Spinor is not the same as a 4-comp. vector; its transformation properties under spatial rotations is different.† If we represent Dirac's ψ as...

$\rightarrow \psi = \begin{pmatrix} \varphi \\ \chi \end{pmatrix}$, φ & χ are each 2-component spinors,

then// $i\hbar \frac{\partial}{\partial t}\psi = \mathcal{H}\psi \Rightarrow i\hbar\frac{\partial}{\partial t}\begin{pmatrix} \varphi \\ \chi \end{pmatrix} = \begin{pmatrix} +mc^2 & c\boldsymbol{\sigma}\cdot\mathbf{p} \\ c(\boldsymbol{\sigma}\cdot\mathbf{p}) & -mc^2 \end{pmatrix}\begin{pmatrix} \varphi \\ \chi \end{pmatrix}$,

or// $\boxed{\begin{aligned} i\hbar\,\partial\varphi/\partial t &= mc^2\varphi + c(\boldsymbol{\sigma}\cdot\mathbf{p})\chi \\ i\hbar\,\partial\chi/\partial t &= -mc^2\chi + c(\boldsymbol{\sigma}\cdot\mathbf{p})\varphi \end{aligned}}$ Get two (coupled) 2-component Dirac eqtns, for a free particle of mass m. (12)

NOTE: if the Dirac eqtn as formulated above is to be parity-invariant -- as it must be to incorporate parity-invariant EM interactions -- then φ & χ must have opposite intrinsic parities. We can see this as follows...

$$\left[\begin{array}{l} \text{Under the parity operation: } P(x_k \rightarrow (-)x_k): \\ mc^2 \rightarrow (+)mc^2, \quad \boldsymbol{\sigma}\cdot\mathbf{p} \rightarrow (-)\boldsymbol{\sigma}\cdot\mathbf{p} \quad \text{since } \mathbf{p}\rightarrow(-)\mathbf{p}\text{, polar vector, } \boldsymbol{\sigma}\rightarrow(+)\boldsymbol{\sigma}\text{, axial vector.} \\ \text{So, Dirac eqtns (12) are P-invariant only if: } \varphi \rightarrow (+)\varphi,\ \chi \rightarrow (-)\chi. \end{array}\right\} \quad (13)$$

The requirement $P\begin{pmatrix} \varphi \\ \chi \end{pmatrix} = \begin{pmatrix} \varphi \\ -\chi \end{pmatrix}$ already shows that $\begin{pmatrix} \varphi \\ \chi \end{pmatrix}$ is not just a 4-vector.

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4) A major benefit gained from Dirac's formulation is that we can identify a (+)ve definite probability density. Get this from a continuity equation, which we derive in the standard fashion. Put $\mathbf{p} = -i\hbar\nabla$ into Dirac's Eqtn, so that...

① $i\hbar\frac{\partial}{\partial t}\psi = mc^2\beta\psi - i\hbar c\,\alpha_k\frac{\partial\psi}{\partial x_k}$, take Hermitian conjugate thru the eqtn. Use $\beta^\dagger = \beta$, and $\alpha_k^\dagger = \alpha_k$.

② $-i\hbar\frac{\partial}{\partial t}\psi^\dagger = mc^2\psi^\dagger\beta + i\hbar c\frac{\partial\psi^\dagger}{\partial x_k}\alpha_k$;

w// $\psi^\dagger = (\psi_1^*, \psi_2^*, \psi_3^*, \psi_4^*)$ a rowmatrix, and: $\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}$ a column matrix. (14)

NOTE: for matrix M: $(M\psi)_\alpha = M_{\alpha\beta}\psi_\beta$. By defn: $(\psi^\dagger M)_\alpha = \psi_\beta^* M_{\beta\alpha}$. If M is

† See L. Landau & E. Lifshitz "QM" (Addison-Wesley, 2nd ed., 1965), Secs. 55-58.

Dirac's Continuity Equation: def$^{\underline{n}}$s of ρ(density) & $\mathbb{J}$(current).

Hermitian, $M_{\beta\alpha} = M^*_{\alpha\beta}$, so: $(\psi^\dagger M)_\alpha = M^*_{\alpha\beta}\psi^*_\beta = (M\psi)^\dagger_\alpha$. This explains why, in ②, we have put $(\beta\psi)^\dagger = \psi^\dagger\beta$, etc. Now, multiply ① on the left by $\psi^\dagger$, and ② on the right by ψ... then subtract the eqtns to get...

$$\rightarrow i\hbar\left[\psi^\dagger\left(\frac{\partial\psi}{\partial t}\right) + \left(\frac{\partial\psi^\dagger}{\partial t}\right)\psi\right] = 0 - i\hbar\left[\psi^\dagger c\alpha_k\left(\frac{\partial\psi}{\partial x_k}\right) + \left(\frac{\partial\psi^\dagger}{\partial x_k}\right)c\alpha_k\psi\right]$$

or $\boxed{\frac{\partial\rho}{\partial t} + \nabla\cdot\mathbb{J} = 0}$ w/ $\rho = \psi^\dagger\psi = \psi^*_\mu\psi_\mu$ ← <u>Dirac probability density</u>,

$\mathbb{J} = \psi^\dagger c\alpha\psi$ ← <u>Dirac probability current</u>. (16)

REMARKS on Dirac's continuity eqtn, Eq. (16).

1. Eq. (16) ensures that the integrated Dirac density is time-independent, as...

$$\rightarrow \frac{\partial}{\partial t}\int_\infty \rho\, d^3x + \int_\infty(\nabla\cdot\mathbb{J})\,d^3x = 0 \Rightarrow \underline{\int_\infty \rho\, d^3x = \text{time-indpt cnst}}. \quad (17)$$

$\oint_S \mathbb{J}\cdot d\mathbb{S} \to 0$, for $\psi \to 0$ on S @ ∞.

2. $\rho = \psi^\dagger\psi$ can be interpreted as a probability density in Schrodinger's sense because: (a) $\rho = \psi^*_\mu\psi_\mu = |\psi_\mu|^2 \geq 0$, is non-negative everywhere, (18)

(b) $\int\rho\, d^3x = 1$, is a possible & appropriate normalization.

3. The Dirac current has 3 components, each with (−)ve parity...

$$\rightarrow J_k = \psi^\dagger c\alpha_k\psi = c(\varphi^\dagger\sigma_k\chi + \chi^\dagger\sigma_k\varphi) \quad \text{has (−) parity, since: } P(\varphi,\chi) = (\varphi,-\chi). \quad (19)$$

So $\mathbb{J}$ can really represent a <u>probability flow</u>, reversing sign when the space cds are inverted [(−)rightward flow ≡ (+) leftward flow]. Finally, by analogy with the Schrodinger & Klein-Gordon Eqtns, we remark that in Dirac's formulation the <u>velocity operator</u> must be $v_k = c\alpha_k$, since...

$$\left.\begin{matrix}\text{Schrodinger}\\ \text{Klein-Gordon}\end{matrix}\right\} J_k = \mathrm{Re}\left[\psi^*\left(\frac{\hbar}{im}\frac{\partial}{\partial x_k}\right)\psi\right] \Rightarrow v_k = \frac{\hbar}{im}\frac{\partial}{\partial x_k} \text{ is velocity operator;}$$

$$\text{Dirac Eqtn: } J_k = \psi^\dagger(c\alpha_k)\psi \Rightarrow v_k = c\alpha_k \text{ is velocity operator.} \quad (20)$$

This has amusing consequences, e.g. $\langle v_k\rangle = \pm c$ for all particles. More, later.

Standard Representation for Dirac Eq. The Gamma Matrices.

5) We shall now put Dirac's Eqtn into the "standard representation", a compact form which facilitates later relativistic maneuvers. We have...

$$\rightarrow \left(i\hbar \frac{\partial}{\partial t} + i\hbar c\, \alpha_k \frac{\partial}{\partial x_k} - mc^2 \beta\right)\psi = 0, \text{ for a free particle;}$$

$$\text{so}/\!\!/ \left(\frac{\partial}{\partial x_4} - i\alpha_k \frac{\partial}{\partial x_k} + \frac{mc}{\hbar}\beta\right)\psi = 0, \quad w/\!\!/ \; x_4 = ict. \qquad (21)$$

Multiply from the left by β, w/ $\beta^2 = 1$, and then write...

$$\rightarrow \underline{\underline{\left(\gamma_\mu \frac{\partial}{\partial x_\mu} + \frac{mc}{\hbar}\right)\psi = 0}}, \quad \text{where: } \underline{\underline{\gamma_4}} = \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \; 4\times 4, \qquad (22)$$

still summing over $\mu = 1$ to 4

$$\underline{\underline{\gamma_k}} = -i\beta\alpha_k = \begin{pmatrix} 0 & -i\sigma_k \\ +i\sigma_k & 0 \end{pmatrix};$$

$$\boxed{(\gamma_\mu p_\mu - imc)\psi = 0}, \quad w/\!\!/ \; p_\mu = -i\hbar\, \partial/\partial x_\mu = \text{4-momentum}. \qquad (23)$$

This last form will be useful when we move from the present free particle case to the presence of an external EM field via $p_\mu \rightarrow p_\mu - (q/c)A_\mu$.

The γ_μ, called the Dirac "gamma matrices", are 4×4 Hermitian matrices formed from α_k & β of Eq. (11), as defined in Eq. (22). All the γ_μ are traceless, and have eigenvalues ± 1. These features can be discovered from their anticommutation rule... just as the set (α_k, β), they obey...

$$\underline{\underline{\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = \{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}}} \qquad (24)$$

{ } denotes anti-commutator: i.e. $\{A, B\} = AB + BA$.

Moving from the (α_k, β) form of the Dirac Eqtn [Eq. (21)] to the (γ_μ) form [Eq. (22)] changes the appearance of the continuity equation. In Eq. (16), recall we had...

$$\rightarrow J_k = c\psi^\dagger \alpha_k \psi, \quad \rho = \psi^\dagger\psi, \quad \text{and: } \nabla\cdot \mathbf{J} + \partial\rho/\partial t = 0. \qquad (25)$$

In the γ_μ rep$^{\underline{n}}$, we have: $\gamma_k = -i\beta\alpha_k$, $\gamma_4 = \beta$. Then (since $\beta^2 = 1 \Rightarrow \beta^{-1} = \beta$): $\alpha_k = i\beta\gamma_k$, and the components in (25) can be written...

$$\rightarrow J_k = ic\,\psi^\dagger \beta\gamma_k\psi, \quad \rho = \psi^\dagger\beta^2\psi = \psi^\dagger\beta\gamma_4\psi. \qquad (26)$$