Lagrangian for a Continuum

{background for Jackson Sec. (12.8), esp. Eq. (12.83); (~ Goldstein "Classical Mechanics, Ch. 11).

8) We have constructed Lagrangians for two discrete systems, viz...

① q in external $\mathbb{E}$ & $\mathbb{B}$ (i.e. ϕ & $\mathbb{A}$), ‖ both Lagrangians $L \Rightarrow$ Lorentz force law.

② q_1 interacting with q_2 (extl. $\mathbb{E}$ & $\mathbb{B} \equiv 0$); ‖ NOTE: the q_i's have discrete positions $\mathbb{r}_i$.

We shall now try something new: viz. construct an L for a continuous system...

③ fields $\mathbb{E}$ & $\mathbb{B}$ as generated by their own sources ρ & $\mathbb{J}$; ‖ want the Lagrangian L(EM flds) $\Rightarrow$ Maxwell Eqtns. NOTE: $\mathbb{E}$ & $\mathbb{B}$ are continuous fcns of position $\mathbb{r}$.

Such an L, for the EM fields per se, not only regurgitates the Maxwell Eqs, but is useful to //

(1) modify the Maxwell system (e.g. adding m(photon) ≠ 0, or magnetic monopoles), because it is ~ easy to see how to add "manifestly covariant" terms;

(2) serve as a model for other field theories (e.g. gravitation);

(3) provide a transition to QM and field quantization via the "Canonical formalism."

The construction of L(EM flds) thus appears worthwhile. Maybe even simple. But there is a catch: for discrete q_i's, the position cds $x_i = x_i(t)$ are discrete [$x_i(t)$ is where q_i is at, at time t], while for continuous $\mathbb{E}$'s, the position cd $\mathbb{r} \neq \mathrm{fcn}(t)$ is a continuous variable ($\mathbb{r}$ gives only one space pt. of $\mathbb{E}$). Then, in going from a discrete (particle) system L to a continuous (field) system L, position variables must play a new role in the Euler–Lagrange equations...

discrete charges q_i at positions $x_i(t)$: $\dfrac{d}{dt}(\partial L/\partial \dot{x}_i) = \dfrac{\partial L}{\partial x_i}$;

Continuous fields $\mathbb{E}(\mathbb{r},t)$, $\mathbb{r} \neq \mathrm{fcn}(t)$: $\dfrac{d}{dt}(\partial L/\partial \dot{?}) = \partial L/\partial ?$

We must find what new variables ? and $\dot{?}$ are needed.

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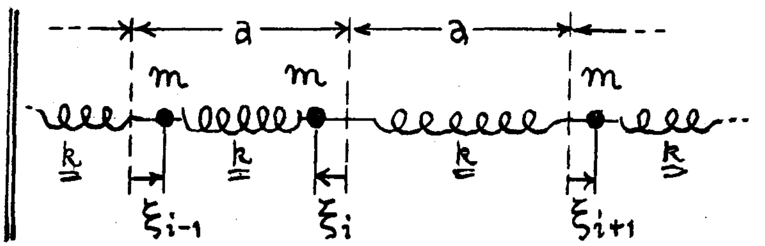
9) As a guide, we shall analyse the continuum limit of a simple (1D) discrete system.

INEAR HAIN { masses m & springs k all identical, equilibrium separations all $\equiv a$; 1D motion with displacements ξ_i.



1D Linear Chain

To specify the linear chain, we need the $\xi_i = \xi_i(t)$. Standard Lagrange method is:

K.E.: $T = \frac{1}{2}\sum_i m\dot{\xi}_i^2$, **P.E.**: $V = \frac{1}{2}\sum_i k(\xi_{i+1}-\xi_i)^2$; ¶

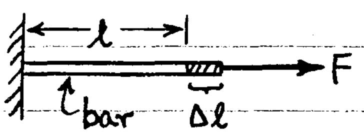
Lagrangian: $L = T - V = \frac{1}{2}\sum_i [m\dot{\xi}_i^2 - k(\xi_{i+1}-\xi_i)^2]$,

or// $L = \frac{1}{2}\sum_i a\left[\mu\dot{\xi}_i^2 - ka\left(\frac{\xi_{i+1}-\xi_i}{a}\right)^2\right]$, $\int$ a = lattice spacing, $\mu = \frac{m}{a}$ = mass/unit length. (1)

Eqtns of Motion $\Big\}$ $\frac{d}{dt}(\partial L/\partial\dot{\xi}_i) - \frac{\partial L}{\partial \xi_i} = 0 \Rightarrow \boxed{\mu\ddot{\xi}_i - ka\left(\frac{\xi_{i+1}-\xi_i}{a^2}\right) + ka\left(\frac{\xi_i-\xi_{i-1}}{a^2}\right) = 0}$.

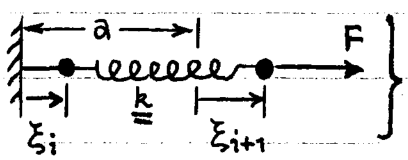
[What we will do now is make this system continuous by passing to the limit $a\to 0$. Question is: how does L and its eqtns-of-motion change?

10) When $a\to 0$, there is no trouble interpreting $\mu = \frac{m}{a} \to \frac{dm}{da} \to$ finite. But what about $\lim_{a\to 0}(ka)$? From mechanics, recall definition of "Young's Modulus" for an elastic bar:



$\Big\}$ $F = Y\left(\frac{\Delta\ell}{\ell}\right)$, Y = Young's Modulus.

For our linear chain...



$\Big\}$ $F = k(\xi_{i+1}-\xi_i) = \underbrace{(ka)}_{\text{analogous to } Y}\overbrace{\left(\frac{\xi_{i+1}-\xi_i}{a}\right)}^{\text{analogous to } \frac{\Delta\ell}{\ell}}$. (2)

Evidently, in the continuum limit: $ka \to Y$. The Lagrangian of Eq. (1) is...

$$\to L = \frac{1}{2}\sum_i \left[\mu\dot{\xi}_i^2 - Y\left(\frac{\xi_{i+1}-\xi_i}{a}\right)^2\right]a, \quad (3)$$

and we must go to $\lim_{a\to 0}$. In this limit, we will have...

ξ_i (discrete) $\to \xi(x)$, a continuous fcn of position x on the chain;

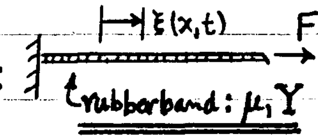
$a \to dx$, $\left(\frac{\xi_{i+1}-\xi_i}{a}\right) \to \frac{\partial\xi}{\partial x}$, and $\sum_i \to \int dx$. (4)

¶ This $V \Rightarrow$ correct force on i^{th} particle: $F_i = -\frac{\partial V}{\partial \xi_i} = k[\underbrace{(\xi_{i+1}-\xi_i)}_{F_i\text{(right)}} - \underbrace{(\xi_i - \xi_{i-1})}_{F_i\text{(left)}}]$.

Passage to Continuum Limit

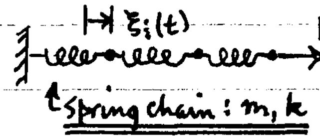
With the prescriptions of Eq. (4), the discrete chain L of Eq. (3) goes over to...

$$\boxed{L = \frac{1}{2}\int\left[\mu\dot{\xi}^2 - Y\left(\frac{\partial\xi}{\partial x}\right)^2\right]dx} \leftarrow \text{for a continuous chain:} \quad (5)$$



Compare with...

$$\longrightarrow L = \frac{1}{2}\sum_i\left[m\dot{\xi}_i^2 - k(\xi_{i+1}-\xi_i)^2\right] \leftarrow \text{for discrete chain:} \quad (1)$$



In comparing these two L's, note that in both cases the cd ξ measures displacement of some part of the chain from an equilibrium position. But the labelling is different. In the discrete m case, the labelling is $\xi \to \xi_i(t)$, with index i denoting a discrete position. In the continuous μ case, the labelling $\xi \to \xi(x,t)$ is by means of a continuous position variable x.

Also, in the continuous case, the (equilibrium) position x is not a dynamical (indpt) variable -- it is just a position label. The dynamical variables for L(continuum) are ξ and $\dot{\xi}$, and they appear as dependent variables in L(continuum).

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11) What happens to the Lagrange eqtn-of-motion in the continuum limit? From Eq. (1)...

$$\mu\ddot{\xi}_i - \frac{Y}{a}\left[\underbrace{\left(\frac{\xi_{i+1}-\xi_i}{a}\right)}_{(\partial\xi/\partial x)|_{at\,x}} - \underbrace{\left(\frac{\xi_i - \xi_{i-1}}{a}\right)}_{(\partial\xi/\partial x)|_{at\,(x-a)}}\right] = 0 \quad \int \text{have put } ka = Y, \text{ as } a\to 0;$$

$$\ddot{\xi}_i \to \ddot{\xi} \qquad \Rightarrow \frac{1}{a}\left(\frac{\partial\xi}{\partial x}\right)\Big|_{x-a}^{x} = \frac{\partial^2\xi}{\partial x^2}, \text{ as } a\to 0;$$

So//

$$\boxed{\mu(\partial^2\xi/\partial t^2) - Y(\partial^2\xi/\partial x^2) = 0}, \ \text{w/ displacement } \xi = \xi(x,t) = \text{fcn}\begin{pmatrix}\text{position } x\\ \text{time } t\end{pmatrix}. \quad (6)$$

This is just the eqtn for elastic waves (at velocity $v = \sqrt{Y/\mu}$) on a "rubberband". Just as L of Eq. (5) is a continuous fcn of ξ and $\dot{\xi}$, the associated eqtns-of-motion now describe a continuous displacement $\xi(x,t)$.

It is ~remarkable that we generate a wave eqtn by this discrete → continuum ruse. But, wave eqtns are the heart of EM theory... so, we can hope to do the same for the EM field. What we will need to do is find the "displacements" $\xi(x,t)$ appropriate to the continuum fields. The ξ's will turn out to be $\propto \phi$ & A, not surprisingly.

Continuum Lagrangian Formulation: Eqtns of Motion

12) We shall now seek general Lagrange eqtns-of-motion for a continuous system. In the above example of the linear chain, we found that the generalized coordinate (a displacement) was $\xi = \mathrm{fcn}(x;t)$. In 3D, evidently $\xi = \mathrm{fcn}(x,y,z;t)$. Then...

$$\left[\begin{array}{l} \xi = \xi(x,y,z;t), \text{ and}/\!/ \text{ Lagrangian}: L = \iiint \mathcal{L}\, dx\, dy\, dz, \\ \text{with } \mathcal{L} = \mathcal{L}(\xi;\xi_x,\xi_y,\xi_z,\xi_t;x,y,z;t) \leftarrow \text{called "Lagrangian Density"}. \end{array}\right. \qquad (7)$$

Here: $\xi_x = \partial\xi/\partial x$, etc. For a 3D elastic medium, we would have:

$$\left[\begin{array}{l} \mathcal{L} = \frac{1}{2}\left[\mu\xi_t^2 - Y(\xi_x^2+\xi_y^2+\xi_z^2)\right], \text{ and...} \\ \text{eq-of-motion}: \quad \mu\xi_{tt} - Y(\xi_{xx}+\xi_{yy}+\xi_{zz}) = 0 \end{array}\right\} \text{for 3D rubberband.} \qquad (8)$$

This particular $\mathcal{L}$ does not depend on position (x,y,z) explicitly -- unless the medium is anisotropic (i.e. μ and/or $Y = \mathrm{fcns}(x,y,z)$). We will carry a possible (x,y,z) variation in $\mathcal{L}$, as well as dependence on $\xi, \xi_x, \ldots, \xi_t$, and t. The program will be:

$$\left[\begin{array}{l} \text{define action}: A(\text{path}) = \int_{t_1}^{t_2} L\, dt = \int_{t_1}^{t_2} dt \iiint_{\text{system}} \mathcal{L}\, dx\, dy\, dz, \\ \text{w}/\!/ \ \mathcal{L} = \mathcal{L}(\xi;\xi_x\cdots\xi_t;x\cdots t), \text{ and impose}: \boxed{\delta A(\text{path}) = 0}. \end{array}\right. \qquad (9)$$

REMARKS

1. Eq. (9) is just Hamilton's Principle again. But now the language is different ($\mathcal{L}$ replaces L, $\xi(x,t)$ replaces $x(t)$), and we will get different Euler-Lagrange eqtns;
2. The generalized cd ξ need not be a spatial displacement... in fact, for the EM field, it turns out $\xi \propto$ potentials ϕ & $\mathbf{A}$. Choice of ξ need only $\Rightarrow$ "acceptable" $\mathcal{L}$;
3. An "acceptable" $\mathcal{L}$ means an $\mathcal{L}$ which gives the correct eqs-of-motion (e.g. Maxwell field eqtns) -- at least in some limit (for a free particle, or source-free region, etc.). $\mathcal{L}$(acceptable) need not $= \mathcal{T}$(KE density) $- \mathcal{V}$(PE density), as for a mechanical system. $\mathcal{L}$ in a field theory is still a kind of energy density, but it may look~ weird.

Euler-Lagrange Equations for a Continuum

13) Now do the variational problem of Eq. (9) for $\mathcal{L}$.

$$\rightarrow \delta A = \int_{t_1}^{t_2} dt \underset{\text{system}}{\iiint} dx_1\, dx_2\, dx_3\; \delta\mathcal{L}(\xi;\, \xi_{x_1} \cdots \xi_t;\, x \cdots t) = 0. \tag{10}$$

but // $$\delta\mathcal{L} = \left(\frac{\partial\mathcal{L}}{\partial\xi}\right)\delta\xi + \overset{①}{\left(\frac{\partial\mathcal{L}}{\partial\xi_t}\right)\delta\xi_t} + \overset{②}{\left(\frac{\partial\mathcal{L}}{\partial\xi_{x_k}}\right)\delta\xi_{x_k}}$$ { Sum on $k = 1, 2, 3$; the x_k & t not varied.

term ① ⇒ integral : $$\int_{t_1}^{t_2} dt \left(\frac{\partial\mathcal{L}}{\partial\xi_t}\right) \underbrace{\delta\left(\frac{\partial\xi}{\partial t}\right)}_{=\partial(\delta\xi)/\partial t} = \underbrace{\left(\frac{\partial\mathcal{L}}{\partial\xi_t}\right)\delta\xi\Big|_{t_1}^{t_2}}_{0,\text{ since } \delta\xi = 0\ @\ t = t_1\ \&\ t_2.} - \int_{t_1}^{t_2} dt\, \frac{\partial}{\partial t}\left(\frac{\partial\mathcal{L}}{\partial\xi_t}\right)\delta\xi. \tag{11}$$

(partial integrate) differentials commute

term ② ⇒ integrals : $$\int dx_k \left(\frac{\partial\mathcal{L}}{\partial\xi_{x_k}}\right) \underbrace{\delta\left(\frac{\partial\xi}{\partial x_k}\right)}_{=\partial(\delta\xi)/\partial x_k} = \underbrace{\left(\frac{\partial\mathcal{L}}{\partial\xi_{x_k}}\right)\delta\xi}_{0,\text{ assuming } \delta\xi = 0\ @\ \text{system boundaries.}} - \int dx_k\, \frac{\partial}{\partial x_k}\left(\frac{\partial\mathcal{L}}{\partial\xi_{x_k}}\right)\delta\xi. \tag{12}$$

[for ∞ systems, assume $\mathcal{L}$ vanishes at $x_k \to \infty$]

Putting results of (11) & (12) into (10), have...

$$\rightarrow \delta A = \int_{t_1}^{t_2} dt \underset{\text{system}}{\iiint} dx_1\, dx_2\, dx_3 \left[\frac{\partial\mathcal{L}}{\partial\xi} - \frac{\partial}{\partial t}\left(\frac{\partial\mathcal{L}}{\partial\xi_t}\right) - \frac{\partial}{\partial x_k}\left(\frac{\partial\mathcal{L}}{\partial\xi_{x_k}}\right)\right]\delta\xi = 0. \tag{13}$$

With the variations $\delta\xi(x_k, t)$ arbitrary, the integral can vanish identically only if the $[\;] \equiv 0$. Then we have a new type of Euler-Lagrange eqtn...

$$\boxed{\frac{\partial}{\partial t}\left(\frac{\partial\mathcal{L}}{\partial\xi_t}\right) + \frac{\partial}{\partial x_k}\left(\frac{\partial\mathcal{L}}{\partial\xi_{x_k}}\right) = \frac{\partial\mathcal{L}}{\partial\xi}} \tag{14}$$

← **LAGRANGE EQS for a CONTINUOUS MEDIUM.**

$\mathcal{L} = \mathcal{L}(\xi;\, \xi_{x_k}, \xi_t;\, x_k, t)$ = Lagrange Density.

Sum on $k = 1, 2, 3$.

REMARKS

1. Eq. (14) replaces: $\frac{d}{dt}(\partial L/\partial\dot{q}_i) = \partial L/\partial q_i$, for a discrete system. It is equivalent to Jkn Eq. (12.83); he would write: $\partial^\beta[\partial\mathcal{L}/\partial(\partial^\beta\xi)] = \partial\mathcal{L}/\partial\xi$, w/ $\partial^\beta = \left(\frac{1}{c}\frac{\partial}{\partial t}, -\nabla\right)$.
2. Discrete system w/ n degrees of freedom ⇒ n Lagrange eqtns. Here, we have a continuous system, w/ ∞ degrees of freedom, but only one Lagrange eqtn. What? Difference is that the ξ's $\propto$ continuously on t and x_k, and Eq. (14) is a PDE in both t and x_k, not an ODE in t.
3. If $\mathcal{L}$ contains more than one continuum cd ξ, say a set of $\xi^{(i)}(x_k, t)$, then we get a set of eqtns like (14): $\frac{\partial}{\partial x_\mu}\left(\partial\mathcal{L}/\partial\xi^{(i)}_{x_\mu}\right) = \partial\mathcal{L}/\partial\xi^{(i)}$, w/ $\mu = 0, 1, 2, 3$, and $x_0 = ct$.

→ NOW WE are PREPARED to TOTALLY LAGRANGIFY the ELECTROMAGNETIC FIELD.