1992 ϕ 507 Problems Assigned: 2/24. Due: 3/2/92. file $\mathcal{H}_6$ ⑥

(20) To approximate the ground state of the simple harmonic oscillator (SHO), use the trial wavefunction: $\phi(x) = A[1-(|x|/\alpha)]$, for $|x| \leq \alpha$, and $\phi(x) \equiv 0$, for $|x| > \alpha$. Here A = cnst and α = variable (length) parameter. Calculate $E(\alpha) = \frac{\langle\phi|\mathcal{H}(\text{SHO})|\phi\rangle}{\langle\phi|\phi\rangle}$ and -- for optimum α -- show that this energy lies less than 10% above the exact value.

---

(21) [Davydov Ch. VII #6, p. 205]. Use the trial wavefunction: $\phi(\alpha, r) = Ae^{-\frac{1}{2}\alpha r^2}$, to estimate the ground state energy of the hydrogen atom. NOTE: here you are approximating the atom's radial motion by that of an "equivalent" 1D SHO.

---

(22) In a QM system with Hamiltonian $\mathcal{H}$, let the eigenfunctions & eigenenergies be ψ_n & E_n, so: $\mathcal{H}\psi_n = E_n\psi_n$. To approximate the ground state energy E_0, suppose you use the trial function: $\psi = \psi_0 + \lambda\phi$, w/ ψ_0 = actual ground state wavefcn, λ is a small (real) parameter, and ϕ is an arbitrary fcn with the expansion $\phi = \sum_n c_n\psi_n$. Show that if the approximate (variational) energy: $E(\lambda) = \langle\psi|\mathcal{H}|\psi\rangle/\langle\psi|\psi\rangle$, is expanded in a power series in λ, viz.: $E(\lambda) = E_0 + \lambda\mathcal{E}_1 + \lambda^2\mathcal{E}_2 + \lambda^3\mathcal{E}_3 + \ldots$, then $\mathcal{E}_1 \equiv 0$, while $\mathcal{E}_2$ is the positive quantity: $\mathcal{E}_2 = \sum_n |c_n|^2(E_n - E_0)$. CONCLUSION: for any perturbation on $\mathcal{H}$ which shifts $\psi_0 \to \psi_0 + \lambda\phi$ by a term first order in some small parameter λ, the ground state energy $E_0 \to E_0 + \lambda^2\mathcal{E}_2$ shift is only a second order correction.

---

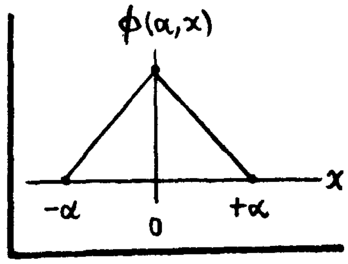
(23) (A) Show (by substitution) that a solution to: $y''(\xi) + \alpha\xi^n y(\xi) = 0$, w/ α & n = cnsts and $\xi \geq 0$, is given by: $y(\xi) = A\sqrt{\xi}\, J_\nu(\zeta)$, w/ A = cnst, $\nu = \frac{1}{n+2}$, $\zeta = \left(\frac{2\sqrt{\alpha}}{n+2}\right)\xi^{\frac{1}{2}(n+2)}$. $J_\nu(\zeta)$ is the Bessel fcn of order ν. (B) Assume the asymptotic form: $y(\xi) \sim \xi^{-k}e^{-a\xi^\ell}$, as $\xi \to \infty$. By proper choice of the cnsts k, ℓ & a, show that as $\xi \to \infty$, this form satisfies the differential eqtn: $y''(\xi) + \alpha\xi^n y(\xi) = \frac{n}{4}\left(\frac{n}{4}+1\right)\xi^{-2}y(\xi) \to 0$.

---

(24) Bessel's ODE is: $y'' + \frac{1}{x}y' + \left(1 - \frac{\nu^2}{x^2}\right)y = 0$, w/ ν = real cnst. Find an approximate soln. for the Bessel fcn $y \simeq J_\nu(x)$ by the WKB method. Find an asymptotic form for $J_\nu(x)$ as $x \to$ "large" (specifically: $x \gg |\nu|$). You may assume $|\nu| \gg \frac{1}{2}$.

ϕ507 Solutions

⑥ Estimate SHO groundstate energy w/ trial wavefcn: $\phi(\alpha,x) = A[1-(|x|/\alpha)]$.



1. A & α = consts, and: $\phi(\alpha,x) = A[1-(|x|/\alpha)]$ for $|x| \leq \alpha$; $\phi \equiv 0$, otherwise. Normalization integral is...

$$\rightarrow \langle\phi|\phi\rangle = A^2 \int_{-\alpha}^{+\alpha} \left(1-\frac{|x|}{\alpha}\right)^2 dx = 2\alpha A^2 \int_0^1 (1-u)^2 du \quad \leftarrow u = x/\alpha.$$

So// $\langle\phi|\phi\rangle = \frac{2}{3}\alpha A^2 = 1 \Rightarrow \underline{\underline{A^2 = 3/2\alpha}}$. (1)

---

2. The SHO Ham^n is; $\mathcal{H}(\text{SHO}) = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}m\omega^2x^2$, where m = SHO mass and ω is its natural freq. Then, with value of A in Eq. (1), the energy for ϕ is

$$\rightarrow E(\alpha) = \langle\phi|\mathcal{H}(\text{SHO})|\phi\rangle = \frac{3}{2\alpha}\int_{-\alpha}^{+\alpha}\left(1-\frac{|x|}{\alpha}\right)\left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}m\omega^2x^2\right]\left(1-\frac{|x|}{\alpha}\right)dx. \quad (2)$$

To evaluate $E(\alpha)$, we need to notice that $\frac{d^2}{dx^2}|x|$ generates a δ-fcn. Because...

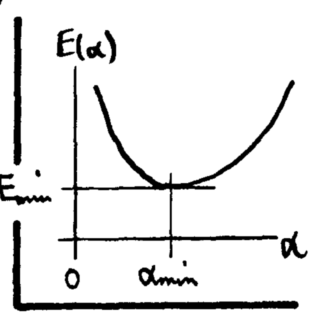
$$\int_{-\epsilon}^{+\epsilon}\left(\frac{d^2}{dx^2}|x|\right)dx = \int_{-\epsilon}^{+\epsilon}\frac{d}{dx}\left(\frac{d|x|}{dx}\right)dx = \left(\frac{d|x|}{dx}\right)\Big|_{x=+\epsilon} - \left(\frac{d|x|}{dx}\right)\Big|_{x=-\epsilon} = (+1)-(-1) = 2,$$

So// $\underline{\underline{\frac{d^2}{dx^2}|x| = 2\delta(x)}}$. (3)

Use this fact in Eq. (2) to calculate...

$$\rightarrow E(\alpha) = \frac{3}{2\alpha}\left\{\int_{-\alpha}^{+\alpha}\left(1-\frac{|x|}{\alpha}\right)\frac{\hbar^2}{2m}\frac{1}{\alpha}\cdot 2\delta(x)\,dx + \frac{1}{2}m\omega^2\int_{-\alpha}^{+\alpha}x^2\left(1-\frac{|x|}{\alpha}\right)^2dx\right\}$$

$$E(\alpha) = \frac{3}{2\alpha}\left\{\frac{\hbar^2}{m\alpha} + m\omega^2\alpha^3\underbrace{\int_0^1 u^2(1-u)^2du}_{=1/30}\right\} = \frac{3}{2}\frac{\hbar^2}{m\alpha^2} + \frac{1}{20}m\omega^2\alpha^2. \quad (4)$$



3. As a fcn of the parameter α, $E(\alpha)$ looks like the graph sketched. The minimum is at...

$$\frac{\partial E}{\partial \alpha} = 0 \Rightarrow \alpha^2 = \sqrt{30}\,(\hbar/m\omega) = \alpha_{min}^2; \quad (5)$$

So// $\boxed{E_{min} = E(\alpha_{min}) = \sqrt{\frac{6}{5}}\left(\frac{1}{2}\hbar\omega\right) = 1.095\,E_0}$. (6)

E_{min} is the best estimate for the groundstate energy $E_0 = \frac{1}{2}\hbar\omega$ for this type of trial ϕ.

(21) Estimate H-atom groundstate energy w/ trial wavefcn: $\phi(\alpha,r) = Ae^{-\frac{1}{2}\alpha r^2}$.

1) Just follow Davydov's Eqs. (51.12)-(51.15), where he does the calculation for the trial fcn $\phi(\alpha,r) = Ae^{-\beta r}$ [a lucky guess!]. We have...

$$\rightarrow E(\alpha) = \langle\phi|\mathcal{H}|\phi\rangle / \langle\phi|\phi\rangle, \text{ w/ } \mathcal{H} = -\frac{\hbar^2}{2m}\nabla^2 - \frac{e^2}{r}, \text{ for H atom.} \qquad (1)$$

For ϕ with radial variation only: $\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)$. Norm integral is:

$$\phi = Ae^{-\frac{1}{2}\alpha r^2} \Rightarrow \langle\phi|\phi\rangle = \int_0^\infty |\phi|^2 \cdot 4\pi r^2 dr = 4\pi|A|^2 \int_0^\infty r^2 e^{-\alpha r^2} dr.$$

$$\text{But: } \int_0^\infty r^2 e^{-\alpha r^2} dr = \frac{\sqrt{\pi}}{4\alpha^{3/2}}, \text{ so: } \underline{\underline{\langle\phi|\phi\rangle = (\pi/\alpha)^{\frac{3}{2}}|A|^2}}. \qquad (2)$$

---

2) $E(\alpha)$ in Eq. (1) is now... (with: $a_0 = \hbar^2/me^2$ = Bohr radius)...

$$\rightarrow E(\alpha) = \left(\frac{\alpha}{\pi}\right)^{\frac{3}{2}} \int_0^\infty 4\pi r^2 dr \cdot e^{-\frac{1}{2}\alpha r^2}\left[-\frac{\hbar^2}{2m}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) - \frac{e^2}{r}\right]e^{-\frac{1}{2}\alpha r^2} \qquad (3)$$

$$= 4\pi e^2\left(\frac{\alpha}{\pi}\right)^{\frac{3}{2}} \int_0^\infty dr\left[-\frac{a_0}{2}e^{-\frac{1}{2}\alpha r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}e^{-\frac{1}{2}\alpha r^2}\right) - re^{-\alpha r^2}\right]$$

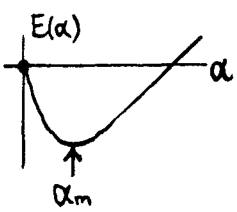
$$= 4\pi e^2\left(\frac{\alpha}{\pi}\right)^{\frac{3}{2}}\left\{\frac{a_0\alpha}{2}\underbrace{\int_0^\infty dr \cdot e^{-\frac{1}{2}\alpha r^2}\frac{\partial}{\partial r}\left(r^3 e^{-\frac{1}{2}\alpha r^2}\right)} - \frac{1}{2\alpha}\right\}$$

(partial integrate) $= -\int_0^\infty dr \cdot \left(r^3 e^{-\frac{1}{2}\alpha r^2}\right)\frac{\partial}{\partial r}\left(e^{-\frac{1}{2}\alpha r^2}\right) = +\alpha\int_0^\infty r^4 e^{-\alpha r^2} dr$

$= \frac{3\sqrt{\pi}}{8}/\alpha^{3/2}$

so/ $$E(\alpha) = 4\pi e^2\left(\frac{\alpha}{\pi}\right)^{3/2}\left\{\frac{a_0\alpha}{2}\cdot\frac{3\sqrt{\pi}}{8\alpha^{3/2}} - \frac{1}{2\alpha}\right\}$$

or/ $$\underline{\underline{E(\alpha) = \frac{2e^2}{\sqrt{\pi}}\left\{\left(\frac{3\sqrt{\pi}a_0}{8}\right)\alpha - \sqrt{\alpha}\right\}}}, \text{ w.r.t. } \phi = Ae^{-\frac{1}{2}\alpha r^2}. \qquad (4)$$



---

3) Minimize $E(\alpha)$ in Eq. (4)...

$$\rightarrow \partial E/\partial\alpha = 0 \Rightarrow \underline{\underline{\alpha = 16/9\pi a_0^2 = \alpha_m}}, \text{ and/ } \boxed{E(\alpha_m) = -\left(\frac{8}{3\pi}\right)\frac{e^2}{2a_0}}. \qquad (5)$$

$E(\alpha_m)$ is the best estimate to $E(\text{gnd}) = -e^2/2a_0$, with a trial fcn $\phi = e^{-\frac{8}{9\pi}(r/a_0)^2}$.

$E(\alpha_m)$ lies <u>above</u> $E(\text{gnd})$ by: $\Delta E/E = [E(\alpha_m) - E(\text{gnd})]/|E(\text{gnd})| = 1 - (8/3\pi) = 0.1512$, i.e. about 15%. So ϕ(SHO) gives a rather poor fit.

(22) For ground state (ψ_0, E_0), $\mathcal{O}(\lambda)$ perturbation on wavefcn $\psi_0 \Rightarrow \mathcal{O}(\lambda^2)$ correction to energy E_0.

1) Calculation is best done by putting in $\phi = \sum_n c_n \psi_n$ at the very end. Straightforwardly:

$$E(\lambda) = \langle\psi|\mathcal{H}|\psi\rangle / \langle\psi|\psi\rangle = \langle\psi_0+\lambda\phi|\mathcal{H}|\psi_0+\lambda\phi\rangle / \langle\psi_0+\lambda\phi|\psi_0+\lambda\phi\rangle$$

or/

$$\rightarrow E(\lambda) = \frac{\overset{①}{\langle\psi_0|\mathcal{H}|\psi_0\rangle} + \lambda[\overset{②}{\langle\psi_0|\mathcal{H}|\phi\rangle} + \overset{③}{\langle\phi|\mathcal{H}|\psi_0\rangle}] + \lambda^2\langle\phi|\mathcal{H}|\phi\rangle}{\underset{④}{\langle\psi_0|\psi_0\rangle} + \lambda\underbrace{[\langle\psi_0|\phi\rangle + \langle\phi|\psi_0\rangle]}_{N} + \lambda^2\langle\phi|\phi\rangle}. \quad (1)$$

We've used λ = real here. Term ① $\equiv E_0$, and in terms ② & ③, use $\langle\psi_0|\mathcal{H} = E_0\langle\psi_0|$ & $\mathcal{H}|\psi_0\rangle = E_0|\psi\rangle$, resp. ($\mathcal{H}$ is Hermitian). Term ④ $\equiv 1$, by normalization. With the shorthand notation $N = \langle\psi_0|\phi\rangle + \langle\phi|\psi_0\rangle$, Eq. (1) becomes...

$$\underline{\underline{E(\lambda) = [(1+\lambda N)E_0 + \lambda^2\langle\phi|\mathcal{H}|\phi\rangle] / [(1+\lambda N) + \lambda^2\langle\phi|\phi\rangle]}}. \quad (2)$$

---

2) In Eq. (2), $\lambda \rightarrow$ small. If we define the quantity: $\kappa = \lambda^2/(1+\lambda N)$, then

$$\left[\begin{array}{l} E(\lambda) = E_0\left[1 + \frac{\kappa}{E_0}\langle\phi|\mathcal{H}|\phi\rangle\right] / \left[1 + \kappa\langle\phi|\phi\rangle\right], \\ \text{w/ } \kappa = \lambda^2/(1+\lambda N) \simeq \lambda^2[1 - \lambda N + (\lambda N)^2 - \dots]. \end{array}\right. \quad (3)$$

The leading term in κ is $\mathcal{O}(\lambda^2)$ in smallness. To $\mathcal{O}(\lambda^2)$, $E(\lambda)$ expands as...

$$E(\lambda) \simeq E_0\left[1 + \frac{\lambda^2}{E_0}\langle\phi|\mathcal{H}|\phi\rangle\right]\left[1 - \lambda^2\langle\phi|\phi\rangle\right] \simeq E_0 + \lambda^2\mathcal{E}_2,$$

$$\text{w/ } \underline{\underline{\mathcal{E}_2 = \langle\phi|\mathcal{H}|\phi\rangle - E_0\langle\phi|\phi\rangle}}. \quad (4)$$

As advertised, the first correction to E_0 is $\mathcal{O}(\lambda^2)$, <u>not</u> $\mathcal{O}(\lambda)$.

---

3) Calculate $\mathcal{E}_2$ in Eq. (4) by putting in $\phi = \sum_n c_n\psi_n$. Since $\{\psi_n\}$ is an orthonormal set: $\langle\psi_m|\psi_n\rangle = \delta_{mn}$, we get...

$$\underline{\underline{\mathcal{E}_2}} = \sum_{m,n} c_m^* c_n [\langle\psi_m|\mathcal{H}|\psi_n\rangle - E_0\langle\psi_m|\psi_n\rangle] = \underline{\underline{\sum_n |c_n|^2(E_n - E_0)}}, \quad (5)$$

as required. $\mathcal{E}_2 \geqslant 0$, since $E_n - E_0 \geqslant 0$. So $E(\lambda)$ in Eq. (4) lies above E_0.

(23) Solution to: $y''+\alpha\xi^n y=0$ for $y=y(\xi)$. Asymptotic form for $\xi\to\infty$.

This problem appears in the WKB turning point problem, for $\alpha=-1$, $n=1$ (Airy's ODE).

(A)

1) Let: $x(\xi)=\sqrt{\xi}\,J_\nu(\zeta)$, w/ $\nu=1/(n+2)$ & $\zeta=\left(\frac{2\sqrt{\alpha}}{n+2}\right)\xi^{\frac{1}{2}(n+2)}$. By direct differentiation...

$$\rightarrow \frac{dx}{d\xi}=\sqrt{\xi}\left(\frac{d\zeta}{d\xi}\right)\frac{d}{d\zeta}J_\nu(\zeta)+\frac{1}{2}\xi^{-\frac{1}{2}}J_\nu(\zeta). \quad (1)$$

But: $\left(\frac{d\zeta}{d\xi}\right)=\sqrt{\alpha}\,\xi^{\frac{n}{2}}$, and: $\frac{d}{d\zeta}J_\nu(\zeta)=-\frac{\nu}{\zeta}J_\nu(\zeta)+J_{\nu-1}(\zeta)$ {Mathews & Walker Eq. (7-54)}. So...

$$\rightarrow \frac{dx}{d\xi}=\sqrt{\alpha}\,\xi^{\frac{1}{2}(n+1)}\left[-\frac{\nu}{\zeta}J_\nu(\zeta)+J_{\nu-1}(\zeta)\right]+\frac{1}{2}\xi^{-\frac{1}{2}}J_\nu(\zeta)$$

$$=\sqrt{\alpha}\,\xi^{\frac{1}{2}(n+1)}J_{\nu-1}(\zeta)-\underbrace{\left[\sqrt{\alpha}\,\xi^{\frac{1}{2}(n+1)}\frac{1/(n+2)}{\left(\frac{2\sqrt{\alpha}}{n+2}\right)\xi^{\frac{1}{2}(n+2)}}\right]}_{=\frac{1}{2}\xi^{-1/2}}J_\nu(\zeta)+\frac{1}{2}\xi^{-\frac{1}{2}}J_\nu(\zeta)$$

cancels

So,
$$\frac{dx}{d\xi}=\sqrt{\alpha}\,\xi^{\frac{1}{2}(n+1)}J_{\nu-1}(\zeta). \quad (2)$$

---

2) The second derivative is calculated as...

$$\rightarrow \frac{1}{\sqrt{\alpha}}\frac{d^2x}{d\xi^2}=\xi^{\frac{1}{2}(n+1)}\left(\frac{d\zeta}{d\xi}\right)\frac{d}{d\zeta}J_{\nu-1}(\zeta)+\frac{n+1}{2}\xi^{\frac{1}{2}(n-1)}J_{\nu-1}(\zeta). \quad (3)$$

Use $\left(\frac{d\zeta}{d\xi}\right)=\sqrt{\alpha}\,\xi^{\frac{n}{2}}$ as above, and: $\frac{d}{d\zeta}J_{\nu-1}(\zeta)=\frac{\nu-1}{\zeta}J_{\nu-1}(\zeta)-J_\nu(\zeta)$ {M & W (7-55)}. So...

$$\frac{1}{\sqrt{\alpha}}\frac{d^2x}{d\xi^2}=\sqrt{\alpha}\,\xi^{n+\frac{1}{2}}\left[\left(\frac{\nu-1}{\zeta}\right)J_{\nu-1}(\zeta)-J_\nu(\zeta)\right]+\frac{n+1}{2}\xi^{\frac{1}{2}(n-1)}J_{\nu-1}(\zeta) \quad (4)$$

$$=-\sqrt{\alpha}\,\xi^n x(\xi)+\left[\underbrace{\sqrt{\alpha}\,\xi^{n+\frac{1}{2}}\left(\frac{\nu-1}{\zeta}\right)}+\frac{n+1}{2}\xi^{\frac{1}{2}(n-1)}\right]J_{\nu-1}(\zeta) \quad (5)$$

$$=\sqrt{\alpha}\,\xi^{n+\frac{1}{2}}\frac{\frac{1}{n+2}-1}{\left(\frac{2\sqrt{\alpha}}{n+2}\right)\xi^{\frac{n}{2}+1}}=-\left(\frac{n+1}{2}\right)\xi^{\frac{1}{2}(n-1)}$$

cancels

So, $\frac{1}{\sqrt{\alpha}}\frac{d^2x}{d\xi^2}=-\sqrt{\alpha}\,\xi^n x(\xi)+\text{zero}$, a// $\boxed{\frac{d^2x}{d\xi^2}+\alpha\xi^n x=0}$, for// $x(\xi)=\sqrt{\xi}\,J_\nu(\zeta)$. (6)

---

3) We have shown that $x(\xi)=\sqrt{\xi}\,J_\nu(\zeta)$, w/ $\nu=1/(n+2)$ & $\zeta=\left(\frac{2\sqrt{\alpha}}{n+2}\right)\xi^{\frac{1}{2}(n+2)}$, satisfies the ODE of interest, viz $x''+\alpha\xi^n x=0$. Then $y(\xi)=Ax(\xi)$ is also a soln, for $A=$ cnst. For the Airy problem: $y''-\xi y=0$, the soln is: $y(\xi)=A\sqrt{\xi}\,J_{1/3}\left(\frac{2i}{3}\xi^{3/2}\right)$.

ϕ 507 Solutions

23 (cont'd)

4) Now assume an asymptotic form: $x(\xi) \sim \xi^{-k} e^{-a\xi^{\ell}}$, as $\xi \to \infty$. Differentiate...

(B)

$$\rightarrow \frac{dx}{d\xi} = -k\xi^{-(k+1)} e^{-a\xi^{\ell}} + \xi^{-k}\left[-a\ell\xi^{\ell-1} e^{-a\xi^{\ell}}\right] = -\left(k\xi^{-1} + a\ell\xi^{\ell-1}\right)x; \quad (7)$$

$$\rightarrow \frac{d^2x}{d\xi^2} = -\left[-k\xi^{-2} + a\ell(\ell-1)\xi^{\ell-2}\right]x + \left[k\xi^{-1} + a\ell\xi^{\ell-1}\right]^2 x$$

... gather terms to get...

$$\rightarrow \frac{d^2x}{d\xi^2} = \underbrace{(a\ell)^2\xi^{2(\ell-1)}}_{①}\left[1 - \underbrace{\left(\frac{\ell-2k-1}{a\ell}\right)}_{②}\xi^{-\ell} + \underbrace{\frac{k(k+1)}{(a\ell)^2}}_{③}\xi^{-2\ell}\right]x. \quad (8)$$

---

5) The parameters k, ℓ, a are free. We fix them by the following choices...

Set factor ① $= -\alpha\xi^n \Rightarrow \begin{cases} 2(\ell-1) = n, \text{ or: } \underline{\ell = \frac{1}{2}(n+2)}; \\ (a\ell)^2 = -\alpha, \text{ or: } \underline{a = 2\sqrt{-\alpha}/(n+2)}. \end{cases}$ (9a)

Set factor ② $\equiv 0 \Rightarrow \underline{k = \frac{1}{2}(\ell-1) = \frac{1}{4}n}$. (9b)

Then factor ③: $k(k+1)/(a\ell)^2 = -\frac{1}{\alpha}\frac{n}{4}\left(\frac{n}{4}+1\right)$. (9c)

With the choices in Eq. (9), Eq. (8) becomes...

$$\frac{d^2x}{d\xi^2} = -\alpha\xi^n\left[1 - 0 - \frac{1}{\alpha}\frac{n}{4}\left(\frac{n}{4}+1\right)\xi^{-(n+2)}\right]. \quad (10)$$

---

6) We can now state that: $x(\xi) = \xi^{-\frac{n}{4}} \exp\left[-\left(\frac{2\sqrt{-\alpha}}{n+2}\right)\xi^{\frac{1}{2}(n+2)}\right]$, satisfies the ODE:

$$\boxed{\frac{d^2x}{d\xi^2} + \alpha\xi^n x = \frac{n}{4}\left(\frac{n}{4}+1\right)\xi^{-2}x(\xi)} \rightarrow 0, \text{ as } \xi \to \infty, \quad (11)$$

as required. $x(\xi)$ is therefore an asymptotic form for $\sqrt{\xi}\, J_\nu(\xi)$ of part (A). For the Airy problem: $x'' - \xi x = 0$, the asymptotic form is as was used in Eq. (39), p. 14 of "Notes on the WKB Method", viz: $\underline{x(\xi) \sim \xi^{-\frac{1}{4}} \exp\left(-\frac{2}{3}\xi^{3/2}\right)}$.

④ Find an asymptotic form for the Bessel fcn $J_\nu(x)$, $x \to$ "large", via WKB.

1) Bessel's Eqtn: $y'' + (1/x)y' + [1 - (\nu^2/x^2)]y = 0$, converts to WKB form, via:

$$\rightarrow y(x) = \psi(x)\exp\left(-\frac{1}{2}\int\frac{dx}{x}\right) = \psi(x)/\sqrt{x},$$

$$\Rightarrow \boxed{\psi'' + k^2(x)\psi = 0, \text{ w/ } k(x) = \left[1 - \frac{1}{x^2}\left(\nu^2 - \frac{1}{4}\right)\right]^{1/2}.} \qquad (1)$$

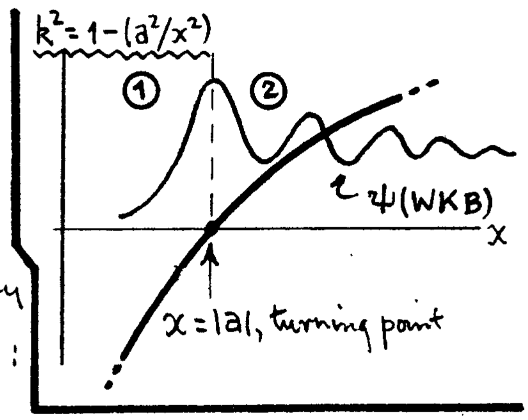
This eqtn is exact. A WKB approxn to $\psi(x)$ [and thus to $y = \psi/\sqrt{x}$] will work at values of x where k is "slowly-varying", i.e.

$$\rightarrow \left|\frac{1}{k^2}(dk/dx)\right| = \left|\frac{1}{k^3x^3}\left(\nu^2 - \frac{1}{4}\right)\right| \ll 1, \text{ or } |(kx)^3| \gg \left|\nu^2 - \frac{1}{4}\right|$$

This works OK when $|x| \to$ "large", so long as i.e. $\left|x^2 - \left(\nu^2 - \frac{1}{4}\right)\right|^{\frac{3}{2}} \gg \left|\nu^2 - \frac{1}{4}\right|$ (2)

ν = some cnst. Then a WKB form for ψ should be good for $\frac{1}{2} \ll |\nu| \ll x \to \infty$.

2) Let $a = \left(\nu^2 - \frac{1}{4}\right)^{1/2}$, so $k(x) = \left[1 - (a^2/x^2)\right]^{\frac{1}{2}}$. $x = |a|$ is a "turning point" for the probm [$k(a) = 0$], and we want ψ(WKB) for $x > |a|$. To be an acceptable solution, ψ should decrease exponentially in region ①, and oscillate in region ②. So we write:



$$\rightarrow \psi(x > |a|) \simeq (A/\sqrt{k})\sin\left(\int_a^x k(\xi)d\xi + \beta\right); \; k \text{ as above, ampl. } A \; \& \text{ phase } \beta = \text{cnsts};$$

but/ $\displaystyle\int_a^x k(\xi)d\xi = \int_a^x \frac{d\xi}{\xi}(\xi^2 - a^2)^{1/2} = (x^2 - a^2)^{1/2} - a\cos^{-1}\left(\frac{a}{x}\right) \simeq x - \frac{\nu\pi}{2}$ } for $x \gg |a|$ & $a \simeq \nu$.

So/ Since $k \simeq 1$ as $x \to$ "large", then: $\psi(x > |a|) \simeq A\sin\left(x - \frac{\nu\pi}{2} + \beta\right)$. (3)

3) Since $y = \psi/\sqrt{x}$, the WKB solution to Bessel's Eqtn, for $\frac{1}{2} \ll |\nu| \ll x \to \infty$, is

$$\boxed{y(x) = J_\nu(x) \simeq \frac{\text{cnst}}{\sqrt{x}}\sin\left(x - \frac{\nu\pi}{2} + \beta\right)} \qquad (4)$$

When the phase $\beta = \pi/4$, this is a standard result; see NBS Math. Handbook #(9.2.1). The phase β can be fixed by the WKB Connection Formulas.