Born Approximation to the Scattering Amplitude. Validity.

5) The first approximation to the scattering amplitude suggested in Eq. (15), where -- in the integral for A, the actual wavefcn $\psi(\mathbf{r}')$ is taken to be the incident wave $\psi(\mathbf{r}') \simeq \phi_b(\mathbf{r}') = \exp(i\mathbf{k}_b\cdot\mathbf{r}')$ -- is a venerable result. It is called the "Born Approximation", and it gives the scattering amplitude:

$$\left[\begin{array}{l} A(b\to a) \simeq -\frac{m}{2\pi\hbar^2}\int_\infty d^3x'\, V(\mathbf{r}')\, e^{i\mathbf{q}\cdot\mathbf{r}'} = A_B(\mathbf{q}) \quad \text{BORN APPROXN} \\[2ex] \text{where: } \hbar\mathbf{q} = \hbar(\mathbf{k}_b - \mathbf{k}_a)\text{, momentum transfer: (b)efore} \to \text{(a)fter.} \end{array}\right\} \tag{16}$$

$A_B(\mathbf{q})$ [Davydov Eq. (106.20)] is correct to first (lowest) order in V, for "weak" scattering events where ψ at any time is never very much changed from the the incident wave ϕ_b. $A_B(\mathbf{q})$ resembles the first (lowest) order transition probability for transitions $m\to k$ [ref. CLASS NOTES, p. tD 5, Eq. (13)]...

$$\to a_k^{(1)}(t) = -(i/\hbar)\int_{t_0}^{t} dt'\, V_{km}(t')\, e^{i\omega_{km}t'}. \tag{17}$$

Here, just as the ψ(scattering) is never much changed from the initial ϕ_b, the initial state amplitude $a_m(t_0)=1$ is "barely depleted" [p. tD 7, Eq. (22)]. <u>The Born Approx^n $A_B(\mathbf{q})$ for the scattering amplitude plays the same role in scattering theory as does the transition amplitude $a_k^{(1)}(t)$ in tD Pert$^{\underline{n}}$ Theory.</u>

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6) How well does it work to take the full scattering amplitude $A(b\to a) \simeq A_B(\mathbf{q})$, per Eq. (16)? In other words, what is the range of validity of the Born Approx^n?

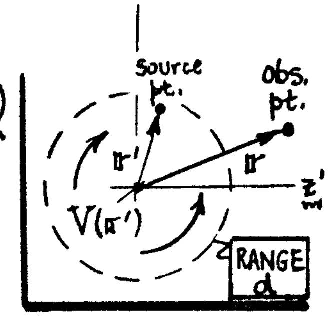
Going back a bit, to Eq. (8) -- which is exact -- the Born approxn amounts to:

$$\to \psi(\mathbf{r}) = \phi_b(\mathbf{r}) - \frac{m}{2\pi\hbar^2}\int_\infty d^3x' \left(\frac{e^{ikR}}{R}\right) V(\mathbf{r}')\,\psi(\mathbf{r}'),\quad R = |\underbrace{\mathbf{r}}_{\text{obs}^{x}\text{n pt.}} - \underbrace{\mathbf{r}'}_{\text{source pt.}}|. \tag{18}$$

approximate ψ in integral by incoming ϕ_b

This stratagem can be "good" only if ψ never deviates very much from ϕ_b, even "in close", where $V(\mathbf{r}')$ is relatively strong. That is, the scattered wave in (18) [2nd term RHS] must be small compared to the incoming wave [1st term RHS]:

Born Approxn Validity at Low Energies.



$$\rightarrow \left|\frac{m}{2\pi\hbar^2}\int_\infty d^3x' \left(\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\right) V(\mathbf{r}')\,\phi_b(\mathbf{r}')\right| \ll |\phi_b(\mathbf{r})| = 1, \quad (19)$$

and this inequality must hold for all $\mathbf{r}$, including $\mathbf{r}\sim\mathbf{r}'$, in the region where $V(\mathbf{r}')$ is "strong". If $V(\mathbf{r}')$ is centered at the origin of coordinates, then take $\mathbf{r}\to 0$ in (19) for the most stringent condition, i.e.

$$\left|\frac{m}{2\pi\hbar^2}\int_\infty d^3x' \left(\frac{e^{ikr'}}{r'}\right) V(\mathbf{r}')\, e^{-i\mathbf{k}_b\cdot\mathbf{r}'}\right| \ll 1, \text{ in close;}$$

... for $\mathbf{k}_b$ along z'-axis, and $|\mathbf{k}_b| = |\mathbf{k}_a| = k$ for elastic collisions...

$$\rightarrow \left|\frac{m}{2\pi\hbar^2}\int_\infty d^3x' \,\frac{V(\mathbf{r}')}{r'}\, e^{ikr'(1-\cos\theta')}\right| \ll 1, \quad \theta' = \text{colatitude} \measuredangle \text{ for } \mathbf{r}'. \quad (20)$$

This inequality depends on the energy E of the incoming beam, because the wave number: $k = \sqrt{2mE}/\hbar$, controls the size of the phase in the $e^{ikr'(\;)}$ factor.

LOW ENERGY : $kr'_{MAX} \rightarrow 0$, and: $e^{ikr'(1-\cos\theta')} \rightarrow 1$.

$$\text{Eq. (20)} \Rightarrow \quad \frac{m}{2\pi\hbar^2}\underbrace{\left|\int_\infty d^3x' \,\frac{V(\mathbf{r}')}{r'}\right|}_{1} \ll 1$$

factor 1 has dimensions of (potential) × (length)^2. Define: $1 = \bar{V}\times 4\pi d^2$. $\bar{V}$ is the average potential in "range" d.

$$\text{so} \quad \frac{m}{2\pi\hbar^2}\cdot \bar{V}\times 4\pi d^2 \ll 1, \quad \text{or} \quad \boxed{\bar{V} \ll \frac{1}{2m}(\hbar/d)^2}. \quad (21)$$

This has an easy interpretation. When the incoming particles are within the range d of V, they are localized to $\Delta x \sim d$, must have momentum components $\Delta p \sim \hbar/\Delta x$, and hence K.E. at least of size $E \sim \frac{1}{2m}(\Delta p)^2 = \frac{1}{2m}\left(\frac{\hbar}{d}\right)^2$. Condition (21) says the Born Approxn is OK at low energies so long as the average potential $\bar{V}$ is small compared to the incident particle K.E. Makes sense... $\bar{V} \ll E_{kin}$ certainly means the scattering is a small perturbation on the incoming beam, and then Ψ cannot differ too much from ϕ_b, as we have used in Eq. (18).

Validity for Spherical Symmetry. High Energy.

7) Validity of the Born Approxn at high energy ($k \to$ large in Eq. (20)) is best preceded by a simplification: we specialize to spherically symmetric potentials. Then the $\measuredangle$ lar part of the integral in Eq. (20) can be done...

$$\int_\infty d^3x' \frac{V(r')}{r'} e^{ikr'(1-\cos\theta')} = \int_\infty \frac{V(r')}{r'} e^{ikr'} e^{-ikr'\cos\theta'} \cdot 2\pi r'^2 dr' \sin\theta' d\theta'$$

$$= 2\pi \int_0^\infty dr'\, r' V(r') e^{ikr'} \left\{ \int_{-1}^{+1} d\mu\, e^{-ikr'\mu} \right\} = \frac{2\pi}{ik} \int_0^\infty dr'\, V(r') [e^{2ikr'} - 1]$$

... so Eq. (20) becomes... (put $\hbar k/m = v$, particle velocity)...

$$\boxed{\left| \int_0^\infty dr'\, V(r') [e^{2ikr'} - 1] \right| \ll \hbar v}. \tag{22}$$

condition for validity of Born Approxn, for spherically symmetric potentials [Davydov (106.16)].

There is no limitation on k here... we can take either the low energy ($k \to 0$) or high energy ($k \to$ large) limit. In the latter case...

HIGH ENERGY: $kr' \gg 1$ for all $r' > 0 \Rightarrow \langle e^{2ikr'} \rangle = 0$ in (22):

So/ $$\boxed{\left| \int_0^\infty V(r')\, dr' \right| \ll \hbar v}. \tag{23}$$

The integral here can be written as $\left| \int_0^\infty V(r')\, dr' \right| = \bar{V} \cdot d$, where $\bar{V}$ is ~ average value of V within its range d, in the manner of Eq. (21). Then (23) requires: $\bar{V} \ll (\hbar/d) v \sim mv^2$, and the validity condition at high energies is once again a requirement that the average potential $\bar{V}$ is small compared to the incident particle K.E.

Since $\bar{V} \ll$ particle K.E. is evidently easier to satisfy as the particle K.E. increases, then the Born Approxn works better at higher energies. We can say that the Born scattering amplitude $A_B(q)$ of Eq. (16) above is a reasonable high-energy approximation to the scattering event, in the sense of Eq. (23) [or Eq. (21), at lower energies].

Nature of $\bar{V}$ & d: Potentials for which Born Approxn works.

ASIDE Nature of the average potential $\bar{V}$ and range d.

1. For spherically symmetric potentials, the condition of Eq.(22) for validity of the Born Approxn has both low & high energy limits (drop the primes, now):

$$\left|\int_0^\infty [e^{2ikr}-1]V(r)\,dr\right| \ll \hbar v \Rightarrow \begin{cases} \left|\int_0^\infty r\,V(r)\,dr\right| \ll \hbar^2/2m, \text{ LOW E } (k\to 0); \\ \left|\int_0^\infty V(r)\,dr\right| \ll \hbar v, \text{ HIGH E } (k\to \text{large}). \end{cases} \tag{24}$$

We have been writing the integrals over V(r) as though they existed, e.g. in Eq. (21), we put $\left|\int_0^\infty r\,V(r)\,dr\right| = \bar{V} d^2$, w/ $\bar{V}$ = average potential within range d. But for some types of potentials, the integrals don't exist... e.g. for a Coulomb potential $V(r) = -Ze^2/r$ (for scattering of an electron from a nucleus Ze), clearly $\left|\int_0^\infty r\,V(r)\,dr\right| \to \infty$, and the Born Approxn doesn't work at any energy.

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2. So, in addition to being restricted to "high energies", the Born Approxn applies only to those potentials for which the integral $\int_0^\infty (e^{2ikr}-1)V(r)\,dr$ exists. $V(r) \sim \frac{1}{r}$ does not meet this criterion, but other potentials do, as demonstrated by Davydov on pp. 455-56.

(a) Exponential Potential: $V(r) = V_0\, e^{-r/r_0}$.

$$\left|\int_0^\infty [e^{2ikr}-1]V(r)\,dr\right| = \left(\frac{2kr_0}{\sqrt{1+(2kr_0)^2}}\right) V_0 r_0 \quad \text{can be written as } \bar{V} d \text{, where } \bar{V}\to V_0 \text{ \& } d \to r_0 \text{ @ high E.}$$

So//

$$\text{Eq}(24) \Rightarrow \left(\frac{2kr_0}{\sqrt{1+(2kr_0)^2}}\right) V_0 r_0 \ll \frac{\hbar^2 k}{m}, \text{ or } \begin{cases} V_0 \ll \hbar^2/2mr_0^2, \text{ at low E;} \\ V_0 \ll \hbar v/r_0, \text{ at high E.} \end{cases} \tag{25}$$

This is a "clean" example... the low & high E criteria closely resemble those developed in Eqs. (21) & (23) above, and r_0 has a reasonably clear interpretation as a "range". It is clearly convenient to have some exponential damping factor in V(r) to ensure the existence of the integral $\int_0^\infty [e^{2ikr}-1]V(r)\,dr$.

ASIDE (cont'd) Nature of $\overline{V}$ & d.

(b) Yukawa (screened Coulomb) Potential : $V(r) = \dfrac{(Z_1 e)(Z_2 e)}{r} e^{-r/r_0}$.

$$\left|\int_0^\infty [e^{2ikr}-1]V(r)dr\right| = \overline{V}d \quad \text{w/} \quad \overline{V} = Z_1 Z_2 e^2/r_0,\ r_0 = \text{screening length}; \quad \text{\&} \quad d = r_0\left[(\ln\sqrt{1+\rho^2})^2 + (\tan^{-1}\rho)^2\right]^{1/2},\ \rho = 2kr_0. \tag{26}$$

As the screening length $r_0 \to \infty$, so that we approach the pure Coulomb potential (i.e. $V(r) \to Z_1 Z_2 e^2/r$), the range $d \to r_0[(\ln\rho)^2 + (\frac{\pi}{2})^2]^{1/2} \simeq r_0 \ln(2kr_0)$ becomes ∞, and the integral $\overline{V}d \to Z_1 Z_2 e^2 \ln(2kr_0)$ also diverges. But these divergences are logarithmically weak, so that a specific value of r_0 which makes the integral finite is not critical. The validity criterion is (22),

i.e. $\longrightarrow Z_1 Z_2 e^2\left[(\ln\sqrt{1+\rho^2})^2 + (\tan^{-1}\rho)^2\right]^{1/2} \ll \hbar v$, w/ $\rho = 2\dfrac{mvr_0}{\hbar}$. (27)

Roughly speaking, this requires : $Z_1 Z_2 e^2 \ll \hbar v$, for (screened) Coulomb scattering between charges $Z_1 e$ & $Z_2 e$ at relative velocity v.

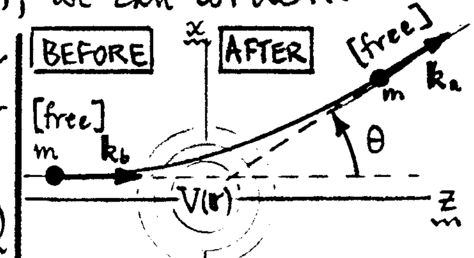
END of ASIDE

8) Now that we understand the range-of-validity of the Born Approxn [i.e. the approxn $\psi(\text{scattering}) \simeq \phi_b(\text{incoming})$ that produced the scattering amplitude $A_B(\mathbf{q})$ of Eq. (16)] we can calculate some actual differential scattering cross-sections $d\sigma/d\Omega$ for "interesting" interaction potentials V.

For elastic scattering (per remark 2 on p. ScT 7): $d\sigma/d\Omega = |A(b\to a)|^2$, and for the Born Approxn $A(b\to a) \simeq A_B(\mathbf{q})$, per Eq. (16), we can write...

$$\frac{d\sigma}{d\Omega} = \left(\frac{m}{2\pi\hbar^2}\right)^2 |\langle\phi_a|V|\phi_b\rangle|^2 \quad \text{w/} \quad \phi_b(\mathbf{r}) = e^{i\mathbf{k}_b\cdot\mathbf{r}},\ \phi_a(\mathbf{r}) = e^{i\mathbf{k}_a\cdot\mathbf{r}}$$

$$\text{w/} \quad \langle\phi_a|V|\phi_b\rangle = \int_\infty V(\mathbf{r})\, e^{i(\mathbf{k}_b-\mathbf{k}_a)\cdot\mathbf{r}}\, d^3x. \tag{28}$$



The matrix element measures the amplitude for a free-free transition $\phi_b \to \phi_a$ induced by m's "collision" with V. All that happens is that m's initial momentum $\hbar\mathbf{k}_b$ changes direction to $\hbar\mathbf{k}_a$; it is "scattered" through $\measuredangle\,\theta$.

$d\sigma/d\Omega$ in Born Approximation

The momentum change $\mathbb{k}_b \to \mathbb{k}_a$ is elastic in that $|\mathbb{k}_b| = k = |\mathbb{k}_a|$, and the scattering ∢ θ (i.e. ∢ between $\mathbb{k}_b$ & $\mathbb{k}_a$) can be brought in as follows...

let $\mathbb{k}_b$ be along z-axis : $\mathbb{k}_b = (0, 0, k)$

let $\mathbb{k}_a$ lie in xz-plane : $\mathbb{k}_a = (k\sin\theta, 0, k\cos\theta)$

So// $\mathbb{k}_b - \mathbb{k}_a = k(-\sin\theta, 0, 1-\cos\theta)$,

and//

$$|\mathbb{k}_b - \mathbb{k}_a| = k\sqrt{\sin^2\theta + (1-\cos\theta)^2} = 2k\sin(\theta/2),\quad \theta = \text{scatt}^{\underline{g}}\ ∢. \tag{29}$$

Scattering by ∢ θ is thus accompanied by a momentum transfer of size $\Delta k = 2k\sin(\theta/2)$ to the scattered particle. Δk ranges from zero at $\theta = 0$ ("forward" scattering, i.e. no scattering at all) to $\Delta k = 2k$ at $\theta = 180°$ (i.e. back-scattered particles). This is in accord with classical notions.

The matrix element in Eq. (28) is the 3D Fourier transform $\tilde{V}(\mathbb{q})$ of the scattering potential $V(\mathbb{r})$ w.r.t. the momentum transfer $\mathbb{q} = \mathbb{k}_b - \mathbb{k}_a$, i.e.

$$\to \langle \phi_a | V | \phi_b \rangle = \int_\infty V(\mathbb{r})\, e^{i\mathbb{q}\cdot\mathbb{r}}\, d^3x = \tilde{V}(\mathbb{q}) \quad \int \mathbb{q} = \mathbb{k}_b - \mathbb{k}_a,\ \text{and:}\quad q = |\mathbb{k}_b - \mathbb{k}_a| = 2k\sin\frac{\theta}{2};$$

and//

$$\frac{d\sigma}{d\Omega} = (m/2\pi\hbar^2)^2\, |\tilde{V}(\mathbb{q})|^2. \tag{30}$$

For spherically symmetric potentials $V(\mathbb{r}) = V(r)$, $\tilde{V}(\mathbb{q})$ can be simplified by doing the ∢ integration. Pick a cd. system with $\mathbb{q}$ along the z-axis. Then:

$$\tilde{V}(q) = \int_\infty V(r)\, e^{iqr\cos\vartheta} \cdot 2\pi r^2 dr \sin\vartheta\, d\vartheta \quad \ldots \text{let } \mu = \cos\vartheta$$

$$= 2\pi \int_0^\infty V(r)\, r^2 dr \int_{-1}^{+1} e^{iqr\mu}\, d\mu = \underline{\underline{\frac{4\pi}{q}\int_0^\infty r\, V(r) \sin qr\, dr}},$$

and//

$$\boxed{\frac{d\sigma}{d\Omega} = (m/2\pi\hbar^2)^2\, |\tilde{V}(q)|^2} \quad q = 2k\sin\frac{\theta}{2},\ \theta = \text{scatt}^{\underline{g}}\ ∢. \tag{31}$$

This the Born Approxn to the differential scattering cross-section for spherically symmetric potentials. It is a useful form for many problems.