Note to Phys. 507 Students   Friday, 6/4/71

1) Attached are the notes on Weisskopf-Wigner theory, and also problem ② on the final exam. Since the material on WW theory ran rather longer than expected, there will be no further problems (e.g. no scattering theory problem). You will have enough to do to cover the attached material.

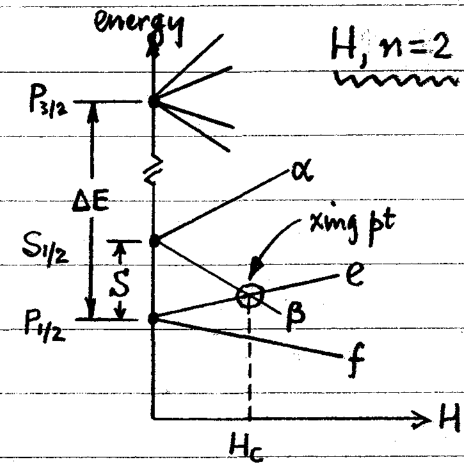
2) Your well thought out and beautifully presented solutions to the final exam, problems ① & ②, are due by 3 P.M. on Friday, 11 June. I will not accept your papers later than this. Since the problems are reasonably involved, particularly ①, I strongly suggest that you at least begin them by early in the week. There is no question that if you wait until Thursday afternoon to begin, you will be Totally Wiped Out.

3) I will hand back to you (via mailbox) corrected tests and final grades on Monday, 14 June. Also, some time this coming week, I will hand back the final problem set (problems ⑧③ & ⑧④).

4) Finally, I urge you to work this test out by yourself -- it is supposed to measure your competence in the course. You are free to consult the literature, and/or any textbooks you wish. But I would prefer that you not consult with each other.

Good luck, and may the Good G★eee! bless you all.

QM 507 Final Exam (Take Home - due 11 June 1971)

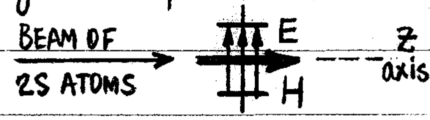


100 pts. **①** The $n=2$ state of atomic hydrogen consists of levels $2^2P_{3/2}$, $2^2S_{1/2}$ and $2^2P_{1/2}$ as shewn. Neglecting hfs, the 2P levels are split by the fs interaction, $\Delta E =$ 10,969 MHz. The degeneracy between the $S_{1/2}$ & $P_{1/2}$ levels is lifted by the Lamb shift, $S = 1058$ MHz, which is a QED effect. In an external magnetic fld H, the levels split as shewn; it is traditional to refer to the $S_{1/2}$ levels as α & β, and the $P_{1/2}$ levels as e & f, as indicated.

a) Using a linear theory of the Zeeman effect, calculate the fld H_c (in gauss) at which the levels β & e cross over (i.e. indicated xing pt.).

b) The 2S levels are "metastable", in that the lifetime for a decay $2S \to 1S$ (by double photon emission) is very long, namely $\sim 1/10$ sec. By contrast, the $2P \to 1S$ decay occurs very rapidly (by dipole radiation), with a lifetime $\tau = 1/\gamma \sim 10^{-9}$ sec. Verify the latter number by calculating the $2P \to 1S$ spontaneous emission rate γ in dipole approx. Also estimate the $2S \to 2P$ lifetime against dipole radiation.

c) Suppose a beam of metastables enters a region where there is maintained a mag. fld. $\vec{H}$ along the



z-axis, and a weak electric fld $\vec{E} \perp$ that axis, as shewn. The 2S and 2P levels are then coupled via a Stark matrix element $V_{PS} = \langle \phi_{2P} | e\vec{E}\cdot\vec{x} | \phi_{2S} \rangle$. Neglect $2S_{1/2}$ coupling to the $2P_{3/2}$ states, which are "far away". Of the remaining possible couplings, shew that for the indicated geometry, $|V| \equiv 0$ for αe and βf coupling, while $|V| = N e a_0 E$ for αf and βe coupling. Calculate the numerical factor N. What relative orientation of $\vec{E}$ & $\vec{H}$ would give αe and βf coupling? What is the selection rule operating here?

d) As a fcn of H, the βe level separation is $E_\beta - E_e = \hbar\omega = hS - g\mu_0 H$, where g is adjusted so that $\omega = 0$ at $H = H_c$. Near H_c, levels β and e are close together, so the Stark coupling is relatively much stronger for βe than for αf. To the extent that αf coupling can be ignored, the βe coupling becomes a two level problem. Assume the state superposition $\psi = a_\beta \phi_\beta e^{-\frac{i}{\hbar}E_\beta t} + a_e \phi_e e^{-\frac{i}{\hbar}E_e t}$, and use the Schrödinger eqtn to write the amplitude eqtns

$$i\hbar \dot{a}_\beta = V^* a_e e^{+i\omega t}, \quad i\hbar \dot{a}_e = V a_\beta e^{-i\omega t} - \tfrac{1}{2} i\hbar\gamma a_e,$$

where a_β & a_e are resp. the time-dpt S & P amplitudes, $V = \langle \phi_e | e\vec{E}\cdot\vec{x} | \phi_\beta \rangle$, and γ is the spontaneous decay rate of the P state. The terms in V follow from the S. eqtn, while the term in γ in the 2^{ND} eqtn is added phenomenologically, in such a way that for $V=0$, the P state amplitude decays as $|a_e|^2 = e^{-\gamma t}$, which is deemed to represent the $2P \to 1S$ spontaneous decay. To relate to the experiment of part c, solve these eqtns with the bndy conditions $a_\beta = 1$, $a_e = 0$ at $t=0$, which is the time of entry of the metastable atom into the E & H fld region. Suppose the coupling $|V| = N e a_0 E$ is "weak", i.e. $|V| \ll \frac{1}{2}\hbar\gamma$ -- which is the natural width of the P level. By examining the time dependence of $|a_\beta|^2$, show that the metastable develops an effective decay rate: $\Gamma \simeq C E^2 \gamma$, by virtue of its coupling to the P level via E. Calculate the proportionality factor C, and show that Γ, plotted vs. H (at cnst E), exhibits a Lorentzian resonance at the xing pt. mag. fld. H_c. What is the half-width of this resonance (in gauss)?

e) Finally, for part c, assume the initial intensity of the 2S beam is B_0, and it enters the E-H fld "transition region" from the left at velocity v (adiabatically!). Suppose E & H are cnst over a length l. Calculate the 2S intensity B to the right of the transition region. Plot B vs. H, and show that it exhibits a resonant decrease at $H = H_c$. For $v = 10^6$ cm/sec & $l = 1$ cm, what E-fld (in volts/cm) "quenchs" 50% of the 2S beam? In terms of an H_c measured thus, what is the Lamb shift, S? To what fraction of a linewidth should H_c be measured to get S to 1 MHz?

QM 507 Final Exam (Take Home – due 11 June 1971).

100 pts. ② Consider a hydrogenlike atom placed in a blackbody radiation field at temperature T. By the interaction of the atomic electron with the field, there are radiative level shifts due to both the zero-pt. vibrations (i.e. Lamb Shift) and the external photons, which are represented by a number fcn $N(k,T)$ -- such that $N(k,T)\rho(k)dk$ is the total # of photons available, at temp. T, between wavenumbers k & $k+dk$, indpt. of polarization.

a) Show that the level shift in state n due to external photons is

$$W_n(T) = \frac{2e^2}{3\pi}\sum_f |\vec{v}_{fn}/c|^2 \, \mathcal{P}\int_0^{K\to\infty} \frac{k\,dk}{k_{nf}-k} N(k,T),$$

in dipole approximation, where the sum is over all final states f for transitions $n \to f$ induced by the external photons, $\vec{v}_{fn} = \frac{1}{m}\langle f|\vec{p}|n\rangle$, and $\mathcal{P}$ denotes a principal value integral.

b) Put in the appropriate form for $N(k,T)$, and examine the integral for "small" T (i.e. $\beta T \ll$ any significant transition energy $|E_f - E_n|$, where β is the Boltzmann cnst). Show in this case that $W_n(T) \simeq -CT^2$, and calculate the proportionality cnst C (Note -- here it is appropriate to use the TRK Sum Rule, Merzbacher, pp. 457-458). Finally, estimate the relative size of $W_n(T)$ and the Lamb Shift (say for the $n=2$ level) at room temperature.

Solution to Phys. 507 Final Exam — 11 June 71

100 pts (4 hrs) 6/11/71

0 a) From problem (70), the Landé g-factors of the levels are

$$2P_{1/2}:\ g_J = 2/3\ ,\quad 2S_{1/2}:\ g_J = 2$$

For a $J=1/2$ level, the magnetic energies are

$$\mathcal{E} = -\vec{\mu}\cdot\vec{H} = +g_J\mu_0\vec{J}\cdot\vec{H} = m_J g_J \mu_0 H = \pm\tfrac{1}{2}g_J\mu_0 H$$

Taking $2P_{1/2}$ as zero of energy, the levels in question have energy

$$\mathcal{E}_\beta = S - \tfrac{1}{2}\times 2\mu_0 H\ ,\quad \mathcal{E}_e = +\tfrac{1}{2}\times\tfrac{2}{3}\mu_0 H$$

$$\therefore\ \mathcal{E}_\beta - \mathcal{E}_e = S - \tfrac{4}{3}\mu_0 H$$

At $H=H_c$, $\mathcal{E}_\beta - \mathcal{E}_e \equiv 0 \Rightarrow H_c = \frac{3}{4}S/\mu_0\ ;\ S = \frac{4}{3}H_c/\mu_0$

$$\left.\begin{array}{l} S = 1058\ \mathrm{MHz} \\ \mu_0 = 1.40\ \mathrm{MHz/Gs}\end{array}\right\}\ H_c = \frac{3}{4}\,1058/1.40 = 566\ \mathrm{Gs}$$

b) From p. 348 of QM 507 notes, the spontaneous decay rate is

$$\Gamma_{i\to f} = (4k^3/3\hbar)\,|\langle f|e\vec{r}|i\rangle|^2$$

The major problem here is in evaluating the matrix element. The final state is the $1S_0$ gnd state, with eigenfcn

$$\psi_f = R_{10}(r)\times\frac{1}{\sqrt{4\pi}}\ ,\quad R_{10}(r) = (Z/a_0)^{\frac{3}{2}}\times 2e^{-Zr/a_0}\ \leftarrow 1S_0$$

The initial state can be any one of $2P_0$, $2P_{\pm1}$, for which

see prob. (57)

$$\left\{\begin{array}{l} \psi_i^{(0)} = R_{21}(r)\times\sqrt{\frac{3}{4\pi}}\cos\vartheta \\ \psi_i^{(\pm1)} = R_{21}(r)\times\sqrt{\frac{3}{8\pi}}\sin\vartheta\, e^{\pm i\varphi}\end{array}\right.\quad R_{21}(r) = \frac{(Z/a_0)^{\frac{3}{2}}}{2\sqrt{6}}\,\frac{Zr}{a_0}\,e^{-\frac{1}{2}Zr/a_0}$$

Since we can write

$$|\vec{r}|^2 = \frac{1}{2}\Big[\underbrace{|x+iy|^2}_{\substack{r\sin\vartheta e^{+i\varphi} \\ ①}} + \underbrace{|x-iy|^2}_{\substack{r\sin\vartheta e^{-i\varphi} \\ ②}}\Big] + \underbrace{|z|^2}_{\substack{r\cos\vartheta \\ ③}} \qquad (1)$$

① is non-vanishing only for $\psi_i^{(-1)}$ against ψ_f
② " " " " " $\psi_i^{(+1)}$ " ψ_f
③ " " " " " $\psi_i^{(0)}$ " ψ_f

With volume element $d^3r = r^2 dr \sin\vartheta\, d\vartheta\, d\varphi$, we calculate...

$$\langle f|z|i\rangle = \int d^3r\, R_{10}(r)\frac{1}{\sqrt{4\pi}}\, r\cos\vartheta\, R_{21}(r)\sqrt{\frac{3}{4\pi}}\cos\vartheta$$

$$= \frac{\sqrt{3}}{4\pi} I \int_0^{\pi} \cos^2\vartheta \sin\vartheta\, d\vartheta \int_0^{2\pi} d\varphi = \sqrt{\tfrac{1}{3}}\, I$$

where : $I = \int_0^{\infty} r^3 R_{10}(r) R_{21}(r)\, dr$ (2)

$$\langle f|x+iy|i\rangle = \int d^3r\, R_{10}(r)\frac{1}{\sqrt{4\pi}}\, r\sin\vartheta e^{+i\varphi} R_{21}(r)\sqrt{\frac{3}{8\pi}}\sin\vartheta e^{-i\varphi}$$

$$= \frac{\sqrt{3/2}}{4\pi} I \int_0^{\pi} \sin^3\vartheta\, d\vartheta \int_0^{2\pi} d\varphi = \sqrt{\tfrac{2}{3}}\, I$$

Clearly $\langle f|x-iy|i\rangle \equiv \langle f|x+iy|i\rangle$ in this case. So we have

$$|\langle f|\vec{r}|i\rangle|^2 = \frac{1}{2}\Big[\frac{2}{3}I^2 + \frac{2}{3}I^2\Big] + \frac{1}{3}I^2 = I^2$$

In forming $\Gamma_{2P\to1S}$, we should take $\frac{1}{3}$ of this, in order to average

$$\frac{2^{15}}{3^9} = \frac{2}{3}\left(\frac{128}{81}\right)^2 = 1.66$$

over initial states. So we have... (9)

$$\Gamma_{2P\to1S} = (4k^3/3\hbar)\times\frac{1}{3}I^2$$

where: $I = \int_0^\infty r^3 R_{10}(r)R_{21}(r)\,dr = \frac{1}{\sqrt{6}}\frac{a_0}{Z}\int_0^\infty x^4 e^{-\frac{3}{2}x}\,dx$

$$= \frac{1}{\sqrt{6}}\left(\frac{a_0}{Z}\right)\frac{2^8}{3^4} \Rightarrow I^2 = \frac{2^{15}}{3^9}(a_0/Z)^2 \quad \bigstar$$

Now, as we show on p. 348 of the QM 507 notes, if we write

$\hbar ck = K mc^2(Z\alpha)^2$, $K = \frac{3}{8}$ for 2P→1S

$|\langle f|e\vec{r}|i\rangle| = Mea_0/Z$, $M = \frac{1}{\sqrt{6}}2^8/3^4 \times \underbrace{\frac{1}{\sqrt{3}}}_{\text{averaging factor}}$ for 2P→1S

$$\therefore \Gamma_{2P\to1S} = \underbrace{\frac{4}{3}K^3M^2}_{2^8/3^7}\underbrace{(Z\alpha)^4}_{\text{take } Z=1}(a_0/c)^{-1} = \underbrace{(4\alpha/9)^4}_{1.111\times10^{-10}}\times\underbrace{\frac{c}{a_0}}_{5.67\times10^{18}/\text{sec}}$$

or/ $\Gamma_{2P\to1S} = 6.25\times10^8\ \text{sec}^{-1} \Rightarrow \tau = 1/\Gamma = 1.60\times10^{-9}$ sec

Which is the "right" answer (see THBK #5, p.1).

For a 2S→2P transition, Γ should be down by a factor of $(S/E)^3$, where $E = 10.2$ eV is the 2P-1S energy separation; this ignores any numerical differences in the dipole matrix element. Since $S = 1058$ MHz $= 4.38\times10^{-6}$ eV, then $(S/E)^3 = 7.9\times10^{-20}$, and $\Gamma_{2S\to2P} \simeq 0.495\times10^{-10} \Rightarrow \tau = 1/\Gamma \sim 2\times10^{10}$ sec $= 635$ years.

Apparently the dipole M.E. $|\langle 2P|\vec{x}|2S\rangle|^2 = 9a_0^2$ rather than above $1.66\,a_0^2$. So $\Gamma_{2S\to2P}$ should be increased by $\sim 5\times \Rightarrow \tau \sim 4\times10^9$ sec (100 yrs)

$\bigstar$ This agrees with eq.(9), p.2 of THBK # 5.

c) Where $\phi_{n\ell}^{m_\ell}$ are the hydrogenic wfcns, the states are -- in the coupled rep^{n} (see prob. (72))...

$\alpha: \phi_{20}^{0}\chi_+$, $\beta: \phi_{20}^{0}\chi_-$ $\Big\{$ $\chi_\pm$ are the spin up and spin down eigenspinors

$e: \sqrt{\frac{1}{3}}\,\phi_{21}^{0}\chi_+ - \sqrt{\frac{2}{3}}\,\phi_{21}^{+1}\chi_-$

$f: \sqrt{\frac{2}{3}}\,\phi_{21}^{-1}\chi_+ - \sqrt{\frac{1}{3}}\,\phi_{21}^{0}\chi_-$

Now with $\vec{E}$ in the plane $\perp$ quantization axis, we write

$$e\vec{E}\cdot\vec{r} = e(E_x x + E_y y)$$

$$= \tfrac{1}{2}e\left[E_+(x-iy) + E_-(x+iy)\right], \quad E_\pm = E_x \pm iE_y$$

$$= \tfrac{1}{2}e\,r\sin\vartheta\left[E_+e^{-i\varphi} + E_-e^{+i\varphi}\right]$$

Thus, the matrix elements of interest are

$$\langle e|e\vec{E}\cdot\vec{r}|\alpha\rangle = \langle\sqrt{\tfrac{1}{3}}\,\phi_{21}^{0}\chi_+ - \sqrt{\tfrac{2}{3}}\,\phi_{21}^{+1}\chi_-|\tfrac{1}{2}e\,r\sin\vartheta[\sim]|\phi_{20}^{0}\chi_+\rangle$$

$$= \sqrt{\tfrac{1}{3}}\,\tfrac{1}{2}e\,\langle\phi_{21}^{0}|r\sin\vartheta[E_+e^{-i\varphi}+E_-e^{+i\varphi}]|\phi_{20}^{0}\rangle \equiv 0$$

Since the ϕ's here have no φ dependence, then this M.E. is $\equiv 0$, because the integration over φ wipes them out ($\int_0^{2\pi} e^{\pm i\varphi}d\varphi \equiv 0$). Similarly...

$$\langle f|e\vec{E}\cdot\vec{r}|\beta\rangle = \langle\sqrt{\tfrac{2}{3}}\,\phi_{21}^{-1}\chi_+ - \sqrt{\tfrac{1}{3}}\,\phi_{21}^{0}\chi_-|\tfrac{1}{2}e\,r\sin\vartheta[\sim]|\phi_{20}^{0}\chi_-\rangle$$

$$= -\sqrt{\tfrac{1}{3}}\,\tfrac{1}{2}e\langle\phi_{21}^{0}|r\sin\vartheta[E_+e^{-i\varphi}+E_-e^{+i\varphi}]|\phi_{20}^{0}\rangle \equiv 0$$

The non-zero couplings are for αf & βe. We have...

$$\langle f|e\vec{E}\cdot\vec{r}|\alpha\rangle = \sqrt{\tfrac{2}{3}}\,\tfrac{1}{2}e\,\langle\phi_{21}^{-1}|\,r\sin\vartheta\left[E_+e^{-i\varphi}+E_-e^{+i\varphi}\right]|\phi_{20}^{0}\rangle$$

Now: $\phi_{21}^{-1} = R_{21}(r)\times\sqrt{\frac{3}{8\pi}}\sin\vartheta\, e^{-i\varphi}$, $\phi_{20}^{0} = R_{20}(r)\times\frac{1}{\sqrt{4\pi}}$

The $e^{-i\varphi}$ here projects out the E_- term in V. So we have

$$V_{\alpha f} = \sqrt{\tfrac{2}{3}}\,\tfrac{1}{2}e\int d^3r\, R_{21}(r)\sqrt{\tfrac{3}{8\pi}}\sin\vartheta\, e^{-i\varphi}\left[r\sin\vartheta\, E_-e^{+i\varphi}\right]R_{20}(r)\frac{1}{\sqrt{4\pi}}$$

$$= (eE_-/8\pi)\int_0^\infty r^3 R_{21}(r)R_{20}(r)\,dr\;\underbrace{\int_0^\pi \sin^3\vartheta\, d\vartheta\int_0^{2\pi} d\varphi}_{\frac{4}{3}\times 2\pi}$$

$$= \frac{1}{3}eE_-\int_0^\infty r^3\left[\frac{(Z/a_0)^{\frac{3}{2}}}{2\sqrt{6}}\left(\frac{Zr}{a_0}\right)e^{-\frac{1}{2}Zr/a_0}\right]\left[\frac{(Z/a_0)^{\frac{3}{2}}}{2\sqrt{2}}\left(2-\frac{Zr}{a_0}\right)e^{-\frac{1}{2}Zr/a_0}\right]$$

$$= \frac{eE_-}{12\sqrt{12}}\,\frac{a_0}{Z}\underbrace{\int_0^\infty x^4(2-x)e^{-x}\,dx}_{-72} = -\sqrt{3}\,eE_-(a_0/Z)$$

For $Z=1$, $|V_{\alpha f}| = \sqrt{3}\,ea_0E_\perp$, where $E_\perp = (E_x^2+E_y^2)^{\frac{1}{2}}$.

The β-e coupling is...

$$\langle e|e\vec{E}\cdot\vec{r}|\beta\rangle = -\sqrt{\tfrac{2}{3}}\,\tfrac{1}{2}e\langle\phi_{21}^{+1}|\,r\sin\vartheta\left[E_+e^{-i\varphi}+E_-e^{+i\varphi}\right]|\phi_{20}^{0}\rangle$$

or $$V_{\beta e} = -\sqrt{\tfrac{2}{3}}\,\tfrac{1}{2}e\int d^3r\,R_{21}(r)\sqrt{\tfrac{3}{8\pi}}\sin\vartheta\, e^{+i\varphi}\left[r\sin\vartheta\, E_+e^{-i\varphi}\right]R_{20}(r)\frac{1}{\sqrt{4\pi}}$$

Evidently, $V_{\beta e} = +\sqrt{3}\, e E_{\perp} a_0/Z$, so $|V_{\beta e}| \equiv |V_{\alpha f}|$. Either can be written

$$V = \mathcal{N} e a_0 E \quad \begin{cases} E = E_{\perp} \\ \mathcal{N} = \sqrt{3} \end{cases}$$

$\vec{E} \perp$ quantization axis $\Rightarrow$ αf & βe coupling, with $\Delta m_J = \pm 1$,
$\vec{E} \parallel$ " " $\Rightarrow$ αe & βf " , " $\Delta m_J = 0$.

d) $\psi = a_\beta \phi_\beta e^{-\frac{i}{\hbar}E_\beta t} + a_e \phi_e e^{-\frac{i}{\hbar}E_e t}$

$$i\hbar \frac{\partial \psi}{\partial t} = (H_0 + V)\psi, \text{ where } \begin{cases} H_0 \phi = E\phi \\ V = \text{Stark perturbation} \end{cases}$$

$$\Rightarrow (i\hbar \dot{a}_\beta + E_\beta a_\beta)\phi_\beta e^{-\frac{i}{\hbar}E_\beta t} + (i\hbar \dot{a}_e + E_e a_e)\phi_e e^{-\frac{i}{\hbar}E_e t} =$$

$$= (H_0 + V)(a_\beta \phi_\beta e^{-\frac{i}{\hbar}E_\beta t} + a_e \phi_e e^{-\frac{i}{\hbar}E_e t})$$

This gives E_β & E_e w.r.t. the ϕ

Operate first with $\langle \phi_\beta | \ \rangle$, then with $\langle \phi_e | \ \rangle$, to get

$$i\hbar \dot{a}_\beta e^{-\frac{i}{\hbar}E_\beta t} = a_e \langle \phi_\beta | V | \phi_e \rangle e^{-\frac{i}{\hbar}E_e t}$$

$$i\hbar \dot{a}_e e^{-\frac{i}{\hbar}E_e t} = a_\beta \langle \phi_e | V | \phi_\beta \rangle e^{-\frac{i}{\hbar}E_\beta t}$$

Here we have assumed the diagonal elements of V are $\equiv 0$.
If we define

$$V = \langle \phi_e | e\vec{E}\cdot\vec{r} | \phi_\beta \rangle, \quad |V| = \mathcal{N} e a_0 E_{\perp} \text{ from above}$$

$$\omega = \frac{1}{\hbar}(E_\beta - E_e), \quad \beta\text{-}e \text{ energy separation}$$

$\therefore\ i\hbar\dot{a}_p = V^* a_e e^{+i\omega t}, \quad i\hbar\dot{a}_e = V a_p e^{-i\omega t} - \frac{1}{2} i\hbar\gamma a_e$

The 1ST terms RHS are the S. eqtn result. The 2ND term RHS in the 2ND eqtn is added phenomenologically, to give $|a_e|^2 = e^{-\gamma t}$ when $V=0$. We still don't really know if this is correct.

Decoupling the eqtns, we find a 2ND order eqtn for a_p...

$$\ddot{a}_p + \left(\frac{\gamma}{2} - i\omega\right)\dot{a}_p + \left(\left|\frac{V}{\hbar}\right|^2\right) a_p = 0$$

or for a_e...

$$\ddot{a}_e + \left(\frac{\gamma}{2} + i\omega\right)\dot{a}_e + \left(\left|\frac{V}{\hbar}\right|^2 + \frac{1}{2} i\omega\gamma\right) a_e = 0$$

Assume a solution for a_p of the form

$$a_p(t) = e^{-\mu t} \Longrightarrow \mu^2 - \left(\frac{\gamma}{2} - i\omega\right)\mu + \Omega^2 = 0$$

where $\Omega^2 = |V/\hbar|^2$. Soln's for μ are

$$\mu_{1,2} = \frac{1}{2}\left[\left(\frac{\gamma}{2} - i\omega\right) \pm \sqrt{\left(\frac{\gamma}{2} - i\omega\right)^2 - 4\Omega^2}\,\right]$$

If $\Omega \ll \left|\frac{\gamma}{2} - i\omega\right|$, i.e. $|V| \ll \frac{1}{2}\hbar\gamma$, then

(+) sign : $\mu_1 \cong \left(\frac{\gamma}{2} - i\omega\right) - \dfrac{\Omega^2}{\frac{\gamma}{2} - i\omega} = \frac{\gamma}{2}(1 - \mathcal{L}) - i\omega(1 + \mathcal{L})$

(−) sign : $\mu_2 \cong \Omega^2 / \left[\frac{\gamma}{2} - i\omega\right] = \frac{\gamma}{2}\mathcal{L} + i\omega\mathcal{L}$

where : $\mathcal{L} = \Omega^2 / \left[\left(\frac{\gamma}{2}\right)^2 + \omega^2\right]$

Obviously, for $\Omega \to 0$, μ_1 is the P state exponent, μ_2 the S state

The general solution for a_β is

$$a_\beta = Ae^{-\mu_1 t} + Be^{-\mu_2 t}$$

Bndy conditions are $a_\beta = 1$, $a_e = 0$ at $t=0$. From the eqtn $i\hbar\dot{a}_\beta = V^* a_e e^{i\omega t}$, $a_e = 0$ at $t=0 \Rightarrow \dot{a}_\beta = 0$. So we have

$$\left.\begin{aligned} a_\beta = 1 &\Rightarrow A + B = 1 \\ \dot{a}_\beta = 0 &\Rightarrow \mu_1 A + \mu_2 B = 0 \end{aligned}\right\} A = \frac{\mu_2}{\mu_2 - \mu_1}, \quad B = \frac{\mu_1}{\mu_1 - \mu_2}$$

Now for large t, the term in $e^{-\mu_1 t} \propto e^{-\frac{\gamma}{2}t}$ quickly damps out, and we have (since $|\mu_2| \ll \mu_1$)

$$a_\beta \simeq Be^{-\mu_2 t} \simeq e^{-(\frac{\gamma}{2}\mathcal{L} + i\omega\mathcal{L})t}$$

$$\therefore |a_\beta|^2 \simeq e^{-\gamma\mathcal{L}t}$$

Evidently, the S state has a decay rate given by

$$\Gamma \simeq \gamma\mathcal{L} = \frac{\Omega^2\gamma}{(\frac{\gamma}{2})^2 + \omega^2} = \frac{|V|^2\gamma}{(E_\beta - E_e)^2 + (\frac{\hbar\gamma}{2})^2}$$

But $|V| = \sqrt{3}\, e a_0 E_\perp$, so we can write

$$\Gamma = CE^2\gamma, \quad C = 3(ea_0)^2 \Big/ \left[(E_\beta - E_e)^2 + \left(\frac{\hbar\gamma}{2}\right)^2\right]$$

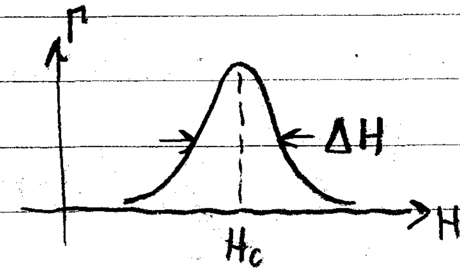
With $E_\beta - E_e = hS - g\mu_0 H$, $g = \frac{4}{3}$ for xing pt at $H = H_c$

or $E_\beta - E_e = g\mu_0(H_c - H)$, $g = \frac{4}{3}$

we can write the S-state decay rate as

$$\Gamma = \left|V/\tfrac{\hbar\gamma}{2}\right|^2 \gamma \Big/ \left[1+\left(\frac{H-H_c}{\Delta H/2}\right)^2\right]$$

where : $\Delta H = \hbar\gamma/g\mu_0$

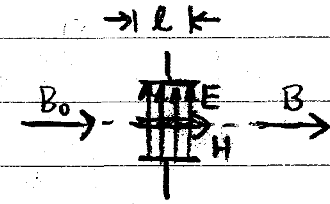


This is a Lorentzian resonance centered at the xing pt fld H_c, with a FWHM of ΔH. Numerically...

$$\Delta H = \gamma \big/ 2\pi g(\mu_0/h) = 53.3 \text{ Gs} \quad (\text{actually} \sim 54 \text{ Gs})$$

γ: 625 MHz μ_0/h: 1.40 MHz/Gs

e) In the transition region, the beam decays as



$$\frac{dB}{dt} = -\Gamma B \Rightarrow B(t) = B_0\, e^{-\int^t \Gamma\, dt}$$

But $dt = dl/v$, where v = beam velocity, and dl is an element of pathlength along the beam. With H & E cnst over length l, so is Γ and the integral is trivially done to give

$$B(l)/B_0 = e^{-\Gamma l/v}$$

Now if 50% of the beam is quenched, this ratio = ½, so

$$\Gamma = \frac{v}{l}\ln 2 \quad \text{for 50\% quenching}$$

Take $H = H_c$, so that $\Gamma = 3\left(\frac{e a_0 E}{\hbar\gamma/2}\right)^2 \gamma = (E/E_0)^2\gamma$, where $E_0 = \frac{\hbar\gamma}{2}\big/\sqrt{3}\, e a_0 = 22.5$ V/cm. Then (with $\tau = 1/\gamma$ the 2P lifetime), the 50% quench level is

$S = \frac{4}{3}\mu_0 H_c \Rightarrow \Delta S/S = \Delta H_c/H_c \sim 10^{-3} \Rightarrow \Delta H_c \sim 10^{-3} H_c \sim 0.5\,\text{Gs}$ 10

This is ~ 1% of linewidth

$$E = E_0 \sqrt{\frac{v\tau}{l} \ln 2}$$

$$\left.\begin{aligned} v &= 10^6 \text{ cm/sec} \\ \tau &= 1.60 \times 10^{-9} \text{ sec} \\ l &= 1 \text{ cm} \end{aligned}\right\} \quad E = \underset{\substack{\uparrow \\ 22.5\,\text{V/cm}}}{E_0} \times \underbrace{\sqrt{1.11 \times 10^{-3}}}_{0.033} = 0.75\ \text{V/cm}$$

100 pts (3 hrs.) 6/12/71 **2** This is essentially PHYS 482 prob. #1 of 12 Mar. 62.

a) Starting from the expression given on QM 507 notes, p. 375, viz

$$\delta_{i\to f} = \frac{1}{\hbar c} \mathcal{P} \int_0^\infty \frac{dk}{k_{if} - k} \left[\sum_\sigma \int_{4\pi} d\Omega_{\hat{k}} \left| \langle f(F) | \mathcal{H}_{int} | i(I) \rangle \right|^2 \rho(k) \right]$$

with $\rho(k) = Vk^2/(2\pi)^3$, $\hbar c k_{if} = E_i - E_f$, we note the general interaction matrix elements are -- in dipole approx (p. 335, 336, 346)

$$\langle f(F) | \mathcal{H}_{int} | i(I) \rangle_A = \sqrt{N_\sigma(k)} \left\langle f \left| \frac{e}{mc} \left(\frac{2\pi\hbar c}{Vk} \right)^{\frac{1}{2}} \hat{\epsilon}_\sigma \cdot \vec{p} \right| i \right\rangle \begin{cases} \text{ABSORPTION} \\ i \to f > i \end{cases}$$

$$\langle f(F) | \mathcal{H}_{int} | i(I) \rangle_E = \sqrt{N_\sigma(k) + 1} \left\langle f \left| \frac{e}{mc} \left(\frac{2\pi\hbar c}{Vk} \right)^{\frac{1}{2}} \hat{\epsilon}_\sigma \cdot \vec{p} \right| i \right\rangle \begin{cases} \text{EMISSION} \\ i \to f < i \end{cases}$$

Summing over all final states f, we get

$$\sum_{f>i} |ABS|^2 + \sum_{f<i} |EMS|^2$$

$$= e^2 \frac{2\pi\hbar c}{Vk} \left[\sum_{f>i} N_\sigma(k) \left| \hat{\epsilon}_\sigma \cdot \frac{\vec{v}_{fi}}{c} \right|^2 + \sum_{f<i} \left(N_\sigma(k) + 1 \right) \left| \hat{\epsilon}_\sigma \cdot \frac{\vec{v}_{fi}}{c} \right|^2 \right]$$

where $\vec{v}_{fi} = \frac{1}{m} \langle f | \vec{p} | i \rangle$, as usual. The term which persists when the external photons are not present (i.e. $N_\sigma(k) \equiv 0$) gives the Lamb shift. The remainder is the radiative shift of interest, for which the $|M_1 E_1|^2$ is just

$$e^2 \frac{2\pi\hbar c}{Vk} \sum_f N_\sigma(k) \left| \hat{\epsilon}_\sigma \cdot \frac{\vec{v}_{fn}}{c} \right|^2$$

in dipole approx. Putting this into the general expression for $S_{i\to f}$, we have the desired

works back on k_{nf} too!

$$W_n = \frac{1}{\hbar c} \mathcal{P} \int_0^\infty \frac{dk}{k_{nf}-k} \sum_\sigma \int_{4\pi} d\Omega_{\hat{k}} \, e^2 \frac{2\pi\hbar c}{Vk} \left(\sum_f N_\sigma(k) \left| \hat{\epsilon}_\sigma \cdot \frac{\vec{v}_{fn}}{c} \right|^2 \right) \frac{Vk^2}{(2\pi)^3}$$

$$= \frac{e^2}{4\pi^2} \sum_f \left(\sum_\sigma \int_{4\pi} d\Omega_{\vec{k}} \left| \hat{\epsilon}_\sigma \cdot \frac{\vec{v}_{fn}}{c} \right|^2 \right) \mathcal{P} \int_0^\infty \frac{k\,dk}{k_{nf}-k} N(k)$$

Here we have assumed $N_\sigma(k)$ is in fact indpt of polarization σ.

By the machinations on pp. 347-348 of The Notes, the sum over $\vec{k}$ is just $(8\pi/3) |\vec{v}_{fn}/c|^2$. So we have

$$\boxed{W_n = \frac{2e^2}{3\pi} \sum_f |\vec{v}_{fn}/c|^2 \, \mathcal{P} \int_0^\infty \frac{k\,dk}{k_{nf}-k} N(k)}$$

as desired. QED.

b) For blackbody radiation, the photon number fcn is ☆

$$N(k) = 1/(e^{\mu k}-1), \quad \mu = \hbar c/\beta T \quad \left\{ \beta = \text{Boltzmann const} \right.$$

Putting this into W_n, we have...

☆ See Leighton, p. 339. We take $g_s = 1$ and $\alpha = 0$ (for a large # of photons).

$$W_n = \frac{2e^2}{3\pi} \sum_f |\vec{v}_{fn}/c|^2 \, \mathcal{P}\int_0^\infty \frac{k\,dk}{e^{\mu k}-1}\left(\frac{1}{k_{nf}-k}\right)$$

Change variables to $x = \mu k \Rightarrow$

$$\mathcal{P}\int_0^\infty \frac{k\,dk}{e^{\mu k}-1}\left(\frac{1}{k_{nf}-k}\right) = -\frac{\beta T}{\hbar c}\,\mathcal{P}\int_0^\infty \frac{x\,dx}{e^x-1}\left(\frac{1}{x+\left(\frac{E_f-E_n}{\beta T}\right)}\right)$$

$$\therefore W_n(T) = -\frac{2\alpha}{3\pi}(\beta T)\sum_f |\vec{v}_{fn}/c|^2 \, J_{fn}(T)$$

$$\text{where}: \; J_{fn}(T) = \mathcal{P}\int_0^\infty \frac{x\,dx}{e^x-1}\left(\frac{1}{x+\left(\frac{E_f-E_n}{\beta T}\right)}\right)$$

It is rumored this integral is convergent (see Anluck & Kothari, Proc. Roy. Soc. A 214, 137 (Mar. 1952)). In any case, if n is the gnd state, then $E_f - E_n > 0$ for all f, and there is no trouble with the denominator. For $T \sim 300°$, $\beta T \sim 0.026\,\text{eV}$, and so $\beta T \ll E_f - E_n$ for a typical atom. Then

$$J_{fn}(T) \simeq \frac{\beta T}{E_f - E_n} \underbrace{\int_0^\infty \frac{x\,dx}{e^x-1}}_{\hookrightarrow\, \zeta(2) = \pi^2/6}, \quad \text{for } \beta T \ll |E_f - E_n|$$

{ Riemann zeta fcn
G & R, p. 325

$$\therefore W_n(T) \simeq -\frac{\pi}{9}\alpha(\beta T)^2 \sum_f \left|\frac{\vec{v}_{fn}}{c}\right|^2 \frac{1}{E_f - E_n}$$

We must try to evaluate the sum. We note (p. 346 of QM 507 notes)

$$\frac{\vec{v}_{fn}}{c} = \frac{1}{mc}\langle f|\vec{p}|n\rangle = +\frac{i}{\hbar c}(E_f - E_n)\langle f|\vec{x}|n\rangle$$

$$\therefore \left|\frac{\vec{v}_{fn}}{c}\right|^2 \frac{1}{E_f - E_n} = \frac{1}{\hbar^2 c^2} |\vec{x}_{fn}|^2 (E_f - E_n), \quad \vec{x}_{fn} = \langle f | \vec{x} | n \rangle$$

↳ dipole matrix elt.

Thus we have ...

$$W_n(T) = -\frac{\pi}{9}\alpha\left(\frac{\beta T}{\hbar c}\right)^2 \sum_f |\vec{x}_{fn}|^2 (E_f - E_n)$$

By use of the Thomas-Reiche-Kuhn sum rule (Merzbacher, p. 458, Bethe & Salpeter, p. 256), we set

$$\sum_f |\vec{x}_{fn}|^2 (E_f - E_n) = \hbar^2/2m, \quad m = \text{electron mass}$$

$$\therefore W_n(T) \simeq -\frac{\pi}{18}\alpha(\beta T)^2/mc^2 \quad \left\{ \begin{array}{l} \text{dipole approx.} \\ \beta T \ll |E_f - E_n| \end{array} \right.$$

This is the final desired expression. We note that in this approximation, all levels (in the atom) are uniformly depressed, so there is no way to detect this shift. There is, however, a relative level shift in the next order of approximation (which is of order $\Delta W \sim \alpha^3 R_y (\beta T / R_y)^3$, where $R_y = \frac{1}{2}\alpha^2 mc^2$ is the Rydberg).

* * *

For a numerical estimate, take

$$\left. \begin{array}{l} T \simeq 300°K \Rightarrow \beta T \simeq 0.026 \text{ eV} \\ mc^2 = 511 \text{ keV}, \ \alpha \simeq 1/137 \end{array} \right\} W(T) \simeq -1.7 \times 10^{-12} \text{ eV} = 408 \text{ cps}$$

This is only 0.4 ppm of the $n=2$ Lamb shift, $S = 1058$ MHz.