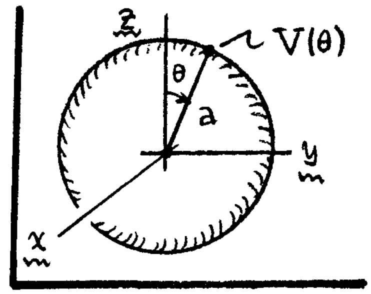
3) What we've done at this point is to construct a general solution to $\nabla^2\phi = 0$ in the case of "azimuthal symmetry" (i.e. no φ-dependence, $m=0$). It is...

$$\boxed{\phi(r,\theta) = \sum_{l=0}^{\infty}\left[A_l r^l + B_l r^{-(l+1)}\right] P_l(\cos\theta)}. \quad (8)$$



Suppose we have a simple problem with both spherical & azimuthal symmetry: we want ϕ everywhere when $\phi(a,\theta) = V(\theta)$ is specified on a sphere of radius a as shown.

<u>inside sphere</u> $(r \leq a)$ } $B_l = 0$, and: $V(\theta) = \sum_{l=0}^{\infty}[A_l a^l] P_l(\cos\theta)$;

so// $A_l a^l = \left(\frac{2l+1}{2}\right)\int_0^{\pi} V(\psi) P_l(\cos\psi)\sin\psi\, d\psi$, by orthogonality of the P_l;

and// $\longrightarrow \phi(r \leq a, \theta) = \sum_{l=0}^{\infty} v_l \left(\frac{r}{a}\right)^l P_l(\cos\theta)$, w// $\underline{v_l = \left(\frac{2l+1}{2}\right)\int_0^{\pi} V(\psi) P_l(\cos\psi)\sin\psi\, d\psi}$. (9A)

<u>outside sphere</u> $(r \geq a)$ } $A_l = 0$, and: $V(\theta) = \sum_{l=0}^{\infty}[B_l a^{-(l+1)}] P_l(\cos\theta)$;

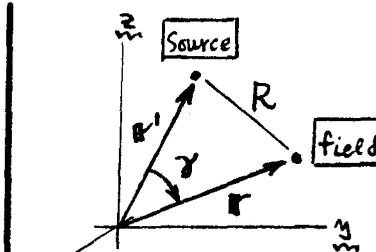
so// $B_l a^{-(l+1)} = v_l$, and// $\longrightarrow \phi(r \geq a, \theta) = \sum_{l=0}^{\infty} v_l \left(\frac{a}{r}\right)^{l+1} P_l(\cos\theta)$. (9B)

The expressions for ϕ & Eqs. (9A) & (9B) give the general solution for all problems [$V(\theta)$ on $r = a$] of this sort. Jackson shows how this works specifically for two hemispheres with $V(\theta) = \begin{cases} +V, & 0 \leq \theta < \frac{\pi}{2}, \\ -V, & \frac{\pi}{2} < \theta \leq \frac{\pi}{2}, \end{cases}$ in his Eq. (3.36).

On <u>axis of (rotational) symmetry</u>, $\theta = 0$, and: $\phi(r,0) = \sum_{l=0}\left[A_l r^l + B_l r^{-(l+1)}\right]$, by Eq. (8). If $\phi(r,0)$ can be found and expanded in powers of r, then the A_l & B_l can be determined. The solution off-axis, $\phi(r,\theta)$, can then be written down just by multiplying each $[\;]_l$ by $P_l(\cos\theta)$. See Jk[n] pp. 91-92.

4) Jackson exploits the trick noted at bottom of last page to expand the point source potential in spherical cds as...

$$\left[\begin{aligned} \frac{1}{|\mathbf{r}-\mathbf{r}'|} &= \sum_{l=0}^{\infty} \frac{1}{r}\left(\frac{r'}{r}\right)^l P_l(\cos\gamma), \text{ when } r > r'; \\ &= \sum_{l=0}^{\infty} \frac{1}{r'}\left(\frac{r}{r'}\right)^l P_l(\cos\gamma), \text{ when } r' > r. \quad (10) \end{aligned}\right.$$



$R = |\mathbf{r}-\mathbf{r}'|$

$\gamma = \measuredangle(\mathbf{r}, \mathbf{r}')$

This expansion will be useful later, when we deal with integrals like: $\phi(\mathbf{r}) = \frac{1}{4\pi}\int_{\text{Sources}} \frac{1}{R}\rho(\mathbf{r}')\,d^3x'$ {Helmholtz' Theorem}. We see that if $r \gg r'$, far outside a source, we may need only a few terms in the series for $1/R$.

In one final application of the axis-of-symmetry trick at bottom of last page, Jackson finds the potential $\phi(r,\theta)$ everywhere for a charged circular ring with axis $\equiv$ z-axis. See Jk^{n} p. 93 & Fig. 3.4. Neat trick!