ϕ 507 Schedule

Mon. 2/24/92

DATE	LECTURE	ASSIGNMENT
24 Feb.	WKB I: Basic Solution & Remarks.	Set #6 (due 3/2)
26 "	WKB II: Accuracy of Solution. The Neumann Problem.	-
28 "	WKB III: Turning Points & Connection Formulas.	-
2 Mar	WKB Applications: QM Tunneling thru a Barrier.	Set #7 (due 3/9)
4 "	WKB Applications: Double-well & double-hump problems.	-
6 "	Stat. State Pertb^{n} Theory I: Basic Recursion Formula.	-
exam cutoff! 9 Mar.	Stat. State Pertb^{n} Theory II: Ψ_k to $O(\lambda)$, E_k to $O(\lambda^2)$.	Set #8 (due 3/23)
11 "	Stat. State Pertb^{n} Theory III: Case of Degeneracy.	-
13 "	Time-dependent Pertb^{n} Theory: Lowest order transitions.	-
16 Mar.	SPRING BREAK	no set (no fret?)
18 "	SPRING BREAK	-
20 "	SPRING BREAK	-
23 Mar.	MIDTERM EXAM (2hr., in-class, open-book).*	Set #9 (due 3/30)
25 "	Time-dependent Pertb^{n} Theory: Fermi Golden Rule #2.	-
27 "	Time-dependent Pertb^{n} Theory: Adiabatic & Sudden Approxns.	-

* The MIDTERM will cover (all) material from lecture # 1 (17 Jan.) thru lecture # 22 (11 Mar.).

WKB: Examples of use.

The WKB Method

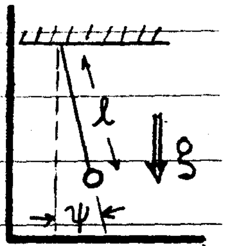
Ref. Davydov, Ch. III; ★ Sakurai, Sec. 2.4.

1) The WKB method is a way of obtaining <u>approximate</u> solutions to 2nd order ODE's of the form of a generalized SHO (simple harmonic oscillator) eqtn:

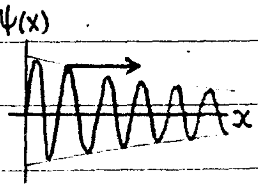
$$\boxed{\frac{d^2\psi}{dx^2} + k^2(x)\,\psi = 0} \qquad (1)$$

$\psi = \psi(x)$ is an "amplitude" of some sort, $k = k(x)$ is a variable "spring cnst".

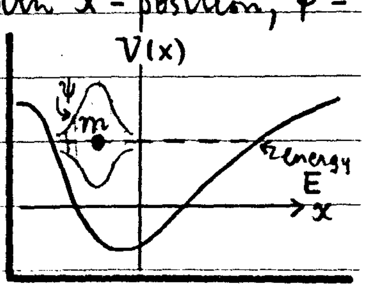
The method works to the extent that $k(x)$ varies "slowly" with x [we will defined "slowly" below]; it works <u>best</u> when $k(x) \to$ cnst. In any case, eqtns of this type arise in many examples in physics, e.g....



A. If $x = t$ (time), and ψ = displacement of a <u>pendulum</u>, then Eq. (1) is the pendulum's eqtn-of-motion, with $k(x) \leftrightarrow \omega(t)$ the natural frequency. $\omega = \omega(t)$ can depend on time if the length of the pendulum changes with t [$\omega = \sqrt{g/l}$ = fcn(t)] (e.g. m on a rubber band).



B. If x = position, and ψ = field amplitude of an <u>EM wave</u>, then Eq. (1) is the space-dependent part of the wave eqtn for the propagation. $k(x)$ is the "wavenumber", related to wavelength λ by $k = 2\pi/\lambda$. $k = k(x)$ if the wave propagates through a medium whose index of refraction n is changing ($\lambda = c/n\nu \Rightarrow k = 2\pi\frac{n\nu}{c}$, $n = n(x)$).



C. The <u>1D Schrödinger Eqtn</u> of QM is of form of Eq. (1), with x = position, ψ = System "wavefcn", and $\hbar k(x)$ the system momentum. For a particle of mass m & energy E moving in an external potential $V(x)$, have: $\left[\!\left[k(x) = \left(\frac{2m}{\hbar^2}[E - V(x)]\right)^{1/2} \right]\!\right]$. Here $\hbar$ = Planck's cnst. Clearly $k = k(x)$ wherever $V(x) \neq$ cnst. Note that in this problem, k can be real or imaginary (if $E \gtrless V(x)$).

WKB $\leftrightarrow$ Wentzel, Kramers, Brillouin... physicists who "popularized" the method in the early days of QM. Method actually invented by Jefferies (British math$^{\underline{n}}$) at < 1900.

WKB (cont'd) Solns: $\psi \sim e^{iS}$; eqtn for S.

D. Finally, from a math standpoint alone, we easily see that <u>any</u> 2nd order homogeneous ODE of the form...

$$\rightarrow y'' + f(x)\,y' + g(x)\,y = 0\,, \quad \text{for } y = y(x)\,; \tag{2}$$

can be cast into the WKB form with the substitution:

$$y(x) = \psi(x) \exp\left[-\frac{1}{2}\int^{x} f(\xi)\,d\xi\right]$$

why doesn't integral have a lower limit? What difference?

$$\Rightarrow \boxed{\psi'' + k^2(x)\,\psi = 0\,, \quad \text{w/ } k(x) = \pm\sqrt{g(x) - \tfrac{1}{2}\left[f'(x) + \tfrac{1}{2}f^2(x)\right]}}\,. \tag{3}$$

So a WKB solution to this problem approximates a very general 2nd order (homogeneous) ODE ... provided $k(x)$ is "slowly varying" with x.

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2) <u>A clue as to how to proceed</u> to solve the WKB eqtn [Eq. (1) above] is found by looking at the solutions when k actually <u>is</u> cnst, say $k = k_0$. Then...

$$\left[\begin{array}{l} k = k_0 = \text{cnst} \Rightarrow \text{WKB eqtn: } \psi'' + k_0^2\,\psi = 0\,; \\ \ldots \text{ solutions are: } \psi(x) \propto e^{\pm i k_0 x} = \exp\left(\pm i \int^{x} k_0\,d\xi\right). \end{array}\right. \tag{4}$$

This suggests that if $k \to k(x)$ varies slowly with x, $\psi(x)$ will resemble:

$$\rightarrow \psi(x) = e^{iS(x)}\,, \quad S(x) \simeq \pm \int^{x} k(\xi)\,d\xi \quad (\text{when } k(\xi) \simeq \text{cnst}). \tag{5}$$

To get a better fix on the "phase" $S(x)$, we change dept. variables by the substitution: $\psi(x) = e^{iS(x)}$. This gives an exact (nonlinear) eqtn for $S(x)$, viz

$$\left[\begin{array}{l} \psi(x) = e^{iS(x)} \text{ into } \psi'' + k^2(x)\,\psi = 0\,; \\ \Rightarrow \boxed{(dS/dx)^2 = k^2(x) + i\,(d^2S/dx^2)} \end{array}\right. \tag{6}$$

This eqtn <u>cannot</u> be solved for S when $k(x)$ is an arbitrary fcn. But...

... if, in this eqtn, $k \sim k_0 = \text{cnst}$, then $S(x) \sim \pm k_0 x$, $dS/dx \sim \pm k_0$, and $S'' \sim 0$. This suggests that when $k(x)$ is "slowly varying", the effect in the eqtn for $S(x)$ will be that S'' is "small"; more specifically: $\underline{\underline{|S''| \ll |k^2(x)|}}$.

WKB (cont'd) Slowly-varying condition; 1st iteration for S.

3) Elaborate on the last idea, that k "slowly varying" $\Rightarrow |S''| \ll |k|^2 \ldots$

$|S''| \ll |k|^2 \Rightarrow$ Eq. (6) is : $\underline{(dS/dx)^2 \simeq k^2(x)}$

... solutions : $S(x) \simeq \pm \int^x k(\xi)\, d\xi$ — in accord with Eq. (5) above. (7)

Now plug this (approximate) solution back into the "slowly varying" condition to find a condition on k for the Whole Approach to be valid...

$|S''| \ll |k|^2$, with : $S(x) \simeq \pm \int^x k(\xi)\, d\xi \Rightarrow |S''| = \left|\frac{dk}{dx}\right|$.

SLOWLY VARYING $\Rightarrow \left|\frac{dk}{dx}\right| \ll |k|^2$, or $\boxed{\left|\frac{1}{k}\left(\frac{dk}{dx}\right)\right| \ll |k|}$. (8)

This says that for a "slowly varying" fcn $k(x)$, the fractional change in the fcn, dk/k, per interval dx, should be small compared to the fcn k itself in that interval. OK... that's intuitive for a weak variation in $k(x)$.

NOTE : condition of Eq. (8) fails whenever $|k| \to 0$ but $|dk/dx| \neq 0$, so the WKB method has big problems when $|k| \to 0$... e.g. it doesn't work.