DEPARTMENT OF PHYSICS

M.S. COMPREHENSIVE / PH. D. QUALIFYING EXAMINATION

NOVEMBER 26, 1984

DEPARTMENT OF PHYSICS

M.S. COMPREHENSIVE and PH.D. QUALIFYING EXAM

MONDAY, 26 NOVEMBER 1984, 8 AM-12 NOON

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Answer each of the following eight (8) questions.

All questions are of equal weight.

Begin your answer to each question on a <u>new</u> sheet of paper. Solutions to different questions must <u>not</u> appear on the same sheet of paper.

Label each page of your answer sheets as follows:

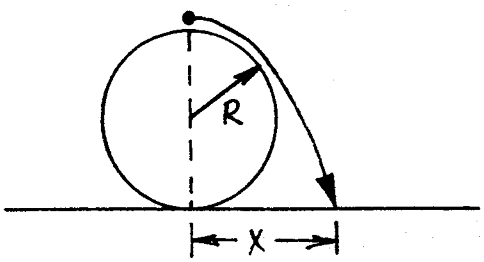
A. Your name in upper left-hand corner.

B. Problem number, and page number <u>for that problem</u>, in upper right-hand corner.

1. A particle starts from rest at the top of a frictionless sphere of radius R and slides off the sphere under the force of gravity.

(a) How far below its starting point does it get before flying off the sphere?

(b) How far horizontally does it land from the sphere (x)?



Questions 1-8 sequentially one after the other with space for figures allocated when fitting several questions per page.

1. ~~(2)~~ A particle starts from rest at the top of a frictionless sphere of radius R and slides off the sphere under the force of gravity.

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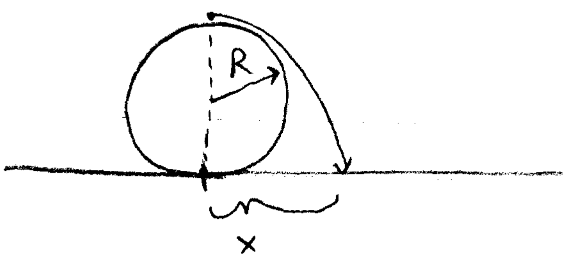
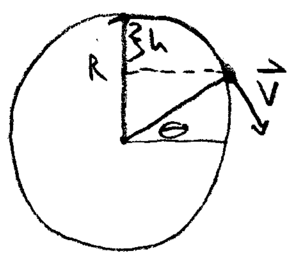


Problem 2 solution

(a) the particle can stay on the surface of the sphere only as long as the normal component of the force is greater than or equal to the mass of the particle times its centripetal acceleration

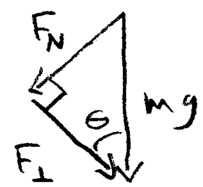


$$\tfrac{1}{2}mv^2 = mgh = mg\,R\,(1-\sin\theta)$$
$$v^2 = 2gR\,(1-\sin\theta)$$

centripetal acceleration needed for particle to continue in circular path

$$\frac{v^2}{R} = 2g\,(1-\sin\theta)$$

now take the normal component of the gravitational force:



$$F_N = mg\sin\theta$$

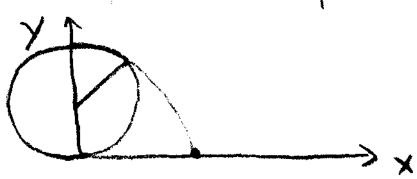
The particle will fly off the sphere when

$$\frac{F_N}{m} = \frac{v^2}{R} \rightarrow \cancel{g}\sin\theta = 2\cancel{g}\,(1-\sin\theta)$$

$$2 = 3\sin\theta \qquad \boxed{\sin\theta = 2/3}\text{, or } \boxed{h = R/3}$$

how far below starting point it flys off

(b) first, let's set up a coordinate system:



The constraint of staying on the sphere's surface vanishes at the point

$$x = R\cos\theta = (\sqrt{5}/3)\,R$$
$$y = 2R - h = (5/3)\,R$$

at that point, $V = \sqrt{2gR/3} \Rightarrow V_x = V\sin\theta = \left(2\sqrt{2gR/3}\right)/3$

$$V_y = -V\cos\theta = -\left(\sqrt{10gR/3}\right)/3$$

we now use these values of x, y, v_x, v_y as initial conditions for the remaining unconstrained free fall of the particle:

$$m a_x = 0$$
$$m a_y = -g$$

so
$$x = x_0 + v_x^0 t$$
$$y = y_0 + v_y^0 t - \tfrac{1}{2} g t^2$$

$$x = (\sqrt{5}/3) R + \frac{2}{3}\sqrt{\frac{2gR}{3}}\, t$$

$$y = \frac{5}{3} R - \sqrt{\frac{10gR}{3}}\left(\frac{1}{3}\right) t - \frac{1}{2} g t^2$$

set $y=0$, solve for t, substitute into $x(t)$ to find x.

$$g t^2 + \frac{2}{3}\sqrt{\frac{10gR}{3}}\, t - \frac{10}{3} R = 0 \qquad t = \frac{1}{2g}\left\{ -\frac{2}{3}\sqrt{\frac{10gR}{3}} + \left[\frac{4}{9}\cdot\frac{10gR}{3} + \frac{40}{3} Rg\right]^{1/2} \right\}$$

(+ because we want $t>0$ root)

$$\boxed{t_{ground} = \frac{1}{3}\left[\frac{10}{\sqrt{3}} - \sqrt{\frac{10}{3}}\right]\sqrt{\frac{R}{g}}}$$

so

$$x = (\sqrt{5}/3) R + \frac{2}{3}\sqrt{\frac{2gR}{3}}\left[\frac{1}{3}\left(\frac{10}{\sqrt{3}} - \sqrt{\frac{10}{3}}\right)\sqrt{\frac{R}{g}}\right]$$

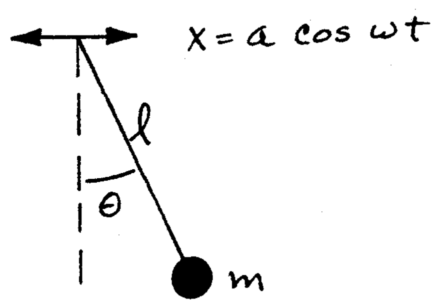
$$\boxed{x = \left(\frac{20\sqrt{2} + 5\sqrt{5}}{27}\right) R \approx \quad R}$$

2. A pendulum bob of mass m is suspended by a string of length l from a moving point of support. The point of support moves to and fro along a horizontal x-axis according to the equation $x = a\cos\omega t$.

Assume that the pendulum swings only in the vertical plane containing the x-axis. Let the position of the pendulum be described by the angle θ which the string makes with a line vertically downward.

(a) Set up the equation of motion for arbitrary amplitude.

(b) Find the steady-state amplitude for small oscillations as a function of m, l, a, and ω



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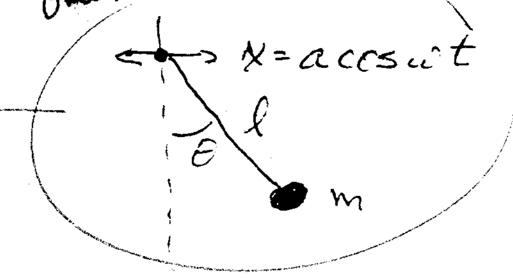
2. A pendulum bob of mass m is suspended by a string of length l from a moving point of support. The point of support moves to and fro along a horizontal x-axis according to the equation $x = a\cos\omega t$.

omega



Assume that the pendulum swings only in the vertical plane containing the x-axis. Let the position of the pendulum be described by the angle θ which the string makes with a line vertically downward

~~a) Set up the Lagrangian function~~

a) Set up the equation of motion for arbitrary amplitude.

b) Find the amplitude of steady-state ~~oscillations~~ for small oscillations; as a ~~function~~ of m, l, a, and ω.

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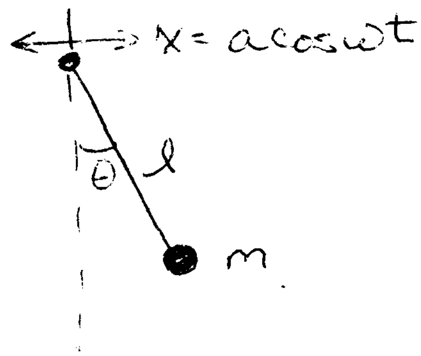


$$x = a\cos\omega t + l\sin\theta$$
$$y = l\cos\theta$$

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) = \frac{1}{2}m\left[(-a\omega\sin\omega t + l\cos\theta\,\dot{\theta})^2 + l^2\sin^2\theta\,\dot{\theta}^2\right]$$

$$= \frac{1}{2}m\left\{a^2\omega^2\sin^2\omega t - 2a\omega l\cos\theta\,\dot{\theta}\sin\omega t + l^2\dot{\theta}^2\right\}$$

$$V = -mgl\cos\theta$$

$$L = T - V = \frac{1}{2}m\left\{a^2\omega^2\sin^2\omega t + l^2\dot{\theta}^2 - 2a\omega l\cos\theta\sin\omega t\,\dot{\theta}\right\} + mgl\cos\theta$$

$$\frac{\partial L}{\partial\dot{\theta}} = ml^2\dot{\theta} - ma\omega l\cos\theta\sin\omega t$$

$$\frac{\partial L}{\partial\theta} = am\omega l\sin\theta\sin\omega t\,\dot{\theta} - mgl\sin\theta$$

$$ml^2\ddot{\theta} + ma\omega l \sin\theta \sin\omega t\,\dot{\theta} - ma\omega^2 l\cos\theta\cos\omega t$$

$$- ma\omega l\,\cancel{\sin\theta}\,\sin\omega t\,\dot{\theta} + mgl\sin\theta = 0$$

5

$$\Rightarrow \boxed{ml^2\ddot{\theta} + mgl\sin\theta = ma\omega^2 l\cos\theta\cos\omega t}$$

b) Small oscillations, set $\sin\theta \simeq \theta$

$\cos\theta \simeq 1$

divide by ml^2

$$\ddot{\theta} + (g/l)\,\theta = a\cos\omega t\left(\frac{\omega^2}{l}\right)$$

use eqn 2.148 - 2.163

Driven Oscillator. ~~Define $\frac{F_0}{m} = a\omega^2/l$ $\quad \omega_0^2 = g/l$~~

~~with $\theta_0 = 0$ (phase of driver)~~

~~$\beta = \pi/2$ ($\gamma = 0$ no damping)~~

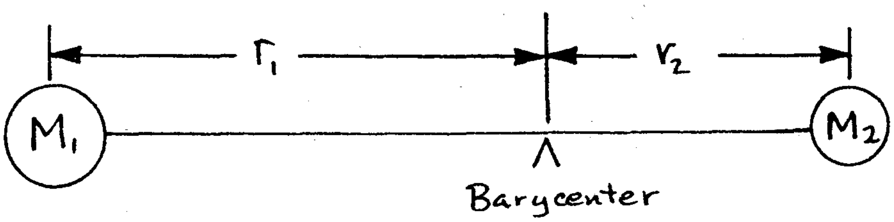
~~$$\theta = \frac{a\omega^2}{l}\left[\frac{1}{(\omega_0^2 - \omega^2)}\right]\cos\omega t$$~~

5

$$\boxed{\theta = \frac{a\omega^2\cos\omega t}{[g - l\omega^2]}}$$

Try $\theta = A\cos\omega t$ for steady state, solve for A gives

3. If two massive bodies revolve about each other under their mutual gravitational force, they each travel along an elliptical path about their common center of mass, the so-called barycenter.



(a) Derive Kepler's third law from Newton's laws for the special case when each orbit is a circle. Do <u>not</u> assume that the mass of one body can be neglected relative to that of the other.

(b) Use Kepler's third law to calculate the mass of the moon given the following data:

$G = 6.672 \times 10^{11}$ N-m^2/kg^2

M(earth) = 5.977×10^{24} kg

R(earth-moon) = 384404 km

T(sidereal of moon) = 27.322 days

(c) How far from the center of the earth is the barycenter?

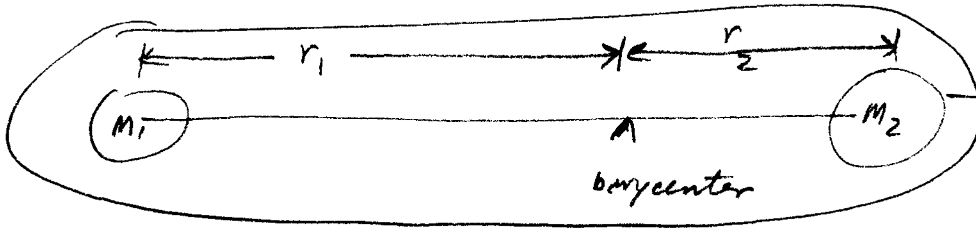
M.S. Comprehensive/Ph.D. Qualifying Exam

Larry D. Kirkpatrick

3.

If two massive bodies revolve about each other under their mutual gravitational force, they each travel along an elliptical path about their common center of mass, the so-called barycenter.

a. Derive Kepler's third law from Newton's laws for the special case when each orbit is a circle. Do **not** assume that the mass of one body can be neglected relative to that of the other.



b. Use Kepler's third law to calculate the mass of the moon given the following data:

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M(earth) = 5.977×10^{24} kg

R(earth-moon) = 384404 km

T(sidereal of moon) = 27.322 days

c. How far from the center of the earth is the barycenter?

Equating the gravitational force to the centripetal force we have

$$\frac{G m_1 m_2}{(r_1+r_2)^2} = \frac{m_1 v_1^2}{r_1} \qquad \text{with} \qquad v_1 = \frac{2\pi r_1}{T}$$

$$= \frac{m_1 4\pi^2 r_1}{T^2} \qquad \Rightarrow \qquad \frac{G m_2}{(r_1+r_2)^2} = \frac{4\pi^2 r_1}{T^2}$$

Similarly

$$\frac{G m_1}{(r_1+r_2)^2} = \frac{4\pi^2 r_2}{T^2}$$

Adding

$$\frac{G(m_1+m_2)}{(r_1+r_2)^2} = 4\pi^2 \frac{(r_1+r_2)}{T^2}$$

or

$$T^2 = \frac{4\pi^2}{G(m_1+m_2)} (r_1+r_2)^3$$

b) $$m_1 + m_2 = \frac{4\pi^2}{GT^2}(r_1+r_2)^3 = 6.031 \times 10^{24}\ kg$$

round off problem

$$m_e = 5.977 \times 10^{24}\ kg \Rightarrow m_m = 0.054 \times 10^{24}\ kg$$

(should be 0.074×10^{24} kg)

c) $m_e r_e = m_m r_m$ and $r_e + r_m = r$

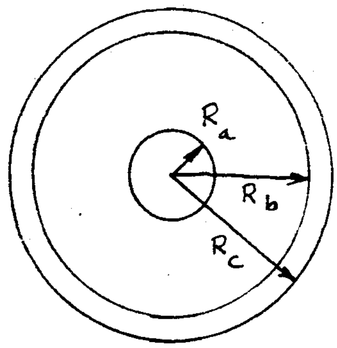
$$\Rightarrow r_e = \frac{m_m}{m_e+m_m} r = 3442\ km$$

(should be 4672 km)

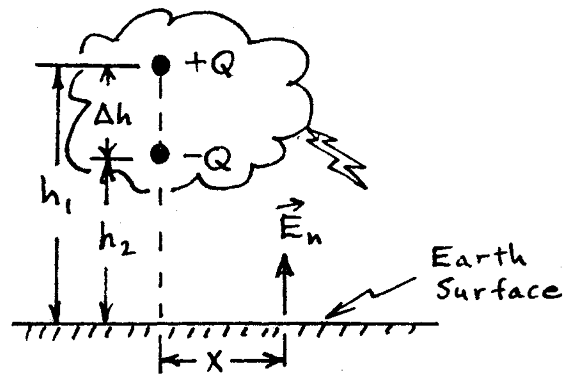
4. An infinitely long insulating rod of radius R_a has a uniform volume charge density $\rho > 0$. The rod has $\varepsilon=\varepsilon_o$. The rod is inside of, and on the axis of, an infinitely long concentric conductor of inner radius $R_b > R_a$ and outer radius R_c. The conductor has a charge density per unit length of $\lambda > 0$.

(a) Find the electric field vector for all r, where r is measured from the axis of symmetry.

(b) Find the values of the surface charge densities on the inner and outer surfaces of the conductor in terms of ρ and λ.



5. Inside a thunderstorm, a separation of charge is generated, with charge + Q at altitude h_1, and (-) Q at $h_2 < h_1$. Assume the sizes of these charge concentrations are small relative to: h_1, h_2 and $\Delta h = h_1 - h_2$. Also, assume the earth's surface can be approximated by a conducting plane. Measure distance x along the surface, with x = 0 directly below the $\pm$ Q charges.



(a) Find the electric field E_n normal to the earth's surface at distance x from the storm.

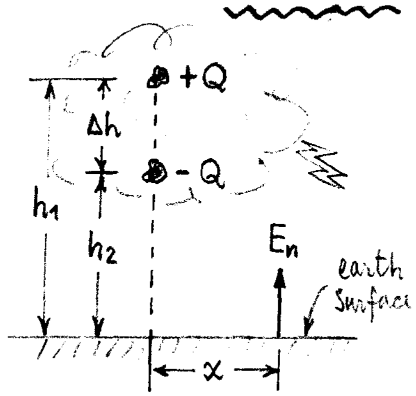
(b) Calculate the charge density induced on the surface directly below the storm (x = 0).

(c) Let $h_{1,2} = h \pm 1/2\,\Delta h$, with $\Delta h \ll h$. Find the x-value for which E_n vanishes. Neglect terms of order $(\Delta h/h)^2$.

M.S. Exam: Nov. 1984 (Robiscoe)

E & M: Electrostatics

6 Nov. 84



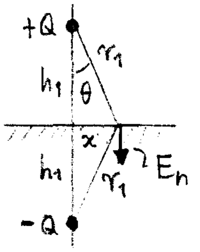
"Inside a thunderstorm, a separation of charge is generated, with charge $+Q$ at altitude h_1, and $(-)Q$ at $h_2 < h_1$. Assume the size of these charge concentrations are small relative to: h_1, h_2 and $\Delta h = h_1 - h_2$. Also, assume the earth's surface can be approximated by a conducting plane. Measure distance x along the surface, with $x=0$ directly below the $\pm Q$ charges.

A. Find the electric field E_n normal to the earth's surface at distance x from the storm.

B. Calculate the charge density induced on the surface directly below the storm ($x=0$).

C. Let $h_{1,2} = h \pm \frac{1}{2}\Delta h$, with $\Delta h \ll h$. Find the x-value for which E_n vanishes. Neglect terms of order $(\Delta h/h)^2$."

Solution: Use method of images. $+Q$ has image $-Q$ at distance h_1 below plane; $-Q$ has image $+Q$ at h_2 below plane; these pairs ensure plane $\equiv$ equipotential.



$$E_n(\text{by } +Q) = -\frac{2kQ}{r_1^2}\cos\theta = -\frac{2kQh_1}{r_1^3}, \quad r_1^2 = h_1^2 + x^2 \ \& \ k = \frac{1}{4\pi\epsilon_0} \text{ (MKS)}$$

Similarly, $$E_n(\text{by } -Q) = +\frac{2kQh_2}{r_2^3}, \quad r_2^2 = h_2^2 + x^2$$

A. Net normal field is: $E_n = E_n(\text{by } +Q) + E_n(\text{by } -Q) \ldots$

$$\boxed{E_n = 2kQ\left(\frac{h_2}{r_2^3} - \frac{h_1}{r_1^3}\right) = 2kQ\left[\frac{h_2}{(h_2^2+x^2)^{3/2}} - \frac{h_1}{(h_1^2+x^2)^{3/2}}\right]}$$

B. Field directly below storm ($x=0$) is...

$$E_{n0} = 2kQ\left(\frac{1}{h_2^2} - \frac{1}{h_1^2}\right) = \frac{2kQ}{h_1^2 h_2^2}(h_1^2 - h_2^2) = 4kQ\frac{h\,\Delta h}{h_1^2 h_2^2} > 0.$$

The charge density is: $\boxed{\sigma_0 = E_{n0}/4\pi k = \frac{Q}{2\pi}\left(\frac{1}{h_2^2} - \frac{1}{h_1^2}\right)} > 0.$ Means, at some x, both σ & E_n vanish.

C. The denominators appearing in part A are ...

$$(h_{1,2}^2 + x^2)^{\frac{3}{2}} = \left[(h \pm \tfrac{1}{2}\Delta h)^2 + x^2\right]^{\frac{3}{2}} = \left[h^2 + x^2 \pm h\Delta h + \cancelto{\text{negl.}}{\mathcal{O}(\Delta h)^2}\right]^{\frac{3}{2}}$$

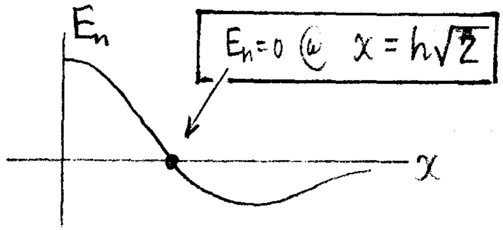
$$\simeq (h^2+x^2)^{\frac{3}{2}}\left[1 \pm \frac{h\Delta h}{h^2+x^2}\right]^{\frac{3}{2}}, \text{ to } \mathcal{O}(\Delta h/h),$$

So, $(h_{1,2}^2 + x^2)^{-\frac{3}{2}} \simeq (h^2+x^2)^{-\frac{3}{2}}\left[1 \mp \frac{3}{2}\frac{h\Delta h}{h^2+x^2}\right]$, by Binomial Expansion

Use this in E_n of part A to find...

$$E_n \simeq \frac{2kQ}{(h^2+x^2)^{3/2}}\left[(h - \tfrac{1}{2}\Delta h)\left(1 + \frac{3}{2}\frac{h\Delta h}{h^2+x^2}\right) - (h + \tfrac{1}{2}\Delta h)\left(1 - \frac{3}{2}\frac{h\Delta h}{h^2+x^2}\right)\right] =$$

so, $$\underline{\underline{E_n \simeq \frac{2kQ\Delta h}{(h^2+x^2)^{3/2}} \cdot \left(\frac{2h^2 - x^2}{h^2+x^2}\right)}}, \text{ to } \mathcal{O}\left(\frac{\Delta h}{h}\right).$$



For large x, this $\sim$ dipole field, as it must. $E_n > 0$ for $0 \leq x < h\sqrt{2}$, $E_n < 0$ for $h\sqrt{2} < x \to \infty$, and the cross-over is @ $x = h\sqrt{2}$, as shown.

6. Find the atmospheric pressure p as a function of altitude z above the earth's surface on the assumption that the temperature, T, decreases with altitude, according to $T = T_o (1-\alpha z)$. At the surface, $z = 0$, $p = p_o$, and $T = T_o$.

Dick Smith
#1 8/84

Maybe too easy for Comp ?

Fall 84

l.c.

(p) l.c.

6. Find the ~~atmospheric~~ l.c. atmospheric pressure, (p) as a function of altitude, (z) above the earth's surface on the assumption that the temperature, T, decreases with altitude, according to $T = T_0(1 - \alpha z)$. At the surface, $z=0$ $P = P_0$ and $T = T_0$.

— " —

Take $T = T_0(1-\alpha z)$ + treat air as ideal gas (Hint ?)

$$pV = RT \quad \text{or} \quad \rho = \frac{M}{V} = \frac{Mp}{RT}$$

Main point ⇒ Static equil. ⇒ $\vec{f} = \nabla p$ and use $\vec{f} = \rho \vec{g}$

So

$$\frac{dp}{dz} = -\rho g = -\frac{Mg}{RT}p$$

or

$$\frac{dp}{p} = -\frac{Mg}{RT_0}\frac{dz}{(1-\alpha z)}$$

Integrate from $P_0 \to P$

Let $u = 1-\alpha z$
$du = -\alpha dz$

$$\ln(P/P_0) = -\frac{Mg}{RT_0}\int_0^z \frac{dz}{(1-\alpha z)}$$

$$= \frac{Mg}{\alpha R T_0}\ln(1-\alpha z)$$

$$\Rightarrow \boxed{p = p_0(1-\alpha z)^{Mg/\alpha R T_0}}$$

7. Evaluate the integral $I = \int_0^{\pi} \frac{\cos\theta \, d\theta}{5 - 4\cos\theta}$ by the contour method.

Math ϕ - GF J

7. Evaluate the integral $I = \int_0^{\pi} \frac{\cos\theta \, d\theta}{5 - 4\cos\theta}$

by contour method

Soln: Note $\cos\theta$ is even fcn. of θ, so $I = \frac{1}{2}\int_0^{2\pi} \frac{\cos\theta \, d\theta}{5 - 4\cos\theta}$

then let $z = e^{i\theta}$, so I becomes a contour integral about the unit circle:

$$dz = ie^{i\theta} d\theta = iz\,d\theta, \quad \text{or } d\theta = -i\frac{dz}{z}$$

$$\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right)$$

$$I = \frac{1}{2i} \oint \frac{dz}{z} \frac{\frac{1}{2}(z + \frac{1}{z})}{5 - 2(z + \frac{1}{z})} = -\frac{1}{4i} \oint \frac{dz\,(z^2+1)}{z\,(2z^2 - 5z + 2)}$$

$$= -\frac{1}{8i} \oint \frac{dz\,(z^2+1)}{z(z-2)(z-\frac{1}{2})} = -\frac{2\pi i}{8i}\left\{\frac{+1}{(-2)(-\frac{1}{2})} + \frac{((\frac{1}{2})^2+1)}{(\frac{1}{2})(-\frac{3}{2})}\right\}$$

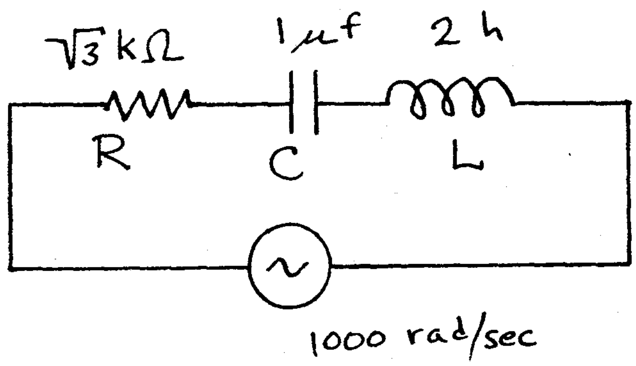
$$= -\frac{\pi}{4}\left\{1 - \frac{5}{3}\right\} = +\frac{\pi}{6}$$

8. You are given an L,C,R, series circuit connected to a generator operating at an angular frequency of 1000 radians per second. L is two Henries, C is one microFarad and R is $\sqrt{3}$ kiloOhm.

(a) Find the impedance seen by the generator.

(b) If the generator current is given by $I_0 \cos\omega t$, what is the phase angle ϕ of the voltage across the generator?

(c) If the generator frequency is allowed to vary, the circuit exhibits a resonance behavior. Discuss the properties of resonance in this circuit, e.g. frequency, voltage, current, phase, width, etc.



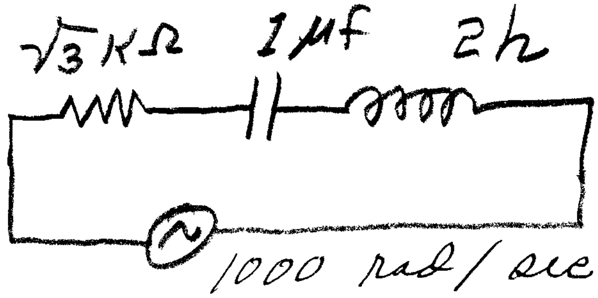
?

8. You are given an L, C, R, series circuit connected to a generator operating at an angular frequency of 1000 radians per second. L is two Henries, C ~~is~~ one microFarad and R is $\sqrt{3}$ kΩ ~~kilo ohms~~.

a) Find the impedance seen by the generator

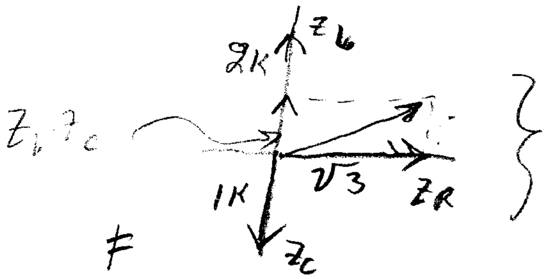
b) ~~Assume that~~ If the generator current is given by $I_0 \cos\omega t$, what is the phase angle ϕ of the voltage across the generator?

c) If the generator frequency is allowed to vary the circuit exhibits a resonance behavior. Discuss the properties of resonance in this circuit, e.g. frequency, voltage, current, phase, width, etc.



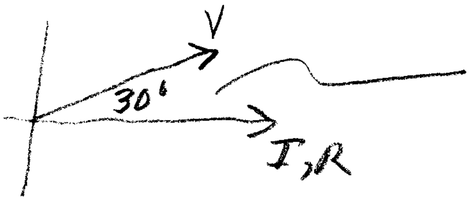
ans

a) $\frac{1}{\omega C} = \frac{1}{10^3 10^{-6}} = 1K\Omega$, $\omega L = 10^3 2 = 2K$

ǂ 

$\Rightarrow |Z| = 2K\Omega$

$\tan\theta = \frac{1}{\sqrt{3}} \rightarrow \theta = 30°$

b) 

30° ahead of I

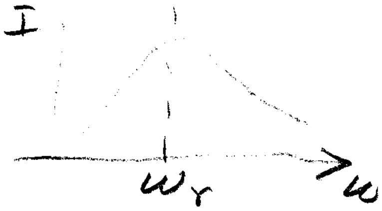
ǂ or $Z = R + \frac{1}{j\omega C} + j\omega L = \sqrt{3}\,10^3 + (-j)10^3 + j\,2\,10^3$

$= [\sqrt{3} + j(+1)]\,10^3$

$|Z| = \sqrt{3+1} = \sqrt{4} = 2\ K\Omega$

$\tan\theta = \frac{1}{\sqrt{3}}$

c) the current goes through a max



and $\omega_r = \frac{1}{\sqrt{LC}}$

$|Z|$ is minimum, resulting from the Z_C + Z_L "balancing" out $\Rightarrow Z = R$, $\phi_r = 0$

The width of the curve is called the band width and is usually defined by the half power points which lead to $\Delta f = R/2\pi L$ etc

DEPARTMENT OF PHYSICS

M.S. COMPREHENSIVE and PH.D. QUALIFYING EXAM

MONDAY, 26 NOVEMBER 1984, 1 PM-5PM

* * * * * * *

Answer each of the following eight (8) questions.

All questions are of equal weight.

Begin your answer to each question on a new sheet of paper. Solutions to different questions must not appear on the same sheet of paper.

Label each page of your answer sheets as follows:

A. Your name in upper left-hand corner.

B. Problem number, and page number for that problem, in upper right-hand corner.

9. (a) Consider a system of spin 1/2. What are the eigenvalues and normalized eigenvectors of the operator $AS_y + BS_z$, where S_y, S_z are the angular momentum operators, and A and B are real constants?

(b) Assume that the system is in a state corresponding to the upper eigenvalue. What is the probability that a measurement of S_y will yield the value $+\hbar/2$?

The Pauli matrices are

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Swenson sol'n. (a) $AS_y + BS_z = \frac{\hbar}{2}\begin{pmatrix} B & -iA \\ iA & -B \end{pmatrix}$ Find e.v. and efns. $\hookrightarrow \lambda\frac{\hbar}{2}$ $\hookrightarrow \phi$

$$\frac{\hbar}{2}\begin{pmatrix} B & -iA \\ iA & -B \end{pmatrix}\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \lambda\frac{\hbar}{2}\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}.$$ sol'ns req $\|\ \|$ coeffs $= 0$.

eigenvalue prob. $\Rightarrow$ $\begin{vmatrix} B-\lambda & -iA \\ iA & -B-\lambda \end{vmatrix} = 0$. $-(B-\lambda)(B+\lambda) + A^2 = 0$

$B^2 - \lambda^2 + A^2 = 0$ Eigenvalues $\boxed{\lambda = \pm(B^2+A^2)^{1/2}}$

$B\phi_1 - iA\phi_2 = \lambda\phi_1$ $\phi_1 = \frac{iA}{B-\lambda}\phi_2$ $\phi_1 = \boxed{\frac{iA}{B \mp \sqrt{B^2+A^2}}\phi_2 \equiv \alpha_\pm \phi_2}$

$\phi_\pm = \begin{pmatrix} \alpha \\ 1 \end{pmatrix}\phi_2$ Normalize: $|\phi_2|^2 (\alpha^* \ 1)\begin{pmatrix} \alpha \\ 1 \end{pmatrix} = 1 = (\alpha\alpha^* + 1)|\phi_2|^2$

$|\phi_2|^2 = \frac{1}{1+\alpha\alpha^*}$; $1 + \alpha\alpha^* = \frac{A^2 + [B \mp \sqrt{B^2+A^2}]^2}{[B \mp \sqrt{B^2+A^2}]^2}$

choose ϕ_2 real $\phi_2 = \boxed{\frac{B \mp \sqrt{B^2+A^2}}{[A^2 + [B \mp \sqrt{B^2+A^2}]^2]^{1/2}} \equiv \beta_\pm}$ Define β, γ

$$\boxed{\gamma_\pm \equiv \alpha_\pm \beta_\pm = \frac{iA}{[A^2 + [B \mp \sqrt{B^2+A^2}]^2]^{1/2}}}$$

Normalized eigenvectors $\underline{\phi_\pm = \begin{pmatrix} \gamma_\pm \\ \beta_\pm \end{pmatrix}}$ $\Longleftrightarrow$ eigenvalues $= \underline{\pm\frac{\hbar}{2}(B^2+A^2)^{1/2}}$

(b) Need eigenvectors of S_y. Found easily from part (a) by setting $A = 1$, $B = 0$. e.vals. $= \pm\frac{\hbar}{2}$ and eigenfunctions are

$$\chi_\pm^y = \frac{1}{\sqrt{2}}\begin{pmatrix} \mp i \\ 1 \end{pmatrix}$$

Told that system in state ϕ_+. Expand ϕ_+ as linear combination $\chi_\pm^y$ and coeff. of χ_+^y is prob. amplitude to measure $\hbar/2$ for S_y.

$\phi_+ = a\chi_+^y + b\chi_-^y$ $|a|^2 =$ probability sought

or

$$\begin{pmatrix} \gamma_+ \\ \beta_+ \end{pmatrix} = \frac{a}{\sqrt{2}}\begin{pmatrix} -i \\ 1 \end{pmatrix} + \frac{b}{\sqrt{2}}\begin{pmatrix} i \\ 1 \end{pmatrix} \Rightarrow \left.\begin{array}{l} \sqrt{2}\gamma_+ = -ia + ib \\ \sqrt{2}\beta_+ = a + b \end{array}\right\} \text{eliminate } b$$

$$a = \frac{1}{\sqrt{2}}(i\gamma_+ + \beta_+)$$

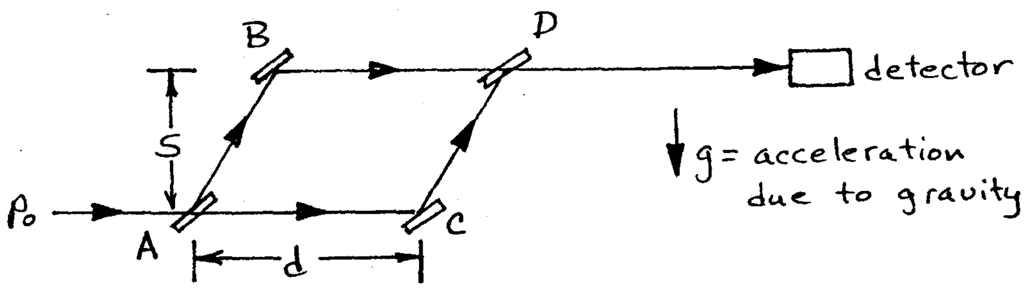
$\Rightarrow$ Probability to find $+\frac{\hbar}{2}$ for S_y $= a^*a = \frac{1}{2}\left(\beta_+^2 + \gamma_+\gamma_+^* + i(\gamma_+ - \gamma_+^*)\beta_+\right)$

$$= \frac{1}{2}\,\frac{B^2 + A^2 - AB + (A-B)\sqrt{B^2+A^2}}{A^2 + B^2 - B\sqrt{B^2+A^2}} = \frac{1}{2}\left[1 + \frac{A(\sqrt{B^2+A^2} - B)}{A^2+B^2-B\sqrt{A^2+B^2}}\right]$$

10. The Colella-Overhauser-Werner experiment (Phys. Rev. Lett. 34, 1472(1975)) measures the effects of gravitation on quantum interference.

A beam of nonrelativistic neutrons with momentum p_o and mass M is split and Bragg reflected through a carefully prepared silicon crystal. The neutrons in the upper beam, ABD, will have a different wavelength over the path BD than the neutrons in the lower beam AC. The difference is due to the difference in gravitational potential between the paths BD and AC. The crystal has been constructed so that the path length ABD is equal to the path length ACD. (The paths ABD and ACD are within the Si crystal.)

During the experiment, interference fringes are observed with the crystal held at various angles to the vertical. When B and C are at the same height (all paths horizontal) the paths ABD and ACD are indistinguishable and there is no phase shift. When the crystal is rotated 90^o about the incident beam axis so that ABCD lies in the vertical plane, the phase shift between the two beams is maximized.



(a) Calculate the phase shift, $\Delta\phi$, between the two beams when the crystal is oriented vertically (as shown above), as measured when the beams recombine at D. You may assume that $p_o^2/2M \gg Mgs$.

(b) If $M = 1.67\text{x}10^{-27}$kg; $g = 9.8$ m/sec^2; $\hbar = 1.05\text{x}10^{-34}$ J. sec; $d = 3$cm , $s = 2$cm, and $p_o = 4\text{x}10^{-24}$ kg m/sec (i.e., $v_o \sim 10^3$ m/sec), then how big is $\Delta\phi$? Is this an observable shift in an interference pattern?

Note: ignore the bending of the neutron beams caused by "g" (This is a higher order effect).

Hirsch?

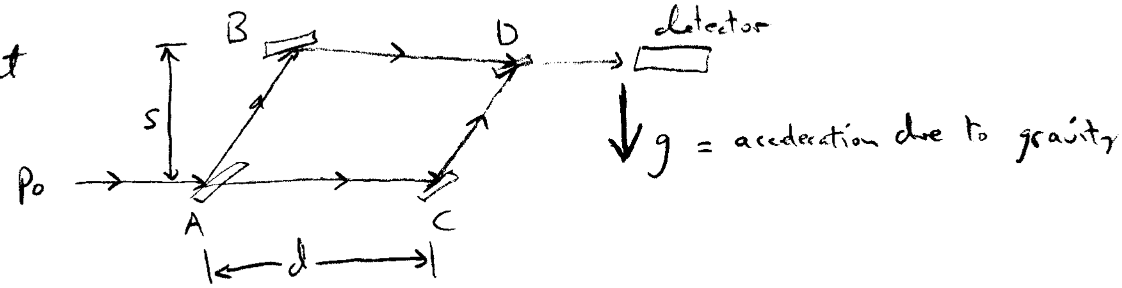
10.

nonrelativistic

~~(3)~~ The Colella - Overhauser - Werner experiment (Phys. Rev. Lett. 34, 1472 (1975)) measures the effects of gravitation on quantum interference.

A beam of neutrons with momentum p_0 (and mass M) is split and Bragg reflected through a carefully prepared silicon crystal. The neutrons in the upper beam, ABD, will have a different wavelength over the path BD than the neutrons in the lower beam AC. The difference is due to the difference in gravitational potential between the paths BD and AC. The crystal has been constructed so that the path length ABD is equal to the path length ACD. (*The paths ABD and ACD are within the Si crystal.*)

During the experiment, interference fringes are observed with the crystal held at various angles to the vertical. When B and C are at the same height (all paths horizontal) the paths ABD and ACD are indistinguishable and there is no phase shift. When the crystal is rotated 90° about the incident beam axis so that ABCD lies in the vertical plane, the phase shift between the two beams is maximized.



(a) Calculate the phase shift, $\Delta\phi$, between the two beams when the crystal is oriented vertically (as shown above), as measured when the beams recombine at D. You may assume that $p_0^2/2M \gg Mgs$

(b) If $M = 1.67 \times 10^{-27}$ kg ; $g = 9.8$ m/sec^2 ; $\hbar = 1.05 \times 10^{-34}$ J.sec; $d = 3$ cm $s = 2$ cm, and $p_0 = 4 \times 10^{-24}$ kg m/sec (i.e., $V_0 \sim 10^3$ m/sec), then how big is $\Delta\phi$? Is this an observable shift in an interference pattern?

Note: ignore the bending of the neutron beams caused by "g" (This is a higher order effect).

(3) Solution

(a) neutrons injected with $p = p_0$ $\quad \lambda_0 = \frac{h}{p_0}$

along upper path, the neutrons kinetic energy is reduced

$$T_{up} = \frac{p_0^2}{2M} - Mgs = p_{up}^2/2M$$

$$\Rightarrow p_{up}^2 = p_0^2 - 2M^2gs \qquad p_{up} = p_0\sqrt{1 - \frac{2M^2gs}{p_0^2}}$$

$$\text{so} \quad \lambda_{up} = \frac{h}{p_0\sqrt{1 - \frac{2M^2gs}{p_0^2}}} = \frac{\lambda_0}{\sqrt{1 - \frac{2M^2gs}{p_0^2}}}$$

The total phase shift will be $\Delta\phi = (\lambda_{up} - \lambda_0) \cdot \frac{d}{\lambda_0} \cdot \frac{2\pi}{\lambda_0}$

$$\boxed{\Delta\phi = \left[\frac{1}{\sqrt{1 - \frac{2M^2gs}{p_0^2}}} - 1\right] \cdot \frac{p_0 d}{\hbar}}$$

or, to first order in $\frac{M^2gs}{p_0^2}$, $\leftarrow$ legit since $\frac{p_0^2}{2M} \gg Mgs$

$$\boxed{\Delta\phi = \frac{M^2g}{\hbar p_0} d \cdot s}$$

(b) $\frac{M^2g}{\hbar p_0} d \cdot s = \Delta\phi$

$$\frac{(1.67\times10^{-27})^2(9.8)(.03)(.02)}{(1.05\times10^{-34})(4\times10^{-24})} = \underline{39 \text{ radians}}$$

or $\Delta N = \frac{\Delta\phi}{2\pi} \sim 6.2$ fringes $\qquad$ easily observable (and it was)

11. A 50 Gev electron hits a proton at rest. Assume a head-on collision.

(a) What is the total energy in the center of mass frame?

(b) Assume further that the collision is perfectly elastic. What are the final energies of the electron and the proton in the lab frame?

Kinnersley

11. Q. A 50 GeV electron hits a proton at rest. Assume a head-on collision.

u.c. ℓ.c.

a) What is the total energy in the C.M. frame?
center of mass

b) Assume further that the collision is perfectly elastic. What are the final energies of the electron and the proton in the lab frame?

A. Before the collision, the momentum 4-vectors are:

$$\vec{P}_P = (0, E_0/c) \qquad E_0 = 940\,\text{MeV}$$
$$P_e = (\gamma m_e v, \gamma m_e c) \approx (E/c, E/c) \qquad E = 50\,\text{GeV}$$

a) Find the CM frame, where the spatial parts of $\vec{P}_P, \vec{P}_e$ are equal and opposite:

$$P_P' = E_0/c\,(-\operatorname{sh}\chi, \operatorname{ch}\chi)$$
$$P_e' = E/c\,(e^{-\chi}, e^{-\chi})$$

$$E e^{-\chi} = E_0 \operatorname{sh}\chi \approx E_0\left(\tfrac{1}{2}e^{\chi}\right)$$
$$e^{\chi} = \sqrt{\frac{2E}{E_0}} \approx 10$$

$P_e' = (50)\left(\frac{1}{10}\right) = 5\,\text{GeV}/c$ for each particle.

Total CM energy $= \boxed{10\,\text{GeV}}$

b) After the collision, the space parts of P_P, P_e change sign:

$$\vec{P}_P{}' = E_0/c\,(+\,sh\,\chi,\ ch\,\chi)$$
$$\vec{P}_e{}' = E/c\,(-e^{-\chi},\ e^{-\chi})$$

Now transform back, using $\chi \to -\chi$:

$$\vec{P}_e{}'' = E/c\,(-e^{-2\chi},\ e^{-2\chi})$$
$$\vec{P}_P{}'' = E_0/c\,(2\,sh\chi\,ch\chi,\ \underbrace{ch^2\chi + sh^2\chi}_{\approx \frac{1}{2}e^{2\chi}})$$

The energies are:

$$E_e'' = (50)\left(\tfrac{1}{10}\right)^2 = 0.5\ \text{GeV}$$ (now moving to the left)

$$E_p'' = (1)\left(\tfrac{1}{2}\right)(100) = 50\ \text{GeV}$$ (actually 49.5)

Note at these ultrarelativistic energies, $m_p \gg m_e$ no longer matters! The particles are both light, and we get a billiard ball effect in which the particles trade energy.

12. Useful integral $\int_0^\infty e^{-x^2} dx = \pi/2$

A box has a very dilute gas of U^{235} and U^{238} atoms. Particle densities are n_{235} and n_{238} respectively. Mass densities are ρ_{235} and ρ_{238} respectively. The gas is at temperature T. A small hole of area S opens into a vacuum. The total rates of escape $\dot{N}_{235}$ and $\dot{N}_{238}$ of U^{235} and U^{238} atoms are given by

$$\dot{N}_{235} = S\, n_{235}\, f_{235} \qquad \text{atoms/sec}$$
$$\dot{N}_{238} = S\, n_{238}\, f_{238} \qquad \text{atoms/sec}$$

(a) Use dimensional analysis to find the dimensions of the f factor.

(b) What is the ratio of f_{235}/f_{238}?

(c) What is the ratio of $\dot{N}_{235}/\dot{N}_{238}$?

(d) Evaluate f using a kinetic theory model in which the probability of an atom having speed v is the Boltzmann distribution

$$P(V) = N \exp\left(-\frac{Mv^2}{2kT}\right) v^2$$

N is a normalization constant. M is the mass of an atom, k is Boltzmann's constant.

#12

Useful Integral $\int_0^\infty e^{-x^2}dx = \sqrt{\pi}/2$

A box has a very dilute gas of U^{235} and U^{238} atoms. Particle densities are n_{235} and n_{238} respectively. Mass densities are ρ_{235} and ρ_{238} respectively. The gas is at temperature T. A small hole of area S opens into a vacuum. The rate of escape of U^{235} and U^{238} atoms is given by

$$\dot{N}_{235} = S\, n_{235}\, f_{235} \quad \text{atoms/sec}$$

$$\dot{N}_{238} = S\, n_{238}\, f_{238} \quad \text{atoms/sec}$$

1. Use dimensional analysis to find the dimensions of the f factor.
2. What is the ratio of f_{235}/f_{238}?
3. What is the ratio of $\dot{N}_{235}/\dot{N}_{238}$?
4.

over

(4) Evaluate f using a kinetic theory model in which the probability of an atom having speed v is the Boltzmann distribution

$$P(v) = N\left(e^{-\frac{Mv^2}{2kT}}\right)v^2$$

N is a normalization constant. M is the mass of an atom, k is Boltzmann's constant.

Answer

$\dot{N} = \frac{1}{T}$ $A = L^2$ $n = \frac{1}{L^3}$

$f = L/T$ a velocity. It must be proportional to typical scale of thermal speed in gas $f \sim \sqrt{\frac{kT}{M}}$

so $\dot{N} \sim n/\sqrt{M}$ $M = \rho/n$

$$\dot{N} \sim n^{3/2}/\rho^{1/2}$$

$$\frac{\dot{N}_{235}}{\dot{N}_{238}} = \left(\frac{n_{235}}{n_{238}}\right)^{3/2}\left(\frac{\rho_{238}}{\rho_{235}}\right)^{1/2}$$

$$\dot{N} = \iiint_V \frac{A\cos\theta}{4\pi r^2}\, d\Omega\, r^2 dr\, P(v)\, d^3v\, n$$

$$\dot{N} = \frac{An}{2}\int_0^{\pi/2} \cos^2\theta\, d\theta \int P(v)\, v\, dv$$

$$= \frac{An\pi}{8}\,\bar{v} \qquad f = \frac{\pi}{8}\,\bar{v} = \sqrt{\frac{\pi kT}{8M}}$$

$$\bar{v} = \int_0^\infty P(v)\, v\, dv = \frac{\int_0^\infty e^{-\alpha v^2} v^3 dv}{\int_0^\infty e^{-\alpha v^2} v^2 dv} = \sqrt{\frac{8kT}{\pi M}}$$

$$= \frac{-\frac{d}{d\alpha}\int v\, dv\, e^{-\alpha v^2}}{-\frac{d}{d\alpha}\int e^{-\alpha v^2} dv}$$

13. A one-dimensional harmonic oscillator of mass m and natural frequency ω is described at t=0 by the wave function

$$\psi(x,o) = (u_o+u_1)/\sqrt{2}$$

where $$u_o = \frac{1}{\pi^{1/4}\sqrt{\Delta}} e^{-x^2/2\Delta^2}$$

$$u_1 = \sqrt{2}\left(\frac{x}{\Delta}\right)u_o$$

$$\Delta = \sqrt{\hbar/m\omega} .$$

Both u_o and u_1 are normalized. For t>0, compute $\langle E\rangle$ and $\langle x\rangle$.

QM J. Hermanson

A one-dimensional harmonic oscillator of mass m and natural frequency ω is described at $t=0$ by the wavefunction

$$\psi(x,0) = (u_0 + u_1)/\sqrt{2} \,,$$

where
$$u_0 = \frac{1}{\pi^{1/4}\sqrt{\Delta}} e^{-x^2/2\Delta^2}$$
$$u_1 = \sqrt{2}\left(\frac{x}{\Delta}\right) u_0$$
$$\Delta = \sqrt{\hbar/m\omega} \,.$$

Both u_0 and u_1 are normalized. For $t>0$, compute $\langle E \rangle$ and $\langle x \rangle$.

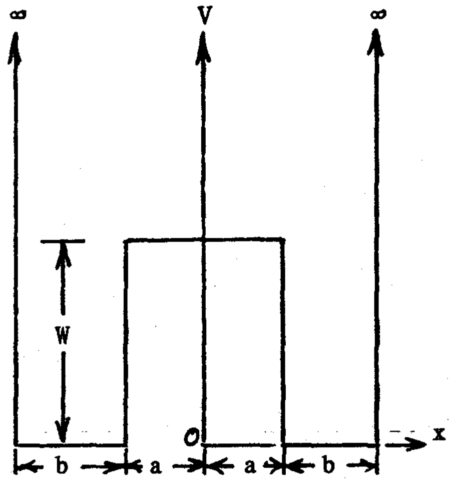
Soln:
$$\psi(x,t) = \left(u_0 e^{-i\omega t/2} + u_1 e^{-\frac{3}{2}i\omega t}\right)/\sqrt{2}$$
$$= a u_0 + b u_1$$

$$\langle E \rangle = |a|^2 \frac{\hbar\omega}{2} + |b|^2 \cdot \frac{3}{2}\hbar\omega$$
$$= \left(\frac{1}{4} + \frac{3}{4}\right)\frac{\hbar\omega}{2} = \hbar\omega \quad \text{[constant!]}$$

$$\langle x \rangle = (a^* b + b^* a)\int (u_0 x) u_1 \, dx$$
$$\left[\text{Since } \int u_0^2 x \, dx = \int u_1^2 x \, dx = 0\right]$$
$$= \frac{1}{2}\left(e^{-i\omega t} + e^{i\omega t}\right) \cdot \int \frac{\Delta}{\sqrt{2}} u_1^2 \, dx$$
$$\left[\int u_1^2 dx = 1\right]$$
$$= \frac{\Delta}{\sqrt{2}} \cos\omega t$$

[oscillates at classical freq!]

14.

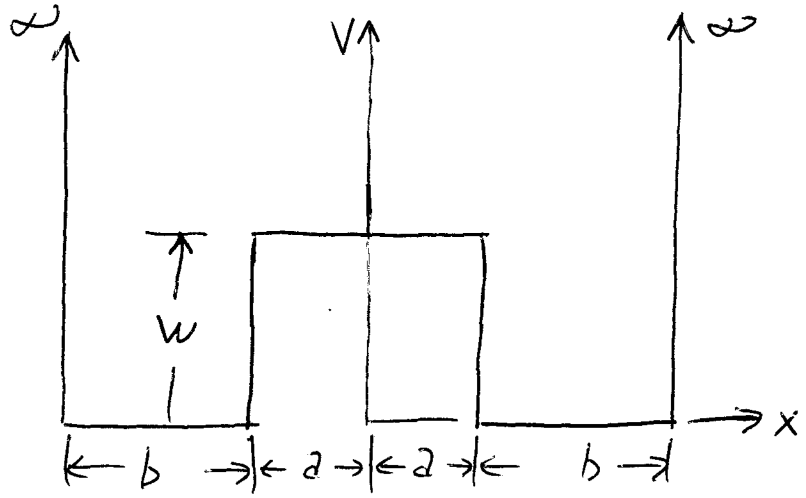


A particle of mass m is in this 1-d well.

a) Find the barrier height W such that the ground state energy $E_o = W$.

b) Find and describe the normalized ground state wave function for all x.

#14 Schmidt: Quantum Mechanics



A particle of mass m is in this 1-d well.

a) Find the barrier height W such that the ground state energy $E_0 = W$.

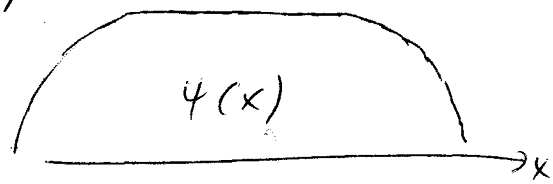
b) Describe the normalized ground state wave function for all x.

Schmidt: Quantum Mechanics Solution

a) Ground state wave function must have zero curvature, be symmetric, and have no zero crossings:

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + V\psi = E\psi \quad \text{for } V=0,\ \psi \text{ is sinusoidal:}$$

$$\psi = A\sin\alpha x' \quad (x' = x+a+b)$$



$$+\frac{\hbar^2\alpha^2}{2m} = E_0 = W$$

Also, to be flat, $\alpha b = \frac{\pi}{2}$, $\boxed{\alpha = \frac{\pi}{2b}}$

$$\boxed{\frac{\hbar^2\pi^2}{8mb^2} = E_0 = W}$$

b) $\int \psi^*\psi\, dx = 1 \qquad A^2 b + 2A^2 a = 1$

$$\boxed{A = \sqrt{\frac{1}{b+2a}}}$$

$-a-b \le x \le -a$: $\quad \psi = A\sin[\alpha(x+a+b)]$

$-a \le x \le a \quad \psi = A$

$a \le x \le a+b \quad \psi = A\sin[\alpha(a+b-x)]$

15. Sodium is an alkali metal with $z=11$.

a) What is the electron configuration of the ground state of an isolated sodium atom?

b) What are the L,S, and J quantum numbers of the state? How are they expressed in spectroscopic notation?

c) What is the most probable first excited state? What J values are possible for it?

#15.

Q. Sodium is an alkali metal with $Z = 11$.

a) What is the electron configuration of the ground state of an isolated sodium atom?

b) What are the L, S, and J quantum numbers of the state? How are they expressed in spectroscopic notation?

c) What is the most probable first excited state? What J values are possible for it

A. (a) $(1s)^2 (2s)^2 (2p)^6 (3s)$

(b) $n=3$ $^2S_{1/2}$ $\quad S=\frac{1}{2},\ L=0,\ J=\frac{1}{2}$

(c) $(3p) \Rightarrow L=1 \Rightarrow J=\frac{1}{2}, \frac{3}{2}$ $\quad {}^2P_{\frac{1}{2}},\ {}^2P_{3/2}$

16. Consider an optical fiber with refractive indices $n_{core}=1.552$ and $n_{cladding}=1.550$.

a) Calculate the critical angle for propagation and illustrate your results with a carefully labeled sketch.

b) For 10 km of this fiber, calculate the difference in propagation time between a ray propagating along the optic axis and one taking the longest path inside the fiber.

c) Comment on the significance of your result in part b for optical fiber communications at high pulse rates.

d) What other effect could limit the performance of an optical fiber system using short pulses?

#!6. Consider an optical fiber with refractive indices $n_{\cancel{core}} = 1.552$ and $n_{\cancel{cladding}} = 1.550$.

a) Calculate the critical angle for propagation and illustrate your results with a carefully labeled sketch.

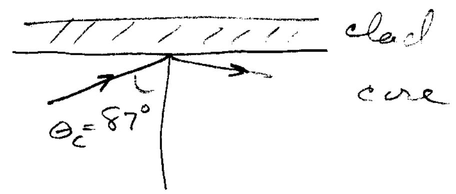
b) For 10 km of this fiber, calculate the difference in propagation time between a ray propagating along the optic axis and one taking the longest path inside the fiber.

c) Comment on the significance of your result in part b for optical fiber communications at high pulse rates.

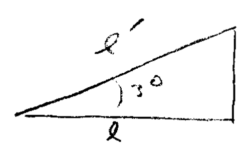
d) What other effect could limit the performance of an optical fiber system using short pulses?

A a) $n_{core} \sin \Theta_c = n_{cladding}$

$$\Theta_c = 87^\circ$$



b) $l' \cos 3^\circ = 10\text{ km}$



$\Delta l = l' - l = 13$ m $\qquad n\frac{\Delta l}{c} = 70$ nsec

c) Pulses must be at least this far apart or they merge together and information is lost. Real systems would be seriously limited by this design.

d) Chromatic dispersion arises from $\frac{dn}{d\lambda}$. Since short pulses contain a spread of λ's, this also leads to pulse stretching.