TO: ϕ 519 students.
FROM: R.T. Robiscoe
RE: Some math references.

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In Jackson's Chs. 1-3 (and my paraphrases thereof) we have touched on some fairly dense math -- particularly solutions to certain 2nd order ODE^s which generate special fcns (like Legendre polynomials) that have specific properties (orthogonality, completeness, etc.). I have referred to certain Great Truths about these ODE^s [viz. Fuchs' Theorem, Sturm-Liouville theory, hypergeometric & confluent hypergeometric series, etc.] which makes working with these special fcns relatively easy.

Maybe it isn't so "easy" for you, if you've forgotten (or never seen) the Great Truths. So what follows is a short list of references you might read to refresh your memory. If you do that, you may earn a license to use the method of Solution by Proclamation. I will refer to the following texts:

① G. Arfken "Math. Methods for ϕst^s" (Academic Press, 3rd ed., 1985).

② S. Hassani "Foundations of Math. ϕ" (Allyn & Bacon, 1st ed., 1991).

③ Mathews & Walker "Math. Methods of ϕ" (Benjamin, 2nd ed., 1970).

④ Morse & Feshbach "Methods of Theoretical ϕ" (McGraw-Hill, 1st ed, 1953).

⑤ Abramovitz & Stegun "(NBS) Handbook of Math. Fcns" (9th printing {Dover 1965}).

⑥ Gradshteyn & Ryzhik "Table of Integrals, etc" (Acad. Press, revised edition, 1980).

Refs. ①-④ are textbooks; I am familiar with all but Hassani.
Refs ⑤-⑥ are handbooks, listing functional properties, integrals, etc.

FUCHS' Theorem

This tells you under what conditions you can get series solutions to 2nd order ODE's [$y'' + a(x)y' + b(x)y = 0$], when the coefficients $a(x)$ & $b(x)$ are singular. Fuchs' Thm mentioned and used in ① [Secs. 8.5 & 8.6], done in some detail in ② [Sec. 9.5.3], finessed in ③ [Sec. 1-2], and treated practically in ④ [pp. 530-539].

STURM-LIOUVILLE Theory

Concerns properties of solutions to: $[p(x)y']' + [q(x) + \lambda w(x)]y = 0$, like discreteness & ordering of eigenvalues λ, completeness & orthogonality of eigenfunctions $y_\lambda(x)$, etc. Absolutely essential to understanding the special fcns of physics in a general way. Treated simply (but not completely) in ① [Ch. 9], treated in more detail and with many examples in ② [Ch. 10], mentioned sketchily in ③ [Secs. 9-2 & 12-3], done nicely (and with a proof of completeness) in ④ [pp. 719-743].

HYPERGEOMETRIC Functions

These fcns are solutions to the following ODE's (both are S-L equations):
(A) $x(1-x)y'' + [\gamma - (1+\alpha+\beta)x]y' - \alpha\beta y = 0$, w/ α, β, γ = cnsts; this is the hypergeometric eqtn; (B) $xy'' + (\gamma - x)y' - \alpha y = 0$, w/ α & γ = cnsts; this is the confluent hypergeometric eqtn. Eq. (A) has regular singular points at $x=0$, 1, and ∞; Eq. (B) has a regular singularity at $x=0$ and an essential singularity at $x=\infty$. Despite their simple appearance, solutions to (A) [denoted ${}_2F_1(\alpha, \beta; \gamma; x)$] and to (B) [denoted ${}_1F_1(\alpha; \gamma; x)$] generate virtually all the special fcns of physics... e.g. for Legendre polynomials:

$P_n(\cos\theta) = {}_2F_1(n+1, -n; 1; \sin^2(\theta/2))$; and for Bessel functions: $J_\nu(x) = [x^\nu e^{-ix}/2^\nu \Gamma(\nu+1)]\,{}_1F_1(\frac{1}{2}+\nu; 1+2\nu; ix)$; etc. Since α, β, γ may be (+)ve, (−)ve, or complex, the series solutions to (A) & (B) are rich in detail, and are well worth studying to discover relevant functional details [like recurrence relations, differentiation formulas, asymptotic behavior as $x \to 0$, or $x \to \infty$, etc.]. ${}_2F_1$ & ${}_1F_1$ are treated briefly in ① [Secs. 13.5 & 13.6]; ② [Secs. 9.5.4 & 9.5.5] gives a concise account based on earlier work in Ch. 9; ③ [Secs. 7-3 & 7-4] is abbreviated, but gives a succinct account of the connection of (A) & (B) with Riemann's eqtn; and ④ [pp. 541–555] is fairly comprehensive, stressing relations to special fcns.

__Use of Tables__

Refs. ⑤ & ⑥ are compilations of a large number of __facts__ which have been proven about ${}_2F_1(\alpha, \beta; \gamma; x)$ and ${}_1F_1(\alpha; \gamma; x)$ in __general__... for ${}_2F_1$, see ⑤ [Ch. 15] or ⑥ [Secs 9.10–9.15, pp. 1039-48]*; for ${}_1F_1$, see ⑤ [Ch. 13] or ⑥ [Secs. 9.20–9.23, pp. 1057–1063]. Refs. ⑤ & ⑥ also compile __facts__ for __specific__ choices of ${}_2F_1$ and ${}_1F_1$ fcns... e.g. for ${}_2F_1 \to$ Legendre fcns, see various relations in ⑤ [Ch. 8] or ⑥ [Secs. 8.70–8.85, pp. 998–1023]; for ${}_1F_1 \to$ Bessel fcns, see ⑤ [Chs. 9, 10, 11] or ⑥ [Secs. 8.40–8.59, pp. 951–991].

Although entries in these tables cost a lot of work and careful analysis, once done they are similar to trig identities.

Incidentally, Ref ⑥ [Gradshteyn & Ryzhik, or just "G & R"] is __highly recommended__ as an integral table. (G & R) > 10 × (CRC).

* Riemann's ODE and solutions are outlined in ⑥ [Secs. 9.16–9.17].