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Φ520 Final Exam (in class, 3 hrs).

Mon. 10 May '93

This exam is open-book, open-notes, and is worth 300 points total. There are 6 problems on 3 pages, with point-values as marked. For each problem, put a box around your answer. Number your solution pages consecutively, write your name on p. 1, and staple the pages together before handing them in.

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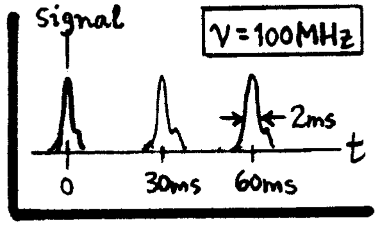
①[50 pts]. Signals from a pulsar in the Crab Nebula, $\simeq 6500$ light years distant from earth, can be detected at radio frequencies: $\nu = 100$ MHz. The signals consist of a steady train of identical pulses, each with temporal width $\Delta t = 2$ ms, repeated at 30 ms intervals.



(A) Assume the pulsewidth Δt is due to velocity dispersion of the pulse in transit -- in turn due to a finite photon mass m. Find an upper limit on m, from above data.

(B) Find a number for your limit on m as a ratio: m/m_e, w/ m_e = electron mass.

It helps to know: $m_e = 9.1\times10^{-28}$ gm, $c = 3.0\times10^{10}\ \frac{\text{cm}}{\text{sec}}$, $h = 6.6\times10^{-27}$ erg-sec.

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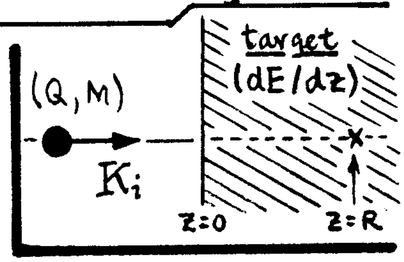
②[50 pts]. A high-energy particle (Q,M) travels along the z-axis, initially with kinetic energy K_i. At $z=0$, it strikes a target, and -- traveling in a straight line -- it penetrates to a depth $z=R$, where it stops. R is called the particle's "range". During $z=0\to R$, (Q,M) loses energy at a rate: $dE/dz = -A\left[(U/K)+(K/U)\right]$, w/ A & U are (+)ve cnsts specific to the target material, and K is (Q,M)'s instantaneous kinetic energy.



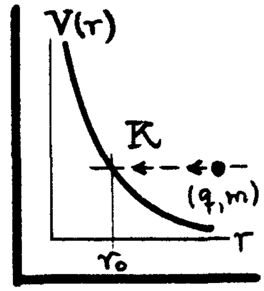
(A) Sketch $\left|\frac{dE}{dz}\right|$ vs. K. What role does U play? Typically, what is the size of U?

(B) Calculate (Q,M)'s range R as a fcn of the incident kinetic energy K_i.

(C) Find asymptotic forms for $R(K_i)$ for low & high energies ($K_i \ll Mc^2$ & $K_i \gg Mc^2$). Make a rough sketch of how $R(K_{in})$ varies with K_{in}.

(D) For a calibrated target material (A & U known), measurements of the range R can be used to measure the energy K_{in}. Is the method more accurate at high or low energy?

③ [50pts]. A non-relativistic particle of charge q & mass m, with initial kinetic energy K, makes a head-on collision with a fixed central force field. The interaction is repulsive, as specified by a potential $V(r)$: $V(r)$ increases as the separation r decreases, and $V(r) > K$ when $r < r_0$. So r_0 = "distance of closest approach" for (q,m) in the event.



(A) Show that the total energy radiated by q during the collision is:

$$\Delta W = \frac{4}{3c}(q/mc)^2 \sqrt{m/2} \int_{r_0}^{\infty} (dV/dr)^2 [K - V(r)]^{-1/2} dr.$$

HINTS: Assume $\Delta W \ll K$. Note Newton II in this case is $\frac{dp}{dt} = -dV/dr$.

(B) Let $V(r) = V_0 \exp(-r/a)$, w/ V_0 = cnst $> K$, and a = cnst (range). Find ΔW of part (A) explicitly for a collision of (q,m) with this field. Integrals are doable.

(C) In part (A), you assumed the radiation ΔW was $\ll$ kinetic energy K. Now show (numerically) that this assumption is justified for $V(r)$ of part (B), when (q,m) is an electron, and the range: $a \sim \hbar^2/me^2 = 0.53 \times 10^{-8}$ cm is of atomic dimensions.

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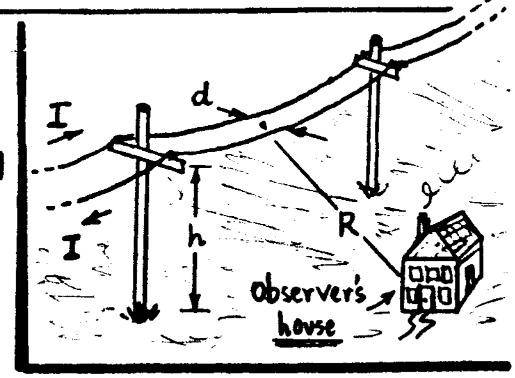
④ [50pts]. In class, we mentioned the phenomenon of "ELF radiation", i.e. the broadcast of Extra Low Frequency EM waves (e.g. from power lines at 60 Hz) which might be a health hazard. Consider the sketch: an observer's house is at distance R~100m from power lines which are at height h~10m. The lines, separated by d~1m, carry current I~100A at 60 Hz, phased so that the currents are (instantaneously) in opposite directions. If the power lines are a major feeder, the current may be delivered at voltage ~7200V.



(A) Show that any EM radiation, as such, is entirely negligible in this system.

(B) The observer is not exposed to EM radiation. What EM fields does he see?

(C) Estimate the size of the fields in part B, for the given geometry and for $\begin{cases} I = 100\text{A} \\ V = 7200\text{V} \end{cases}$.

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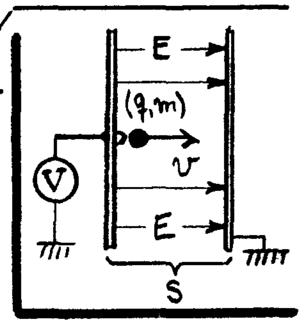
⑤ [50 pts]. A relativistic charged particle (q, m) moves in an external EM field specified by a 4-potential $A^\alpha = (\phi, \mathbf{A})$. In a Lagrange formalism that treats the particle's 4-position x^α and 4-velocity u^α as generalized coordinates, Hamilton's principle yields the Euler equations for the particle Lagrangian L:

$$\boxed{\frac{d}{d\tau}(\partial L/\partial u^\alpha) = \partial_\alpha L}$$ τ = particle propertime, $\partial_\alpha = \partial/\partial^\alpha = (\partial/\partial^0, \nabla)$ {covariant del}.

Evidently, L must be a Lorentz scalar for these eqtns to be covariant.

(A) Show that for (q, m) coupled to the field A^α, a choice for L that gives the correct equation-of-motion is: $\underline{L = \frac{1}{2} m u_\alpha u^\alpha + (q/c) u_\alpha A^\alpha}$.

(B) Find the canonical momenta P_α for the Lagrangian L of part (A). Show that the Hamiltonian $\mathcal{H}$ is this formulation is a Lorentz scalar, and find its value. How could this $\mathcal{H}$ be used in a quantum-mechanical context?



⑥ [50 pts]. To try to "see" relativistic effects, try accelerating electrons in a parallel plate capacitor. Your capacitor (with an electron source at one plate) can be placed in a vacuum, so that a beam can be fired across the gap. The capacitor plate separation, S, can be adjusted from $\underline{S = 0.1 \text{ cm to } S = \text{few cm}}$. To accelerate the beam across the gap, you have available a DC voltage supply with a maximum output of $\underline{V_M = 100\,\text{kV}}$ (kilovolts). You also have some fast electronics, capable of resolving electron pulse arrival times to $\sim \underline{0.1\,\text{ns}}$. The idea is to "see" relativistic transit time effects.

(A) If the electron pulse starts from the LH plate at time $t = 0$, at velocity $v = 0$, find relativistic forms for $v(t)$ and the distance traveled at time t.

(B) By adjusting V & S, find the maximum value of $\beta = \frac{v}{c}$ possible for this expt.

(C) Find a relation between gap transit times t_r & t_{nr}, t_r for the relativistic case & t_{nr} calculated non-relativistically. By what % does t_r differ from t_{nr} for the max. β found in part (B)? Are you likely to "see" this difference?

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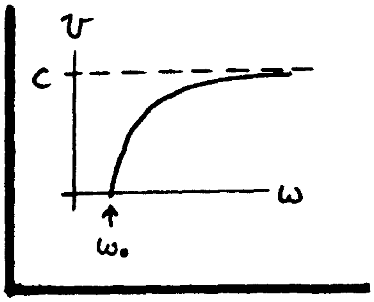
① [50 pts]. Photon mass limit from pulsar data.

1. If the photon has mass m, then it obeys a dispersion relation:

$$\rightarrow \omega = \sqrt{k^2c^2 + \omega_0^2} \quad \left\{ \begin{array}{l} \omega = 2\pi\nu = \text{frequency},\ k = \text{wave \#},\ c = \text{light speed}, \\ \text{and: } \omega_0 = mc^2/\hbar,\ \hbar = \text{Planck cnst.} \end{array} \right. \tag{1}$$

The photon group velocity is then...



$$\rightarrow v = \frac{\partial\omega}{\partial k} = c \cdot \frac{kc}{\sqrt{k^2c^2+\omega_0^2}} = c\sqrt{1-(\omega_0/\omega)^2}\ ; \tag{2}$$

so, $v \simeq c\left[1 - \frac{1}{2}(\omega_0/\omega)^2\right]$, for $\omega_0 \ll \omega$;

and, $$\frac{\partial v}{\partial\omega} \simeq c\,\frac{\omega_0^2}{\omega^3}, \text{ for } \omega_0 \ll \omega. \tag{3}$$

Signals at frequencies in a range $\Delta\omega$ about ω thus show a velocity dispersion Δv of size:

$$\Delta v \simeq (\partial v/\partial\omega)\,\Delta\omega, \quad \text{i.e.,} \quad \underline{\underline{\frac{\Delta v}{c} \simeq \left(\frac{\omega_0}{\omega}\right)^2 \frac{\Delta\omega}{\omega}}}. \tag{4}$$

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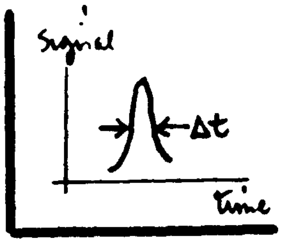
2. If the signal pulse is spread out in time by Δt by the velocity dispersion just calculated, then -- since the pulse has been in transit for time D/c, where D is the distance to the source -- we can write



$$\rightarrow \frac{D}{c}\Delta v \leqslant c\Delta t, \quad \text{or,} \quad \frac{\Delta v}{c} \leqslant \Delta t/(D/c),$$

so, $$\text{Eq.(4)} \Rightarrow \boxed{\left(\frac{\nu_0}{\nu}\right)^2 \frac{\Delta\nu}{\nu} \leqslant \left(\frac{\Delta t}{D/c}\right)}. \tag{5}$$

We have converted to linear freq. $\nu = \frac{\omega}{2\pi}$. Now $\nu_0 = mc^2/h$.

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3. The frequency spread in Eq. (5) is $\Delta\nu \simeq 1/\Delta t$ (per Fourier Thm), and so the upper limit on the photon mass term is

$$\boxed{\nu_0 = mc^2/h \lesssim (\nu\Delta t)\sqrt{\nu/(D/c)}.} \qquad (6)$$

If $\nu = 100$ MHz, $\Delta t = 2$ ms, and $D = c \times 6500$ years†

$$\rightarrow \nu_0 = \frac{mc^2}{h} \lesssim 10^8 \times 2\times10^{-3}\sqrt{\frac{10^8}{6500\times 3.156\times10^7}} = \underline{\underline{4.42\times10^3 \text{ Hz}}}. \qquad (7)$$

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4. In (7): $\nu_0 = \frac{m}{m_e}(m_e c^2/h)$, and for the electron...

$$m_e c^2/h = c/(h/m_e c) = \frac{3\times10^{10}\text{ cm/sec}}{2.43\times10^{-10}\text{cm}} = 1.23\times10^{20}\text{ Hz}. \qquad (8)$$

So Eq. (7) reads...

$$1.23\times10^{20}\text{ Hz} \times \frac{m}{m_e} \lesssim 4.42\times10^3\text{ Hz}$$

or $$\boxed{m/m_e \lesssim 3.6\times10^{-17}}, \text{ at } \nu = 100\text{ MHz}. \qquad (9)$$

This limit is ~4 orders of magnitude <u>less</u> sensitive than that established from geophysical data. To compete with the geophysical data, the pulse measurements here would have to be pushed down to frequencies $\nu \simeq 200$ kHz. Earth's ionosphere prevents measurements below $\nu \sim 10$ MHz, and so at best $m/m_e < 10^{-18}$ from pulsars.

† 1 year = 3.156×10^7 sec.

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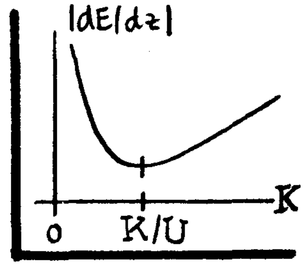
② [50 pts]. Find range of (Q, M) stopping @ $dE/dz = -A[(U/K)+(K/U)]$, K = kinetic energy.

(A) $|dE/dz| = A[(U/K)+(K/U)]$ vs. K, is sketched at right. The significance of U is that K = U locates the minimum of energy loss curve. From Jkⁿ Fig. 13.4 or 13.5, the minimum of the curve occurs at kinetic energy $U = K \simeq Mc^2$, the particle rest energy.



(B) Since $K = E - Mc^2$, the (relativistically correct) energy loss eqtn can be written:

$$\rightarrow \frac{dK}{dz} = -A\left[\frac{U}{K} + \frac{K}{U}\right], \text{ w/ } \lambda \frac{dk}{dz} = -\left(\frac{1}{k} + k\right) \text{ w/ } k = K/U, \ \lambda = U/A. \qquad (1)$$

This eqtn is easily integrated from $k = k_{in}$ @ $z = 0$ to a general interior point:

$$\lambda \int_{k_{in}}^{k} \frac{k\,dk}{1+k^2} = -\int_0^z dz \Rightarrow z(k) = \frac{\lambda}{2} \ln\left(\frac{1+k_{in}^2}{1+k^2}\right). \qquad (2)$$

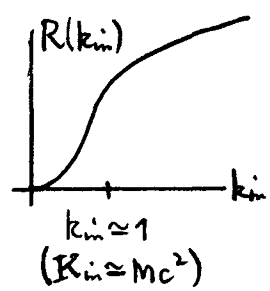
$\lambda = U/A$ has dimensions energy / $\frac{\text{energy}}{\text{length}}$ = length, and $z(k)$ is the distance that (Q, M) has traveled by the time its K.E. has dropped to $k \leq k_{in}$ (in units of U). The particle stops when $k \to 0$, so its "range" is $R = z(0)$, or...

$$\boxed{R(k_{in}) = \frac{\lambda}{2} \ln(1 + k_{in}^2)}, \text{ w/ } \lambda = \frac{U}{A},\ k_{in} = \frac{K_{in}}{U}. \qquad (3)$$

(C) At low incident K.E.'s, $K_{in} \ll Mc^2 \simeq U$, so in Eq. (3) $k_{in} \ll 1$ and $\ln(1+k_{in}^2) \simeq k_{in}^2$. At $K_{in} \gg Mc^2$, $\ln(1+k_{in}^2) \simeq 2\ln k_{in}$.

So/

$$R(k_{in}) \simeq \begin{cases} \frac{1}{2}\lambda k_{in}^2, & \text{for low energies } (K_{in} \ll Mc^2), \\ \lambda \ln(k_{in}), & \text{for high energies } (K_{in} \gg Mc^2). \end{cases} \qquad (4)$$



(D) From the asymptotic forms in Eq. (4), we easily find the fractional errors...

$$\left[\begin{array}{l} \text{@ low energy } (K_{in} \ll Mc^2): \ \frac{dk_{in}}{k_{in}} = \frac{1}{2}(dR/R), \\ \text{@ high energy } (K_{in} \gg Mc^2): \ \frac{dk_{in}}{k_{in}} = (\ln k_{in}) \frac{dR}{R}. \end{array}\right. \qquad (5)$$

For a given fractional error in the range determination, the fractional error in energy is smaller at low energies. Method is better at low K_{in}.

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③ [50 pts]. Radiation during a (nonrelativistic) scattering event.

(A) Total energy radiated is: $\Delta W = \int_{-\infty}^{\infty} P\,dt$, w/ $P = (2q^2/3c^3)|dv/dt|^2$ the Larmor radiation rate. But the acceleration $dv/dt = \frac{1}{m}(dp/dt)$, and since the collision is head-on (along r-coordinate only), then by Newton II: $|dp/dt| = |(-)dV/dr|$. Hence: $P = (2q^2/m^2c^3)|dV/dr|^2$, and

$$\longrightarrow \Delta W = (2q^2/3m^2c^3)\int_{t=-\infty}^{t=+\infty}[dV/dr]^2\,dt. \qquad (1)$$

Assume the radiation loss ΔW is small compared to the incident energy K. Then mechanical energy is conserved, so that the particle velocity at any r, i.e. $v = dr/dt$ is such that...

$$\frac{1}{2}mv^2 + V(r) = K \Rightarrow v = \frac{dr}{dt} = \sqrt{\frac{2}{m}}\,[K - V(r)]^{1/2}$$

$$\xrightarrow{\text{soy}} dt = \sqrt{\frac{m}{2}}\,dr/[K - V(r)]^{1/2}. \qquad (2)$$

Use this to convert Eq. (1) to an integration over r, noting that the collision is symmetric in time about the closest approach: $-\infty \leq t \to 0 \Rightarrow \infty \geq r \geq r_0$, and: $0 \leq t \leq \infty \Rightarrow r_0 \leq r \leq \infty$. Then, as required:

$$\Delta W = \frac{2q^2}{3m^2c^3}\cdot 2\int_{r_0}^{\infty}[dV/dr]^2\sqrt{\frac{m}{2}}\,dr/[K - V(r)]^{1/2},$$

soy
$$\boxed{\Delta W = \frac{4}{3c}(q/mc)^2\sqrt{\frac{m}{2}}\int_{r_0}^{\infty}\left(\frac{dV}{dr}\right)^2[K - V(r)]^{-1/2}\,dr.} \qquad (3)$$

The closest approach distance is defined by $V(r_0) = K$.

) For: $V(r) = V_0 e^{-r/a}$, have: $dV/dr = -\frac{1}{a}V(r)$. The integral in (3) is:

$$J(K) = \frac{1}{a^2}\int_{r_0}^{\infty}\{[V(r)]^2/\sqrt{K - V(r)}\}\,dr. \qquad (4)$$

Define a new variable $z = K - V(r)$, so that...

$$r = r_0 \Rightarrow z = 0, \quad r = \infty \Rightarrow z = K;$$

$$dz = -(dV/dr)\,dr = \frac{1}{a}V(r)\,dr, \quad \text{so}/\!/ \quad dr = [a/V(r)]\,dz;$$

and/

$$J(K) = \frac{1}{a^2}\int_{z=0}^{z=K} \left\{[V(r)]^2/\sqrt{z}\right\}\frac{a}{V(r)}\,dz = \frac{1}{a}\int_0^K \frac{dz}{\sqrt{z}}\{K - z\}$$

$$= \frac{1}{a}\left\{K\int_0^K \frac{dz}{\sqrt{z}} - \int_0^K \sqrt{z}\,dz\right\}$$

by/

$$J(K) = \frac{1}{a}\left\{2K\sqrt{z}\Big|_0^K - \frac{2}{3}z^{3/2}\Big|_0^K\right\} = \frac{4}{3a}K^{3/2}. \tag{5}$$

With this result, the radiation loss of Eq. (3) is...

$$\Delta W = \frac{4}{3c}(q/mc)^2\sqrt{\frac{m}{2}}\cdot\frac{4}{3a}K^{3/2}$$

by/

$$\boxed{\Delta W = \frac{8}{9c}\left(\frac{r_0}{a}\right)\sqrt{\frac{2}{m}}\,K^{3/2}}, \quad r_0 = q^2/mc^2 = \text{classical charge radius of } (q/m). \tag{6}$$

(C) If (q,m) is an electron, and $a \sim \hbar^2/me^2$ (Bohr radius) is of atomic dimensions, then in (6): $r_0/a \sim (e^2/mc^2)/(\hbar^2/me^2) = \alpha^2$, where $\alpha = e^2/\hbar c \simeq 1/137$ is the fine structure constant. And if $K = \frac{1}{2}mv^2$ (at ∞ separation), then $\sqrt{2/m}\,K^{1/2} = v$. Consequently, the ratio...

$$\Delta W/K = \frac{8}{9c}(r_0/a)\sqrt{\frac{2}{m}}\,K^{1/2} \sim \frac{8}{9}\alpha^2\frac{v}{c}$$

i.e./

$$\boxed{\Delta W/K \sim (4.74\times 10^{-5})\,\frac{v}{c}}. \tag{7}$$

Since $v < c$ (in fact $v \ll c$ for this nonrelativistic calculation), then certainly: $\Delta W/K < 50$ ppm. So, indeed ΔW is negligible w.r.t. K, as assumed in Eq. (2) above.

④ [50 pts]. Analyse ELF "radiation".

A. At $\omega = 2\pi f$, $f = 60$ Hz, wavelength is : $\lambda = c/f = \frac{3\times 10^{10}}{60} = 5000$ km.

We are in the static zone (Jkⁿ p.392), where : $d\left(\substack{\text{system}\\ \text{size}}\right) \ll R\left(\substack{\text{obsᵗⁿ}\\ \text{distance}}\right) \ll \lambda\left(\substack{\text{wave}\\ \text{length}}\right)$.

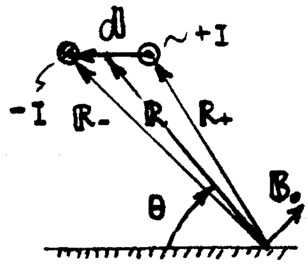
For the field of a single charge [Jkⁿ Eq. (14.14), for non-relativistic motion]:

$$\rightarrow \mathbb{E} = \underbrace{e\left[\frac{\hat{n}}{R^2}\right]}_{\text{static fld}} + \underbrace{\frac{e}{c}\left[\frac{\hat{n}\times(\hat{n}\times\dot{\beta})}{R}\right]}_{\text{radiation fld}} \Rightarrow \left|\frac{E(\text{radⁿ})}{E(\text{static})}\right| \sim \left|\frac{e\dot{\beta}}{cR}\Big/\frac{e}{R^2}\right| = \left|\frac{R\dot{\beta}}{c}\right|. \quad (1)$$

Since $\dot{\beta} = \omega\beta$ for the current motion : $\left|\frac{E(\text{radⁿ})}{E(\text{static})}\right| \sim 2\pi\beta\frac{R}{\lambda}$. But $\frac{R}{\lambda} = 2\times 10^{-5}$, and also $\beta \ll 1$ ($\beta \sim 10^{-5}$ perhaps), so E(radⁿ) is entirely negligible. The observer (i.e. victim) will at most "see" the static fields.

B. As noted above, the observer can at most see "static" $\mathbb{E}$ & $\mathbb{B}$ fields -- which oscillate harmonically at $e^{-i\omega t}$. The $\mathbb{E}$-fld vanishes because the system is overall charge neutral. That leaves the $\mathbb{B}$-fld... observer will see : $\mathbb{B}(R,t) = \mathbb{B}_0(R)\,e^{-i\omega t}$, where $\mathbb{B}_0(R)$ is generated by the wires.

C. $\mathbb{B}_0$ is generated by the two-wire system as shown. Its magnitude at the observer is [cf Jkⁿ Eq. (5.6), p. 171]:



$$B_0 = \frac{2I}{c}\left(\frac{1}{R_+} - \frac{1}{R_-}\right), \quad \text{w/ } \mathbb{R}_\pm = \mathbb{R} \mp \tfrac{1}{2}\mathbb{d}, \quad d \ll R. \quad (2)$$

So/ $R_\pm = \left(R^2 \mp \mathbb{R}\cdot\mathbb{d} + \frac{1}{4}d^2\right)^{\frac{1}{2}} = R\left(1 \mp \frac{d}{R}\cos\theta + \frac{1}{4}\cancel{\frac{d^2}{R^2}}\right)^{\frac{1}{2}}$ ← negligible

or/ $R_\pm \simeq R\left(1 \mp \frac{1}{2}\frac{d}{R}\cos\theta\right)$, to 1ˢᵗ order in $\frac{d}{R}$. (3)

and/ $\boxed{B_0(R) \simeq \frac{2I}{c}\left(\frac{d\cos\theta}{R^2}\right), \text{ likewise}}$, and : $B_0 < \frac{2I}{c}(d/R^2)$

$I = 100$ A [MKS] ↔ $I = 3\times 10^{11}$ statA [CGS, Jkⁿ p.820]. Then for $d = 1$m, $R = 100$ m, get:

$B_0 < (2\times 3\times 10^{11}/3\times 10^{10})(1/10^6) = 20\times 10^{-6}$ Gauss, less than $10^{-5}\times$ Bearth.

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⑤ [50pts] Work out $(q,m) \leftrightarrow$ field coupling via optional Lagrange formalism.

(A) $L = \frac{1}{2} m u_\alpha u^\alpha + \frac{q}{c} u_\beta A^\beta$, into $\frac{d}{d\tau}(\partial L/\partial u^\alpha) = \partial_\alpha L$ gives...

$$\rightarrow \frac{d}{d\tau}\left(m u_\alpha + \frac{q}{c} A_\alpha\right) = \frac{q}{c}(\partial_\alpha A_\beta)\, u^\beta, \qquad (1)$$

where we have used $G_\sigma H^\sigma = H_\sigma G^\sigma$ for 4-vectors G & H. The 1st term on the LHS is the Minkowski force: $\frac{d}{d\tau}(m u_\alpha) = dp_\alpha/d\tau = f_\alpha$. For the 2nd term LHS, use the Chain Rule: $\frac{d}{d\tau} = (\partial x^\beta/\partial\tau)\frac{\partial}{\partial x^\beta} = (\partial_\beta)\, u^\beta$. Then write

$$f_\alpha + \frac{q}{c}(\partial_\beta A_\alpha)\, u^\beta = \frac{q}{c}(\partial_\alpha A_\beta)\, u^\beta,$$

or// $$f_\alpha = \frac{q}{c}(\partial_\alpha A_\beta - \partial_\beta A_\alpha)\, u^\beta. \qquad (2)$$

Again use $G_\sigma H^\sigma = H_\sigma G^\sigma$ on the β-sum, and change the covariant index α to contravariant [see Jk^n Eq. (11.75)]. Then...

$$\boxed{f^\alpha = \frac{d}{d\tau}(m u^\alpha) = \frac{q}{c} u_\beta F^{\alpha\beta}}, \quad F^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha \quad \text{field tensor Jk}^n\ (11.136) \qquad (3)$$

This is the correct covariant form of the Lorentz force law (Jk^n Eq. (11.144)).

(B) The canonical momenta are: $P_\alpha = \partial L/\partial u^\alpha = m u_\alpha + (q/c) A_\alpha$, and the Hamiltonian is [see Jk^n Sec. (12.1)]

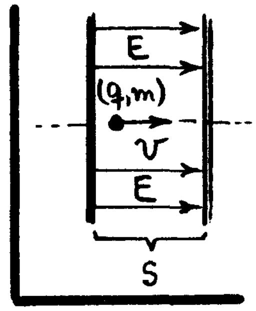
(cancel)

$$\mathcal{H} = P_\alpha u^\alpha - L = \left(m u_\alpha + \cancel{\frac{q}{c} A_\alpha}\right) u^\alpha - \left(\frac{1}{2} m u_\alpha u^\alpha + \cancel{\frac{q}{c} u_\beta A^\beta}\right)$$

or// $$\mathcal{H} = \frac{1}{2} m u_\alpha u^\alpha = +\frac{1}{2} m c^2 \qquad (4)$$ this is a Lorentz scalar, as required.

We have used: $u_\alpha u^\alpha = +c^2$, for the 4-velocity. If $\mathcal{H}$ were to be used in a QM formalism, we would write it in terms of the <u>canonical</u> momenta P_α, for which: $m u_\alpha = P_\alpha - \frac{q}{c} A_\alpha$, so that: $\mathcal{H} = \frac{1}{2m}(P_\alpha - \frac{q}{c} A_\alpha)(P^\alpha - \frac{q}{c} A^\alpha)$. We would then impose the QM condition: $P_\alpha = -i\hbar\,\partial_\alpha$. See Jk^n Eq. (12.29).

ϕ520 Final Exam Solutions (1993)



⑥ [50 pts]. Try a relativistic check for an electron in a capacitor.

(A) In transit between the plates, the charge has the (relativistic) eqtn-of-motion: $\frac{d}{dt}(\gamma m v) = qE = \text{cnst}$ $\left(\gamma = \frac{1}{\sqrt{1-\beta^2}},\ \beta = \frac{v}{c},\ \text{as usual}\right)$,

$$\xrightarrow{\text{so}} \frac{d}{dt}\left(\frac{\beta}{\sqrt{1-\beta^2}}\right) = \frac{qE}{mc} = \underline{\underline{\Omega}},\ \text{cnst} \Rightarrow \underline{\underline{\beta/\sqrt{1-\beta^2} = \Omega t}} \quad (1)$$

The field $E = \frac{1}{s}V$(applied voltage), and the solution is for release from the lefthand plate at time $t=0$, when $v=0$. The velocity & distance traveled at time t are:

$$\rightarrow \underline{\underline{\beta(t) = \frac{\Omega t}{\sqrt{1+(\Omega t)^2}}}};\ \underline{\underline{D(t) = \int_0^t c\beta(t')\,dt' = \frac{c}{\Omega}\left(\sqrt{1+(\Omega t)^2} - 1\right)}}. \quad (2)$$

$\underline{\text{NOTE}}$, for $\Omega t \ll 1$: $D(t) = \frac{1}{2}at^2\left[1 - \frac{1}{4}(\Omega t)^2 + \dots\right]$ ✓ $a = c\Omega = qE/m$ is the Newtonian accel^{n}. (3)

(B) With $\Omega t = \beta/\sqrt{1-\beta^2}$ from Eq. (1), D of Eq. (2) can be written in terms of β:

$$\rightarrow D(\beta) = (\gamma - 1)\frac{c}{\Omega},\ \text{w/}\ \frac{c}{\Omega} = \frac{mc^2}{qV}s \leftarrow \text{have used } E = V/s \quad (4)$$

$\left.\begin{matrix}\text{max } \beta \text{ when}\\ D(\beta) = s\end{matrix}\right\} \Rightarrow \underline{\underline{\gamma = 1 + \frac{qV}{mc^2}}}$ ‖ note that this γ is $\underline{\text{indpt}}$ of capacitor plate sep^{n} s. (5)

If (q,m) is an electron (w/ $mc^2 = 511$ keV), and $V = 100$ kV, then $\frac{qV}{mc^2} = \frac{100}{511} = 0.1957$. So $\gamma = 1.1957$, and: $\underline{\beta = \sqrt{1-(1/\gamma^2)} = 0.548}$. Mildly relativistic.

(C) Non-relativistic transit time t_{nr} is found from: $s = \frac{1}{2}at_{nr}^2$, w/ $a = \frac{qE}{m}$ [per Eq (3)], i.e. $\underline{t_{nr} = \sqrt{2s/a}}$. Relativistic transit time is found from Eq. (2):

$$\rightarrow s = \frac{c}{\Omega}\left(\sqrt{1+(\Omega t_r)^2} - 1\right) \Rightarrow t_r = t_{nr}\sqrt{1+(\Omega s/2c)},\ \text{w/}\ t_{nr} = \sqrt{\frac{2s}{c\Omega}}. \quad (6)$$

$$\dots \text{but}: \Omega s/2c = qV/2mc^2,\ \text{so}: \boxed{t_r = t_{nr}\sqrt{1 + (qV/2mc^2)}} \quad (7)$$

The specific relativistic correction here [i.e. term in $(qV/2mc^2)$] is again indept of the plate separation s -- as in Eq. (5). For (q,m) = electron thru $V = 100$ kV: $\underline{t_r = 1.0478\, t_{nr}}$, and: $\underline{t_{nr}} = \frac{s}{c}\sqrt{2mc^2/qV} = \underline{(0.107\ \text{nsec}) \times s}$, s in cm. We'd need time resolution to about 5 psec to "see" relativity here. Not plausible.