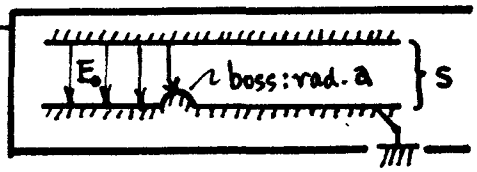
⑭ [Jackson Prob. (1.12)]. Prove Green's Reciprocation Theorem: if potential ϕ is due to volume & surface charge densities ρ & σ in a volume V enclosed by surface S, and if ϕ' is generated by ρ' & σ' in the same V enclosed by S, then:

$$\boxed{\int_V \rho\phi' d^3x + \oint_S \sigma\phi' da = \int_V \rho'\phi\, d^3x + \oint_S \sigma'\phi\, da\ .}$$

⑮ Consider the ODE (ordinary differential equation): $\mathcal{D}(u)=0$, where $\mathcal{D}$ is the operator: $\mathcal{D} = p_2(x)\dfrac{d^2}{dx^2} + p_1(x)\dfrac{d}{dx} + p_0(x)$, and the interval is $x \in [a,b]$. $\mathcal{D}$ is <u>not</u> self-adjoint unless $p_1 = dp_2/dx$; this is rarely the case at first glance. Show, however, that a function $\mu(x)$ can be constructed [from the $p_i(x)$] such that $\tilde{\mathcal{D}} = \mu(x)\mathcal{D}$ <u>is</u> self-adjoint. Find $\mu(x)$ explicitly. What conditions on the $p_i(x)$ are required for $\mu(x)$ to exist?

⑯ [Jackson Prob. (2.6)] [15 pts]. On one plate of a large ‖ plate capacitor (of separation s) there is a small hemi-spherical boss of radius $a \ll s$, as shown. This plate is grounded. The other plate is at a potential V such that far from the boss the inter-plate electric field is $E_0 = \dfrac{V}{s}$ = Cnst.

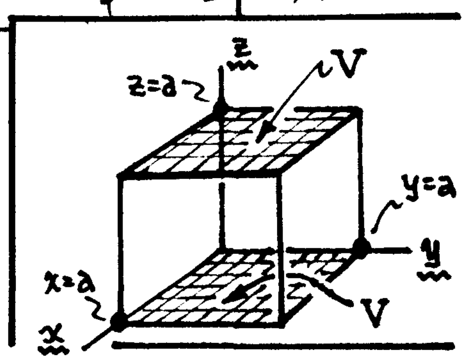


(A) After finding the potential between the plates (<u>HINT</u>: use spherical polar cds), calculate the surface-charge densities on the boss, and at an arbitrary pt. on the plane.

(B) Show that the total charge on the boss has magnitude: $\frac{3}{4}E_0 a^2$.

(C) If, instead of the other plate charged to potential V, a pt. charge q were placed at distance $d > a$ above the center of the boss, show that the charge induced on the boss is: $\tilde{q} = (-)q\{1-[(1-\epsilon^2)/\sqrt{1+\epsilon^2}\}$, w/ $\epsilon = \frac{a}{d}$. Show $\tilde{q} \simeq -\frac{3}{2}q\epsilon^2$, for $\epsilon \ll 1$.

⑰ [Jackson Prob. (2.13)]. Find the potential everywhere inside a hollow conducting cube of side a, when four sides are held at $\phi = 0$, and the top and bottom faces are at $\phi = V$ = cnst. Etc. Do this problem as stated in text.



(14) [Jackson Prob. (1.12)]. Prove Green's Reciprocation Theorem.

1) Use Green's Theorem [Jk^n Eq. (1.35)] with 1st fcn $= \phi$, 2nd fcn $\psi = \phi'$...

$$\rightarrow \int_V (\phi \nabla^2 \phi' - \phi' \nabla^2 \phi)\, dV = \oint_S \left(\phi \frac{\partial \phi'}{\partial n} - \phi' \frac{\partial \phi}{\partial n}\right) da. \qquad (1)$$

But ϕ & ϕ' generated by ρ & ρ' $\Rightarrow$ $\nabla^2\phi = -4\pi\rho$ & $\nabla^2\phi' = -4\pi\rho'$, in V. Also, on S the normal derivatives are proportional to the surface charge densities; in fact: $\partial\phi/\partial n = +4\pi\sigma$, $\partial\phi'/\partial n = +4\pi\sigma$ [note the + sign... this results because n is the outward unit normal on S, while σ is defined by the local E(normal) pointing toward the interior of V]. Put these expressions into Eq. (1) and rearrange a few terms...

$$\int_V (-\cancel{4\pi}\rho'\phi + \cancel{4\pi}\rho\phi')\, dV = \oint_S (\cancel{4\pi}\sigma'\phi - \cancel{4\pi}\sigma\phi')\, da$$

or $$\boxed{\int_V \rho'\phi\, dV + \oint_S \sigma'\phi\, da = \int_V \rho\phi'\, dV + \oint_S \sigma\phi'\, da.} \quad \underline{\text{QED}} \qquad (2)$$

---

2) NOTE: Applied to set of n conductors, first with charge Q_j & potential ϕ_j on the j^{th} conductor, then with a new assignment Q_j' & ϕ_j' on the the j^{th} conductor (and with the surface S at ∞), Eq. (2) yields:

$$\rightarrow \sum_{j=1}^{n} Q_j' \phi_j = \sum_{j=1}^{n} Q_j \phi_j'. \qquad (3)$$

If all but two conductors, 1 & 2, are grounded, and if all the charges are zero except Q_1 & Q_2', then: $Q_2'\phi_2 = Q_1\phi_1'$. Now set $Q_2' = Q_1 = 1$, so that: $\phi_2\left(\begin{smallmatrix}\text{for unit charge}\\ \text{on conductor on \#1}\end{smallmatrix}\right) = \phi_1\left(\begin{smallmatrix}\text{for unit charge}\\ \text{on conductor \#2}\end{smallmatrix}\right)$. This implies that the potential of an uncharged conductor acted on by a unit charge at point P is the same as the potential at P due to a unit charge placed on the conductor. This is why the theorem is called a "reciprocity theorem."

(15) $\mathcal{D}(u)=0$, w/ $\mathcal{D} = p_2 \frac{d^2}{dx^2} + p_1 \frac{d}{dx} + p_0$, $a \leq x \leq b$. Find μ so that $\mu\mathcal{D}$ is self-adjoint.

1) If $p_1 \neq p_2'$, $\mathcal{D}$ is not self-adjoint, in which case look at

$$\rightarrow \tilde{\mathcal{D}} = \mu(x)\,\mathcal{D} = \tilde{p}_2 \frac{d^2}{dx^2} + \tilde{p}_1 \frac{d}{dx} + \tilde{p}_0, \quad \tilde{p}_i = \mu p_i \; (i=0,1,2). \tag{1}$$

$\tilde{\mathcal{D}}$ will be self-adjoint iff $\tilde{p}_1 = \frac{d}{dx}\tilde{p}_2$, i.e. we want μ such that

$$\rightarrow \mu p_1 = \frac{d}{dx}(\mu p_2), \quad \text{or} \quad \frac{1}{\mu}\left(\frac{d\mu}{dx}\right) = \frac{1}{p_2}\left(p_1 - \frac{dp_2}{dx}\right). \tag{2}$$

The differential eqtn for μ is easily integrated to give

$$\ln\left[\frac{\mu(x)}{\mu(a)}\right] = \int_a^x \frac{d\xi}{p_2}\left(p_1 - \frac{dp_2}{d\xi}\right),$$

$$\text{or} \quad \mu(x) = \mu(a)\exp\left\{\int_a^x d\xi\,[p_1(\xi)/p_2(\xi)] - \int_a^x \frac{dp_2}{p_2}\right\}. \tag{3}$$

2) Set $\mu(a)=1$ for convenience [a scale factor is unimportant in $\tilde{\mathcal{D}}(u)=0$], and integrate the term $\int dp_2/p_2$ on RHS of Eq. (3). Then...

$$\boxed{\mu(x) = \frac{p_2(a)}{p_2(x)}\, e^{\int_a^x d\xi\,[p_1(\xi)/p_2(\xi)]}}, \tag{4}$$

is the desired self-adjoint factor. If $p_1 = p_2'$, then $\mu(x) \equiv 1$ on $[a,b]$, as should be. Otherwise, in order for $\mu(x)$ to exist, $p_2(x)$ must be non-zero on $[a,b]$, and $p_1(x)$ can at most have a finite number of singularities.

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3) Check $\tilde{\mathcal{D}}$ for self-adjointhood; i.e. does $\int_a^b v\,\tilde{\mathcal{D}}(u)\,d\xi = \int_a^b u\,\tilde{\mathcal{D}}(v)\,d\xi$, for u & v any solns to the diff eq? By two partial integrations...

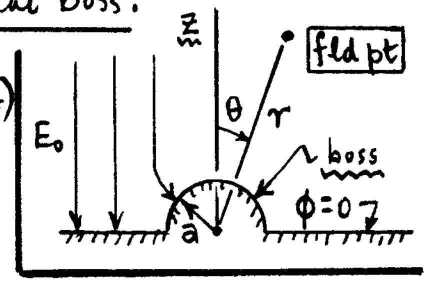
$$\int_a^b v\tilde{\mathcal{D}}(u)\,d\xi = \int_a^b v\left[\frac{d}{d\xi}(\tilde{p}_2 u') + \tilde{p}_0 u\right]d\xi = \underbrace{\cancel{(v\tilde{p}_2 u')}}_{0}\Big|_a^b - \int_a^b(\tilde{p}_2 v')u'\,d\xi + \int_a^b v\tilde{p}_0 u\,d\xi$$

$$= -\underbrace{\cancel{(u\tilde{p}_2 v')}}_{0}\Big|_a^b + \int_a^b u\frac{d}{d\xi}(\tilde{p}_2 v')\,d\xi + \int_a^b u\tilde{p}_0 v\,d\xi = \int_a^b u\,\tilde{\mathcal{D}}(v)\,d\xi. \; \{\text{OK}$$

⑯ [Jkⁿ #(2.6)] [15 pts]. ||-plate capacitor with hemispherical boss.

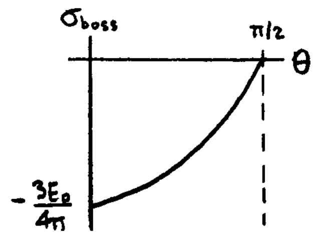
(A) The potential between the plates appears in Jkⁿ Eq.(2.14), viz.

$$\boxed{\phi(r,\theta) = E_0\left(r - \frac{a^3}{r^2}\right)\cos\theta} \quad \left\{ \begin{array}{l} 0 \leq \theta \leq \frac{\pi}{2}, \\ r \geq a; \end{array} \right. \qquad (1)$$



with cds as shown. This ϕ is the potential for a conducting sphere of radius a in a uniform field E_0, and it meets the requisite B.C. ($\phi = 0$ @ $r = a$ and $\phi = 0$ @ $\theta = \frac{\pi}{2}$). The first term is due to the plates, and the second term is generated by the boss. Note: E_0(here) = $-E_0$(Jkⁿ Fig. 2.6).

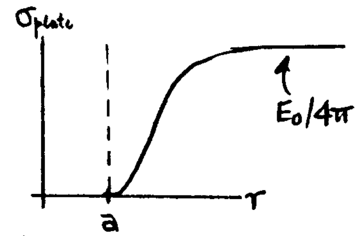
The charge density on the boss is...



$$\rightarrow \sigma_{boss}(\theta) = -\frac{1}{4\pi}\left(\frac{\partial\phi}{\partial r}\right)\Big|_{r=a} = -\left(\frac{3E_0}{4\pi}\right)\cos\theta, \qquad (2)$$

... and on the grounded plane...

$$\rightarrow \sigma_{plane}(r) = -\frac{1}{4\pi r}\left(\frac{\partial\phi}{\partial\theta}\right)\Big|_{\theta=\frac{\pi}{2}} = \frac{E_0}{4\pi}\left(1 - \frac{a^3}{r^3}\right). \qquad (3)$$



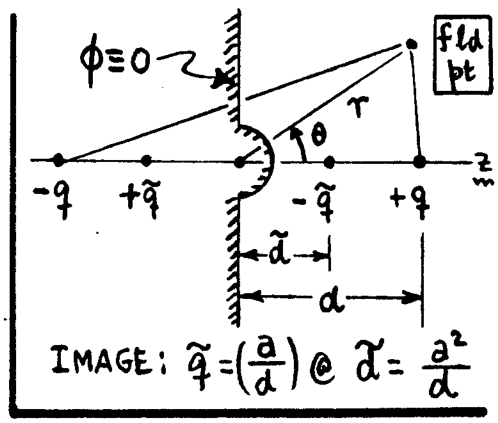
Note that $\sigma_{plate} \rightarrow \frac{E_0}{4\pi}$, cnst, at r = few × a.

(B) The charge on the boss is...

$$\rightarrow Q_{boss} = \int_0^{\pi/2} \sigma_{boss} \cdot 2\pi a^2 \sin\theta \, d\theta = -\frac{3E_0}{4\pi} \cdot 2\pi a^2 \int_{\pi/2}^{0} \cos\theta \, d\cos\theta = -\frac{3E_0 a^2}{4}. \qquad (4)$$

So $\boxed{|Q_{boss}| = \frac{3}{4}|E_0| a^2}$, as required.

(C) Do this problem by image method. If pt. q is at d in front of plane, image $(-)q$ at d in back of plane renders the plane an equipotential surface ($\phi = 0$). For the boss, a charge $-\tilde{q} = -(a/d)q$ located at $\tilde{d} = a^2/d$ [$\tilde{d}$ is actually inside the boss; drawing at right is not accurate in that regard] puts the hemi–



(16)(C) (cont'd)

Sphere at $\phi = 0$ in presence of the exterior $+q$ [see Jackson Sec 2.2, Eqs (2.4)]. The equipotential on the plane is restored if $-\tilde{q}$ is balanced by its image $+\tilde{q}$ at $\tilde{d}$ in back of plane. The potential at some field pt. (r, θ) in front of plane is then just the problem of four charges; $\pm q, \pm\tilde{q}$; straightforwardly...

$$\rightarrow \phi(r,\theta) = \frac{q}{\sqrt{r^2+d^2-2rd\cos\theta}} - \frac{q}{\sqrt{r^2+d^2+2rd\cos\theta}} - \frac{\tilde{q}}{\sqrt{r^2+\tilde{d}^2-2r\tilde{d}\cos\theta}} + \frac{\tilde{q}}{\sqrt{r^2+\tilde{d}^2+2r\tilde{d}\cos\theta}}. \tag{5}$$

Put in $\tilde{q} = \left(\frac{a}{d}\right) q$ & $\tilde{d} = \frac{a^2}{d}$ to obtain...

$$\rightarrow \phi(r,\theta) = q\left\{\left[r^2+d^2-2rd\cos\theta\right]^{-\frac{1}{2}} - \left[a^2+\frac{d^2}{a^2}r^2-2rd\cos\theta\right]^{-\frac{1}{2}} - \left[r^2+d^2+2rd\cos\theta\right]^{-\frac{1}{2}} + \left[a^2+\frac{d^2}{a^2}r^2+2rd\cos\theta\right]^{-\frac{1}{2}}\right\}. \tag{6}$$

On the boss $(r=a)$, the 1st & 2nd, and 3rd & 4th terms are the same. The charge density on the boss is...

$$\rightarrow \sigma(\theta) = -\frac{1}{4\pi}\left(\frac{\partial\phi}{\partial r}\right)\Big|_{r=a} = -\frac{q}{4\pi}\left(\frac{d^2-a^2}{a}\right)\left\{\frac{1}{(d^2+a^2-2ad\cos\theta)^{3/2}} - \frac{1}{(d^2+a^2+2ad\cos\theta)^{3/2}}\right\}, \tag{7}$$

... and the charge on the boss is...

$$\rightarrow Q_{boss} = 2\pi a^2 \int_0^{\pi/2} \sigma(\theta)\sin\theta\, d\theta = -\frac{qa}{2}(d^2-a^2)\int_0^1 d\mu\left\{\frac{1}{(d^2+a^2-2ad\mu)^{3/2}} - \frac{1}{(d^2+a^2+2ad\mu)^{3/2}}\right\}$$

take (+) ve sq. rt.

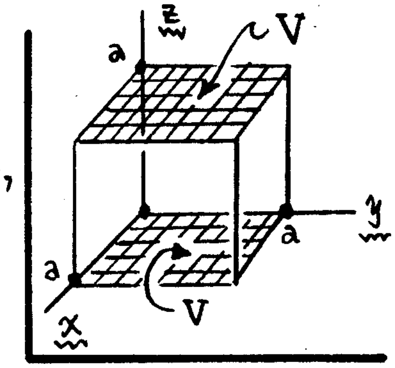
$$\text{or}\quad Q_{boss} = -\frac{qad}{\cancel{2}}\left(\frac{d^2-a^2}{d}\right)\left[\frac{\cancel{2}}{-2ad}\frac{1}{\sqrt{d^2+a^2-2ad\mu}} - \frac{\cancel{2}}{+2ad}\frac{1}{\sqrt{d^2+a^2+2ad\mu}}\right]\Bigg|_{\mu=1}^{\mu=0}$$

$$\text{or}\quad \boxed{Q_{boss} = -q\left\{1-\left(\frac{d^2-a^2}{d}\right)\frac{1}{\sqrt{d^2+a^2}}\right\} = -q\left\{1-\frac{(1-\epsilon^2)}{\sqrt{1+\epsilon^2}}\right\}},\quad \epsilon = \frac{a}{d} \leq 1. \tag{8}$$

Since $(1/\sqrt{1+\epsilon^2}) \simeq 1-\frac{1}{2}\epsilon^2$ for $\epsilon \ll 1$, indeed: $Q_{boss} \simeq -\frac{3}{2}q\epsilon^2$, when $d \gg a$. So when the pt. source is far away from the boss, it induces a very small Q_{boss}. Compare this with Eq. (4), where Q_{boss} is cnst when $E_0 =$ cnst.

ϕ519 Solutions

⑰ [Jackson Prob. (2.13)]. ϕ(inside) for hollow conducting cube.



(A) The B.C. that $\phi = 0$ @ $x = 0$ & a and $y = 0$ & a demand that, in a series like Jk$^{\underline{n}}$ Eq. (2.56), the x and y variations go as $(\sin \alpha_n x)$ and $(\sin \beta_m y)$, w/ $\underline{\underline{\alpha_n}} = \frac{n\pi}{a}$, $\beta_m = \frac{m\pi}{a} = \alpha_m$, (and $n, m = 1, 2, 3, \ldots$). The solution must look like

$$\rightarrow \phi(x,y,z) = \sum_{n,m=1}^{\infty} [\sin \alpha_n x][\sin \alpha_m y] \{ A_{nm} \sinh \gamma_{nm} z + B_{nm} \cosh \gamma_{nm} z \}, \quad (1)$$

where $\gamma_{nm} = \sqrt{\alpha_n^2 + \alpha_m^2} = \frac{\pi}{a}\sqrt{n^2 + m^2}$. The A_{nm} & B_{nm} are fixed by the B.C. that $\phi = V$ (= cnst) @ $z = 0$ & a, viz...

$\underline{\underline{\phi = V \text{ @ } z = 0}} \Rightarrow V = \sum_{n,m} B_{nm} [\sin \alpha_n x][\sin \alpha_m x] \leftarrow$ project B_{nm} by orthogonality

so// $B_{nm} = \left(\frac{2}{a}\right)^2 \int_0^a dx [\sin \alpha_n x] \int_0^a dy [\sin \alpha_m y] V = \frac{4V}{a^2} \frac{1}{\alpha_n \alpha_m} [1 - \cos n\pi][1 - \cos m\pi]$,

$$\xrightarrow{\text{or}} B_{nm} = \frac{4V}{\pi^2 nm}\left(1 - [(-)^m + (-)^n] + (-)^{m+n}\right) = \begin{cases} 16V/\pi^2 nm, & n \,\&\, m \text{ both odd;} \\ 0, & \text{otherwise.} \end{cases} \quad (2)$$

$\underline{\underline{\phi = V \text{ @ } z = a}} \Rightarrow V = \sum_{n,m = \text{odd}} \{ A_{nm} \sinh \gamma_{nm} a + B_{nm} \cosh \gamma_{nm} a \} [\sin \alpha_n x][\sin \alpha_m y]$.

Same procedure as above $\Rightarrow \{ \; \} = B_{nm}$, i.e. $A_{nm} = \left(\frac{1 - \cosh \gamma_{nm} a}{\sinh \gamma_{nm} a}\right) B_{nm}$. $\quad (3)$

The potential everywhere inside the cube is then...

$$\phi(x,y,z) = \frac{16V}{\pi^2} \sum_{n,m = \text{odd}} \frac{1}{nm} \left\{ \left(\frac{1 - \cosh \gamma_{nm} a}{\sinh \gamma_{nm} a}\right) \sinh \gamma_{nm} z + \cosh \gamma_{nm} z \right\} [\sin \alpha_n x][\sin \alpha_m y],$$

where: $\alpha_n = \frac{\pi n}{a}$, $\gamma_{nm} = \frac{\pi}{a}\sqrt{n^2 + m^2}$, and $n, m = 1, 3, 5, \ldots \quad (4)$

The $\{ \; \}$ here can be put in much more palatable form by some hyperbolic trig. Write $\{\text{Eq.}(4)\} = \frac{1}{\sinh \gamma_{nm} a} \{ \sinh \gamma_{nm} z - (\sinh \gamma_{nm} z \cosh \gamma_{nm} a - \cosh \gamma_{nm} z \sinh \gamma_{nm} a) \} =$

$= \frac{1}{\sinh \gamma_{nm} a} \{ \sinh \gamma_{nm} z - \sinh \gamma_{nm} (z - a) \} = \ldots = \frac{1}{\cosh(\gamma_{nm} a/2)} \cosh \gamma_{nm}\left(z - \frac{a}{2}\right)$. Then

ϕ519 Solutions $\qquad \alpha_n = \frac{\pi}{a} n, \quad \gamma_{nm} = \frac{\pi}{a}\sqrt{n^2+m^2}$

(17)(A)(cont'd)

$$\boxed{\phi(x,y,z) = \frac{16V}{\pi^2} \sum_{n,m=\text{odd}} \frac{1}{nm} \left\{ \frac{\cosh \gamma_{nm}(z-\frac{a}{2})}{\cosh(\gamma_{nm} a/2)} \right\} [\sin \alpha_n x][\sin \alpha_m y]} \qquad (5)$$

In this form, evidently ϕ gives the same series @ $z=0$ & a; both sum to V.

(B) At the center of the cube $x=y=z=a/2$, have $\alpha_n x = \frac{\pi}{2} n$, with $n = \text{odd} = 2k+1$ ($k=0,1,2,\ldots$). So: $\sin \alpha_n x = (-)^k$, at center. Similarly $\sin \alpha_m y = (-)^\ell$. Then

$$\rightarrow \phi(\text{ctr}) = \frac{16V}{\pi^2} \sum_{k,\ell=0}^{\infty} (-)^{k+\ell} \Big/ (2k+1)(2\ell+1) \cosh\left(\frac{\pi}{2}\sqrt{(2k+1)^2+(2\ell+1)^2}\right). \qquad (6)$$

or $$\frac{\phi(\text{ctr})}{V} = \frac{16}{\pi^2} \Bigg\{ \underbrace{\frac{1}{\cosh\frac{\pi}{2}\sqrt{2}}}_{k=0=\ell} - \underbrace{\frac{2}{3}\frac{1}{\cosh\frac{\pi}{2}\sqrt{10}}}_{(k,\ell)=(1,0)\,\&\,(0,1)} + \Bigg[\underbrace{\frac{2}{5}\frac{1}{\cosh\frac{\pi}{2}\sqrt{26}}}_{(k,\ell)=(2,0)\,\&\,(0,2)} + \underbrace{\frac{1}{9}\frac{1}{\cosh\frac{\pi}{2}\sqrt{18}}}_{k=1=\ell} \Bigg] - \cdots \Bigg\}$$

$$= 0.347\,546 - 0.015\,048 + [0.000431 + 0.000460] - \cdots$$

$$= 0.333\,389 - \Theta(10^{-5}), \text{ through terms with } k+\ell = 2. \qquad (7)$$

The next terms (w/ $k+\ell=3$) are $\Theta(10^{-5})$. Since this is an alternating series of decreasing terms, the value will not change by more than the last term ignored. Hence: $\phi(\text{ctr}) = [0.33339 - \Theta(10^{-5})]V$, and 3 figure accuracy is attained by just 6 of the terms in the series in Eq. (6). To within this accuracy, we see $\phi(\text{ctr}) = \phi_{Av}(\text{walls}) = 2V/6 = 0.33333\,V$.

(C) Using ϕ per Eq. (5), we calculate the surface charge density on plane $z=a$:

$$\rightarrow \sigma(x,y) = -\frac{1}{4\pi}\left(\frac{\partial \phi}{\partial z}\right)\Big|_{z=a} = (-)\frac{4V}{\pi^3} \sum_{n,m=\text{odd}} \frac{\gamma_{nm}}{nm} \left\{ \tanh\left(\frac{\gamma_{nm} a}{2}\right) \right\} [\sin \alpha_n x][\sin \alpha_m y].$$

or $$\boxed{\sigma(x,y) = (-)\frac{4V}{\pi^2 a} \sum_{n,m=\text{odd}} \sqrt{\frac{1}{n^2}+\frac{1}{m^2}} \left\{ \tanh(\gamma_{nm} a/2) \right\} [\sin \alpha_n x][\sin \alpha_m y].} \qquad (8)$$

This series appears to be conditionally convergent.