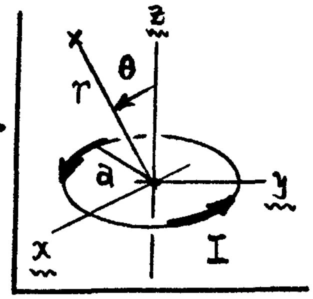
Magnetostatics (cont'd)

5) The simplest non-trivial B-field problem is to find the field everywhere in space for a plane (circular) current loop of radius a carrying current I. As $a \to 0$ & $I \to \infty$ (in such a way as to make $m = \frac{1}{c} I \pi a^2 \to$ cnst), the B-field generated this way is the "elemental" magnetic field, from a source as close as we can get to the point-charge of electrostatics.



Jackson does the problem in his Sec. 5.5. Highlights are ..

current density $\Big\}$ $J_\varphi = \frac{I}{a}\delta(r'-a)\sin\theta'\,\delta(\cos\theta')$ $\int$ in xy-plane only (at $\theta' = \pi/2$),

$$\text{so}/\!/\ \mathbb{J} = J_\varphi(-\hat{e}_x \sin\varphi' + \hat{e}_y\cos\varphi'),\ \text{in } \varphi'\text{-direction;} \tag{21.1}$$

vector potential $\Big\}$ $\mathbb{A}(\mathbf{r}) = \frac{1}{c}\int \frac{d^3x'}{|\mathbf{r}-\mathbf{r}'|}\mathbb{J}(\mathbf{r}') \to A_\varphi$ only,

$$\xrightarrow{\text{and}/\!/} A_\varphi(r,\theta) = \frac{Ia}{c}\int_0^{2\pi}\frac{\cos\varphi'\,d\varphi'}{\sqrt{r^2+a^2-(2ra\sin\theta)\cos\varphi'}}. \tag{21.2}$$

The integral would be trivial (and zero) if we had a $\sin\varphi'$ rather than $\cos\varphi'$ in numerator. As it is, we need <u>Elliptic Integrals</u> to get A_φ...

<u>Elliptic Integrals</u> (Legendre form -- see Mathews & Walker, Sec. 3-4) (21.3)

solution to "simple pendulum" →

1st kind: $F(\beta,k) = \int_0^\beta d\alpha/\sqrt{1-k^2\sin^2\alpha}$; complete: $F(\beta=\frac{\pi}{2},k) = K(k)$;

2nd kind: $E(\beta,k) = \int_0^\beta d\alpha\sqrt{1-k^2\sin^2\alpha}$; complete: $E(\beta=\frac{\pi}{2},k) = E(k)$;

3rd kind: $\Pi(\beta,n,k) = \int_0^\beta \frac{d\alpha/\sqrt{1-k^2\sin^2\alpha}}{1-n\sin^2\alpha}$; complete: $\Pi(\beta=\frac{\pi}{2},n;k) = \Pi(n,k)$

Here: k is the "modulus"; $|k| \leq 1$ for most problems. These integrals join the ranks of special fcns, and are tabulated, with asymptotic expansions etc. <u>More information in G&R, Sec. 8.110 or A&S, Chap. 17.</u>

Magnetostatics (cont'd)

Anyway, the vector potential in Eq. (21.2) can be expressed as...

$$\rightarrow A_\varphi(r,\theta) = \frac{(4Ia/c)}{\sqrt{r^2+a^2+2ra\sin\theta}}\cdot\frac{1}{k^2}\left[(2-k^2)K(k)-2E(k)\right], \int \text{Jackson Eq. (5.37)}$$

$$\text{w/ } k^2 = 4ra\sin\theta/(r^2+a^2+2ra\sin\theta) \lesssim 1. \qquad \underline{(21.4)}$$

This relation is exact. The field is now gotten -- in spherical cds -- via $\mathbb{B} = \nabla\times\mathbb{A}$. Details of $\mathbb{B}$ are complicated and not interesting except in extreme cases where $k^2\to$ small, i.e.

$k^2\to$ small [for $a \ll r$ (or $r \ll a$), or near axis, $\theta\to 0$]*

$$B_r = \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(A_\varphi\sin\theta) \simeq \frac{2m\cos\theta}{r^3}\left\{\frac{1+\frac{1}{2}\epsilon\sin\theta+\epsilon^2}{[1+2\epsilon\sin\theta+\epsilon^2]^{5/2}}\right\},$$

$$B_\theta = -\frac{1}{r}\frac{\partial}{\partial r}(rA_\varphi) \simeq \frac{m\sin\theta}{r^3}\left\{\frac{1-\epsilon\sin\theta-2\epsilon^2}{[1+2\epsilon\sin\theta+\epsilon^2]^{5/2}}\right\}, \qquad \underline{(21.5)}$$

and/ $B_\varphi = 0$ (clear, by symmetry); w/ $m = \frac{I\pi a^2}{c}$ (magnetic dipole moment), $\epsilon = \frac{a}{r}$.

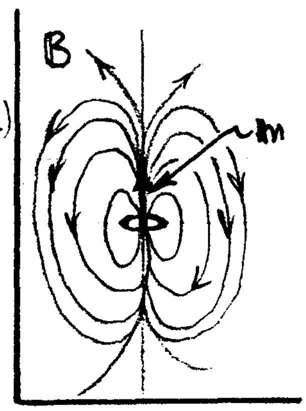
When $\epsilon \ll 1$ ("far" away from a "small" loop)...

See Ch. 4 Jk^{n} (4.12)

$$B_r \simeq \frac{2m\cos\theta}{r^3}\left\{1-\frac{9}{2}\left(\frac{a}{r}\right)\sin\theta+\cdots\right\} \rightarrow \frac{2m\cos\theta}{r^3}$$

$$B_\theta \simeq \frac{m\sin\theta}{r^3}\left\{1-\frac{11}{2}\left(\frac{a}{r}\right)\sin\theta+\cdots\right\} \rightarrow \frac{m\sin\theta}{r^3}$$

DIPOLE FIELD $\underline{(21.6)}$



This is the basic behavior of the elemental magnetic field source (a current loop)... it generates a dipole field, whose strength is measured by a "magnetic dipole moment" of size: $m = \frac{1}{c}I\times(\text{loop area})$.

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* $k^2 \ll 1 \Rightarrow$

$$K(k) = \frac{\pi}{2}\left[1+(1/2)^2k^2+(1\cdot3/2\cdot4)^2k^4+\cdots\right]$$

$$E(k) = \frac{\pi}{2}\left[1-(1/2)k^2-\frac{1}{3}(1\cdot3/2\cdot4)^2k^4-\cdots\right]$$