Notion of 4-Vectors

SRT: 4-Vectors, Kinematics & Dynamics; Special Math.

SRT arises from the search for and identification of kinematic (e.g.// space $\mathbf{r}$, time t) and dynamic (e.g.// momentum $\mathbf{p}$, energy E) quantities which -- in transforming event coordinates between relatively moving observers -- preserves the lightspeed $c \equiv$ cnst, and may also conserve other relations (e.g. $\mathbb{F} = d\mathbf{p}/dt$). A language to emphasize such preserved/conserved, or invariant/covariant characteristics of the theory is thus to be desired. That language is <u>the language of 4-vectors</u> & <u>4-tensors</u>.

1) The notion of a 4-vector, with an associated conserved characteristic (length), originates in the <u>invariance of the spacetime interval</u>, i.e. the fact that

$$\left.\begin{array}{l} \tilde{x} = (x_0, x_1, x_2, x_3) = (x_0, \mathbf{x}), \quad x_0 = ct, \quad \text{or// } d\tilde{x} = (dx_0, d\mathbf{x}), \\[6pt] \text{have Lorentz-invariant "lengths"} \left\{\begin{array}{l} \tilde{x}^2 = x_0^2 - \mathbf{x}^2 = \underline{\underline{\text{SAME}}} \\ (d\tilde{x})^2 = (dx_0)^2 - (d\mathbf{x})^2 = \underline{\underline{\text{SAME}}} \end{array}\right\rangle \text{in all inertial frames,} \\[6pt] \text{...by virtue of their transforming between frames, } K \to K', \text{ via Lorentz transf}^{\underline{n}}\text{...} \\[6pt] x_0 \to x_0' = \gamma(x_0 - \boldsymbol{\beta}\cdot\mathbf{x}); \quad x_\parallel \to x_\parallel' = \gamma(x_\parallel - \beta x_0), \\ \text{w// } \gamma = 1/\sqrt{1-\beta^2}. \qquad \mathbf{x}_\perp \to \mathbf{x}_\perp' = \mathbf{x}_\perp; \end{array}\right\} (1)$$

$\parallel$ & $\perp$ refer to comps. of position $\mathbf{x}$ which are $\parallel$ & $\perp$ motion $\boldsymbol{\beta}$.

Evidently the "4-vector" $\tilde{x} = (x_0, \mathbf{x})$, used -- say -- to track a particle's motion, not only contains the relevant time (x_0) & position ($\mathbf{x}$) information, but it also carries with it a Lorentz <u>invariant</u> of the motion, viz. $(x_0^2 - \mathbf{x}^2)$. We can call this the vector's "length" [which one might expect would be $(x_0^2 + \mathbf{x}^2)$] if we just remember to put in the $(-)$ sign between the TIMELIKE part x_0^2 and SPACELIKE part $\mathbf{x}^2$. We will formalize this in a moment.

The 4-vector position $\tilde{x} = (x_0, \mathbf{x})$ seems useful. Are there other such qtys?

SRT Math: 4-vector velocity

2) Begin by defining a <u>4-vector $\tilde{A} = (A_0, \mathbf{A})$</u> as any collection of 4 components which has the same properties as does $\tilde{x} = (x_0, \mathbf{x})$, viz.

$$\tilde{A} = (A_0, \mathbf{A}),\ \text{transforms via LT as:}\ \boxed{\begin{aligned} A_0' &= \gamma(A_0 - \boldsymbol{\beta}\cdot\mathbf{A}), \\ A_\parallel' &= \gamma(A_\parallel - \beta A_0)\ \&\ \mathbf{A}_\perp' = \mathbf{A}_\perp; \end{aligned}}$$

and $\tilde{A}$ has the Lorentz invariant length: $\underline{\underline{\tilde{A}^2 = A_0^2 - \mathbf{A}^2}}$. (2)

While we are at it, we can generate another Lorentz invariant, i.e.

→ if $\tilde{A}$ & $\tilde{B}$ are 4-vectors, then: $\underline{\underline{\tilde{A}\cdot\tilde{B} = A_0B_0 - \mathbf{A}\cdot\mathbf{B}}}$, is Lorentz inv. (3)

Proof is straightforward (left to reader). This is our new version of "scalar product."

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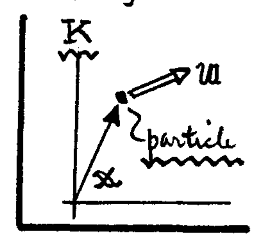
3) 4-vectors, with their Lorentz-invariant lengths, plus the notion of an invariant <u>proper time</u> τ associated with every particle's motion, are <u>indispensable</u> in deciding how to write down an acceptable version of kinematics & dynamics. Suppose a particle is moving at velocity $\mathbf{u}$ and is instantaneously located at position $\mathbf{x}(t)$ in an observer's frame K [note: $\mathbf{u}$, $\mathbf{x}$, and t are all cds as measured by K]. <u>Invent a "4-velocity"</u>:



4-velocity → $\tilde{u} = \dfrac{d\tilde{x}}{d\tau}$, w/ $d\tau = dt\sqrt{1 - u^2/c^2}$ ← $d\tau$ = particle proper time [p. SRT 10 Eq. (28)].

↳ $\tilde{u}$ <u>is</u> a 4-vector, because $\begin{cases} \text{position } \tilde{x} = (x_0, \mathbf{x}) \text{ is a 4-vector,} \\ d\tau \text{ is a Lorentz-invariant scalar.} \end{cases}$ ★ (4)

$\tilde{u}$ should have an invariant length. Easily seen by putting $\dfrac{d}{d\tau} = \gamma_u \dfrac{d}{dt}$ in (4):

$$\underline{\underline{\tilde{u} = \gamma_u \frac{d}{dt}(ct, \mathbf{x}) = \gamma_u(c, \mathbf{u})}}\ \Big\}\ \text{w/}\ \begin{aligned} \gamma_u &= 1/\sqrt{1 - u^2/c^2} \\ \mathbf{u} &= d\mathbf{x}/dt \end{aligned}\ \text{in K coordinates;}$$

So $\tilde{u}^2 = \gamma_u^2(c^2 - \mathbf{u}^2) = c^2$ ← Lorentz invariant length: 4-velocity. (5)

★ Any (4-vector) × (Lorentz-invariant scalar) is also a 4-vector. This is easily seen as a consequence of the <u>linearity</u> of the Lorentz transformation cited in Eq. (2).

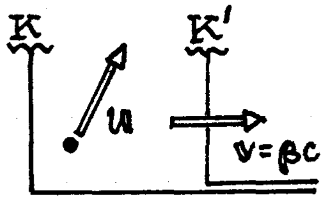
SRT Math: Velocity Addition

4) Since $\tilde{u}$ is an authentic 4-vector, we immediately know how to do a <u>Lorentz transform</u> from K (where $\tilde{u}$ is defined) to a frame K′ moving at $v = \beta c$ w.r.t. K. This is just the LT cited in Eq. (2), i.e.

$$\left.\begin{array}{l} \rightarrow K \rightarrow K'(\text{at } \beta) \Rightarrow \tilde{u} = (\gamma_u c, \gamma_u \mathbf{u}) \rightarrow \tilde{u}' = (\gamma_u' c, \gamma_u' \mathbf{u}'), \\ \text{w/ } 0^{\text{th}} \text{ comp.}: (\gamma_u' c) = \gamma_v[(\gamma_u c) - \beta\cdot(\gamma_u \mathbf{u})], \quad \text{w/ } \underline{\gamma_v = 1/\sqrt{1-\beta^2}}, \\ \text{\&/ } (\gamma_u' u_\parallel') = \gamma_v[(\gamma_u u_\parallel) - \beta(\gamma_u c)], \quad \gamma_u' \mathbf{u}_\perp' = \gamma_u \mathbf{u}_\perp . \end{array}\right\} \quad (6)$$

The transform on the 0^{th} comp. yields: $\gamma_u' = \gamma_v \gamma_u (1 - \frac{1}{c}\beta\cdot\mathbf{u})$, and this can be used to eliminate γ_u' in the $u_\parallel$ & $u_\perp$ parts of the remainder.¶ Result is:

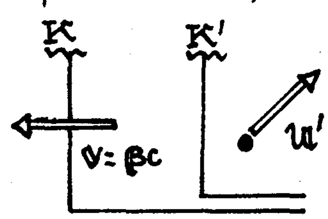
$$\boxed{\begin{array}{l} u_\parallel' = (u_\parallel - v)/(1 - \frac{1}{c}\beta\cdot\mathbf{u}) \\ \mathbf{u}_\perp' = \mathbf{u}_\perp / \gamma_v (1 - \frac{1}{c}\beta\cdot\mathbf{u}) \end{array}}$$



[Particle 3-velocity is $\mathbf{u}$ in K. Apparent velocity is $(u_\parallel', \mathbf{u}_\perp')$ in passing frame K′.] (7)

The inverse transform is (interchange primes & non-primes, send $\beta \rightarrow (-)\beta$)...

$$\boxed{\begin{array}{l} u_\parallel = (u_\parallel' + v)/(1 + \frac{1}{c}\beta\cdot\mathbf{u}') \\ \mathbf{u}_\perp = \mathbf{u}_\perp' / \gamma_v (1 + \frac{1}{c}\beta\cdot\mathbf{u}') \end{array}}$$



[Particle 3-velocity is $\mathbf{u}'$ in K′. Apparent velocity is $(u_\parallel, \mathbf{u}_\perp)$ in passing frame K.] (8)

Eq. (8) is the usual form of the <u>Velocity-Addition Formula</u> as quoted.

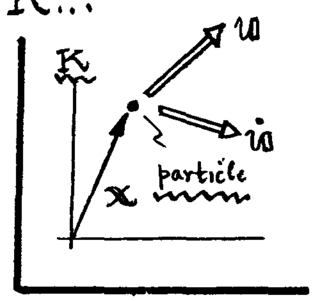
NOTES

1. Once we got an authentic 4-vector $\tilde{u}$, the velocity-addition formula is "easy".
2. The 3-velocity $\mathbf{u}$ does <u>not</u> have any special LT properties; it is $\gamma_u \mathbf{u}$ that does.
3. From (8), velocities v & $u_\parallel' = v$ add up to $2v/(1+\beta^2) < 2v$. If $v \rightarrow c$, then we "add" to find: $c + c = c$. This strange algebra <u>protects</u> c as a limit velocity.

¶ Important to note 3 differents γ's in Eq. (6): $\gamma_u = 1/\sqrt{1-u^2/c^2}$ and $\gamma_u' = 1/\sqrt{1-u'^2/c^2}$ are the γ's for the <u>particle's</u> motion, as seen by K and K′; $\gamma_v = 1/\sqrt{1-v^2/c^2}$ is that for $K \xrightarrow{LT} K'$

SRT Math: 4-Acceleration & 4-Momentum

5) We continue the kinematic description of a particle's motion by defining a <u>4-acceleration</u>: $\tilde{a} = d\tilde{u}/d\tau$. This must be a 4-vector, because $\tilde{u}$ is, and we have only divided $d\tilde{u}$ by the Lorentz-invariant proper time $d\tau$. If the particle is moving at (instantaneous) velocity $\mathbf{u}(t)$ in the observer's frame K...



4-acceleration

$$\tilde{a} = \frac{d\tilde{u}}{d\tau} = \gamma_u \frac{d}{dt}(\gamma_u c, \gamma_u \mathbf{u}), \quad \text{w/ } \underline{\gamma_u = 1/\sqrt{1-u^2/c^2}}.$$

$$\ldots \text{ use: } \frac{d\gamma_u}{dt} = \gamma_u^3 \frac{u}{c^2}\left(\frac{du}{dt}\right) \ldots$$

So
$$\tilde{a} = \gamma_u^2\left(\alpha, \frac{d\mathbf{u}}{dt} + \alpha\frac{\mathbf{u}}{c}\right), \quad \text{w/ } \alpha = \gamma_u^2 \frac{u}{c}\left(\frac{du}{dt}\right) = \frac{1}{c}\gamma_u^2\, \mathbf{u}\cdot(d\mathbf{u}/dt)$$

or
$$\underline{\underline{\tilde{a} = \gamma_u^2\left(\frac{1}{c}\gamma_u^2\, \mathbf{u}\cdot\mathbf{a},\ \mathbf{a} + \frac{1}{c}\gamma_u^2(\mathbf{u}\cdot\mathbf{a})\frac{\mathbf{u}}{c}\right)}}, \quad \text{w/ } \mathbf{a} = \frac{d\mathbf{u}}{dt}. \tag{9}$$

$\mathbf{a} = d\mathbf{u}/dt$ is the (old) Newtonian 3-acceleration as observed by K... now, relativistically, we have the much more complicated expression...

$$\rightarrow \mathbf{a}(\text{Newton}) = \frac{d\mathbf{u}}{dt} \rightarrow \mathbf{a}(\text{SRT}) = \gamma_u^2\left[\mathbf{a} + \frac{1}{c}\gamma_u^2(\mathbf{u}\cdot\mathbf{a})\frac{\mathbf{u}}{c}\right]. \tag{10}$$

$\mathbf{a}$ picks up a component $\parallel \mathbf{u}$ (even when no forces are acting $\parallel \mathbf{u}$). <u>How can you write $\mathbf{F} = m\mathbf{a}$ now</u>? Evidently, mechanics is <u>really</u> modified by SRT!

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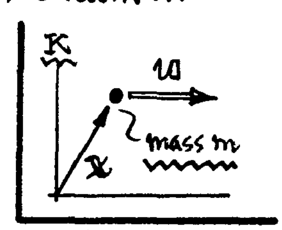
6) Continue with 4-vectors and Lorentz invariant scalars into doing relativistic particle dynamics. A natural place to start is to define a 4-momentum...

4-momentum

$$\tilde{p} = m\tilde{u} = (\gamma_u mc, \gamma_u m\mathbf{u}), \quad \text{w/ } \gamma_u = 1/\sqrt{1-u^2/c^2}. \tag{11}$$



Since the 4-velocity $\tilde{u}$ is a 4-vector, $\tilde{p}$ will be also -- <u>but only if the "mass" m here is a Lorentz-invariant scalar</u>. We posit the existence of such a mass m for the particle, as an intrinsic particle descriptor, and we call it the particle "rest mass" -- imagining that it can be measured most conveniently (and for all time) as an inertial reaction when the particle is brought to rest in our frame. We'll see if this is right a posteriori.

SRT Math: 4-Dynamics

For the 4-momentum $\tilde{p}$, one usually writes...

$$\left[\!\left[\ \tilde{p} = (\gamma_u mc, \mathbb{p}), \quad \mathbb{p} = \gamma_u m \mathbb{u} \ \leftarrow \text{relativistic version of 3-momentum.} \right.\right. \tag{12}$$

The following facts are relevant... $\tilde{u}^2 = c^2$, from Eq. (5)

1. Conserved length of $\tilde{p}$: $\tilde{p}^2 = m^2\tilde{u}^2 \Rightarrow (\gamma_u mc)^2 - \mathbb{p}^2 = m^2c^2$;

by $\boxed{\mathcal{E}^2 = (\mathbb{p}c)^2 + (mc^2)^2}$ ← relativistic momentum-energy relation

w/ $\mathcal{E} = \gamma_u mc^2 = mc^2/\sqrt{1-u^2/c^2} = \mathcal{E}(u)$. (13)

2. $\mathcal{E}(u)$ has the dimensions of an energy, and for $u \ll c$ has the expansion:

$$\rightarrow \mathcal{E}(u) = \mathcal{E}(0) + \tfrac{1}{2}mu^2\Big[1 + \underbrace{\tfrac{3}{4}(u/c)^2 + \Theta(u/c)^4 + \cdots}\Big] \tag{14}$$

"rest energy" Newton's K.E. SRT corrections (negligible for $u \ll c \to \infty$).

in particle rest frame, $u=0$, and: $\boxed{\mathcal{E}(0) = mc^2}$. (15)

So, by positing the existence of a "rest mass" m, we get the "rest energy" for free. $\mathcal{E}(u)$, including both $\mathcal{E}(0)$ and the K.E., is called the total energy of m.

3. We now write the 4-momentum as: $\tilde{p} = (\mathcal{E}/c, \mathbb{p})$ w/ $\mathbb{p}$ & $\mathcal{E}$ as in Eqs. (12) & (13). (16)

4. A relativistic version of kinetic energy is now defined as...

$$\rightarrow K(u) = \mathcal{E}(u) - \mathcal{E}(0) = (\gamma_u - 1)mc^2 = \tfrac{1}{2}mu^2\Big[1 + \overbrace{\tfrac{3}{4}(u/c)^2 + \cdots}^{\text{SRT terms}}\Big]. \tag{17}$$

Newton

5. And a USEFUL RELATION:

$$\left.\begin{aligned} \mathbb{p} &= \gamma_u m \mathbb{u} \\ \mathcal{E} &= \gamma_u mc^2 \end{aligned}\right\} \Rightarrow \boxed{\mathbb{u} = c^2\mathbb{p}/\mathcal{E}} \tag{18}$$

[NOTE: $\mathcal{E} = c^2 p/u$, and when $u \to c$, $\mathcal{E} \to pc$. Comparison with Eq. (13) shows $u \to c$ is possible only for particles of zero rest mass m (photons!).]

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7) There is still the question [below Eq. (10)] of how you write $\mathbb{F} = m\mathbb{a}$ in SRT. This can be done by defining a "Minkowski Force" $\tilde{F} = d\tilde{p}/d\tau$, and then setting $\tilde{F} = m\tilde{a}$, where m=(invariant) rest mass, and $\tilde{a}$ is the 4-acceleration of Eq. (9). What this procedure leads to (it has its moments) is left to the problems as an exercise for the reader.