DEPARTMENT OF PHYSICS

M.S. COMPREHENSIVE / PH. D. QUALIFYING EXAMINATION

MARCH 30, 1987

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MONDAY, MARCH 30, 1987, 8-12 AM

Answer each of the following questions, all of which carry equal weight. Begin your answer to each question on a new sheet of paper; solutions to different questions must not appear on the same sheet. Each sheet of paper must be labeled with your name and the problem number, in the upper right hand corner. When more than one sheet is submitted for a problem, be sure the pages are ordered properly. [Problems 1-8].

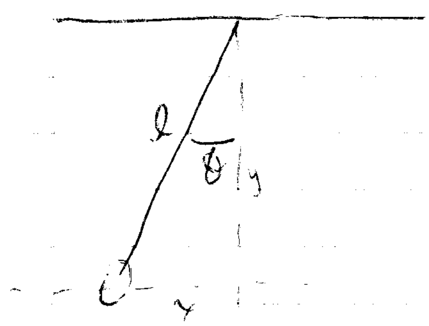
1. Consider a simple pendulum swinging in a vertical plane and consisting of a mass m attached to a string of length ℓ. After the pendulum is set into motion the length of the string is shortened at a constant rate. The suspension point remains fixed. Compute the Lagrangian and Hamiltonian functions. Compare the Hamiltonian and the total energy and discuss the conservation of energy for the system.

Pre-lim Class. Mech. Drumheller

(1)

Consider a simple ~~plane~~ pendulum, swinging in a vertical plane, which consists of a mass m attached to a string of length l. After the pendulum is set into motion the length of the string is shortened at a constant rate. The suspension point remains fixed. Compute the Lagrangian and Hamiltonian functions. Compare the Hamiltonian and total energy and discuss the conservation of energy for the system.

Sol'n



$$\frac{dl}{dt} = -\alpha = \text{constant}$$

$$\therefore \quad l = -\alpha t + \beta$$

$$x = l\sin\theta \qquad \dot{x} = \dot{l}\sin\theta + l\cos\theta\,\dot{\theta}$$

$$y = l\cos\theta \qquad \dot{y} = +\dot{l}\cos\theta - l\sin\theta\,\dot{\theta}$$

$$\dot{x} = -\alpha\sin\theta + l\cos\theta\,\dot{\theta}$$

$$\dot{y} = -\alpha\cos\theta - l\sin\theta\,\dot{\theta}$$

$$T = \tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2) = \tfrac{1}{2}m(\alpha^2 + l^2\dot{\theta}^2)$$

$$V = -mgy = -mgl\cos\theta$$

$$\boxed{L = \tfrac{1}{2}m(\alpha^2 + l^2\dot{\theta}^2) + mgl\cos\theta}$$

$$P_\theta = \frac{\partial L}{\partial\dot{\theta}} = ml^2\dot{\theta} \quad \text{or} \quad \dot{\theta} = P_\theta/ml^2 \qquad P_l = \frac{\partial L}{\partial \dot{l}} = c$$

$$H = P_\theta\dot{\theta} - L = \frac{P_\theta^2}{ml^2} - \frac{m}{2}\left(\alpha^2 + l^2\frac{P_\theta^2}{m^2 l^4}\right) - mgl\cos\theta$$

$$\boxed{H = \frac{P_\theta^2}{2ml^2} - \frac{m\alpha^2}{2} - mgl\cos\theta}$$

not equal

$$E_{total} = T + V = \frac{m}{2}(\alpha^2 + l^2\dot{\theta}^2) - mgl\cos\theta$$

$$\boxed{E_{total} = \frac{P_\theta^2}{2ml^2} + \frac{m\alpha^2}{2} - mgl\cos\theta}$$

The total energy is not conserved because the velocities can be written $\dot{x} = \dot{l}\sin\theta + l\cos\theta\,\dot{\theta} = -\alpha\sin\theta \mp \alpha t\cos\theta\,\dot{\theta} - \beta$

$\dot{y} = -\alpha\cos\theta + \alpha t\sin\theta\,\dot{\theta} + \beta$

which are time dependent.

2. A first-year graduate student is neglecting his studies and learning to ski at Bridger Bowl. He observes that his maximum velocity on the beginner slope (tilted 5^o to the horizontal) is only 20 km/hr, so he gathers his courage and ascends the Bridger lift to the steeper slopes (25^o to the horizontal). Assume that the only frictional force is linearly proportional to both the normal force of the skier on the slope and the skier's velocity; $f = \alpha N v$. What is the maximum speed of the skier down the steeper slope?

Neglect air resistance; assume the student has not yet learned to turn his skis, and hence accelerates straight down the hill. Also assume that the coefficient of sliding friction, α, is the same on the beginner and advanced slopes.

Prelim

(2) Yeckmans [illegible]

A first-year graduate student is neglecting his studies and learning to ski at Bridger Bowl. He observes that his maximum velocity on the beginner slope (tilted 5° to the horizontal) is only 20 km/hr, so he gathers his courage and ascends the Bridger lift to the steeper slopes (25° to the horizontal). Assume that the only frictional force is linearly proportional to both the normal force of the skier on the slope and the skier's velocity; $f = \alpha N v$. What is the maximum speed of the skier down the steeper slope?

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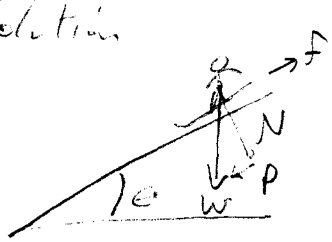
OK JLC

OK [illegible]

②

Solution



W = weight of skier.

N = normal force on slope $= W\cos\theta$

P = parallel force to slope $= W\sin\theta$

f = frictional force $= \alpha N v$

when the skier reaches maximum velocity, his acceleration is zero

$$a = \frac{1}{M}\left(P - \alpha N v\right) = 0 \Rightarrow V_{max} = \frac{P}{\alpha N} = \frac{1}{\alpha}\tan\theta$$

on the beginner's slope $V_{max} = 20$ km/hr

so

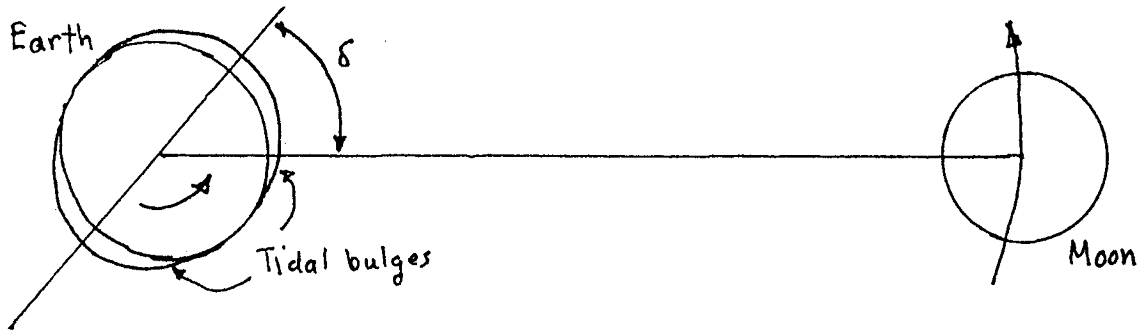
$$20\text{ km/hr} = \frac{1}{\alpha}\tan(5^\circ) \rightarrow \alpha = 4.37\times10^{-3}\ \frac{\text{hr}}{\text{km}}$$

Then the maximum velocity on the steeper slope will be

$$V_{max} = \frac{1}{\alpha}\tan(25^\circ) = \frac{\tan(25^\circ)}{\tan(5^\circ)}(20\text{ km/hr})$$

$$= (5.33)(20\text{ km/hr}) = \boxed{106.6\text{ km/hr}}$$

3. The Earth rotates (period = 24 hours) faster than the Moon orbits around the Earth (period = 28 days). Viscosity in the Earth causes the tidal bulges created on the Earth by the Moon to be rotated ahead of the line connecting the centers of the Earth and Moon:



Because the tidal lag angle, δ, is not zero, the Earth experiences a torque from the gravitational attraction of the Moon. This results in a slow transfer of angular momentum from the Earth's rotational motion to the Moon's orbital motion. The Apollo astronauts placed mirrors on the surface of the Moon so that the distance to the Moon could be accurately measured (to search for a gravitational anomaly known as the <u>Nordtvedt</u> effect); these lunar laser ranging measurements have shown that the mean distance to the Moon from the Earth is increasing at a rate of about 4 cm/year.

Assuming that the total angular momentum of the Earth–Moon system is conserved, calculate the rate at which the length of the day is currently increasing, dT/dt, where T is the length of the day. Note that dT/dt is a dimensionless number. By how much will the length of the day increase during one century?

You may assume that the Moon's orbit lies in the Earth's equatorial plane; also assume that the Moon's orbit is circular.

$G = 6.67 \times 10^{-11}\ m^3 sec^{-2} kg^{-1}$

Mass of Moon = 7.34×10^{22} kg

Mass of Earth = 5.98×10^{24} kg

Radius of Earth = 6.38×10^{6} m

Earth–Moon distance = 3.84×10^{8} m

Pre lim

(7) MECHANICS

OK - GT

HISCOCK

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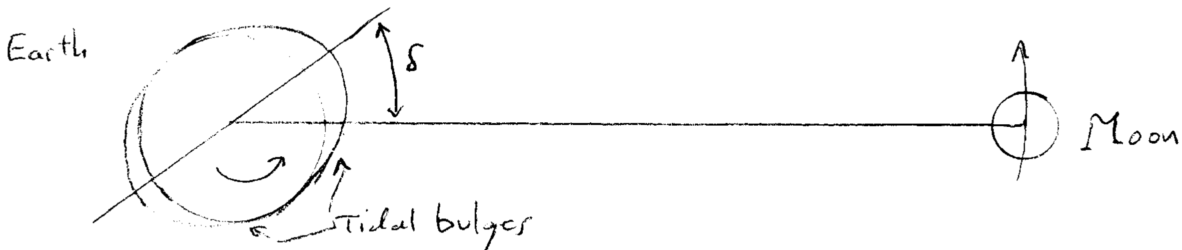

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Mass of Earth $= 5.98 \times 10^{24}$ kg

Radius of Earth $= 6.38 \times 10^{6}$ m

Earth-Moon distance $= 3.84 \times 10^{8}$ m

2

Solution:

(1) Let the orbital angular momentum of the Moon be L

$$L = mvr$$

where the orbital velocity v, is determined by $v^2 = GM/r$ [M = mass of Earth, m = mass of moon]

so

$$L = m\sqrt{GMr}$$

(2) The spin angular momentum of the Earth is given by

$$S = I\omega = \frac{2}{5}MR^2 \frac{2\pi}{T} = \frac{4\pi}{5}MR^2 T^{-1}$$

where T is the length of the "day".

(3) Conservation of angular momentum: the total angular momentum of the Earth-moon system is constant:

$$J = L + S \quad ; \quad \partial J/\partial t = 0$$

but:

$$\frac{\partial J}{\partial t} = \frac{\partial L}{\partial t} + \frac{\partial S}{\partial t} = 0$$

$$\frac{\partial L}{\partial t} = \frac{\partial L}{\partial r}\frac{dr}{dt} \qquad \frac{\partial S}{\partial t} = \frac{\partial S}{\partial T}\frac{dT}{dt}$$

so:

$$\frac{\partial J}{\partial t} = \frac{1}{2} m\sqrt{\frac{GM}{r}}\frac{dr}{dt} - \frac{4\pi}{5}MR^2 T^{-2}\frac{dT}{dt} = 0$$

so

$$\boxed{\frac{dT}{dt} = \frac{5}{8\pi}\frac{mT^2}{R^2}\sqrt{\frac{G}{Mr}}\frac{dr}{dt}}$$

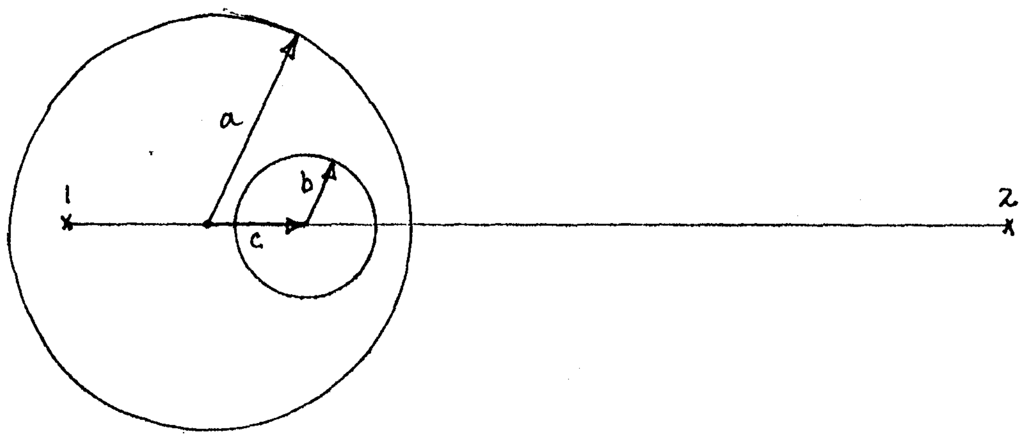
$$= \frac{5}{8\pi}\frac{(7.34\times10^{22}\,kg)(8.64\times10^4\,sec)^2}{(6.38\times10^6\,m)^2}\left[\frac{6.67\times10^{-11}\,m^3 sec^{-2} kg^{-1}}{(5.97\times10^{24}\,kg)(3.84\times10^8\,m)}\right]^{1/2}\left(\frac{.04\,m}{3.15\times10^7\,sec}\right)$$

$$\boxed{\frac{dT}{dt} = 5.80\times10^{-13}}$$

in one century (3.15×10^9 sec)

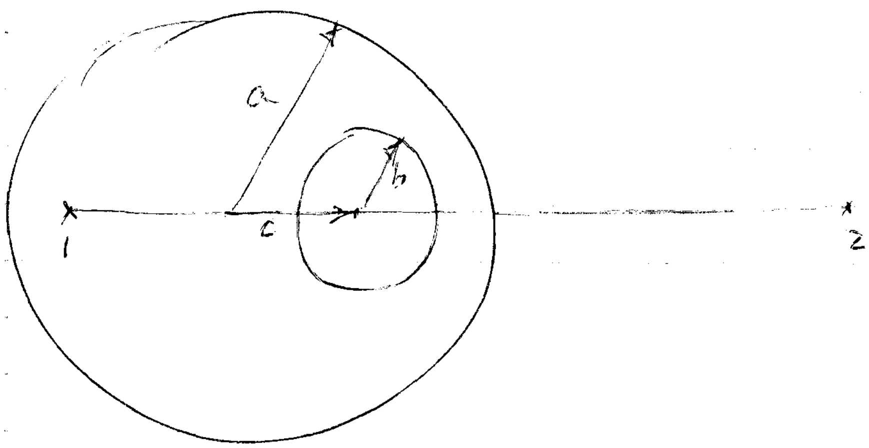
$$\boxed{\frac{\Delta T}{\Delta t} = 1.83\times10^{-3}\,sec/century}$$

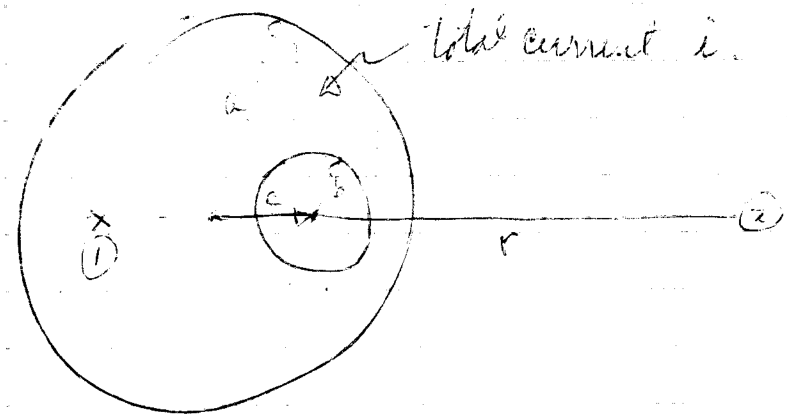
4. A long straight wire of radius a has a circular hole of radius b parallel to the axis at a distance c from the center ($b<c$) and carries a total current i. Calculate the magnetic field at points 1 and 2, assuming the current density is uniform.



(4)

A long straight wire of radius a has a circular hole of radius b parallel to the axis at a distance c from the center ($b < a$) and carries a total current i. Calculate the magnetic field at points 1 & and 2, assuming the current density is uniform





Current density $J = \dfrac{i}{\pi(a^2-b^2)}$

Same as two wires carrying same current density in opposite directions.

For pt. # 2: $\boxed{B = \dfrac{\mu_0\, J A}{2\pi\, r}}$ for $\hat{r} > a$, $\hat{r} > b$ from Ampere's law.

$$B_2 = \frac{\mu_0}{2\pi}\,\frac{i\,\pi a^2}{\pi(a^2-b^2)}\,\frac{1}{r} - \frac{\mu_0}{2\pi}\,\frac{i\,\pi b^2}{\pi(a^2-b^2)}\,\frac{1}{r-c}$$

$$= \frac{\mu_0 i}{2\pi}\left(\frac{1}{a^2-b^2}\right)\left(\frac{a^2}{r} - \frac{b^2}{r-c}\right)$$

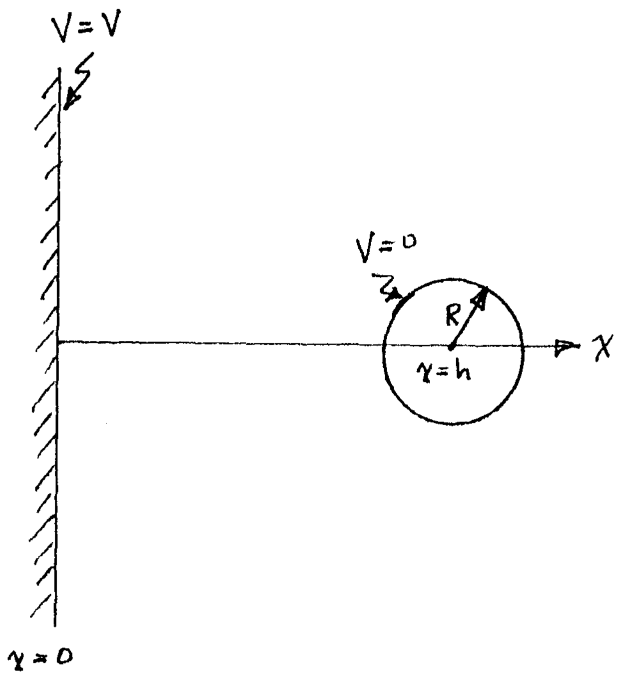
For pt. # 1 $\quad B = \dfrac{\mu_0\, J A\, r}{2\pi\, a^2} \quad r < a$

$$B_1 = \frac{\mu_0}{2\pi}\,\frac{i\,\pi a^2}{\pi(a^2-b^2)}\,\frac{r}{a^2} - \frac{\mu_0}{2\pi}\,\frac{i\,\pi b^2}{\pi(a^2-b^2)}\,\frac{1}{r+c}$$

$$= \frac{\mu_0}{2\pi}\,\frac{i}{a^2-b^2}\left(r - \frac{b^2}{r+c}\right)$$

5. Consider a conducting plate of area L^2 held at a voltage V and placed with its surface parallel to the y-z plane and located at x=0. A conducting tube of radius R and length L is grounded so that its potential is zero, and is placed parallel to the y axis at x=h.

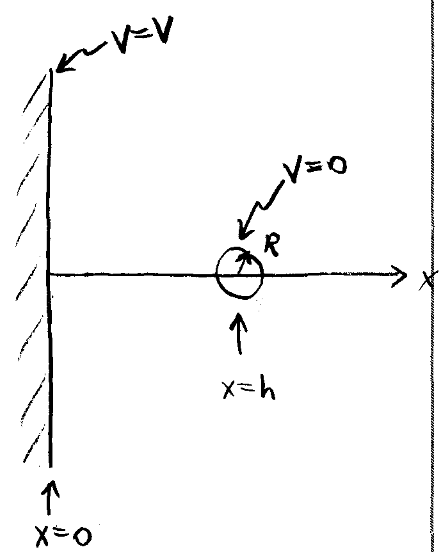
With R<<h<<L find an approximate value for the electric field on the x-axis near the surface of the plate.



E+M

(5) Consider a conducting plate of area L^2 held at a voltage V and placed with it's surface parallel to the y-z plane and located at $x=0$. A conducting tube of radius R and length L is grounded so that it's potential $V=0$ and is placed parallel to the y axis at $x=h$.

With $R \ll h \ll L$ find an approximate value for the electric field on the x-axis near the surface of the plate.



OK · GT

solution

Since $L \gg h$ we can assume for an approximate solution that the plate and tube are infinite.

Assume that the charge per unit length on the tube is $-\lambda$. To hold the plate at voltage V, we place an image tube at $x=-h$ with a charge per unit length of $+\lambda$.

Since $R \ll h$ we can approximate the electric field on the x-axis as that of an isolated tube. For an isolated tube

$$E(2\pi r)L = \frac{\lambda L}{\epsilon_0}$$

$$\rightarrow E = \frac{-\lambda}{2\pi r \epsilon_0}$$

Thus for the tube and it's image, the E-field on the positive x-axis between the plate and tube is

$$\vec{E} = \frac{\lambda}{2\pi\epsilon_0}\left[\frac{1}{(h+x)} + \frac{1}{(h-x)}\right]\hat{x}$$

To relate to the voltage we integrate for $x=0$ to $x=h-R$.

$$V(0) - V(h-R) = -\int_{h-R}^{0} \vec{E}\cdot d\vec{x} = \int_{0}^{h-R}\left(\frac{\lambda}{2\pi\epsilon_0}\right)\left[\frac{1}{(h+x)} + \frac{1}{(h-x)}\right]dx$$

$$= V = \frac{\lambda}{2\pi\epsilon_0}\Big[\ln(h+x) - \ln(h-x)\Big]_{x=0}^{x=h-R}$$

E+M soln (cont)

$$V = \frac{\lambda}{2\pi\epsilon_0}\left\{ \ln\left[\frac{(h+x)}{(h-x)}\right]\right\}_{x=0}^{x=h-R}$$

$$= \frac{\lambda}{2\pi\epsilon_0}\left[\ln\left(\frac{2h-R}{R}\right) - \ln(1)\right]$$

$$\approx \frac{\lambda}{2\pi\epsilon_0}\ln\left(2h/R\right) \qquad \text{since } R \ll h$$

Thus

$$\frac{\lambda}{2\pi\epsilon_0} = \frac{V}{\ln(2h/R)}$$

Now we can solve for $\vec{E}\,(x=0)$

$$\vec{E}\,(x=0) = \frac{\lambda}{2\pi\epsilon_0}\left(\frac{2}{h}\right)\hat{x}$$

$$\vec{E}(x=0) = \frac{2V\;\hat{x}}{h\ln(2h/R)}$$

6. A hot cathode plate produces electrons that traverse a short distance d to a second plate. The potential across the two parallel plates is V_o. Under conditions that result in a constant current density across the gap derive a relationship between the current density and V_o. Assume that the electrons move as independent particles.

compr? – GT 2

(6) A hot cathode plate produces electrons that traverse a short distance d to a second plate. The potential across the two parallel plates is V_0. Under conditions that result in a constant current density across the gap ~~comput~~ derive a relationship between the current density and V_0. Assume that the electrons move as independent particles.

Approach:

① $J = nqv$

$F = ma = qE$ $\qquad V_0 = \int_0^d E(x)\,dx$

$\nabla \cdot E = \rho/\varepsilon_0 = nq/\varepsilon_0$

② Find differential equation for n or v.

②' Invent trial soln.

③ Solve for $E(x)$, and V_0.

⑥

def. $J = q\,n(x)\,v(x) = \text{const.}$

F=ma $m \dfrac{dv}{dt} = q E(x)$

Gauss's law $\nabla \cdot E = \rho/\varepsilon_0$

$$m v \frac{dv}{dx} = q E(x) \qquad \left(v = \frac{dx}{dt} \right)$$

$$\frac{dE}{dx} = \frac{nq}{\varepsilon_0} = \frac{J}{\varepsilon_0} \frac{1}{v}$$

$$\frac{dE}{dx} = \left(\frac{m}{q}\right)\left(\frac{dv}{dx}\right)^2 + \frac{m}{q} v \frac{d^2v}{dx^2} = \frac{J}{\varepsilon_0} \frac{1}{v}$$

or $\left(\dfrac{m\varepsilon_0}{qJ}\right)\left[v\left(\dfrac{dv}{dx}\right)^2 + v^2 \dfrac{d^2v}{dx^2} \right] = 1$

let $v = A x^h$. $\quad v' = h A x^{h-1} \quad v'' = h(h-1) A x^{h-2}$

$$\left(\frac{m\varepsilon_0}{qJ}\right)\left[A x^h\, h^2 A^2 x^{2h-2} + A^2 x^{2h}\, h(h-1) A x^{h-2} \right] = 1$$

$$A^3 \left(\frac{m\varepsilon_0}{qJ}\right) \left[h^2 x^{3h-2} + h(h-1)\, x^{3h-2} \right] = 1$$

Solution exists for $h = \frac{2}{3}$ since then $x^{3h-2} = 1$.

$$h^2 = \frac{4}{9} \qquad h(h-1) = \frac{2}{3}\left(-\frac{1}{3}\right) = -\frac{2}{9}$$

$$A = \left[\frac{qJ}{m\varepsilon_0} \left(+\frac{9}{2}\right) \right]^{1/3}$$

$$v = \left(\frac{qJ}{m\varepsilon_0} \frac{9}{2} \right)^{1/3}$$

6

$$\frac{dE}{dx} = \frac{J}{\varepsilon_0}\left(\frac{m\varepsilon_0}{qJ}\frac{2}{9}\right)^{1/3} x^{-2/3}$$

$$E = \int_0^x dx\,\frac{dE}{dx} = \frac{J}{\varepsilon_0}\left(\frac{2m\varepsilon_0}{9qJ}\right)^{1/3} x^{1/3}\;3$$

Finally

$$V_0 = \int_0^d E\cdot dx$$

$$= \left(\frac{2m\varepsilon_0}{9qJ}\frac{J^3}{\varepsilon_0^3}\right)^{1/3} 3\,x^{4/3}\,\frac{3}{4}\Big|_0^d$$

$$V_0 = \left(\frac{2m}{9q\varepsilon_0^2}\frac{9^3}{4^3}\right)^{1/3} d^{4/3}\,J^{2/3}$$

$$V_0 = (\quad)\,J^{2/3}$$

$$\frac{81}{9} \qquad 729$$

7. In 1913 Bohr discovered that he could explain the emission spectrum of hydrogen if he quantized the orbital angular momentum of the orbiting electron. Show how this quantization leads to a quantization of the total energy of the electron plus proton. Derive an explicit expression for the quantized energy levels of hydrogen. For simplicity you may assume the mass of the proton is large compared to the mass of the electron.

CARLSTEN

(7) QM

Oh-GES

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Show how this quantization leads to a quantization of the total energy of the electron plus proton. Derive an explicit expression for the quantized energy levels of hydrogen. For simplicity you may assume the mass of the proton is large compared to the mass of the electron.

8. Consider a spinless particle of mass m moving in one-dimension, subject to a potential of the form:

$$V(z) = \begin{cases} -V_o \text{ for } |z|<a \\ 0 \text{ for } |z|>a \end{cases},$$

where $V_o > 0$. We will consider the eigenvalue problem for this potential.

1. Show that the eigenfunctions $\psi(z)$ of this potential can always be chosen to have a definite parity.

2. Obtain the transcendental equations from which the energy E of the bound states of either parity are to be formed.

3. Sketch the wave functions of the lowest three bound states, and indicate their parity.

4. Obtain a condition for the product $V_o a^2$ such that the well barely binds the first mode of odd parity (i.e., the parameters of the well must be such that the energy E of the first odd mode is vanishing small).

5. Is there always at least one mode of even parity? (A graphical discussion of the eigenvalue equation is useful to answer this question.)

Adolfo Prelim

⑧ QUANTUM MECHANICS OK - GT

Consider a spinless particle of mass m moving in one-dimension, subject to a potential of the form:

$$V(z) = \begin{cases} -V_0 & \text{for } |z| < a \\ 0 & \text{for } |z| > a \end{cases},$$

where $V_0 > 0$. We will consider the eigenvalue problem for this potential.

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5. Is there always at least one mode of even parity? (A graphical discussion of the eigenvalue equation is useful to answer this question.)

8

<u>Solution</u>:

1) $$-\frac{\hbar^2}{2m}\frac{d^2}{dz^2}\psi(z) + V(z)\psi(z) = E\psi(z)$$

Since $V(z) = V(-z) \implies$ if $\psi(z)$ is a solution for a given energy E, so is $\psi(-z)$. Moreover,

$$\psi_{even}(z) = \psi(z) + \psi(-z)$$

and

$$\psi_{odd}(z) = \psi(z) - \psi(-z)$$

are also solutions. It is in fact convenient to search for $\psi_{even}(z)$ and $\psi_{odd}(z)$ directly.

2) <u>Even solutions</u>

$$\psi(z) = \begin{cases} A\, e^{K(z+a)} & \text{for } z < -a \\ B \cos kz & \text{for } |z| < a \\ A\, e^{-K(z-a)} & \text{for } z > a. \end{cases}$$

where

$$K = \left[\frac{2m}{\hbar^2}|E|\right]^{1/2}$$

(E is negative.)

$$k = \left[\frac{2m}{\hbar^2}(V_0 - |E|)\right]^{1/2}$$

(8)

B Cs. at $z=+a$:
$$B \cos ka = A$$
$$-kB \sin ka = -\kappa A$$

$\Rightarrow$ $\boxed{k \tan ka = \kappa}$ even-mode eigenvalue eq.

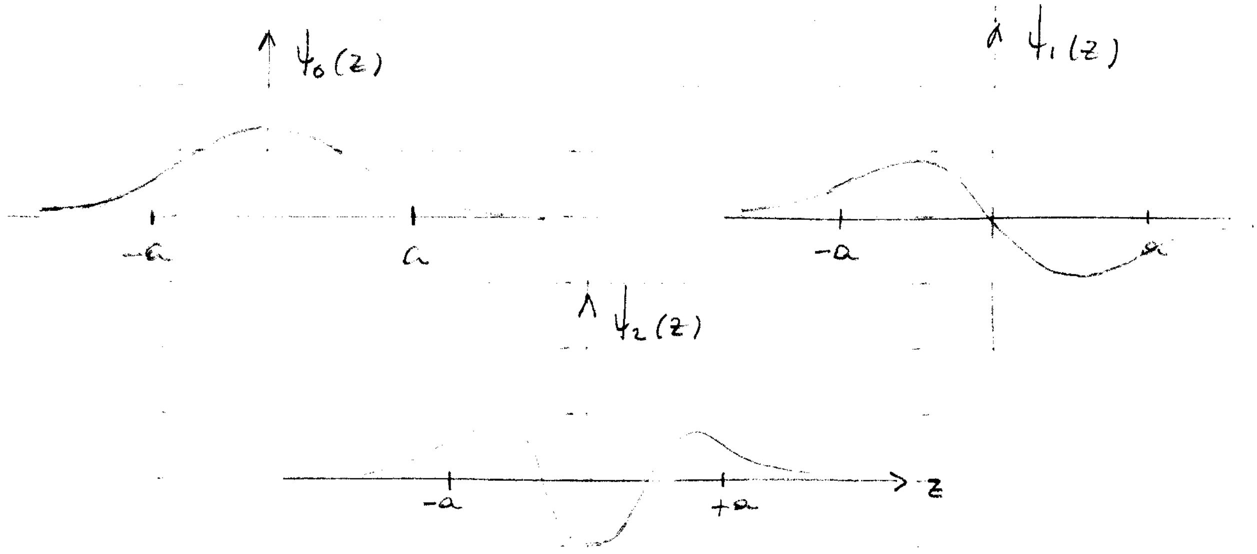
Odd solutions

$$\psi(z) = \begin{cases} A \; e^{\kappa(z+a)} & \text{for } z<-a \\ B \; \sin kz & \text{for } |z|<a \\ -A \; e^{-\kappa(z-a)} & \text{for } z>a \end{cases}$$

B.Cs. at $z=+a$.
$$B \sin ka = -A$$
$$Bk \cos ka = \kappa A$$

$\Rightarrow$ $\boxed{k \cot ka = -\kappa}$ odd-mode eigenvalue eq.

3)



8

4) If $|E| \to 0 \Rightarrow K \to 0$

$\Rightarrow k \cot ka \to 0 \qquad \Rightarrow \quad ka \to \frac{\pi}{2}$

For $|E| \to 0 \qquad k \to \left(\frac{2m}{\hbar^2} V_0\right)^{1/2}$

$$\Rightarrow \quad \left(\frac{2m}{\hbar^2} V_0\right)^{1/2} a = \frac{\pi}{2}$$

or $$\frac{2m}{\hbar^2} V_0 a^2 = \frac{\pi^2}{4}$$

$$\Rightarrow \quad V_0 a^2 = \frac{\pi^2 \hbar^2}{8m}$$

5) $$k \tan ka = K$$

$$K^2 = \frac{2m}{\hbar^2}|E| \qquad k^2 = \frac{2m}{\hbar^2}(V_0 - |E|)$$

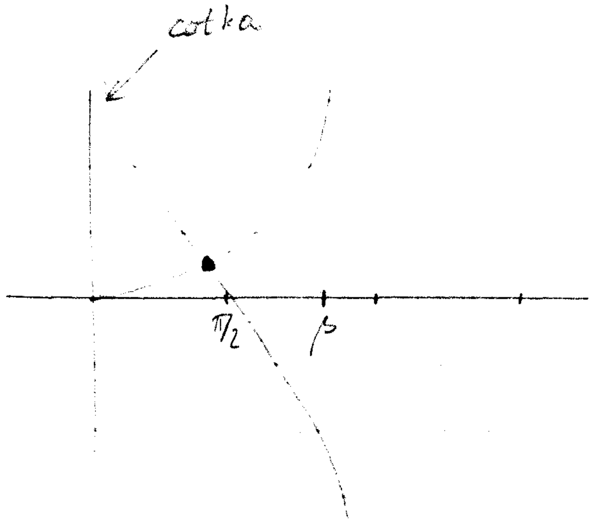
$$= \frac{2m}{\hbar^2} V_0 - K^2$$

$$\to \quad K^2 = \frac{2mV_0}{\hbar^2} - k^2$$

$$\cot ka = \frac{k}{K} = \frac{k}{\left[\frac{2mV_0}{\hbar^2} - k^2\right]^{1/2}}$$

$$= \frac{ka}{\left[\beta^2 - (ka)^2\right]^{1/2}} \quad ; \quad \beta^2 = \frac{2mV_0 a^2}{\hbar^2}$$

(8)



Always a solution, no matter what the value of $\beta = \frac{2mV_0}{\hbar^2} a^2$

DEPARTMENT OF PHYSICS

M.S. COMPREHENSIVE/PH.D. QUALIFYING EXAMINATION

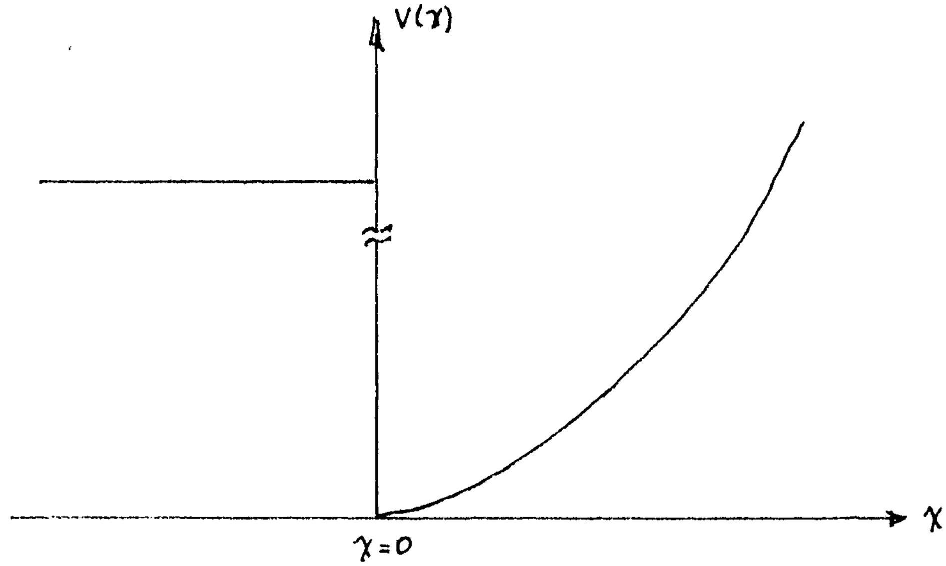
MONDAY, MARCH 30, 1987, 1-5 PM

Answer each of the following questions, all of which carry equal weight. Begin your answer to each question on a new sheet of paper; solutions to different questions must not appear on the same sheet. Each sheet of paper must be labeled with your name and the problem number, in the upper right hand corner. When more than one sheet is submitted for a problem, be sure the pages are ordered properly. [Problems 9-16].

9. Consider the one-dimensional potential

$$v(x) = \begin{cases} \frac{m}{2}\omega^2 x^2 & \text{for } x>0 \\ +\infty & \text{for } x<0 \end{cases}$$

Find the energy eigenvalues for this potential.



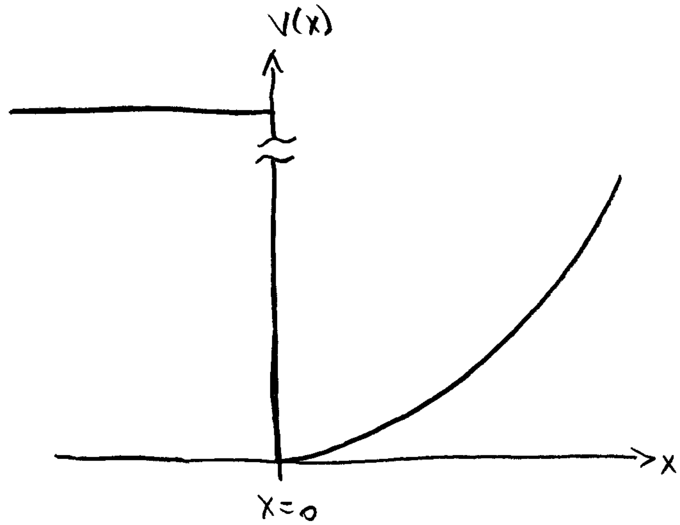
OK JEY
OK GT
O.K. (A.E)
OK - GT

(9)

Consider the one-dimentional potential

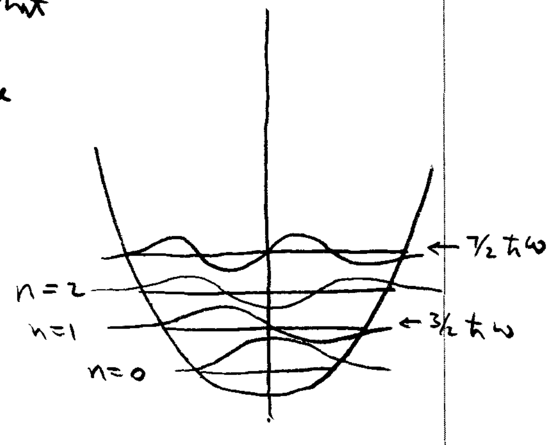
$$V(x) = \begin{cases} \frac{m}{2}\omega^2 x^2 & \text{for } x>0 \\ +\infty & \text{for } x<0 \end{cases}$$

Find the energy eigenvalues for this potential.



Note that this potential is the same as the harmonic oscillator for $x > 0$. Thus we expect similar solutions except that for the present case we can use only those solutions which are zero at the origin since $V = \infty$ for $x < 0$.

Recall that the harmonic oscillator energy eigenvalues are $E_n = (n + 1/2)\hbar\omega$ (see figure). Note that only odd values of n are zero at the origin. Thus these are the allowed wave functions for the one-sided harmonic oscillator. Therefore the allowed energy levels are



$$E_m = \frac{(4m+3)}{2}\hbar\omega \qquad m = 0, 1, 2, \ldots$$

10. a) A particle is in the orbital angular momentum superposition state $\sqrt{\frac{1}{3}}\,(Y_0^0 + Y_1^0 + Y_1^1)$ where the $Y_\ell^m(\theta,\phi)$'s are spherical harmonics. Find the expectation values of L^2 and L_z.

b) Consider a system of 3 particles, each with orbital angular momentum quantum number $\ell = 1$. How many linearly independent states can be forms with <u>total</u> orbital angular momentum quantum number $\ell=1$?

(10) Q. Mech GT

(a) A particle is in the orbital angular momentum superposition state $\frac{1}{\sqrt{3}}\left(Y_0^0 + Y_1^0 + Y_1^1\right)$ where the $Y_l^m(\theta,\phi)$'s are spherical harmonics. Find the expectation values of L^2 and L_z

(b) ~~Three pt~~ Consider a system of 3 particles, each with orbital angular momentum quantum number $l=1$. ~~Find~~ How many linearly independent states can be forms with total orbital angular momentum quantum number $l=1$?

Ans

(a) $\langle L^2\rangle = \frac{1}{3}\hbar^2\left\{0 + 1\cdot 2 + 1\cdot 2\right\} = \frac{4}{3}\hbar^2$

$\langle L_z\rangle = \frac{1}{3}\hbar\left\{0 + 0 + 1\right\} = \frac{\hbar}{3}$

(b) For two particles together, ~~there are~~ one finds $l = 2, 1,$ or 0. Adding a third gives $l = 3,2,1$ or $2,1,0$ or 1. That is, there are 3 $l=1$ multiplets, and the total # of indp. states is $3\cdot 3$ or $\underline{\underline{9}}$.

11. Evaluate the following integral:

$$I = \int_0^\infty dx \; \frac{x \sin kx}{x^2+a^2}$$

where k and a are real numbers, with $k > 0$.

Adolfo Dalm...

(11)

MATHEMATICAL PHYSICS

Evaluate the following integral:

$$I = \int_0^\infty dx \; \frac{x \sin kx}{x^2 + a^2} ,$$

where k and a are real numbers, with $k > 0$.

Solution

Since the integrand is an even function of x $\Rightarrow$

$$I = \frac{1}{2} \int_{-\infty}^{+\infty} dx \; \frac{x \sin kx}{x^2+a^2}$$

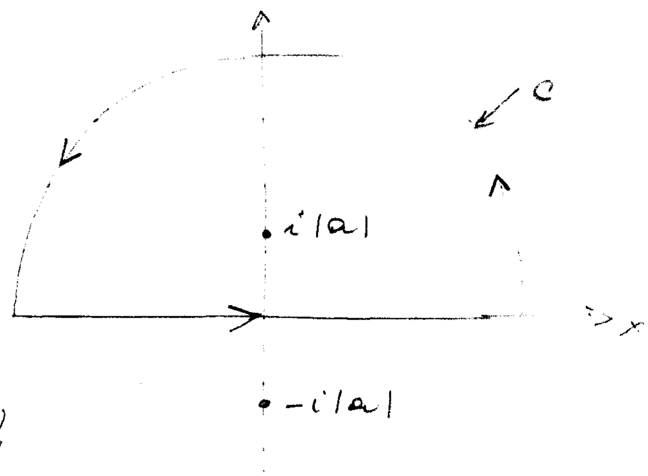
$$= \frac{1}{2} \int_{-\infty}^{+\infty} dx \; \frac{x \; \mathrm{Im}\, e^{ikx}}{x^2+a^2}$$

$$= \frac{1}{2} \, \mathrm{Im} \int_{-\infty}^{+\infty} dx \; \frac{x \, e^{ikx}}{x^2+a^2}$$

In the complex plane: $x = x_r + i x_i$

$$e^{ikx} = e^{ikx_r} \times e^{-kx_i} \longrightarrow 0 \quad \text{for } x_i > 0$$

$\therefore$



$$I = \frac{1}{2} \, \mathrm{Im} \oint_C dx \; \frac{x \, e^{ikx}}{(x+i|a|)(x-i|a|)}$$

$$= \frac{1}{2} \, \mathrm{Im} \left\{ 2\pi i \; \text{Res at } x = i|a| \right\}$$

$$= \frac{1}{2} \, \mathrm{Im} \; 2\pi i \; \frac{i|a| \, e^{ik i|a|}}{2i|a|} = \frac{\pi}{2} \, \mathrm{Im} \left\{ i \, e^{-k|a|} \right\}$$

$$\Rightarrow \quad I = \frac{\pi}{2} \, e^{-k|a|}$$

12. A quantity of monatomic ideal gas at the surface of the earth is at temperature T_o and pressure p_o. Suppose it rises to altitude h, expanding adiabatically as it goes. What is its temperature at this altitude?

<u>Hint</u>: In an adiabatic change of ideal gas, pv^γ = const., where $\gamma = 5/3$ for the monatomic gas. Further, $\frac{dp}{dh} = -\frac{mg}{RT}p$, where g is the gravitational constant and m the molar mass.

(12) Thermo

GT

A quantity of monatomic ideal gas at the surface of the earth is at temperature T_0 and pressure p_0. Suppose it rises to altitude h, expanding adiabatically as it goes. What is its temperature at this altitude?

Hint: In an adiabatic change of ideal gas, $pV^{\gamma} = \text{const.}$, where $\gamma = 5/3$ for the monatomic gas. Further, ~~[illegible]~~

$\frac{dp}{dh} = -\frac{mg}{RT} p$, where g is the gravitational constant and m the molar mass.

Ans Since $pV^{\gamma} = cst$ & $pV = nRT$ for an ideal gas,

$p^{1-\gamma} T^{\gamma} = cst$ for the adiabatic change.

Then $\frac{dT}{dh} = \frac{dT}{dp}\frac{dp}{dh}$. But $T \propto p^{-\frac{1-\gamma}{\gamma}}$

so $\frac{dT}{dp} = -\frac{1-\gamma}{\gamma} \frac{T}{p}$

& $\frac{dT}{dh} = -\frac{1-\gamma}{\gamma} \frac{T}{p}\left(-\frac{mg}{RT}\right) p = +\frac{mg}{R} \frac{1-\gamma}{\gamma}$

Thus $T - T_0 = +\frac{mg}{R}\frac{(1-\gamma)}{\gamma} h.$

$T = T_0 + \frac{mg}{R}\left(\frac{1-\gamma}{\gamma}\right) h.$

13. The Helmholtz free energy is defined as $F = -kT \ln Z$ where Z is the canonical partition function.

a. Calculate the free energy for a quantum mechanical harmonic oscillator of frequency ω.

b. What is F in this case when $kT \gg h\omega$?

c. From F calculate the entropy.

Hint: $$\sum_{n=0}^{\infty} e^{-na} = \frac{1}{1-e^{-a}}$$

MS Comp / Ph.D Prelim Stat. Mech. Drumheller

(1)

The free energy is defined for a quantum system as $F = -kT \ln Z$ where Z is the canonical partition function.

a. Calculate the free energy for a q.m. harmonic oscillator of frequency ω.

b. What is F in this case when $kT \gg \hbar\omega$?

c. ~~Calculate the~~ From the free energy, calculate the entropy.

Hint: $\sum_{i=0}^{\infty} e^{-ia} = \frac{1}{1-e^{-a}}$

Sol'n.

(13)

$\vdots$		$\vdots$
————	3	$3\hbar\omega$
————	2	$2\hbar\omega$
————	1	$\hbar\omega$
————	0	0
	n	ε_n

$$Z = e^{0} + e^{-\hbar\omega/\tau} + e^{-2\hbar\omega/\tau} + \cdots$$

$$\tau \equiv kT$$

$$\therefore Z = \sum_{n=0}^{\infty} e^{-n\hbar\omega/\tau} = \frac{1}{1 - e^{-\hbar\omega/\tau}}$$

a/ $$F = -kT \ln\left(\frac{1}{1-e^{-\hbar\omega/\tau}}\right) = kT \ln\left(1 - e^{-\hbar\omega/\tau}\right)$$

b/ $kT \gg \hbar\omega$ so $e^{-\hbar\omega/\tau} = 1 - \frac{\hbar\omega}{\tau} + \cdots$

$$F = kT \ln\left(1 - \left(1 - \frac{\hbar\omega}{\tau} + \cdots\right)\right) = kT \ln \frac{\hbar\omega}{\tau}$$

c/ $$\sigma = -\left(\frac{\partial F}{\partial \tau}\right)_V = -\frac{1}{k}\left(\frac{\partial F}{\partial T}\right)_V$$

$$= -\tau \frac{1}{1 - e^{-\hbar\omega/\tau}} \frac{\hbar\omega}{\tau^2} + \ln\left(1 - e^{-\hbar\omega/\tau}\right)$$

$$= \frac{\hbar\omega/kT}{e^{-\hbar\omega/kT} - 1} + \ln\left(1 - e^{-\hbar\omega/kT}\right)$$

14. Explain why the total electronic spin of the helium atom in its ground state is zero. What is the total spin in the 1st excited state?

(14) Quantum Mechanics GT

Explain why the total electronic spin of the helium atom in its ground state is zero. What is the total spin in the 1st excited state?

Ans: He has a filled 1s shell - that is both electrons have $l=0$ (symmetric spatial state) and so must be in an antisymmetric spin state ($S_{tot}=0$). In the 1st excited state, one electron moves to a state with principle quantum number 2, and [illegible] an antisymmetric spatial state [illegible]. The antisymmetric state reduces the electron-electron Coulomb energy, as compared to the symmetric state, and so is favored. This requires a symmetric (or $S=1$, triplet) state of spin.

15. NEWS FLASH: A supernova was observed exploding in the Large Magellanic Cloud on February 27, 1987. This is the first local supernova since the time of Galileo. A rumor is circulating that a burst of 7 MeV neutrinos was detected by the Mt. Blanc detector on the same day. Assuming that the supernova neutrinos arrived at Earth within 10 hours of the photons, what upper limit for the mass of the neutrino is implied by these observations?

Neglect photon scattering in the supernova atmosphere.

Distance to Large Magellanic Cloud $\simeq$ 150,000 light years.

(New) Prelim / US Comp OK-GT ✓

1 Relativity Hiscock

NEWS FLASH: A supernova was observed exploding in the Large Magellanic cloud on <u>February 27, 1987</u>. This is the first local supernova since the time of Galileo. A <u>rumor</u> is circulating that a burst of 7 MeV neutrinos was detected by the Mt. Blanc detector on the same day. Assuming that the supernova neutrinos arrived at Earth within <u>10 hours</u> of the photons, what <u>upper limit</u> for the mass of the neutrino is implied by these observations?

Neglect photon scattering in the supernova atmosphere.

Distance to Large Magellanic cloud $\simeq$ 150,000 light years.

Solution:

$E = 7\,MeV = \gamma m c^2$ where m is the (hypothetical) mass of the neutrino

If the neutrinos arrive within 10 hours of the photons after a 150,000 light-year trip, then the velocity of the neutrinos can differ from c by no more than:

$$1 - \frac{v}{c} \lesssim \frac{10 \text{ hours}}{150{,}000 \text{ years}} = 7.61 \times 10^{-9} .$$

$$\gamma = \frac{1}{\sqrt{1 - (v/c)^2}} \gtrsim \frac{1}{\sqrt{1 - (1 - 7.61 \times 10^{-9})^2}} \simeq 8106$$

$$\boxed{mc^2 = \frac{E}{\gamma} \le \frac{E}{8106} \simeq 864 \text{ eV}}$$

16. Laue back reflection diffraction (Bragg scattering) is commonly used to determine crystal structure. Describe how this measurement works.

[Expt φ] W107
[Sol. St.]

✓

Laue Back Reflection diffraction (Bragg scattering) is commonly used to determine crystal structure. Describe ~~in detail~~ how this measurement works.

See Kittel

SHOULD WE EXPECT ~~THAT~~ ~~THEY~~ will have had this?
? JED

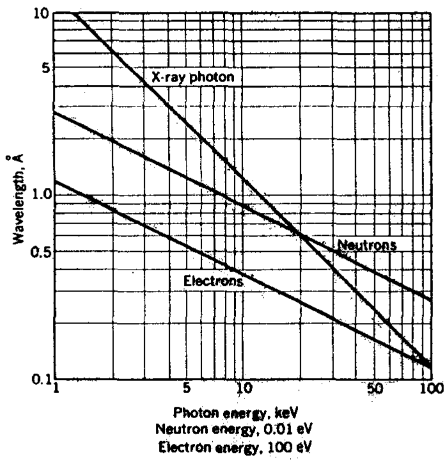


Figure 2 Wavelength versus particle energy, for photons, neutrons, and electrons.

barded by electrons has a strong line Cu$K\alpha$ at 1.5418 Å, in the middle of the important range. Copper makes a good target: it is an excellent heat conductor with a high melting point. Nuclei, because of their heavy mass, do not scatter x-rays effectively: x-rays see the electrons.

Neutrons. The energy of a neutron is related to its de Broglie wavelength λ by $\epsilon = h^2/2M_n\lambda^2$, where $M_n = 1.675 \times 10^{-24}$ g is the mass of the neutron. We recall that $\epsilon = p^2/2M_n$, and the wavelength λ is related to the momentum p by $\lambda = h/p$. In laboratory units,

$$\lambda(\text{Å}) \cong \frac{0.28}{[\epsilon(\text{eV})]^{1/2}} , \tag{2}$$

where ϵ is the neutron energy in eV. We have $\lambda = 1$ Å for $\epsilon \cong 0.08$ eV. Because of their magnetic moment, neutrons can interact with the magnetic electrons of a solid, and neutron methods are valuable in structural studies of magnetic crystals. In nonmagnetic materials the neutron interacts only with the nuclei of the constituent atoms.

Electrons. The energy of an electron is related to its de Broglie wavelength λ by $\epsilon = h^2/2m\lambda^2$, where $m = 0.911 \times 10^{-27}$ g is the mass of the electron. In laboratory units,

$$\lambda(\text{Å}) \cong \frac{12}{[\epsilon(\text{eV})]^{1/2}} . \tag{3}$$

Electrons interact strongly with matter because they are charged; they penetrate a relatively short distance into a crystal.

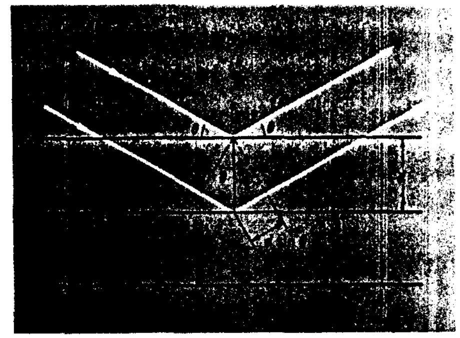


Figure 3 Derivation of the Bragg equation $2d \sin \theta = n\lambda$; here d is the spacing of parallel atomic planes and $2\pi n$ is the difference in phase between reflections from successive planes. The reflecting planes have nothing to do with the surface planes bounding the particular specimen.

Bragg Law

W. L. Bragg[1] presented a simple explanation of the diffracted beams from a crystal. Suppose that the incident waves are reflected specularly[2] from parallel planes of atoms in the crystal, with each plane reflecting only a very small fraction of the radiation, like a lightly silvered mirror. The diffracted beams are found when the reflections from parallel planes of atoms interfere constructively, as in Fig. 3. We treat elastic scattering in which the energy of the x-ray is not changed on reflection. Inelastic scattering, with the excitation of elastic waves, is discussed at the end of the chapter.

Consider parallel lattice planes spaced d apart, Fig. 4. The radiation is incident in the plane of the paper. The path difference for rays reflected from adjacent planes is $2d \sin \theta$, where θ is measured from the plane. Constructive interference of the radiation from successive planes occurs when the path difference is an integral number n of wavelengths λ, so that

$$2d \sin \theta = n\lambda . \tag{4}$$

This is the Bragg law. Although the reflection from each plane is specular, for only certain values of θ will the reflections from all parallel planes add up in phase to give a strong reflected beam. Of course, if each plane were perfectly reflecting, only the first plane of a parallel set would see the radiation and any wavelength would be reflected. But each plane reflects 10^{-3} to 10^{-5} of the incident radiation.

[1] W. L. Bragg, Proc. Cambridge Phil. Soc. **17**, 43 (1913). The Bragg derivation is simple but is convincing only because it reproduces the correct result.
[2] In specular (mirrorlike) reflection the angle of incidence is equal to the angle of reflection.

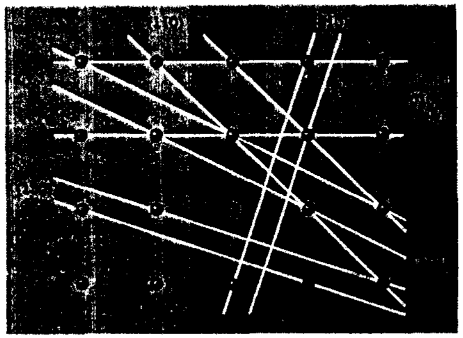


Figure 4 Several types of reflecting planes in a simple cubic crystal lattice. The planes shown are labeled by their indices. We have shown in each case a set of two parallel planes. The closest distance between parallel planes tends to decrease as the indices increase; thus high index reflections require shorter wavelengths.

The Bragg law is a consequence of the periodicity of the lattice. The law does not refer to the arrangement of atoms in the basis associated with each lattice point. The composition of the basis determines the relative intensity of the various orders n of diffraction from a given set of parallel planes. Bragg reflection can occur only for wavelength $\lambda \leq 2d$. This is why we cannot use visible light.

PERIMENTAL DIFFRACTION METHODS

The Bragg law (4) requires that θ and λ be matched: monochromatic x-rays of wavelength λ striking a three-dimensional crystal at an arbitrary angle of incidence will not in general be reflected. To satisfy the Bragg law requires an accident, and to create the accident it is necessary to scan in either wavelength or angle. The standard methods of diffraction used in crystal structure analysis are designed expressly to accomplish this. We describe three simple, older methods, still used by physicists; but for professional crystallography these techniques have been replaced by complicated precession camera methods.

Laue Method

In the Laue method (Fig. 5), a single crystal is stationary in a beam of x-ray or neutron radiation of continuous wavelength. The crystal selects and diffracts the discrete values of λ for which planes exist of spacing d and incidence angle θ satisfying the Bragg law. A source is used that produces a beam of x-rays over a wide range of wavelengths, perhaps from

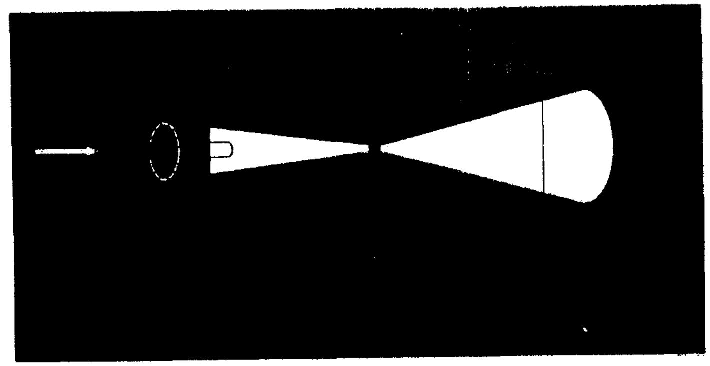


Figure 5 A flat plate camera. With a continuous spectrum x-ray beam and a single crystal specimen, the camera produces Laue patterns. The adjustable mount is convenient for the orientation of single crystals needed in other solid state experiments. The film B is used for back-reflection Laue patterns. (Courtesy of Philips Electronic Instruments.)

Figure 6 Laue pattern of a silicon crystal in approximately the [100] orientation. Note that the pattern is nearly invariant under a rotation of $2\pi/4$. The invariance follows from the fourfold symmetry of silicon about a [100] axis. The black center is a cut out in the film. (Courtesy of J. Washburn.)

0.2 Å to 2 Å. A pinhole arrangement produces a well-collimated beam. The dimensions of the single-crystal specimen need not be greater than 1 mm. Flat film receives the diffracted beams. The diffraction pattern consists of a series of spots, Fig. 6. The pattern will show the symmetry of the crystal: if a crystal has a fourfold axis of symmetry parallel to the beam, the Laue pattern will show fourfold symmetry. The Laue method is widely used to orient crystals for solid state experiments.