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Φ507 Final Exam (in class, 3 hrs.) — Wed. 13 May 1992

This exam is open-book, open-notes, and is worth 270 points total. There are seven problems on 3 pages, with point values as marked. For each problem, put a $\boxed{\text{box}}$ around your answer. Number your solution pages consecutively, write your name on page 1, and staple the pages together before handing them in.

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① [20 pts.]. In a certain QM system, it is found that the eigenfunction $u_n(x)$ of energy E_n is translationally invariant, i.e. if $u_n(x)$ is a solution to $\mathcal{H}u_n = E_n u_n$, then so is $u_n(x+\Delta x)$, w/ Δx = arbitrary displacement in the position x.

(A) As a consequence of this invariance, show that the system's linear momentum (operator) p must commute with the Hamiltonian $\mathcal{H}$, i.e. $[\mathcal{H}, p] = 0$.

(B) From the fact that $[\mathcal{H}, p] = 0$, identify the "QM system" you are dealing with.

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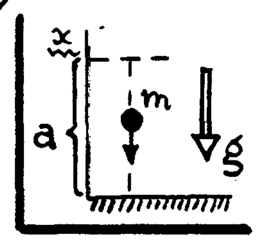
② [35 pts.]. Use the Bohr-Sommerfeld quantization rule, namely $\oint p(x)\,dx = (n+\frac{1}{2})h$, to quantize the allowed energy levels for a ball of mass m bouncing elastically and vertically in a uniform gravitational field of acceleration g.



(A) Show that the maximum bounce height a is quantized, and find its quantum form. (B) As $n \to$ large, find the incremental distance Δa_n between adjacent a_n-values. Show that any measurement of Δa_n (i.e. attempt to fix n) destroys the "orbit". How does this claim depend on n?

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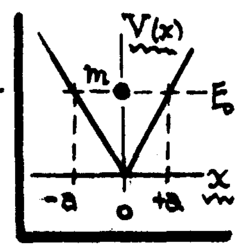
③ [45 pts.]. A particle of mass m moves in an attractive potential well $V(x) = G|x|$, w/ G = const > 0. Required: estimate the ground state energy E_0.



(A) Estimate E_0 by means of the WKB approximation.

(B) Estimate E_0 by the variational method, using as a trial function (with A = norm const, and b = variable parameter): $\phi(x) = A[1-(|x|/b)]$, $|x| \leq b$; $\phi = 0$, otherwise.

(C) Which of E_0(WKB) & E(Var^{n}l) lies closer to the true value of E_0?

(next page)

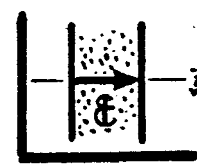
④ [25 pts.]. For a hydrogenlike atom (w/ Coulomb potential: $V(r) = -Ze^2/r$), the expectation value of $1/r^2$ (r = radial coordinate) in state $|n\ell m\rangle$ is given as:

$$\underline{\langle 1/r^2\rangle = \langle n\ell m|\tfrac{1}{r^2}|n\ell m\rangle = (Z/a_0)^2/n^3(\ell+\tfrac{1}{2})}, \text{ w/ } a_0 = \hbar^2/me^2.$$

Use this result to show that $\langle 1/r^3\rangle$ in the same state is given by:

$$\underline{\langle 1/r^3\rangle = \langle n\ell m|\tfrac{1}{r^3}|n\ell m\rangle = (Z/a_0)^3/n^3\ell(\ell+1)(\ell+\tfrac{1}{2})}.$$

Do <u>not</u> use explicit wavefunctions. Instead, look for assistance in the equation-of-motion for an electron in orbit.

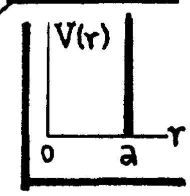


⑤ [40 pts]. A sample of ground state H atoms is placed in a parallel-plate capacitor. A time-dependent but spatially uniform electric field is applied as: $\underline{\mathbb{E}(t) = 0, \text{ for } t<0}$; $\underline{\mathbb{E}(t) = \mathcal{E}\, e^{-t/\tau}, \text{ for } t>0}$. τ = cnst >0, and $\mathcal{E}$ = cnst vector.

(A) Use first-order time-dept. perturbation theory to estimate the probability of finding excited states n in the sample [i.e. transitions: g(ground)$\to n$(excited)] at times $t \gg \tau$.

(B) Finesse the relevant matrix element $\langle n|(\text{coupling})|g\rangle$ by setting it equal to a numerical coefficient $\mathcal{N}\times$ relevant scale factors. What is the $g\to n$ transition probability of part (A) when τ is "large" (say $\tau \sim 1\times 10^{-9}$ sec).

(C) If $|\mathcal{E}| \sim 10^6$ volts/cm, about how big is the probability calculated in <u>part (B)</u>?



⑥ [45 pts]. A particle of mass m and energy E is incident on a fixed center with scattering potential: $V(r) = V_0 a\,\delta(r-a)$, w/ r = 3D radial coordinate, and V_0 & a = (+)ve cnsts. Treat the scattering by first Born Approximation.

(A) What condition on V_0 is needed so that the Born Approximation is valid at <u>all</u> energies E? Assume this condition is satisfied in what follows.

(B) Find the differential scattering cross-section $(d\sigma/d\Omega)$ as a fcn of momentum transfer q. Sketch $(d\sigma/d\Omega)$ vs. q over the allowed range of q. $(d\sigma/d\Omega)$ <u>vanishes</u> at certain q-values. Is there any physics in this?

(C) Express the total scattering cross-section σ as an integral over q. Find the leading terms in σ (including E-dependence) in the low energy limit.

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(7) [60 pts.]. The neutral scalar field $\phi(\mathbf{r},t)$ obeys the Klein-Gordon equation: $[\nabla^2 - \frac{1}{c^2}(\partial^2/\partial t^2) - \mu^2]\phi = 0$, w/ $\mu = mc/\hbar$. Since ϕ represents a spinless particle, and obeys a wave equation similar to those for EM fields, then it can be made into a quantized field by the same techniques we have used in class to quantize the EM field. The result is:

$$\boxed{\begin{aligned}&\phi(\mathbf{r},t) = \sum_{\mathbf{k}} \left(c\sqrt{\hbar/2\omega V}\right)\left[a_{\mathbf{k}}(t)e^{i\mathbf{k}\cdot\mathbf{r}} + a_{\mathbf{k}}^{\dagger}(t)e^{-i\mathbf{k}\cdot\mathbf{r}}\right],\\ &\text{w/}\quad \omega = c\sqrt{\mathbf{k}^2+\mu^2}, \quad a_{\mathbf{k}}(t) = a_{\mathbf{k}}(0)e^{-i\omega t},\\ &\text{\&/}\quad [a_{\mathbf{k}}, a_{\mathbf{k}'}] \equiv 0;\ [a_{\mathbf{k}}^{\dagger}, a_{\mathbf{k}'}^{\dagger}] \equiv 0;\ [a_{\mathbf{k}}, a_{\mathbf{k}'}^{\dagger}] = \delta_{\mathbf{k}\mathbf{k}'}.\end{aligned}}$$

V is the volume of a "box" in which ϕ obeys periodic boundary conditions.

The companion field: $\pi(\mathbf{r},t) = \frac{1}{c^2}\frac{\partial}{\partial t}\phi(\mathbf{r},t)$, is a sort of generalized momentum if ϕ is considered as a generalized displacement. For ϕ as defined above, and $\pi = \frac{1}{c^2}(\partial\phi/\partial t)$, <u>prove</u> the equal-time commutation relation:

$\rightarrow [\phi(\mathbf{r},t), \pi(\mathbf{r}',t)] = i\hbar\,\delta(\mathbf{r}-\mathbf{r}')$.

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① [20 pts.]. Analyse consequences of translational invariance in a QM system.

(A) 1. We are given that:

$$\mathcal{H}\, u_n(x) = E_n\, u_n(x), \quad \text{and}\quad \mathcal{H}\, u_n(x+\Delta x) = E_n\, u_n(x+\Delta x). \qquad (1)$$

Suppose $\Delta x \to$ infinitesimal, and expand $u_n(x+\Delta x)$ by Taylor series...

$$u_n(x+\Delta x) = u_n(x) + \Delta x \left(\frac{\partial u_n}{\partial x}\right)\Big|_{\Delta x=0} + \cdots \leftarrow \frac{\partial}{\partial x} = \frac{i}{\hbar}\, p \text{ (operator)}$$

so, $$u_n(x+\Delta x) = u_n(x) + \frac{i\Delta x}{\hbar}\, p\, u_n(x) + \cdots \qquad (2)$$

The second of Eqs. (1) then gives...

$$\mathcal{H}\left[u_n(x) + \frac{i\Delta x}{\hbar}\, p\, u_n(x) + \cdots\right] = E_n\left[u_n(x) + \frac{i\Delta x}{\hbar}\, p\, u_n(x) + \cdots\right] \qquad (3)$$

terms cancel

or, $$\cancel{\left(\frac{i\Delta x}{\hbar}\right)} \mathcal{H}\, p\, u_n(x) = \cancel{\left(\frac{i\Delta x}{\hbar}\right)} p \underbrace{E_n\, u_n(x)}_{=\mathcal{H} u_n(x)}$$ since E_n commutes with p.

i.e., $$[\mathcal{H} p - p\mathcal{H}]\, u_n(x) = 0, \quad \text{so,}\quad \underline{\underline{[\mathcal{H}, p] = 0}}, \text{ as required.} \qquad (4)$$

(B) 2. Since $[\mathcal{H}, p] = 0$, then the momentum p is a constant of the motion, as is the total energy $E_n = (p^2/2m) + V$. So the potential is at most a cnst, which can be set to zero. Then $E = p^2/2m$.

The "QM system" under discussion is a free particle.

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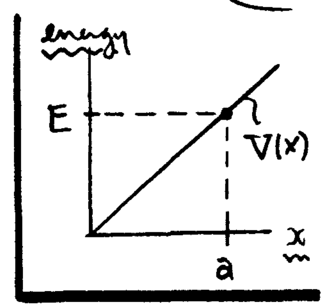
② [35 pts.]. Quantum mechanics of a bouncing ball.



(A) **1.** m is moving in a potential $V(x) = mgx$, where x is its vertical coordinate, and its total energy is $E = mga$, where a is the height of its bounce. Turning points are at $x=0$ & $x=a$, so that

$$\rightarrow \int_0^a \sqrt{2m[E-V(x)]}\,dx = \sqrt{2m^2g}\int_0^a (a-x)^{\frac{1}{2}}dx = (n+\tfrac{1}{2})\pi\hbar\,;\ \underline{\underline{n=0,1,2,\ldots}}\,; \quad (1)$$

by the Bohr-Sommerfeld rule [p. WKB 18, Eq. (52)]. The bounce height is quantized via Eq. (1), i.e. $a \rightarrow a_n$.

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2. The integral in (1) is easily done...

$$(a-x)^{\frac{1}{2}}dx = -\tfrac{2}{3}d(a-x)^{\frac{3}{2}} \Rightarrow \int_0^a (a-x)^{\frac{1}{2}}dx = \tfrac{2}{3}(a-x)^{\frac{3}{2}}\Big|_{x=a}^{x=0} = \frac{2}{3}a^{3/2}, \quad (2)$$

$$\text{So, } \sqrt{2m^2g}\cdot\frac{2}{3}a^{3/2} = (n+\tfrac{1}{2})\pi\hbar, \text{ or } \boxed{a_n = \left[\frac{9\pi^2\hbar^2}{8m^2g}(n+\tfrac{1}{2})^2\right]^{1/3}}. \quad (3)$$

As $n \rightarrow$ large ($n \gg 1/2$), quantized bounce heights are

$$\underline{\underline{a_n = A\,n^{2/3}}}, \text{ where: } \underline{\underline{A}} = \left[\frac{9\pi^2}{8}(c^2/g)(\hbar/mc)^2\right]^{1/3}. \quad (4)$$

With $\hbar/mc$ = m's Compton wavelength, it's easy to see that A has dim^{n}s of length.

(B) **3.** For large n, the bounce height difference between $(n+1)$ & n is:

$$\rightarrow \Delta a_n = A[(n+1)^{2/3} - n^{2/3}] = An^{2/3}[(1+\tfrac{1}{n})^{\frac{2}{3}} - 1] \simeq \tfrac{2}{3}A/n^{1/3}. \quad (5)$$

Location to $\Delta x = \Delta a_n \Rightarrow$ momentum uncertainty $\Delta p \sim \hbar/\Delta x$, and hence:

$$\rightarrow \Delta E \sim \frac{1}{2m}(\Delta p)^2 \simeq \frac{\hbar^2}{2m}(1/\Delta a_n)^2 = \frac{9\hbar^2}{8m}\cdot\frac{n^{2/3}}{A^2}. \quad (6)$$

The total energy is $E_n = mga_n = mgAn^{2/3}$, so the comparison is:

$$\rightarrow \boxed{\Delta E/E_n \sim (9\hbar^2/m^2g)\frac{1}{A^3} = \frac{1}{\pi^2}}. \quad (7)$$

ΔE is comparable to E_n, for all n.

The measurement ~ destroys the orbit, by boosting n by a factor $\sim (1+\frac{1}{\pi^2})^{\frac{3}{2}} = 1.16$.

FE3

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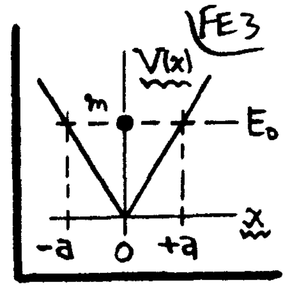

③ [45 pts.]. Estimate groundstate energy E_0 for $V(x) = G|x|$.

(A) **1.** The turning points are at $x = \pm a$, w/ $E_0 = Ga$. The WKB estimate proceeds from the Bohr-Sommerfeld quantization rule [class notes, p. WKB 18]:

$$\rightarrow \int_{-a}^{+a} \sqrt{2m[E_0 - V(x)]}\,dx = 2\int_0^a \sqrt{2mG(a-x)}\,dx = (n+\tfrac{1}{2})\pi\hbar \Big|_{n=0} \quad \text{n=0 for } E_0 \text{ the groundstate energy} \tag{1}$$

i.e./ $$2\sqrt{2mG}\,\underbrace{\int_0^a (a-x)^{\frac{1}{2}}dx}_{(2/3)a^{3/2}} = \tfrac{1}{2}\pi\hbar \Rightarrow \underline{\underline{a^{3/2} = \tfrac{3}{8}\pi\hbar/\sqrt{2mG}}}. \tag{2}$$

But $a = E_0/G$. Use this in (2) $\Rightarrow$ $$\boxed{E_0(\text{WKB}) = \left(\frac{9\pi^2}{128}\cdot\frac{\hbar^2 G^2}{m}\right)^{\frac{1}{3}} = 0.8853\left(\frac{\hbar^2 G^2}{m}\right)^{\frac{1}{3}}} \tag{3}$$

(B) **2.** For the variational estimate use: $\phi(x) = A\left(1 - \frac{|x|}{b}\right)$, $|x| \leq b$, and $\phi = 0$, otherwise.

Norm: $$\langle\phi|\phi\rangle = A^2\int_{-b}^{b}\left(1-\frac{|x|}{b}\right)^2 dx = 2A^2 b\underbrace{\int_0^1 (1-u)^2 du}_{1/3} = 1 \Rightarrow \underline{\underline{A^2 = 3/2b}}. \tag{4}$$

Variational Energy $\}$ $$E_0(b) = \langle\phi|\mathcal{H}|\phi\rangle = A^2\int_{-b}^{b}\left(1-\frac{|x|}{b}\right)\left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + G|x|\right]\left(1-\frac{|x|}{b}\right)dx. \tag{5}$$

... important to note that (recall prob. ⑳): $\frac{d^2}{dx^2}|x| = 2\delta(x)$ ...

So/ $$E_0(b) = \underset{= 3/2b}{A^2}\left\{\frac{\hbar^2}{mb}\int_{-b}^{b}\left(1-\frac{|x|}{b}\right)\delta(x)\,dx + 2Gb^2\underbrace{\int_0^1 u(1-u)^2 du}_{1/12}\right\}$$

or/ $$\underline{\underline{E_0(b) = \frac{3}{2}\frac{\hbar^2}{mb^2} + \frac{1}{4}Gb}} \tag{6}$$

Now minimize $E_0(b)$ w.r.t. width parameter b...

$$\frac{\partial E_0(b)}{\partial b} = 0 \Rightarrow b = \left(\frac{12\hbar^2}{mG}\right)^{1/3}, \text{ and: } E_0(b) = \left[\underbrace{\left(\frac{3}{128}\right)^{1/3}}_{0.2862} + \underbrace{\left(\frac{3}{16}\right)^{1/3}}_{0.5724}\right]\left(\frac{\hbar^2 G^2}{m}\right)^{1/3} \tag{7}$$

i.e./ $$\boxed{E_0(\text{Var}^{n}\text{l}) = 0.8586\,(\hbar^2 G^2/m)^{\frac{1}{3}}} \tag{8}$$

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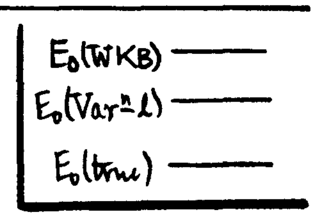
3. $E_0(\text{Var}^{n}\text{l})$ must always lie <u>above</u> E_0(true). Since E_0(WKB) $= 1.0311 \times E_0(\text{Var}^{n}\text{l})$ lies above $E_0(\text{Var}^{n}\text{l})$, then <u>$E_0(\text{Var}^{n}\text{l})$ is the better approximation to E_0(true)</u>.



FE4

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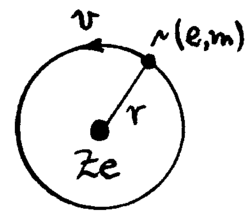
④ [25 pts.]. For H-like atom, manufacture $\langle 1/r^3\rangle$ from $\langle 1/r^2\rangle$.



1. The equation of the electron orbit at r, viz...

$$\rightarrow\ mv^2/r = Ze^2/r^2, \qquad (1)$$

can be written in terms of the orbital ∡ momentum $L = mvr$ as:

$$\rightarrow\ L^2/r^3 = Zme^2/r^2. \qquad (2)$$

Quantum-mechanically, Eq. (2) will hold in an expectation-value sense (by Ehrenfest's Theorem: Sakurai, p.87) and so in the state $|nlm\rangle$

$$\rightarrow\ \langle nlm|\frac{L^2}{r^3}|nlm\rangle = \hbar^2\frac{Z}{a_0}\langle nlm|\frac{1}{r^2}|nlm\rangle,\quad a_0 = \hbar^2/me^2. \qquad (3)$$

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2. In Eq. (3), L^2 is an <u>operator</u>, which operates on the ∡ cds of $|nlm\rangle$, and which has the eigenvalue $l(l+1)\hbar^2$ in that state. Then (3) reads...

$$\rightarrow\ l(l+1)\langle nlm|\frac{1}{r^3}|nlm\rangle = \frac{Z}{a_0}\langle nlm|\frac{1}{r^2}|nlm\rangle$$

So $\langle nlm|\frac{1}{r^3}|nlm\rangle = \frac{Z/a_0}{l(l+1)}\langle nlm|\frac{1}{r^2}|nlm\rangle$

$$= \underline{\underline{(Z/a_0)^3/n^3\, l(l+1)(l+\tfrac{1}{2})}}, \qquad (4)$$

as required.

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⑤ [40 pts.]. Estimate H-atom excitations by a pulsed electric field $\mathbb{E} = \mathcal{E} e^{-t/\tau}$.

(A) 1. By 1st order t-dept. perturbation theory, the amplitude for $g \to n$ is [Class notes p. tD5, Eq. (13)]:

$$\rightarrow a(t) = -\frac{i}{\hbar}\int_{t_0}^{t} V_{ng}(t')\, e^{i\omega_{ng}t'}\,dt', \quad \text{w/ } V_{ng}(t') = \langle \underset{\text{excited}}{n} | V(x,t') | \overset{\text{ground}}{g}\rangle. \tag{1}$$

Here, V is a Stark coupling $e\mathbb{E}\cdot\mathbf{r}$, and so -- with the given $\mathbb{E} = \mathcal{E}e^{-t/\tau}$ @ $t>0$:

$$a(\infty) = -\frac{i}{\hbar}\langle n|e\mathcal{E}\cdot\mathbf{r}|g\rangle \int_0^\infty e^{-t/\tau}\, e^{i\omega_{ng}t}\,dt = -\frac{i}{\hbar}\langle n|e\mathcal{E}\cdot\mathbf{r}|g\rangle \frac{\tau}{1 - i\omega_{ng}\tau}$$

and// $$\boxed{|a(\infty)|^2 = \frac{\tau^2/\hbar^2}{1+\omega_{ng}^2\tau^2}|\langle n|e\mathcal{E}\cdot\mathbf{r}|g\rangle|^2}. \tag{2}$$

$|a(\infty)|^2$ is the probability for g(ground) $\to n$(excited) @ $t \gg \tau$.

(B) 2. The matrix element in Eq. (2) is $\langle n|e\mathcal{E}\cdot\mathbf{r}|g\rangle = e\mathcal{E}\cdot\langle n|\mathbf{r}|g\rangle$, and we put this equal to $e\mathcal{E}\cdot\mathcal{N}a_0$, where $a_0 = \hbar^2/me^2$ is the Bohr radius. The numerical coefficient $\mathcal{N}$ contains geometry of $\mathcal{E}$ as well as the "strength" of the dipole matrix element $\langle n|\mathbf{r}|g\rangle$. The transition probability of Eq. (2) is

$$\rightarrow |a(\infty)|^2 = \mathcal{N}^2\left[\frac{(\omega_{ng}\tau)^2}{1+(\omega_{ng}\tau)^2}\right](e\mathcal{E}a_0/\hbar\omega_{ng})^2 \rightarrow \underline{\underline{\mathcal{N}^2(e\mathcal{E}a_0/E_{ng})^2}} \tag{3}$$

$E_{ng} = (E_n - E_g)$ is the $g \to n$ transition energy, and the expression on the far RHS of Eq. (3) is valid when $\omega_{ng}\tau \gg 1$. Since the first possible transition is $g(1S) \to n(2P)$, with $\omega_{ng} = (10.2\,\text{eV})/\hbar = 2\pi \times 2.5\times10^{15}$ Hz, then certainly this expression is valid for "large" $\tau \sim 10^{-9}$ sec. At this point, the $g \to n$ transition probability is actually independent of τ.

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3. The $g \to n$ probability just calculated in part (B) can be written as:

$$\rightarrow |a(\infty)|^2 = \mathcal{N}^2(\mathcal{E}/\mathcal{E}_{ng})^2, \quad \text{w/ } \mathcal{E}_{ng} = E_{ng}/ea_0 \text{ (an electric field)}. \tag{4}$$

For the first possible transition: $g(1S) \to n(2P)$, $E_{ng} = 10.2$ eV, and the scale field: $\mathcal{E}_{ng} \simeq 10\,\text{volts}/a_0 = 2\times10^9$ volts/cm. If $\mathcal{E} \sim 10^6$ volts/cm, then the transition probability is: $\boxed{|a(\infty)|^2 \sim \frac{1}{4}\mathcal{N}^2\times10^{-6}}$, certainly < 1 ppm.

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⑥ [45 pts.]. Analyse scattering from potential $V(r) = V_0 a\, \delta(r-a)$, via Born Approxn.

(A) 1. Let $k = \sqrt{2mE/\hbar^2}$ be m's incident wave#. Born Approxn validity requires:

$$\rightarrow \left| \int_0^\infty [e^{2ikr} - 1]\, V(r)\, dr \right| \ll \hbar v = \hbar^2 k/m \quad \text{Davydov Eq. (106.16), or Class notes p. ScT 10, Eq(22).} \qquad (1)$$

... for $V(r) = V_0 a\,\delta(r-a)$, Eq (1) $\Rightarrow$ $\underline{\underline{\left(\frac{\sin ka}{ka}\right) V_0 \ll \frac{1}{2m}(\hbar/a)^2}}$. (2)

Born Approxn is good at all energies (even $E \to 0$) if $\boxed{V_0 \ll \frac{1}{2m}(\hbar/a)^2}$. (3)

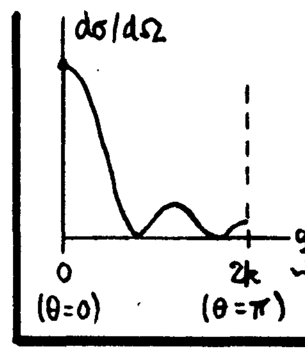
(B) 2. By class notes p. ScT (13), Eq. (31), the differential scattering cross-section is:

$$\rightarrow \frac{d\sigma}{d\Omega} = \left(\frac{m}{2\pi\hbar^2}\right)^2 |\tilde{V}(q)|^2, \text{ w/ } q = 2k\sin(\theta/2) \quad \text{q = momentum transfer, } \theta = \text{scattering angle.} \qquad (4)$$

and w/ $\tilde{V}(q) = \frac{4\pi}{q}\int_0^\infty r\, V(r) \sin qr\, dr = [4\pi V_0 a^3]\left(\frac{\sin qa}{qa}\right)$ for the given: $V(r) = V_0 a\,\delta(r-a)$. (5)

In (5), $[4\pi V_0 a^3] = \int_0^\infty V(r) \cdot 4\pi r^2 dr = \underline{\underline{\Lambda}}$, the "volume" of $V(r)$. So we get...

$$\boxed{\frac{d\sigma}{d\Omega} = \left(\frac{m\Lambda}{2\pi\hbar^2}\right)^2 \left(\frac{\sin qa}{qa}\right)^2} \quad \text{w/ } \Lambda = 4\pi V_0 a^3,\ \underline{q = 2k\sin(\theta/2)}. \qquad (6)$$



By the inequality in (3), the coefficient $(m\Lambda/2\pi\hbar^2)^2 \ll a^2$. $(d\sigma/d\Omega)$ vs. q is sketched at right -- the scattering vanishes when $qa = n\pi$, $n = 1, 2, \ldots$ (and $q \leq 2k$). At these points, there is a sort of resonance condition, where an integral # of half-wavelengths of q fit inside the scattering potential, i.e. $\underline{\underline{n \cdot \frac{1}{2}(2\pi/q) = a}}$, and $V(r)$ appears to be transparent.

(C) 3. The solid ∡ $d\Omega = 2\pi \sin\theta\, d\theta = (2\pi/k^2)\, q\, dq$ [probl. 44], so the total cross-section is:

$$\rightarrow \sigma = \int_{4\pi} (d\sigma/d\Omega)\, d\Omega = \left(\frac{m\Lambda}{2\pi\hbar^2}\right)^2 \frac{2\pi}{k^2} \int_0^{2k} \left(\frac{\sin qa}{qa}\right)^2 q\, dq = \frac{2\pi}{k^2 a^2}\left(\frac{m\Lambda}{2\pi\hbar^2}\right)^2 \int_0^{2ka} \frac{dx}{x} \sin^2 x. \qquad (7)$$

The integral is not an elementary fcn. When $a \to 0$ ($ka \ll 1$), put $\sin^2 x \simeq [x(1 - \frac{x^2}{6})]^2$ so that $\int_0^{2ka} (\sin^2 x)\frac{dx}{x} \simeq \int_0^{2ka} x(1 - \frac{x^2}{3})\, dx = 2(ka)^2[1 - \frac{2}{3}(ka)^2]$. Then leading terms in σ.

$$\boxed{\sigma \simeq 4\pi (m\Lambda/2\pi\hbar^2)^2 \left[1 - \tfrac{2}{3}k^2 a^2\right]} \qquad (8)$$

σ falls off slowly with energy (at low energy).

ϕ 507 Final Exam Solutions (1992)

⑦ [60 pts.]. Calculate a field commutator for neutral scalar field $\phi(\mathbf{r},t)$.

1. With: $\phi(\mathbf{r},t) = \sum_{\mathbf{k}} c\sqrt{\hbar/2\omega V}\,[a_{\mathbf{k}}(t)e^{i\mathbf{k}\cdot\mathbf{r}} + a^{\dagger}_{\mathbf{k}}(t)e^{-i\mathbf{k}\cdot\mathbf{r}}]$, and $a_{\mathbf{k}}(t) = a_{\mathbf{k}}(0)e^{-i\omega t}$:

$$\rightarrow \pi(\mathbf{r},t) = \frac{1}{c^2}(\partial\phi/\partial t) = (-)i\sum_{\mathbf{k}}\frac{1}{c}\sqrt{\hbar\omega/2V}\,[a_{\mathbf{k}}(t)e^{i\mathbf{k}\cdot\mathbf{r}} - a^{\dagger}_{\mathbf{k}}(t)e^{-i\mathbf{k}\cdot\mathbf{r}}]. \quad (1)$$

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2. In forming the commutator $[\phi(\mathbf{r},t),\pi(\mathbf{r}',t)]$, we should change the summation variable for π from $\mathbf{k}$ to $\mathbf{k}'$, and do the double sum $\sum_{\mathbf{k}}\sum_{\mathbf{k}'}$. But all the a's commute except for $\mathbf{k}'=\mathbf{k}$, and so we get back to just the single sum (via $\delta_{\mathbf{k}\mathbf{k}'}$). Thus...

$$[\phi(\mathbf{r},t),\pi(\mathbf{r}',t)] = \phi(\mathbf{r},t)\pi(\mathbf{r}',t) - \pi(\mathbf{r}',t)\phi(\mathbf{r},t)$$

$$= (-)i\left(\frac{\hbar}{2V}\right)\sum_{\mathbf{k}}[a_{\mathbf{k}}e^{i\mathbf{k}\cdot\mathbf{r}} + a^{\dagger}_{\mathbf{k}}e^{-i\mathbf{k}\cdot\mathbf{r}}][a_{\mathbf{k}}e^{i\mathbf{k}\cdot\mathbf{r}'} - a^{\dagger}_{\mathbf{k}}e^{-i\mathbf{k}\cdot\mathbf{r}'}] +$$

$$+ i\left(\frac{\hbar}{2V}\right)\sum_{\mathbf{k}}[a_{\mathbf{k}}e^{i\mathbf{k}\cdot\mathbf{r}'} - a^{\dagger}_{\mathbf{k}}e^{-i\mathbf{k}\cdot\mathbf{r}'}][a_{\mathbf{k}}e^{i\mathbf{k}\cdot\mathbf{r}} + a^{\dagger}_{\mathbf{k}}e^{-i\mathbf{k}\cdot\mathbf{r}}] \quad (2)$$

$$= (i\hbar/2V)\sum_{\mathbf{k}}\Big\{\overset{①}{\cancel{a_{\mathbf{k}}a_{\mathbf{k}}e^{i\mathbf{k}\cdot(\mathbf{r}'+\mathbf{r})}}} + \overset{②}{a_{\mathbf{k}}a^{\dagger}_{\mathbf{k}}e^{i\mathbf{k}\cdot(\mathbf{r}'-\mathbf{r})}} - \overset{③}{a^{\dagger}_{\mathbf{k}}a_{\mathbf{k}}e^{-i\mathbf{k}\cdot(\mathbf{r}'-\mathbf{r})}} -$$

$$- \overset{④}{\cancel{a^{\dagger}_{\mathbf{k}}a^{\dagger}_{\mathbf{k}}e^{-i\mathbf{k}\cdot(\mathbf{r}'+\mathbf{r})}}} - \overset{⑤}{\cancel{a_{\mathbf{k}}a_{\mathbf{k}}e^{i\mathbf{k}\cdot(\mathbf{r}+\mathbf{r}')}}} + \overset{⑥}{a_{\mathbf{k}}a^{\dagger}_{\mathbf{k}}e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}} -$$

$$- \overset{⑦}{a^{\dagger}_{\mathbf{k}}a_{\mathbf{k}}e^{-i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}} + \overset{⑧}{\cancel{a^{\dagger}_{\mathbf{k}}a^{\dagger}_{\mathbf{k}}e^{-i\mathbf{k}\cdot(\mathbf{r}+\mathbf{r}')}}}\Big\}. \quad (3)$$

Terms ①&⑤ cancel, as do terms ④&⑧. Combine ⑥&③, and ②&⑦ to get...

$$\rightarrow [\phi(\mathbf{r},t),\pi(\mathbf{r}',t)] = \left(\frac{i\hbar}{2V}\right)\sum_{\mathbf{k}}\Big\{\underbrace{[a_{\mathbf{k}},a^{\dagger}_{\mathbf{k}}]}_{\equiv 1}e^{i\mathbf{k}\cdot\mathbf{R}} + \underbrace{[a_{\mathbf{k}},a^{\dagger}_{\mathbf{k}}]}_{\equiv 1}e^{-i\mathbf{k}\cdot\mathbf{R}}\Big\}, \quad (4)$$

w/ $\mathbf{R} = \mathbf{r}-\mathbf{r}'$. Each commutator $\equiv 1$ as noted. Also $\sum_{\mathbf{k}} e^{-i\mathbf{k}\cdot\mathbf{R}} = \sum_{\mathbf{k}} e^{+i\mathbf{k}\cdot\mathbf{R}}$. Then

$$\rightarrow [\phi(\mathbf{r},t),\pi(\mathbf{r}',t)] = i\hbar\left\{\frac{1}{V}\sum_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{R}}\right\}. \quad (5)$$

density of modes: p. QF 13, Eq. (43)

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3. With V the "box" for periodic B.C: $\frac{1}{V}\sum_{\mathbf{k}} \rightarrow \frac{1}{V}\int_{\infty}\left[\frac{V}{(2\pi)^3}\right]d^3k = \frac{1}{(2\pi)^3}\int_{\infty} d^3k$. So

$$\rightarrow (1/V)\sum_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{R}} \rightarrow \left(\frac{1}{2\pi}\int_{-\infty}^{\infty}dk_x\,e^{ik_xX}\right)\left(\frac{1}{2\pi}\int_{-\infty}^{\infty}dk_y\,e^{ik_yY}\right)\left(\frac{1}{2\pi}\int_{-\infty}^{\infty}dk_z\,e^{ik_zZ}\right) = \quad (6)$$

$$= \delta(X)\,\delta(Y)\,\delta(Z).$$

So//

$$\boxed{[\phi(\mathbf{r},t),\pi(\mathbf{r}',t)] = i\hbar\,\delta(\mathbf{r}-\mathbf{r}')}. \quad (7)$$

QED