[redacted] $\left\{\begin{array}{l}\text{Set \#⑦ : Probs. 23-25.} \\ \underline{\text{Assigned : 11/4/88; due 11/14/88.}}\end{array}\right.$ ⑦ P9

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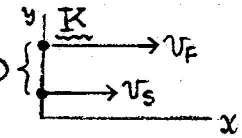
(23) [Jackson Prob. (11.5)]. In frame K, a F(ast) & S(low) runner line up @ D{ distance D along the y-axis, for a race down the x-axis. Two starters, one beside each runner, fire their starting pistols at slightly different times, in order to handicap the faster runner. The handicap time (difference) in K is T.



(A) For what range of T-values can there be a frame K′, moving at $v<c$ w.r.t. K, where the handicap vanishes? For what range of T will K′ <u>always</u> see a handicap $T'>0$?

(B) Find the Lorentz transform $K \to K'$ for each of the cases in part (A). Specify both the $K\to K'$ relative velocity (& direction), and the runners' positions in K′. Who wins?

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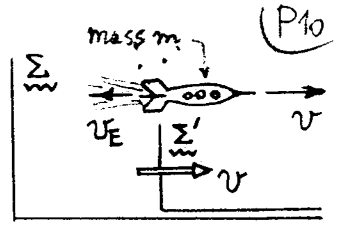
(24) [Jackson Prob. (11.7)]. An ∞simal Lorentz transform and inverse is represented by :

$\underline{x'^{\alpha} = (g^{\alpha\beta} + \epsilon^{\alpha\beta})\, x_{\beta}}$, $\underline{x^{\alpha} = (g^{\alpha\beta} + \epsilon'^{\alpha\beta})\, x'_{\beta}}$. $(g^{\alpha\beta})$ is the metric tensor [Jkⁿ Eq. (11.70)], and the ϵ's are ∞simals (⇒ retain first order only). (A) Show, from the definition of the inverse, that : $\epsilon'^{\alpha\beta} = (-)\,\epsilon^{\alpha\beta}$. (B) Show, from invariance of the norm, that $\underset{\sim}{\epsilon}$ is antisymmetric : $\epsilon^{\alpha\beta} = (-)\,\epsilon^{\beta\alpha}$. (C) Write the transform with <u>contravariant</u> components on both sides of the eqtn. Show that $(\epsilon^{\alpha\beta})$ is equivalent to $\underset{\sim}{L}$ in Jkⁿ Eq. (11.93).

[redacted]



(25) [redacted]. A rocket starts from rest in reference frame Σ (earth) at time $t=0$, when its rest mass is m_0. Later, it is moving at velocity v w.r.t. Σ, with mass (as measured on-board) $m < m_0$, since it has burned some fuel. Assume the exhaust velocity of the burning fuel is v_E = constant, relative to the rocket.

(A) When the rocket velocity is v in Σ, consider frame Σ' -- also moving at v in Σ -- instantaneously at rest w.r.t. rocket. In the next instant, the rocket ejects a fuel increment dm (at v_E w.r.t. Σ'; note: m decreases in time), and increases its velocity by dv' w.r.t. Σ'. Use momentum conservation in Σ' for this situation, to relate dm, v_E, dv', and the rocket mass m. Keep only 1st order terms in the small quantities dm & dv'.

(B) The increment dv' in Σ' transforms to dv in Σ, so -- after the acceleration in part (A) -- the rocket velocity is $(v+dv)$ in Σ. Use SRT velocity addition to find dv in terms of dv'; retain only 1st order terms in dv'. Use this, with momentum conservation in part (A), to find the rocket eqtn-of-motion in Σ: $\boxed{m(dv/dm)+v_E(1-\beta^2)=0}$, w/ $\beta=v/c$.

(C) Solve the eqtn-of-motion of part (B) for β in terms of m, m_0 & v_E. Show the solution is: $\boxed{\beta=(1-f^{2\beta_E})/(1+f^{2\beta_E})}$, w/ $\beta_E=\frac{v_E}{c}$ is the exhaust velocity (units of c), and $f=\frac{m}{m_0}$ is the fractional mass remaining when the rocket attains $v=\beta c$. What is the "best" β_E -- i.e. the β_E giving the largest rocket β for a given fractional fuel burn $(1-f)$?

(D) If τ is the rocket's proper time, show: $\boxed{m(d\beta/d\tau)+\beta_E(1-\beta^2)\frac{dm}{d\tau}=0}$ is its eqtn-of-motion. Assume an exponential burn scheme onboard the rocket: $m=m_0e^{-\alpha\tau}$. Use this in the eqtn-of-motion, transform back to the Σ-frame, and solve the resulting eqtn for β as a fcn of Σ (earth) time t. At what time t (earth) does the rocket reach $\beta=n\beta_E$, n = some #? If $n\beta_E \ll 1$, find the mass remaining @ $\beta=n\beta_E$. What happens for a photon rocket, $\beta_E=1$?

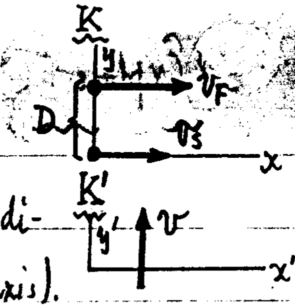
(E) For the exponential burn of part (D), find the acceleration felt by the rocket crew.

(F) If 99% of the initial rocket mass m_0 is fuel, calculate β at burnout. Assume $\beta_E \ll 1$.

(G) Calculate how far the rocket has traveled w.r.t. Σ (earth) at the burnout in part (F).

[redacted]

(23) Analyse handicapped race from standpoint of moving observer K'.



11/11/84

(a) Evidently, K' must move along the y-axis of K, in order to get modifications of the starters' delay time T (arranged by signals along the y-axis). Then, assuming K' velocity is $v\hat{y}$ in K, the handicap time in K' is...

$$T' = \gamma\left(T - \frac{v\Delta y}{c^2}\right) = \gamma\left(T - \beta\frac{D}{c}\right), \text{ since } \Delta y = D \text{ between starters } \Big\{ \text{and: } \beta = v/c,\ \gamma = 1/\sqrt{1-\beta^2}.$$

T' vanishes if $T = \beta\frac{D}{c}$. With $0<\beta<1$, T' can vanish over the range: $\boxed{0 < T < \frac{D}{c}}$.

If there is to be a true handicap in K', then: $T' = \gamma\left(T - \beta\frac{D}{c}\right) > 0 \Rightarrow T > \beta D/c$.

For $0<\beta<1$, this is always true only when $\boxed{T \geqslant D/c}$, i.e. when the starter's pistol shots are connected in a causal (time-like) way.

(b) The $K \rightarrow K'$ transform is: $t' = \gamma\left(t - \frac{v}{c^2}y\right)$, $y' = \gamma(y - vt)$. For the two cases...

A. $T' = 0 \Rightarrow T = \beta\frac{D}{c}$, or// $\underline{\underline{\beta = cT/D}}$, and//: $t' = \gamma\left(t - \frac{y}{D}T\right)$, $y' = \gamma\left(y - \left(\frac{cT}{D}\right)ct\right)$

B. $T' > 0 \Rightarrow T > \beta\frac{D}{c}$, or $\underline{\underline{\beta < cT/D}}$, and// $\left.\begin{array}{l} t' = \gamma\left(t - \epsilon\frac{y}{D}T\right) \\ y' = \gamma\left(y - \epsilon\left(\frac{cT}{D}\right)ct\right) \end{array}\right\}$ for: $\beta = \epsilon(cT/D)$ $(0<\epsilon<1)$.

In both cases, K' moves at v along the (+)ve y-axis of K, i.e. $\vec{\beta} = \beta\hat{y}$.

Assume the runners have cnst speeds: $v_S = V$, and: $v_F = V + \Delta V$, in K. Then their K cds are: $(x_S = Vt,\ y_S = 0)$, and $(x_F = (V+\Delta V)(t-T),\ y_F = D)$, for $t \geqslant T$. In K'...

$$\left[\begin{array}{l} x_S' = x_S = Vt, \text{ or: } \underline{\underline{x_S'}} = Vt'/\gamma \ ;\ y_S' = -\gamma v t = -vt', \text{ i.e. } \underline{\underline{y_S'}} = -vt'; \\ x_F' = x_F = (V+\Delta V)(t-T), \text{ or: } \underline{\underline{x_F'}} = (V+\Delta V)\left(\frac{t'}{\gamma} + \frac{vD}{c^2} - T\right), \text{ for: } t' > \gamma\left(T - \frac{vD}{c^2}\right); \\ y_F' = \gamma(D - vt) \leftarrow t = \frac{t'}{\gamma} + \frac{vD}{c^2}, \text{ gives: } \underline{\underline{y_F'}} = \frac{D}{\gamma} - vt', \text{ for: } t' \text{ as above } \Big\{ \text{note correct Lorentz contraction.} \end{array}\right.$$

Who wins the race? Either F or S runner may... but point is: K & K' will agree who wins.

[redacted]

11/17/84

Explore details of ∞simal Lorentz transform $\begin{cases} x'^{\alpha} = (g^{\alpha\beta} + \epsilon^{\alpha\beta})\, x_\beta \\ x^{\alpha} = (g^{\alpha\beta} + \epsilon'^{\alpha\beta})\, x'_\alpha \end{cases}$

(a) From the defns, and Jackson's Eq. (11.73), have: $x'^{\alpha} = x^{\alpha} + \epsilon^{\alpha\beta} x_\beta$, $x^{\alpha} = x'^{\alpha} + \epsilon'^{\alpha\beta} x'_\beta$.

This implies: $(x'^{\alpha} - x^{\alpha}) = \epsilon^{\alpha\beta} x_\beta = (-)\,\epsilon'^{\alpha\beta} x'_\beta = (-)\,\epsilon'^{\alpha\beta} g_{\beta\sigma} x'^{\sigma}$, by (11.72).

But: $x'^{\sigma} = (g^{\sigma\tau} + \epsilon^{\sigma\tau})\, x_\tau$, so: $\epsilon^{\alpha\beta} x_\beta = (-)\,\epsilon'^{\alpha\beta} g_{\beta\sigma} (g^{\sigma\tau} + \cancel{\epsilon^{\sigma\tau}})\, x_\tau$ ← ignore, 2^{ND} order in ϵ,

Use (11.71): $g_{\beta\sigma} g^{\sigma\tau} = \delta^{\tau}_{\beta}$, so: $\epsilon^{\alpha\beta} x_\beta = (-)\,\epsilon'^{\alpha\beta} \delta^{\tau}_{\beta} x_\tau = (-)\,\epsilon'^{\alpha\beta} x_\beta$.

This can only hold for all x_β if in fact: $\boxed{\epsilon'^{\alpha\beta} = (-)\,\epsilon^{\alpha\beta}}$, as required.

(b) Impose invariance of the norm: $x'_\sigma x'^{\sigma} = x_\mu x^{\mu}$. Then calculate, to 1^{ST} order in ϵ:

$$x'_\sigma x'^{\sigma} = g_{\sigma\tau} x'^{\tau} x'^{\sigma} = g_{\sigma\tau} (g^{\tau\mu} + \epsilon^{\tau\mu})\, x_\mu\, (g^{\sigma\nu} + \epsilon^{\sigma\nu})\, x_\nu$$

$$= \underbrace{(g_{\sigma\tau} g^{\tau\mu})}_{\delta^{\mu}_{\sigma},\ \text{sum } \tau} x_\mu g^{\sigma\nu} x_\nu + g_{\sigma\tau} \epsilon^{\tau\mu} x_\mu g^{\sigma\nu} x_\nu + \underbrace{(g_{\sigma\tau} g^{\tau\mu})}_{\delta^{\mu}_{\sigma},\ \text{sum } \tau} x_\mu \epsilon^{\sigma\nu} x_\nu$$

$$= g_{\tau\sigma} g^{\sigma\nu} = \delta^{\nu}_{\tau},\ \text{sum } \sigma$$

so, $x'_\sigma x'^{\sigma} = x_\mu (g^{\mu\nu} x_\nu) + x_\mu \epsilon^{\nu\mu} x_\nu + x_\mu \epsilon^{\mu\nu} x_\nu$

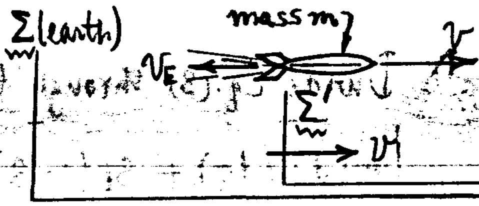
$= x_\mu x^{\mu} + x_\mu (\epsilon^{\nu\mu} + \epsilon^{\mu\nu})\, x_\nu$ { If: $x'_\sigma x'^{\sigma} = x_\mu x^{\mu}$, then 2^{ND} term must $\equiv 0$. So: $\boxed{\epsilon^{\nu\mu} = (-)\,\epsilon^{\mu\nu}}$

(c) Write: $x'^{\alpha} = (g^{\alpha\beta} + \epsilon^{\alpha\beta})\, x_\beta = (g^{\alpha\beta} + \epsilon^{\alpha\beta})\, g_{\beta\gamma} x^{\gamma} = (\delta^{\alpha}_{\gamma} + \epsilon^{\alpha}_{\gamma})\, x^{\gamma}$,

since: $g^{\alpha\beta} g_{\beta\gamma} = \delta^{\alpha}_{\gamma}$, and with: $\epsilon^{\alpha}_{\gamma} = \epsilon^{\alpha\beta} g_{\beta\gamma}$. But: $\delta^{\alpha}_{\gamma} x^{\gamma} = x^{\alpha}$, is the identity, and since ϵ is a 1^{ST} order ∞simal: $x'^{\alpha} = [\exp(\epsilon^{\alpha}_{\gamma})]\, x^{\gamma}$.

Symbolically: $\boxed{\tilde{x}' = [\exp(\underset{\sim}{\epsilon})]\, \tilde{x}}$, is the Lorentz transform, where $\tilde{x}$ & $\tilde{x}'$ are the cd 4-vectors. Then $\underset{\sim}{\epsilon} = \underset{\sim}{L}$ of Eqs (11.87) & (11.93), when $\underset{\sim}{L}$ is an ∞simal. Strictly speaking: $(\epsilon^{\alpha}_{\gamma}) = (L^{\alpha}_{\gamma})$ here, but: $(\epsilon^{\alpha\beta}) = (L^{\alpha\beta})$, also.

[redacted]



(25) [redacted]. Analyse trip in a relativistic rocket.

11/12/84

A. In Σ', the momentum gained by the rocket in ejecting an amount of fuel $(-)dm$ [note that dm is negative] is: $(m-(-dm))\,dv' = m\,dv'$, to 1st order ∞simals. The usual γ factor here is $=1$, to 1st order in dv'. This momentum change is equal & opposite to the momentum of the ejected fuel, which is: $v_E(-dm)$, with the γ_E factor absorbed into dm to make it the effective rest mass increment in Σ', rather than the increment in a frame moving at v_E w.r.t. Σ'. Thus, in Σ'...

$$m\,dv' = v_E(-dm), \text{ or: } \boxed{m\,dv' + v_E\,dm = 0}, \tag{1}$$

represents momentum conservation, with m the (instantaneous) rest mass in Σ'.

B. dv' in $\Sigma' \leftrightarrow dv$ in Σ, and -- by the velocity addition formula -- to 1st order qtys

$$v + dv = (v + dv')/\left(1 + \frac{v\,dv'}{c^2}\right) \Rightarrow \boxed{dv = (1-\beta^2)\,dv'}, \text{ where } \beta = \frac{v}{c}. \tag{2}$$

Using this in Eq. (1), we immediately get the eqtn-of-motion in Σ...

$$\boxed{m(dv/dm) + v_E(1-\beta^2) = 0}, \text{ in } \Sigma(\text{earth}) \text{ frame}. \tag{3}$$

C. Reorganize the eqtn-of-motion, Eq. (3), with: $\beta_E = v_E/c$, and: $f = m/m_0$:

$$\frac{d\beta}{1-\beta^2} + \beta_E \frac{df}{f} = 0 \begin{Bmatrix} \text{integrate over} \\ 0 \to \beta \Rightarrow 1 \to f \end{Bmatrix} \frac{1}{2}\ln\left|\frac{1+\beta}{1-\beta}\right| + \beta_E \ln f = 0,$$

$$\text{so//}\quad \frac{1-\beta}{1+\beta} = f^{2\beta_E}, \quad \text{or//}\quad \boxed{\beta = (1-f^{2\beta_E})/(1+f^{2\beta_E})}, \text{ with } \begin{cases} \beta_E = v_E/c, \\ f = m/m_0. \end{cases} \tag{4}$$

For fixed f, β is a monotonically increasing fcn of β_E, so the "best" β_E is the largest. This is $\beta_E = 1$, i.e. $v_E = c$, or a "photon rocket", with the fuel ejected as radiation. So far, NASA has not been able to do this, due to lack of funding.

[redacted]

D. Divide Eq. (3) through by $d\tau$ (proper time), and reorganize slightly to get...

$$\boxed{m(d\beta/d\tau) + \beta_E(1-\beta^2)\,\frac{dm}{d\tau} = 0}, \text{ so } \frac{d\beta}{d\tau} = \alpha\beta_E(1-\beta^2), \text{ for: } m = m_0 e^{-\alpha\tau}, \quad (5)$$

in the rocket frame. The Σ(earth) time increment: $dt = d\tau/\sqrt{1-\beta^2}$, so...

$$\frac{d\beta}{dt} = \alpha\beta_E(1-\beta^2)^{\frac{3}{2}} \Rightarrow \boxed{\beta = \alpha\beta_E t/[1+(\alpha\beta_E t)^2]^{\frac{1}{2}},} \quad (6)$$

for the rocket's exptl. burn scheme, as viewed from Σ(earth). Solving Eq. (6) for t...

$$\alpha\beta_E t = \beta/\sqrt{1-\beta^2}, \text{ so: } \beta = n\beta_E \text{ @ } \boxed{t = \frac{n}{\alpha}[1-(n\beta_E)^2]^{-\frac{1}{2}}} \simeq \frac{n}{\alpha}\left[1+\frac{1}{2}(n\beta_E)^2\right] \quad (7)$$

From Eq. (4): $2\beta_E \ln f + \ln\left(\frac{1+\beta}{1-\beta}\right) = 0$, so $\beta_E \ln f + \beta\left[1+\frac{1}{3}\beta^2+\ldots\right] = 0$, when $\beta \ll 1$.

So $f \simeq e^{-(\beta/\beta_E)[1+\frac{1}{3}\beta^2]}$, for $\beta \ll 1$, and $\beta = n\beta_E \ll 1 \Rightarrow \boxed{f \simeq e^{-n[1+\frac{1}{3}(n\beta_E)^2]}}$ (8)

For a photon rocket: $\beta_E = 1$, and we must limit $0 \leq n \leq 1$. For $n \ll 1$, Eqs. (7) & (8) go thru as before. But as $n \to 1$: $t(\text{earth}) = \frac{n}{\alpha}/\sqrt{1-n^2} \to \infty$, and: $f = \sqrt{\frac{1-n}{1+n}} \to 0$.

E. From Eq. (1), the on-board acceleration is: $a' = \frac{dv'}{d\tau} = -v_E\left[\frac{1}{m}\left(\frac{dm}{d\tau}\right)\right]$. For the exponential burn scheme, this is just: $\boxed{a' = v_E\alpha = \text{cnst}}$. This a' could be adjusted to give some convenient fraction of g, so as to please the passengers.

F. By simple identities, the expression for β in Eq. (4) can be written...

$$\beta = \tanh[\beta_E \ln(1/f)] \simeq \beta_E \ln(1/f), \text{ for } \beta_E \ll 1 \text{ and } \ln(1/f) \text{ not too large.} \quad (9)$$

For $f = 0.01 \left\{\begin{matrix}1\% \text{ payload}\\ 99\% \text{ fuel}\end{matrix}\right\}$, get: $\boxed{\beta \simeq \beta_E \ln 100 = 4.61\,\beta_E}$ at burnout. Unimpressive!

G. Distance traveled (in Σ(earth)): $D(t) = \int_0^t c\beta(t')\,dt'$, with β of Eq. (6). Easily get

$$D(t) = \frac{c}{\alpha\beta_E}\left[\sqrt{1+(\alpha\beta_E t)^2} - 1\right], \text{ or (by Eq. (7)): } D(@\beta = n\beta_E) = \frac{c}{\alpha\beta_E}\left[\frac{1}{\sqrt{1-(n\beta_E)^2}} - 1\right]. \quad (10)$$

So: $D \simeq \frac{1}{2}n^2 v_E/\alpha$, if $n\beta_E \ll 1$, and $n = 4.61 \left\{\begin{matrix}\text{from}\\ \text{Eq. (9)}\end{matrix}\right\} \Rightarrow \boxed{D = 10.6\,\frac{v_E}{\alpha}}$. Not very far!