Wave-Particle Duality & Quantization

A key to understanding why quantum mechanics is necessary is what we may call "wave-particle duality"... i.e. the experimental fact that an entity--such as a photon of light -- may behave like a wave in some experiments (diffraction, interference, etc), and behave like a particle in other experiments (BB radⁿ, Compton effect, etc). The entity -- which will soon include electrons and all basic units of matter as well as photons -- thus has a dual nature: it is not exclusively a wave, nor is it a particle... any theory describing that entity must allow for both wave and particle features, and those features must be related. We have already done that for photons in Intro. p.8, Eq.(22)...

$$\rightarrow \text{photons}: \underbrace{(\omega, \mathbf{k})}_{\text{wave}} \longleftrightarrow \underbrace{(\mathcal{E}, \mathbf{P})}_{\text{particle}} = \underset{\uparrow \text{ new cnst.}}{\hbar}(\omega, \mathbf{k}), \quad \text{w/ } \hbar = h/2\pi. \tag{1}$$

This was accomplished at the expense of introducing a new, non-classical, natural cnst h = Planck cnst. That is a big step beyond classical physics.

We will now show that by simultaneously considering both wave-like and particle-like properties of photons, we get a new quantum condition.

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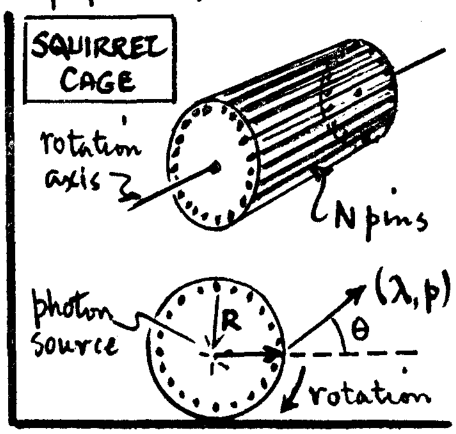
1) Consider the following Gedanken experiment. The apparatus is a "squirrel cage": a hollow drum with a large number N of equally spaced pins around the circumference, joining the edges of two discs. The drum can rotate about the axis shown. A photon source inside the drum and on-axis projects a beam of light radially outward. The beam falls on a small portion of the drum circumference -- but a portion big enough to contain a "large" number of pins. The pins act like a diffraction



Quantization of Angular Momentum

grating, so if the incident photon is considered as a <u>wave</u>, it diffracts per...

$$\left[\begin{array}{l} d\sin\theta = n\lambda \;,\; d = \text{inter-pin spacing } \& \; n = 1,2,3,\ldots \\ \ldots \text{but} : d = 2\pi R/N \;,\; \text{so/} \; \underline{\underline{\sin\theta = n \times (N\lambda/2\pi R)}} \;\int \text{photon diffraction at } \measuredangle\, \theta. \end{array}\right. \tag{2}$$

Now, at the same time, think of the photon as a <u>particle</u>. It collides elastically with one of the pins, transfers momentum, and imparts an angular momentum to the drum, which begins to rotate. The $\measuredangle$ momentum transferred is...

$$\rightarrow \underline{\underline{L_{drum} = pR\sin\theta}} \;,\; \text{w/}\; p = h/\lambda,\; \theta = \text{scattering } \measuredangle. \tag{3}$$

Assume the photon's wave-like & particle-like properties are acting together, so we can combine Eqs. (2) & (3) to get...

$$\left[\begin{array}{l} \underline{\underline{L_{drum} = \frac{h}{\lambda}\cdot R \cdot \left(\frac{nN\lambda}{2\pi R}\right) = n(N\hbar)}} \;\int N = \text{an integer},\; \hbar = h/2\pi,\; n = 1,2,3,\ldots \\ \text{so/ drum's } \measuredangle \text{ momentum is } \underline{\text{quantized}} \text{ in units of } N\hbar. \end{array}\right. \tag{4}$$

Quantization of the drum's $\measuredangle$ momentum thus follows from the photon's duality -- the need to be able to describe its passage through the pins by both wave-like means [Eq. (2)] and particle-like means [Eq. (3)]. In general, we shall see that wave-particle duality (for electrons, etc. as well as photons) is the root physical "cause" of all standard quantization.

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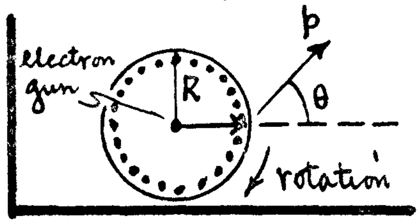
2) It is conceivable that the drum can acquire $\measuredangle$ momentum states other than those specified by : $L_{drum} = n(N\hbar)$, in Eq. (4). But it is hard to see why such states would not be excited by photons... they just provide a convenient way of transferring momentum to the drum, and $L_{drum} = n(N\hbar)$ holds for <u>any</u> photon momentum $p = h/\lambda$. In fact, the quantization of L_{drum} in Eq. (4) does not depend on any specific photon property like its energy or momentum... it just depends on the photon's dual wave-particle character.

deBroglie relation for electrons. DUALITY $\longleftrightarrow$ QUANTIZATION.

Assume for the moment that we have discovered a new law of nature -- namely, that the drum $\measuredangle$ momentum is always quantized in units of $N\hbar$, no matter how the drum is set into rotation. With this assumption, we can show that all particles -- electrons, He atoms, beachballs, etc. -- will be scattered by the pins as though they had wavelike properties. This follows simply from the conservation of $\measuredangle$ momentum. Replace the photon source on the drum axis by a electron gun, which shoots a beam of electrons radially. The e^s collide elastically with the pins, are deflected, and set the drum into rotation as before. If the electron momentum is p and the scattering $\measuredangle$ is θ, then for a single event



$$\rightarrow L_{electron} = pR\sin\theta = L_{drum} = nN\hbar \Rightarrow \underline{\underline{\sin\theta = n(N\hbar/pR)}}. \qquad (5)$$

This result for $\sin\theta$ is a prototype diffraction eqtn (i.e. $\sin\theta = n \times$ const, w/ $n = 1, 2, 3, \ldots$ giving the preferred scattering $\measuredangle$s). It implies that the electron scattering is really a diffraction effect. If that is so, we can associate a wavelength λ with the electron at momentum p, by identifying (5) & (2)

$$\left[\begin{array}{ccc} n(N\hbar/pR) = \sin\theta = n(N\lambda/2\pi R) \end{array}\right\} \boxed{\lambda = h/p} \qquad (6)$$

↑ Eq.(5): e-scattering — PARTICLE; ↑ Eq.(2): e-diffraction — WAVE; wavelength for an electron

This is de Broglie's relation, ascribing a wave-like character to a particle. Photons already obey this [p. Intro. 8, Eq. (20)]; now also material particles! For photons, it was wave-particle duality that implied quantization; now we have quantization implying wave-particle duality. So we have learned:

① WAVE-PARTICLE DUALITY $\Longleftrightarrow$ QUANTIZATION of DYNAMIC VARIABLES.

② Mechanical $\measuredangle$ momentum is (probably) quantized in units of $\hbar$.

③ The deBroglie relation $\lambda = h/p$ (apparently) holds for all particles.

QM (microscopic domain) $\longrightarrow$ CM (macroscopic domain).

Correspondence Principle

1) With wave-particle duality & quantization established, at least for photons (and later for all particles), and with the incipient breakdown of the classical description of matter, it is clear that a substantially new & different "quantum" mechanics must be invented to replace the old classical mechanics. For photons, even the venerable Maxwell Equations must be modified in order to incorporate the quantum aspects of EM radiation. More on that, later.

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2) In inventing the new theory, what should we save from the old? Presumably we want conservation laws (for energy, linear & angular momentum, etc.), simply because they work everywhere in classical physics. And we should expect that when we go from microscopic systems (individual photons, electrons, atoms) -- where quantum effects are clearly important -- to the macroscopic domain (beachballs, planets, etc.) -- where all relevant variables are in effect completely continuous -- the laws of our new QM will reduce to the well-known and well-documented classical results. So it is fair to require that in the macroscopic limit our new QM should produce Newton's Laws and the standard form of Maxwell's Equations.

If this is the case [i.e. QM (microscopic) $\rightarrow$ CM (macroscopic)], then we should retain such classical ideas as wave amplitudes and particle trajectories in developing QM. Although such ideas must be combined in some new way for microscopic systems (to accommodate wave-particle duality), they must regain their identity in the macroscopic limit.

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3) This way of thinking, that there is a gradual transition from QM on a microscopic level, to CM on a macroscopic level, and that we should be guided

by classical physics in formulating QM, was first enunciated by Bohr (1923). It can be stated as follows...

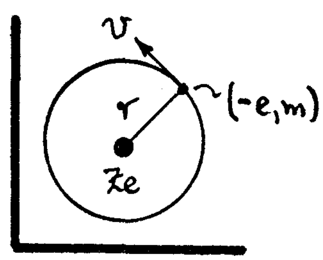
BOHR'S CORRESPONDENCE PRINCIPLE

> "In the classical limit of macrospic systems (with a large number of quanta and/or large quantum numbers) the laws of quantum physics must reduce to those of classical physics, on the average." (7)

This rather flabby sounding statement has some real physics in it. To show that, we will now quantize the hydrogen atom, using only the ideas that: (A) the photon has a discrete energy, (B) the radiation from a "large" hydrogen atom obeys a simple classical law.

ASIDE Quantization of H-atom via Correspondence Principle.

1. For an electron $(-e,m)$ in a circular orbit of radius r about a nucleus Ze:



total orbit energy: $E = \frac{1}{2}mv^2 - Ze^2/r$;

force balance: $mv^2/r = Ze^2/r^2$;

so// $mv^2 r = Ze^2$, and// $Ze^2/r = mv^2$

orbital ∡ momentum: $L = mvr$;

so// $vL = Ze^2$, and// $E = -\frac{1}{2}mv^2 = -\frac{1}{2}m(Ze^2)^2/L^2$ (8)

2. The atom radiates (a photon) when the electron changes its orbit from one where $L = L_1$ to one where $L = L_2$. The photon, at freq. ν, has a discrete energy...

$$\rightarrow h\nu = \Delta E = -\frac{1}{2}m(Ze^2)^2\left[\frac{1}{L_1^2} - \frac{1}{L_2^2}\right], \text{ for orbit transition } L_1 \rightarrow L_2. \quad (9)$$

That the orbit energy change ΔE appears as a discrete photon, $h\nu$, is the only quantum condition imposed on this otherwise classical calculation. In particular, the ∡ momenta L_1 & L_2 in Eq. (9) are not assumed quantized. Now, assume large orbits for the atom, so that it is nearly classical. For large orbits, $v \to 0$, so $L = Ze^2/v \to$ large. Assume $\Delta L = L_1 - L_2$ in Eq. (9) is fractionally small, and expand $\left[\frac{1}{L_1^2} - \frac{1}{L_2^2}\right]$ to 1st order in ΔL. Then...

$$h\nu = m(Ze^2)^2 \Delta L / L^3, \text{ as } L \to \text{large}. \tag{10}$$

3. Now enters the Correspondence Principle. Classically, the only frequency that the electron can radiate is the $\nu = v/2\pi r$ corresponding to its orbit frequency (the electron has no other time scale available). Use this in (10) to get

$$\rightarrow \hbar v/r = m(Ze^2)^2 \Delta L / L^3, \text{ for the radiative transition.} \tag{11}$$

But, from Eq. (8): $v = Ze^2/L$, so: $\frac{v}{r} = mv^2/mvr = mv^2/L = m\left(\frac{Ze^2}{L}\right)^2/L$, or: $v/r = m(Ze^2)^2/L^3$. Use this in (11), so that

$$\rightarrow \hbar \cdot m(Ze^2)^2/L^3 = m(Ze^2)^2 \Delta L/L^3 \Rightarrow \boxed{\Delta L = \hbar} \text{ allowed } \measuredangle \text{ momentum change during transition.} \tag{12}$$

This is curious (and even true!): the atom's orbital $\measuredangle$ momentum changes in discrete units of $\hbar$ during a radiative transition.

4. Now in (12), if L can only change in units of $\hbar$, it must be of the form:

$$\rightarrow L = n\hbar + \text{cnst}, \text{ w/ } n = \text{an integer, and: } \Delta n = 1 \text{ during a transition.} \tag{13}$$

A restriction on the cnst here follows from claiming that in the atom the electron can be found orbiting in a clockwise sense just as often as in a counterclockwise sense... i.e. if $+L$ is observed, then $-L$ must also be possible. Then we have to admit a series of possible L values, of form (set $\hbar = 1$):

$$\left[\underbrace{-L, -L+1}_{\text{CW rotation}}, \ldots, \underbrace{+L-1, +L}_{\text{CCW rotation}} \Rightarrow \text{an } \underline{\text{integral}} \text{ \# of allowed } L\text{-values, namely } 2L+1. \right. \tag{14}$$

But $2L+1$ = integer means, in (13), that the cnst can only be 0 or half-integral. For simplicity (and to agree with the observed H-atom spectrum), we choose the cnst $\equiv 0$. Then, from (13), we get a quantized $\measuredangle$ momentum:

H-atom Quantization (concluded)

$\text{cnst} \equiv 0 \Rightarrow \boxed{L = L_n = n\hbar}$ only w/ $n = 1, 2, 3, \ldots$ (15)

Although L is quantized in this way, the "graininess" ($\Delta n = 1$) will not be apparent for a classically large atom ($n \rightarrow$ large).

5. With L quantized per Eq. (15), all of the atom's dynamical variables become quantized. For $n = 1, 2, 3, \ldots$ these discrete values are...

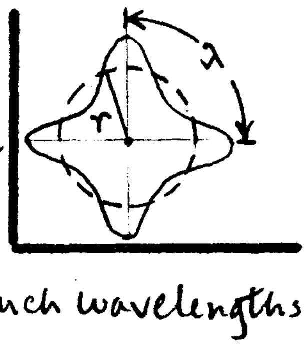
$$\begin{cases} \text{ENERGY}: & E_n = -\frac{1}{2}m(Ze^2)^2/L_n^2 = -\frac{1}{2}m(Ze^2/\hbar)^2/n^2; \\ \left.\begin{matrix}\text{ORBITAL}\\ \text{VELOCITY}\end{matrix}\right\} & v_n = Ze^2/L_n = (Ze^2/\hbar)\frac{1}{n}; \\ \left.\begin{matrix}\text{ORBITAL}\\ \text{RADIUS}\end{matrix}\right\} & r_n = L_n/mv_n = (\hbar^2/Zme^2)\,n^2. \end{cases} \tag{16}$$

All these are correct values for a non-relativistic, spinless H-atom.

The quantization in Eq. (16) results from assuming only that the photon is a discrete particle, and that a "large" H-atom behaves classically. In particular, we never did assume L was quantized a priori (i.e. that $mvr = n\hbar$), as Bohr did in his original theory of the H-atom. The atom's quantization is thus a consequence of the photon's quantization.

In passing, we note that the electron momentum is quantized. Write it as

$\rightarrow p = mvr/r = n\hbar/r$, w/ $\underline{p = h/\lambda}$, if/ $\underline{n\lambda = 2\pi r}$ (17)



In this form, i.e. that of the deBroglie relation, we see that if we are to associate a wavelength λ with the electron, then quantization implies that an integral number of such wavelengths fit around an orbit circumference... presumably this means that a stable orbit exists only when the λ corresponds to a standing wave.

END of ASIDE