Need to consider relativistic single-q radiation.

Radiation from a Single, Relativistic q

$\int$ Jackson Secs. 14.1 – 14.3

1) Our previous notes on simple radiating systems, treating radn from monochromatic current sources (pp. Rad 1–7), had two shortcomings, viz.

A. All the radiating q's had to move in unison at frequency ω; their accelerations were not arbitrary, but instead specified by $\ddot{x} = -\omega^2 x$ (Hooke's Law);

B. The conveniently-made dipole approxn [d (system size) $\ll \lambda$ (radiated wavelength)] implies that the charge velocities $v \sim \omega d = ckd = c \cdot 2\pi(d/\lambda) \ll c$ are nonrelativistic.

We now want to relax these restrictions, so we consider a single q which experiences an arbitrary acceleration $\mathbf{a}$, and whose velocity v may be $\sim c$.

Since this is a putatively relativistic problem, it is appropriate to do it in a covariant fashion. The first step is to obtain the 4-potential $A^\alpha = (\phi, \mathbf{A})$ from the 4-current $J^\alpha = (c\rho, \mathbf{J})$ generated by q's motion. Second step is to get the fields by $\mathbf{E} = -\nabla\phi - \frac{1}{c}(\partial\mathbf{A}/\partial t)$, $\mathbf{B} = \nabla \times \mathbf{A}$ (i.e. we find the field tensor $F^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha$); in this stage, we hope to find fields which fall off with distance R from q as $1/R$, so that they can carry off radiation energy to ∞. Third step, is to calculate the radiated energy per unit time & area via the Poynting Vector: $\mathbf{S} = \frac{c}{4\pi}(\mathbf{E} \times \mathbf{B})$; then we can find radiation rates for a given acceleration $\mathbf{a}$, calculate $\measuredangle$lar distributions of the radiated energy, etc.

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2) We can do step ① above w/o specializing to single q's. The 4-vector wave eqtn relates the potential A^α to its sources, so we want a solution to:

$$\left.\begin{aligned} &\Box A^\alpha = (4\pi/c) J^\alpha, \quad \text{in Lorentz gauge } (\partial_\alpha A^\alpha = 0), \\ &\text{i.e.}\ \left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)(\phi, \mathbf{A}) = -\frac{4\pi}{c}(c\rho, \mathbf{J}), \quad \text{w/}\ \frac{1}{c}\frac{\partial\phi}{\partial t} + \nabla\cdot\mathbf{A} = 0. \end{aligned}\right\} \quad (1)$$

Jackson does this in Eqs. (12.123) – (12.137), w/ covariant flags flying. We don't need all that, however, because we have already solved the wave eqtns for ϕ & $\mathbb{A}$ -- see class notes, pp. ME 14-18, particularly Eq. (12), p. ME 17. There, among other things, we discovered the notion of retarded & advanced times $t'(\text{source}) = t(\text{observer}) \mp \frac{1}{c}\,|\mathbf{r}(\text{observer}) - \mathbf{r}'(\text{source})|$, and we found the solutions:

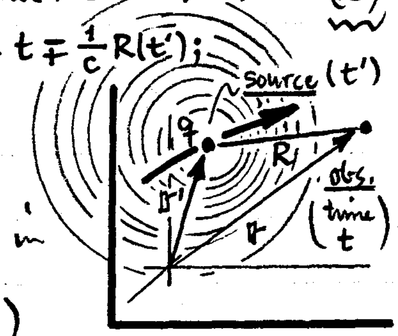
$$\left.\begin{aligned} \phi(\mathbf{r},t) &= \phi_0(\mathbf{r},t) + \frac{1}{c}\int_{\text{sources}} d^3x' \int dt'\; G^{(\pm)}(\mathbf{r},t;\mathbf{r}',t')\; c\rho(\mathbf{r}',t'), \\ \mathbb{A}(\mathbf{r},t) &= \mathbb{A}_0(\mathbf{r},t) + \frac{1}{c}\int_{\text{sources}} d^3x' \int dt'\; G^{(\pm)}(\mathbf{r},t;\mathbf{r}',t')\; \mathbb{J}(\mathbf{r}',t'); \end{aligned}\right\} \quad (2)$$

w/ ϕ_0 & $\mathbb{A}_0$ solns to homogeneous wave eqtn: $\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)[\phi_0, \mathbb{A}_0] = 0$;

& $\longrightarrow$ $$G^{(\pm)} = \frac{1}{R}\,\delta\left[t' - \left(t \mp \frac{1}{c}R(t')\right)\right] \quad (3)$$

retarded (+) - advanced (-) Green's fcns, pick out times: $t' = t \mp \frac{1}{c}R(t')$;

$R = |\mathbf{r} - \mathbf{r}'|$, source-observer distance (per sketch).



The Green's fcn G used here satisfies the point-source eqtn in both space & time: $\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)G = -4\pi\,\delta(\mathbf{r}-\mathbf{r}')\,\delta(t-t')$ [see Jkn Eqs. (6.63)-(6.66)], and it represents a sharply defined EM wave-front moving out at velocity c from the disturbance at $(\mathbf{r}', t')$.

We can just add Eqs. (2) to begin a covariant description. W/ $A^\alpha = (\phi, \mathbb{A})$, the 4-potential is generated from the 4-current via

$$\rightarrow A^\alpha(x) = A_0^\alpha(x) + \frac{4\pi}{c}\int_{\text{sources}} d^4x' \left[\frac{1}{4\pi c}\, G^{(\pm)}\right] J^\alpha(x'); \quad (4)$$

w/ $x = (ct, \mathbf{r})$ & $x' = (ct', \mathbf{r}')$ = 4-vector positions $\left\{\begin{array}{l} x \leftrightarrow \text{observer}, \\ x' \leftrightarrow \text{source}; \end{array}\right.$

& $\longrightarrow$ $\int d^4x' = \int d^3x' \int c\,dt'$, integrals over "hypervolume" in Eqs (2).

$A^\alpha(x)$ is a solution to $\Box A^\alpha = \frac{4\pi}{c} J^\alpha$ in Eq. (1). Since, in Eq. (4), the hypervolume d^4x' is invariant, and A^α & J^α are 4-vectors, then $\left[\frac{1}{4\pi c} G^{(\pm)}\right]$ must be a Lorentz invariant.

Lorentz-invariant G's for the $\Box A^\alpha = (4\pi/c) J_\alpha$ problem.

3) We can demonstrate that $G^{(\pm)}$ of Eq. (3) is a <u>Lorentz invariant</u> by rewriting it as:

$$\rightarrow \frac{1}{4\pi c} G^{(\pm)} = \frac{1}{4\pi R}\,\delta[(x_0'-x_0)\pm R], \text{ w/ } x_0 = ct \quad \{\text{have used: } \delta(T) = c\,\delta(cT)\},$$

$$= \frac{1}{4\pi R}\,\delta[(x_0-x_0')\mp R], \text{ w/ } R = |\mathbf{r}-\mathbf{r}'| \quad \{\text{have used: } \delta(X) = \delta(-X)\},$$

$$\left\{\begin{array}{l} (+) \Rightarrow \underline{\text{retarded}} \text{ Green's fcn; contributes @ : } t'(\text{source}) = t \overset{\text{past}}{-} \frac{R}{c} < t(\text{observer}), \\ (-) \Rightarrow \underline{\text{advanced}} \text{ Green's fcn; contributes @ : } t'(\text{source}) = t \underset{\text{future}}{+} \frac{R}{c} > t(\text{observer}). \end{array}\right. \tag{5}$$

To flag the time ordering, insert step functions θ...

$$\left[\!\left[\frac{1}{4\pi c} G^{(\pm)} = \left\{\begin{array}{l} \frac{1}{4\pi R}\,\theta(x_0-x_0')\,\delta[(x_0-x_0')-R] \underset{\text{def}^{\text{n}}}{\equiv} D_{ret}(x-x') \\ \frac{1}{4\pi R}\,\theta(x_0'-x_0)\,\delta[(x_0-x_0')+R] \equiv D_{adv}(x-x') \end{array}\right. \right\| \begin{array}{l}\text{def}^{\text{ns}} \text{ of D's in} \\ \text{Jk}^{\text{n}} \text{ Eq. (12.131)} \\ \text{\& (12.132).}\end{array} \tag{6}$$

[of course: $\theta(\xi) = \left\{\begin{array}{l} 1, \text{ for } \xi > 0 \\ 0, \text{ for } \xi < 0 \end{array}\right.$]. Now the δ-fcns in (6) can be rewritten, by using:

$$\rightarrow \delta[(\xi-a)(\xi-b)] = \frac{1}{|a-b|}[\delta(\xi-a) + \delta(\xi-b)]. \tag{7a}$$

... consider observer-source spacetime interval: $(x-x')^2 = (x_0-x_0')^2 - R^2$, ...

$$\rightarrow \delta[(x-x')^2] = \frac{1}{2R}\{\underbrace{\delta[(x_0-x_0')-R]}_{\text{contributes only when } (x_0-x_0')>0} + \underbrace{\delta[(x_0-x_0')+R]}_{\text{contributes only when } (x_0-x_0')<0}\} \tag{7b}$$

Using (7b) in (6), we obtain the retarded & advanced Green's fcns:

$$\boxed{D_{ret}(x-x') = \frac{1}{2\pi}\theta(x_0-x_0')\,\delta[(x-x')^2], \quad D_{adv} = \frac{1}{2\pi}\theta(x_0'-x_0)\,\delta[(x-x')^2].} \tag{8}$$

In this form, it is easy to see that these versions of the Green's fcns are in fact <u>Lorentz invariant</u>, because: <u>(A)</u> the spacetime interval $(x-x')^2$ is invariant, <u>(B)</u> the time-ordering $(x_0-x_0')>0$ for D_{ret} or $(x_0-x_0')<0$ for D_{adv} cannot be changed causally.

Manifestly covariant solutions for A^α. Current J^α for single q.

4) Our wave solutions in Eq. (4) [solⁿs to: $\Box A^\alpha = (4\pi/c)J^\alpha$] can now be written in a manifestly covariant fashion; viz...

$$\boxed{A^\alpha(x) = \underbrace{A_-^\alpha(x)}_{\text{incoming}} + \frac{4\pi}{c}\int_{\text{sources}} d^4x'\, D_{ret}(x-x')\,J^\alpha(x')}$$

$A_-^\alpha(x)$ = homogeneous solⁿ as $t \to (-)\infty$ [distant past], when $J^\alpha = 0$. It is an "incoming wave". (9a)

or//

$$\boxed{A^\alpha(x) = \underbrace{A_+^\alpha(x)}_{\text{outgoing}} + \frac{4\pi}{c}\int_{\text{sources}} d^4x'\, D_{adv}(x-x')\,J^\alpha(x')}$$

$A_+^\alpha(x)$ = homogeneous soln as $t \to (+)\infty$ [distant future], when $J^\alpha = 0$. It is an "outgoing wave". (9b)

These are the most general solutions for the Maxwell field [they fully specify the field tensor $F^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha$] in the Lorentz gauge [$\partial_\alpha A^\alpha = 0$], in a non-medium [$\epsilon = 1$, $\mu = 1$], and on an ∞ domain -- where $J^\alpha(x')$ vanishes when $|t'| \to \infty$ and/or $|\mathbf{r}'| \to \infty$.

A radiation field is by definition a field which escapes the source system; it is the difference between the above outgoing & incoming waves. By subtraction...

$$\left[\begin{array}{l} A_{rad}^\alpha(x) = A_+^\alpha(x) - A_-^\alpha(x) = \dfrac{4\pi}{c}\displaystyle\int d^4x'\, D_{rad}(x-x')\,J^\alpha(x'), \\ \text{w/}\ D_{rad}(x-x') = D_{ret}(x-x') - D_{adv}(x-x') = \dfrac{\pm 1}{2\pi}\,\delta[(x-x')^2],\ \text{for } t \gtrless t'. \end{array}\right. \tag{10}$$

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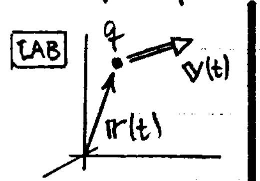
5) An important application of Eqs. (9) can now be made to the arbitrary motion of a point charge q. If q is located at $\mathbf{r}(t)$ and moving at $\mathbf{v}(t)$ in lab...



$$\left.\begin{array}{l} \rho(\mathbf{r}',t) = q\,\delta(\mathbf{r}'-\mathbf{r}(t)) \\ \mathbf{J}(\mathbf{r}',t) = q\mathbf{v}(t)\,\delta(\mathbf{r}'-\mathbf{r}(t)) \end{array}\right\} J_{LAB}^\alpha = q\,(c, \mathbf{v}(t)) \times \delta(\mathbf{r}'-\mathbf{r}(t)). \tag{11}$$

If $u^\alpha = \gamma(c, \mathbf{v})$, w/ $\gamma = 1/\sqrt{1 - v^2/c^2}$, is q's 4-velocity, then we write...

$$\rightarrow J_{lab}^\alpha = q\,\frac{1}{\gamma}\,u^\alpha(t)\,\delta(\mathbf{r}'-\mathbf{r}(t)) = q\int_{-\infty}^{\infty} u^\alpha(t')\left[\delta(\mathbf{r}'-\mathbf{r}(t'))\,\delta(t-t')\right]\frac{dt'}{\gamma}. \tag{12}$$

But dt' is source time, and $dt'/\gamma = d\tau$ is q's propertime. Then (12) ⇒

4-potential for arbitrary motion of a point charge q.

$$\boxed{J^\alpha(x') = qc\int_{-\infty}^{\infty} d\tau\, u^\alpha(\tau)\,\delta^{(4)}[x'-r(\tau)]}$$

w/ $\delta^{(4)}(x'-r(\tau)) = \delta(\mathbf{r}'-\mathbf{r}(\tau))\,\delta(c(t'-\tau))$, 4D point-source function. (13)

This is Jkⁿ Eq. (12.139). These maneuvers have made the 4-current J^α generated by q's motion manifestly covariant. q's motion is arbitrary.

Next step is to put $J^\alpha(x')$ of (13) into Eq. (9a), to find the causal [retarded time] potential generated by q's motion. Set $A^\alpha_-(x) \equiv 0$, and get...

$$A^\alpha(x) = \frac{4\pi}{c}\int d^4x'\, D_{ret}(x-x')\cdot qc\int_{-\infty}^{\infty} d\tau\, u^\alpha(\tau)\,\delta^{(4)}[x'-r(\tau)]$$

$$= 4\pi q\int_{-\infty}^{\infty} d\tau\, u^\alpha(\tau)\int d^4x'\, D_{ret}(x-x')\,\delta^{(4)}[x'-r(\tau)]$$

$$= 4\pi q\int_{-\infty}^{\infty} d\tau\, u^\alpha(\tau)\, D_{ret}[x-r(\tau)]$$

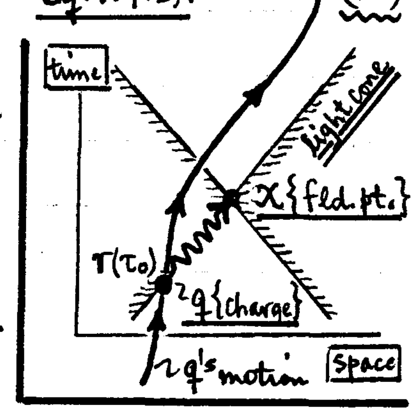
NOTE: $x=(ct,\mathbf{r})$ is the field point, $r(\tau)=(c\tau,\mathbf{r}_q(\tau))$ is q's location. (14)

Put in D_{ret} of Eq. (8) to get...

$$A^\alpha(x) = 2q\int_{-\infty}^{\infty} d\tau\, u^\alpha(\tau)\,\theta[x_0-r_0(\tau)]\,\delta[(x-r(\tau))^2].$$

Jackson Eq. (14.3). (15)

This integral over q's proper time τ can contribute to A^α at the field point (at time t) from only one point in time $\tau=\tau_0$, when q was situated (instantaneously) on the light cone centered on x, as indicated at right. In fact this is demanded by the δ-fcn in (15); we need:



LIGHT-CONE CONDITION $\boxed{[x-r(\tau_0)]^2 = 0}$. (16)

Also, because of the θ-fcn in (15) [causality], we need $x_0 > r_0(\tau_0)$ ··· this picks out the contribution q(radiation) $\rightarrow$ x (fld. pt.) from the backward light cone centered on x rather than the forward light cone (so we get signals from the past, not the future).

With that, $A^\alpha(x)$ in Eq. (15) is easily evaluated. Result is...