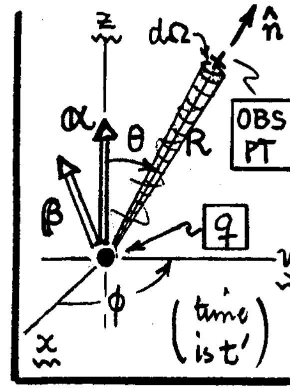
# Radiated power. Larmor's Formulas for $\beta \ll 1$.




**8)** Everything on the RHS of $\mathbb{S}_{rad}$ in Eq. (23) is evaluated at the retarded (source) time $t' = t - \frac{1}{c}R(t')$; the actual radiant energy reaches the obs$^{\underline{v}}$n point at the later time $t$. But nothing prevents us from calculating the power radiated at time $t'$ that will eventually reach the observer at time $t = t' + \frac{1}{c}R(t')$. So, we can calculate the radiated energy/time per solid ∡ $d\Omega = \sin\theta\, d\theta\, d\phi$ at $t'$...

$$\rightarrow \frac{dP}{d\Omega} = R^2 |\mathbb{S}_{rad}| = \frac{q^2}{4\pi c}\left[\frac{\hat{n}\times[(\hat{n}-\beta)\times\alpha]}{(1-\hat{n}\cdot\beta)^3}\right]^2 \quad \text{all quantities in this expression eval. @ } t'. \tag{25}$$

... and the total radiated power (into all $4\pi$ solid ∡s)...

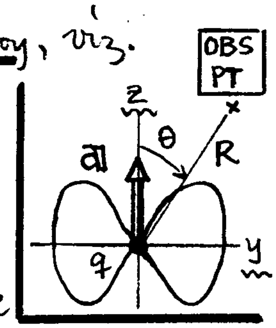
$$\rightarrow P_{total}(t') = \int_{4\pi} (dP/d\Omega)\, d\Omega = \frac{q^2}{4\pi c}\int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta\, d\theta \left[\frac{\hat{n}\times[(\hat{n}-\beta)\times\alpha]}{(1-\hat{n}\cdot\beta)^3}\right]^2. \tag{26}$$

This ∡ integration is not pretty. Just the numerator is...

$$\leadsto \left(\hat{n}\times[(\hat{n}-\beta)\times\alpha]\right)^2 = [(1-\hat{n}\cdot\beta)\alpha + \beta]^2 - (\hat{n}\cdot\alpha)^2 - $$
$$-2(1-\hat{n}\cdot\beta)(1-\hat{n}\cdot\alpha)(\alpha\cdot\beta) - [1-(\hat{n}\cdot\alpha)^2]\beta^2, \tag{27}$$

... and it requires knowing ∡$(\hat{n},\beta)$ & ∡$(\alpha,\beta)$ as well as ∡$(\hat{n},\alpha)$.

So, to get anything ~ pleasing/informative out of Eqs. (25)–(26), we need some approximation. The simplest thing to do is a nonrelativistic ploy, viz.




$$\underline{\beta \ll 1} \Rightarrow \begin{cases} (1-\hat{n}\cdot\beta) \simeq 1, \text{ in denoms of Eqs. (25)-(26);} \\ \left(\hat{n}\times[(\hat{n}-\beta)\times\alpha]\right)^2 \simeq \left(\hat{n}\times(\hat{n}\times\alpha)\right)^2 = \alpha^2 \sin^2\theta. \end{cases} \tag{28}$$

$q$'s acceleration $\mathbf{a} = c\alpha$ is instantaneously ∥ z-axis. Above qtys are

$$\boxed{\begin{aligned} \frac{dP}{d\Omega} &\simeq (q^2/4\pi c^3)|\mathbf{a}|^2 \sin^2\theta, \\ P_{total} &\simeq (2q^2/3c^3)|\mathbf{a}|^2. \end{aligned}} \tag{29}$$

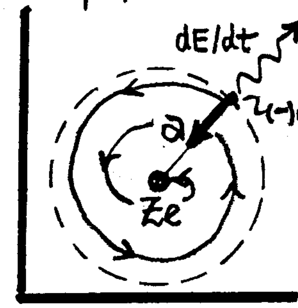
LARMOR RADIATION FORMULAS [Jackson Eqs. (14.21) & (14.22)]

These formulas are OK for arbitrary accel$^{\underline{n}}$ $\mathbf{a}$, so long as $q$'s velocity remains $\ll c$.

# Lienard's relativistic generalization of P(Larmor).

**REMARKS** re Larmor radiation, Eq.(29).

1. Again, $q$ does not radiate unless it accelerates : $\frac{dP}{d\Omega} > 0$ only if $|\mathbf{a}| > 0$.




2. Radiative collapse of the classical atom: the (-)e in orbit is accelerated per : $m|\mathbf{a}| = Ze^2/r^2$, w/ $r = r(t)$ its instantaneous radius. If $E(t)$ is its orbit energy:

$$\rightarrow \frac{dE}{dt} = -\frac{2e^2}{3c^3}\left(\frac{Ze^2}{m}\right)^2 \frac{1}{r^4} \leftarrow \text{rate of orbit energy loss due to radiation.} \tag{30}$$

The electron spirals into the nucleus; its orbit collapses due to radiation. Time required is $< 10^{-10}$ sec (see problems for details). This calculation ⇒ death to any classical model of the atom, with $e$'s in planetary orbits.

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3. In: $P_{total} = (2q^2/3c^3)|\mathbf{a}|^2$, the statement that the acceleration $\mathbf{a}$ can be "large", so long as $q$'s velocity $v = \beta c$ remains "small" ($\ll c$) is eventually self-contradictory; $\beta \to 1$, as $\mathbf{a}$ acts indefinitely. Clearly, what is needed is a relativistic generalization of Larmor's $P_{total}$.

$P = d\mathcal{E}(\text{rad}^n)/dt(\text{lab}) \rightarrow d\mathcal{E}(\text{rad}^n)/d\tau(\text{proper})$ is a Lorentz scalar, since it is the ratio of time-like components of two 4-vectors. So we look for a Lorentz scalar which reduces to P(Larmor) when $\beta \ll 1$, and which at most contains $\beta$ & $\dot{\beta}$. Jackson does this in his Eqs. (14.23)–(14.26), as follows

use $\left\{ E = \gamma mc^2,\ \mathbf{p} = \gamma m\mathbf{v} \right.$, to calculate: $\frac{dE}{d\tau} = c\beta \frac{dp}{d\tau}$

$$\rightarrow P(\text{Larmor}) = \frac{2q^2}{3c^3}\left(\frac{1}{m}\frac{d\mathbf{p}}{dt}\right)\cdot\left(\frac{1}{m}\frac{d\mathbf{p}}{dt}\right) \rightarrow (-)\frac{2q^2}{3m^2c^3}\left(\frac{dp}{d\tau}\right)_\mu \left(\frac{dp}{d\tau}\right)^\mu = P(\text{rel.}) \quad \text{Minkowski force}$$

$$\ldots \left(\frac{dp}{d\tau}\right)_\mu \left(\frac{dp}{d\tau}\right)^\mu = \frac{1}{c^2}\left(\frac{dE}{d\tau}\right)^2 - \left(\frac{d\mathbf{p}}{d\tau}\right)^2 = \beta^2\left(\frac{dp}{d\tau}\right)^2 - \left(\frac{d\mathbf{p}}{d\tau}\right)^2 \ldots$$

So// $P(\text{rel.}) = \frac{2q^2}{3c}\left[\left(\frac{d}{d\tau}\gamma\boldsymbol{\beta}\right)^2 - \beta^2\left(\frac{d}{d\tau}\gamma\beta\right)^2\right]$,

or//

$$\boxed{P(\text{rel.}) = \frac{2q^2}{3c}\gamma^6\left[\dot{\boldsymbol{\beta}}^2 - (\boldsymbol{\beta}\times\dot{\boldsymbol{\beta}})^2\right]} \tag{31}$$

This is Jk$^n$ Eq.(14.26). The "dot" means $d/dt$ (lab time). As usual: $\gamma = 1/\sqrt{1-\beta^2}$. Result due to Lienard (1898!).

# $\phi$520 End Game

25 Apr. 94

| DATE | LECTURE | REMARKS |
|---|---|---|
| Mon. 4/25 | HOLIDAY (Maxwell's birthday). | – |
| Wed. 4/27 | Accelerator radiation: power, ∡ distrib$^n$, pp. q Rad 11-14 | – |
| Fri. 4/29 | finish accel$^r$ rad$^n$. Ultra-relativistic q, pp. 14-17. | Set # P13 due. Set # P14 assigned. |
| Mon. 5/2 | Synchrotron Rad$^n$: freq.-angle distrib$^n$, pp. 18-20. | – |
| Wed. 5/4 | q's eqtn-of-motion. Radiation reaction. (pp. RR 1-11.) | (final preview?) |
| Fri. 5/6 | Orthodox disasters. Dirac's equation. (pp. RR 1-11.) | Set # 14 due. |
| Mon. 5/9 | EXAM WEEK | – |
| Wed. 5/11 | EXAM WEEK | – |
| Fri. 5/13 | EXAM WEEK | – |

The $\phi$520 Final Exam is scheduled for 4-6 PM on Tuesday, 10 May, in room AJM 230.

I will try to extend the exam time by one hour -- to either 3-6 PM, or 4-7 PM -- and will inform you of the change ASAP.

Dick Robiscoe