ϕ507 Final Exam Preview 5/3/95

The ϕ507 Final Exam will be given 2-5 PM on Wed. 5/10/95 in AJM 230.

Exam questions will cover material related to the following topics: (1) time-dependent perturbation theory, (2) QM angular momentum, (3) magnetic interactions in atoms, (4) QM scattering per Born approximation, (5) QM of identical particles, (6) free-particle Dirac equation, (7) field quantization via the SHO formalism. The topics are at the level & content of your Xerox class notes, but an additional QM reference text may prove helpful.

The exam has 7 problems worth 300 pts. Specific problems are (in same order as above):

(1) Analysis of "detailed balancing" via time-dependent pert$^{\underline{b}}$n theory.
(2) Fundamental properties of the ladder operators $J_\pm$ for $\measuredangle$ momentum.
(3) Spectroscopic signature of an exotic variation of hydrogen.
(4) Scattering from a crystal lattice.
(5) Accounting possible QM states for indistinguishable particles.
(6) Properties of the free neutrino field.
(7) A field commutator for the neutral scalar field.

The exam is open-book, open-notes. You may bring to the exam:

1. Xerox copies of your class notes, assigned problems & solutions.
2. A copy of one QM reference text of your choice.
3. A math reference table, a calculator, and a dictionary.

May you learn to deal with most of the ZitterBewegung in your life.

Good Luck // Dick Robiscoe

ϕ 507 Final Exam (3 hr., in class) Wed. 10 May 1995

This exam is open-book, open-notes, and is worth 300 pts. total. For each of the 7 problems, $\boxed{\text{box}}$ the answer on your solution sheet. Number your solution pages in order, write your name on p.1, and staple the pages together before handing in.

⟨1⟩ [40 pts]. Start out with a stationary QM system that has eigenenergies E_n and eigenstates $|n\rangle$. At time $t=0$, turn on a time-dependent perturbation $V(x,t)$ that lasts over $0 \leq t \leq T$. Assuming V is "weak", we can use first-order time-dependent perturbation theory to calculate the probability $P(m \to k, T)$ the V induces a transition from an initial state m to a final state $k \neq m$ in the QM system. Similarly, we can find $P(k \to m, T)$ for the inverse transition.

(A) Show that under "normal" circumstances: $P(k \to m, T) = P(m \to k, T)$, which implies that the QM system shows equal absorption & emission rates. This result is known as the "principle of detailed balancing."

(B) Under what circumstances will detailed balancing fail to hold?

⟨2⟩ [35 pts.]. A QM ∡ momentum operator $\mathbb{J} = (J_x, J_y, J_z)$, obeying the usual commutators: $[J_x, J_y] = i J_z$, etc. (with $\hbar = 1$), has eigenfcns $|jm\rangle$ with the usual eigenvalues: $\mathbb{J}^2 |jm\rangle = j(j+1)|jm\rangle$, $J_z|jm\rangle = m|jm\rangle$. By examining appropriate commutators, it is not hard to show that the "ladder operators" $J_\pm = J_x \pm i J_y$ have the effect: $J_\pm |jm\rangle \propto |j\, m\pm 1\rangle$, i.e. $J_\pm$ steps the m-values by $\Delta m = \pm 1$. Here, we wish to find the constants of proportionality.

(A) If: $J_+|jm\rangle = A|j\, m+1\rangle$, show how the constant A is determined.

(B) If: $J_-|jm\rangle = B|j\, m-1\rangle$, show how the constant B is determined.

$\boxed{\text{NOTE}}$ This problem requires a derivation. It is not enough just to quote the "well-known" results for A & B.

(next page)

⟨3⟩ [45 pts.]. The mu meson, μ^+, is an elementary particle with charge $+e$, mass $= 207\, m_e$ (m_e = mass of electron), spin $\frac{1}{2}\hbar$, a "normal" Dirac g-value ($g_\mu = 2$), and a lifetime (for $\mu^+ \to e^+ + \nu_e + \bar{\nu}_\mu$) of 2.2×10^{-6} sec in its rest frame. During its short life, the muon μ^+ can capture an electron e^- to form a bound system μ^+e^-, called "muonium"; this is an exotic H-atom, with μ^+ replacing the proton.

(A) For a normal H-atom, the light emitted during an $n = 3 \to 2$ transition is the Balmer α line at wavelength: $\lambda_\alpha = 656.3$ nm. What is λ_α for muonium?

(B) For a normal H-atom, the ground state hyperfine splitting (in freq. units) is $\Delta\nu(\text{hfs}) = 1420$ MHz. What is $\Delta\nu(\text{hfs})$ for the ground state of muonium? ★

⟨4⟩ [45 pts.]. If a QM scattering potential $V(\mathbf{r})$ has the periodicity property that: $V(\mathbf{r}+\mathbf{a}) = V(\mathbf{r})$, for $\mathbf{a}$ a given constant vector, show that in the first Born approx$^{\underline{n}}$ the scattering of an incident particle vanishes unless: $\mathbf{q}\cdot\mathbf{a} = 2n\pi$, w/ $n = 0, 1, 2, \ldots$. Here: $\mathbf{q} = \mathbf{k}(\text{before}) - \mathbf{k}(\text{after})$, is the momentum transfer.

⟨5⟩ [40 pts.]. Consider two identical QM particles (either bosons or fermions). Each particle can be in one of N distinct quantum states ($N \geq 2$). Show that for the two-particle system:

(A) The number of possible exchange-symmetric states is $\frac{1}{2}N(N+1)$, while the number of possible exchange-antisymmetric states is $\frac{1}{2}N(N-1)$. †

(B) If each particle has spin S, the ratio of symmetric to antisymmetric spin states is: $(S+1)/S$. How does this check out for $S = \frac{1}{2}$?

(next page)

★ You should be able to get $\Delta\nu(\text{hfs})$ for muonium by simple scaling arguments.

† A combinatorial truth: the number of different ways to choose m objects at a time from a selection of $n \geq m$ objects is $n!/m!(n-m)!$

⟨6⟩ [45 pts.]. Dirac's wave equation for a free, massless, spin $\frac{1}{2}$ particle (i.e. a neutrino) is: $\underline{c(\boldsymbol{\sigma}\cdot\mathbb{p})\psi = i\hbar\frac{\partial\psi}{\partial t}}$, w/ $\boldsymbol{\sigma} = \{$Pauli matrices and $\mathbb{p}$ = linear momentum operator.

(A) The particle has an intrinsic ∡ momentum $\mathbb{S} = \frac{\hbar}{2}\boldsymbol{\sigma}$, and also might have an orbital ∡ momentum $\mathbb{L} = \mathbb{r}\times\mathbb{p}$ about a given center. Is either $\mathbb{S}$ or $\mathbb{L}$ a conserved quantity for the particle's motion? If not, what ∡ momentum is conserved? Support your answers with calculations showing just what is conserved.

(B) Show that the spin of this particle in a positive energy state is parallel to its momentum, while the spin in a negative energy state is anti-parallel to $\mathbb{p}$.

⟨7⟩ [50 pts.]. A neutral scalar field $\phi(\mathbb{r},t)$ obeys the Klein-Gordon equation, i.e. with $\mu = mc/\hbar$: $\underline{[\nabla^2 - \frac{1}{c^2}(\partial^2/\partial t^2) - \mu^2]\phi = 0}$. Since ϕ represents a spinless particle, and satisfies a wave equation similar to those for EM fields, then we can convert it into a quantized field by the same techniques used in class to quantize the EM field. The mass term in μ is "hidden" in a dispersion relation $\omega = \omega(k)$, and if V is the volume of a "box" where ϕ obeys periodic boundary conditions, the result of the SHO quantization procedure is

$$\boxed{\begin{array}{l}\phi(\mathbb{r},t) = \sum_{\mathbb{k}} \left(c\sqrt{\hbar/2\omega V}\right)\left[a_{\mathbb{k}}(t)e^{i\mathbb{k}\cdot\mathbb{r}} + a_{\mathbb{k}}^\dagger(t)e^{-i\mathbb{k}\cdot\mathbb{r}}\right], \text{ w/ } \omega = c\sqrt{\mathbb{k}^2+\mu^2} \\ \text{and } a_{\mathbb{k}}(t) = a_{\mathbb{k}}(0)e^{-i\omega t}, \text{ w/ } [a_{\mathbb{k}}, a_{\mathbb{k}'}] = 0,\ [a_{\mathbb{k}}^\dagger, a_{\mathbb{k}'}^\dagger] = 0,\ [a_{\mathbb{k}}, a_{\mathbb{k}'}^\dagger] = \delta_{\mathbb{k}\mathbb{k}'}.\end{array}}$$

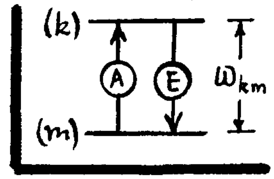
ϕ is now an operator which by its presence can create or destroy other spinless particles. The companion field: $\underline{\pi(\mathbb{r},t) = \frac{1}{c^2}\frac{\partial}{\partial t}\phi(\mathbb{r},t)}$, is a sort of generalized momentum operator, if ϕ is considered as a generalized displacement.

Problem: for ϕ as defined in the box above, and for $\pi = \frac{1}{c^2}(\partial\phi/\partial t)$, prove the equal-time commutation relation for these wave-fields:

→ $[\phi(\mathbb{r},t), \pi(\mathbb{r}',t)] = i\hbar\,\delta(\mathbb{r}-\mathbb{r}')$.

ϕ507 Final Exam Solutions (1995)

1 [40pts]. "Detailed balancing" via first-order time-dependent pert$^{\text{b}}$n theory.



(A) **1.** From $\genfrac{}{}{0pt}{}{\text{CLASS}}{\text{NOTES}}$, p. tD5, Eq. (13); the (first-order) amplitude for an A (absorption) process $m \to k$ induced by $V(x,t) = \hbar\,\Omega(x,t)$ is

$$\rightarrow a^{(1)}_{m\to k}(T) = -i\int_0^T \Omega_{km}(\tau)\, e^{i\omega_{km}\tau}\, d\tau \quad \text{w/}\ \omega_{km} = \frac{1}{\hbar}(E_k - E_m),\quad \Omega_{km}(\tau) = \frac{1}{\hbar}\langle k|V(x,\tau)|m\rangle. \tag{1}$$

This is for V acting over time $0 \to T$, and the probability of finding state k at time T is: $\underline{P(m\to k, T) = |a^{(1)}_{m\to k}(T)|^2}$. Similarly, the probability of finding state m after an E (emission) process $k \to m$ induced by V over $0 \to T$ is: $\underline{P(k\to m, T) = |a^{(1)}_{k\to m}(T)|^2}$, where the inverse amplitude is...

$$\rightarrow a^{(1)}_{k\to m}(T) = -i\int_0^T \Omega_{mk}(\tau)\, e^{i\omega_{mk}\tau}\, d\tau. \tag{2}$$

2. Compare the amplitudes in Eqs. (1) & (2)... specifically, look at the conjugate...

$$\rightarrow \{a^{(1)}_{k\to m}(T)\}^* = +i\int_0^T \Omega^*_{mk}(\tau)\, e^{-i\omega^*_{mk}\tau}\, d\tau. \tag{3}$$

Under "normal" circumstances, the energy ω_{mk} is <u>real</u>, so $\omega^*_{mk} = \omega_{mk} = (-)\omega_{km}$, and the perturbation $V(x,t)$ is <u>Hermitian</u>, so $\Omega^*_{mk} = \Omega_{km}$. Then...

$$\rightarrow \{a^{(1)}_{k\to m}(T)\}^* = +i\int_0^T \Omega_{km}(\tau)\, e^{+i\omega_{km}\tau}\, d\tau = (-)\, a^{(1)}_{m\to k}(T). \tag{4}$$

This result immediately gives: $|a^{(1)}_{k\to m}(T)|^2 = |a^{(1)}_{m\to k}(T)|^2$, i.e. the detailed balancing: $\underline{\underline{P(k\to m, T)\big|_{E} = P(m\to k, T)\big|_{A}}}$, as desired. This lowest order result can be generalised to hold at all orders of pert$^{\text{b}}$n theory (w/ same assumptions).

(B) **3.** Above proof depends on system energies being real and pert$^{\text{b}}$n V being Herm$^{\text{n}}$. The <u>system</u> is Hermitian, and conserves particles; no account is taken of the "photons" destroyed or created by the A & E processes. When those "photons" are accounted for, $\omega_{km} \to$ complex (it acquires a net decay rate $\propto$ spontaneous decay γ_k for upper state) and Eq. (4) isn't true. $P(k\to m, T) > P(m\to k, T)$, and detailed balancing "fails."

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⟨2⟩ [35 pts.]. For ∡ mom^{m} ladder operators : $J_\pm|jm\rangle = \left\{\begin{smallmatrix}A\\B\end{smallmatrix}\right\}|j\,m\pm1\rangle$, find cnsts A & B.

1. We must first recall that $\mathbb{J} = (J_x, J_y, J_z)$ is a Hermitian operator; each component J_k is self-adjoint : $J_k^\dagger = J_k$. This follows from the requirement that the ∞smal rotation operator : $R_k(\delta\phi) = 1 - i(\delta\phi)J_k$, for a rotation by $\delta\phi$ about the k^{th} axis, is unitary [i.e. $R_k^\dagger(\delta\phi) = 1 + i(\delta\phi)J_k^\dagger$ is such that $R_k^\dagger R_k = 1$; then $R_k^\dagger(+\delta\phi) = R_k(-\delta\phi)$ is just the inverse rotation, with the same $J_k = J_k^\dagger$]. It follows that although $J_\pm = J_x \pm iJ_y$ are not Hermitian, they are in fact the adjoints of each other, i.e.

$$\rightarrow J_+^\dagger = (J_x + iJ_y)^\dagger = J_x^\dagger - iJ_y^\dagger = J_x - iJ_y = J_-, \text{ and}/\!/ \; J_-^\dagger = J_+. \qquad (1)$$

A) **2.** Now, assume $J_+|jm\rangle = A|j\,m+1\rangle$, and that the eigenstates $|jm\rangle$ are orthonormal. Consider a matrix element which isolates A, i.e....

$$\langle jm|J_-J_+|jm\rangle = \langle \overset{J_+}{J_-^\dagger} jm|J_+jm\rangle = \langle J_+jm|J_+jm\rangle = |A|^2\underbrace{\langle jm+1|jm+1\rangle}_{1}$$

$$\text{i.e.}/\!/ \rightarrow |A|^2 = \langle jm|J_-J_+|jm\rangle. \qquad (2)$$

But : $J_-J_+ = (J_x - iJ_y)(J_x + iJ_y) = \overbrace{J_x^2 + J_y^2}^{\mathbb{J}^2 - J_z^2} + i\overbrace{[J_x, J_y]}^{iJ_z} = \mathbb{J}^2 - J_z^2 - J_z$. So...

$$|A|^2 = \langle jm|\mathbb{J}^2 - J_z^2 - J_z|jm\rangle = j(j+1) - m^2 - m = (j-m)(j+m+1)$$

$$\text{and}/\!/ \quad \boxed{J_+|jm\rangle = \sqrt{(j-m)(j+m+1)}\,|j\,m+1\rangle}. \qquad (3)$$

The desired proportionality cnst A = the √ here, to within a phase factor.

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3) **3.** For $J_-|jm\rangle = B|j\,m-1\rangle$, carry out a similar procedure to get : $|B|^2 = \langle jm|J_+J_-|jm\rangle$, and : $J_+J_- = \mathbb{J}^2 - J_z^2 + J_z$. Then $|B|^2 = j(j+1) - m^2 + m = (j+m)(j-m+1)$, so that :

$$\boxed{J_-|jm\rangle = \sqrt{(j+m)(j-m+1)}\,|j\,m-1\rangle}. \qquad (4)$$

B = the √ here, again to within an arbitrary (uniform) phase factor.

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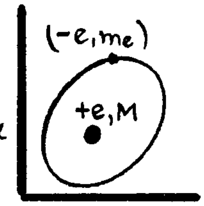

(3) [45 pts]. Some spectroscopic features of $\mu^+ e^-$ (muonium).

(A) 1) For purposes of this problem the μ^+ acts (so long as it lives) just like a replacement proton, except it is lighter ($M_\mu = 207\,m_e$ vs. $M_p = 1836\,m_e$), and it has a normal Dirac g-value ($g_\mu = 2$ vs $g_p = 2\times 2.79$). The Bohr energy levels for any such bound $(-e, m_e) \leftrightarrow (+e, M)$ system are:

$$\underline{\underline{E_n = -\tfrac{1}{2}\alpha^2 mc^2/n^2}} \quad \int \alpha = e^2/\hbar c \simeq 1/137;\ n = 1,2,3,\ldots;\ \text{and:} \quad \underline{m = m_e/[1+(m_e/M)]} \leftarrow \text{electron reduced mass.} \tag{1}$$

The only thing that changes here, upon replacing p^+ by μ^+, is the mass M. For the Balmer α transition $n = 3 \to 2$, the emitted energy & photon wavelength are:

$$\Delta E_\alpha = E_3 - E_2 = \frac{5}{72}\alpha^2 mc^2, \quad \lambda_\alpha = \frac{hc}{\Delta E_\alpha} = (72/5\alpha^2)\frac{h}{mc}$$

So//

$$\underline{\underline{\lambda_\alpha = \left[1 + \frac{m_e}{M}\right]\cdot(72/5\alpha^2)(h/m_e c)}} \quad \int \text{upon putting in reduced mass } m \text{ of Eq. (1).} \tag{2}$$

Then λ_α for muonium and λ_α for normal hydrogen are related by...

$$\frac{\lambda_\alpha(\text{muonium})}{\lambda_\alpha(\text{hydrogen})} = \frac{1+(m_e/M_\mu)}{1+(m_e/M_p)} = \frac{1+(1/207)}{1+(1/1836)} = 1.004\,284$$

NOTE: the difference $\Delta\lambda_\alpha = 2.8$ nm is readily detected.

... if $\underline{\lambda_\alpha(H) = 656.3\text{ nm}}$, then: $\boxed{\lambda_\alpha(\mu) = 659.1\text{ nm}}$. (3)

(B) 2) Recall [from ϕ507 prob^m #61] that the ground state hyperfine splitting for a hydrogenic atom, with a spin-$\frac{1}{2}$ nucleus characterized by g-value g_n, was...

$$\rightarrow \Delta\nu_{hfs} = \frac{8}{3}|g_n|\alpha^2 c R_\infty, \quad R_\infty = \text{Rydberg cnst for } \underline{\text{infinite}} \text{ mass nucleus.} \tag{4}$$

In replacing p^+ by μ^+, the only parameter that changes is $|g_n|$. Important: the way g_n is defined, it includes the mass ratio: $g_n = g(\text{nucleus})\cdot(m_e/M)$. So: $|g_n|_{\text{proton}} = 2\times 2.79\cdot(m_e/M_p)$, $|g_n|_{\text{muon}} = 2\times 1\cdot(m_e/M_\mu)$, and the ratio is: $|g_n|_{\text{muon}}/|g_n|_{\text{proton}} = (1/2.79)(M_p/M_\mu) = 3.179$. Then, for the hfs interval...

$$\rightarrow \frac{\Delta\nu_{hfs}(\mu)}{\Delta\nu_{hfs}(H)} = \frac{|g_n|_{\text{muon}}}{|g_n|_{\text{proton}}} = 3.179, \quad \text{and}// \quad \boxed{\Delta\nu_{hfs}(\mu) = 3.179\cdot\Delta\nu_{hfs}(H) = 4514\text{ MHz}} \tag{5}$$

ϕ507 Final Exam Solutions (1995)

(4) [45 pts.]. Scattering from a periodic potential: $V(\mathbf{r}+\mathbf{a}) = V(\mathbf{r})$.

1. In Born Approxn, the diff'l scattering cross-section is [class notes, p. ScT 12, Eq (28)]:

$$\rightarrow \frac{d\sigma}{d\Omega} = \left(\frac{m}{2\pi\hbar^2}\right)^2 |\tilde{V}(\mathbf{q})|^2, \quad \text{w/}\ \underline{\tilde{V}(\mathbf{q})} = \int_\infty V(\mathbf{r}')e^{i\mathbf{q}\cdot\mathbf{r}'}d^3x' \quad \mathbf{q} = \mathbf{k}_{before} - \mathbf{k}_{after} = \text{momentum transfer}. \quad (1)$$

The required scattering periodicity (i.e. scattering only at $\mathbf{q}\cdot\mathbf{a} = 2n\pi$) must be a feature of the Fourier transform $\tilde{V}(\mathbf{q})$ of a periodic $V(\mathbf{r})$.

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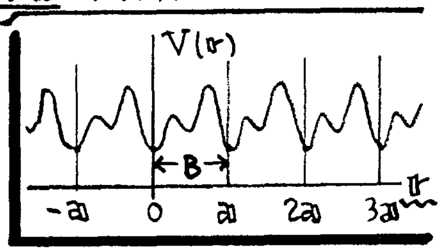
2. A periodic $V(\mathbf{r})$ is defined in a basic interval B (i.e. $0 \leq \mathbf{r} \leq \mathbf{a}$, symbolically); it is zero outside B, but repeats itself so that $V(\mathbf{r}+\lambda\mathbf{a}) = V(\mathbf{r})$, for $\lambda = 0, \pm1, \pm2, \ldots$ In fact we can represent such a fcn by the ∞ sum...



$$V(\mathbf{r}) = \sum_{\lambda=-\infty}^{\lambda=+\infty} V(\mathbf{r}+\lambda\mathbf{a}). \quad (2)$$

For this representation, it is easy to show that: $V(\mathbf{r}+\mathbf{a}) = V(\mathbf{r})$, so the periodicity condition is OK.

Using Eq (2) for $\tilde{V}(\mathbf{q})$ in (1):

$$\rightarrow \tilde{V}(\mathbf{q}) = \sum_{\lambda=-\infty}^{\lambda=+\infty} \int_\infty V(\mathbf{r}'+\lambda\mathbf{a})e^{i\mathbf{q}\cdot\mathbf{r}'}d^3x'$$

change integration variables to $\mathbf{r} = \mathbf{r}'+\lambda\mathbf{a}$; note: $V(\mathbf{r})$ vanishes outside interval B.

$$= \sum_{\lambda=-\infty}^{\lambda=+\infty} e^{-i\lambda\mathbf{q}\cdot\mathbf{a}} \int_B V(\mathbf{r})e^{i\mathbf{q}\cdot\mathbf{r}}d^3x = \underline{\tilde{V}_B(\mathbf{q})\, S(\mathbf{q}\cdot\mathbf{a})}, \quad (3)$$

$$\text{w/}\ S(\phi) = \sum_{\lambda=-\infty}^{\lambda=+\infty} e^{-i\lambda\phi} = \sum_{\lambda=0}^{\infty}(e^{i\phi})^\lambda + \sum_{\lambda=0}^{\infty}(e^{-i\phi})^\lambda - 1, \quad \text{w/}\ \phi = \mathbf{q}\cdot\mathbf{a} \quad (4)$$

$\tilde{V}_B(\mathbf{q})$ is $V(\mathbf{r})$'s Fourier Transform over its basic interval; the sum $S(\phi) \Rightarrow$ periodicity.

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3. Clearly, $S(\phi)\to\infty$ when $\phi = 2n\pi$ ($n = 0, 1, 2, \ldots$), for then it is an ∞ series of ones. When $\phi \neq 2n\pi$, use the geometric series $\left[\sum_{\lambda=0}^{N} r^\lambda = (1-r^{N+1})/(1-r)\right]$ to sum Eq. (4):

$$\rightarrow S(\phi) = \lim_{N\to\infty}\left\{\frac{1-e^{i(N+1)\phi}}{1-e^{i\phi}} + \frac{1-e^{-i(N+1)\phi}}{1-e^{-i\phi}} - 1\right\} = \lim_{N\to\infty}\left\{\frac{\cos N\phi - \cos(N+1)\phi}{1-\cos\phi}\right\}$$

$$\text{or/}\ \underline{S(\phi) = \lim_{N\to\infty}\left\{\sin\left[(N+\tfrac{1}{2})\phi\right]/\sin\tfrac{\phi}{2}\right\}}, \quad (5)$$

When $\phi \neq 2n\pi$, $S(\phi)$ is well-behaved, but tends to zero because of the rapidly oscillating numerator. Then, indeed:

$$\boxed{\frac{d\sigma}{d\Omega} \propto |\tilde{V}_B(\mathbf{q})\, S(\mathbf{q}\cdot\mathbf{a})|^2 \cong 0, \text{ unless } \mathbf{q}\cdot\mathbf{a} = 2n\pi.} \quad (6)$$

The Born Approxn $\Rightarrow$ "strong" (∞) scattering when $\mathbf{q}\cdot\mathbf{a} = 2n\pi$.

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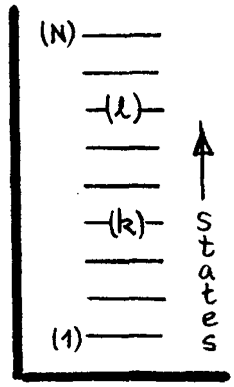
⑤ [40 pts.]. Counting exchange symmetric & antisymmetric states for 2 particles.



(A) 1. Let the indices k & l, for "first" and "second" particle, run from 1 to N. And denote the eigenfcn of the n^{th} state by ϕ_n, w/ $1 \leq n \leq N$ also. For the exchange-symmetric case, we can obviously form N system wavefcns out of simple products like $\phi_n(1)\phi_n(2)$, w/ both particles (1) & (2) in the same eigenstate n (i.e. $k=n$ & $l=n$). As well, when k & l are different eigenstates, we can form exchange-symmetric combn's: $\frac{1}{\sqrt{2}}[\phi_k(1)\phi_l(2)+\phi_l(1)\phi_k(2)]$. The number of ways that two (distinct) objects, here k & l, can be chosen from a set of N (distinct) objects is: $N!/2!(N-2)! = \frac{1}{2}N(N-1)$. So...

$$\rightarrow \#\text{ symmetric states} \Big\}\; \underbrace{N}_{\text{states w: } k=l} + \underbrace{\tfrac{1}{2}N(N-1)}_{\text{states w: } k\neq l} = \underline{\underline{\tfrac{1}{2}N(N+1)}}. \qquad (1)$$

2. For the exchange-antisymmetric case, the states $\phi_n(1)\phi_n(2)$ are not possible; this deletes the N on the LHS of Eq.(1). The states w/ $k \neq l$ are possible, with system wavefcn: $\frac{1}{\sqrt{2}}[\phi_k(1)\phi_l(2) - \phi_l(1)\phi_k(2)]$. As above, the $k \neq l$ choices are $\frac{1}{2}N(N-1)$ in number, and so...

$$\rightarrow \#\text{ antisymmetric states} \Big\}\; 0 + \tfrac{1}{2}N(N-1) = \underline{\underline{\tfrac{1}{2}N(N-1)}}. \qquad (2)$$

NOTE: if ϕ_n is the entire wavefcn, then we'll find only $\begin{cases}\text{bosons in Eq.(1) states,}\\ \text{fermions in Eq.(2) states.}\end{cases}$

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3. If each particle has spin S, the # states available is $N = 2S+1$, i.e. just the # of distinct m-values. Then, for $N = 2S+1$, above results give...

$$\left[\begin{array}{ll}\#\text{symmetric spin states: } & (S+1)(2S+1) \\ \#\text{antisymmetric spin states: } & (S)\cdot(2S+1)\end{array}\right\rangle\; \underline{\underline{\text{ratio} = (S+1)/S}}. \qquad (3)$$

For $S=\frac{1}{2}$, this gives 3 symm. states & 1 antisymm. state, a well-known result.

ϕ507 Final Solutions (1995)

⑥ [45 pts]. Analyse Dirac Eqtn for a (free) neutrino.

(A) **1.** When the particle is free & massless, the Dirac Hamiltonian is $\mathcal{H} = c(\boldsymbol{\sigma}\cdot\mathbb{p})$.
For the angular momentum dynamics, look at the commutators ... for spin ...

$$\rightarrow [\mathcal{H}, \boldsymbol{\sigma}]_j = c[\sigma_i p_i, \sigma_j] = c p_i [\sigma_i, \sigma_j] \leftarrow \text{use } [\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k \,\bigstar$$

$$= 2ic\epsilon_{ijk} p_i \sigma_k = -2ic\epsilon_{jik} p_i \sigma_k = -2ic(\mathbb{p}\times\boldsymbol{\sigma})_j,$$

so $\underline{[\mathcal{H}, \tfrac{\hbar}{2}\boldsymbol{\sigma}] = +i\hbar c(\boldsymbol{\sigma}\times\mathbb{p})}$, spin is <u>not</u> a constant of the motion; (1)

... and for orbital ∡ momentum $\mathbb{L} = \mathbb{r}\times\mathbb{p}$ ...

$$\rightarrow [\mathcal{H}, \mathbb{L}]_k = c[\sigma_i p_i, \epsilon_{kij} x_i p_j] = c\epsilon_{kij}\sigma_i [p_i, x_i p_j]$$

$$= -i\hbar c\epsilon_{kij}\sigma_i \left\{\frac{\partial}{\partial x_i}(x_i p_j) - (x_i p_j)\frac{\partial}{\partial x_i}\right\} = -i\hbar c\epsilon_{kij}\sigma_i p_j$$

so $\underline{[\mathcal{H}, \mathbb{L}] = -i\hbar c(\boldsymbol{\sigma}\times\mathbb{p})}$, orbital ∡ momentum is <u>not</u> constant. (2)

Neither $\mathbb{L}$ nor spin $\mathbb{S} = \frac{\hbar}{2}\boldsymbol{\sigma}$ is separately cnst, but the <u>total</u> ∡ momentum $\boxed{\mathbb{J} = \mathbb{L} + \mathbb{S}}$ <u>is</u> a cnst of the motion for a (free) neutrino.

(B) **2.** In an eigenstate of energy E, have $i\hbar\,\partial\psi/\partial t = E\psi$, so Dirac's Eqtn is:

$$\rightarrow E\psi = c(\boldsymbol{\sigma}\cdot\mathbb{p})\psi = (cp\cos\phi)\psi, \quad \phi = \measuredangle(\boldsymbol{\sigma}, \mathbb{p}). \tag{3}$$

ϕ is the fixed ∡ between the particle's spin $\frac{\hbar}{2}\boldsymbol{\sigma}$ and its momentum $\mathbb{p}$; this definition is permissible since for fixed E the momentum $\mathbb{p}$ is a cnst of the motion. In fact, $E^2 = c^2p^2$ for this massless particle, so its energy can be either $E = +cp$ or $E = -cp$. Then Eq. (3) requires ...

$$\left[\begin{array}{l} (+)\text{energy}: E = +cp \Rightarrow \cos\phi = +1 \;\&\; \phi = 0^\circ, \text{ so: spin } \boldsymbol{\sigma} \text{ is } \parallel \mathbb{p}; \\ (-)\text{energy}: E = -cp \Rightarrow \cos\phi = -1 \;\&\; \phi = 180^\circ, \text{ so: spin } \boldsymbol{\sigma} \text{ is anti-}\parallel \mathbb{p}. \end{array}\right. \tag{4}$$

★ Davydov, Eq. (59.15). $\epsilon_{ijk} = \begin{cases} +1, & \text{when } ijk = \widehat{123} \\ -1, & \text{when } ijk = \widehat{132} \end{cases}$, and $\epsilon_{ijk} \equiv 0$, otherwise.

" $\underline{\underline{\mathbb{R} = \mathbb{r} - \mathbb{r}'}}$. Each commutator = 1 as noted. Also $\sum_{\mathbb{k}} e^{\cdots} = \sum_{\mathbb{k}} e^{\cdots}$, then

$$\rightarrow [\phi(\mathbb{r},t), \pi(\mathbb{r}',t)] = i\hbar\left\{\frac{1}{V}\sum_{\mathbb{k}} e^{i\mathbb{k}\cdot\mathbb{R}}\right\}. \tag{5}$$

density of modes: p. QF13, Eq. (43)

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3. With V the "box" for periodic B.C: $\frac{1}{V}\sum_{\mathbb{k}} \rightarrow \frac{1}{V}\int_\infty \left[\frac{V}{(2\pi)^3}\right] d^3k = \frac{1}{(2\pi)^3}\int_\infty d^3k$. So

$$\rightarrow (1/V)\sum_{\mathbb{k}} e^{i\mathbb{k}\cdot\mathbb{R}} \rightarrow \left(\frac{1}{2\pi}\int_{-\infty}^{\infty} dk_x\, e^{ik_x X}\right)\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} dk_y\, e^{ik_y Y}\right)\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} dk_z\, e^{ik_z Z}\right) = \delta(X)\delta(Y)\delta(Z). \tag{6}$$

so $\boxed{[\phi(\mathbb{r},t), \pi(\mathbb{r}',t)] = i\hbar\,\delta(\mathbb{r}-\mathbb{r}')}$. (7) QED