Condition for Lorentz covariance of Dirac Eqtn.

Dirac Equation: Lorentz Covariance

For an acceptable relativistic theory of the electron, which is what the Dirac Eqtn purports to offer, it is essential that we demonstrate the Lorentz covariance of the theory (i.e. that the Dirac Eqtn exhibits the same form in all inertial frames which are connected by Lorentz transformations). We do that now, for free particles.

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1) The (free particle) Dirac Eqtn in some reference inertial frame K is written:

$$\left(\gamma_\mu \frac{\partial}{\partial x_\mu} + k\right)\psi(x) = 0 \,, \quad \text{w/ } k = mc/\hbar \quad \begin{cases} \{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}, \text{ for Dirac matrices,} \\ x \leftrightarrow x_\mu = (\mathbf{r}, ict), \text{ 4-position.} \end{cases} \tag{1}$$

Now, make a Lorentz transformation from K to a new inertial frame K':

$K \to K'$ via Lorentz transf^n $\Lambda = (\Lambda_{\mu\nu})$: $x_\mu \to x'_\mu = \Lambda_{\mu\nu} x_\nu$,

so/ $\partial/\partial x_\mu = (\partial x'_\nu / \partial x_\mu)\, \partial/\partial x'_\nu = \Lambda_{\nu\mu}\, \partial/\partial x'_\nu$

and/ $$\left(\Lambda_{\nu\mu}\gamma_\mu \frac{\partial}{\partial x'_\nu} + k\right)\psi(x') = 0 \,. \quad \longleftarrow \text{This is Eq.(1), directly transformed from } K \text{ to } K' \text{ cds.} \tag{2}$$

NOTE: the Dirac matrices γ_μ do not depend on $\mathbf{r}$ & t, and hence do not change under Λ. The γ_μ, once chosen, should be the same for observers K & K'.

Now... Lorentz covariance demands that the K' version of the Dirac Eqtn looks the same as the K version. That is, K' should write -- in his/her cds x'...

$$\left(\gamma_\nu \frac{\partial}{\partial x'_\nu} + k\right)\psi'(x') = 0 \,. \quad \longleftarrow \begin{array}{l}\text{This is covariant form of Dirac Eq. in } K'. \\ \text{The wavefcn } \psi \text{ in } K \text{ has become } \psi' \text{ in } K'.\end{array} \tag{3}$$

The covariance of the Dirac Eqtn is manifest if we can reconcile Eqs. (2) & (3), i.e. the version of the K' eqtn as written by K, and K''s own version of the eqtn. This reconciliation is readily possible if we can find a matrix S such that:

$$\boxed{\psi'(x') = S\,\psi(x') \,, \quad \text{and/} \quad S^{-1}\gamma_\nu S = \Lambda_{\nu\mu}\gamma_\mu} \quad \text{needed for Lorentz covariance of Dirac Eq.} \tag{4}$$

This follows by direct substitution into Eq.(3). S is independent of $\mathbf{r}$ & t.

Character of the covariance matrix $\mathcal{S}$ in Eq. (4).

REMARKS on the required covariance matrix $\mathcal{S}$ in Eq. (4).

1. The 4x4 matrices $\mathcal{S}$ & γ_ν operate on components of the 4-spinor ψ, while the 4x4 Lorentz transform matrix Λ operates on components of 4-vectors (e.g. $x=(\mathbf{r}, ict)$). $\mathcal{S}$ & γ_ν, and Λ, are defined in different spaces, so Λ commutes with $\mathcal{S}$ & γ_ν.

2. That the desired $\mathcal{S}$ exists is ensured by Pauli's Theorem [p. DE10, Eq. (33)]. Note:

... if $\gamma'_\alpha = \Lambda_{\alpha\mu}\gamma_\mu$, then since $\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}$ ...

$$\rightarrow \{\gamma'_\alpha, \gamma'_\beta\} = \Lambda_{\alpha\mu}\Lambda_{\beta\nu}\{\gamma_\mu, \gamma_\nu\} = 2\Lambda_{\alpha\mu}\Lambda_{\beta\mu} = 2\delta_{\alpha\beta} \quad \text{(5)}$$

using fact that Λ is orthogonal, i.e. $\Lambda\tilde{\Lambda}=1$.

So the γ'_α obey the same anticommutation rule as the γ_μ. Per Pauli, there does exist a nonsingular $\mathcal{S}$ (unique up to a phase) such that: $\gamma'_\nu = \Lambda_{\nu\mu}\gamma_\mu = \mathcal{S}^{-1}\gamma_\nu\mathcal{S}$.

3. Note that $\mathcal{S}$ is not unitary in general. From Eq. (4) as definition...

$$\gamma_\nu \mathcal{S} = \Lambda_{\nu\mu}\mathcal{S}\gamma_\mu \quad \longleftarrow \text{have commuted } \mathcal{S} \text{ \& } \Lambda\text{. Mult. on left by } \mathcal{S}^\dagger. \quad \text{(6)}$$

so $\mathcal{S}^\dagger\gamma_\nu\mathcal{S} = \Lambda_{\nu\mu}(\mathcal{S}^\dagger\mathcal{S})\gamma_\mu$ — take Herm. conjugate of both sides of Eq. (6). Mult on right by $\mathcal{S}$ (and use $\gamma_\nu^\dagger = \gamma_\nu$)...

and $\mathcal{S}^\dagger\gamma_\nu\mathcal{S} = \Lambda^*_{\nu\mu}\gamma_\mu(\mathcal{S}^\dagger\mathcal{S})$. If $\mathcal{S}$ were unitary, $\mathcal{S}^\dagger\mathcal{S}=1$, the last two eqtns would demand that $\Lambda^*_{\nu\mu} = \Lambda_{\nu\mu}$, i.e. that the Lorentz transform Λ be pure real. But this is true for us only for pure spatial rotations [p. DE3, Eq. (4)]. A velocity boost $\Rightarrow$ our Λ has imaginary components Λ_{k4} & Λ_{4k}. Hence the proposition that $\mathcal{S}$ is unitary in general fails.

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2) It turns out that $\mathcal{S}$ is "almost unitary". What this means can be discovered by looking at the adjoint Dirac Eqtn. The Dirac eqtn and its adjoint are related by:

$$\frac{\partial}{\partial x_\mu}(\gamma_\mu\psi) + k\psi = 0 \leftrightarrow \frac{\partial}{\partial x_\mu}(\bar{\psi}\gamma_\mu) - k\bar{\psi} = 0, \quad \text{w/ } \bar{\psi} = \psi^\dagger\gamma_4 \quad \text{(7)}$$

(adjoint spinor) — Eq (27) p. DE9

(Get the adjoint eqtn by taking the Hermitian conjugate of original eqtn, then multiply on the right by γ_4). The Lorentz-transformed adjoint eqtn (like (2) above) is:

Covariance of the adjoint Dirac Eqtn. 4-vectorhood for the current J_μ.

$$x_\mu \to x'_\mu = \Lambda_{\mu\nu} x_\nu \Rightarrow \frac{\partial}{\partial x'_\nu} \bar{\Psi}(x') \underbrace{\Lambda_{\nu\mu}\gamma_\mu}_{= S^{-1}\gamma_\nu S, \text{ by Eq.(4)}} - k\bar{\Psi}(x') = 0$$

... mult. on right by S^{-1}, so...

$$\frac{\partial}{\partial x'_\nu} \bar{\Psi}'(x')\gamma_\nu - k\bar{\Psi}'(x') = 0, \quad \text{if} \; \boxed{\bar{\Psi}'(x') = \bar{\Psi}(x') S^{-1}}. \qquad (8)$$

Eq.(8) is the required covariant form of the adjoint eqtn. Now, upon $x \to x' = \Lambda x$, have:

$$\left[\!\!\left[\begin{array}{l} \text{Dirac Eqtn is covariant if [Eq.(4)]: } \psi(x) \to \psi'(x') = S\psi(x') \\ \text{Adjoint Eq. " " " [Eq.(8)]: } \bar{\Psi}(x) \to \bar{\Psi}'(x') = \bar{\Psi}(x') S^{-1} \end{array}\right\} \text{w/ } \bar{\Psi} = \psi^\dagger\gamma_4. \qquad (9)$$

These two requirements are self-consistent if...

$$\bar{\Psi}'(x') = [\psi'(x')]^\dagger \gamma_4 = [S\psi(x')]^\dagger\gamma_4 = [\psi(x')]^\dagger S^\dagger \gamma_4$$

$$= [\psi(x')]^\dagger \gamma_4 S^{-1} \quad \underline{\text{equate}} \Rightarrow \boxed{S^\dagger\gamma_4 S = \gamma_4} \text{ w/ } \gamma_4 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \qquad (10)$$

We see that S is "almost unitary", to the extent that γ_4 is "almost unity". We will actually derive what S is below (see next page), but first... note in passing...

ASIDE Dirac probability current $J_\mu(x)$ as a 4-vector.

Recall [p. DE9, Eq.(29)]: $J_\mu(x) = ic\bar{\Psi}(x)\gamma_\mu\psi(x)$, w/ $\partial J_\mu(x)/\partial x_\mu = 0$, in frame K.
A Lorentz transform $K \to K'$ (i.e. $x_\mu \to x'_\mu = \Lambda_{\mu\nu} x_\nu$) transforms $J_\mu(x)$ to ...

$$\to J'_\mu(x') = ic\bar{\Psi}'(x')\gamma_\mu\psi'(x') = ic\bar{\Psi}(x') \underbrace{S^{-1}\gamma_\mu S}_{= \Lambda_{\mu\nu}\gamma_\nu \text{ by using Eq.(4)}} \psi(x') \quad \text{by using Eq.(9)}$$

i.e. $$J'_\mu(x') = ic\Lambda_{\mu\nu}\bar{\Psi}(x')\gamma_\nu\psi(x') = \Lambda_{\mu\nu}J_\nu(x'). \qquad (11)$$

So $J_\mu \to J'_\mu = \Lambda_{\mu\nu}J_\nu$ indeed transforms as an authentic 4-vector (x' in Eq.(11) just fixes a given spacetime point; it happens to be labelled in K' cds).

Now J_μ as a 4-vector $\Rightarrow$ the probability density, i.e. (note $\bar{\Psi} = \psi^\dagger\gamma_4$)...

$$\to \rho = J_4/ic = \bar{\Psi}\gamma_4\psi = \psi^\dagger\psi, \qquad (12)$$

must transform as the 4th (time-like) component of a 4-vector. So, for a Lorentz boost at velocity β along the x_1-axis, say, we'll have $\underline{\rho \to \rho' = \gamma\rho}$, w/ $\gamma = 1/\sqrt{1-\beta^2}$ the usual dilation factor. Then, we see that since the 3-volume element contracts by the same factor, i.e. $d^3x \to d^3x' = (dx_1/\gamma)\,dx_2 dx_3 = \frac{1}{\gamma} d^3x$, the total probability measured by ρ is a Lorentz invariant:

$$\rightarrow \int \rho\, d^3x \to \int \rho'\, d^3x' = \int (\gamma\rho)\cdot \frac{1}{\gamma} d^3x = \int \rho\, d^3x \quad \surd \text{ total probability is a Lorentz invariant.} \tag{13}$$

That is reassuring... we don't lose track of particles in Dirac theory.

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3) We shall now <u>derive the covariance matrix $\mathcal{S}$</u> for an ∞smal Lorentz transform:[¶]

$$\left[\begin{array}{l} \underline{\Lambda_{\mu\nu} = \delta_{\mu\nu} + \epsilon_{\mu\nu}} \quad \surd \text{ } \infty\text{smal Lorentz transform} \leftrightarrow O(\epsilon^2) \text{ negligible;} \\ \qquad\qquad\qquad\qquad (\epsilon_{\mu\nu}) \text{ is an \underline{antisymmetric} matrix}: \ \epsilon_{\nu\mu} = -\epsilon_{\mu\nu}; \\ \text{w/ } \Lambda_{\mu\alpha}\Lambda_{\mu\beta} = \delta_{\alpha\beta} = \Lambda_{\alpha\mu}\Lambda_{\beta\mu} \leftarrow \Lambda \text{ is an \underline{orthogonal} matrix.} \end{array}\right. \tag{14}$$

$\mathcal{S}$ is defined by the two covariance requirements:

$$\left[\begin{array}{ll} \mathcal{S}^{-1}\gamma_\mu \mathcal{S} = \Lambda_{\mu\nu}\gamma_\nu, & \text{for covariance of Dirac Eqtn [Eq.(4)],} \\ \mathcal{S}^\dagger \gamma_4 \mathcal{S} = \gamma_4, & \text{for covariance of adjoint Dirac Eq. [Eq.(10)].} \end{array}\right\} \tag{15}$$

Clearly, when $\epsilon_{\mu\nu} = 0$ and $\Lambda_{\mu\nu} = \delta_{\mu\nu}$, we must have $\mathcal{S} \equiv 1$. So we try...

$$\rightarrow \underline{\underline{\mathcal{S} = 1 + \epsilon_{\mu\nu} T_{\mu\nu}}} \quad \surd \ \underline{\mathcal{S}}, \underline{1} \ \& \ \underline{T_{\mu\nu}} \text{ are } 4\times4 \text{ matrices. } \epsilon_{\mu\nu} \text{ is just a number [element of } (\epsilon_{\mu\nu})]. \text{ Sum over } \mu,\nu = 1 \text{ to } 4. \tag{16}$$

Since $\epsilon_{\nu\mu} = -\epsilon_{\mu\nu}$ (and $\epsilon_{\mu\mu} \equiv 0$), there are only 6 independent terms in this sum. We could write it as: $\underline{\mathcal{S}} = \underline{1} + \sum_{\nu,\mu > \nu} \epsilon_{\mu\nu} \underline{T'_{\mu\nu}}$, w/ $\underline{T'_{\mu\nu}} = \underline{T_{\mu\nu}} - \underline{T_{\nu\mu}}$, a set of 6 indept 4×4 <u>antisymmetric</u> matrices. In any case, it is clear that...

$$\rightarrow \underline{\underline{\mathcal{S}^{-1} = 1 - \epsilon_{\alpha\beta} T_{\alpha\beta}}}, \ \text{so/ } \underline{\mathcal{S}}^{-1}\underline{\mathcal{S}} = \underline{\mathcal{S}}\underline{\mathcal{S}}^{-1} = \underline{1}, \text{ neglecting } O(\epsilon^2). \tag{17}$$

This form of $\mathcal{S}^{-1}$ is the <u>inverse</u> of $\mathcal{S}$ in Eq.(16), at least for ∞smal Lorentz Λ's.

¶ See problem 11.7, p. 564 of J.D. Jackson "Classical Electrodynamics" (Wiley, 1975).

Form of Covariance Matrix $\underline{S}$ for ∞smal Lorentz Transform.

With the S & S^{-1} forms in Eqs. (16) & (17), the first of the defining eqs. (15) is...

$$\underline{S}^{-1}\gamma_\mu\underline{S} = \Lambda_{\mu\nu}\underline{\gamma_\nu} \Rightarrow (\underline{1}-\epsilon_{\alpha\beta}\underline{T_{\alpha\beta}})\cdot\underline{\gamma_\mu}(1+\epsilon_{\rho\sigma}\underline{T_{\rho\sigma}}) = (\delta_{\mu\nu}+\epsilon_{\mu\nu})\underline{\gamma_\nu}$$

or// $\boxed{\epsilon_{\alpha\beta}[\underline{\gamma_\mu}, \underline{T_{\alpha\beta}}] = \epsilon_{\mu\nu}\underline{\gamma_\nu}}$, to 1st order in ϵ. (18)

This relation defines the 6 antisymmetric covariance matrices $\underline{T_{\alpha\beta}}$ ($\beta\neq\alpha$) that are needed [per Eq. (16)] for the ∞smal Lorentz transform in Eq. (14). Evidently the $\underline{T_{\alpha\beta}}$ are related to the $\underline{\gamma_\mu}$; in fact they must be of order $\underline{\gamma_\mu}^2$, so that we get terms linear in $\underline{\gamma_\mu}$ on both sides of Eq. (18). A reasonable guess is...

$$\underline{T_{\alpha\beta}} = \begin{cases} 0, \text{ when } \alpha=\beta; \\ K\underline{\gamma_\alpha}\underline{\gamma_\beta}, \alpha\neq\beta. \end{cases} \quad \text{with } \{\gamma_\alpha,\gamma_\beta\}=0, \text{ for } \alpha\neq\beta, \text{ this Ansatz} \Rightarrow 6 \text{ indept } \underline{T_{\alpha\beta}}, \text{ obeying: } \underline{T_{\beta\alpha}} = -\underline{T_{\alpha\beta}} \quad (19)$$

With this Ansatz, the commutator in (18) is...

$$\rightarrow \frac{1}{K}[\underline{\gamma_\mu}, \underline{T_{\alpha\beta}}] = (\underline{\gamma_\mu}\underline{\gamma_\alpha})\underline{\gamma_\beta} - \underline{\gamma_\alpha}(\underline{\gamma_\beta}\underline{\gamma_\mu})$$

$$= (2\delta_{\mu\alpha}\underline{1} - \underline{\gamma_\alpha}\underline{\gamma_\mu})\underline{\gamma_\beta} - \underline{\gamma_\alpha}(2\delta_{\beta\mu}\underline{1} - \underline{\gamma_\mu}\underline{\gamma_\beta})$$

cancel

$$= 2(\delta_{\mu\alpha}\underline{\gamma_\beta} - \delta_{\beta\mu}\underline{\gamma_\alpha}) \quad (20)$$

Use this result in Eq. (18) to obtain...

$$\epsilon_{\mu\nu}\underline{\gamma_\nu} = 2K\epsilon_{\alpha\beta}(\delta_{\mu\alpha}\underline{\gamma_\beta} - \delta_{\beta\mu}\underline{\gamma_\alpha}) = 2K(\epsilon_{\mu\beta}\underline{\gamma_\beta} - \underbrace{\epsilon_{\alpha\mu}}_{\text{set } = -\epsilon_{\mu\alpha}}\underline{\gamma_\alpha})$$

i.e.// $\epsilon_{\mu\nu}\underline{\gamma_\nu} = 4K\epsilon_{\mu\alpha}\underline{\gamma_\alpha}$ ← this eqtn is an identity if $\underline{\underline{K = 1/4}}$. (21)

Now, with $K=\frac{1}{4}$, and the Ansatz of Eq. (19), we have the desired T-matrices:

$$\rightarrow \underline{\underline{\underline{T_{\mu\nu}} = \frac{1}{4}\gamma_\mu\gamma_\nu, \text{ for } \nu\neq\mu;\ \underline{T_{\mu\mu}} \equiv 0.}}$$

So// $\boxed{\underline{S} = \underline{1} + \frac{1}{4}\epsilon_{\mu\nu}\underline{\gamma_\mu}\underline{\gamma_\nu}}$ is the desired covariance matrix (to $O(\epsilon)$) for the ∞smal Lorentz transform $\Lambda_{\mu\nu} = \delta_{\mu\nu}+\epsilon_{\mu\nu}$. (22)

The "unitarity" condition $S^\dagger\gamma_4 S = \gamma_4$ in Eq. (15) then restricts the $\epsilon_{\mu\nu}$. It is ~ easy to show that: $\epsilon_{k\ell}^* = \epsilon_{k\ell}$ (real), $\epsilon_{k4}^* = -\epsilon_{k4}$ (imag.), satisfied for proper Λ's.

Phase freedom in $\underset{\sim}{S}$. Rewrite of $\underset{\sim}{S}$ in terms of (spin) matrices $\underline{\sigma_{\mu\nu}}$.

REMARKS on the covariance matrix $\underset{\sim}{S}$ of Eq. (22).

1. $\underset{\sim}{S}$ in Eq. (22) is not unique, because the γ_μ are not unique [p. DE9, Eqs. (30)-(32)]. As well, $\underset{\sim}{S}$ may contain an arbitrary phase factor, as we see by the following.

Let: $\underset{\sim}{S} \rightarrow \underset{\sim}{S}' = (1+\epsilon_{\mu\nu}\lambda_{\mu\nu})\underset{\sim}{S}$, to $O(\epsilon)$ $\surd$ the $\epsilon_{\mu\nu}$ are numbers specifying the ∞smal LT, Eq. (14). The $\lambda_{\mu\nu}$ are numbers ~ 1. (23)

Then $\underset{\sim}{S}'$ obeys the first of Eqs. (15), i.e. $\underset{\sim}{S}'^{-1}\underline{\gamma_\mu}\,\underset{\sim}{S}' = \Lambda_{\mu\nu}\underline{\gamma_\nu}$ (neglect $O(\epsilon^2)$).

The second of the defining eqtns in Eq. (15) then relates $\underset{\sim}{S}'$ & $\underset{\sim}{S}$, as...

$$\underset{\sim}{S}'^{\dagger}\underline{\gamma_4}\,\underset{\sim}{S}' = \underline{\gamma_4} \Rightarrow [1+(\epsilon_{\mu\nu}\lambda_{\mu\nu})^*]\underbrace{\underset{\sim}{S}^{\dagger}\underline{\gamma_4}\,\underset{\sim}{S}}_{=\underline{\gamma_4},\text{ by 2nd of Eqs. (15)}}[1+(\epsilon_{\mu\nu}\lambda_{\mu\nu})] = \underline{\gamma_4}$$

so,

$$[1+(\epsilon_{\mu\nu}\lambda_{\mu\nu})+(\epsilon_{\mu\nu}\lambda_{\mu\nu})^*] = 1, \text{ to } O(\epsilon) \Rightarrow (\epsilon_{\mu\nu}\lambda_{\mu\nu}) \text{ is pure imag}^y = i\delta.$$

Then: $\underset{\sim}{S}' = (1+i\delta)\underset{\sim}{S} = (e^{i\delta})\underset{\sim}{S}$, to 1st order in $\delta = -i\,\epsilon_{\mu\nu}\lambda_{\mu\nu}$. (24)

This analysis shows that if $\underset{\sim}{S}$ obeys the defining eqtns in Eq. (15), so does $\underset{\sim}{S}e^{i\delta}$, with δ = arbitrary phase (to $O(\epsilon)$). This phase can be adjusted to give the same effective $\underset{\sim}{S}$ when and if the $\underline{\gamma_\mu}$'s are shifted. In what follows, we will not do this... we'll use $\delta=0$ and the standard representation of the $\underline{\gamma_\mu}$'s.

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2. It is convenient to rewrite $\underset{\sim}{S}$ of Eq. (22) in terms of a new matrix $\underline{\sigma_{\mu\nu}}$, i.e.:

$$\underline{\sigma_{\mu\nu}} = -\tfrac{1}{2}i(\underline{\gamma_\mu}\,\underline{\gamma_\nu} - \underline{\gamma_\nu}\,\underline{\gamma_\mu}) = \begin{cases} -i\,\underline{\gamma_\mu}\,\underline{\gamma_\nu}, & \text{for } \nu\neq\mu, \\ 0, & \text{for } \nu=\mu. \end{cases} \quad \text{antisymmetric: } \underline{\sigma_{\nu\mu}} = (-)\underline{\sigma_{\mu\nu}} \qquad (25)$$

All the nonzero $\underline{\sigma_{\mu\nu}}$'s exhibit the property:

$$\rightarrow \underline{\sigma_{\mu\nu}}^2 = -(\underline{\gamma_\mu}\,\underline{\gamma_\nu})(\underline{\gamma_\mu}\,\underline{\gamma_\nu}) = +(\underline{\gamma_\mu}\,\underline{\gamma_\mu})(\underline{\gamma_\nu}\,\underline{\gamma_\nu}) = \underline{1}\times\underline{1} = \underline{1}. \qquad (26)$$

In terms of the $\underline{\sigma_{\mu\nu}}$, we can write our Dirac covariance matrix of Eq. (22) as...

$$\boxed{\underset{\sim}{S} = \underline{1} + \tfrac{1}{4}i\,\epsilon_{\mu\nu}\,\underline{\sigma_{\mu\nu}}}, \text{ for ∞smal LT: } \Lambda_{\mu\nu} = \delta_{\mu\nu} + \epsilon_{\mu\nu}. \qquad (27)$$

The matrices $\underline{\sigma_{\mu\nu}}$ relate to how the Dirac particle's spin transform under LT's $\underline{\Lambda}$.

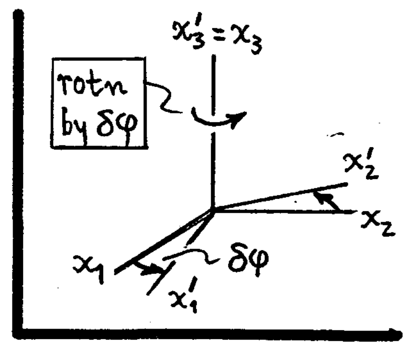
Explicit form of $\underline{S}$ for a pure spatial rotation.

4) We shall now construct the covariance matrices $\underline{S}$ explicitly for 2 specific LT's. Two things change during the LT: the cds $x \to x' = \Lambda x$, and the wavefcn $\psi \to \psi' = \underline{S}\psi$:

$$\boxed{\begin{array}{l}\text{under}: x_\mu \to x'_\mu = \Lambda_{\mu\nu} x_\nu, \text{ w/ } \Lambda_{\mu\nu} = \delta_{\mu\nu} + \epsilon_{\mu\nu} \text{ [to 1st order in } \epsilon], \text{ have:} \\ \psi(x) \to \psi'(x') = \underline{S}\psi(x'), \text{ w/ } \underline{S} = \underline{1} + \frac{i}{4}\epsilon_{\mu\nu}\underline{\sigma_{\mu\nu}}, \text{ the } \underline{\sigma_{\mu\nu}} \text{ per Eq. (25).}\end{array}} \quad (28)$$

We will look at explicit $\underline{S}$'s for: (A) a pure space rotation about the x_3-axis, (B) a pure Lorentz boost along the x_3-axis (both to $O(\epsilon)$). These exercises will show more clearly how the LT properties of the Dirac 4-spinors ψ differ from 4-vectors.

(A) Pure space rotation by ∡ $\delta\varphi$ about x_3-axis.



The ∞smal LT is in this case...

$$\to \underline{\Lambda} = \left[\begin{array}{cc|cc} 1 & +\delta\varphi & 0 & 0 \\ -\delta\varphi & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \quad \text{so, the } \epsilon\text{'s are:} \quad \underline{\underline{\epsilon_{12} = \delta\varphi = -\epsilon_{12}}}, \quad \text{all other } \epsilon_{\mu\nu} = 0. \quad (29)$$

The prescribed covariance matrix is...

$$\underline{S_3}(\delta\varphi) = \underline{1} + \tfrac{1}{4} i \left[(+\delta\varphi)\underline{\sigma_{12}} + (-\delta\varphi)\underbrace{\underline{\sigma_{21}}}_{\text{put this } = (-)\underline{\sigma_{12}}}\right] = \underline{1} + \tfrac{1}{2} i\,\delta\varphi\,\underline{\sigma_{12}}$$

or// $\underline{S_3}(\delta\varphi) = \underline{1}\exp\left(\frac{1}{2} i\,\delta\varphi\,\underline{\sigma_{12}}\right)$, to 1st order in $\delta\varphi$;

and// $\boxed{\underline{S_3}(\varphi) = \underline{1}\exp\left(\frac{1}{2} i\varphi\,\underline{\sigma_{12}}\right)}$, for a finite rotation φ about x_3-axis. ¶ (30)

$\underline{S_3}(\varphi)$ can be put in trig form as follows...

$$\underline{S_3}(\varphi) = \underline{1}\, e^{\frac{1}{2} i\varphi\underline{\sigma_{12}}} = \underline{1}\left\{\underline{1} + i\left(\frac{\varphi}{2}\right)\underline{\sigma_{12}} - \frac{1}{2!}\left(\frac{\varphi}{2}\right)^2 \overbrace{\underline{\sigma_{12}^2}}^{=\underline{1}} - \frac{i}{3!}\left(\frac{\varphi}{2}\right)^3 \overbrace{\underline{\sigma_{12}^3}}^{=\underline{\sigma_{12}}} + \ldots\right\}$$

or// $\boxed{\underline{S_3}(\varphi) = \underline{1}\cos(\varphi/2) + i\,\underline{\sigma_{12}}\sin(\varphi/2)}$. (31)

¶ Prove Eq. (30) by claiming rotations about the x_3-axis (alone) are associative, so...

$\underline{S}(\varphi+\delta\varphi) = \underline{S}(\varphi)\underline{S}(\delta\varphi) \Rightarrow \underline{S}(\varphi) + \delta\varphi(d\underline{S}/d\varphi) = \underline{S}(\varphi)[\underline{1} + \frac{1}{2} i\,\delta\varphi\,\underline{\sigma_{12}}]$,

or// $d\underline{S}/d\varphi = \frac{1}{2} i\,\underline{S}\,\underline{\sigma_{12}}$ ← a differential eqtn for $\underline{S} = \underline{S}(\varphi)$.

Solution is Eq. (30): $\underline{S}(\varphi) = \underline{S}(0)\exp\left(\frac{1}{2} i\varphi\,\underline{\sigma_{12}}\right)$, with choice $\underline{S}(0) = 1$ obvious.

Form & character of covariance matrix $\mathbb{S}$ for spatial rotations.

$\boxed{\text{REMARKS}}$ on: $\underline{\mathbb{S}_3}(\varphi) = \underline{1}\cos(\varphi/2) + i\,\underline{\sigma_{12}}\sin(\varphi/2)$, of Eq. (31).

1. An immediate surprise: upon a 360° spatial rotation, $\underline{\mathbb{S}_3}(360°) = (-)\,\underline{1}$ is not the identity. We need a rotation of $\underline{2}\times 360°$ to get back to where we started.

2. We calculate $\underline{\sigma_{12}}$ explicitly as follows...

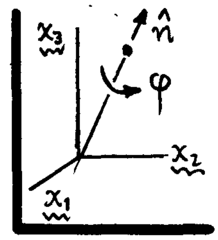
$$\rightarrow \underline{\sigma_{12}} = -i\,\underline{\gamma_1}\,\underline{\gamma_2} = -i\begin{pmatrix} 0 & -i\sigma_1 \\ +i\sigma_1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i\sigma_2 \\ +i\sigma_2 & 0 \end{pmatrix} = -i\begin{pmatrix} \sigma_1\sigma_2 & 0 \\ 0 & \sigma_1\sigma_2 \end{pmatrix} \quad \text{the } \sigma_k = \text{Pauli matrices} \tag{32}$$

... but: $\sigma_1\sigma_2 = i\sigma_3$, so: $\boxed{\underline{\sigma_{12}} = \begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}}$. Generally: $\sigma_{ij} = \epsilon_{ijk}\begin{pmatrix} \sigma_k & 0 \\ 0 & \sigma_k \end{pmatrix}$.

Then, for a rotation by $\measuredangle\varphi$ about the x_3-axis, the covariance matrix is...

$$\underline{\mathbb{S}_3}(\varphi) = \underline{1}\cos(\varphi/2) + i\,\underline{\Sigma_3}\sin(\varphi/2) = \exp[i\varphi(\underline{\Sigma_3}/2)], \quad \text{w/ } \underline{\Sigma_3} = \begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}. \tag{33}$$

3. An obvious generalization of Eq. (33) for rotation by $\measuredangle\varphi$ about an arbitrary axis specified by unit vector $\hat{n}$ is...



$$\underline{\mathbb{S}_n}(\varphi) = \exp[i\varphi(\underline{\hat{n}}\cdot\underline{\Sigma}/2)] = \underline{1}\cos(\varphi/2) + i(\hat{n}\cdot\underline{\Sigma})\sin(\varphi/2), \tag{34}$$

w/ $\underline{\Sigma} = \begin{pmatrix} \sigma & 0 \\ 0 & \sigma \end{pmatrix}$, $\hat{n} =$ unit vector along rotation axis, and: $(\hat{n}\cdot\underline{\Sigma})^2 = \underline{1}$.

This means that the Dirac wavefcn ψ will transform under space rotations as...

$$\psi = \begin{pmatrix} \phi \\ \chi \end{pmatrix} \xrightarrow[\text{about axis } \hat{n}]{\text{rotn by } \measuredangle\varphi} \psi' = \underline{\mathbb{S}_n}(\varphi)\,\psi = \begin{pmatrix} [\exp(i\varphi\hat{n}\cdot\sigma/2)]\,\phi \\ [\exp(i\varphi\hat{n}\cdot\sigma/2)]\,\chi \end{pmatrix}. \tag{35}$$

4. An important feature of Eq. (35): Under a space rotation by $\measuredangle\varphi$ about axis $\hat{n}$, any scalar wavefcn ψ transforms via an angular momentum operator $\mathbb{J}$ as‡:

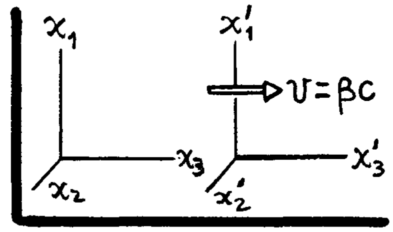
$$\psi \xrightarrow[\text{about } \hat{n}]{\measuredangle\varphi} \psi' = [\exp(i\varphi\hat{n}\cdot\mathbb{J}/\hbar)]\,\psi \quad \mathbb{J} = \measuredangle \text{ momentum operator, obeying: } \mathbb{J}\times\mathbb{J} = i\hbar\mathbb{J}. \tag{36}$$

All 4 of the scalar entries in Dirac's ψ in Eq. (35) undergo such a (unitary) transform under rotation by φ, if we identify $\underline{\mathbb{J} = (\hbar/2)\sigma}$. So we confirm:

$\boxed{\text{Particles described by the Dirac wave equation possess an \underline{intrinsic} angular momentum: } \mathbb{J} = (\hbar/2)\sigma. \text{ This is the particle's spin, w/ eigenvalues } \pm\hbar/2.}$ (37)

‡ Davydov, Sec. II. 18(c); Schiff "QM" (3rd ed), Sec. 27; etc.

Explicit form of covariance matrix $\underline{S}$ for pure Lorentz boost.



(B) Pure Lorentz boost along x_3-axis.

The ∞smal LT for this case is (β=∞smal, $\gamma = 1/\sqrt{1-\beta^2} \to 1$)...

$$\to \Lambda = \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & +\delta\theta \\ 0 & 0 & -\delta\theta & 1 \end{array}\right] \quad \sqrt{\text{the ∞smals are now:}} \quad \underline{\underline{\epsilon_{34} = \delta\theta = i\beta = -\epsilon_{43}}}, \qquad (38)$$

all other $\epsilon_{\mu\nu} = 0$

Compare with Eq.(29). We now have a "rotation" in the 34 rather than 12 plane. By analogy with the treatment in Eqs.(30) & (31), the covariance matrix is:

$$\underline{S_3}(\beta) = \underline{1}\exp\left(\tfrac{1}{2}i\theta\,\underline{\sigma_{34}}\right) = \underline{1}\cos(\theta/2) + i\,\underline{\sigma_{34}}\sin(\theta/2)$$

or// $$\boxed{\underline{S_3}(\beta) = \underline{1}\exp\left(-\tfrac{1}{2}\varphi\,\underline{\sigma_{34}}\right) = \underline{1}\cosh(\varphi/2) - \underline{\sigma_{34}}\sinh(\varphi/2)}, \quad \text{w/ } \varphi = i\theta, \text{ and: } \underline{\tanh\varphi = \beta}. \qquad (39)$$

For σ_{34}, we calculate: $\left[\!\left[\sigma_{34} = -i\gamma_3\gamma_4 = -\begin{pmatrix} 0 & -i\sigma_3 \\ i\sigma_3 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & \sigma_3 \\ \sigma_3 & 0 \end{pmatrix}\right]\!\right]$, and so:

$$\left.\begin{array}{l} \underline{\underline{\underline{S_3}(\beta) = \begin{pmatrix} 1\cosh(\varphi/2) & -\sigma_3\sinh(\varphi/2) \\ -\sigma_3\sinh(\varphi/2) & 1\cosh(\varphi/2) \end{pmatrix}}}, \quad \tanh\varphi = \beta; \\ \text{w/}\ \cosh\varphi = \gamma = E/mc^2 \Rightarrow \underline{\cosh(\varphi/2)} = \sqrt{\tfrac{1}{2}(\cosh\varphi + 1)} = \sqrt{(E+mc^2)/2mc^2}, \\ \text{and/}\ \underline{\sinh(\varphi/2)} = \sqrt{\cosh^2(\varphi/2) - 1} = \sqrt{(E-mc^2)/2mc^2}. \end{array}\right\} \qquad (40)$$

REMARKS on $\underline{S_3}(\beta)$.

1. Now the Dirac wavefcn transforms under a Lorentz boost along the x_3 axis as:

$$\left[\!\left[\begin{pmatrix} \phi \\ \chi \end{pmatrix} \xrightarrow[\text{along } x_3\text{-axis}]{\text{boost } \beta} \begin{pmatrix} \phi' \\ \chi' \end{pmatrix} = \begin{pmatrix} 1\cosh(\varphi/2) & -\sigma_3\sinh(\varphi/2) \\ -\sigma_3\sinh(\varphi/2) & 1\cosh(\varphi/2) \end{pmatrix}\begin{pmatrix} \phi \\ \chi \end{pmatrix}\right.. \qquad (41)$$

2. In contrast to the pure space rotation [Eq.(35)] (rot^n in 12 plane), the <u>spacetime</u> rotation [Eq.(41)] (rot^n in 34 plane) <u>mixes</u> components of the otherwise independent bispinors ϕ & χ. There is a distant analogy with 4-vectors here... a pure space rotation does <u>not</u> mix space & time components, while a spacetime rotation <u>does</u>. (This suggests that Dirac's ψ must have at least 4 components). But ψ by itself is <u>not</u> a 4-vector; it doesn't transform that way [Remark 1, p. DE37]. On the other hand, $\bar{\psi}\gamma_\mu\psi$ <u>is</u> a 4-vector, as we showed in Eq.(11).