ϕ519 Final Exam { in class, 3 hr limit } Mon. 16 Dec. 1991

This exam is open-book, open-notes. It consists of 7 problems (point values marked) worth 225 points. At end of each problem, box or underline your answer. Number your solution pages sequentially, write your name on p.1, and staple pages together before handing in.

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① [30 pts.]. For the electromagnetic field $(\mathbb{E}, \mathbb{B})$ in vacuum, verify the conservation law:

$$\boxed{\nabla\cdot\mathbb{Q} + \frac{\partial\xi}{\partial t} = 0} \quad \text{w/} \quad \mathbb{Q} = \mathbb{E}\times\dot{\mathbb{E}} + \mathbb{B}\times\dot{\mathbb{B}}, \quad \xi = \mathbb{E}\cdot(\nabla\times\mathbb{E}) + \mathbb{B}\cdot(\nabla\times\mathbb{B}).$$

The dot means $\frac{\partial}{\partial t}$. Discuss this "continuity equation" for a linearly polarized light wave, where -- during propagation -- $\mathbb{E}$ & $\mathbb{B}$ maintain fixed space directions.

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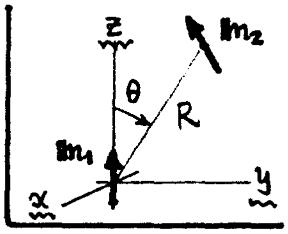
② [35 pts.]. Two small magnets, of dipole moments $\mathbb{m}_1$ & $\mathbb{m}_2$, interact at distance R between their centers; $R \gg$ magnet dimensions. $\mathbb{m}_1$ is held fixed at the origin, pointing along the z-axis, while $\mathbb{m}_2$ is oriented arbitrarily, at position (R, θ) as shown in sketch.



(A) Calculate the radial component of the force, F_R, exerted by $\mathbb{m}_1$ on $\mathbb{m}_2$.

(B) If $\mathbb{m}_2$ is held $\parallel \mathbb{m}_1$, show that the radial force F_R can be attractive or repulsive, depending on $\mathbb{m}_2$'s location. Find ranges of θ for attraction vs. repulsion.

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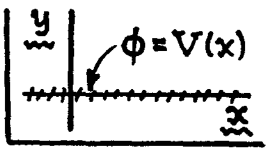
③ [25 pts.]. Let $\phi(x,y)$ be the electrostatic potential for a 2D problem. Suppose $\phi(x,0) = V(x)$ is specified on the x-axis, and ϕ has the symmetry $\phi(x,-y) = \phi(x,y)$. Show: $\underline{\phi(x,y) = \sum_{n=0}^{\infty} A_n y^{2n}[\partial^{2n}V(x)/\partial x^{2n}]}$, $A_n = \frac{(-)^n}{(2n)!}$.



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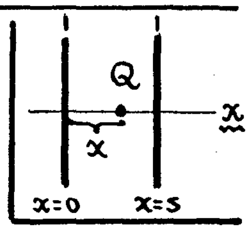
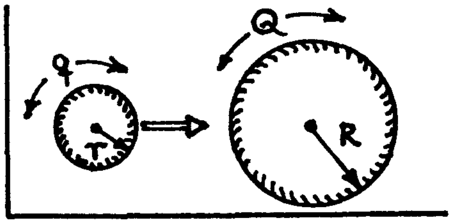
④ [35 pts.]. Two arbitrarily large, grounded, conducting planes are parallel to one another, intersecting the x-axis at $x=0$ & $x=s$. A point charge Q is placed at distance x from the left-hand plane $(0<x<s)$.



(A) Find the force $F(x)$ acting on Q in the form of an infinite series.

(B) Sketch a graph of $F(x)$ vs. x over $0 \le x \le s$. Use symmetry arguments to find a numerical value for the series: $\sum_{n=1}^{\infty} \frac{n}{(4n^2-1)^2}$.

(5) [40 pts.]. Consider two conducting spherical shells, of radii r and $R \gg r$, which -- except for the charge transfers described -- are isolated from their surroundings. Initially the R-shell is uncharged ($Q=0$), and the r-shell bears charge q. The r-shell is now brought into <u>contact</u> with the R-shell, which thereby acquires some charge ($Q>0$). The r-shell is then removed to a remote place, charged back up to total charge q, and again brought back to contact the R-shell. This charging procedure is repeated ad infinitum.



(A) After 1st contact, show the R-shell acquires charge: $Q = \beta q$, w/ $\beta = R/(R+r)$.

(B) Find a recursion relation between the R-shell charges Q_{n+1} (after $n+1$ contacts) and Q_n (after n contacts). Here $n = 0, 1, 2, 3, \ldots$ and $Q_0 = 0$.

(C) By iterating the relation in part (B), find Q_{n+1} explicitly in terms of q, r & R. For $n \to \infty$, what is the charge and potential of the R-shell? **COMMENT.**

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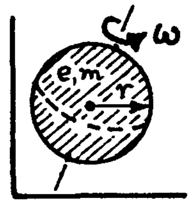
(6) [30 pts.]. Picture the electron as a uniform sphere of mass m, charge e and radius r, which is spinning about an axis at constant angular velocity ω.



(A) Calculate the magnetic moment m_s for this spinning charge.

(B) The measured electron moment is $\simeq e\hbar/2mc$ [$\hbar$ = Planck const]. Equate this to m_s of part (A), put in the classical electron radius $r = e^2/mc^2$, and find a <u>number</u> for the electron's equatorial velocity $v = r\omega$. Is v "reasonable"? **HINT**: $\alpha = e^2/\hbar c \simeq 1/137$.

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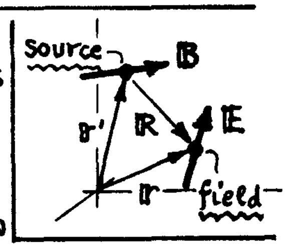
(7) [30 pts.]. Consider a region V of space where the magnetic field changes with time as $\dot{\mathbb{B}}(\mathbf{r},t)$ [the dot $\Rightarrow \partial/\partial t$]. Let $\mathbb{R} = \mathbf{r}\binom{\text{field}}{\text{point}} - \mathbf{r}'\binom{\text{source}}{\text{point}}$.



(A) Treat time t as "just" a parameter. Starting from Faraday's Law, show that the induced electric field is: $\underline{\underline{\mathbb{E}(\mathbf{r},t) = \frac{1}{4\pi c}\int_V \frac{d^3x'}{R^3}\left[\mathbb{R}\times\dot{\mathbb{B}}(\mathbf{r}',t)\right]}}$.

(B) In part (A), $\mathbb{E}$ & $\dot{\mathbb{B}}$ are running on the <u>same</u> clock-time t. Comment on why this must be so, or why this cannot be so.

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● [30 pts.]. For EM fields in vacuo, show: $\nabla\cdot\mathbf{Q} + \dfrac{\partial\xi}{\partial t} = 0 \left\{ \begin{array}{l} \mathbf{Q} = \mathbf{E}\times\dot{\mathbf{E}} + \mathbf{B}\times\dot{\mathbf{B}}, \\ \xi = \mathbf{E}\cdot(\nabla\times\mathbf{E}) + \mathbf{B}\cdot(\nabla\times\mathbf{B}). \end{array}\right.$

1) Maxwell's Eqs. for $\mathbf{E}$ & $\mathbf{B}$ fields in vacuo (sources ρ & $\mathbf{J} \equiv 0$) are:

$$\left[\begin{array}{ll} \nabla\cdot\mathbf{E} = 0, & \nabla\times\mathbf{E} = -\frac{1}{c}\dot{\mathbf{B}} \\ \nabla\cdot\mathbf{B} = 0, & \nabla\times\mathbf{B} = +\frac{1}{c}\dot{\mathbf{E}} \end{array}\right\} \text{the dot} \Rightarrow \frac{\partial}{\partial t}. \tag{1}$$

From these, we can form the quantity...

$$\xi = \mathbf{E}\cdot(\nabla\times\mathbf{E}) + \mathbf{B}\cdot(\nabla\times\mathbf{B}) = \frac{1}{c}\left[-\mathbf{E}\cdot\dot{\mathbf{B}} + \mathbf{B}\cdot\dot{\mathbf{E}}\right]$$

$$\xrightarrow{\text{So}} \frac{\partial\xi}{\partial t} = -\frac{1}{c}(\mathbf{E}\cdot\ddot{\mathbf{B}} - \ddot{\mathbf{E}}\cdot\mathbf{B}). \tag{2}$$

2) With use of the identity: $\nabla\cdot(\mathbf{M}\times\mathbf{N}) = \mathbf{N}\cdot(\nabla\times\mathbf{M}) - \mathbf{M}\cdot(\nabla\times\mathbf{N})$, the other quantity in the required conservation law is...

$$\nabla\cdot\mathbf{Q} = \nabla\cdot(\mathbf{E}\times\dot{\mathbf{E}} + \mathbf{B}\times\dot{\mathbf{B}}) = \cancel{\dot{\mathbf{E}}\cdot\underbrace{(\nabla\times\mathbf{E})}_{=-\frac{1}{c}\dot{\mathbf{B}}}} - \mathbf{E}\cdot(\nabla\times\dot{\mathbf{E}}) + \tag{3}$$

1st & 3rd terms RHS cancel, and so...

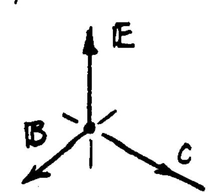
$$+ \cancel{\dot{\mathbf{B}}\cdot\underbrace{(\nabla\times\mathbf{B})}_{=+\frac{1}{c}\dot{\mathbf{E}}}} - \mathbf{B}\cdot(\nabla\times\dot{\mathbf{B}}).$$

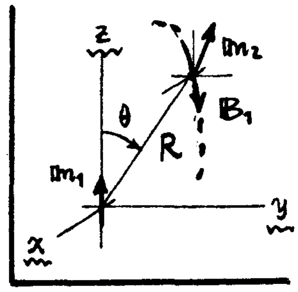
$$\rightarrow \nabla\cdot\mathbf{Q} = -\mathbf{E}\cdot\frac{\partial}{\partial t}\underbrace{(\nabla\times\mathbf{E})}_{=-\frac{1}{c}\dot{\mathbf{B}}} - \mathbf{B}\cdot\frac{\partial}{\partial t}\underbrace{(\nabla\times\mathbf{B})}_{=+\frac{1}{c}\dot{\mathbf{E}}} = \frac{1}{c}(\mathbf{E}\cdot\ddot{\mathbf{B}} - \mathbf{B}\cdot\ddot{\mathbf{E}}). \tag{4}$$

Comparing Eqs. (4) & (2), we see we have the desired identity...

$$\boxed{\nabla\cdot\mathbf{Q} + \frac{\partial\xi}{\partial t} = 0} \int^{\text{w/}} \begin{array}{l} \mathbf{Q} = \mathbf{E}\times\dot{\mathbf{E}} + \mathbf{B}\times\dot{\mathbf{B}}, \\ \xi = \mathbf{E}\cdot(\nabla\times\mathbf{E}) + \mathbf{B}\cdot(\nabla\times\mathbf{B}). \end{array} \tag{5}$$

3) For a linearly polarized wave, both $\mathbf{E}$ & $\dot{\mathbf{E}}$ and $\mathbf{B}$ & $\dot{\mathbf{B}}$ are collinear, so $\mathbf{Q} \equiv 0$. Then: $\xi = \frac{1}{c}(\dot{\mathbf{E}}\cdot\mathbf{B} - \mathbf{E}\cdot\dot{\mathbf{B}})$ is conserved. In fact $\xi \equiv 0$ in this case, since $\mathbf{E}$ is $\perp$ $\mathbf{B}$ & $\dot{\mathbf{B}}$, and $\mathbf{B}$ is $\perp$ $\mathbf{E}$ & $\dot{\mathbf{E}}$. For a circularly polarized, however, $\mathbf{Q}$ and ξ are both nontrivial.





[35 pts.]. Analyse interaction between two small magnets.

(A) 1) $\mathbf{m}_1 = m_1\hat{z}$ generates a dipole field $\mathbf{B}_1$ [Jk^n Eq.(5.56)], which -- at the site of $\mathbf{m}_2$ -- can be written...

$$\rightarrow \mathbf{B}_1 = \frac{m_1}{R^3}\left[(3\cos\theta)\hat{n} - \hat{z}\right], \qquad (1)$$

where $\hat{n}$ is a unit vector $\mathbf{R}/R$ between centers. We've used $\mathbf{m}_1\cdot\hat{n} = m_1\cos\theta$.

2) The energy of $\mathbf{m}_2$ in the field $\mathbf{B}_1$ is given by [Jk^n Eq.(5.72)]...

$$\rightarrow U_2 = -\mathbf{m}_2\cdot\mathbf{B}_1 = -\frac{m_1}{R^3}\left[(3\cos\theta)\,\hat{n}\cdot\mathbf{m}_2 - \hat{z}\cdot\mathbf{m}_2\right]. \qquad (2)$$

The force on $\mathbf{m}_2$ by $\mathbf{m}_1$ is $\mathbf{F}_{1on2} = -\nabla U_2 = +\nabla(\mathbf{m}_2\cdot\mathbf{B}_1)$ [Jk^n Eq.(5.69)], and the radial part is $F_R = \frac{\partial}{\partial R}(\mathbf{m}_2\cdot\mathbf{B}_1)$. By Eq.(2), this is...

$$\boxed{F_R = -\frac{3m_1}{R^4}\left[(3\cos\theta)\,\hat{n}\cdot\mathbf{m}_2 - \hat{z}\cdot\mathbf{m}_2\right]}. \qquad (3)$$

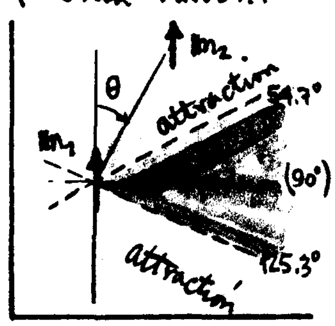
We've used the fact that there is no R dependence in the [ ] in Eq.(2). The factors $(\hat{n}\cdot\mathbf{m}_2)$ & $(\hat{z}\cdot\mathbf{m}_2)$ can depend only on the $\measuredangle$ orientation of $\mathbf{m}_2$ w.r.t. $\mathbf{m}_1$, and they will determine the sign of F_R, i.e. whether F_R is attractive or repulsive.

(B) 3) If $\mathbf{m}_2$ is $\parallel$ $\mathbf{m}_1$, then $\mathbf{m}_2 = m_2\hat{z}$, and the radial force in Eq.(3) becomes...

$$F_R = -\frac{3m_1m_2}{R^4}\left[3\cos^2\theta - 1\right] = \begin{cases}\text{attractive for: } \cos\theta > 1/\sqrt{3},\\ \text{repulsive for: } \cos\theta < 1/\sqrt{3}.\end{cases} \qquad (4)$$

So the magnets can attract or repel, depending on the $\measuredangle$ position θ of $\mathbf{m}_2$ relative to $\mathbf{m}_1$. The key $\measuredangle$ is $\cos\theta = 1/\sqrt{3} \Rightarrow \theta = 54.7°$. Then have...

$$F_R = \begin{cases}\text{attractive for: } 0 \leq \theta < 54.7°,\ 125.3 < \theta \leq 180°;\\ \text{zero for } \theta = 54.7° \ \&\ \theta = 125.3°;\\ \text{repulsive for: } 54.7 < \theta < 125.3°.\end{cases} \qquad (5)$$



This is for $\mathbf{m}_2 \parallel \mathbf{m}_1$, and the $\measuredangle$ θ as sketched above.

[25 pts.]. Find (verify) a solution to a 2D potential problem.

1) The profferred solution, viz.

$$\rightarrow \phi(x,y) = \sum_{n=0}^{\infty} A_n\, y^{2n}\,[\partial^{2n}V(x)/\partial x^{2n}], \quad \underline{\underline{A_n}} = (-)^n/(2n)! \tag{1}$$

obeys the symmetry $\phi(x,-y) = \phi(x,y)$, and it also satisfies the boundary condition: $\phi(x,0) = A_0[\partial^0 V(x)/\partial x^0] = V(x)$, since $A_0 = 1$.

2) To show that ϕ of Eq. (1) is actually a solution to the 2D problem, we need only show that it satisfies Laplace' Eqtn: $\nabla^2\phi = 0$.* So we look at...

$$\rightarrow \nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = \sum_n A_n\, y^{2n}\frac{\partial^{2n+2}}{\partial x^{2n+2}}V(x) + \sum_n A_n\cdot 2n(2n-1)\,y^{2n-2}\frac{\partial^{2n}}{\partial x^{2n}}V(x). \tag{2}$$

The 2nd sum RHS contributes zero for $n=0$. We step the summation variable by one: $n=m+1$, w/ $m=0,1,2,\ldots$ and collect like terms. Then

$$\rightarrow \nabla^2\phi = \sum_{n=0}^{\infty}\left[A_n + (2n+2)(2n+1)A_{n+1}\right] y^{2n}\frac{\partial^{2n+2}}{\partial x^{2n+2}}V(x). \tag{3}$$

3) $\nabla^2\phi = 0$ if the coefficient $[\;] = 0$ for all n. If we were constructing the solution, the condition $[\;] = 0$ would serve as a recursion relation defining the A_n's. As it is, with A_n given per Eq. (1)...

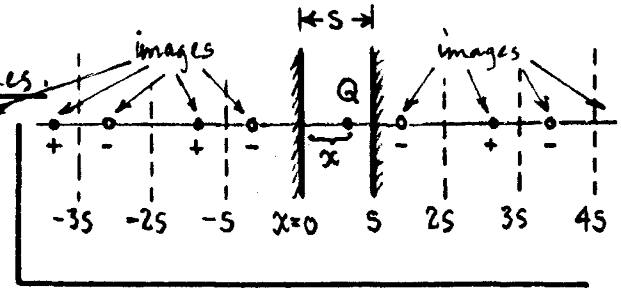
$$\rightarrow [\text{Eq.}(3)] = \frac{(-)^n}{(2n)!} + (2n+2)(2n+1)\frac{(-)^{n+1}}{(2n+2)!} = \frac{(-)^n}{(2n)!}\left\{1 - \frac{(2n+2)(2n+1)(2n)!}{(2n+2)!}\right\} = 0. \tag{4}$$

So, in Eq. (3), $\underline{\underline{\nabla^2\phi = 0}}$, and $\phi(x,y)$ is indeed the (unique) solution to this particular 2D potential problem.

QED

* It is tacitly assumed there are no free charges present, i.e. $\rho(\text{surface}) \equiv 0$.

(1) [35 pts.]. Force on Q between ∞ conducting planes.

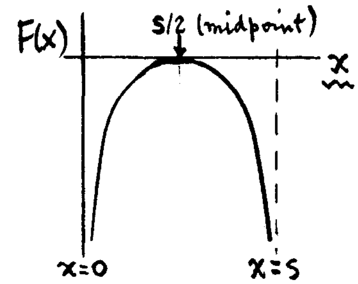


(A) Do problem by images. Q and its images comprise an ∞ number of pairs of charges $\pm Q$, located at positions: $x_n = 2ns \pm x$, w/ $n = 0, \pm 1, \pm 2, \pm 3 \ldots$

The $+Q$ images at $2s+x, 4s+x, 6s+x, \ldots$ act on the original Q (at x) with a force F_1 directed to the left (i.e. (−) direction), w/ $F_1 = (-)\sum_{n=1}^{\infty} Q^2/(2ns)^2$. This is exactly cancelled by the force F_2 acting on Q to the right due to the $+Q$ images at $-2s+x, -4s+x, -6s+x, \ldots$ since $F_2 = +\sum_{n=1}^{\infty} Q^2/(2ns)^2$. At this point, the $+Q$ images drop out of the problem... the $+Q$ images exert no net force on Q.

The $-Q$ images do exert a net force on Q... those at $2s-x, 4s-x, 6s-x, \ldots$ exert a force $F_3 = +\sum_{n=1}^{\infty} Q^2/(2ns-2x)^2$, while $F_4 = (-)\sum_{n=0}^{\infty} Q^2/(2ns+2x)^2$ is exerted by the $-Q$ images at $-x, -2s-x, -4s-x, \ldots$ The net force on Q is...

$$F(x) = F_3 + F_4 = -\frac{Q^2}{4x^2} + \sum_{n=1}^{\infty}\left[\frac{Q^2}{(2ns-2x)^2} - \frac{Q^2}{(2ns+2x)^2}\right],$$



or $$\boxed{F(x) = -\frac{Q^2}{4x^2} + Q^2 s x \sum_{n=1}^{\infty} \frac{n}{[(ns)^2 - x^2]^2}} \quad (1)$$

(B) F(x) evidently diverges to (−) ∞ as $x \to 0$, and -- since Q has no way of distinguishing between the left hand ($x=0$) plane and righthand ($x=s$) plane, we must also have $F(x) \to (-)\infty$ as $x \to s$. For this reason, F(x) must be symmetric about the midpoint ($x = s/2$). Also, when Q is at $x = s/2$, the (−) Q images to the left and to the right are symmetrically disposed, so $F(x = \frac{s}{2}) = 0$. The resulting F(x) vs. x is sketched above. Note that $x = s/2$ is an unstable equilibrium point.

The fact that $F(x=\frac{s}{2}) = 0$ has an amusing consequence. In Eq. (1), it means...

$$\rightarrow F(x=\tfrac{s}{2}) = -\frac{Q^2}{s^2} + \frac{Q^2 s^2}{2}\sum_{n=1}^{\infty}\frac{n}{[(ns)^2 - \frac{1}{4}s^2]^2} = 0, \text{ or } \boxed{\sum_{n=1}^{\infty}\frac{n}{(4n^2-1)^2} = \frac{1}{8}}. \quad (2)$$

As a piece of arithmetic, this is even true (see Gradshteyn & Ryzhik, # (0.236.4)).

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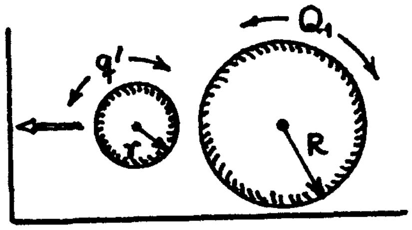
③ [35 pts]. Analyse repetitive charging a la Van de Graaf.



(A) After the 1st contact, R-shell has charge $Q_1 > 0$, and r-shell has $q' < q$. By conservation of charge, clearly: $q' + Q_1 = q$. Also, the shell potentials must be equal; at large separations: $q'/r = Q_1/R$.† Solution of these two equations simultaneously gives...

$$\left.\begin{array}{l} q' + Q_1 = q \\ q'/r = Q_1/R \end{array}\right\} \boxed{Q_1 = \beta q}, \text{ w/ } \underline{\underline{\beta}} = R/(R+r). \quad (1)$$

(B) Continuing with the ideas in part (A), after the $(n+1)^{st}$ contact, R-shell goes from Q_n to Q_{n+1}, while r-shell goes from q to q'_{n+1}. Then write...

$$\left.\begin{array}{l} \underline{\text{Charge conservation}}: q'_{n+1} + Q_{n+1} = q + Q_n \\ \underline{\text{potential equalization}}: q'_{n+1}/r = Q_{n+1}/R \end{array}\right\} \boxed{Q_{n+1} = \beta(q + Q_n)}. \quad (2)$$

Here $n = 0$ (no contacts), 1, 2, 3, ..., and evidently $Q_0 = 0$ as an initial condition.

(C) Iterate Eq. (2)...

$$Q_{n+1} = \beta(q + Q_n) = \beta[q + \beta(q + Q_{n-1})] = \beta[(1+\beta)q + \beta Q_{n-1}]$$

$$= \beta[(1+\beta)q + \beta^2(q + Q_{n-2})] = \beta[\underbrace{(1+\beta+\beta^2+\cdots+\beta^n)}_{= (1-\beta^{n+1})/(1-\beta) \text{ [Geometric Series]}}q + \beta^n \cancel{Q_0}^{\,0}]$$

$$\boxed{Q_{n+1} = \frac{\beta q}{1-\beta}(1-\beta^{n+1}) = \frac{R}{r}(1-\beta^{n+1})\, q}. \quad (3)$$

This is the charge on the R-shell after $(n+1)$ charging contacts. When $n \to \infty$, $\beta^{n+1} \to 0$ (since $\beta = R/(R+r) < 1$). The final condition of the R-shell is:

$$\boxed{\begin{array}{l} \text{charge}: Q_\infty = (R/r)q \gg q, \\ \text{potential}: V_\infty = Q_\infty/R = q/r. \end{array}} \quad (4)$$

For $R \gg r$, the R-shell can acquire a very large charge, and large potential discharge energy $Q_\infty^2/2R$. Van de Graaff's work this way.

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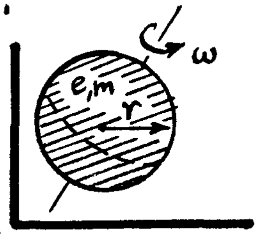
† Can think of shells joined by long thin wire, with a switch in it, rather than moving.

[30 pts.]. A classical model of the electron magnetic moment.



(A) All parts of this uniform-sphere electron model have the same charge-to-mass ratio e/m, so the magnetic moment generated by the rotation at ∡ velocity ω can be calculated by Jk$^{\underline{n}}$ Eq. (5.59):

$\rightarrow$ $\mathbf{m}_s = (e/2mc)\,\mathbf{L}$, or: $m_s = (e/2mc)\,L$, w/o attention to direction.† (1)

The rotational ∡ momentum is $L = I\omega$, where the moment of inertia of a solid sphere about a diameter is: $I = \frac{2}{5}mr^2$.★ Then...

$$\boxed{m_s = \frac{e}{2mc}\cdot\frac{2}{5}mr^2\omega = \frac{e}{5c}r^2\omega}. \quad (2)$$

This is the desired classical electron magnetic moment for a solid sphere.

(B) Per instruction...

$$\left.\begin{array}{l} m_s = e\hbar/2mc \\ r = e^2/mc^2 \end{array}\right\} \text{Eq. (2)} \Rightarrow \frac{e\hbar}{2mc} = \frac{e}{5c}\left(\frac{e^2}{mc^2}\right)^2\omega,$$

so $\omega = \frac{5}{2}\left(\frac{mc^2}{\hbar}\right)\frac{1}{\alpha^2}$, where: $\alpha = \frac{e^2}{\hbar c} \simeq \frac{1}{137}$. (3)

The equatorial velocity is...

$$v = r\omega = \frac{e^2}{\cancel{mc}}\cdot\frac{5}{2}\left(\frac{\cancel{mc}}{\hbar}\right)\frac{1}{\alpha^2} = \frac{5}{2}c/\alpha$$

so $\boxed{v/c = \frac{5}{2}/\alpha \simeq 343.}$ (4)

For all models of this sort, $v/c \sim$ several hundred, which is physically unreasonable. The classical electron (even with relativistic corrections) cannot rotate fast enough to produce its own magnetic moment. Spin is other-worldly.

★ For a spherical shell: $I = \frac{2}{3}mr^2$. The numerical factor is not important.

† Since electron charge $e = -|e|$ is negative, actually $\mathbf{m}_s$ is anti-∥ to $\mathbf{L}$.

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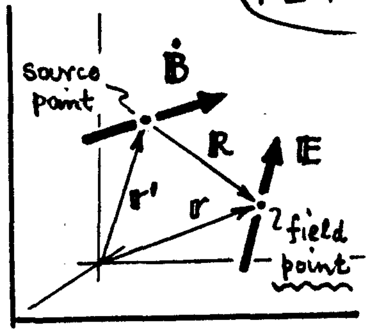

● [30pts.]. Show how $\partial\mathbb{B}/\partial t$ induces an $\mathbb{E}$-field.

1) Faraday's Law prescribes: $\nabla\times\mathbb{E} = -\frac{1}{c}\dot{\mathbb{B}}$, w/ dot $\Rightarrow \partial/\partial t$. The RHS of this equation is in effect a current density $\mathbb{J} = -\frac{1}{c}\dot{\mathbb{B}}$ which generates $\mathbb{E}$.

2) From Helmholtz' Theorem as proved in class, we know that the solution to $\nabla\times\mathbb{E} = \mathbb{J}$ is: $\mathbb{E} = \nabla\times\left(\frac{1}{4\pi}\int_V \frac{d^3x'}{R}\mathbb{J}\right)$, where $R = |\mathbb{r}-\mathbb{r}'|$ is the distance between field point & source point. The $\nabla\phi$ (scalar potential) part of $\mathbb{E}$ vanishes because there is no free charge density present. Putting $\mathbb{J} = -\frac{1}{c}\dot{\mathbb{B}}$ into the Helmholtz solution for $\mathbb{E}$, we can write

$$\longrightarrow \mathbb{E}(\mathbb{r},t) = (-)\frac{1}{4\pi c}\nabla\times\int_V \frac{d^3x'}{R}\dot{\mathbb{B}}(\mathbb{r}',t). \qquad (1)$$

3) The ∇ in Eq. (1) operates on the field points (x_i), not the source points (x_i'). When ∇ is moved inside the integral, we have...

$$\nabla\times\left(\frac{1}{R}\dot{\mathbb{B}}\right) = \underbrace{\left(\nabla\frac{1}{R}\right)}_{(-)\mathbb{R}/R^3 \text{ well-known identity;}}\times\dot{\mathbb{B}} + \frac{1}{R}\underbrace{\nabla\times\dot{\mathbb{B}}}_{0 \text{ because } \nabla \text{ doesn't operate on } \mathbb{r}'.} \qquad (2)$$

So//

$$\boxed{\mathbb{E}(\mathbb{r},t) = \frac{1}{4\pi c}\int_V \frac{d^3x'}{R^3}\left[\mathbb{R}\times\dot{\mathbb{B}}(\mathbb{r}',t)\right]}, \text{ as desired.} \qquad (3)$$

4) In this solution, both $\mathbb{E}$ (field) & $\dot{\mathbb{B}}$ (source) are running on the <u>same</u> time t... this carries over from Faraday's Law in differential form (local in space & time): $\nabla\times\mathbb{E}(\mathbb{r},t) = -\frac{1}{c}\dot{\mathbb{B}}(\mathbb{r},t)$; t is treated as just a parameter in getting to Eq. (3). For $\dot{\mathbb{B}}$ as a global source, causally related to $\mathbb{E}$, there is a <u>time delay</u> between the source & field points... if $\dot{\mathbb{B}}$ runs on time t', then $\mathbb{E}$ runs on time $t = t' + \frac{R}{c}$. Consequently, there must be corrections to Eq. (3).