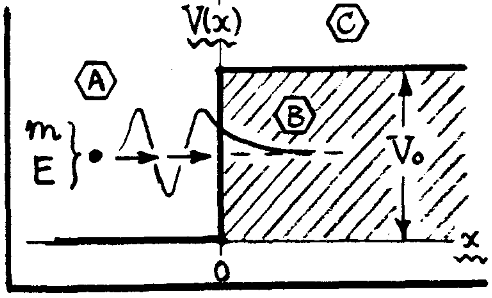
1992

ϕ507 MidTerm Exam (in class, 3 hrs.) Mon. 23 Mar. 1992

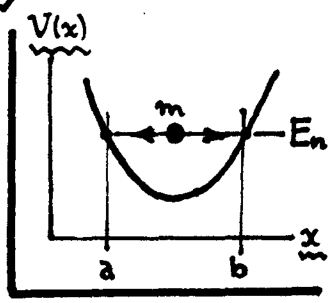
This exam is open-book, open-notes, and is worth 150 points total. There are 5 problems in all, with individual point values as marked. For each problem, $\boxed{\text{box}}$ your answer (when appropriate). Number your solution pages consecutively, write your name on the cover sheet, and staple the pages together before handing them in.

[30 pts]. A particle of mass m and total energy E is incident from a field-free region Ⓐ $[x<0,\ E>0]$ onto a potential step $V_0>E$ at $x>0$. Region Ⓑ $[x>0,\ E<V_0]$ is classically inaccessible to m, since there its kinetic energy would be negative. However, m's wavefunction ψ(in B) does not vanish, so then $|\psi(\text{in B})|^2>0$ represents a finite probability that m is in region Ⓑ. Construct a QM argument to support the following claim: even though $\psi(\text{in B})\neq 0$, any attempt to actually locate m in region Ⓑ will boost m to the classically accessible region Ⓒ $[x>0,\ E>V_0]$; thus m cannot [redacted]ever be detected in region Ⓑ.



[30 pts]. Let $\psi(\vec{r},t)$ be a solution to the free-particle Klein-Gordon equation for a particle of mass m. Transform to a new wavefunction ϕ via: $\psi(\vec{r},t)=\phi(\vec{r},t)\,e^{-\frac{i}{\hbar}mc^2t}$. Under what condition(s) will ϕ satisfy the nonrelativistic Schrödinger equation? Interpret the condition(s) when ψ is a plane-wave solution.

[30 pts.]. A particle of mass m is bound at energy E_n in a 1D potential well $V(x)$ as shown; its turning points are at $x=a$ & b. Show that when $n\to$ large, the spacing between adjacent energy levels near E_n is: $\boxed{\Delta E_n \simeq \hbar\omega_n}$, where ω_n is the natural vibration frequency of m in level E_n. HINTS: (1) If $v_n(x)$ is the velocity of m's motion in level E_n, then by definition: $2\pi/\omega_n = 2\int_a^b dx/v_n(x)$; (2) Try considering the Bohr-Sommerfeld quantization rule as a function of n.



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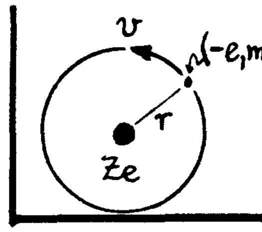
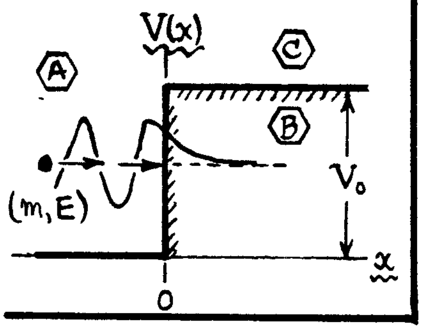


[30 pts.]. A hydrogen-like atom [potential: $V(r) = -Ze^2/r$] is in its ground state, with total energy given by the Bohr formula: $E_1 = -Z^2e^2/2a_0$, w/ $a_0 = \hbar^2/me^2$. Calculate (i.e. get a <u>number</u> for) the probability that the electron will be found at a distance from the nucleus which is <u>larger</u> than its energy would permit from a classical standpoint. <u>HINT</u>: fix the distance in question by equating E_1 to its classical counterpart.

[30 pts.]. "Muonium" is a hydrogen-like atom, μ^+e^-, formed (for a few μsec) as a bound state of a positively-charged mu meson μ^+ and an ordinary electron e^-. Except for its finite lifetime, the mu behaves in all respects like a heavy electron: it has charge $|e|$, spin $\frac{1}{2}$, a "Dirac" g-value: $g_\mu = 2$, but mass $m_\mu = 207\,m_e$. Calculate the hyperfine structure interval $\Delta\nu_\mu$ in the ground state of muonium. <u>HINT</u>: the ground state interval in ordinary hydrogen, p^+e^-, is: $\Delta\nu_H = 1420$ MHz.

ϕ507 Mid Term Solutions

(1) [30 pts]. Explain why a QM particle won't be found in a "forbidden" place.



1. Regions (A) & (C) are classically accessible, for total particle energies $E>0$ & $E>V_0$, resp. In these regions, m's propagation wave # is: $k=\sqrt{(2m/\hbar^2)[E-V]}$, with $V=0$ for $x<0$, and $V=V_0$ for $x>0$. m is quasi-free, and $\psi \sim e^{\pm ikx}$.

2. In region (B), when $E<V_0$, the wave# becomes $\underline{\underline{\kappa=\sqrt{(2m/\hbar^2)[V_0-E]}}}$, and the wavefunction goes as $\psi(\text{in } B)\sim e^{-\kappa x}$. Although $\psi(\text{in } B)$ declines rapidly for $x>0$, it is nonzero over a distance $\underline{\underline{\delta x \sim 1/\kappa}}$.

3. Any attempt to actually locate m in region (B) with $E<V_0$ (and (−)ve K.E., etc.) must localize m to $\underline{\underline{\Delta x \leq 1/\kappa}}$. But -- by the Uncertainty Principle -- this act of measurement must generate momentum components for m of the order

$$\rightarrow \Delta p \sim \hbar/\Delta x \gtrsim \hbar\kappa = \sqrt{2m[V_0-E]}. \tag{1}$$

These momentum components amount to a boost in m's kinetic energy of size

$$\rightarrow \Delta E \sim (\Delta p)^2/2m \gtrsim V_0 - E. \tag{2}$$

4. And so the act of trying to locate m in region (B) increases its energy from $E<V_0$ to a new energy, viz

$$\left[\!\left[\, E \rightarrow E' = E+\Delta E \gtrsim E+(V_0-E), \text{ i.e. } \boxed{E'>V_0}, \text{ after localization.} \tag{3}\right.\right.$$

But $E'>V_0$ means m has been boosted into the classically accessible region (C).

$\boxed{\text{CONCLUSION}}$: m will never actually be detected in the "forbidden" place (B); measurements will locate it only in the "allowed" places (C) (or (A)).

ϕ507 MidTerm Solutions

② [30 pts.]. Find Schrödinger limit on Klein-Gordon plane waves.

1. The free-particle KG eqtn is: $\left[\nabla^2 - \frac{1}{c^2}(\partial^2/\partial t^2) - (mc/\hbar)^2\right]\psi(\vec{r},t) = 0$ $\int$ class notes p. fs 15.

If we substitute $\psi = \phi \exp(-\frac{i}{\hbar} mc^2 t)$, a straightforward calculation shows that

$$\rightarrow \frac{1}{c^2}(\partial^2\psi/\partial t^2) = \left[\frac{1}{c^2}(\partial^2\phi/\partial t^2) - \frac{2im}{\hbar}(\partial\phi/\partial t) - (mc/\hbar)^2\phi\right] e^{-\frac{i}{\hbar}mc^2 t}. \quad (1)$$

Plugging this into the free-particle KG eqtn for ψ, we find ϕ must satisfy

$$\rightarrow \left(\nabla^2 + \frac{2im}{\hbar}\frac{\partial}{\partial t}\right)\phi = \frac{1}{c^2}(\partial^2\phi/\partial t^2), \text{ for KG } \phi. \quad (2)$$

2. The Schrödinger equation for a free particle of mass m and wavefcn φ is:

$$\left[\!\left[-\frac{\hbar^2}{2m}\nabla^2\varphi = i\hbar\frac{\partial\varphi}{\partial t}, \text{ or } \left(\nabla^2 + \frac{2im}{\hbar}\frac{\partial}{\partial t}\right)\varphi = 0. \right.\right. \quad (3)$$

Comparison with Eq. (2) shows that the KG ϕ will satisfy the Schrödinger equation only if $\frac{1}{c^2}(\partial^2\phi/\partial t^2) \rightarrow$ negligible. More precisely, we need this term to be negligibly small compared the others... in particular:

$$\rightarrow \left|\frac{1}{c^2}\left(\frac{\partial^2\phi}{\partial t^2}\right)\right| \ll \left|\frac{2im}{\hbar}\left(\frac{\partial\phi}{\partial t}\right)\right|, \text{ or } \underline{\underline{\left|\frac{1}{\dot\phi}(\partial\dot\phi/\partial t)\right| \ll mc^2/\hbar}}. \quad (4)$$

3. A plane-wave solution to the free-particle KG eqtn is $\psi(\vec{r},t) = e^{\frac{i}{\hbar}(\vec{p}\cdot\vec{x} - Et)}$, where $\vec{p}$ is the relativistic particle momentum and E is the total (relativistic) energy. Then the plane-wave version of $\phi = \psi \exp(\frac{i}{\hbar}mc^2 t)$ is:

$$\rightarrow \phi(\vec{r},t) = \exp\left[\frac{i}{\hbar}(\vec{p}\cdot\vec{x} - \mathcal{E}t)\right], \quad \mathcal{E} = E - mc^2. \quad (5)$$

$\mathcal{E}$ is the "conventional" (actually relativistic) kinetic energy for the particle. For ϕ of Eq. (5), the condition in Eq. (4) prescribes

$$\boxed{\mathcal{E} \ll mc^2}, \text{ for KG } \phi \text{ to } \simeq \text{ satisfy Schrödinger Equation}. \quad (6)$$

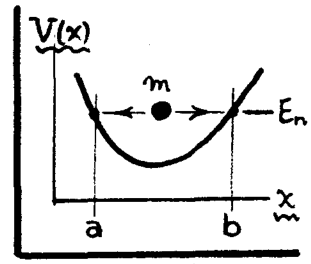
Only very slowly moving m's, @ $v \ll c$, will qualify as Schrödinger-like.

ϕ 507 Mid Term Solutions

MT3

③ [30 pts.]. Find the spacing ΔE_n of WKB bound-state energy levels ($n \to$ large).

1. The bound state energies E_n are found from the Bohr-Sommerfeld rule:



$$\rightarrow \int_a^b \sqrt{2m[E_n - V(x)]}\, dx = (n + \tfrac{1}{2})\pi\hbar . \qquad (1)$$

When $n \to$ large, E_n and n become quasi-continuous functions (e.g. $\Delta n/n \to 0$, for unit steps), so we differentiate (1) by $\frac{\partial}{\partial n}$ to get...

$$\int_a^b \frac{1}{2}\left(2m[E_n - V(x)]\right)^{-\frac{1}{2}} \cdot 2m\left(\frac{\partial E_n}{\partial n}\right) dx \simeq \pi\hbar, \text{ for } n \to \text{large};$$

$$\xrightarrow{\text{or}} \underline{\underline{m\left(\frac{\partial E_n}{\partial n}\right)\int_a^b \frac{dx}{p_n(x)} \simeq \pi\hbar}}, \text{ w/ } p_n(x) = \sqrt{2m[E_n - V(x)]}. \qquad (2)$$

$p_n(x)$ is the momentum of m in level E_n.

2. The natural period of the (quasi-oscillatory) motion of m in level E_n is $\tau_n = 2\int_a^b dx/v_n(x)$, with $v_n(x) = m$'s velocity. Set $v_n(x) = p_n(x)/m$, and put $\tau_n = 2\pi/\omega_n$, where ω_n is the (4 lar) oscillation frequency. Then...

$$\rightarrow \frac{2\pi}{\omega_n} = 2\int_a^b \frac{dx}{p_n(x)/m}, \text{ so } \underline{\underline{m\int_a^b \frac{dx}{p_n(x)} = \frac{\pi}{\omega_n}}}. \qquad (3)$$

3. Using Eq. (3) in Eq. (2), we obtain...

$$\rightarrow \left(\frac{\partial E_n}{\partial n}\right)\cdot\frac{\pi}{\omega_n} \simeq \pi\hbar, \text{ or } \underline{\underline{\frac{\partial E_n}{\partial n} \simeq \hbar\omega_n}}. \qquad (4)$$

Then, to a first approximation (and for $n \to$ large), the spacing between adjacent levels, $\Delta n = 1$ around energy E_n, is given by

$$\boxed{\Delta E_n \simeq (\partial E_n/\partial n)\,\cancelto{1}{\Delta n} \simeq \hbar\omega_n}, \qquad (5)$$

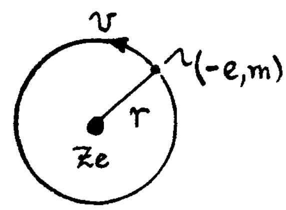
where the frequency ω_n is defined in Eq. (3). n must be large enough here (i.e. terms of $O(1/n) \to$ negligible) to justify the derivatives take in Eq. (2). The result of Eq. (5) certainly does not work for the low-n states.

ϕ507 MidTerm Solutions

④ [30 pts.]. Find "non-classical" orbit probability for hydrogenlike atom.

1. The Bohr energy in the $n=1$ ground state is : $\underline{\underline{E_1 = -Z^2 e^2/2a_0}}$, $a_0 = \frac{\hbar^2}{me^2}$ (ref. Davydov, ¶ 38). Classically, for the electron in orbit of radius r:

$$\frac{mv^2}{r} = \frac{Ze^2}{r^2}, \text{ and total orbit energy is :}$$



$$E(\text{class}) = \text{K.E.} + \text{P.E.} = \tfrac{1}{2}mv^2 - \frac{Ze^2}{r} = -\frac{Ze^2}{2r};$$

So $E(\text{class}) = E_1 \Rightarrow \underline{\underline{r = a_0/Z = r_0}}$. (1)

r_0 is the classical orbit radius. We must calculate the probability that the electron is found at distances $\geq r_0$.

2. For the hydrogenlike atom, the (normalized) ground-state wavefunction is (ref Davydov, Table 8, p. 155) :*

$$\rightarrow \psi_1(r) = (Z^3/\pi a_0^3)^{\frac{1}{2}} e^{-Zr/a_0}. \qquad (2)$$

The probability that the electron is found at some $r \geq r_0$ is then:

$$P(r_0) = \int_{r_0}^{\infty} |\psi_1(r)|^2 \cdot 4\pi r^2 dr = 4\pi\left(\frac{Z^3}{\pi a_0^3}\right) \int_{r_0}^{\infty} r^2 e^{-\left(\frac{2Z}{a_0}\right)r} dr \qquad (3)$$

← let $x = 2Zr/a_0$

by $P(r_0) = \frac{1}{2}\int_{x_0}^{\infty} x^2 e^{-x} dx$, w/ $x_0 = (2Z/a_0) r_0 = 2$, for classical orbit. (4)

The integral is tabulated (or can be done by partial integration). Result is:

$$\rightarrow P(r_0) = \tfrac{1}{2}(x_0^2 + 2x_0 + 2)e^{-x_0} \qquad (5)$$

NOTE: $P(r_0 = 0) = 1$, so norm is correct.

3. For the classical orbit, $r \geq r_0 = a_0/Z$, have $x_0 = 2$, so the probability is

$$\boxed{P(r \geq a_0/Z) = \tfrac{1}{2}(10)e^{-2} = 0.6767.} \qquad (6)$$

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* All you need to know is that : $\psi_1(r) = N e^{-Zr/a_0}$, N = cnst. Then, for proper norm, divide $P(r_0)$ in Eq. (3) on the RHS by $\langle \psi_1 | \psi_1 \rangle = \int_0^\infty |\psi_1(r)|^2 \cdot 4\pi r^2 dr$.

⑤ [30 pts.]. Calculate the ground-state hfs interval $\Delta\nu_\mu$ for muonium, μ^+e^-.

1. The ground-state hfs interval for ordinary hydrogen, p^+e^-, is (ref Prob. ⑯):

$$\rightarrow \Delta\nu_H = \frac{16}{3}(\mu_p/\mu_0)\,\alpha^2 c R_\infty \left(1+\frac{m_e}{m_p}\right)^{-3} = \underline{1420\ \mathrm{MHz}}. \qquad (1)$$

$\mu_0 = e\hbar/2m_ec$ is the Bohr magneton, α = fs cnst, c = light speed, R_∞ = Rydberg. We have set the nuclear g-value: $|g_n| = 2\times(\mu_p/\mu_0)$, μ_p = proton magnetic moment, and we have included the reduced mass correction (factor in m_e/m_p)-- this is not essential.* The only things that change in $\Delta\nu$ when we go from p^+e^- to μ^+e^- [i.e. replace the proton (spin $\frac{1}{2}$) with a muon (spin $\frac{1}{2}$)] are: (A) the moment $\mu_p \rightarrow \mu_\mu$, (B) the mass $m_p \rightarrow m_\mu$. The magnetic moments are

$$\left[\begin{array}{l} \mu_p = \underbrace{2.793}_{\text{anomalous}}\cdot\frac{m_e}{m_p}\cdot\mu_0 \\ \mu_\mu = \underbrace{1.000}_{\text{Dirac}}\cdot\frac{m_e}{m_\mu}\cdot\mu_0 \end{array}\right]$$

so $$\mu_\mu/\mu_p = \frac{1}{2.973}(m_p/m_\mu) = \frac{1}{2.973}\left(\frac{1836}{207}\right) = \underline{3.176}. \qquad (2)$$

2. The hfs intervals in μ^+e^- and p^+e^- are in the ratio

$$\frac{\Delta\nu_\mu}{\Delta\nu_H} = \left(\frac{\mu_\mu}{\mu_p}\right)\left[\frac{1+(m_e/m_\mu)}{1+(m_e/m_p)}\right]^{-3} = 3.176\,[1-\overbrace{0.01274}^{\text{reduced mass correction}}]$$

so $$\boxed{\Delta\nu_\mu = 4510\,[1-0.01274]\,\mathrm{MHz} = 4452\ \mathrm{MHz}.} \qquad (3)$$

The measured value is $\Delta\nu_\mu = 4463$ MHz (to a few ppm), so the estimate in Eq. (3) is ~0.25% low. The difference is due to relativistic corrections not included in the theoretical $\Delta\nu$ of Eq. (1) above. In fact $\Delta\nu_\mu$ (theory) and $\Delta\nu_\mu$ (expt) agree at present to better than 1 ppm.

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* The reduced mass correction enters in the way shown in Eq. (1) because $\Delta\nu_H \propto \langle 1/r^3\rangle \propto 1/a_0^3$, and $a_0 = \hbar^2/m_ee^2 \rightarrow \frac{\hbar^2}{m_ee^2}\left(1+\frac{m_e}{m_p}\right)$ when m_e is corrected for finite m_p.