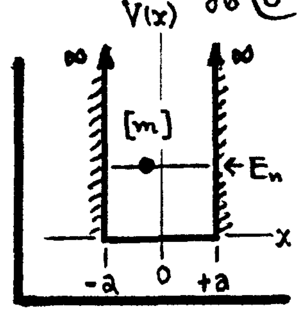
ϕ 506 Problems



(24) [15 pts]. A mass m is contained in a 1D "box" with ∞ly steep potential walls at $x = \pm a$ as shown (i.e. $V(x) = 0$, for $|x| < a$; $V(x) = \infty$, for $|x| > a$). m is in an eigenstate of energy E_n.

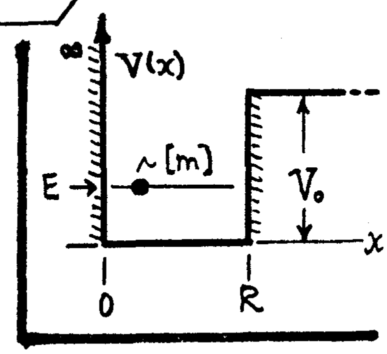
(A) Calculate m's mean position $\langle x \rangle$ and its variance $(\Delta x)^2 = \langle (x - \langle x \rangle)^2 \rangle$.

(B) Calculate m's mean momentum $\langle p \rangle$ and its variance $(\Delta p)^2 = \langle (p - \langle p \rangle)^2 \rangle$.

(C) What is the uncertainty product $\Delta x \Delta p$ in state n? Comment.

---

(25) For mass m contained in the 1D "box" specified in problem (24), find the average force exerted by the particle on one wall of the well. Compare your QM result for $\langle F \rangle$ with the corresponding classical expression. HINT: do the QM problem for very large but finite wall height V_0; then let $V_0 \to \infty$.



(26) Consider the 1D potential $V(x)$ in the sketch, i.e.

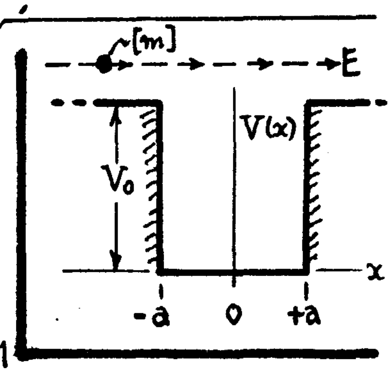
$$V(x) = \begin{cases} \infty, & \text{for } x < 0; \\ 0, & \text{for } 0 < x < R; \\ V_0, \text{ cnst}, & \text{for } x > R. \end{cases}$$

and consider a particle of mass m bound in $V(x)$, i.e. m at energy E such that $0 < E < V_0$. (This is a crude model of nuclear binding, w/ x = radial cd).

(A) Find the condition which determines the bound state energies E.

(B) What is the minimum V_0 for which a level is just barely bound (@ $E = V_0-$)?

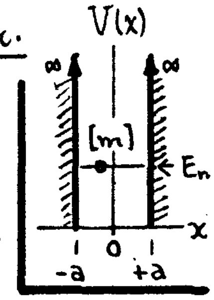
(C) At what value of V_0 does a second bound level appear?



(27) [15 pts] A mass m with energy $E > V_0$ is incident on the rectangular potential well shown. To fix ideas, let m come in from the left at unit incident amplitude. Find the reflected intensity R, the transmitted intensity T, and show: $R + T = 1$.

ϕ506 Solutions

(24) [15 pts]. For m in a 1D box, in state n, find $\langle x\rangle$ & $\langle p\rangle$ and Δx & Δp, etc.



(A) 1. From CLASS NOTES, p. Sol$^{\underline{ns}}$ 4, Eqs. (10) & (11), the eigenstates for the box are:

$$\rightarrow \psi_n(x) = \begin{cases} A\cos k_n x, & \text{for } n=\text{odd}=1,3,5\ldots \\ B\sin k_n x, & \text{for } n=\text{even}=2,4,6,\ldots \end{cases} \Bigg\| \text{ w/ } k_n = n\pi/2a, \text{ and energies: } E_n = \hbar^2 k_n^2/2m \text{ (for all } n) \quad (1)$$

For normalization, $\langle\psi_n|\psi_n\rangle = 1$, we need : $A = 1/\sqrt{a} = B$ (for all n). In any case, it is clear that $\langle x\rangle = \langle\psi_n|x\psi_n\rangle = 0$; the average location of m in the box is at its center, since m has no reason to prefer the LHS or RHS of the box. It is also true that $\langle x\rangle = \int_{-a}^{+a} x|\psi_n|^2 dx = 0$, since the integrand is an odd fcn of x. The same reasoning $\Rightarrow$ $\langle p\rangle = 0$; m has no reason to be preferentially traveling right or left.

---

2. The variance Δx in position is just...

$$\rightarrow (\Delta x)^2 = \langle x^2\rangle = \int_{-a}^{+a} x^2|\psi_n|^2 dx = \frac{1}{a}\cdot\begin{cases} \int_{-a}^{+a} x^2\cos^2 k_n x\,dx, & n=\text{odd}; \\ \int_{-a}^{+a} x^2 \sin^2 k_n x\,dx, & n=\text{even}. \end{cases} \quad (2)$$

and

$$\left.\begin{aligned} (\Delta x)^2\Big|_{n=\text{odd}} &= \frac{2}{k_n^3 a}\int_0^{k_n a} u^2\cos^2 u\,du = \frac{2}{k_n^3 a}\left[\frac{u^3}{6} + \left(\frac{u^2}{4}-\frac{1}{8}\right)\sin 2u + \frac{u\cos 2u}{4}\right]\Bigg|_{u=0}^{u=\frac{n\pi}{2}} = \frac{a^2}{3}, \\ (\Delta x)^2\Big|_{n=\text{even}} &= \frac{1}{a}\int_{-a}^{+a} x^2(1-\cos^2 k_n x)\,dx = \frac{1}{a}\int_{-a}^{+a} x^2 dx - (\Delta x)^2\Big|_{n=\text{odd}} = \frac{a^2}{3}. \end{aligned}\right\} \quad (3)$$

The position variance is the same in all states, viz. $\Delta x = a/\sqrt{3}$.

(B) 3. With $\langle p\rangle = 0$, the momentum variance is...

$$\rightarrow (\Delta p)^2 = \langle p^2\rangle = \int_{-a}^{+a}\psi_n^*\left\{\left(-i\hbar\frac{\partial}{\partial x}\right)^2\right\}\psi_n\,dx = (-)\frac{\hbar^2}{a}\cdot\begin{cases} \int_{-a}^{+a}\cos k_n x(\partial^2/\partial x^2)\cos k_n x\,dx, & n=\text{odd}; \\ \int_{-a}^{+a}\sin k_n x(\partial^2/\partial x^2)\sin k_n x\,dx, & n=\text{even}. \end{cases} \quad (4)$$

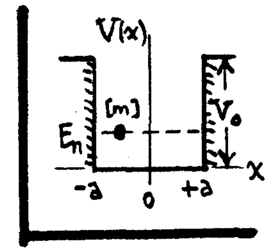
But $(\partial^2/\partial x^2)\cos k_n x = -k_n^2\cos k_n x$, and $(\partial^2/\partial x^2)\sin k_n x = -k_n^2\sin k_n x$, so (4) $\Rightarrow$

$$\rightarrow (\Delta p)^2 = \frac{1}{a}(\hbar k_n)^2\cdot\begin{cases}\int_{-a}^{+a}\cos^2 k_n x\,dx \\ \int_{-a}^{+a}\sin^2 k_n x\,dx\end{cases} = (\hbar k_n)^2, \text{ for all } n\ldots \text{ so } \Delta p = n\left(\frac{\pi\hbar}{2a}\right). \quad (5)$$

The momentum variance is not the same in all states; Δp increases with n.

(C) 4. The variance product, from (3) & (5) above, is state-dependent, as...

$\Delta x\,\Delta p = n\left(\frac{\pi}{2\sqrt{3}}\right)\hbar = 0.907\,n\hbar$; $n=1,2,3,\ldots$ (6) The uncertainty limit $\Delta x\,\Delta p \geq \frac{\hbar}{2}$ is not violated. What happens here is that $\Delta p = \hbar k_n$ is always as big as p itself.



(25) m in a 1D box at energy E_n; find force on wall.

1. First, let box walls have a finite step $V_0 \gg E_n$. Classically the force on the RH wall is $F = dV/dx = V_0\,\delta(x-a)$; (this is by m, on wall, and it results in a perfect reflection). QMly, we want the expectation value of F, i.e. if ψ_n = eigenfcn for state n :

$$\rightarrow \langle F\rangle_n = \int_{-\infty}^{\infty} \psi_n^* F \psi_n\, dx = V_0 |\psi_n(a)|^2. \qquad (1)$$

Thus we need a (normalized) value of $|\psi_n(a)|^2$ as $V_0 \to \infty$.

---

2. In the well (CLASS NOTES, pp. Solns 1-4), with $\alpha = \left[\frac{2m}{\hbar^2}E\right]^{1/2}$ & $\beta = \left[\frac{2m}{\hbar^2}(V_0 - E)\right]^{1/2}$, solns are:

$$\left[\begin{array}{l} \text{CLASS I}: \psi(x) = A\cos\alpha x, \text{ w/ } \alpha\tan\alpha a = +\beta, \text{ for continuity;} \\ \text{CLASS II}: \psi(x) = B\sin\alpha x, \text{ w/ } \alpha\,\mathrm{ctn}\,\alpha a = -\beta, \text{ for continuity.} \end{array}\right\} \qquad (2)$$

As $V_0 \to \infty$, ψ vanishes outside the well, $E_n \to n^2\left(\frac{\pi^2\hbar^2}{8ma^2}\right)$, and $\alpha \to \alpha_n = \frac{n\pi}{2a}$.

The normzn conditions, that $\int |\psi_n(x)|^2 dx = 1$, then require...

$$\left[\begin{array}{l} \text{CLASS I}: \; A^2 \int_{-a}^{+a} \cos^2\alpha_n x\, dx = 1 \;\Rightarrow\; A = 1/\sqrt{a}, \\ \text{CLASS II}: \; B^2 \int_{-a}^{+a} \sin^2\alpha_n x\, dx = 1 \;\Rightarrow\; B = 1/\sqrt{a}. \end{array}\right\} \qquad (3)$$

This norm was already discovered in prob. (24). As $V_0 \to \infty$, the normed eigenfcns are: $\psi_n \to (1/\sqrt{a})\cos\alpha x$ & $\psi_n \to (1/\sqrt{a})\sin\alpha x$, w/ $\alpha \to \alpha_n = n\pi/2a$.

---

3. Values for $\cos\alpha x$ & $\sin\alpha x$, at $x = a$ (& $V_0 \to \infty$) are still needed. From Eq. (2):

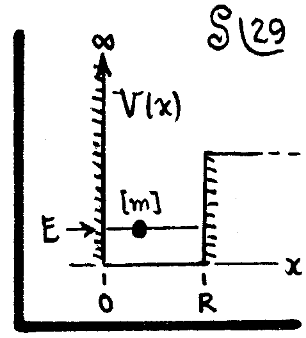
$$\left[\begin{array}{l} \text{CLASS I}: \alpha\sin\alpha a = +\beta\cos\alpha a \Rightarrow \cos\alpha a = \alpha/\sqrt{\alpha^2+\beta^2} = \sqrt{E/V_0}, \\ \text{CLASS II}: \alpha\cos\alpha a = -\beta\sin\alpha a \Rightarrow \sin\alpha a = \alpha/\sqrt{\alpha^2+\beta^2} = \sqrt{E/V_0}. \end{array}\right. \qquad (4)$$

For both classes, we get $\psi(a) = \sqrt{E/aV_0}$, for $V_0 \gg E$ & $E \to E_n$. Then in (1)...

$$\rightarrow \langle F\rangle_n = V_0|\psi_n(a)|^2 \to V_0 \cdot (E_n/aV_0), \text{ i.e. } \boxed{\langle F\rangle_n \to E_n/a} \;(\text{as } V_0 \to \infty). \qquad (5)$$

This is the final result; it is independent of V_0 as $V_0 \to \infty$. Classically, for m at velocity v : $F\left(\begin{smallmatrix}\text{on one}\\ \text{wall}\end{smallmatrix}\right) = 2mv\left(\begin{smallmatrix}\text{momentum}\\ \text{reversal}\end{smallmatrix}\right) \times \frac{v}{4a}\left(\begin{smallmatrix}\text{time between}\\ \text{collisions}\end{smallmatrix}\right) = \frac{\frac{1}{2}mv^2}{a} = \frac{E}{a}$, also.

φ506 Solutions



(26) m in 1D potential well shown; find bound states, etc.

(A) 1. In the various regions, the system wavefcn Ψ looks like...

$$\left[\begin{array}{l} x<0\ ;\ \Psi \equiv 0\ ; \\ 0<x<R\ ;\ \Psi = A e^{+ikx} + B e^{-ikx},\ \text{w/}\ \underline{k=\sqrt{2mE/\hbar^2}},\ \text{and } A \,\&\, B = \text{cnsts}; \\ x>R\ :\ \Psi = C e^{-\kappa x},\ \text{w/}\ \underline{\kappa = \sqrt{(2m/\hbar^2)(V_0 - E)}},\ \text{and } C = \text{cnst}. \end{array}\right. \qquad (1)$$

At $x=0$, Ψ' is discontinuous (since $V\to\infty$), but Ψ must be continuous, so...

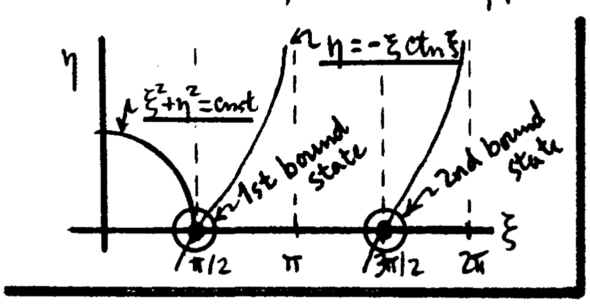
$$\rightarrow \Psi(0-) = \Psi(0+) \Rightarrow 0 = A+B,\ \text{w/}\ B=-A\ \text{and/}\ \underline{\underline{\Psi(x) = 2iA\sin kx}}\ @\ x>0. \qquad (2)$$

Continuity in <u>both</u> Ψ & Ψ' at $x=R$ (where $V=V_0$, finite) then requires...

$$\left[\begin{array}{l} 2iA\sin kR = Ce^{-\kappa R} \\ 2ikA\cos kR = -\kappa C e^{-\kappa R} \end{array}\right\} \text{so/}\ k\,\text{ctn}\, kR = -\kappa,\ \text{or/}\ \boxed{\eta = -\xi\,\text{ctn}\,\xi}\ \Big\{ \text{w/}\ \underline{\xi = kR},\ \underline{\eta = \kappa R}\ \&/\ \xi^2+\eta^2 = \frac{2m}{\hbar^2}V_0R^2. \qquad (3)$$

The boxed eqtn here determines the bound state energies of the system. It can be analysed by the graphical method appearing in CLASS NOTES, p. Sol^{ns} 3.

(B) 2. If m is just barely bound, then $E \simeq V_0$ and $\kappa \to 0$. Thus, from (3), we have $\text{ctn}\, kR \to 0$, which happens for the first time at $\xi = kR = \pi/2$, i.e.



$$kR \to \sqrt{2mV_0/\hbar^2}\,R = \pi/2$$

$$\text{or/}\ \boxed{V_0 \geqslant \pi^2\hbar^2/8mR^2}\ \Big\{ \text{for 1st bound energy @ } E \geqslant 0. \qquad (4)$$

This is the min. well depth needed for one level.

---

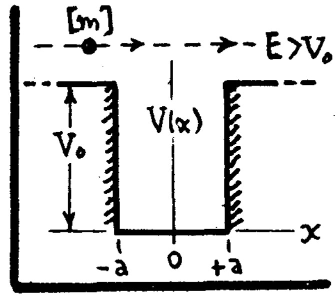
3. From the sketch in part (B), a second bound level can appear in the well when $\xi = kR = 3\pi/2$, i.e. when...

$$\sqrt{2mV_0/\hbar^2}\,R = 3\pi/2 \Rightarrow \boxed{V_0 \geqslant 9(\pi^2\hbar^2/8mR^2)}\ \Big\{ \text{for 2nd bound energy @ } E\geqslant 0. \qquad (5)$$

In this way, the well "fills up" according to its depth.

ϕ506 Solutions

(27) [15 pts.]. Reflection & transmission for m incident on rect^r well @ $E > V_0$.



1. m is never bound, but its wave# changes, as...

$$\rightarrow \underline{\underline{k}} = \left[\frac{2m}{\hbar^2}(E-V_0)\right]^{1/2} @ |x|>a, \quad \underline{\underline{K}} = \left[\frac{2m}{\hbar^2}E\right]^{1/2} @ |x|<a. \quad (1)$$

Both k & K here are real #s. The corresponding wavefcns are

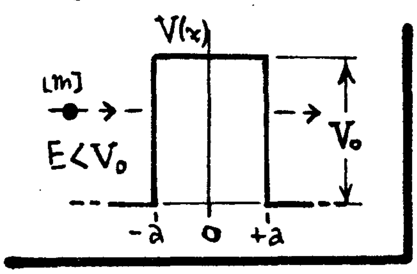
$$\left[\begin{array}{l} x<(-)a : \psi_1(x) = Ae^{+ikx} + Be^{-ikx}; \\ |x|<a : \psi_2(x) = Ce^{+iKx} + De^{-iKx}; \\ x>+a : \psi_3(x) = Ee^{+ikx} + \cancel{F}e^{-ikx}. \end{array}\right\| \text{ w/ } k \text{ \& } K \text{ as defined in Eq. (1).} \Bigg\} (2)$$

$\hookrightarrow$ 0, wave is travelling to right only.

We want reflection & transmission coefficients $R = |B|^2/|A|^2$ & $T = |E|^2/|A|^2$.

---

2. We have already solved the rect^r barrier problem (CLASS NOTES, pp. Sol^{ns} 5-9):



$$\text{w/ } \underline{\underline{k'}} = \left[\frac{2m}{\hbar^2}E\right]^{1/2} @ |x|>a, \quad \underline{\underline{K'}} = \left[\frac{2m}{\hbar^2}(V_0-E)\right]^{1/2} @ |x|<a;$$

$$\text{\& } \left\| \begin{array}{l} \psi_1(x) = Ae^{ik'x} + Be^{-ik'x}, \text{ for } x<-a; \\ \psi_2(x) = Ce^{-K'x} + De^{+K'x}, \text{ for } |x|<a; \\ \psi_3(x) = Ee^{ik'x} \text{ (rightward only), for } x>+a. \end{array} \right\} (3)$$

The arithmetic involved in matching ψ & ψ' at the boundaries $x = \pm a$ will be just the same for the ψ-fcns in Eq. (2), and those in Eq. (3). In fact, the results must be identical if we make the follow replacements...

$$\left[\begin{array}{l} \text{replace } k' \text{ in Eq. (3) by } k \text{ in Eq. (2);} \\ \text{replace } K' \text{ in Eq. (3) by } -iK \text{ in Eq. (2).} \end{array}\right\| \begin{array}{l} \text{then the barrier problem in} \\ \text{Eq. (3) is formally identical} \\ \text{to the well problem in Eq (2)} \end{array} \quad (4)$$

∴ We can now take over the results of the barrier problem. For example...

$$\rightarrow E/A = e^{-2ika}/\left[\cosh(-2iKa) + \tfrac{1}{2}i\lambda\sinh(-2iKa)\right], \text{ w/ } \lambda = \frac{(-iK)}{k} - \frac{k}{(-iK)}. \quad (5)$$

This is Eq. (10) on p. Sol^{ns}7, and the k & K in this expression for E/A are now those in Eq. (1) above. Since $\lambda = -i\left(\frac{K}{k}+\frac{k}{K}\right)$ now, and $\cosh(-iu) = \cos u$, while $\sinh(-iu) = -i\sin u$, we can write Eq. (5) as...

$$\rightarrow E/A = e^{-2ika}/[\cos 2ka - i\rho\sin 2ka], \text{ w/ } \underline{\rho} = \frac{1}{2}\left(\frac{K}{k}+\frac{k}{K}\right), \tag{6}$$

and where: $k = \left[\frac{2m}{\hbar^2}(E-V_0)\right]^{1/2}$, and: $K = \left[\frac{2m}{\hbar^2}E\right]^{1/2}$. Transmission coefficient is:

$$\boxed{T = |E/A|^2 = 1/[\cos^2 2ka + \rho^2\sin^2 2ka]}. \tag{7}$$

---

4. We treat the reflected wave similarly. From Eq. (11) of p. Sol^{ns}7, have...

$$B/A = -\frac{1}{2}i\mu(E/A)\sinh(-2ika), \text{ w/ } \mu = -i\left(\frac{K}{k}-\frac{k}{K}\right); \tag{8}$$

$$\xrightarrow{\text{so}} B/A = -i\sigma(E/A)\sin 2ka, \text{ w/ } \underline{\sigma} = \frac{1}{2}\left(\frac{K}{k}-\frac{k}{K}\right). \tag{9}$$

Using $T = |E/A|^2$ from Eq. (7), we find the reflection coefficient as...

$$\boxed{R = |B/A|^2 = \sigma^2\sin^2 2ka/[\cos^2 2ka + \rho^2\sin^2 2ka]}. \tag{10}$$

---

5. T & R take simpler forms if we notice: $\sigma^2 + 1 = \rho^2$. Then (7) & (10) are

$$\boxed{\begin{array}{l|l} \text{transmission coefficient}\ \} \ T = 1/(1+\sigma^2\sin^2 2ka) & \sigma = \frac{1}{2}\left(\frac{K}{k}-\frac{k}{K}\right), \text{ and:} \\ \text{reflection coefficient}\ \} \ R = \dfrac{\sigma^2\sin^2 2ka}{1+\sigma^2\sin^2 2ka} & k = \left[\frac{2m}{\hbar^2}(E-V_0)\right]^{1/2} \text{ \& } K = \left[\frac{2m}{\hbar^2}E\right]^{1/2} \end{array}} \tag{11}$$

These forms make it clear that $R + T = 1$ (as must be).

Note the following peculiarity: when $2ka = n\pi$ (i.e. at incident energies $E_n = n^2(\pi^2\hbar^2/8ma^2)$), we have $T \to 1$ & $R \to 0$. At these energies -- which are the same as the bound states in an ∞ly deep well of width $2a$ -- m passes over the well as though the well didn't exist!