DEPARTMENT OF PHYSICS

2009 COMPREHENSIVE EXAM

PROBLEMS

Begin your answer to each question on a new sheet of paper; solutions to different questions must not appear on the same sheet. Label each sheet of paper with your identification number in the upper left hand corner and problem number in the upper right hand corner. When more than one sheet is submitted for a problem, be sure the pages are ordered properly.

PHYSICAL CONSTANTS

Quantity	Symbol	Value
acceleration due to gravity	g	9.8 m s^{-2}
gravitational constant	G	6.67×10^{-11} N m^2 kg^{-2}
permittivity of vacuum	ε_o	8.85×10^{-12} C^2 N^{-1} m^{-2}
permeability of vacuum	μ_o	$4\pi \times 10^{-7}$ N A^{-2}
speed of light in vacuum	c	3.00×10^{8} m s^{-1}
elementary charge	e	1.602×10^{-19} C
mass of electron	m_e	9.11×10^{-31} kg
mass of proton	m_p	1.673×10^{-27} kg
Planck constant	h	6.63×10^{-34} J s
Avogadro constant	N_A	6.02×10^{23} mol^{-1}
Boltzmann constant	k	1.38×10^{-23} J K^{-1}
molar gas constant	R	8.31 J mol^{-1} K^{-1}
standard atmospheric pressure		1.013×10^{5} Pa

1. Consider a two-state quantum system with a time-independent Hamiltonian:

$$H|\phi_1\rangle = \varepsilon_1|\phi_1\rangle \quad \text{and} \quad H|\phi_2\rangle = \varepsilon_2|\phi_2\rangle$$

where the real constants obey $\varepsilon_1 < \varepsilon_2$ and both states $|\phi_1\rangle$ and $|\phi_2\rangle$ are normalized.

Define two other states of the system

$$|\Phi_1\rangle = \frac{1}{\sqrt{2}}\left(|\phi_1\rangle + i|\phi_2\rangle\right) \quad \text{and} \quad |\Phi_2\rangle = \frac{1}{\sqrt{2}}\left(|\phi_1\rangle - i|\phi_2\rangle\right)$$

a. Prove that $|\phi_1\rangle$ and $|\phi_2\rangle$ are orthogonal using definitions of Hermitian adjoints and operators.

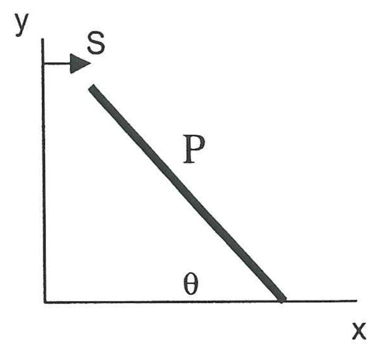
b. Explicitly demonstrate that $|\Phi_1\rangle$ and $|\Phi_2\rangle$ are also orthogonal and normalized.

c. At time *t=0*, the state of the system is known to be $|\Phi_1\rangle$. Find the time interval between a minimum and the subsequent maximum of the probability that the system will be found in the state $|\Phi_2\rangle$. Calculate the product of this interval and the uncertainty in energy. Does this product satisfy a Heisenberg uncertainty relation? If not, explain why not. If so, discuss its physical significance considering that there is no "time operator" in quantum mechanics.

Recall that, for an observable represented as an operator A the uncertainty in the expectation value $\langle A\rangle$ is defined as $\sqrt{\langle A^2\rangle - \langle A\rangle^2}$

2. A ladder of length ℓ and mass m leans against a frictionless wall at an angle θ_o . The bottom of the ladder is on a frictionless floor. As the ladder falls, it will at some angle $\theta<\theta_o$ lose contact with the wall.

a. Using S and θ as the generalized coordinates, find the Lagrangian for the ladder. S is the distance between the left end of the ladder and the vertical wall. The moment of inertia of the rod about its center of mass is $\frac{1}{12}m\ell^2$.



b. Use the method of Lagrange multipliers to find the angle where the ladder will lose contact with the wall.

c. To find a specific angle for your answer, assume the initial angle θ_o=45°. Feel free to use the sine tables in the CRC.

3. Some system is described by two chemical reagents whose (dimensionless) concentrations are denoted g and h. Each concentration must be non-negative. They vary according to the coupled differential equations

$$\frac{dg}{dt} = h - 2g^2$$

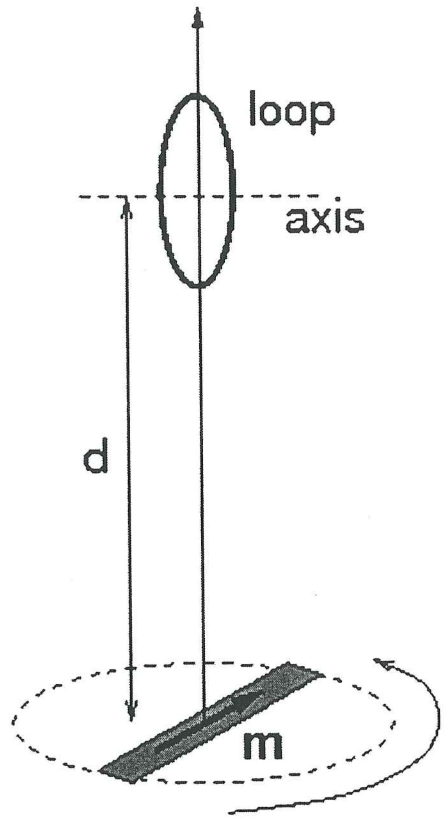
$$\frac{dh}{dt} = 4g^2 - 2g - gh$$

a. Find a non-trivial steady state, (g_0, h_0) for this system. (i.e. one other than the trivial solution $g=h=0$.)
b. Consider small perturbations, $\delta g = g - g_0$ and $\delta h = h - h_0$, to the steady state found above. Write down the system of equations governing the evolution of the these perturbations. The equations should be *linear* in δg and δh.
c. Is the equilibrium (g_0, h_0) **stable** or **unstable**? (It must be one or the other). In other words, if initialized to some arbitrary state *very near* the equilibrium, will the system approach the equilibrium or diverge from it?
d. What is the characteristic time scale for the approaching or diverging identified in part (c)? (i.e. the time for e-folding).

4. A Buckminsterfullerene, or "buckyball", is a spherically shaped molecule composed entirely of carbon atoms. The most common, naturally occurring buckyball is C_{60}, which contains 60 carbon atoms and has a diameter of roughly 1 nm. We consider here the interference of these massive C_{60} molecules as they pass through a double-slit apparatus (for future reference, there is an account of this in O. Nairz, M. Arndt, and A. Zeilinger, *Am. J. Phys.* **71** (4), 2003). For this experiment, assume a beam of molecules with an average speed of 200 m/s, a slit separation of 100 nm, and a slit-to-detector distance of 1 m. Ignore the single-slit interference pattern.

a. To one significant figure, estimate the spatial separation between interference maxima near the undeviated beam in the observation plane. Which is larger: the spatial separation between interference maxima, or the size of the molecules? Describe the observational consequences if the reverse had been true.
b. The velocity distribution of the molecules incident on the slits has a width in both the longitudinal and the transverse directions. Quantitatively characterize how each of these widths affects the interference pattern.
c. Beginning with a supply of a powder of the C_{60} molecules, draw a block diagram of an experimental apparatus that you might use to create and observe a double slit diffraction pattern of these molecules. Include experimental components that minimize the effects discussed in part (b). Label each component in your diagram and describe its basic function.

5. A very small bar magnet with magnetic dipole moment $\mathbf{m}$ spins about its center on a vertical axle ($\hat{\mathbf{z}}$) at angular frequency ω. At a distance d directly above the center of the magnet is a wire loop with area A whose axis (the normal to the area) is oriented horizontally ($\hat{\mathbf{x}}$, see figure; the loop lies in the y-z plane, the magnet spins in the x-y plane). At time $t=0$, $\mathbf{m}$ is oriented parallel to the axis of the wire loop (i.e. $\mathbf{m} \parallel \hat{\mathbf{x}}$; the figure shows a time $t>0$). The loop is very small compared to the separation $A \ll d^2$ and the bar magnetic is also small enough that it may be treated as a point dipole (the figure is not to scale). The loop has a total resistance R in comparison to which its inductance may be neglected (at frequency ω) and you may furthermore assume $\omega d \ll c$, the speed of light.



a. What is the current in the wire loop as a function of time?
b. The wire's current creates a magnetic field at the magnet. What is the resulting torque on the magnet as a function of time?
c. What is the **average** power which must be supplied to keep the magnet spinning at constant frequency ω.

6. Most of the mirrors that are used for focusing X-rays rely on the grazing incidence of the X-rays on polished surfaces. The reason for this is that the angle of incidence (measured relative to the polished surface) needs to be below a critical angle, below which X-rays are reflected totally (total external reflection). Snell's law suggests that for total external reflection to happen the index of refraction of the material must be less than 1. Imagine a polished metallic surface such as gold with an AC (alternating current) electrical conductance σ and plasma frequency ω_p. Imagine an X-ray of angular frequency $\omega > \omega_p$, falling onto this surface at a grazing angle of θ. Assume that for this X-ray the index of refraction of air is 1 while the index of refraction of gold is n. Assume that the number of conduction electrons per unit volume of gold is ρ and that they are free to move within the volume of gold. Taking into account the definition of $n = ck/\omega$,

a. Determine the index of refraction, n, of gold in terms of ω and ω_p, and confirm that $n < 1$.

b. Determine the critical angle for total external reflection of these X rays from the gold surface. Give a numerical value for a typical synchrotron X-ray energy of ~100 eV and the typical gold plasmon energy of ~3.8 eV.

(Hint: Consider the motion of conduction electrons under the influence of electromagnetic fields of the traveling wave: the X-rays incident on the gold surface. Ignore the contribution of the stationary positive ion background that holds the conduction electrons in the volume as well as the Lorentz force exerted by the X-ray's ***B***-field on a moving electron in metal relative to that exerted by the ***E***-field (you will get a bonus point if you explain why). Otherwise electrons are assumed to be free. Remember that Maxwell's equations in vacuum in Gaussian units are given by

$$\vec{\nabla}.\vec{E} = -4\pi\rho e, \quad \vec{\nabla}.\vec{B} = 0, \quad \vec{\nabla}\times\vec{E} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t}, \quad \vec{\nabla}\times\vec{B} = \frac{1}{c}\frac{\partial\vec{E}}{\partial t} + \frac{4\pi}{c}\vec{J}, \quad \vec{J} = \sigma\vec{E}$$

Also remember

$\vec{\nabla}\times\vec{\nabla}\times\vec{V} = \vec{\nabla}(\vec{\nabla}.\vec{V}) - \nabla^2\vec{V}$ and the definition of the plasma frequency of a metal: $\omega_p = \frac{4\pi\rho e^2}{m}$

7. As a hypothetical exercise, suppose that a photon of angular frequency ω with wave number $k = 2\pi/\lambda$ were to have a mass $m_\gamma > 0$.

a. Using the relativistic mass-energy relation, show that

$$\omega^2 = k^2c^2 + \omega_0^2$$

where ω_0 is a constant.

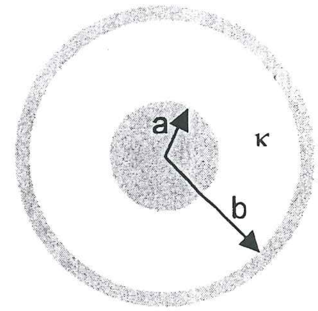
b. Calculate ω_0 if $m_\gamma = 4 \times 10^{-48}$ g.

c. Find the photon's group velocity v_g.

d. Sketch v_g vs ω

e. At what wavelength λ does v_g fall below $0.9999c$?

8. A common cable used in the lab is RG-58 A/U which is most often used with BNC connectors you see on many laboratory instruments such as the oscilloscope. The cable is made of an inner wire of radius *a* surrounded by a dielectric of radius *b* that is then surrounded by a metal mesh as the outer conductor. The outer mesh is typically covered by a thin black plastic jacket. Assume that the dielectric constant κ for the polyethylene insulator between the inner wire and the outer shield is 2.09. Assume the inner wire has a diameter of 0.035 inches (20 AWG) and the outer diameter of the insulation is 0.114 inches.



a) Find an expression for the capacitance per unit length of the coaxial cable in terms of *a*, *b* and κ.

b) Now use the numbers given above to get a numerical estimate for the capacitance per unit length. Since you don't have a calculator, just make an order of magnitude estimate. Feel free to use the log tables in the CRC.

c) Convert your answer in b) to units of "puffs per foot" (picofarads per foot).

9. A pond of water at a uniform temperature of 0 °C is covered with a layer of ice that is 4.0 cm thick. If the air temperature stays constant at -10.0 °C, and the ice thickness grows from 4.0 cm to 8.0 cm in a period of 10 hours, determine the latent heat of solidification of water. Clearly state any assumptions you use in solving this problem. (The density of ice is given as 1 gm/cm^3 and the thermal conductivity of ice is 2 W/m-K.)

10. Consider the operator

$$\hat{A} = i\frac{\partial}{\partial r}$$

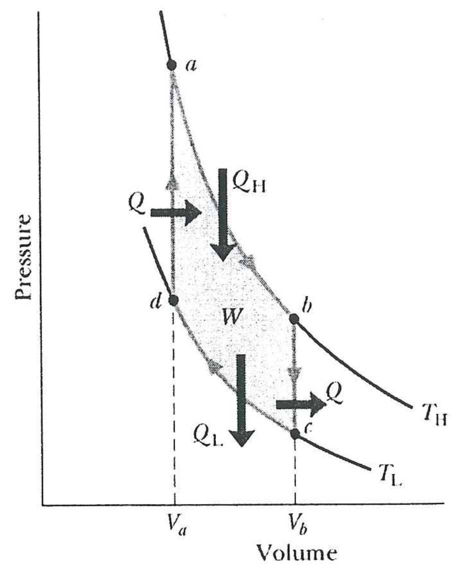
where r is the radius in spherical polar coordinates.

a. What is its Hermitian conjugate, $\hat{A}^{\dagger}$?
b. Is the operator

   $$\hat{B} = \tfrac{1}{2}\hat{A} + \tfrac{1}{2}\hat{A}^{\dagger}$$

   Hermitian? Please justify your answer.
c. Does $\hat{B}$ commute with the Hamiltonian of a free particle?
d. Write down an un-normalized eigenstate of $\hat{B}$.

11. The Stirling heat engine is diagramed on the pressure volume plot shown. For n moles of an ideal monatomic gas, find the following. (As you can see from the figure, the Stirling engine has isothermal paths from a to b and c to d, and has isovolumetric paths from b to c and d to a.)



a. Find an expression for the efficiency ε of the Stirling heat engine, defined as $\varepsilon = W/|Q_{in}|$, where $|Q_{in}| = |Q_H| + |Q|$, as labeled in the diagram.

b. Find a numerical estimate for the efficiency ε using your expression in a) with n=2.0, T_H=850K, T_L=300K, V_a=2.0L and V_b=4L. Express your answer as a percentage (%). Feel free to use the log tables in the CRC.

c. Find a numerical estimate for the efficiency ε_{carnot} (also as a percentage) for a Carnot engine running between the same two temperatures as in b). (The ideal Carnot engine has isothermal paths from a to b and c to d also, but has adiabatic paths from b to c and d to a.)

12. A sledge hammer consists of a small iron head of mass M at the end of thin shaft of equal mass and length L. (The moment of inertia of the shaft about its center is $I_{cm} = ML^2/12$.) The hammer hangs vertically from a hook at the end of its handle (the end of the shaft opposite the head) about which it pivots frictionlessly. It is at rest when it is struck at by a projectile at *the shaft's midpoint*. The sledge hammer delivers *to the projectile* an impulse

$$\int \mathbf{F}_{sp}(t)\,dt = \Delta p\,\hat{\mathbf{x}}$$

which is purely horizontal (i.e. perpendicular to the shaft).

a. Where is the center of mass of the sledge hammer (i.e. how far is it from the hook), and what is the moment of inetria about the pivot?
b. What is the impulse delivered by the hook to the sledge hammer?
c. Is it possible for a projectile to deliver the same impulse to the shaft, but to strike at a different point so that no impulse is delivered from the hook? If so, how far from the hook must it strike?

13. An empty cylinder of radius R and height h was open to the atmosphere at sea level before it was sealed and set into a spinning motion with an angular velocity of ω around its symmetry axis. Answer the following questions:

a. Determine the pressure inside the cylinder as a function of radius r from the axis of rotation assuming that the air inside the cylinder has reached a mechanical equilibrium within the cylinder and a thermal equilibrium with the air outside.

b. Determine the pressure, in atmospheres, at $r=0$ assuming that $m\omega^2 R^2 / 2 \simeq k_B T$ where k_B is the Boltzmann constant and m is the mass of an air molecule.

14. For a one-dimensional potential barrier of finite height and width (barrier width a and height V) determine the minimum energy such that *all* incident particles of mass m with that energy are transmitted through the barrier. Determine the energy in terms of the parameters a, V, and m and define the particle energy and the barrier height from the same zero of energy.

(For electrons scattering off noble gas atoms, this is called the Ramsauer effect, for neutrons scattering off nuclei this is called size resonance.)

15. Consider the electrostatic potential $\Phi(\mathbf{r})$ at an arbitrary field point $\mathbf{r}$ due to a positive point charge q located above the xy plane at $x=y=0$, $z=a$ and a negative point charge $-q$ located below the xy plane at $x=y=0$, $z=-a$. Treat the vector $\mathbf{r}$ in terms of spherical coordinates: r is the distance of the field point from the origin, θ is the polar angle (down from the +z axis), and ϕ is the azimuthal angle counterclockwise about the +z axis).

a. Determine the exact expression for $\Phi(r,\theta,\phi)$ assuming that the potential is zero as $\rightarrow \infty$. Explain why $\Phi(\mathbf{r})$ is independent of ϕ.
b. For $r >> a$ determine the first two non-zero terms in the series expansion of $\Phi(\mathbf{r})$ in powers of a/r for arbitrary θ. Explain the physical significance of these first two terms.