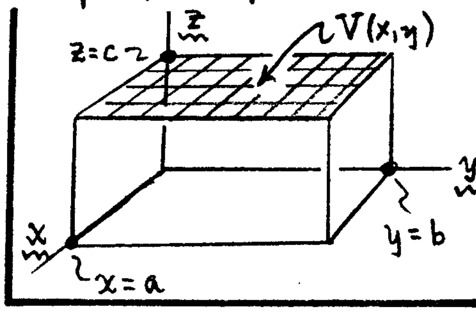
7) As a specific application of Eq. (33) [soln to $\nabla^2\phi=0$ in rect^r cds], Jk^n considers a hollow rect^r box with one corner at $(x,y,z)=(0,0,0)$, and sides a,b,c (see Jk^n Fig. 2.9, p. 70). Here the top face of the box (i.e. plane at $z=c$) has a specified potential $V(x,y)$, while the other 5 sides are at $\phi\equiv 0$. In particular, the potential vanishes along the x-direction at $x=0$ & $x=a$, and also along the y-direction at $y=0$ & $y=b$. These B.C. are easily accommodated in the above general soln by taking:



$$\rightarrow U(x)=\sin\alpha_n x,\quad \alpha_n a=n\pi;\quad V(y)=\sin\beta_m y,\quad \beta_m b=m\pi; \tag{34}$$

where $n,m=1,2,3,\ldots$, independently. As for $W(z)$, it must be zero on the plane $z=0$, so: $W(z)=\sinh\gamma_{nm}z$, only, w/ $\gamma_{nm}=\sqrt{\alpha_n^2+\beta_m^2}$. Then

$$\boxed{\phi(x,y,z)=\sum_{n,m} C_{nm}[\sin\alpha_n x][\sin\beta_m y][\sinh\gamma_{nm}z]}\ \int \alpha_n=\frac{n\pi}{a},\ \beta_m=\frac{m\pi}{b},\ \gamma_{nm}=\sqrt{\alpha_n^2+\beta_m^2}. \tag{35}$$

This is a soln to $\nabla^2\phi=0$ in the box which satisfies the B.C. $\phi=0$ on 5 faces of the box, all except the face $z=c$. The coefficients C_{nm} are now chosen (actually forced) to fit the B.C. on $z=c$, viz

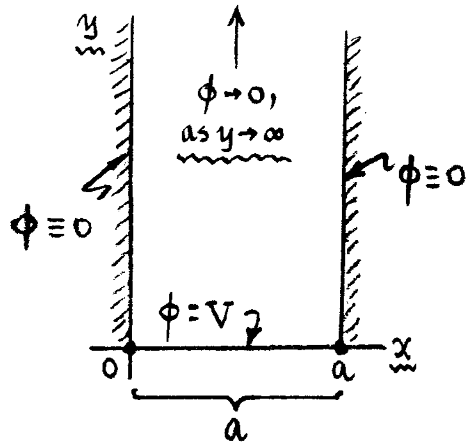
$$\rightarrow V(x,y)=\sum_{n,m}[C_{nm}\sinh\gamma_{nm}c]\sin\alpha_n x\sin\beta_m y,\ @\ z=c \tag{36}$$

The $\sin\alpha_n x$ & $\sin\beta_m y$ are orthogonal fcns on $[0,a]$ & $[0,b]$ resp., so you can project out the coefficients in the usual fashion, with result...

$$[C_{nm}\sinh\gamma_{nm}c]=\left(\frac{2}{a}\right)\int_0^a dx\,\sin\alpha_n x\cdot\left(\frac{2}{b}\right)\int_0^b dy\,\sin\beta_m y\,V(x,y). \tag{37}$$

The factors $(2/a)$ & $(2/b)$ are normalization factors. Eqs. (35) & (37) together are a complete solution to the $\nabla^2\phi=0$ problem in this box, satisfying all the B.C. by now.

10) The somewhat clumsy feature of the soln just achieved is that $\phi =$ $\underline{\underline{\infty}}$ series (whose rate-of-convergence may be very slow). In some problems (not all!), the series may be summed to a closed analytic form for ϕ. Jackson gives such an example in his Sec. (2.10). Obviously:



$$\left[\!\!\left[\begin{aligned} &\phi(x,y) = \sum_{n=1}^{\infty} A_n\left[\sin\left(\frac{n\pi x}{a}\right)\right] e^{-n\pi y/a}, \\ &\text{w/}\quad A_n = \frac{2}{a}\int_0^a dx\left[\sin\left(\frac{n\pi x}{a}\right)\right]\phi(x,0). \end{aligned}\right.\right. \tag{38}$$

Note that "a" is the only scale length in the problem; it enters the x-variation naturally, while it's used in the y-variation because $\beta_n = \alpha_n$ here.

If $\phi(x,0) = V = $ cnst, the A_n's are trivial: $A_n = \begin{cases} 4V/n\pi, & n = \text{odd} \\ 0, & n = \text{even} \end{cases}$

$$\xrightarrow{\text{So}} \phi(x,y) = \frac{4V}{\pi}\sum_{n=\text{odd}}^{\infty}\frac{1}{n}\left[\sin(n\pi x/a)\right]e^{-n\pi y/a}. \tag{39}$$

Another (boring!) ∞ series. This one can be summed, however, as Jackson shows on pp. 73-74. Result is...

$$\boxed{\phi(x,y) = \frac{2V}{\pi}\tan^{-1}\left(\frac{\sin kx}{\sinh ky}\right), \quad k = \frac{\pi}{a}.} \tag{40}$$

With this analytic form, it's "easy" to draw graphs, as in Jkn Fig. (2.11).

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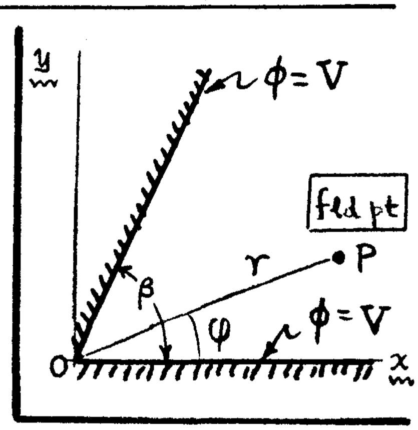
11) Finally, in Sec. (2.11), Jkn discusses the problem of finding the potential near a vertex (or corner) in a conductor -- where (as pictured at right) two conducting planes, (more or less) at the same potential V, intersect. We shall look at the 2D problem only; it is convenient to use plane polar cds (r, φ).



See dgms in $Jk^{\underline{n}}$ Fig.(2.13), p. 77. E: → small in corners; $\sigma \to$ large at edges.

In 2D plane polar cds, the Laplace problem (for $0 \leq \varphi \leq \beta$) is...

$$\rightarrow \nabla^2\phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2}\left(\frac{\partial^2\phi}{\partial\varphi^2}\right) = 0. \quad (41)$$

...separate: $\phi(r,\varphi) = R(r)\,\Psi(\varphi)$... put into $\frac{1}{\phi}\nabla^2\phi = 0$ to get...

$$\frac{r}{R}\frac{d}{dr}\left(r\frac{dR}{dr}\right) = \nu^2, \quad \frac{1}{\Psi}\frac{d^2\Psi}{d\varphi^2} = -\nu^2; \quad \nu = \text{separation cnst};$$

So//

$$\left[\!\left[R_\nu(r) = \begin{cases} a r^{\nu} + b r^{-\nu}, & \nu \neq 0 \\ a_0 + b_0 \ln r, & \nu = 0 \end{cases} ; \quad \Psi_\nu(\varphi) = \begin{cases} A\cos\nu\varphi + B\sin\nu\varphi, & \nu \neq 0 \\ A_0 + B_0\varphi, & \nu = 0 \end{cases} \right. . \quad (42)$$

The a's & b's, A's & B's are arbitrary (integration) cnsts at this point. Per B.C. they will (usually) be quantized to a set $\{a_n, b_n; A_n, B_n\}$.★ For the present problem (dgm, last page), in order to have the potential $\phi(r,\varphi) = V$ at $\varphi = 0$ & $\varphi = \beta$, for all $r \geq 0$, we throw out the $\nu = 0$ solns, and -- by superposition -- use the product fcns $R_\nu(r)\Psi_\nu(\varphi)$ as a sum, in the form...

$$\left[\!\left[\phi(r,\varphi) = V + \sum_{n=1}^{\infty} a_n r^{\nu_n}\sin(\nu_n\varphi), \quad \underline{\underline{\nu_n}} = n\pi/\beta; \quad (\beta = \text{wedge} \measuredangle) \right.$$

$$\text{so//} \quad \phi = V \text{ @ } \varphi = 0 \;\&\; \varphi = \beta; \text{ and// } \phi \text{ finite for all } r \geq 0. \quad (43)$$

The coefficients a_n depend on whatever (finite) values of ϕ are defined at "distant" points. If we assume ϕ(distant) does not make $a_1 \to 0$, then...

$$\rightarrow \boxed{\phi(r,\varphi) \simeq V + a_1 r^{(\pi/\beta)}\sin(\pi\varphi/\beta)}, \text{ for } n = 1 \text{ (dominates as } r \to 0);$$

$$\left[\!\left[\text{so// } (E_r, E_\varphi) = -\left(\frac{\partial}{\partial r}, \frac{1}{r}\frac{\partial}{\partial\varphi}\right)\phi = -(\pi a_1/\beta)\, r^{\frac{\pi}{\beta}-1}\left\{\begin{matrix}\sin\\ \cos\end{matrix}(\pi\varphi/\beta)\right\}, \right. \quad (44)$$

$$\text{and// surface charge: } \sigma(r) = \frac{1}{4\pi}E_\varphi(r, \varphi = 0 \text{ or } \beta) = -(a_1/4\beta)\, r^{(\pi/\beta)-1}.$$

NOTE: for $\beta < \pi$ & $r \to 0$, $\sigma \to 0$; for $\beta = \pi$, σ = cnst; for $\beta > \pi$, $\sigma \to$ large as $r \to 0$, etc.

★ And, as $Jk^{\underline{n}}$ notes on p. 76, one can write a general soln to $\nabla^2\phi = 0$ in 2D polar cds as: $\phi(r,\varphi) = (a_0 + b_0\ln r) + \sum_{n=1}^{\infty}\left[a_n r^n \sin(n\varphi + \alpha_n) + b_n r^{-n}\sin(n\varphi + \beta_n)\right]$.

from J. D. Jackson "Classical Electrodynamics" (Wiley, 2nd ed, 1975)

and the general solution becomes

$$\Phi(\rho, \phi) = V + \sum_{m=1}^{\infty} a_m \rho^{m\pi/\beta} \sin(m\pi\phi/\beta) \tag{2.72}$$

The still undetermined coefficients a_m depend on the potential remote from the corner at $\rho = 0$. Since the series involves positive powers of $\rho^{\pi/\beta}$, for small enough ρ only the first term in the series will be important.* Thus, *near* $\rho = 0$, the potential is approximately

$$\boxed{\Phi(\rho, \phi) \simeq V + a_1 \rho^{\pi/\beta} \sin(\pi\phi/\beta)} \tag{2.73}$$

The electric field components are

$$\left.\begin{aligned} E_\rho(\rho, \phi) &= -\frac{\partial \Phi}{\partial \rho} \simeq -\frac{\pi a_1}{\beta} \rho^{(\pi/\beta)-1} \sin(\pi\phi/\beta) \\ E_\phi(\rho, \phi) &= -\frac{1}{\rho}\frac{\partial \Phi}{\partial \phi} \simeq -\frac{\pi a_1}{\beta} \rho^{(\pi/\beta)-1} \cos(\pi\phi/\beta) \end{aligned}\right\} \tag{2.74}$$

The surface charge densities at $\phi = 0$ and $\phi = \beta$ are equal and are approximately

$$\sigma(\rho) = \frac{E_\phi(\rho, 0)}{4\pi} \simeq -\frac{a_1}{4\beta} \rho^{(\pi/\beta)-1} \tag{2.75}$$

The components of the field and the surface charge density near $\rho = 0$ all vary with distance as $\rho^{(\pi/\beta)-1}$. This dependence on ρ is shown for some special cases in

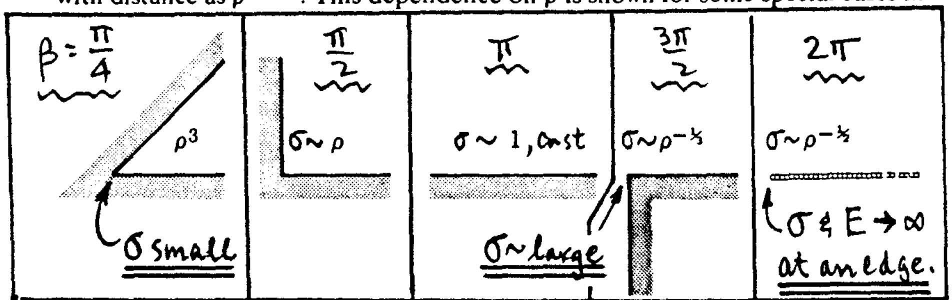


Fig. 2.13 Variation of the surface charge density (and the electric field) with distance ρ from the "corner" or edge for opening angles $\beta = \pi/4$, $\pi/2$, π, $3\pi/2$, and 2π.

Fig. 2.13. For a very deep corner (small β) the power of ρ becomes very large. Essentially no charge accumulates in such a corner. For $\beta = \pi$ (a flat surface), the field quantities become independent of ρ, as is intuitively obvious. When $\beta > \pi$, the two-dimensional corner becomes an edge and the field and the surface charge density become singular as $\rho \to 0$. For $\beta = 2\pi$ (the edge of a thin sheet) the singularity is as $\rho^{-1/2}$. This is still integrable so that the charge within a finite distance from the edge is finite, but it implies that field strengths become very large at the edges of conducting sheets (or, in fact, for any configuration where $\beta > \pi$).

The singular behavior of the fields near sharp edges is the reason for the effectiveness of lightning rods. In the idealized situation discussed here the field strength increases without limit as $\rho \to 0$, but for a thin sheet of thickness d with a smoothly rounded edge it can be inferred that the field strength at the surface will be proportional to $d^{-1/2}$. For small enough d this can be very large. In absolute vacuum such field strengths are possible, but in air electrical breakdown and a discharge will occur if the field strength exceeds a certain value (depending on the exact shape of the electrode, its proximity to the other electrodes, etc., but greater than about 2.5×10^4 volts/cm for air at N.T.P., sometimes by a factor of four). In thunderstorms, with large potential differences between the ground and the thunderclouds, a grounded sharp conducting edge, or better, a point (see Section 3.4), will have breakdown occur around it first and will then provide one end of the jagged conducting path through the air along which the lightning discharge travels.