ϕ507 Solutions

(86) [15 pts]. Find properties of Dirac matrices, defined by $\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}$.

1. For $\nu = \mu$, the anticommutation rule yields: $\gamma_\mu \gamma_\mu = 1$, the unit matrix, so each γ_μ is its own inverse: $\gamma_\mu^{-1} = \gamma_\mu$. If also each γ_μ is Hermitian, i.e. $\gamma_\mu^\dagger = \gamma_\mu$, then $\gamma_\mu^{-1} = \gamma_\mu^\dagger$, so that the γ_μ's must be unitary matrices. But, the eigenvalues of a unitary matrix are ± 1 (well-known), so the γ_μ eigenvalues are ± 1.

Alternatively, let ϕ be an eigenvector of γ_μ, i.e. $\gamma_\mu \phi = \Gamma \phi$, w/ Γ = eigenvalue. Then, multiply through this eqtn by γ_μ and use $\gamma_\mu^2 = 1$ to get...

$$\rightarrow \gamma_\mu^2 \phi = \Gamma \gamma_\mu \phi \Rightarrow 1\phi = \Gamma^2 \phi, \text{ so// } \Gamma^2 = 1 \text{ \&// } \underline{\Gamma = \pm 1}. \tag{1}$$

We remark (from the general theory of matrices) that each γ_μ can be brought to its canonical form via a similarity transform S, i.e. $\gamma_\mu \rightarrow \gamma_\mu' = S\gamma_\mu S^{-1}$. In the canonical form, the eigenvalues appear on the diagonal, i.e. $\gamma_\mu' = \begin{pmatrix} +1 & & 0 \\ & -1 & \\ 0 & & +1 \\ & & & \ddots \end{pmatrix}$.

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2. Now look at the trace of γ_μ, i.e.: $Tr(\gamma_\mu) = \Sigma$ (diagonal entries). Choose any of the other matrices γ_ν, with $\nu \neq \mu$. Then, since $\gamma_\nu^2 = 1$, and matrix multiplication is associative, we can write...

$$\rightarrow Tr(\gamma_\mu) = Tr(\gamma_\mu \gamma_\nu^2) = Tr[(\gamma_\mu \gamma_\nu)\gamma_\nu]. \tag{2}$$

Now use the fact that the Tr is not sensitive to multiplicative order, viz. $Tr(BA) = Tr(AB)$ for any square matrices A & B.* So Eq. (2) yields...

$$\rightarrow Tr(\gamma_\mu) = Tr[(\gamma_\mu \gamma_\nu)\gamma_\nu] = Tr[(\gamma_\nu \gamma_\mu)\gamma_\nu]. \tag{3}$$

But for $\nu \neq \mu$, we know that: $\gamma_\nu \gamma_\mu = (-)\gamma_\mu \gamma_\nu$. Put this into Eq. (3) to get

$$\rightarrow Tr(\gamma_\mu) = (-)Tr[(\gamma_\mu \gamma_\nu)\gamma_\nu] = (-)Tr(\gamma_\mu \gamma_\nu^2) = (-)Tr(\gamma_\mu). \tag{4}$$

Eq. (4) implies that $2Tr(\gamma_\mu) = 0$, or that: $\boxed{Tr(\gamma_\mu) \equiv 0}$, as advertised.

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* $Tr(AB) = \sum_\kappa \left(\sum_\lambda A_{\kappa\lambda} B_{\lambda\kappa}\right)$; $Tr(BA) = \sum_\alpha \left(\sum_\beta B_{\alpha\beta} A_{\beta\alpha}\right) = Tr(AB)$ $\int$ upon setting $\alpha = \lambda$, $\beta = \kappa$.

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3. The facts that γ_μ has eigenvalues ± 1, and $\mathrm{Tr}(\gamma_\mu) = 0$, allow us to claim...

(1) The canonical form of γ_μ, viz. $\gamma'_\mu = S\gamma_\mu S^{-1} = \begin{pmatrix} +1 & & 0 \\ & -1 & \\ 0 & & +1 \ldots \end{pmatrix}$ must have just as many -1's as $+1$'s along the diagonal; the ± 1's "pair off." This follows from the fact that: $\mathrm{Tr}(\gamma'_\mu) = \mathrm{Tr}(S\gamma_\mu S^{-1}) = \mathrm{Tr}(S^{-1}S\gamma_\mu) = \mathrm{Tr}(\gamma_\mu) = 0$, i.e. the Tr is invariant under the similarity transform that produces the diagonal form.

(2) Since the ± 1 eigenvalues "pair off" in γ'_μ, then the γ_μ's must be matrices of <u>even</u> rank. The rank of the γ_μ's could be 2, 4, 6...

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4. If the γ_μ rank were 2, then the γ_μ would have to be related to the Pauli matrices σ_k, $k=1,2,3$; these are the only 2nd rank matrices that obey the required anticommutation rule: $\{\sigma_k, \sigma_l\} = 2\delta_{kl}$. But there are <u>four</u> independent γ_μ's, while there are only <u>three</u> independent σ_k's. Adding the 2×2 identity matrix to the σ_k's doesn't help, because the identity does not anticommute with the other three σ_k. So, the four required γ_μ's <u>cannot</u> be of rank 2.

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5. The next possible rank of the γ_μ's is 4, and this turns out to be OK. There <u>are</u> four γ_μ's of rank 4 which obey $\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}$, which have eigenvalues ± 1, are traceless, and are Hermitian. They can be written as...*

$$\rightarrow \gamma_k = \begin{pmatrix} 0 & -i\sigma_k \\ i\sigma_k & 0 \end{pmatrix},\ k=1,2,3;\quad \gamma_4 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}; \qquad (5)$$

w/ $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ } Pauli matrices; $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ } identity.

This representation is not unique... the γ_μ's of Eq. (5) can be replaced by the set $\gamma'_\mu = S\gamma_\mu S^{-1}$, where S is any nonsingular matrix.

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* J. J. Sakurai "Advanced QM" (Addison-Wesley, 1967), Appendix B, p. 305.

(87) Continuity Eqtn for a Dirac particle in an external field $\tilde{A} = (\mathbf{A}, i\phi)$.

1. In the field $(A_\mu) = (A_k, i\phi)$, the Dirac eqtn for particle (q, m) can be written:

$$\rightarrow \left(\frac{\partial}{\partial x_k} - \frac{iq}{\hbar c} A_k\right)\gamma_k \psi + \left(\frac{\partial}{\partial x_4} + \frac{q\phi}{\hbar c}\right)\gamma_4 \psi + (mc/\hbar)\psi = 0. \quad [k=1,2,3]. \qquad (1)$$

Take the Hermitian conjugate of this eqtn, noting $(\partial/\partial x_4)^* = -(\partial/\partial x_4)$, to get...

$$\rightarrow \left(\frac{\partial}{\partial x_k} + \frac{iq}{\hbar c} A_k\right)\psi^\dagger \gamma_k - \left(\frac{\partial}{\partial x_4} - \frac{q\phi}{\hbar c}\right)\psi^\dagger \gamma_4 + (mc/\hbar)\psi^\dagger = 0. \qquad (2)$$

Multiply thru Eq. (2) by γ_4 on the right. Use $\gamma_k\gamma_4 = (-)\gamma_4\gamma_k$ (from $\{\gamma_\mu, \gamma_\nu\} = 0$ for $\mu \neq \nu$), and denote the adjoint $\bar{\psi} = \psi^\dagger\gamma_4$. Then, with $A_4 = i\phi$, Eq (2) yields:

$$-\left(\frac{\partial}{\partial x_k} + \frac{iq}{\hbar c}\right)\bar{\psi}\gamma_k - \left(\frac{\partial}{\partial x_4} + \frac{iq}{\hbar c}A_4\right)\bar{\psi}\gamma_4 + (mc/\hbar)\bar{\psi} = 0,$$

or// $$\underline{\underline{\left(\frac{\partial}{\partial x_\mu} + \frac{iq}{\hbar c}A_\mu\right)\bar{\psi}\gamma_\mu - (mc/\hbar)\bar{\psi} = 0.}} \qquad (3)$$

As advertised, (3) is Dirac's eqtn for the adjoint wavefcn $\bar{\psi} = \psi^\dagger\gamma_4$.

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2. For a continuity eqtn, multiply (1) on the left by $\bar{\psi}$, and (3) on the right by ψ...

$$\begin{cases} \bar{\psi}\gamma_\mu(\partial\psi/\partial x_\mu) - \frac{iq}{\hbar c}A_\mu\bar{\psi}\gamma_\mu\psi + (mc/\hbar)\bar{\psi}\psi = 0, \\ (\partial\bar{\psi}/\partial x_\mu)\gamma_\mu\psi + \frac{iq}{\hbar c}A_\mu\bar{\psi}\gamma_\mu\psi - (mc/\hbar)\bar{\psi}\psi = 0, \end{cases} \qquad (4)$$

... and add these eqtns to get a simple conserved current eqtn...

$$\bar{\psi}\gamma_\mu(\partial\psi/\partial x_\mu) + (\partial\bar{\psi}/\partial x_\mu)\gamma_\mu\psi = 0,$$

or// $$\frac{\partial}{\partial x_\mu}(\bar{\psi}\gamma_\mu\psi) = 0, \text{ i.e.// } \boxed{\partial J_\mu/\partial x_\mu = 0, \text{ w/ } J_\mu = ic\bar{\psi}\gamma_\mu\psi}. \qquad (5)$$

Here, in an external field (A_μ), the Dirac current J_μ is exactly the same form as we used for a free particle... see CLASS NOTES, p. DE9, Eq. (29). The field (A_μ) does not appear explicitly in J_μ. But, of course, $A_\mu \neq 0$ does change J_μ, insofar as it changes ψ. See also Sakurai "Advanced QM" (Addison-Wesley, 1967), p. 107, Eq. (3.199).

(88) Use Ham$^{\underline{n}}$ form of Dirac Eq$^{\underline{tn}}$ in an ext$^{\underline{l}}$ field to show $\mathbf{p} \to (-)\mathbf{p}$ when $q \to (-)q$.

$\underline{1.}$ With $(A_\mu) = (A_k, i\phi)$, the Dirac standard form in Prob. (88) is:

$$\to \left(\frac{\partial}{\partial x_k} - \frac{iq}{\hbar c} A_k\right)\gamma_k \psi + \left(\frac{\partial}{\partial x_4} + \frac{q}{\hbar c}\phi\right)\gamma_4 \psi + (mc/\hbar)\psi = 0 \leftarrow \text{put in } p_k = -i\hbar \frac{\partial}{\partial x_k},\ x_4 = ict, \text{ \& rearrange...}$$

so// $$c\left(p_k - \frac{q}{c}A_k\right) i\gamma_k \psi + mc^2 \psi = \left(i\hbar \frac{\partial}{\partial t} - q\phi\right)\gamma_4 \psi \leftarrow \text{insert } \beta = \gamma_4,\ \beta\alpha_k = i\gamma_k, \text{ mult. on left by } \beta, \text{ w/ } \beta^2 = 1\ldots$$

so// $$c\left(p_k - \frac{q}{c}A_k\right)\alpha_k \psi + \beta mc^2 \psi = \left(i\hbar\frac{\partial}{\partial t} - q\phi\right)\psi \leftarrow \text{rearrange terms...}$$

or// $$\underline{\underline{i\hbar\, \partial\psi/\partial t = \mathcal{H}\psi}},\ \text{w/ } \underline{\underline{\mathcal{H}(q, \mathbf{p}) = c\left(\mathbf{p} - \frac{q}{c}\mathbf{A}\right)\cdot\boldsymbol{\alpha} + q\phi + \beta mc^2}}, \text{ as desired.} \quad (1)$$

$\underline{2.}$ Treat $\mathbf{p}$ as a real eigenvalue. Take complex conjugate of (1), noting $\beta^* = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}^* = \beta\ldots$

$$-i\hbar\frac{\partial\psi^*}{\partial t} = \left[c\left(\mathbf{p} - \frac{q}{c}\mathbf{A}\right)\cdot\boldsymbol{\alpha}^* + q\phi + \beta mc^2\right]\psi^* \leftarrow \text{mult. on left by } \gamma_2 \text{ (in search of } \psi_c = \gamma_2\psi^*)$$

so// $$-i\hbar\frac{\partial\psi_c}{\partial t} = \left[c(\gamma_2\boldsymbol{\alpha}^*)\cdot\left(\mathbf{p} - \frac{q}{c}\mathbf{A}\right) + q\phi\,\gamma_2 + (\gamma_2\beta)mc^2\right]\psi^*,\ \underline{\psi_c = \gamma_2\psi^*}. \quad (2)$$

Now work out the matrix products, using $\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}$, $\{\alpha_k, \alpha_\ell\} = 2\delta_{k\ell}$, and $\{\boldsymbol{\alpha}, \beta\} = 0$. With $\gamma_2 = -i\beta\alpha_2$ & $\gamma_4 = \beta$, clearly $\gamma_2\beta = -\beta\gamma_2$. A bit more algebra shows that $\gamma_2\boldsymbol{\alpha}^* = +\boldsymbol{\alpha}\gamma_2$ for the first term RHS.★ So Eq.(2) reads:

$$-i\hbar\frac{\partial\psi_c}{\partial t} = \left[c\boldsymbol{\alpha}\cdot\left(\mathbf{p} - \frac{q}{c}\mathbf{A}\right)\gamma_2 + q\phi\gamma_2 - \beta mc^2\gamma_2\right]\psi^*$$

or// $$\to i\hbar\frac{\partial\psi_c}{\partial t} = (-)\left[c\left(\mathbf{p} - \frac{q}{c}\mathbf{A}\right)\cdot\boldsymbol{\alpha} + q\phi - \beta mc^2\right]\psi_c,\ \text{w/ } \underline{\psi_c = \gamma_2\psi^*}. \quad (3)$$

$\underline{3.}$ Define $\tilde{q} = -q$, $\tilde{\mathbf{p}} = -\mathbf{p}$. Then (3) can be written, for $\psi_c = \gamma_2\psi^*\ldots$

$$\underline{i\hbar\,\partial\psi_c/\partial t = \tilde{\mathcal{H}}\psi_c},\ \text{w/ } \underline{\underline{\tilde{\mathcal{H}}(\tilde{q}, \tilde{\mathbf{p}}) = c\left(\tilde{\mathbf{p}} - \frac{\tilde{q}}{c}\mathbf{A}\right)\cdot\boldsymbol{\alpha} + \tilde{q}\phi + \beta mc^2}}. \quad (4)$$

Clearly $\tilde{\mathcal{H}}(\tilde{q}, \tilde{\mathbf{p}}) \equiv \mathcal{H}(-q, -\mathbf{p})$, by comparing Eqs. (1) & (4). So, if ψ is the wave fcn for $(q, \mathbf{p})$ moving in the field $(\mathbf{A}, i\phi)$, then the charge conjugate wavefcn $\psi_c = \gamma_2\psi^*$ describes $(-q, -\mathbf{p})$ moving in that field; $\mathbf{p} \to (-)\mathbf{p}$, when $q \to (-)q$.

★ Since $\boldsymbol{\alpha} = \begin{pmatrix}0 & \boldsymbol{\sigma}\\ \boldsymbol{\sigma} & 0\end{pmatrix}$ & $\boldsymbol{\sigma} = \left\{\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}, \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\right\}$ [CLASS NOTES, p. DE10] then α_1 & α_3 are real; α_2 is imag.

(89) [15 pts] Analyse $p_k = mc\alpha_k$ as a Dirac momentum operator for a free particle.

1. As suggested, let $p_k = imc\beta\gamma_k$, so that the expectation value is...

$$\rightarrow \langle p_k \rangle = \langle \psi^\dagger p_k \psi \rangle = imc \langle \bar{\psi} \gamma_k \psi \rangle, \text{ w/ } \bar{\psi} = \psi^\dagger \beta \text{ (adjoint)}. \tag{1}$$

Evidently, this p_k is $\propto$ components of the Dirac current $J_k = ic\bar{\psi}\gamma_k\psi$. Notice that $p_k = imc\beta\alpha_k$ is a Hermitian operator,¶ so it will have real eigenvalues: $\langle p_k \rangle^* = \langle p_k \rangle$.

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2. For the charge-conjugate wavefcn $\psi_c = \gamma_2 \psi^*$, the expectation value will be:

$$\rightarrow \langle p_k \rangle_c = imc \langle \bar{\psi}_c \gamma_k \psi_c \rangle = imc \langle \psi_c^\dagger \gamma_4 \gamma_k \psi_c \rangle \quad \Big\{ \text{here } \beta = \gamma_4 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \text{ and } \gamma_4 \text{ is real.}$$

$$= imc \langle (\gamma_2 \psi^*)^\dagger \gamma_4 \gamma_k (\gamma_2 \psi^*) \rangle$$

$$= imc \langle (\psi^{*\dagger} \gamma_2) \gamma_4 \gamma_k \gamma_2 \psi^* \rangle, \text{ since } \gamma_2^\dagger = \gamma_2$$

$$= (-) imc \langle (\psi^{\dagger *} \gamma_4) \gamma_2 \gamma_k \gamma_2 \psi^* \rangle, \text{ since } \gamma_2\gamma_4 = (-)\gamma_4\gamma_2$$

$$= (-) imc \langle (\psi^\dagger \gamma_4)^* \gamma_2 \gamma_k \gamma_2 \psi^* \rangle, \text{ since } \gamma_4^* = \gamma_4 \text{ (real)}$$

$$= (-) imc \langle \bar{\psi}^* (\gamma_2 \gamma_k \gamma_2) \psi^* \rangle. \tag{2}$$

Now we claim that in Eq. (2), the matrix $\underline{(\gamma_2\gamma_k\gamma_2) = \gamma_k^*}$. This works because γ_2 is real and $\gamma_2^2 = 1$; then for $k=2$, the relation is an identity, and for $k = 1$ & 3, where $\gamma_k^* = -\gamma_k$, the relation expresses $\{\gamma_2, \gamma_k\} = 0$. So we can write Eq. (2) in the form...

$$\rightarrow \langle p_k \rangle_c = -imc \langle \bar{\psi}^* \gamma_k^* \psi^* \rangle = \left(imc \langle \bar{\psi} \gamma_k \psi \rangle \right)^* = \langle p_k \rangle^*. \tag{3}$$

But $\langle p_k \rangle^* = \langle p_k \rangle$, as noted above, so we've shown: $\underline{\langle \mathbf{p} \rangle_c = + \langle \mathbf{p} \rangle}$, i.e. the momentum in the charge-conjugate state is unchanged, if $\mathbf{p} = mc\boldsymbol{\alpha}$.

¶ $p_k^\dagger = -imc\alpha_k^\dagger \beta^\dagger = -imc\alpha_k\beta = +imc\beta\alpha_k = p_k$, since α_k & β are Hermitian, and anti-commute.

3. Since charge conjugation: $\psi \to \psi_c = \gamma_2 \psi^*$, is supposed to change the sign of each of charge q, energy E, and momentum $\mathbf{p}$, i.e. $(q, E, \mathbf{p}) \to (q, E, \mathbf{p})_c = -(q, E, \mathbf{p})$,¶ then -- since $\langle \mathbf{p} \rangle_c = + \langle \mathbf{p} \rangle$ is incompatible with this -- it must be that $\mathbf{p} = mc\boldsymbol{\alpha}$ is not a correct defⁿ for the momentum operator for a Dirac particle. In fact, we will show later that $\mathbf{v} = c\boldsymbol{\alpha}$ is not even a cnst of the motion (not even for a free particle)*, so $\mathbf{p} = mc\boldsymbol{\alpha}$ must automatically fail as a defⁿ of momentum for a free particle.

4. What does work? We need $\langle \mathbf{p} \rangle_c = (-)\langle \mathbf{p} \rangle$ at the very least, and we know (non-relativistically) that $p_k = -i\hbar\,\partial/\partial x_k$. Try the latter defⁿ for the Dirac case, i.e. look at expectation values...

$$\langle p_k \rangle = \langle \psi^\dagger (-i\hbar\,\partial/\partial x_k)\psi \rangle = -i\hbar \left\langle \psi^\dagger \frac{\partial}{\partial x_k} \psi \right\rangle,$$

and//

$$\langle p_k \rangle_c = -i\hbar \left\langle \psi_c^\dagger \frac{\partial}{\partial x_k} \psi_c \right\rangle = -i\hbar \left\langle (\gamma_2 \psi^*)^\dagger \frac{\partial}{\partial x_k} (\gamma_2 \psi^*) \right\rangle. \tag{4}$$

The integrand of $\langle p_k \rangle_c$ can be rewritten ($\because \gamma_2^\dagger = \gamma_2$, $\gamma_2^2 = 1$, & γ_2 real):

$$(\psi^{*\dagger} \gamma_2)\gamma_2 \frac{\partial}{\partial x_k} \psi^* = \left(\psi^\dagger (\partial/\partial x_k)\psi\right)^* \tag{5}$$

By using this in Eq. (4), we get...

$$\langle p_k \rangle_c = -i\hbar \langle \psi^\dagger (\partial/\partial x_k)\psi \rangle^* = -\langle \psi^\dagger (-i\hbar\,\partial/\partial x_k)\psi \rangle^*,$$

or//

$$\underline{\langle \mathbf{p} \rangle_c = -\langle \mathbf{p} \rangle^*, \text{ for : } \mathbf{p} = -i\hbar\,\partial/\partial \mathbf{r}.} \tag{6}$$

Now, for a free particle, $\langle \mathbf{p} \rangle$ is real (and a cnst of the motion), so that $\langle \mathbf{p} \rangle^* = \langle \mathbf{p} \rangle$. Then $\underline{\langle \mathbf{p} \rangle_c = -\langle \mathbf{p} \rangle}$, as required by charge conjugation, so that $\mathbf{p} = -i\hbar\,\partial/\partial \mathbf{r}$ at least passes this elementary test.

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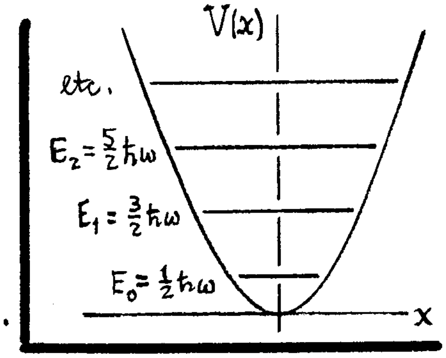
* In fact, $\mathbf{v} = c\boldsymbol{\alpha}$ is connected with the particle's Zitter Bewegung; see pp. DE 24-27.

¶ CLASS NOTES: p. DE 12, Eq. (43).



(69) Bosons & fermions in a SHO well.

Any single particle in the SHO well $V(x) = \frac{1}{2}m\omega^2 x^2$ will be in an eigenstate of energy $E_n = (n+\frac{1}{2})\hbar\omega$, w/ $n = 0, 1, 2, \ldots$ Each such state is non-degenerate (unique).

(A) If 2N non-interacting bosons are put into V(x), any number of them can occupy any given E_n state, since it is always possible to construct simple sums of product states like $|1\rangle_{n_1} \cdots |i\rangle_{n_i} |j\rangle_{n_j} \cdots |2N\rangle_{n_{2N}}$ that have the required (+) symmetry upon exchange of any pair. In particular, the exchange symmetry is satisfied for all these particles in the same state. So all the bosons can sink (or "condense") into the lowest state E_0, where they exhibit a ground state energy $\boxed{\mathcal{E}(\text{bosons}) = 2N \times E_0 = N\hbar\omega}$. (1)

(B) 2N non-interacting fermions, each of which (by exchange symmetry) requires a unique wavefcn not shared with any sibling, will fit into the nondegenerate levels of V(x) as follows: 2 into E_0, one with spin up & one with spin down; 2 into E_1, spin up & down; ...; 2 into E_n, spin up & down. The lowest energy (i.e. the ground state energy) for such an arrangement is...

$$\mathcal{E}(\text{fermions}) = 2\cdot\tfrac{1}{2}\hbar\omega + 2\cdot\tfrac{3}{2}\hbar\omega + \cdots = \sum_{n=0}^{N-1} 2\cdot(n+\tfrac{1}{2})\hbar\omega = \hbar\omega\sum_{n=0}^{N-1}(2n+1)$$

or// $\boxed{\mathcal{E}(\text{fermions}) = N^2\hbar\omega = N\times\mathcal{E}(\text{bosons})}$. (2)

For $N \to$ large, the system energy is much larger for fermions than bosons, by reqts of exchange symmetry. This could be relieved if the fermions "paired-off".

(C) For the SHO: $\langle n|x^2|n\rangle = (n+\frac{1}{2})\hbar/m\omega$ [Davydov (1991), Eq. 26.14], So...

$$\begin{cases} \text{bosons } (n=0 \text{ only}): & x_{rms}^{(B)} = \sqrt{\hbar/2m\omega}, \\ \text{fermions } (n=0\to N-1): & x_{rms}^{(F)} = x_{rms}^{(B)}\sqrt{2N-1}. \end{cases} \quad (3)$$

The fermion system is actually bigger in space as well as energy.

)[30 pts]. In prob. 66, you showed that for electron scattering from a charge distribution $\rho(\mathbf{r})$, the transform of the scattering potential important for the Born approx$^{\underline{n}}$ was: $\tilde{V}(\mathbf{q}) = -(4\pi e/q^2)\int_\infty \rho(\mathbf{r})e^{i\mathbf{q}\cdot\mathbf{r}}\,d^3x$, w/ $\mathbf{q} = \mathbf{k}(\text{in}) - \mathbf{k}(\text{out}) = \text{mom}^{\underline{m}}$ transfer.

) ts.] (A) Put: $\underline{\rho(\mathbf{r}) = e\,\delta(\mathbf{r}) - e|\psi(r)|^2}$, for e-scattering from a neutral H-atom, with the bound electron in a spherically symmetric eigenstate $\psi(r)$. By inverting the transform $\tilde{V}(\mathbf{q})$, show that the actual scattering potential is: $V(\mathbf{r}) = -e^2\left[\frac{1}{r} - \int_\infty \frac{d^3x'}{|\mathbf{r}-\mathbf{r}'|}|\psi(r')|^2\right]$. Interpret the terms contributing to $V(\mathbf{r})$. Next, find $V(\mathbf{r})$ explicitly for the H-atom ground state: $\psi(r') = (1/\sqrt{\pi a_0^3})\,e^{-r'/a_0}$, w/ $a_0 = \hbar^2/me^2$. If $\rho = r/a_0$, you should get $\underline{\underline{V(r) = -\frac{e^2}{a_0}\left(1+\frac{1}{\rho}\right)e^{-2\rho}}}$. $\boxed{\text{HINT}}$ Use the $1/|\mathbf{r}-\mathbf{r}'|$ expansion per Jackson's Eq. (3.38).

) ts.] (B) For the $V(r)$ in part (A), evaluate the "validity criterion" for use of the Born approx$^{\underline{n}}$ [CLASS NOTES, p. ScT 10, Eq. (22)]. It is convenient to define and use the dimensionless energy parameter: $\underline{\lambda = k^2 a_0^2 = E/E_H}$, w/ E = incident electron energy & w/ $E_H = e^2/2a_0$ = H-atom binding energy. Show that the Born approx$^{\underline{n}}$ fails at low energies, i.e. $\lambda \to 0$. Estimate a lower bound on λ, above which the Born approx$^{\underline{n}}$ is ~ OK.

) ts.] (C) Assume the differential cross-section for $e \to$ H atom scattering as cited in prob. 66 is correct (i.e. Sakurai's version): $\underline{d\sigma/d\Omega = (4a_0^2/Q^4)[1-16/(Q^2+4)^2]^2}$, w/ $Q = qa_0$, $q = 2k\sin\frac{\theta}{2}$, θ = scattering $\measuredangle$. From this, find the $\underline{\text{total}}$ cross-section $\sigma(\lambda)$. $\boxed{\text{HINT}}$: develop & use the relation: $d\Omega = (2\pi/k^2a_0^2)\,Q\,dQ$. Compare your result for $\sigma(\lambda)$ with the following known facts for e-H scattering: (1) as the result of experiment: $\sigma(\lambda\to 0) = (30\pm 5)\pi a_0^2$; (2) at high energy: $\sigma(\lambda \gg 1) \simeq 7\pi/3k^2$, per Landau & Lifshitz "QM" (1965), p. 535. Comment on the agreement.

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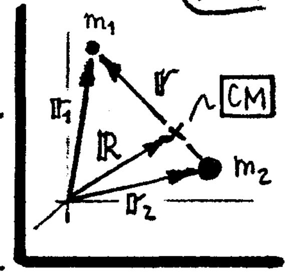
69 A gaggle of $2N$ identical particles (w/ $N \geq 1$) finds itself in a 1D SHO potential: $V(x) = \frac{1}{2}m\omega^2 x^2$. Ignore interactions between the particles.

(A) What is the total ground state energy of the system if the particles are $\underline{\text{bosons}}$?

(B) What " " " " " " " " if " " " $\underline{\text{fermions}}$?

(C) Compare the size (i.e. spatial extent) of systems (A) & (B). Define "size" by: $x_{rms} = \sqrt{\langle x^2\rangle}$, for the highest state occupied. Look up $\langle x^2\rangle$ in any convenient QM text.

ϕ507 Problems



(51) A QM system consists of two particles, masses m_1 & m_2. Express the operators for total momentum $\mathbb{P} = \mathbb{p}_1 + \mathbb{p}_2$ and total ∡ momentum $\mathbb{L} = \mathbb{l}_1 + \mathbb{l}_2$ in terms of the relative coordinate $\mathbb{r} = \mathbb{r}_1 - \mathbb{r}_2$ and center-of-mass coordinate $\mathbb{R} = \frac{m_1 \mathbb{r}_1 + m_2 \mathbb{r}_2}{m_1 + m_2}$. You must find a transformation $(\nabla_1, \nabla_2) \rightarrow (\nabla_R, \nabla_r)$ between gradient operators acting on the cds $\mathbb{r}_1$ & $\mathbb{r}_2$, and those acting on the cds $\mathbb{R}$ & $\mathbb{r}$. Show that the K.E. part of the Hamiltonian, viz: $\hat{K} = -\frac{\hbar^2}{2m_1}\nabla_1^2 - \frac{\hbar^2}{2m_2}\nabla_2^2$, can be written as: $\hat{K} = -\frac{\hbar^2}{2M}\nabla_R^2 - \frac{\hbar^2}{2\mu}\nabla_r^2$, w/ $M = m_1 + m_2$ & $\mu = \frac{m_1 m_2}{m_1 + m_2}$. Why is this useful?

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(52) [15 pts]. Consider the central potential: $V(r) = -\frac{B}{r} + \frac{A}{r^2}$; B & A are (+)ve cnsts.

(A) Sketch V(r) vs. r. What physical system might exhibit a potential of this sort?

(B) Write the radial wave eqtn in dimensionless form ["atomic units" are: length $a_0 = \frac{\hbar^2}{mB}$, energy $E_0 = \frac{B}{a_0}$]. Find the radial wavefcn $R(\rho)$, and show the bound state energies are $\boxed{E_{n\ell} = -\frac{1}{2}E_0/(n+\Delta_\ell)^2}$, w/ $n = 1, 2, 3, \ldots$ & $\ell = 0, 1, \ldots, (n-1)$, just as for H-atoms. The "quantum defect" Δ_ℓ lifts the ℓ-degeneracy. Obtain an exact expression for Δ_ℓ.

(C) Assume A is "small" and expand $E_{n\ell}$ to terms of $\mathcal{O}(A)$. In a given n-state, how are the ℓ-states arranged? Sketch the energies for $n = 1, 2, 3$. What is the splitting in state n?

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(53) Refer to CLASS NOTES on ∡ momentum, p. ∡4. Supply the missing steps between Eqs. (12) & (14), i.e. show: $[J_x, J_y] = i\hbar J_z$, follows from the geometrical statement in Eq. (12) re the rotation operators $\mathcal{R}(\varphi_k)$. Retain terms to $\mathcal{O}(\varphi^2)$, and respect ordering.

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(54) The Pauli matrices $\sigma = (\sigma_x, \sigma_y, \sigma_z)$ for a spin $\frac{1}{2}$ particle obey the commutation rule: $[\sigma_\alpha, \sigma_\beta] = 2i\sigma_\gamma$, w/ $\alpha\beta\gamma$ = cyclic permutation of xyz [Sakurai, Sec. (3.2)]. The standard form of the σ_k's is: $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

(A) Prove the <u>anti</u>-commutation rule: $\{\sigma_\alpha, \sigma_\beta\} = \sigma_\alpha\sigma_\beta + \sigma_\beta\sigma_\alpha = 2\delta_{\alpha\beta}$.

(B) If $\mathbb{A}$ & $\mathbb{B}$ are any two vector operators that commute with σ, use $[\sigma_\alpha, \sigma_\beta]$ and $\{\sigma_\alpha, \sigma_\beta\}$ to prove the Dirac identity: $\boxed{(\sigma\cdot\mathbb{A})(\sigma\cdot\mathbb{B}) = \mathbb{A}\cdot\mathbb{B} + i\sigma\cdot(\mathbb{A}\times\mathbb{B})}$.

(53) Show $[J_x, J_y] = i\hbar J_z$ follows from relations between rotation operators.

1. Set $\hbar=1$ for convenience. From cited CLASS NOTES, the rotation operator about the k^{th} axis is:

$$\rightarrow R(\varphi_k) = e^{-i\varphi_k J_k} = 1 - i\varphi_k J_k - \frac{1}{2}\varphi_k^2 J_k^2 + \ldots \quad (k = x, y, \text{ or } z), \tag{1}$$

to order φ_k^2. To 2^{nd} order again, a sequence of rotations like φ_y & φ_x gives:

$$\rightarrow R(\varphi_x)R(\varphi_y) = 1 - i(\varphi_x J_x + \varphi_y J_y) - \left(\frac{1}{2}\varphi_x^2 J_x^2 + \frac{1}{2}\varphi_y^2 J_y^2 + \varphi_x\varphi_y J_x J_y\right). \tag{2}$$

The inverse -- φ_x followed by φ_y -- is given by (2) with labels x & y interchanged:

$$\rightarrow R(\varphi_y)R(\varphi_x) = 1 - i(\varphi_x J_x + \varphi_y J_y) - \left(\frac{1}{2}\varphi_x^2 J_x^2 + \frac{1}{2}\varphi_y^2 J_y^2 + \varphi_x\varphi_y J_y J_x\right). \tag{3}$$

Clearly $R(\varphi_y)R(\varphi_x) \neq R(\varphi_x)R(\varphi_y)$, to $\Theta(\varphi^2)$, unless $[J_x, J_y] = 0$.

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2. In fact, per cited CLASS NOTES, $R(\varphi_y)R(\varphi_x) \neq R(\varphi_x)R(\varphi_y)$, but instead (NOTES $\sqrt{}$ p. 44 Eq. (12)):

$$\left[\begin{array}{l} R(\varphi_x)R(\varphi_y) = R(\varphi_z)[R(\varphi_y)R(\varphi_x)], \\ \text{w/ } R(\varphi_z) = 1 - i\varphi_z J_z, \text{ to } \Theta(\varphi^2), \text{ with } \varphi_z = \varphi_x\varphi_y. \end{array}\right. \tag{4}$$

In terms of $R(\varphi_x)R(\varphi_y)$ of Eq. (2), $R(\varphi_y)R(\varphi_x)$ of Eq. (3), and to terms of $\Theta(\varphi^2)$, this geometrical statement reads [put $\varphi_x\varphi_y = \varphi_z$]...

$$\rightarrow 1 - i(\varphi_x J_x + \varphi_y J_y) - \left(\frac{1}{2}\varphi_x^2 J_x^2 + \frac{1}{2}\varphi_y^2 J_y^2 + \varphi_z J_x J_y\right) =$$

$$= (1 - i\varphi_z J_z)\left[1 - i(\varphi_x J_x + \varphi_y J_y) - \left(\frac{1}{2}\varphi_x^2 J_x^2 + \frac{1}{2}\varphi_y^2 J_y^2 + \varphi_z J_y J_x\right)\right] \tag{5}$$

... many cancellations later...

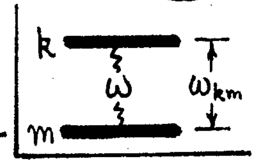
$$-\varphi_z J_x J_y = -\varphi_z J_y J_x - i\varphi_z J_z$$

or// $\cancel{\varphi_z}[J_x, J_y] = i\cancel{\varphi_z} J_z$, for $\varphi_z \neq 0$

i.e.// $\boxed{[J_x, J_y] = i J_z}$. **QED** (6)

This result holds to $\Theta(\varphi^2)$; to $\Theta(\varphi)$ -- for ∞smal rotations (which commute), we have $[J_x, J_y] = 0$, by comment below Eq. (3). For $\Theta(\varphi^3)$ and higher, (6) also holds, but with higher powers of such commutators.

(48) [20 pts]. A pulsed harmonic perturbation: $V_{ij}(t) = 2\hbar\,\Omega_{ij}\cos\omega t$, is applied to a QM system during time $t = 0 \to T$. The matrix element Ω_{ij} is time-independent, and the frequency ω approaches an exact resonance: $\nu = (\omega_{km} - \omega) \to 0$. We remarked in class [NOTES, p. tD6] that the first-order amplitude $a_k^{(1)} \simeq -i\,\Omega_{km} t$ cannot be correct as $t \to$ large. To see what actually happens near resonance, we do a new version of the $m \to k$ problem: we make an exactly solvable 2-level model of $m \to k$.



(A) When $\nu = (\omega_{km} - \omega) \to 0$, only the states m & k participate in transitions, to good approximation. By ignoring all other off-resonance states, show that the "exact" eqtns for the amplitudes are: $\boxed{i\dot{a}_k = \Omega_{km}\, a_m\, e^{i\nu t},\ \ i\dot{a}_m = \Omega_{mk}\, a_k\, e^{-i\nu t}}$; a 2-level problem.

(B) By decoupling the eqtns in part (A), one can find exact forms for the amplitudes a_k & a_m. Find $a_k(t)$ & $a_m(t)$, for $0 < t < T$, assuming the system was initially in state m: $a_m(0) = 1$, $a_k(0) = 0$. Define and use the quantity: $Q = [1 + (2|\Omega_{km}|/\nu)^2]^{1/2}$.

(C) Sketch the $m \to k$ transition probability $|a_k|^2$ vs ν. Now, what happens as $\nu \to 0$?

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(49) A QM system with unperturbed eigenstates ϕ_n & energies $E_n = \hbar\omega_n$ is subjected to a time-dep$^{\underline{t}}$ perturbation: $V(t) = (\hbar A/\tau\sqrt{\pi})\exp(-t^2/\tau^2)$, over $-\infty \leq t \leq +\infty$. τ is a scale time, and the (dimensionless) operator A is independent of time.

(A) If initially (@ $t = -\infty$) the system is in its ground state ϕ_0, use 1$^{\underline{st}}$ order t-dep$^{\underline{t}}$ pert$^{\underline{n}}$ theory to show that the probability amplitude for the transition $0 \to k \neq 0$ @ $t = +\infty$ is: $\boxed{a_k^{(1)}(\infty) = -i\,A_{k0}\exp\left(-\tfrac{1}{4}\omega_{k0}^2\tau^2\right)}$, w/ $A_{k0} = \langle \phi_k | A | \phi_0 \rangle$, and $\omega_{k0} = (E_k - E_0)/\hbar$.

(B) For an impulsive perturbation, $\tau \to 0$ (and $V(t) \to \hbar A\,\delta(t)$). Show then that the probability P_{out} of the system making any transition out of the ground state is: $P_{out} = [\langle 0|A^2|0\rangle - \langle 0|A|0\rangle^2]$. HINT: Evaluate $\sum_{k\neq 0} |a_k^{(1)}(\infty)|^2$ for $\tau \to 0$.

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(50) A 1D SHO, w/ mass m & spring cnst k, is initially in its ground state, w/ normalized wavefcn: $\phi(x) = (\alpha/\pi)^{1/4}\, e^{-\frac{1}{2}\alpha x^2}$, w/ $\alpha = \sqrt{mk}/\hbar$. The spring cnst is changed suddenly, from k to Nk, w/ $N > 0$ some numerical factor. Find the probability P_0 that the SHO will remain in its (new) ground state. Calculate P_0 for $N = 2$ & $N = 1/2$. Over what range of N-values will P_0 exceed 50%?

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(49) $V(t) = (\hbar A/\tau\sqrt{\pi})\, e^{-t^2/\tau^2}$. Show: $P(\text{gnd}\to\text{out}) = (A^2)_{00} - (A_{00})^2$, as $\tau \to 0$.

(A) 1. By 1st order t-dept perturbation theory, the $m \to k$ transition amplitude is

$$\to a_k^{(1)}(\infty) = -\frac{i}{\hbar}\langle k|\hbar A/\tau\sqrt{\pi}|m\rangle \int_{-\infty}^{\infty} e^{-t^2/\tau^2} e^{i\omega_{km}t}\,dt$$

$$= -i A_{km} \exp\left(-\tfrac{1}{4}\omega_{km}^2\tau^2\right), \text{ w/ } A_{km} = \langle k|A|m\rangle. \quad (1)$$

This is from CLASS NOTES, p. tD5, Eq. (13), and it covers full exposure to the (Gaussian) pulse, from time $t_0 = -\infty$ to $t \to +\infty$. If $|m\rangle = |0\rangle$ is the ground state, then...

$$a_k^{(1)}(\infty) = -i A_{k0} \exp\left(-\tfrac{1}{4}\omega_{k0}^2\tau^2\right), \text{ w/ } A_{k0} = \langle k|A|0\rangle \;\&\; \omega_{k0} = \frac{1}{\hbar}(E_k - E_0). \quad (2)$$

For any finite $\tau \geq 0$, all states k are excited for which $A_{k0} \neq 0$.

(B) 2. When $\tau \to 0$, $V(t)$ becomes sharply peaked near $\tau = 0$, with width $\Delta t \sim \tau$ and height $\propto 1/\tau$. Since: $\int_{-\infty}^{\infty} V(t)\,dt = \hbar A$ is indept of τ, then indeed we get the delta-fcn behavior: $V(t) \to \hbar A\,\delta(t)$, as $\tau \to 0$. This is clearly a "sudden" perturbation, and the $0 \to k$ transition probability from Eq. (2) becomes...

$$\to |a_k^{(1)}(\infty)|^2 = |A_{k0}|^2 e^{-\frac{1}{2}\omega_{km}^2\tau^2} \to |A_{k0}|^2, \text{ as } \tau \to 0. \quad (3)$$

The net transition probability for any transition out of the ground state, i.e. $0 \to k \neq 0$ is just the sum...

$$\to P_{out} = \sum_{k\neq 0} |a_k^{(1)}(\infty)|^2 = \sum_{k\neq 0} A_{k0}^* A_{k0} = \sum_{k=0}^{\infty} A_{0k}A_{k0} - A_{0k}A_{k0}\Big|_{k=0}$$

$$= \sum_{k=0}^{\infty} \langle 0|A|k\rangle\langle k|A|0\rangle - (A_{00})^2$$

$$\text{or}/\!/ \quad P_{out} = \langle 0|A^2|0\rangle - \langle 0|A|0\rangle^2. \quad \textbf{QED} \quad (4)$$

In doing Eq. (4), we've assumed A is Hermitian $A_{km}^* = A_{mk}$, and we've assumed the eigenfcns $|k\rangle$ are complete: $\sum_k |k\rangle\langle k| = 1$.