SHO Model for Electrical Conductivity

4) In addition to expressions for the attenuation $\alpha(\omega)$ & phase velocity $v_{ph}(\omega)$, our SHO model for $\epsilon(\omega)$ also yields a reasonable expression for the <u>conductivity $\sigma(\omega)$</u> of the medium. Since conduction (of internal currents) is due to free or nearly free electrons, we want to look at $\epsilon(\omega)$ for those electrons which have binding energy $\hbar\omega_j \to 0$. From Eq. (6), we can write

$$\epsilon(\omega) = 1 + \omega_p^2 \sum_{j=1}^{Z} g_j/[\omega_j^2 - \omega(\omega + i\gamma_j)] = 1 + i\left(\frac{4\pi N e^2}{m}\right)\sum_{j=1}^{Z} \frac{f_j}{(\gamma_j - i\omega)\omega + i\omega_j^2},$$

or/

$$\rightarrow \epsilon(\omega) = \underbrace{\epsilon_B(\omega)}_{\text{from bound electrons}} + \underbrace{\lim_{\omega_0\to 0}\left[i\left(\frac{4\pi N e^2}{m}\right)\frac{f_0}{(\gamma_0 - i\omega)\omega + i\omega_0^2}\right]}_{\text{from fraction } f_0 \text{ of nearly free electrons } [@ (\omega_0, \gamma_0)]} \qquad \text{(12A)}$$

Assume: $\omega\left(\begin{smallmatrix}\text{EM wave}\\ \text{frequency}\end{smallmatrix}\right) \gg \omega_0\left(\begin{smallmatrix}\text{binding for}\\ \text{nearly free } e\text{'s}\end{smallmatrix}\right)$, and neglect ω_0 in 2nd term. Then:

$$\rightarrow \underline{\underline{\epsilon(\omega) = \epsilon_B(\omega) + i\left[\frac{4\pi N e^2 f_0}{m(\gamma_0 - i\omega)\omega}\right]}} \qquad \text{(12B)}$$

✓ interpret γ_0 as a damping cnst for free electron motion (due to lattice collisions, etc).

This form of $\epsilon(\omega)$ can be connected with the medium's conductivity as follows:

use $\nabla\times\mathbb{H} = \frac{1}{c}(\partial\mathbb{D}/\partial t) + \frac{4\pi}{c}\mathbb{J}$ ✓ impose <u>OHM'S LAW: $\mathbb{J} = \sigma\mathbb{E}$</u>; (13)

put: $\mathbb{D} = \epsilon_B(\omega)\mathbb{E}$, for normal polarization;

with $\mathbb{E} = \mathbb{E}_0 e^{-i\omega t}$, same harmonic t-dependence as assumed in SHO model;

so $$\underline{\underline{\nabla\times\mathbb{H} = -i\frac{\omega}{c}\left\{\epsilon_B(\omega) + i\left[\frac{4\pi\sigma}{\omega}\right]\right\}\mathbb{E}}} \qquad \text{(14)}$$

the { } here is the <u>effective</u> $\epsilon(\omega)$ including a conduction term in σ.

Had we assumed zero conduction (i.e. set $\mathbb{J}=0$), the { } in (14) would be $\equiv \epsilon(\omega)$. With conduction present, we identify the { } in (14) with $\epsilon(\omega)$ in (12B) to get:

$$\{\,\} = \epsilon(\omega) \Rightarrow \boxed{\sigma(\omega) = ne^2/m(\gamma_0 - i\omega)} \qquad \text{(15)}$$

✓ w/ $n = Nf_0 = \#$ free e's/unit volume.

This is called the Drude conductivity -- it works when there truly are ~ free electrons. **<u>NOTE</u>**: in cgs system, σ has units of <u>frequency</u> (i.e. Hz).

REMARKS on: $\sigma = ne^2/m(\gamma_0 - i\omega)$, at low frequencies.

1. When $\omega \to 0$, have: $\boxed{\sigma \to \sigma_{DC} = ne^2\tau_0/m}$ w/ $\tau_0 = \frac{1}{\gamma_0} = \left\{\begin{array}{l}\text{collision time for}\\ \text{conduction electrons.}\end{array}\right.$ (16)

2. If, in Eqs. (12), we do not pass strictly to the limit of free electrons...

$$\left[\!\left[\sigma = \lim_{\omega_j \to 0} \sum_j \left(\frac{n_j e^2}{m}\right) \Big/ \left[(\gamma_j - i\omega) + i(\omega_j/\omega)\,\omega_j\right]\right.\quad \begin{array}{l} n_j = Nf_j = \#\text{ of } e\text{'s} \\ \text{weakly bound @ } \hbar\omega_j. \end{array} \tag{17}$$

GOOD CONDUCTOR: $\omega_j = 0$ is possible $\left(\begin{smallmatrix}e\text{'s can}\\ \text{be free}\end{smallmatrix}\right) \Rightarrow \sigma \simeq \frac{ne^2}{m}/(\gamma_0 - i\omega)$. (18A)

Typically (for Cu): $\gamma_0 \sim 3\times10^{13}$ Hz $\simeq 2\pi \times 5000$ GHz. Then σ is real and freq.-independent through μwave region: $f_\mu \sim 1000$ GHz. Beyond f_μ, $\sigma \to$ complex.

POOR CONDUCTOR: $\omega_j \geq \omega_0$, min. $\left(\begin{smallmatrix}e\text{'s are}\\ \sim\text{bound}\end{smallmatrix}\right) \Rightarrow \sigma \simeq \frac{n_0 e^2}{m} \Big/ \left[(\gamma_0 - i\omega) + i\left(\frac{\omega_0}{\omega}\right)^2 \omega_0\right]$. (18B)

$\omega_0 > 0$ significantly affects σ even for very weak binding: $\hbar\omega_0 = 0.01\,\text{eV} \leftrightarrow \omega_0 = 1.5\times10^{13}$ Hz. This, and the fact that the density n_0 is "small" for poor conductors, makes $\sigma \sim$ negligible until perhaps $\omega \sim$ optical frequencies.

3. This Drude-type estimate of $\sigma \Rightarrow$ there is no important difference between conductors & insulators as $\omega \to$ large. The conductivity term in: $\epsilon = \epsilon_B + i\left(\frac{4\pi\sigma}{\omega}\right)$, appears as a low-freq. "resonance" (@ $\omega_j = 0$ or ω_0) in both cases, and vanishes as $\omega \to \infty$.

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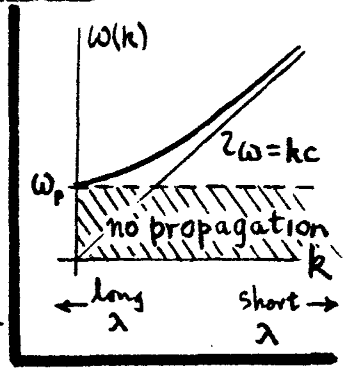
5) In the high-freq. limit: $\omega \gg$ all ω_j, our $\epsilon(\omega)$ model in Eq. (12A) predicts:

$\omega \gg \omega_j \Rightarrow \boxed{\epsilon(\omega) = 1 - (\omega_p^2/\omega^2)}$ w/ $\omega_p = \sqrt{4\pi NZe^2/m}$ = "plasma frequency". (19)

This result is uniform for all media: they all behave like a collection of quasi-free electrons. The EM wave frequency ω is so high that any bound electron orbital motion is negligibly small during one oscillation of the EM field. In a manner of speaking, the e's are frozen-in-place during passage of the wave; they respond more to the wave than to their parent atom.

Another feature of the high-freq. limit is the dispersion relation. With $\epsilon(\omega)$ of

Plasma dispersion relation. Size of ω_p. Plasma phase & group velocities.



Eq. (19), and for a non-permeable medium with $\mu=1$...

$$\rightarrow k = \frac{\omega}{c}\sqrt{\mu\epsilon} \Rightarrow ck = \sqrt{\omega^2-\omega_p^2}\,, \text{ or } \boxed{\omega = \sqrt{\omega_p^2 + k^2c^2}} \quad (20)$$

The boxed equation is known as the "plasma dispersion relation." When it is obeyed (at high ω), there is no wave propagation allowed at low freq.: $0 \leq \omega \leq \omega_p$; ω_p is a cutoff frequency.

ASIDE Relative size of ω_p and the (bound) orbital freqs ω_j.

Let: $\hbar\omega_j = \beta\,\frac{e^2}{2a_0}$, $0<\beta<1$ & $\frac{e^2}{2a_0} = 13.6\,eV$ (H-atom binding) for bound e's.

$$\text{So}\quad \frac{\omega_p^2}{\omega_j^2} = \frac{(\hbar\omega_p)^2}{\beta^2(e^2/2a_0)^2} = \frac{4a_0^2}{\beta^2e^4}\cdot\hbar^2\,\frac{4\pi NZe^2}{m} = 16\pi\,\frac{NZ}{\beta^2}\,a_0^2\cdot\frac{\hbar^2}{me^2}$$

... but $\hbar^2/me^2 = a_0 = 0.53\times10^{-8}$ cm (Bohr radius), so...

$$\omega_p^2/\omega_j^2 = 16\pi a_0^3\cdot\frac{NZ}{\beta^2}\,, \text{ or } \underline{\underline{\frac{\omega_p}{\omega_j} \simeq \frac{2.73}{\beta}\sqrt{n}}}$$

$n = NZ$ = # electrons per cubic Angstrom.

In a solid or liquid, $n\sim10$ atoms/cubic Å, and β may be $\leftrightarrow$ 2eV; then $\omega_p\sim$ 50-60$\times\omega_j$. Even in a gas, with $n\sim0.1$ & $\beta\leftrightarrow$ 3eV, have $\omega_p\sim4\omega_j$. So the limit $\omega\gg\omega_p$ in the plasma dispersion relation is well above any binding frequency ω_j in the system -- the e's really are "stationary."

<u>NOTE</u> In a "real" plasma, may have n as low as $10^9/\text{cm}^3$ (glow discharge), but then the e's barely interact, so $\beta\to0$ and still ω_p(plasma) $\gg \omega_j$(interaction).

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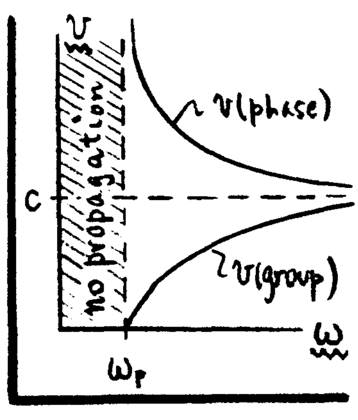

Notice -- for the dispersion relation in (20) -- the wave velocities:

$$\left.\begin{array}{l} \underline{\text{PHASE VELOCITY}}:\ v_{ph} = \frac{\omega}{k} = \left(\frac{\omega}{\sqrt{\omega^2-\omega_p^2}}\right)c\,, \\ \underline{\text{GROUP VELOCITY}}:\ v_{gr} = \partial\omega/\partial k = \left(\frac{\sqrt{\omega^2-\omega_p^2}}{\omega}\right)c\,; \end{array}\right\}\ v_{ph}\,v_{gr} = c^2. \quad (21)$$

As $\omega\to\omega_p$, the wave just slows down and <u>stops</u>, when $v_{gr}\to0$.

Plasma dispersion relation at "low frequency". Attenuation coeff.

The high-frequency dispersion relation: $\omega = \sqrt{\omega_p^2 + k^2c^2}$, holds approximately in dielectrics only at EM wave frequencies $\omega \gg$ largest bound orbit freq. ω_j. BUT, in an ionized gas (i.e. a plasma, like the earth's ionosphere), most of the electrons are not bound at all; also, one finds essentially free electrons in a metallic conductor. For these cases, the present "high-frequency" discussion ($\omega \geq \omega_p$) holds down to relatively "low" freqs. $\omega_p \sim \omega_j$. Note, in particular...

$$\left.\begin{matrix}\text{for ionosphere}\\ \text{(or metals)}\end{matrix}\right\} \; k = \frac{1}{c}\sqrt{\omega^2 - \omega_p^2} = i\,\frac{\omega_p}{\omega}\left[1-(\omega/\omega_p)^2\right]^{\frac{1}{2}}, \text{ for } \underline{\underline{\omega < \omega_p}};$$

$$\text{so: EM wave ampl.} \propto e^{ikz} = e^{-\frac{\alpha}{2}z}, \quad \boxed{\alpha = \frac{2\omega_p}{c}\left[1-\left(\frac{\omega}{\omega_p}\right)^2\right]^{\frac{1}{2}}} \text{ attenuation coefficient.} \qquad (22)$$

Typical plasma frequencies are: $\omega_p \sim 6\times10^{10-12}$ Hz [for electron densities $n \sim 10^{12-16}$ per cm^3], so typical attenuation lengths for EM waves @ $\omega < \omega_p$ are: $\alpha^{-1} \sim 250 - 0.25$ cm. Waves @ $\omega < \omega_p$ in the interior of the plasma fall off exponentially with propagation distance, and waves @ $\omega < \omega_p$ incident from the outside are reflected*. Metallic conductors behave in like manner.

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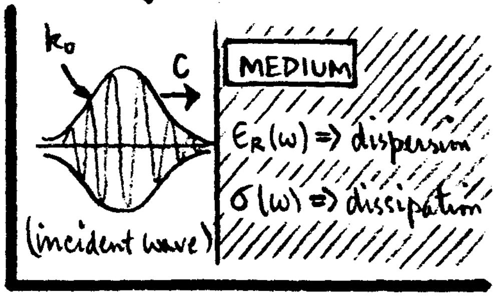
6) We skip Jackson's Sec. 7.6 on "propagation in the ionosphere" for the moment, and move on to Sec. 7.7 -- which treats propagation in a conducting medium. Sec. 7.7 is a rephrasing of the above material on $\mathrm{Re}\,\epsilon(\omega)$ & $\mathrm{Im}\,\epsilon(\omega)$, couched in practical terms that an electrical engineer might use. Although phenomenological in nature, it contacts the reality of published engineering data.

Start from...

$$\text{wave \#}: k = \frac{\omega}{v} = k_0\sqrt{\mu\epsilon}, \quad \text{w/ } k_0 = \omega/c; \qquad (23)$$

$$\text{diel. cnst}: \epsilon = \epsilon_R + i(4\pi\sigma/\omega) \quad \epsilon_R = \mathrm{Re}\,\epsilon(\omega), \text{ and: } \epsilon_I = \frac{4\pi\sigma}{\omega}, \ \sigma \text{ is real.}$$



$$/\!/ \quad \underline{\underline{k = k_0\sqrt{\mu\epsilon_R}\,(1+i\gamma)^{1/2}}}, \quad \text{w/ } \gamma = 4\pi\sigma/\omega\epsilon_R \quad (\sigma \text{ \& } \epsilon_R \text{ can be fcns of } \omega). \qquad (24)$$

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* E.g. the earth's ionosphere, @ $n \simeq 10^6\, e^-$'s/cm^3 $\Rightarrow \frac{\omega_p}{2\pi} \simeq 10^7$ Hz. All $\omega < \omega_p$ is reflected.

Propagation in a conducting medium, per size of $r = \frac{4\pi\sigma}{\omega\epsilon_R}$.

The wave conduction properties (i.e. the behavior of e^{ikz}) evidently depend critically on the ratio r, and will be quite different -- at a given ω -- for good conductors ($\sigma \to$ large) vs. poor conductors ($\sigma \to$ small).

As before [Eq. (9)], we write...¶

$$k = \beta + i\frac{\alpha}{2} \Rightarrow \text{from (24)} \begin{cases} \beta = k_0\sqrt{\mu\epsilon_R}\left[\frac{1}{2}\left(\sqrt{1+r^2}+1\right)\right]^{1/2}, \\ \frac{\alpha}{2} = k_0\sqrt{\mu\epsilon_R}\left[\frac{1}{2}\left(\sqrt{1+r^2}-1\right)\right]^{1/2}; \end{cases} \tag{25}$$

w/ $r = 4\pi\sigma/\omega\epsilon_R$. There are two limiting cases...

① **POOR CONDUCTOR : $r = 4\pi\sigma/\omega\epsilon_R \ll 1$.**

So/ $\left[\begin{aligned} \beta &\simeq k_0\sqrt{\mu\epsilon_R}\left(1+\tfrac{1}{8}r^2 - \cdots\right), \\ \frac{\alpha}{2} &\simeq \frac{2\pi\sigma}{c}\sqrt{\mu/\epsilon_R}\left(1-\tfrac{1}{8}r^2 + \cdots\right). \end{aligned}\right.$ $\;\Big|\;$ $\frac{\alpha}{2}/\beta \simeq \frac{r}{2} \ll 1$ ∫ attenuation per wavelength relatively small. Also: α depends only weakly on ω. (26A)

② **GOOD CONDUCTOR : $r = 4\pi\sigma/\omega\epsilon_R \gg 1$.**

So/ $\left[\begin{aligned} \beta &\simeq \frac{1}{\delta}\left(1+\frac{1}{2r}+\cdots\right) \\ \frac{\alpha}{2} &\simeq \frac{1}{\delta}\left(1-\frac{1}{2r}-\cdots\right) \end{aligned}\right.$ $\boxed{\delta = \frac{c}{\sqrt{2\pi\mu\sigma\omega}}}$ $\;\Big|\;$ $\frac{\alpha}{2} \simeq \beta \Rightarrow$ attenuation per wavelength is relatively large. Both $\frac{\alpha}{2}$ & β depend on $\sqrt{\omega}$. (26B)

The parameter δ, w/ units of length, is the conductor's "skin depth".

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7) For both good & poor conductors, the plane wave fields are...

$$\left[\begin{aligned} \mathbf{E} &= \mathbf{E}_0\left(e^{-\frac{\alpha}{2}\hat{n}\cdot\mathbf{r}}\right)e^{i(\beta\hat{n}\cdot\mathbf{r}-\omega t)} \\ \mathbf{H} &= \mathbf{H}_0\left(e^{-\frac{\alpha}{2}\hat{n}\cdot\mathbf{r}}\right)e^{i(\beta\hat{n}\cdot\mathbf{r}-\omega t)} \end{aligned}\right. \quad \Big| \quad \begin{aligned} &\hat{n} = \text{propagation direction}; \\ &\mathbf{H}_0 = \frac{c}{\mu\omega}k(\hat{n}\times\mathbf{E}_0) \text{ ∫ Faraday's Law}. \end{aligned} \tag{27}$$

¶ Note, in general: $\sqrt{x \pm iy} = \sqrt{\frac{1}{2}(\rho+x)} \pm i\sqrt{\frac{1}{2}(\rho-x)}$, where: $\rho = \sqrt{x^2+y^2}$.

Plane wave in a Conducting Medium. Skin Depth.

Since k is complex, $\mathbb{E}$ & $\mathbb{H}$ are automatically out of phase. Write...

$$\underline{\underline{k = |k|e^{i\phi}}} \int \left.\begin{aligned} |k| &= k_0\sqrt{\mu\epsilon_R}\,(1+r^2)^{1/4}, \\ \phi &= \tfrac{1}{2}\tan^{-1}(r), \end{aligned}\right\} r = \frac{4\pi\sigma}{\omega\epsilon_R}. \qquad (28A)$$

The magnetic & electric field amplitudes are then related by...

$$\mathbb{H}_0 = \sqrt{\frac{\epsilon_R}{\mu}}\,(1+r^2)^{\frac{1}{4}}\,e^{i\phi}\,(\hat{n}\times\mathbb{E}_0)$$ phase shift

So $$\underline{|\mathbb{H}_0|/|\mathbb{E}_0| = \sqrt{\frac{\epsilon_R}{\mu}}\,(1+r^2)^{1/4}}. \qquad (28B)$$

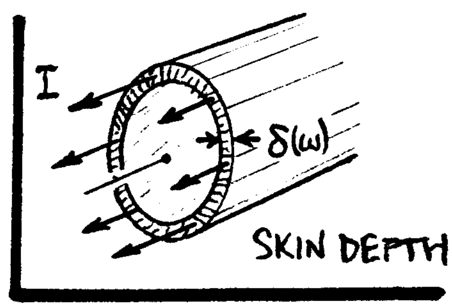
In a good conductor, $r \gg 1$, and the EM wave's magnetic field is dominant... it is larger than the electric field by a factor $\sqrt{\epsilon_R r/\mu} = \sqrt{4\pi\sigma/\mu\omega}$. The wave does not propagate very far in a good conductor, but while it does, it stores most of its energy in $\mathbb{H}$, not $\mathbb{E}$.

Finally, the reason for the name "Skin depth" for the parameter δ in Eq. (26B) above is that it is a characteristic damping length for penetration (into a good conductor) by an EM wave at freq. ω...

$$\left.\begin{array}{c}\text{SKIN DEPTH}\\ \text{(good conductor)}\end{array}\right\} \boxed{\delta = \frac{2}{\alpha} = c/\sqrt{2\pi\mu\sigma\omega}} \qquad (29)$$

$\int$ EM wave ampl. at depth x goes as : $\exp(-x/\delta)$.

A high freq. wave (e.g. a current wave at right) only penetrates the "skin" of a solid conductor, to a depth $\sim \delta$. For copper...



$$\left[\begin{array}{lll} \delta = 8.5\,\text{mm} @ & \omega = 2\pi\times 60\,\text{Hz}; & \\ \quad "\;\; 2.1\,\text{mm} @ & \quad " \quad\;\; " \;\; 1000\,\text{Hz}; & \\ \quad "\;\; 0.22\,\text{mm} @ & \quad " \quad\;\; " \;\; 100\,\text{kHz}. & \end{array}\right. \qquad (30)$$