## When does $\Psi(x) = \sum_n c_n \psi_n(x)$ hold?

## Dirac's Expansion Postulate and Completeness of an Eigenfcn Set $\{\psi_n\}$

Several times in the recent past, we have used Dirac's expansion postulate (# $\boxed{5}$ on p. Prop. 27 of these notes), i.e. the claim that if a QM system is characterized by a set of orthonormal eigenfcns $\{\psi_n(x)\}$, then an arbitrary state $\Psi(x)$ of the system (like the wavepackets we have been using) can be written as the superposition: $\Psi(x) = \sum_n c_n \psi_n(x)$, and $|c_n|^2$ then measures the probability that the $n^{\text{th}}$ eigenstate will be found present in a measurement on state $\Psi$. When all this can be done, the set $\{\psi_n(x)\}$ is said to be "Complete" -- i.e. it can completely specify any arbitrary $\Psi(x)$ relevant to the system.

Here we want to sharpen our notion of "completeness" for an eigenfcn set $\{\psi_n(x)\}$ and also to address the question of completeness and normalization for free particle eigenfcns $\exp(+\frac{i}{\hbar}\mathbf{p}\cdot\mathbf{r})$.

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1) Start with the eigenfcns $\{\psi_n\}$ generated by some Hermitian operator $Q$, via

$$Q\psi_n(x) = q_n \psi_n(x) \quad \begin{cases} \text{the } \{q_n\} = \{\text{cnsts}\} \text{ are eigenvalues;} \\ \text{the } \{\psi_n\} \text{ are orthonormal: } \langle\psi_m|\psi_n\rangle = \delta_{mn}. \end{cases} \tag{1}$$

Now try expanding an arbitrary state $\Psi$ of $Q$ in terms of the $\{\psi_n\}$, i.e.

$$\Psi(x) = \sum_n c_n \psi_n(x) \leftarrow \text{if this is possible, } \{\psi_n\} \text{ is a "complete set."} \tag{2}$$

The expansion coefficients $c_n$ in Eq. (2) are readily found...

$$\rightarrow \langle\psi_m|\Psi\rangle = \sum_n c_n \langle\psi_m|\psi_n\rangle = \sum_n c_n \delta_{mn} = c_m$$

so, $$c_n = \langle\psi_n|\Psi\rangle = \int \psi_n^*(x')\Psi(x')\,dx'. \tag{3}$$

Now we demand self-consistency: we should be able to put $c_n$ of Eq. (3) back into the RHS of Eq. (2) and thereby regenerate $\Psi(x)$. We find...

## Closure Relation. The spectral coefficients $\{|c_n|^2\}$.

$$\rightarrow \Psi(x) = \sum_n \left[\int \psi_n^*(x')\,\Psi(x')\,dx'\right]\psi_n(x) = \int \Big[\underbrace{\sum_n \psi_n^*(x')\,\psi_n(x)}_{\text{must act like } \delta(x'-x)}\Big]\,\Psi(x')\,dx' \qquad (4)$$

This is an identity iff the last [ ] behaves like a Dirac delta fcn. Then, indeed: $\int[\delta(x'-x)]\,\Psi(x')\,dx' = \Psi(x)$. Claim:

**CLOSURE RELATION**

The expansion $\Psi(x) = \sum_n c_n \psi_n(x)$ is possible if and only if the $\{\psi_n\}$ obey: $\underline{\sum_n \psi_n^*(x')\,\psi_n(x) = \delta(x'-x)}$. The $\{\psi_n\}$ are then a complete set. (5)

The Closure Relation is the principal criterion for completeness of the $\{\psi_n\}$.

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**2)** Normalization of $\Psi$ proceeds apace...

$$\rightarrow \langle\Psi|\Psi\rangle = 1 \Rightarrow \Big\langle \sum_m c_m\psi_m \Big| \sum_n c_n\psi_n \Big\rangle = \sum_{m,n} c_m^* c_n \overbrace{\langle\psi_m|\psi_n\rangle}^{\delta_{mn}} = \boxed{\sum_n |c_n|^2 = 1}. \qquad (6)$$

For discrete states, this is the analogue of Parseval's Theorem (see CLASS NOTES, p. Sch. 12). In fact we have parallel constructions...

**FREE PARTICLE** (w/ momentum $q$ = continuous variable). (7A)

| | |
|---|---|
| $\underline{\Psi(x)} = \frac{1}{\sqrt{2\pi}}\int \phi(q)\,e^{iqx}\,dq$; | Norm: $\int|\Psi(x)|^2dx = 1 = \int|\phi(q)|^2dq$. |
| $\Rightarrow$ spectral fcn $\phi(q)$ is: | Interpretation: $|\phi(q)|^2dq$ = probability of |
| $\underline{\phi(q)} = \frac{1}{\sqrt{2\pi}}\int e^{-iqx}\,\Psi(x)\,dx$. | finding $\Psi$ with momentum in $dq$ at $q$. |

**GENERAL QM SYSTEM** (w/ eigenvalues $q_n$ discrete). (7B)

| | |
|---|---|
| $\underline{\Psi(x)} = \sum_n c_n\psi_n(x)$; | Norm: $\int|\Psi(x)|^2dx = 1 = \sum_n |c_n|^2$. |
| $\Rightarrow$ Spectral coefficients $c_n$ are: | Interpretation: $|c_n|^2$ = probability of finding |
| $\underline{c_n} = \int \psi_n^*(x)\,\Psi(x)\,dx$. | $\Psi$ with eigenvalue $q_n$ in state $\psi_n$. |

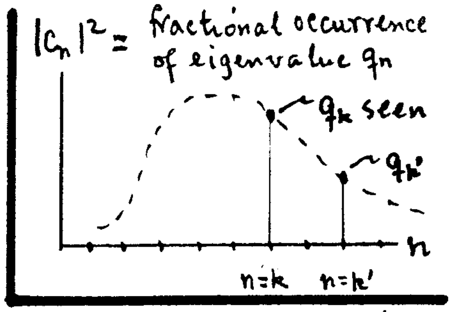
This analogy makes plausible Dirac's postulate # $\boxed{6}$ (p. Prop. 27), namely that the $\{|c_n|^2\}$ are a probability distribution for the eigenvalues $q_n$.

The $\{|c_n|^2\}$ as a probability distribution. Completeness for SHO $\{\psi_n\}$.

In fact, the expectation value of $Q$ in the composite state $\Psi$ is...

$$\langle Q\rangle = \langle\Psi|Q|\Psi\rangle = \sum_{m,n} c_m^* c_n \langle\psi_m|Q\psi_n\rangle \quad \int \text{use: } Q\psi_n = q_n\psi_n \text{ for eigenstates, then use } \langle\psi_m|\psi_n\rangle = \delta_{mn}$$

$$= \sum_{m,n} c_m^* c_n q_n \langle\psi_m|\psi_n\rangle, \quad \text{so} \quad \boxed{\langle Q\rangle = \sum_n |c_n|^2 q_n}. \tag{8}$$




A given single measurement of $Q$ in the state $\Psi$ cannot yield anything other than some eigenvalue $q_k$ (because these are the only observables -- recall work on p. Prop. 19). The next measurement of $\langle Q\rangle$ may yield $q_{k'}$, then $q_{k''}$, $q_{k'''}$ etc. The probability of any given $q_n$ showing up -- when working with $\Psi = \sum_n c_n\psi_n$ -- is just $|c_n|^2$. Then the value $\langle Q\rangle = \sum_n |c_n|^2 q_n$ in Eq. (8) has an obvious meaning as the average of all these measurements.

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## 3) ASIDE Completeness for the SHO eigenfcns.

We shall show that the SHO eigenfcns $\psi_n(x)$ are complete, by demonstrating that the closure relation [Eq. (5) above] holds: $\sum_n \psi_n^*(x)\psi_n(x') = \delta(x-x')$.

1. Eq. (40) p. Sol^{ns} 18 $\Big\}$ $\psi_n(x) = \left(\dfrac{\alpha/\sqrt{\pi}}{2^n n!}\right)^{\frac{1}{2}} e^{-\frac{1}{2}\xi^2} H_n(\xi)$, w/ $\xi = \alpha x$ & $\alpha = \sqrt{\dfrac{m\omega}{\hbar}}$; (A1)

so $$\underline{\underline{\text{SUM}}} = \sum_n \psi_n^*(x)\psi_n(x') = \frac{\alpha}{\sqrt{\pi}} e^{-\frac{1}{2}(\xi^2+\xi'^2)} \sum_{n=0}^{\infty} \frac{1}{2^n n!} H_n(\xi)H_n(\xi'). \tag{A2}$$

2. We process the SUM in Eq. (A2) by using Rodrigues' formula...

$$\rightarrow H_n(\xi') = (-)^n e^{+\xi'^2}(d/d\xi')^n e^{-\xi'^2}, \tag{A3}$$

... and the tabulated integral (from $\int_{-\infty}^{\infty} e^{-au^2 \pm bu}\,du = \sqrt{\frac{\pi}{a}}\,e^{b^2/4a}$)...

$$\rightarrow \int_{-\infty}^{\infty} e^{-\frac{1}{4}\sigma^2 - i\xi'\sigma}\,d\sigma = 2\sqrt{\pi}\,e^{-\xi'^2}. \tag{A4}$$

Use the $e^{-\xi'^2}$ in (A4) on the RHS of (A3), i.e. form...

Completeness for the SHO eigenfcns (cont'd)

$$\rightarrow H_n(\xi') = (-)^n e^{\xi'^2}\left(\frac{d}{d\xi'}\right)^n \cdot \frac{1}{2\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-\frac{1}{4}\sigma^2 - i\xi'\sigma}\, d\sigma$$ ✓ differentiate under the integral

$$= \frac{i^n}{2\sqrt{\pi}} e^{\xi'^2}\int_{-\infty}^{\infty} \sigma^n e^{-\frac{1}{4}\sigma^2 - i\xi'\sigma}\, d\sigma. \qquad \text{(A5)}$$

This is evidently an integral representation of the Hermite polynomials

3. Use (A5) in the SUM of (A2) to write...

$$\underline{\underline{\text{SUM}}} = \frac{\alpha}{\sqrt{\pi}} e^{-\frac{1}{2}(\xi^2+\xi'^2)} \sum_{n=0}^{\infty} \frac{H_n(\xi)}{2^n n!} \cdot \frac{i^n}{2\sqrt{\pi}} e^{\xi'^2}\int_{-\infty}^{\infty} \sigma^n e^{-\frac{1}{4}\sigma^2 - i\xi'\sigma}\, d\sigma. \qquad \text{(A6)}$$

... interchange order of $\sum_{n=0}^{\infty}$ and $\int_{-\infty}^{\infty}$, and rearrange factors...

$$\underline{\underline{\text{SUM}}} = \frac{\alpha}{2\pi} e^{-\frac{1}{2}(\xi^2-\xi'^2)} \int_{-\infty}^{\infty} \underbrace{\left[\sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{i\sigma}{2}\right)^n H_n(\xi)\right]} e^{-\frac{1}{4}\sigma^2 - i\xi'\sigma}\, d\sigma \qquad \text{(A7)}$$

generating fcn for the $H_n$ is : $g(s,\xi) = \sum_{n=0}^{\infty} \frac{s^n}{n!} H_n(\xi) = e^{-s^2+2\xi s}$ ✓ Eq. (30) p. Sol$^{\underline{ns}}$ 17

so// $[\quad] = g(s,\xi)\big|_{s=\frac{i\sigma}{2}} = e^{\frac{1}{4}\sigma^2 + i\xi\sigma}$. (A8)

4. Use of this result in (A7) simplifies things considerably, viz.

$$\underline{\underline{\text{SUM}}} = \alpha e^{-\frac{1}{2}(\xi^2-\xi'^2)} \cdot \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i(\xi-\xi')\sigma}\, d\sigma. \qquad \text{(A9)}$$

the integral is $\delta(\xi-\xi')$

The remaining integral is just a rep$^{\underline{n}}$ of Dirac's delta fcn $\delta(\xi-\xi')$, w/ $\xi = \alpha x$. So we can write (A9), using $\delta(\alpha u) = \frac{1}{\alpha}\delta(u)$, as...

$$\underline{\underline{\text{SUM}}} = e^{-\frac{1}{2}\alpha^2(x^2-x'^2)} \cdot \delta(x-x') = \delta(x-x'). \qquad \text{(A10)}$$

The exponential factor is just $\equiv 1$ in effect, since the RHS is nonzero only when $x = x'$. Altogether then, we have shown closure for the SHO $\psi_n$'s:

$$\boxed{\sum_n \psi_n^*(x)\,\psi_n(x') = \delta(x-x') \Longleftrightarrow \text{the SHO } \{\psi_n\} \text{ is complete.}} \qquad \text{(A11)}$$

Evidently, the analytic demonstration of closure for a set $\{\psi_n\}$ is intricate. Other tests for completeness can be used (e.g. "convergence-in-the-mean" for $\sum_n c_n\psi_n$).

# Proof that QM doesn't exist. Free-particle eigenfcns.

**4)** Not all eigenfcns and eigenvalues work into our normalization-orthogonality-completeness scheme so neatly. For example, consider free-particle wavefcns, which are eigenfcns of momentum. The following catastrophe seems plausible...

## PROOF THAT $\hbar = 0$

**1.** Let $\phi$ be an eigenfcn of the (Hermitian) momentum operator, i.e.

$$p_{op}\phi = p\phi \,, \text{ w/ } p_{op} = -i\hbar\,\partial/\partial x \text{ and/ } p = \text{real cnst} \tag{9a}$$

**2.** Consider the commutator $i\hbar = [x, p_{op}]$. Take expectation values on both sides

$$i\hbar\langle\phi|\phi\rangle = \langle\phi|[x,p_{op}]|\phi\rangle \leftarrow \text{use norm } \langle\phi|\phi\rangle = 1 \text{ on LHS...} \tag{9b}$$

$$\text{so/ } i\hbar = \langle\phi|x p_{op}|\phi\rangle - \langle\phi|p_{op}x|\phi\rangle \leftarrow \text{use Hermitian property of } p_{op} \text{ in 2nd term RHS...} \tag{9c}$$

$$\text{then/ } i\hbar = \langle\phi|x|p_{op}\phi\rangle - \langle p_{op}\phi|x|\phi\rangle \leftarrow \text{use } p_{op}\phi = p\phi \ldots \tag{9d}$$

$$\text{and/ } \underline{\underline{i\hbar = p\langle\phi|x|\phi\rangle - p^*\langle\phi|x|\phi\rangle \equiv 0}}, \text{ since } p \text{ is real.} \tag{9e}$$

**3.** (9e) $\Rightarrow \hbar \equiv 0$. **QED**. Corollary: QM $\rightarrow$ 0 ( $[x, p_{op}] = 0$, etc.). (9f)

Something must be <u>wrong</u> with this proof, even though we've done "orthodox" operations ( $\phi$ normed, $p_{op}$ Hermitian, etc.).

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There are two things <u>wrong</u> with this proof, namely the assumptions in (9b) & (9c) that <u>$\langle\phi|\phi\rangle = 1$</u> & <u>$p_{op}$ is Hermitian w.r.t. to the $\phi$-fcns</u>. Both of these "orthodox" assumptions fail because free-particle wavefcns have the (unique) peculiar feature that they do <u>not</u> vanish at $\infty$. To see this, solve (9a)...

$$-i\hbar\frac{\partial}{\partial x}\phi = p\phi \Rightarrow \underline{\underline{\phi = Ce^{\frac{i}{\hbar}px} = \phi_p(x)}} \int \text{free particle momentum eigenfcn; } C = \text{"norm" cnst.} \tag{10}$$

$$\text{so/ } \langle\phi_p|\phi_p\rangle = |C|^2\int_{-\infty}^{\infty} e^{-(i/\hbar)px}\cdot e^{+(i/\hbar)px}\,dx = |C|^2\int_{-\infty}^{\infty} dx \rightarrow \infty\,; \tag{11}$$

and/ (next page)

## Box normalization for free-particle eigenfcns.

$$\rightarrow \langle \phi | p_{op}(x\phi) \rangle = \langle p_{op}\phi | x\phi \rangle - i\hbar \underbrace{(x|\phi|^2)\Big|_{x=-\infty}^{x=+\infty}}_{|C|^2[(+)\infty - (-)\infty] = \infty.} \qquad (12)$$

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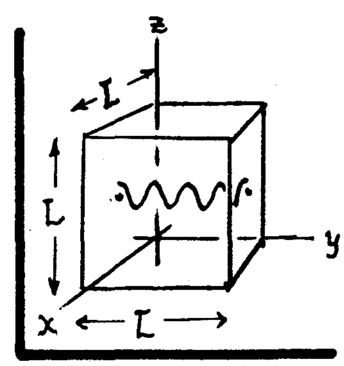
**5)** By the example of Eqs. (9)-(12), it is clear that the procedures for normalization, orthogonality, and completeness for (free-particle) momentum eigenfcns must be treated as a special case. In 3D, these eigenfcns are...

$$-i\hbar\nabla\phi = \mathbf{p}\phi \Rightarrow \phi = C_p\, e^{(i/\hbar)\mathbf{p}\cdot\mathbf{r}} = \phi_p(\mathbf{r})$$

... put: $\underline{\underline{\mathbf{p} = \hbar\mathbf{k}}}$ (deBroglie), so: $\underline{\underline{\phi_k(\mathbf{r}) = C_k\, e^{i\mathbf{k}\cdot\mathbf{r}}}}$;

... and: $\phi_k(x,y,z) = C_k\, e^{ik_x x}\, e^{ik_y y}\, e^{ik_z z}$, in rect$^{\underline{r}}$ cds. (13)

<u>One way of normalizing the $\phi_k(\mathbf{r})$</u> is to imagine the space they are in to be a very large cubical box of side $L \to \infty$, and to impose "<u>periodic boundary conditions</u>", i.e. with the L-box repeated indefinitely...




$\rightarrow \phi_k(x+L, y, z) = \phi_k(x,y,z)$, and similarly for $y$ & $z$;

so// $\mathbf{k} = (k_x, k_y, k_z) = \frac{2\pi}{L}(n_x, n_y, n_z)$ ✓ the $n_i$ are integers: $0, \pm1, \ldots$ (14)

We did a similar procedure in our analysis off BB radiation (see NOTES, p. Intro. 2). $\mathbf{k}$ is now <u>discrete</u>, and so is the energy $E_k = \hbar^2\mathbf{k}^2/2m$, but the spacing $\Delta k = 2\pi/L$ between adjacent k-values can be made as small as desired by letting $L \to \infty$; that is the way we recover the <u>continuum</u> of k-values (and $E_k$-values) that characterize a free particle.

The main reason why -- in Eq. (14) -- we require that $\phi_k(\mathbf{r})$ <u>repeat</u> itself on the walls, rather than requiring $\phi_k(\mathbf{r}) \equiv 0$ there (as for the BB radiation case), is that we want to be able to recover the case of a truly free particle, where $|\phi_k|^2$ need <u>not</u> vanish at $\infty$ (i.e. $L \to \infty$). The requirement $\phi_k \equiv 0$ on the walls would imply that the walls provided an actual physical containment -- they would act as though there were a potential $V \to \infty$ at the boundaries.

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**6)** With the so-called "box normalization" in Eq. (14), we can carry out the standard procedures for the free-particle eigenfcns $\phi_k(\mathbf{r})$, viz...

NORMALIZATION

$$\rightarrow \langle \phi_k | \phi_k \rangle = |C_k|^2 \int_{box} (e^{-i\mathbf{k}\cdot\mathbf{r}})(e^{+i\mathbf{k}\cdot\mathbf{r}})\, d^3r = |C_k|^2 \int_{box} d^3r = |C_k|^2 L^3;$$

so// $\langle \phi_k | \phi_k \rangle = 1 \Rightarrow C_k = 1/L^{3/2}$

and// $$\boxed{\phi_k(\mathbf{r}) = (1/L^{3/2})\, e^{i\mathbf{k}\cdot\mathbf{r}}} \quad \checkmark \text{ normalized free-particle eigenfcn (in a cubical box of side } L). \tag{15}$$

ORTHOGONALITY

This depends explicitly on the quasi-discrete nature of $\mathbf{k}$, as follows...

$$\rightarrow \langle \phi_k | \phi_l \rangle = \frac{1}{L^3} \int_{box} e^{-i(\mathbf{k}-\mathbf{l})\cdot\mathbf{r}}\, d^3r$$

$$= \frac{1}{L^3} \int_{-L/2}^{+L/2} e^{-i(k_x - l_x)x}\, dx \times \begin{pmatrix}\text{similar integral}\\ \text{in } y\end{pmatrix} \times \begin{pmatrix}\text{similar integral}\\ \text{in } z\end{pmatrix}$$

$$= \frac{1}{L^3}\left[\frac{2}{k_x - l_x} \sin(k_x - l_x)\frac{L}{2}\right] \times \begin{bmatrix}\text{similar result}\\ \text{in } y\end{bmatrix} \times \begin{bmatrix}\text{similar result}\\ \text{in } z\end{bmatrix}. \tag{16}$$

But: $k_x - l_x = (2\pi/L)\Delta n_x$, where $\Delta n_x$ = some integer $= 0, \pm1, \pm2, \ldots$ So...

$$\langle \phi_k | \phi_l \rangle = \left[\frac{\sin \pi \Delta n_x}{\pi \Delta n_x}\right] \times \left[\frac{\sin \pi \Delta n_y}{\pi \Delta n_y}\right] \times \left[\frac{\sin \pi \Delta n_z}{\pi \Delta n_z}\right] = \begin{cases} 0, & \text{if any } \Delta n_i \neq 0; \\ 1, & \text{only if all } \Delta n_i = 0. \end{cases} \tag{17}$$

So $\langle \phi_k | \phi_l \rangle = 0$, unless $\mathbf{l} \equiv \mathbf{k}$. The shorthand for this orthogonality is...

$$\boxed{\langle \phi_k | \phi_l \rangle = \delta_{kl}} \quad \checkmark \text{ orthonormality for free-particle eigenfcns (still in a cubical box).} \tag{18}$$

COMPLETENESS

This can be demonstrated by converting the closure sum to an integral, as...

## Completeness for free-particle $\phi_k(\mathbf{r})$'s. δ-fcn normalization.

$$\rightarrow \sum_k \phi_k^*(\mathbf{r}')\,\phi_k(\mathbf{r}) = \frac{1}{L^3}\sum_{n_x=-\infty}^{\infty}\sum_{n_y=-\infty}^{\infty}\sum_{n_z=-\infty}^{\infty} e^{\frac{2\pi i}{L}[n_x(x-x')+n_y(y-y')+n_z(z-z')]} \quad (19)$$

... as $L\to\infty$, replace $\sum_{n_x=-\infty}^{\infty}$ by $\int_{-\infty}^{\infty} dn_x = (L/2\pi)\int_{-\infty}^{\infty} dk_x$ $\int$ a different way of counting. Then...

$$\rightarrow \underline{\underline{\sum_k \phi_k^*(\mathbf{r}')\,\phi_k(\mathbf{r})}} = \frac{1}{(2\pi)^3}\cdot\int_{-\infty}^{\infty}dk_x\int_{-\infty}^{\infty}dk_y\int_{-\infty}^{\infty}dk_z\; e^{i[k_x(x-x')+k_y(y-y')+k_z(z-z')]}$$

$$= \delta(x-x')\,\delta(y-y')\,\delta(z-z') = \underline{\underline{\delta(\mathbf{r}-\mathbf{r}')}}. \quad (20)$$

**QED**: Momentum eigenfcns $\phi_k(\mathbf{r}) = (1/L^{3/2})\,e^{i\mathbf{k}\cdot\mathbf{r}}$, subject to periodic boundary conditions in a cubical box of side $L$, are both <u>orthonormal</u> and <u>complete</u>. That means we can describe the most general free particle motion by: $\Psi(\mathbf{r}) = \sum_{\mathbf{k}} a_k\,\phi_k(\mathbf{r}) \rightarrow \int_{-\infty}^{\infty} d^3k\, A(\mathbf{k})\,e^{i\mathbf{k}\cdot\mathbf{r}}$. We took this Fourier-type wavepacket as a given when we worked with free-particle motion.

An alternate approach to free-particle eigenfcns is sometimes taken in a "box" which is already ∞ in size. We put...

$$\rightarrow \phi_k(\mathbf{r}) = (1/2\pi)^{3/2}\,e^{i\mathbf{k}\cdot\mathbf{r}}, \text{ in an } \infty \text{ box};$$

say $$\underline{\underline{\langle\phi_k|\phi_l\rangle}} = \int_\infty \phi_k^*(\mathbf{r})\,\phi_l(\mathbf{r})\,d^3r = \frac{1}{(2\pi)^3}\int_\infty e^{-i(\mathbf{k}-\mathbf{l})\cdot\mathbf{r}}\,d^3r$$

$$= \left[\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i(k_x-l_x)x}\,dx\right]\times\left[\begin{matrix}\text{similar}\\\text{integral}\\\text{in } y\end{matrix}\right]\times\left[\begin{matrix}\text{similar}\\\text{integral}\\\text{in } z\end{matrix}\right] = \underline{\underline{\delta(\mathbf{k}-\mathbf{l})}}. \quad (21)$$

In this case, we get a <u>Dirac</u> delta instead of the <u>Kronecker</u> $\delta_{\mathbf{k}\mathbf{l}}$ in Eq. (18). The procedure here is called "delta-fcn normalization". The completeness condition is:

$$\rightarrow \int_\infty \phi_k^*(\mathbf{r}')\,\phi_k(\mathbf{r})\,d^3k = (1/2\pi)^3\int_\infty e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}\,d^3k = \delta(\mathbf{r}-\mathbf{r}'). \quad (22)$$

The closure sum $\sum_k \phi_k^*(\mathbf{r}')\,\phi_k(\mathbf{r})$ on the LHS has become an integral, a priori.

For the interested student: stay in a box with the free-particle $\phi_k(x)$'s, and see what the "proof" in Eqs. (9a)-(9f) really does say.