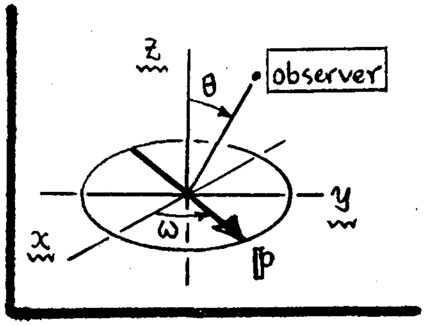
4) [40pts]. An electric dipole moment of strength $\mathbf{p}$ lies in the xy-plane and rotates about the z-axis at constant angular velocity $\omega$, as shown in the sketch. An observer is situated above the plane, in a position inclined at $\measuredangle\,\theta$ with respect to the z-axis. Assume the observer is "far away" from $\mathbf{p}$, i.e. at a distance $\gg$ the wavelength radiated by $\mathbf{p}$.

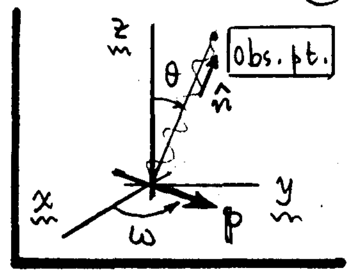



(A) Find the power radiated per solid $\measuredangle$ (i.e. the angular distribution $dP/d\Omega$) at the observer's location. Average $dP/d\Omega$ over a period of $\mathbf{p}$'s motion to obtain a compact form $\langle dP/d\Omega\rangle$. Is there any $\measuredangle\,\theta$ for which $\langle dP/d\Omega\rangle = 0$?

(B) Find the total (time-averaged) power radiated by the rotating $\mathbf{p}$.


Unbeknownst to most historians, Einstein started down the road of professional basketball before an ankle injury diverted him into science.

$\phi$ 520 MidTerm Solutions (1994, cont'd)




**4** [40 pts.]. Analyse radiation from a spinning EDM.

(A) $r$ (observer distance) $\gg \lambda$ (radiated wavelength) $\Rightarrow$ observer is in the radiation zone, where the leading-term expressions in Eqs. (16) & (17) of CLASS NOTES, p. Rad. 7 are valid. The radiated power/solid $\measuredangle$ is: $dP/d\Omega = \frac{1}{4\pi c^3}|\hat{n}\times\ddot{\mathbb{p}}|^2$, w/ $\hat{n}$ = unit vector in direction of observer. Now: $\hat{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$, w/ $\theta$: colatitude $\measuredangle$ & $\phi$ azimuthal $\measuredangle$ in the cd. system shown. As for $\ddot{\mathbb{p}}$, note...

$$\mathbb{p} = (p\cos\omega t, p\sin\omega t, 0) \Rightarrow \ddot{\mathbb{p}} = -\omega^2 p(\cos\omega t, \sin\omega t, 0), \text{ for spinning EDM.} \quad (1)$$

So/
$$\hat{n}\times\ddot{\mathbb{p}} = -\omega^2 p \begin{bmatrix} \hat{x} & \hat{y} & \hat{z} \\ \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\omega t & \sin\omega t & 0 \end{bmatrix} = -\omega^2 p \begin{pmatrix} -\cos\theta\sin\omega t \\ \cos\theta\cos\omega t \\ \sin\theta\cos\phi\sin\omega t - \sin\theta\sin\phi\cos\omega t \end{pmatrix}, \quad (2)$$

And/
$$|\hat{n}\times\ddot{\mathbb{p}}|^2 = \omega^4 p^2\Big[\cos^2\theta\overbrace{(\sin^2\omega t + \cos^2\omega t)}^{1} + \sin^2\theta(\cos^2\phi\sin^2\omega t + \sin^2\phi\cos^2\omega t) - $$
$$- 2\sin^2\theta\cos\phi\sin\phi\sin\omega t\cos\omega t\Big]. \quad (3)$$

Averaging over a period of the rotation, we have: $\langle\sin^2\omega t\rangle = \frac{1}{2} = \langle\cos^2\omega t\rangle$, and: $\langle\sin\omega t\cos\omega t\rangle = 0$. Then, on average...

$$\rightarrow \langle|\hat{n}\times\ddot{\mathbb{p}}|^2\rangle = \omega^4 p^2\left[\cos^2\theta + \tfrac{1}{2}\sin^2\theta\right] = \tfrac{1}{2}\omega^4 p^2(1+\cos^2\theta)$$

So/ angular distribution $\Big\}$
$$\boxed{\langle dP/d\Omega\rangle = \frac{1}{4\pi c^3}\langle|\hat{n}\times\ddot{\mathbb{p}}|^2\rangle = \frac{\omega^4 p^2}{8\pi c^3}(1+\cos^2\theta).} \quad (4)$$

The $\measuredangle$ distribution here, viz. $\frac{1}{2}(1+\cos^2\theta)$, is quite different than the $\sin^2\theta$ distribution found for a linear (oscillating) dipole oriented along the z-axis. For the spinning dipole, there is **no** $\measuredangle$ $\theta$ at which the radiation vanishes.

---

(B) Since the solid $\measuredangle$ $d\Omega = 2\pi\sin\theta\, d\theta$ (w/ no $\phi$ dependence), the total power is

$$\langle P\rangle = \int_0^\pi \langle dP/d\Omega\rangle \cdot 2\pi\sin\theta\, d\theta = \frac{\omega^4 p^2}{4c^3}\int_{-1}^{+1}(1+\mu^2)\,d\mu = \boxed{\frac{2\omega^4 p^2}{3c^3}} \quad (5)$$

The energy loss per cycle is $\Delta E = \langle P\rangle\cdot\frac{2\pi}{\omega} = \frac{4\pi}{3}(\omega/c)^3 p^2$. We can keep $\omega \simeq$ cnst only if $\Delta E \ll$ mechanical rotation energy at any given time. Gradually, $\mathbb{p}$ slows down.