[redacted] { Set #⑥: Probs. 19-22.
Assigned: 10/28/88; due 11/4/88.

⑥ P8

⑲ In Sec. (6.12), at the bottom of p. 252, Jackson remarks that if there is a magnetic charge on the nucleon, it is $|q_m(\text{nucleon})| < 2\times10^{-24}\,e$. Further, Jackson claims this limit on $|q_m|$ comes from "knowing that the average magnetic field at the earth's surface is not more than $B_s \sim 1$ Gauss." Show how $B_s \sim 1$G limits $|q_m|$, and verify Jackson's number. [redacted]: assume the earth is made out of germanium.

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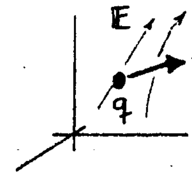
⑳ [redacted]. The standard Lorentz law for the force on charge q moving at velocity $\mathbb{v}$ through fields $\mathbb{E}$ & $\mathbb{B}$ is: $\underline{\mathbb{F}_L = q\left(\mathbb{E} + \frac{\mathbb{v}}{c}\times\mathbb{B}\right)}$.



In class, it was mentioned that $\mathbb{F}_L$ is incomplete, because it does not include an effective retarding force due to the EM radiation by q. Here we wish to look at a (crude) modification to $\mathbb{F}_L$ due to this "radiation damping."

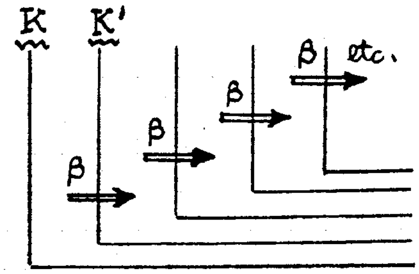
6pts (A) A charged particle (q,m), accelerated non-relativistically at $\mathbb{a}$, radiates energy at a rate: $\underline{d\mathcal{E}/dt = -\frac{2}{3}(q^2/c^3)|\mathbb{a}|^2}$ [Jk^n Eq. (14.22)]. Show that if q moves randomly such that $\mathbb{a}$ and its velocity $\mathbb{v}$ are uncorrelated (on average), then this loss can be attributed to a radiation reaction force: $\boxed{\mathbb{f} = m\tau\dot{\mathbb{a}}}$, where $\tau = \frac{2}{3}(q^2/mc^3)$. Calculate τ in sec. for an electron ([redacted]: $\tau \sim$ time for a light signal to travel across the classical electron radius: $r_0 = e^2/mc^2$.)

6pts (B) Write q's eqtn-of-motion as: $m\mathbb{a} = \mathbb{F}_L + \mathbb{f}$, and assume $\mathbb{f}$ is "small" compared to $\mathbb{F}_L$. Consider a frame in which q is instantaneously at rest (i.e. $\mathbb{v}=0$) and iterate the eqtn-of-motion to show: $\underline{\mathbb{f} = q\tau[\dot{\mathbb{E}} + (q/mc)\,\mathbb{E}\times\mathbb{B}]}$. Compare each term in $\mathbb{f}$ to the main accelerating force $q\mathbb{E}$. For an electron, at what $\mathbb{E}$-field frequencies will $\mathbb{f} \sim \mathbb{F}_L$? At what $\mathbb{B}$-field strengths will $\mathbb{f} \sim \mathbb{F}_L$? Find numbers in each case.

5pts (C) $\tau \sim e^2/mc^3$ is a purely electromagnetic quantity and is very small. Compare τ to the time scale on which you expect quantum effects to be important. What do you conclude?

3pts (D) In the absence of external $\mathbb{E}$ & $\mathbb{B}$ fields: $m\mathbb{a} = \mathbb{f}$. Solve this equation for the motion of (q,m). You should notice something strange. Be concerned, but not alarmed.

(21) [Jackson Prob. (11.2)]. Show explicitly that two successive Lorentz transformations in the same direction (at velocity β_1, followed by β_2) are equivalent to a single Lorentz transformation @ $\boxed{\beta = \frac{\beta_1 + \beta_2}{1 + \beta_1 \beta_2}}$, w/ $\beta = \frac{v}{c}$. This is relativistic velocity addition.



(22) Initially, K' is moving at velocity $\beta_1 = \beta$ (w/ $0 < \beta < 1$) in reference system K. To boost his velocity, K' boards a convenient system moving by him at relative velocity β. By the addition formula in prob. (21), K' is now moving relative to K' @ $\beta_2 = 2\beta/(1+\beta^2)$. K' continues the process -- each time boarding a system moving by him at β. Show that after $(n-1)$ such boardings, the K' velocity relative to K is : $\boxed{\beta_n = (1-\epsilon^n)/(1+\epsilon^n)}$, with $0 < \epsilon < 1$, and find ϵ in terms of β. Can you get to $v = c$ by a finite number of finite accelerations ?

⑲ Set limit on nucleon magnetic charge q_m, from fact: your compass doesn't work well.

10/24/84 1. Earth data: $R(\text{radius}) = 6370$ km, $M(\text{mass}) = 5.98\times10^{24}$ kg, $\rho\left(\begin{smallmatrix}\text{mean}\\ \text{density}\end{smallmatrix}\right) = 5.52$ gm/cm^3, $B\left(\begin{smallmatrix}\text{surface}\\ \text{mag. fld.}\end{smallmatrix}\right) < 1$ G. } CRC Hndbk

Since nucleon mass: $m \simeq 1.67\times10^{-27}$ kgm, then earth contains:

$$N = M/m = 5.98\times10^{24}/1.67\times10^{-27} = 3.58\times10^{51} \text{ nucleons.} \quad (1)$$

The density ρ is about that of Germanium (is the earth really just a huge semiconductor?), for which we note: Z (# protons) = 32, A (atomic weight) = 73.

2. Let: $\underline{q_m(\text{proton}) = \alpha e}$, and assume (worst case): $q_m(\text{neutron}) = -q_m(\text{proton})$. Then, with: $\mathbb{B} = (\alpha e/r^2)\,\hat{r}$ from an individual proton, field at earth's surface is:

$$B_{surf} = (N_p - N_n)\,\alpha e/R^2 \quad \left\{\begin{array}{l} N_p = \#\text{ protons} = \mu N \\ N_n = \#\text{ neutrons} = (1-\mu)N \end{array}\right\} \mu = \frac{Z}{A} \text{ for avg nucleus inside earth}$$

So// $$B_{surf} = (2\mu - 1)\,N\alpha e/R^2 \Rightarrow |\alpha| = \left(\frac{A}{A-2Z}\right)\frac{R^2}{Ne}\,|B_{surf}| \quad (2)$$

This is just Gauss' Law for a spherical (symmetric) distribution of monopoles. The #'s: $R = 6.37\times10^8$ cm, N of Eq. (1), and $e = 4.80\times10^{-10}$ esu, give ...

$$|\alpha| \leq \left(\frac{A}{A-2Z}\right)\cdot 2.36\times10^{-25}\,|B_{max}| \quad (3)$$

B_{max}, in G, is the maxm observable field at surface.

3. The correction factor $A/(A-2Z)$ out in front on the RHS of Eq. (3) is present because we've assumed $q_m(\text{neutron}) = -q_m(\text{proton})$; it would be just 1 had we assumed q_m's of the same sign. If $B_{max} = 1$ G, and for germanium ...

$$|\alpha| \leq \frac{73}{9}\times 2.36\times10^{-25} = 1.91\times10^{-24} \Leftrightarrow \boxed{|q_m(\text{nucleon})| \leq (2\times10^{-24})\,e}. \quad (4)$$

This verifies Jackson's number.

10/27/88

(20) [20 pts]. Construct & evaluate (classical) radiation reaction force $\mathbf{f}$.

(A) If $\mathbf{f}$ is the force responsible for the radiation loss, then: $\mathbf{f}\cdot\mathbf{v} = +d\mathcal{E}/dt$, by definition. If we assume: $\mathbf{f} = m\tau\dot{\mathbf{a}}$, then (with $\tau = \frac{2}{3}q^2/mc^3$)...

$$\mathbf{f}\cdot\mathbf{v} = m\tau(\mathbf{v}\cdot\dot{\mathbf{a}}) = \frac{2}{3}(q^2/c^3)\left[\frac{d}{dt}(\mathbf{v}\cdot\mathbf{a}) - |\mathbf{a}|^2\right]\text{, by a simple identity,}$$

$$\text{or } \mathbf{f}\cdot\mathbf{v} = (+)\frac{d\mathcal{E}}{dt} + \left[\cancel{\frac{2}{3}(q^2/c^3)\frac{d}{dt}(\mathbf{v}\cdot\mathbf{a})}\right]^{\to 0} \begin{cases}\text{for: } \mathbf{f} = m\tau\dot{\mathbf{a}},\\ \text{and: } d\mathcal{E}/dt = -\frac{2}{3}(q^2/c^3)|\mathbf{a}|^2.\end{cases} \quad (1)$$

The [ ] term is zero (on average), because if $\mathbf{v}$ & $\mathbf{a}$ are not correlated, then the avg. value $\langle\mathbf{v}\cdot\mathbf{a}\rangle = 0$, and $\frac{d}{dt}\langle\mathbf{v}\cdot\mathbf{a}\rangle = 0$. Actually, all we need is $\langle\frac{d}{dt}(\mathbf{v}\cdot\mathbf{a})\rangle = 0$; this will be true of periodic motion as well as random. What Eq. (1) shows then is that $\boxed{\mathbf{f} = m\tau\dot{\mathbf{a}} \text{ is consistent w/ } \mathbf{f}\cdot\mathbf{v} = d\mathcal{E}/dt.}$

The scale time τ for an electron ($q = e$) can be written as...

$$\tau = \frac{2}{3}r_0/c, \text{ w/ } r_0 = e^2/mc^2 = 2.82\times10^{-13}\text{ cm} \begin{cases}\text{classical}\\ \text{electron}\\ \text{radius}\end{cases}$$

$$\text{So// } \tau = \frac{2}{3}\cdot\frac{2.82\times10^{-13}}{3\times10^{10}} = \underline{6.27\times10^{-24}\text{ sec}}. \quad (2)$$

Evidently τ(electron) is very small, and τ is even smaller for protons.

(B) If, in $m\mathbf{a} = \mathbf{F}_L + \mathbf{f}$, the radiation reaction force $\mathbf{f}$ is "small", then to first approxn: $m\mathbf{a} = q(\mathbf{E} + \frac{\mathbf{v}}{c}\times\mathbf{B})$, so that by taking d/dt of both sides...

$$m\dot{\mathbf{a}} = q\left[\dot{\mathbf{E}} + \underset{(1)}{\frac{\dot{\mathbf{v}}}{c}}\times\mathbf{B} + \underset{(2)}{\frac{\mathbf{v}}{c}}\times\dot{\mathbf{B}}\right] \simeq q\left[\dot{\mathbf{E}} + (q/mc)\,\mathbf{E}\times\mathbf{B}\right]. \quad (3)$$

In term (1), we've used $m\dot{\mathbf{v}} = q\mathbf{E}$ in the frame where q is instantaneously at rest ($\mathbf{v} = 0$); in term (2), we've just set $\mathbf{v} = 0$ in that frame. Then, per ad

$$\boxed{\mathbf{f} = m\tau\dot{\mathbf{a}} \simeq q\tau\left[\dot{\mathbf{E}} + (q/mc)\,\mathbf{E}\times\mathbf{B}\right]}. \quad (4)$$

If the radiation reaction $\mathbb{f}$ is to be truly small w.r.t. Lorentz force $\mathbb{F}_L$, then both terms on the RHS of Eq. (4) should be small compared to $q\mathbb{E}$ (the major part of $\mathbb{F}_L$ for a non-relativistic particle). Thus...

$$\mathbb{f}\ \text{small}\begin{cases} q\tau\dot{E} \ll qE, \text{ or: } \dot{E}/E \ll \frac{1}{\tau} \sim 2\pi\times 2.53\times 10^{22}\,\text{Hz} \\ q\tau\left(\frac{q}{mc}\right)EB \ll qE, \text{ or: } B \ll \frac{mc}{q\tau} \sim 5.4\times 10^{16}\,\text{G} \end{cases} \quad (5)$$

numbers are for an electron

The E-field frequencies at which $f \sim F_L$ are thus $\sim \underline{10^{22}\,\text{Hz}}$, and the B-fields at which $f \sim F_L$ are $\sim \underline{10^{16}\,\text{G}}$. Such frequencies & fields are many orders of magnitude beyond anything attainable on earth.

5pts// (C) In fact, QM effects must become operative long before the time scale $\tau \sim e^2/mc^3$ is reached. If a particle of mass m is to be detected at all, in any of its manifestations, its energy must be uncertain to less than $\Delta E \sim mc^2$, and -- by the Uncertainty Principle -- the time available for detection is: $\Delta t > \hbar/\Delta E$. So $\boxed{\tau_{QM} \sim \hbar/mc^2}$ is a fundamental time limit for detection of m, according to QM. Now, comparing τ & τ_{QM}...

$$\left.\begin{array}{l} \text{rad. reaction time scale}: \tau \sim e^2/mc^3 \\ \text{QM detection (limit) time}: \tau_{QM} \sim \hbar/mc^2 \end{array}\right\} \quad \tau \sim \left(\frac{e^2}{\hbar c}\right)\tau_{QM} = \frac{\tau_{QM}}{137}. \quad (6)$$

fine structure cnst, again.

CONCLUSION: What happens on the time scale τ is not detectable, by basic QM limits. Among other thing, this $\Rightarrow$ our model for $\mathbb{f}$ is not directly verifiable.

3pts// (D) In the absence of fields: $m\mathbb{a} = \mathbb{f}$, or: $\mathbb{a} = \tau\dot{\mathbb{a}}$. Solutions to this eqtn are //

$$\longrightarrow \mathbb{a}(t) = \begin{cases} 0, \text{ for all } t; \text{ or} \\ \mathbb{a}(0)\exp(+t/\tau). \end{cases} \quad (7)$$

The first solution would not have surprised Newton (or anyone else), but the second solution is seriously weird -- at the slightest initial acceln $\mathbb{a}(0)$, q runs away, exponentially, to ∞ (w/o fields!). We will return to this dilemma, later.

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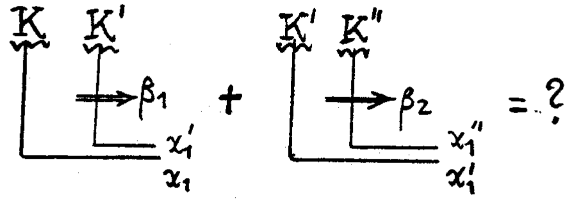

S28

10/28/84

(21) Find equivalent velocity for two successive Lorentz transf^{n}s.

1) Two successive Lorentz transf^{n}s (along x_1-axis) yield, with $x_0 = ct$...

$$\underline{\underline{K \to K'(\beta_1)}} \begin{cases} x_0' = \gamma_1(x_0 - \beta_1 x_1), \\ x_1' = \gamma_1(x_1 - \beta x_0); \end{cases} \Bigg\| \quad \underline{\underline{K' \to K''(\beta_2)}} \begin{cases} x_0'' = \gamma_2(x_0' - \beta_2 x_1'), \\ x_1'' = \gamma_2(x_1' - \beta_2 x_0'). \end{cases} \tag{1}$$

2) Plug the x_0' & x_1' values from the $K \to K'$ transform into the x_0'' & x_1'' eqtns to get,

$$x_0'' = (1+\beta_1\beta_2)\gamma_1\gamma_2\left[x_0 - \left(\frac{\beta_1+\beta_2}{1+\beta_1\beta_2}\right)x_1\right],$$

$$x_1'' = (1+\beta_1\beta_2)\gamma_1\gamma_2\left[x_1 - \left(\frac{\beta_1+\beta_2}{1+\beta_1\beta_2}\right)x_0\right]. \tag{2}$$

3) Now work out the algebraic identity...

$$(1+\beta_1\beta_2)\gamma_1\gamma_2 = \left[\frac{(1+\beta_1\beta_2)^2}{\underbrace{(1-\beta_1^2)(1-\beta_2^2)}_{= 1+(\beta_1\beta_2)^2 - (\beta_1^2+\beta_2^2) = 1+2\beta_1\beta_2+(\beta_1\beta_2)^2-(\beta_1^2+2\beta_1\beta_2+\beta_2^2)}}\right]^{\frac{1}{2}} = \left[\frac{(1+\beta_1\beta_2)^2}{(1+\beta_1\beta_2)^2-(\beta_1+\beta_2)^2}\right]^{\frac{1}{2}} = \left[\frac{1}{1-\left(\frac{\beta_1+\beta_2}{1+\beta_1\beta_2}\right)^2}\right]^{\frac{1}{2}}$$

i.e., $(1+\beta_1\beta_2)\gamma_1\gamma_2 = \gamma$, where: $\gamma = \dfrac{1}{\sqrt{1-\beta^2}}$, and: $\beta = (\beta_1+\beta_2)/(1+\beta_1\beta_2)$. (3)

4) The overall $K \to K''$ transf^n of Eq. (2) can now be written as...

$$x_0'' = \gamma(x_0 - \beta x_1), \quad x_1'' = \gamma(x_1 - \beta x_0),$$

with: $\boxed{\beta = (\beta_1+\beta_2)/(1+\beta_1\beta_2)}$, and: $\gamma = 1/\sqrt{1-\beta^2}$. (4)

This is a standard Lorentz transf^n for $K \to K''(\beta)$, at the advertised value of β. Velocities do <u>not</u> add linearly, as per Galileo. While K' <u>thinks</u> he boosts his velocity by β_2 in boarding K'', he only gets: $\beta - \beta_1 = \left(\frac{1-\beta_1^2}{1+\beta_1\beta_2}\right)\beta_2 < \beta_2$, w.r.t. K.

[redacted]

(22) Calculate aggregate **velocity** for a succession of Lorentz transf^{n}s at β.

10/28/84

1) After $(n-1)$ boardings, the K-K' relative **velocity will be**...

$$\beta_n = \frac{\beta + \beta_{n-1}}{1+\beta\beta_{n-1}} \;;\; n \geqslant 1, \text{ and: } \beta_0 = 0,\ \beta_1 = \beta,\ \beta_2 = \frac{2\beta}{1+\beta^2}, \text{ etc} \tag{1}$$

In principle, this can be iterated for β_n in terms of $\beta_1 = \beta$. The first few terms are...

$$\beta_1 = \beta,\quad \beta_2 = \frac{2\beta}{1+\beta^2},\quad \beta_3 = \frac{3\beta+\beta^3}{1+3\beta^2},\quad \beta_4 = \frac{4\beta+4\beta^3}{1+6\beta^2+\beta^4} \tag{2}$$

$$\beta_5 = \frac{5\beta+10\beta^3+\beta^5}{1+10\beta^2+5\beta^4},\quad \beta_6 = \frac{6\beta+20\beta^3+6\beta^5}{1+15\beta^2+15\beta^4+\beta^6},\quad \beta_7 = \frac{7\beta+35\beta^3+21\beta^5+\beta^7}{1+21\beta^2+35\beta^4+7\beta^6}, \text{ etc.}$$

2) Consulting a table of binomial coefficients, it is apparent these results follow from:

$$\beta_n = \sum_{k=0}^{[n/2]} \binom{n}{2k+1}\beta^{2k+1} \Big/ \sum_{k=0}^{[n/2]}\binom{n}{2k}\beta^{2k} \quad \begin{cases} [n/2] = \text{greatest integer in } n/2, \\ \binom{n}{m} = n!/m!(n-m)! \end{cases} \tag{3}$$

Indeed, this Ansatz satisfies the recursion relation in Eq. (1). Next, note that:

$$(1+\beta)^n = \sum_{m=0}^{n}\binom{n}{m}\beta^m = \binom{n}{0}\beta^0 + \binom{n}{1}\beta^1 + \binom{n}{2}\beta^2 + \binom{n}{3}\beta^3 + \cdots + \binom{n}{n}\beta^n$$

$$= \mathcal{D} + \mathcal{N}, \text{ where: } \mathcal{D} = \sum_{k=0}^{[n/2]}\binom{n}{2k}\beta^{2k}, \text{ and } \mathcal{N} = \sum_{k=0}^{[n/2]}\binom{n}{2k+1}\beta^{2k+1}. \tag{4}$$

Similarly...

$$(1-\beta)^n = \mathcal{D} - \mathcal{N} \quad \text{so}// \quad \begin{cases} \mathcal{D} = \frac{1}{2}[(1+\beta)^n + (1-\beta)^n], \\ \mathcal{N} = \frac{1}{2}[(1+\beta)^n - (1-\beta)^n]. \end{cases} \tag{5}$$

3) With these identities, we can form the aggregate velocity of Eq. (3)...

$$\rightarrow \beta_n = \mathcal{N}/\mathcal{D} = (1-\epsilon^n)/(1+\epsilon^n), \text{ with: } \boxed{\epsilon = (1-\beta)/(1+\beta)} \tag{6}$$

With $0<\beta<1 \Rightarrow 0<\epsilon<1$, and $0<\beta_n<1$ for any finite # of accelerations. For $n \to$ large: $\beta_n \simeq 1 - 2\epsilon^n$; we can at most approach $v = c$ from below; $v \leq c$ always.