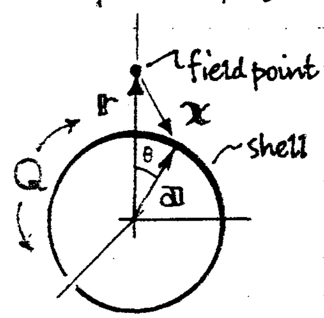
ϕ 519 Problems $\left\{\begin{array}{l}\text{Set \#2 : Probs. 4-6.}\\ \text{Assigned 9/30/88; due 10/7/88.}\end{array}\right.$

2

P2

4 [redacted]. What happens if Coulomb's Law is not inverse square? Suppose that for point charges q & q' separated by distance r, the force law were: $\mathbb{F} = [qq' f(r)]\hat{r}$, where $f(r)$ decreases with r and vanishes at ∞, but is otherwise arbitrary. $\mathbb{F}$ is still radial, so we can define a potential (why?) at distance x from q: $V(x) = q\int_x^\infty f(\xi)\,d\xi$, or: $V = \sum_i dq_i \int_{x_i}^\infty f(\xi)\,d\xi$, for an assembly of charges dq_i. Now consider a uniformly charged conducting spherical shell of radius a and surface charge density: $\sigma = \frac{Q}{4\pi a^2}$. Let the field point be at $\mathbb{r}$ on the z-axis as shown; we can treat $r > a$ (outside shell) and $r < a$ (inside) separately. With x the distance from field point to shell charge element, the potential at r is:



$$\rightarrow V(r) = \iint_{\text{shell}} \sigma \left\{ \int_x^\infty f(\xi)\,d\xi \right\} a^2 \sin\theta\, d\theta\, d\phi.$$

(5pts) (A) Do the ϕ integration for $V(r)$. Use: $x^2 = a^2 + r^2 - 2ar\cos\theta$ (converts θ to x-integration) to show that:

$$\boxed{V(r) = \frac{Q}{2ar}\int_L^U \left\{ \int_x^\infty f(\xi)\,d\xi \right\} x\,dx.}$$

The upper & lower limits U & L depend on r & a. Find U & L for both $r > a$ (outside shell) & $r < a$ (inside shell).

(5pts) (B) Put $f(\xi) = \frac{1}{\xi^2} h(\xi)$ in part (A). Show that if $h(\xi) = $ cnst, then the shell looks like a point charge outside ($r > a$), and the potential inside ($r < a$) is everywhere constant. Conversely, if $h(\xi) \neq$ cnst, then neither of these well-known inverse-square-law results holds.

(7pts) (C) Suppose the Coulomb departure were: $h(\xi) = (\lambda/\xi)^\delta$, with λ a characteristic length, and the exponent $|\delta| \ll 1$. Show that to first order in δ, $V(r)$ of part (A) is...

$$\rightarrow V(r) \simeq \frac{Q}{2ar}\left\{ (U - L) - \delta\left[U \ln\left(\frac{U}{\lambda}\right) - L \ln\left(\frac{L}{\lambda}\right) \right] \right\}.$$

Form: $\Delta V(r) = V(r)\big|_{\delta=0} - V(r)\big|_{\delta\neq 0}$; and find limiting forms for $\Delta V(r)$ when $r \ll a$, $r = a$, and $r \gg a$. Sketch a graph of $\Delta V(r)$ vs. r over $0 \leq r \rightarrow \infty$.

(8pts) (D) Inside the shell, write: $V(r) \simeq \frac{Q}{a}[1 - \delta g(\rho)]$, with $\rho = \frac{r}{a}$. Specify $g(\rho)$. Suppose you had a laboratory shell of radius $a = 50$ cm, charged up to 10 kV potential, and you could detect potential differences to an accuracy of $\pm 1\,\mu$V over distances $\Delta r \sim 1$ cm. What limits could you establish on δ and/or λ? [redacted]!

φ519 Problems

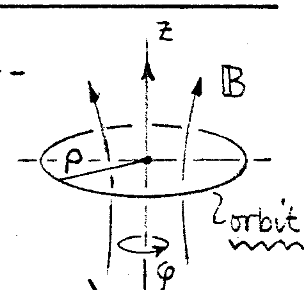
⑤ Details of EM "duality". If -- in addition to electric monopoles (with charge & current densities ρ_e & $\mathbb{J}_e$) -- we had to account for magnetic monopoles (ρ_m & $\mathbb{J}_m$), Maxwell's Eqs. would assume the more symmetric form [Jackson, Eq. (6-150)]:

$$\boxed{\begin{array}{ll} \nabla\cdot\mathbb{D} = 4\pi\rho_e, & \nabla\cdot\mathbb{B} = 4\pi\rho_m, \\ (-)\nabla\times\mathbb{E} = \frac{1}{c}(\partial\mathbb{B}/\partial t) + \frac{4\pi}{c}\mathbb{J}_m; & \nabla\times\mathbb{H} = \frac{1}{c}(\partial\mathbb{D}/\partial t) + \frac{4\pi}{c}\mathbb{J}_e. \end{array}}$$

This is in a medium: $\mathbb{E}$ & $\mathbb{B}$ are the "true" electric & magnetic fields; $\mathbb{D} = \epsilon\mathbb{E}$ and $\mathbb{H} = (1/\mu)\mathbb{B}$ are the "augmented" fields which include polarization effects. c is the free-space light velocity. Now consider the field-source "duality transformations" given in Jackson Eqs. (6-151) & (6-152); these transforms mix electric & magnetic quantities in an almost arbitrary fashion.

(A) Verify that $\mathbb{E}\times\mathbb{H}$ [Poynting Vector] and $(\mathbb{E}\cdot\mathbb{D} + \mathbb{B}\cdot\mathbb{H})$ [EM field energy] are form invariant under a duality transform (i.e. $\mathbb{E}'\times\mathbb{H}' \equiv \mathbb{E}\times\mathbb{H}$, where $\mathbb{E}'$ & $\mathbb{H}'$ are the duals, etc).

(B) Verify that the (above) Maxwell Eqs. are also form invariant under duality.



⑥ Suppose a magnetic (induction) field $\mathbb{B}$ is axially symmetric -- i.e. $\mathbb{B}$ is not dependent on the azimuthal ∡ φ about the axis z. A charged particle (assume any charge q & mass m) is orbiting in $\mathbb{B}$ in a circle of radius ρ. Use cylindrical cds (ρ, z, φ).

(A) Show that if $\mathbb{B}$ changes in time through the plane of the orbit, the particle is accelerated by an electric field: $\underline{\mathbb{E} = \hat{\varphi}(\dot{\Phi}/2\pi c\rho)}$, where Φ is the magnetic flux through the orbit. What happens to the (−) sign in Faraday's Law?

(B) Assume the particle motion is non-relativistic. Show that during an acceleration period, when $\dot{\mathbb{B}} \neq 0$, the orbit radius ρ can be held constant if Φ is designed so that: $\boxed{\dot{\Phi} = 2\pi\rho^2\dot{B}_z}$, where B_z is the z-component of $\mathbb{B}$ at the orbit. This is called the "betatron condition".

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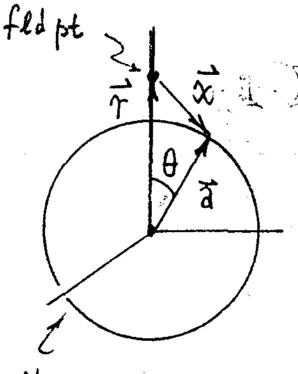

4 [redacted] Find Shell potential for non-Coulombic force law.

9/26/84

A. In the given form for $V(r)$, do the ϕ integration $(0 \leq \phi \leq 2\pi)$,

(5pts.) and put in $\sigma = Q/4\pi a^2$. Then...

$$V(r) = \frac{Q}{2}\int_0^\pi \left(\int_x^\infty f(\xi)\,d\xi\right)\sin\theta\,d\theta. \quad (1)$$

Here, $x = x(\theta)$, via ; $x^2 = a^2 + r^2 - 2ar\cos\theta$. Since both a & r are fixed during the integration over the Shell, in fact ; $x\,dx = ar\sin\theta\,d\theta$ (differential form of law of cosines). Using this in Eq. (1), have immediately...

$$\boxed{V(r) = \frac{Q}{2ar}\int_L^U \left(\int_x^\infty f(\xi)\,d\xi\right) x\,dx} \quad \begin{cases} U, L = r \pm a, & \text{for } r > a \text{ (outside)}, \\ U, L = a \pm r, & \text{for } r < a \text{ (inside)}. \end{cases} \quad (2)$$

The limits U & L are the max & min values of x, corresponding to the range $\pi \geq \theta \geq 0$.

B. Insert $f(\xi) = \frac{1}{\xi^2}h(\xi)$ in Eq. (2), and partial integrate...

(5pts.)
$$\int_x^\infty f(\xi)\,d\xi = \frac{1}{x}h(x) + \int_x^\infty \frac{d\xi}{\xi}h'(\xi),$$

So// $$V(r) = \frac{Q}{2ar}\int_L^U \left[h(x) + x\int_x^\infty \frac{d\xi}{\xi}h'(\xi)\right]dx. \quad (3)$$

Suppose $h(\xi) = C$, cnst, so that $h'(\xi) \equiv 0$. Then the potential is...

$$V(r) = \frac{QC}{2ar}\int_L^U dx = \frac{QC}{2ar}(U-L) = \begin{cases} QC/r, & r > a \text{ (outside)}, \\ QC/a, & r < a \text{ (inside)}. \end{cases} \quad (4)$$

Of course we would choose $C = 1$ in usual units. Then $V = Q/r$ = point charge potential, everywhere outside, while $V = Q/a$ = cnst, everywhere inside Shell. These are the characteristic features of an inverse-square law force.

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C. If $h(\xi)=(\lambda/\xi)^{\delta} \Rightarrow f(\xi)=\lambda^{\delta}/\xi^{2+\delta}$, two trivial integrations give...

(7pts.) $$\int_x^{\infty} f(\xi)\,d\xi = \frac{1}{1+\delta}\,\frac{1}{x}\left(\frac{\lambda}{x}\right)^{\delta},$$

and// $$V(r)=\frac{Q}{2ar}\left(\frac{1}{1-\delta^2}\right)\left[\left(\frac{\lambda}{U}\right)^{\delta}U-\left(\frac{\lambda}{L}\right)^{\delta}L\right]. \qquad (5)$$

For $|\delta|<<1$; $N^{\delta}=e^{\delta \ln N}\simeq 1+\delta\ln N$, to 1st order in δ (and useful even if N varies over many orders of magnitude). Then, to $O(\delta)$, as advertised...

$$\boxed{V(r)\simeq\frac{Q}{2ar}\left\{(U-L)-\delta\left[U\ln(U/\lambda)-L\ln(L/\lambda)\right]\right\}} \qquad (6)$$

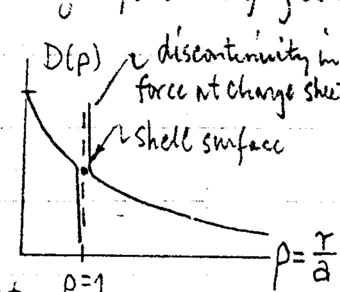
The "departure fcn": $\Delta V(r)=V(r)|_{\delta=0}-V(r)|_{\delta\neq 0}$ is easily found to be...

$$\Delta V(r)=\frac{Q\delta}{2a}D(r),\quad D(r)=\frac{1}{r}\left[U\ln(U/\lambda)-L\ln(L/\lambda)\right], \begin{cases} U,L=r\pm a, \text{ outside;}\\ U,L=a\pm r, \text{ inside.}\end{cases}$$

i.e.// $$D(\rho)=\begin{cases}\left(1+\frac{1}{\rho}\right)\ln\left(\frac{\rho+1}{n}\right)-\left(1-\frac{1}{\rho}\right)\ln\left(\frac{\rho-1}{n}\right), & \underline{\underline{\rho}}=r/a>1; \qquad (7)\\[2ex] \left(\frac{1}{\rho}+1\right)\ln\left(\frac{1+\rho}{n}\right)-\left(\frac{1}{\rho}-1\right)\ln\left(\frac{1-\rho}{n}\right), & \rho=r/a<1;\ \text{and}\ \underline{\underline{n}}=\frac{\lambda}{a}.\end{cases}$$

Using: $\lim_{x\to 0} x\ln x=0$, and $\ln(1\pm\epsilon)\simeq\pm\epsilon$ as $\epsilon\to 0$, we straightforwardly get:

$$\begin{cases} D(\rho)\simeq 2\left[\ln(1/n)+1\right]-\frac{1}{3}\rho^2, \ \text{for}\ \rho<<1; \qquad (8)\\ D(\rho)\simeq\frac{2}{\rho}\left[\ln(\rho/n)+1\right],\ \rho>>1;\ D(\rho=1)=2\ln\left(\frac{2}{n}\right).\end{cases}$$



D(ρ) decreases with ρ; with usual D′(ρ) discontinuity at the sheet.

D. With reference to Eq.(7) above, and inside the shell, we can write...

(8 pts.) $$V(r)\simeq\frac{Q}{a}\left[1-\delta g(\rho)\right],\quad \underline{g(\rho)}=\frac{1}{2}D(\rho)=\frac{1}{2}\left[\left(\frac{1}{\rho}+1\right)\ln\left(\frac{1+\rho}{n}\right)-\left(\frac{1}{\rho}-1\right)\ln\left(\frac{1-\rho}{n}\right)\right], \qquad (9)$$

where: $\rho=r/a\leq 1$, and: $n=\lambda/a$. If we measure potential differences ΔV over

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distances Δr which are "small" relative to the shell radius a, then we can determine the ratio...

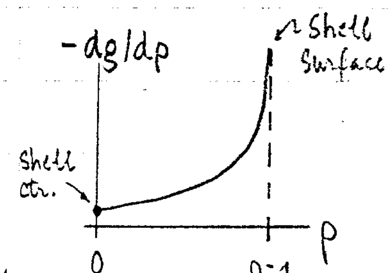
$$\Delta V/\Delta r \simeq -\frac{Q\delta}{a}(\Delta g/\Delta r) \simeq -\frac{V_s\,\delta}{a}(dg/d\rho), \tag{10}$$

where: $V_s = Q/a$, is the potential at the shell surface. Notice that this measurement is indpt of the scale distance λ, since the derivative -- calculated via

$$g(\rho) = \frac{1}{2}\left[\frac{1}{\rho}\ln\left(\frac{1+\rho}{1-\rho}\right) + \ln\left(\frac{1-\rho^2}{n^2}\right)\right], \quad 0 \leq \rho \leq 1;$$

is//

$$\frac{dg}{d\rho} = \frac{1}{\rho}\left[1 - \frac{1}{2\rho}\ln\left(\frac{1+\rho}{1-\rho}\right)\right] \simeq \left\{\begin{array}{l} -\rho/3, \text{ as } \rho \to 0, \\ \ln\sqrt{1-\rho}, \text{ as } \rho \to 1. \end{array}\right\} \text{ indpt. of } \lambda. \tag{11}$$



$dg/d\rho \to (-)\infty$ as $\rho \to 1-$ reflects the discontinuity in the force as we pass through the charge sheet. But here of course we are working over finite intervals Δr, so we cannot pass to $\rho = 1$; in fact, we can get down to a "fineness" of perhaps $\rho = 0.99$, so the max value of $-(dg/d\rho)$ is $-(dg/d\rho)|_{\rho=0.99} = 1.69$. At the other end, $\rho \to 0$, we can to $\rho = 0.01$ (i.e. resolve $\Delta r \simeq 1/2$ cm out of $a = 50$ cm), so the min is $-(dg/d\rho)|_{\rho=0.01} = 0.01/3$. Now solve Eq. (10) for δ, to get...

$$\rightarrow \delta \simeq \left(\frac{\Delta V}{V_s}\right)\frac{a}{\Delta r}\Big/\left[-(dg/d\rho)\right]. \tag{12}$$

We see that for max. sensitivity, we should work near the sphere center, where $-(dg/d\rho)$ is smallest. With $V_s = 10$ KV, $\Delta V = \pm 1\,\mu V$, $a = 50$ cm and $\Delta r = 1$ cm, and the $-(dg/d\rho)|_{\rho=0.01} = 0.01/3$ estimate from above, this gives $\delta \simeq \pm 1.5 \times 10^{-10}$ as a limit of sensitivity for this expt. Plimpton & Lawton (Phys. Rev. 50, 1066 (1936)) did about this well, ~50 years ago.

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● Examine Maxwell's Eqs. under a Duality Transform.

9/25/84

1. Although $\vec{E}$ & $\vec{H}$, $\vec{E}'$ & $\vec{H}'$ etc. are actually vectors, they are related under a duality transform as though they were components of vectors, acted on by a rotation matrix: $\underset{\sim}{R}(\xi) = \begin{pmatrix} \cos\xi & \sin\xi \\ -\sin\xi & \cos\xi \end{pmatrix}$. E.g. $\begin{pmatrix}\vec{E}\\ \vec{H}\end{pmatrix} = \underset{\sim}{R}(\xi)\begin{pmatrix}\vec{E}'\\ \vec{H}'\end{pmatrix}$ is the first of Jackson's Eqs (6-151); the second is $\begin{pmatrix}\vec{D}\\ \vec{B}\end{pmatrix} = \underset{\sim}{R}(\xi)\begin{pmatrix}\vec{D}'\\ \vec{B}'\end{pmatrix}$. In any case, it is obvious the inverses are provided via $[\underset{\sim}{R}(\xi)]^{-1} = \underset{\sim}{R}(-\xi)$, if needed, and it is semi-obvious that since $\underset{\sim}{R}$ preserves scalar & vector products, there will be conserved products.

2. Checking this out, look at the total field energy. Drop vector signs → Then

$$E'\cdot D' = (E\cos\xi - H\sin\xi)\cdot(D\cos\xi - B\sin\xi) = E\cdot D\cos^2\xi + H\cdot B\sin^2\xi \quad \begin{matrix} -E\cdot B\cos\xi\sin\xi \\ -D\cdot H\cos\xi\sin\xi \end{matrix}$$

$$H'\cdot B' = (E\sin\xi + H\cos\xi)\cdot(D\sin\xi + B\cos\xi) = E\cdot D\sin^2\xi + H\cdot B\cos^2\xi \quad \begin{matrix} +E\cdot B\cos\xi\sin\xi \\ +D\cdot H\cos\xi\sin\xi \end{matrix}$$

$\Rightarrow \vec{E}\cdot\vec{D}' + \vec{H}'\cdot\vec{B}' \equiv \vec{E}\cdot\vec{D} + \vec{H}\cdot\vec{B}$, ∴ <u>field energy invariant under duality.</u>

Look also at the Poynting vector (energy flow). Sufficient to look at one component:

$$(E'\times H')_x = E_y'H_z' - E_z'H_y' = (E_y\cos\xi - H_y\sin\xi)(E_z\sin\xi + H_z\cos\xi) -$$

$$-(E_z\cos\xi - H_z\sin\xi)(E_y\sin\xi + H_y\cos\xi) = \cdots = E_yH_z - E_zH_y$$

So, $\vec{E}'\times\vec{H}' \equiv \vec{E}\times\vec{H} \Rightarrow$ <u>Poynting vector invariant under duality.</u>

3. "Add" the first two Maxwell Eqs. in Jackson's (6-150), to get: $\vec{\nabla}\cdot\begin{pmatrix}\vec{D}\\ \vec{B}\end{pmatrix} = 4\pi\begin{pmatrix}\rho_e\\ \rho_m\end{pmatrix}$. Operate with $\underset{\sim}{R}^{-1}(\xi)$, noting: $\underset{\sim}{R}^{-1}\begin{pmatrix}\vec{D}\\ \vec{B}\end{pmatrix} = \begin{pmatrix}\vec{D}'\\ \vec{B}'\end{pmatrix}$, and: $\underset{\sim}{R}^{-1}\begin{pmatrix}\rho_e\\ \rho_m\end{pmatrix} = \begin{pmatrix}\rho_e'\\ \rho_m'\end{pmatrix}$. Then have $\vec{\nabla}\cdot\begin{pmatrix}\vec{D}'\\ \vec{B}'\end{pmatrix} = 4\pi\begin{pmatrix}\rho_e'\\ \rho_m'\end{pmatrix}$, so the duality invariance is obvious. Invariance for the 2ND pair of eqtns is shown similarly, after noting: $\underset{\sim}{R}^{-1}\begin{pmatrix}\vec{H}\\ -\vec{E}\end{pmatrix} = \begin{pmatrix}\vec{H}'\\ -\vec{E}'\end{pmatrix}$. <u>POINT</u>: Semi-arbitrary admixtures (i.e. labelling) of E & B fields & sources don't change Maxwell's description.

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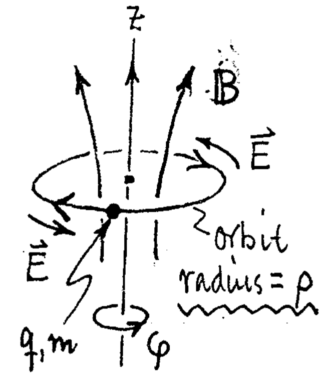
10/1/84

⑥ Analyse the dynamics of betatron acceleration.

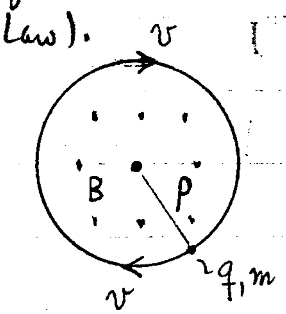


A. A (+)vely charged particle will orbit in the $(-)\hat{\varphi}$ direction, as shown. This loop has a directed surface normal along the $(-)z$ axis, and so Faraday's Law in integral form reads...

$$\oint \vec{E}\cdot d\vec{l} = -\frac{1}{c}\frac{\partial}{\partial t}\int \vec{B}\cdot(-d\vec{S}) = \frac{1}{c}\dot{\Phi} \quad \left\{ \begin{array}{l} \Phi = \int \vec{B}\cdot d\vec{S} = \text{mag flux}; \\ \text{Faraday's } (-) \text{ sign is gone.} \end{array} \right. \tag{1}$$

By symmetry, $\vec{E}$ lies along the $(\pm)\hat{\varphi}$ direction, and is constant along the entire circular orbit. Then $\oint \vec{E}\cdot d\vec{l} = \mp E\cdot 2\pi\rho$, so we have--as required

$$\boxed{\vec{E} = \hat{\varphi}(\pm\dot{\Phi}/2\pi\rho c)} \quad \left\{ \begin{array}{l} \text{Choice of } (\pm) \text{ depends on how } \Phi \text{ is defined. If } \Phi = \\ \int \vec{B}\cdot(+\hat{z})\,dS, \text{ as above, then choose } (-) \text{ sign, so that} \\ \dot{\Phi} > 0 \text{ accelerates a (+)ve } q \text{ (by Lenz' Law).} \end{array} \right. \tag{2}$$



B. Suppose the particle in orbit at radius ρ has charge q, mass m, and is moving at tangential velocity v. The centripetal acceleration is provided by the Lorentz force:

$$mv^2/\rho = \frac{q}{c}vB \Rightarrow \text{momentum}: p = mv = \frac{q}{c}\rho B$$

so// inertial reaction: $$dp/dt = \frac{q}{c}(\rho\dot{B} + B\dot{\rho}) \tag{3}$$

B here is B_z, on the orbit, and we are looking for the condition where $\dot{\rho} = 0$. The accelerating force due to $\dot{B} \neq 0$ is $F = qE$, with E given by Eq. (2)...

$$qE = dp/dt \Rightarrow q(\dot{\Phi}/2\pi c\rho) = \frac{q}{c}(\rho\dot{B}_z + B_z\underbrace{\dot{\rho}}_{\to 0})$$

... cancel q & c $\Rightarrow$ $$\boxed{\dot{\Phi} = 2\pi\rho^2\dot{B}_z} \leftarrow \text{betatron condition.} \tag{4}$$

This condition ensures ρ = cnst during the acceleration. It is often stated as follows...

$\Phi\binom{\text{whole}}{\text{orbit}} = B_{AV}\cdot\pi\rho^2 \Rightarrow B_{AV}\binom{\text{orbit}}{\text{plane}} = 2\times B\binom{\text{at}}{\text{orbit}}$, i.e. avg B fld = 2B at orbit.