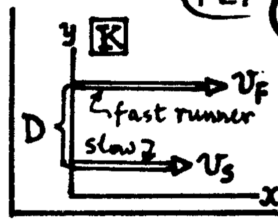
1992 ϕ520 Problems **Assigned: 3/19. Due: 3/27/92.**



(69) [Jk^n # (11.5)]. In frame $\underset{\sim}{K}$, a F(ast) and S(low) runner line up-- separated by distance D along the y-axis-- for a race down the x-axis. Two starters, one beside each runner, signal GO at slightly different times, so as to handicap the F runner. The handicap time difference in $\underset{\sim}{K}$ is T.

(A) For what range of T-values can there be a frame $\underset{\sim}{K}'$, moving (any which way) @ $v<c$ w.r.t. $\underset{\sim}{K}$, where the handicap vanishes? For what T-values will K' always see $T'>0$?

(B) Find the Lorentz transform $\underset{\sim}{K} \rightarrow \underset{\sim}{K}'$ for both cases in part (A). Specify both the $\underset{\sim}{K} \rightarrow \underset{\sim}{K}'$ relative velocities (and directions), and the runner's positions in $\underset{\sim}{K}'$. Who wins the race?

● [Jk^n # (11.7)]. An infinitesimal Lorentz transform and its inverse is represented by: $x'^{\alpha} = (g^{\alpha\beta} + \epsilon^{\alpha\beta})x_{\beta}$, $x^{\alpha} = (g^{\alpha\beta} + \epsilon'^{\alpha\beta})x'_{\beta}$. $(g^{\alpha\beta})$ is the metric tensor [Jk^n Eq. (11.70)], and the ϵ's are ∞simal. Retain only 1st order terms in ϵ. (A) Show, from the definition of the inverse, that: $\epsilon'^{\alpha\beta} = (-)\epsilon^{\alpha\beta}$. (B) Show, from invariance of the norm, that $\underset{\sim}{\epsilon}$ is antisymmetric: $\epsilon^{\alpha\beta} = (-)\epsilon^{\beta\alpha}$. (C) Write transform with contravariant x's on both sides of both equations. Show that $\underset{\sim}{\epsilon}$ is equivalent to $\underset{\sim}{L}$ in Jk^n Eq. (11.93).

● [20 pts.]. Ref. notes on "Relativistic Rocket Trip" (RRT), pp. 1-7, Eqs. (1)-(16).

(A) Prove Eq. (2): $dv = (1-\beta^2)\,du$, by use of the velocity addition formula (w/ $\beta = \frac{v}{c}$).

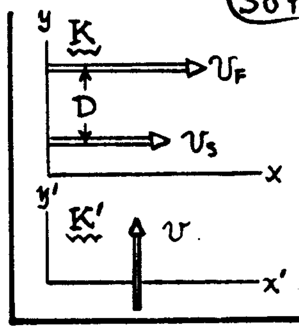
(B) Prove Eq. (9): $m(d\beta/dm) + \frac{v_E}{c}(1-\beta^2)$, by any means you choose.

(C) Consider a one-way trip, and as a quality factor, define: $Q(\tau) = D(\tau)/R(\tau)$, w/ τ = rocket time, $R(\tau) = m(0)/m(\tau)$ the "burn ratio", and $D(\tau)$ the distance travelled as measured by earth observers. At given τ, larger $Q(\tau)$ values imply a more efficient trip. For the case of onboard acceleration A = cnst and rocket exhaust velocity v_E = cnst, analyse a plot of $Q(\tau)$ vs. τ and find the maximum @ $\tau = \tau_m$. For $\epsilon = \frac{v_E}{c} < 1$, find explicit forms for $\beta(\tau_m)$, $D(\tau_M)$, and earth time $t(\tau_m)$.

(D) Keep the onboard acceleration A = cnst, but allow the exhaust velocity $v_E = v_E(\tau)$. Find a general expression for the burn-ratio $R(\tau)$. Choose $\epsilon(\tau) = \epsilon_0 + (\epsilon_1 - \epsilon_0)(1 - e^{-\omega\tau})$ w/ $\epsilon_1 > \epsilon_0$, and show that: $R(\tau) = R_0(\tau)/f(\tau)$, w/ $R_0(\tau) = \exp\{A\tau/c\epsilon_0\}$. Find $f(\tau)$ explicitly. How does this choice improve the trip? (NOTE: $\epsilon_0 < \epsilon_1 < 1$, here).

\S87 **21**

ϕ520 Solutions



(9) [Jk^n # (11.5)]. Analyse handicapped race from moving frame K'.

(A) 1. Clearly, K' must move along y-axis of K in order to (most efficiently) record changes in the delay time T, which is arranged by signals along the y-axis. So, let K' move at $v \parallel y$ & y' axes. With $\beta = v/c$, $\gamma = 1/\sqrt{1-\beta^2}$ and separation of events $\Delta y = D$ in K, K' records a delay time:

$$\rightarrow T' = \gamma\left(T - \frac{v}{c^2}\Delta y\right) = \gamma\left(T - \beta\frac{D}{c}\right). \quad (1)$$

T' vanishes if $T = \beta D/c$. Since $0<\beta<1$, T' can vanish over a range: $\boxed{0 < T < \frac{D}{c}}$. This is acausal, since $D > cT$. On the other hand, if K' sees a true handicap, then $T' > 0$ in Eq. (1) and so $T > \beta D/c$. For $0<\beta<1$, this is always true only if $\boxed{T \geqslant D/c}$, i.e. when $D \leqslant cT$, so the start signals can be causally connected.

(B) 2. The $K \rightarrow K'$ transform is: $t' = \gamma(t - \frac{v}{c^2}y)$, $y' = \gamma(y - vt)$. For the above cases:

I. $T' = 0 \Rightarrow T = \beta\frac{D}{c}$, or $\beta = cT/D$, so $t' = \gamma(t - \frac{y}{D}T)$, $y' = \gamma[y - (\frac{cT}{D})ct]$. (2)

II. $T' > 0 \Rightarrow T > \beta\frac{D}{c}$, or $\beta < cT/D$. Let $\beta = \epsilon(cT/D)$, w/ $0<\epsilon<1$. Then...

$$K \rightarrow K': \; t' = \gamma\left(t - \epsilon\frac{y}{D}T\right), \; y' = \gamma\left[y - \epsilon\left(\frac{cT}{D}\right)ct\right]. \quad (3)$$

In both cases, K' moves at v along the positive y-axis of K (i.e. $\boldsymbol{\beta} = \beta\hat{y}$).

3. Assume the runners have cnst speeds: $v_S = V$, and $v_F = V + \Delta V$, in K. Then their K cds are: $\{x_S = Vt, y_S = 0\}$ & $\{x_F = (V+\Delta V)(t-T), y_F = D\}$, for $t \geqslant T$. In K', cds are:

(SLOW) $x_S' = x_S = Vt$, or: $x_S' = Vt'/\gamma$; $y_S' = -\gamma v t = -vt'$ $\int$(K' & K origins coincide at beginning of race...). (4)

(FAST) $x_F' = x_F = (V+\Delta V)(t-T)$, or: $x_F' = (V+\Delta V)\left(\frac{t'}{\gamma} + \frac{vD}{c^2} - T\right)$ $\int$(good for times t' so $t' > \gamma(T - \frac{\beta}{c}D)$);

and $y_F' = \gamma(D - vt)$. $\leftarrow$ plug in: $t = \frac{t'}{\gamma} + \frac{vD}{c^2}$, to get: $y_F' = \frac{D}{\gamma} - vt'$ $\int$(shows correct Lorentz contraction). (5)

In K: $\Delta x = x_F - x_S = [\Delta V(t-T) - VT]$, while in K': $\Delta x' = \frac{1}{\gamma}[\Delta V(t' - T') - VT']$.

Either F or S runner may win (depending on length of race), but K & K' will agree on who won.

(68) [Jk^{n} #(11.7)]. Investigate details of ∞smal Lorentz transform $\begin{cases} x'^{\alpha} = (g^{\alpha\beta} + \epsilon^{\alpha\beta}) x_\beta, \\ x^{\alpha} = (g^{\alpha\beta} + \epsilon'^{\alpha\beta}) x'_\beta. \end{cases}$

(A) 1. From the definitions, and Jk^{n} Eq. (11.73), have: $\underline{x'^{\alpha} = x^{\alpha} + \epsilon^{\alpha\beta} x_\beta}$, $\underline{x^{\alpha} = x'^{\alpha} + \epsilon'^{\alpha\beta} x'_\beta}$.

These imply: $(x'^{\alpha} - x^{\alpha}) = \epsilon^{\alpha\beta} x_\beta = (-)\,\epsilon'^{\alpha\beta} x'_\beta = (-)\,\epsilon'^{\alpha\beta} g_{\beta\sigma} x'^{\sigma}$ [use Eq. (11.72)].

But $x'^{\sigma} = (g^{\sigma\tau} + \epsilon^{\sigma\tau}) x_\tau$ is given, so we get the relation between $\underset{\sim}{\epsilon}$ and $\underset{\sim}{\epsilon}'$:

$$\rightarrow \epsilon^{\alpha\beta} x_\beta = (-)\,\epsilon'^{\alpha\beta} g_{\beta\sigma} (g^{\sigma\tau} + \underbrace{\cancel{\epsilon^{\sigma\tau}}}_{\text{ignore } O(\epsilon^2)}) x_\tau = (-)\,\epsilon'^{\alpha\beta} \underbrace{(g_{\beta\sigma} g^{\sigma\tau})}_{\delta^{\tau}_{\beta},\ \text{by (11.71)}} x_\tau = (-)\,\epsilon'^{\alpha\beta} x_\beta. \quad (1)$$

Have $\epsilon^{\alpha\beta} x_\beta = (-)\epsilon'^{\alpha\beta} x_\beta$. Good for all x_β only if $\boxed{\epsilon'^{\alpha\beta} = (-)\,\epsilon^{\alpha\beta}}$, as req'd. (2)

(B) 2. Invariance of the norm requires: $x'_\sigma x'^{\sigma} = x_\mu x^{\mu}$. Then calculate (to $O(\epsilon)$ only):

$$x'_\sigma x'^{\sigma} = g_{\sigma\tau} x'^{\tau} x'^{\sigma} = g_{\sigma\tau} (g^{\tau\mu} + \epsilon^{\tau\mu}) x_\mu (g^{\sigma\nu} + \epsilon^{\sigma\nu}) x_\nu \leftarrow \text{ignore } O(\epsilon^2) \quad (3)$$

$$= \underbrace{(g_{\sigma\tau} g^{\tau\mu})}_{\delta^{\mu}_{\sigma},\ \underline{\underline{\text{now sum } \sigma}}} x_\mu g^{\sigma\nu} x_\nu + \underbrace{g_{\sigma\tau} \epsilon^{\tau\mu} x_\mu g^{\sigma\nu}}_{\delta^{\nu}_{\tau},\ \underline{\underline{\text{now sum } \tau}}} x_\nu + \underbrace{(g_{\sigma\tau} g^{\tau\mu})}_{\delta^{\mu}_{\sigma},\ \underline{\underline{\text{now sum } \sigma}}} x_\mu \epsilon^{\sigma\nu} x_\nu$$

So//

$$x'_\sigma x'^{\sigma} = x_\mu (g^{\mu\nu} x_\nu) + x_\mu \epsilon^{\nu\mu} x_\nu + x_\mu \epsilon^{\mu\nu} x_\nu$$

or//

$$x'_\sigma x'^{\sigma} = x_\mu x^{\mu} + x_\mu (\epsilon^{\nu\mu} + \epsilon^{\mu\nu}) x_\nu, \text{ to first order in } \epsilon. \quad (4)$$

If $x'_\sigma x'^{\sigma} \equiv x_\mu x^{\mu}$, then 2nd term RHS in (3) must vanish. So: $\boxed{\epsilon^{\nu\mu} = (-)\,\epsilon^{\mu\nu}}$ (5)

(C) 3. Write: $x'^{\alpha} = (g^{\alpha\beta} + \epsilon^{\alpha\beta}) x_\beta = (g^{\alpha\beta} + \epsilon^{\alpha\beta}) g_{\beta\gamma} x^{\gamma}$, i.e. $\underline{x'^{\alpha} = (\delta^{\alpha}_{\gamma} + \epsilon^{\alpha}_{\gamma}) x^{\gamma}}$, using $g^{\alpha\beta} g_{\beta\gamma} = \delta^{\alpha}_{\gamma}$, and defining: $\epsilon^{\alpha}_{\gamma} = \epsilon^{\alpha\beta} g_{\beta\gamma}$ ($\frac{\text{mixed}}{\underset{\sim}{\epsilon}\ \text{tensor}}$). But $\delta^{\alpha}_{\gamma} x^{\gamma} = x^{\alpha}$ is the identity, and since ϵ is a first order ∞smal: $1 + \epsilon = e^{\epsilon}$. Then we can write above relation: $x'^{\alpha} = x^{\alpha} + \epsilon^{\alpha}_{\gamma} x^{\gamma}$, in 4-vector form as the Lorentz transform:

$$\boxed{\tilde{x}' = [\exp(\underset{\sim}{\epsilon})]\, \tilde{x}} \quad (6)$$

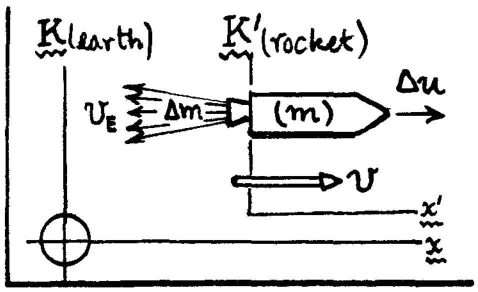
As such, $\underset{\sim}{\epsilon}$ is the ∞smal form of Jk^{n}'s $\underset{\sim}{L}$ in Eqs. (11.87) & (11.93). Strictly speaking: $\epsilon^{\alpha}_{\gamma} = L^{\alpha}_{\gamma}$ here, but $\epsilon^{\alpha\gamma} = L^{\alpha\gamma}$ also.

(8) [20 pts]. Investigate further details of travel on a relativistic rocket.

(A) 1. The rocket increments its velocity $v \rightarrow v + du$ by -- in effect -- jumping into a cd. system traveling @ du w.r.t. itself. This velocity addition is seen by an earth observer as an increment dv in v such that

$$(v + dv)_{earth} = \left[(v + du)/\left(1 + \frac{v\,du}{c^2}\right)\right]_{rocket} \Rightarrow \boxed{dv = [1 - (v^2/c^2)]\,du}. \qquad (1)$$

This relation is given to first order in the infinitesimals du & dv.



(B) 2. For the dynamics, consider what happens in a frame K'(rocket) moving at v relative to K(earth) and instantaneously at rest w.r.t. rocket. In the next instant of time, the rocket ejects fuel Δm at velocity v_E (relative to itself), and increments its velocity by Δu (in rocket frame). Conservation of momentum in K' requires : $m\Delta u = -v_E\,\Delta m$; Δm is negative. There are no dilation corrections to this expression because the new rocket frame K'', moving at Δu w.r.t. K', has $\gamma_{\Delta u} = 1/\sqrt{1 - (\Delta u/c)^2} \rightarrow 1$, to 1st order ∞smals. The only relativistic correction is to relate Δu (in K') to a Δv (in K). This is already done by Eq. (1): $\Delta u = \Delta v/[1 - (v/c)^2]$. Hence:

$$m\,\Delta u + v_E\,\Delta m = 0 \ [\text{in } K'] \longrightarrow m\,\Delta v/[1 - (v/c)^2] + v_E\,\Delta m = 0 \ [\text{in } K];$$

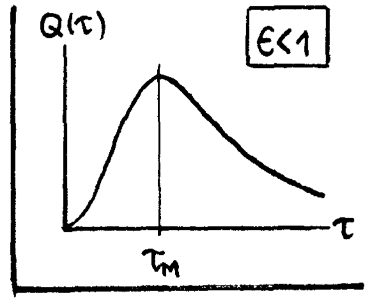
$$\text{or} \left\{\begin{array}{l} \Delta v \rightarrow dv \\ \Delta m \rightarrow dm \\ \beta = v/c \end{array}\right\} \boxed{m\frac{d\beta}{dm} + \frac{v_E}{c}(1 - \beta^2) = 0}. \qquad (2)$$

(C) 3. For the case of $v_E\left(\substack{\text{exhaust}\\ \text{velocity}}\right) = \text{cnst}$, and $A\left(\substack{\text{onboard}\\ \text{acceleration}}\right) = \text{cnst}$, we have shown in the "Relativistic Rocket Trip" class notes that in terms of rocket time τ...

$$\left[\begin{array}{l} \text{Eq. (8): distance traveled (ref. to earth frame): } D(\tau) = \frac{c^2}{A}[\cosh(A\tau/c) - 1]; \\ \text{Eq. (11): burn-ratio (}^{w/}\ m_0 = m(0)\text{, initial mass): } R(\tau) = \frac{m_0}{m(\tau)} = \exp(A\tau/\epsilon c). \end{array}\right. \qquad (3)$$

Here $\epsilon = \frac{v_E}{c} \leq 1$. The asymptotic limits for the quality factor Q are ...

$$Q(\tau) = \frac{D(\tau)}{R(\tau)} \rightarrow \begin{cases} \frac{1}{2}A\tau^2[1-(A\tau/\epsilon c)+\cdots], & \tau\to 0; \\ (c^2/2A)\exp\left[-\left(\frac{1-\epsilon}{\epsilon}\right)\frac{A\tau}{c}\right], & \tau\to\infty. \end{cases} \quad (4)$$



For $\epsilon<1$, $Q(\tau)$ vs. τ goes through a maxm at $\tau=\tau_M$...

$$\rightarrow \frac{\partial Q}{\partial \tau} = 0 \Rightarrow \frac{1}{D}\frac{\partial D}{\partial \tau} = \frac{1}{R}\frac{\partial R}{\partial \tau}, \text{ or } \left(\frac{\sinh\phi}{\cosh\phi-1}\right)\Big|_{\tau=\tau_M} = \frac{1}{\epsilon}, \text{ w/ } \phi=\frac{A\tau}{c}. \quad (5)$$

Eq.(5) can be rewritten: $\tanh(\phi_m/2)=\epsilon$, or: $\underline{\tanh\phi_m = 2\epsilon/(1+\epsilon^2)}$, by hyperbolic trig identities. Then τ_m is found from:*

$$\rightarrow \tau_M = \frac{c}{A}\ln\left(\frac{1+\epsilon}{1-\epsilon}\right) = \frac{2c}{A}\left(\epsilon+\tfrac{1}{3}\epsilon^3+\tfrac{1}{5}\epsilon^5+\cdots\right), \ \epsilon<1; \quad (6)$$

So

$$\left.\begin{aligned} &\text{Eq.(7) of notes} \Rightarrow \text{velocity}: \beta(\tau_m) = \tanh\phi_m = 2\epsilon/(1+\epsilon^2); \\ &\text{Eq.(6) of notes} \Rightarrow \text{earthtime}: t(\tau_m) = \frac{c}{A}\sinh\phi_m = (2v_E/A)/(1-\epsilon^2); \\ &\text{Eq.(8) of notes} \Rightarrow \text{distance}: D(\tau_m) = \frac{c^2}{A}[\cosh\phi_m - 1] = \frac{2v_E^2}{A}/(1-\epsilon^2). \end{aligned}\right\} \quad (7)$$

This is all for $\epsilon<1$. When $\epsilon\to 1$, $\tau_m\to\infty$, along with $t(\tau_m)$ and $D(\tau_m)$.

(D) **4.** When $A=$ cnst, the distance $D(\tau) = \frac{c^2}{A}[\cosh\phi-1]$, w/ $\phi=\frac{A\tau}{c}$, is as in Eq.(3) above and the velocity $\beta(\tau)=\tanh\phi$. The eqtn-of-motion, (2) above, can be rewritten as:

$$\rightarrow \frac{v_E}{c}\left(\frac{dR}{R}\right) = \frac{d\beta}{1-\beta^2}, \text{ or } \frac{1}{R}\left(\frac{dR}{d\tau}\right) = \frac{c}{v_E}\left(\frac{d\phi}{d\tau}\right) \Rightarrow \boxed{R(\tau) = \exp\left\{\frac{A}{c}\int_0^\tau \frac{d\tau'}{\epsilon(\tau')}\right\}}. \quad (8)$$

Now $\epsilon(\tau)=\frac{1}{c}v_E(\tau)$ can be a fcn of τ, chosen to suppress R. If we choose: $\epsilon(\tau) = \epsilon_1-(\epsilon_1-\epsilon_0)e^{-\omega\tau}$ [so $\epsilon(0)=\epsilon_0 \to \epsilon(\infty)=\epsilon_1$], then we calculate...

$$R(\tau) = \exp\left\{\frac{A}{c}\int_0^\tau \frac{d\tau'}{\epsilon_1-(\epsilon_1-\epsilon_0)e^{-\omega\tau'}}\right\} = R_0(\tau)/f(\tau), \text{ w/ } R_0(\tau) = e^{A\tau/c\epsilon_0} \left(\begin{matrix}\text{burn ratio}\\ \text{at steady } \epsilon_0\end{matrix}\right)$$

And

$$f(\tau) = \exp\left\{\frac{A\tau}{c}\left[\left(\frac{1}{\epsilon_0}-\frac{1}{\epsilon_1}\right) - \frac{1}{\epsilon_1\omega\tau}\ln|\epsilon_1-(\epsilon_1-\epsilon_0)e^{-\omega\tau}|\right]\right\}. \quad (9)$$

The burn suppression factor $f(\tau)$ increases the quality $Q_0(\tau)$ [Eq.(4)] by same amt.

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* $\tanh\phi = (e^{2\phi}-1)/(e^{2\phi}+1) = s \Rightarrow e^{2\phi} = (1+s)/(1-s)$, or: $\phi = \ln\sqrt{(1+s)/(1-s)}$.