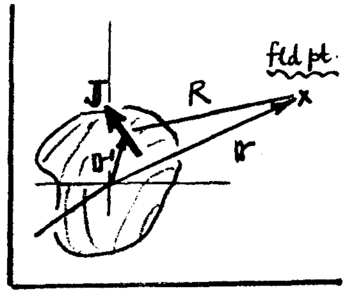
<u>Magnetostatics</u> (cont'd)

6) The magnetic moment idea for generation of $\mathbb{B}$ can be divorced from details (like plane circular loops) of the current distribution by simply expanding the fundamental solution for $\mathbb{A}$... it becomes clear that <u>the first non-zero term in $\mathbb{A}$ contributes a dipole term to $\mathbb{B}$.</u> In the process, we can generalize the definition of the magnetic dipole moment $\mathbb{m}$ of the system which generates $\mathbb{B}$.



$$\mathbb{A}(\mathbb{r}) = \frac{1}{c}\int \frac{d^3x'}{|\mathbb{r}-\mathbb{r}'|}\,\mathbb{J}(\mathbb{r}'), \text{ in general}$$

$$\ldots\ \frac{1}{|\mathbb{r}-\mathbb{r}'|} = \frac{1}{r} + \frac{\mathbb{r}\cdot\mathbb{r}'}{r^3} + \cdots, \text{ for } \begin{matrix} r\,(\text{observer}) \gg \\ r'\,(\text{source size}) \end{matrix};$$

So//

$$\underbrace{A_i}_{i^{\text{th}}\text{ component}}(\mathbb{r}) = \frac{1}{cr}\int J_i(\mathbb{r}')\,d^3x' + \frac{1}{cr^3}\int (\mathbb{r}\cdot\mathbb{r}')\,J_i(\mathbb{r}')\,d^3x' + \cdots \qquad (22)$$

↑ this term vanishes (would be monopole) ↑ this is first nonzero contribution

(for details see Jk^n Sec. (5.6).)

The first term RHS in (22) vanishes when $\nabla'\cdot\mathbb{J}=0$, which is what we are working with in magneto<u>statics</u>.¶ As for the second term, one uses...

$$(\mathbb{r}\cdot\mathbb{r}')\,\mathbb{J} = (\mathbb{r}\cdot\mathbb{J})\,\mathbb{r}' - \mathbb{r}\times(\mathbb{r}'\times\mathbb{J}) \to 0 - \tfrac{1}{2}\,\mathbb{r}\times(\mathbb{r}'\times\mathbb{J}), \qquad (23)$$

plus some manipulation [see Jk^n p. 181] to arrive at...

$$\mathbb{A}(\mathbb{r}) = 0 - \frac{1}{cr^3}\,\frac{1}{2}\,\mathbb{r}\times\int[\mathbb{r}'\times\mathbb{J}(\mathbb{r}')]\,d^3x' + \cdots$$

or//

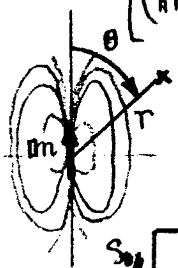
$$\boxed{\mathbb{A}(\mathbb{r}) = \frac{\mathbb{m}\times\mathbb{r}}{r^3} + \cdots, \text{ w/ } \mathbb{m} = \frac{1}{2c}\int[\mathbb{r}'\times\mathbb{J}(\mathbb{r}')]\,d^3x'}. \qquad (24)$$

This <u>leading</u> term in $\mathbb{A}$ (for $r \gg r'$) falls off with distance as $1/r^2$; the associated field $B \propto \partial A/\partial r$ goes as $1/r^3$... it is a <u>dipole field</u>, in general. The magnetic moment $\mathbb{m}$ is now calculable for <u>any</u> current distrib$^{\underline{n}}$ $\mathbb{J}$.

¶ This term does <u>not</u> vanish for time-dependent fields; in fact it gives EM <u>radiation</u>.

7) The magnetic field which from (the dipole approxn to) $\mathbb{A}$ is...

$$\mathbb{B}_{dipole} = \nabla\times\left(\frac{\mathbb{m}\times\mathbb{r}}{r^3}\right) = \mathbb{m}\underbrace{\left(\nabla\cdot\frac{\mathbb{r}}{r^3}\right)}_{4\pi\,\delta(\mathbb{r})} - \frac{\mathbb{r}}{r^3}\cancelto{0}{(\nabla\cdot\mathbb{m})} + \cancelto{0\ (\mathbb{m}=\text{cnst})}{\left(\frac{\mathbb{r}}{r^3}\cdot\nabla\right)\mathbb{m}} - (\mathbb{m}\cdot\nabla)\frac{\mathbb{r}}{r^3}$$



$$\left[(\mathbb{m}\cdot\nabla)\frac{\mathbb{r}}{r^3}\right]_i = \left(m_k\frac{\partial}{\partial x_k}\right)\frac{x_i}{r^3} = \frac{m_i}{r^3} + m_k x_i\frac{\partial}{\partial x_k}\left(\frac{1}{r^3}\right)$$

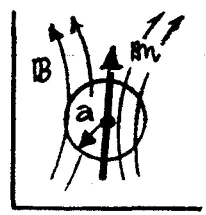
$$= \left(\frac{\mathbb{m}}{r^3}\right)_i - 3x_i\frac{m_k x_k}{r^5} = \frac{1}{r^3}\left[\mathbb{m} - \frac{3}{r^2}\mathbb{r}(\mathbb{m}\cdot\mathbb{r})\right]_i$$

So,
$$\boxed{\mathbb{B}_{dipole} = \frac{1}{r^3}\left[3(\mathbb{m}\cdot\hat{n})\hat{n} - \mathbb{m}\right] + 4\pi\,\mathbb{m}\,\delta(\mathbb{r})},\quad \hat{n} = \frac{\mathbb{r}}{r}. \qquad (25)$$

this term is WRONG, unfortunately.

This is a dipole field -- just like $\mathbb{E}_{dipole}$ of Jkⁿ Eq. (4.13). Unfortunately, the $\delta(\mathbb{r})$ coefficient is wrong... because we have derived it from the potential $\mathbb{A} = (\mathbb{m}\times\mathbb{r})/r^3$, which holds only at "large" r. As $r\to 0$, it fails.

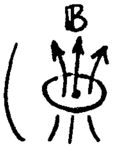
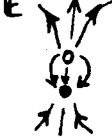
The term in $\delta(\mathbb{r})$ should be present, however, if there really is a magnetic "point source" (∞ly small current loop) at the origin. Jackson shows how to get the right coefficient in his Eqs. (5.60)-(5.64). The correct form depends on the idea that if all the current sources $\mathbb{J}$ which generate $\mathbb{m}$ are contained in a sphere of "small" radius a at the origin, then the volume integral: $\int_{r<a}\mathbb{B}\,d^3x' = \frac{2}{3}\cdot 4\pi\,\mathbb{m}$. As a consequence...



$$\left[\!\left[\mathbb{B}_{dipole} = \frac{1}{r^3}\left[3(\mathbb{m}\cdot\hat{n})\hat{n} - \mathbb{m}\right] + \frac{8\pi}{3}\mathbb{m}\,\delta(\mathbb{r})\right]\!\right]. \qquad (26)$$

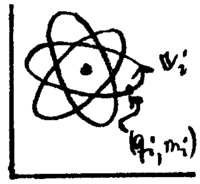
By comparison, the formula for a point electric dipole is Jkⁿ Eq. (4.20)

$$\rightarrow \mathbb{E}_{dipole} = \frac{1}{r^3}\left[3(\mathbb{p}\cdot\hat{n})\hat{n} - \mathbb{p}\right] - \frac{4\pi}{3}\mathbb{p}\,\delta(\mathbb{r}). \qquad (27)$$

The $\frac{8\pi}{3}\rightarrow(-)\frac{4\pi}{3}$ is due to source structure differences ( vs ).

Magnetostatics (cont'd)

8) The system magnetic moment $\mathbb{m} = \frac{1}{2c}\int[\mathbb{r}' \times \mathbb{J}(\mathbb{r}')]\,d^3x'$ in Eq.(24) can be written for a collection of discrete particles (e.g. in an atom)...



$$\mathbb{J} = \sum_i q_i \mathbb{v}_i\, \delta(\mathbb{r} - \mathbb{r}_i) \leftarrow \text{dropped primes on source position } \mathbb{r}'$$

$$\xrightarrow{\text{So}} \mathbb{m} = \frac{1}{2c}\sum_i q_i(\mathbb{r}_i \times \mathbb{v}_i) = \sum_i \left(\frac{q_i}{2m_i c}\right)\mathbb{L}_i, \tag{28}$$

where: $\mathbb{L}_i = m_i \mathbb{r}_i \times \mathbb{v}_i$ = ∢ momentum of m_i's orbit.

If the $\{q_i\}$ have cnst charge/mass ratios $q_i/m_i = e/m$, then

$$\mathbb{m} = (e/2mc)\,\mathbb{L}, \quad \mathbb{L} = \sum_i \mathbb{L}_i. \tag{29}$$

ASIDE Size of things in atoms

(~atom radius)

e, m = electron charge, mass ; $a_0 = \hbar^2/me^2 \simeq 0.53 \times 10^{-8}$ cm = Bohr radius

magnetic dipole } $|\mathbb{L}| \sim \hbar \Rightarrow |\mathbb{m}| \sim e\hbar/2mc$ ← called "Bohr magneton";

electric dipole } $|\Delta r| \sim a_0 \Rightarrow |\mathbb{p}| \sim e a_0$ (at most);

So relative dipole strengths } $$|\mathbb{m}| \div |\mathbb{p}| \sim \frac{e\hbar}{2mc}\frac{1}{ea_0} = \frac{e^2}{\hbar c} \simeq \frac{1}{137} \tag{30}$$

$e^2/\hbar c$ ← fine structure cnst ← dimensionless #

In atoms, evidently, <u>magnetic interactions are intrinsically much weaker than electric interactions</u>. In fact they are relative strength α^2...

ELECTRON BINDING ENERGY : $\mathcal{E}_{\text{electronic}} \sim e^2/a_0$ ($\alpha^2 mc^2$, by the way)

MAGNETIC INTERACTION (dipole-dipole coupling) : $\mathcal{E}_{\text{magnetic}} \sim |\mathbb{m}| \cdot \frac{|\mathbb{m}|}{a_0^3}$ (spin-orbit energy)

$$\text{So } \frac{\mathcal{E}_{\text{magnetic}}}{\mathcal{E}_{\text{electronic}}} \sim \frac{|\mathbb{m}|^2}{a_0^3} \Big/ \frac{(ea_0)^2}{a_0^3} = \left(\frac{|\mathbb{m}|}{|\mathbb{p}|}\right)^2 \sim \alpha^2 \simeq \frac{1}{20{,}000}. \tag{31}$$

This small ratio ⇒ rarely necessary in atoms to take $\mathbb{B}$ terms beyond dipole.

11/20/91

Table 6. Summary of the 1986 recommended values of the fundamental physical constants.

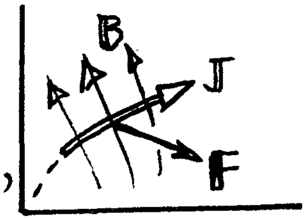
An abbreviated list of the fundamental constants of physics and chemistry based on a least-squares adjustment with 17 degrees of freedom. The digits in parentheses are the one-standard-deviation uncertainty in the last digits of the given value. Since the uncertainties of many of these entries are correlated, the full covariance matrix must be used in evaluating the uncertainties of quantities computed from them.

Quantity	Symbol	Value	Units	Relative uncertainty (ppm)
speed of light in vacuum	c	299 792 458	$\mathrm{m\,s^{-1}}$	(exact)
permeability of vacuum	μ_0	$4\pi\times10^{-7}$	$\mathrm{N\,A^{-2}}$	
		=12.566 370 614...	$10^{-7}\,\mathrm{N\,A^{-2}}$	(exact)
permittivity of vacuum	ϵ_0	$1/\mu_0 c^2$		
		=8.854 187 817...	$10^{-12}\,\mathrm{F\,m^{-1}}$	(exact)
Newtonian constant of gravitation	G	6.672 59(85)	$10^{-11}\,\mathrm{m^3\,kg^{-1}\,s^{-2}}$	128
Planck constant	h	6.626 0755(40)	$10^{-34}\,\mathrm{J\,s}$	0.60
$h/2\pi$	$\hbar$	1.054 572 66(63)	$10^{-34}\,\mathrm{J\,s}$	0.60
elementary charge	e	1.602 177 33(49)	$10^{-19}\,\mathrm{C}$	0.30
magnetic flux quantum, $h/2e$	Φ_0	2.067 834 61(61)	$10^{-15}\,\mathrm{Wb}$	0.30
electron mass	m_e	9.109 3897(54)	$10^{-31}\,\mathrm{kg}$	0.59
proton mass	m_p	1.672 6231(10)	$10^{-27}\,\mathrm{kg}$	0.59
proton-electron mass ratio	m_p/m_e	1836.152 701(37)		0.020
fine-structure constant, $\frac{1}{2}\mu_0 ce^2/h$	α	7.297 353 08(33)	10^{-3}	0.045
inverse fine-structure constant	α^{-1}	137.035 9895(61)		0.045
Rydberg constant, $\frac{1}{2}m_e c\alpha^2/h$	R_∞	10 973 731.534(13)	$\mathrm{m^{-1}}$	0.0012
Avogadro constant	N_A, L	6.022 1367(36)	$10^{23}\,\mathrm{mol^{-1}}$	0.59
Faraday constant, $N_A e$	F	96 485.309(29)	$\mathrm{C\,mol^{-1}}$	0.30
molar gas constant	R	8.314 510(70)	$\mathrm{J\,mol^{-1}K^{-1}}$	8.4
Boltzmann constant, R/N_A	k	1.380 658(12)	$10^{-23}\,\mathrm{J\,K^{-1}}$	8.5
Stefan–Boltzmann constant, $(\pi^2/60)k^4/\hbar^3c^2$	σ	5.670 51(19)	$10^{-8}\,\mathrm{W\,m^{-2}\,K^{-4}}$	34
	Non-SI units used with SI			
electron volt, $(e/\mathrm{C})\,\mathrm{J} = \{e\}\,\mathrm{J}$	eV	1.602 177 33(49)	$10^{-19}\,\mathrm{J}$	0.30
(unified) atomic mass unit, $1\,\mathrm{u} = m_u = \frac{1}{12}m(^{12}\mathrm{C})$	u	1.660 5402(10)	$10^{-27}\,\mathrm{kg}$	0.59

from// CODATA BULLETIN # 63 [Pergamon, Nov. 1986]
E.R. Cohen & B.N. Taylor "The 1986 Adjustment of the Fundamental Physical Constants."

Magnetostatics (cont'd)

9) The (Lorentz) magnetic force on a current density $\mathbb{J}$ is



$$\mathbb{F} = \sum_i \frac{q_i}{c} \mathbb{v}_i \times \mathbb{B}_i = \frac{1}{c}\int d^3x \left[\sum_i q\mathbb{v}_i\, \delta(\mathbb{r}-\mathbb{r}_i)\right] \times \mathbb{B}(\mathbb{r}),$$

$$\xrightarrow{\text{So}} \mathbb{F} = \frac{1}{c}\int d^3x\, \mathbb{J}(\mathbb{r}) \times \mathbb{B}(\mathbb{r}), \quad \underline{\mathbb{J} \text{ in external } \mathbb{B}}. \qquad (32)$$

note $\mathbb{J}\times\mathbb{B}$ volume force.

This can be worked around to $\mathbb{F} = \nabla(\mathbb{m}\cdot\mathbb{B})$, by mathematical artifice, as Jackson shows in his Eqs (5.65)–(5.69). A bit different approach is the following. Recall, for an arbitrary electric charge distribution, that the interaction energy with an external field was [Jk$^{\text{n}}$ Eq. (4.24)]...

$$\rightarrow W_{elec} = q\,\phi(0) - \mathbb{p}\cdot\mathbb{E} - \frac{1}{6}\sum_{i,j} Q_{ij} \frac{\partial}{\partial x_i} E_j(0) + \cdots$$

$$W_{ED} = -\mathbb{p}\cdot\mathbb{E} \left\{ \text{interaction energy} : \text{elec. dipole } \mathbb{p} \leftrightarrow \text{ext}^{\text{l}} \text{ field } \mathbb{E}. \right.$$

By analogy: $\boxed{W_{MD} = (-)\,\mathbb{m}\cdot\mathbb{B}}$ $\left\{ \text{interaction energy} : \text{magn dipole } \mathbb{m} \leftrightarrow \text{ext}^{\text{l}} \text{ field } \mathbb{B}. \right.$ (33)

Again, this can be verified by fundamental notions of what is meant by dipoles and fields. The force then is easy...

$$\boxed{\mathbb{F}_{MD} = (-)\nabla W_{MD} = +\nabla(\mathbb{m}\cdot\mathbb{B})}, \quad \underline{\mathbb{m} \text{ in external } \mathbb{B}}. \qquad (34)$$

This expression is equivalent to Eq. (32), with: $\mathbb{m} = \frac{1}{2c}\int [\mathbb{r}\times\mathbb{J}(\mathbb{r})]\, d^3x$.

For the record, we note the torque acting on a ("point-source") $\mathbb{m}$ by $\mathbb{B}$...

$$\mathbb{F} = \int d^3x'\, \mathcal{F}(\mathbb{r}), \quad \mathcal{F}(\mathbb{r}) = \frac{1}{c}\mathbb{J}(\mathbb{r})\times\mathbb{B}(\mathbb{r}) = \text{force/vol. on } \mathbb{J};$$

do 5 min on plasma pinch effect.

$$\mathbb{T} = \int d^3x\, \tau(\mathbb{r}), \quad \tau = \mathbb{r}\times\mathcal{F} = \text{torque/vol. on } \mathbb{J};$$

$$\xrightarrow{\text{So}} \mathbb{T} = \frac{1}{c}\int d^3x\, \mathbb{r}\times[\mathbb{J}\times\mathbb{B}] \rightarrow \mathbb{m}\times\mathbb{B}(0), \text{ at site of } \mathbb{m} \qquad (35)$$

Jk$^{\text{n}}$ Eq. (5.71).