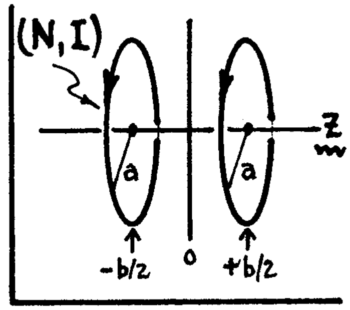
Φ 519 Problems   Assigned 11/18. Due 11/25/91.

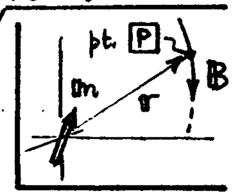


(35) [15 pts]. A Helmholtz coil is a device to produce a relatively uniform $\mathbb{B}$-field in a given volume. The coil consists of two identical circular loops, radius a with N turns carrying current I, sharing a common axis, and separated by distance b. Let the loop axis be the z-axis, and situate the loops at $z = \pm b/2$, as shown in sketch.

(A) It is possible to choose b so that the axial field near the center of the coil (i.e. $z=0$) goes as: $\underline{\underline{B_z(z) = B_z(0)\left[1 - k(z/a)^4 + \cdots\right]}}$. Find the condition on b which makes this so, and calculate the numerical constant k.

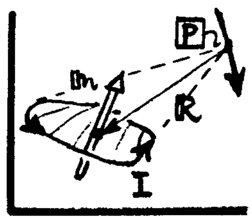
(B) Find the radial field B_ρ close to the axis and near coil center for B_z of part (A).

(C) Within a small volume at coil center, the total field $B = \sqrt{B_z^2 + B_\rho^2}$ is quite uniform. Picture this region as a cylinder, and find its dimensions and volume if -- inside the cylinder -- ΔB_z & ΔB_ρ are both $\leqslant \epsilon B_z(0)$, with $\epsilon \ll 1$. Calculate the coil radius a if $\epsilon = 1/10^4$ homogeneity is required over $\Delta z = 1$ cm.

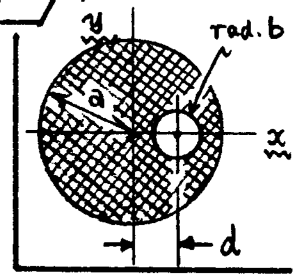


(36) Consider a magnetic dipole field: $\mathbb{B} = \frac{1}{r^5}\left[3(\mathbf{m}\cdot\mathbf{r})\mathbf{r} - r^2\mathbf{m}\right]$, at $r > 0$.

(A) Show that $\mathbb{B} = -\nabla\psi$, where $\underline{\underline{\psi = \mathbf{m}\cdot\mathbf{r}/r^3}}$ is a $\underline{\text{scalar}}$ potential.

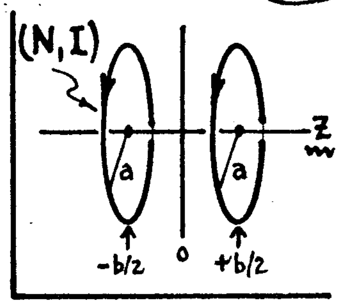


(B) Suppose $\mathbf{m}$ is generated by a current loop I as shown. Consistent with the dipole approximation ($R \gg$ loop size), show that the potential of part (A) is: $\psi = -(I/c)\,\Omega$, where Ω is the solid ∡ subtended by the loop at the field point P. Hence: $\mathbb{B} = (I/c)\nabla\Omega$, as quoted in Jk^{n} Prob. (5.1). **HINT**: $\Omega = \int \frac{1}{R^2}\hat{R}\cdot d\mathbb{A}$, with the integral over the loop surface area.



(37) [Jackson #(5.3)]. A cylindrical conductor of radius a has a hole of radius b bored ∥ to and at distance d from its axis ($d+b<a$). The current density is uniform and ∥ axis throughout the rest of the cylinder. If the conductor carries total current I, find the magnitude and direction of the $\mathbb{B}$-field in the hole. You may assume $\nabla\cdot\mathbb{B} = 0$.

ϕ 519 Solutions



7) [15 pts]. Analyse Helmholtz coil field for homogeneity.

(A) The double-loop configuration generates an axial field:

$$\rightarrow B_z(z) = \frac{2\pi NI}{ca}\left\{\left[1+(\zeta-\beta)^2\right]^{-\frac{3}{2}} + \left[1+(\zeta+\beta)^2\right]^{-\frac{3}{2}}\right\}, \qquad (1)$$

where $\zeta = z/a$ & $\beta = (b/2)/a$. Evidently $B_z(-z) = B_z(+z)$, so a Taylor expansion of $B_z(z)$ about $z=0$ will contain only even derivatives: $B_z(z) = B_z(0) + \frac{1}{2}z^2 B_z''(0) + \frac{1}{24}z^4 B_z^{IV}(0) + \ldots$. The condition for a z^4 field is $B_z''(0)=0$, i.e.

$$\rightarrow B_z''(0) = -\frac{3B_0}{a^2}\left\{\frac{1-4(\zeta-\beta)^2}{[1+(\zeta-\beta)^2]^{7/2}} + \frac{1-4(\zeta+\beta)^2}{[1+(\zeta+\beta)^2]^{7/2}}\right\}\Bigg|_{z=0} = -\frac{6B_0}{a^2}\frac{1-4\beta^2}{(1+\beta^2)^{7/2}} = 0. \qquad (2)$$

Here $B_0 = \frac{2\pi NI}{ca}$, and $B_z''(0)=0$ if $\beta = b/2a = \frac{1}{2}$, i.e. if $\boxed{b=a}$. With this choice, the central field is $B_z(0) = (16/5^{3/2})\,B_0$, and near $z=0$ we'll have

$$\rightarrow B_z(z) = B_z(0)\left[1 - k\zeta^4 + \ldots\right], \text{ for } b=a, \text{ with } \zeta = z/a. \qquad (3)$$

The cnst $k = (-)\frac{a^4}{24} B_z^{IV}(0)/B_z(0)$. After some arithmetic: $\boxed{k = 144/125}$.

(B) By the method used in Prob. 34: $\nabla\cdot\mathbb{B} = 0 \Rightarrow \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho B_\rho) \simeq -(\partial B_z/\partial z)\big|_{\rho=0}$, or: $B_\rho \simeq -\frac{1}{2}\rho(\partial B_z/\partial z)_{axis}$, good to 1st order in ρ. Use Eq. (3) for $\left(\frac{\partial B_z}{\partial z}\right)_{axis}$ to get:

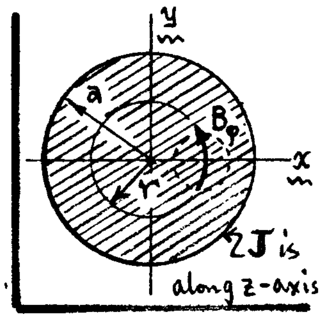
$$\boxed{B_\rho \simeq 2k\,B_z(0)\,\zeta^3\,\rho/a}, \text{ for } \zeta = \frac{z}{a} \ll 1 \ \& \ \rho \ll a. \qquad (4)$$

(C) For a cylinder with ends at $\pm z$, the max. axial field change is: $\Delta B_z = k\zeta^4 B_z(0)$, so $\Delta B_z \leq \epsilon B_z(0)$ defines a cylinder of length: $L = 2a(\epsilon/k)^{1/4}$. From Eq. (4), the radial field change is max. at the ends also, and $\Delta B_\rho \leq \epsilon B_z(0)$ gives the cylinder radius: $\rho = \frac{a}{2}(\epsilon/k)^{\frac{1}{4}}$. So the "homogeneity" cylinder is...

$$\text{CYLINDER}\begin{cases}\text{length}: L = 2a(\epsilon/k)^{1/4}\\ \text{radius}: \rho = \frac{1}{2}a(\epsilon/k)^{1/4}\end{cases} \Big\rangle \text{ volume}: V = \pi\rho^2 L = \frac{\pi}{2}a^3(\epsilon/k)^{3/4}. \qquad (5)$$

If $\epsilon = 1/10^4$, $(\epsilon/k)^{1/4} = 0.09652$. And if $L \geq 1$ cm, the coil radius must be $a \geq 5.18$ cm. The cylinder $\rho = \frac{1}{4}L = 0.25$ cm, and $V = 0.196\ \text{cm}^3$.

(7) $\mathbb{B}$-field in a cylindrical hole in a cylindrical conductor.



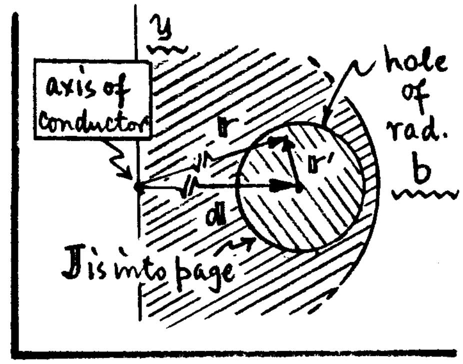
1. First, consider the conductor without the hole (i.e. hole filled in). At any $r \leq a$, the interior field will be in the φ (azimuthal) direction and of a size B_φ calculable by Ampère's Law, i.e.

$$\rightarrow \nabla \times \mathbb{B} = \frac{4\pi}{c}\mathbb{J} \Rightarrow \oint_{loop} \mathbb{B}\cdot d\mathbb{l} = \frac{4\pi}{c} I(\text{enclosed});$$

So // circular loop of radius r $\Big\}$ $B_\varphi \cdot 2\pi r = \frac{4\pi}{c} J\cdot \pi r^2$, $\underline{\underline{\mathbb{B} = \left(\frac{2\pi}{c}J\right)\hat{z}\times\mathbb{r}}}$. (1)

J is the const current density through the conductor, and $\hat{z}$ a unit vector $\parallel$ $\hat{z}$-axis

2. Now, recreate the hole by adding an oppositely directed current density J to the hole region (hole of rad. b centred at $r=d$ on x-axis, etc.). Then the hole has net $J=0$, and the $J_{opposite}$ generates a magnetic field of the form of Eq. (1), viz.



$$\underline{\underline{\mathbb{B}' = -\left(\frac{2\pi}{c}J\right)\hat{z}\times\mathbb{r}'}}, \quad \mathbb{r}' \text{ from hole center.} \tag{2}$$

3. The net field in the hole is the superposition of $\mathbb{B}$[Eq.(1)] & $\mathbb{B}'$[Eq.(2)], i.e.

$$\rightarrow \mathbb{B}_{hole} = \mathbb{B} + \mathbb{B}' = \frac{2\pi J}{c}\hat{z}\times(\mathbb{r}-\mathbb{r}'). \tag{3}$$

But $\mathbb{r}-\mathbb{r}' = \mathbb{d} = \hat{x}\,d$, where d is the distance from conductor axis to hole axis. And $\hat{z}\times\hat{x} = \hat{y}$ lies along the y-axis, so that Eq. (3) becomes...

$$\rightarrow \mathbb{B}_{hole} = \left(\frac{2\pi J d}{c}\right)\hat{y}. \tag{4}$$

4. Finally (with hole present), conductor will be carrying current $I = J\cdot\pi(a^2-b^2)$. So the field in Eq. (4) is

$$\boxed{\mathbb{B}_{hole} = \left[\frac{2Id}{c}\left(\frac{1}{a^2-b^2}\right)\right]\hat{y}}. \tag{5}$$

(36) Achieve a representation of $\mathbb{B}_{dipole}$ via a scalar potential $\psi = \frac{\mathbf{m}\cdot\mathbf{r}}{r^3} = (-)\frac{I}{c}\Omega$.

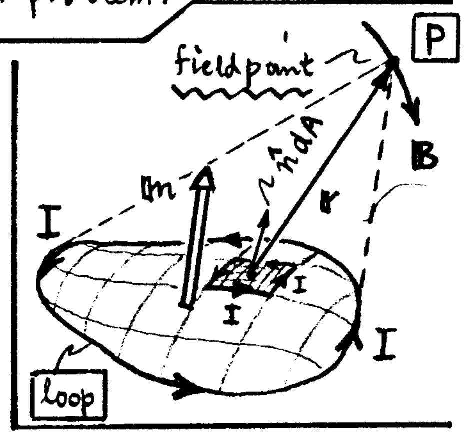
(A) By the identity: $\nabla(\mathbf{a}\cdot\mathbf{b}) = (\mathbf{a}\cdot\nabla)\mathbf{b} + (\mathbf{b}\cdot\nabla)\mathbf{a} + \mathbf{a}\times(\nabla\times\mathbf{b}) + \mathbf{b}\times(\nabla\times\mathbf{a})$, with $\mathbf{a} = \mathbf{m} = $ cnst, and $\mathbf{b} = \mathbf{r}/r^3$, we have...

$$\rightarrow \nabla(\mathbf{m}\cdot\mathbf{r}/r^3) = (\mathbf{m}\cdot\nabla)\frac{\mathbf{r}}{r^3} + 0 + \mathbf{m}\times\left(\nabla\times\frac{\mathbf{r}}{r^3}\right) + 0. \qquad (1)$$

But: $\nabla\times(\mathbf{r}/r^3) = [\nabla(1/r^3)]\times\mathbf{r} + \frac{1}{r^3}\cancelto{0}{(\nabla\times\mathbf{r})} = 0$, since $\nabla\left(\frac{1}{r^3}\right) = -\frac{3}{r^5}\mathbf{r}$ for $r\neq 0$, and the 1st term RHS vanishes because $\mathbf{r}\times\mathbf{r} = 0$. Then, w/ $\psi = \mathbf{m}\cdot\mathbf{r}/r^3$...

$$\rightarrow (-)\nabla\psi = -(\mathbf{m}\cdot\nabla)\frac{\mathbf{r}}{r^3} = \frac{1}{r^3}\left[3(\mathbf{m}\cdot\hat{r})\hat{r} - \mathbf{m}\right] = \mathbb{B}_{dipole} \qquad (2)$$

The details of $(\mathbf{m}\cdot\nabla)\mathbf{r}/r^3$ were worked out in class. Indeed the dipole field can be generated from a scalar potential: $B_{dipole} = -\nabla\psi$, w/ $\psi = (\mathbf{m}\cdot\mathbf{r})/r^3$. This should not be astonishing, since $\nabla\times\mathbb{B} = 0$ for this problem.



(B) With $\mathbf{m}$ generated by a loop current I, divide a surface through the loop into many small circuits by a mesh, as shown. Let each microloop carry current I; then, because the currents cancel in the common branches of adjacent loops, the net effect of the microloops is the same as that of the main loop carrying current I around its periphery only. Each microloop generates a moment: $d\mathbf{m} = \frac{I}{c}\hat{n}\,dA$, where dA is the loop area and $\hat{n}$ the local unit normal to the surface [cf. Jk$^{\underline{n}}$ Eq. (5.57)]. The total loop moment can then be represented as: $\mathbf{m} = \frac{I}{c}\int_{loop}\hat{n}\,dA$.

At dipole distances, r changes negligibly as the vector $\mathbb{R} = (-)\mathbf{r}$, from pt. P to the loop ranges over the loop surface. The potential ψ of part (A) can be gotten from:

$$\rightarrow \psi = \int_{loop}(\mathbf{r}\cdot d\mathbf{m})/r^3 = -\frac{I}{c}\int_{loop}(\mathbb{R}/R^3)\cdot\hat{n}\,dA = -\frac{I}{c}\Omega,\quad \Omega = \int_{loop}\frac{1}{R^2}\hat{R}\cdot d\mathbf{A}. \qquad (3)$$

Ω is the solid $\measuredangle$ subtended by the loop at the field point P. Then, as required, the field at P is: $\boxed{\mathbb{B} = (I/c)\nabla\Omega}$. Same result is quoted in Jk$^{\underline{n}}$ Prob. (5.1).