,(\cos\omega t\cos\phi + \sin\omega t\sin\phi)\right)$$

$$= \gamma f\left(1 - \frac{v}{c}\sin\theta\cos(\omega t - \phi)\right)$$

Thus,

$$\Phi_E(\tau) = \int^{\tau} 2\pi f_E\, d\tau$$

$$= 2\pi f\gamma\tau - 2\pi f\frac{R}{c}\sin\theta\sin(\omega\gamma\tau - \phi)$$

$$\therefore \quad \boxed{A_E(\tau) = A_0\cos\left(2\pi f\gamma\tau - 2\pi f\, R/c\,\sin\theta\sin(\omega\gamma\tau - \phi)\right)}$$

(b)

$$\Delta\Phi_E(t) = \Phi_E(\theta+\Delta\theta,\ \phi+\Delta\phi,\ t) - \Phi_E(t)$$

$$\simeq \frac{\partial\Phi_E}{\partial\theta}\Delta\theta + \frac{\partial\Phi_E}{\partial\phi}\Delta\phi \qquad R/c \simeq 500 \text{ seconds}$$

For $\Delta t = 1$ day (and similarly for $\Delta\phi$)

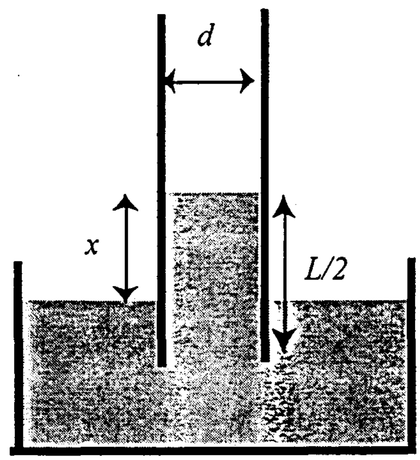
$$\frac{\partial\Phi_E}{\partial\theta} \simeq -2\pi f\frac{R}{c}\cos\theta\cos(\omega t - \phi)\,\omega\Delta t, \qquad \omega\Delta t = \frac{2\pi}{365}$$

Resolution depends on θ, ϕ. Is best when $\theta \simeq 0$, $\omega t \simeq \phi$

Setting $\Delta\Phi = 1$ $\Rightarrow$ $\boxed{\Delta\theta_{max} \simeq \dfrac{365}{4\pi^2 f\, R/c} \simeq 1.8\times10^{-11}}$ $\simeq 3.8$ micro arc seconds

9

A parallel plate capacitor with square plates of side L and a separation d between the plates is charged to a potential V and disconnected from the battery. It is then vertically inserted into a large reservoir of dielectric liquid with a relative dielectric constant ε and a density ρ until the liquid fills half the volume between the capacitor plates as shown at right.



(a) What is the capacitance of the system?

(b) What is the electric field strength between the capacitor plates?

(c) What is the distribution of charge density over the plates?

(d) What is the difference in vertical height x between the level of liquid within the capacitor plates and that in the external reservoir?

Electricity & Magnetism
Parallel Plate Capacitor in a Dielectric Fluid Bath

a) The total capacitance is just the sum of two capacitors in parallel (SI units)

$$C_T = C_1 + C_2 = \frac{\varepsilon_0 A_1}{d} + \frac{\varepsilon A_2}{d} = \frac{\varepsilon_0 L\times(L/2)}{d} + \frac{\varepsilon\, L\times(L/2)}{d}$$

$$= \frac{L^2}{2d}(\varepsilon+\varepsilon_0) = \underline{\underline{C_0\left(\frac{\varepsilon+\varepsilon_0}{2\varepsilon_0}\right)}} \qquad \left(C_0 = \frac{\varepsilon_0 L^2}{d}\right)$$

b) The capacitor was charged to a potential V_0 then disconnected and immersed. The charge remains constant, but potential changes

$$Q_0 = C_0 V \qquad \text{before}$$

$$Q_0 = C_T V' \qquad \text{after}$$

$$\rightarrow \quad V' = \frac{Q_0}{C_T} = \frac{C_0 V}{C_T} = \left(\frac{2\varepsilon_0}{\varepsilon+\varepsilon_0}\right) V$$

and the electric field strength is

$$|E| = \frac{|V|}{d} = \underline{\underline{\left(\frac{2\varepsilon_0}{\varepsilon+\varepsilon_0}\right)\frac{V}{d}}}$$

c) To get the charge distribution use maxwell's eq.

$$\vec{\nabla}\cdot\vec{D} = \rho \qquad \text{and} \qquad \vec{D} = \varepsilon\vec{E}$$

c) (cont.) Maxwell's eq. can be rewritten as

$$\vec{D}\cdot\hat{n} = \sigma \quad \rightarrow \quad \vec{D} = \sigma\hat{n} \quad \text{or} \quad |\vec{D}| = \sigma$$

where $\hat{n}$ is a unit vector normal to the capacitor plates.

$$\boxed{\begin{aligned} \rightarrow \sigma_{\substack{in\\bath}} &= \varepsilon|\vec{E}| = \frac{2\varepsilon\varepsilon_0}{\varepsilon+\varepsilon_0}\frac{V}{d} \\ \sigma_{\substack{in\\air}} &= \varepsilon_0|\vec{E}| = \frac{2\varepsilon_0^2}{\varepsilon+\varepsilon_0}\frac{V}{d} \end{aligned}}$$

d) The energy of the capacitor, U_c, is lowered by adding the dielectric ($V' < V$ for $\varepsilon > \varepsilon_0$), but the gravitational energy, U_g, is increased.

Let the fluid rise to height h, then

$$U_c = \frac{1}{2}\frac{Q_0^2}{C_T} \quad \text{but} \quad C_T = \frac{\varepsilon_0 L\times(L-h)}{d} + \frac{\varepsilon L h}{d}$$

$$\rightarrow U_c = \frac{1}{2}\frac{d\,Q_0^2}{L[\varepsilon_0 L + (\varepsilon-\varepsilon_0)h]}$$

The weight of fluid balances $F_z = -\left.\frac{\partial U_c}{\partial h}\right|_{h=L/2}$ (electrostatic force)

$$\rightarrow \rho g L d x = + \frac{2d\,Q_0^2(\varepsilon-\varepsilon_0)}{L^3\,(\varepsilon+\varepsilon_0)^2}$$

$$\boxed{x = \frac{2\varepsilon_0^2 V^2(\varepsilon-\varepsilon_0)}{\rho g d^2(\varepsilon+\varepsilon_0)^2}}$$

using $Q_0^2 = C_0^2 V^2 = \frac{\varepsilon_0^2 L^4}{d^2}V^2$

Permafrost is a layer of earth that remains frozen year round. At depth z, measured downward from the ground, the temperature $T(z,t)$ satisfies the heat equation

$$\partial T / \partial t = D\nabla^2 T \qquad (1)$$

where D is the thermal diffusion coefficient. Temperature at ground level (z=0) undergoes the periodic (i.e., seasonal) variation

$$T(0,t) = \overline{T} - \Delta T \cos(\omega t)$$

where $\overline{T}$ is the average annual temperature, ΔT is the difference between summer high and winter low and $\omega = 2\pi / \text{year} = 2 \times 10^{-7}$ rad/sec is the annual frequency. At great depth the ground should have no effect so

$$\partial T / \partial z \to 0 \quad \text{as} \quad z \to \infty .$$

(a) Solve equation (1) subject to the boundary conditions provided to yield the full temperature profile $T(z,t)$.

(b) Assuming $\overline{T} < 0$ in degrees Celsius find an expression for the depth of the permafrost layer: namely the depth below which $T(z,t) < 0$ for all time.

10

Solution

a. The upper boundary condition consists of two pieces: a constant term, $\bar{T}$, and an oscillatory piece with frequency ω. The solution must therefore consist of two pieces. It can be easily seen that $T(z,t) = \bar{T} =$ constant satisfies the diffusion equation, so all the remains is to determine the oscillatory piece of the solution. Assuming a solution of the form

$$T(z,t) = \bar{T} + \mathrm{Re}\left[f(z)\, e^{-i\omega t}\right] ,$$

and replacing it into (1) gives

$$-i\omega f(z) = Df''(z) .$$

The general solution to this is $f(z) = Ae^{-kz}$ where

$$k = \pm\sqrt{-i\omega/D} = \pm\sqrt{\omega/2D}\,(1-i)$$

Only the upper sign is acceptable as $z \to \infty$. Matching the boundary condition at $z = 0$ yields a complete solution

$$T(z,t) = \bar{T} - \Delta T e^{-\sqrt{\omega/2D}z} \cos(\omega t - \sqrt{\omega/2D}z) \tag{2}$$

b. The maximum of expression (2)

$$\max T = \bar{T} + \Delta T e^{-\sqrt{\omega/2D}z}$$

which is assumed when $\omega t - \sqrt{\omega/2D}z = \pi$. Setting the maximum to freezing, $T = 0$, gives the expression for permafrost depth

$$z_{\mathrm{pf}} = \sqrt{2D/\omega}\ln(-\Delta T/\bar{T}) \tag{3}$$

which is a real number only when $\bar{T} < 0$, and is positive only when $\Delta T > |\bar{T}|$. Thus there can be no permafrost where the mean temperature is above freezing, and when the maximum temperature is below freezing the permafrost level is $z_{\mathrm{pf}} = 0$: the ground (i.e. everything is frozen all year).

* Note that temperature reaches freezing at the permafrost level at

$$\omega t = \pi + \ln(-\Delta T/\bar{T}) = \pi + z_{\mathrm{pf}}\sqrt{\omega/2D} .$$

The seasonal variation is defined so that $t = 0$ is the coldest day in winter and $t = \pi/\omega$ is the warmest day in summer. Since $-\Delta T/\bar{T} > 1$ the thaw always occurs *after* the warmest day, however, it is possible for the delay to be more than one year for very deep permafrost.

Do one of the following problems.

The first papers about the MASER were published in 1954 independently by Charles Townes and by Dr. Basov and Dr. Prochorov at the Lebedev Institute in Moscow, resulting in the Nobel Prize in 1964 for all three. Describe how the ammonia MASER works.

a) Beginning with ammonia, NH_3, as a pyramid shaped molecule with a base of three hydrogen atoms and nitrogen at the apex, draw a rough sketch of the potential $V(z)$ experienced by the nitrogen atom if $z = 0$ is the plane passing through the 3 hydrogen atoms.

b) Draw a sketch of the ground state wavefunction and the first excited state wavefunction for this potential.

c) What is the polarization state of the radiation emitted from the ammonia maser?

OR

Hall Effect: Consider a bar at uniform temperature in a static uniform electric field E_x in the x-direction and a static uniform magnetic field B_z in the z-direction. Charge carriers moving through the material will create a voltage across the width of the bar, termed the Hall voltage.

a) Using the Drude theory of metals, according to which electrons undergo free acceleration punctuated by randomizing collisions with the lattice, calculate the magnetoresistance $r(H) = E_x/j_x$ and the Hall coefficient $R_H = E_y/(j_x H)$, in the steady state.

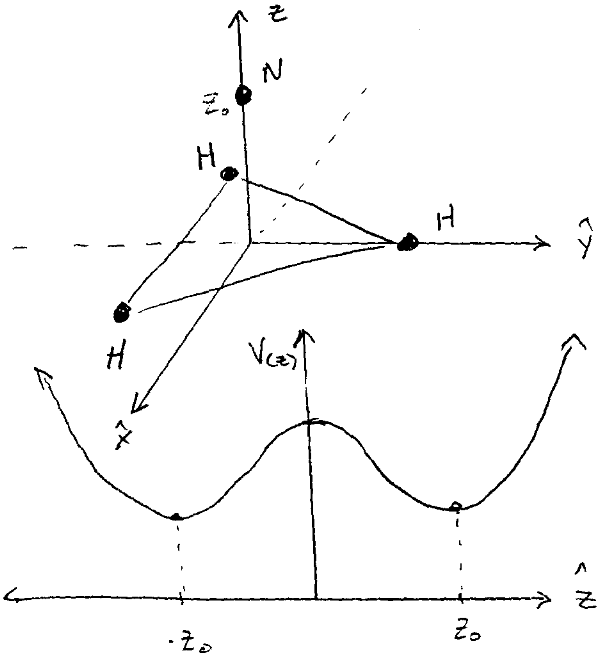
b) What is the sign of the Hall coefficient if the charge carriers are positively charged? Negatively charged?

#11a CMP ①

Condensed Matter Physics

MASER: Draw the potential $V(z)$ for the N atom

a)

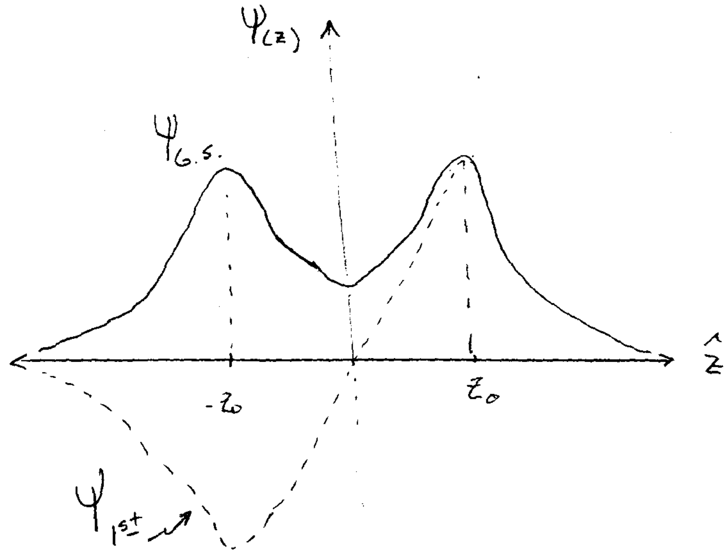


The hydrogen atoms form a triangular base, with N atom at the apex at the position $z = z_0$.

This potential has 2 minimums at $z = z_0$ and $z = -z_0$.

[Looks like doubly degenerate G.S.]

b) The ground state has the lowest energy.



$\psi_{G.S.}$ has equal probability to be at $z = z_0$ or $-z_0$ and is symmetric.

$\psi_{1^{st}}$ excited state also has equal probability to be at $z = z_0$ and $z = -z_0$ but is antisymmetric. This state has more curvature (higher energy) than the ground state.

c) The G.S. is symmetric $\rightarrow l =$ even and in fact $l = 0$.

The 1^{st} excited state therefore has $l = 1$

$\therefore \Delta l = +1$ (photon has $l = 1 \rightarrow$ cons. of l)

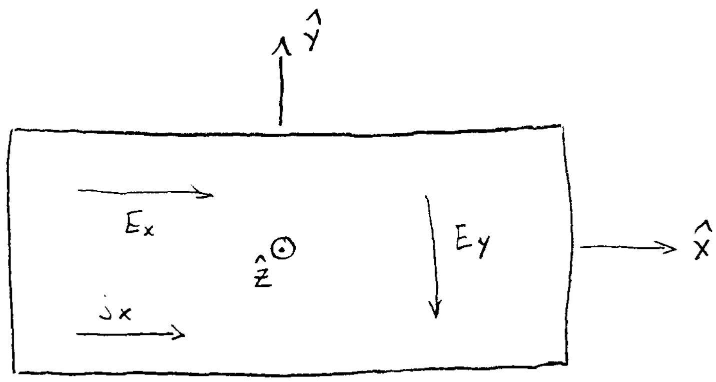
Both G.S and 1^{st} exc. state have same azimuthal symmetry

$\rightarrow \Delta m = 0$ $\boxed{\therefore \text{Linear polarized photon}}$

Condensed Matter Physics

Hall Effect:

a)



The Lorentz force

$$\vec{F} = q\left(\vec{E} + \frac{\vec{v}}{c} \times \vec{B}\right)$$

where $\vec{E} = E_x \hat{x}$

$\vec{B} = B_z \hat{z}$

Define $\rho(H) = \dfrac{E_x}{j_x}$

and $R_H = \dfrac{E_y}{j_x H}$

The Drude Model states that an electron will experience a randomizing collision on average in a time τ.

The momentum per electron, $\vec{p}(t)$, changes due to the action of the Lorentz force over an infinitesimal time, dt

$$\vec{p}(t+dt) = \left(\vec{p}(t) + \vec{f}(t)\,dt\right) \times \left(1 - \frac{dt}{\tau}\right)$$

where $\left(1 - \frac{dt}{\tau}\right)$ represents the fraction of electrons which do <u>not</u> suffer a collision,

$$\rightarrow \vec{p}(t+dt) - \vec{p}(t) = -\left(\frac{dt}{\tau}\right)\vec{p}(t) + \vec{f}(t)\,dt + O(dt)^2$$

$$\frac{d\vec{p}(t)}{dt} = -\frac{\vec{p}(t)}{\tau} + \vec{f}(t)$$

Which is just a force and a frictional damping term from the collisions.

Substituting in the lorentz Force

$$\frac{d\vec{p}}{dt} = q\left(\vec{E} + \frac{\vec{p}}{mc} \times \vec{H}\right) - \frac{\vec{p}}{\tau}$$

At steady state

$$\frac{dp_x}{dt} = 0 = qE_x + \frac{qHp_y}{mc} - \frac{p_x}{\tau}$$

and $$\frac{dp_y}{dt} = 0 = qE_y - \frac{qHp_x}{mc} - \frac{p_y}{\tau}$$

If we multiply both equations by $\frac{nq\tau}{m}$

and let $\sigma_0 = \frac{nq^2\tau}{m}$; $\vec{j} = nq\vec{v}$; $\omega_c = \frac{qH}{mc}$

$$\sigma_0 E_x = -\omega_c \tau j_y + j_x$$
$$\sigma_0 E_y = +\omega_c \tau j_x + j_y$$

At steady state $j_y = 0$

$\rightarrow \sigma_0 E_x = j_x$ or $\boxed{\rho(H) = \frac{E_x}{j_x} = \sigma_0}$ independent of H!

And

$$\sigma_0 E_y = \omega_c \tau j_x$$

$\rightarrow \boxed{R_H = \frac{E_y}{j_x H} = \frac{1}{nqc}}$ independent of τ!

b) If the charge carriers are positive

q is positive and $\underline{R_H > 0}$

If q is negative, $\underline{R_H < 0}$.

12

We wish to solve the second order linear differential equation $L\left[u(x)\right]=\ln(x)$ with

$$L=\frac{d}{dx}\left[x\frac{d}{dx}\right]-\frac{1}{x}$$

on the interval [0,1], with boundary conditions $u(0) = 0$ and $u(1) = 1$.

(a) Find the Green function $G(x,x')$ for L, for homogeneous boundary conditions (that is, satisfying $G(0,x') = G(1,x') = 0$).

(b) Find the solution $u(x)$ using the Green function of part (a).

#12

12

- Solution

(a) We first need to find the linearly independent solutions of the homogeneous equation $L[u] = 0$ satisfying the homogeneous boundary conditions $u(0) = u(1) = 0$. We can write this equation as $xu'' + u' - u/x = 0$, and notice that a power x^α will be a solution, provided α satisfies the indicial equation $\alpha(\alpha-1) + \alpha - 1 = 0$. The solutions to this equation are just $\alpha = \pm 1$, so we can take $u_1 = x$, $u_2 = 1/x$. Now we form G in two pieces, $G_<(x,x')$ for $x < x'$ and $G_>(x,x')$ for $x > x'$:

$$G_<(x,x') = c_1 x + c_2/x \qquad\qquad G_>(x,x') = d_1 x + d_2/x$$

Since $G_<(0,x')=0$, it is necessary to take $c_2= 0$, while the condition $G_>(1,x') = 0$ forces $d_1= -d_2$. Next the two solutions must be matched at $x = x'$. In the usual way we demand that

$$G_>(x',x') = G_<(x',x')$$

and since G satisfies $L[G] = \delta(x-x')$, a single integration across the point $x=x'$ gives

$$G_>'(x',x') - G_<'(x',x') = 1/x'$$

These translate into

$$c_1 x' = d_1(x' - 1/x') \qquad\qquad d_1(x'^2+1)/x'^2 - c_1 = 1/x'$$

or

$$c_1 = ½(x'^2-1)/x' \qquad \text{and} \qquad d_1 = ½\, x'$$

So

$$G_<(x,x') = ½(x'^2-1)(x/x') \qquad \text{and} \qquad G_>(x,x') = ½(x^2-1)(x'/x)$$

(b) We can use the Green's function to find a part of solution to the inhomogeneous equation satisfying the homogeneous BC's, and to this we need to add a linear combination of u_1 and u_2 to satisfy the inhomogeneous BC's. But by inspection the latter is just x itself. So

$$u(x) = x + \int_0^x dx'\, G_>(x,x')\, ln(x') + \int_x^1 dx'\, G_<(x,x')\, ln(x') +$$

$$= x + ½[(x^2-1)/x] \int_0^x dx'\, x'\, ln(x') + ½\, x \int_x^1 dx'\, (x'-1/x')\, ln(x')$$

$$= x + ½[(x^2-1)/x][\, ½\, x^2 (ln(x) - ½\,)] + [½x][-¼ - ½x^2(ln(x)-½) + ½(ln(x))^2]$$

$$= x\, [1 - ¼\, ln(x) + ¼\, (ln(x))^2]$$

13

Prove that the relation $PV^{\gamma} = constant$, where $\gamma = (f+2)/f$ and f is the number of degrees of freedom per molecule for partitioning the energy, holds for an adiabatic expansion of an ideal gas. How would you go about deriving an analogous relation $p = f(V)$ for an adiabatic expansion of a non-ideal gas, such as a van der Waals gas, whose equation of state and the total excess energy due to intermolecular interactions are given below? Do not attempt to solve the differential equation.

Assume for the non-ideal gas that

$$(p + \frac{N^2 a}{V^2})(V - Nb) = NkT$$

and

$$\Delta U = -\frac{N^2 a}{V}$$

13

Prove that the relation PV^{γ} = *constant*, where $\gamma = (f+2)/f$ and f is the number of degrees of freedom per molecule for partitioning the energy, holds for an adiabatic expansion of an ideal gas. How would you go about deriving an analogous relation $p = f(V)$ for an adiabatic expansion of a non-ideal gas, such as a van der Waals gas, whose equation of state and the total excess energy due to intermolecular interactions are given below? Do not attempt to solve the differential equation.

$$((p+\frac{N^2a}{V^2})(V-Nb) = NkT \text{ and } \Delta U = -\frac{N^2a}{V})$$

Solution: Total energy of an ideal gas: $U = \underbrace{Nf}_{\text{total degrees of freedom}}\underbrace{(\tfrac{1}{2}kT)}_{\text{energy per degree of freedom}}$

The first law: $dU = đQ + đW$

For a reversible adiabatic process: $đQ = 0, \quad đW = -p\,dV$

Combining the first equation with the last one:

$$dU = -p\,dV = Nf\,\tfrac{1}{2}k\,dT \quad \text{----- (1)}$$

Ideal gas law: $pV = NkT$, taking differential of both sides

yields: $$dp\,V + p\,dV = Nk\,dT \quad \text{----- (2)}$$

from (1)

$$\frac{2}{f}\,p\,dV = -Nk\,dT \quad \text{---- (3)}$$

(2) + (3) yields: $V\,dp + \left(\frac{f+2}{f}\right)p\,dV = 0$

$$\frac{dp}{p} + \gamma\frac{dV}{V} = 0 \Rightarrow \ln p + \ln V^{\gamma} = \ln(\text{const.})$$

$$\Rightarrow \boxed{pV^{\gamma} = \text{Const}}, \text{ where } \gamma = \frac{f+2}{f}$$

(Cont'd)

#13b

Van der Waals gas:

For this part we follow a very similar procedure to that of ideal gas:

Total energy of the gas: $U = Nf\left(\frac{1}{2}kT\right) - \frac{aN^2}{V}$

$\Rightarrow dU = Nf\frac{1}{2}k\,dT + a\left(\frac{N}{V}\right)^2 dV$, this can be related to $p\,dV$

using the first law: $dU = \text{đ}Q + \text{đ}W$. For an adiabatic expansion (assuming reversible quasistatic process) $\text{đ}Q = 0$, $\text{đ}W = -p\,dV$

combining the latter relation with the one above it one obtains;

$$\boxed{-Nk\,dT = \frac{2}{f}\left(p + a\left(\frac{N}{V}\right)^2\right)dV} \quad \text{(4)}$$

Now $Nk\,dT$ term can be eliminated by differentiating the Van der Waals equation:

$$\boxed{Nk\,dT = \left(dp - \frac{2aN^2}{V^3}dV\right)(V - Nb) + \left(p + a\left(\frac{N}{V}\right)^2\right)dV} \quad \text{(5)}$$

(4)+(5) yields:

$$\left(dp - 2a\frac{N^2}{V^3}dV\right)(V - Nb) + \gamma\left(p + a\left(\frac{N}{V}\right)^2\right)dV = 0$$

where $\gamma = 1 + \frac{2}{f}$

This equation can be re-arranged:

$$\boxed{p' + \frac{\gamma\left(p + a\left(\frac{N}{V}\right)^2\right)}{(V - Nb)} = 2a\frac{N^2}{V^3}} \quad ; \text{ where } p' = \frac{dp}{dV} \quad \text{(6}$$

This is what I expected from the exam. However, this differential equation can easily be solved (though it may not look like it).

To solve the equation (6) set $q = p + a\left(\frac{N}{V}\right)^2$

$$\Rightarrow \quad p' = q' + 2a\,\frac{N^2}{V^3}\,, \quad \text{insert this in equation (6):}$$

$$q' + \cancel{2a\,\frac{N^2}{V^3}} + \frac{\gamma q}{(V-Nb)} = \cancel{2a\,\frac{N^2}{V^3}}$$

$$\Rightarrow \quad \frac{dq}{q} + \frac{\gamma\, dV}{(V-Nb)} = 0$$

set $V - Nb = u$, $\quad dV = du$

This yields a result very similar to the ideal gas cas:

$$\ln q + \ln u^{\gamma} = \ln c \qquad \text{or}$$

$$\boxed{q\,u^{\gamma} = \text{Const.}}$$ where q and u are given by:

$$q = p + a\left(\frac{N}{V}\right)^2$$
$$u = V - Nb$$

Notice that van der Waals equation is very similar to the ideal gas equation if we use the reduced variables:

$$\boxed{q\,u = NkT}$$

— o —

A certain linear-filter circuit provides a 2-second delay and low-pass filtering. When the input voltage is a perfect spike $V_{in}(t) = \delta(t)$ the output is the filter's Green function

$$V_{out}(t) = G(t) = \exp[-(t-2)] \qquad \text{for} \quad t > 2$$

whose Fourier transform is

$$\hat{G}(\omega) = \int_{-\infty}^{\infty} G(t) e^{i\omega t} dt = e^{2i\omega} / (1 - i\omega)$$

A signal is fed into the filter whose Fourier transform is

$$\hat{V}_{in}(\omega) = e^{i\omega} / ((\omega + 2i)^2 - 1)$$

(a) What is the Fourier transform of the output voltage $\hat{V}_{out}(\omega)$?

(b) Using the theory of residues show that the output $V_{out}(t)$ is zero for $t < t_0$. Find t_0 and be specific about the contour used and the locations of all poles.

(c) Find the output $V_{out}(t)$ at times $t >> t_0$.

(14)

#14

Solution

a. A simple application of the Green's function gives

$$V_{\text{out}}(t) \;=\; \int_{-\infty}^{\infty} G(t-t')V_{\text{in}}(t')\,dt' \;,$$

which is a convolution. The Fourier tranform of a convolution is the product of the Fourier transforms so

$$\hat{V}_{\text{out}}(\omega) \;=\; \hat{G}(\omega)\cdot\hat{V}_{\text{in}}(\omega) \;=\; -\frac{1}{i}\,\frac{e^{3i\omega}}{(\omega+i)(\omega+2i-1)(\omega+2i+1)} \qquad (1)$$

b. Performing the inverse Fourier transform of (1) gives

$$\begin{aligned} V_{\text{out}}(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{V}_{\text{out}}(\omega)e^{-i\omega t}\,d\omega \\ &= -\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{e^{-i(t-3)\omega}}{(\omega+i)[\omega-(1-2i)][\omega-(-1-2i)]}\,d\omega \;. \qquad (2) \end{aligned}$$

To apply the theory of residues the path of integration must be a closed contour. The integral along the real axis may be closed along a curve either in the upper half-plane, $\mathrm{Im}(\omega) > 0$ or the lower half-plane $\mathrm{Im}(\omega) < 0$. If the additional piece makes negligible contribution to the integral than the contour integral gives the inverse Fourier transform. Due to the factor $e^{-i(t-3)\omega}$ this will occur as long as $\mathrm{Im}[(t-3)\omega] < 0$ and the imaginary part of ω is extremely large. Thus the contour is closed in the upper-half plane for $t < 3$ and in the lower half-plane for $t > 3$.

The integrand has single poles at $\omega = -i$, $\omega = 1-2i$ and $\omega = -1-2i$, all in the lower half-plane. Therefore the countour closed in the upper half-plane encloses no poles and $V_{\text{out}}(t) = 0$ for $t < t_0 = 3$.

c. From the results of part b. we know that

$$V_{\text{out}}(t) \;=\; -\frac{1}{2\pi i}\oint_C \frac{e^{-i(t-3)\omega}}{(\omega+i)[\omega-(1-2i)][\omega-(-1-2i)]}\,d\omega \;,$$

where the contour C encloses all three poles, $\omega = -i$, $\omega = 1-2i$ and $\omega = -1-2i$, in the clockwise sense. The integral is therefore $-2\pi i$ times the sum of all three residues. Due to the factor $e^{-i\omega t}$, the long-time limit will be dominated by the pole with the least negative imaginary part: $\omega = -i$. Using only this residue yields

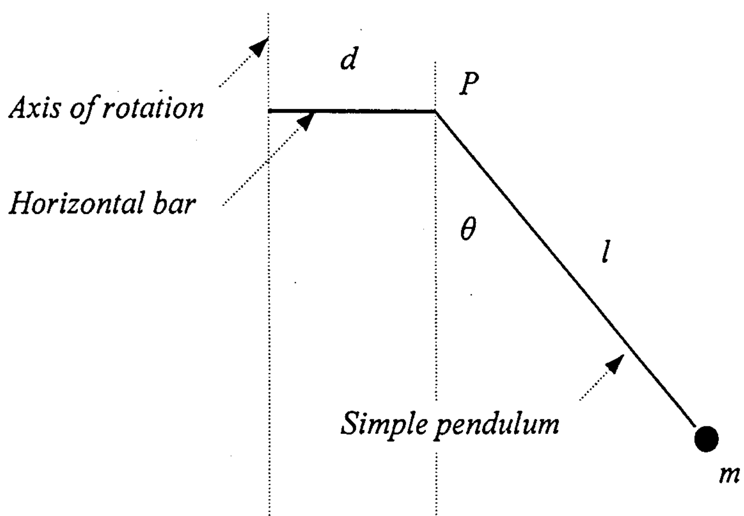
$$V_{\text{out}}(t) \;=\; \operatorname*{Res}_{\omega=-i}\,\frac{e^{-i(t-3)\omega}}{(\omega+i)[\omega-(1-2i)][\omega-(-1-2i)]} \;=\; -\frac{e^{-(t-3)}}{2} \qquad (3)$$

15

A simple model of some carnival rides can be represented by a simple pendulum consisting of a massless rigid rod of length l and a point mass m attached to one end, pivoted about point P located at the end of a horizontal bar of length d. The bar rotates with constant angular frequency ω around a vertical axis as shown in the figure. Answer the following questions:

(a) Find the equation from which the angle θ_o for stable equilibrium can be determined. Show that for $d = 0$ this angle is given by $\cos\theta_0 = \frac{g}{l\omega^2}$ provided that $\omega^2 \geq \frac{g}{l}$.

(b) For $d = 0$ and for small deviations of θ from θ_o, obtain the equation of motion of the pendulum and show that it is identical to the equation of simple harmonic motion with angular frequency $\varpi = \omega \sin\theta_o$.



15

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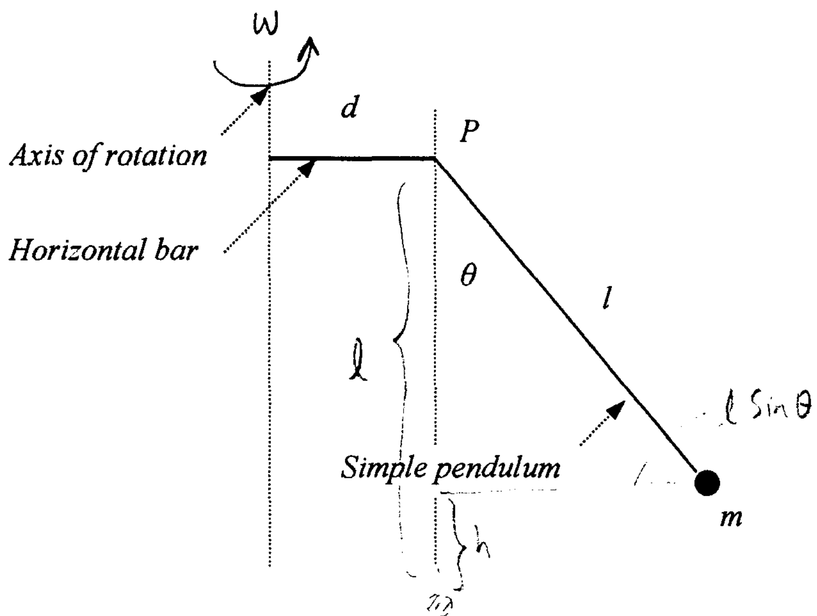
(a) Find the equation from which equilibrium angle θ_o can be determined. Show that for $d = 0$ this angle is given by $Cos\theta_o = \frac{g}{l\omega^2}$ provided that $\omega^2 \geq \frac{g}{l}$.

(b) For $d = 0$ and for small deviations of θ from θ_o, obtain the equation of motion of the pendulum and show that it is identical to the equation of simple harmonic motion with angular frequency $\varpi = \omega \sin\theta_o$.



Solution: (a) Let us use Lagrangian approach:

$$L = T - V, \quad \text{where } T = \frac{1}{2} m l^2 \dot{\theta}^2 + \frac{1}{2} m w^2 (d + l\sin\theta)^2,$$

and $V = mgh = mgl(1 - \cos\theta)$

$$\Rightarrow \boxed{L = \frac{1}{2} m l^2 \dot{\theta}^2 + \frac{1}{2} m w^2 (d + l\sin\theta)^2 + mgl(\cos\theta - 1)}$$

L-equation: $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$

L - equation yields:

$$m\ell^2\ddot{\theta} - m\omega^2(d+\ell\sin\theta)\ell\cos\theta + mg\ell\sin\theta = 0$$

by devide by $m\ell^2$ and re-arrange:

$$\boxed{\ell\ddot{\theta} - \omega^2(d+\ell\sin\theta)\cos\theta + g\sin\theta = 0}$$

At equilibrium $\ddot{\theta} = 0$

$$\Rightarrow \quad g\sin\theta_0 = \omega^2(d+\ell\sin\theta_0)\cos\theta_0$$

for $d=0$ $\quad \boxed{\cos\theta_0 = \frac{g}{\omega^2\ell}}$ $\quad$ since $\cos\theta_0 \leq 1$

$$\Rightarrow \quad \boxed{\omega^2 \geq \frac{g}{\ell}}$$

This is a very counter-intuitive result, in that ω must be above a certain minimum before $\theta_0 \neq 0$ can be had.

(b) For $d=0$

$$\boxed{\ell\ddot{\theta} - \omega^2\ell\sin\theta\cos\theta + g\sin\theta = 0}$$

set $\theta = \theta_0 + \phi$ where ϕ is small (i.e. $\sin\phi \approx \phi$)

The equation can be rewritten using new variable;

$$l\ddot{\phi} - \frac{\omega^2 l}{2} \sin 2(\theta_0+\phi) + g \sin(\theta_0+\phi) = 0 \quad \ldots \quad (1)$$

Use $\sin(\theta_0+\phi) \approx \sin\theta_0 + \cos\theta_0 \, \phi$

$\sin 2(\theta_0+\phi) \approx \sin 2\theta_0 + \cos 2\theta_0 \, (2\phi)$

Insert in (1), set $\omega^2 l = \dfrac{g}{\cos\theta_0}$ (from part (a))

$$l\ddot{\phi} - \frac{g}{2\cos\theta_0}\left(\sin 2\theta_0 + \cos 2\theta_0 \, 2\phi\right) + g\sin\theta_0 + g\cos\theta_0 \, \phi = 0$$

$$\Rightarrow l\ddot{\phi} - \cancel{g\sin\theta_0} - \frac{g\cos 2\theta_0}{\cos\theta_0}\phi + g\cos\theta_0\,\phi + \cancel{g\sin\theta_0} = 0$$

$$l\ddot{\phi} + \overset{\omega^2 l}{\frac{g}{\cos\theta_0}}\left(\cos^2\theta_0 - \underbrace{\cos 2\theta_0}_{\cos^2\theta_0 - \sin^2\theta_0}\right)\phi = 0$$

$$l\ddot{\phi} + l\omega^2\sin^2\theta_0\,\phi = 0 \Rightarrow \boxed{\ddot{\phi} + (\omega\sin\theta_0)^2\phi = 0}$$

This is a simple harmonic oscillator equation with an angular frequency $\boxed{\bar{\omega} = \omega\sin\theta_0}$

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