Wave ⟷ Helmholtz Eqtn : Kirchoff's Solution.

Diffraction Theory

Ref. J.D. Jackson "Classical Electrodynamics" (Wiley, 2nd ed., 1975), Sec. 9.8, et seq.

1) Consider any monochromatic wave disturbance $\Psi(\mathbf{r},t)=\psi(\mathbf{r})e^{\pm i\omega t}$, at frequency ω, which obeys a wave eqtn...

$$\left(\nabla^2-\frac{1}{c^2}\partial^2/\partial t^2\right)\Psi(\mathbf{r},t)=0 \longleftrightarrow \underline{\underline{(\nabla^2+k^2)\psi(\mathbf{r})=0}} \quad \left\{ \begin{array}{l} k=\frac{\omega}{c}=\text{wave \#}, \\ c=\text{phase velocity}; \end{array}\right. \tag{1}$$

... for (transverse) EM waves: c = light speed, Ψ = any compt of $\mathbb{E}$ or $\mathbb{B}$;
... for (longitudinal) sound waves: c = sound speed, Ψ = pressure p, or density ρ.

The character of the wave (light or sound, etc.) does not matter... diffraction effects are a property (involving wave interference) of solutions to the above homogeneous Helmholtz Eqtn. The solution of interest,* first derived by Kirchoff,

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* See Jackson Sec. 6.8. In general, we can solve the inhomogeneous Helmholtz Eqtn:

$$\left.\begin{array}{l} (\nabla^2+k^2)\psi(\mathbf{r})=-F(\mathbf{r}),\ k^2=\text{cnst}; \\ \text{\& } (\nabla^2+k^2)G(\mathbf{r},\mathbf{r}')=-\alpha\,\delta(\mathbf{r}-\mathbf{r}'),\ \alpha=\text{cnst},\ G(\mathbf{r},\mathbf{r}')=\text{Green's fcn}; \end{array}\right\} \text{①}$$

by the usual procedure... mult. 1st eqtn on left by G, 2nd on left by ψ, subtract:

$$\Rightarrow\ G\nabla^2\psi-\psi\nabla^2G=-GF(\mathbf{r})+\alpha\psi(\mathbf{r})\delta(\mathbf{r}-\mathbf{r}'),$$

$$\text{or}\ \ \alpha\psi(\mathbf{r})\delta(\mathbf{r}-\mathbf{r}')=G(\mathbf{r},\mathbf{r}')F(\mathbf{r})+\nabla\cdot(G\nabla\psi-\psi\nabla G).$$

Interchange labels $\mathbf{r}'$ & $\mathbf{r}$ and claim G is symmetric: $G(\mathbf{r}',\mathbf{r})=G(\mathbf{r},\mathbf{r}')$...

$$\Rightarrow\ \alpha\psi(\mathbf{r}')\delta(\mathbf{r}-\mathbf{r}')=G(\mathbf{r},\mathbf{r}')F(\mathbf{r}')+\nabla'\cdot(G\nabla\psi'-\psi\nabla'G),\ \text{by Green's identity.}$$

Now integrate thru this eqtn (over a finite domain V) by $\int_V d^3x'$, and use the Divergence Theorem to convert the 2nd integral RHS to an integral over surface $\mathbb{S}'$ enclosing V:

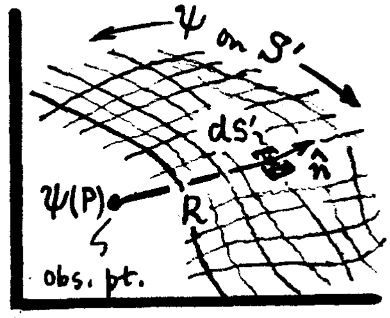
$$\boxed{\alpha\psi(\mathbf{r})=\int_V d^3x'\,G(\mathbf{r},\mathbf{r}')F(\mathbf{r}')+\oint_{S'}d\mathbb{S}'\cdot\left[G(\mathbf{r},\mathbf{r}')\nabla'\psi(\mathbf{r}')-\psi(\mathbf{r}')\nabla'G(\mathbf{r},\mathbf{r}')\right]} \text{②}$$

② is the general sol$^{\underline{n}}$ to the inhomogeneous system ① above. If ① is homogeneous, w/ source term $F(\mathbf{r})\equiv 0$, then only the surface term survives, and we have...

$$(\nabla^2+k^2)\psi(\mathbf{r})=0 \Rightarrow \underline{\underline{\psi(\mathbf{r})=\frac{1}{\alpha}\oint_{S'}d\mathbb{S}'\cdot\left[G(\mathbf{r},\mathbf{r}')\nabla'\psi(\mathbf{r}')-\psi(\mathbf{r}')\nabla'G(\mathbf{r},\mathbf{r}')\right]}}, \text{③}$$

w/ $(\nabla^2+k^2)G(\mathbf{r},\mathbf{r}')=-\alpha\,\delta(\mathbf{r}-\mathbf{r}')$. With $\alpha=4\pi$ & $G(\mathbf{r},\mathbf{r}')=\frac{1}{R}e^{ikR}$, $R=|\mathbf{r}-\mathbf{r}'|$ [see Jkn Eq. (6.62)], ③ is the "Kirchoff Solution" used in Eq. (2) next page.

is,
$$\boxed{\psi(P) = \frac{1}{4\pi}\oint_{S'} d\mathbf{S}'\cdot\left[\left(\frac{e^{ikR}}{R}\right)\nabla'\psi - \psi\,\nabla'\left(\frac{e^{ikR}}{R}\right)\right]}. \quad (2)$$



"P" denotes "observation point", and R is the distance from pt. P to the directed surface element $d\mathbf{S}'$. This surface term is always discarded in solutions for ψ on an ∞ domain. Here it provides all the fun -- $\psi(P)$ is completely determined by the ψ-values on the boundary surface S' (NOTE: S' should be a <u>closed</u> surface).

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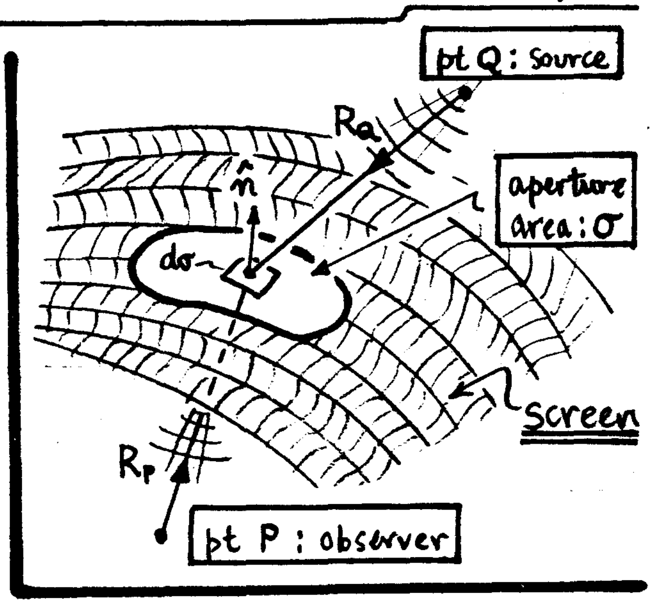
2) Kirchoff applied Eq. (2) to the problem sketched at right: a source at pt. Q broadcasts light or sound waves through an <u>aperture</u> of area σ to a "listening point" at pt. P. All other communication between P & Q is blocked by a "screen". Assume:

(1) ψ & $\nabla\psi$ vanish everywhere on the screen;

(2) ψ & $\nabla\psi$ in the aperture are the same as their free-space values (in absence of screen).



Then Eq. (2), for the "sound" received at pt. P from the broadcast at Q, is

$$\rightarrow \psi(P) = \frac{1}{4\pi}\int_{\text{aperture}} \hat{n}\,d\sigma\cdot\left[\left(\frac{e^{ikR_P}}{R_P}\right)\nabla_Q\psi(\mathbf{R}_Q) - \psi(\mathbf{R}_Q)\,\nabla_P\left(\frac{e^{ikR_P}}{R_P}\right)\right] \quad (3)$$

... assignment of subscripts: ① disturbances ψ & $\nabla\psi$ originate at pt. Q, propagate to aperture; ② spherical wavelets on aperture propagate via R_P to pt. P.

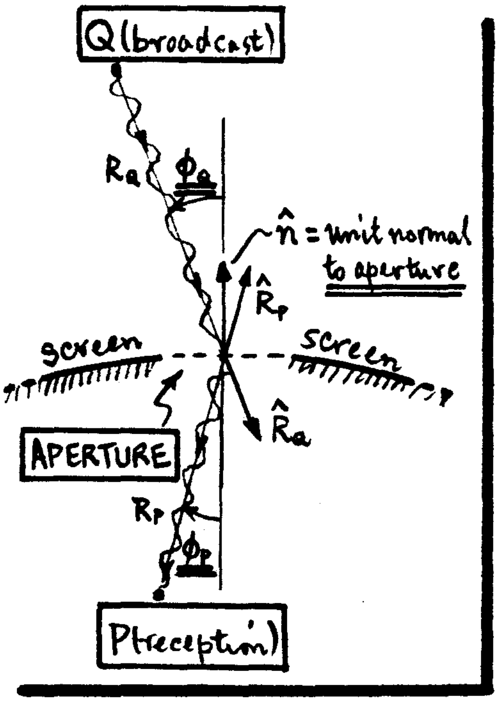
Assume <u>point</u> source at Q [no $\measuredangle$ dependence for $\psi(\mathbf{R}_Q)$]. Then calculate...

$$\rightarrow \nabla_Q\psi(\mathbf{R}_Q) = \hat{R}_Q(\partial\psi/\partial R_Q), \quad \nabla_P\left(\frac{e^{ikR_P}}{R_P}\right) = \hat{R}_P\left(ik - \frac{1}{R_P}\right)\frac{e^{ikR_P}}{R_P};$$

So,
$$\psi(P) = \frac{1}{4\pi}\int_{\text{aperture}} d\sigma\left[(\hat{n}\cdot\hat{R}_Q)\frac{\partial\psi(R_Q)}{\partial R_Q} - (\hat{n}\cdot\hat{R}_P)\left(ik - \frac{1}{R_P}\right)\psi(R_Q)\right]\frac{e^{ikR_P}}{R_P}. \quad (4)$$

$\hat{R}_P$ & $\hat{R}_Q$ are unit vectors <u>from</u> pts P & Q <u>to</u> a point on the aperture. We can replace $(\hat{n}\cdot\hat{R}_Q)$ & $(\hat{n}\cdot\hat{R}_P)$ by $\measuredangle$'s ϕ_Q & ϕ_P such that...

Derivation of Kirchoff's Formula (cont'd). Remarks.



$$\left[\!\left[\begin{array}{ll}\hat{n}\cdot\hat{R}_Q = -\cos\phi_Q, & \phi_Q = \measuredangle \text{ of incidence on aperture;} \\ \hat{n}\cdot\hat{R}_P = +\cos\phi_P, & \phi_P: \measuredangle \text{ of "refraction" at aperture.}\end{array}\right.\tag{5}$$

Use these in Eq. (4), and factor out ik, so as to write:

$$\rightarrow \psi(P) = \frac{k}{4\pi i}\int_{\text{aperture}} d\sigma \left[\left(1-\frac{1}{ikR_P}\right)\psi(R_Q)\cos\phi_P + \right.$$
$$\left. + \frac{\cos\phi_Q}{ik}\left(\frac{\partial\psi(R_Q)}{\partial R_Q}\right)\right]\frac{e^{ikR_P}}{R_P}. \tag{6}$$

The "sound" at pt. P, viz. $\psi(P)$, is now specified once the values of ψ & $\partial\psi/\partial R$ on the aperture are known. To supply that information, Kirchoff used assumption (2) [above Eq. (3) on last page]: ψ & $\partial\psi/\partial R$ on the aperture are the free-space values; as broadcast from a pt. Q

$$\left[\!\left[\begin{array}{c}\text{point-source} \\ \text{broadcast}\end{array}\right\} \psi(R_Q) = \frac{e^{ikR_Q}}{R_Q}, \quad \frac{\partial}{\partial R_Q}\psi(R_Q) = ik\left(1-\frac{1}{ikR_Q}\right)\frac{e^{ikR_Q}}{R_Q}. \tag{7}$$

Then Eq. (6) yields Kirchoff's Diffraction Formula:

$$\boxed{\psi_k(P) = \frac{k}{4\pi i}\int_{\text{aperture}} d\sigma\left[\left(1-\frac{1}{ikR_P}\right)\cos\phi_P + \left(1-\frac{1}{ikR_Q}\right)\cos\phi_Q\right]\frac{e^{ik(R_P+R_Q)}}{R_P R_Q}} \tag{8}$$

This formula is quoted in Jkⁿ Eq. (9.132), with "obliquity factor" Θ given by the Kirchoff approximation: $\Theta(\phi_P,\phi_Q) = \frac{1}{2}(\cos\phi_P + \cos\phi_Q)$, and the terms in $1/kR \sim \lambda(\text{wave-length})/R(\text{distance to screen}) \ll 1$ ignored.

REMARKS On Kirchoff's Formula, Eq. (8).

1. On the LHS of (8), we have subscripted ψ with the wave # k to flag the fact that ψ_k is the solution for a monochromatic source, at freq. $\omega = kc$. If Q broadcasts over a frequency spectrum, specified by some amplitude $A(k)$, then -- because we can use superposition for the Helmholtz Eqtn (a linear PDE) -- the solution is: $\psi(P) = \int_{-\infty}^{+\infty} A(k)\psi_k(P)\,dk$.

REMARKS Kirchoff's Eq. (8) [cont'd].

2. Kirchoff's result is actually an approximation, based principally on the assumption [# (1) on p. DT 2] that ψ & $\nabla\psi$ vanish everywhere outside the aperture. This is mathematically weak, as discussed by Jackson on pp. 429-31, But the remedies make very little difference in the final result. With the usual (reasonable) assumptions...

A "Short" wavelengths : $1/kR = \frac{1}{2\pi}(\lambda/R) \ll 1$ (negligible). (9A)

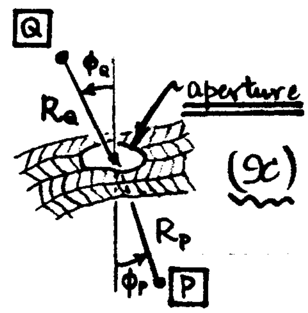
→ The broadcast wavelength λ is negligible compared to system size R;¶

B "Small" apertures : $R\left(\begin{smallmatrix}\text{pt. P to screen}\\ \text{pt. Q to screen}\end{smallmatrix}\right) \gg$ characteristic aperture size. (9B)

→ Then R_P & R_Q (and ϕ_P & ϕ_Q) change negligibly in integration over aperture;

Kirchoff's formula in Eq. (8) simplifies to...

$$\boxed{\psi_k(P) = \frac{k}{2\pi i}\,\Theta(\phi_P,\phi_Q)\,\frac{1}{R_P R_Q}\int_{\text{aperture}} d\sigma\, e^{ik(R_P+R_Q)}, \quad \text{w/ obliquity factor: } \Theta(\phi_P,\phi_Q) = \tfrac{1}{2}(\cos\phi_P + \cos\phi_Q).}$$ (9C)



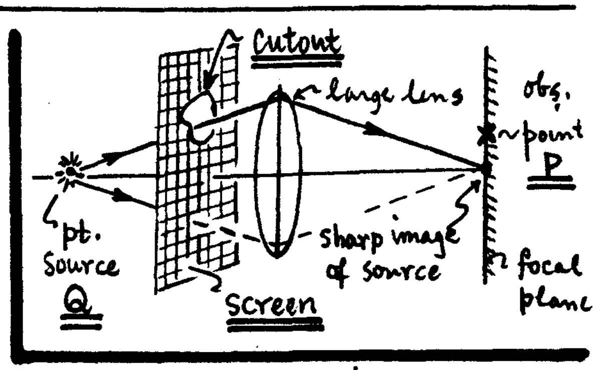
As noted by Jackson, the remedies to Kirchoff's approximation (w/ assumptions A & B above) at most change the obliquity factor Θ by ~ negligible amounts. The diffraction in all cases is mainly determined by the phase integral in (9C), viz. $\int d\sigma\, e^{ik(R_P+R_Q)}$, and all such approx^{n}s work best when the broadcast wavelength $\lambda \ll$ aperture size d (they fail when $\lambda \sim d$). Hierarchy is :

→ $\lambda\left(\begin{smallmatrix}\text{broadcast}\\ \text{wavelength}\end{smallmatrix}\right) \ll d\left(\begin{smallmatrix}\text{aperture}\\ \text{size}\end{smallmatrix}\right) \ll R\left(\begin{smallmatrix}\text{source or observer}\\ \text{distance to screen}\end{smallmatrix}\right)$. (10)

3. The integrand in (8), or the expression for $\psi_k(P)$ in (9C), is entirely symmetric under exchange of the labels P & Q. This ⇒ reciprocity if we interchange the broadcast & reception points : $\psi(\text{at P from Q}) \equiv \psi(\text{at Q from P})$. This holds for a

¶ This works better for light ($\lambda \sim 5\times10^{-7}$m, visible) than for sound ($\lambda \sim 0.3$m @ 1000Hz).

point-to-point $P \leftrightarrow Q$ interchange, but does not guarantee that the two diffraction patterns [Q(broadcast) → P(reception) vs. P(broadcast) → Q(reception)] will be the same; those patterns involve the neighborhoods of points P & Q. So your loudspeakers may in fact sound better from the northeast corner of your room than from the southwest. Try it and see...



4. Babinets Principle applies: the diffraction pattern due to the screen-with-cutout is the same (in intensity) as that produced by a screen in the form of the cutout (except for the direct line of sight between source & obs'n point). In other words, interchanging screen and aperture does not change the diffraction pattern intensity. Reasoning is...

(a) Suppose $\psi(P)$ at point P for screen-with-cutout (as above)
(b) " $\psi'(P)$ " " " screen & cutout interchanged

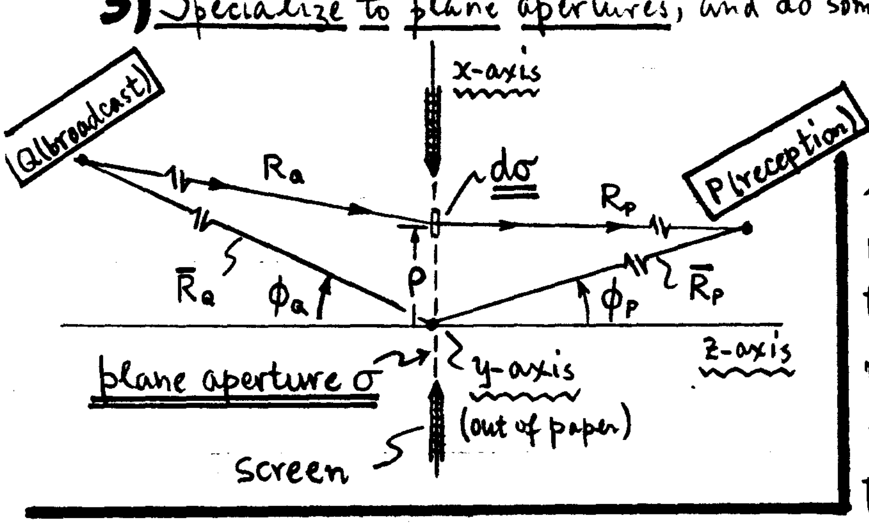
$$\left.\right\} \quad \underline{\underline{\psi(P) + \psi'(P) = 0}}. \qquad (11)$$

We have a zero on the RHS of (11) because: pt. P would be dark (off lens axis) if the screen were removed in case (a); pt. P would again be dark if the screen were replaced in case (b) [thus blocking all light from Q]. So: $\psi'(P) = -\psi(P)$, and the light pattern intensities are the same: $|\psi'(P)|^2 = |\psi(P)|^2$, for (a) & (b). The screens in cases (a) & (b) are called "complementary screens."

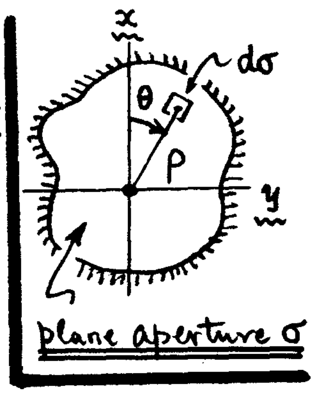
5. Eqs. (8), or (9C), for $\psi_k(P)$ are appropriate for a scalar diffraction theory; we have solved Helmholtz' Eqtn $(\nabla^2 + k^2)\psi_k(\mathbf{r}) = 0$ for a scalar field ψ_k. This is OK for the scalar fields $\psi_k \sim$ pressure or density for sound, but some subtle modifications occur when the theory is generalized to vector fields, such as the $\mathbb{E}$ & $\mathbb{B}$ fields contained in light waves. Jackson discusses the differences in his Secs. (9.9)–(9.12). In this brief survey, we cannot dwell on such details -- we will continue with the scalar theory, keeping in mind that it describes sound diffraction quite well, but that light diffraction may differ in some details. What we have done is to have reduced the prototype diffraction problem to an evaluation of the phase integral $\int d\sigma\, e^{ik(R_P + R_Q)}$ in Eq. (9c).

Kirchoff's Formula for a Plane Aperture: Leading Terms.

3) Specialize to plane apertures, and do some geometry...



NOTE: P & Q need not lie in the xz-plane, in general. We will track the area elements $d\sigma$ in the aperture by plane polar cds ρ & θ, as above. We assume $\bar{R}_{P,Q} \gg \rho \gg \lambda$, per Eq. (10).



For either R_P or R_Q, by the law of cosines...

$$\rightarrow R^2 = \bar{R}^2 + \rho^2 - 2\bar{R}\rho\cos\left(\tfrac{\pi}{2} - \phi\right) \Rightarrow R = \bar{R}\left[1 - 2(\rho/\bar{R})\sin\phi + (\rho/\bar{R})^2\right]^{1/2}. \tag{12}$$

$\bar{R}$ is the distance from the aperture center to pt. P or Q; $\bar{R}$ = cnst during $\int_{\text{aperture}} d\sigma$.
Since $\bar{R} \gg \rho$ by assumption, we can expand the square root in (12) as...

$$\rightarrow \underline{R \simeq \bar{R} - \rho\sin\phi + \tfrac{1}{2}(\rho^2/\bar{R})\cos^2\phi}, \text{ neglecting relative order } (\rho/\bar{R})^3. \tag{13}$$

This is for $R_{P,Q}$ & $\phi_{P,Q}$. For $\int_{\text{aperture}} d\sigma$ of Eq. (9C), form the quantity...

$$\rightarrow R_P + R_Q = (\bar{R}_P + \bar{R}_Q) - \Delta(P,Q)$$

$$\text{w/} \left[\!\left[\Delta(P,Q) \simeq \underbrace{\rho(\sin\phi_P + \sin\phi_Q)}_{\text{Frahoufer term}} - \underbrace{\tfrac{1}{2}\rho^2\left(\frac{\cos^2\phi_P}{\bar{R}_P} + \frac{\cos^2\phi_Q}{\bar{R}_Q}\right)}_{\text{Fresnel term (see Eqs. (17))}} + \cdots \right]\!\right] \tag{14}$$

The ( )'s RHS in (14) are cnst during the integration $\int_{\text{aperture}} d\sigma$. In Eq. (9C), the factor outside the integral $1/R_P R_Q \rightarrow 1/\bar{R}_P \bar{R}_Q$, evidently, and -- since there is no θ dependence in the integrand -- $\int d\sigma = \int \rho\, d\rho \int_0^{2\pi} d\theta = 2\pi \int \rho\, d\rho$, so $\psi_k(P)$ is...

$$\boxed{\psi_k(P) \simeq \frac{k}{i}\,\Theta(\phi_P, \phi_Q)\cdot\frac{e^{ik(\bar{R}_P + \bar{R}_Q)}}{\bar{R}_P \bar{R}_Q}\cdot\int_{\text{aperture}} \rho\, d\rho\, e^{-ik\Delta(P,Q)}}. \tag{15}$$

Without specifying the ρ-dependence of the aperture, (15) is the simplest form that Kirchoff's diffraction solution reduces to, under the assumptions of pt. Q (broadcast) $\rightarrow$ pt. P (reception), and λ (broadcast wavelength) $\ll$ aperture size $\ll R_{P,Q}$ (source, observer distances).