Magnetostatics

Jackson Ch. 5: problems involving static $\mathbb{B}$-fields, in most of their glory.

1) If you believe Maxwell's Equations, viz. (in a non-material):

$$\left[\begin{array}{l|l} \nabla\cdot\mathbb{E} = 4\pi\rho & \nabla\cdot\mathbb{B} = 0 \\ \nabla\times\mathbb{E} = -\frac{1}{c}(\partial\mathbb{B}/\partial t) & \nabla\times\mathbb{B} = +\frac{1}{c}(\partial\mathbb{E}/\partial t) + \frac{4\pi}{c}\mathbb{J} \end{array}\right] \quad (1)$$

... for electrostatics ...	... for magnetostatics ...
$\partial\mathbb{B}/\partial t = 0$	$\partial\mathbb{E}/\partial t = 0$
and/ $\boxed{\nabla\cdot\mathbb{E} = 4\pi\rho \quad \nabla\times\mathbb{E} = 0}$	and// $\boxed{\nabla\cdot\mathbb{B} = 0 \quad \nabla\times\mathbb{B} = \frac{4\pi}{c}\mathbb{J}}$ (2)
So// $\mathbb{E} = -\nabla\phi$ {via Helmholtz; ϕ = scalar potential;	So// $\mathbb{B} = \nabla\times\mathbb{A}$ {via Helmholtz; $\mathbb{A}$ = vector potential;
w// $\underline{\underline{\nabla^2\phi = -4\pi\rho}}$.	w// $\underline{\underline{\nabla\times(\nabla\times\mathbb{A}) = \frac{4\pi}{c}\mathbb{J}}}$. (3)
basic eqtn of electrostatics	basic eqtn of magnetostatics

REMARKS

1. If electrostatics is an exercise in solving a 2nd order scalar PDE, magnetostatics will involve (much more complicated) solns to 2nd order vector PDE's. With the usual vector identity [curl curl = grad div $-\nabla^2$], (3) is...

$$\rightarrow \nabla^2\mathbb{A} - \nabla(\cancelto{0}{\nabla\cdot\mathbb{A}}) = -\frac{4\pi}{c}\mathbb{J}. \quad (4)$$

2. The solution to Eq. (4) would be ~ hopeless if it were not for the following remarkable fact: we can set $\nabla\cdot\mathbb{A} = 0$. The argument goes as follows:

$$\rightarrow \mathbb{B} = \nabla\times\mathbb{A}, \text{ is unchanged under: } \mathbb{A}\rightarrow\tilde{\mathbb{A}} = \mathbb{A} + \nabla\psi \quad \left\{\text{Gauge transform}\right. . \quad (5)$$

Magnetostatics (cont'd)

This is true since curl grad $\equiv 0$; Ψ is an arbitrary (twice differentiable) fcn. Now if $\nabla\cdot\mathbb{A}\neq 0$, just use $\tilde{\mathbb{A}}$, after having imposed...

$$\nabla\cdot\tilde{\mathbb{A}} = \nabla\cdot\mathbb{A} + \nabla^2\psi = 0, \quad \text{i.e.}/\!/ \quad \nabla^2\psi = -\nabla\cdot\mathbb{A} \quad \text{this fixes } \psi \; *$$

$$\text{So}/\!/ \quad \boxed{\nabla^2\tilde{\mathbb{A}} = -\frac{4\pi}{c}\mathbb{J}}, \quad \text{w}/\!/ \; \nabla\cdot\tilde{\mathbb{A}} = 0 \;(\text{Coulomb Gauge}). \qquad (6)$$

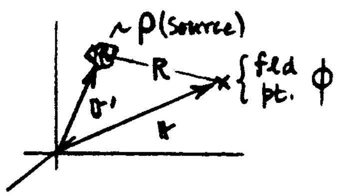
3. This last eqtn is at least manageable. Drop the tilde. Then...

electrostatics

$$\nabla^2\phi = -4\pi\rho$$

$$\xrightarrow{\text{So}/\!/} \phi(\mathbf{r}) = \int_\infty \frac{1}{R}\rho(\mathbf{r}')\,d^3x',$$

w/ $R = |\mathbf{r}-\mathbf{r}'|\ldots$



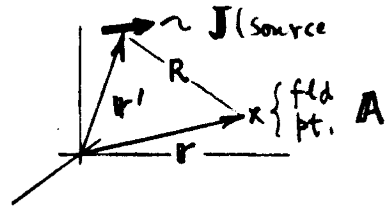
and: $\underline{\underline{\mathbb{E} = -\nabla\phi}}$.

magnetostatics

$$\nabla^2\mathbb{A} = -\frac{4\pi}{c}\mathbb{J}$$

$$\xrightarrow{\text{So}/\!/} \mathbb{A}(\mathbf{r}) = \frac{1}{c}\int_\infty \frac{1}{R}\mathbb{J}(\mathbf{r}')\,d^3x', \qquad (7)$$

w/ $R = |\mathbf{r}-\mathbf{r}'|\ldots$



and: $\underline{\underline{\mathbb{B} = \nabla\times\mathbb{A}}}$. (8)

4. Another condition is implicit in the <u>statics</u> problem, viz.

$$\nabla\cdot\left\{\nabla\times\mathbb{B} = \frac{1}{c}\frac{\partial\mathbb{E}}{\partial t} + \frac{4\pi}{c}\mathbb{J}\right\} \Rightarrow 0 = \frac{4\pi}{c}\left[\frac{\partial}{\partial t}\underbrace{\left(\frac{\nabla\cdot\mathbb{E}}{4\pi}\right)}_{\rho} + \nabla\cdot\mathbb{J}\right]$$

$$\text{i.e.}/\!/ \quad \frac{\partial\rho}{\partial t} + \nabla\cdot\mathbb{J} = 0 \quad \left\{\begin{array}{l}\text{CONTINUITY}\\ \text{EQUATION}\end{array}\right. \left(\begin{array}{c}\text{charge}\\ \text{cons}^{\text{v}}\text{n}\end{array}\right) \qquad (9)$$

"Statics" $\Rightarrow$ no t-dependence $\Rightarrow$ $\boxed{\begin{array}{c}\partial\rho/\partial t = 0\\ \nabla\cdot\mathbb{J} = 0\end{array}}$ for both $\left\{\begin{array}{c}\text{electro}\\ \text{magneto}\end{array}\right\}$ statics.

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* For the solution in (7): $\nabla\cdot\mathbb{A} = \frac{1}{c}\int_\infty d^3x'\,\mathbb{J}(\mathbf{r}')\cdot\nabla\left(\frac{1}{R}\right) = -\frac{1}{c}\int_\infty d^3x'\,\mathbb{J}(\mathbf{r}')\cdot\nabla'\left(\frac{1}{R}\right)$

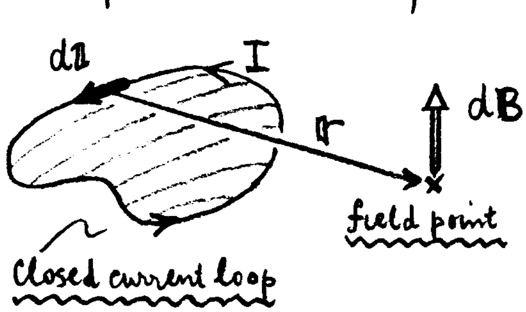
$= -\frac{1}{c}\int_\infty d^3x'\left[\nabla'\cdot(\mathbb{J}/R) - \frac{1}{R}\nabla'\cdot\mathbb{J}\right] \equiv 0$ $\left\{\begin{array}{l}\text{surface term}\to 0\text{, and}\\ \nabla'\cdot\mathbb{J}(\mathbf{r}')=0\text{, by (9)}\end{array}\right.$. So $\nabla^2\psi = 0$, actually.

Magnetostatics (Cont'd)

2) So much for the similarities between the electrostatics & magnetostatics problems. Now for the differences. The basic field laws are very different:

$$\left[\begin{array}{lll} \nabla\cdot\mathbb{E} = 4\pi\rho & \longleftrightarrow & \mathbb{E} = (q_E/r^2)\,\hat{r}\,, \text{ for electric monopole } q_E\,; \\ \nabla\cdot\mathbb{B} = 0 & \longleftrightarrow & \mathbb{B} = (q_m/r^2)\,\hat{r} \equiv 0\,, \text{ magnetic monopole } q_m \equiv 0\,. \end{array}\right\} \quad (10)$$

For $\mathbb{B}$-fields, nothing like a Coulomb law exists; there are no magnetic monopoles (and nobody knows why). The closest thing to it is...



$$\boxed{d\mathbb{B} = kI\,\frac{d\boldsymbol{\ell}\times\mathbb{r}}{r^3}} \quad [\sim\text{Ampere Law}], \qquad (11)$$

w/ k = proportionality cnst = $1/c$ (choice)

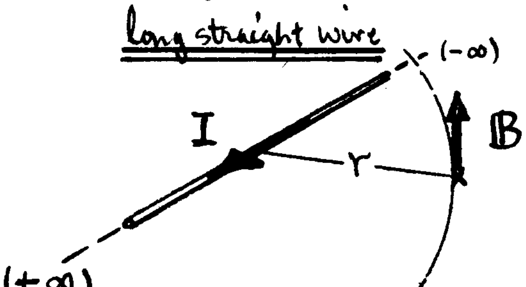
Gaussian units $\begin{cases} c = 3\times10^{10}\ \text{cm/sec} \\ I \text{ in statamps } (3\times10^9\,\text{stat A} = 1\,\text{A})^{*} \\ B \text{ in Gauss } (10^4\,\text{G} = 1\,\text{Tesla}) \end{cases}$

REMARKS

1. $|d\mathbb{B}|$ does fall off as $1/r^2$, with r = distance from source, but the vector dependence is much different. In fact $|d\mathbb{B}| \equiv 0$ if $d\boldsymbol{\ell}$ is $\parallel \mathbb{r}$.

2. Eq. (11) can be integrated for some simple geometries, e.g. (put $k = 1/c$)...

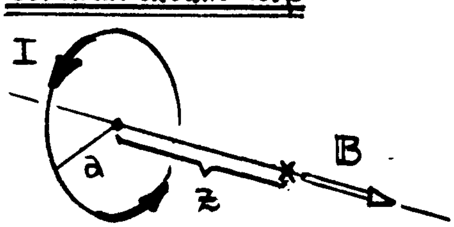
Long straight wire



$\mathbb{B}$ in concentric circles about wire:

$$\boxed{B = 2I/cr}$$

Circular current loop



$\mathbb{B}$ along loop axis:

$$\boxed{B = (2\pi I/ca)\Big/\left(1+\frac{z^2}{a^2}\right)^{3/2}} \qquad (12)$$

$*$ Related to $\left\{\dfrac{e(\text{Gaussian}) = 4.8\times10^{-10}\ \text{esu}}{e(\text{MKS}) = 1.6\times10^{-19}\ \text{Coul.}}\right. = 3\times10^9$, units ratio.

Magnetostatics (cont'd)

3. In MKS units, choose $k = \mu_0/4\pi$ rather than $k = 1/c$ (CGS), so the MKS formulas for $\mathbf{B}$ are gotten by replacing $1/c$ by $\mu_0/4\pi$, e.g.:
$B\left(\substack{\text{long}\\\text{wire}}\right) = \frac{1}{c}(2I/r)\,[\text{CGS}] \rightarrow \frac{\mu_0}{4\pi}(2I/r) = \mu_0 I/2\pi r\,[\text{MKS}]$.

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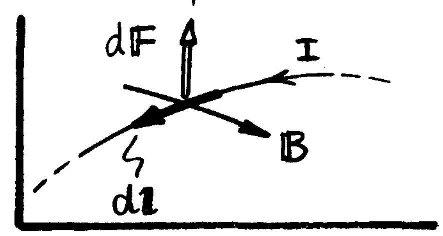
3) Eq. (11) [~ Ampere's Law] shows how a source $I\,d\mathbf{l}$ generates a magnetic field $d\mathbf{B}$, but we still need to know how $I\,d\mathbf{l}$ couples to an already existing field $\mathbf{B}$. Answer is...

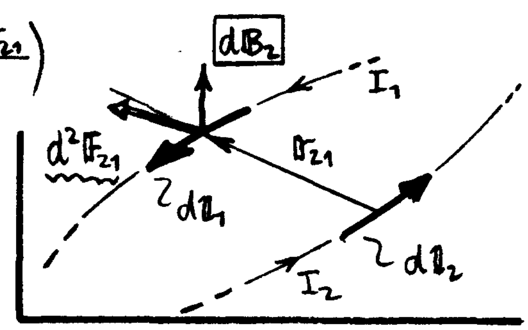
$$\boxed{d\mathbf{F} = \frac{1}{c} I\,(d\mathbf{l}\times\mathbf{B})} \quad [\text{Lorentz' Law}]. \tag{13}$$



If I is due to the motion of a single charge Δq, then $I\,d\mathbf{l} = (\Delta q)\mathbf{v}$, and: $\Delta\mathbf{F} = (\Delta q/c)\,\mathbf{v}\times\mathbf{B}$, which is Lorentz' Law.

Now, consider the magnetic interaction between two elemental sources $I_1\,d\mathbf{l}_1$ & $I_2\,d\mathbf{l}_2$...

$$d^2\mathbf{F}_{2\,\text{on}\,1} = \frac{1}{c} I_1 (d\mathbf{l}_1 \times d\mathbf{B}_2), \qquad d\mathbf{B}_2 = \frac{I_2}{c}\left(\frac{d\mathbf{l}_2\times\mathbf{r}_{21}}{r_{21}^3}\right)$$



i.e.,// $$d^2\mathbf{F}_{21} = \frac{I_1 I_2}{c^2}\left[\frac{d\mathbf{l}_1\times(d\mathbf{l}_2\times\mathbf{r}_{21})}{r_{21}^3}\right]$$

$$\xrightarrow{\text{or}} d^2\mathbf{F}_{21} = \frac{I_1 I_2}{c^2 r_{21}^2}\left[(d\mathbf{l}_1\cdot\hat{r}_{21})\,d\mathbf{l}_2 - (d\mathbf{l}_1\cdot d\mathbf{l}_2)\,\hat{r}_{21}\right]. \tag{14}$$

This is the force by $I_2\,d\mathbf{l}_2$ on $I_1\,d\mathbf{l}_1$. Reversing the roles...

$$d^2\mathbf{F}_{12} = \frac{I_2 I_1}{c^2 r_{12}^2}\left[(d\mathbf{l}_2\cdot\hat{r}_{12})\,d\mathbf{l}_1 - (d\mathbf{l}_2\cdot d\mathbf{l}_1)\,\hat{r}_{12}\right] \quad \begin{cases}\hat{r}_{12} = (-)\,\hat{r}_{21}\\ r_{12} = r_{21}\end{cases}$$

So// $$\left[\!\left[d^2\mathbf{F}_{12} + d^2\mathbf{F}_{21} = \frac{I_1 I_2}{c^2 r_{21}^2}\left[(d\mathbf{l}_1\cdot\hat{r}_{21})\,d\mathbf{l}_2 - (d\mathbf{l}_2\cdot\hat{r}_{21})\,d\mathbf{l}_1\right] \neq 0. \right]\!\right] \tag{15}$$

A seeming Disaster... by Newton III, should have: $d^2\mathbf{F}_{12} + d^2\mathbf{F}_{21} \equiv 0$. What has been left out is that both $I_k\,d\mathbf{l}_k$ are parts of current loops.