The 9 matrix elements of $\vec{T}$.

Matrix Elements of $\vec{T}$-Vectors

refs. Sakurai, Sec. 3.10; Landau & Lifshitz "QM", ¶ 29.

1) We know that for any $\vec{T}$-vector: $\langle \alpha j m | \vec{T} | \alpha' j' m' \rangle \equiv 0$, unless $j' = j, j \pm 1$ and $m' = m, m \pm 1$. So only **9 non-zero matrix elements are possible**. These can be inter-related, and their m-dependence specified, by exploiting various commutation relations between $\vec{T}$ and its companion $\measuredangle$ momentum $\vec{J}$.

Our first reduction of the problem is to write $\vec{T}$ in the form...

$$\vec{T} = \hat{x}\, T_x + \hat{y}\, T_y + \hat{z}\, T_z \quad \ldots (\hat{x}, \hat{y}, \hat{z}) = \text{coordinate unit vectors},$$

or

$$\vec{T} = \tfrac{1}{2}(\hat{x} - i\hat{y})\, T^+ + \tfrac{1}{2}(\hat{x} + i\hat{y})\, T^- + \hat{z}\, T_z, \quad T^\pm = T_x \pm i T_y. \tag{1}$$

We do this in order to use the selection rules...

$$T^\pm = 0 \text{ unless } m' = m \mp 1, \quad T_z = 0 \text{ unless } m' = m;$$

So

$$\left.\begin{aligned} \langle \alpha j m | \vec{T} | \alpha' j' m' \rangle = {} & \tfrac{1}{2}(\hat{x} - i\hat{y})\, \delta_{m', m-1} \langle \alpha j m | T^+ | \alpha' j' m' \rangle + \\ & + \tfrac{1}{2}(\hat{x} + i\hat{y})\, \delta_{m', m+1} \langle \alpha j m | T^- | \alpha' j' m' \rangle + \\ & + \hat{z}\, \delta_{m' m} \langle \alpha j m | T_z | \alpha' j' m' \rangle, \ \underline{\text{and}}\ j' = j, j \pm 1. \end{aligned}\right\} \tag{2}$$

$\boxed{9 \text{ M.E.'s}}$

Furthermore, we can get the T^+ matrix elements from the T^-. Since $\vec{T}$ is Hermitian, then $(T^+)^\dagger = T^-$, and we can write...

$$\rightarrow \langle \alpha j m | T^+ | \alpha' j' m' \rangle = \left(\langle \alpha' j' m' | T^- | \alpha j m \rangle \right)^*. \tag{3}$$

The problem is reduced to calculating appropriate matrix elements of T^- & T_z. Then we will have all possible matrix elements of $\vec{T}$, as desired.

Vector Matrix Elements : non-zero M.E.'s of T^-.

2) Certain T_- matrix elements follow from the commutator identity... (4)

→ $[J^-, T^-] = 0$ ← proof is straightforward,

$$\text{soll}\quad \langle \alpha j\, m-1 | J^- T^- | \alpha' j' m' \rangle = \langle \alpha j\, m-1 | T^- J^- | \alpha' j' m' \rangle,$$

$$\text{or}\quad \langle J^+(\alpha j\, m-1) | T^- | \alpha' j' m' \rangle = \langle \alpha j\, m-1 | T^- | J^-(\alpha' j' m') \rangle.$$

→ But : $J^{\pm} | \alpha j \mu \rangle = \sqrt{(j \mp \mu)(j \pm \mu + 1)}\, | \alpha j\, \mu \pm 1 \rangle$,

$$\Rightarrow \sqrt{(j-m+1)(j+m)}\, \langle \alpha j m | T^- | \alpha' j' m' \rangle = \tag{4}$$

$$= \sqrt{(j'+m')(j'-m'+1)}\, \langle \alpha j\, m-1 | T^- | \alpha' j'\, m'-1 \rangle.$$

By the T^- selection rule, both matrix elements here are $\equiv 0$, unless $m' = m+1$. Impose this, and rewrite Eq. (4) in the form...

$$\rightarrow \frac{\langle \alpha j m | T^- | \alpha' j'\, m+1 \rangle}{\sqrt{(j'+m+1)(j'-m)}} = \frac{\langle \alpha j\, m-1 | T^- | \alpha' j' m \rangle}{\sqrt{(j+m)(j-m+1)}}. \tag{5}$$

This must hold for all j & j', and in particular $j' = j$. Impose this, and also define : $\mu = m-1$, on the RHS of the eqtn. Then...

$$\rightarrow \frac{\langle \alpha j m | T^- | \alpha' j\, m+1 \rangle}{\sqrt{(j+m+1)(j-m)}} = \frac{\langle \alpha j \mu | T^- | \alpha' j\, \mu+1 \rangle}{\sqrt{(j+\mu+1)(j-\mu)}},\quad \mu = m-1. \tag{6}$$

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3) A semi-amazing fact now emerges : this ratio must be independent of the m-value involved, because it retains its form and remains the same as we step m through all its values : $m \to \mu = m-1 \to \mu' = \mu - 1 = m-2 \to$ etc. We therefore define this m-independent ratio as

Vector Matrix Elements : reduced M.E. $\langle\alpha j\|\mathcal{T}\|\alpha' j\rangle$.

a new entity $\langle\alpha j\|\mathcal{T}\|\alpha' j\rangle$, called a "reduced matrix element" and write

$$\boxed{\langle\alpha j m|T^-|\alpha' j\, m+1\rangle = \sqrt{(j-m)(j+m+1)}\,\langle\alpha j\|\mathcal{T}\|\alpha' j\rangle.} \tag{7}$$

Remarks

1. $\langle\alpha j\|\mathcal{T}\|\alpha' j\rangle$ is called "reduced" because its m-dependence has been extracted.
2. Nothing more can be done about calculating $\langle\alpha j\|\mathcal{T}\|\alpha' j\rangle$ until an explicit form for $\vec{T}$ is given. However, we do know that $\langle\alpha j\|\mathcal{T}\|\alpha' j\rangle$ is independent of m, and may calculate it from Eq. (7) for any convenient m-value ... e.g. for $m=0$: $\langle\alpha j\|\mathcal{T}\|\alpha' j\rangle = \langle\alpha j 0|T^-|\alpha' j 1\rangle/\sqrt{j(j+1)}$.
3. $\mathcal{T}$ is a Hermitian operator if $\vec{T}$ is -- we shall show this shortly.

Assuming $\mathcal{T}$ is Hermitian, and knowing that $T^+ = (T^-)^\dagger$, we can use Eq. (7) to write...

$$\langle\alpha j m|T^\mp|\alpha' j\, m\pm 1\rangle = \sqrt{(j\mp m)(j\pm m+1)}\,\langle\alpha j\|\mathcal{T}\|\alpha' j\rangle,$$

$\Delta m = \pm 1$ so,

$$\boxed{\langle\alpha j m|\vec{T}|\alpha' j\, m\pm 1\rangle = \tfrac{1}{2}(\hat{x}\pm i\hat{y})\sqrt{(j\mp m)(j\pm m+1)}\,\langle\alpha j\|\mathcal{T}\|\alpha' j\rangle.} \tag{8}$$

This gives 2 of the 9 matrix elements we are looking for.

4) Next, we relate T_z (for $j'=j$, $m'=m$) to $\langle\alpha j\|\mathcal{T}\|\alpha' j\rangle$. We use...

$\rightarrow [J^+, T^-] = 2T_z$ ← by straightforward arithmetic,

so,

$$2\langle\alpha j m|T_z|\alpha' j m\rangle = \langle J^-(\alpha j m)|T^-|\alpha' j m\rangle - \langle\alpha j m|T^-|J^+(\alpha' j m)\rangle$$

$$= \sqrt{(j+m)(j-m+1)}\,\langle\alpha j\, m-1|T^-|\alpha' j m\rangle - \sqrt{(j-m)(j+m+1)}\,\langle\alpha j m|T^-|\alpha' j\, m+1\rangle$$

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Vector Matrix Elements: 3rd M.E. of $\vec{T}$ for $\Delta j = 0$.

These result from plugging in T^- matrix elements of Eq. (7) above.

w// $$2\langle \alpha j m | T_z | \alpha' j m \rangle = \Big[\underbrace{\left(\sqrt{(j+m)(j-m+1)}\right)^2 - \left(\sqrt{(j-m)(j+m+1)}\right)^2}_{=2m}\Big]\langle \alpha j \| \mathcal{T} \| \alpha' j \rangle$$

and //

$\Delta m = 0$

$$\boxed{\begin{aligned} &\langle \alpha j m | T_z | \alpha' j m \rangle = m \langle \alpha j \| \mathcal{T} \| \alpha' j \rangle, \\ \Rightarrow\ &\langle \alpha j m | \vec{T} | \alpha' j m \rangle = \hat{z}\, m \langle \alpha j \| \mathcal{T} \| \alpha' j \rangle \end{aligned}} \quad (9)$$

Again, we can calculate $\langle \alpha j \| \mathcal{T} \| \alpha' j \rangle$ from T_z for any convenient m-value (except $m=0$, which gives an indeterminate expression). Eqs. (8) & (9) together give all 3 non-zero $\vec{T}$-vector matrix elements for $j' = j$.

Eq. (9) allows an easy demonstration that $\mathcal{T}$ is Hermitian if T_z is -- as was assumed in Eq. (8). Start from...

$$\longrightarrow \quad \langle \alpha j \| \mathcal{T} \| \alpha' j \rangle = \frac{1}{m}\langle \alpha j m | T_z | \alpha' j m \rangle, \quad m \neq 0$$

So // $$\langle \alpha j \| \mathcal{T} \| \alpha' j \rangle^* = \frac{1}{m}\langle \alpha j m | T_z | \alpha' j m \rangle^* = \frac{1}{m}\langle \alpha' j m | \underbrace{T_z^\dagger}_{= T_z} | \alpha j m \rangle$$

$$= \frac{1}{m}\langle \alpha' j m | T_z | \alpha j m \rangle = \langle \alpha' j \| \mathcal{T} \| \alpha j \rangle. \quad (10)$$

So $\mathcal{T}_{AB}^* = \mathcal{T}_{BA}$, for matrix elements, and $\mathcal{T}$ is Hermitian if T_z is.

5) The calculation of the remaining 6 matrix elements of $\vec{T}$, for $j' = j \pm 1$ and $m' = m, m \pm 1$, proceeds in a similar fashion -- using certain semi-clever commutation rules, plus the selection rules, to pick out the desired matrix elements of T^- & T_z. Details are in Condon & Shortley, pp. 59-64. There are no big surprises, just a great deal of algebra.

Vector Matrix Elements: Summary of all non-zero $\langle\alpha j m|\vec{T}|\alpha' j' m'\rangle$. ∡

Just as the 3 $\vec{T}$-vector matrix elements for $j'=j$ could all be related to a single reduced matrix element $\langle\alpha j\|\mathcal{T}\|\alpha' j\rangle$, so to are the 3 elements for $j'=j+1$ related to $\langle\alpha j\|\mathcal{T}\|\alpha' j+1\rangle$, and the 3 for $j'=j-1$ related to $\langle\alpha j\|\mathcal{T}\|\alpha' j-1\rangle$. These three reduced matrix elements are in general different, and we shall denote them by...

$$\boxed{P=\langle\alpha j\|\mathcal{T}\|\alpha' j+1\rangle\,,\quad Q=\langle\alpha j\|\mathcal{T}\|\alpha' j\rangle\,,\quad R=\langle\alpha j\|\mathcal{T}\|\alpha' j-1\rangle}\,. \tag{11}$$

With that, the final results for all possible non-zero $\vec{T}$-vector matrix elements may be reduced to the following table...

① $j'=j+1$

$$\langle\alpha j m|\vec{T}|\alpha', j+1, m\pm 1\rangle = \mp\tfrac{1}{2}(\hat{x}\pm i\hat{y})\sqrt{(j\pm m+1)(j\pm m+2)}\,[P],$$

$$\langle\alpha j m|\vec{T}|\alpha', j+1, m\rangle = \hat{z}\sqrt{(j+1)^2-m^2}\,[P];$$

② $j'=j$

$$\langle\alpha j m|\vec{T}|\alpha', j, m\pm 1\rangle = \tfrac{1}{2}(\hat{x}\pm i\hat{y})\sqrt{(j\mp m)(j\pm m+1)}\,[Q], \tag{12}$$

$$\langle\alpha j m|\vec{T}|\alpha', j, m\rangle = \hat{z}\,m\,[Q];$$

③ $j'=j-1$

$$\langle\alpha j m|\vec{T}|\alpha', j-1, m\pm 1\rangle = \pm\tfrac{1}{2}(\hat{x}\pm i\hat{y})\sqrt{(j\mp m)(j\mp m-1)}\,[R],$$

$$\langle\alpha j m|\vec{T}|\alpha', j-1, m\rangle = \hat{z}\sqrt{j^2-m^2}\,[R].$$

These relations hold in any QM system where ∡ momentum $\vec{J}$ is well-defined.

6) This table represents an enormous simplification for calculating matrix elements involving vector coupling between any two states of well-defined ∡ momenta... this is the usual case in atomic & molecular systems, where the states do have a well-defined $\vec{J}$, and the couplings go like $(e\vec{r})\cdot\vec{E}$ [electric dipole] or $\vec{\mu}\cdot\vec{B}$ [magnetic dipole]. The table reduces the calculational task to finding just 3 reduced matrix elements: P, Q, R.

As an example of the use of the table, consider electric dipole transitions: $(\alpha' j\, m') \rightarrow (\alpha\, j+1, m)$, driven by a coupling: $V = \vec{\mathcal{E}}\cdot\vec{r}$, with $\vec{\mathcal{E}}$ = an applied electric field. Since $\vec{\mathcal{E}}$ will be cnst over the dimensions $\vec{r}$ of the atom, the key matrix element of the coupling is...

$$\mathcal{M} = \langle \alpha, j+1, m | V | \alpha' j m' \rangle = \vec{\mathcal{E}} \cdot \langle \alpha, j+1, m | \vec{r} | \alpha' j m' \rangle \qquad (13)$$

Now use ③ in the table (with j shifted up one unit) to write...

$$\left.\begin{array}{l} \mathcal{M}(m' = m \pm 1) = \pm \frac{1}{2}(\mathcal{E}_x \pm i\,\mathcal{E}_y)\sqrt{(j \mp m)(j \mp m + 1)}\;[R], \\ \mathcal{M}(m' = m) = \mathcal{E}_z \sqrt{(j+1)^2 - m^2}\;[R], \quad R = \langle \alpha, j+1 \| r \| \alpha' j \rangle. \end{array}\right\} \qquad (14)$$

In this case, R is called the radial matrix element for the system. The transition rate (or line strength) for the coupling is $\Gamma \propto |\mathcal{M}|^2$, so...

$$\left[\!\!\left[\begin{array}{l} \Gamma(m' = m \pm 1) = \frac{1}{4}\,\mathcal{E}_\perp^2 |R|^2 (j \mp m)(j \mp m + 1), \quad \Gamma(m' = m) = \mathcal{E}_\parallel^2 |R|^2 [(j+1)^2 - m^2], \\ \text{where: } \mathcal{E}_\perp = (\mathcal{E}_x^2 + \mathcal{E}_y^2)^{\frac{1}{2}}, \; \mathcal{E}_\parallel = \mathcal{E}_z, \text{ are components of } \vec{\mathcal{E}} \perp \,\&\, \parallel \text{ z-axis.} \end{array} \right]\!\!\right] \qquad (15)$$

Notice that you can get the ratios of the transition rates without knowing R. For elementary particle decays, the "branching ratios" are sometimes gotten this way.