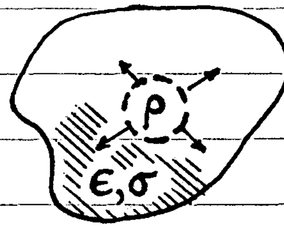
Φ519 Final Exam (in class, 3 hr. limit)

Mon. 12 Dec. 1988

This exam is open-book, open-notes, and is worth 150 points total. For each problem, put your answer in a $\boxed{\text{box}}$ on your solution sheets. $\boxed{\text{Number}}$ your solution sheets, write your $\boxed{\text{name}}$ on sheet #1, and $\boxed{\text{staple}}$ the sheets together before handing them in.

---

(1) [redacted]. A very large sample of material is homogeneous and isotropic; it has permittivity ϵ and conductivity σ, and Ohm's Law is obeyed: $\mathbf{J} = \sigma \mathbf{E}$. At time $t=0$, there is a given free-charge density $\rho_0(\mathbf{r})$ localized in the interior of the material.

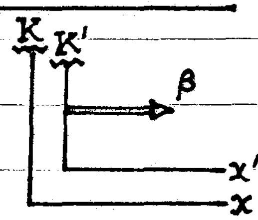


(A) Show that ρ_0 will decay exponentially in time, as: $\rho(\mathbf{r},t) = \rho_0(\mathbf{r}) e^{-\nu t}$, for $t \geq 0$. Calculate the "relaxation time" $\tau = 1/\nu$ in terms of ϵ and σ.

(B) If the initial distribution is spherically symmetric: $\rho_0(\mathbf{r}) = \rho_0(r)$, find the current density $\mathbf{J}$ as a fcn of $\mathbf{r}, t$. Is the solution causally peculiar?

---

(2) [redacted]. Coordinate frames K & K$'$ are related by a Lorentz boost along the x-axis. That is, for $x_0 = ct$, $\beta = \frac{v}{c}$, $\gamma = 1/\sqrt{1-\beta^2}$:

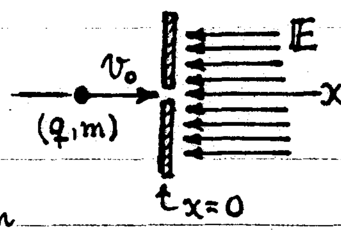


$$\boxed{x' = \gamma(x - \beta x_0)\ ,\quad x_0' = \gamma(x_0 - \beta x)}.$$

By writing out the derivatives explicitly, verify that the D'Alembertian (wave operator): $\Box = (\partial/\partial x_0)^2 - (\partial/\partial x)^2$, is a Lorentz invariant.

---

(3) [redacted]. A particle of charge q and mass m travels at relativistic velocity v_0 along the x-axis of the lab frame. At $x=0$, q passes through a small hole in one plate of a capacitor (fixed in lab) and encounters a constant electric field $\mathbf{E} = -E\hat{x}$ which opposes its motion.



(A) Find the distance s (in lab) which q travels -- to the right of the plate -- before it stops.

(B) If you had done this problem nonrelativistically, would your estimate of q's stopping distance s have been smaller or larger than the result of part (A)?

ϕ519 Final Exam (cont'd)

(12/12/88)

④ [redacted]. In empty space, the wave eqtns for the electric & magnetic fields $\mathbb{E}$ & $\mathbb{B}$ are: $[\nabla_r^2 - \frac{1}{c^2}(\partial^2/\partial t^2)]\{\mathbb{E},\mathbb{B}\} = 0$. It is straightforward to show that particular solutions to these eqtns can be written in the form, where $\mathbb{E}_0$ & $\mathbb{B}_0$ are constant amplitudes:

$$\boxed{\mathbb{E}(\mathbf{r},t) = \mathbb{E}_0 f(\phi), \quad \mathbb{B}(\mathbf{r},t) = \mathbb{B}_0 g(\phi)}, \text{ w/ } \phi = \mathbb{k}\cdot\mathbf{r} - \omega t \text{ (a "phase").}$$

$\mathbb{k}$ is a constant wavevector, and ω is another (scalar) constant with dimensions of frequency. f & g are arbitrary (twice differentiable) scalar fcns. Use Maxwell's Eqs to show:

(A) $\mathbb{k}$, $\mathbb{E}_0$ and $\mathbb{B}_0$ must be mutually orthogonal; [redacted]: $\frac{\partial}{\partial q}F(\theta(q)) = \left(\frac{\partial\theta}{\partial q}\right)\frac{dF}{d\theta}$.

(B) The frequency ω and wavenumber $k = |\mathbb{k}|$ are related by: $\omega^2 = k^2c^2$.

---

⑤ [redacted]. The Maxwell field tensor $\underset{\sim}{F}$ is given in its covariant & contravariant forms by Jackson's Eqs. (11.138) & (11.137), resp.

$$\longrightarrow (F_{\alpha\beta}) = \begin{bmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & -B_3 & B_2 \\ -E_2 & B_3 & 0 & -B_1 \\ -E_3 & -B_2 & B_1 & 0 \end{bmatrix}, \quad (F^{\mu\nu}) = \begin{bmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & -B_3 & B_2 \\ E_2 & B_3 & 0 & -B_1 \\ E_3 & -B_2 & B_1 & 0 \end{bmatrix}.$$

With the summation convention in effect (i.e. sum over repeated indices):

(A) Show that $F_{\alpha\beta}F^{\alpha\beta}$ is a Lorentz-invariant scalar;

(B) Calculate $F_{\alpha\beta}F^{\alpha\beta}$ explicitly (in terms of $\mathbb{E}$ & $\mathbb{B}$) to find the field invariant.

[redacted]: if $\underset{\sim}{P}$ & $\underset{\sim}{Q}$ are tensors: $(\underset{\sim}{P}\underset{\sim}{Q})_\alpha^{\ \lambda} = P_{\alpha\kappa}Q^{\kappa\lambda}$, then $(\underset{\sim}{P}\underset{\sim}{Q})_\alpha^{\ \alpha} = P_{\alpha\kappa}Q^{\kappa\alpha} = \mathrm{Tr}(\underset{\sim}{P}\underset{\sim}{Q})$.

---

⑥ [redacted]. Consider the Lagrange density $\mathcal{L}$ for a free scalar field $\phi = \phi(x^\nu)$:

$$\boxed{\mathcal{L} = \frac{1}{2}[(\partial_\alpha\phi)(\partial^\alpha\phi) - \mu^2\phi^2] = \frac{1}{2}\left[\left(\frac{\partial\phi}{c\,\partial t}\right)^2 - (\nabla\phi)^2 - \mu^2\phi^2\right].}$$

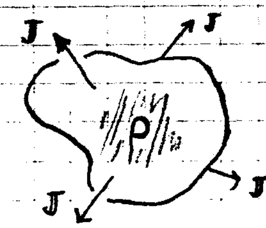
Here, μ = cnst. And ϕ is, of course, a continuous fcn of the space-time coordinates x^ν.

(A) Find the "equation-of-motion" (i.e. the wave-type eqtn) which ϕ obeys.

(B) Now add a term (on RHS) $4\pi\rho\phi$ to above $\mathcal{L}$. How does this change the wave eqtn for ϕ? How do you interpret ρ in the new ϕ eqtn? What role does μ play?

Φ 519 Final Exam Solutions

(1) [redacted]. Calculate the charge relaxation time for a conducting medium.



A. 1) The current density $\mathbb{J}$ is related to the charge density ρ by the continuity eqtn: $\nabla\cdot\mathbf{J} = -\partial\rho/\partial t$. Also, ρ generates an $\mathbb{E}$-field by Gauss' Law: $\nabla\cdot\mathbb{E} = (4\pi/\epsilon)\rho$; it is this $\mathbf{E}$ which drives $\mathbf{J}$ according to Ohm's Law: $\mathbb{J} = \sigma\mathbb{E}$. Altogether, we have the system...

11/19/88

$$\rightarrow -\partial\rho/\partial t = \nabla\cdot\mathbb{J} = \nabla\cdot(\sigma\mathbb{E}) = \sigma\nabla\cdot\mathbb{E} = (4\pi\sigma/\epsilon)\rho,$$

$$\text{or}\quad \frac{\partial\rho}{\partial t} = (-)\nu\rho \Rightarrow \boxed{\rho(\mathbf{r},t) = \rho(\mathbf{r},0) = \rho_0(\mathbf{r})\,e^{-\nu t}},\quad \text{w/}\ \nu = \frac{4\pi\sigma}{\epsilon}. \qquad (1)$$

So the initial distribution $\rho_0(\mathbf{r})$ does decay exponentially in time; the time cnst is:

$$\boxed{\tau = 1/\nu = \epsilon/4\pi\sigma}. \qquad (2)$$

B. 2) If the initial distribution is spherically symmetric, $\rho_0(\mathbf{r}) = \rho_0(r)$, then the continuity eqtn $\nabla\cdot\mathbb{J} = \nu\rho$ implies the current is radial: $\mathbb{J} = \hat{r}J_r$. Thus in spherical polar coordinates, we have [for $\nabla\cdot$ in sph. cds, see Jackson's back cover]:

$$\rightarrow \nabla\cdot\mathbb{J} = \nu\rho \Rightarrow \frac{1}{r^2}\frac{\partial}{\partial r}(r^2J_r) = \nu\rho_0(r)e^{-\nu t},$$

$$\text{or}\quad \frac{\partial}{\partial r}(r^2J_r) = (\nu e^{-\nu t})\,r^2\rho_0(r). \qquad (3)$$

The current at $r=0$ should be zero, so one integration of Eq. (3) gives...

$$\boxed{J_r(r,t) = \frac{\nu e^{-\nu t}}{r^2}\int_0^r x^2\rho_0(x)\,dx}. \qquad (4)$$

In order for this solution to make sense, t & r must be causally connected -- e.g. $t > r/v$, where v is the average velocity of charge transport to r.

ϕ 519 Final Exam Solutions (cont'd)

(2) [redacted]. Establish Lorentz invariance of wave operator under 1D Lorentz boost.

1) Write the D'Alembertian as...

11/19/88

$$\Box = \frac{\partial^2}{\partial x_0^2} - \frac{\partial^2}{\partial x^2} = \left(\frac{\partial}{\partial x_0} + \frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial x_0} - \frac{\partial}{\partial x}\right), \quad x_0 = ct. \tag{1}$$

2) x_0 and x are each fcns of both x_0' and x'. Thus, for the transformation $\underline{x' = \gamma(x - \beta x_0),\ x_0' = \gamma(x_0 - \beta x)}$, the derivatives go as...

$$\frac{\partial}{\partial x_0} = \left(\frac{\partial x_0'}{\partial x_0}\right)\frac{\partial}{\partial x_0'} + \left(\frac{\partial x'}{\partial x_0}\right)\frac{\partial}{\partial x'} = \gamma\left(\frac{\partial}{\partial x_0'} - \beta\frac{\partial}{\partial x'}\right),$$

$$\frac{\partial}{\partial x} = \left(\frac{\partial x_0'}{\partial x}\right)\frac{\partial}{\partial x_0'} + \left(\frac{\partial x'}{\partial x}\right)\frac{\partial}{\partial x'} = \gamma\left(-\beta\frac{\partial}{\partial x_0'} + \frac{\partial}{\partial x'}\right); \tag{2}$$

... and the combinations which appear in Eq. (1) are...

$$\frac{\partial}{\partial x_0} + \frac{\partial}{\partial x} = \gamma(1-\beta)\left(\frac{\partial}{\partial x_0'} + \frac{\partial}{\partial x'}\right),$$

$$\frac{\partial}{\partial x_0} - \frac{\partial}{\partial x} = \gamma(1+\beta)\left(\frac{\partial}{\partial x_0'} - \frac{\partial}{\partial x'}\right). \tag{3}$$

3) The D'Alembertian in primed coordinates is the product of the two factors in Eq. (3), i.e. under the given Lorentz transform $K \to K'$

$$\longrightarrow \Box = \frac{\partial^2}{\partial x_0^2} - \frac{\partial^2}{\partial x^2} \to \Box' = \left[\gamma(1-\beta)\left(\frac{\partial}{\partial x_0'} + \frac{\partial}{\partial x'}\right)\right]\cdot\left[\gamma(1+\beta)\left(\frac{\partial}{\partial x_0'} - \frac{\partial}{\partial x'}\right)\right]$$

i.e. $$\Box' = \underbrace{\gamma^2(1-\beta^2)}_{\equiv 1,\ \text{since } \gamma = 1/\sqrt{1-\beta^2}.}\left(\frac{\partial}{\partial x_0'} + \frac{\partial}{\partial x'}\right)\left(\frac{\partial}{\partial x_0'} - \frac{\partial}{\partial x'}\right) = \frac{\partial^2}{\partial x_0'^2} - \frac{\partial^2}{\partial x'^2}. \tag{4}$$

$\boxed{\text{So, under a Lorentz boost: } \Box \to \Box' \equiv (\Box)'}$; the <u>form</u> of $\Box$ is unchanged, and the elements of $\Box$ are just a matter of labelling. <u>Thus $\Box$ is a Lorentz invariant</u>; K & K' cannot be distinguished by the $\Box \to \Box'$ transform.

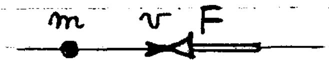
ϕ519 Final Exam Solutions (cont'd)

3 [redacted]. Calculate braking distance for relativistic q

A. 1) The easiest way to solve this problem is to use the relativistic work-energy theorem (from Xerox class notes of 2 Nov. 88). If a lab force $\mathbb{F}$ acts on a particle of mass m moving at velocity $\mathbb{V}$, then...

11/22/88

$$\rightarrow \mathbb{F}\cdot\mathbb{V} = \frac{d}{dt}(\gamma m c^2), \quad \text{w/ } \gamma = 1/\sqrt{1-\beta^2},\ \beta = \frac{v}{c}. \qquad (1)$$



2) In the present case, $\mathbb{V}$ is along the (+) x-axis, $\mathbb{F} = -q\mathbb{E}$ opposes $\mathbb{V}$ along the (−) x-axis, so that -- with $v = dx/dt$ -- Eq. (1) becomes:

$$-qE\frac{dx}{dt} = mc^2\frac{d\gamma}{dt}, \quad \text{or } \int dx = -(mc^2/qE)\int d\gamma. \qquad (2)$$

Integrate from: $x = 0,\ \gamma = \gamma_0 = 1/\sqrt{1-\beta_0^2}$, at entry, to: $x = S,\ \gamma = 1$ when q stops to obtain the desired stopping distance...

$$\boxed{S = -(mc^2/qE)\int_{\gamma_0}^{1} d\gamma = \frac{mc^2}{qE}(\gamma_0 - 1)}, \quad \gamma_0 = 1\Big/\sqrt{1-(v_0/c)^2}. \qquad (3)$$

B. 3) Nonrelativistically, q would stop at that position S_{nr} where the work done opposing its motion just equalled its KE, i.e.

$$qES_{nr} = \frac{1}{2}mv_0^2 \Rightarrow S_{nr} = \frac{1}{2}\frac{mv_0^2}{qE} = \frac{1}{2}\left(\frac{mc^2}{qE}\right)\beta_0^2. \qquad (4)$$

The comparison between the actual S of Eq. (3) and the nonrel. estimate is:

$$\boxed{S/S_{nr} = \frac{2}{\beta_0^2}(\gamma_0 - 1).} \qquad (5)$$

Since $\gamma_0 = 1/\sqrt{1-\beta_0^2} \simeq 1 + \frac{1}{2}\beta_0^2 + \frac{3}{8}\beta_0^4 + \ldots \Rightarrow \gamma_0 - 1 \simeq \frac{\beta_0^2}{2}\left(1 + \frac{3}{4}\beta_0^2 + \ldots\right)$, then the ratio in Eq. (5) is

$$S/S_{nr} \simeq 1 + \frac{3}{4}\beta_0^2 + \ldots > 1, \quad \text{i.e. } S_{nr} < S. \qquad (6)$$

The nonrelativistic calculation <u>underestimates</u> the stopping distance.

ϕ 519 Final Exam Solutions (cont'd)

(4) [redacted]. Analyse planewave solutions by vacuum Maxwell Eqtns.

A N 1) In vacuo, the Maxwell Eqs. are: $\nabla\cdot\{\mathbb{E},\mathbb{B}\}=0$, $\nabla\times\{\mathbb{E},\mathbb{B}\}=\frac{1}{c}\{-\dot{\mathbb{B}},\dot{\mathbb{E}}\}$, where "$\cdot$" $\Rightarrow \frac{\partial}{\partial t}$. Impose $\nabla\cdot\mathbb{E}=0$ on $\mathbb{E}=\mathbb{E}_0 f(\phi)$, $\phi=\mathbb{k}\cdot\mathbb{r}-\omega t$, to find...

$$\nabla\cdot\mathbb{E}=\nabla\cdot(\mathbb{E}_0 f)=f(\overbrace{\cancel{\nabla\cdot\mathbb{E}_0}}^{0,\ \mathbb{E}_0\text{ is cnst}})+\mathbb{E}_0\cdot\nabla f=\mathbb{E}_0\cdot\frac{\partial}{\partial\mathbb{r}}f(\phi)=0. \tag{1}$$

11/24/88

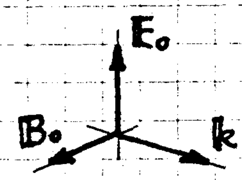
But $(\partial/\partial\mathbb{r})f(\phi)=(\partial\phi/\partial\mathbb{r})\frac{df}{d\phi}=\mathbb{k}f'(\phi)$, by implicit differentiation. So:

$$\nabla\cdot\mathbb{E}=(\mathbb{E}_0\cdot\mathbb{k})f'(\phi)=0 \Rightarrow (\mathbb{E}_0\cdot\mathbb{k})=0, \text{ or } \boxed{\mathbb{k}\text{ is }\perp\mathbb{E}_0}. \tag{2}$$

In the same way, we use $\nabla\cdot\mathbb{B}=0$ on $\mathbb{B}=\mathbb{B}_0 g(\phi)$ to get:

$$\nabla\cdot\mathbb{B}=(\mathbb{B}_0\cdot\mathbb{k})g'(\phi)=0 \Rightarrow (\mathbb{B}_0\cdot\mathbb{k})=0, \text{ or } \boxed{\mathbb{k}\text{ is }\perp\mathbb{B}_0}. \tag{3}$$

2) Next, impose Faraday's Law on the given $\mathbb{E}$ & $\mathbb{B}$ solutions...



$$\nabla\times(\mathbb{E}_0 f)=-\frac{1}{c}\frac{\partial}{\partial t}(\mathbb{B}_0 g) \Rightarrow (\mathbb{k}\times\mathbb{E}_0)f'=+\frac{\omega}{c}\mathbb{B}_0 g',$$

i.e. $\mathbb{B}_0=(cf'/\omega g')\,\mathbb{k}\times\mathbb{E}_0$, or $\boxed{\mathbb{B}_0\perp\text{ plane of }\mathbb{k}\ \&\ \mathbb{E}_0}$. (4)

Together with Eqs. (2) & (3), this means that $\mathbb{k}$, $\mathbb{E}_0$ & $\mathbb{B}_0$ form an orthogonal triad, as indicated in the sketch.

B N 3) To get information on ω, we use the last Maxwell Eq. (Ampere's Law)...

$$\nabla\times(\mathbb{B}_0 g)=\frac{1}{c}\frac{\partial}{\partial t}(\mathbb{E}_0 f) \Rightarrow (\mathbb{k}\times\mathbb{B}_0)g'=-\frac{\omega}{c}\mathbb{E}_0 f',$$

i.e. $\mathbb{k}\times\mathbb{B}_0=-(\omega f'/cg')\,\mathbb{E}_0$. (5)

At the same time: $\mathbb{B}_0=(cf'/\omega g')\,\mathbb{k}\times\mathbb{E}_0$, from Eq. (4). Plug this into Eq. (5), and use the vector triple product rule to get...

$$(-)\frac{\omega^2}{c^2}\mathbb{E}_0=\mathbb{k}\times(\mathbb{k}\times\mathbb{E}_0)=(\overbrace{\cancel{\mathbb{k}\cdot\mathbb{E}_0}}^{0,\text{ by Eq.(2)}})\mathbb{k}-k^2\mathbb{E}_0, \text{ or } \boxed{\omega^2=k^2c^2}. \tag{6}$$

The solutions $\mathbb{E}=\mathbb{E}_0 f(\phi)$, $\mathbb{B}=\mathbb{B}_0 g(\phi)$ are called "planewave solutions".

ϕ 519 Final Exam Solutions (cont'd)

(5) [redacted]. Extract a field invariant (E^2-B^2) from the field tensor.

A. 1) Since $F_{\alpha\beta}$ and $F^{\alpha\beta}$ are qualified covariant & contravariant tensors, they transform by the rules quoted in Jackson's Eqs. (11.64) & (11.63). In a new (primed) Lorentz frame:

$$\rightarrow F'_{\alpha\beta}F'^{\alpha\beta} = \left[\left(\frac{\partial x^\gamma}{\partial x'^\alpha}\right)\left(\frac{\partial x^\delta}{\partial x'^\beta}\right)F_{\gamma\delta}\right]\left[\left(\frac{\partial x'^\alpha}{\partial x^\kappa}\right)\left(\frac{\partial x'^\beta}{\partial x^\lambda}\right)F^{\kappa\lambda}\right] = F_{\gamma\delta}\, a^\gamma_\kappa\, a^\delta_\lambda\, F^{\kappa\lambda}, \quad (1)$$

11/24/88

where: $a^\gamma_\kappa = (\partial x^\gamma/\partial x'^\alpha)(\partial x'^\alpha/\partial x^\kappa)$, $a^\delta_\lambda = (\partial x^\delta/\partial x'^\beta)(\partial x'^\beta/\partial x^\lambda)$, by rearranging terms. The summation convention is in effect, so: $a^\gamma_\kappa = (\partial x^\gamma/\partial x^\kappa) = \delta^\gamma_\kappa$, the Kronecker delta (this follows from orthogonality of the x^γ cds). Similarly, $a^\delta_\lambda = \delta^\delta_\lambda$, and then -- on summing over the indices λ & κ -- Eq. (1) gives...

$$\rightarrow F'_{\alpha\beta}F'^{\alpha\beta} = F_{\gamma\delta}\,\delta^\gamma_\kappa\,\delta^\delta_\lambda\,F^{\kappa\lambda} = F_{\gamma\delta}\,\delta^\gamma_\kappa\,F^{\kappa\delta} = F_{\gamma\delta}F^{\gamma\delta}. \quad (2)$$

So $F_{\alpha\beta}F^{\alpha\beta}$ is a Lorentz invariant scalar; it is the same in all Lorentz frames.

B. 2) Since the field tensor $\underset{\sim}{F}$ is totally antisymmetric, then $F^{\alpha\beta} = (-)F^{\beta\alpha}$, and it is clear that the quantity of interest is...

$$\rightarrow F_{\alpha\beta}F^{\alpha\beta} = (-)F_{\alpha\beta}F^{\beta\alpha} = (-)\left[\underset{\sim}{F}(\text{covar.})\cdot\underset{\sim}{F}^T(\text{contra.})\right]_{\alpha\alpha} = (-)\mathrm{Tr}\left[\underset{\sim}{F}(\text{cov.})\cdot\underset{\sim}{F}^T(\text{con.})\right] \quad (3)$$

$\underset{\sim}{F}^T$(con.) is the transpose of the contravariant $\underset{\sim}{F}$, $\underset{\sim}{F}$(cov.) is the covariant $\underset{\sim}{F}$, and Tr $\Rightarrow$ trace, or sum of the diagonal elements. So we want...

$$\mathrm{Tr}[\;] = \mathrm{Tr}\begin{bmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & -B_3 & B_2 \\ -E_2 & B_3 & 0 & -B_1 \\ -E_3 & -B_2 & B_1 & 0 \end{bmatrix}\begin{bmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & B_3 & -B_2 \\ -E_2 & -B_3 & 0 & B_1 \\ -E_3 & B_2 & -B_1 & 0 \end{bmatrix} = \mathrm{Tr}\begin{bmatrix} -E^2 & & \boxed{\text{stuff}} & \\ & -E_1^2+B_3^2+B_2^2 & & \\ & & -E_2^2+B_3^2+B_1^2 & \\ \boxed{\text{stuff}} & & & -E_3^2+B_2^2+B_1^2 \end{bmatrix}$$

So//

$$(-)\mathrm{Tr}[\;] = E^2 + (E_1^2-B_2^2-B_3^2) + (E_2^2-B_1^2-B_3^2) + (E_3^2-B_1^2-B_2^2) = 2(E^2-B^2),$$

i.e.,

$$\boxed{F_{\alpha\beta}F^{\alpha\beta} = 2(E^2-B^2)}. \quad (4)$$

The required field invariant is thus (E^2-B^2). Notice that we calculated only the diagonal elements of $\underset{\sim}{F}(\text{cov.})\underset{\sim}{F}^T(\text{con.})$; they were all we needed.

ϕ519 Final Exam Solutions (cont'd)

⑥ [redacted]. Analyse Lagrange density: $\mathcal{L} = \frac{1}{2}[(\partial_\alpha \phi)(\partial^\alpha \phi) - \mu^2 \phi^2]$, for scalar field ϕ.

A. 1) The eqtn-of-motion derived in class (see notes of 12/2/88) for a continuum generalized coordinate ϕ was... [we sum over $k=1,2,3$]...

$$\frac{\partial}{\partial x_k}(\partial \mathcal{L}/\partial \phi_{x_k}) + \frac{\partial}{\partial t}(\partial \mathcal{L}/\partial \phi_t) = \partial \mathcal{L}/\partial \phi. \qquad (1)$$

11/28/88

Since the given $\mathcal{L}$ can be written as...

$$\mathcal{L} = \frac{1}{2}\left[\frac{1}{c^2}\phi_t^2 - \phi_{x_k}^2 - \mu^2\phi^2\right], \qquad (2)$$

... then a direct plug-in to Eq. (1) yields...

$$-\frac{\partial}{\partial x_k}\phi_{x_k} + \frac{1}{c^2}\frac{\partial}{\partial t}\phi_t = -\mu^2\phi$$

$$\text{or}/\!/ \quad \boxed{(\Box + \mu^2)\phi = 0}, \quad \text{w}/\!/ \quad \Box = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2. \qquad (3)$$

Eq. (3) is the desired free-field wave eqtn for a ϕ with above $\mathcal{L}$.

B. 2) If $\mathcal{L} \to \mathcal{L} + 4\pi\rho\phi$, the only term which changes in the above calculation is $\partial\mathcal{L}/\partial\phi$ in Eq. (1)... evidently: $\partial\mathcal{L}/\partial\phi \to (\partial\mathcal{L}/\partial\phi) + 4\pi\rho$. The term in ρ is left on the RHS of Eq. (3), so that the new wave eqtn for ϕ is

$$\boxed{(\Box + \mu^2)\phi = 4\pi\rho}. \qquad (4)$$

ρ evidently introduces a "source density" for the ϕ field. Since ρ drives the ϕ eqtn, it can be interpreted as a "charge" per unit volume which generates the ϕ-field.

3) Eq. (4) is analogous to the Proca Eqs... Jackson Eq. (12.93). It is permissible to interpret μ there, and here, as a "mass term": $\mu = m_\phi c/\hbar$, where m_ϕ is the mass of the quantum which propagates the ϕ-field.