DEPARTMENT OF PHYSICS

2015 COMPREHENSIVE EXAM

10 PROBLEMS & MATH FORMULAE (6 PAGES)

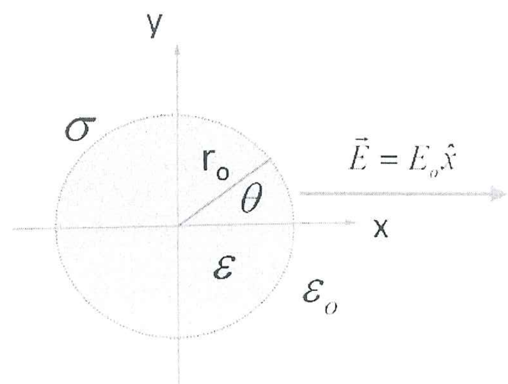
Begin your answer to each question on a new sheet of paper; solutions to different questions must not appear on the same sheet. Label each sheet of paper with your identification number in the upper left hand corner and problem number in the upper right hand corner. When more than one sheet is submitted for a problem, be sure the pages are ordered properly.

PHYSICAL CONSTANTS

Quantity	Symbol	Value
acceleration due to gravity	g	9.807 m s^{-2}
gravitational constant	G	6.674×10^{-11} N m^2 kg^{-2}
permittivity of vacuum	ε_o	8.854×10^{-12} C^2 N^{-1} m^{-2}
permeability of vacuum	μ_o	$4\pi \times 10^{-7}$ N A^{-2}
speed of light in vacuum	c	2.998×10^{8} m s^{-1}
elementary charge	e	1.602×10^{-19} C
mass of electron	m_e	9.109×10^{-31} kg
mass of proton	m_p	1.673×10^{-27} kg
Planck constant	h	6.626×10^{-34} J s
Avogadro constant	N_A	6.022×10^{23} mol^{-1}
Boltzmann constant	k	1.381×10^{-23} J K^{-1}
molar gas constant	R	8.314 J mol^{-1} K^{-1}
standard atmospheric pressure		1.013×10^{5} Pa

The figure below shows the cross section of a long insulating full cylinder of radius r_o and dielectric ε. The axis of the cylinder is aligned with the z-direction. The dielectric cylinder has a uniform surface charge density of σ and was subjected to uniform electric field $\vec{E} = E_o\hat{x}$ along the x-axis. Answer the following questions.

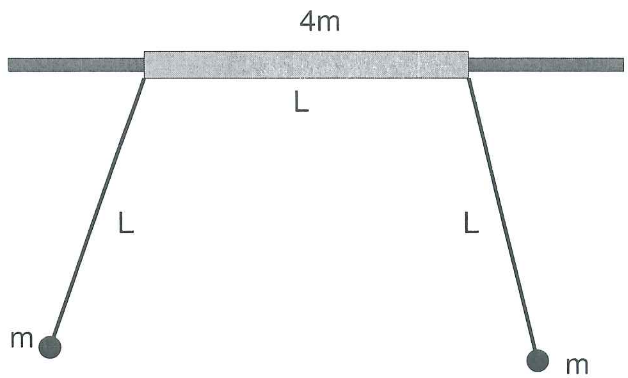
1. First ignore the uniform surface charge distribution on the dielectric cylindrical surface and determine the electrostatic potential distribution in all space.
2. Now introduce the uniform surface charge distribution on the insulating cylindrical surface and determine the electrostatic potential distribution in all space. Explain the reasoning in how you arrived at this result.



2. An electron is bound in the harmonic potential $V = (1/2)m\omega^2x^2$. A weak, constant, electric field $\mathbf{E} = E\mathbf{i}$ is present. Write down the resulting Hamiltonian for this system and determine the new ground state wave function. Show that your wave function satisfies the Schödinger equation, and determine the new ground state energy. Make a sketch illustrating the potential without and with the electric field, and the ground state solution for each. Finally, express the resulting wave function in terms of the dipole moment $\mathbf{p}$.

3.)

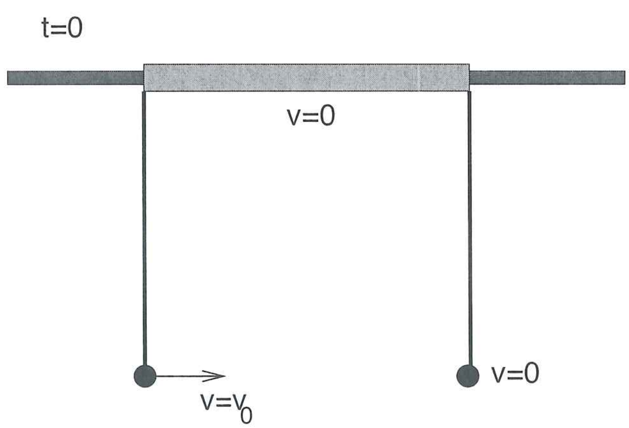
A pipe of mass $4m$ and length L slides frictionlessly along a horizontal bar. From each end of the pipe hangs an identical pendulum consisting of a massless, inextensible rod of length L and a small bobs, of mass m.



a. Define generalized coordinates for the system and write down its Lagrangian. This should be valid for arbitrary angles.

b. For small perturbations to equilibrium, find **all** normal modes and their corresponding eigenfrequencies.

c. At $t = 0$, with the system in equilibrium, the left mass is given small horizontal velocity v_0, while the pipe and right mass are initially unperturbed. Write down the position of **the pipe** for all times $t > 0$.



4) **Quantum Orbit**

A spin-$\frac{1}{2}$ particle is constrained to move on a surface of a ball with radius R, so that its wave function depends only on polar (θ) and azimuthal (ϕ) angles.

(a) First, consider Hamiltonian that does not depend on spin degree of freedom:

$$\hat{\mathcal{H}}_0 = -\frac{\hbar^2}{2mR^2}\nabla^2_{\theta,\phi} \qquad \text{where} \qquad \nabla^2_{\theta,\phi} = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta\frac{\partial}{\partial\theta}\right] + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$$

is the angular part of the Laplacian. What is the energy spectrum of this Hamiltonian and degeneracies of the energy levels, including the spin degeneracy.

(b) Write value of the energy for state whose angular dependence is given by $\psi(\theta,\phi) = Const \times \cos\theta$. Denote this energy E_1. In ket-notation $|\,\ell, \ell_z; s, s_z\,\rangle$ write all states that have the same energy E_1, using standard notations for orbital (ℓ) and spin (s) quantum numbers.

(c) Now look at Hamiltonian with spin-orbit interaction $\hat{V} = A\,\hat{\mathbf{L}}\cdot\hat{\mathbf{S}}$ where A is a constant, $\hat{\mathbf{L}}$ is orbital angular momentum operator, and $\hat{\mathbf{S}}$ is spin angular momentum operator,

$$\hat{\mathcal{H}} = -\frac{\hbar^2}{2mR^2}\nabla^2_{\theta,\phi} + A\,\hat{\mathbf{L}}\cdot\hat{\mathbf{S}}$$

Classify the new eigenstates and their degeneracies using appropriate quantum numbers, provide the energies of these states; indicate how degeneracy of the E_1 states was lifted: completely, partially, not lifted at all?

(d) Confirm part (c) by treating $\hat{V}$ as a perturbation and finding the change in energies of E_1 levels to first order in constant A. (Note: a symmetric matrix that you should get consists on 80% from zeros - use orthogonality of basis states in part (b) to eliminate many matrix elements. Bring the matrix to block-diagonal form to simplify calculations further.)

Hint: for part (d) write

$$\hat{\mathbf{L}}\cdot\hat{\mathbf{S}} = \frac{1}{2}\left[\hat{L}_+\hat{S}_- + \hat{L}_-\hat{S}_+\right] + \hat{L}_z\hat{S}_z$$

and use the following properties of all (orbital and spin) angular momentum operators:

$$\begin{aligned} \hat{J}_+ &= \hat{J}_x + i\hat{J}_y \qquad & \hat{J}_+|\,j, j_z\,\rangle &= \hbar\sqrt{j(j+1) - j_z(j_z+1)}\,|\,j, j_z+1\,\rangle \\ \hat{J}_- &= \hat{J}_x - i\hat{J}_y \qquad & \hat{J}_-|\,j, j_z\,\rangle &= \hbar\sqrt{j(j+1) - j_z(j_z-1)}\,|\,j, j_z-1\,\rangle \\ & & \hat{J}_z|\,j, j_z\,\rangle &= \hbar j_z|\,j, j_z\,\rangle \end{aligned}$$

Briefly answer the following questions. Your 5 best answers out of the 7 will be used to evaluate your performance:

a. What are the fundamental assumptions (postulates) of quantum statistical mechanics? You will get full credit if you state one of these fundamental assumptions (postulates) correctly.

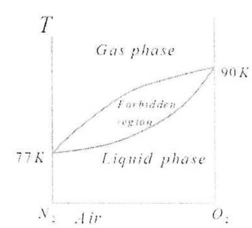
b. Can *p, V, and N* be the independent variables of a fundamental equation in a single-component system? Explain why or why not.

c. Determine the ratio of ^{4}He atoms occupying the ground state ($\varepsilon_0 = 0$) to that of ^{4}He atoms occupying the first excited state at $\sim 1\ K$. The energy splitting between the ground state and the first excited state, ε_1, is $\sim 10^{-14}$ K. The chemical potential of ^4He at $\sim 1\ K$ is $\mu \approx -\frac{kT}{N} = -10^{-22}\ K$.

d. At quantum concentrations the average distance between two nearest particles is comparable to their De Broglie wavelength. At such concentrations and above bosons and fermions behave very differently. At a given temperature which is larger, the quantum concentration of electrons or the quantum concentration of neutrons? Why? Can you use this information to explain why the electrons and protons must combine to form the core of a neutron star?

e. Prove that $\left(\frac{\partial V}{\partial T}\right)_{p,N} = -\left(\frac{\partial S}{\partial p}\right)_{T,N}$

f. The boiling point of pure liquid O_2 is ~90 K, while that of pure liquid N_2 is ~77 K. On the other hand, the boiling point of air (~78% N_2, 21% O_2 and ~1% Ar) is ~78 K. Explain very briefly why, when the temperature of air is lowered to just below 90 K, all of the oxygen in the air does not turn into liquid O_2, leaving behind pure N_2 gas. Refer to the phase diagram of $(1-x)N_2 + xO_2$ on the right.



g. We know that when a monotonic ideal gas is allowed to expand freely its temperature remains the same. What happens to the temperature of a monotonic van der Waals gas when it is allowed to expand freely? Explain why.

6.)

Consider two long concentric cylindrical shells of radii $r = a$ and $r = b$ (with $b > a$ and we use cylindrical coordinates (r, θ, z) with r the distance perpendicular to the z-axis). Both cylinders are centered along the z-axis. Each cylinder is wrapped with a wire (winding in the $\hat{e}_\theta$ direction) with N turns of the wire per unit length such that $Nb \gg 1$. The wire wrapping the outer cylinder (radius b) carries current $I(t)$, and that wrapping the inner cylinder (radius a) carries current $-I(t)$, where the current be an oscillatory function of time, $I(t) = I_0 \cos \omega t$. (Imagine that the wire connects from the outer to the inner cylinder at $z = +\infty$, with a current source at $z = -\infty$). The current $I(t)$ is clearly slowly-varying in time, so you can work to leading, non-trivial order in an expansion in $b\omega/c \ll 1$.

(a) Find $\vec{B}$ and $\vec{E}$.

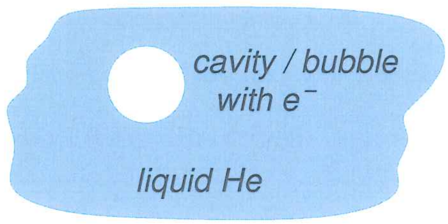
(b) Find the magnetic field energy per unit length, and the associated self-inductance per unit length of the system.

(c) Find the energy flux through the boundaries of the region between the shells, and verify energy conservation.

(d) Compute the angular momentum stored in the fields.

7) **Electron Bubble**

Some low-temperature experiments involve injecting electrons into liquid Helium. Inside the liquid, about a thousand of He atoms get displaced and an empty cavity is formed containing a single electron. Your goal is to calculate the numerical value for minimal size (radius R) of such a bubble.



(a) Treating the inside of the bubble as spherically symmetric potential well with infinitely high walls, calculate first three energy states of the electron.

(b) In the lowest energy state find the pressure (force per unit area) exerted by electron on the surface of the bubble. Is it a positive pressure that expands the bubble, or negative, that collapses the bubble?

(c) To find the size of the bubble use the fact that the electron's pressure is balanced by surface tension pressure only, whose magnitude is $P_{st} = 2\sigma/R$. The coefficient of surface tension of Helium is $\sigma \approx 1.5 \cdot 10^{-4}$ N/m.

8.)

(a) Consider a binary neutron star system, with stars of masses m_1 and m_2, in a circular orbit of radius r_0 with period τ. Let us assume that r_0 is very large, such that General Relativity effects can be ignored, for now. Thus, the binary is in a circular orbit entirely thanks to the attraction of classical, Newtonian gravity. Derive Kepler's third law relation between r_0 and τ.

(b) Suppose that the masses of the binary system are $m_1 = M_\odot$ and $m_2 = 1.5M_\odot$ (where $M_\odot$ is the solar mass), and that $r_0 = 4AU$ (four times the distance between the Earth and the Sun). Find the orbit period, τ, in units of years.

(c) Some mischievous, super-advanced aliens, let's call them Romulans, are playing with their new weapon/toy. By the push of a button, they can cause the two neutron stars to suddenly stop moving around each other. How long do the aliens now have to wait, in Earth years, before they can enjoy watching the two neutron stars collide? You might, or might not, be interested to know that

$$\int_0^1 du\,(u^{-1} - 1)^{-1/2} = 2\int_0^{\pi/2} \sin^2\theta\, d\theta = \frac{\pi}{2} \ ,$$

where $u = \sin^2\theta$.

(d) Is such a Romulan toy possible with current Earth technology? Does it violate any physical principle? Can you suggest a way to make it work, without violating anything? Just a few comments here suffice.

(e) The Romulans have another older toy (from a previous war with the Federation) that, instead of suddenly stopping the neutron stars from moving around each other, can deliver a single, impulse $\Delta\vec{p}$ of arbitrary magnitude instantaneously to either one of the neutron stars. They wish to use this tool to bring about a collision between the neutron stars. What is the minimum impulse (i.e. minimum magnitude) they must deliver and to which neutron star should they deliver it? You may express your result in r_0 and $M_\odot$. How does the amount of time (in years) that the Romulans have to wait to see the neutron stars collide compare to the amount of time computed in part (c)?

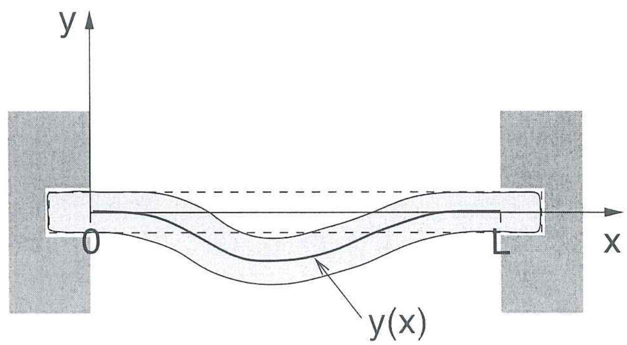
(f) If the Romulan's hold their fire, the neutron stars will eventually collide due to the emission of gravitational waves, a General Relativity phenomenon. Just like electromagnetic waves, gravitational waves carry energy and angular momentum away from the binary system, forcing the orbit to decay. Let us assume that as the binary decays, the stars remain in a circular orbit whose radius decreases as they lose energy at the rate $\dot{E} = -(32/5)(G^4/c^5)(m^3\mu^2/r_0^5)$. Calculate the amount of time it will now take for the binary neutron star system of part (b) to collide. You may approximate the neutron stars as point particles and define collision as the instant when their radial separation vanishes.

9. In a quantum-mechanical rotator, such as a diatomic molecule, the angular momentum is always quantized in nature. Consider such a rotator, whose energy levels are given by $E_J = ((h/2\pi)^2/2I)J(J + 1)$, where I is the moment of inertia, J is the rotational quantum number, and the degeneracy is 2J+1. Determine the partition function of this oscillator, and use it to find the heat capacity at constant volume C_V in the two limits: (a) $k_BT >> ((h/2\pi)^2/2I$ and (b) $k_BT << ((h/2\pi)^2/2I$. Explain why your two answers make physical sense.

10.)

A stiff metal beam is held in a nearly horizontal position by supports at its ends, $x = 0$ and $x = L$ (see figure). It sags slightly under its own weight, leading to a vertical deflection from horizontal written $y(x)$, for $0 \leq x \leq L$. The supports impose the boundary conditions

$$y(0) = y'(0) = y(L) = y'(L) = 0 \quad . \tag{1}$$



The beam has linear mass density $\lambda(x)$ and potential energy given by

$$V = \int_0^L \left[\frac{\alpha}{24} \left(\frac{d^2 y}{dx^2} \right)^2 + g\lambda(x)\, y(x) \right] dx \ ,$$

where g is the gravitational acceleration and α is a constant quantifying the beam's stiffness.

a. Use the calculus of variations to derive the equation for the beam configuration whose potential energy is the **minimum**. This is the equilibrium.

b. Solve the equilibrium equation from a. for a beam with uniform mass density, $\lambda(x) = \lambda_0$, subject to the boundary conditions given by eq. (1). [HINT: The solution will be a polynomial.]

c. Now find the equilibrium when the uniform beam has a mass m suspended from its midpoint

$$\lambda(x) = \lambda_0 + m\,\delta(x - L/2) \ ,$$

where $\delta(x)$ is Dirac's δ-function.

MATHEMATICAL FORMULAE:

Trigonometric functions

x	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π	3π	4π	$n\pi$
$\sin(x)$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	$+1$	$\frac{1}{\sqrt{2}}$	0	$-\frac{1}{\sqrt{2}}$	-1	$-\frac{1}{\sqrt{2}}$	0	0	0	0
$\cos(x)$	$+1$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	$-\frac{1}{\sqrt{2}}$	-1	$-\frac{1}{\sqrt{2}}$	0	$\frac{1}{\sqrt{2}}$	$+1$	-1	$+1$	$(-1)^n$

$$\sin^2 x + \cos^2 x = 1 \quad , \quad 1 + \tan^2 x = \sec^2 x \quad , \quad 1 + \cot^2 x = \csc^2 x$$

$$e^{ix} = \cos x + i \sin x \quad , \quad \cos(x) = \tfrac{1}{2}(e^{ix} + e^{-ix}) \quad , \quad \sin(x) = -i\tfrac{1}{2}(e^{ix} - e^{-ix})$$

$$\sin(a+b) = \sin a \cos b + \cos a \sin b \quad , \quad \sin(2a) = 2 \sin a \cos a$$

$$\cos(a+b) = \cos a \cos b - \sin a \sin b$$

$$\cos(2a) = \cos^2 a - \sin^2 a = 1 - 2\sin^2 a = 2\cos^2 a - 1$$

Hyperbolic functions

$$\cosh(x) = \tfrac{1}{2}(e^x + e^{-x}) \quad , \quad \sinh(x) = \tfrac{1}{2}(e^x - e^{-x}) \quad , \quad e^x = \cosh(x) + \sinh(x)$$

$$\cosh^2 x - \sinh^2 x = 1 \quad , \quad 1 - \tanh^2 x = \mathrm{sech}^2 x \quad ,$$

$$\frac{d\cosh(x)}{dx} = \sinh(x) \quad , \quad \frac{d\sinh(x)}{dx} = \cosh(x) \quad , \quad \frac{d\tanh(x)}{dx} = \mathrm{sech}^2(x) \quad ,$$

$$\cos(ix) = \cosh(x) \quad , \quad \sin(ix) = i\sinh(x) \quad , \quad \tan(ix) = i\tanh(x)$$

Indefinite Integrals (anti-derivatives)

$$\int \sin(x)\cos(x)\,dx = -\tfrac{1}{4}\cos(2x)$$

$$\int \sin^2(x)\,dx = \tfrac{1}{2}x - \tfrac{1}{4}\sin(2x) \quad , \quad \int \cos^2(x)\,dx = \tfrac{1}{2}x + \tfrac{1}{4}\sin(2x)$$

$$\int x\sin(ax)\,dx = \frac{1}{a^2}\sin(ax) - \frac{x}{a}\cos(ax)$$

$$\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1}(x) \quad , \quad \int \frac{dx}{1+x^2} = \tan^{-1}(x) \quad ,$$

$$\int \frac{x\,dx}{\sqrt{x^2+a^2}} = \sqrt{x^2+a^2} \quad , \quad \int \frac{dx}{\sqrt{x^2+a^2}} = \ln\left(x + \sqrt{x^2+a^2}\right)$$

$$\int \frac{x\,dx}{x^2+a^2} = \tfrac{1}{2}\ln(x^2+a^2)$$

$$\int \frac{dx}{(x^2+a^2)^{3/2}} = \frac{x}{a^2\sqrt{x^2+a^2}} \quad , \quad \int \frac{x\,dx}{(x^2+a^2)^{3/2}} = -\frac{1}{\sqrt{x^2+a^2}}$$

Definite Integrals (m, n are integers > 0)

$$\int_0^a \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi x}{a}\right)dx = \tfrac{1}{2}a\delta_{mn} \quad , \quad \int_0^a \sin^2\left(\frac{m\pi x}{a}\right)\sin^2\left(\frac{n\pi x}{a}\right)dx = \frac{a}{8}[2+\delta_{mn}]$$

$$\int_{-\infty}^{\infty} e^{-u^2/2}\,du = \int_{-\infty}^{\infty} u^2 e^{-u^2/2}\,du = \sqrt{2\pi} \quad , \quad \int_0^{\infty} u^n e^{-\alpha u}\,du = \alpha^{-n-1}\,n!$$

$$\int_0^{2\pi} \cos^{2n}\theta\,d\theta = 2\pi\,\frac{1\cdot 3\cdots(2n-1)}{2^n\,n!} \quad .$$

Sums & Series:

$$\frac{1-x^n}{1-x} = \sum_{j=0}^{n} x^n = 1 + x + \cdots + x^n$$

$$e^x = \sum_{j=0}^{\infty} \frac{x^n}{n!} = 1 + x + \tfrac{1}{2}x^2 + \cdots \quad , \quad \ln(1+x) = x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 + \cdots$$

$$\cos(x) = 1 - \tfrac{1}{2}x^2 + \tfrac{1}{4!}x^4 + \cdots \quad , \quad \sin(x) = x - \tfrac{1}{3!}x^3 + \tfrac{1}{5!}x^5 + \cdots$$

$$\frac{1}{\sqrt{1-x}} = 1 + \tfrac{1}{2}x + \tfrac{3}{8}x^2 \cdots + \frac{(2n)!}{4^n(n!)^2}x^n \cdots$$

Fourier transforms:

$$\int_{-\infty}^{\infty} e^{ikx}\,dx = 2\pi\delta(k) \quad , \quad \int_{-\infty}^{\infty} e^{-x^2/\lambda^2}e^{ikx}\,dx = \sqrt{\pi}\lambda e^{-k^2\lambda^2/4}$$

$$\int_{-\infty}^{\infty} e^{-|x|/\lambda}e^{ikx}\,dx = \frac{2\lambda}{1+\lambda^2k^2} \quad , \quad \int_{-\infty}^{\infty} \mathrm{sech}(x/\lambda)e^{ikx}\,dx = 2\lambda\,\mathrm{sech}(\lambda k/2)$$

$$\int_{-\infty}^{\infty} \mathrm{Ai}(x/\lambda)e^{ikx}\,dx = \lambda\,e^{-i\lambda^3k^3/3} \quad , \quad \frac{1}{\pi}\int_0^{\pi} e^{iz\cos\theta}\cos(n\theta)\,d\theta = i^n\,J_n(z)$$

Miscellaneous

$$\det\begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc \quad , \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Vector Calculus

Vectors:

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \mathbf{b}\cdot(\mathbf{c}\times\mathbf{a}) \quad , \quad \mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = \mathbf{b}\,(\mathbf{a}\cdot\mathbf{c}) - \mathbf{c}\,(\mathbf{a}\cdot\mathbf{b}) \ ,$$

$$\nabla\cdot(f\mathbf{a}) = \nabla f\cdot\mathbf{a} + f\nabla\cdot\mathbf{a} \ , \ \nabla\times(f\mathbf{a}) = \nabla f\times\mathbf{a} + f\nabla\times\mathbf{a}$$

$$\nabla\cdot(\mathbf{a}\times\mathbf{b}) = \mathbf{b}\cdot\nabla\times\mathbf{a} - \mathbf{a}\cdot\nabla\times\mathbf{b}$$

$$\nabla\times(\mathbf{a}\times\mathbf{b}) = \mathbf{a}(\nabla\cdot\mathbf{b}) - \mathbf{b}(\nabla\cdot\mathbf{a}) - (\mathbf{a}\cdot\nabla)\mathbf{b} + (\mathbf{b}\cdot\nabla)\mathbf{a}$$

$$\nabla(\mathbf{a}\cdot\mathbf{b}) = \mathbf{a}\times(\nabla\times\mathbf{b}) + \mathbf{b}\times(\nabla\times\mathbf{a}) + (\mathbf{a}\cdot\nabla)\mathbf{b} + (\mathbf{b}\cdot\nabla)\mathbf{a}$$

Cylindrical Coordinates: (r,ϕ,z) , $x = r\cos\phi$, $y = r\sin\phi$

$$\hat{\mathbf{r}}\times\hat{\boldsymbol{\phi}} = \hat{\mathbf{z}} \ , \ d\mathbf{l} = dr\,\hat{\mathbf{r}} + r\,d\phi\,\hat{\boldsymbol{\phi}} + dz\,\hat{\mathbf{z}} \ , \ dV = r\,dr\,d\phi\,dz \ ,$$

$$\nabla f = \frac{\partial f}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial f}{\partial\phi}\hat{\boldsymbol{\phi}} + \frac{\partial f}{\partial z}\hat{\mathbf{z}}$$

$$\nabla\times\mathbf{v} = \left[\frac{1}{r}\frac{\partial v_z}{\partial\phi} - \frac{\partial v_\phi}{\partial z}\right]\hat{\mathbf{r}} + \left[\frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r}\right]\hat{\boldsymbol{\phi}} + \left[\frac{\partial(rv_\phi)}{\partial r} - \frac{\partial v_r}{\partial\phi}\right]\frac{\hat{\mathbf{z}}}{r}$$

$$\nabla\cdot\mathbf{v} = \frac{1}{r}\frac{\partial(rv_r)}{\partial r} + \frac{1}{r}\frac{\partial v_\phi}{\partial\phi} + \frac{\partial v_z}{\partial z} \ , \ \nabla^2 f = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial\phi^2} + \frac{\partial^2 f}{\partial z^2}$$

Spherical Coordinates: (r,θ,ϕ) , $x = r\sin\theta\,\cos\phi$, $y = r\sin\theta\,\sin\phi$, $z = r\cos\theta$

$$\hat{\mathbf{r}}\times\hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\phi}} \ , \ d\mathbf{l} = dr\,\hat{\mathbf{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,d\phi\,\hat{\boldsymbol{\phi}} \ , \ dV = r^2\sin\theta\,dr\,d\theta\,d\phi \ ,$$

$$\nabla f = \frac{\partial f}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial f}{\partial\theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial\phi}\hat{\boldsymbol{\phi}}$$

$$\nabla\times\mathbf{v} = \left[\frac{\partial(\sin\theta\,v_\phi)}{\partial\theta} - \frac{\partial v_\theta}{\partial\phi}\right]\frac{\hat{\mathbf{r}}}{r\sin\theta} + \left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial\phi} - \frac{\partial(rv_\phi)}{\partial r}\right]\frac{\hat{\boldsymbol{\theta}}}{r} + \left[\frac{\partial(rv_\theta)}{\partial r} - \frac{\partial v_r}{\partial\theta}\right]\frac{\hat{\boldsymbol{\phi}}}{r}$$

$$\nabla\cdot\mathbf{v} = \frac{1}{r^2}\frac{\partial(r^2\,v_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(\sin\theta\,v_\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi}$$

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2}$$

Special functions

Bessel functions — $\Omega_m(x)$ stands for either $J_m(x)$ or $Y_m(x)$

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{df}{dr}\right) - \frac{m^2}{r^2}f = -k^2 f \quad\Longrightarrow\quad f(r) = J_m(kr) \;\;\text{or}\;\; f(r) = Y_m(kr)$$

$$J_{-m}(x) = (-1)^m J_m(x) \quad , \quad m \text{ integer}$$

$$\Omega_{m-1}(x) + \Omega_{m+1}(x) = \frac{2m}{x}\Omega_m(x) \quad , \quad \Omega_{m-1}(x) - \Omega_{m+1}(x) = 2\Omega_m'(x)$$

$$J_m(x) = \frac{1}{m!}\left(\frac{x}{2}\right)^m + \cdots \quad , \quad x \ll 1$$

$$Y_0(x) = \frac{2}{\pi}\ln(x) + \cdots \quad , \quad Y_m(x) = -\frac{(m-1)!}{\pi}\left(\frac{x}{2}\right)^{-m} + \cdots \quad , \quad m \geq 1$$

$$\int_0^1 J_m(j_{m,n}x)\, J_m(j_{m,n'}x)\, x\, dx \;=\; \tfrac{1}{2}J_{m+1}^2(j_{m,n})\,\delta_{n,n'} = \tfrac{1}{2}\Big[J_m'(j_{m,n})\Big]^2\delta_{n,n'} \quad , \quad J_m(j_{m,n}) = 0$$

n	$j_{0,n}$	$J_1(j_{0,n})$	$j_{1,n}$	$J_2(j_{1,n})$	$j_{2,n}$	$J_3(j_{2,n})$	$j_{3,n}$	$J_4(j_{3,n})$	$j_{4,n}$	$J_5(j_{4,n})$
0	2.40	0.519	3.83	0.403	5.14	0.340	6.38	0.298	7.59	0.268
1	5.52	-0.340	7.02	-0.300	8.42	-0.271	9.76	-0.249	11.06	-0.232
2	8.65	0.271	10.17	0.250	11.62	0.232	13.02	0.218	14.37	0.206
3	11.79	-0.232	13.32	-0.218	14.80	-0.207	16.22	-0.196	17.62	-0.188

Modified Bessel functions

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{df}{dr}\right) - \frac{m^2}{r^2}f = +k^2 f \quad\Longrightarrow\quad f(r) = I_m(kr) \;\;\text{or}\;\; f(r) = K_m(kr)$$

$$I_m(x) = \frac{1}{m!}\left(\frac{x}{2}\right)^m + \cdots \quad , \quad x \ll 1$$

$$K_0(x) = -\ln(x) + \cdots \quad , \quad K_m(x) = \tfrac{1}{2}(m-1)!\left(\frac{x}{2}\right)^{-m} + \cdots \quad , \quad m \geq 1$$

$$I_m(x) \;\to\; \frac{e^x}{\sqrt{2\pi x}} \quad , \quad K_m(x) \;\to\; e^{-x}\sqrt{\frac{\pi}{2x}} \quad , \quad x \gg m+1$$

$$I_m(x) = i^{-m}J_m(ix) \quad , \quad K_m(x) = \frac{\pi}{2}i^{m+1}\Big[J_m(ix) + i\,Y_m(ix)\Big]$$

$$I_{-m}(x) = I_m(x) \quad , \quad K_{-m}(x) = K_m(x)$$

Spherical Bessel functions — $z_\ell(x)$ stands for either $j_\ell(x)$ or $n_\ell(x)$

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{df}{dr}\right) - \frac{\ell(\ell+1)}{r^2}f = -k^2 f \quad\Longrightarrow\quad f(r) = j_\ell(kr) \;\;\text{or}\;\; f(r) = n_\ell(kr)$$

$$j_\ell(x) \;=\; (-x)^\ell\left(\frac{1}{x}\frac{d}{dx}\right)^\ell\left(\frac{\sin x}{x}\right) \quad , \quad n_\ell(x) \;=\; -(-x)^\ell\left(\frac{1}{x}\frac{d}{dx}\right)^\ell\left(\frac{\cos x}{x}\right)$$

$$j_0(x) \;=\; \frac{\sin x}{x} \quad , \quad j_1(x) \;=\; \frac{\sin x}{x^2} - \frac{\cos x}{x} \quad , \quad j_\ell(x) \;=\; \frac{x^\ell}{1\cdot 3\cdots(2\ell+1)} + \cdots \quad x \ll 1$$

$$n_0(x) = -\frac{\cos x}{x} \quad , \quad n_1(x) = -\frac{\cos x}{x^2} - \frac{\sin x}{x} \quad , \quad n_\ell(x) = -\frac{1\cdot 3\cdots(2\ell-1)}{x^{\ell+1}} + \cdots \quad x \ll 1$$

$$\frac{2\ell+1}{x} z_\ell(x) = z_{\ell-1}(x) + z_{\ell+1}(x) \quad , \quad (2\ell+1)\, z'_\ell(x) = \ell\, z_{\ell-1}(x) - (\ell+1) z_{\ell+1}(x)$$

Zeroes of $j_\ell(x)$ and $j'_\ell(x)$:

	$j_\ell(x_{\ell,n}) = 0$				$j'_\ell(x'_{\ell,n}) = 0$		
n	$x_{0,n}$	$x_{1,n}$	$x_{2,n}$	$x_{3,n}$	$x'_{0,n}$	$x'_{1,n}$	$x'_{2,n}$
1	3.1416	4.4934	5.7635	6.9880	0.0000	2.0816	3.3421
2	6.2832	7.7253	9.0950	10.4171	4.4934	5.9404	7.2900
3	9.4248	10.9041	12.3229	13.6980	7.7253	9.2059	10.6139
4	12.5664	14.0662	15.5146	16.9236	10.9041	12.4045	13.8461

Legendre polynomials

$$\frac{d}{dx}\left[(1-x^2)\frac{dP_\ell}{dx}\right] = -\ell(\ell+1)\, P_\ell(x) \quad , \quad -1 < x < 1$$

$$P_0(x) = 1 \quad , \quad P_1(x) = x \quad , \quad P_2(x) = \tfrac{3}{2}x^2 - \tfrac{1}{2} \quad , \quad P_3(x) = \tfrac{5}{2}x^3 - \tfrac{3}{2}x \quad , \quad P_4(x) = \tfrac{35}{8}x^4 - \tfrac{30}{8}x^2 + \tfrac{3}{8}$$

$$P_\ell(-x) = (-1)^\ell P_\ell(x) \quad , \quad P_\ell(1) = 1 \quad , \quad P_\ell(0) = (-1)^{\ell/2}\frac{1\cdot 3\cdots(\ell-1)}{2^{\ell/2}(\ell/2)!} \quad , \quad \ell \text{ even}$$

$$P_\ell(x) = \frac{1}{2^\ell \ell!}\frac{d^\ell}{dx^\ell}(x^2-1)^\ell \quad , \quad \int_{-1}^{1} P_\ell(x) P_{\ell'}(x)\, dx = \frac{2}{2\ell+1}\delta_{\ell,\ell'}$$

$$P'_{\ell+1} - P'_{\ell-1} = (2\ell+1)P_\ell \quad , \quad (\ell+1)P_{\ell+1} = (2\ell+1)xP_\ell - \ell P_{\ell-1}$$

$$P'_{\ell+1} = xP'_\ell + (\ell+1)P_\ell \quad , \quad (x^2-1)P'_\ell = \ell\,[x\,P_\ell - P_{\ell-1}]$$

Associated Legendre functions

$$\frac{d}{dx}\left[(1-x^2)\frac{dP_\ell^m}{dx}\right] - \frac{m^2}{1-x^2}P_\ell^m = -\ell(\ell+1)\, P_\ell^m(x) \quad , \quad |m| \le \ell \quad , \quad -1 < x < 1$$

$$P_\ell^m(x) = (-1)^m (1-x^2)^{m/2}\frac{d^m}{dx^m}P_\ell \quad , \quad P_\ell^{-m}(x) = (-1)^m \frac{(\ell-m)!}{(\ell+m)!}P_\ell^m(x)$$

$$\int_{-1}^{1} P_\ell^m(x) P_{\ell'}^m(x)\, dx = \frac{2}{2\ell+1}\frac{(\ell-m)!}{(\ell+m)!}\delta_{\ell,\ell'}$$

Spherical harmonics

$$\nabla^2 Y_{\ell m} = -\frac{\ell(\ell+1)}{r^2} Y_{\ell m}$$

$$Y_{\ell m}(\theta,\phi) = \sqrt{\frac{(2\ell+1)}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}}\, P_\ell^m(\cos\theta) e^{im\phi} \quad , \quad Y_{\ell,-m}(\theta,\phi) = (-1)^m Y^*_{\ell m}(\theta,\phi)$$

$$\int_0^{2\pi}\int_0^{\pi} Y_{\ell m}(\theta,\phi)\, Y^*_{\ell' m'}(\theta,\phi)\,\sin\theta\, d\theta\, d\phi \;=\; \delta_{\ell,\ell'}\,\delta_{m,m'}$$

ℓ	$m = 0$	$m = 1$	$m = 2$
0	$Y_{00} = 1/\sqrt{4\pi}$		
1	$Y_{10} = \sqrt{\frac{3}{4\pi}}\cos\theta$	$Y_{11} = -\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\phi}$	
2	$Y_{20} = \sqrt{\frac{5}{4\pi}}(\frac{3}{2}\cos^2\theta - \frac{1}{2})$	$Y_{21} = -\sqrt{\frac{15}{8\pi}}\sin\theta\,\cos\theta\, e^{i\phi}$	$Y_{22} = \sqrt{\frac{15}{32\pi}}\sin^2\theta\, e^{2i\phi}$