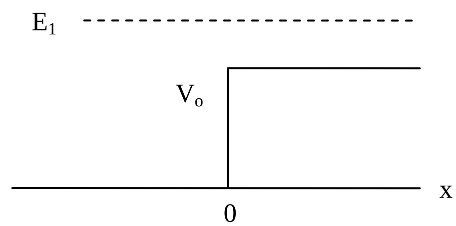
1)

Consider the 1-D scattering of a quantum particle of mass m and at energy E_1 by a step potential $V_o > 0$ as shown in the figure. Assume that the potential energy to the left of the step potential (x<0) is zero. Also assume that $E>V_o$ for this 1-D scattering process.



The incident wave function, traveling from left to right, is given by:

$$\Psi_1(x,t) = Ae^{+ik_1x-i\omega_1t}$$

where the wave vector k_1 and the angular frequency ω_1 of this incident traveling wave are both positive and real.

a) For this incident quantum wave, use Schrödinger's equation to find B/A, the ratio of the reflection amplitude B to the incident amplitude A.

b) Then use Schrödinger's equation to find the transmission amplitude C/A, the ratio of the transmission amplitude C to the incident amplitude A.

c) Finally, find the reflection coefficient R and the transmission coefficient T and verify that R and T will indeed sum to unity.

2.

If a plasma contains a small amount of Helium, then a fraction of it, f_0, will be neutral, a fraction, f_1, will be singly ionized, and the remaining fraction, f_2, will be doubly ionized. (Obviously, the fractions sum to unity: $f_0 + f_1 + f_2 = 1$). Helium may change between charges states through the processes of ionization, at rate I_i, or recombination (its inverse) at a rate R_i. The fractions thus evolve according to

$$\frac{df_0}{dt} = R_1 f_1 - I_0 f_0 \tag{1}$$

$$\frac{df_1}{dt} = I_0 f_0 + R_2 f_2 - (I_1 + R_1) f_1 \tag{2}$$

$$\frac{df_2}{dt} = I_1 f_1 - R_2 f_2 \tag{3}$$

Let us simplify things by taking all rates to be equal: $I_0 = I_1 = R_1 = R_2 = \nu$ and denoting that single rate ν.

Consider the case where Helium is introduced all at once, at $t = 0$, and in an entirely neutral state: $f_0 = 1$, $f_1 = f_2 = 0$. Write down the fraction of doubly ionized Helium $f_2(t)$ for $t > 0$.

[Hint: You might write eqs. (1)–(3) using a matrix, and then find a general solution in terms of eigenvectors and eigenvalues of that matrix.]

3)

Answer the following questions:

a. Imagine that an electron is a spherical ball of radius r charged uniformly with $-e$. Determine its minimum radius (classical radius) using convincing physical reasoning.

b. The center of a star is dense and mostly fully ionized hydrogen. Estimate the time in years for a photon to escape from the center of the star out to a radius of $R = 10^8$ m, using a mean plasma density of $\rho = 1.5\text{x}10^5$ kg/m^3. Assume only elastic scattering off of the electrons and a photon with a wavelength much smaller than the mean free path between the elastic scattering events.

(Hints: The capacitance of a spherical capacitor is $C = \dfrac{4\pi\varepsilon_o}{(1/r - 1/R)}$, $m_p = 1.7 \times 10^{-27}$ kg, $m_e = 9.1 \times 10^{-31}$kg. The Thomson elastic scattering cross section is $\sigma_T = 6.7 \times 10^{-29}$ m^2, the mean free path between collisions is $\ell = (n\sigma)^{-1}$, and in a random walk the root mean square distance traveled from the starting point is given by $\sqrt{< x^2 >} = \sqrt{N}\ell$).

4)

The Saros Cycle. The orbital plane of the moon around Earth is tilted by 5.15 deg with respect to the orbital plane of the Earth around the Sun (the so-called *ecliptic*) as schematically depicted below (not to scale!). If you view the Earth-Moon system as a spinning top, the Sun exhibits tidal forces on that system that try to force the lunar orbit into the ecliptic.

Make a *leading-order* estimate of the time-averaged torque and the precession period of the moon-Earth system due to the Sun's influence. Give the precession period in years.

Side note: the relative position of Earth, Sun and moon governs the occurrence of both solar and lunar eclipses. The precession period that you will calculate corresponds to a naturally occurring cycle in the occurrence of solar and lunar eclipses, the so-called *Saros Cycle* that was already observationally known to the ancient Babylonians!

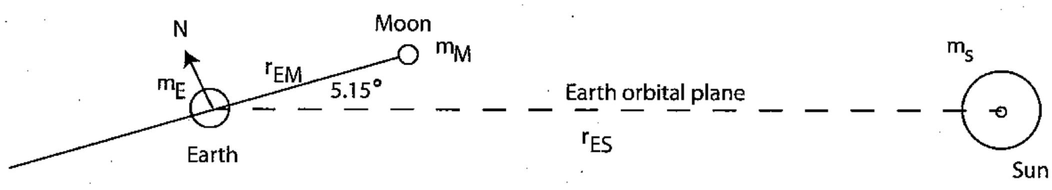
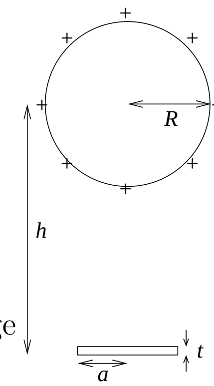


FIG. 1: Sketch of the setup of the problem.

5. A small, circular, copper disk, of mass density ρ, radius a, and thickness $t \ll a$, lies on a flat, non-conducting table. A glass sphere of radius R and uniformly distributed positive charge Q is suspended with its center a distance $h > R$ above the disk. Assume that $h \gg a$ and $h \gg t$.



   (a) Explain why we can take the electric field due to the sphere as approximately uniform over the volume of the disk.

   (b) Assuming that the field is uniform, determine the total charge induced on the upper surface of the disk due to the presence of the sphere. Why must its sign be negative?

   (c) Consider the interaction between the upper surface charges from part (b) and those on the sphere. Determine the value of h such that their electrostatic attraction equals the magnitude of the force of gravity on the disk.

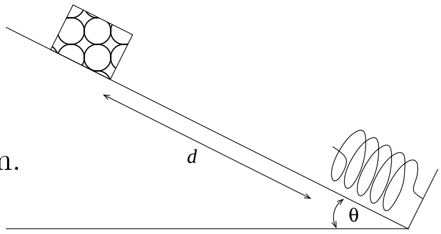
   (d) In the laboratory, you find that the disk begins to rise from the table at a value of h that is much smaller than you predicted in your analysis in part (c). Explain qualitatively how your analysis can be modified to give a better description of your observations.

6)

This problem investigates the contribution of the rotational motion of a diatomic molecule (such as CO, H_2, or O_2) or a linear molecule (such as CO_2) to the specific heat of a gas made up one variety of such molecules. The allowed rotational energies of such molecules are given by $E_j = \frac{\hbar^2 j(j+1)}{2I} = j(j+1)\varepsilon$ where $\varepsilon = \frac{\hbar^2}{2I}$ and I is the moment of inertia of the molecule. Answer the following questions.

1. Write down the rotational partition function of one such molecule assuming that *N* identical molecules are confined in a volume *V* at temperature *T*.
2. Evaluate the partition function
    a. for a CO molecule assuming $kT >> \varepsilon$ and
    b. for an H_2 molecule assuming $kT << \varepsilon$ and the total nuclear spin state of the H_2 molecule is *J=0* (a singlet parahydrogen state, with a symmetric wave function).
3. Determine the high-temperature and low-temperature specific heats of these molecules as described in Part 2a & 2b above, respectively. (Hints: $\bar{E} = -\frac{\partial \ln Z}{\partial \beta}$, where $\beta = 1/kT$, for low-temperature specific heat. Ignore higher-order terms after the first-order temperature-dependent term.)

7. A block of mass M is released from rest on a frictionless surface inclined at angle θ from horizontal, a distance d from the end of massless spring of spring constant κ, which is in equilibrium.



(a) Derive an expression for the compression of the spring at the instant that the block has its maximum speed in two independent ways:

- using Newton's Laws of Motion accompanied by the relevant free-body diagram, and
- using conservation of energy.

Discuss the salient features of your two results. Using the values $M = 10$ kg, $d = 4.0$ m, $\kappa = 250$ N/m, and $\theta = 30°$, calculate the value of the compression to one significant figure.

(b) Derive an expression for the maximum compression of the spring.

(c) Now, instead, the surface is *not* frictionless, and the block slides with a coefficient of kinetic friction μ. Do you expect the spring compressions from parts (a) and (b) to increase, decrease, or remain the same? Rederive each of your expressions to include the effects of friction, and show them to be consistent with your qualitative expectations.

8)

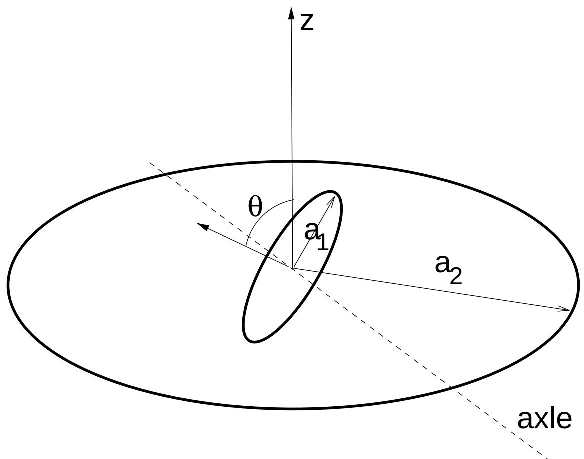
Asymptotic Expansions. Find the first term in the asymptotic expansion of the following integral (i.e. the behavior of the integral in the limit $x \gg 1$):

$$I = \frac{1}{\pi} \int_0^{\pi} \left(t^4 + 2t^6\right)^{1/2} e^{x \cos t} \cos nt \, dt \,, \tag{1}$$

where n is a constant. Show *all* your work.

9.

A small circular wire loop of radius a_1 sits inside a larger circular wire loop of radius $a_2 \gg a_1$. The two loops have the same center. The outer loop is fixed to lie in the x–y plane while the inner loop is mounted on an axle, parallel to $\hat{\mathbf{y}}$, which allows the angle between the loop normals, θ, to change. The outer loop has resistance R_2.



a. Compute the mutual inductance between the two loops for arbitrary angle θ.

b. Steady current I_1 is driven through the inner loop, and its axle is rotated steadily: $\theta = \omega t$. Compute the current flowing in the outer loop. (You should neglect its self-inductance.)

c. Find the **torque** on the inner loop as a function of time.

d. Work must be done against the above torque in order to keep the inner loop spinning at constant speed. Compute the fraction of this work that is dissipated resistively in the outer loop.

10)

A superposition quantum state can be written as:

$$|\Psi(t)\rangle = \sum_n c_n |n\rangle e^{-iE_n t/\hbar}$$

where the $|n\rangle$ are the eigenstates of the time independent Hamiltonian with eigenvalues of the energy E_n and coefficients c_n that satisfy the normalization requirement:

$$\sum_n |c_n|^2 = 1$$

Now consider the interaction of the eigenstates $|n\rangle$ with a time dependent perturbation given by

$$H'(t) = \begin{cases} V\cos(\omega t), & t \ge 0 \\ 0, & t < 0 \end{cases},$$

where V is the spatial part of the perturbation that leads to coupling between the states. This time dependent perturbation leads to a coupling between the coefficients c_n that can be shown via Schrödinger's equation to be given by:

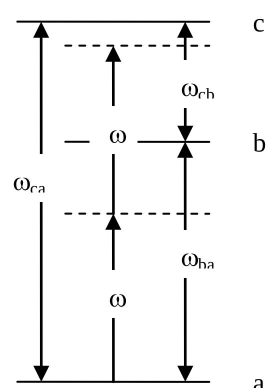
$$c_n(t) = -\frac{i}{\hbar}\int_0^t H'_{nm}(t') e^{i\omega_{nm}t'} c_m(t')\,dt' + c_n(0)$$

where $H'_{ba}(t) = \langle b|H'(t)|a\rangle$ and the frequency is related to the energies by $\omega_{ij} = \dfrac{E_i - E_j}{\hbar}$.

Now let this perturbation interact with a three level system as shown in the figure. We assume that only these three states are accessible. Assume the initial conditions on the coefficients for the superposition state are:

$c_a(0) = 1,\ c_b(0) = 0$ and $c_c(0) = 0$.

Also assume that state c has the same symmetry as state a so that V_{ac}=0 but that both V_{ab} and V_{bc} are nonzero. Here the notation is $V_{ba} = \langle b|V|a\rangle$.



a) Use first order time dependent perturbation theory to find the time dependent coefficient for the first excited state b $c_b^{(1)}(t)$. Simplify your answer by dropping the non-resonant term (some books call this the the Rotating Wave Approximation).

b) Now use your simplified result from a) and also second order time dependent perturbation theory to find the time dependent coefficient for the second excited state $c_c^{(2)}(t)$. Again, use the Rotating Wave Approximation to drop the non-resonant terms.

c) Now let $|\omega_{ca} - 2\omega| \ll |\omega_{cb} - \omega|$. Using this approximation, find an expression for the value of the product $|V_{cb}||V_{ba}|$ needed to get the maximum probability $|c_c^{(2)}|^2$ will be equal to 0.01.

d) For the case in c), find an expression for the time t_{max} to reach the maximum probability $|c_c^{(2)}|^2 = 0.01$ for the first time.

11. One of the fundamental characteristics of the non-relativistic superfluid state is that the system behaves as if a non-zero fraction of the atoms is described by the same time-independent wave function $\psi(\vec{r})$ in non-relativistic quantum mechanics. Here, we explore some of the surprising properties of this macroscopic quantum fluid.

    (a) Explain why the general form of this wave function can be written $\psi(\vec{r}) = |\psi(\vec{r})|e^{iS(\vec{r})}$, where S is a purely real, scalar function.

    (b) Explain why both ψ and $\vec{\nabla}\psi$ are required to be continuous, single-valued functions within a confining cylindrical volume. From these conditions, determine the mathematical constraints on S.

    (c) Recall that the *probability current density* for a particle of mass m and charge q in a real vector potential $\vec{A}$ is defined

    $$\vec{j} \equiv \frac{1}{2m}\left[\psi^* \left(-i\hbar\vec{\nabla} - \frac{q}{c}\vec{A}\right)\psi + \psi\left(+i\hbar\vec{\nabla} - \frac{q}{c}\vec{A}\right)\psi^*\right]$$

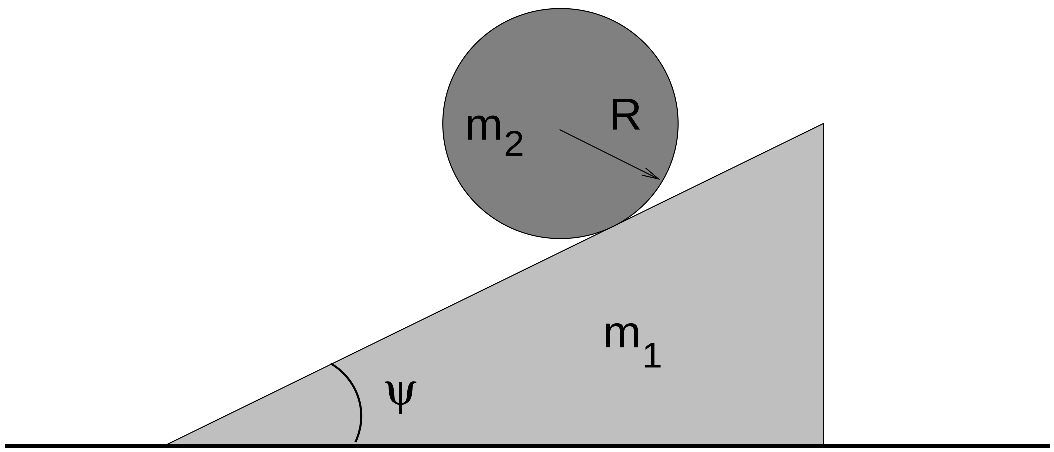
    in Gaussian units. We can further introduce the velocity field $\vec{v}$ such that $\vec{j} = |\psi|^2\vec{v}$. Determine a manifestly real expression for $\vec{v}$ in terms of the modulus and phase of ψ. For the case that the magnetic field is zero, show that $\vec{\nabla}\times\vec{v} = 0$, *i.e.*, that the velocity field has zero *vorticity*, except at special singular points.

    (d) The *circulation* of any vector field is defined by its closed-path integral. For a superfluid, the circulation is maintained with zero dissipation. Evaluate the circulation of your general expression for $\vec{v}$ about an arbitrary point. Show that it is quantized in units of $2\pi\hbar/m$ for the case of an electrically neutral superfluid, *e.g.*, atomic helium in a terrestrial lab or neutrons in a neutron star. This results in the appearance of persistent vortices when the system is put in rotation.

    (e) If, instead, the superfluid is *charged*, show that an additional term containing the *magnetic flux* appears in the circulation. The magnetic flux quantum is important in describing *superconductivity*.

12.

A wedge of mass m_1 slides frictionlessly along a table. Its upper surface is inclined from the horizontal by ψ. A solid ball of radius R, mass m_2, and rotational inertia $I = (2/5)m_2R^2$, rolls along the upper surface **without slipping**. The wedge is at rest and the ball is released from rest near the top of the slope as shown.



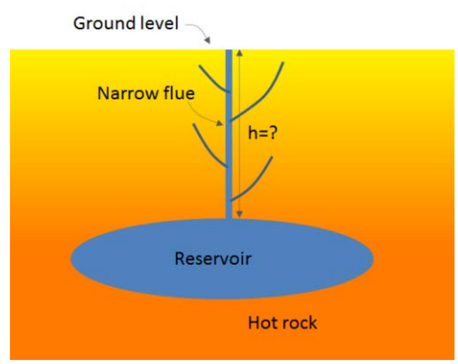
a. Identify clearly a set of generalized coordinates. (This should include a sketch.) Write down the Lagrangian of the system using these.

b. Find the acceleration **of the wedge**. Your answer should depend only on R, m_1, m_2, g and ψ.

13)

The Clausius-Clapeyron (or vapor pressure) equation describes the p vs. T phase diagram at thermal equilibrium between the liquid and gas phases of a substance such as water. The slope of the p vs. T curve is given by $\frac{dp}{dT} = \frac{L}{T\Delta v}$ where L is the latent heat to convert one mole of the substance from the liquid to the gas phase and Δv is the change in the volume per mole of the substance when it transforms from the liquid to the gas phase. Using this relation and assuming that L is independent of temperature in our temperature region of interest answer the following questions:

1. Prove that $p(T) = p_o \exp(-L/RT)$. Assume that the substance is water and use $L \approx 40\,kJ$ per mole to obtain an order of magnitude estimation for p_o.
2. Assume that the famous Old Faithful geyser in Yellowstone National Park is modeled as a large underground reservoir at depth h from the ground level (see figure below). A narrow flue connects the reservoir to the ground level. The surrounding rock is hot because of volcanic activity and the temperature increases linearly with depth as $T = T_s + \alpha h$, where T_s is the surface temperature at the ground level. As you know, Old Faithful erupts about every 90 mins expelling tons of steam and water within less than five minutes. Assume that $\alpha \approx 1\,^{o}$C/m. Show how you would go about determining the depth h of the reservoir and explain the eruption mechanism of Old Faithful to a nonphysicist.



14)

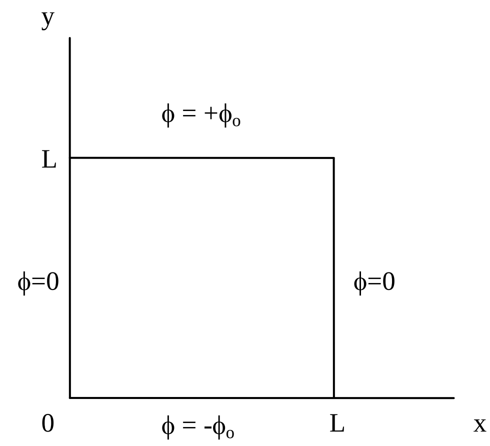
Find the potential ϕ at all points <u>inside</u> a region given by $0<x<L$ and $0<y<L$, and $-\infty<z<\infty$. The potentials for the four boundary planes are given by:

$$\phi(0,y,z)=0$$

$$\phi(L,y,z)=0$$

$$\phi(x,0,z)=-\phi_o$$

$$\phi(x,L,z)=+\phi_o$$



15)

Null Rays around a Black Hole. The trajectory of a null ray (light) in the geometry of a Schwarzschild black hole, i.e. one with $SO(3)$ symmetry, can be written as $r(\phi)$, where $u \equiv GM/(c^2 r(\phi))$ satisfies the equation

$$\frac{d^2u}{d\phi^2} + u = 3u^2 \,, \tag{1}$$

with $r(\phi)$ is the distance from the black hole to the photon ray and M the black hole mass. Denote by b the distance of closest approach, i.e. the periholium, and consider a non-grazing collision so that $GM/(c^2 b) \ll 1$. Given this:

(i) Draw a diagram!

   (ia) Imagine first that the black hole mass is tiny, and draw the trajectory of the "unperturbed" ray, i.e. a ray that travels unaffected by the presence of the black hole. Label this trajectory r_0 or u_0.

   (ib) Imagine now that the black hole mass is not negligible, so that the light ray is deflected. Draw the trajectory for the "perturbed" path, so that both the perturbed and unperturbed rays are *initially* parallel to each other. Denote this trajectory by r_1 or u_1.

   (ic) Denote the angle between the unperturbed and the perturbed path $\delta\phi$. Indicate this angle in your diagram, as well as the locations of $r = \infty$ at $t = -\infty$ and $t = +\infty$, the distance b and the angle $\delta\phi$. Why are there no additional coordinates needed to solve this problem?

(ii) Solve for the unperturbed trajectory of light, i.e. for the leading-order in M/b solution? Interpret this solution physically, i.e. explain it in words, and compare it to your expectation from part (i).

(iii) Now solve for the perturbed trajectory to next-order in $GM/(c^2 b)$. What is the angular deflection, $\delta\phi$ of a photon from its unperturbed path as it passes the spherical body? You may assume that $\delta\phi$ is small and solve for it to leading order in $GM/(c^2 b)$.