DEPARTMENT OF PHYSICS

1999 COMPREHENSIVE EXAM

Answer each of the following questions, all of which carry equal weight. Begin your answer to each question on a new sheet of paper; solutions to different questions must not appear on the same sheet. Each sheet of paper must be labeled with your name and the problem number in the upper right hand corner. When more than one sheet is submitted for a problem, be sure the pages are ordered properly.

PHYSICAL CONSTANTS

Quantity	Symbol	Value
acceleration due to gravity	g	$9.8\ \mathrm{m\ s^{-2}}$
gravitational constant	G	$6.67 \times 10^{-11}\ \mathrm{N\ m^2\ kg^{-2}}$
permittivity of vacuum	ε_o	$8.85 \times 10^{-12}\ \mathrm{C^2\ N^{-1}\ m^{-2}}$
permeability of vacuum	μ_o	$4\pi \times 10^{-7}\ \mathrm{N\ A^{-2}}$
speed of light in vacuum	c	$3.00 \times 10^{8}\ \mathrm{m\ s^{-1}}$
elementary charge	e	$1.602 \times 10^{-19}\ \mathrm{C}$
mass of electron	m_e	$9.11 \times 10^{-31}\ \mathrm{kg}$
mass of proton	m_p	$1.673 \times 10^{-27}\ \mathrm{kg}$
Planck constant	h	$6.63 \times 10^{-34}\ \mathrm{J\ s}$
Avogadro constant	N_A	$6.02 \times 10^{23}\ \mathrm{mol^{-1}}$
Boltzmann constant	k	$1.38 \times 10^{-23}\ \mathrm{J\ K^{-1}}$
molar gas constant	R	$8.31\ \mathrm{J\ mol^{-1}\ K^{-1}}$
standard atmospheric pressure		$1.013 \times 10^{5}\ \mathrm{Pa}$

Section 1 M 9-12	1. A	QM well (Avci)
	2. L	back emf (Adams)
	3. N	pendulum motion (Longcope)
Section 2 M 2-4	4. D	PV diagram (Avci)
	5. J	contour integral (Tsuruta)
Section 3 T 9-5	6. B	perturbation theory (Link)
	7. E	LS coupling (Adams)
	8. I	minimum energy (Longcope)
Section 4 T 2-4	9. F	Fermi energy (Hermanson)
	10. K	lightning bolt (Robiscoe)
Section 5 W 9-12	11. C	2 particles in well (Hermanson)
	12. G	relativistic collision (Link)
	13. M	image charge (Avci)
Section 6 W 2-4	14. H	deuteron bound state (Robiscoe
	15. O	pulley and scale (Adams)

Question 1

Consider a particle of mass m confined in a one-dimensional infinite potential well (shown below) where $V(x) = 0$ *if* $0 < x < a$, *and* $V(x) = \infty$ otherwise. The purpose of this problem is to compare the classical motion of the particle with its quantum mechanical motion. For the classical motion, assume that the particle is initially at the origin, makes elastic collisions with the walls and is free to move in the region between the walls.

(a) Make a sketch of x vs. t of the particle during a period of its classical motion.

Now consider the quantum mechanical aspect of the motion. Assume that at t=0 the state of the particle is given by

$$\psi(0) = \frac{1}{\sqrt{2}}\left[|\phi_1\rangle + |\phi_2\rangle\right]$$

where $|\phi_1\rangle$ and $|\phi_2\rangle$ are the ground state and the first excited state of the particle in the infinite well, respectively.

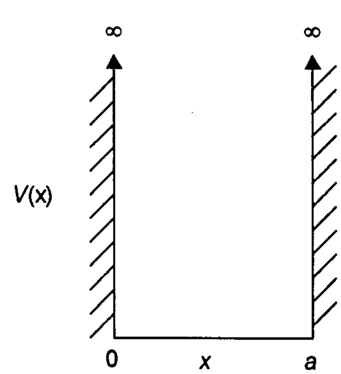
(b) Determine the time evolution, $\psi(t)$, of the particle.

(c) Calculate the time-dependent mean value $\langle x \rangle$ of the position of the particle and sketch $\langle x \rangle$ vs. t to compare the result with that sketched in part (a) above.

Hint: Useful integrals

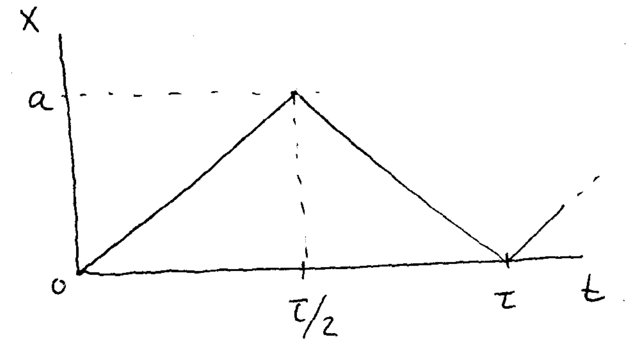
$$\int x \cos \beta x \, dx = \frac{1}{\beta^2} \cos \beta x + \frac{x}{\beta} \sin \beta x$$

$$\int x \sin^2 \beta x \, dx = \frac{x^2}{4} - \frac{x \sin 2\beta x}{4\beta} - \frac{\cos 2\beta x}{8\beta^2}$$



Avci, Quantum qa:

(a)



(b)

$\phi(x) = \alpha \sin kx$, @ $x = 0$ & a $\phi(x) = 0$

$\Rightarrow k_n a = n\pi$ where $n = 1, 2, \ldots$

$\langle \phi_n | \phi_m \rangle = \delta_{mn} \Rightarrow \alpha^2 \int_0^a \sin^2 kx\, dx = \alpha^2 \int_0^a \frac{1}{2}(1 - \cos 2kx)\, dx = 1$

$\Rightarrow \frac{a\alpha^2}{2} = 1 \Rightarrow \boxed{\alpha = \sqrt{\frac{2}{a}}}$

$\Rightarrow$ Stationary States: $\underline{\phi_n = \sqrt{\frac{2}{a}} \sin k_n x, \text{ where } k_n = \frac{n\pi}{a}}$

Time evolution:

$$\boxed{\begin{gathered}\psi(t) = \frac{1}{\sqrt{2}} \left\{ e^{-\frac{iE_1}{\hbar}t} |\phi_1\rangle + e^{-\frac{iE_2}{\hbar}t} |\phi_2\rangle \right\} \\ \text{where } E_n = \frac{\hbar^2 k_n^2}{2m}\end{gathered}}$$

(c) $\langle x \rangle = \langle \Psi(t) | x | \Psi(t) \rangle$

$$\langle x \rangle = \frac{1}{2} \Big\{ \langle \phi_1 | x | \phi_1 \rangle + \langle \phi_2 | x | \phi_2 \rangle + 2 \cos\left(\frac{E_2 - E_1}{\hbar}\right)t \, \langle \phi_1 | x | \phi_2 \rangle \Big\},$$

where we have used the fact that $|\phi_1\rangle$ & $|\phi_2\rangle$ are real and $\langle \phi_1 | x | \phi_2 \rangle = \langle \phi_2 | x | \phi_1 \rangle$.

Using the integrals given in the hint:

$$\langle \phi_n | x | \phi_n \rangle = \frac{2}{a} \int_0^a x \sin^2 k_n x = \frac{2}{a} \times \frac{a^2}{4} = \frac{a}{2} \quad \text{for all } n.$$

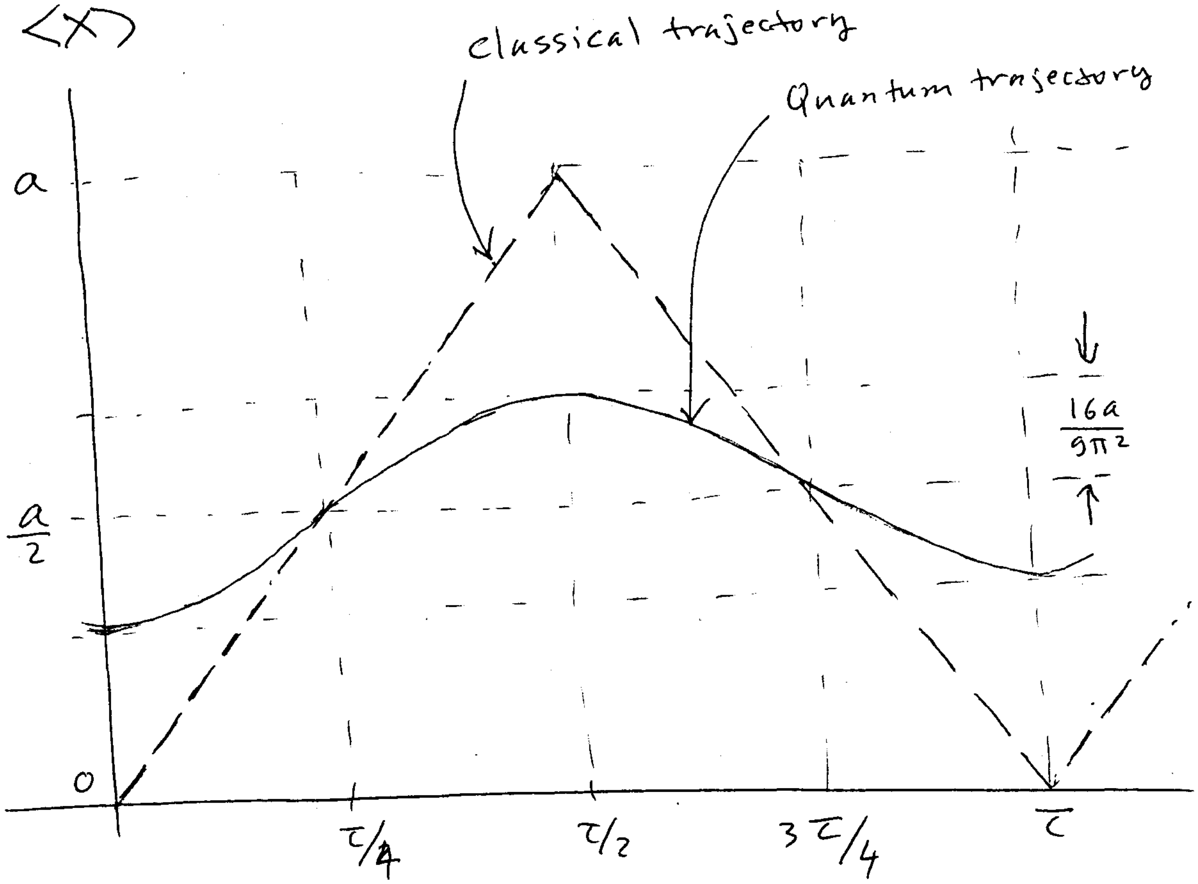
$$\langle \phi_1 | x | \phi_2 \rangle = \frac{2}{a} \int_0^a x \sin k_1 x \sin k_2 x \, dx$$

$$= \frac{2}{a} \int_0^a \frac{1}{2} \left(\cos(k_1 - k_2)x - \cos(k_1 + k_2)x \right) dx$$

$$= \frac{1}{a} \left\{ \frac{1}{(k_1 - k_2)^2} \cos(k_1 - k_2)x \Big|_0^a - \frac{1}{(k_1 + k_2)^2} \cos(k_1 + k_2)x \Big|_0^a \right\}$$

$$\langle \phi_1 | x | \phi_2 \rangle = \frac{1}{a} \left\{ \frac{1}{\frac{\pi^2}{a^2}} (-2) - \frac{1}{\frac{(3\pi)^2}{a^2}} (-2) \right\} = -\frac{16a}{9\pi^2}$$

$$\Rightarrow \quad \boxed{\langle x \rangle = \frac{a}{2} - \frac{16a}{9\pi^2} \cos \omega t, \text{ where } \omega = \frac{E_2 - E_1}{\hbar}}$$



⟨x⟩
classical trajectory
Quantum trajectory
a
a/2
0
τ/4
τ/2
3τ/4
τ
16a/9π²

Question 2

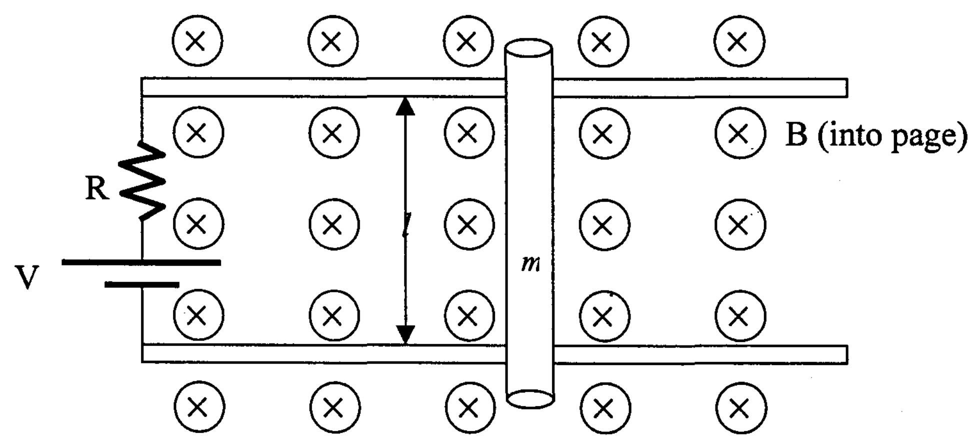
A rod of mass m, initially at rest, sits atop a pair of horizontal conducting rails in a uniform magnetic field as shown. Assume that the resistance R represents the concentrated resistance of the entire circuit. The frictional force between the rod and the rails has a constant value f.

(a) When the rod is initially released, which direction will the rod move? Explain.

(b) What is the maximum value of the frictional force f for which the rod will move? (We will assume that f is below this critical value.)

(c) As the rod accelerates, the back EMF induced by its motion will cause the rod's velocity to asymptotically approach a constant value. Find this equilibrium velocity v_{eq}

(d) Find an expression for the velocity of the rod as a function of time from the moment it is released. Express your answer in terms of v_{eq} and a time constant τ, which in turn should be expressed in terms of V, R, m, B, f, and l.



#2

a) Current is $\downarrow$ B field is in. RHR gives force to the right. $\therefore$ rod moves right.

b) Initially $i = V/R$

$$\therefore \quad F_{mag} = i\ell B = \frac{V\ell B}{R}$$

$$\boxed{f < \frac{V\ell B}{R}}$$

c) $\Phi = B\ell x$

$$\frac{d\Phi}{dt} = V_{back} = B\ell v$$

$$i = \frac{V - B\ell v}{R}$$

$$F_{mag} = \frac{V\ell B - B^2\ell^2 v}{R}$$

at Equilibrium $F_{mag} = f$

$$f = \frac{V\ell B - B^2\ell^2 v_{eq}}{R}$$

$$\ell^2 B^2 v_{eq} = \ell B V - fR$$

$$\boxed{v_{eq} = \frac{V}{\ell B} - \frac{fR}{(\ell B)^2}}$$

#2

d) $F_{NET} = ma = \dfrac{V\ell B - B^2\ell^2 v}{R} - f$

$$\frac{dv}{dt} = \left(\frac{V\ell B}{R} - f\right) - \left(\frac{\ell^2 B^2}{R}\right)v$$

$$v(0) = 0$$

$\therefore$ sol^{n} should look like $v(t) = V_{eq}\left(1 - e^{-t/\tau}\right)$

Or, we could just guess $v(t) = A + Be^{-t/\tau}$

$$\frac{dv}{dt} = -\frac{B}{\tau} e^{-t/\tau}$$

$$\frac{dv}{dt} = -\frac{\cancel{B}}{\tau}\left(\frac{v - A}{\cancel{B}}\right)$$

$$\frac{dv}{dt} = \frac{A}{\tau} - \frac{v}{\tau}$$

$$\therefore \boxed{\tau = \frac{R}{(\ell B)^2}}$$

$$\frac{A}{\tau} = \frac{V\ell B}{R} - f$$

$$A = \left(\frac{\ell B V}{R} - f\right)\frac{R}{\ell^2 B^2}$$

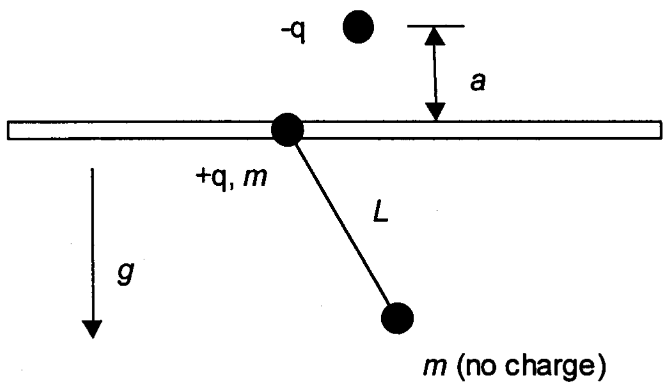
$$= \frac{V}{\ell B} - \frac{fR}{(\ell B)^2} = V_{eq}$$

$v(0) = 0 \Rightarrow 0 = A + B \quad \therefore B = -A$

$$v(t) = A\left(1 - e^{-t/\tau}\right) = V_{eq}\left(1 - e^{-t/\tau}\right)$$

Question 3

A pendulum is built from two equal masses (m) connected by a rigid rod of length L (see figure).



The pivot (the upper mass) slides frictionlessly on a horizontal bar. It has an electrical charge +q, while the bob (the lower mass) is uncharged. A second opposing electric charge (-q) is located a distance a from the top of the horizontal bar. Find the eigenfrequencies of small oscillations about equilibrium. (It is not necessary to find the normal modes).

Solution #3

Using coordinate x for the pivot and θ for the rod, the cartesian coordinates for each pivot (p) and bob (b) are

$$\begin{aligned} x_p &= x \quad , \quad y_p = 0 \\ x_b &= x + L\sin(\theta) \quad , \quad y_b = -L\cos(\theta) \end{aligned}$$

The potential energy of each mass/charge is

$$V_p = -\frac{q^2}{\sqrt{a^2+x^2}} \tag{1}$$

$$V_b = -mgL\cos(\theta) \tag{2}$$

Thus the total energy is

$$V(x,\theta) = V_p + V_b = -\frac{q^2}{\sqrt{a^2+x^2}} - mgL\cos(\theta) \ . \tag{3}$$

Equilibrium occurs where

$$\frac{\partial V}{\partial x} = \frac{q^2 x,}{(a^2+x^2)^{3/2}} = 0 \tag{4}$$

$$\frac{\partial V}{\partial \theta} = mgL\sin(\theta) = 0 \tag{5}$$

The equilibrium is therefore $x = 0$ and $\theta = 0$. The matrix of second derivatives is

$$\underline{\underline{v}} = \begin{bmatrix} q^2/a^3 & 0 \\ 0 & mgL \end{bmatrix} \tag{6}$$

The velocities are

$$\begin{aligned} \dot{x}_p &= \dot{x} \quad , \quad & \dot{y}_p &= 0 \\ \dot{x}_b &= \dot{x} + L\cos(\theta)\dot{\theta} \quad , \quad & \dot{y}_b &= L\sin(\theta)\dot{\theta} \end{aligned}$$

The kinetic energies are

$$\begin{aligned} T_p &= \tfrac{1}{2}m(\dot{x}_p^2 + \dot{y}_p^2) = \tfrac{1}{2}m\dot{x}^2 \ , \\ T_b &= \tfrac{1}{2}m(\dot{x}_b^2 + \dot{y}_b^2) = \tfrac{1}{2}m\dot{x}^2 + \tfrac{1}{2}mL^2\dot{\theta}^2 + mL\cos(\theta)\dot{x}\dot{\theta} \ , \end{aligned}$$

And thus the total kinetic energy is

$$T = T_b + T_p = m\dot{x}^2 + \tfrac{1}{2}mL^2\dot{\theta}^2 + mL\cos(\theta)\dot{x}\dot{\theta} \ , \tag{7}$$

The mass matrix is then

$$\underline{\underline{m}} = \begin{bmatrix} 2m & mL \\ mL & mL^2 \end{bmatrix} \tag{8}$$

The equation for normal modes is then

$$(\underline{\underline{v}} - \omega^2\underline{\underline{m}}) \cdot \underline{\rho} = 0 \ . \tag{9}$$

The secular equation

$$\begin{aligned} \det|\underline{\underline{v}} - \omega^2\underline{\underline{m}}| &= \det\begin{vmatrix} q^2/ma^3 - 2\omega^2 & -L\omega^2 \\ -L\omega^2 & gL - L^2\omega^2 \end{vmatrix} \\ &= (q^2/ma^3 - 2\omega^2)(gL - L^2\omega^2) - L^2\omega^4 \\ &= L^2\omega^4 - (2gL + q^2L^2/ma^3)\omega^2 + q^2gL/ma^3 = 0 \end{aligned} \tag{10}$$

The solution to this bi-quadratic is

$$\omega^2 = \left(\frac{g}{L} + \frac{q^2}{2ma^3}\right) \pm \sqrt{\left(\frac{g}{L}\right)^2 + \left(\frac{q^2}{2ma^3}\right)^2} \tag{11}$$

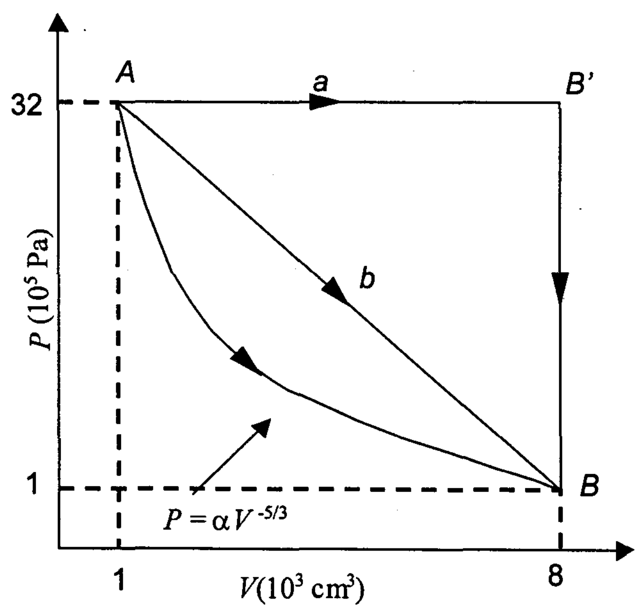
Question 4

The pressure P of a certain amount of gas is found to change with its volume V according to the relation $P = \alpha V^{-5/3}$ (where α is a constant) during a quasi-static process $A \rightarrow B$ in which no heat is exchanged with the environment, as shown in the figure below.

Find (1) the work done on the gas, (2) the net heat absorbed by the gas, and (3) the change in the entropy of the gas in each of the following processes, each of which takes the system from state A to state B. The gas is *not* necessarily an ideal gas.

(a) The system expands from its original volume to its final volume, heat being added to maintain the pressure constant (*path a*). The volume is then kept constant, and heat is extracted to reduce the pressure to 10^5 *Pa*.

(b) The volume is increased and heat is supplied to cause the pressure to decrease linearly with the volume (*path b*).



Solution:

(a) (1) $W_a = -\int_{\text{Path } a} P\,dV$ = Area under Path a

$= -P_i(V_f - V_i) = 32\times10^5\,\text{Pa} \times 7\times10^{-3}\,\text{m}^3$

$\Rightarrow \boxed{W_a = -22{,}400\,\text{J}}$

(2) Heat absorbed by the gas:

$Q_a = \Delta U - W_a$, where ΔU is the change in internal energy U. This we can obtain from $P = \alpha V^{-5/3}$ process because ΔU is independent of path and only depends on the end points. This is because U like entropy S are state functions and for quasi-static processes these functions depend only on the end points. Along the adiabatic path $đQ = 0$ hence $dS = 0$, therefore

$$\Delta U = W = -\int_{V_i}^{V_f} P\,dV = -\int_{V_i}^{V_f} \alpha V^{-5/3}\,dV$$

$$= -\alpha \frac{1}{(-\frac{5}{3}+1)} V^{-5/3} V \Big|_{V_i}^{V_f} = \frac{3}{2}\Big(\underbrace{\alpha V_f^{-5/3}}_{P_f} V_f - \underbrace{\alpha V_i^{-5/3}}_{P_i} V_i\Big)$$

$$\Delta U = \frac{3}{2}(P_f V_f - P_i V_i) = \frac{3}{2}(10^5\,\text{Pa} \times 8\times10^{-3}\,\text{m}^3 - 32\times10^5\,\text{Pa}\times10^{-3}\,\text{m}^3)$$

$\Rightarrow \boxed{\Delta U = -3600\,\text{J}}$

(3) Change in entropy $\Delta S = ?$

$\Delta S = 0$ because, as explained above, S is a state function and depends only on the end points as long as system is in equilibrium. Since points A & B lie on an adiabatic proces any other process (quasi-static) that connects these two points will preserve the entropy.

Hence $\Delta S = S(B) - S(A) = 0$

(b) (1) $W_b = -$ Area under path b.

$$W_b = -\left\{\left(\frac{32-1}{2}\right)\times 10^5\,Pa \times (8-1)\times 10^{-3}\,m^3 + 1\times 10^5\,Pa \times 7\times 10^{-3}\,m^3\right\}$$

$$W_b = -\left(\frac{31\times 7}{2} + 7\right)\times 10^2 = \boxed{-11,550\,J}$$

(2) $Q_b = \Delta U - W_b = -3600 + 11550$

$$\boxed{Q_b = +7,950\,J}$$

(3) $\Delta S = 0$ as explained above.

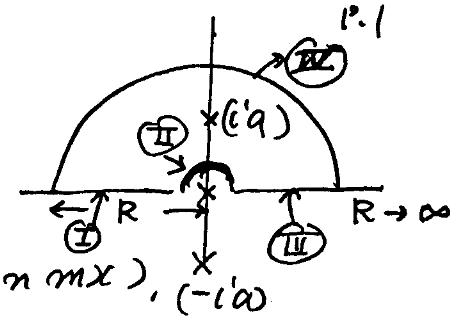
Question 5

Using contour integration evaluate:

$$I(m,a) = \int_0^{\infty} \frac{\sin(mx)}{x(x^2+a^2)}\,dx;$$ assume $m > 0$.

#5

Comp/99 MP Tsuruta **Key** P.1



(a) $I = \int_0^{\infty} \frac{\sin(mx)}{x(x^2+a^2)}\,dx$. (1)

consider $m>0$ (If $m<0$, take $-\sin mx$),
Integrate "even" fn of x.

$\therefore \frac{1}{2}\int_{-\infty}^{+\infty} \frac{\sin mx}{x(x^2+a^2)}\,dx \equiv I$. (2)

must find complex function reducing to (2) along real axis,

$f(z) = \frac{e^{imz}}{z(z^2+a^2)} = \frac{e^{imz}}{z(z-ia)(z+ia)}$.

If $\sin mz$ taken $\to (e^{imz} - e^{-imz})/2$
$\hookrightarrow \infty$ for $z \to \infty$.

$\oint_c f(z)dz = \underset{\uparrow}{P} \int_{-\infty}^{+\infty} \frac{e^{imx}}{x(x^2+a^2)}\,dx - \underbrace{\frac{2\pi i}{2} \text{Res}(z=0)}_{\text{(\#2)}} + \underbrace{\oint_{R\to\infty}(\quad)}_{\hookrightarrow = 0 \ (\#1)}$. (3)

(means "principal" part of integral)

$\frown = \frac{1}{2}\bigcirc$

(#2) $\text{Res}(z=0) = R(0) = \frac{1}{a^2}$

(#1) Use Jordan's Lemma.
$\int e^{imz} g(z)dz \to 0$ for $R\to\infty$ if $g(z)=0$
(IV) $\leq \int e^{-\sin\theta} M d\theta \leq \int e^{-\theta} M d\theta$

also $\oint_c f(z)dz = 2\pi i \,\text{Res}(f(z), z=ia) \underset{(\#3)}{=} 2\pi i \frac{e^{-ma}}{(-\cancel{2}a^2)}$
$= -\frac{\pi i e^{-ma}}{a^2}$ (4)

(#3) $\text{Res}(z=ia) = R(ia) = \frac{e^{-ma}}{ia(2ia)} = \frac{e^{-ma}}{-2a^2}$

$\therefore$ (3) = (4) gives

$P\int_{-\infty}^{+\infty} \frac{\cos mx + i\sin mx}{x(x+a^2)} = -\frac{i\pi e^{-ma}}{a^2} + \frac{\pi i}{a^2}$.

But $\int_{-\infty}^{+\infty} \frac{\cos mx}{x(x^2+a^2)} = 0$ because odd fn.

So $\boxed{I = \frac{1}{2}\int_{-\infty}^{+\infty} \frac{\sin mx}{x(x^2+a^2)} = \frac{1}{2}\frac{\pi}{a^2}(1 - e^{-ma})}$ // Ans.

Question 6

A particle of mass m and charge q is moving in a one-dimensional potential $V(x) = m\omega^2 x^2 / 2$. A weak electric field E in the x direction is then applied, causing the spectrum of eigenenergies to change (we are neglecting radiation associated with acceleration of the charge).

(a) Calculate in perturbation theory the lowest-order non-vanishing correction to the n-th energy level. The following relationships for the harmonic oscillator should prove useful:

$$\hat{a}^\dagger |n\rangle = \sqrt{n+1}\,|n+1\rangle$$
$$\hat{a}|n\rangle = \sqrt{n}\,|n-1\rangle$$
$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}\left(\hat{a} + \hat{a}^\dagger\right)$$

where $|n\rangle$ denotes the harmonic oscillator eigenstate of occupation number n in the absence of a field, $\hat{a}$ is an annihilation operator, $\hat{a}^\dagger$ is a creation operator, and $\hat{x}$ is the position operator.

(b) Obtain a constraint on the field strength required for the perturbation expansion to be justified.

#6

Solution

In perturbation theory the eigenenergies are given to second order by

$$E_n = E_n^{(0)} + \langle n|\hat{H}'|n\rangle + \sum_{i\neq n} \frac{|\langle n|\hat{H}'|i\rangle|^2}{E_n - E_i}.$$

The perturbation Hamiltonian in this case is $\hat{H}' = -F\hat{x} = -q\mathcal{E}\hat{x}$. The only non-vanishing matrix elements are

$$\begin{aligned}\langle n|\hat{H}'|n+1\rangle &= -F\sqrt{\frac{\hbar(n+1)}{2m\omega}} \\ \langle n|\hat{H}'|n-1\rangle &= -F\sqrt{\frac{\hbar n}{2m\omega}}.\end{aligned}$$

The first-order correction to the energy is

$$E_n^{(1)} = \langle n|\hat{H}'|n\rangle = 0.$$

The second-order correction is

$$E_n^{(2)} = \sum_{i\neq n} \frac{|\langle n|\hat{H}'|i\rangle|^2}{E_n - E_i} = \frac{\hbar F^2}{2m\omega}\left[\frac{n+1}{-\hbar\omega} + \frac{n}{\hbar\omega}\right] = -\frac{F^2}{2m\omega^2},$$

so all energy levels are shifted down by the same amount. For the perturbation expansion to be valid, we require $|E_n^{(2)}| \ll \hbar\omega$, giving

$$\mathcal{E}^2 \ll \frac{2\hbar m\omega^3}{q^2}.$$

Question 7

In LS coupling, optically active electrons are considered to couple their individual spin angular momenta and orbital angular momenta independently according to:

$$\mathbf{S} = \mathbf{S}_1 + \mathbf{S}_2 + \ldots$$
$$\mathbf{L} = \mathbf{L}_1 + \mathbf{L}_2 + \ldots .$$

These total spin and orbital angular momenta are quantized in the usual ways:

$$L = \sqrt{l(l+1)}\hbar \qquad\qquad S = \sqrt{s(s+1)}\hbar$$

This leads to multiple degenerate states with each defined by the quantum numbers s and l. The spin-orbit interaction couples these two vectors such that the magnitude of the total angular momentum J ($\mathbf{J} = \mathbf{L} + \mathbf{S}$) is constant and is also quantized according to

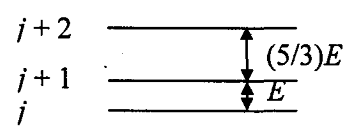
$$J = \sqrt{j(j+1)}\hbar$$

The spin-orbit interaction also removes the degeneracy of states with the same s and l but different j. The resulting set of energy levels is called a multiplet.

(a) Given that the spin-orbit interaction energy is proportional to $\mathbf{S}\cdot\mathbf{L}$, show that it has an expectation value given by $\overline{\Delta E} = K\left[j(j+1) - l(l+1) - s(s+1)\right]$. Assume that the constant K has the same value for all levels within a single multiplet.

(b) Show that the energy separation between the state j and state j+1 is proportional to $j+1$. (This is called the Landé interval rule.)

(c) Measurements of the line spectrum of a certain atom show that the separation between adjacent energy levels of increasing energy to be in the ratio 3 to 5 as illustrated at right. Use the Landé interval rule to assign quantum numbers s, l, and j to these levels.



#7

a) $\underline{J}' = \underline{L}' + \underline{S}'$

$$\underline{J}'\cdot\underline{J}' = \underline{L}'\cdot\underline{L}' + \underline{S}'\cdot\underline{S}' + 2\,\underline{S}'\cdot\underline{L}'$$

$$j'(j'+1)\hbar = l'(l'+1)\hbar + s'(s'+1)\hbar + 2\,\underline{S}'\cdot\underline{L}'$$

$$\underline{S}'\cdot\underline{L}' = \frac{\hbar}{2}\left[j'(j'+1) - l'(l'+1) - s'(s'+1) \right]$$

Since $\overline{\Delta E} \propto \underline{S}'\cdot\underline{L}'$ we can write

$$\overline{\Delta E} = K\left[j'(j'+1) - l'(l'+1) - s'(s'+1) \right]$$

b) let $\mathcal{E}$ = energy separation

$$\mathcal{E} = \overline{\Delta E}_{j+1} - \overline{\Delta E}_{j}$$

$$= K\left\{ \left[(j'+1)(j'+2) - l'(l'+1) - s'(s'+1) \right] - \left[(j')(j'+1) - l'(l'+1) - s'(s'+ \right.\right.$$

same l' + s' for all members of multiplet

$$\therefore\ \mathcal{E} = K\left(j'^2 + 3j' + 2 - (j'^2 + j') \right)$$
$$= K(2j' + 2)$$
$$= 2K(j'+1)$$

So $\mathcal{E}$ is proportional to $j'+1$, the total angular momentum quantum # of the upper state.

#7

c) For lower transition $\mathcal{E} = 2K(j'+1)$

for upper transition $\frac{5}{3}\mathcal{E} = 2K(j'+2)$

solve these $\Rightarrow$ $2K(j'+1) = \frac{6}{5}K(j'+2)$

$$2j'+2 = 6/5\,j' + \frac{12}{5}$$

$$\frac{4}{5}j' = 2/5$$

$$j' = 1/2$$

$\therefore$ the multiplet consists of levels $j' = 1/2, 3/2, 5/2$.

$j'_{max} = l' + s' = 5/2$

$j'_{min} = |l' - s'| = 1/2$

assume $l' > s'$ $\Rightarrow$ $l' + s' = 5/2$ ①

$l' - s' = 1/2$ ②

adding ① + ② $2l' = 6/2$

$l' = 3/2$

l' must be an integer so $l' < s'$

$\Rightarrow$ $l' + s' = 5/2$ ③

$s' - l' = 1/2$ ④

adding ③ + ④ $\Rightarrow$ $2s' = 3$; $s' = 3/2$

subtracting ③ - ④ $\Rightarrow$ $2l' = 2$; $l' = 1$

$\therefore$ $l' = 1$ $s' = 3/2$ $j' = 1/2, 3/2, 5/2$

Question 8

Consider continuous real functions $y(x)$, defined on $x \geq 0$, satisfying the boundary condition y(0)=0, and the normalization

$$\int_0^\infty y^2(x)\,dx = 1.$$

(a) Find a differential equation for that function $y(x)$ which minimizes the energy

$$E\{y(x)\} = \int_0^\infty \left[(y')^2 + xy^2\right] dx.$$

(b) Show that the differential equation you derived in part (a) is of Sturm-Liouville type. What plays the role of an eigenvalue? Say clearly what this means for the uniqueness of the solution $y(x)$, and for possible eigenvalues.

(c) Using a trial function of the form $\rho(x) = Axe^{-x}$, estimate the minimum energy E.

Solution #8

(a) Multiplying the normalization constraint by the undetermined multiplier λ, and adding the product to the energy gives the energy

$$E'\{y(x)\} = \int_0^\infty [(y')^2 + xy^2 - \lambda y^2]\,dx + \lambda \ . \tag{1}$$

Variation provides the integral

$$\delta E' = 2\int_0^\infty [-y'' + xy - \lambda y]\,\delta y\,dx \ .$$

Setting this integral to zero for all $\delta y(x)$ gives the Sturm-Liouville equation

$$y'' - xy = -\lambda y \ , \tag{2}$$

where λ is an eigenvalue.

(b) The Sturm-Liouville equation (2) has multiple solutions, each with a unique eigenvalue λ. Multiplying (2) by y and integrating gives

$$\int_0^\infty [(y')^2 + xy^2]\,dx = \lambda = E \ . \tag{3}$$

Since Sturm-Liouville equations have distinct, non-degenerate eigenvalues which are bounded from below, the minimum eigenvalue is unique. The minimum energy state is also therefore unique.

#4

(c) The eigenvalue can be estimated from the ratio

$$R\{\rho\} = \frac{\int_0^\infty [(y')^2 + xy^2]\, dx}{\int_0^\infty y^2\, dx} .$$

Placing the (un-normalized) trial function $\rho(x) = xe^{-x}$ into the numerator and denominator gives

$$\begin{aligned} \text{numerator} &= \int_0^\infty [(1-x)^2 + x^3]\, e^{-2x}\, dx \\ &= \tfrac{1}{2}0! - 2\tfrac{1}{4}1! + \tfrac{1}{8}2! + \tfrac{1}{16}3! = \tfrac{5}{8} \\ \text{denominator} &= \int_0^\infty x^2 e^{-2x}\, dx = \tfrac{1}{8}2! = \tfrac{1}{4} \end{aligned}$$

The upper bound on the eigenvalue is then

$$E = \lambda \le R\{xe^{-x}\} = \frac{5}{2} = 2.5 . \qquad (4)$$

(*) The actual solution to (2) is

$$y(x) = \mathrm{Ai}(x - \lambda) .$$

The eigenvalues are found from the requirement $y(0) = \mathrm{Ai}(-\lambda) = 0$. The smallest zero of $\mathrm{Ai}(x)$ is $a_1 = -2.33$ making the minimum energy $E = 2.33$.

Question 9

N electrons are contained in a cubical box of side L, bounded by the six planes

$$x = 0, L; \ \ y = 0, L; \ \ z = 0, L$$

Assuming that the electrons do not interact with each other, and that the walls of the box are infinitely hard, determine

(a) the allowed energy levels E Single e

(b) the maximum energy E_{max} when the temperature T=0 (Hint: consider the allowed wavevectors and assume that N is very large).

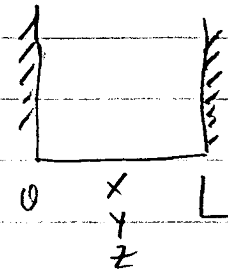
At T=0, find

(c) the average energy per electron, expressed as a fraction of E_{max}.

(d) the total energy of the electron gas.

(e) the pressure exerted on the walls by the electron gas.

Solution



Let $V = L^3$

a) energies:

$\psi \sim \sin(k_x x)\sin(k_y y)\sin(k_z z)$

vanishes on the walls $x=0;\ y=0;\ z=0$

To satisfy $\psi = 0$ on walls $x=L;\ y=L;\ z=L$

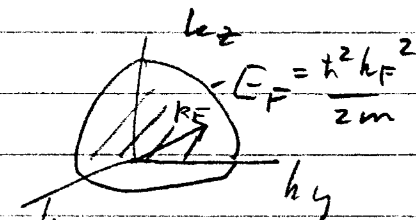
we require
$$k_x L = l\pi$$
$$k_y L = m\pi \quad [\text{integers } l, m, n]$$
$$k_z L = n\pi$$

Thus

$$\psi \sim \sin\left(\frac{l\pi x}{L}\right)\sin\left(\frac{m\pi y}{L}\right)\sin\left(\frac{n\pi z}{L}\right)$$

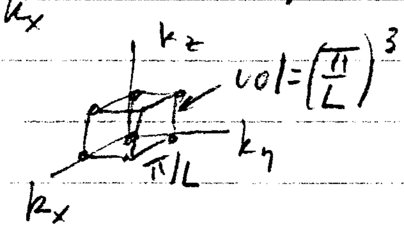
$$E = \frac{\hbar^2}{2m}\left(k_x^2 + k_y^2 + k_z^2\right)$$

$$\boxed{E = \frac{\hbar^2}{2m}\cdot\frac{\pi^2}{L^2}\left(l^2 + m^2 + n^2\right)}$$



b) maximum energy E_F:

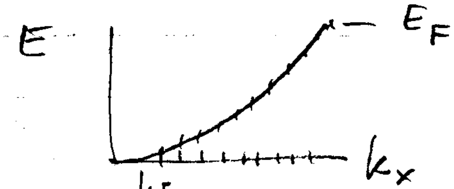
$$\left(\frac{1}{8}\right)\frac{4\pi}{3}k_F^3 \cdot \frac{2}{(\pi/L)^3} = N$$



$$k_F^3 = \frac{3\pi^2 N}{L^3} = 3\pi^2\frac{N}{V}$$

$$\boxed{E_F = \frac{\hbar^2}{2m}\left(3\pi^2\frac{N}{V}\right)^{2/3}}$$

c). average energy in the band:



$$\langle E \rangle = \frac{\int_0^{k_F} 4\pi k^2 dk\, E_k}{\int_0^{k_F} 4\pi k^2 dk} \quad ; E_k = \frac{\hbar^2 k^2}{2m}$$

$$= \frac{\hbar^2}{2m} \frac{\int_0^{k_F} k^4 dk}{\int_0^{k_F} k^2 dk}$$

$$= \frac{\hbar^2}{2m} \frac{k_F^5/5}{k_F^3/3} = \frac{3}{5} \frac{\hbar^2 k_F^2}{2m} = \frac{3}{5} E_F$$

d). total energy

$$E = N \cdot \langle E \rangle = \boxed{\frac{3}{5} \frac{\hbar^2}{2m} \left(3\pi^2 \frac{N}{V}\right)^{2/3}} \times N$$

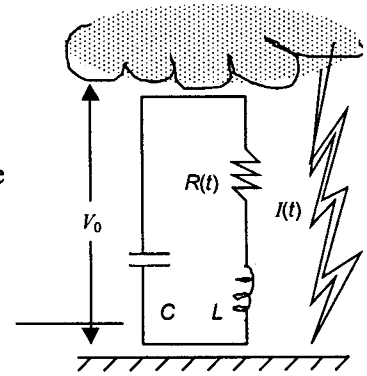
$$= \frac{3\hbar^2}{10m} (3\pi^2 N)^{2/3} V^{-2/3} \times N$$

e). pressure

$$p = -\frac{\partial E}{\partial V} = \frac{2}{3} \frac{E}{V} = \boxed{\frac{\hbar^2}{5m} (3\pi^2 N)^{2/3} V^{-5/3}} \times N$$

i.e., $pV^{5/3} = \text{const}$ at $T=0$

Question 10



Cloud-to-ground lightning bolts are short, intense current pulses $I=I(t)$, driven by large cloud/ground voltage differences V_0. A pulse resembles a transient current moving in a *CRL* circuit (per sketch) with C the cloud/ground capacitance, R the resistance the bolt encounters in passing, and L the self-inductance the bolt develops. During a single bolt, at time $t \geq 0$, assume:

(1) V_0 & $C \approx$ constants (the cloud is large and can generate many lightning bolts);
(2) $R=R(t)$ decreases with t, then increases (a brief dielectric breakdown of the air);
(3) L varies slowly in time compared to R (the bolt radius a goes from 0, to a peak, then back to 0. Meanwhile: $L \propto \ln(1/a)$, but $R \propto 1/a^2$ varies much more rapidly);
(4) $R(t)$ & L are $\approx$ the same over the length of the bolt (avoid partial differential equations just yet);
(5) the bolt is "overdamped": $\langle RC \rangle >> \langle L/R \rangle$, on average (ensures a *single* pulse).

(a) (2)&(5) imply $I(t)$ is controlled by the circuit R &L, and is of duration $\approx$ the time-scale of the $R(t)$ variation. Let C be so large that, during the pulse, the return current through C is negligible [or, by *(1)*, let V_0 = constant]. **PROBLEM**: Write a circuit equation for the current $I(t)$ thru R &L, with $R=R(t)$ [by*(2)*] and L $\approx$constant [by*(3)*].

(b) Your circuit equation should look like: $dI/dt+\Omega(t)I=A$, with $\Omega(t)$ and A = constant fixed by circuit parameters. Find your own expressions for $\Omega(t)$ and A. Such ordinary differential equations are solved by use of an integrating factor: $\mu(t) = \exp(\int\Omega(t)dt)$, and the particular integral of interest here—which obeys the initial condition: $I(0) = 0$—is: $I(t) = [A/\mu(t)] \int\mu(t')dt'$. **PROBLEM**: Verify this result as a solution to your circuit equation for the bolt.

(c) Now let: $R(t)=(R_0/2)[(\tau/t)+(t/\tau)]$; τ & R_0=constants. At $t>0$, this $R(t)$ satisfies *(2)*, and its minimum is $R(\tau)=R_0$. **PROBLEM**: show, for this $R(t)$, that the integrating factor $\mu(t)$ in part (b) can be evaluated *explicitly*. The integrand for the $I(t)$ pulse is now explicit. NOTATION: let $x = t/\tau$ be a dimensionless time.

(d) The $I(t)$ integral in part (c) is doable if τ(breakdown) and L/R_0 (time-constant) are simply related. Let $\tau=nL/R_0$, with n=constant. **PROBLEM**: pick n so that the integral is doable and gives a simple form for $I(x)$. How does this form for $I(x)$ behave for $x<<1$, $x>>1$? Where is its peak, approximately? Sketch $I(x)$ vs. x, for $x>0$. How does the behavior of $I(x)$ vs. x follow $R(x)$ vs. x?

(A) Under the circumstances, C acts like a battery that maintains a ~ const voltage $V_0 = Q/C$ across R & L. The cloud's (huge) charge Q changes negligibly during a single bolt. Kirchoff's voltage law holds instantaneously (conservation of energy!) and the required circuit eqtn is:

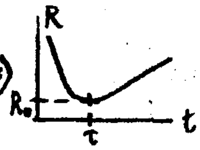
→ $\underline{\underline{L\dot{I} + RI = V_0}}$, w/ V_0 = cnst and $R = R(t)$ & $L = L(t)$ permitted.

(B) We have: $\underline{\underline{dI/dt + \Omega(t) I = \text{cnst} = V_0/L}}$, w/ $\Omega(t) = R(t)/L$, L = cnst.

Calling for an immediate solution may seem cruel. BUT, the solution appears in Gradshteyn & Ryzhik, Sec. (16.316), so it can be looked up.

For $I(0)=0$, the desired solution to our circuit equation is:

$$\boxed{I(t) = \frac{1}{\mu(t)}\,\frac{V_0}{L}\int_0^t \mu(t')\,dt'} \quad \text{w/} \quad \underline{\underline{\mu(t) = \exp\left[\frac{1}{L}\int R(t)\,dt\right]}}.$$

(C) If $R(t) = \frac{R_0}{2}\left[(\tau/t)+(t/\tau)\right]$ $\Big(\Rightarrow$  $\Big)$, then -- in $\mu(t)$ -- get

$$\rightarrow \int R(t)\,dt = \frac{R_0}{2}\int\left[\frac{\tau}{t}+\frac{t}{\tau}\right]dt = \frac{R_0\tau}{2}\left[\ln(t/\tau) + \tfrac{1}{2}(t/\tau)^2\right].$$

Let $\underline{\underline{x = t/\tau}}$ be a dimensionless time. The integrating factor in (B) is:

$$\rightarrow \underline{\underline{\mu(t) = \exp\left[\frac{R_0\tau}{2L}\left(\ln x + \tfrac{1}{2}x^2\right)\right] = x^K e^{\frac{1}{2}Kx^2}}} \quad \text{w/} \quad \underline{\underline{x = t/\tau,\ K = R_0\tau/2L}}.$$

So $\mu(t)$ $\underline{\underline{\text{is}}}$ explicit, and semi-simple. The remaining $I(t)$ integral is:

$$\boxed{I(t) = \frac{V_0\tau/L}{x^K e^{\frac{1}{2}Kx^2}}\int_0^x y^K e^{\frac{1}{2}Ky^2}\,dy}$$

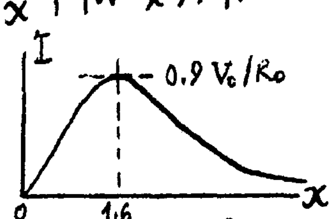
(D) If $\tau = nL/R_0$, then $K = n/2$, and the $I(t)$ integral in (C) is

$$\rightarrow I(t) = \frac{nV_0}{R_0}\left(\frac{1}{x}\right)^{\frac{n}{2}} e^{-\frac{n}{4}x^2}\int_0^x y^{\frac{n}{2}}\, e^{\frac{1}{4}ny^2}\,dy.$$

The integral is doable if $\underline{\underline{n=2}}$ $\left(\int y e^{\frac{1}{2}y^2}dy = e^{\frac{1}{2}y^2}\right)$; we get the pulse...

$$\boxed{I(t) = \frac{2V_0}{R_0}\left(\frac{1}{x}\right)\left[1-e^{-\frac{1}{2}x^2}\right], \ \text{w/}\ x = \frac{t}{\tau}} \sim \begin{cases} (V_0/R_0)\,x, & \text{for } x \ll 1; \\ (2V_0/R_0)\,\frac{1}{x}, & \text{for } x \gg 1. \end{cases}$$

Asymptotically, $I\left(\sim x \rightarrow \frac{1}{x}\right)$ follows the $\underline{\text{inverse}}$ of $R\left(\sim \frac{1}{x} \rightarrow x\right)$. The peak occurs $\underline{\text{later}}$ than R_{min}:



$$\rightarrow dI/dx = 0 \Rightarrow (1+x^2)\,e^{-\frac{x^2}{2}} = 1 \Rightarrow x \simeq 1.6, \text{ and } I_{peak} \simeq 0.9\, V_0/R_0.$$

Question 11

Two identical spin-zero particles of mass m move in a one-dimensional box with hard walls at $x = 0$ and $x = a$.

(a) Determine the ground-state energy and wave function if the particles do not interact with each other.

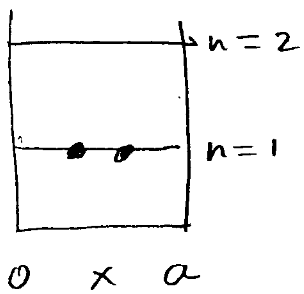
(b) Suppose the particles interact according to the potential energy $V = A\delta\left(x_1 - x_2\right)$, where A is a real constant. Find the first-order shift of the ground-state energy due to V.

(c) Determine the ground-state energy of four identical spin-1/2 particles in this same box if the particles do not interact with each other.

(d) What is the total spin S of the ground-state in (c)? Is the ground state degenerate?

Solution

(a)



For one particle,

$$\phi_1 = \sqrt{\frac{2}{a}} \sin \frac{\pi x}{a} \qquad (S=0)$$

$$E_1 = \frac{\pi^2 \hbar^2}{2ma^2}$$

The g.s. of the two-particle system is

$$\psi = \phi_1(x)\,\phi_2(x) = \frac{2}{a} \sin \frac{\pi x_1}{a} \sin \frac{\pi x_2}{a} = \text{symmetric} \checkmark \quad (S=0)$$

$$E_{tot} = 2E_1 = \frac{\pi^2 \hbar^2}{ma^2}$$

(b) Given the perturbation $V = A\,\delta(x_1 - x_2)$,

$$\Delta E = \langle \psi | A\,\delta(x_1 - x_2) | \psi \rangle \quad \text{in first order}$$

$$= \frac{4}{a^2} \int dx_1 dx_2 \sin^2 \frac{\pi x_1}{a} \sin^2 \frac{\pi x_2}{a} A\,\delta(x_1 - x_2)$$

$$= \frac{4A}{a^2} \int_0^a dx_1 \sin^4 \frac{\pi x_1}{a} \quad ; \text{ let } x = \frac{\pi x_1}{a}$$

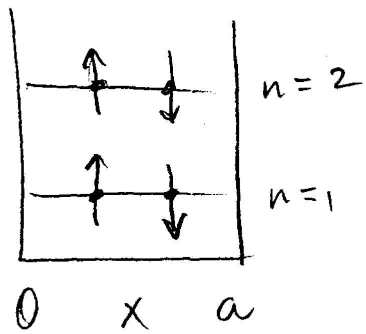
$$= \frac{4A}{a^2} \cdot \frac{a}{\pi} \int_0^\pi dx \cdot \sin^4 x$$

$$= \frac{4A}{\pi a} \cdot \frac{3}{4} \int_0^\pi dx \sin^2 x \quad [\text{handbook}]$$

$$= \frac{\cancel{4}A}{\cancel{\pi} a} \cdot \frac{3}{\cancel{4}} \cdot \frac{\cancel{\pi}}{2} = \frac{3A}{2a} \quad \text{units} \checkmark$$

#11

(c) g.s. of 4 particles with $s = 1/2$ (fermions)



$$E_2 = 2^2 E_1 = \frac{2\pi^2\hbar^2}{ma^2}$$

$$E_1 = \frac{\pi^2\hbar^2}{2ma^2}$$

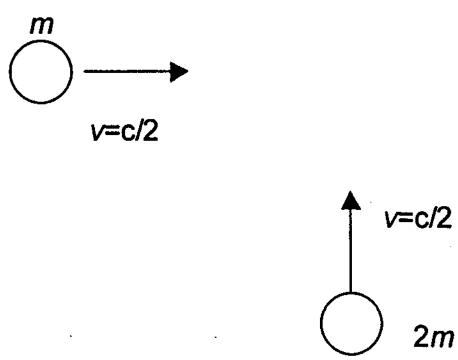
$$E_{tot} = 2E_1 + 2E_2 = \frac{5\pi^2\hbar^2}{ma^2}$$

(d) Because spins are paired in each subshell,

$S = 0$; nondegenerate g.s.

Question 12

Two balls of sticky 'putty' of rest masses m and $2m$, each moving at lab velocity c/2, collide at right angles and stick together to form a blob.



(a) Calculate the rest mass of the blob immediately after the collision.

(b) Find the velocity vector of the blob after the collision.

(c) Your calculation of part (a) should give a rest mass that is greater than the sum of the rest masses of the original blobs ($3m$). If the blobs are ordinary matter (i.e., baryons and electrons) the collision cannot produce more putty since baryon number is conserved. Discuss what accounts for the extra rest mass.

#12

Solution

Take blob 1 to be moving in the x direction and blob 2 in the y direction. The four-momenta are

$$\mathbf{p_1} = m\gamma\left(1, \frac{1}{2}, 0, 0\right) \qquad \mathbf{p_2} = 2m\gamma\left(1, 0, \frac{1}{2}, 0\right),$$

where $\gamma = 2/\sqrt{3}$. Four-momentum is conserved in the collision, so the final four-momentum is

$$\mathbf{p_f} = \mathbf{p_1} + \mathbf{p_2} = m\gamma\left(3, \frac{1}{2}, 1, 0\right).$$

The final rest mass is

$$m_f^2 = -\mathbf{p_f}\cdot\mathbf{p_f} = \frac{31}{4}m^2\gamma^2 = \frac{31}{3}m^2.$$

The final four-velocity is

$$\mathbf{u_f} = \frac{\mathbf{p_f}}{m_f} = \frac{6}{\sqrt{31}}\left(1, \frac{1}{6}, \frac{1}{3}, 0\right),$$

so the velocity vector is

$$\frac{\vec{v}}{c} = \frac{1}{6}\hat{x} + \frac{1}{3}\hat{y}.$$

The collision produces extra mass-energy in the form of thermal excitations and possibly non-conserved particles (*e.g.*, photons, electrons, positrons, pions, etc.).

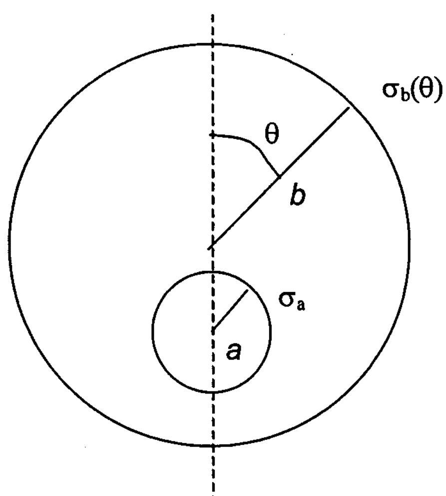
Question 13

Consider two infinitely thin non-conducting spherical shells of radii a and b as shown below. The centers of the spheres are separated by a distance y. The small sphere is completely inside the large one. The small sphere has a uniform charge density of $\sigma_a = Q/(4\pi a^2)$ while the large sphere has a non-uniform charge density of $\sigma_b = \sigma_b(\theta)$, which is to be found. Answer the following questions:

(a) Explain briefly and precisely how you will go about determining $\sigma_b = \sigma_b(\theta)$ so that the electric potential is zero everywhere outside the large sphere r>b. (Hint: Consider a grounded conducting spherical shell with a point charge inside.)

(b) Determine $\sigma_b = \sigma_b(\theta)$. (Hint: $q' = -(b/y)\ q,\ and\ \ y' = b^2/y$)

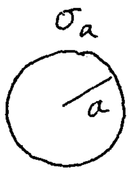
(c) Determine the energy required to form $\sigma_b(\theta)$ in the presence of σ_a.



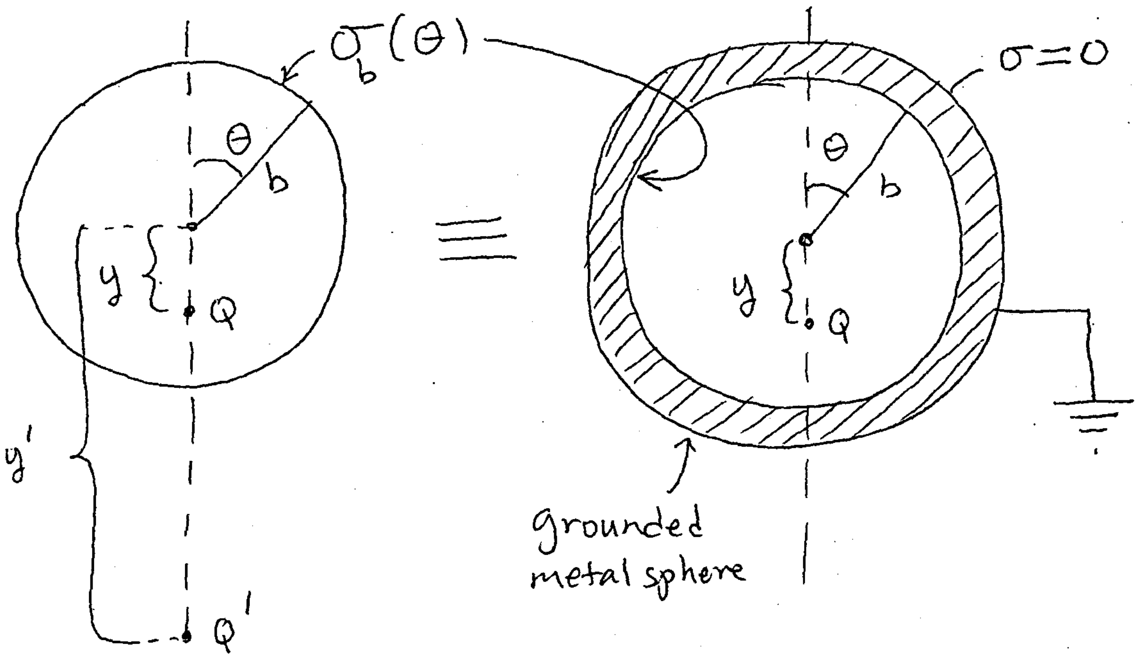
Solution

(a) Consider the following tools:

For the spherical charge distribution over the small sphere:



$= r/Q \Rightarrow V_a = \begin{cases} \frac{Q}{a} & r<a \\ \frac{Q}{r} & r>a \end{cases}$



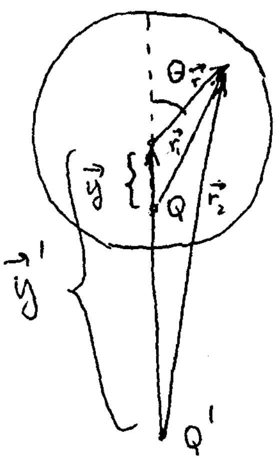
$\sigma_b(\theta)$ is equivalent to the charge distribution on the inner surface of the grounded spherical metal shell of radius b with a point charge placed y-distance below the center. Problem can be solved using image charge method by first finding the Q' and then finding the electric potential on the surface of the sphere using principle of superposition, and finally by $\sigma(\theta) = \frac{1}{4\pi}\frac{\partial \phi}{\partial r}$ (Note + sign is because we use inner surface of the sphere)

(b) Image charge that represents the $\sigma_b(\theta)$ can be obtained from:

$$Q' = -\frac{b}{y}Q \quad \text{located at} \quad y' = \frac{b^2}{y}$$

Electric potential at an arbitrary point inside the sphere $(r<b)$ can be simply given by:

$$\phi(r) = \frac{Q}{r_1} + \frac{Q'}{r_1}$$



$$\vec{r}_1 = \vec{r} + \vec{y} \Rightarrow r_1 = (r^2+y^2+2ry\cos\theta)^{1/2}$$

$$\vec{r}_2 = \vec{r} + \vec{y}' \Rightarrow r_2 = (r^2+y'^2+2ry'\cos\theta)^{1/2}$$

Note: $$\frac{y'}{b} = \frac{b}{y}$$

$$\phi(r) = \frac{Q}{(r^2+y^2+2ry\cos\theta)^{1/2}} - \frac{(b/y)\,Q}{(r^2+y'^2+2ry'\cos\theta)^{1/2}}$$

Charge density in the <u>inner</u> surface of the metal can be found from

$$\sigma_b(\theta) = \frac{1}{4\pi}\frac{\partial\phi}{\partial r}\Big|_{r=b} = -\frac{Q}{4\pi}\left\{\frac{(b+by\cos\theta)}{\underbrace{(b^2+y^2+2by\cos\theta)^{3/2}}_{y^3\left(1+\left(\frac{b}{y}\right)^2+2\left(\frac{b}{y}\right)\cos\theta\right)^{3/2}}} - \frac{(b/y)(b+by'\cos\theta)}{\underbrace{(b^2+y'^2+2by'\cos\theta)^{3/2}}_{b^3\left(1+\left(\frac{y'}{b}\right)^2+2\left(\frac{y'}{b}\right)\cos\theta\right)}}\right\}$$

$$\left(\frac{y'}{b}\right)^2 = \left(\frac{b}{y}\right)^2 \qquad \frac{y'}{b} = \left(\frac{b}{y}\right)$$

$$\Rightarrow \sigma_b(\theta) = -\frac{Q}{4\pi}\left\{\frac{\frac{b}{y^3} + \cancel{\frac{b}{y^2}\cos\theta} - \frac{1}{by} - \cancel{\frac{b}{y^2}\cos\theta}}{\left(1+\left(\frac{b}{y}\right)^2+2\left(\frac{b}{y}\right)\cos\theta\right)^{3/2}}\right\}$$

$$\boxed{\sigma_b(\theta) = -\frac{Q}{4\pi b^2}\,\frac{\left(\frac{b}{y}\right)\left(\left(\frac{b}{y}\right)^2-1\right)}{\left(1+\left(\frac{b}{y}\right)^2+2\left(\frac{b}{y}\right)\cos\theta\right)^{3/2}}}$$

Note that the majori of charge i accumulate around $\theta \approx \pi$ region.

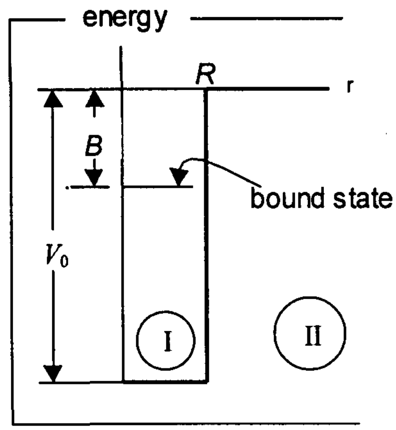
(c) Amount of energy required to to form $\sigma_b(\theta)$ in the presence of σ_a is equivalent to the energy required to bring Q' to location y' in the presence of Q located at y:

$$\boxed{U = \frac{QQ'}{|\vec{y}' - \vec{y}|} = \frac{-\frac{b}{y}Q^2}{\left(\frac{b^2}{y} - y\right)} = -\frac{bQ^2}{(b^2 - y^2)}}$$

PS: One can only use image charge to calculate the fields INSIDE the large sphere. Fields outside must be calculated taking account the charge distribution $\sigma_b(\theta)$ and using integration.

Question 14

The "deuteron" (a deuterium nucleus) is a system of proton (p) & neutron (n) just barely bound by the nuclear force. Photo-disintegration experiments ($\gamma + H^2 \rightarrow n + H^1$) yield a binding energy $\underline{B=2.23 \text{ Mev}}$, and $p \rightarrow n$ scattering data imply the bound state has a radius $\underline{R \approx 2\times10^{-13} \text{ cm}}$.



As a simple model of the deuteron, assume the $p-n$ binding is provided by the radially symmetric nuclear potential well as sketched at right (a so-called " square well"):

$V(r) = -V_0$, a constant, for: $0 \le r \le R$; and
$V(r) = 0$, otherwise.

r is the internucleon distance; the bound level B is shown. Assume: (1) the bound level is an <u>S–state</u> (in fact it is $^3S_1$) with total orbital angular momentum $l = 0$; (2) the nucleons p & n have essentially the same mass, namely: $\underline{M = 939 \text{ Mev/c}^2}$.

(a) Solve the (radial) Schrodinger equation for the system wavefunction $\psi(r)$ both inside ($r<R$) and outside ($r>R$) the well. $\psi(r)$ need <u>not</u> be normalized. Briefly discuss and justify the boundary conditions that you impose upon $\psi(r)$. *HINTS*: *(1)* include the"reduced mass"in the wave equation; *(2)* it helps to use $u(r) = r\psi(r)$ in the radial equation.

(b) By suitably matching ψ & ψ' (or u & u') at $r=R$, derive a relation (involving B, V_0, R etc.) required to ensure continuity. If $B \ll V_0$ (by assumption), use this relation to find an (approximate) expression for the quantity $V_0 R^2$. This limits the "size" of the nuclear well.

(c) Using given data (for R,M, etc.), compute a <u>number</u> for the nuclear well-depth V_0; use units of Mev. The *size* of V_0 indicates the strength of the nuclear attraction between p & n at distances $\approx 10^{-13}$ cm.

#14

(A) The radial Schrödinger eqtn (no ∡ momentum) is -- in spherical cds...

$$\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{2\mu}{\hbar^2}E\right]\psi(r) = 0 .$$

The reduced mass $\mu = M_1M_2/(M_1+M_2) = M/2$ in this case. E is the (equivalent) particle energy. The substitution $\psi(r) = \frac{1}{r}u(r)$ converts the eqtn to

$$\frac{d^2u}{dr^2} + K^2u = 0, \quad \text{w/} \quad K^2 = \frac{M}{\hbar^2}E .$$

See (OVER)

In region I, $E = V_0 - B$, and: $u_I'' + K^2 u_I = 0$, w/ $K = \sqrt{\frac{M}{\hbar^2}(V_0 - B)}$.

Soln is: $u_I(r) = a \sin Kr + b \cos Kr$, a & b = consts. Put $b = 0$ to ensure $\psi(r) = \frac{1}{r} u(r)$ is finite at $r = 0$. Then: $\boxed{u_I(r) = a \sin Kr}$.

In region II, $E = -B$, and $u_{II}'' - k^2 u_{II} = 0$, w/ $k = \sqrt{MB/\hbar^2}$. Soln is: $u_{II}(r) = c e^{-kr} + d e^{+kr}$. Put $d = 0$ to make $\psi(r) \to 0$ as $r \to \infty$. So $\boxed{u_{II}(r) = c e^{-kr}}$.

(B) ψ & $\frac{d\psi}{dr}$ must be <u>continuous</u> at all r. In particular, at $r = R$...

$$\left.\begin{array}{l} \psi_I(R) = \psi_{II}(R) \Rightarrow a \sin KR = c e^{-kR} \\ \psi_I'(R) = \psi_{II}'(r) \Rightarrow Ka \cos KR = -kc e^{-kR} \end{array}\right\}$$

divide 1st eq. by 2nd eq.. $\tan KR = -K/k$

The required continuity relation is: $\tan KR = -\left(\frac{V_0}{B} - 1\right)^{1/2}$. If $B \ll V_0$, then $KR \gtrsim \frac{\pi}{2}$, which yields: $\boxed{V_0 R^2 \gtrsim (\pi/2)^2 \frac{\hbar^2}{M}}$.

(C) From (B): $V_0 \gtrsim \left(\frac{\pi}{2}\right)^2 \hbar^2 / MR^2$, w/ $R = 2 \times 10^{-13}\,\text{cm} = \frac{1}{1.41} e^2/mc^2$ (classical electron rad.)

$$\text{So...}\ V_0 \gtrsim \left(\frac{1.41\pi}{2}\right)^2 \frac{1}{M} \left(\frac{\hbar}{e^2/mc^2}\right)^2 = \left(\frac{1.41\pi}{2}\underbrace{\frac{\hbar c}{e^2}}_{\frac{1}{\alpha} = 137}\right)^2 \overbrace{\left(\frac{m}{M}\right)}^{1/1839} \underbrace{mc^2}_{0.511\,\text{MeV}} = 25.6\,\text{MeV}$$

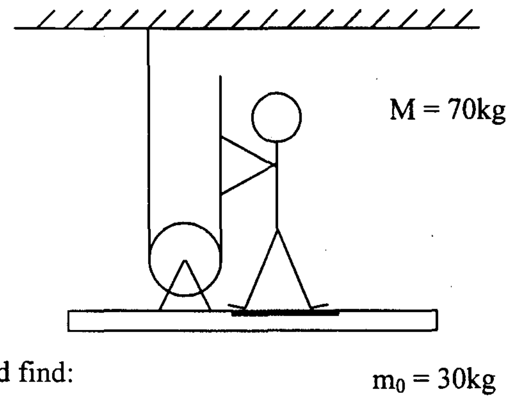
(1) Prob^m is non-relativistic because V_0 & B are $\ll Mc^2$.

(2) Quadrupole moment not possible in $^3S_1$ state. Must be a $^3D_1$ admixture.

(3) $^3S_1$ as ground state of H_2, would be preferred--except it violates Pauli.

Question 15

A person with mass $M = 70$ kg stands on a light bathroom scale that sits on a platform with mass $m_0 = 30$ kg. The person pulls on a massless string giving it a tension $T = 800$ N. The end of the rope is firmly anchored in the ceiling and passes around a pulley attached to the platform. The person pulls up on the rope as shown. Both sections of the rope are vertical and the pulley spins on frictionless bearings. Initially consider the case where the pulley has negligible mass and find:



(a) the acceleration of the person, and

(b) the reading on the bathroom scale.

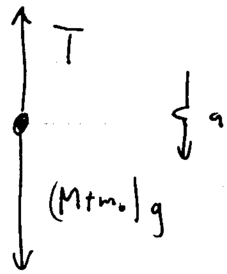
Now, consider the case where the pulley has uniform mass $m = 10$ kg and sufficient friction that the rope does not slip against the pulley's surface.

(c) Find a general expression for the person's acceleration and show that it reduces to the result from (a) in the case $m = 0$.

(d) Find the reading on the bathroom scale.

#15

a)

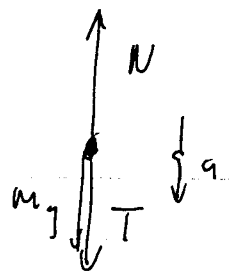


$$(M+m_0)\,a = (M+m_0)\,g - T$$

$$a = \frac{100(9.8)-800}{100}$$

$$= 1.8\ m/s^2\ [\downarrow]$$

b)



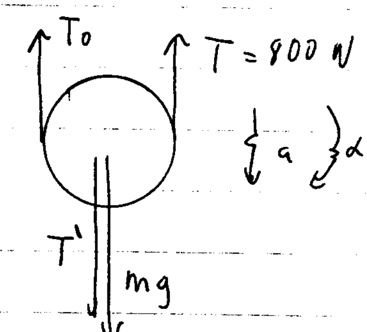
$$Ma = T + Mg - N$$

$$N = M(g-a) + T$$

$$= 70(9.8-1.8) + 800$$

$$= 1360\ N$$

c) pulley:



$$ma = mg + T' - T_0 - T \quad ①$$

$$I\alpha = (T_0 - T)\,r$$

$$I = \tfrac{1}{2} m r^2$$

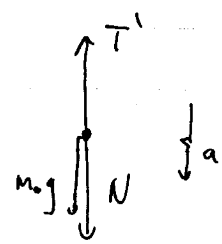
$$\tfrac{1}{2} m r^{\cancel{2}} \alpha = (T_0 - T)\,\cancel{r}$$

$$r\alpha = a$$

$$\tfrac{1}{2} m a = T_0 - T$$

$$T_0 = \tfrac{1}{2} m a + T \quad ②$$

platform:

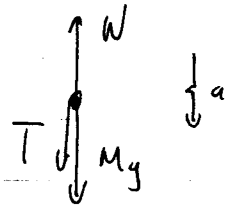


$$N + m_0 g - T' = m_0 a \quad ③$$

add 1 + 3 $\quad (m_0 + m)a = (m_0 + m)g - T_0 - T + N \quad$ (4)

This is just what you get if you consider the pulley + platform system

person:



$Ma = T + Mg - N \quad$ (5)

add (4) + (5) $\Rightarrow \quad (M + m_0 + m)a = (M + m_0 + m)g - T_0$

This is equation for person + platform + pulley system.

use (2) to eliminate T_0

$$(M + m_0 + m)a = (M + m_0 + m)g - \tfrac{1}{2}ma - T$$

$$(M + m_0 + \tfrac{3}{2}m)a = (M + m_0 + m)g - T$$

$$\boxed{a = \frac{(M + m_0 + m)g - T}{(M + m_0 + \frac{3}{2}m)}}$$

$a = 2.42\ m/s^2\ [\downarrow]$

for $m = 0 \qquad a = \dfrac{(M + m_0)g - T}{M + m_0}$ as before.

d) from (5) $\quad N = T + Mg - Ma = 800 + 70(9.8 - 2.42)$

$= 1317\ N$