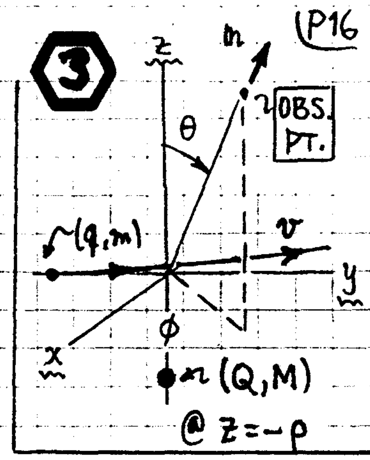
[redacted] { Set #3 : probs 42-44.
Assigned : 1/27/89 ; due : 2/3/89.

(P16



(42) In Prob. (41) you found the radiation energy loss/solid ∡ in a "soft" Coulomb collision between (q,m) [initially at velocity $v \ll c$] and (Q,M) [initially at rest]. For $M \gg m$, and $\frac{1}{2}mv^2 \gg qQ/\rho$ (w/ ρ = impact parameter), and w/ $d\Omega = \sin\theta\, d\theta\, d\phi$, the result was

$$\boxed{\frac{d\mathcal{E}}{d\Omega} = \frac{q^2}{32vc^3\rho^3}(qQ/m)^2\left[4 - n_y^2 - 3n_z^2\right]}.$$

n_y & n_z are the components (in cd. system shown) of $\mathbf{n}$ = unit vector to the OBS. PT. Find the total energy $\mathcal{E}$ radiated by q during the collision, and compare $\mathcal{E}$ with q's initial K.E. If q = electron at 100 keV, Q = proton, and ρ is fixed at 1 Å, how many "soft" collisions can q have before losing 10% of its energy? Is such radiation an important loss mechanism?

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(43) [redacted]]. Do Jackson's Prob. (14.15), p. 698. [redacted]INT'S: (±) helicity ⇒ $\binom{\text{left}}{\text{right}}$ circular polarization of the light [Jk^n p. 274]; polarization is: $\mathcal{P} = \dfrac{d^2I_\parallel - d^2I_\perp}{d^2I_\parallel + d^2I_\perp}$.

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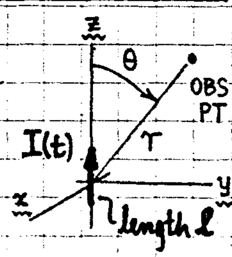
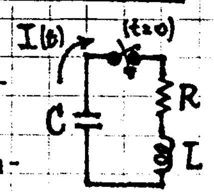
(44) [redacted] pts]. Consider a 1D arc discharge along the z-axis: a current pulse $I(t)$ -- which begins at time $t=0$ -- flows along a path of length ℓ. An observer, situated at position (r,θ) [with $r \gg \ell$] detects the arc radiation.



5pts (A) Start from the arc's Poynting vector derived in class [notes of 1/27/89]. Show that the arc's frequency-angle spectrum at the observer pt. is:

$$\boxed{\frac{d^2I}{d\omega\, d\Omega} = \left(\frac{\sin^2\theta}{8\pi^2c^3}\right)\ell^2\omega^2\left|\int_0^\infty I(t)\,e^{-i\omega t}\,dt\right|^2.}$$



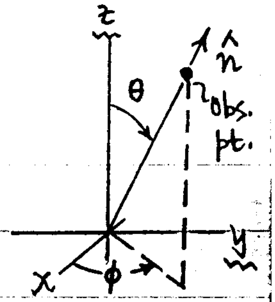
3pts (B) A (crude) model of the arc's $I(t)$ is the discharge of a capacitor C (at voltage V_0 initially, and switched on at $t=0$) through a resistance-inductance combination R & L (both consts). Then: $I(t) = (V_0/L\Gamma)\,e^{-\gamma t}\sinh\Gamma t$, w/ $\gamma = \frac{R}{2L}$ & $\Gamma = \sqrt{\gamma^2 - (1/LC)}$, for the overdamped case. Sketch $I(t)$ vs. t, roughly indicating the pulse risetime & duration.

4pts (C) Calculate the arc spectrum $d^2I/d\omega\, d\Omega$ for the model of part (B). Sketch the spectrum as a fcn of ω. Over what range of frequencies is the arc detectable?

3pts (D) Calculate the total energy radiated by the arc. Compare it with $\int I^2R\,dt$ = discharge energy.

[redacted]NT: see R. Robiscoe & Z. Sui, J. Appl. Phys. 64, 4364 (Nov. 1988).

[redacted]



Prob. (42) Analyse energy loss by electrons in "soft" collisions.

1. In the cd. system indicated: $n_z = \cos\theta$, $n_y = \sin\theta\sin\phi$, and the solid ∢: $d\Omega = \sin\theta\, d\theta\, d\phi$. Total energy radiated is then:

$$\mathcal{E} = \int (d\mathcal{E}/d\Omega)\, d\Omega = \left[\frac{q^2}{32 v c^3 \rho^3}\left(\frac{qQ}{m}\right)^2\right]\int_0^{\pi}\sin\theta\, d\theta\int_0^{2\pi}\{1+(3-\sin^2\phi)\sin^2\theta\}\, d\phi \qquad \underline{(1)}$$

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or $\mathcal{E} = [\ ]\int_0^{\pi}\sin\theta\, d\theta\,\{2\pi + 5\pi\sin^2\theta\} = [\ ]\cdot\frac{32\pi}{3}$,

so $\boxed{\mathcal{E} = \frac{\pi q^2}{3 v c^3 \rho^3}\left(\frac{qQ}{m}\right)^2}$ or $\mathcal{E} = \frac{\pi}{3}\frac{mc^2}{\beta}\left(\frac{r_0}{\rho}\right)^3\left(\frac{Q}{q}\right)^2 \int \beta = v/c, \ r_0 = \frac{q^2}{mc^2}$. $\qquad\underline{(2)}$

This is the required result: $\mathcal{E}$ = total energy radiated by q in a "soft" collision. In the second form in Eq. (2), r_0 has dimensions of length (it is the classical EM radius of q), so $\mathcal{E} \propto mc^2$ is manifestly an energy.

2. The ratio of $\mathcal{E}\left(\begin{smallmatrix}\text{radn}\\\text{loss}\end{smallmatrix}\right)$ to $\frac{1}{2}mv^2\left(\begin{smallmatrix}\text{initial}\\\text{energy}\end{smallmatrix}\right)$ is ...

$$\rightarrow R = \mathcal{E}/\tfrac{1}{2}mv^2 = \frac{2\pi}{3}\left(\frac{Q}{q}\right)^2\left(\frac{r_0}{\beta\rho}\right)^3 \qquad \underline{(3)}$$

NOTE This whole approach fails when $\beta \to 0$, since then it is no longer possible to satisfy the "soft" collision condition: $\frac{1}{2}mv^2 \gg qQ/\rho \Rightarrow \beta^2 \gg (2Q/q)(r_0/\rho)$. But if $Q(\text{proton}) = e = |q(\text{electron})|$, then $r_0 = e^2/mc^2 = 2.82\times10^{-5}$ Å, and if $\rho = 1$ Å, then: $r_0/\rho = 2.82\times10^{-5}$, and we only need $\beta \gg 0.0075$. For an electron @ kinetic energy: $K = 100$ keV...

$$\beta = \frac{\sqrt{2k+k^2}}{1+k}, \quad k = \frac{K}{mc^2} = \frac{100}{511}, \quad \text{so}\ \underline{\beta = 0.548}\ (e\ @\ 100\ \text{keV}). \qquad \underline{(4)}$$

With this, Eq. (3) gives loss ratio: $R = \frac{2\pi}{3}(r_0/\beta\rho)^3 = 2.85\times10^{-13}$, for a single collision, and $R = 0.1$ only after $\underline{N \sim 3.5\times10^{11}}$ such collisions. A 100 keV electron does not radiate much energy by soft collisions in a material.

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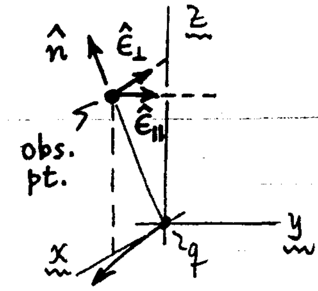
Prob$^{\underline{m}}$ 43 [[redacted], Jk$^{\underline{n}}$ #(14.15)]. Analyse polarization of synchrotron radiation.



(a) To keep track of the "helicity" of the synchrotron radiation, we
5pts project the contributions to $d^2I/d\Omega\,d\omega$ [energy per unit solid ∡ and unit frequency interval] on the directions...

$$\hat{\epsilon}_{L,R} = \frac{1}{\sqrt{2}}(\hat{\epsilon}_\perp \mp \hat{\epsilon}_\parallel) \leftrightarrow \hat{\epsilon}_{\perp,\parallel} = \frac{1}{\sqrt{2}}(\hat{\epsilon}_L \pm \hat{\epsilon}_R),$$

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w/ $\hat{\epsilon}_L$ rotating @ (+) helicity (left circ. polarization), $\hat{\epsilon}_R$ rotating @ (−) helicity (right circ. polarization). (1)

Then the integrand of $d^2I/d\Omega d\omega$ in Jk$^{\underline{n}}$ #(14.78) is...

$$-\hat{\epsilon}_\parallel A_\parallel + \hat{\epsilon}_\perp A_\perp = \frac{1}{\sqrt{2}}\left[(A_\perp + A_\parallel)\hat{\epsilon}_L + (A_\perp - A_\parallel)\hat{\epsilon}_R\right]$$

So// $|-\hat{\epsilon}_\parallel A_\parallel + \hat{\epsilon}_\perp A_\perp|^2 = \frac{1}{2}|A_\parallel \pm A_\perp|^2$ { (+) for (+)ve helicity [left circ. polz$^{\underline{n}}$]; (−) for (−)ve helicity [right circ. polz$^{\underline{n}}$] } (2)

The rest of the calculation in Eqs. (14.78) → (14.83) goes through, giving quoted result//

$$\boxed{\frac{d^2I_\pm}{d\Omega d\omega} = \frac{q^2}{6\pi^2 c}\left(\frac{\omega\rho}{c}\right)^2\left(\theta^2+\frac{1}{\gamma^2}\right)^2\left[K_{\frac{2}{3}}(\xi) \pm \frac{\theta}{\sqrt{\theta^2+1/\gamma^2}}K_{\frac{1}{3}}(\xi)\right]^2}$$

for (±) helicity, $\xi = \left(\frac{\omega\rho}{3c}\right)\left(\theta^2+\frac{1}{\gamma^2}\right)^{\frac{3}{2}}$. (3)

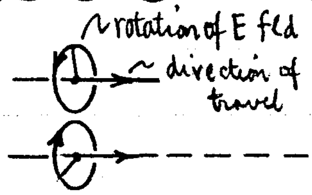
(b) Looking now for linear polarization (along $\hat{\epsilon}_\parallel$ & $\hat{\epsilon}_\perp$) rather than circular
7pts polarization (along $\hat{\epsilon}_L$ & $\hat{\epsilon}_R$), follow the derivation on Jk$^{\underline{n}}$ p. 674 to see that:
$d^2I = d^2I_\parallel + d^2I_\perp$, where: $d^2I_\parallel \propto A_\parallel^2 \propto K^2_{\frac{2}{3}}(\xi)$, $d^2I_\perp \propto A_\perp^2 \propto \left(\frac{\theta^2}{\theta^2+1/\gamma^2}\right)K^2_{\frac{1}{3}}(\xi)$.

Then the desired (net fractional) polarization is defined by...

$$\boxed{P(\theta) = \frac{d^2I_\parallel - d^2I_\perp}{d^2I_\parallel + d^2I_\perp} = \frac{(\theta^2+\gamma^{-2})K^2_{2/3}(\xi) - \theta^2 K^2_{1/3}(\xi)}{(\theta^2+\gamma^{-2})K^2_{2/3}(\xi) + \theta^2 K^2_{1/3}(\xi)}} \quad (4)$$

This is exact. The 3 approximate cases of interest examine the high frequency polarization (@ $\omega > \omega_c = 3\gamma^3(c/\rho)$), and low freq ($\omega < \omega_c$) at large & small θ.

See Jk$^{\underline{n}}$, p. 274... (±) helicity ⇒ (left/right) circular polarization of the light



{ left circularly polarized light ⇒ (+) helicity;
{ right circularly polarized light ⇒ (−) helicity.

[redacted]

43 (cont'd)

[1] $\omega \gg \omega_c \Rightarrow \xi \gg 1$, and $K_\nu^2(\xi) \simeq \frac{\pi}{2\xi} e^{-2\xi}$ (for both $\nu = \frac{1}{3}$ & $\frac{2}{3}$)●. Then...

$$\boxed{P(\theta) \simeq 1/(1+2\gamma^2\theta^2)}, \text{ for } \omega > \omega_c \text{ and all } \measuredangle\text{s } \theta. \tag{5}$$

This also covers the case $\omega \sim \omega_c$ and $\theta \gg 1/\gamma$.

Since $\xi = \frac{\omega}{\omega_c}(1+\gamma^2\theta^2)^{3/2}$, then if $\theta \to$ large, ξ will still be large if $\omega \sim \omega_c$. So $P(\theta)$ of Eq.(5) also covers the case $\omega \lesssim \omega_c$, but $\gamma\theta \gg 1$. Thus...

[2] $\omega \lesssim \omega_c$, $\theta \gg 1/\gamma \Rightarrow$

$$\boxed{P(\theta) \simeq 1/[1+2(\gamma\theta)^2]}, \quad \theta \gg 1/\gamma. \tag{6}$$

Finally if $\omega < \omega_c$, and $\theta \to$ small ($\theta \ll 1/\gamma$), $\xi \to$ small. Then, in Eq.(4), $K_\nu(\xi) \simeq \frac{1}{2}\Gamma(\nu)[2/\xi]^\nu$, by NBS # (9.6.9). The polarization becomes...

[3] $\omega < \omega_c$, $\theta \ll 1/\gamma$ (near orbit plane)

$$\boxed{P(\theta) \simeq \left[A\left(\frac{\omega}{\omega_c}\right)^{\frac{2}{3}} - \gamma^2\theta^2\right] \Big/ \left[A\left(\frac{\omega}{\omega_c}\right)^{\frac{2}{3}} + \gamma^2\theta^2\right]}, \quad \theta \ll 1/\gamma, \tag{7}$$

w/ $A = 2^{\frac{1}{3}}\Gamma(\frac{2}{3})/\Gamma(\frac{1}{3}) = 0.4056$.

This $\Rightarrow P(\theta) \simeq 1 - \frac{2}{A}(\gamma\theta)^2\left(\frac{\omega_c}{\omega}\right)^{\frac{2}{3}}$, or $\sim 100\%$ polarization near orbit plane.

(c) From P. Joos (Phys. Rev. Lett. 4 558 (1960)), the electron energy was 700 MeV,
3pts so $\gamma = 700/0.511 = 1370$. The measured θ range was $0 \lesssim \theta \lesssim 6$ mrad (referred to orbit plane), so that: $0 \lesssim \gamma\theta \lesssim 8.22$. Joos measured separately: $d^2I_\parallel \propto (1+\gamma^2\theta^2)K_{2/3}^2(\xi)$, and: $d^2I_\perp \propto \theta^2 K_{1/3}^2(\xi)$; see Joos' Fig. 2. The agreement between experiment & theory was within 25%.

● $K_\nu(\xi) \sim \sqrt{\frac{\pi}{2\xi}}\, e^{-\xi}\left\{1 + \frac{4\nu^2-1}{8\xi} + \dots\right\}$, by NBS Handbook # (9.7.2)

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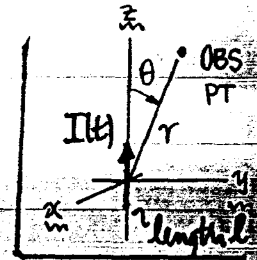
(44) [redacted] Analyse freq. spectrum for a 1D arc discharge.



5pts (A) Poynting vector as derived in class [Eq. (18) of notes of 1/27/89]:

$$\longrightarrow S(r,t) = \left(\frac{\sin^2\theta}{4\pi r^2}\right)\frac{1}{c^3}\left[\dot{I}(t')\,l\right]^2 \leftarrow \text{energy/unit time \& area at observer.} \tag{1}$$

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The energy/unit time & solid ∡ is $r^2 S$, and if we integrate this over all t, we get

$$\longrightarrow \frac{d\mathcal{E}}{d\Omega} = \int_{-\infty}^{\infty} r^2 S(r,t)\,dt = \left(\frac{\sin^2\theta}{4\pi c^3}\right) l^2 \int_{-\infty}^{\infty}\left[\dot{I}(t')\right]^2 dt' \leftarrow \begin{matrix}\text{radiated energy}\\ \text{per unit solid ∡.}\end{matrix} \tag{2}$$

The integral can be converted to an integration over a freq. variable ω by means of Parseval's Theorem for Fourier Integrals*, which states...

$$\longrightarrow \int_{-\infty}^{\infty}|F(t)|^2 dt = \int_{-\infty}^{\infty}|f(\omega)|^2 d\omega, \text{ if } f(\omega) = (1/\sqrt{2\pi})\int_{-\infty}^{\infty}F(t)\,e^{-i\omega t}dt. \tag{3}$$

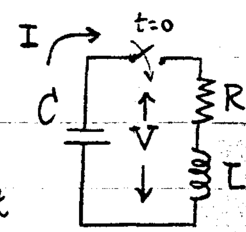
We identify F(t) in Eq. (3) with $\dot{I}(t)$ in Eq. (2), so we can write...

$$\longrightarrow \frac{d\mathcal{E}}{d\Omega} = \int_{-\infty}^{\infty}\sigma(\omega)\,d\omega, \quad \sigma(\omega) = \left(\frac{\sin^2\theta}{4\pi c^3}\right)\frac{l^2}{2\pi}\left|\int_{-\infty}^{\infty}\dot{I}(t)\,e^{-i\omega t}dt\right|^2. \tag{4}$$

σ(ω) is evidently the radiated energy per unit solid ∡, per unit frequency; it is what Jackson calls the frequency-angle spectrum. Thus we have, as desired...

$$\boxed{\frac{d^2 I}{d\omega\, d\Omega} = \sigma(\omega) = \left(\frac{\sin^2\theta}{8\pi^2 c^3}\right) l^2\omega^2\left|\int_0^{\infty} I(t)\,e^{-i\omega t}dt\right|^2.} \tag{5}$$

Two details in going Eq. (4) → (5): the {F.T. of $\dot{I}(t)$} → iω {F.T. of I(t)}, by partial integration (for any I(t) which vanishes as t → ±∞); and the lower limit on the integral is put to zero (for I(t)'s which vanish @ t < 0).



3pts (B) For the passive CRL cct described, the cct eqtns are: $I = -C\dot{V}$, $V = RI + L\dot{I}$. It is easily verified that the solution for the current

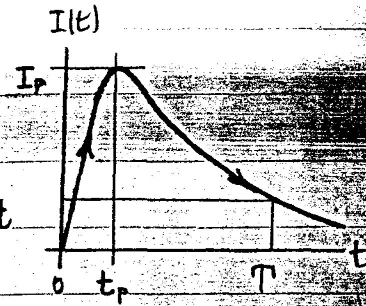
* G. Arfken "Math Methods for Physicists" (Academic Press, 3rd ed, 1985), Eq. (15.55).

* Since we integrate over all times, the distinction between t & $t' = t - \frac{r}{c}$ is unimportant.

[redacted] most of the material in parts (A)-(C) is worked out in R. Robiscoe & Z. Sui, J. Appl. Phys. 64, 4364 (1988).

I which results from the initial conditions: $V(0)=V_0$, $I(0)=0$, is just the given:

$$\rightarrow I(t) = \frac{V_0}{L\Gamma} e^{-\gamma t} \sinh \Gamma t \quad \int \gamma = R/2L, \text{ and } \Gamma = \sqrt{\gamma^2 - (1/LC)}. \qquad (6)$$



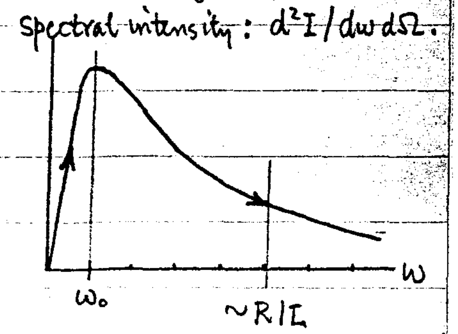
$I(t)$ shows no oscillations so long as Γ is real, i.e. so long as $CR^2/L > 4$; this is the condition for overdamping. The small-t behavior is: $I(t) \sim (V_0/2L)t$, while as $t \to$ large we have: $I(t) \sim (V_0/2L\Gamma) e^{-(\gamma-\Gamma)t}$. Roughly speaking: $I(t)$ goes through a peak ($I_P \sim V_0/R$) at time $\boxed{t_P \sim L/R}$ (which is the "risetime"), then falls off exponentially in a characteristic time (i.e. "duration") $\boxed{T \sim RC}$. The overall behavior is sketched.

4pts (C) For $I(t) = (V_0/L\Gamma) e^{-\gamma t} \sinh \Gamma t$, the spectrum of Eq. (5) requires the F.T.:

$$\int_0^\infty I(t) e^{-i\omega t} d\omega = \frac{V_0}{L\Gamma} \int_0^\infty e^{-\gamma t}\left(\frac{e^{\Gamma t} - e^{-\Gamma t}}{2}\right) e^{-i\omega t} dt = \frac{V_0/L}{(\omega_0^2 - \omega^2) + 2i\gamma\omega}, \qquad (7)$$

where $\omega_0 = 1/\sqrt{LC}$ is the cct natural frequency. Taking the absolute square, find...

$$\boxed{\frac{d^2 I}{d\omega d\Omega} = \left(\frac{\sin^2\theta}{8\pi^2 c^3}\right) \frac{\ell^2 V_0^2}{R^2} \left[\frac{4\gamma^2\omega^2}{(\omega_0^2-\omega^2)^2 + 4\gamma^2\omega^2}\right]}. \qquad (8)$$



As a fcn of ω, the spectrum peaks @ $\omega = \omega_0$, then falls off slowly [as $\sim(\omega_0/\omega)^2$]. Beyond $\omega = \omega_0$, the spectrum does not fall to $1/n$ of its peak value until $\omega \sim 2\sqrt{n}(R/L)$; if $n = 10$ (i.e. if $d^2I/d\omega d\Omega$ is detectable out to 10% of its peak value), then spectrum freq. range is $0 \leq \omega \lesssim 7(R/L)$.

3pts (D) From Eq. (2), with $d\Omega = 2\pi \sin\theta\, d\theta$, the total arc radiation energy is...

$$\rightarrow \mathcal{E}_{rad} = \int_{4\pi} d\Omega \int_{-\infty}^{\infty} r^2 S\, dt = \frac{\ell^2}{2c^3} \int_0^\pi \sin^3\theta\, d\theta \int_{-\infty}^{\infty} [\dot{I}(t)]^2 dt = \frac{\ell^2}{2c^3} \cdot \frac{4}{3} \cdot \int_0^\infty [\dot{I}(t)]^2 dt. \qquad (9)$$

To get MKS units, the RHS must be multiplied by $(1/4\pi\epsilon_0)$. Then, for the pulse: $I(t) = (V_0/L\Gamma) e^{-\gamma t} \sinh\Gamma t$, calculate: $\int_0^\infty \dot{I}^2 dt = \frac{1}{4\gamma}(V_0/L)^2$, so that

$= 1/\sqrt{\mu_0\epsilon_0}$ (MKS) use

$$\rightarrow \mathcal{E}_{rad} = \left(\frac{1}{4\pi\epsilon_0}\right) \cdot \frac{2\ell^2}{3c^3} \cdot \frac{1}{4\gamma}(V_0/L)^2 = \ldots, \quad \text{w/} \quad \boxed{\mathcal{E}_{rad} = \frac{1}{12\pi} \frac{Z_0}{R} \left(\frac{\omega_0 \ell}{c}\right)^2 C V_0^2}. \qquad (10)$$

Here $Z_0 = \sqrt{\mu_0/\epsilon_0} = 377\,\Omega$, and $\omega_0 = 1/\sqrt{LC}$. The total discharge energy $\mathcal{E}_{dis} = \int_0^\infty R I^2 dt = \frac{1}{2} C V_0^2$ (clearly), so: $\underline{\mathcal{E}_{rad}/\mathcal{E}_{dis} = (Z_0/6\pi R)(\omega_0 \ell/c)^2}$. This ratio $< 10^{-6}$, typically.