DEPARTMENT OF PHYSICS

PH. D. COMPREHENSIVE EXAMINATION

SEPTEMBER 21-22, 1987

DEPARTMENT OF PHYSICS

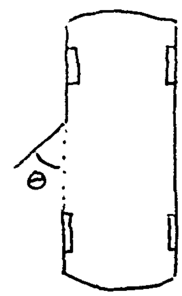
Ph.D. COMPREHENSIVE EXAMINATION

MONDAY, SEPTEMBER 21, 1987, 9 AM - 12 PM

Answer each of the following questions, all of which carry equal weight. Begin your answer to each question on a new sheet of paper solutions to different questions must not appear on the same sheet of paper. Each sheet of paper must be labeled with your name and the problem number in the upper right hand corner of the page. If more than one sheet is submitted for a problem, be sure the pages are ordered properly.

1. Devise a simple, practical method for determining the center of mass of an automobile. Be sure to describe the measurements you would make and the calculations you would perform.

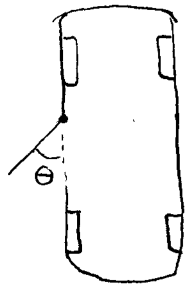
2. A car's door is initially open ($\theta = \pi/2$) and at rest ($\dot{\theta} = 0$). The car is accelerated at the uniform rate, a, starting at t=0. As the car accelerates, the door closes. Compute how long it takes for the door to completely close ($\theta = 0$). Assume the door has mass m, and its center of mass is located a distance ℓ from the hinge. Let I be the moment of inertia of the door about its hinges and $r_0^2 \equiv I/m$, the radius of gyration. Neglect wind resistance.



$\boxed{\text{Mechanics}}$

Lindblom

2.



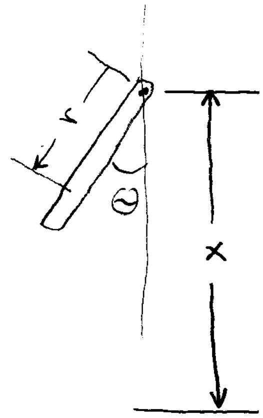
A car's door is initially open ($\theta = \pi/2$) and at rest ($\dot\theta = 0$). The car is accelerated at the uniform rate, a, starting at $t=0$. As the car accelerates the door closes. Compute how long it takes for the door to completely close ($\theta = 0$). Assume the door has mass m, having its center of mass located a distance l from the hinges and let I be the moment of inertia of the door about its hinges, and $r_0^2 \equiv I/m$ is the radius of gyration.

OK - JLe

O.K. (perhaps a hint should be given?) A.L.

good - JH

(2)



Let $\rho(r)$ be the mass per unit length along the car door. The kinetic energy is:

$$T = \frac{1}{2}\int \rho(r)\left\{\left[\dot{x} + r\dot{\theta}\sin\theta\right]^2 + \left[r\dot{\theta}\cos\theta\right]^2\right\}dr$$

$$= \frac{1}{2}\int \rho(r)\left\{\dot{x}^2 + 2r\dot{x}\dot{\theta}\sin\theta + r^2\dot{\theta}^2\right\}dr$$

$$= \frac{1}{2}\dot{x}^2\int \rho(r)\,dr + \dot{x}\dot{\theta}\sin\theta\int \rho(r)\,r\,dr + \frac{1}{2}\dot{\theta}^2\int \rho(r)r^2\,dr$$

$$= \frac{1}{2}m\dot{x}^2 + ml\dot{x}\dot{\theta}\sin\theta + \frac{1}{2}\dot{\theta}^2 m r_0^2$$

The system is constrained to move so that $x = \frac{1}{2}at^2$, so

$$T = \frac{1}{2}ma^2t^2 + mlat\dot{\theta}\sin\theta + \frac{1}{2}\dot{\theta}^2 m r_0^2$$

Lagranges equations for this system are given by:

$$0 = \frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right) - \frac{\partial L}{\partial\theta} = \frac{d}{dt}\left[+mlat\sin\theta + mr_0^2\dot{\theta}\right] - mlat\dot{\theta}\cos\theta$$

$$\boxed{0 = mr_0^2\ddot{\theta} + mla\sin\theta}$$

We now integrate this equation:

$$0 = \frac{1}{2}r_0^2\frac{d}{dt}(\dot{\theta})^2 - al\frac{d}{dt}\cos\theta$$

③

$$\frac{1}{2} r_0^2 \left[\dot{\theta}^2 - \dot{\theta}_0^2 \right] = a\ell \left[\cos\theta - \cos\theta_0 \right]$$

By assumption, for our problem $\theta_0 = \pi/2$, $\dot{\theta}_0 = 0$

$$\Rightarrow \qquad \frac{1}{2} r_0^2 \dot{\theta}^2 = a\ell \cos\theta$$

$$\Rightarrow \qquad t - t_0 = \left[\frac{1}{2} \frac{r_0^2}{a\ell} \right]^{1/2} \int_{\frac{\pi}{2}}^{\theta} \frac{d\theta'}{[\cos\theta']^{1/2}}$$

$$\int \frac{d\theta}{[\cos\theta]^{1/2}} = \int \frac{\sin\theta \, d\theta}{[\cos\theta(1-\cos^2\theta)]^{1/2}} = \int_0^{\cos\theta} \frac{dx}{[x(1-x^2)]^{1/2}}$$

$$= 2 \int_0^{\cos\theta} \frac{dy}{(1-y^4)^{1/2}}$$

The final position of the door is $\theta = 0$, so:

$$t - t_0 = 2 \left[\frac{1}{2} \frac{r_0^2}{a\ell} \right]^{1/2} \int_0^1 \frac{dy}{(1-y^4)^{1/2}}$$

$$\boxed{t - t_0 = 2 \left[\frac{1}{2} \frac{r_0^2}{a\ell} \right]^{1/2} \frac{\sqrt{\pi}}{4} \frac{\Gamma(1/4)}{\Gamma(3/4)}}$$

3. A particle of mass m, initially at rest, is struck by a photon of energy $h\nu$. If the photon is scattered through an angle of 90° (in the lab frame),

a) What is the scattering angle θ' in the center-of-mass system?

b) What is the recoil energy of the particle in the lab frame?

Treat the problem relativistically.

ok - L.A.L.
OK - JLC
O.K. A.E.

Classical Mechanics — Hermanson

3 A particle of mass m, initially at rest, is struck by a photon of energy $h\nu$. If the photon is scattered through an angle of 90^0 (in the lab frame),

a) what is the scattering angle θ' in the center-of-mass system;

b) what is the recoil energy of the particle in the lab frame?

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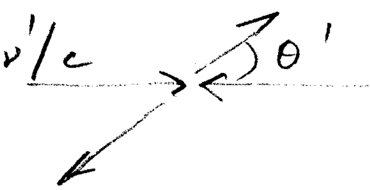
a) In the CM the photon 4-momentum is

$$p' = \frac{h\nu'}{c}\begin{pmatrix} 1 \\ \cos\theta' \\ \sin\theta' \\ 0 \end{pmatrix}$$



with $\nu' = \gamma(1-\beta)\nu$

In the lab it is

$$P = \begin{pmatrix} \gamma & \gamma\beta & & \\ \gamma\beta & \gamma & & \\ & & 1 & \\ & & & 1 \end{pmatrix} p' = \frac{h\nu'}{c}\begin{pmatrix} \gamma(1+\beta\cos\theta') \\ \gamma(\beta+\cos\theta') \\ \sin\theta' \\ 0 \end{pmatrix}$$

Now $\tan\theta = p_y/p_x = \dfrac{\sin\theta'}{(\beta+\cos\theta')} = \infty$ since $\theta = 90^0$

$$\boxed{\cos\theta' = -\beta}$$

To determine β, the CM velocity/c,

$$\beta = \frac{Pc}{E} = \frac{h\nu}{h\nu + mc^2}$$

$$\boxed{\cos\theta' = -\frac{h\nu}{h\nu + mc^2}} = -\frac{1}{1 + mc^2/h\nu}$$

b) The *final* photon energy is

$$h\nu_f = \gamma h\nu'(1 + \beta\cos\theta') \qquad \text{from } p \text{ above}$$

$$= \gamma^2 h\nu (1-\beta)(1-\beta^2)$$

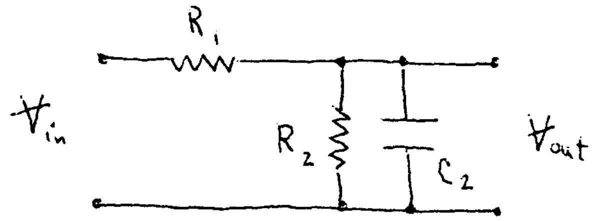
$$= h\nu(1-\beta)$$

$$\text{Recoil energy} = h\nu - h\nu_f = \beta h\nu$$

$$= \frac{h\nu}{1 + \frac{mc^2}{h\nu}}$$

4. Consider the complex representation e^{st} for a time varying voltage source, where $s = \sigma + j\omega$. [Note: $j=\sqrt{-1}$].

a) What is the significance of σ and ω?



b) For the circuit shown, with $V_{in} \sim e^{st}$

(1) Find the system function

$$H(s) = \frac{V_{out}}{V_{in}}$$

(2) Sketch $\log |H(j\omega)|$ and $\measuredangle H(j\omega)$ vs $\log \omega$ for the case $\sigma = 0$.

(3) Find H for $\sigma = \omega = 0$.

(4) Find the asymptotic form of $|H(j\omega)|$ for large ω. (Assume $\sigma = 0$)

(5) What <u>useful</u> application would such a circuit have?

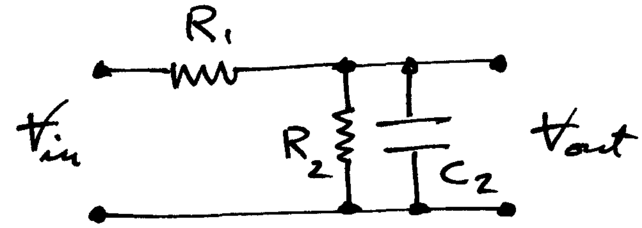
(6) How can the circuit be "compensated" so that $H(j\omega)$ becomes independent of ω? What useful application would such a circuit have?

#1 Experimental / Core

4 Consider the complex representation e^{st} for a time varying voltage source, where $s = \sigma + j\omega$.
[Note: $j \equiv \sqrt{-1}$]

(a) What is the significance of σ & ω?

(b) For the circuit shown, with $v_{in} \propto e^{st}$



① Find the system function $H(s) \equiv \frac{v_{out}}{v_{in}}$.

② Sketch $\log|H(j\omega)|$ and $\measuredangle H(j\omega)$ vs $\log \omega$ for case $\sigma = 0$.

③ Find H for $\sigma = \omega = 0$.

④ Find the asymptotic form of $|H(j\omega)|$ for large ω. (assume $\sigma = 0$)

⑤ What useful application would such a circuit have?

⑥ How can the circuit be "compensated", so that $H(j\omega)$ becomes independent of ω? What useful application would such a circuit have?

- OK JLC

- Seems very specialized LAL

- too specialized perhaps - JH

Solution

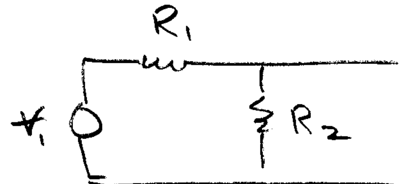
(a) σ describes exponential growth ($\sigma > 0$) or decay ($\sigma < 0$). When $\sigma = 0$, the amplitude is constant.

ω describes oscillatory behavior: $\mathrm{Re}(e^{j\omega t}) = \cos \omega t$

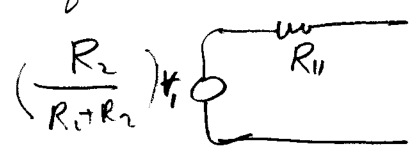
(b) ① Use voltage divider relation : $H(s) = \dfrac{Z_2(s)}{Z_1(s) + Z_2(s)}$

<u>Easiest</u> way

replace



with Thevenin equivalent

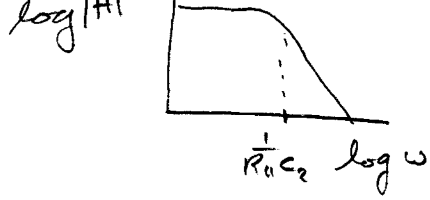


$$R_{\parallel} = \frac{R_1 R_2}{R_1 + R_2}$$

Then $$H(s) = \left(\frac{R_2}{R_1+R_2}\right) \frac{1/sC_2}{R_{\parallel} + 1/sC_2} = \frac{R_2}{R_1+R_2} \frac{1}{1 + R_{\parallel} C_2 s}$$

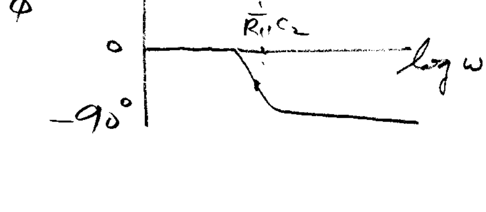
which has a pole at $\omega = -\dfrac{1}{R_{\parallel} C_2}$, giving it "low pass filter" characteristics as shown below

②



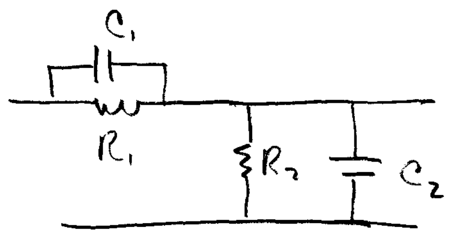
③ $H(0) = \dfrac{R_2}{R_1 + R_2}$

④ $H(j\omega) = \dfrac{R_2}{R_1+R_2} \dfrac{1}{j\omega R_{\parallel} C_2}$



(5) It gives a combination of two useful functions, – low pass filtering reduces noise on signals
– attenuation

It might thus be used to monitor a noisy high voltage signal

(6) 

When $R_1 C_1 = R_2 C_2$ poles & zeros cancel

Then $H(j\omega) = \frac{R_2}{R_1 + R_2}$ independent of ω

A 10X oscilloscope probe is built this way.

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MONDAY, SEPTEMBER 21, 1987, 1 PM - 4 PM

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5. The spherical harmonics $Y_\ell^m(\theta\varphi)$ form a complete and orthonomal set of functions for $0 \leq \varphi \leq 2\pi$ and $0 \leq \theta \leq \pi$. Obtain the coefficients $a_{\ell m}$ in the following expansion

$$\sin^2\theta \cos^2\varphi = \sum_{\ell m} a_{\ell m} Y_\ell^m(\theta\varphi)$$

Note: This type of expansion occurs frequently in the theory of angular momentum in quantum mechanics.

$$Y_0^0(\theta\varphi) = \frac{1}{\sqrt{4\pi}}$$

$$Y_1^0(\theta\varphi) = \sqrt{\frac{3}{4\pi}} \cos\theta$$

$$Y_1^{\pm 1}(\theta\varphi) = \mp\sqrt{\frac{3}{8\pi}} e^{\pm i\varphi} \sin\theta$$

$$Y_2^0(\theta\varphi) = \sqrt{\frac{5}{16\pi}} (3\cos^2\theta - 1)$$

$$Y_2^{\pm 1}(\theta\varphi) = \mp\sqrt{\frac{15}{8\pi}} e^{\pm i\varphi} \cos\theta \sin\theta$$

$$Y_2^{\pm 2}(\theta\varphi) = \sqrt{\frac{15}{32\pi}} e^{\pm 2i\varphi} \sin^2\theta$$

ok - LAL.
ok - [illegible]
OK - JH

5 Mathematical Physics. A.E.

The spherical harmonics $Y_\ell^m(\theta\varphi)$ form a complete and orthonormal set of functions for $0 \le \varphi \le 2\pi$ and $0 \le \theta \le \pi$. Obtain the coefficients $a_{\ell m}$ in the following expansion

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$$Y_2^0(\theta\varphi) = \sqrt{\frac{5}{16\pi}} \, (3\cos^2\theta - 1)$$

$$Y_2^{\pm1}(\theta\varphi) = \mp\sqrt{\frac{15}{8\pi}} \, e^{\pm i\varphi} \cos\theta \sin\theta$$

$$Y_2^{\pm2}(\theta\varphi) = \sqrt{\frac{15}{32\pi}} \, e^{\pm 2i\varphi} \sin^2\theta$$

<u>Solution</u>

Using the orthogonality of the $\{Y_\ell^m\}$ we obtain the result that

$$a_{\ell m} = \int d\Omega \left(Y_\ell^m(\theta\varphi)\right)^* \sin^2\theta \cos^2\varphi .$$

We avoid evaluating any integrals by using the table of the $\{Y_\ell^m\}$. We make implicit use of the fact that the set $\{a_{\ell m}\}$ is unique. Thus it does not matter how we find it.

We have that

$$Y_2^2 + Y_2^{-2} = 2\left(\frac{15}{32\pi}\right)^{1/2} \sin^2\theta \cos 2\varphi$$

$$\cos 2\varphi = \cos^2\varphi - \sin^2\varphi = 2\cos^2\varphi - 1$$

$\therefore$

$$Y_2^2 + Y_2^{-2} = 4\left(\frac{15}{32\pi}\right)^{1/2} \sin^2\theta \cos^2\varphi$$
$$- 2\left(\frac{15}{32\pi}\right)^{1/2} \sin^2\theta$$

$\therefore$

$$\sin^2\theta \cos^2\varphi = \frac{1}{4}\left(\frac{32\pi}{15}\right)^{1/2} \left[Y_2^2 + Y_2^{-2} + 2\left(\frac{15}{32\pi}\right)^{1/2} \sin^2\theta \right]$$

All that remains to be done is to find the expansion of $\sin^2\theta$ in terms of the Y_l^m. Now:

$$Y_2^0 = \left(\frac{5}{16\pi}\right)^{1/2} \left[3(1-\sin^2\theta) - 1 \right]$$

$$= \left(\frac{5}{16\pi}\right)^{1/2} \left[2 - 3\sin^2\theta \right]$$

$$Y_0^0 = \frac{1}{\sqrt{4\pi}}$$

$$= \frac{1}{\sqrt{4\pi}} \left(\frac{16\pi}{5}\right)^{1/2} \left(\frac{5}{16\pi}\right)^{1/2}$$

$$= 2 \left(\frac{5}{16\pi}\right)^{1/2} \frac{1}{5^{1/2}}$$

$\therefore$

$$Y_2^0 - \sqrt{5}\, Y_0^0 = \left(\frac{5}{16\pi}\right)^{1/2} (2 - 3\sin^2\theta) - 2\left(\frac{5}{16\pi}\right)^{1/2}$$

$$= -3\left(\frac{5}{16\pi}\right)^{1/2} \sin^2\theta$$

$\therefore$

$$\sin^2\theta = -\frac{1}{3}\left(\frac{16\pi}{5}\right)^{1/2} \left(Y_2^0 - \sqrt{5}\, Y_0^0 \right)$$

Thus:

$$\sin^2\theta \cos^2\varphi = \left(\frac{2\pi}{15}\right)^{1/2} \Big[Y_2^{2} + Y_2^{-2}$$
$$- \frac{2}{3} \left(\frac{15 \times 16 \times \pi}{5 \times 32 \times \pi}\right)^{1/2} (Y_2^0 - \sqrt{5}\, Y_0^0) \Big]$$

or,

$$\sin^2\theta \cos^2\varphi = \left(\frac{2\pi}{15}\right)^{1/2} \Big[Y_2^{2} + Y_2^{-2}$$
$$- \sqrt{\frac{2}{3}} \left(Y_2^0 - \sqrt{5}\, Y_0^0 \right) \Big]$$

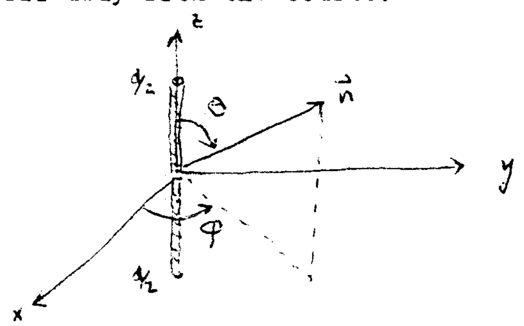
$$\Rightarrow \qquad a_{0,0} = \left(\frac{2\pi}{15}\right)^{1/2} \left(\frac{10}{3}\right)^{1/2} = \left(\frac{2 \times \cancel{10}^{\,2} \times \pi}{3 \times \cancel{15}_{\,3}}\right)^{1/2} = \frac{2}{3}\sqrt{\pi}$$

$$a_{2,2} = \left(\frac{2\pi}{15}\right)^{1/2}$$

$$a_{2,-2} = \left(\frac{2\pi}{15}\right)^{1/2}$$

$$a_{2,0} = \left(\frac{2\pi}{15}\right)^{1/2} (-) \left(\frac{2}{3}\right)^{1/2} = - \left(\frac{2 \times 2 \times \pi}{5 \times 9}\right)^{1/2} = -\frac{2}{3}\sqrt{\frac{\pi}{5}}$$

6. Compute the vector potential caused by a sinusoidal current density $\vec{J}(\vec{x},t) = I e^{-i\omega t} \sin\left[\frac{\omega}{c}\left(\frac{d}{2} - |z|\right)\right]\delta(x)\delta(y)\,\vec{e}_z$ in a center fed linear antenna in the "wave zone" far away from the source.

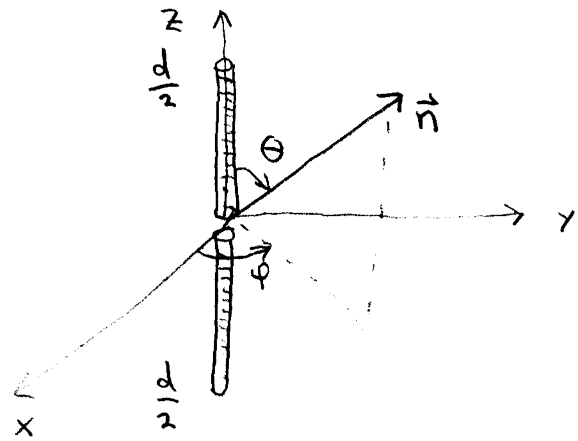


Use this vector potential to compute the power radiated per unit solid angle as a function of the direction $\vec{n}$ by computing the Poynting vector.

Lidblom

1 **Electromagnetism**

6. Compute the vector potential caused by a sinusoidal current density $\vec{J}(\vec{x},t) = I e^{-i\omega t} \sin\left[\frac{\omega}{c}\left(\frac{d}{2} - |z|\right)\right] \delta(x)\,\delta(y)\,\vec{e}_z$ in a center fed linear antenna in the "wave zone" far away from the source.



Use this vector potential to compute the power radiated per unit solid angle as a function of the direction $\vec{n}$ by computing the Poynting vector.

1st part easy, 2nd hard - JH

seems long - JK

O.K. together with John's coil problem.

The vector potential is given by the integral:

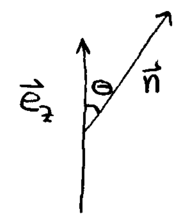
$$\vec{A}(\vec{x},t) = \frac{1}{c}\int d^3x' \int dt' \; \frac{\vec{J}(\vec{x}',t')}{|\vec{x}-\vec{x}'|}\, \delta\left(t'-t+\frac{|\vec{x}-\vec{x}'|}{c}\right)$$

For sinusoidal currents $\vec{J}(\vec{x}',t') = \vec{J}(\vec{x}')\,e^{-i\omega t'}$, and limiting attention to the wave zone $|\vec{x}-\vec{x}'| \simeq r - \vec{n}\cdot\vec{x}'$ this expression

$$\vec{A}(\vec{x},t) \simeq \frac{e^{-i\omega t}}{c}\int d^3x' \; \frac{\vec{J}(\vec{x}')\, e^{+i\frac{\omega}{c}(r-\vec{n}\cdot\vec{x}')}}{r-\vec{n}\cdot\vec{x}'}$$

$$\simeq \frac{1}{rc}\, e^{i\omega(\frac{r}{c}-t)} \int d^3x' \; \vec{J}(\vec{x}')\, e^{-i\frac{\omega}{c}\vec{n}\cdot\vec{x}'}$$

Now consider a current density



$$\vec{J}(\vec{x}') = I\sin\left[\frac{\omega}{c}\left(\frac{d}{2}-|z'|\right)\right]\delta(x')\delta(y')\,\vec{e}_z$$

$$\vec{A}(\vec{x},t) = \frac{I}{rc}\, e^{i\omega(\frac{r}{c}-t)}\,\vec{e}_z \int_{-d/2}^{d/2} \sin\left[\frac{\omega}{c}\left(\frac{d}{2}-|z'|\right)\right] e^{-i\frac{\omega}{c}z'\cos\theta}\, dz'$$

$$= \frac{I}{rc}\, e^{i\omega(\frac{r}{c}-t)}\,\vec{e}_z\, 2\int_0^{d/2} \sin\left[\frac{\omega}{c}\left(\frac{d}{2}-z'\right)\right]\cos\left[\frac{\omega}{c}\cos\theta\, z'\right] dz'$$

$$= \frac{2I}{rc}\, e^{i\omega(\frac{r}{c}-t)}\,\vec{e}_z \int_0^{d/2} dz' \cos\left[\frac{\omega}{c}\cos\theta\, z'\right]\left\{\sin\frac{\omega d}{2c}\cos\frac{\omega}{c}z' - \cos\frac{\omega d}{2c}\sin\frac{\omega}{c}z'\right\}$$

$$= \frac{2I}{rc}\, e^{i\omega(\frac{r}{c}-t)}\,\vec{e}_z \left\{ \sin\frac{\omega d}{2c}\left[\frac{\sin\left[\left(\frac{\omega}{c}\cos\theta-\frac{\omega}{c}\right)\frac{d}{2}\right]}{2\frac{\omega}{c}(\cos\theta-1)} + \frac{\sin\left[\left(\frac{\omega}{c}\cos\theta+\frac{\omega}{c}\right)\frac{d}{2}\right]}{2\frac{\omega}{c}(\cos\theta+1)}\right]\right.$$

$$\left. + \cos\frac{\omega d}{2c}\left[\frac{\cos\left[\frac{d\omega}{2c}(\cos\theta-1)\right]-1}{2\frac{\omega}{c}(1-\cos\theta)} + \frac{\cos\left[\frac{\omega d}{2c}(1+\cos\theta)\right]-1}{2\frac{\omega}{c}(1+\cos\theta)}\right]\right\}$$

$$\vec{A}(\vec{x},t) = \frac{2I}{rc} e^{i\omega\left(\frac{r}{c}-t\right)} \vec{e}_z \left(\frac{2\omega}{c}\right)^{-1} \times$$

$$\left\{ (1-\cos\theta)^{-1} \left[\cos\frac{\omega d}{2c} \cos\left(\frac{\omega d}{2c}(1-\cos\theta)\right) - \cos\frac{\omega d}{2c} \right.\right.$$

$$\left. + \sin\frac{\omega d}{2c} \sin\left(\frac{\omega d}{2c}(1-\cos\theta)\right)\right]$$

$$+ (1+\cos\theta)^{-1} \left[\cos\frac{\omega d}{2c} \cos\left(\frac{\omega d}{2c}(1+\cos\theta)\right) - \cos\frac{\omega d}{2c} \right.$$

$$\left.\left. + \sin\frac{\omega d}{2c} \sin\left(\frac{\omega d}{2c}(1+\cos\theta)\right)\right] \right\}$$

$$= \frac{I}{\omega r} e^{i\omega\left(\frac{r}{c}-t\right)} \vec{e}_z \left\{ \frac{\cos\left[\frac{\omega d}{2c}\cos\theta\right] - \cos\frac{\omega d}{2c}}{1-\cos\theta} + \frac{\cos\left[\frac{\omega d}{2c}\cos\theta\right] - \cos\frac{\omega d}{2c}}{1+\cos\theta} \right\}$$

$$\boxed{\vec{A}(\vec{x},t) = \frac{2I}{\omega r} e^{i\omega\left(\frac{r}{c}-t\right)} \vec{e}_z \, \frac{\cos\left[\frac{\omega d}{2c}\cos\theta\right] - \cos\frac{\omega d}{2c}}{\sin^2\theta}}$$

Next compute the Electric and Magnetic Fields:

$$\vec{B} = \vec{\nabla}\times\vec{A}$$

$$\frac{1}{c}\partial_t \vec{E} = -\frac{i\omega}{c}\vec{E} = \vec{\nabla}\times\vec{B} \quad \Rightarrow \quad \vec{E} = \frac{ic}{\omega}\vec{\nabla}\times\vec{B}$$

The vector potential has the form: $A^x = A^y = 0$

$$A^z = \frac{1}{r} f(\cos\theta) e^{i\omega\left(\frac{r}{c}-t\right)}$$

and $\cos\theta = \frac{z}{r}$

④

To leading order in $\frac{1}{r}$: (the "wave zone" approximation)

$$\partial_x A^z = \frac{i\omega}{c} A^z \frac{x}{r} \qquad \partial_y A^z = \frac{i\omega}{c} A^z \frac{y}{r}$$

$$\Rightarrow \quad \vec{B} = \vec{\nabla}\times\vec{A} = \left(\partial_y A^z, -\partial_x A^z, 0\right) = \frac{i\omega}{c} A^z \frac{1}{r}(y,-x,0)$$

Similarly $\vec{E} = \frac{ic}{\omega}\vec{\nabla}\times\vec{B} = \left(-\partial_z B^y, \partial_z B^x, \partial_x B^y - \partial_y B^x\right)$

$$= -\frac{i\omega}{c} A^z \frac{1}{r^2}\left(xz, yz, -x^2-y^2\right)$$

The energy flux of the electromagnetic field is given by the Poynting vector:

$$\vec{S} = \frac{c}{4\pi}\vec{E}\times\vec{B}$$

$$= \frac{c}{4\pi}\frac{\omega^2}{c^2}(A^z)^2\frac{1}{r^3}\left(xz, yz, z^2-r^2\right)\times(y,-x,0)$$

$$= \frac{\omega^2}{4\pi c}(A^z)^2\frac{1}{r^3}\left[+x(z^2-r^2), y(z^2-r^2), -zx^2-zy^2\right]$$

$$= \frac{\omega^2}{4\pi c}(A^z)^2\frac{\vec{r}}{r}(\cos^2\theta - 1)$$

$$= \frac{1}{4\pi c}\frac{4I^2}{r^2}\left\{\cos\left[\frac{\omega d}{2c}\cos\theta\right] - \cos\frac{\omega d}{2c}\right\}^2 \frac{e^{2i\omega(\frac{r}{c}-t)}}{\sin^2\theta}$$

To find the energy flux per solid angle take the real part, time average and multiply the radial component by r^2.

$$\boxed{\frac{dP}{d\Omega} = \frac{I^2}{2\pi c}\left[\cos\left[\frac{\omega d}{2c}\cos\theta\right] - \cos\frac{\omega d}{2c}\right]^2 / \sin^2\theta}$$

7. Consider a nonlinear medium which has a nonlinear polarization given by $P = \chi^{(1)}E + \chi^{(2)}EE$. Now consider an incident plane wave $E_1(x,t) = E_1 e^{ik_1x - i\omega_1 t}$.

a) Calculate the second harmonic intensity generated if the medium has a length L, where the second harmonic field is given by $E_2(x,t) = E_2(x)e^{ik_2x - i\omega_2 t}$ and $\omega_2 = 2\omega_1$. Let the phase velocity at ω_1 be c/n_1 and at ω_2 be c/n_2. Hint: you may assume that $\frac{\partial}{\partial x}E_2(x) \ll k_2E_2(x)$ and that $|E_2| \ll |E_1|$.

b) What is the minimum thickness of the medium that will give the maximum second harmonic signal?

7 Consider a nonlinear medium which has a polarization given by $P = \chi^{(1)} E + \chi^{(2)} EE$. Now consider an incident plane wave $E_1(x,t) = E_1 e^{ik_1 x - i\omega_1 t}$. a) Calculate the second harmonic intensity generated if the medium has a length L, where the second harmonic field is given by $E_2(x,t) = E_2(x) e^{ik_2 x - i\omega_2 t}$ and $\omega_2 = 2\omega_1$. Let the phase velocity at ω_1 be c/n_1 and at ω_2 be c/n_2.

Hint: You may assume that $\frac{\partial}{\partial x} E_2(x) \ll k_2 E_2(x)$ and that $|E_2| \ll |E_1|$.

b) What is the minimum thickness of the medium that will give the maximum second harmonic signal.

Solution

$$P_2 = \chi^{(2)} E_1 E_1$$

We put this into Maxwell's wave equation w/ $\omega_2 = 2\omega_1$

$$\frac{\partial^2}{\partial x^2} E_2 - \frac{1}{v_2^2}\frac{\partial^2 E_2}{\partial t^2} = \frac{4\pi}{c^2}\frac{\partial^2}{\partial t^2} P_2 = \frac{4\pi}{c^2}\frac{\partial^2}{\partial t^2}\chi^{(2)} E_1 E_1 e^{2ik_1 x - 2i\omega_1 t}$$

let $E_2(x,t) = E_2(x)\, e^{ik_2 x - i\omega_2 t}$ $\qquad k_2 E_2 >> \frac{\partial}{\partial x} E_2$

neglect

then $$\cancel{-k_2^2} E_2(x) e^{ik_2 x - i\omega_2 t} + 2ik_2\left[\frac{\partial}{\partial x} E_2(x)\right] e^{ik_2 x - i\omega_2 t} + \left[\cancel{\frac{\partial^2}{\partial x^2} E_2(x)}\right] e^{ik_2 x - i\omega_2 t} + \cancel{\frac{\omega_2^2}{v_2^2}} E_2(x) e^{i\ldots}$$

$$= -\frac{4\pi}{c^2}(2\omega_1)^2 \chi^{(2)} E_1 E_1 e^{2ik_1 x - 2i\omega_1 t}$$

$$\rightarrow \quad \frac{\partial}{\partial x} E_2(x) = \frac{2\pi i}{c^2}\omega_2^2 \chi^{(2)} E_1^2 e^{2ik_1 x - ik_2 x}$$

let $2k_1 - k_2 = \Delta k$

$$E_2(x) = \int_0^L \frac{2\pi i}{c^2}\omega_2^2 \chi^{(2)} E_1^2 e^{i\Delta k x} = \frac{2\pi i\, \omega_2^2 \chi^{(2)} E_1^2}{c^2}\frac{(e^{i\Delta k L} - 1)}{i\Delta k}$$

$$I_2(x) \propto |E_2(x)|^2 = \left|\frac{2\pi\, \omega_2^2 \chi^{(2)} E_1^2}{c\, \Delta k}\right|^2 \left[2 - 2\cos(\Delta k L)\right]$$

a) $$I_2(x) \propto \frac{\omega_2^4 |\chi^{(2)}|^2 I_1^2}{c^2} \frac{\sin^2\left(\frac{\Delta k L}{2}\right)}{(\Delta k)^2}$$

b) First maximum will occur at $\frac{\Delta k L}{2} = \pi/2 \rightarrow L = \pi/\Delta k$

w/ $\Delta k = 2k_1 - k_2 = 2n_1\frac{\omega_1}{c} - n_2\frac{\omega_2}{c}$

8. Consider a single coil in the x-y plane centered at $(x,y,z)=(0,0,0)$. A second coil is placed parallel to the first and centered at $(x,y,z)=(0,0,d)$. Let the coils have a radius b and carry a current I.

a) Calculate the magnetic field at $z=d/2$.

b) Find the ratio of b/d that will give the most uniform field at $z=d/2$.

8 Consider a single coil in the x-y plane centered at $(x,y,z) = (0,0,0)$. A second coil is placed parallel to the first and centered at $(x,y,z) = (0,0,d)$. Let the coils have a radius b and carry a current I.

a) Calculate the magnetic field at $z = d/2$.

~~b) Calculate~~ $\cancel{\partial/\partial z\, B_z}$ ~~at~~ $\cancel{z = d/2}$

~~c) Calculate~~ $\cancel{\partial^2/\partial^2 z\, B_z}$ ~~at~~ $\cancel{z = d/2}$ ~~when~~ $\cancel{b = d}$

~~d) What do the results of b) and c) tell you about the magnetic field between the two coils at~~ $\cancel{z = d/2}$~~?~~

b) Find the ratio of b/d that will give the most uniform field at $z = d/2$.

b) follows from $\nabla \cdot B = 0$ and symmetry trivially. L.A.L.
Problem seems very simple.

OK - JH

Solution

a) the magnetic field due to one coil can be obtained from the Biot-Savart law

$$d\vec{B} = \frac{\mu_0}{4\pi} I \frac{d\vec{l} \times \hat{r}}{r^2}$$

Since we want the field on the z-axis, we know by symetry that $\vec{B} = B_z \hat{z}$.

thus

$$dB_z = \frac{\mu_0}{4\pi} I \frac{dl}{(b^2+z^2)} \frac{b}{(b^2+z^2)^{1/2}}$$

So

$$B_z = \frac{\mu_0}{4\pi} I \frac{2\pi b^2}{(b^2+z^2)^{3/2}}$$

thus for the two coils

$$B_z = \frac{\mu_0 I b^2}{2}\left[\frac{1}{(b^2+z^2)^{3/2}} + \frac{1}{(b^2+(z-d)^2)^{3/2}}\right]$$

a) thus at $z = d/2$

$$\boxed{B_z = \frac{\mu_0 I b^2}{(b^2 + d^2/4)^{3/2}}}$$

b) solving for $\partial/\partial z\, B_z$ we find

$$\frac{\partial}{\partial z} B_z = \frac{\mu_0 I b^2}{2}\left[-\frac{3z}{(b^2+z^2)^{5/2}} - \frac{3(z-d)}{(b^2+(z-d)^2)^{5/2}}\right]$$

thus at $z = d/2$

$$\boxed{\left.\frac{\partial}{\partial z} B_z\right|_{z=d/2} = 0}$$

Solution (cont)

Solving for $\partial^2/\partial z^2 \, B_z$ we find

$$\frac{\partial^2}{\partial z^2} B_z = -3\mu_0 I b^2 \left[\frac{1}{(b^2+z^2)^{5/2}} - \frac{5z^2}{(b^2+z^2)^{7/2}} + \frac{1}{(b^2+(z-d)^2)^{5/2}} - \frac{5(z-d)^2}{(b^2+(z-d)^2)^{7/2}} \right]$$

evaluating this at $z = d/2$ we find that

that $$\left.\frac{\partial^2}{\partial z^2} B_z\right|_{z=d/2} = \frac{-6\mu_0 I b^2}{(b^2+d^2/4)^{5/2}} \left[1 - \frac{5\,d^2/4}{(b^2+d^2/4)} \right]$$

when $b = d$

$$\left.\frac{\partial^2}{\partial z^2} B_z\right|_{\substack{z=d/2 \\ b=d}} = \cdots \left[1 - \frac{5d^2/4}{5d^2/4} \right] = 0$$

Since both the first and second derivatives of B_z are zero at the center of the coil when $b=d$, we expect the magnetic field to be very uniform in this region. This arrangement is called a Helmholtz coil.

DEPARTMENT OF PHYSICS

Ph.D. COMPREHENSIVE EXAMINATION

TUESDAY, SEPTEMBER 22, 1987, 9 AM - 12 PM

Answer each of the following questions, all of which carry equal weight. Begin your answer to each question on a new sheet of paper solutions to different questions must not appear on the same sheet of paper. Each sheet of paper must be labeled with your name and the problem number in the right hand corner of the page. If more than one sheet is submitted for a problem, be sure the pages are ordered properly.

9. Derive the Clebsch-Gordan coefficients

$$\langle L = 1, M_L, S = 1/2, M_S | L, S, J, M_J \rangle$$

Conc

9. <u>Clebsch-Gordon Coefficients</u> $\langle L=1\, M_L\, S=\tfrac{1}{2}\, M_S \,|\, L S J M_J\rangle$

① Coupling 1 & $\frac{1}{2}$ gives $J = \frac{3}{2}, \frac{1}{2}$; $-J \le M_J \le J$

② Use more compact notation $\left.\right\}$ $\langle M_L M_S \,|\, J M_J\rangle$
for this special case

③ Most of the credit will be received for correct <u>procedure</u>.

There is only 1 possibility for $\langle 1\, \tfrac{1}{2} \,|\, \tfrac{3}{2}\, \tfrac{3}{2}\rangle \equiv 1$

From that starting point, use $J_- \equiv L_- + S_-$
where $J_- \,|J\, M_J\rangle = \sqrt{J(J+1) - m(m-1)}\; |J\, M_J - 1\rangle$

For example,

$$J_- \left|\tfrac{3}{2}\, \tfrac{3}{2}\right\rangle = \sqrt{\tfrac{3}{2}\left(\tfrac{5}{2}\right) - \left(\tfrac{3}{2}\right)\left(\tfrac{1}{2}\right)}\; \left|\tfrac{3}{2}\, \tfrac{1}{2}\right\rangle = \underline{\sqrt{3}\; \left|\tfrac{3}{2}\, \tfrac{1}{2}\right\rangle}$$

$$= L_- \left|1\, \tfrac{1}{2}\right\rangle + S_- \left|1\, \tfrac{1}{2}\right\rangle$$

$$= \sqrt{1(2) - 0}\; \left|0\, \tfrac{1}{2}\right\rangle + \sqrt{\tfrac{1}{2}\left(\tfrac{3}{2}\right) - \tfrac{1}{2}\left(-\tfrac{1}{2}\right)}\; \left|1\, -\tfrac{1}{2}\right\rangle$$

$$= \underline{\sqrt{2}\; \left|0\, \tfrac{1}{2}\right\rangle + 1 \left|1\, -\tfrac{1}{2}\right\rangle}$$

So $\left|\tfrac{3}{2}\, \tfrac{1}{2}\right\rangle = \underbrace{\sqrt{\tfrac{1}{3}}}_{\langle 1\, -\frac{1}{2} | \frac{3}{2}\, \frac{1}{2}\rangle} \left|1\, -\tfrac{1}{2}\right\rangle + \underbrace{\sqrt{\tfrac{2}{3}}}_{\langle 0\, \frac{1}{2} | \frac{3}{2}\, \frac{1}{2}\rangle} \left|0\, \tfrac{1}{2}\right\rangle$

Similarly,
$$\langle 0\ -\tfrac{1}{2} \mid \tfrac{3}{2}\ -\tfrac{1}{2} \rangle = \sqrt{\tfrac{2}{3}}$$
$$\langle -1\ \ \tfrac{1}{2} \mid \tfrac{3}{2}\ -\tfrac{1}{2} \rangle = \sqrt{\tfrac{1}{3}}$$

$$\langle -1\ -\tfrac{1}{2} \mid \tfrac{3}{2}\ -\tfrac{3}{2} \rangle = 1 \qquad \text{(also by inspection)}$$

For $J = \frac{1}{2}$

$$\langle 1\ -\tfrac{1}{2} \mid \tfrac{1}{2}\ \tfrac{1}{2} \rangle = \sqrt{\tfrac{2}{3}}$$
$$\langle 0\ \ \tfrac{1}{2} \mid \tfrac{1}{2}\ \tfrac{1}{2} \rangle = -\sqrt{\tfrac{1}{3}}$$

and

$$\langle 0\ -\tfrac{1}{2} \mid \tfrac{1}{2}\ -\tfrac{1}{2} \rangle = \sqrt{\tfrac{1}{3}}$$
$$\langle -1\ \ \tfrac{1}{2} \mid \tfrac{1}{2}\ -\tfrac{1}{2} \rangle = -\sqrt{\tfrac{2}{3}}$$

10. An electron in a one-dimensional world is subject to the potential

$$V = (1/2)kx^2$$

Determine the transition rate for spontaneous emission of radiation when the electron is in the first excited state.

Q. Mech. Hermanson

ok - JLC

10 An electron in a one-dimensional world is subject to the potential

$$V = \frac{1}{2} k x^2 .$$

Determine the transition rate for spontaneous emission of radiation when the electron is in the first excited state.

$$\Gamma = \frac{4e^2 \omega_{kn}^3}{3\hbar c^3} \left| \langle k | \underset{\sim}{r} | n \rangle \right|^2$$

~~could provide the formula?~~ no JLC

$$= \frac{4e^2 \omega^3}{3\hbar c^3} \left| \langle 0 | x | 1 \rangle \right|^2 \quad \text{with } \omega = \sqrt{k/m}$$

Here $\langle 0 | x | 1 \rangle = \langle 0 | \sqrt{\frac{\hbar}{2m\omega}} (\cancel{a^+} + a) | 1 \rangle$ since $\langle 0 | a | 1 \rangle = 1$

$$= \sqrt{\frac{\hbar}{2m\omega}}$$

$$\Gamma = \frac{4e^2\omega^3}{3\hbar c^3} \cdot \frac{\hbar}{2m\omega}$$

$$\boxed{\Gamma = \frac{2}{3} \frac{e^2 \omega^2}{mc^3}}$$

11. Consider an electron bound by a one-dimensional harmonic oscillator potential of frequency ω_0. For $t<0$ the electron is assumed to be in the ground state $|0\rangle$ of the harmonic oscillator. Suppose that at $t=0$ we switch on a time-dependent electric field of the form $E_0 \cos\omega t$. (The field is collinear with the harmonic oscillator.)

a) Using time-dependent perturbation theory calculate, to lowest order, the probability of finding the electron at a later time t in the first excited state, $|1\rangle$, of the harmonic oscillator. Discuss the behavior of this probability as a function of ω for $\omega \to \omega_0$ for a fixed value of t.

b) What is the lowest non-vanishing order for the transition $|0\rangle \to |2\rangle$ to occur where $|2\rangle$ is the second excited state of the harmonic oscillator?

Suggestion. Use the interaction picture. The relation between the state vector and the coupling Hamiltonian in the interaction and Schrodinger pictures is given by the equations

$$|\Psi_I(t)\rangle = e^{i/\hbar \hat{H}_0 t} |\Psi_S(t)\rangle$$

and

$$\hat{U}_I(t) = e^{i/\hbar \hat{H}_0 t} \hat{U}(t) e^{-i/\hbar \hat{H}_0 t},$$

where H_0 is the unperturbed Hamiltonian.

Note: $$\hat{x} = \sqrt{\frac{\hbar}{2m\omega_0}} (a^+ + a)$$

$$a^+ |n\rangle = \sqrt{n+1} |n+1\rangle$$

$$a |n\rangle = \sqrt{n} |n-1\rangle$$

OK - JH

OK. JLC

ok. LAL

AE

11 **Quantum Mechanics.**

Consider an electron bound by a one-dimensional harmonic oscillator potential of frequency ω_0. For $t<0$ the electron is assumed to be in the ground state $|0\rangle$ of the harmonic oscillator.

Suppose that at $t=0$ we switch on a time-dependent electric field of the form $\mathcal{E}_0 \cos\omega t$. (The field is collinear with the harmonic oscillator.)

a) Using time-dependent perturbation theory calculate, to lowest order, the probability of finding the electron at a later time t in the first excited state, $|1\rangle$, of the harmonic oscillator. Discuss the behavior of this probability as a function of ω for $\omega \to \omega_0$ for a fixed value of t.

b) What is the lowest non-vanishing order for the transition $|0\rangle \to |2\rangle$ to occur, where $|2\rangle$ is the second excited state of the harmonic oscillator?

Suggestion. Use the interaction picture. The relation between the state vector and the coupling Hamiltonian in the interaction and Schrödinger pictures is given by the equations

$$|\psi_I(t)\rangle = e^{\frac{i}{\hbar}\hat{H}_0 t}|\psi_S(t)\rangle$$

and

$$\hat{U}_I(t) = e^{\frac{i}{\hbar}\hat{H}_0 t}\,\hat{U}(t)\,e^{-\frac{i}{\hbar}\hat{H}_0 t},$$

where $\hat{H}_0$ is the unperturbed Hamiltonian.

Note :

$$\hat{x} = \left(\frac{\hbar}{2m\omega_0}\right)^{1/2} (a^+ + a)$$

$$a^+ |n\rangle = \sqrt{n+1}\ |n+1\rangle$$

$$a |n\rangle = \sqrt{n}\ |n-1\rangle$$

Solution.

a) The equation of motion for the state $|\psi_I(t)\rangle$ is readily obtained from the Schrödinger equation satisfied by $|\psi_S(t)\rangle$. We have that

$$i\hbar \frac{d}{dt} |\psi_I(t)\rangle = \hat{U}_I(t) \, |\psi_I(t)\rangle \,.$$

To first order in $\hat{U}_I(t)$ we have that

$$|\psi_I(t)\rangle = |0\rangle + \frac{1}{i\hbar} \int_0^t dt' \, \hat{U}_I(t') \, |0\rangle \,.$$

In the present case the coupling Hamiltonian is

$$\hat{U}_S(t) = \begin{cases} -\underset{\sim}{\mathcal{E}} \cdot (-e \underset{\sim}{\hat{x}}) = +e\mathcal{E}_0 \cos\omega t \, \hat{x} \,, & t > 0 \\ 0 & t < 0 \end{cases}$$

$\therefore$

$$|\psi_I(t)\rangle = |0\rangle + \frac{e\mathcal{E}_0}{i\hbar} \int_0^t dt' \cos\omega t' \, e^{\frac{i}{\hbar}\hat{H}_0 t'} \, \hat{x} \, e^{-\frac{i}{\hbar}\hat{H}_0 t'} |0\rangle$$

Then

$$\langle 1 | \psi_I(t)\rangle = \frac{e\mathcal{E}_0}{i\hbar} \int_0^t dt' \cos\omega t' \, e^{i\omega_0 t'} \quad \langle 1 | \hat{x} | 0\rangle$$

Thus :

$$\mathcal{P}_{0\to 1}(t) = |\langle 0|\psi_S(t)\rangle|^2 = |\langle 0|\psi_I(t)\rangle|^2$$

$$= \left(\frac{e\mathcal{E}_0}{\hbar}\right)^2 \left|\int_0^t dt' \cos\omega t' \, e^{i\omega_0 t'}\right|^2 |\langle 1|\hat{x}|0\rangle|^2$$

Now :

$$\langle 1|\hat{x}|0\rangle = \left(\frac{\hbar}{2m\omega_0}\right)^{1/2} \langle 1| a + a^{+} |0\rangle = \left(\frac{\hbar}{2m\omega_0}\right)^{1/2}$$

$$\boxed{\mathcal{P}_{0\to 1}(t) = \frac{e^2\mathcal{E}_0^2}{2m\hbar\omega_0} \left|\int_0^t dt' \, e^{i\omega_0 t'} \cos\omega t'\right|^2}$$

Integrating by parts (or writing $\cos\omega t = \frac{e^{i\omega t} + e^{-i\omega t}}{2}$) we find that

$$\int dt' \, e^{i\omega_0 t'} \cos\omega t' =$$

$$= \frac{e^{i\omega_0 t'}}{\omega^2 - \omega_0^2} \left(i\omega_0 \cos\omega t + \omega \sin\omega t \right)$$

$\therefore$

$$\int_0^t dt'\, e^{i\omega_0 t'} \cos\omega t' =$$

$$= \frac{1}{\omega^2 - \omega_0^2}\left[e^{i\omega_0 t}\left(i\omega_0 \cos\omega t + \omega \sin\omega t \right) - i\omega_0 \right]$$

Thus, for $\omega \to \omega_0$

$$P_{0\to 1}(t) \sim \left| \frac{1}{\omega^2 - \omega_0^2} \right|$$: quantum mechanical resonance.

b) Consider the matrix element

$$\langle 2 | \hat{x} | 0 \rangle = \left(\frac{\hbar}{2m\omega_0} \right)^{1/2} \langle 2 | a + a^\dagger | 0 \rangle$$

$$= 0 .$$

$\Rightarrow$ there is no first-order transition from $|0\rangle$ to $|2\rangle$.

The second-order transition is governed by products of matrix elements of the form

$$\langle 2 | \hat{x} | n \rangle \langle n | \hat{x} | 0 \rangle ,$$

where the "intermediate" state $|n\rangle$ is any of the excited states, including $|n=1\rangle$. Since the expression

$$\langle 2|\hat{x}|1\rangle\langle 1|\hat{x}|0\rangle =$$

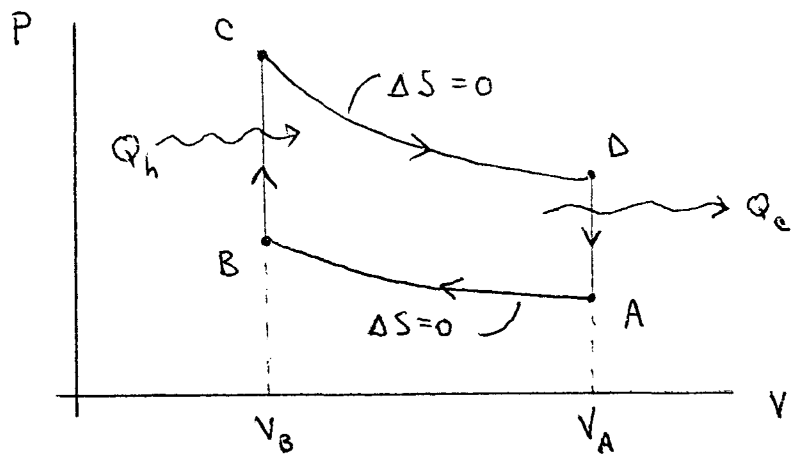
$$= \left(\frac{\hbar}{2m\omega_0}\right) \langle 2|a+a^+|1\rangle\langle 1|a+a^+|0\rangle$$

$$= \left(\frac{\hbar}{2m\omega_0}\right) \sqrt{2}\langle 2|2\rangle\langle 1|1\rangle$$

$$= \sqrt{2}\left(\frac{\hbar}{2m\omega_0}\right)$$

: non-zero, the transition $|0\rangle \rightarrow |2\rangle$ is allowed to second order.

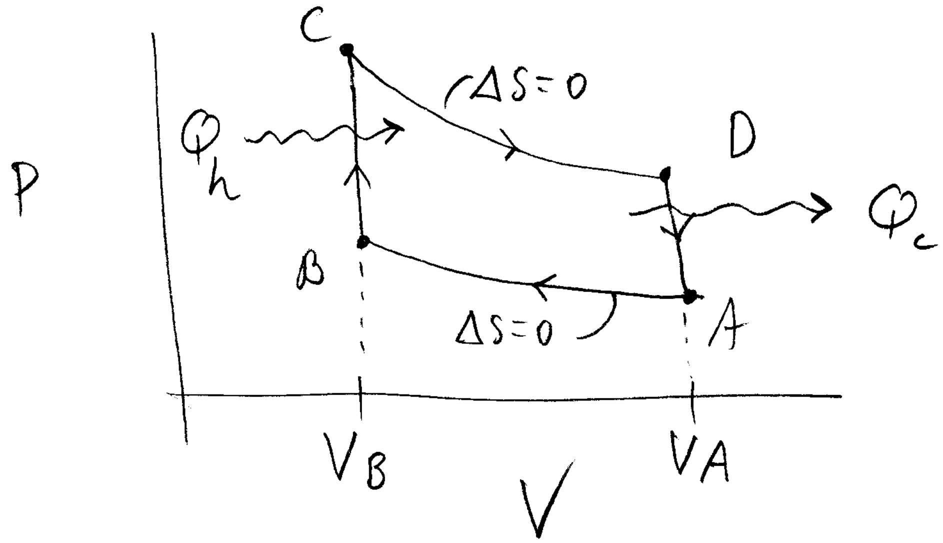
12. One mole of an ideal monatomic gas is taken through the four-step cycle shown below. Processes A → B and C → D take place at constant entropy, while the other processes occur at constant volume, V_A and V_B as indicated. Compute the engine efficiency in terms of V_A and V_B.



Thermodynamics Hermanson

ok-ne

12 One mole of an ideal monatomic gas is taken through the four-step cycle shown below. Processes $A \to B$ and $C \to D$ take place at constant entropy, while the other processes occur at constant volume, V_A and V_B as indicated. Compute the engine efficiency in terms of V_A and V_B.



The efficiency $\varepsilon = \frac{W}{Q_h} = 1 - \frac{Q_c}{Q_h}$

$$Q_c = C_v (T_D - T_A) \quad ; \quad Q_h = C_v (T_C - T_B)$$

$$\varepsilon = 1 - \frac{T_D - T_A}{T_C - T_B}$$

Now on the adiabats $A \to B$ and $C \to D$,

$$PV^{5/3} = \text{const} \quad \text{or} \quad \underline{TV^{2/3} = \text{const}}$$

$A \to B$: $\quad T_B V_B^{2/3} = T_A V_A^{2/3}$

$C \to D$: $\quad T_C \underset{V_B}{V_C}^{2/3} = T_D \underset{V_A}{V_D}^{2/3}$

Subtracting the 1st from the 2nd eqn,

$$(T_C - T_B) V_B^{2/3} = (T_D - T_A) V_A^{2/3}$$

$$\frac{T_D - T_A}{T_C - T_B} = \frac{V_B^{2/3}}{V_A^{2/3}}$$

$$\boxed{\varepsilon = 1 - \left(\frac{V_B}{V_A}\right)^{2/3}}$$

D.K. (Standard) A.E.
Fine L.A.L.

DEPARTMENT OF PHYSICS

Ph.D. COMPREHENSIVE EXAMINATION

TUESDAY, SEPTEMBER 22, 1987, 1 PM - 4 PM

Answer each of the following questions, all of which carry equal weight. Begin your answer to each question on a new sheet of paper solutions to different questions must not appear on the same sheet of paper. Each sheet of paper must be labeled with your name and the problem number in the upper right hand corner of the page. If more than one sheet is submitted for a problem, be sure the pages are ordered properly.

13. Consider the motion of a point particle of mass M in a region of space containing a fixed potential energy $\varphi(x)$. Find the differential equation for the path joining points $\vec{x}_1$ and $\vec{x}_2$ along which the particle should be moved if one wishes to minimize the time average of the square magnitude of the "external" force (i.e. not including that produced by the potential φ) applied to the particle. Assume that the velocities at the two endpoints of the path, $\vec{V}_1$ and $\vec{V}_2$ are to be given and fixed, and that the beginning and ending times t_1 and t_2 are given and fixed. Find at least one second integral of this differential equation.

Paths of this type might be used as the most economical orbits for navigating spacecraft in the solar system, since they minimize the amount of non-gravitational force (e.g. fuel) need to guide the craft to a desired destination.

Lindblom

Mechanics

13. Consider the motion of a point particle of mass m in a region of space containing a fixed potential energy $\varphi(\vec{x})$. Find the differential equation for the path joining points $\vec{x}_1$ and $\vec{x}_2$ along which the particle should be moved if one wishes to minimize the time average of the square magnitude of the "external" force (i.e. not including that produced by the potential φ) applied to the particle. Assume that the velocities at the two endpoints of the path, $\vec{v}_1$ and $\vec{v}_2$ are to be given and fixed, and that the beginning and ending times t_1 and t_2 are given and fixed. Find at least one second integral of this differential equation.

Paths of this type might be used as the most economical orbits for navigating spacecraft in the solar system, since they minimize the amount of non-gravitational force (e.g. fuel) needed to guide the craft to a desired destination.

(1)

Solution:

The non gravitational force applied to a particle moving along a path $\vec{x}(t)$ is simply:

$$\vec{F} = m \frac{d^2\vec{x}}{dt^2} + \vec{\nabla}\varphi$$

We wish to minimize the integral of the norm of this quantity:

$$Q = \frac{1}{2}\int_{t_1}^{t_2} \vec{F}\cdot\vec{F}\, dt$$

$$\delta Q = \int_{t_1}^{t_2} \vec{F}\cdot\left\{ m \frac{d^2\delta\vec{x}}{dt^2} + \sum_i \left(\vec{\nabla}\frac{\partial\varphi}{\partial x_i}\right)\delta x^i \right\} dt$$

$$= \int_{t_1}^{t_2} \frac{d}{dt}\left(m\vec{F}\cdot\frac{d\delta\vec{x}}{dt}\right) dt - \int_{t_1}^{t_2} m\frac{d\vec{F}}{dt}\cdot\frac{d\delta\vec{x}}{dt}\, dt$$

$$+ \int_{t_1}^{t_2} \vec{F}\cdot\left(\sum_i \vec{\nabla}\frac{\partial\varphi}{\partial x^i}\right)\delta x^i\, dt$$

$$= \int_{t_1}^{t_2} \frac{d}{dt}\left\{ m\vec{F}\cdot\frac{d\delta\vec{x}}{dt} - m\frac{d\vec{F}}{dt}\cdot\delta\vec{x} \right\} dt$$

$$+ \int_{t_1}^{t_2} \sum_i \left\{ m\frac{d^2F^i}{dt^2} + \sum_j F^j \partial_j\partial_i\varphi \right\} \delta x^i\, dt$$

(2)

The boundary conditions require that the position and velocity of the particle be fixed at the endpoints so:

$$\delta\vec{x} = 0 \quad \text{at } t_1 \text{ and } t_2$$

$$\delta ? \rightsquigarrow \frac{d\delta\vec{x}}{dt} = 0 \quad \text{at } t_1 \text{ and } t_2$$

Thus the first integral vanishes so:

$$\delta Q = \int_{t_1}^{t_2} \sum_i \left\{ m \frac{d^2F^i}{dt^2} + \sum_j F^j \partial_j \partial_i \varphi \right\} \delta x^i \, dt$$

The minimum of Q occurs when $\delta Q = 0$ for all nearby paths so

$$\begin{cases} m \dfrac{d^2F^i}{dt^2} + \sum_j F^j \partial_j \partial_i \varphi = 0 \\ F^i = m \dfrac{d^2x^i}{dt^2} + \partial_i \varphi \end{cases}$$

is the fourth order differential equation for the desired path.

A simple second integral of this system is $F^i = 0$ or

$$m \frac{d^2x^i}{dt^2} + \partial_i \varphi = 0$$

This is just the classical path without external forces!

14. Consider a one-dimensional crystal composed of N atoms. The interatomic separation is a_o. In the harmonic approximation, the vibrational properties of this crystal are described by a collection of linear harmonic oscillators (phonons) of frequency $\omega(k)$, where k is a wave vector.

a) Write down the vibrational energy density of the crystal at temperature T. From it write down a general equation for the specific heat at constant volume.

b) In the limit of low temperatures, one can make the approximation that the relevant oscillators in your result for C_v are those for which $\omega(k) = vk$, where v is a constant. Use this approximation to obtain the temperature dependence of C_v for $T \to 0$ K.

~~Qual~~ → comp

Solid State

AE

14 Consider a one-dimensional crystal composed of N atoms. The interatomic separation is a_0. In the harmonic approximation, the vibrational properties of this crystal are described by a collection of linear harmonic oscillators (phonons) of frequency $\omega(k)$, where k is a wave vector.

1. Write down the vibrational energy density of the crystal at temperature T. From it write down a general equation for the specific heat at constant volume.

2. In the limit of low temperatures, one can make the approximation that the relevant oscillators in your result for c_v are those for which $\omega(k) = ck$, where c is a constant. Use this approximation to obtain the temperature dependence of c_v for $T \to 0K$

O.K. (A.e.g)

Too hard?, OK for Comps JED → comp

good comp prob JLC

OK for qual - GT.

ok L.A.L.

RC - Change the const. c in $\omega(k) = ck$ to avoid confusion

OK - JH

Solution.

a) For a collection of harmonic oscillators of frequency $\omega(k)$ we have that the vibrational energy at temperature T is given by

$$E = \sum_k \hbar\omega(k) \left(n_k + \tfrac{1}{2}\right) \quad ,$$

where

$$n_k = \frac{1}{e^{\beta\hbar\omega(k)} - 1} \quad , \qquad \beta = \frac{1}{k_B T}$$

is the boson occupation number.

We then have that

$$\frac{E}{L} = \frac{1}{L} \sum_k \hbar\omega(k) \left(n_k + \tfrac{1}{2}\right) \quad , \qquad L = Na_0 .$$

Thus:

$$C_v = \frac{\partial}{\partial T} \frac{1}{L} \sum_k \hbar\omega(k) \left(n_k + \tfrac{1}{2}\right)$$

$$= \frac{\partial}{\partial T} \int_0^\infty \frac{dk}{\pi} \frac{\hbar\omega(k)}{e^{\beta\hbar\omega(k)} - 1}$$

b) For $T \to 0K$ the main contribution to the integral comes from small wave vectors. Thus we make the approximation

$$\omega(k) = ck$$

for all wave vectors. (The linear relation only holds for small wave vectors.)

Then:

$$C_v \cong \frac{\hbar c}{\pi} \frac{\partial}{\partial T} \int_0^\infty dk \; \frac{k}{e^{\beta \hbar c k} - 1}$$

Set : $\beta \hbar c k = x \quad \rightarrow \quad dk = \dfrac{dx}{\beta \hbar c}$

$\therefore$

$$\int_0^\infty dk \; \frac{k}{e^{\beta \hbar c k} - 1} = \frac{1}{\beta^2 \hbar^2 c^2} \int_0^\infty dx \; \frac{x}{e^x - 1}$$

$$= \frac{k_B^2 T^2}{\hbar^2 c^2} \int_0^\infty dx \; \frac{x}{e^x - 1}$$

We then have that

$$C_v \xrightarrow[T \to 0]{} a T$$

where

$$a = \frac{2 \, k_B^2}{\pi \hbar c} \int_0^\infty dx \; \frac{x}{e^x - 1}$$

15. Consider a free atom.

a) Find the allowed L-S terms (spectroscopic states) for each electron configuration listed below. Briefly explain, carefully stating any general principles which you use. Specify the states by the total angular momentum L and the total spin S.

(1) np^2

(2) $nsn'l \qquad l = 0, 1, 2 \ldots \qquad (n \neq n')$

(3) $np^5n's \qquad (n \neq n')$

b) What interaction causes each L-S term to have a different energy?

c) For case a-1, what is the lowest LSJ state? Why?

$$\vec{J} \equiv \vec{L} + \vec{S}$$

d) When $n' \gg n$ for case a-2 or a-3, what kind of functional dependence on n' might one expect for the energy of the state? What useful "chemical" information can be extracted from that dependence for a gas of these atoms?

for spring? A.E.

OK for comp JLC

OK - JH

Atomic Molecular / Core

15. Consider a free atom.

(a) Find the allowed energy levels ~~L–S terms~~ for each electron configuration listed below. Briefly explain, carefully stating any general principles which you use. Specify the energy levels by the total angular momentum L and the total spin S.

(1) np^2

(2) $ns\,n'l$ $\qquad l = 0, 1, 2, \cdots \qquad (n \neq n')$

(3) $np^5 n's$ $\qquad (n \neq n')$

(b) What ~~interaction~~ causes each level ~~L–S term~~ to have a different energy?

(c) For case a-1, what is the lowest LSJ state? Why? $\qquad \vec{J} \equiv \vec{L} + \vec{S}$

(d) What kind of energy ~~functional~~ dependence on n might one expect in case a-2 or a-3 when $n' \gg n$? What useful "chemical" information can be extracted from that dependence for a gas of these atoms?

Solution

(a) For case 1, the electrons are "equivalent" and we must obey the Pauli Exclusion Principle — no two electrons can have the same set of quantum numbers — by carefully examining $m_{\ell 1}\ m_{\ell 2}\ m_{s1}\ m_{s2}$.

① np^2	$M_S =$ 1	0	−1
$M_L =$ 2		(1^+1^-)	
1	(1^+0^+)	$(1^+0^-)\ (1^-0^+)$	(1^-0^-)
0	(1^+-1^+)	$(1^+-1^-)(1^--1^+)(0^+0^-)$	(1^--1^-)
−1	(0^+-1^+)	$(0^+-1^-)(0^--1^+)$	(0^--1^-)
−2		(-1^+-1^-)	

∴ 1 term with $L=2$ $S=0$ ^1D
1 term with $L=1$ $S=1$ ^3P
1 term with $L=0$ $S=0$ ^1S

② since $n \neq n'$ all $m_s\ m_\ell$ combinations are allowed

$$\begin{cases} L = 0+\ell & \text{for each value of } \ell \\ S = 1, 0 & \text{for each value of } \ell \end{cases}$$

③ since p^5 has one "hole", the results are identical to ② (except n & n' have been interchanged

(b) electron electron repulsion $\frac{e^2}{r_{ij}}$

(c) less than half-filled shell since $2 < \frac{2(2\ell+1)}{2}$ for p
so smallest J is low (rule arises from spin orbit coupling)
The allowed values of J are $|L-S| \leq J \leq L+S$
The lowest level is thus $^3P_0$ $(J=0)$

(d) When $n' >> n$ the electron labeled n' interacts weakly with the "core". Thus, we expect a Rydberg series, which can be used to find the ionization potential(s).

16. Discuss the potential impacts of recent advances insuperconductivity on the construction of the so-called superconducting super collider. Your discussion should be at the level of articles in The Physics Teacher, a journal intended for teachers of high school and undergraduate physics. Briefly discuss such topics as (1) the basic physics of superconductors, (2) what advances have been made, (3) what problems must be solved before these advances can be utilized in the construction, (4) the potential impacts on the construction, and (5) what physics might be done with a super collider.