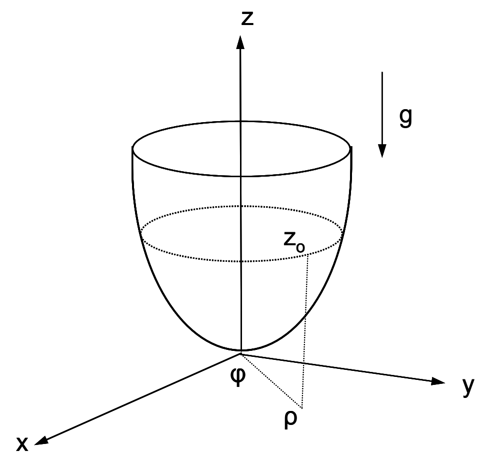
A particle of mass *m* moves without friction on the inside surface of an axially symmetric container described by $z = \frac{1}{2} b\,(x^2 + y^2)$, where *b* is a constant and *z* is the vertical direction, as shown in the figure on the right.



a. Initially the particle is moving in a circular orbit at height $z=z_o$. Find the energy and the angular momentum of the particle in terms of m, b, z_o and g, the gravitational constant.
b. While the particle is moving in the horizontal circular orbit it is poked downwards slightly. Obtain the frequency of oscillation about the unperturbed orbit for a very small oscillation amplitude.

Solution:

a. For a constant z and hence a constant ρ the total energy E and total angular momentum L are given by $E = \frac{1}{2}m\,v^2 + mgz_o = \frac{1}{2}m\,(\rho_o\dot{\varphi})^2 + mgz_o$ and $L = mv\rho = m\dot{\varphi}\rho^2$, where the only unknown is $\dot{\varphi}$. This can be found using Langrange equations in cylindrical coordinates. Setting up $x = \rho\, Cos\varphi$, $y = \rho\, Sin\varphi$ and z=z lets us write the Lagrangian of the particle:

$$L = T - V = \frac{1}{2}m\,(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - mgz$$

$$= \frac{1}{2}m\,((\dot{\rho}\, Cos\varphi - \dot{\varphi}\rho Sin\varphi)^2 + (\dot{\rho}\, Sin\varphi + \dot{\varphi}\rho Cos\varphi)^2 + \dot{z}^2) - mgz, \text{ where } z = \frac{1}{2}b\,\rho^2.$$

When z is eliminated in the last equation the Lagrangian reduces to $L = \frac{1}{2}m\,(\dot{\rho}^2 + \rho^2\dot{\varphi}^2 + b^2\rho^2\dot{\rho}^2) - \frac{1}{2}mgb\rho^2$. The Lagrange equations for each of ρ and φ can be determined from the generalized form $\frac{d}{dt}(\frac{\partial L}{\partial \dot{q}}) - (\frac{\partial L}{\partial q}) = 0$, where q is one of ρ and φ. The Lagrange equations for ρ and φ can then be found:

$\frac{d}{dt}(\frac{\partial L}{\partial \dot{\rho}}) - (\frac{\partial L}{\partial \rho}) = 0$ yields $\ddot{\rho}\,(1 + b^2\rho^2) + b^2\rho\dot{\rho}^2 - \rho\,(\dot{\varphi}^2 - gb) = 0,$ and

$\frac{d}{dt}(\frac{\partial L}{\partial \dot{\varphi}}) - (\frac{\partial L}{\partial \varphi}) = 0$ yields $\frac{d}{dt}(m\rho^2\dot{\varphi}) = \frac{dL}{dt} = 0$ (conservation of angular momentum).

For $\rho = \rho_o$= const., $\dot{\rho} = 0$ and $\ddot{\rho} = 0$, using the ρ-equation immediately yields $\dot{\varphi} = \sqrt{gb}$, from which we determine that

$E = \frac{1}{2}m\,(\rho_o\dot{\varphi})^2 + mgz_o = m\,g(\frac{1}{2}b\rho_o{}^2) + mgz_o = 2mgz_o$. Results can be summarized as: $E = 2mgz_o,$ and $L = 2mz_o\sqrt{\frac{g}{b}},$ where $\dot{\varphi} = \sqrt{bg},$ and $z_o = \frac{1}{2}b\rho_o{}^2$ are used.

b. Let us assume that for a small-amplitude oscillation ρ is expected to stay near its equilibrium value $\rho_o = \sqrt{\frac{2z_o}{b}}$. Set $\rho = \rho_o + r,$ where $r \ll \rho_o.$ Substituting this into the ρ-equation above we obtain:

$\ddot{r}\,(1 + b^2\,(\rho_o + r)^2) + b^2(\rho_o + r)\dot{r}^2 - (\rho_o + r)\,(\dot{\varphi}^2 - gb) = 0$. Now we need to eliminate $\dot{\varphi}$, which can be done by noting that the conservation of angular momentum asserts $m\rho^2\dot{\varphi} = m\rho_o{}^2\sqrt{gb},$ or $\dot{\varphi} = (\frac{\rho_o}{\rho})^2\sqrt{gb};$ or $\dot{\varphi}^2 = (1 - 4\frac{r}{\rho_o})gb$, for which we used the approximation $(1+x)^n \simeq 1 + nx$ if x $\ll 1$. Now using the approximation $\rho = \rho_o + r \simeq \rho_o$ where appropriate and ignoring the higher order $\dot{r}^2$ terms we obtain

$\ddot{r}\,(1+b^2\,\rho_o{}^2)+4gbr\simeq 0,$ which can be written as $\ddot{r}+\dfrac{4gb}{1+b^2\,\rho_o{}^2}r\simeq 0$, which is the equation for a simple harmonic oscillator with an angular frequency of $\omega=\sqrt{\dfrac{4gb}{1+b^2\,\rho_o{}^2}}=2\sqrt{\dfrac{gb}{1+2b\,z_o}},$ where $z_o=\dfrac{1}{2}b\rho_o{}^2$ is used.

An infinite one-dimensional square-well potential defined as

$$V(x) = 0, \quad 0 \le x \le a,$$
$$V(x) = \infty, \quad 0 > x > a$$

has well-defined normalized energy eigenfunctions given by:

$\varphi_n(x) = \sqrt{\dfrac{2}{a}} Sin(nk_1 x),$ where $k_1 = \dfrac{\pi}{a}$. Answer the following questions:

a. For the 3rd excited state (n=3) what is the probability $P_n(x)$ that a particle is located between the interval at $x = \dfrac{a}{3}$ and $x + dx$, where $dx = \dfrac{a}{1000}$?
b. Now consider the momentum space representation of the wave functions for the one-dimensional square-well potential described above and let us call these functions $\phi_n(p)$, where p is the linear momentum of the particle in the nth excited state. Determine $\phi_n(p)$, and explain briefly the physical meaning of $\phi_n(p)$.
c. Determine the probability of finding a particle in the 3rd excited state with a momentum between $p = 2\hbar k_1$ and $p + dp$, where $dp = \dfrac{\hbar k_1}{1000}$.

Solution:

a. In quantum mechanics $P_n(x) = |\varphi_n(x)|^2$ represents the probability density which, when multiplied by small increment *dx*, gives the probability of finding a particle, described by $\varphi_n(x)$, in the narrow interval between *x* and *x*+*dx*. Therefore, $P_3(\frac{a}{3}) = \left|\sqrt{\frac{2}{a}}Sin(\cancel{3}\frac{\pi}{\cancel{a}}\frac{\cancel{a}}{\cancel{3}})\right|^2 = 0$, which means that the probability of finding a particle in the 3rd excited state in the vicinity of $x = \frac{a}{3}$ is zero.

b. To determine the normalized momentum space representation, $\phi_n(p)$, of the eigenfunctions we have to find the Fourier transform of the real space wave functions, $\varphi_n(x)$. This is simply done by $\phi_n(p) = (\frac{1}{2\pi\hbar})^{1/2}\int_{-\infty}^{+\infty}\varphi_n(x)e^{(\frac{ipx}{\hbar})}dx$.

Because $\varphi_n(x)$ is zero everywhere except $0 \le x \le a$, the above is reduced to $\phi_n(p) = (\frac{1}{2\pi\hbar})^{1/2}\int_0^a\varphi_n(x)e^{(\frac{ipx}{\hbar})}dx = (\frac{1}{a\pi\hbar})^{1/2}\int_0^a Sin(nk_1x)\, e^{\frac{ipx}{\hbar}}dx$. This integration is readily performed by noting that $Sin(nk_1x) = \frac{e^{ink_1x} - e^{-ink_1x}}{2i}$, thus

$$\phi_n(p) = (\frac{1}{a\pi\hbar})^{1/2}\frac{1}{2i}\int_0^a e^{i(\frac{p}{\hbar}+nk_1)x} - e^{i(\frac{p}{\hbar}-nk_1)x}dx = (\frac{1}{a\pi\hbar})^{1/2}\frac{1}{2i}\left[\left(\frac{e^{i(\frac{p}{\hbar}+nk_1)a}-1}{i(\frac{p}{\hbar}+nk_1)}\right) - \left(\frac{e^{i(\frac{p}{\hbar}-nk_1)a}-1}{i(\frac{p}{\hbar}-nk_1)}\right)\right]$$

Noting that $e^{\pm ink_1a} = e^{\pm in\pi} = (-1)^n$, $\phi_n(p)$ can easily be reduced to $\phi_n(p) = (\frac{\pi a}{\hbar})^{1/2}\frac{n(1-(-1)^n e^{\frac{ipa}{\hbar}})}{(n\pi)^2 - (\frac{pa}{\hbar})^2}$. The physical meaning of $\phi_n(p)$ can be related to probability density $Q_n(p) = |\phi_n(p)|^2$ in momentum space, which can be related to the probability of finding a particle in the nth energy eigenstate with a momentum between p and p+dp, which is given by $Q_n(p)\, dp$.

c. For *n=3* and $p = 2\hbar k_1$ we can determine the probability density easily: $Q_n(p) = |\phi_n(p)|^2 = \left|(\frac{\pi a}{\hbar})^{1/2}\frac{n(1-(-1)^n e^{\frac{ipa}{\hbar}})}{(n\pi)^2-(\frac{pa}{\hbar})^2}\right|^2 = (\frac{6}{5})^2\frac{1}{\pi^2}(\frac{a}{\pi\hbar})$; therefore, the probability of finding a particle in the 3rd excited state with a momentum between $p = 2\hbar k_1$ and $p + dp$, where $dp = \dfrac{\hbar k_1}{1000}$, is given by

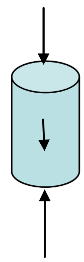
$$Q_n(p)\,dp = (\frac{6}{5})^2 \frac{1}{\pi^2} (\frac{\cancel{a}}{\cancel{\pi}\cancel{\hbar}}) \frac{\cancel{\hbar}\cancel{\pi}}{1000\cancel{a}} = 1.46 \times 10^{-4}.$$

This problem investigates the variation of temperature and pressure as a function of altitude in Earth's atmosphere, which is assumed to be an ideal gas with an average molecular weight of $M=29$ g/mo. It is further assumed that the gravitational acceleration of g=9.8 m/s^2 does not vary with altitude for the heights of interest in this problem. Answer the following questions:

a. Show that differential pressure and height are related to each other by $\frac{dp}{p} = -\frac{Mg}{RT}dz$, where z is the altitude measured from sea level, $p = p(z)$ is the pressure at altitude z, and $R=8.31$ J/mol K.
b. Suppose that the pressure decrease in atmosphere is due to the adiabatic expansion of an ideal gas governed by $pV^{\gamma} = \text{const.}$, and assume $\gamma \simeq 1.4$, determine that $\frac{dT}{dz} = (\frac{1}{\gamma} - 1)\frac{Mg}{R}$.
c. Determine the relative pressure $p(z)/p_o$ and the temperature at a typical cruising altitude of $z=35,000$ ft, assuming that the pressure at sea level is one atmosphere and the temperature is about 27^o C.

Solution:

a. The mechanical equilibrium of a cylindrical column of air (of unit area) at altitude z, as shown in the figure on the right, can be represented as $p(z) = \rho g\, dz + p(z+dz)$, where the force per unit area by definition is the pressure, and ρ is the density of the air in the column and is defined by $\rho = \frac{nM}{V}$. Using the ideal gas law, $pV = nRT$, immediately yields



$\rho = \frac{p}{RT} M$. Using the definition $dp = p(z+dz) - p(z) = -\rho g dz$ and inserting ρ from the above equation into the last equation immediately yields the desired result: $\frac{dp}{p} = -\frac{Mg}{RT} dz$.

b. Eliminating V from the ideal gas law $pV = nRT$ using the adiabatic relation $pV^{\gamma} = \text{const.}$ immediately yields $pT^{\frac{\gamma}{1-\gamma}} = \text{ constant}$. Differentiating the latter relation yields $dp\, T^{\frac{\gamma}{1-\gamma}} + \frac{\gamma}{1-\gamma} T^{(\frac{\gamma}{1-\gamma}-1)} p\, dT = 0$. This last result can be reduced to $\frac{dT}{T} = \frac{\gamma - 1}{\gamma} \frac{dp}{p}$ Combining the result of (a) above with this last relation yields the desired relation: $\frac{dT}{dz} = (\frac{1}{\gamma} - 1)\frac{Mg}{R}$.

c. Solution of the simple differential equations in (b) yields the variation of temperature with altitude: $T(z) = T_o + (\frac{1}{\gamma} - 1)\frac{Mg}{R} z$. By inserting T(z) into the equation in (a) we reduce it to $\frac{dp}{p} = -\frac{Mg}{RT_o + (\frac{1-\gamma}{\gamma})Mg\, z)} dz$. Changing the variable to $\eta = RT_o + (\frac{1-\gamma}{\gamma})Mg\, z$, and using $dz = \frac{d\eta}{(\frac{1}{\gamma} - 1)\, Mg}$ the last differential equation can be reduced to $\frac{dp}{p} = -\frac{1}{(\frac{1}{\gamma} - 1)} \frac{d\eta}{\eta}$. The solution of this simple equation is $\ln \frac{p(z)}{p_o} = -\frac{1}{(1 - \frac{1}{\gamma})} \ln \frac{\eta}{\eta_o}$, where $\frac{\eta(z)}{\eta_o} = 1 + (\frac{1}{\gamma} - 1)\frac{Mgz}{RT_o}$. We can now determine the numerical values for $T(z)$ and $p(z)$ by inserting $T_o=300$ K, $M=0.029$ kg/mol, $R=8.31$ J/K Mol, $g=9.8$ m/s^2 and $z=35{,}000$ ft * 0.3048 m/ft=$10{,}668$ m.

We obtain $T = 300 + (\frac{1}{1.4} - 1)\frac{0.029 \times 9.8 \times 35{,}000 \times 0.3048}{8.31} \simeq 196K = -77^oC$;

similarly $\ln\frac{p}{p_o} = \frac{1}{(1-\frac{1}{1.4})}\ln(1 + (\frac{1}{1.4} - 1)\frac{0.029 \times 9.8 \times 10{,}668}{8.31 \times 300}) = -1.49$, or

$p = 0.22\,p_o = 0.22$ atm.

Our universe is filled with black body radiation at a temperature of T= 3 K. This radiation is thought to be a relic from the "big bang" now filling the continuously expanding and cooling universe. Answer the following questions:

a. Express the photon number density analytically in terms of T, universal constants and numerical cofactors.
b. Now determine n numerically in terms of photons/cm^3.

(Hint: The Bose-Einstein distribution for photons is given by $\frac{1}{e^{\beta\hbar\omega}-1}$, the integral $\int_0^\infty \frac{x^2dx}{e^x-1} \simeq 2.4$, and $d^3\boldsymbol{n} = \frac{V}{(2\pi)^3}\, d^3\boldsymbol{k}$)

Solution:

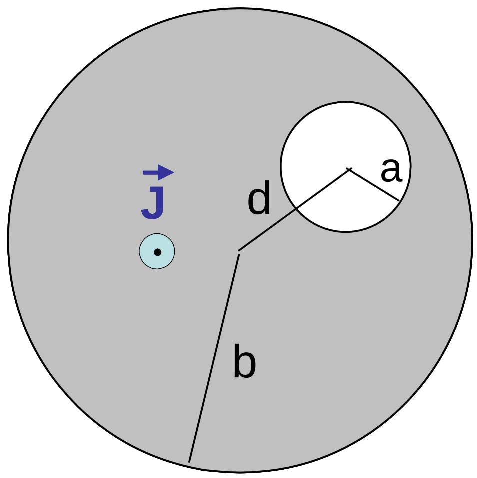
a. The Bose-Einstein distribution for photons is given by $< n >= \dfrac{1}{e^{\beta\hbar\omega} - 1}$, where $\beta = 1/kT$. This is better known as the *Planck distribution function* and <n> simply gives the average number of photons per *mode* $\hbar\omega$ in volume V. The total number of photons in the volume can be found by integrating over all modes. $N = 2\int < n > d^3\boldsymbol{n}$, where $d^3\boldsymbol{n}$ is the number of modes within a small volume $d^3\boldsymbol{k}$ in $\boldsymbol{k}$-space for a given polarization and is given by $d^3\boldsymbol{n} = \dfrac{V}{(2\pi)^3}\, d^3\boldsymbol{k}$, where V is the volume of the universe. The factor 2 in front of the integral is due to there being two polarizations per $\hbar\omega$. Using the relation $\omega = ck$ and converting to spherical coordinates in $\boldsymbol{k}$-space we immediately find $d^3\boldsymbol{k} = 4\pi k^2 dk = 4\pi \dfrac{\omega^2}{c^2}\dfrac{d\omega}{c} = \dfrac{4\pi\omega^2}{c^3} d\omega$. Setting $x = \beta\hbar\omega$ and arranging the terms in the integral we obtain $N = 2\dfrac{V}{\pi^2}\left(\dfrac{k_B}{\hbar c}\right)^2 I\, T^3$, where $I = \int_0^\infty \dfrac{x^2 dx}{e^x - 1} \simeq 2.4$, and $x = \beta\hbar\omega$. The number density, n, can be obtained from the last relations:

$$n = \frac{N}{V} \simeq \frac{1}{\pi^2}\left(\frac{k_B}{\hbar c}\right)^3 I\, T^3 = 0.24\left(\frac{k_B}{\hbar c}\right)^3 T^3$$

b. Setting $k_B = 1.38 \times 10^{-23}$ J/K, $\hbar = 1.05 \times 10^{-34}$ J.s, $c = 3.0 \times 10^8$ m/s, and $T = 3$ K in the last equation for n, we obtain

$n \simeq 1.84 \times 10^8$ photons/m^3 $\simeq 200$ photons/cm^3.

An off-centered hole of radius *a* is bored parallel to the axis of a long metallic cylinder of radius *b (b>a)*. With the exception of the bored hole the cylinder is assumed to be full. The two axes are at a distance *d* apart as shown in the figure below. A uniform current *I* with a current density ***J*** flows in the cylinder out of the plane of the paper and perpendicular to the paper as shown in figure. What are the magnitude and direction of the magnetic field ***B*** at the center of the hole?



Solution:

The direction of magnetic field ***B*** is easily determined using the right-handed screw rule; the direction is shown in the figure below. The magnitude of ***B*** can easily be determined using the principle of superposition, which is implied by the linearity of Maxwell's equations, in particular Ampere's Law $\oint \vec{B}.d\vec{s} = \mu_o I = \mu_o \iint \vec{J}.d\vec{S}$. Let us assume that current *I* has two components, I_1 and I_2, where I_1 flows through a solid cylinder of radius *b* while I_2 flows in the opposite direction through a solid cylinder of radius *a*, located inside the bore hole. The superposition of the two currents must be equal to the current flowing through the cylinder with the bore hole, $I = I_1 + I_2$, where $I_1 = \iint \vec{J}_1.d\vec{S} = \pi b^2 J_1$ and $I_2 = \iint -\vec{J}_2.d\vec{S} = -\pi a^2 J_2$, where $I = \pi(b^2 J_1 - a^2 J_2)$. Furthermore, in order to produce zero current in the bore-hole region J_1 must be equal to J_2, which must be equal to $J = J_1 = J_2 = \dfrac{I}{\pi(b^2 - a^2)}$. Now applying the principle of superposition we assert that the magnetic field at the center of the bore hole has contributions from $\boldsymbol{J_1}$ and from $\boldsymbol{J_2}$. Let us call these fields $\boldsymbol{B_1}$ and $\boldsymbol{B_2}$, respectively: the net magnetic field at the center is then given by $\boldsymbol{B} = \boldsymbol{B_1} + \boldsymbol{B_2}$.

It is clear that $\boldsymbol{B_2} = \mathbf{0}$ because the current through the arbitrarily small area ($I' = J\pi r^2 = \dfrac{I\, r^2}{(b^2 - a^2)} \to 0$ as $r \to 0$) can be made zero, which produces zero $\boldsymbol{B_2}$.

$\boldsymbol{B_1}$ can easily be calculated from

$\oint \vec{B}_2.d\vec{s} = \mu_o \iint \vec{J}.d\vec{S}$, which yields $2\pi d B_2 = \dfrac{\mu_o \pi d^2 I}{\pi(b^2 - a^2)} \to B_2 = \dfrac{\mu_o I\, d}{2\pi(b^2 - a^2)}$.

Therefore, $B = B_1 + B_2 = B_2 = \dfrac{\mu_o I\, d}{2\pi(b^2 - a^2)}$.