DEPARTMENT OF PHYSICS

2004 COMPREHENSIVE EXAM

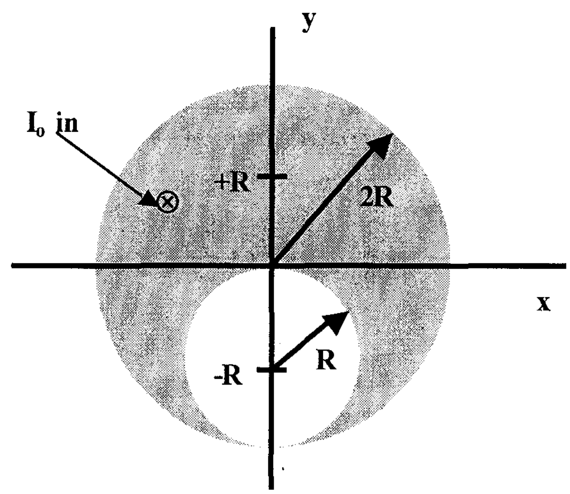
Answer each of the following questions, all of which carry equal weight. Begin your answer to each question on a new sheet of paper; solutions to different questions must not appear on the same sheet. Each sheet of paper must be labeled with your name and the problem number in the upper right hand corner. When more than one sheet is submitted for a problem, be sure the pages are ordered properly.

PHYSICAL CONSTANTS

Quantity	Symbol	Value
acceleration due to gravity	g	9.8 m s^{-2}
gravitational constant	G	6.67×10^{-11} N m^2 kg^{-2}
permittivity of vacuum	e_0	8.85×10^{-12} C^2 N^{-1} m^{-2}
permeability of vacuum	μ_0	$4\pi \times 10^{-7}$ N A^{-2}
speed of light in vacuum	c	3.00×10^8 m s^{-1}
elementary charge	e	1.602×10^{-19} C
mass of electron	m_e	9.11×10^{-31} kg
mass of proton	m_p	1.673×10^{-27} kg
Planck constant	h	6.63×10^{-34} J s
Avogadro constant	N_A	6.02×10^{23} mol^{-1}
Boltzmann constant	k	1.38×10^{-23} J K^{-1}
molar gas constant	R	8.31 J mol^{-1} K^{-1}
standard atmospheric pressure		1.013×10^5 Pa

Problem #1

A very long straight conductor has a circular cross section of radius 2R and carries a current I_o (into the page). Inside the conductor there is a cylindrical hole of radius R whose axis is parallel to the axis of the conductor. The axis of the cylindrical hole is at y = -R. Assume that the current density is uniform in the conductor. Find the B field vector on the y-axis at y = +R.

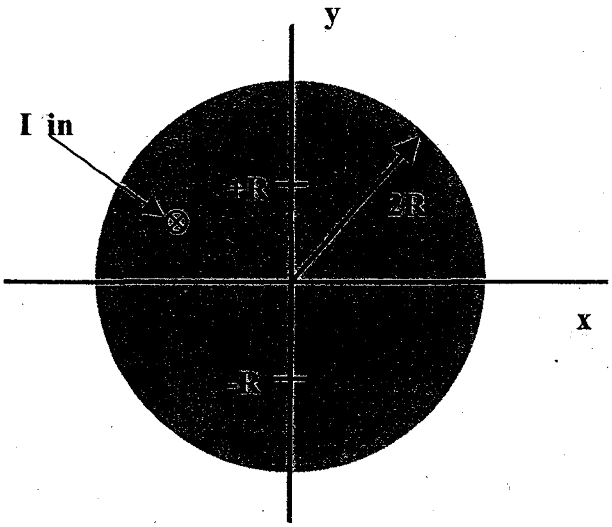


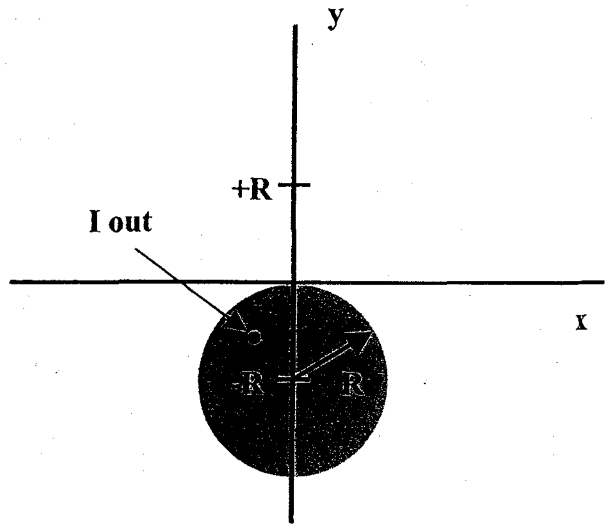
#1

Solution

Ampere's law and superposition can be used to solve this problem.

Break the problem into two wires with identical current densities flowing in opposite directions . Then using Ampere's law, find the B field at y=+R on the y-axis for each wire. Then add the two B fields together to get the answer.





First find the current density in the wire accounting for the area of the cylindrical hole.

$$J = \frac{I_o}{\left[\pi(2R)^2 - \pi R^2\right]} = \frac{I_o}{3\pi R^2}$$

Now to find B_1 for the large wire, we use an amperian curve of radius R centered at the origin.

$$\oint \vec{B} \cdot d\vec{l} = B_1(2\pi R) = \mu_o I_{thru} \qquad I_{thru} = J\pi R^2 = \frac{I_o}{3}$$

$\vec{B}_1 = \dfrac{\mu_o I_o}{6\pi R}\hat{x}$ The direction is given by the right hand rule.

Now to find the B_2 for the small wire, we use an Amperian curve of radius 2R centered on the small wire.

$$\oint \vec{B} \cdot d\vec{l} = B_2(2\pi 2R) = \mu_o I_{thru} \qquad I_{thru} = J\pi R^2 = \frac{I_o}{3}$$

$$\vec{B}_2 = -\frac{\mu_o I_o}{12\pi R}\hat{x}$$

The direction reverses since the "current" in the small wire is opposite the large wire.

By adding the two fields we can obtain the B field vector at y=R on the y-axis due to the original wire with the cylindrical hole.

$$\boxed{\vec{B}_{tot} = +\frac{\mu_o I_o}{12\pi R}\hat{x}}$$

Problem #2

Normally, we are given a potential and asked to solve for its eigenstates. It turns out often the opposite is true as well.

This problem has three parts. A nonrelativistic particle of mass *m* moves in a three dimensional central potential *V(r)* which vanishes at large *r*. We are given that an exact eigenstate of this potential is

$$\tilde{\theta}(r) = C r^{\sqrt{3}} e^{-\alpha r} \cos\theta$$

Where C and α are constants.

a) What is the angular momentum of this state?
b) What is the energy?
c) What is *V(r)*?

a) $\tilde{\Theta}(r) = C r^{\sqrt{3}} e^{-\alpha r} \cos\Theta$

The eigenstates are written in terms of Legandre polynomials $P_\ell(\cos\Theta)$

$\cos\Theta$ Factor indicates $\ell = 1$

$\rightarrow$ angular momentum $\ell = 1$.

b) To determine the energy, the Schroedinger eq.

$$H\tilde{\Theta} = E\tilde{\Theta}$$

$$-\frac{\hbar^2}{2m}\nabla^2\tilde{\Theta} + V(r)\tilde{\Theta} = E\tilde{\Theta}$$

Now, For $\ell = 1$

$$\nabla^2\Theta = \left\{\frac{1}{r}\frac{d^2}{dr^2}r - \frac{2}{r^2}\right\} C r^{\sqrt{3}} e^{-\alpha r}\cos\Theta$$

$$= \left[\alpha^2 - \frac{2\alpha(1+\sqrt{3})}{r} + \frac{1+\sqrt{3}}{r^2}\right]\tilde{\Theta}(r)$$

At large r

$$E = -\frac{\hbar^2\alpha^2}{2m}$$

c) To determine the potential $V(r)$

$$V(r)\tilde{\Theta}(r) = \left[E + \frac{\hbar^2\nabla^2}{2m}\right]\tilde{\Theta}(r)$$

$$V(r) = (1+\sqrt{3})\frac{\hbar^2}{2m}\left[-\frac{2\alpha}{r} + \frac{1}{r^2}\right]$$

Problem #3

Consider Earth's atmosphere to be an ideal gas of effective molecular weight $M_{air} = 29$ and constant temperature $T = 300$ K.

(a) Find the variation of the atmospheric pressure with altitude h relative to the sea level, if the air pressure at sea level is $p_0 = 1$ atm (~ 100 kN/m^2).

(b) What is the air pressure in Bozeman (about 1500 m above the sea level)?

(c) Estimate what is the boiling temperature of water in Bozeman.

Hint: Phase transitions of the first kind obey the Clapeyron-Clausius equation:

$$\frac{dP}{dT} = \frac{L}{T(v_{vapor} - v_{liquid})}, \qquad (1)$$

where L is the latent heat (for water - vapor transition, $L = 2{,}270$ kJ/kg) and v_{vapor} and v_{liquid} are the specific volume (volume per 1 gm of mass) of the two phases.

Universal gas constant, $R = 8.31$ J/K mol.

(a)

$$dp = -\frac{dm \cdot g}{S} \; ; \quad dm = dh \cdot S \cdot \text{density} = dh \cdot S \cdot \frac{\text{molar mass } M_{air}}{\text{molar volume } V}$$

$$pV = RT \Rightarrow \text{molar volume } V = \frac{RT}{P}$$

$$dp = -\frac{dh \cdot \cancel{S} \cdot M_{air} P g}{\cancel{S} \cdot RT} = -\frac{dh\, M_{air} \cdot P g}{RT}$$

$$\frac{dp}{dh} = -p \cdot \frac{M_{air}\, g}{RT} \Rightarrow p(h) = P_0 \, e^{-\frac{M_{air} \cdot h g}{RT}}$$

$$= 1\,atm \cdot e^{-\frac{29\,g/mol \cdot 9.8\, m/s^2\; 1500\,m}{8.3\, J/K \cdot mol \cdot 300\,K}} \approx 1 \cdot e^{-0.165} \approx 0.85\ atm.$$

(b) For water $v_{vapor} \gg v_{liquid} \Rightarrow$ Clapeyron-Clausius formula can be rewritten as:

$$\frac{dP}{dT} = \frac{L}{T\, v_{vapor}} \; ; \quad v_{vapor} = \frac{V(1\text{ mole of } H_2O)}{18} = \frac{RT}{18 \cdot P}$$

$$\frac{dP}{dT} = \frac{L \cdot 18 P}{RT^2}$$

Water starts to boil when equilibrium pressure of water vapor P equals external (atmospheric) pressure. At $P = 1\,atm$ the boiling temperature is $T_b = 373\,°K$;
At altitude $h = 1500\,m$ the pressure change is

$$\Delta P = P_0 - P(h) = 0.15\ atm$$

$$\frac{\Delta P}{\Delta T} \approx \frac{L \cdot 18 P}{RT^2} \Rightarrow \Delta T = \frac{\Delta P}{P} \cdot \frac{RT^2}{L \cdot 18} = -0.15 \cdot \frac{8.3\, J/K\,mol \cdot (373K)^2}{2{,}270\, J/g \cdot 18}$$

$$= -4{,}2\,°K$$

Water boils in Bozeman at $\approx 96\,°C$

Problem #4

Consider a hot air balloon designed for three passengers. The balloon is made of a light fabric and supports a light bamboo basket that holds a propane tank and large burner. For the density of air at 0 °C use 1.29 kg/m^3.

(a) Estimate the mass of the passengers.
(b) Estimate the mass of the balloon's fabric, the basket, and the heating equipment.
(c) Estimate the mass of air in the balloon for neutral buoyancy.
(d) Estimate the temperature difference required for neutral buoyancy.
(e) Estimate the temperature difference required to give the balloon a vertical acceleration equal to that of a sports car which can accelerate from 0 to 60 mph in 8 seconds (neglect drag).

#4

Serway Mechanics Problem – this one is really a first year physics problem that requires the students to make some realistic estimations to find answers. Knowledge of the buoyancy force, kinematics, application of Newton's second law and thermodynamics is tested.

Consider a hot air balloon designed for three passengers. The balloon is made of a light fabric and supports a light bamboo basket that holds a propane tank and large burner. The density of air at 0 °C use 1.29 kg/m^3.

a. Estimate the mass of the passengers.
b. Estimate the mass of the balloon's fabric, the basket, and the heating equipment.
c. Estimate the mass of air in the balloon for neutral buoyancy.
d. Estimate the temperature difference required for neutral buoyancy.
e. Estimate the temperature difference required to give the balloon a vertical acceleration equal to that of a sports car which can accelerate from 0 to 60 mph in 8 seconds (neglect drag).

Solution

a. I used 75 kg/person --- 225 kg.

b. 80 kg

c. The buoyancy force is $B = \rho_{air}gV$. In order to find the mass of air in the balloon, we need to equate all downward forces to the buoyant force which is found by considering the weight of the displaced air. Using a radius of 5 m (assumming it is spherical) we get $V = 523$ m^3. The total downward force is (305 kg)g+m_{air}g. The weight of the displaced air is (1.29 kg/m^3)(523 m^3)g. Setting these equal, cancelling g, and solving for m_{air} we find $m_{air} = 370$ kg.

d. Using the ideal gas law, $\rho_2/\rho_1 = T_1/T_2$ since the pressures and volumes are identical. From the result in part (c), $\rho_2 = 0.707$ kg/m^3. This gives $T_2 = 498$ K and $\Delta T = 225$ °C.

e. The acceleration is 3.38 m/s^2. As in part (c), one needs to equate the forces (Buoyant force up, mass of balloon+occupants+contained air down) to the net upward force via Newton's second law. The mass of air in the balloon enters both sides of the equation. Solving for this yields $m_{air} = 271$ kg which is smaller than in part (c). This gives a density $\rho_2 = 0.517$ kg/m^3. Using the method in part (d) a temperature $T_2 = 681$ K is obtained which yields a temperature difference of 408 K.

Problem #5

Consider an electron in the n=2 state of the hydrogen atom. We ignore relativistic corrections, so that the 2s and 2p states are initially degenerate. Now we apply a weak electric field $\vec{\mathbf{E}}$ in the z-direction ($\vec{\mathbf{E}} = |\mathbf{E}|\,\hat{\mathbf{z}}$). Calculate how these two n=2 states are altered by this perturbation to lowest order in powers of $|\mathbf{E}|$.

This is the Stark Effect. (look it up)

For the $n=2$ state of hydrogen using $|LM\rangle$ notation, the four orbital states are

one s-state, $\ell=0$ $\quad |00\rangle$

three p-states, $\ell=1$ $\quad |1\,-1\rangle, \; |1\,0\rangle, \; |1\,1\rangle$

For the perturbation $V(z) = -e|E|z$

we need to determine $\langle LM|(-e|E|z)|L'M'\rangle$

for all L, M, L', M'. Only one state gives a non-zero contribution

$$\xi = -e|E| \langle 00|z|10\rangle$$

The four degenerate states split

$$
\begin{array}{llll}
 & & \text{———} \quad E_{2s} + \xi & 1 \text{ state} \\
4\,\text{states} \quad \underset{\text{———}}{E_{2s}} & \dashrightarrow & \text{———} \quad E_{2s} & 2 \text{ states} \\
 & & \text{———} \quad E_{2s} - \xi & 1 \text{ state}
\end{array}
$$

To determine these energies exactly (we need the explicit representations

$$|00\rangle = \frac{1}{\sqrt{8\pi a_0^3}}\, e^{-r/2a_0}\left(1 - \frac{r}{2a}\right) \;;\; |10\rangle = \frac{z}{\sqrt{32\pi a_0^5}}\, e^{-r/2a_0}$$

$$\xi = -e|E| \frac{2\pi}{16\pi a_0^4} \int_0^\infty dr\, r^4 e^{-r/a_0}\left(1 - \frac{r}{2a_0}\right) \int_0^\pi d\theta \sin\theta \cos^2\theta$$

$$s = \frac{r}{a_0} \qquad = -\frac{e|E|a_0}{12} \int_0^\infty ds\, s^4 \left(1 - \frac{s}{2}\right) e^{-s} = 3e|E|a_0$$

Problem #6 The wavefunction for the 1s state of the hydrogen atom is

$$\Psi_{1s} = \frac{1}{\sqrt{\pi a_0^3}} e^{-r/a_0}$$

where a_o is the Bohr radius.

(a) What is the most probable radius to find the electron?
(b) What is the average value of the radius?
(c) What is the probability of finding the electron beyond the first Bohr radius?

#6

$$\psi = \frac{1}{\sqrt{\pi}}\left(\frac{1}{a_0}\right)^{3/2} e^{-r/a_0}$$

<u>Part A</u> $P(r) = r^2 \psi^* \psi = \frac{1}{\pi}\frac{1}{a_0^3} r^2 e^{-2r/a_0}$

$$\frac{dP(r)}{dr} = \frac{1}{\pi a_0^3}\left(2r e^{-2r/a_0} - \frac{2}{a_0} r^2 e^{-2r/a_0}\right) = 0$$

$$= \frac{2r}{\pi a_0^3} e^{-2r/a_0}\left(1 - r/a_0\right) = 0$$

$$\therefore \quad 1 - r/a_0 = 0 \quad \text{or} \quad r = a_0$$

<u>Part B</u> $\langle r \rangle = \langle \psi | r | \psi \rangle = \iiint \frac{1}{\pi}\frac{1}{a_0^3} r e^{-2r/a_0} r^2 \sin\theta \, dr\, d\theta\, d\phi$

$$= \frac{1}{\pi}\frac{1}{a_0^3} \underbrace{\int d\phi \int \sin\theta\, d\theta}_{=4\pi} \int_{r=0}^{\infty} r^3 e^{-2r/a_0}\, dr$$

$$= \frac{4}{a_0^3}\int_0^{\infty} r^3 e^{-2r/a_0}\, dr$$ from CRC Tables

$$= \frac{4}{a_0^3}\left[\frac{3!}{(2/a_0)^4}\right] = \frac{4\cdot 3\cdot 2\cdot 1}{2\cdot 2\cdot 2\cdot 2} a_0$$

$$\langle r \rangle = \frac{3}{2} a_0$$

<u>Part C</u> $P = \int_{a_0}^{\infty} \psi^*\psi \, r^2 \sin\theta\, dr\, d\theta\, d\phi = \frac{1}{\pi}\frac{1}{a_0^3}\underbrace{\int d\phi \int \sin\theta\, d\theta}_{=4\pi} \int_{a_0}^{\infty} r^2 e^{-2r/a_0}\, dr$

$$= \frac{4}{a_0^3}\int_{a_0}^{\infty} r^2 e^{-2r/a_0}\, dr$$

$$P = \frac{4}{a_0^3}\left\{ e^{-2r/a_0}\left[\frac{r^2}{(-\frac{2}{a_0})} - \frac{2r}{(-2/a_0)^2} + \frac{2}{(-2/a_0)^3}\right]\right\}\Bigg|_{a_0}^{\infty}$$

$$= \frac{4}{a_0^3}\left\{ e^{-2r/a_0}\left[-\frac{r^2 a_0}{2} - \frac{r a_0^2}{2} - \frac{a_0^3}{4}\right]\right\}\Bigg|_{a_0}^{\infty}$$

$$= \frac{4}{a_0^3}\left\{(0) - e^{-2}\left(-\frac{a_0^3}{2} - \frac{a_0^3}{2} - \frac{a_0^3}{4}\right)\right\}$$

$$= \frac{\cancel{4}}{\cancel{a_0^3}} e^{-2}\left(\frac{5\cancel{a_0^3}}{\cancel{4}}\right) = 5e^{-2} = .677$$

$$P = 67.7\%$$

Problem #7

Sometimes the heat capacity of solids or molecular systems display a large peak at temperatures near 1 K due to internal nuclear effects resulting from the protons and neutrons depopulating excited energy states. This is often called a "nuclear Schottky anomaly". This type of effect can be modeled by considering a system of N weakly interacting particles that have only two energy states ε_1 and ε_2.

(a) Without doing any calculations, plot the average energy of this system as a function of temperature. Clearly indicate the value of the average energy at very low temperature and at very high temperature. At what approximate temperature would you expect the crossover from low to high temperature behavior to occur.

(b) Using the result of part (a), make a qualitative plot of the heat capacity at constant volume C_V as a function of T.

(c) Write down the partition function for this system.

(d) Using the result of part (c) calculate explicitly the average energy and heat capacity C_V as a function of T. Do the resulting expressions agree with your results from parts (a) and (b) If $(\varepsilon_2 - \varepsilon_1)/kT$ is small, what is the temperature dependence of C_V?

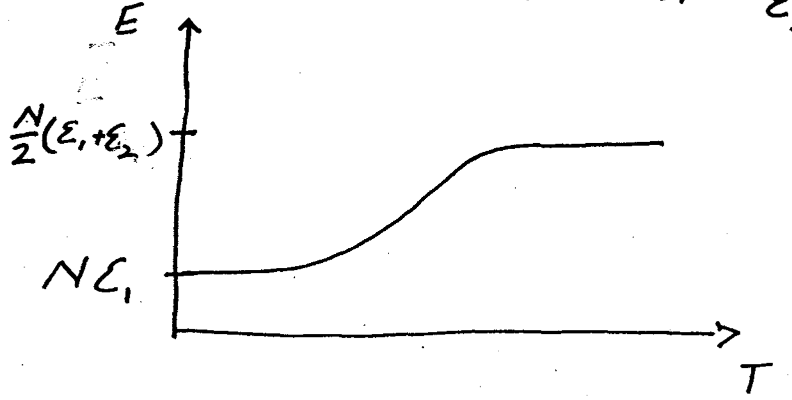
#7

a) The Boltzmann factor is $P_1 \sim e^{-\varepsilon_1/kT}$. At low temperature, only ε_1 will be occupied. At high temperature, both will be occupied w/ equal probability. As a result

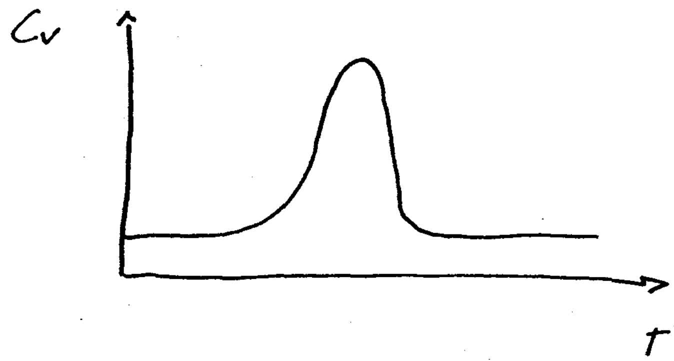
$$\bar{E} = \frac{N}{2}(\varepsilon_1 + \varepsilon_2) \quad \text{at high } T$$

$$\bar{E} = N\varepsilon_1 \quad \text{at low } T$$

The crossover occurs when $\varepsilon_2 - \varepsilon_1 \sim kT$



b) $C_V = \left(\frac{\partial E}{\partial T}\right)_V$



c) $Z = \sum_i e^{-\beta\varepsilon_i} = e^{-\beta\varepsilon_1} + e^{-\beta\varepsilon_2} \qquad \beta = \frac{1}{kT}$

for one particle

d)

$$\frac{\bar{E}}{N} = -\frac{\partial \ln Z}{\partial \beta} = \frac{1}{e^{-\beta\varepsilon_1}+e^{-\beta\varepsilon_2}} \times \left(\varepsilon_1 e^{-\beta\varepsilon_1} + \varepsilon_2 e^{-\beta\varepsilon_2}\right)$$

so

$$\bar{E} = \frac{N\varepsilon_1 e^{-\beta\varepsilon_1}}{e^{-\beta\varepsilon_1}+e^{-\beta\varepsilon_2}} + \frac{N\varepsilon_2 e^{-\beta\varepsilon_2}}{e^{-\beta\varepsilon_1}+e^{-\beta\varepsilon_2}}$$

This agrees w/ the limiting forms discussed in (a)

$$C_V = \frac{\partial \bar{E}}{\partial T} = -\frac{1}{kT^2}\frac{\partial \bar{E}}{\partial \beta} \qquad \left(\frac{\partial \beta}{\partial T} = -\frac{1}{kT^2}\right)$$

$$\frac{\partial \bar{E}}{\partial \beta} = N\left[\frac{(\varepsilon_1-\varepsilon_2)(\varepsilon_2-\varepsilon_1)e^{-\beta(\varepsilon_2-\varepsilon_1)}}{(1+e^{-\beta(\varepsilon_2-\varepsilon_1)})^2}\right] \quad \left(\text{with a bit of work!}\right)$$

so

$$C_V = \frac{N}{kT^2}\,\frac{(\varepsilon_2-\varepsilon_1)^2 e^{-\beta(\varepsilon_2-\varepsilon_1)}}{(1+e^{-\beta(\varepsilon_2-\varepsilon_1)})^2}$$

$C_V \sim \frac{1}{T^2}$ if $(\varepsilon_2-\varepsilon_1)/kT$ is small.

Problem #8

Consider two parallel conducting plates, cathode and anode, separated by a distance d. The cathode plate is grounded (potential $V=0$), while the anode plate is held at a constant positive potential , V_0. Transverse size of the plates is much larger than the distance between the plates, d.

(a) One single electron is emitted from the cathode with zero initial velocity. Find the velocity of the electron as it arrives at the anode.

(b) Consider now that at the cathode there is an unlimited supply of electrons at zero initial velocity that are accelerated by the electric field towards the anode. The electrons in the anode-cathode gap form a space charge density,

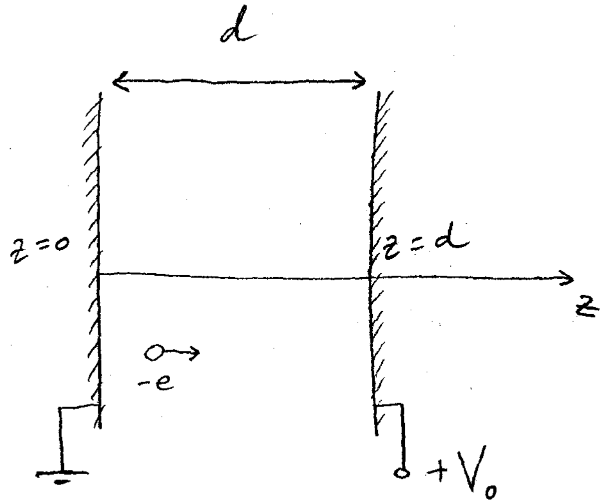
$$\rho = \frac{J}{v}, \tag{1}$$

where J is the current density v the electron velocity. Write down the equation for electrostatics potential in the region between the plates, if $v << c$. Explain qualitatively, how will the electrons leaving the cathode modify the electric field?

(c) Show that maximum space-charge limited current is expressed as (S.I. units),

$$I_{\max} = \frac{4\varepsilon_0 A}{9d^2}\sqrt{\frac{2e}{m}}V^{\frac{3}{2}}, \tag{2}$$

where A is the area of the cathode and e is the magnitude of the electron charge and m the electron mass.



(a) At the anod $\frac{m_e v^2}{2} = eV_0 \Rightarrow v = \sqrt{\frac{2eV_0}{m_e}}$

(b) $\nabla^2 \Phi(x,y,z) = -\rho/\varepsilon_0$: Poisson equation for potential in 3 dimensions.

This reduces to 1-dimensional problem:

$$\frac{d^2\Phi(z)}{dz^2} = -\frac{\rho(z)}{\varepsilon_0} \quad ; \quad \rho(z) = \frac{J}{v(z)} \quad ; \quad v(z) = \sqrt{\frac{2e\Phi(z)}{m_e}}$$

$$\frac{d^2\Phi(z)}{dz^2} = \frac{|J|}{\varepsilon_0}\sqrt{\frac{m_e}{2e\Phi(z)}} \Rightarrow \frac{d^2\Phi}{dz^2} = C\frac{1}{\sqrt{\Phi}} \quad ; \quad C = \frac{|J|}{\varepsilon_0}\sqrt{\frac{m_e}{2e}}$$

Solve the diff. equation by assuming $\Phi(z) = az^b$ form of the solution:

$$ab(b-1)z^{b-2} = ca^{-\frac{1}{2}}z^{-b/2} \Rightarrow b-2 = -b/2 \Rightarrow b = 4/3$$

$$a \cdot \frac{4}{3} \cdot \frac{1}{3} = c\,a^{-1/2} \Rightarrow a^{3/2} = \frac{9}{4}c \Rightarrow \Phi(z) = \left(\frac{9}{4}c\right)^{2/3} z^{4/3}$$

At $z = d$

$$\Phi(z) = V_0 = \left(\frac{9}{4}\frac{|J|}{\varepsilon_0}\sqrt{\frac{m_e}{2e}}\right)^{2/3} \cdot d^{4/3}$$

$$V_0^{3/2} = \frac{9}{4}\frac{|J|}{\varepsilon_0}\sqrt{\frac{m_e}{2e}} \cdot d^2 \Rightarrow |J|_{max} = \frac{4}{9}\varepsilon_0\sqrt{\frac{2e}{m_e}}\frac{1}{d^2}V_0^{3/2}$$

Maximum current $I_{max} = J_{max} \cdot A = \frac{4\varepsilon_0 A}{9d^2}\sqrt{\frac{2e}{m_e}}V_0^{3/2}$

Problem #9

a) *Relativistic Dynamics.* An antineutrino of momentum pc = 1.8 Mev strikes a proton initially at rest. Is it possible for the reaction $\bar{\nu} + p \rightarrow n + e^{+}$ to occur? [Take the rest masses to be: $m_{\nu}c^2 = 0,\ m_p c^2 = 938.3\ MeV,\ m_n c^2 = 939.6\ MeV,\ and\ m_e c^2 = 0.5\ MeV$]

a) *Story Problem.* At 12:00 noon on April 1, 2004, John Dough was abducted by aliens from the planet Skyron and carried away toward their home planet at a speed of 4/5 c. He immediately explained to his captors that they would have to return him to earth because he had had income in 2004 and had not yet filed his 2004 federal tax return. But the Skyronians informed him that they had all of his financial records in their computer memory banks and that they would be able to quickly fill out his return and sent it by radio FAX back to Earth. After five full days of computer calculations, they finally completed his return and sent it off. On what Earth date did it arrive at the IRS radio FAX machine?

Problem #9

a) All discrete conserved quantities are conserved in

$$\nu^0 + p \to n + e^+$$

Is energy conserved?

$$E_{initial} = E_\nu + E_p = 1.8 + 938.3 = 940.1 \text{ MeV}$$

$$E_{final} = \sqrt{m_n^2 + p_n^2} + \sqrt{m_e^2 + p_e^2} \approx 939.6\left(1 + \frac{p_n^2}{2m_n^2}\right) + 0.5\left(1 + \frac{p_e^2}{2m_e^2}\right)$$

$$= 940.1 + \frac{1}{2}\frac{p_n^2}{m_n} + \frac{1}{2}\frac{p_e^2}{m_e}$$

$\Rightarrow$ E is conserved if $p_n = p_e = 0$

Is momentum conserved? $p_{initial} = 1.8$ MeV

$p_{final} = 0$ (see above)

so No!

b) The Skyronians work on the taxes for 5 spaceship days.

This is $\frac{5}{\sqrt{1 - v^2/c^2}} = 5 \cdot \frac{5}{3} = \frac{25}{3} = 8.33$ days

(This is longer because moving Skyronian clocks run slow).

During this time, the ship has moved $D = \frac{4}{5}c \cdot \frac{25}{3} = \frac{20}{3}$ c·days

The radio fax will travel $\frac{20}{3}$ c·days in $\frac{20}{3}$ days.

The time it takes to get the fax after the departure is

$$\frac{25}{3} + \frac{20}{3} = \frac{45}{3} = 15 \text{ days.} \Rightarrow \text{April 16, noon.}$$

John Dough will have to pay a penalty.

Problem #10

Find the volume of the largest rectangular parallelepiped (that is, box), with edges parallel to the axes, inscribed in the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1. \tag{1}$$

Hint: Let the point (x,y,z) be the corner in the first octant where the box touches the ellipsoid.

If the $\overbrace{\text{point}}^{\text{corner}}$ (x, y, z) is set in the first octant, then the volume of the box is

$$V = 8xyz,$$

where x, y, z are related by the ellipsoid equation

$$\phi(x,y,z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

By the method of Lagrange multipliers, we write

$$F(x,y,z) = f + \lambda\phi = 8xyz + \lambda\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right)$$

$$\left.\begin{aligned} x \cdot \Big|\ \frac{\partial F}{\partial x} &= 8yz + 2\lambda\frac{x}{a^2} = 0 \\ y \cdot \Big|\ \frac{\partial F}{\partial y} &= 8xz + 2\lambda\frac{y}{b^2} = 0 \\ z \cdot \Big|\ \frac{\partial F}{\partial z} &= 8xy + 2\lambda\frac{z}{c^2} = 0 \end{aligned}\right\}$$

Solve these equation together with the ellipsoid equation.

$$3 \cdot 8xyz + 2\lambda\underbrace{\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right)}_{=1} = 0 \Rightarrow 24xyz + 2\lambda = 0$$

$$\lambda = -12xyz$$

$$\frac{1}{yz} \cdot \Big|\ 8yz - 24\frac{x^2yz}{a^2} = 0 \Rightarrow 3x^2 = a^2$$

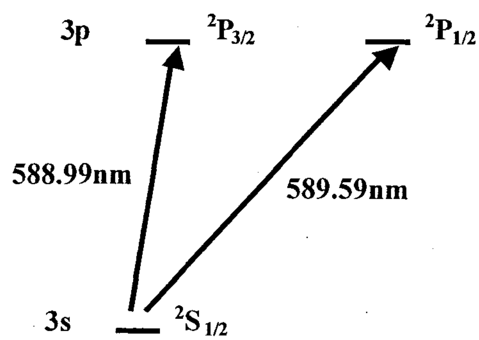
$$\frac{1}{xz} \cdot \Big|\ 8xz - 24\frac{y^2xz}{b^2} = 0 \Rightarrow 3y^2 = b^2$$

$$\frac{1}{xy} \cdot \Big|\ 8xy - 24\frac{z^2xy}{c^2} = 0 \Rightarrow 3z^2 = c^2$$

Maximum volume is: $V_{max} = \dfrac{8abc}{3\sqrt{3}}$

Problem #11

This problem will deal with sodium, which has 11 electrons. The lowest three energy levels of sodium are shown in the energy level diagram. The wavelengths of the yellow-orange sodium D lines are shown for the two transitions from the 3p $^2P_{1/2}$ and the 3p $^2P_{3/2}$ excited levels to the 3s $^2S_{1/2}$ ground state.



(a) Write down the electron configuration for sodium ground state.
[As an example, to show the notation wanted, the electron configuration for nitrogen is given by $(1s)^2(2s)^2(2p)^3$].

(b) Describe in words the meaning of each of the numbers and letters in the notation 3p $^2P_{3/2}$ of this excited state in sodium.

(c) If the ratio of the population in the upper two levels shown is $N(^2P_{3/2})/ N(^2P_{1/2}) = 1.915$ in a sodium lamp, find the temperature of the lamp.

#11

Solution

a) The key here is to remember that the s shells have a maximum of two electrons and the p shells have a maximum of six electrons. Since there are 11 electrons in sodium, the electron configuration is given by

$$(1s)^2(2s)^2(2p)^6(3s)^1$$

b) The notation for the excited state 3p $^2P_{3/2}$ is as follows. The 3p gives the energy level of the optically active electron, which has been excited from the 3s energy level to the 3p energy level. The 3 in the 3p corresponds to the principal quantum number n and the p corresponds to the second quantum number l, which is related to the orbital angular momentum of the state. The 2 in the $^2P_{3/2}$ indicates that the states are doublets, when the spin angular momentum is accounted for and is given by 2s+1, where s is the spin (1/2 for one optically active electron in this case). The P in the $^2P_{3/2}$ indicates redundantly that the state is in l=1. The 3/2 in the $^2P_{3/2}$ is the value of the third quantum number j, which is related to the total angular momentum, a combination of both spin and angular momenta.

c) To find the temperature of the discharge lamp we will use the discrete form of the Maxwell-Boltzmann distribution for the population n_i in the level of energy E_i.

$$n_i = g_i e^{-\frac{E_i}{kT}}$$

where g_i is the degeneracy of the each state and is given by 2j+1. Labeling the $^2P_{3/2}$ state 2 and the $^2P_{1/2}$ state 1, we get

$$N_2 = (2\frac{3}{2}+1)e^{-\frac{E_2}{kT}} \quad \text{and} \quad N_1 = (2\frac{1}{2}+1)e^{-\frac{E_1}{kT}}$$

Thus for the ratio we get

$$\frac{N_2}{N_1} = 2e^{-\frac{(E_2-E_1)}{kT}} = 1.915$$

where we have put in the ratio given in the problem. Now solving for the temperature T we obtain

$$T = \frac{(E_2 - E_1)}{k\ln(1.044)}$$

Now we need to get the energies from the wavelengths

$$E_2 - E_1 = h(\nu_2 - \nu_1) = hc(\frac{1}{\lambda_2} - \frac{1}{\lambda_1})$$

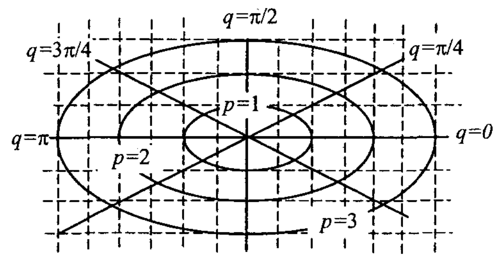
Thus the temperature is given by

$$T = \frac{hc\left(\frac{1}{\lambda_2} - \frac{1}{\lambda_1}\right)}{k\ln(1.044)} = \frac{(6.626x10^{-34}\,J\cdot s)(2.998x10^{8}\,m/s)\left(\frac{1}{(588.99x10^{-9}\,m)} - \frac{1}{(589.59x10^{-9}\,m)}\right)}{(1.381x10^{-23}\,J/T)\ln(1.044)}$$

$$\boxed{T = 577.2K}$$

Problem #12 Two-dimensional elliptical coordinates can be expressed in terms of Cartesian coordinates by

$$p = \sqrt{(x/2)^2 + y^2}$$
$$q = \tan^{-1}\left(\frac{2y}{x}\right),$$



the inverse transformation being

$$x = 2p\cos q$$
$$y = p\sin q.$$

A few of the coordinate lines are shown solid above, with some Cartesian coordinate lines shown as dashed lines.

a) Find the (covariant) basis vectors in the elliptical coordinate system, expanded in the Cartesian $\{\hat{\boldsymbol{i}}, \hat{\boldsymbol{j}}\}$ basis.

b) Draw these basis vectors on the diagram at the point $p = 3$, $q = \pi/4$.

c) Find the covariant components of the elliptical coordinate metric tensor g_{ij}.

d) A vector field $\vec{A}$ has Cartesian components given by $A_x = xy$, $A_y = -xy$. Find the contravariant components of $\vec{A}$ in the new coordinate system.

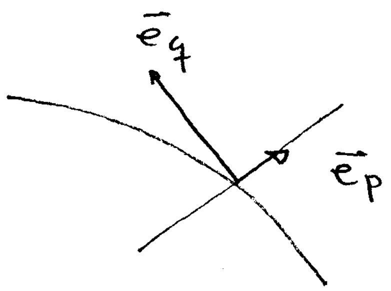
[Remember $\frac{d}{dz}(\tan^{-1} z) = \frac{1}{1+z^2}$]

(1) a) Basis vectors point along coordinate lines

$$\vec{e}_p = \frac{\partial \vec{R}}{\partial p} = \frac{\partial x}{\partial p}\hat{\imath} + \frac{\partial y}{\partial p}\hat{\jmath} = 2\cos q\,\hat{\imath} + \sin q\,\hat{\jmath}$$

$$\vec{e}_q = \frac{\partial x}{\partial q}\hat{\imath} + \frac{\partial y}{\partial q}\hat{\jmath} = -2p\sin q\,\hat{\imath} + p\cos q\,\hat{\jmath}$$

(2) b)



at $p=3$ $q=\pi/4$

$$\vec{e}_p = \sqrt{2}\,\hat{\imath} + \frac{\sqrt{2}}{2}\hat{\jmath} = \sqrt{2}\left(\hat{\imath} + \tfrac{1}{2}\hat{\jmath}\right)$$

$$\vec{e}_q = -3\sqrt{2}\,\hat{\imath} + 3\frac{\sqrt{2}}{2}\hat{\jmath} = 3\sqrt{2}\left(-\hat{\imath} + \tfrac{1}{2}\hat{\jmath}\right)$$

(3) c) $\cancel{g_{pp} = \vec{e}_p \vec{e}_p = 4p^2\cos^2 q + 1}$

$$g_{pp} = \vec{e}_p \cdot \vec{e}_p = 4\cos^2 q + \sin^2 q \qquad g_{qq} = \vec{e}_q \cdot \vec{e}_q = 4p^2\sin^2 q + p^2\cos^2 q$$

$$g_{pq} = g_{qp} = -4p\sin q\cos q + p\sin q\cos q = -3p\sin q\cos q$$

(4) d)

$$A^p = \frac{\partial p}{\partial x}A_x + \frac{\partial p}{\partial y}A_y = \frac{x/4}{\sqrt{x^2/4+y^2}}\,xy + \frac{y}{\sqrt{x^2/4+y^2}}\,(-xy)$$

$$A^p = \frac{xy}{p}\left(x/4 - y\right) = 2p^2\cos q\sin q\left(\frac{\cos q}{2} - \sin q\right) = \underline{p^2\cos q\sin q\left(\cos q - \frac{\sin q}{2}\right)}$$

$$A^q = \frac{\partial q}{\partial x}A_x + \frac{\partial q}{\partial y}A_y = \frac{1}{1+\frac{4y^2}{x^2}}\left[\frac{-2y}{x^2}(xy) + \frac{2}{x}(-xy)\right]$$

$$A^q = \frac{1/4\,x^2}{p^2}\left[-\frac{2y^2}{x} - 2y\right] = -\frac{1/2\,xy}{p^2}\left[y+x\right] = \cancel{-p\sin q\cos q(2\cos q + \sin}$$

$$= \underline{-p\sin q\cos q\,(2\cos q + \sin q)}$$

Problem #13 Consider a chain of 6 atoms of mass M spaced by a distance a and connected by springs with an effective spring constant C. Assume that only nearest neighbors interact.

(a) What are the allowed phonon wavevectors (k-values)?
(b) Starting from the equation of motion, derive the dispersion relationship.
(c) Plot the phonon energy as a function of the phonon wavevector.
(d) For the probability distribution

$$P(E) = \begin{matrix} 1 & E < 1.5\hbar\sqrt{\dfrac{C}{M}} \\ 0 & otherwise \end{matrix}$$

what is the phonon contribution to the energy of the system?

#13

Part A The allowed values of K are

$$K = 0, \pm \frac{2\pi}{L}, \pm \frac{4\pi}{L}, \pm \frac{6\pi}{L}$$
$$= 0, \pm \frac{\pi}{3a}, \pm \frac{2\pi}{3a}, \pm \frac{\pi}{a}$$

Part B The equation of motion is:

$$M \frac{d^2 u_s}{dt^2} = C(u_{s+1} + u_{s-1} - 2u_s)$$ ⇐ Newton's Law, nearest neighbor interaction

assume a solution $u_s = u e^{i(kx_s - \omega t)}$ (with $x_s = sa$

we have

$$-M\omega^2 u e^{i(ksa - \omega t)} = \left\{ C u e^{i(k(s+1)a - \omega t)} + C u e^{i(k(s-1)a - \omega t)} - 2Cu e^{i(ksa - \omega t)} \right\}$$

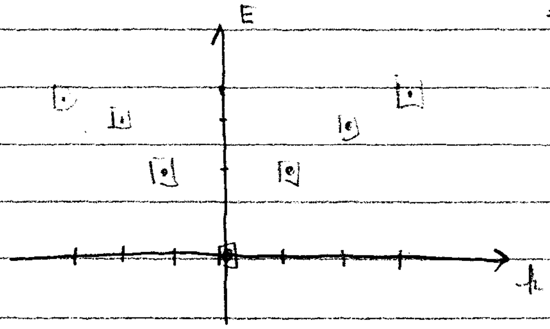
$$-M\omega^2 = C\left[e^{ika} + e^{-ika} - 2\right]$$

$$\omega^2 = -\frac{2C}{M}\left[\frac{e^{ika} + e^{-ika}}{2} - 1\right]$$

$$= \frac{2C}{M}(1 - \cos ka)$$

$$\omega^2 = \frac{4C}{M} \sin^2\left(\frac{ka}{2}\right)$$

Part C $E = \hbar\omega = 2\hbar\sqrt{\frac{C}{M}} \sin^2 \frac{1}{2}ka$
$= 0$ for $K = 0$
$= \hbar\sqrt{\frac{C}{M}}$ for $K = \pm\frac{\pi}{3a}$
$= 1.75\hbar\sqrt{\frac{C}{M}}$ for $K = \pm\frac{2\pi}{3a}$
$= 2\hbar\sqrt{\frac{C}{M}}$ for $K = \pm\frac{\pi}{a}$



Part D $E = (1)(1)(0) + (2)(1)\hbar\sqrt{\frac{C}{M}} + (2)(0)\,1.73\hbar\sqrt{\frac{C}{M}} + (2)(0)\,2\hbar\sqrt{\frac{C}{M}} = 2\hbar$

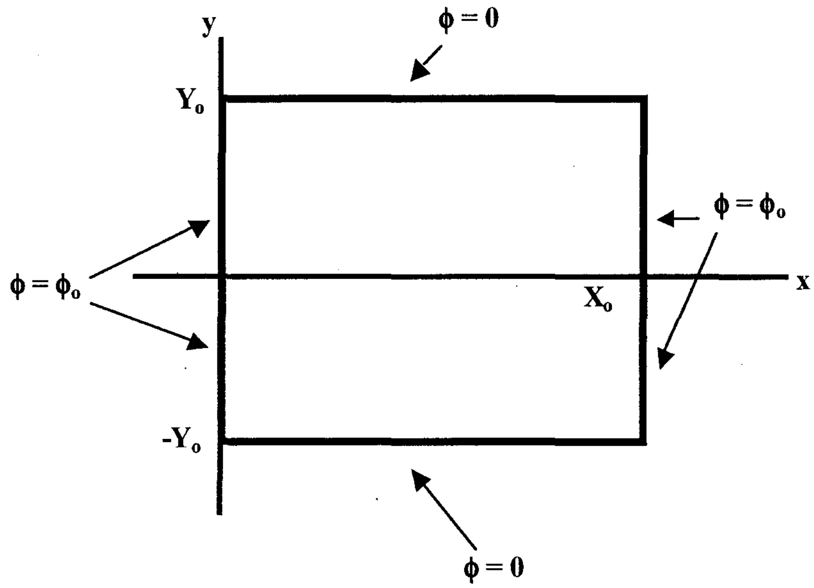
Problem #14

Find the electric potential ϕ inside the region defined by $-\infty<z<+\infty$, $0<x<X_o$, $-Y_o<y<Y_o$. The potential for the four boundary planes is given by

$$\phi(x,+Y_0,z) = \phi(x,-Y_0,z) = 0$$

$$\phi(0,y,z) = \phi(X_0,y,z) = +\phi_0$$

Note that the potential does not depend on the z coordinate.



Solution

This is a boundary value problem in rectangular coordinates. Laplace's equation in these coordinates leads to the solution :

$$\phi = \sum (a_1 e^{\alpha x} + a_2 e^{-\alpha x})(b_1 e^{\beta y} + b_2 e^{-\beta y})$$

where we have ignored the z terms since there is no dependence in the z direction.

Also for this to be a valid solution of Laplace's equation:

$$\alpha^2 + \beta^2 = 0$$

Thus one of the coordinates will be imaginary. Looking at the boundary conditions we expect that this imaginary term will be associated with the x-axis with terms that will look like cosh since the exponential terms of the cosh will be needed to curve the potential up to the non zero boundary value. In the y direction we must expect a simple cos behavior since the potential goes to zero at the boundaries. Thus we are led to write our solution in the form:

$$\phi = \sum (a_1 e^{\alpha x} + a_2 e^{-\alpha x})(b_1 e^{i\alpha y} + b_2 e^{-i\alpha y})$$

where we expect α to be real.

Now consider the boundaries at $y=+Y_o$ and $y=-Y_o$. Note that the potential is symmetric for these boundaries since $\phi(Y_o)=\phi(-Y_o)=0$, so that $b_1=b_2=b/2$ and the y part of the solution can be written as:

$$\phi = \sum (a_1 e^{\alpha x} + a_2 e^{-\alpha x})\frac{b}{2}(e^{i\alpha y} + e^{-i\alpha y}) = \sum (a_1 e^{\alpha x} + a_2 e^{-\alpha x}) b\cos(\alpha y)$$

Also since $y(\pm Y_o)=0$, $\cos(\alpha Y_o)=0$ and we will need $\alpha Y_o = n\pi/2$, with n odd. Substituting this for α, our solution becomes:

$$\phi = \sum_{n\ \mathrm{odd}} (a_1 e^{\frac{n\pi}{2Y_o}x} + a_2 e^{-\frac{n\pi}{2Y_o}x}) b\cos(\frac{n\pi}{2Y_o} y)$$

Now we proceed to the boundary conditions for $x=0$ and $x=X_o$. The potential is also symmetric for these boundaries but the symmetry axis is not $x=0$ but is $x=+X_o/2$. Thus we let:

$$a_1 = \frac{a}{2}\exp(\frac{-n\pi}{2Y_o}\frac{X_o}{2}) \quad \text{and} \quad a_2 = \frac{a}{2}\exp(\frac{+n\pi}{2Y_o}\frac{X_o}{2})$$

Thus our solution becomes:

$$\phi = \sum_{n\ odd} \frac{a}{2}\left(e^{\frac{n\pi}{2Y_o}(x-\frac{X_o}{2})} + e^{-\frac{n\pi}{2Y_o}(x-\frac{X_o}{2})}\right) b\cos(\frac{n\pi}{2Y_o}y) = \sum_{n\ odd} a\cosh\left[\frac{n\pi}{2Y_o}(x-\frac{X_o}{2})\right] b\cos(\frac{n\pi}{2Y_o}y)$$

Combining the coefficients a and b into A (that will depend on n), we get:

$$\phi = \sum_{n\ odd} A_n \cosh\left[\frac{n\pi}{2Y_o}(x-\frac{X_o}{2})\right]\cos(\frac{n\pi}{2Y_o}y)$$

Finally we need to find A_n. To do this we again use the boundary condition at $x=X_o$.

$$\phi(X_o) = \phi_o = \sum_{n\ odd} A_n \cosh\left[\frac{n\pi}{2Y_o}(\frac{X_o}{2})\right]\cos(\frac{n\pi}{2Y_o}y)$$

To find the A_n, we multiply both sides of the equation by:

$\cos(\frac{m\pi}{2Y_o}y)$ and integrate over y to get:

$$\int_{-Y_o}^{Y_o} \phi_o \cos(\frac{m\pi}{2Y_o}y)dy = \sum_{n\ odd} A_n \cosh\left[\frac{n\pi}{2Y_o}(\frac{X_o}{2})\right]\int_{-Y_o}^{Y_o} \cos(\frac{n\pi}{2Y_o}y)\cos(\frac{m\pi}{2Y_o}y)dy$$

The integral on the right has orthogonal functions in the integrand that give a Kronecker delta. Thus we get:

$$\int_{-Y_o}^{Y_o} \phi_o \cos(\frac{m\pi}{2Y_o}y)dy = \sum_{n\ odd} A_n \cosh\left[\frac{n\pi}{2Y_o}(\frac{X_o}{2})\right]\left(\frac{2Y_o}{2}\right)\delta_{nm} = A_{m,odd}\cosh\left[\frac{m\pi}{2Y_o}(\frac{X_o}{2})\right]Y_o$$

When the integral on the left is evaluated we get:

$$\int_{-Y_o}^{Y_o} \phi_o \cos(\frac{m\pi}{2Y_o}y)dy = \phi_o\left(\frac{2Y_o}{m\pi}\right)\sin\left(\frac{m\pi}{2Y_o}y\right)\Bigg|_{-Y_o}^{Y_o} = \left(\frac{4\phi_o Y_o}{m\pi}\right)\sin\left(\frac{m\pi}{2}\right) = \left(\frac{4\phi_o Y_o}{m\pi}\right)(-1)^{\frac{m+3}{2}}$$

where the last step is true because m is odd only.

Combining we can write an expression for A_m:

#14 end

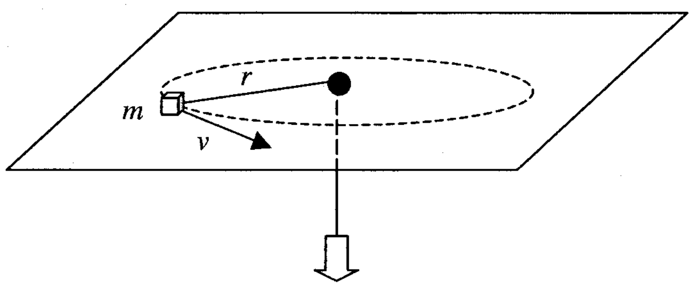
$$A_m = \frac{\left(\frac{4\phi_o}{m\pi}\right)(-1)^{\frac{m+3}{2}}}{\cosh\left(\frac{m\pi X_0}{4Y_o}\right)}$$

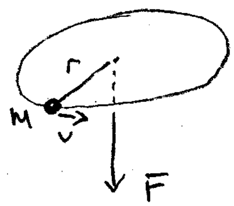
Finally we can write down the complete solution:

$$\boxed{\phi(x,y) = \sum_{m\text{ odd}} (-1)^{\frac{m+3}{2}} \left(\frac{4\phi_o}{m\pi}\right) \frac{\cosh\left[\frac{m\pi}{2Y_o}(x - \frac{X_o}{2})\right]}{\cosh\left[\frac{m\pi X_o}{4Y_o}\right]} \cos\left(\frac{m\pi}{2Y_o} y\right)}$$

Problem #15

A mass m is attached to the end of a string. The mass moves on the frictionless surface of a horizontal table, and the string passes through a hole on the table (see Figure) under which someone is pulling on the string to keep it taut at all times. The mass is initially moving in a circle of radius r and has a kinetic energy E_0. The string is then slowly pulled until the radius of the circle is halved ($r/2$). How much work was done by the person pulling on the string? (Assume the string has zero mass).





To remain in circular motion

$$F = m\frac{v^2}{r}$$

Pulling on the rope exerts no torque and the angular momentum is constant.

$\ell = mvr$

$$\rightarrow \quad F = \frac{\ell^2}{mr^3}$$

The work needed to move the mass from $r = R$ to $r = \frac{R}{2}$ is

$$W = -\int_R^{R/2} \vec{F} \cdot d\hat{r} = \frac{\ell^2}{2mr^2}\Big|_R^{R/2}$$

$$= \frac{\ell^2}{2m}\left(\frac{4}{R^2} - \frac{1}{R^2}\right) = \frac{3\ell^2}{2mR^2}$$

In terms of $E_0 = \frac{1}{2}mv^2 = \frac{\ell^2}{2mR^2}$

$$\rightarrow \quad W = 3E_0$$