QM Problems

(35) Consider a QM state described by the wave function

Prob. #(40) ϕ507 (Apr. 92)

$$\Psi_n(x,t) = \left[u_n(x)\, e^{-\frac{i}{\hbar}E_n t}\right] e^{-\frac{1}{2}\Gamma_n t}, \quad t \geqslant 0.$$

This may be interpreted as a quasi-stationary bound state of energy E_n and lifetime $\Delta t_n = 1/\Gamma_n$. By Fourier analysing $\Psi_n(x,t)$ with respect to time, show that the energy spectrum of the decay contains energy components of width $\Delta E_n = \hbar\Gamma_n$ about E_n. What is the most probable energy associated with the decay?

(20pts.) (36) A particle of mass m is bound in a double well δ-fcn potential: $V(x) = -C[\delta(x+a)+\delta(x-a)]$, $C = \text{cnst} > 0$. For such a symmetric potential, we can choose solutions ψ_e & ψ_o which exhibit even & odd parity, resp. Do this. Also, write the bound state energies as $E = -\hbar^2\kappa^2/2m$ (this defines κ).

a) Obtain the transcendental equations (in terms of κ, etc.) which give the two allowed energy levels of the system. Solve them approximately for large a, and compare the solutions with the well-known bound state energy of a single well (viz. $\mathcal{E} = -\frac{1}{2}mC^2/\hbar^2$). What is the energy splitting? Which state (ψ_e or ψ_o) is more tightly bound? What happens for $a \to 0$?

b) Sketch both ψ_e and ψ_o vs. x. What is the physical significance of states described by the linear combinations $\psi_\pm = (\psi_e \pm \psi_o)/\sqrt{2}$?

c) Suppose the system is in state ψ_+ at $t=0$. The wave fcn at some later time is then $\Psi = (\psi_e\, e^{-\frac{i}{\hbar}E_e t} + \psi_o\, e^{-\frac{i}{\hbar}E_o t})/\sqrt{2}$, where E_e and E_o are the eigenenergies. By rewriting Ψ in terms of ψ_+ and ψ_-, show that the system oscillates between the states ψ_+ and ψ_- at a characteristic frequency $\Omega = \frac{1}{2}(E_o - E_e)/\hbar$. Interpret this oscillation physically.

QM Problems

prob. #(13) φ507 (Jan.'93) (37) Shew that the unit step fcn may be represented by

$$\theta(\tau) = \lim_{\epsilon \to 0_+} \frac{i}{2\pi} \int_{-\infty}^{+\infty} (\omega + i\epsilon)^{-1} e^{-i\omega\tau} d\omega = \begin{cases} 1, & \text{for } \tau > 0 \\ 0, & \text{for } \tau < 0 \end{cases}$$

by evaluating an appropriate contour integral in the complex ω-plane. Shew that if $\epsilon \to 0_-$, then the integral generates $\theta(\tau) - 1$, the well known out-of-step fcn. What is the corresponding integral for $\delta(\tau)$?

prob. #(10) φ507 (Jan.'93) (38) Let the Hamiltonian $H = H_0 + V$, where H_0 describes a free particle, and V includes all interactions. Let ξ be the space-time pt. (x,t). Then the S. eqtn is

$$\left(i\hbar \frac{\partial}{\partial t'} - H_0'\right)\psi(\xi') = \hbar \rho(\xi'), \quad \rho(\xi') = \frac{1}{\hbar} V(\xi')\psi(\xi').$$

ρ acts as a source fcn for the otherwise free propagation of ψ. Let G_0 be the free particle propagator (Green's fcn) which satisfies the pt. source eqtn

$$\left(i\hbar \frac{\partial}{\partial t'} - H_0'\right) G_0(\xi', \xi) = \hbar\, \delta(\xi' - \xi).$$

Shew that the general solution to the Schrodinger problem may be written

$$\psi(\xi') = \psi_0(\xi') + \int G_0(\xi', \xi)\rho(\xi)\, d\xi$$

(12)→(9) where ψ_0 is a free particle wave-fcn.

prob. #(12) φ507 (Jan.'93) (39) A free particle in 1D has mean momentum k_0, and is initially localized within $\Delta x \sim \delta$, so that its wavefcn at t=0 is $\psi(x,0) = A e^{ik_0 x} e^{-x^2/2\delta^2}$. By integrating $\psi(x,0)$ over the free propagator K_0, show that at t>0, ψ is given by

$$\psi(x,t) = A(1+i\tau)^{-\frac{1}{2}} e^{i(k_0 x - \omega_0 t)} e^{-(1-i\tau)(x - v_0 t)^2/2\delta^2(1+\tau^2)}, \quad \tau = \frac{\hbar t}{m\delta^2}$$

where $v_0 = \hbar k_0/m$ & $\omega_0 = \hbar k_0^2/2m$. Interpret this result physically.

QM Problems

prob. # 14
ϕ507 (Jan. 93)

(40) Shew that the perturbation series for the propagator G, namely

$$G = G_0 + \int G_0 \Omega G_0 + \iint G_0 \Omega G_0 \Omega G_0 + \cdots$$

(G_0 = free propagator, Ω = interaction) may be formally summed to yield

$$G(\xi',\xi) = G_0(\xi',\xi) + \int d\xi_1\, G_0(\xi',\xi_1)\,\Omega(\xi_1)\,G(\xi_1,\xi).$$

What does this correspond to physically (in terms of ψ propagation)? Use this result to verify the Lippmann-Schwinger eqtn. (also proved in problem (38)):

$$\psi(\xi') = \psi_0(\xi') + \int d\xi_1\, G_0(\xi',\xi_1)\,\Omega(\xi_1)\,\psi(\xi_1).$$

20 pts

prob. # 15
ϕ507 (Jan. 93)

(41) a) For a bound state problem, the propagator G may be represented by

$$G(\vec{x}',t';\vec{x},t) = -i\,\theta(t'-t)\sum_n u_n^*(\vec{x})\,u_n(\vec{x}')\,e^{-i\omega_n(t'-t)}, \quad \omega_n = E_n/\hbar.$$

Using the integral form for θ (prob. (37)), shew G can be Fourier analysed as

$4\frac{1}{2} \rightarrow$
$$G(\vec{x}',t';\vec{x},t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} G_\omega(\vec{x}',\vec{x})\,e^{-i\omega(t'-t)}\,d\omega, \quad G_\omega(\vec{x}',x) = \sum_n \frac{u_n^*(\vec{x})\,u_n(\vec{x}')}{(\omega-\omega_n)+i\epsilon}.$$

b) G_ω, with $\lim_{\epsilon\to 0+}$ understood, is known as the stationary propagator for energy $E=\hbar\omega$. With H the system Hamiltonian, shew G_ω obeys the pt. source eqtn: $(E-H')G_\omega(\vec{x}',\vec{x}) = \hbar\delta(\vec{x}'-\vec{x})$. Use this to shew that the Lippmann-Schwinger eqtn for a stationary state at energy E, with time-indpt interaction Ω, is

$$\psi_E(\vec{x}') = \varphi_E(\vec{x}') + \int d^3x\, G_{0\omega}(\vec{x}',\vec{x})\,\Omega(\vec{x})\,\psi_E(\vec{x}),$$

$9\frac{1}{2} \rightarrow$ with φ_E a free particle wfcn, and $G_{0\omega}(\vec{x}',\vec{x})$ the free stationary propagator.

c) By evaluating the sum (and appropriate contour integrals), derive an expression for $G_{0\omega}(x',x)$, the free stationary propagator in 1D. Compare your

$12\frac{1}{2} \rightarrow$ answer with that in P.B. James, Am. J. Phys. 38, 1319 (Nov. 1970).

QM Problems

(42) In our derivation of $S^{(n)}_{\beta\alpha}$, the n^{th} order scattering term in the S-matrix, we used the relations for a free-particle wfcn ϕ_β and propagator G_0

Prob. #(16) ϕ507 (Jan.'93)

$$\phi_\beta(\xi') = i\int dx\, G_0(\xi',\xi)\,\phi_\beta(\xi)\,, \quad \phi^*_\beta(\xi') = i\int dx\, \phi^*_\beta(\xi)\,G_0(\xi,\xi')\,,$$

where $(\xi)=(x,t)$ as usual. The first relation is "true" by definition. "Prove" the second. (Hint: note that for a free particle: $\phi^*_\beta(\xi)=\phi_\beta(-\xi)$. Is this always true? Write the propagation eqtn for $\phi_\beta(-\xi)$, and then look at the behavior of G_0 under a space-time reflection: $\xi\to-\xi$, $\xi'\to-\xi'$).

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(43) Show that in a QM state ψ with a well-defined value of the z-component of $\measuredangle$ momentum, L_z, i.e. $L_z\psi = m\hbar\psi$, the expectation values of the x and y components, L_x and L_y, are identically zero (Hint: use the commutation relations and the Hermitianity of L_z).

(44) In a given QM system, the energy level E is doubly degenerate, i.e. there are two orthonormal eigenfcns ψ_i such that $H\psi_i = E\psi_i$. Suppose there is an operator Q which commutes with H, i.e. $[H,Q]=0$, and which has the following matrix elements w.r.t. eigenfcns ψ_i

$$Q_{11} = \langle\psi_1|Q|\psi_1\rangle\text{, real} \qquad Q_{12} = \langle\psi_1|Q|\psi_2\rangle = |Q_{21}|e^{-i\theta}$$

$$Q_{22} = \langle\psi_2|Q|\psi_2\rangle \equiv Q_{11} \qquad Q_{21} = \langle\psi_2|Q|\psi_1\rangle \equiv Q^*_{12}$$

By taking suitable linear combinations of the ψ_i, explicitly construct wfcns ϕ_1 & ϕ_2 which are simultaneously eigenfcns of H and Q, and show that the ϕ_j are orthonormal. What are the eigenvalues of Q? Note: H and Q are a "complete set of commuting observables" here, as they completely specify the quantum numbers E, q_j associated with the by now distinct ϕ_j.

QM Problems

(45) Suppose, in order to take into account the past history of a state, the exponential decay law were modified from $dP(t)/dt = -\Gamma_0 P(t)$ to

$$\frac{d}{dt}P(t) = -\Gamma_0 \frac{1}{\tau}\int_{t-\tau}^{t} P(x)\,dx \quad , \quad \tau = \text{cnst}$$

The integral represents an average of $P(t)$ over a time interval τ just prior to time t. Γ_0 is the "natural" decay rate, i.e. the decay rate for $\tau = 0$.

a) Show that the original law is recovered in the limit $\tau \to 0$.

b) Show that the above law is non-local in time by converting it to a second-order differential eqtn. So what?

c) Assume a solution of the form $P(t) = P_0 e^{-\Gamma t}$. Derive the (transcendental) eqtn which relates Γ to Γ_0 & τ. Show that for suitably small τ, $\Gamma > \Gamma_0$, but that there is a critical τ_c such that $\tau > \tau_c \Rightarrow$ no solution for Γ (i.e. decay is no longer exponential). Estimate τ_c in terms of Γ_0.

d) What is the nature of the $P(t)$ solution for $\tau > \tau_c$? ($64 question!)

(46) If $\vec{r}$ & $\vec{p}$ do <u>not</u> commute, show that the QM ∡ momentum operator is

$$\vec{L}^2 = (\vec{r}\times\vec{p})^2 = r^2\vec{p}^2 + \hbar^2 \frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\right),$$

rather than the simple "classical" guess: $\vec{L}^2 = r^2\vec{p}^2 + \hbar^2\left(r\frac{\partial}{\partial r}\right)^2$

(47) Redo problem (33) (α-decay) for ∡ momentum $\neq 0$. Assume low energies ($E \ll V(r_0)$), and a relatively weak centrifugal barrier: $B(r) = \ell(\ell+1)\hbar^2/2mr^2 \ll V(r)$, $r > r_0$. Show that for $\ell \neq 0$, the transmission coefficient $T = e^{-\gamma}$ is specified by

$$\gamma \simeq \frac{2\pi C}{\hbar v}\left[1 - \frac{4}{\pi}\sqrt{\frac{E}{V(r_0)}}\left(1 - \frac{1}{2}\sigma\right)\right], \quad \sigma = B(r_0)/V(r_0) \ll 1, \quad v = \sqrt{2E/m}.$$

Calculate numbers, including a lifetime, for: $U^{238} \to Th^{234} + \alpha$, $E = 4.25$ MeV.

QM Problems

(48) a) By considering a finite rotation $\vec{\alpha}$ to be made up of a very large number of ∞smal rotations $\delta\vec{\alpha}$ performed in succession, shew that the rotation operator $R(\vec{\alpha})$ which takes a fcn $F(\vec{r})$ into $F(\vec{r}\,') = R(\vec{\alpha})F(\vec{r})$, upon rotation of the coordinate system by $\measuredangle\,\vec{\alpha}$, is: $R(\vec{\alpha}) = \exp(-\frac{i}{\hbar}\vec{\alpha}\cdot\vec{L})$, where $\vec{L}$ is the QM $\measuredangle$ momentum operator.

b) By the same token, shew that for a finite translation of the coordinate system by $\vec{\epsilon}$ (i.e. $\vec{r} \to \vec{r}\,' = \vec{r} - \vec{\epsilon}$), the appropriate translation operator is $T(\vec{\epsilon}) = \exp(-\frac{i}{\hbar}\vec{\epsilon}\cdot\vec{p})$, where $\vec{p}$ is the QM linear momentum operator.

#⟨2⟩ ϕ507('94) Final (5/12/94)

(49) With regard to $\measuredangle$ momentum $\vec{L}$, make the following assumptions:
(1) Space is isotropic, i.e. the x, y, and z axes are all equivalent.
(2) The possible values of any one component of $\vec{L}$ are $m\hbar$, where m ranges over the $2l+1$ values $-l, -l+1, \ldots 0, \ldots, +l$ (l an integer).
(3) All m-values occur with equal a priori probability.
From these, shew that the average value of $\vec{L}^2$ must be $\overline{L^2} = l(l+1)\hbar^2$.
(Hint: (1) $\Rightarrow$ $\overline{L^2} = 3\overline{L_z^2} = 3\overline{m^2}\hbar^2$. Now calculate $\overline{m^2}$ from (2) & (3).)

(50) Consider the set of fcns $u_n(x) = x^n$, where $n = 0, 1, 2, 3, \ldots$, and the domain of definition is restricted to the interval $-1 \leq x \leq +1$. Clearly the $u_n(x)$ are not orthogonal (or normalized) over this interval. They may be orthogonalized by the Schmidt procedure (with a judicious choice of sign): $\{u_n(x)\} \to \{v_n(x)\}$, orthogonal. Shew that, up to normalization constants, the $v_n(x)$ are just the Legendre polynomials $P_n^0(x)$, for the cases $n=0$ to 3. Can you prove $v_n(x) \propto P_n^0(x)$ generally? In any case, the $P_n^0(x)$ are sometimes called the fundamental set of orthogonal polynomials on $-1 \leq x \leq +1$.

QM Problems

20 pts. (51) In a space where $\vec{J}$ is the $\measuredangle$ momentum operator, so that a finite rotation of the cd. system by $\measuredangle\ \vec{\alpha}$ is represented by rotation operator $R(\vec{\alpha}) = \exp(-i\vec{\alpha}\cdot\vec{J})$ (problem (48)), start out with a vector $\vec{A}$ along the x-axis. A rotation by a small but finite $\measuredangle\ \Delta\alpha_y$ around the y-axis, then by $\Delta\alpha_x$ around the x-axis, is represented by $R(\Delta\alpha_x)\,R(\Delta\alpha_y)$. By looking at the final relative position of $\vec{A}$, show that this operation is equivalent to the sequence $R(\Delta\alpha_z)\,R(\Delta\alpha_y)\,R(\Delta\alpha_x)$, where -- to 2^{ND} order in small quantities, $\Delta\alpha_z \simeq \Delta\alpha_x\,\Delta\alpha_y$. That is, establish the equality

$$R(\Delta\alpha_x)R(\Delta\alpha_y) = R(\Delta\alpha_z)R(\Delta\alpha_y)R(\Delta\alpha_x);\ \Delta\alpha_z = \Delta\alpha_x\Delta\alpha_y,\ \text{to } \mathcal{O}(\Delta\alpha)^2.$$

By plugging in the exp forms for the R's, and expanding them <u>individually</u> to $\mathcal{O}(\Delta\alpha)^2$ (note: $e^P e^Q \neq e^{P+Q}$ for $[P,Q]\neq 0$; see Merzbacher, p. 167), establish the commutation rule $[J_x, J_y] = iJ_z$.

(52) Using the relations for the creation and annihilation operators $J_\pm$, obtain the most general matrix elements of J_x and J_y. That is, calculate the quantities $\langle j'm'|\,J_x \text{ and } J_y\,|jm\rangle$, with appropriate "selection rules" on j and m.

Prob. #(6) ϕ507 (Jan. 1992)

(53) Consider the $\measuredangle$ momentum operator $\vec{S}$ for a spin $\frac{1}{2}$ particle. If we define $\vec{\sigma} = (2/\hbar)\vec{S}$, then $\vec{\sigma}$ obeys $\vec{\sigma}\times\vec{\sigma} = 2i\vec{\sigma}$. The components of $\vec{\sigma}$ can be represented by 2×2 matrices. If S_z is chosen to have eigenvalues $\pm\frac{\hbar}{2}$, then $\sigma_z = \begin{pmatrix} +1 & 0 \\ 0 & -1 \end{pmatrix}$. Using the commutation relations, and this choice for σ_z, derive the matrices for σ_x and σ_y (choosing the "simplest" rep$^{\underline{n}}$). Show that $\vec{S}^2$ is a diagonal matrix, with eigenvalue $\frac{3}{4}\hbar^2$. Note: the σ_i here are called the Pauli matrices for spin $\frac{1}{2}$.

QM Problems

Prob. #① φ507 (Jan. 1992)

(54) a) By direct substitution, shew that a series solution to the differential eqtn

$$\left[x\frac{d^2}{dx^2}+(b-x)\frac{d}{dx}-a\right]F(x)=0 \quad \begin{cases} a \,\&\, b = \text{cnsts} \\ b \neq 0 \text{ or } (-)\text{ve integer} \end{cases}$$

is $F(x) = {}_1F_1(a,b;x) = \sum_{k=0}^{\infty}[(a)_k/(b)_k](x^k/k!)$, $(a)_k = a(a+1)\cdots(a+k-1)$.

b) Suppose $x\to+\infty$. By examining the leading terms in the series for ${}_1F_1$, Shew (using "appropriate" approximations for the Γ-fcns) that for k "large", the k^{th} term is $\sim[\Gamma(b)/\Gamma(a)]\,x^k/(k-(a-b))!$ Argue from this that for large +ve x, ${}_1F_1(a,b;+x)\sim[\Gamma(b)/\Gamma(a)]\,x^{a-b}e^x$. Next, using the Kummer transform, Shew ${}_1F_1(a,b;-x)\sim[\Gamma(b)/\Gamma(b-a)]\,x^{-a}$ for large (−)ve x.

Prob. #④ φ507 (Jan. 1992)

(55) Consider a central potential of the form $V(r)=\frac{A}{r^2}-\frac{B}{r}$, A & B cnsts. Draw V(r). What physical situation might be represented by this interaction? Write the radial eqtn in dimensionless variables (N.B. "atomic units" for this problem are: length $a=\hbar^2/mB$, energy $\mathcal{E}=B/a$). Solve for the radial wavefcn $u(\rho)$, and shew the bound state energies are $E_{n\ell}=-\frac{1}{2}\mathcal{E}/(n+\Delta_\ell)^2$, where -- as for the H atom -- the principal quantum # $n=N+\ell+1$, with $N=0,+1,+2,\ldots$. Give an exact expression for Δ_ℓ. For a given n, which ℓ state is most tightly bound? Approximate $E_{n\ell}$ for $mA/\hbar^2\ll1$, and draw the energy spectrum, including both n & ℓ dependence.

(56) Consider the radial eqtn for the 3D isotropic oscillator, with $V(r)=\frac{1}{2}m\omega^2r^2$ ("atomic units" are: $a=\sqrt{\hbar/m\omega}$, $\mathcal{E}=\hbar\omega$). Extract the asymptotic behavior of $u(\rho)$ as $\rho=r/a\to0$ & ∞, and convert the u eqtn to a confluent hypergeometric eqtn by a change of variables, $z=\rho^2$. Solve this, and shew the bound state energies are $E_\Lambda=(\Lambda+\frac{3}{2})\mathcal{E}$, where $\Lambda=2N+\ell$, $N=0,1,2,\ldots$. What states (N,ℓ) are possible for $\Lambda=0$ to 3? What is the parity & degeneracy of level E_Λ?

QM Problems

(57) The H-atom eigenfcns are $|nlm\rangle = R_{nl}(r)\,Y_{lm}(\vartheta,\varphi)$. Give explicit expressions, including normalization cnsts, for all R_{nl} from $(n,l)=(1,0)$ to $(3,2)$, and for all Y_{lm} from $(l,m)=(0,0)$ to $(2,\pm 2)$. Put your results in tabular form! Write the R_{nl} as $\propto$ polynomials in $x=(2Z/na_0)r$, and the Y_{lm} as $\propto$ expressions in $\cos\vartheta$ & $\sin\vartheta$. What are $|2,1,\pm1\rangle$ and $|3,0,0\rangle$ in terms of r,ϑ,φ?

(58) a) Apply the variational method to the ground state of a particle moving in an attractive Yukawa (or screened Coulomb) potential: $V(r) = -V_0 \frac{a}{r} e^{-r/a}$, where V_0 & a are (+)ve cnsts. Use $R(r) = e^{-\beta(r/a)}$ as a trial wavefcn, with β the variation parameter. By minimizing the energy $\langle E\rangle$, establish a relation between β and the cnst $\gamma^2 = 2mV_0a^2/\hbar^2$. Derive an expression for $\langle E\rangle_{min}$ which depends on this "best" value of β alone.
b) Use these results to describe the binding of a deuteron (n-p system). Assume the "range" of the Yukawa potential is $a = 1.40\times10^{-13}$ cm (pion Compton wavelength). For what value of V_0 (in MeV) is $\langle E\rangle_{min} = -2.226$ MeV, which is the observed deuteron binding energy? With these parameters, what is the rms radius (i.e. $\langle r^2\rangle^{\frac{1}{2}}$) of the deuteron?
c) Show that in the limit $a\to\infty$, but $V_0 a \to$ finite, the correct energy and wavefcn are obtained for the ground state of the Coulomb potential.

(59) Consider the H-atom problem (i.e. Coulomb potential: $V(r) = -Ze^2/r$) for (+)ve energy $E \geq 0$. This relates to e-p scattering, or ionization, or (with $Z\to -Z$) to an $\bar{e}$-p interaction. By examining the asymptotic behavior of the fcn which goes like e^{-ikr} as $r\to 0$, show that the "normalized" radial wavefcn as $r\to\infty$ is

$$R_{kl}(r) \simeq \sqrt{2/\pi}\,\frac{1}{r}\sin\left[\left(kr - \frac{l\pi}{2}\right) + (Z/ka_0)\ln 2kr + \delta_l(k)\right],$$

where: $k = \sqrt{2mE/\hbar^2}$, $a_0 = \hbar^2/me^2$, $\delta_l(k) = \arg\Gamma(l+1-(iZ/ka_0))$.

QM Problems

(60) a) Starting from Bessel's eqtn: $[x^2 \frac{d^2}{dx^2} + x\frac{d}{dx} + (x^2-\nu^2)]\,J_\nu(x) = 0$, show that the solutions are related to the confluent hypergeometric fcn ${}_1F_1$ by: $J_\nu(x) = [(x/2)^\nu/\Gamma(\nu+1)]\,e^{-ix}\,{}_1F_1(\nu+\frac{1}{2}, 2\nu+1; 2ix)$. What are the asymptotic forms for $J_\nu(x)$ as $x\to 0$ and $x\to\infty$?

b) For a free particle (mass m, energy E) in 3D sph. polar cds, the radial eigenfcns are $R_{k\ell}(r) = A_k\, j_\ell(kr)$, where $k=\sqrt{2mE/\hbar^2}$, j_ℓ is the sph. Bessel fcn of order ℓ, and A_k is a norm. cnst. Show that the choice $A_k = \sqrt{2/\pi}\,k$ normalizes $R_{k\ell}(r)$ so that $\int_0^\infty R^*_{k'\ell}(r)\,R_{k\ell}(r)\,r^2dr = \delta(k'-k)$. (Hint: use the asymptotic form for j_ℓ as $r\to\infty$). If this integral were normalized to $\delta(E'-E)$ instead, what would the norm. cnst be?

20 pts. (61) Given: an $N\times N$ Hermitian matrix $\underset{\sim}{H} = (H_{k\ell})$. The problem is to diagonalize $\underset{\sim}{H}$ in principle, i.e. find a matrix $\underset{\sim}{U} \ni$ by a similarity transform (S.T.), $\underset{\sim}{U}\underset{\sim}{H}\underset{\sim}{U}^\dagger = \underset{\sim}{H}'$ is diagonal. To this end, consider the eigenvalue eqtn $\underset{\sim}{H}\vec{u}_k = E_k\vec{u}_k$, $k = 1$ to N.

a) Write the "secular eqtn", which gives the eigenvalues E_k from the $H_{k\ell}$. Assume zero degeneracy, i.e. all E_k distinct. Show the E_k are real, and the eigenvectors $\vec{u}_k$ are orthogonal (Note: the vector product here is $\vec{A}^\dagger\cdot\vec{B} = \sum_i A_i^* B_i$).

b) Let the $\vec{u}_k$ be normalized to unit vectors $\hat{u}_k$ (why is this possible?). Let $\underset{\sim}{U}$ be a matrix with the $\hat{u}_k^\dagger$ as rows, i.e. $U_{k\ell} = u^*_{k\ell}$ ($u_{k\ell} = \ell^{\text{th}}$ comp. of $\hat{u}_k$). Show $\underset{\sim}{U}$ is unitary. Next, show $\underset{\sim}{U}\underset{\sim}{H}\underset{\sim}{U}^\dagger = \underset{\sim}{H}'$ is diagonal, so $\underset{\sim}{U}$ is the desired diagonalization matrix. If the S.T. were written $\underset{\sim}{V}^\dagger\underset{\sim}{H}\underset{\sim}{V} = \underset{\sim}{H}'$, what would $\underset{\sim}{V}$ be?

c) Find $\underset{\sim}{H}^{-1}$ in terms of the E_k and $u_{k\ell}$ (write an explicit form for $H^{-1}_{k\ell}$). What if some $E_k = 0$? Use the result to solve the eqtn $\underset{\sim}{H}\vec{A} = \vec{B}$ for $\vec{A}$ in terms of $\vec{B}$.

d) Apply a S.T. by $\underset{\sim}{U}$ to the eqtn $\underset{\sim}{H}\vec{A} = \vec{B}$. How are the comps. of $\vec{A}' = \underset{\sim}{U}\vec{A}$ and $\vec{B}' = \underset{\sim}{U}\vec{B}$ related? What is the effect of a S.T. by $\underset{\sim}{U}$ on the eigenvalue eqtn? What does $\underset{\sim}{U}$ do to the vectors, and the cd. system? What does $\underset{\sim}{V}$ do?

QM Problems

(62) a) A representation of the Dirac delta-fcn may be made in terms of a suitably normalized Gaussian, $\delta(x) \propto e^{-x^2/\sigma^2}$, in the limit that the width $\sigma \to 0$. Find the Gaussian representation for $\delta(x-x')$.

b) Consider the Hamiltonian operator: $H(x) = -(\hbar^2/2m)\, d^2/dx^2 + V(x)$. In the co-ordinate representation, this generates a matrix of elements $H_{xx'} = H(x)\,\delta(x-x')$. Use the above form for $\delta(x-x')$ to explicitly calculate $H_{xx'}$, and draw a "picture" of this matrix (e.g. looking up the diagonal). Is the matrix "diagonal"? What happens as $\sigma \to 0$?

(63) Let A, B, and C be QM operators, and consider the eqtn $f(A,B)=C$, with f an arbitrary bilinear form. Shew that the same eqtn holds for the matrix rep$^{\underline{n}}$s of these operators with respect to a set of basis fcns $\{\phi_i\}$. That is, with $\underset{\sim}{A}$ a matrix of components $A_{mn} = \int dx\, \phi_m^* A\, \phi_n$, and similarly for $\underset{\sim}{B}$ and $\underset{\sim}{C}$, Shew that $f(\underset{\sim}{A}, \underset{\sim}{B}) = \underset{\sim}{C}$.

(64) With $\underset{\sim}{H}''$ a matrix rep$^{\underline{n}}$ of the operator $H(x)$ on the basis $\{v_k(x)\}$, the matrix $\underset{\sim}{W}$ which diagonalizes $\underset{\sim}{H}''$ (via a similarity transf$^{\underline{n}}$) has entries $W_{k\lambda} = \int dx\, u_k^*(x)\, v_\lambda(x)$, where the $\{u_k(x)\}$ are the eigenfcns of $H(x)$. By considering the integrals involved, shew explicitly that $\underset{\sim}{W}$ is unitary, and that $\underset{\sim}{H}' = \underset{\sim}{W}\,\underset{\sim}{H}''\,\underset{\sim}{W}^\dagger$ is diagonal. Next, transform the eigenvalue eqtn $\underset{\sim}{H}''\vec{a}_k = E_k\vec{a}_k$ by $\underset{\sim}{W}$, to $\underset{\sim}{H}'\vec{a}_k' = E_k\vec{a}_k'$, and shew that the $\vec{a}_k'$ are single component unit vectors.

(65) Try converting the time-dependent S. eqtn, $\mathcal{H}\psi = i\hbar\,\partial\psi/\partial t$, to a matrix problem. Suppose the Hamiltonian can be written $\mathcal{H}(x,p,t) = H(x,p) + V(x,t)$, and that you have at your disposal the basis set $\{u_n(x)\}$ of eigenfcns of H, the time-independent part of $\mathcal{H}$. Can you diagonalize anything relevant?

QM Problems

\# ⑨ (66) Given a set of basis fcns $\{u_k(x)\}$, and general QM operators A and B,
ϕ507 establish the matrix eqtn: $\langle k|AB|l\rangle = \sum_m \langle k|A|m\rangle\langle m|B|l\rangle$ directly,
(1992) by looking at the integrals $\langle k|A|m\rangle = \int dx\, u_k^*(x)\, A\, u_m(x)$, etc.

(67) a) For an operator Ω with eigenvalues ω_k, show that the projection operator for the k^{th} eigenstate is $P_k = \prod_{j\neq k}(\Omega-\omega_j)/(\omega_k-\omega_j)$. What is the easiest way, in this case, to verify the conditions $\sum_k P_k = 1$, $P_k P_l = \delta_{kl} P_k$?

b) Define the parity operator $\mathcal{P}$ by its effect on an arbitrary wavefcn $\psi(x)$, namely $\mathcal{P}\psi(x) = \psi(-x)$. Show that $\mathcal{P}$ is Hermitian, with eigenvalues ± 1. Find the corresponding projection operators $P_\pm$, and use them to show that $\psi(x)$ can be written $\psi_+(x) + \psi_-(x)$, where ψ_+ & ψ_- resp. have even & odd parity.

(68) Consider a two-state QM system, with Hamiltonian $\mathcal{H}(x) = H(x) + V(x)$. Let
Prob. # (35) H generate eigenfcns $\langle x|k\rangle = \phi_k(x)$ with eigenenergies E_k, and consider V
ϕ507 to be a "perturbation" on H. The problem is to find the "perturbed" eigenfcns
(1992) $\varphi_\mu(x)$ and eigenenergies $\mathcal{E}_\mu$ of $\mathcal{H}$ by using matrix methods (w.r.t. basis $\{\phi_k\}$).

a) Representing $\mathcal{H}$ as a matrix, and the φ_μ as vectors in the k-repⁿ, write the eigenvalue problem for $\mathcal{H}$ as a matrix eqtn. Find the $\mathcal{E}_\mu$ exactly in terms of the matrix elements $\mathcal{H}_{kl} = \langle k|\mathcal{H}|l\rangle$, and show they can be written

$$\mathcal{E}_1 = E_1' + \Delta,\quad \mathcal{E}_2 = E_2' - \Delta \quad \begin{cases} E_k' = E_k + V_{kk}, \quad \Delta = \frac{1}{2}(Q-1)(E_1'-E_2'), \\ Q = \left[1 + \left(2|V_{12}|/(E_1'-E_2')\right)^2\right]^{1/2}. \end{cases}$$

How do the $\mathcal{E}_\mu$ behave in the case of V "small" and "large"? What happens in the case of **degeneracy** (i.e. $E_1 = E_2$), with $V_{kk} = 0$, but $V_{12} \neq 0$?

b) Solve for the eigenvectors $\vec{a}_\mu = \begin{pmatrix} a_{\mu 1} \\ a_{\mu 2}\end{pmatrix}$ of $\mathcal{H}$, and find the φ_μ in terms of the ϕ_k. To 1st order terms in V, what are the $\vec{a}_\mu$ as $V \to$ "small"? Why is V_{12} called a "coupling term"? What happens in the above degenerate case?

QM Problems

20 pts. (69) In problem (68), suppose the initial state separation is $\Delta E = E_1 - E_2 > 0$, and the V matrix elements are $V_{11} = -\alpha U$, $V_{22} = +\beta U$, $V_{12} = \gamma U$. Here α, β & γ are +ve cnsts, and U is an external field strength parameter which varies from 0 to ∞. Note that if the coupling γ were zero, the state energy levels $\mathcal{E}_\mu$ would cross one another at $U = U_0 = \Delta E/(\alpha+\beta)$; U_0 is called a "crossing point".

~~Prob. # (36)~~

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(Mar. 1992)

a) How do the energy levels behave for $U \ll U_0$? What happens as $U \to U_0$? What is the level separation at U_0? For $\gamma \neq 0$, can the levels ever cross over?

b) What are the levels for $U \gg U_0$? Which level belongs to which state?

c) Draw a diagram clearly indicating both the low field ($U \ll U_0$) and high field ($U \gg U_0$) behavior of the levels as a fcn of U. Find the slopes $\partial\mathcal{E}_\mu/\partial U$ at high field in terms of the parameter $Q_\infty = \left[1 + (2\gamma/(\alpha+\beta))^2\right]^{\frac{1}{2}}$.

d) What is the approximate distance of closest approach of the levels in terms of V_{12}? This is called the Wigner-Von Neumann Xing Pt. Theorem.

20 pts (70) Consider an atom where the total ∡ momentum $\vec{L}$ and spin $\vec{S}$ combine to form a resultant $\vec{J} = \vec{L} + \vec{S}$. The associated magnetic moments $\vec{\mu}_L = -g_L \mu_0 \vec{L}$ and $\vec{\mu}_S = -g_S \mu_0 \vec{S}$ will couple to form the total magnetic moment $\vec{\mu}_J = \vec{\mu}_L + \vec{\mu}_S$. Using the vector model (i.e. averaging over the mutual precession), show that in an expectation value sense, we can write $\vec{\mu}_J = -g_J \mu_0 \vec{J}$, where g_J -- the Lande g-factor -- is given by

~~# (12)~~

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(1992)

$$g_J = \left[\frac{j(j+1) + \ell(\ell+1) - s(s+1)}{2j(j+1)}\right] g_L + \left[\frac{j(j+1) + s(s+1) - \ell(\ell+1)}{2j(j+1)}\right] g_S$$

Here j, ℓ, s are the $\vec{J}, \vec{L}, \vec{S}$ quantum numbers. Calculate g_J-values for the states $2P_{J=3/2}$, $2P_{J=1/2}$, $2S_{J=1/2}$ in the hydrogen atom. What is the maximum observable μ_J in each state? Suppose you applied an external magnetic field $\vec{H}$. How would these states behave as a fcn of H?

QM Problems

#(14) φ507 (1992)

(71) a) The magnetic moment of a spin $\frac{1}{2}$ particle may be written $\vec{\mu} = -\mu_0 \vec{\sigma}$, where $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ are the Pauli matrices of problem (53). Suppose the particle is placed in an external magnetic field $\vec{H} = (H_x, H_y, H_z)$. Using the standard repⁿ of $\vec{\sigma}$ (e.g. Merzbacher, p. 270), write out the Hamiltonian $\mathcal{H}$ for the system in matrix form. What are the allowed eigenenergies for the particle?

b) With $\mathcal{H}$ a 2×2 matrix, the system eigenstates are two-component "spinors" $\binom{a}{b}$. Assume that $\vec{H}$ is given in spherical polar coordinates: $\vec{H} = H(\sin\vartheta\cos\varphi, \sin\vartheta\sin\varphi, \cos\vartheta)$. Find the eigenspinors corresponding to the eigenenergies calculated above. What are they if $\vec{H}$ is along the z-axis, i.e. $\vartheta = \varphi = 0$?

20 pts. #(11) φ507 (1992)

(72) Consider a single-electron atom, where $\vec{L}$ & $\vec{S}$ couple to form $\vec{J} = \vec{L} + \vec{S}$, with eigenvalues $j = l \pm \frac{1}{2}$. To go from the uncoupled to coupled repⁿ, note that with $m_s = \pm\frac{1}{2}$ only, there are only two m_l values for a given m_j, viz. $m_l = m_j \mp \frac{1}{2}$. If we let $\alpha = |s=\frac{1}{2}, m_s = +\frac{1}{2}\rangle$ & $\beta = |s=\frac{1}{2}, m_s = -\frac{1}{2}\rangle$ represent the spin-up & spin-down eigenfcns, then the coupled repⁿ has eigenfcns with only two terms, i.e.

$$|n; l, s, j, m_j\rangle = C_1(j)\,|n; l, m_l = m_j - \tfrac{1}{2}\rangle\,\alpha + C_2(j)\,|n; l, m_l = m_j + \tfrac{1}{2}\rangle\,\beta .$$

The Clebsch-Gordon transfⁿ here thus amounts to finding just two pairs of cnsts, one for each of $j = l \pm \frac{1}{2}$. By using the J^- operator, calculate the $C_i(j)$. Explicitly calculate the $2P_{3/2}$ & $2P_{1/2}$ eigenfcns of hydrogen in terms of 2P eigenfcns and α & β.

20 pts. #(39) φ507 (Apr '92)

(73) In problem (68), let $V_{11} = V_{22} = 0$, but $V_{12} = V \neq 0$. A general state of the system is $\psi(x,t) = \sum_k a_k(t)\,\phi_k(x)\,e^{-i\omega_k t}$, $\omega_k = E_k/\hbar$. Use the time-dpt. S. eqtn to derive diff. eqtns for the $a_k(t)$, and solve them with the boundary conditions $a_1(0) = 1$, $a_2(0) = 0$ (i.e. system is in state #1 at $t=0$). What is the probability that a transition to state #2 occurs at $t > 0$? Interpret your results physically.

QM Problems

(74) a) A solution to the Dirac eqtn for a Coulomb potential $V(r) = -Ze^2/r$ gives

$$\mathcal{E}_{nj} = mc^2\left(\left[1+\left(\frac{Z\alpha}{N+\gamma}\right)^2\right]^{-\frac{1}{2}} - 1\right) \begin{cases} \gamma = \sqrt{(j+\frac{1}{2})^2 - (Z\alpha)^2}\,;\ j = \text{total } \measuredangle \text{ mom. q \#,} \\ N = 0,1,2,\ldots;\ \text{principal q \#: } n = N + j + \frac{1}{2}. \end{cases}$$

for the allowed eigenenergies. Expand $\mathcal{E}_{nj}$ to $O(Z\alpha)^4$ and compare it with the calculation from the S. eqtn, including spin-orbit and relativistic corrections.
b) Consider the fine structure of the $2P_j$ levels of atomic hydrogen. Calculate the splitting in frequency units (i.e. $\Delta\nu = \Delta\epsilon/h$) and compare it with the number measured by Lamb et al. in Phys. Rev. <u>89</u>, 106 (1953).

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20 pts. (75) The Klein-Gordon (KG) eqtn for a particle of mass m in a time-indpt potential V is "derived" by letting E and $\vec{p}$ be the usual QM operators in the relativistic mass-energy relation: $(E-V)^2 = (\vec{p}c)^2 + (mc^2)^2$. Here E is the <u>total</u> particle energy, including rest energy. The resulting operator eqtn is (as usual) to be multiplied on the right by a wavefcn ψ.

done in fs lectures (pp. fs 14-19) <u>φ507 (1992)</u>

a) For a Coulomb potential, $V(r) = -Ze^2/r$, show that the KG radial eqtn is

$$\left\{\frac{d^2}{dr^2} + \left[\frac{E^2-(mc^2)^2}{(\hbar c)^2} + \left(\frac{2EZ\alpha}{\hbar c}\right)\frac{1}{r} - \frac{\ell(\ell+1)-(Z\alpha)^2}{r^2}\right]\right\} u(r) = 0$$

assuming the separation $\psi(\vec{r},t) = \frac{1}{r}u(r)\,Y_{\ell m}(\theta,\varphi)\,e^{-\frac{i}{\hbar}Et}$. Let $E = mc^2 + \mathcal{E}$, and show that for $c\to\infty$, this reduces to the S. radial eqtn for energy $\mathcal{E}$ in a Coulomb potential.
b) Assume bound states ($\mathcal{E}<0$), and show that the allowed eigenenergies are

$$\mathcal{E}_{n\ell} = mc^2\left(\left[1+\left(\frac{Z\alpha}{N+\frac{1}{2}+\delta}\right)^2\right]^{-\frac{1}{2}} - 1\right) \begin{cases} \delta = \sqrt{(\ell+\frac{1}{2})^2 - (Z\alpha)^2}\,;\ \ell = \text{orbital } \measuredangle \text{ mom q.\#,} \\ N = 0,1,2,\ldots;\ \text{principal q \#: } n = N + \ell + 1. \end{cases}$$

for the state $|n\ell m\rangle$. (<u>Note</u>: the arithmetic here is ~ that of problem (55)).
c) Expand $\mathcal{E}_{n\ell}$ to $O(Z\alpha)^4$ and compare with the Dirac result. Draw the allowed energy levels for $n=2$, labelling each with the proper q.#'s. What is the predicted "fine structure" as compared with the Dirac result?

QM Problems

#15 ϕ507 (1992)

(76) The dipole-dipole interaction can be written $V(1,2) = g_1 g_2 (\mu_0^2/r_{12}^3)\Sigma_{12}$, where $\Sigma_{12} = \vec{S}_1 \cdot \vec{S}_2 - 3(\vec{S}_1 \cdot \hat{r}_{12})(\vec{S}_2 \cdot \hat{r}_{12})$ in the usual notation. Suppose both particles have spin $\frac{1}{2}$. Show that $\Sigma_{12} = \frac{1}{2}[\vec{S}^2 - 3(\vec{S}\cdot\hat{r}_{12})^2]$, where $\vec{S} = \vec{S}_1 + \vec{S}_2$ is total spin. Now average Σ_{12} over all relative $\measuredangle$ lar orientations of $\vec{S}$ and $\hat{r}_{12}$, and find $(\Sigma_{12})_{av}$ for both singlet and triplet states.

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20 pts. #16 ϕ507 (1992)

(77) Hyperfine structure (hfs) of atomic energy levels is caused by the coupling of the nuclear magnetic moment $\vec{\mu}_n = -g_n \mu_0 \vec{I}$ to the magnetic field $\vec{H}e$ generated by the electrons (due to both orbital motion and spin) at the nucleus. Using the vector model, show that the hfs energies are $\mathcal{E}_{hfs} = A_{hfs}(\vec{I}\cdot\vec{J})$, and specify A_{hfs}. Sketch the hfs energy levels allowed in a state of given $\vec{I}$ & $\vec{J}$, which is an eigenstate of the total system $\measuredangle$ momentum $\vec{F} = \vec{I} + \vec{J}$. For a single-electron atom, in a state with $l \neq 0$, calculate A_{hfs} explicitly.

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(78) a) Three identical weakly-interacting particles are described by a Ham^{n} $\mathcal{H}(1,2,3) = \sum_{k=1}^{3} H(k)$, where $H(k) = (\vec{p}_k^{\,2}/2m) + V(k)$ is the total energy of the $k^{\underline{th}}$ particle in an external potential V. Let $\phi_\lambda(k)$ be an eigenfcn of $H(k)$ with energy E_λ. Suppose the particles are in the distinct states $\phi_\alpha, \phi_\beta, \phi_\gamma$. Construct the properly symmetrized eigenfcn $\Psi(1,2,3)$ of $\mathcal{H}$ from the ϕ_λ for both the fermion and boson case.

b) Extend this situation to N identical fermions. Show that a suitable eigenfcn is: $\Psi(1,2,\ldots,N) = A \det \underset{\sim}{\phi}$, where $\underset{\sim}{\phi}$ is a square matrix with entries $(\underset{\sim}{\phi})_{\lambda k} = \phi_\lambda(k)$. You should show that Ψ is in fact an eigenfcn of the total system energy, is normalizable (find the norm cnst A), satisfies exchange symmetry, and is $\equiv 0$ if two or more fermions are in the same state. This Ψ is called a "Slater determinant". What is Ψ for N bosons?

QM Problems

(79) a) In non-degenerate stationary-state perturbation theory (abbr. NDSSPT), shew that the energy E_k to $O(V^2)$ is obtainable from the wavefcn ψ_k to $O(V)$ by calculating the appropriate expectation value.
b) Extend the results of NDSSPT to calculate the $O(V^2)$ correction $\psi_k^{(2)}$ to ψ_k, and the $O(V^3)$ correction $E_k^{(3)}$ to E_k.

Prob. # (30)
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(80) A 1D simple harmonic oscillator has Ham$^{\underline{n}}$: $H_0 = \frac{p^2}{2m} + \frac{1}{2}kx^2$. Suppose H_0 is perturbed by the addition of a term $V(x) = \frac{1}{2}qx^2$. The H_0+V problem can of course be solved exactly. Here, however, we wish to use NDSSPT. Calculate the perturbed energy in the $n^{\underline{th}}$ state to terms of $O(q^2)$ (Note: it is "convenient" to use matrix methods to get matrix elements of x^2 from those of x). Now compare the perturbation result with an expansion of the exact energy to $O(q^2)$.

(81) To place an upper limit on the size of $E_k^{(2)}$, the $O(V^2)$ energy correction in NDSSPT, start from the (obvious) inequality: $|E_k^{(2)}| \leqslant \sum_n{}' |V_{nk}|^2 / |E_k^{(0)} - E_n^{(0)}|$. Replace the energy denominator by $|\Delta E_k^{(0)}|_{AV}$, which is a sort of average energy gap between level k and all others. Does this strengthen or weaken the inequality? Proceed to shew: $|E_k^{(2)}| \lessapprox (\Delta V)_k^2 / |\Delta E_k^{(0)}|_{AV}$, where $(\Delta V)_k$ is the rms deviation of V in state k (i.e. by def$^{\underline{n}}$: $(\Delta V)^2 = \langle V^2\rangle - \langle V\rangle^2$).

Prob. # (34)
ϕ507
(Mar.'92)

(82) Suppose the 1D SHO of prob. (80) is perturbed by $V(x) = \frac{1}{2}kx^2\left(\frac{x}{b}\right)^2$. Calculating only the $1^{\underline{st}}$ order correction, shew that the energy of the $n^{\underline{th}}$ state becomes: $E_n \simeq (n+\frac{1}{2})\hbar(\omega + \delta\omega) + n^2\hbar\delta\omega$, where ω is the SHO natural frequency, and $\delta\omega$ depends on b (find $\delta\omega$!). (Hint: evaluate $\langle n|x^4|n\rangle$ by use of the SHO step operators $a^\dagger$ & a, writing $x = \sqrt{\hbar/2m\omega}(a^\dagger + a)$, and $a^{\dagger n}|0\rangle = \sqrt{n!}\,|n\rangle$).

QM Problems

(83) An ion with spin $S=1$ in a crystal experiences magnetic interactions which are represented by the Ham^n: $H = AS_z^2 + B(S_x^2 - S_y^2)$, with A & B cnsts, and $A \gg B$. The small term in B can be considered a perturbation V on the main contribution $H_0 = AS_z^2$ to the ion crystal field energy.

a) Using as a basis the spinor eigenfcns $\psi_k^{(0)}$ of S_z, and the appropriate matrix for S_z (e.g. Schiff, p. 203), calculate the expectation values $E_k^{(0)}$ of H_0 in each spin state. Draw the energy spectrum. Which states are degenerate?

b) Shew that the inclusion of V lifts the degeneracy, and calculate the perturbed energies E_k and eigenfcns ψ_k. Draw and label the new spectrum. Is your solution to this problem approximate, or exact, or what?

20 pts (84) In our derivation of the S-matrix (lecture 1/29/71), we developed an expansion technique to give $S_{\beta\alpha} = \sum_{n=0}^{\infty} S_{\beta\alpha}^{(n)}$ as a series of terms of successively higher order in the interaction $\Omega = V/\hbar$. Suppose Ω is time-independent, so that the eigenstates $\phi_\alpha(x,t) = \varphi_\alpha(x) e^{-\frac{i}{\hbar}E_\alpha t}$ have a separable time-dependence. Suppose also that the final state $\beta \neq$ initial state α.

a) Shew that the above perturbation series for $S_{\beta\alpha}$ may be written as

$$S_{\beta\alpha} = -2\pi i\,\delta(E_\beta - E_\alpha)\left[\langle\beta|V|\alpha\rangle + \sum_n \frac{\langle\beta|V|n\rangle\langle n|V|\alpha\rangle}{E_\alpha - E_n + i\epsilon} + \dots\right],$$

where $\lim_{\epsilon\to 0+}$ is understood, and $\langle\beta|V|\alpha\rangle = \int dx\, \varphi_\beta^*(x) V(x) \varphi_\alpha(x)$, etc.

b) For $\beta \neq \alpha$, the T-matrix is defined by: $S_{\beta\alpha} = -2\pi i\,\delta(E_\beta - E_\alpha)\langle\beta|T|\alpha\rangle$; its elements have the expansion in [ ] derived above. Interpreting $|S_{\beta\alpha}|^2$ as the probability of a transition $\alpha \to \beta$ induced by V, shew that the transition probability per unit time is given by the Fermi Golden Rule

$$W(\alpha\to\beta) = \frac{2\pi}{\hbar}|\langle\beta|T|\alpha\rangle|^2\,\delta(E_\beta - E_\alpha).$$

(Hint: use integral for δ-fcn, and define an integrated interaction time)