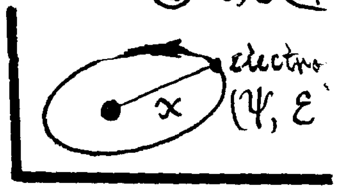
$\boxed{\text{Ex.}}$ Schrödinger Eqtn for the Hydrogen Atom.



If $\psi(x)$ is the wavefunction for the orbiting electron...

$$\rightarrow \psi'' + \underbrace{(2/x)}_{f(x)}\psi' + \underbrace{\left[\mathcal{E} + \frac{2}{x} - \frac{\ell(\ell+1)}{x^2}\right]}_{g(x)}\psi = 0$$

w/ $x \propto$ orbit radius, $\mathcal{E} = (-)$ cnst $\propto$ total orbit energy, $\ell =$ integer $\geqslant 0$. (60)

Do the substitution indicated in Eq.(59)...

$$\psi(x) = u(x)\psi_1(x), \text{ w/ } \psi_1(x) = \exp\left[-\tfrac{1}{2}\int f\,dx\right] = \exp\left[-\int\frac{dx}{x}\right] = \frac{1}{x}.$$

So// $\psi(x) = \frac{1}{x}u(x)$ into Eq.(60) yields...

$$u'' + \left[g - \tfrac{1}{4}(f^2 + 2f')\right]u = 0, \text{ w/ } f = \frac{2}{x} \Rightarrow f^2 + 2f' \equiv 0;$$

and//

$$\longrightarrow u'' + \left[\mathcal{E} + \frac{2}{x} - \frac{\ell(\ell+1)}{x^2}\right]u = 0. \qquad (61)$$

The u'' eqtn is simpler than the ψ'' eqtn, by the elimination of the 1st derivative term. It is also easier to handle, as we shall see below.

⑬ Factoring the Asymptotic Behavior

A useful trick to find the nature of the solutions to a specific ODE, lik $y'' + f(x)y' + g(x)y = 0$ with $f(x)$ & $g(x)$ given explicitly, is to analyze the eqtn in various limits, e.g. as $x \to 0$ or $x \to \infty$. Often, certain terms drop out of the ODE, leaving a simple enough eqtn so that the behavior of $y(x\to 0)$ or $y(x\to\infty)$ can easily be found. Once these "asymptotic limits"¶ are factored out of the required exact $y(x)$ -- by substitution of dept variables -- the remaining ODE (over $0 < x < \infty$) is often made simpler, or recognizable as tabulated.

¶ The functional behavior of $y(x\to 0)$ & $y(x\to\infty)$ is often found as a single term th represents how $y(x)$ behaves over a limited range of x. Such terms are "asymptotic

Asymptotic nature of H-atom wavefcns.

Ex. Asymptotic character of Schrödinger's H-atom Wavefcn.

1. With $\psi = \frac{1}{x}u$, start from u'' eqtn, Eq.(61) above. At large x...

$u'' + \mathcal{E}u \simeq 0$, as $x\to\infty$ (terms in $1/x$ & $1/x^2$ are negligible);

... put $\mathcal{E} = -\kappa^2$, since the binding energy $\mathcal{E} < 0$....

Soll $u'' - \kappa^2 u \simeq 0 \Rightarrow u(x) \simeq Ae^{-\kappa x} + Be^{+\kappa x}$, as $x\to\infty$. (62)

2. Impose a physical condition: $u(x)\to 0$ as $x\to\infty$ (electron not found at $x=\infty$) $\Rightarrow B\equiv 0$ in (62).

So $u(x) \propto e^{-\kappa x}$ as $x\to\infty$. Factor out this behavior by substituting...

$u(x) = e^{-\kappa x}v(x)$, $v(x)$ a new fcn to be found;

Soll $u' = e^{-\kappa x}(v' - \kappa v)$, $u'' = e^{-\kappa x}(v'' - 2\kappa v' + \kappa^2 v)$. (63)

Put these forms into the exact u'' eqtn, Eq.(61), to get... w/ $\mathcal{E} = -\kappa^2$...

$$\underline{\underline{v'' - 2\kappa v' + \left[\frac{2}{x} - \frac{\ell(\ell+1)}{x^2}\right]v = 0.}} \tag{64}$$

3. This eqtn is an exact version of the ψ'' eqtn (60), with $\psi = \frac{1}{x}e^{-\kappa x}v(x)$. A 1st derivative term has reappeared, but the v'' eqtn has simpler coefficient than the ψ'' eqtn. To simplify further, look at what happens as $x\to 0$...

$v'' - [\ell(\ell+1)/x^2]v \simeq 0$, as $x\to 0$ (the $1/x^2$ term dominates as $x\to 0$);

Soll solns are: $v(x) \propto x^{\ell+1}$ & $v(x) \propto x^{-\ell}$, as $x\to 0$. (65)

4. Impose more physics: we don't want $v(x)\to\infty$ as $x\to 0$ (electrons are finite) $\Rightarrow$ throw out the soln $v(x)\propto x^{-\ell}$. Factor out the $x^{\ell+1}$ behavior by substituting...

$v(x) = x^{\ell+1}w(x)$, $w(x)$ a new fcn to be found. (66)

Soll $v' = x^{\ell}[xw' + (\ell+1)w]$, etc.

Plug v' & v'' into the exact eqtn (64), to get an exact eqtn for w...

Asymptotic Behavior, Power Series Sol^{n}s & Fuchs' Theorem

$$\boxed{\begin{array}{l} w'' + 2\left[\frac{\ell+1}{x} - \kappa\right]w' + \frac{2}{x}\left[1-(\ell+1)\kappa\right]w = 0, \\ \text{with: } \psi(x) = \frac{1}{x}u(x) = \cdots = \underbrace{(x^\ell)}_{\text{as } x\to 0}\underbrace{(e^{-\kappa x})}_{\text{as } x\to\infty}\,w(x). \end{array}} \tag{67}$$

5. The w'' eqtn is non-trivial, but (superficially) it is no more complicated than the original ψ'' eqtn, Eq.(60). The point is that we have discovered the limit behaviors: $\psi(x) \sim x^\ell$ as $x\to 0$, and $\psi(x)\sim e^{-\kappa x}$ as $x\to\infty$. Such limits place constraints on how $w(x)$ must behave... $w(x)$ can at most $\to$ cnst as $x\to 0$, and $w(x)$ must be $< e^{-\kappa x}$ as $x\to\infty$. These constraints are sufficient to fix the required solutions $w(x)$ to Eq.(67) as polynomials in x of finite degree. The required $w(x)$'s are Laguerre polynomials; such sol^{n}s to Eq.(67) are tabulated.

(14) Power Series Sol^{n}s to 2nd order ODEs.

As noted just above, one is often interested in finding sol^{n}s to 2nd order ODEs in the form of a polynomial, of finite (or even infinite) degree. The method for doing this is called "solution by power series" or the "method of Frobenius." One begins by classifying the ODE according to its "singular points", by means of Fuchs' Theorem. We consider an ODE of the form...

$$y'' + P(x)y' + Q(x)y = 0. \tag{68}$$

FUCHS' THEOREM

A. If $P(x)$ & $Q(x)$ are both finite as $x\to x_0$, the point x_0 is called an "ordinary (or regular) point" of the ODE. Near an ordinary point, a sol^n to the ODE can be written as a power series, viz...

$$\rightarrow y(x) = \sum_{\lambda=0}^{\infty} a_\lambda (x-x_0)^\lambda, \quad \text{w/ } y(x_0) = a_0. \tag{69}$$

(next page)

Fuchs' Theorem (cont'd)

In fact, there will be two, independent series of this sort as sol$^{\underline{n}}$s near x_0. Details of the coefficients $\{a_\lambda\}$ depend on details of P & Q.

B. If $P(x)$ or $Q(x) \to \infty$ when $x \to x_0$, but $(x-x_0)P(x)$ and $(x-x_0)^2 Q(x)$ remain finite [i.e. $P(x)$ & $Q(x)$ do not diverge faster than $\frac{1}{(x-x_0)}$ & $\frac{1}{(x-x_0)^2}$] then x_0 is called a "regular singular point" of the ODE. Near such an x_0, there is always at least one particular sol$^{\underline{n}}$ of the form...

$$\rightarrow y(x) = (x-x_0)^k \sum_{\lambda=0}^{\infty} a_\lambda (x-x_0)^\lambda. \qquad (70)$$

k may be (+)ve or (−)ve. The 2$^{\underline{nd}}$ solution generated by the series method in this case usually contains a logarithmically divergent term $\sim \ln(x-x_0)$

C. If $P(x)$ or $Q(x) \to \infty$ when $x \to x_0$, and either $(x-x_0)P(x)$ or $(x-x_0)^2 Q(x)$ also diverges, then x_0 is called an "irregular (or essential) singularity" of the ODE. Near such an x_0, no power series sol$^{\underline{n}}$ exists in general.

We shall not prove these claims, but we remark that they can be justified by plugging $y = \sum_\lambda a_\lambda (x-x_0)^{k+\lambda}$ into $y'' + Py' + Qy = 0$, together with the series expansions of P & Q near $x = x_0$, and then deciding how P & Q must behave in order to get a set of finite $\{a_\lambda\}$.

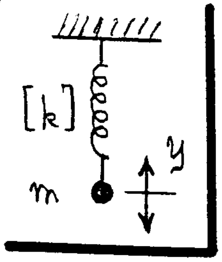
Fuchs' Thm is an existence thm... it says when sol$^{\underline{n}}$s $y = \sum_\lambda a_\lambda (x-x_0)^{k+\lambda}$ to $y'' + Py' + Qy = 0$ are possible or not possible, but of course it doe not give you the set $\{a_\lambda\}$. E.g. for Legendre's Eq [M&W (1-49)]...

$$\left[\begin{matrix}\text{Legendre} \\ \text{Equation}\end{matrix}\right\} \; y'' + P(x)y' + Q(x)y = 0, \;\; \text{w/}\; \begin{aligned} P(x) &= -2x/(1-x^2), \\ Q(x) &= n(n+1)/(1-x^2). \end{aligned} \qquad (71)$$

$x = 0$ is an ordinary point, so two series sol$^{\underline{n}}$s $\sum_\lambda a_\lambda x^\lambda$ are possible there. $x = \pm 1$ are regular singular points, and just one sol$^{\underline{n}}$ $\sum_\lambda b_\lambda (x\pm 1)^{k+\lambda}$ can be done.

Power Series Method for SHO Eqtn

$\boxed{Ex.}$ Power series sol$^{\underline{n}}$ to simple harmonic oscillator eqtn.



k = spring cnst, y = displacement of mass m, $\omega = \sqrt{\frac{k}{m}}$ = natural frequency

Newton II $\Rightarrow$ $m\ddot{y} = -ky$, or// $\underline{\ddot{y} + \omega^2 y = 0}$ $\quad (\dot{y} = dy/dt)$

or// $\underline{\underline{y'' + y = 0}}$, w// $x = \omega t$ & $y = y(x)$. $\quad$ (72)

1. We know the sol$^{\underline{n}}$s are $y \propto \sin x, \cos x$; let's see how the power series method verifies this. We note that for $y'' + y = 0$, $P(x) = 0$ & $Q(x) = 1$ are regular everywhere, so <u>all</u> points x are ordinary points for the ODE. In particular, $x = 0$ is ordinary, so Fuchs' Thm $\Rightarrow$ two indep$^{\underline{t}}$ series sol$^{\underline{n}}$s of the form:

$\rightarrow$ $y(x) = \sum_{\lambda=0}^{\infty} a_\lambda x^{k+\lambda}$, k included here to adjust leading power of x;

and// $y' = \sum_\lambda (k+\lambda) a_\lambda x^{k+\lambda-1}$, $\quad y'' = \sum_\lambda (k+\lambda)(k+\lambda-1) a_\lambda x^{k+\lambda-2}$. $\quad$ (73)

Put these forms for y & y'' into the ODE: $y'' + y = 0$, to obtain...

$$\sum_{\lambda=0}^{\infty} (k+\lambda)(k+\lambda-1) a_\lambda x^{k+\lambda-2} + \sum_{\lambda=0}^{\infty} a_\lambda x^{k+\lambda} = 0,$$

...isolate $\lambda = 0$ & $\lambda = 1$ terms in this sum...

or// $\rightarrow k(k-1) a_0 x^{k-2} + (k+1) k a_1 x^{k-1} +$

$$+ \sum_{\lambda=0}^{\infty} [(k+\lambda+2)(k+\lambda+1) a_{\lambda+2} + a_\lambda] x^{k+\lambda} = 0. \quad (74)$$

---

2. Eq.(74) holds for all x only if the coefficient of <u>each</u> power of x vanishes. Imposing this condition gives a way of finding the $\{a_\lambda\}$, and allowed k's.

- Last term LHS in (74) $\equiv 0 \Rightarrow$ $\underline{\underline{a_{\lambda+2} = (-) a_\lambda / (k+\lambda+2)(k+\lambda+1)}}$ $\sqrt{}$ recursion relation. (75)

This "recursion relation" gives $a_2 \propto a_0$, $a_4 \propto a_2 \propto a_0$, etc., or $a_3, a_5, \ldots$ in terms of a_1. Of all the $\{a_\lambda\}$, only a_0 & a_1 are free cnsts.

- a_0 & a_1 can be taken as the two free integration cnsts required for the general sol$^{\underline{n}}$ of the 2$^{\text{nd}}$ order ODE in Eq.(72). But we have an additiona degree of freedom here, in our choice of k in Eq. (74). We can do the following in order to make the first two terms LHS in (74) vanish...

Let : $a_1 = 0$, $a_0 \neq 0$, and choose k so that...

$$k(k-1) = 0 \Rightarrow k = 0, \text{ or } 1 \quad \text{indicial equation.} \tag{76}$$

Then both of the first two terms LHS in (74) are zero, as needed. Notice that in this series method for a 2$^{\underline{nd}}$ order ODE, the indicial eqtn is quadratic in general (the y'' series gives this -- see Eq. (73)), so k shows two solutions. The k sol$^{\underline{n}}$s may be degenerate, however.

- Now, for the choices in (76), we can write out two distinct series...

→ $a_1 = 0$, $a_0 = A \neq 0$, and $\underline{\underline{k=0}}$ $\Rightarrow$ $\underline{a_{\lambda+2} = (-)\,a_\lambda/(\lambda+2)(\lambda+1)}$;

so// $a_2 = -A/2!$, $a_4 = +A/4!$, ... $a_{2n} = (-)^n A/(2n)!$

and// $$\underline{\underline{y_1(x)}} = A\left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right] = \underline{\underline{A\cos x}}. \tag{77A}$$

The series is instantly recognizable as the Taylor series for $\cos x$, about $x=0$. Likewise, the 2$^{\underline{nd}}$ series gives $\sin x$, as...

→ $a_1 = 0$, $a_0 = B \neq 0$, and $\underline{\underline{k=1}}$ $\Rightarrow$ $\underline{a_{\lambda+2} = (-)\,a_\lambda/(\lambda+3)(\lambda+2)}$;

so// $a_2 = -B/3!$, $a_4 = +B/5!$, ... $a_{2n} = (-)^n B/(2n+1)!$

and// $$\underline{\underline{y_2(x)}} = B\left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right] = \underline{\underline{B\sin x}}. \tag{77B}$$

---

3. The series method thus gives the known sol$^{\underline{n}}$s $y \propto \sin x, \cos x$ to the SHO eqtn, and so is consistent with what we know. Now think of this... every OD $y'' + Py' + Qy = 0$ that can be solved this way generates unique series $y(x) = \sum_\lambda a_\lambda x^{k+\lambda}$, whose $\{a_\lambda\}$ depend on P & Q. These series may be regarded as Taylor expansions of the "special fcns" $y(x)$ that are defined by the ODE.

Power Series Method for Bessel's Eqtn

Ex. Power series sol$^{\underline{n}}$ to Bessel's Eqtn: $\underline{x^2y'' + xy' + (x^2 - \nu^2)y = 0}$ $\sqrt{}\ \nu = \text{cnst}$ (78)

For the SHO Eq., $y'' + y = 0$, we got two indep$^{\underline{t}}$ series sol$^{\underline{n}}$s with no trouble; this is always possible at ordinary points of the 2$^{\underline{nd}}$ order ODE. Things are not as easy at a singular point, as we now show by analyzing the above Bessel's Eqtn.

1. Bessel's ODE has: $P(x) = \frac{1}{x}$, $Q(x) = 1 - \frac{\nu^2}{x^2}$, so $x = 0$ is a regular singular pt. By Fuchs' Thm, there should be one power series solution about $x = 0$, i.e.

$$\rightarrow y(x) = \sum_{\lambda=0}^{\infty} a_\lambda x^{k+\lambda}, \quad \text{w/ } y' = \sum_\lambda (k+\lambda) a_\lambda x^{k+\lambda-1}, \quad y'' = \text{etc.} \tag{79}$$

Put these forms for y, y' & y'' into Eq. (78), and rearrange terms...★

$$\sum_{\lambda=0}^{\infty} (k+\lambda)(k+\lambda-1) a_\lambda x^{k+\lambda} + \sum_{\lambda=0}^{\infty} (k+\lambda) a_\lambda x^{k+\lambda} + \sum_{\lambda=0}^{\infty} (x^2 - \nu^2) a_\lambda x^{k+\lambda} = 0,$$

or//

$$\left[\!\left[(k^2 - \nu^2) a_0 x^k + [(k+1)^2 - \nu^2] a_1 x^{k+1} + \right.\right.$$
$$\left.\left. + \sum_{\lambda=0}^{\infty} \left\{ [(k+2+\lambda)^2 - \nu^2] a_{\lambda+2} + a_\lambda \right\} x^{k+2+\lambda} = 0. \right.\right. \tag{80}$$

---

2. As before, Eq. (80) holds for all x only if the coefficient of each power of x is 0.†

So//

- coefficient of x^k vanishes if $a_0 \neq 0$, but $\underline{\underline{k = \pm\nu}}$ $\sqrt{}$ indicial equation
- for $k = \pm\nu$, coefficient of x^{k+1} vanishes only if $a_1 = 0$. So choose $a_1 = 0$, and look at series sol$^{\underline{n}}$s for $a_0 \neq 0$, and $k = \pm\nu$.
- coefficient of $x^{k+2+\lambda}$ in (80) vanishes $\Rightarrow$ a $\underline{\underline{\text{recursion relation}}}$, viz.

$$\rightarrow a_{\lambda+2} = (-) a_\lambda / [(k+2+\lambda)^2 - \nu^2]. \tag{81}$$

This recursion relation is not as simple as in Eq. (75) for the SHO eqtn. But it, too, can be iterated. As follows...

---

† The series $\sum_{n=0}^{\infty} c_n z^{n+k} \equiv 0$ for all z if and only if all $c_n \equiv 0$.

★ The "rearrangement" is to gather together all terms with like powers of x.

Power Series Method for Bessel's Eqtn (cont'd)

↳ first series solⁿ: $a_1 = 0$, $a_0 = A \neq 0$, and $\underline{k = +\nu}$.

$$\Rightarrow \underline{a_{\lambda+2} = (-)a_\lambda / [(\nu+2+\lambda)^2 - \nu^2] = (-)\, a_\lambda / (2\nu+2+\lambda)(\lambda+2)}. \quad (82A)$$

So// $a_2 = -A/2^2(\nu+1) = -A\nu!/2^2 \cdot 1!\,(\nu+1)!$ } see note ★ below, for defⁿ of $(\nu+1)!$

$a_4 = +A\nu!/2^4 \cdot 2!\,(\nu+2)!$

⋮

$a_\lambda = (-)^{\frac{\lambda}{2}} A\nu!/2^\lambda \cdot (\frac{\lambda}{2})!\,(\nu+\frac{\lambda}{2})!$, for $\lambda = 2, 4, 6, \ldots =$ even integer.

Set $\lambda = 2\mu$ for convenience ($\mu = 0, 1, 2, \ldots$). The first solⁿ is...

$$\underline{\underline{y_1(x)}} = A x^\nu \sum_{\mu=0}^{\infty} [(-)^\mu \nu!/2^{2\mu} \mu!(\nu+\mu)!]\, x^{2\mu} = \underline{\underline{A\nu!\, J_\nu(x)}}. \quad (82B)$$

The series (w/ $A = 1$) is a representation of the "Bessel Fcn of the first kind" usually denoted by $J_\nu(x)$. The series converges for all x, even at the regular singular point, $x = 0$, and also for any value of $\nu \geq 0$.

↳ Second series solⁿ: $a_1 = 0$, $a_0 = B \neq 0$, and $\underline{k = -\nu}$.

$$\Rightarrow \underline{a_{\lambda+2} = (-)\, a_\lambda / [(-\nu+2+\lambda)^2 - \nu^2] = (-)\, a_\lambda / (2+\lambda-2\nu)(\lambda+2)}. \quad (82C)$$

This recursion relation is the same as (82A), but with ν replaced by $-\nu$. The iteration can be done as above, and a series similar to (82B) results, so a second solⁿ can be written: $\underline{y_2(x) = B J_{-\nu}(x)}$. This solⁿ diverges $\sim x^{-\nu}$ as $x \to 0$, but it is OK everywhere else, so long as $\nu \neq n$, an integer. When $\nu \to n$, the denominator in the recursion relation vanishes at $\lambda = 2(n-1)$, and all $a_{\lambda+2} \to \infty$ for $\lambda \geq 2(n-1)$. So the series $x^{-\nu} \sum_\lambda a_\lambda x^\lambda$ does not exist, and $y_2(x) = B J_{-n}(x)$ is worthless. This shows how we can "lose" the 2ⁿᵈ series at a singular point, as "predicted" by Fuch's Theorem.

★ For an integer N, by defⁿ: $N! = N(N-1)(N-2)\cdots 2 \cdot 1$. When $N = \nu \neq$ integer, define $\nu! = \int_0^\infty z^\nu e^{-z}\, dz = \Gamma(\nu+1)$, w/ $\Gamma(\nu)$ is the "gamma fcn". See M&W, Sec. 3-4.

Hypergeometric & Confluent Hypergeometric Series

ASIDE Power Series Sol$^{\underline{n}}$s for Special Fcns

Sol$^{\underline{n}}$s to 2$^{\underline{nd}}$ order ODEs by power series is a widely used method, and virtually all the "special fcns" used in physics start life as a power series. The method need not be done case-by-case (as we've done for the SHO Eqtn on p. ODE 21, and Bessel's Eqtn on p. ODE 23); it can be studied in general. That study is greatly aided by Fuchs' Thm, which notes that series sol$^{\underline{n}}$s to $y'' + P(x)y' + Q(x)y = 0$ do not exist at a point $x = x_0$ unless $P(x)$ behaves no worse than $1/(x-x_0)$, and $Q(x)$ no worse than $1/(x-x_0)^2$. These conditions greatly restrict the possible choices of functional behavior for P & Q, but still allow a great variety of sol$^{\underline{n}}$s $y(x)$.

Two types of general 2$^{\underline{nd}}$ order ODEs that satisfy Fuchs' conditions on P & Q for finite x have been solved by series this way. They are...

(1) HYPERGEOMETRIC EQ: $x(1-x)y'' + [c-(a+b+1)x]y' - aby = 0$. (83A)

a, b & c are cnsts (which may be complex). This eqtn has 3 regular singular points, at $x = 0, 1$ & ∞. One series sol$^{\underline{n}}$, convergent @ $x=0$, for $c>0$, is...

$$\rightarrow y_1(x) \propto F(a,b;c;x) = 1 + \frac{ab}{c}\frac{x}{1!} + \frac{a(a+1)b(b+1)}{c(c+1)}\frac{x^2}{2!} + \dots \qquad (83B)$$

Each particular choice of (a,b,c) generates a new series, and among the sol$^{\underline{n}}$ are: $F \propto (1+x)^n$, $\ln x$, $\sin$ & $\cos x$, e^x, $\sinh$ & $\cosh x$, Legendre fcns, elliptic integrals, etc.

(2) CONFLUENT HYPERGEOM. EQ: $xy'' + (c-x)y' - ay = 0$. (84A)

a & c are cnsts, and this eqtn has a regular singularity at $x=0$, plus an irregu singularity at $x=\infty$. A series sol$^{\underline{n}}$, convergent @ $x=0$, for $c>0$, is...

$$\rightarrow y_1(x) = F(a;c;x) = 1 + \frac{a}{c}\frac{x}{1!} + \frac{a(a+1)}{c(c+1)}\frac{x^2}{2!} + \frac{a(a+1)(a+2)}{c(c+1)(c+2)}\frac{x^3}{3!} + \dots \qquad (84B)$$

Various (a,c) choices $\Rightarrow F \propto$ trig fcns, Bessel fcns, Laguerre & Hermite fcns, etc.

We should study details of Eqs (83) & (84) later.

(15) Miscellaneous Series Solutions

The power series method of solution described on pp. ODE 19–25 gives a "complete" series $y(x) = \sum_\lambda a_\lambda x^{k+\lambda}$, in that all the coefficients can be discovered readily by recursion relations. Series can be found iteratively, however, and they are useful when we need solⁿs only near a particular point x. Following are two examples of series solⁿs by iteration.

I. The 1st method works just by assuming the solⁿ has a Taylor series. Suppose we want to solve the 1st order ODE...

$$\rightarrow dy/dx = y' = f(x,y), \text{ w/ initial value: } y = y_0 \text{ @ } x = x_0, \text{ given.} \quad (85)$$

Assume $x = x_0$ is a nonsingular point, so $y(x)$ has a Taylor series about x_0

$$\begin{cases} y(x) = y_0 + (x-x_0)y_0' + \frac{1}{2}(x-x_0)^2 y_0'' + \dots + \frac{1}{n!}(x-x_0)^n y_0^{(n)} + \dots \\ \text{w/ } y_0 = y(x_0), \quad y_0^{(n)} = (\partial/\partial x)^{n-1} f(x,y)\Big|_{x=x_0,\, y=y_0}, \text{ given.} \end{cases} \quad (86)$$

The derivatives $y_0^{(n)}$ are calculable from the known $f(x,y)$. As many terms in the series as needed can be found -- patience permitting.

$\boxed{\text{Ex.}}$ Solve: $y' = \exp(kxy)$, near $x=0$, with $y(0)=0$. (k = cnst).

Note: $y_0' = 1$ @ $x_0 = 0$ & $y_0 = 0$. Then $y(x) \sim x$ is 1st term in Taylor series. Now

$$\begin{cases} y'' = e^{kxy} \cdot k(xy'+y) = y' \cdot k(xy'+y) \\ \text{So } y_0'' = 0 \text{ @ } x_0 = 0 \text{ \& } y_0 = 0, \text{ and: } y(x) \sim x + O(x^3) \end{cases} \quad (87)$$

no term of $O(x^2)$

To get the $O(x^3)$ coefficient, find y_0'''...

$$\begin{cases} y''' = k(xy'+y) \cdot y'' + k(xy''+2y')y', \text{ w/ } y_0'' = 0 \text{ \& } y_0' = 1; \\ \text{So } y_0''' = 2k \text{ @ } x_0 = 0 \text{ \& } y_0 = 0, \text{ and: } y(x) \sim x + \frac{1}{3}kx^3 + O(x^5). \end{cases} \quad (87)$$

Etc. The series for $y(x)$ will contain only odd powers of x. Why?

Misc. Uses of Series (cont'd)

II. The 2nd method works for $y' = f(x,y)$ when $f(x,y)$ has some special property, like $f(x,y) \to 0$ as $x \to \infty$. It is a pure iteration.

$\boxed{\text{Ex.}}$ Solve: $y' = e^{-xy}$, as $x \to \infty$.

Assume $y > 0$, to ensure $y' \to 0$ as $x \to \infty$.

Then $y' \to 0 \Rightarrow$ $y \sim \text{cnst} = k > 0$, as $x \to \infty$. (0^{th} approxn)

1st approxn: $y \simeq k$, and $y' = e^{-xy} \simeq e^{-kx}$, for large x.

But $y' = \frac{dy}{dx} \simeq e^{-kx} \Rightarrow y \simeq \int e^{-kx}dx = k - \frac{1}{k}e^{-kx}$. (88A)

2nd approxn: use (88A) in the original eqtn, $y' = e^{-xy}$, to write...

$$\rightarrow y' \simeq \exp\left[-x\left(k - \tfrac{1}{k}e^{-kx}\right)\right] = e^{-kx}\exp\left[\tfrac{x}{k}e^{-kx}\right].$$

In the exp here, the [ ] → small as x → large, so expand the exp...

$$\rightarrow y' \simeq e^{-kx}\left\{1 + \frac{x}{k}e^{-kx}\right\} = e^{-kx} + \frac{1}{k}xe^{-2kx}$$

Integrate this last result...

$$y \simeq k - \frac{1}{k}e^{-kx} + \frac{1}{k}\int xe^{-2kx}dx$$

or/ $$y(x) \simeq \underbrace{k}_{0^{th}\text{ approxn}} \underbrace{- \frac{1}{k}e^{-kx}}_{1^{st}\text{ iteration}} + \underbrace{\frac{1}{2k^2}\left(x + \frac{1}{2k}\right)e^{-2kx}}_{2^{nd}\text{ iteration}} + \ldots \qquad (88B)$$

The cnst $k > 0$ is fixed by whatever value is required for $y(\infty)$. The series itself is certainly not a Taylor series -- it is an asymptotic series, good only when $kx \gg 1$. It is no easy task to find out how accurately the series portrays the actual $y(x)$ at a given finite x.