4) Now we assume the "slowly-varying" condition of Eq. (8), and seek to improve the approximate solution $S(x) \simeq \pm \int^x k(\xi)\,d\xi$ of Eq. (7) by <u>iteration</u>. Have...

$$(S')^2 = k^2 + iS'' \leftarrow \text{exact, Eq. (6)}$$

$$\text{approx. soln}: \; S(x) \simeq S_0(x) = \pm \int^x k(\xi)\,d\xi \leftarrow \text{approx., Eq. (7)}$$

... for small term on RHS of <u>exact</u> eqtn, put: $S'' \simeq S_0'' = \pm\left(\frac{dk}{dx}\right)$...

So// $$(S')^2 \simeq k^2 + iS_0'' = k^2\left[1 \pm i\,\frac{1}{k^2}\left(\frac{dk}{dx}\right)\right],$$

or// $$dS/dx \simeq \pm k\Big[1 \pm i\,\underbrace{\frac{1}{k^2}\left(\frac{dk}{dx}\right)}_{\text{"small" by Eq. (8)}}\Big]^{1/2} \underset{\text{Binomial Expansion}}{\simeq} \pm k + \frac{i}{2}\underbrace{\frac{1}{k}\left(\frac{dk}{dx}\right)}_{=\frac{d}{dx}\ln k}. \tag{9}$$

WKB (cont'd) The WKB solution. Remarks on accuracy.

This last eq'n is easily integrated to give an improved solution for $S(x)$, viz:

$$\rightarrow S(x) \simeq S_0(x) + S_1(x) \int \text{w/} \quad S_0(x) = \pm \int^x k(\xi)\,d\xi \leftarrow \text{soln of Eq. (7)}$$

$$S_1(x) = \frac{i}{2}\ln k(x) + \text{cnst} \leftarrow \text{new correction} \qquad (10)$$

In Eq. (6), the solution proposed for $\psi'' + k^2\psi = 0$ was $\psi = e^{iS}$, so we form

$$\rightarrow \psi = e^{iS} \simeq e^{i(S_0+S_1)} = \exp\left[\pm i\int^x k(\xi)\,d\xi\right] \times \underbrace{\exp\left[-\tfrac{1}{2}\ln k(x) + \text{cnst}\right]}_{= \text{cnst}'/\sqrt{k(x)}} \ldots$$

... and we can state...

WKB SOLUTION

$$\boxed{\psi(x) = \left(\frac{\text{cnst}}{\sqrt{k(x)}}\right)\exp\left[\pm i\int^x k(\xi)\,d\xi\right], \text{ is an approximate solution to: } (d^2\psi/dx^2) + k^2(x)\,\psi = 0, \text{ provided: } \left|\frac{1}{k^2}\left(\frac{dk}{dx}\right)\right| \ll 1.} \qquad (11)$$

This form of ψ is called the WKB solution to the problem $\psi'' + k^2\psi = 0$.

REMARKS on WKB solution, Eq. (11).

A. The WKB solution for ψ in Eq. (11) is approximate in that it doesn't quite solve $\psi'' + k^2\psi = 0$; in fact...

$$\left[\begin{array}{l} \psi(x) = \left(\frac{\text{cnst}}{\sqrt{k(x)}}\right)\exp\left[\pm i\int^x k(\xi)\,d\xi\right], \text{ obeys: } \underline{\psi'' + k^2(1-\epsilon)\,\psi = 0}, \\ \text{where: } \underline{\underline{\epsilon(x)}} = \frac{3}{4}\left[\frac{1}{k^2}\left(\frac{dk}{dx}\right)\right]^2 - \frac{1}{2k^3}\left(\frac{d^2k}{dx^2}\right). \end{array}\right. \qquad (12)$$

The WKB version of ψ is "good" only if $|\epsilon(x)| \ll 1$. The 1st term of $\epsilon(x)$ is small (by assumption) because of the "slowly-varying" condition of Eq. (8). The 2nd term of $\epsilon(x)$, involving k'', will usually be small if k' is small. More precisely, note that: $\frac{d}{dx}(k'/k^2) = \frac{1}{k^2}k'' - \frac{2}{k^3}(k')^2$, and rewrite...

$$\rightarrow \epsilon(x) = (-)\left[f + \frac{1}{k}\left(\frac{d}{dx}\right)\right]f, \quad \text{w/} \quad f(x) = \frac{1}{2k^2}\left(\frac{dk}{dx}\right) = \frac{1}{k}\frac{d}{dx}(\ln\sqrt{k}). \qquad (13)$$

For $|\epsilon(x)| \ll 1$, we need both (k'/k^2) and its derivative $\frac{d}{dx}(k'/k^2) \rightarrow$ small.

REMARKS (cont'd)

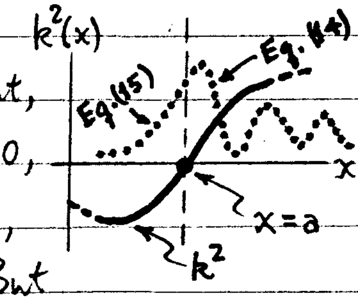
B. To provide the required two independent solutions to $\psi'' + k^2\psi = 0$, choose the + & − exponents in Eq. (11), and -- with A & B = integration cnsts -- form

$$\rightarrow \psi(x) = \frac{1}{\sqrt{k(x)}}\left[A \exp\left(+i\int k(x)\,dx\right) + B\exp\left(-i\int k(x)\,dx\right)\right],\ k^2 > 0. \quad (14)$$

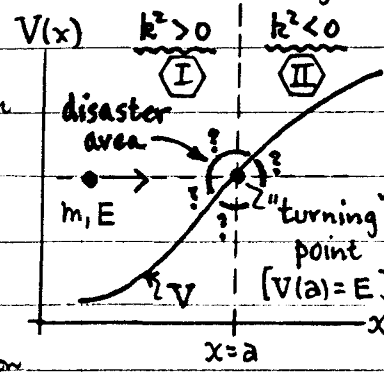
This general WKB solution is evidently oscillatory when k is real, i.e. when $k^2 > 0$. In some problems, however, it may be that $k^2 < 0$ over all or part of the range of x. If, say, $k^2(x) = (-)\kappa^2(x)$, then the appropriate square root is $k = \pm i\kappa$, and the above oscillatory solution becomes exponential:

$$\rightarrow \psi(x) = \frac{1}{\sqrt{\kappa(x)}}\left[C \exp\left(+\int \kappa(x)\,dx\right) + D\exp\left(-\int \kappa(x)\,dx\right)\right],\ k^2 = (-)\kappa^2 < 0. \quad (15)$$

This is the general WKB solution to $\psi'' - \kappa^2\psi = 0$, with C & D = arb^y cnsts. In both cases, WKB is "good" if: $|k'/k^2| \ll 1$ (for (14)), $|\kappa'/\kappa^2| \ll 1$ (for (15)).



C. If $k^2(x)$ is a fcn which passes through zero at some point, say $x = a$ as shown at right, then: (1) @ $x < a$, when $k^2 < 0$, use the exponential solution of Eq. (15) above, (2) @ $x > a$, when $k^2 > 0$, use the oscillatory solution of Eq. (14). But in the neighborhood $a - \delta \leq x \leq a + \delta$, $\delta \to 0$, we run into Big Trouble ... because $|k(x)| \to 0$ @ $x = a$, both types of WKB solns -- which $\propto \frac{1}{\sqrt{k(x)}}$ -- diverge at $x = a$. To boot, the "slowly-varying" condition $|k'/k^2| \ll 1$ is no good.



This annoyance occurs frequently in QM, where (recall from p. WKB 1): $\hbar k(x) = \sqrt{2m[E - V(x)]}$. When m approaches $x = a$, where the potential $V(a) = E$, then $k(x) \to 0$, and any WKB solution to this problem breaks down. While Eq. (14) holds in region (I) ($x < a$ & $k^2 > 0$), and Eq. (15) is OK in region (II) ($x > a$ & $k^2 < 0$), we have no WKB soln near $x = a$. The disaster area $x \sim a$ is called a "turning point", since a classical m would reverse its motion there.