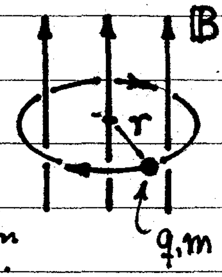
Φ 520 MidTerm Exam (in-class, 2 hrs.) — Sat. 18 Feb. 1989

This exam is open-book, open-notes, and is worth 100 points total. For each problem, put your answer in a $\boxed{\text{box}}$ on your solution sheets. $\boxed{\text{Number}}$ your solution pages, put your $\boxed{\text{name}}$ on page 1, and $\boxed{\text{staple}}$ the pages together, in order, before handing them in.



(1) [35 pts]. A particle of charge q and mass m moves (relativistically) in a plane $\perp$ to a static uniform magnetic field $\mathbb{B}$, as shown.

(A) Calculate the total power P radiated by q. Your answer for P should contain only q, m, B, γ and constants; $\gamma = 1/\sqrt{1-\beta^2}$ = dilation factor.

(B) At time $t=0$, let q's total energy be: $E_0 = \gamma_0 mc^2$, $\gamma_0 \gg 1$. Show that q's energy is: $E = \gamma mc^2$, at time: $t = \frac{3}{2c}(mc^2/q^2)(mc^2/qB)^2\left[\frac{1}{\gamma} - \frac{1}{\gamma_0}\right]$, provided $\gamma_0 > \gamma \gg 1$.

(C) If q is initially nonrelativistic, with kinetic energy K_0 at $t=0$, what is its kinetic energy K at time $t>0$? How does q's orbit radius r change with time?

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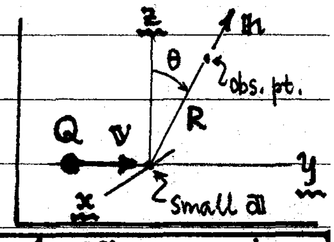
(2) [30 pts]. A magnetic monopole (MM) of charge g, traversing matter, loses energy by electron collisions (as does an "ordinary" charge $Q = ze$). Use Jk^{n} Secs. (13.1)→(13.3), and relevant collision dynamics, to show the MM $\frac{\text{energy loss}}{\text{unit length}}$ is given by Bethe's formula [Eq. (13.44)]

(A) $\boxed{(dE/dx)_{MM} \simeq (\omega_p^2/c^2)\, g^2 \ln(2\gamma^2 m v^2/\hbar\langle\omega\rangle)}$ w/ $\omega_p^2 = 4\pi N Z e^2/m$ (plasma freq), and: $Q = ze$ is replaced by βg.

(B) By Dirac's condition: $g = \frac{ne}{2\alpha}$, w/ $\alpha = \frac{e^2}{\hbar c} \simeq \frac{1}{137}$. Consider the minimal MM, with $n=1$. As $\beta \to 1$, how much faster does a MM lose energy than a particle with $Q=e$? On the same graph, sketch $\left(\frac{dE}{dx}\right)_{MM}$ and $\left(\frac{dE}{dx}\right)_{Q=69e}$ vs. energy. How do the tracks differ in appearance?

HINT: Very little detailed calculation is needed to solve this problem.



(3) [35 pts]. An ultrarelativistic charge Q moves along the y-axis at velocity $\mathbb{v}$. Near the origin, Q suffers a small acceleration $\mathbb{a}$ -- so small that $\mathbb{v}$ changes negligibly [i.e. $\mathbb{v}(\text{after}) = \mathbb{v}(\text{before})$]. But, since $\mathbb{a} \neq 0$, Q radiates. Calculate the Fourier component: $\boxed{\tilde{\mathbb{E}}(\mathbb{r},\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathbb{E}(\mathbb{r},t)\, e^{i\omega t}\, dt}$ of the radiation electric field seen by a distant observer (@ (R,θ), in direction $\mathbb{n}$, above).

HINTS: $\mathbb{E}$ is the Lienard-Wiechert field for Q. Treat the retarded time carefully.

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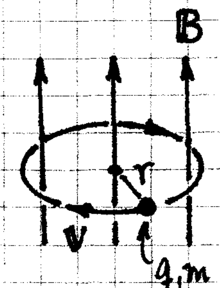
(1) [35 pts]. Analyse radiative energy loss for rel$^{\underline{c}}$ q in cyclotron orbit.



(A) From Jk$^{\underline{n}}$ Eq. (14.46) for q in "instantaneously circular motion"...

total power loss by radiation $\Big\}$ $P = \frac{2}{3}(q^2/c^3)\gamma^4|\dot{\mathbf{v}}|^2$. (1)

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But $\dot{\mathbf{v}}$ is provided by the Lorentz force: $\gamma m\dot{\mathbf{v}} = \frac{q}{c}\mathbf{v}\times\mathbf{B}$, so $|\dot{\mathbf{v}}| = (qB/\gamma m)\beta$, and -- in the particle's own frame -- radiative loss is

$$P = \frac{2}{3}(q^2/c^3)\gamma^4[(qB/\gamma m)\beta]^2, \text{ w/ } \boxed{P = \frac{2}{3}(q^4B^2/m^2c^3)(\gamma^2-1).} \quad (2)$$

(B) If $\gamma = E/mc^2 \gg 1$, q loses energy at a rate (in its own frame)...

$$\rightarrow dE = (-)P\,dt \simeq -\frac{2}{3}(q^4B^2/m^2c^3)\left[\frac{E}{mc^2}\right]^2 dt = -kE^2dt, \quad (3)$$

w/ $k = \frac{2}{3}(q^4B^2/m^4c^7)$. This eqtn is easily integrated over $\left\{\begin{matrix}0\rightarrow t\\ E_0\rightarrow E\end{matrix}\right.$ ...

$$\int_0^t k\,dt \simeq (-)\int_{E_0}^{E} dE/E^2 \Rightarrow t \simeq \frac{1}{k}\left(\frac{1}{E}-\frac{1}{E_0}\right) = \frac{1}{kmc^2}\left(\frac{1}{\gamma}-\frac{1}{\gamma_0}\right),$$

so/ E_0 drops to $E < E_0$ in time $\boxed{t \simeq \frac{3}{2}(m^3c^5/q^4B^2)\left(\frac{1}{\gamma}-\frac{1}{\gamma_0}\right)}$, $\gamma \gg 1$ (4)

(C) If q is initially non-relativistic, the power loss in Eq. (2) manifests itself in a loss of q's kinetic energy K...

$$dK = (-)P\,dt = -\frac{2}{3}(q^4B^2/m^2c^3)\frac{\beta^2}{1-\beta^2}dt \overset{\beta\ll 1}{\simeq} -\frac{2}{3}(q^4B^2/m^2c^5)v^2dt$$

but/ $v^2 = \frac{2K}{m} \Rightarrow \frac{dK}{K} = (-)\frac{dt}{T}$, w/ $\underline{\underline{T = \frac{3}{4}(m^3c^5/q^4B^2)}}$

A simple integration $\Rightarrow$ $\boxed{K(t) = K(0)\,e^{-t/T}}$, w/ T as above. (5)

For the orbit radius, impose the cyclotron condition (nonrel$^{\underline{c}}$ case)...

$$mv^2/r = \frac{q}{c}vB \Rightarrow r = (mc/qB)v = (mc/qB)\sqrt{\frac{2}{m}}K^{1/2} \quad \text{use } K(t) \text{ from Eq. (5).}$$

so/ $\boxed{r(t) = r(0)\,e^{-(t/2T)}}$, w/ $r(0) = \left(\frac{mc}{qB}\right)v(0)$. (6) Both q's K.E. & orbit decay exponentially.

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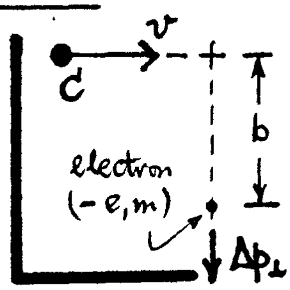
② [30pts]. Worry about stopping power for a MM (magnetic monopole) in matter.



(A) If some "Charge" C -- which couples to electrons by appropriate fields -- undergoes the collision sketched at right, we begin the calculation of stopping power $\frac{dE}{dx}$ by finding the transverse momentum $\Delta p_\perp$ given the electron during the collision. Two calculated cases are...

(1) C = electric monopole Q : $\Delta p_\perp = 2Qe/bv$ ← Jk^n Eq. (13.1); $\Delta p_\perp$ in plane of paper;

(2) C = magnetic monopole g : $\Delta p_\perp = 2ge/bc$ ← Jk^n Eq. (6.155); $\Delta p_\perp$ $\perp$ plane of paper.

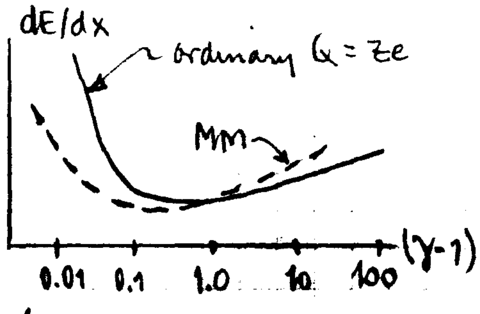
Comparing these $\Delta p_\perp$'s: they are identical in size if we set $\boxed{g\beta = Q}$. With this substitution, all the calculations in Jk^n Secs. (13.1)→(13.3) go through as before (because they deal only with the scalar energy transfer $\Delta E = (\Delta p_\perp)^2/2m$), and they again give Bethe's formula, Jk^n Eq. (13.44). So, with $Q = ze \to \beta g$, have...

$$\boxed{(dE/dx)_{MM} = (\omega_P^2/c^2)\, g^2 \left[\ln(2\gamma^2 m v^2/\hbar\langle\omega\rangle) - \overset{\text{neglect}}{\cancel{\beta^2}}\right]} \quad \int \begin{array}{l}\omega_P^2 = 4\pi N Z e^2/m,\\ m = \text{electron mass},\\ \beta \,\&\, \gamma \leftrightarrow \text{those of MM.}\end{array} \qquad (1)$$

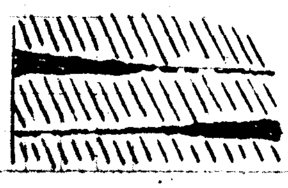
(B) Dirac quantization: $g = ne/2\alpha$, $\alpha \simeq 1/137$ [Jk^n Eq. (6.153)], and $n = 1 \Rightarrow g \simeq$ 69e for minimal MM. This g is equivalent to a bare Thulium nucleus, and it couples to matter very strongly. The MM energy loss compared to some Q = e is...

$$(dE/dx)_{MM}/(dE/dx)_Q = \left(\frac{g\omega_P}{c}\right)^2 \Big/ \left(\frac{Q\omega_P}{v}\right)^2 = \beta^2 (g/Q)^2. \qquad (2)$$

When $\beta \to 1$, this ratio is $(g/Q)^2 \to (69e/e)^2 \sim 4760$, for minimal g vs Q = e. So the MM loses energy ~ 4800× faster than a proton, at $\beta \simeq 1$.



Principal difference between $\left(\frac{dE}{dx}\right)_{MM}$ & $\left(\frac{dE}{dx}\right)_{Q=69e}$ is at low velocities ($\beta \to$ small); they differ by β^2 in denom. ∴ $\beta \to 0$, M loses little energy; Q loses a lot. Situation reverses as $\beta \to 1$ ⇒ tracks (in emulsion) look like... ‖ MM → Q →



↑ dgm ~ Jk^n Fig (13.4), p 629

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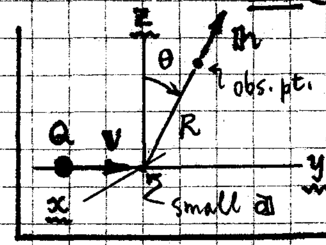


③ [35 pts]. Calculate Fourier $\tilde{\mathbb{E}}_{rad}$ from Q hitting a bump.

1. The Lienard-Wiechert $\mathbb{E}_{rad}$ is Jk^{n} Eq. (14.14)...

$$\rightarrow \mathbb{E}(\mathbf{r},t) = \frac{Q}{cR}\left\{ \mathbf{n}\times\left[(\mathbf{n}-\beta)\times\frac{\mathbf{a}}{c}\right]/(1-\mathbf{n}\cdot\beta)^3\right\}\Big|_{ret}, \text{ "ret"} \leftrightarrow \text{evaluation at retarded time: } t' = t - \tfrac{1}{c}R(t'). \quad (1)$$

At large distances: $R(t') \simeq R - \mathbf{n}\cdot\mathbf{r}(t')$, where R is the fixed distance between the origin and the observer, and $\mathbf{r}(t') = \mathbf{v}t'$ is the position of Q. Thus...

$$\left[\!\left[t' \simeq t - \frac{1}{c}(R - \mathbf{n}\cdot\mathbf{v}t') \Rightarrow t = (1-\mathbf{n}\cdot\beta)t' + \frac{R}{c}. \right]\!\right] \quad (2)$$

2. In forming $\tilde{\mathbb{E}} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathbb{E}\,e^{i\omega t}\,dt$, we replace the t-integration by an integration over t', with $dt = (1-\mathbf{n}\cdot\beta)\,dt'$, so we have...

$$\rightarrow \tilde{\mathbb{E}} = \frac{1}{2\pi}\left(\frac{Q}{c^2R}\right)\int_{-\infty}^{\infty}\left\{\mathbf{n}\times[(\mathbf{n}-\beta)\times\mathbf{a}]/(1-\mathbf{n}\cdot\beta)^2\right\}e^{i\omega\left[(1-\mathbf{n}\cdot\beta)t' + \frac{R}{c}\right]}dt'. \quad (3)$$

But β = cnst, by assumption, in this calculation, while the acceleration $\mathbf{a}$ is by now a fcn of t'. Thus, taking cnsts out of the integrand...

$$\rightarrow \tilde{\mathbb{E}} = \frac{Q}{2\pi c^2 R}\cdot\frac{1}{(1-\mathbf{n}\cdot\beta)^2}\,e^{i\left(\frac{\omega}{c}\right)R}\int_{-\infty}^{\infty}\left\{\mathbf{n}\times[(\mathbf{n}-\beta)\times\mathbf{a}(t')]\right\}e^{i\Omega t'}dt',$$

where: $\Omega = \omega(1-\mathbf{n}\cdot\beta)$. (4)

3. In Eq. (4), the only active element left in the integrand is $\mathbf{a}(t')$. We recognize the Fourier transform of $\mathbf{a}(t')$, and if we define...

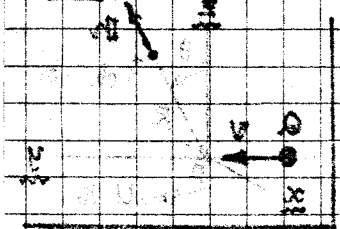
$$\mathbb{A}(\Omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathbf{a}(t')\,e^{i\Omega t'}dt', \quad \text{w/} \quad \Omega = \omega(1-\mathbf{n}\cdot\beta)$$

then//

$$\boxed{\tilde{\mathbb{E}}_\omega = \frac{Q}{c^2}\,\frac{1}{(1-\mathbf{n}\cdot\beta)^2}\left(\frac{e^{i(\omega/c)R}}{R}\right)\mathbf{n}\times[(\mathbf{n}-\beta)\times\mathbb{A}(\Omega)].} \; * \quad (5)$$

The $\frac{1}{R}e^{i(\omega/c)R}$ dependence ⇒ spherical outgoing wave, as expected. $\tilde{\mathbb{E}}_\omega$ cannot be reduced further until we have detailed information on the $\mathbf{a}(t')$ "bump."

* See OVER.



In Eq. (4) overleaf, set $\mathbf{a}(t') = c\,\frac{d\boldsymbol{\beta}}{dt'}$ in integrand,
and take out $\mathbf{n}\times(\mathbf{n}-\boldsymbol{\beta}) = \text{cnst}\ldots$

$$\tilde{\mathbf{E}} = \left[\frac{Q}{2\pi cR}\cdot\frac{1}{(1-\mathbf{n}\cdot\boldsymbol{\beta})^2}\,e^{i(\omega/c)R}\right]\mathbf{n}\times(\mathbf{n}-\boldsymbol{\beta})\times\int_{-\infty}^{\infty}\left(\frac{d\boldsymbol{\beta}}{dt'}\right)e^{i\Omega t'}\,dt'$$

← partial integrate

or $$\tilde{\mathbf{E}} = [\;\;]\,\mathbf{n}\times(\mathbf{n}-\boldsymbol{\beta})\times\left\{\boldsymbol{\beta}\,e^{i\Omega t'}\Big|_{t=-\infty}^{t=+\infty} - i\Omega\int_{-\infty}^{\infty}\boldsymbol{\beta}\,e^{i\Omega t'}\,dt'\right\}$$

If $\boldsymbol{\beta}$ is really cnst, then

$$\tilde{\mathbf{E}} = [\;\;]\,\mathbf{n}\times(\mathbf{n}-\boldsymbol{\beta})\times\left\{2i\,\boldsymbol{\beta}\lim_{T\to\infty}\sin\Omega T - 2\pi i\Omega\,\boldsymbol{\beta}\,\delta(\Omega)\right\}$$

or $$\tilde{\mathbf{E}} = [\;\;]\,\mathbf{n}\times\left[(\mathbf{n}-\boldsymbol{\beta})\times\boldsymbol{\beta}\right]\cdot 2i\left\{\lim_{T\to\infty}\sin\Omega T - \Omega\,\delta(\Omega)\right\}$$

What do you make of that, Bremsstrahlung fans?