ϕ519 Problems — Assigned 9/13/91. Due 9/20/91.

⑥ $\mathbb{F}(\mathbf{r})$ is a vector force field in 3D space. Show that $\mathbb{F}$ is a "conservative field" (i.e. work $\int_A^B \mathbb{F}\cdot d\mathbf{r}$ is independent of path taken between pts A & B) iff $\nabla\times\mathbb{F}=0$. NOTE: iff means "if and only if"... (i.e. $\nabla\times\mathbb{F}=0$ is a necessary and sufficient condition for path-independence of $\int_A^B \mathbb{F}\cdot d\mathbf{r}$).

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⑦ An "elementary particle" has total charge e and a spherically symmetric charge distribution with volume density $\rho(r) \propto e^{-r/a}$, where r is the radial distance from the charge center and $a \geq 0$ is a scale length. (A) Normalize ρ so that in fact $\int_\infty \rho\, dV = e$. (B) Calculate the electric field $E(R)$ at radial distance R. Find asymptotic forms for $E(R)$ when $R \gg a$ and when $R \ll a$. (C) Sketch $E(R)$ vs. R over $0 \leq R \to \infty$. Your $E(R)$ should be finite everywhere. At what (approx.) R-value is $E(R)$ maximum? (D) Calculate the self-energy $W_E = \int_\infty (E^2/8\pi)\, dV$ for this charge. What happens when $a \to 0$? Comment.

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⑧ [Jackson Prob. (1.10)]. Prove the Mean Value Theorem for an electrostatic potential ϕ, viz.: In a charge-free space, the value of ϕ at any point equals the average of ϕ over any sphere centered on that point.

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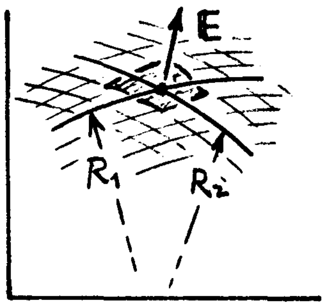
⑨ [Jackson Prob. (1.11)]. Use Gauss' Law to show that at the surface of a curved charged conductor the normal derivative of the electric field obeys: $\boxed{\frac{1}{E}(\partial E/\partial n) = -\left(\frac{1}{R_1}+\frac{1}{R_2}\right)}$, where R_1 & R_2 are the radii-of-curvature of the surface.



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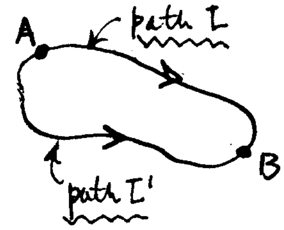
⑩ Interlude. Finish the following limerick. First prize: 10 point bonus.

"An outstanding young man named James Clerk
Wrote down four equations that work.
He published them quickly,
...

⑥ Show $\mathbb{F}$ is a conservative force field iff $\nabla\times\mathbb{F}=0$.

1) First show that if $\int_A^B \mathbb{F}\cdot d\mathbf{r}$ is path-independent, then $\nabla\times\mathbb{F}=0$.



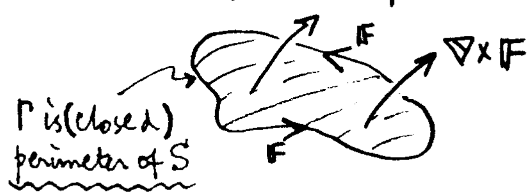
Assume: $\int_{A\,[\text{on } I']}^{B} \mathbb{F}\cdot d\mathbf{r} = \int_{A\,[\text{on } I]}^{B} \mathbb{F}\cdot d\mathbf{r}$. But $\int_{A\,[\text{on } I]}^{B} \mathbb{F}\cdot d\mathbf{r} = (-)\int_{B\,[\text{on } I]}^{A} \mathbb{F}\cdot d\mathbf{r}$,

by def^{n} of the integral. So path-independence implies:

$$\rightarrow \int_{A\,[\text{on } I']}^{B} \mathbb{F}\cdot d\mathbf{r} + \int_{B\,[\text{on } I]}^{A} \mathbb{F}\cdot d\mathbf{r} = \oint_{\text{loop}} \mathbb{F}\cdot d\mathbf{r} \equiv 0, \quad \underline{\text{any}}\text{ loop containing pts } A \,\&\, B. \qquad (1)$$

Now invoke Stokes' Thm. If S is <u>any</u> surface enclosed by the loop Γ...

$$\left[\!\left[\; \underbrace{\oint_\Gamma \mathbb{F}\cdot d\mathbf{r}}_{\text{circulation of } \mathbb{F}} = \underbrace{\int_S (\nabla\times\mathbb{F})\cdot d\boldsymbol{\sigma}}_{\text{current due to } \mathbb{F}} . \;\right]\!\right] \qquad (2)$$



By the assumption of path-independence, we have $\oint_\Gamma \mathbb{F}\cdot d\mathbf{r}=0$, from Eq. (1). So, we have $\int_S(\nabla\times\mathbb{F})\cdot d\boldsymbol{\sigma}=0$, also, for <u>any</u> loop Γ enclosing the (arbitrary) surface S. The only way this last result can be true, for <u>all</u> loops Γ and surfaces S in $\mathbb{F}$ is if $\nabla\times\mathbb{F}\equiv 0$. <u>Thus $\int\mathbb{F}\cdot d\mathbf{r}$ path-indpt $\Rightarrow \nabla\times\mathbb{F}\equiv 0$.</u>

2) For proof in opposite direction, $\nabla\times\mathbb{F}=0 \Rightarrow \oint_\Gamma \mathbb{F}\cdot d\mathbf{r}=0$, by Eq. (2). The loop integral can be decomposed as in Eq. (1), so we have

$$0 = \int_S(\nabla\times\mathbb{F})\cdot d\boldsymbol{\sigma} = \oint_\Gamma \mathbb{F}\cdot d\mathbf{r} = \int_{A\,[\text{on } I']}^{B} \mathbb{F}\cdot d\mathbf{r} + \int_{B\,[\text{on } I]}^{A} \mathbb{F}\cdot d\mathbf{r}$$

$$\text{So,}\quad \int_{A\,[\text{on } I']}^{B} \mathbb{F}\cdot d\mathbf{r} = -\int_{B\,[\text{on } I]}^{A} \mathbb{F}\cdot d\mathbf{r} = +\int_{A\,[\text{on } I]}^{B} \mathbb{F}\cdot d\mathbf{r}, \qquad (3)$$

<u>and thus $\nabla\times\mathbb{F}=0 \Rightarrow \int\mathbb{F}\cdot d\mathbf{r}$ is path-indpt.</u>

3) Altogether: $\int\mathbb{F}\cdot d\mathbf{r}$ path-indpt $\Longleftrightarrow \nabla\times\mathbb{F}=0$. <u>QED</u>

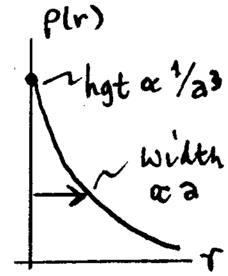
Clearly, this result is just an application of Stokes' Thm, in two directions.

⑦ Analyse "elementary" charge density $\rho(r) \propto e^{-r/a}$.

(A) Let $\rho(r) = N e^{-r/a}$, N = norm$^{\underline{n}}$ cnst. Then for spherical symmetry...

$$\rightarrow e = \int_\infty \rho\, dV = \int_0^\infty N e^{-r/a} \cdot 4\pi r^2 dr = 4\pi N a^3 \underbrace{\int_0^\infty x^2 e^{-x} dx}_{2}$$

$$\Rightarrow \underline{\underline{N = e/8\pi a^3}}, \quad \rho(r) = \left(\frac{e}{8\pi a^3}\right) e^{-r/a}. \quad (1)$$



As $a \to 0$, $\rho(r)$ becomes sharply peaked near $r=0$.

(B) By Gauss' Law, for a spherically symmetric ρ, field at radial distance R is...

$$\rightarrow E(R) = \frac{1}{R^2} Q(\text{inside } R) = \frac{1}{R^2}\int_0^R \rho(r) \cdot 4\pi r^2 dr = \frac{4\pi N}{R^2}\int_0^R r^2 e^{-r/a} dr$$

$$\text{or//} \quad \boxed{E(R) = \frac{e}{R^2}\left[1 - \left(1 + x + \frac{x^2}{2}\right)e^{-x}\right]}, \quad x = \frac{R}{a}. \quad (2)$$

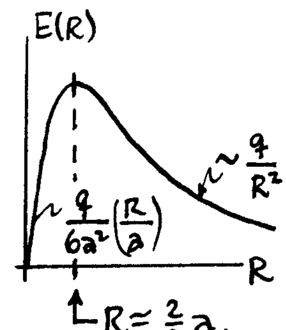
Clearly $\underline{E(R) \sim e/R^2 \text{ as } R \to \infty}$; this is just a standard Coulomb result. When $R \ll a$, $x \to$ small, and $e^{-x} \simeq 1 - x + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \ldots$ in Eq. (2). One must go to $O(x^3)$ to get a non-vanishing physical E, in which case...

$$\rightarrow E(R) \sim \frac{e}{R^2} \cdot \frac{x^3}{6} = \frac{1}{6}\frac{e}{a^2}\left(\frac{R}{a}\right), \text{ when } R \ll a. \quad (3)$$

(C) By part (B), $E \sim 1/R^2$ at large R, and $E \sim R$ when $R \to 0$. The graph is sketched at right. Evidently E is maximum at

$$\frac{\partial E}{\partial R} = 0 \Rightarrow \left(1 + x + \frac{x^2}{2} + \frac{x^3}{4}\right)e^{-x} = 1, \quad x = R/a. \quad (4)$$

Numerical solution is $x \simeq 0.67 \Rightarrow E$ is max @ $R \simeq \frac{2}{3}a$.



(D) For E of Eq. (2), the self-energy is...

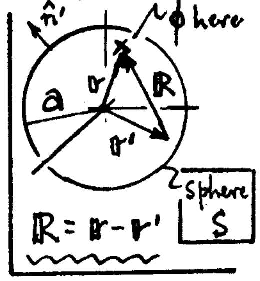
$$\rightarrow W_E = \frac{1}{2}\int_0^\infty \frac{e^2}{R^4}\left[1 - \left(1 + x + \frac{x^2}{2}\right)e^{-x}\right]^2 R^2 dR = \left(\frac{e^2}{2a}\right)\overbrace{\int_0^\infty \frac{dx}{x^2}\left[1 - \left(1 + x + \frac{x^2}{2}\right)e^{-x}\right]^2}^{J}. \quad (5)$$

The integrand of J is finite for all $x \geq 0$; in fact $J = \frac{5}{16}$, and: $\boxed{W_E = \frac{J}{2}\left(\frac{e^2}{a}\right)}$.

W_E diverges as $a \to 0$, even though the particle's field is everywhere finite. The $1/a$ divergence is characteristic of all such models of the charge.

(8) Prove the Mean Value Theorem for electrostatic potential ϕ.

1. Consider Jk$^{\underline{n}}$ Eq.(1.36) for ϕ anywhere inside volume V enclosed by surface S



$$\rightarrow \phi(\mathbf{r}) = \int_V \frac{d^3x'}{R}\rho(\mathbf{r}') + \frac{1}{4\pi}\oint_S dS'\left[\frac{1}{R}\left(\frac{\partial\phi}{\partial n'}\right) - \phi\frac{\partial}{\partial n'}\left(\frac{1}{R}\right)\right]. \quad (1)$$

By hypothesis, V is charge-free, so $\rho \equiv 0$ in V and the first term vanishes. $\rho = 0$ also ensures that over the surface of the sphere: $\oint_S (\mathbb{E}\cdot\hat{n}')dS' = \int_V (\nabla\cdot\mathbb{E})\,d^3x' = 4\pi Q_{in} \equiv 0$.

2. Now shift sphere center to $\mathbf{r} = 0$, so we are calculating ϕ at the center. Since now $R = r' = a$ (sphere radius), Eq. (1) gives:

$$\rightarrow \phi(\text{center}) = \frac{1}{4\pi}\oint_{\text{sphere}:a} dS'\left[\frac{1}{a}\{\hat{n}'\cdot\nabla'\phi\} - \phi(\mathbf{r}')\left\{\hat{n}'\cdot\nabla'\left(\frac{1}{R}\right)\right\}\right]. \quad (2)$$

We have set $\partial/\partial n' = \hat{n}'\cdot\nabla'$ (by def$^{\underline{n}}$). For the first term RHS in Eq. (2), note that $\nabla'\phi = -\mathbb{E}'$, so the integral is

$$\frac{1}{4\pi}\oint_{\text{sphere}:a} dS'\left[\frac{1}{a}\{\hat{n}'\cdot\nabla'\phi\}\right] = (-)\frac{1}{4\pi a}\oint(\mathbb{E}'\cdot\hat{n}')\,dS' \equiv 0, \quad (3)$$

since $\rho \equiv 0$ in V. As for the second term RHS in Eq. (2), note...

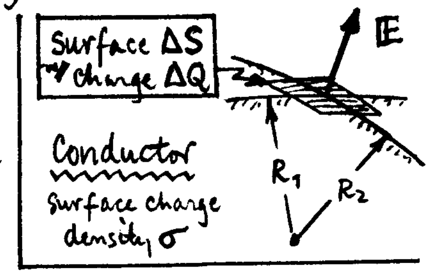
$$\hat{n}'\cdot\nabla'\left(\frac{1}{R}\right) = +\,\hat{n}'\cdot\frac{\mathbb{R}}{R^3} = (-)\frac{1}{r'^2}, \quad \text{since } \mathbb{R} = 0 - \mathbf{r}' \ \& \ \hat{n}' = \frac{\mathbf{r}'}{r'}. \quad (4)$$

But $r' = a$ on S, so this result is: $\hat{n}'\cdot\nabla'\left(\frac{1}{R}\right) = -1/a^2$. Putting it in Eq. (2), we get the desired Mean Value Theorem:

$$\boxed{\phi(\text{center}) = \frac{1}{4\pi a^2}\oint_S \phi(\mathbf{r}')\,dS'}. \qquad \underline{\underline{\text{QED}}} \quad (5)$$

In a charge-free region (where $\nabla^2\phi = 0$), the value of ϕ at the center of a sphere of arbitrary radius a equals its average value on the sphere.

⑨ Find normal derivative $\partial E/\partial n$ at surface of a charged conductor.



1. Gauss' Law: $\oint \mathbb{E}\cdot d\mathbb{S} = 4\pi Q_{in}$, applied to a pillbox with one face inside the conductor (where $\mathbb{E} \equiv 0$) and one face just outside immediately $\Rightarrow$ $\mathbb{E}$ is normal to the conducting surface, and of size $\underline{E = 4\pi\sigma}$, where σ is the local surface charge density. For a small element of surface area ΔS bearing charge ΔQ: $E = 4\pi\,\Delta Q/\Delta S$. This can be related to the curvatures R_1 & R_2 by noting: $\Delta S = (R_1\Delta\theta_1)(R_2\Delta\theta_2)$, if $\Delta\theta_1$ & $\Delta\theta_2$ are the angular extents of ΔS [orthogonality assumed]. So, altogether...

$$\rightarrow E = \frac{1}{R_1 R_2}\left(4\pi\,\Delta Q/\Delta\theta_1\Delta\theta_2\right). \tag{1}$$

Notice that for small changes in R_1 & R_2, this gives: $\frac{\partial E}{\partial R_i} = (-)\frac{E}{R_i}$.

2. At the center of ΔS (now imagined to be infinitesimal), the unit normal $\hat{n} = \frac{1}{2}(\hat{R}_1 + \hat{R}_2)$, w/ $\hat{R}_i = \mathbb{R}_i/R_i$. So the normal derivative of E is

$$\frac{\partial E}{\partial n} = \hat{n}\cdot\nabla E = \frac{1}{2}(\hat{R}_1+\hat{R}_2)\cdot\left[\hat{R}_1\frac{\partial E}{\partial R_1} + \hat{R}_2\frac{\partial E}{\partial R_2}\right]$$

on $$\frac{\partial E}{\partial n} = \frac{1}{2}(1+\hat{R}_1\cdot\hat{R}_2)\left[\frac{\partial E}{\partial R_1} + \frac{\partial E}{\partial R_2}\right]. \tag{2}$$

But, near the center of ΔS, $\hat{R}_1$ & $\hat{R}_2$ are $\sim$ parallel, so $\hat{R}_1\cdot\hat{R}_2 = 1$. Also, $\partial E/\partial R_i = -E/R_i$, by above analysis. Putting this together, get

$$\boxed{\frac{1}{E}\left(\frac{\partial E}{\partial n}\right) = -\left(\frac{1}{R_1}+\frac{1}{R_2}\right)}, \text{ and } E = 4\pi\sigma. \quad \underline{\underline{\textbf{QED}}} \tag{3}$$

Near a sharp point on a conducting surface, where $R_1 \sim R_2 \sim R \rightarrow 0$, enormous electric field gradients exist. This "explains" lightning rods.