(32) [15 pts]. Calculate the expectation value of momentum $\langle p \rangle$ for a SHO wave-packet: $\Psi(x,t) = \sum_{n=0}^{\infty} c_n \psi_n(x) \exp\left(-\frac{i}{\hbar} E_n t\right)$; this Ψ is the same form used in class (NOTES: p. Sol$^{\underline{n}}$s 21, Eq. (48)) to calculate position $\langle x \rangle$.

(A) With your result for $\langle p \rangle$, show that: $\langle p \rangle = m\, d\langle x \rangle / dt$, exactly.

(B) By making the approximations appropriate to $n \to$ large, find $\langle p \rangle$ in the Correspondence Principle regime. Compare this form of $\langle p \rangle$ to the classical motion.

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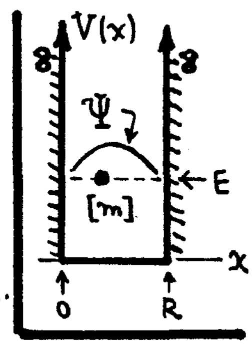
(33) Show that the expectation value of potential energy, $\langle n | V | n \rangle$, in the n^{th} eigenstate of a SHO is $\frac{1}{2} E_n$. HINT: in evaluating $\langle n | x^2 | n \rangle$, it is helpful to recall the identity used in class (NOTES, p. Sol$^{\underline{n}}$s 21, Eq. (52)): $x\,\psi_n(x) = (1/\alpha\sqrt{2})\left[\sqrt{(n+1)}\,\psi_{n+1}(x) + \sqrt{n}\,\psi_{n-1}(x)\right]$, $\alpha = \sqrt{m\omega/\hbar}$; this follows from the identity you proved in problem (30)(B).



(34) [20 pts]. A particle of mass m is in an ∞ly deep potential well as sketched at right. m is in a composite state, described by a wavefcn: $\Psi(x) = \mathcal{N} x (R - x)$, $\mathcal{N}$ = cnst. Clearly, $\Psi \neq$ eigenfcn.

(A) Show that the normalized eigenfcns are: $\psi_n(x) = \sqrt{\frac{2}{R}} \sin(n\pi x / R)$, $n = 1, 2, 3, \ldots$

(B) Normalize Ψ. Find the coefficients $\{c_n\}$ in the eigenfcn expansion: $\Psi(x) = \sum_n c_n \psi_n(x)$. Verify that $\sum_n |c_n|^2 = 1$.

(C) $f(n) = |c_n|^2$ is a probability distribution for finding m (in Ψ) actually in state ψ_n. What is that probability for the ground state, ψ_1? For all other excited states ψ_n, $n > 1$? Why do the numbers turn out that way?

(D) Find the average energy: $\bar{E} = \sum_{n=1}^{\infty} |c_n|^2 E_n$, for m in the state Ψ. Finally, find the energy dispersion $\Delta E = \left(\overline{E^2} - \bar{E}^2\right)^{1/2}$ in state Ψ; Compare ΔE w/ $\bar{E}$.

HINT: if $\lambda(n) = \sum_{k=1}^{\infty} \frac{1}{(2k-1)^n}$, then: $\lambda(2) = \frac{\pi^2}{8}$, $\lambda(4) = \frac{\pi^4}{96}$, $\lambda(6) = \frac{\pi^6}{960}$.

㉜ [15 pts]. Analyse $\langle p\rangle$ for a SHO wavepacket.

1. With $\Psi(x,t)=\sum_{n=0}^{\infty} c_n \psi_n(x)\, e^{-\frac{i}{\hbar}E_n t}$, and $E_n=(n+\frac{1}{2})\hbar\omega$, the exp^n value is:

(A)
$$\langle p\rangle=\int_{-\infty}^{\infty}\Psi^*(x,t)\left\{-i\hbar\frac{\partial}{\partial x}\right\}\Psi(x,t)\,dx=-i\hbar\sum_{n,k=0}^{\infty}c_n^* c_k\, e^{i(n-k)\omega t}\cdot J_{nk},$$

$$\text{where: } J_{nk}=\int_{-\infty}^{\infty}\psi_n^*(x)\frac{\partial}{\partial x}\psi_k(x)\,dx. \tag{1}$$

2. We must evaluate J_{nk}. Recall (CLASS NOTES, p. Sol^ns 18, Eq. (40))...

$$\rightarrow \psi_k(x)=\mathcal{N}_k\, e^{-\frac{1}{2}\xi^2}H_k(\xi) \;\checkmark\; \mathcal{N}_k=(\alpha/2^k k!\sqrt{\pi})^{1/2}, \tag{2}$$
$$\xi=\alpha x,\text{ and: }\alpha=\sqrt{m\omega/\hbar};$$

so//
$$\frac{\partial}{\partial x}\psi_k(x)=\alpha\mathcal{N}_k\frac{\partial}{\partial\xi}\left[e^{-\frac{1}{2}\xi^2}H_k(\xi)\right]$$
$$=\alpha\mathcal{N}_k\left[-\xi e^{-\frac{1}{2}\xi^2}H_k(\xi)+e^{-\frac{1}{2}\xi^2}\overbrace{\frac{\partial}{\partial\xi}H_k(\xi)}^{=2kH_{k-1}(\xi)}\right]$$
$$=-\alpha\xi\,\psi_k(x)+2\alpha k\,\mathcal{N}_k\, e^{-\frac{1}{2}\xi^2}H_{k-1}(\xi). \tag{3}$$

But: $\mathcal{N}_{k-1}=(\alpha/2^{k-1}(k-1)!\sqrt{\pi})^{1/2}=\sqrt{2k}\,\mathcal{N}_k$, so// $\mathcal{N}_k=\frac{\mathcal{N}_{k-1}}{\sqrt{2k}}$, and in (3)...

$$\rightarrow\frac{\partial}{\partial x}\psi_k(x)=-\alpha^2 x\,\psi_k(x)+\alpha\sqrt{2k}\,\psi_{k-1}(x). \tag{4}$$

3. Use of (4) in J_{nk} of (1) now produces...

$$\rightarrow J_{nk}=-\alpha^2\underbrace{\int_{-\infty}^{\infty}\psi_n^*(x)\,x\,\psi_k(x)\,dx}_{\text{NOTES, p. Sol}^{ns}\text{ 21, Eq. (52)}}+\alpha\sqrt{2k}\underbrace{\int_{-\infty}^{\infty}\psi_n^*(x)\psi_{k-1}(x)\,dx}_{\text{use orthogonality}}$$

$$=-\alpha^2\left[\frac{1}{\alpha}\sqrt{\frac{n}{2}}\,\delta_{n-1,k}+\frac{1}{\alpha}\sqrt{\frac{n+1}{2}}\,\delta_{n+1,k}\right]+\alpha\sqrt{2k}\,\delta_{n+1,k}$$

$$=-\alpha\sqrt{\frac{n}{2}}\,\delta_{n-1,k}+\alpha\sqrt{\frac{n+1}{2}}\,\delta_{n+1,k}. \tag{5}$$

4. Now use this result for J_{nk} back in $\langle p\rangle$ of Eq. (1) to write...

$$\rightarrow\langle p\rangle=i\hbar\alpha\sum_{n,k=0}^{\infty}c_n^* c_k\, e^{i(n-k)\omega t}\left[\sqrt{\frac{n}{2}}\,\delta_{n-1,k}-\sqrt{\frac{n+1}{2}}\,\delta_{n+1,k}\right], \tag{6}$$

(next page)

$$\xrightarrow{\text{or}} \langle p\rangle = i\hbar\alpha\Big[\underbrace{\sum_{n=1}^{\infty}}_{\text{here, } n=0 \text{ term vanishes}}\sqrt{\frac{n}{2}}\,c_n^* c_{n-1}\,e^{+i\omega t} - \underbrace{\sum_{n=0}^{\infty}}_{\text{step summand up by one}}\sqrt{\frac{n+1}{2}}\,c_n^* c_{n+1}\,e^{-i\omega t}\Big]$$

$$= i\hbar\alpha\sum_{n=1}^{\infty}\sqrt{\frac{n}{2}}\left(c_n^* c_{n-1}\,e^{i\omega t} - c_{n-1}^* c_n\,e^{-i\omega t}\right). \qquad (7)$$

Let $c_n = |c_n|e^{i\phi_n}$, and define $\Delta\phi_n = \phi_n - \phi_{n-1}$. Then (7) gives, exactly...

$$\boxed{\langle p\rangle = (-)\sqrt{2m\hbar\omega}\sum_{n=0}^{\infty}\sqrt{n}\,|c_n c_{n-1}|\,\sin(\omega t - \Delta\phi_n)}. \qquad (8)$$

Here, we have put in $\alpha = \sqrt{m\omega/\hbar}$.

---

5. Now recall the result for $\langle x\rangle$ (NOTES, p. Sol^ns 22, Eq.(55)), written as...

$$\langle x\rangle = \frac{1}{m\omega}\sqrt{2m\hbar\omega}\sum_{n=0}^{\infty}\sqrt{n}\,|c_n c_{n-1}|\cos(\omega t - \Delta\phi_n), \qquad (9)$$

$$\text{so } \underline{\underline{m\frac{d}{dt}\langle x\rangle = (-)\sqrt{2m\hbar\omega}\sum_{n=0}^{\infty}\sqrt{n}\,|c_n c_{n-1}|\sin(\omega t - \Delta\phi_n) = \langle p\rangle.}} \qquad (10)$$

So, indeed, Newton II is obeyed exactly for the SHO, in an exp^n value sense.

---

6. When $n \to$ large, $\langle p\rangle$ of Eq.(8) can be processed in the same way as was

(B) $\langle x\rangle$ -- NOTES, p. Sol^ns 22, Eqs (55)-(58). That goes as...

$$\rightarrow n \rightarrow \text{large}: \; E_n \simeq n\hbar\omega,\; c_{n-1}\simeq c_n,\; \Delta\phi_n \simeq \Delta\phi \;(\text{indep't of } n) \qquad (11)$$

$$\text{so } \langle p\rangle \simeq (-)\sqrt{2m}\underbrace{\Big(\sum_{n=0}^{\infty}\sqrt{E_n}\,|c_n|^2\Big)}_{\langle\sqrt{E}\rangle}\sin(\omega t - \Delta\phi) = (-)\langle\sqrt{2mE}\rangle\sin(\omega t-\Delta\phi). \qquad (12)$$

... with:

$$x_0 = \frac{1}{m\omega}\sqrt{2mE} = \text{classical maximum amplitude},$$

$$\boxed{\langle p\rangle \simeq (-)\,m\omega\langle x_0\rangle\sin(\omega t - \Delta\phi)}. \qquad (13)$$

This reproduces the classical motion "exactly", since if $x = x_0\cos(\omega t - \Delta\phi)$, then $p = m\frac{d}{dt}x = -m\omega x_0\sin(\omega t - \Delta\phi)$.

(33) Find the expectation value of the P.E. $V(x) = \frac{1}{2} m\omega^2 x$ in the n^{th} state of a SHO.

1. We want to evaluate the expectation value:

$$\rightarrow \langle V \rangle_n = \langle n | V | n \rangle = \tfrac{1}{2} m\omega^2 \langle n | x^2 | n \rangle, \tag{1}$$

where $|n\rangle$ is an eigenstate of the SHO. We have the result (Solns p.21, Eq.(52)):

$$\rightarrow x|n\rangle = (1/\alpha\sqrt{2})\{\sqrt{n+1}\,|n+1\rangle + \sqrt{n}\,|n-1\rangle\}, \quad \text{w/ } \alpha = \sqrt{m\omega/\hbar}, \tag{2}$$

and also...

$$\rightarrow \langle n | x | k \rangle = (1/\alpha\sqrt{2})\{\sqrt{n}\,\delta_{n-1,k} + \sqrt{n+1}\,\delta_{n+1,k}\}. \tag{3}$$

---

2. Multiply through Eq. (2) by x; then take $\langle n|$ through the result...*

$$x^2|n\rangle = \frac{1}{\alpha\sqrt{2}}\{\sqrt{n+1}\; x|n+1\rangle + \sqrt{n}\; x|n-1\rangle\},$$

$$\xrightarrow{\text{So}} \langle n | x^2 | n \rangle = \frac{1}{\alpha\sqrt{2}}\{\sqrt{n+1}\,\langle n|x|n+1\rangle + \sqrt{n}\,\langle n|x|n-1\rangle\}. \tag{4}$$

Now, by use of (3), we find that the terms on the RHS of (4) are...

$$\left[\!\left[\langle n|x|n+1\rangle = \frac{1}{\alpha}\sqrt{\frac{n+1}{2}}, \text{ and: } \langle n|x|n-1\rangle = \frac{1}{\alpha}\sqrt{\frac{n}{2}}. \right.\right. \tag{5}$$

Upon inserting the results of Eq.(5) into Eq.(4), we find...

$$\rightarrow \langle n | x^2 | n \rangle = \frac{1}{\alpha^2}\left(\frac{n+1}{2}\right) + \frac{1}{\alpha^2}\left(\frac{n}{2}\right) = (n + \tfrac{1}{2})\,\hbar/m\omega. \tag{6}$$

---

3. The total energy in the n^{th} state of the SHO is $E_n = (n+\frac{1}{2})\hbar\omega$, and so the result of Eq.(6) can be quoted as...

$$\underline{\underline{\langle n | x^2 | n \rangle = E_n / m\omega^2}}. \tag{7}$$

When this is used in Eq.(1), we find the required result...

$$\boxed{\langle V \rangle_n = \langle n | V | n \rangle = \tfrac{1}{2} E_n}. \quad \text{QED} \tag{8}$$

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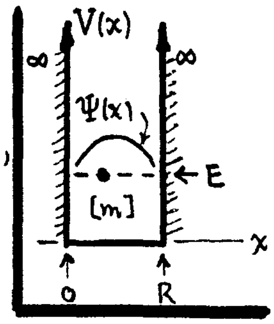
* "Taking $\langle n|$ through the eqtn" means operating by $\int_{-\infty}^{\infty} dx\, \psi_n^*(x)$, from the left.

ϕ506 Solutions



(34) [20 pts]. Composite state $\Psi(x) = \mathcal{N} x(R-x)$ in an ∞ well.

(A) The eigenfcns must obey: $\psi'' + k^2 \psi = 0$, inside the well $(0 \leqslant x \leqslant R)$, with $k^2 = 2mE/\hbar^2$, w/ E: total (kinetic) energy. Solns are clearly $\psi(x) \propto \cos kx$ & $\sin kx$, and we must have $\psi(0) = 0 = \psi(R)$, at the walls, where V becomes ∞. The $\cos kx$ solns are ruled out because they do not vanish at $x=0$. For the remaining $\sin kx$ solns, $\psi(R) \propto \sin kR = 0$ requires $k \to k_n = n\pi/R$, w/ $n = 1, 2, 3, \ldots$. So $\psi_n(x) = A_n \sin(n\pi x/R)$ is an eigenfcn, w/ A_n a norm cnst. From prob^m (24), we know the norm cnst $A_n = \sqrt{1/(\text{well width}/2)}$, or $A_n = \sqrt{2/R}$ in this case. The normalized eigenfcns are then:

$$\boxed{\psi_n(x) = \sqrt{2/R}\,\sin(n\pi x/R)}, \quad n = 1, 2, 3, \ldots \tag{1}$$

(B) To norm the composite state $\Psi(x) = \mathcal{N} x(R-x)$, we want $\mathcal{N}$ such that...

$$\rightarrow \int_0^R |\Psi(x)|^2 dx = \mathcal{N}^2 \int_0^R x^2 (R-x)^2 dx = 1 \Rightarrow \mathcal{N}^2 = 30/R^5;$$

So// $\Psi(x) = \mathcal{N} x(R-x)$, w/ $\mathcal{N} = \sqrt{30/R^5}$, is normed composite state. (2)

If Ψ is expanded as: $\Psi(x) = \sum_{n=1}^{\infty} c_n \psi_n(x)$, then the coefficients are...

$$c_n = \int_0^R \psi_n^*(x)\Psi(x)\,dx = \mathcal{N}\sqrt{\frac{2}{R}} \int_0^R x(R-x)\sin(n\pi x/R)\,dx \quad \text{let } z = \frac{x}{R}$$

$$= \sqrt{60} \int_0^1 z(1-z)\sin n\pi z\,dz = \sqrt{60} \cdot \frac{2}{(n\pi)^3}\left[1 - \cos n\pi\right]$$

w/ $$c_n = \frac{\sqrt{240}}{(n\pi)^3}\left[1 - (-)^n\right] = \begin{cases} [\sqrt{240}/(n\pi)^3] \cdot 2, & \text{for } n = 1, 3, 5, \ldots \\ 0, & \text{for } n = 2, 4, 6, \ldots \text{ (all even } n). \end{cases} \tag{3}$$

The sum over all $|c_n|^2$ values is -- with the help of the given sums...

$$\rightarrow \sum_{n=1}^{\infty} |c_n|^2 = \frac{240 \cdot 4}{\pi^6} \sum_{k=1}^{\infty} 1/(2k-1)^6 = \frac{960}{\pi^6} \lambda(6) = 1. \tag{4}$$

This result verifies Parseval's Theorem in this case, namely that if -- in the expansion $\Psi = \sum_n c_n \psi_n$ -- Ψ & $\{\psi_n\}$ are both normed, then $\sum_n |c_n|^2 = 1$.

(C) For odd integers, $n = 2k-1$, w/ $k = 1, 2, 3, \ldots$ the sol^n in Eq. (3) shows that:

$$\rightarrow |c_{2k-1}|^2 = \frac{960}{\pi^6} / (2k-1)^6 = 0.998\,555 / (2k-1)^6. \tag{5}$$

$k=1$ ($n=1$) $\Rightarrow$ the ground state $\psi_1(x) = \sqrt{2/R}\,\sin(\pi x/R)$ being found in the composite state $\Psi(x) = Nx(R-x)$. Then Eq. (5) gives for that probability

$$\boxed{|c_1|^2 = 0.9986}, \text{ probability for the ground state.} \tag{6}$$

Since $\sum_{n=1}^{\infty} |c_n|^2 = 1$, the probability for all states at $n>1$ is

$$\sum_{n>1}^{\infty} |c_n|^2 = 1 - |c_1|^2 = 0.0014, \text{ probability for all states @ } n>1. \tag{7}$$

This implies that $\Psi(x) = Nx(R-x)$ is predominantly a ground state wavefcn; that happens because $\Psi(x)$ closely resembles $\psi_1(x) = \sqrt{2/R}\,\sin(\pi x/R)$, in vanishing at $x=0$ & $x=R$, having no nodes in $0<x<R$, etc.

(D) The energy levels corresponding to the eigenstates found in part (A) are $E_n = \frac{\hbar^2 k_n^2}{2m}$, i.e. $E_n = (\pi^2\hbar^2/2mR^2)\, n^2$. With the C_n's of Eq. (3), the average energy in the composite state $\Psi(x) = Nx(R-x)$ is -- by definition...

$$\rightarrow \langle E\rangle = \sum_{n=1}^{\infty} |c_n|^2 E_n = \sum_{n=1,3,5}^{\infty} \left(\frac{960}{\pi^6}\right)\frac{1}{n^6}\cdot\left(\frac{\pi^2\hbar^2}{2mR^2}\right) n^2 = \frac{480}{\pi^4}\,\frac{\hbar^2}{mR^2}\,\underbrace{\sum_{k=1}^{\infty}\frac{1}{(2k-1)^4}}_{=\pi^4/96}$$

i.e. $\boxed{\langle E\rangle = 5\cdot(\hbar^2/mR^2)}$ (8)

$\langle E\rangle$ is quite close to the actual ground state energy $E_1 = \frac{\pi^2}{2}\left(\frac{\hbar^2}{mR^2}\right) = 4.94\left(\frac{\hbar^2}{mR^2}\right)$, because Ψ closely resembles ψ_1, as noted above. The average (energy)^2 value is found by: $\langle E^2\rangle = \sum_{n=1}^{\infty} |c_n|^2 E_n^2$, and -- by the same sort of calculation as in Eq. (8), we find: $\underline{\langle E^2\rangle = 30(\hbar^2/mR^2)^2}$. We then get the energy dispersion...

$$\rightarrow \Delta E = \left(\langle E^2\rangle - \langle E\rangle^2\right)^{1/2} = \sqrt{5}\,(\hbar^2/mR^2) \tag{9}$$

This dispersion ΔE is $\sim(1/\sqrt{5})\times E_1 = 0.447\times E_1$. So, even though $\Psi(x)$ resembles the ground state $\psi_1(x)$, this composite state is not really stationary at energy E_1.