DEPARTMENT OF PHYSICS

2005 COMPREHENSIVE EXAM

Answer each of the following questions, all of which carry equal weight. Begin your answer to each question on a new sheet of paper; solutions to different questions must not appear on the same sheet. Each sheet of paper must be labeled with your name and the problem number in the upper right hand corner. When more than one sheet is submitted for a problem, be sure the pages are ordered properly.

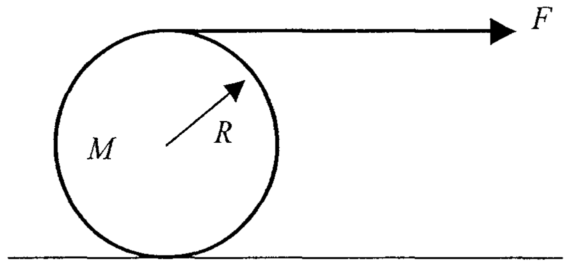
PHYSICAL CONSTANTS

Quantity	Symbol	Value
acceleration due to gravity	g	9.8 m s^{-2}
gravitational constant	G	$6.67 \times 10^{-11} \text{ N m}^2 \text{ kg}^{-2}$
permittivity of vacuum	e_0	$8.85 \times 10^{-12} \text{ C}^2 \text{ N}^{-1} \text{ m}^{-2}$
permeability of vacuum	μ_0	$4\pi \times 10^{-7} \text{ N A}^{-2}$
speed of light in vacuum	c	$3.00 \times 10^{8} \text{ m s}^{-1}$
elementary charge	e	$1.602 \times 10^{-19} \text{ C}$
mass of electron	m_e	$9.11 \times 10^{-31} \text{ kg}$
mass of proton	m_p	$1.673 \times 10^{-27} \text{ kg}$
Planck constant	h	$6.63 \times 10^{-34} \text{ J s}$
Avogadro constant	N_A	$6.02 \times 10^{23} \text{ mol}^{-1}$
Boltzmann constant	k	$1.38 \times 10^{-23} \text{ J K}^{-1}$
molar gas constant	R	$8.31 \text{ J mol}^{-1} \text{ K}^{-1}$
standard atmospheric pressure		$1.013 \times 10^{5} \text{ Pa}$

Problem #1

A roll of toilet paper of mass M and radius R is pulled across the floor without slipping by a force F applied to the paper coming off the roll. The floor has a coefficient of static friction μ.

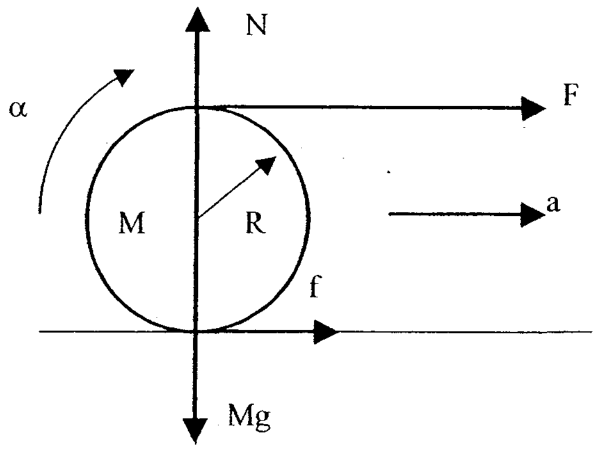
a) Draw a vector diagram where all the forces acting on the roll are shown (a free body diagram). Label each of the forces. Also mark the direction of linear acceleration "a" and angular acceleration α.



b) Write down Newton's second law (equations of motion) for the linear and angular accelerations. Find the maximum acceleration possible without slipping. Assume that the roll has the moment of inertia of a solid disk with mass M and radius R.

Solution

a) Free body diagram. Forces are the normal force N, the force of gravity Mg, the pulling force F and the static friction for f. Unless we are really smart we probably don't know the direction of the frictional force f yet so we just guess it is in the same direction as F. Later we will be able to tell if we are correct when we see if it comes out positive in our solution. Also shown is the linear acceleration direction and the angular acceleration direction.



b) To find the acceleration "a" we use Newton's second law for both the linear motion and the angular motion.

For the linear motion, we have in the horizontal direction $\sum F = F + f = Ma$

For the angular motion, we have $\sum \tau = FR - fR = I\alpha = \frac{1}{2}MR^2\left(\frac{a}{R}\right)$

Combining the two equations, we solve for the acceleration "a" in terms of the friction force f. $a = \frac{4f}{M}$ This equation also shows us that the frictional force is indeed in the direction we assumed.

To find the friction force, we note that for maximum acceleration, the frictional force is a maximum that is given by $f = \mu N = \mu Mg$ since there is no acceleration in the vertical direction.

Thus we obtain for the maximum acceleration $a = \frac{4f}{M} = \frac{4\mu Mg}{M}$

Therefore $\boxed{a = 4\mu g}$

Problem #2

Consider the magnetic dipole field generated by a magnetic dipole $\vec{m}$ (see figure 2a).

$$\vec{B} = \frac{1}{r^5}\left[3\left(\vec{m} \bullet \vec{r}\right)\vec{r} - r^2\vec{m}\right], \quad for \;\; r > 0.$$

a) Show that $\vec{B} = -\vec{\nabla}\Psi$, with $\Psi = \left(\vec{m} \bullet \vec{r}\right)/r^3$.

b) Let $\vec{m}$ be generated by a current loop with current I (as shown in figure 2b). With the dipole approximation ($r >>$ loop size), show that the potential of part (a) is;

$$\Psi = -\frac{I}{c}\Omega$$

where Ω is the solid angle subtended by the loop at the field point P, and hence

$$\vec{B} = \frac{I}{c}\,\vec{\nabla}\Omega$$

[HINT: You may wish to use the fact that the solid angle at the field point P can be given by $\Omega = \int \frac{1}{r^3}\vec{r} \bullet \overrightarrow{da}$, with the integral done over the loop surface area a.]

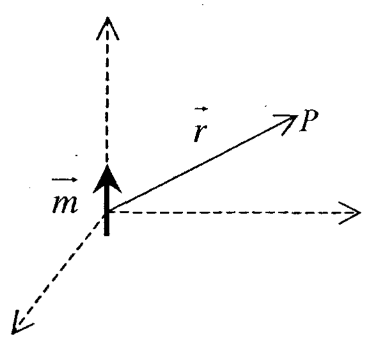


Figure 2a

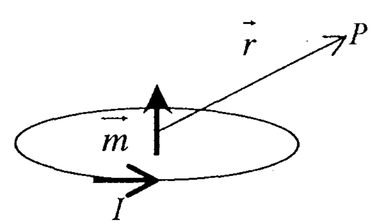


Figure 2b

Key.

$\Psi = \bar{m}\cdot\bar{r} / r^3$ given. ①

$\bar{B} = \frac{1}{r^5}\left[3(\bar{m}\cdot\bar{r})\bar{r} - r^2 \bar{m}\right]$ ② given

(a) $-\bar{\nabla}\Psi = -\bar{\nabla}\left[(\bar{m}\cdot\bar{r})/r^2\right]$

$= -\left[(\bar{m}\cdot\bar{\nabla})\frac{\bar{r}}{r^3} + \bar{m}\times\underbrace{\left(\bar{\nabla}\times\frac{\bar{r}}{r^3}\right)}\right.$

$\underbrace{\frac{\bar{r}}{r^3} = \bar{\nabla}\left(-\frac{1}{r}\right)}$

so $-\bar{m}\times\left(\bar{\nabla}\times\bar{\nabla}\left(\frac{1}{r}\right)\right) = 0$

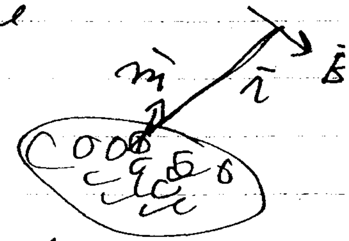
So $\bar{B} = -\bar{\nabla}\Psi = -(\bar{m}\cdot\bar{\nabla})\frac{\bar{r}}{r^3}$

$= -\left[m_x\frac{\partial}{\partial x} + m_y\frac{\partial}{\partial y} + m_z\frac{\partial}{\partial z}\right]\frac{x\hat{i} + y\hat{j} + z\hat{k}}{(x^2+y^2+z^2)^{3/2}}$

$= -\frac{m_x\hat{i} + m_y\hat{j} + m_z\hat{z}}{(x^2+y^2+z^2)^{3/2}} + \frac{3}{2}\left[2xm_x\frac{\bar{r}}{r^5} + 2ym_y\frac{\bar{r}}{r^5} + 2zm_z\frac{\bar{r}}{r^5}\right]$

$= -\frac{\bar{m}}{r^3} + 3(\bar{m}\cdot\bar{r})\frac{\bar{r}}{r^5} = \bar{B}$ given // QED.

(b). As in the figure shown, imagine there are many tiny loops with area $d\bar{a}$ within the big loop. $d\bar{a}$ has $d\bar{m} = \frac{I}{c}\,da\,\hat{n}$ ⊛ ($\hat{n}$ normal to the loop) since $\bar{B}$ to $d\bar{a}$ is directed inward $-\bar{r}$.



Then $\Psi = -\int\frac{(d\bar{m})\cdot\bar{r}}{-r^3} = -\int\hat{n}\left(\frac{1}{c}I\,da\right)\cdot\frac{\bar{R}}{R}$

$= -\frac{I}{c}\int\frac{\bar{r}}{r^3}\cdot d\bar{a}$ But $\Omega = \int\frac{\bar{r}}{r^3}\cdot d\bar{a}$ given.

So $\boxed{\Psi = -\frac{I}{c}\Omega}$ // QED

But $\bar{B} = -\bar{\nabla}\Psi$. So $\bar{B} = \frac{I}{c}\bar{\nabla}\Omega$.

⊛ note: Jackson 2nd ed. has $\frac{1}{c}$. Jackson 3rd ed. has not $\frac{1}{c}$. Difference of definition.

Problem #3

The energy levels E_n of a one dimensional harmonic oscillator are described by

$$E_n = \left(n + \frac{1}{2}\right) \hbar\omega$$

where ω is the oscillator frequency and n is the quantum number.

a) Find the energy levels of the Schrodinger equation for a particle which moves in the following potential;

$$V(x) = \begin{Bmatrix} \frac{m}{2}\omega^2 x^2 & & for \;\; x > 0 \\ \frac{m}{2}(1+\alpha)\omega^2 x^2 & & for \;\; x \leq 0 \end{Bmatrix}$$

for the case where $\alpha = 0$, and $\alpha = \infty$.

b) Make a qualitative description of the energy levels for the region $0 < \alpha < \infty$.

c) Qualitatively describe what happens for the case $\alpha = -1$.

#3

Solution. Quantum mechanics.

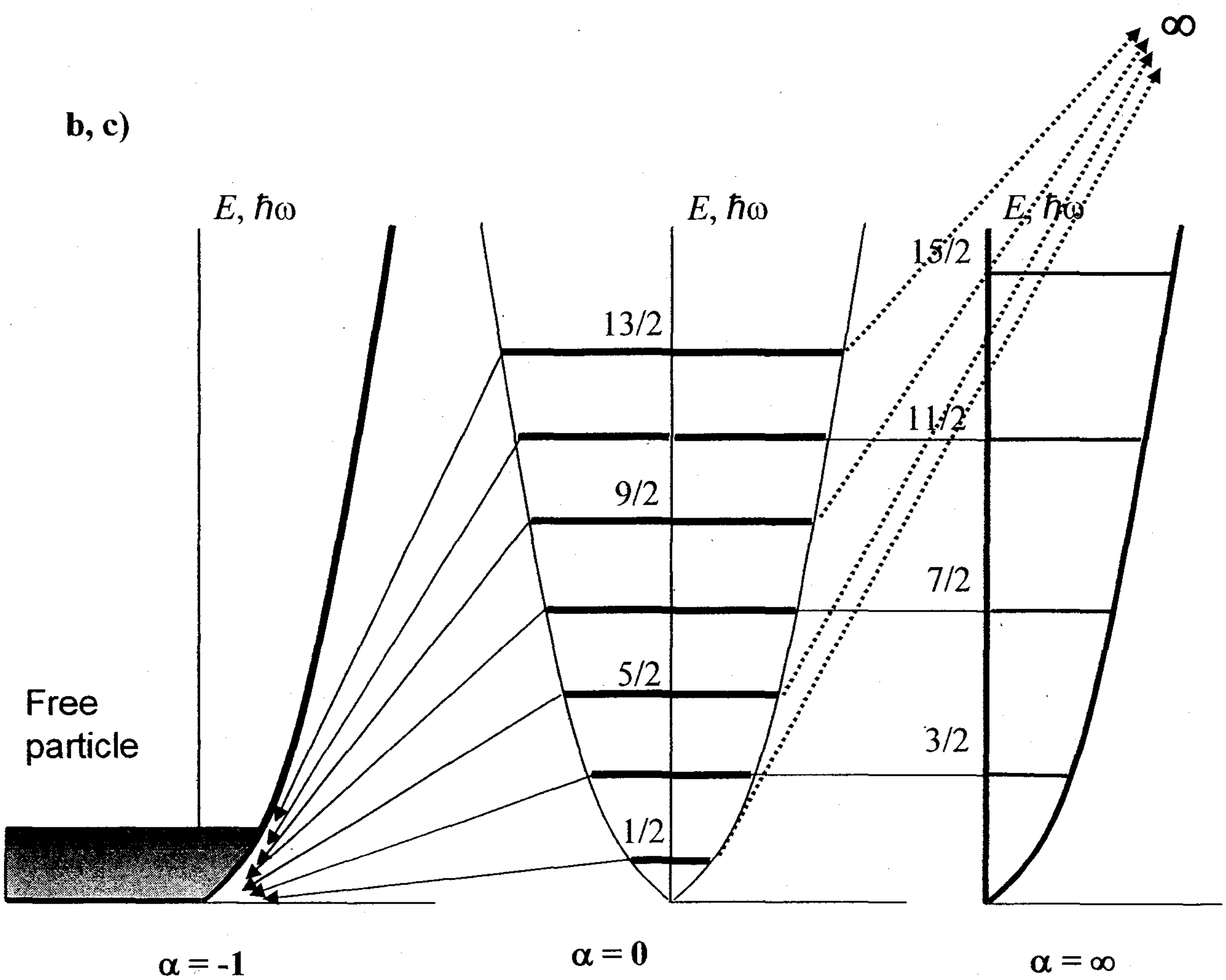
a) In the area of $x > 0$ the wave functions of the particle are described by the same Schrödinger equation as the harmonic oscillator. However, the acceptable wave functions must be equal to zero at $x = 0$. Therefore, only the odd (antisymmetric) wave functions describe the particle movement. The energy levels, which correspond to the odd wave functions of the harmonic oscillator, have the quantum numbers

$$n = 1, 3, 5, \ldots, 2N+1, \ldots \qquad \boxed{E_n = (n + \frac{1}{2})\hbar\omega}$$

It means that the energy levels of the Schrödinger equation for the particle are

$$\boxed{E_N = (2N + \frac{3}{2})\hbar\omega, \qquad N = 0, 1, 2, \ldots}$$

b, c)



Problem #4

A simple harmonic oscillator has energy eigenstates $|n\rangle$, $n = 0, 1, \ldots$, with energies $E_n = \hbar\omega(n + \frac{1}{2})$, where ω is the classical oscillator frequency. A newly discovered property of such a system is its *banality*, characterized by the Hermitian operator, $\hat{B}$. The *only* non-vanishing elements of this operator are

$$\langle 0|\hat{B}|0\rangle = 2 \quad , \quad \langle 0|\hat{B}|2\rangle = i\sqrt{3}$$

and of course, $\langle 2|\hat{B}|0\rangle$. At time $t = 0$ the system's banality was determined to be its minimum.

(a) What are the probabilities of finding this minimum-banality particle in each energy state?

(b) Find an expression for the expected banality, $\langle \hat{B} \rangle$, for all subsequent times.

#4

a. Measuring the banality places the system into one of the *eigenstates* of $\hat{B}$. Since $\hat{B}|n\rangle = 0$ when $n \neq 0, 2$, these are all eigenstates with $B = 0$. The other two eigenstaes must be linear combinations of $\{|0\rangle, |2\rangle\}$. In this basis the banality operator is the 2×2 matrix

$$\hat{B} = \begin{bmatrix} \langle 0|\hat{B}|0\rangle & \langle 0|\hat{B}|2\rangle \\ \langle 2|\hat{B}|0\rangle & \langle 2|\hat{B}|2\rangle \end{bmatrix} = \begin{bmatrix} 2 & i\sqrt{3} \\ -i\sqrt{3} & 0 \end{bmatrix} ,$$

after recognizing that $\langle 2|\hat{B}|0\rangle = \langle 0|\hat{B}|2\rangle^* = -i\sqrt{3}$ since $\hat{B}$ is Hermetian. The eigenvalues and normalized eigenvectors of this matrix are

$$B_+ = 3 \ , \ |+\rangle = \tfrac{1}{2}\left[i\sqrt{3}|0\rangle + |2\rangle\right]$$

$$B_- = -1 \ , \ |-\rangle = \tfrac{1}{2}\left[i|0\rangle - \sqrt{3}|2\rangle\right]$$

The minimum eigenvalue possible is therefore $B = -1$, so this is the initial state of the system:

$$|\psi(0)\rangle = |-\rangle = \tfrac{1}{2}[i|0\rangle - \sqrt{3}|2\rangle] \qquad (1)$$

Since this consists of only the $|0\rangle$ and $|2\rangle$ energy states, $E_0 = \hbar\omega/2$ and $E_2 = 5\hbar\omega/2$ are the only possible measurements. Their probabilities are

$$P_0 = |\langle 0|-\rangle|^2 = \frac{1}{4} \ , \ P_2 = |\langle 2|-\rangle|^2 = \frac{3}{4}$$

b. The time-dependent ket is found by mutliplying each energy eigenstate in the initial state, eq. (1), by the time-dependent phase factor $e^{-iE_nt/\hbar}$:

$$|\psi(t)\rangle = \tfrac{1}{2}\left[i|0\rangle e^{-i\omega t/2} - \sqrt{3}|2\rangle e^{-5i\omega t/2}\right] .$$

The corresponding bra is simply

$$\langle\psi(t)| = \tfrac{1}{2}\left[-i\langle 0|e^{i\omega t/2} - \sqrt{3}\langle 2|e^{5i\omega t/2}\right]$$

The expected banality is found from the expectation

$$\langle\hat{B}\rangle = \langle\psi|\hat{B}|\psi\rangle = \frac{1}{4}[\langle 0|\hat{B}|0\rangle + i\sqrt{3}\langle 0|\hat{B}|2\rangle e^{-2i\omega t} - i\sqrt{3}\langle 2|\hat{B}|0\rangle e^{2i\omega t} + 3\langle 2|\hat{B}|2\rangle]$$

Using the matrix elements, including $\langle 2|\hat{B}|0\rangle = -i\sqrt{3}$ and $\langle 2|\hat{B}|2\rangle = 0$ gives

$$\langle\hat{B}\rangle = \frac{1}{4}[2 - 3e^{-2i\omega t} - 3e^{2i\omega t}] = \tfrac{1}{2} - \tfrac{3}{2}\cos(2\omega t)$$

Problem #5

Using contour integration techniques, evaluate:

$$I(k,a) = \int_{-\infty}^{+\infty} \frac{\cos(kx)}{x^2 + a^2}\, dx$$

along the real axis, where k and a are real and positive. You must explicitly specify the contour(s) you used.

$I(k,a) = \int_0^\infty \frac{\cos kx}{x^2+a^2}\,dx.$ k and a real and >0.

write $I = \frac{1}{2}\int_{-\infty}^{+\infty} \frac{\cos kx}{x^2+a^2}\,dx = \frac{1}{2}\,\mathrm{Re}\int_{-\infty}^{+\infty} \frac{e^{ikx}}{x^2+a^2}\,dx.$ (1)

(note: integrand is even.)

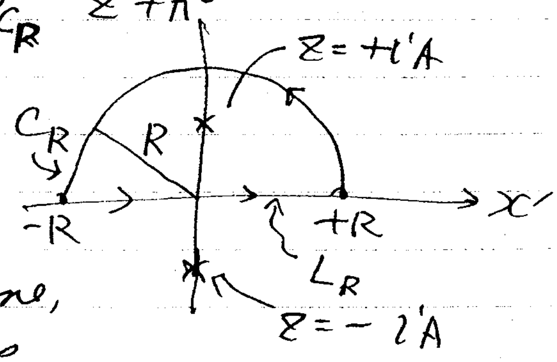
Let $\xi = kx$, and call the integration variable x'.

(1) → $I = \frac{1}{2k}\,\mathrm{Re}\int_{-\infty}^{+\infty} \frac{e^{i\xi}d\xi}{(\xi/k)^2+a^2} = \frac{k}{2}\,\mathrm{Re}\int_{-\infty}^{+\infty} \frac{e^{ix'}dx'}{x'^2+A^2}$

where $A \equiv ka > 0$.

Now, consider the contour integral

$$\oint_{L_R+C_R} e^{iz}dz/(z^2+A^2) = \int_{-R}^{+R} \frac{e^{ix'}dx'}{x'^2+A^2} + \int_{C_R} \frac{e^{iz}dz}{z^2+A^2}$$



C_R is large semi-circle in the upper half plane. The integral has simple poles at $z = \pm iA$, and we choose to close the contour in the upper half plane, so that $e^{iz} = e^{ix}e^{-y}$ vanishes as $y \to +\infty$. In fact, as $R \to \infty$, the $\int_{C_R}$ vanishes altogether (at least as fast as $1/R^2$). So,

$$\int_{-\infty}^{+\infty} \frac{e^{ix}dx}{x^2+A^2} = \oint_{L_\infty+R_\infty} e^{iz}dz/(z^2+A^2)$$

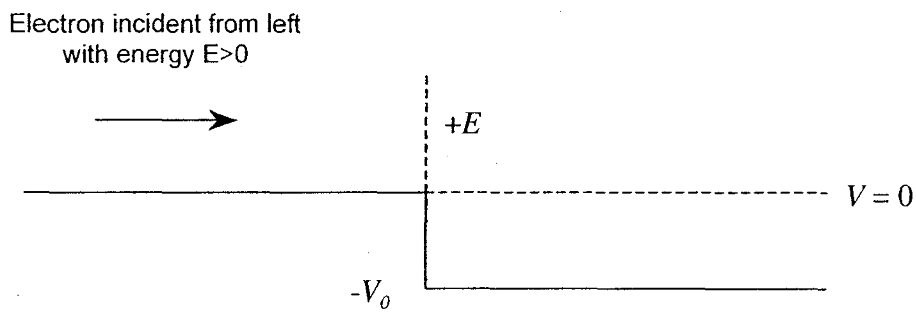
$$= 2\pi i\,\mathrm{Res}\left\{\frac{e^{iz}}{z^2+A^2}\ +iA\right\} = \pi\,\frac{e^{-A}}{A}.$$

So $I(k,a) = \int_0^\infty \frac{\cos kx}{x^2+a^2}\,dx = \left(\frac{\pi}{ka}\right)e^{-ka}$ // ans.

$\left.\begin{matrix}a\\k\end{matrix}\right\} > 0$

Problem #6

An electron with positive energy E is normally incident on a metal and we want to determine the probability that this electron will make it into the metal. We will approximate this real problem as a particle scattering from an infinitely thick barrier (a potential step). The potential (the metal) is a step barrier where the vacuum side on the left (where the electron is coming from) the potential is zero, $V = 0$, and inside the metal on the right the potential is negative, $V = -V_0$. The step in the barrier occurs at $x = 0$. A diagram for this problem is shown below. Determine the REFLECTION co-efficient for this barrier.



This is a scattering problem for a step barrier worked out in any undergraduate quantum mechanics textbook.

Break the problem into two regions. REGION I is x < 0. REGION II is x > 0. In REGION I there is an incident wavefunction from left to right and a reflected wavefunction from right to left. In REGION II there is only a transmitted wavefunction from left to right. We will match two boundary conditions at x = 0 for the two solutions to determine the relative constants in the wavefunctions.

The solutions have the form of traveling waves

$$REGION\ I \qquad \psi_I = Ae^{ikx} + Be^{-ikx} \qquad where\ \ k = \frac{\sqrt{2mE}}{h}$$

$$REGION\ II \qquad \psi_{II} = Ce^{ik'x} + De^{-ik'x} \qquad where\ \ k' = \frac{\sqrt{2m(E+V_0)}}{h}$$

Since REGION II has no left traveling wave, $D = 0$. We now use boundary conditions to determine the other constants in the wavefunctions. We have two B.C.s and three unknowns so we can only solve for the ratio of the constants (or set the incident flux to 1 by setting $A = 1$).

B.C.s: The wavefunction and its derivative must be continuous at x = 0.

$$BC\#1 \qquad \psi_I(x=0^-) = \psi_{II}(x=0^+) \qquad or \qquad A + B = C$$

$$BC\#2 \qquad \psi'_I(x=0^-) = \psi'_{II}(x=0^+) \qquad or \qquad k(A-B) = k'C$$

From these two equations we find that

$$B = \frac{k-k'}{k+k'}A \quad and \quad C = \frac{2k}{k+k'}A$$

Now, the reflection co-efficient, R, is given by

$$R = \frac{\#\ particles\ \ reflected}{\#\ particles\ \ incident} = \frac{BB^*}{AA^*} = \left|\frac{B}{A}\right|^2$$

Substituting in A and B and then for k and k' and simplifying gives

$$R = \left(\frac{k-k'}{k+k'}\right)^2 = \left(\frac{\sqrt{E}-\sqrt{E+V_0}}{\sqrt{E}+\sqrt{E+V_0}}\right)^2 = \left(\frac{1-\sqrt{1+\frac{V_0}{E}}}{1+\sqrt{1+\frac{V_0}{E}}}\right)^2$$

As a check, note that as V_0 --> 0, R --> 0 and as V_0 --> -infinity, R --> 1.

Problem #7

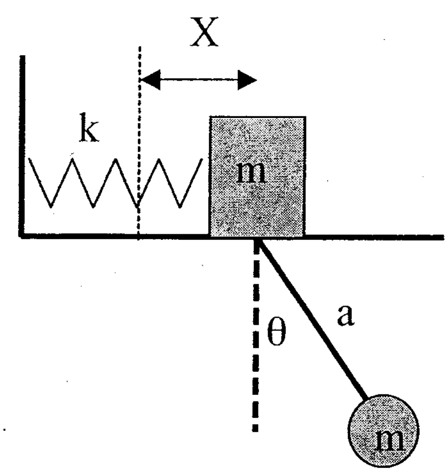
A simple pendulum is made from a massless rod of length R with a ball of mass m on the end. The pendulum is hung from a block, also of mass m, that can slide on the frictionless surface. The block is connected by a spring of spring constant k that is attached to the wall. Let X be the generalized coordinate measured from the relaxed spring position.

a. Find the Lagrangian L for the system (set the gravitational potential to zero at the equilibrium point).

b. Find the Lagrange equation for the generalized coordinate θ.

c. Find the Lagrange equation for the generalized coordinate X.

d. Now use the small angle approximation and drop higher order terms. Then use these two coupled equations from b and c to find the frequencies of oscillation for the two normal modes.



#7

a) To find the Lagrangian L, we start by writing the kinetic energy of the block mass.

This is easy since it only depends on X. $T_b = \frac{1}{2}m\dot{X}^2$

To find the kinetic energy of the pendulum mass, we write the x and y components of the position. $x_p = X + R\sin\theta \quad y_p = -R\cos\theta$

Taking the time derivatives we get: $\dot{x}_p = \dot{X} + R\cos\theta\dot{\theta} \quad \dot{y}_p = R\sin\theta\dot{\theta}$

Thus the kinetic energy of the pendulum mass is:

$$T_b = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) = \frac{1}{2}m\dot{X}^2 + \frac{1}{2}mR^2\dot{\theta}^2 + mR\dot{X}\dot{\theta}\cos\theta$$

The total kinetic energy is:

$$T = m\dot{X}^2 + \frac{1}{2}mR^2\dot{\theta}^2 + mR\dot{X}\dot{\theta}\cos\theta$$

Now we find the potential energy. We need to include both the effect of gravity and the spring.

For gravity we have: $V_g = mg(R + y_p) = mgR(1-\cos\theta)$

Now for the spring we have: $V_s = \frac{1}{2}kX^2$

Thus the Lagrangian becomes:

$$\boxed{L = m\dot{X}^2 + \frac{1}{2}mR^2\dot{\theta}^2 + mR\dot{X}\dot{\theta}\cos\theta - mgR(1-\cos\theta) - \frac{1}{2}kX^2}$$

b) Now we find Lagrange's equation for θ.

$$\frac{\partial L}{\partial\theta} - \frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right) = 0$$

$$\frac{\partial L}{\partial\theta} = -mR\dot{X}\dot{\theta}\sin\theta - mgR\sin\theta$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right) = mR^2\ddot{\theta} + mR\ddot{X}\cos\theta - mR\dot{X}\dot{\theta}\sin\theta$$

$$\boxed{mgR\sin\theta + mR^2\ddot{\theta} + mR\ddot{X}\cos\theta = 0}$$

c) Now we find Lagrange's equation for X.

$$\frac{\partial L}{\partial X} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{X}}\right) = 0$$

$$\frac{\partial L}{\partial X} = -kX$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{X}}\right) = 2m\ddot{X} + mR\ddot{\theta}\cos\theta - mR\dot{\theta}^2\sin\theta$$

$$\boxed{-kX - 2m\ddot{X} - mR\ddot{\theta}\cos\theta + mR\dot{\theta}^2\sin\theta = 0}$$

d) Now we combine the coupled equations to find the frequencies of oscillation. First we make the small angle approximation and drop the higher order terms.

$$g\theta + R\ddot{\theta} + \ddot{X} = 0$$

$$-kX - 2m\ddot{X} - mR\ddot{\theta} = 0$$

Now assume a solution of the form $X = X_o\cos(\omega t) \qquad \theta = \theta_o\cos(\omega t)$

Plugging this solution into our equations, we get

$$\left(g - \omega^2 R\right)\theta_o - \omega^2 X_o = 0$$

$$m\omega^2 R\theta_o + \left(-k + 2m\omega^2\right)X_o = 0$$

To solve, set the determinant=0

$$\begin{vmatrix} \left(g-\omega^2 R\right) & -\omega^2 \\ m\omega^2 R & \left(-k+2m\omega^2\right) \end{vmatrix} = 0$$

After some algebra we get $\quad mR\omega^4 - \left(2mg + Rk\right)\omega^2 + gk = 0$

Solving the quadratic we get $\quad \omega^2 = \dfrac{\left(2mg + Rk\right) \pm \sqrt{4m^2g^2 + R^2k^2}}{2mR}$

Thus our two (positive) frequencies for the normal modes are

$$\boxed{\omega_1 = \sqrt{\frac{\left(2mg+Rk\right)+\sqrt{4m^2g^2+R^2k^2}}{2mR}}} \qquad \boxed{\omega_2 = \sqrt{\frac{\left(2mg+Rk\right)-\sqrt{4m^2g^2+R^2k^2}}{2mR}}}$$

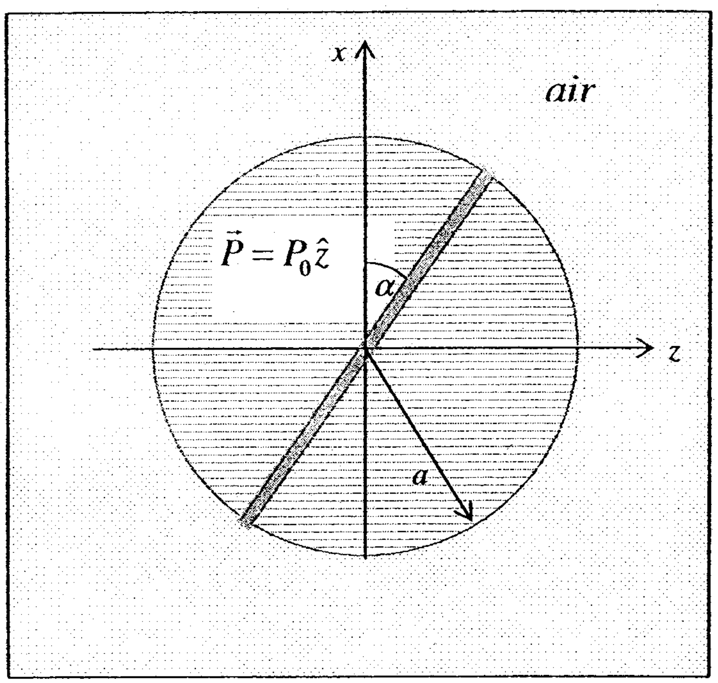
Problem # 8

A sphere of radius a is made of electret material, i.e. a dielectric with a permanent electrical polarization. Let the polarization P_0 be constant and aligned in the z-direction (see figure).

a) Find the electrostatic field inside and outside the sphere.

b) A thin air gap splits the sphere into two hemispheres, such that the dividing plane passes through the y-axis. Determine the force with which the hemispheres are acting on each other if the gap is cut perpendicular to the z-axis ($\alpha = 0$).

c) For an arbitrary angle α, find the force acting parallel to the gap surface. Outer surfaces of the hemisphere may be ignored.



Maxwell stress tensor:

$$T_{\alpha\beta} = \varepsilon_0[(E_\alpha E_\beta + c^2 B_\alpha B_\beta) - \frac{1}{2}(\vec{E}\cdot\vec{E} + c^2\vec{B}\cdot\vec{B})\delta_{\alpha\beta}]$$

Electret (formed of *elektr-* from "electricity" and *-et* from "magnet") is material that has been permanently electrically charged (polarised). The magnetic equivalent is a permanent magnet. Oliver Heaviside coined this term. Polarised material consists of atoms or molecules with electric dipole moment. It can be produced by cooling down material composed of long molecule chains with electric dipole moment so that so-called domains are formed. Electret materials have recently found commercial and technical interest. For example, they are used in one form of microphone. *(From Wikipedia online)*

(a) Need to find el. static potential that satisfies $\nabla^2 \Phi(\vec{r}) = 0$ and boundary conditions.

In spherical coordinates:

$$\Phi(r,\theta) = \sum_{l=0}^{\infty} \left[A_l r^l + B_l r^{-(l+1)} \right] P_l(\cos\theta)$$

Inside : $\vec{P} = P_0 \hat{z} = P_0 \cos\theta \cdot \hat{e}_r - P_0 \sin\theta \, \hat{e}_\theta$) $\to$ only $l=1$ term is present

$$\Phi_{in}(r,\theta) = A_1 r \cos\theta \quad ; \quad \Phi_{out} = B_1 \frac{1}{r^2} \cos\theta$$

Boundary conditions:

(I) Tangential E is continuous ($E_\parallel$) $\quad \left(\frac{\partial \Phi_{in}}{\partial \theta}\right)_{r=a} = \left(\frac{\partial \Phi_{out}}{\partial \theta}\right)_{r=a}$

(II) Normal E is discontinuous by the amount of polarization surface charge $\sigma_s = \vec{P}\cdot\vec{n} = P_0 \cos\theta$

$$E_\perp^{out} - E_\perp^{in} = \frac{1}{\varepsilon_0} P_0 \cos\theta$$

$$-\left(\frac{\partial \Phi^{out}}{\partial r}\right)_{r=a} + \left(\frac{\partial \Phi^{in}}{\partial r}\right)_{r=a} = \frac{1}{\varepsilon_0} P_0 \cos\theta$$

(I) $-A_1 a \sin\theta = -B_1 \frac{1}{a^2} \sin\theta$

(II) $2 \frac{1}{a^3} B_1 \cos\theta + A_1 \cos\theta = \frac{P_0}{\varepsilon_0} \cos\theta$

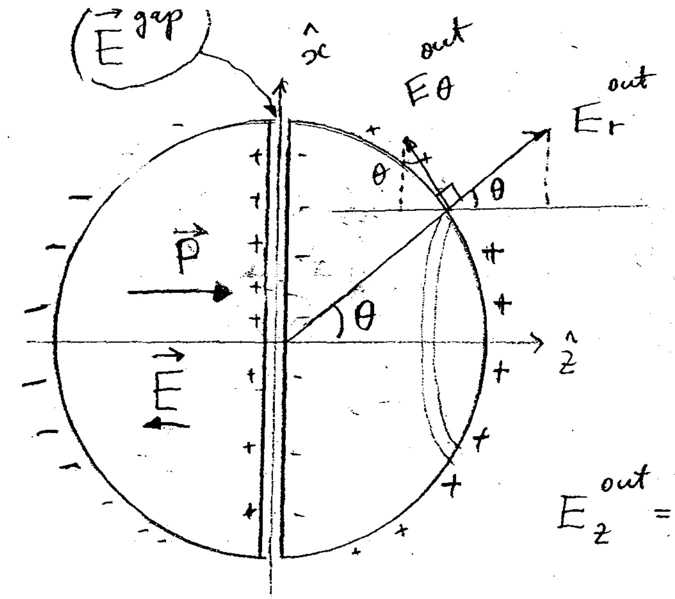
$$A_1 = \frac{1}{3} \frac{1}{\varepsilon_0} P_0 \qquad B_1 = \frac{a^3}{3} \frac{1}{\varepsilon_0} P_0$$

$$\Phi^{in}(r,\theta) = \frac{1}{3} \frac{P_0}{\varepsilon_0} r \cos\theta ; \quad \Phi^{out}(r,\theta) = \frac{1}{3} \frac{P_0}{\varepsilon_0} \frac{a^3}{r^2} \cos\theta$$

$$E_r^{in} = -\frac{1}{3} \frac{P_0}{\varepsilon_0} \cos\theta \quad ; \quad E_r^{out} = \frac{2}{3} \frac{P_0}{\varepsilon_0} \left(\frac{a}{r}\right)^3 \cos\theta$$

$$E_\theta^{in} = \frac{1}{3} \frac{P_0}{\varepsilon_0} \sin\theta ; \quad E_\theta^{out} = \frac{1}{3} \frac{P_0}{\varepsilon_0} \left(\frac{a}{r}\right)^3 \sin\theta$$

$$\boxed{\vec{E}^{in} = -\frac{1}{3} \frac{P_0}{\varepsilon_0} \cdot \hat{z}}$$



$$\vec{E}^{in} = -\frac{1}{3}\frac{P_0}{\varepsilon_0}\hat{z}$$

$$E_r^{out} = \frac{2}{3}\frac{P_0}{\varepsilon_0}\left(\frac{a}{r}\right)^3\cos\theta$$

$$E_\theta^{out} = \frac{1}{3}\frac{P_0}{\varepsilon_0}\left(\frac{a}{r}\right)^3\sin\theta$$

$$E_z^{out} = E_r^{out}\cdot\cos\theta - E_\theta^{out}\cdot\sin\theta.$$

(b) If the electric field vector $\vec{E}$ immediately outside of the surface is known, then the force (acting on one hemisphere) can be calculated from Maxwell stress tensor. E-field inside gap between the hemispheres is given by ($\alpha = 0$)

$$(E^{gap})_z - (E^{in})_z = \frac{P_0}{\varepsilon_0} \Rightarrow (E^{gap})_z = -\frac{1}{3}\frac{P_0}{\varepsilon_0} + \frac{P_0}{\varepsilon} = \frac{2}{3}\frac{P_0}{\varepsilon_0}$$

In the gap $\quad T_{zz}^{in} = \varepsilon_0\left[E_zE_z - \frac{1}{2}\vec{E}^2\right] = \frac{1}{2}\varepsilon_0\left(\frac{2}{3}\frac{P_0}{\varepsilon_0}\right)^2$

Force acting on the right hemisphere in the gap

$$\vec{F}_1 = \int_S (T_{zz}^{in}\cdot\hat{n}_z)\,da = -\frac{2}{9}\frac{P_0^2}{\varepsilon_0}\cdot\pi a^2$$

On the outer surface, forces in x and y direction cancel to 0.
Force in $\hat{z}$-direction

$$\vec{F}_2 = \hat{z}\cdot F_2 = \int_S (T_{zz}^{out}\cdot\hat{n}_z)\,da = 2\pi a\int_0^{\pi/2} d\theta\,\sin\theta\, T_{zz}^{out}$$

$da = d\theta\cdot 2\pi a^2\sin\theta$

$$T_{zz}^{out} = \varepsilon_0\left[E_z^2 - \frac{1}{2}E^2\right] = \varepsilon_0\left[\left(\frac{2}{3}\frac{P_0}{\varepsilon_0}\cos^2\theta - \frac{1}{3}\frac{P_0}{\varepsilon_0}\sin^2\theta\right)^2 - \frac{1}{2}\left(\frac{4}{9}\left(\frac{P_0}{\varepsilon_0}\right)^2\cos^2\theta + \frac{1}{9}\left(\frac{P_0}{\varepsilon_0}\right)^2\sin^2\theta\right)\right] = \frac{P_0^2}{\varepsilon}\left[(\cos^2\theta - 1)^2 - \frac{1}{2}\left(\frac{1}{3}\cos^2\theta + 1\right)\right] = \frac{P_0^2}{\varepsilon_0}\left[\cos^4\theta - 2\cos^2\theta + 1 - \frac{1}{6}\cos^2\theta - \frac{1}{2}\right]$$

$$\frac{18-65+45}{90} = \frac{18-20}{90} = -\frac{2}{90} = -\frac{1}{45}$$

#8-3

$$\int_0^{\pi/2} \left[\cos^4\theta - \frac{13}{6}\cos^2\theta + \frac{1}{2} \right] \sin\theta \, d\theta$$

$$= -\int_0^{\pi/2} \left[\cos^4\theta - \frac{13}{6}\cos^2\theta + \frac{1}{2} \right] d(\cos\theta)$$

$d(\cos\theta) = dx$

$\cos\theta = x$

$0 \Rightarrow x = 1$

$\pi/2 \Rightarrow x = 0$

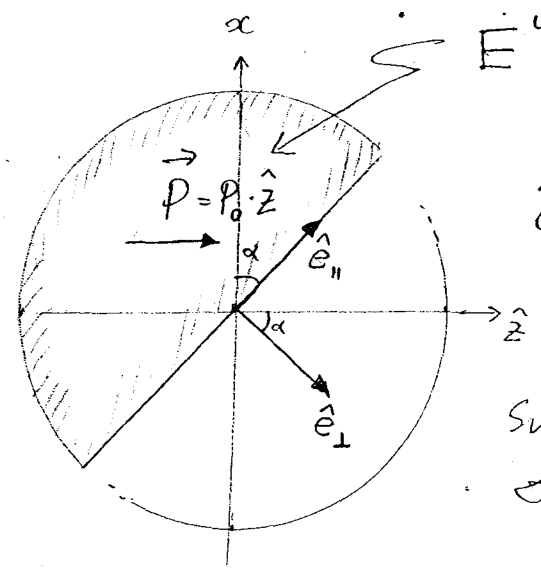
$$= \int_0^1 \left(x^4 - \frac{13}{6}x^2 + \frac{1}{2} \right) dx = \left[\frac{x^5}{5} - \frac{13}{6\cdot 3}x^3 + \frac{x}{2} \right]_0^1 = \frac{1}{5} - \frac{13}{18} + \frac{1}{2} = -\frac{1}{45}$$

$$F_2 = 2\pi a^2 \cdot \frac{P_0^2}{\varepsilon_0}\left(-\frac{1}{45}\right)$$

Total force acting on the right hemisphere is:

$$\vec{F} = -\frac{2\pi a^2 P_0^2}{\varepsilon_0}\left(\frac{1}{9} + \frac{1}{45}\right)\hat{z} = \underline{\underline{-\frac{\pi a^2 P_0^2}{\varepsilon_0} \cdot \frac{4}{15}\,\hat{z}}}$$

(c)



$\hat{e}_\parallel = \cos\alpha\,\hat{x} + \sin\alpha\,\hat{z}$
$\hat{e}_\perp = -\sin\alpha\,\hat{x} + \cos\alpha\,\hat{z}$

Surface polarization charge
$\sigma_p(\alpha) = \vec{P}\cdot\hat{e}_\perp = P_0 \cos\alpha$

Boundary conditions:

(i) $\parallel$- component of E is continuous $\quad \vec{E}^{(in)}\cdot\hat{e}_\parallel = \vec{E}^{(gap)}\cdot\hat{e}_\parallel$

$$E_\parallel^{(gap)} = -\frac{1}{3}\frac{P_0}{\varepsilon_0}\sin\alpha$$

(ii) $\perp$ - component changes by amount $\propto$ surface charge

$$E_\perp^{(gap)} - E_\perp^{(in)} = \frac{P_0}{\varepsilon_0}\cos\alpha$$

$$E_\perp^{(gap)} = -\frac{1}{3}\frac{P_0}{\varepsilon_0}\cos\alpha + \frac{P_0}{\varepsilon_0}\cos\alpha = \frac{2}{3}\frac{P_0}{\varepsilon_0}\cos\alpha$$

Maxwell Stress tensor in the gap ($\hat{e}_\perp, \hat{e}_\parallel, \hat{e}_y$ coordinates)

$$\hat{T} = \begin{pmatrix} T_{\perp\perp} & T_{\perp\parallel} & 0 \\ T_{\perp\parallel} & T_{\parallel} & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \begin{aligned} F_\perp &= \int T_{\perp\perp}\,da = \pi a^2\, T_{\perp\perp} \\ F_\parallel &= \int T_{\perp\parallel}\,da = \pi a^2\, T_{\perp\parallel} \end{aligned}$$

$$T_{\perp\perp} = \varepsilon_0\left[E_\perp^2 - \frac{1}{2}E^2\right] = \varepsilon_0\frac{P_0^2}{\varepsilon_0^2}\left[\left(\frac{2}{3}\cos\alpha\right)^2 - \frac{1}{2}\frac{4}{9}\cos^2\alpha \quad - \frac{1}{2}E_\parallel^2\Big/\left(\frac{P_0}{\varepsilon_0}\right)^2\right]$$

$$= \frac{P_0^2}{\varepsilon_0}\left[\frac{2}{9}\cos^2\alpha - \frac{1}{2\cdot 9}\sin^2\alpha\right] = \frac{P_0^2}{\varepsilon_0}\left[\frac{2}{9}\cos^2\alpha + \frac{1}{18}\cos^2\alpha - \frac{1}{18}\right]$$

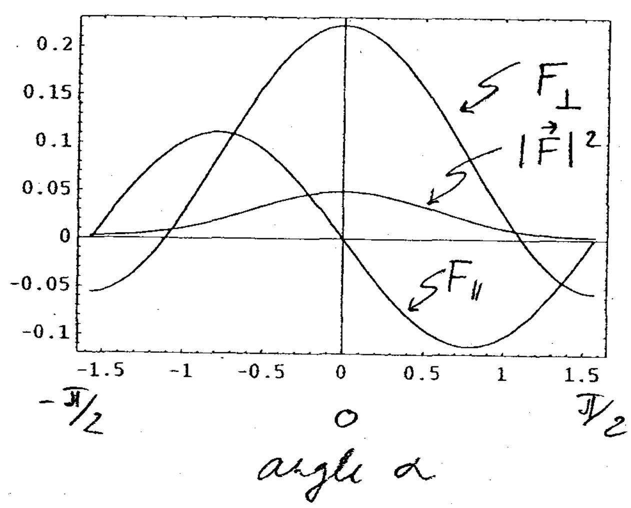
$$T_{\parallel\perp} = \varepsilon_0 \, E_{\parallel} E_{\perp} = \frac{P_0^2}{\varepsilon_0}\left[-\frac{1}{3}\sin\alpha \cdot \frac{2}{3}\cos\alpha\right]$$

Force acting (on the left hemisphere) parallel to the cut surface in the gap

$$\underline{\underline{F_{\parallel} = -\frac{\pi a^2 P_0^2}{\varepsilon_0}\,\frac{2}{9}\sin\alpha\cos\alpha}}$$

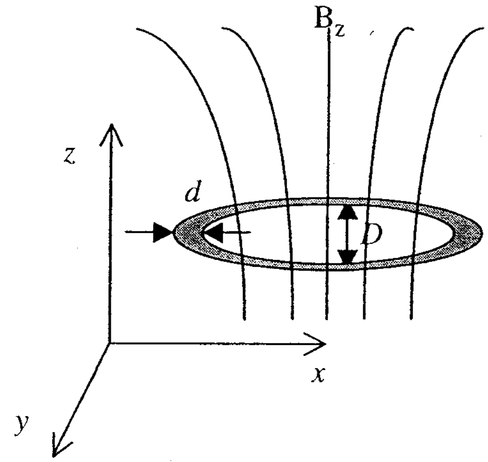
Force (acting on the left hemisphere) perpendicular to the cut surface:

$$F_{\perp} = \pi a^2 \frac{P_0^2}{\varepsilon_0}\left[\frac{5}{18}\cos^2\alpha - \frac{1}{18}\right]$$



Problem # 9

A conducting circular loop made of thin wire of diameter d, resistivity ρ, and mass density ρ_m is falling from a great height in a magnetic field **B** with a component $B_z = B_0 (1 + C z)$, where C is a constant. The loop diameter D is always parallel to the **x-y** plane. Disregarding air resistance, and assuming g is constant, find the terminal velocity of the loop.



#9

The magnetic force acting upwards on the loop is proportional to its magnetic moment, which is proportional to the current flowing through the loop. The current, *I*, is proportional to the rate of change of the flux through the loop since $I = E_{emf} / R$, where E_{emf} is the electromotive force and R is the resistance. When this upward force balances the downward force due to gravity, we have a constant (terminal) velocity.

The magnetic flux is given by

$$\Phi = B \times Area = B_0(1 + Cz) \times \left(\pi \frac{D^2}{4}\right).$$

And the emf is therefore

$$E_{emf} = -\frac{d\Phi}{dt} = -\frac{B_0 C \pi D^2}{4} \frac{dz}{dt} = -\frac{B_0 C \pi D^2}{4} v_z.$$

Working out the forces is a bit tricky. An easier method is energy balance. The loss in gravitational potential energy during steady-state fall is equal to the joule heating in the loop.

$$mg\Delta z = I^2 R \Delta t \quad \text{(remember } I^2 R \text{ is power, not work)}$$

Rewriting,

$$\frac{\Delta z}{\Delta t} = v_z = \frac{I^2 R}{mg} = \frac{E_{emf}}{Rmg} = \frac{\left(B_0 C \pi D^2\right)^2}{16Rmg} v_z{}^2$$

Solving for *R* and *m* as,

$$R = \rho \frac{Length}{Area} = \rho \frac{\pi D}{\pi d^2 / 4} \quad and \quad m = \rho_m V = \rho_m \frac{\pi d^2}{4} \pi D,$$

we find that

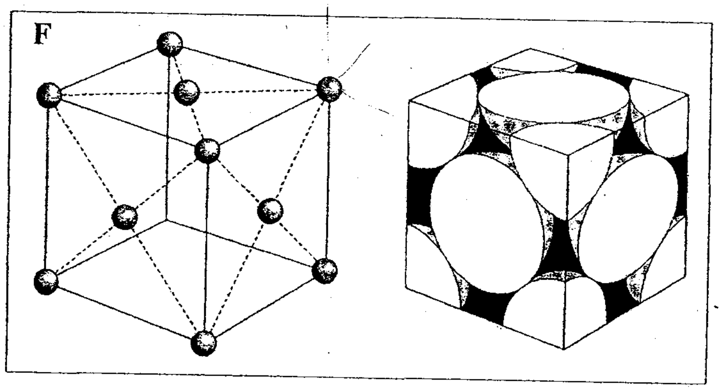
$$v_z = \frac{16 \rho \rho_m g}{B_0{}^2 C^2 D^2},$$ which surprisingly is independent of *d*.

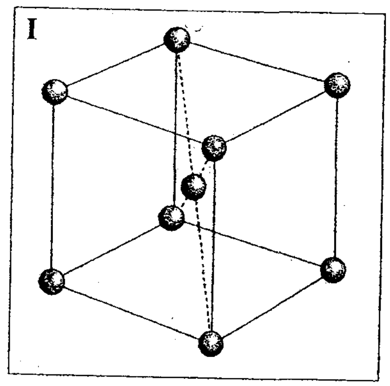
But that is fine because as *d* increases, *m* increases but *R* decreases leaving the combination $R \times m$ constant.

Problem #10

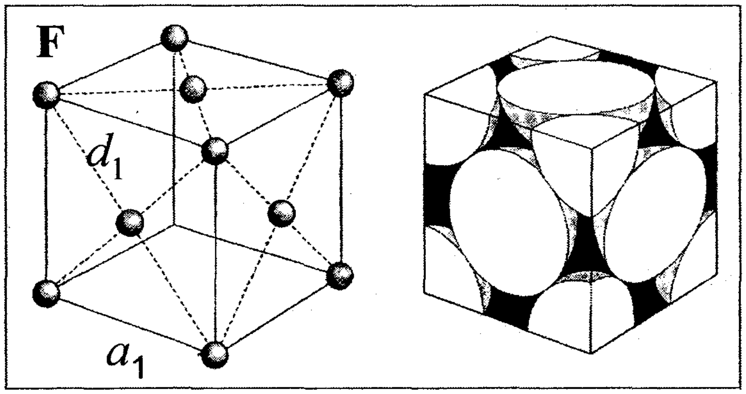
Many metals can be face-centered cubic lattices (F) as well as body-centered cubic lattices (I) as shown in the figures below (note that in figure F, three of the atoms on the back faces of the face-centered cubic lattice are not shown). For some materials there is a transition where the structure changes from one to the other while keeping the mass density of the sample constant.

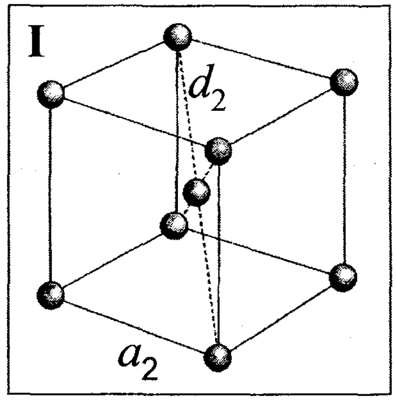
This transition can be modeled as a geometric problem of packing hard spheres. Find the ratio of the diameters of the two hard spheres d_F/d_I while maintaining the constant mass density of the sample. [HINT: You may need to take into account the difference in numbers of atoms per unit cell.]





Solution. Solid State Physics.





If a_1 and a_2 are the corresponding lattice constants then according to the figure

$$\boxed{d_1 = \frac{\sqrt{2}}{2} a_1, \quad d_2 = \frac{\sqrt{3}}{2} a_2,} \tag{1}$$

The unit cell of the face-centred crystal lattice has four atoms, whereas the unit cell of the body-centred crystal lattice has two atoms. Since the mass density ρ of the sample does not change at the transition from one structure to another one, we can write

$$\boxed{\begin{array}{ll} \rho_1 = 4m / V_{cell}(1), & \rho_2 = 2m / V_{cell}(2), \\ V_{cell}(1) = a_1^3, & V_{cell}(2) = a_2^3, \end{array}}$$

From $\boxed{\rho_1 = \rho_2}$ follows $\boxed{a_1^3 / 4 = a_2^3 / 2}$

or

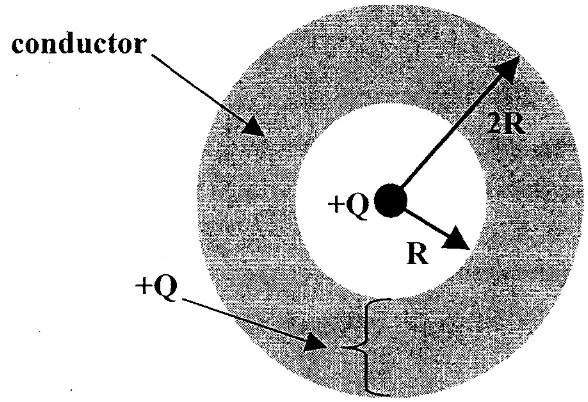
$$\boxed{a_1 / a_2 = \sqrt[3]{2}} \tag{2}$$

Combining (1) and 2 we obtain

$$\boxed{\frac{d_1}{d_2} = (2)^{1/3} \cdot \sqrt{\frac{2}{3}} = 1.029}$$

Problem # 11

A spherical conducting shell of inner radius R and outer radius $2R$ has a net charge $+Q$ placed on it. A point charge of $+Q$ is placed at the center of the cavity inside the spherical shell.



a) Find the magnitude of the electric field in the region $0 < r < R$.

b) Find the magnitude of the electric field in the region $R < r < 2R$.

c) Find the surface charge density on the conductor at $r = R$.

d) Find the surface charge density on the conductor at $r = 2R$.

e) Make a sketch of the magnitude of the electric field for the region $0 < r < 4R$. Make sure the relative magnitudes of the electric field are correct for the points $R/2$, R, $2R$, and $4R$.

f) Make a sketch of the magnitude of the electric potential for the region $0 < r < 4R$. Make sure the relative magnitudes of the electric potential are correct for the points $R/2$, R, $2R$, and $4R$.

#11

Solution

a) Since we have spherical symmetry, Gauss' law is the easiest way to solve this problem.

$$\oint \vec{E} \cdot d\vec{A} = \frac{q_{in}}{\varepsilon_o}$$

Applying Gauss' law in the region 0<r<R, we obtain

$$\oint \vec{E} \cdot d\vec{A} = E(4\pi r^2) = \frac{q_{in}}{\varepsilon_o} = \frac{+Q}{\varepsilon_o}$$

Solving for the magnitude E we get

$$\boxed{E = \frac{+Q}{\varepsilon_o(4\pi r^2)}}$$

b) To find the magnitude of the electric field inside the region R<r<2R, we use the fact that the electric field inside a conductor in static equilibrium is always zero.

$$\boxed{E = 0}$$

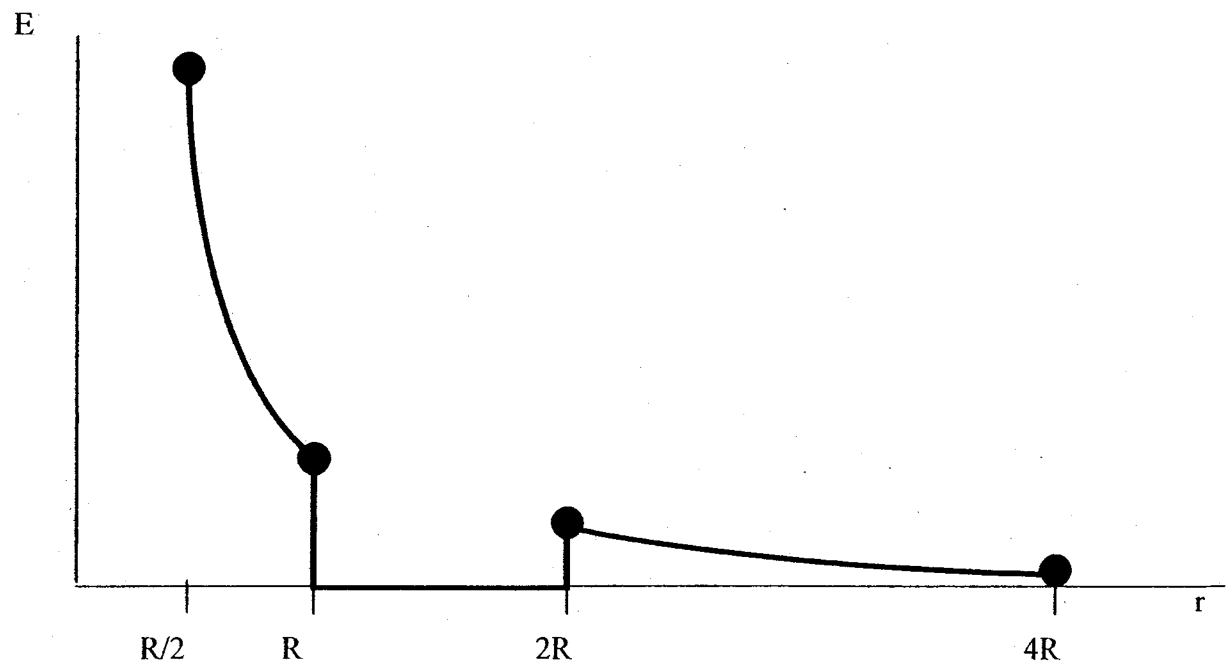
c) To find the surface charge density at r=R, we note that the enclosed charge for R<r<2R must be zero since the electric field is zero and we have spherical symmetry. Thus the charge on the inner surface at r=R must be –Q. Thus the surface charge density must be

$$\boxed{\sigma_R = \frac{-Q}{4\pi R^2}}$$

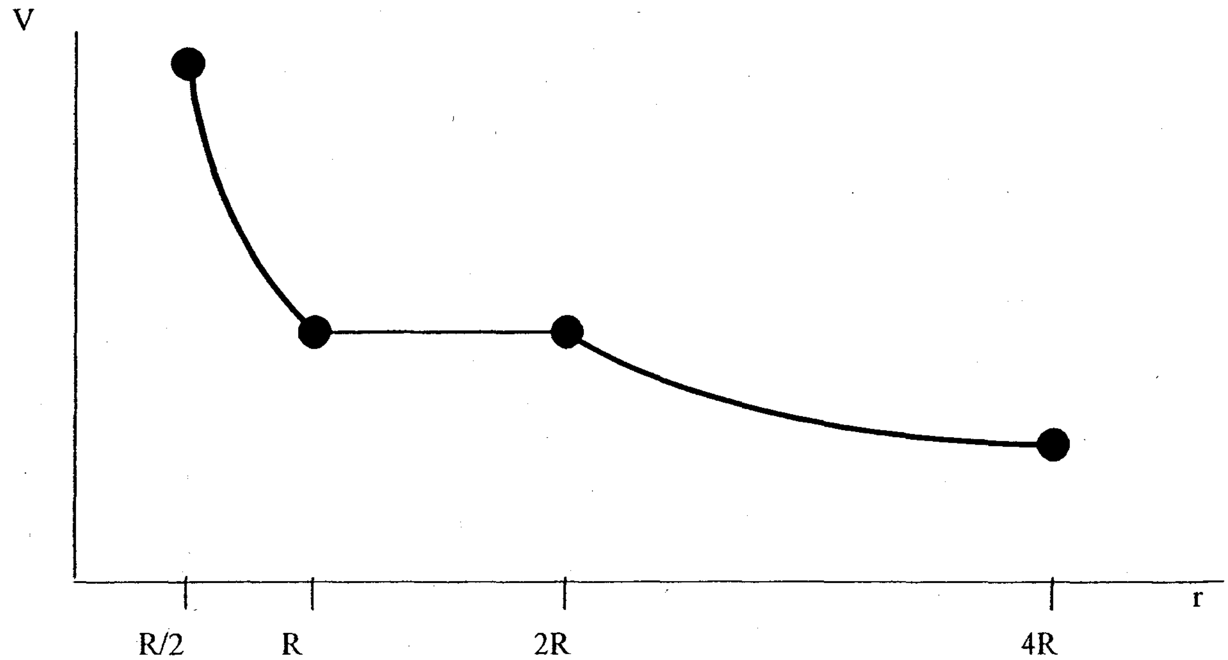
d) Since the charge on the inner surface is –Q, the charge on the outer surface must be +2Q since the net charge on the conductor is +Q. Thus the surface charge density must be

$$\boxed{\sigma_{2R} = \frac{+2Q}{4\pi(2R)^2} = \frac{Q}{8\pi R^2}}$$

e) To sketch the magnitude of the electric field we note that the field goes as $1/r^2$ so that as the distance doubles the magnitude falls by a factor of four. Also, the points inside r=R will have half the charge. Thus the four points shown will be in the ratio of 32:8:4:1. The connecting lines are roughly drawn in.



f) To sketch the magnitude of the electric potential we note that the potential goes as 1/r for spherical symmetry, so that as the distance doubles the potential falls by a factor of two. Also, the potential will be constant across the conductor. Thus the four points shown will be in the ratio of 4:2:2:1. The connecting lines are roughly drawn in.



Problem #12

A metal disk, of mass M and moment of inertia I, has a threaded hole bored through its center. It is screwed onto a threaded rod mounted vertically on a base. The assembly is so well lubricated that the disk spins *frictionlessly* along the threaded rod. The threads are such that one complete rotation advances the disk a distance w along the rod.

(a) Write down a Lagrangian for the disk/rod system.

(b) The disk is released from rest from a height of h (above the base). How long does it take the disk to descend to the base?

(c) What torque is exerted *by the rod* on the disk? How does it vary as the disk descends?

#12

Problem 12

a. Express by z and ϕ the height of the disk and its angle relative to a fixed direction. In terms of these two coordinates the Lagrangian is

$$L(z,\phi,\dot{z},\dot{\phi}) = \tfrac{1}{2}I\dot{\phi}^2 + \tfrac{1}{2}M\dot{z}^2 - Mgz \qquad (1)$$

The threading relates the height and the angle according to the relation

$$z = z_0 + \frac{w\phi}{2\pi}$$

where z_0 is the height when the disk is in its reference orientation. This can be used to eliminate ϕ entirely $\dot{\phi} = (2\pi/w)\dot{z}$ giving the one degree of freedom Lagrangian

$$L(z,\dot{z}) = \tfrac{1}{2}\left[\frac{4\pi^2 I}{w^2} + M\right]\dot{z}^2 - Mgz \qquad (2)$$

b. The equation of motion from Lagrangian (2) is

$$\left[\frac{4\pi^2 I}{w^2} + M\right]\ddot{z} = -Mg \qquad (3)$$

The solution to this, satisfying $z(0) = h$ and $\dot{z}(0) = 0$ is

$$z(t) = h - \tfrac{1}{2}\frac{Mgt^2}{(4\pi^2 I/w^2) + M}$$

Setting $z(t) = 0$ gives the time to hit the bottom

$$t_f = \sqrt{\frac{2h}{g}}\sqrt{\frac{4\pi^2 I}{Mw^2} + 1}$$

c. A direct calculation of the torque follows from the torque on the disk

$$N = I\ddot{\phi} = \frac{2\pi I}{w}\ddot{z} = -\frac{2\pi I w M g}{4\pi^2 I + Mw^2}$$

This is constant over the descent.

The second method involves a Lagrange multiplier, λ, whereby (1) becomes

$$L(z,\phi,\dot{z},\dot{\phi}) = \tfrac{1}{2}I\dot{\phi}^2 + \tfrac{1}{2}M\dot{z}^2 - Mgz + \lambda(z - z_0 - w\phi/2\pi)$$

The new Euler-Lagrange equations are

$$\begin{aligned} I\ddot{\phi} &= -\lambda(w/2\pi) \\ M\ddot{z} &= -Mg + \lambda \end{aligned}$$

These may be combined to eliminaate λ

$$(2\pi/w)I\ddot{\phi} + M\ddot{z} = -Mg \ .$$

Using the constraint to eliminate $\ddot{\phi}$ yields (3).

The torque is now found from the Lagrange multiplier

$$N = -\lambda(w/2\pi) = (w/2\pi)(M\ddot{z} + Mg)$$

Problem # 13

Consider surface, $\sigma(u,v) := \{u \cos v, u \sin v, v\}$, where $0 < u < 1$ and $0 < v < 2\pi$.

(a) Make a sketch of this surface;

(b) What are <u>tangential vectors</u> to this surface?

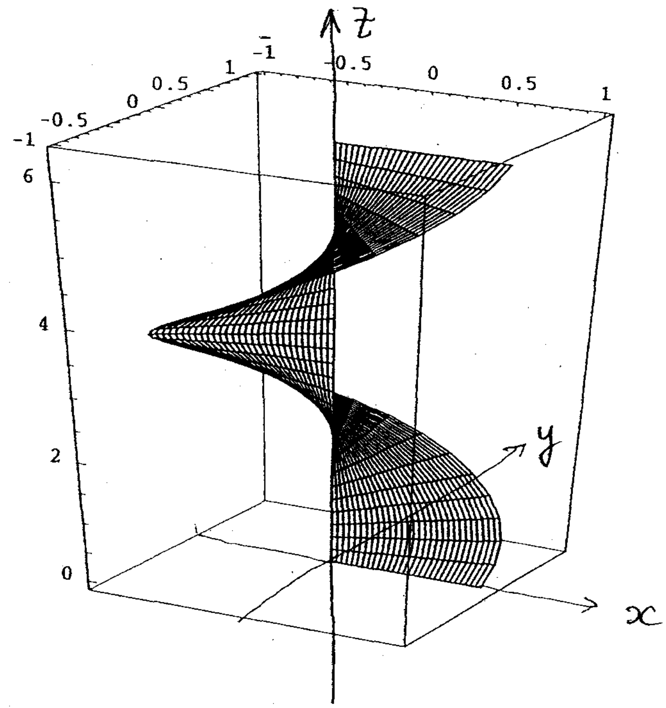
(c) What is <u>normal vector</u> to this surface?

(d) What is the area of this surface?

(e) Consider vector function, $\vec{A} = \{y, -x, 0\}$. What is the <u>flux</u> Φ of $\vec{A}$ through the surface σ?

(Math)

(a)



(b) Tangential vectors

$$\vec{T}_u = \frac{\partial}{\partial u}\{u\cos v;\ u\sin v;\ v\} = \{\cos v,\ \sin v,\ 0\}$$

$$\vec{T}_v = \frac{\partial}{\partial v}\{\ \dots\ \} = \{-u\sin v,\ u\cos v,\ 1\}$$

(c) Normal vector $\vec{N} = \vec{T}_u \times \vec{T}_v = \{\sin v, -\cos v, +u\}$

$$|\vec{N}| = \sqrt{1+u^2}$$

(d) Surface area $S = \int_\sigma |\vec{T}_u \times \vec{T}_v| \, du \, dv$

$$= \int_0^1 \int_0^{2\pi} \sqrt{1+u^2} \, du \, dv = \pi \left(\sqrt{2} + \text{ArcSinh}[1] \right) = \pi \left[\sqrt{2} + \ln(\sqrt{2}+1) \right]$$

(e) $\Phi = \int_\sigma (\vec{N} \cdot \vec{A}) \, du \, dv = \int_\sigma u \, du \, dv = 2\pi \cdot \frac{1}{2} = \underline{\underline{\pi}}$

$$\{\sin v, -\cos v, u\} \cdot \underbrace{\{y, -x, 0\}}_{= \{u \sin v, -u \cos v, 0\}} = u \sin^2 v + u \cos^2 v = u$$

Problem #14

As a graduate student at MSU, you are told that you will be using a new instrument that, as a by-product of its operation, generates X-rays. As the operator of this instrument you would be 5 m away from it and, without any shielding, you would be exposed to 100 times the minimum allowable radiation dosage of X-rays. Since this is not safe, you need to add some lead shielding (which for these X-rays has an absorption cross section of 139 barn, where 1 barn is 10^{-24} cm^2).

a) What is the thickness of lead shielding required to reduce the dosage that the operator would receive down to the minimum allowable radiation dosage? (You may need to make some estimates for the properties of lead.)

b) What dosage would the operator receive if twice the thickness of lead, determined in part (a), was used as shielding? (Report your answer as a percentage of the minimum allowable radiation dosage.)

c) Assuming that the lead shielding was the same thickness as determined in part (a), how far should the operator be seated from the instrument to receive the dosage determined in part (b)?

#14

a) The absorption of X-rays by materials is determined from

$$I_{transmitted} = I_{incident}\ e^{-d/\lambda} \qquad (1)$$

where d is the thickness of material and λ is the inelastic-mean-free-path of the X-ray in that material. The inelastic-mean-free-path is determined (as dimensional analysis indicates) from the atomic volume (volume per atom) by dividing by the atomic absorption coefficient, σ, $\lambda = V_{atom}/\sigma$. You need to estimate the volume per atom. One estimate is to assume that each Pb atom is separated from its neighbor by 1 Å (0.1 nm). This means that if we approximate the Pb atom as a sphere, it will have a radius of radius 0.5 Å, giving a volume per atom somewhere around 0.5 Å^3 (or $0.5\text{x}10^{-24}$ cm^3).

[Another way to get the volume is to make estimates for the density (~10 gm/cm^3) and the atomic weight (~200 gm/mole) with the knowledge of Avagadro's number (6.02 x 10^{23} atom/mole) for a slightly different volume of $0.8\text{x}10^{-24}$ cm^3.]

Therefore $\lambda = (0.5\text{x}10^{-24}\ \text{cm}^3) / (139\text{x}10^{-22}\ \text{cm}^2) = 0.36$ cm. From eq (1) above

$$d = -\ln\left(\frac{I_t}{I_0}\right) \times \lambda = \ln(100) \times 0.36 = 1.6\ \ cm$$

b) If we double the thickness of the Pb then we simply find

$$I'_t = I_0 \times e^{-2d/\lambda} = I_0 \times e^{-d/\lambda} \times e^{-d/\lambda} = I_0 \times \frac{1}{10{,}000}$$

which is 1% of the minimum allowable dosage (mad).

c) Instead of doubling the thickness of the Pb, we can simply move further away because the flux from this (assumed) point source falls as $1/r^2$ (if you don't assume a point source it falls off slower than $1/r^2$). If at 5.0 m the flux is 1 mad, at what distance, s, is the flux 0.01 mad?

$$\frac{5^2}{s^2} = \frac{1}{0.01} \qquad or \qquad s = 5 \times \sqrt{100} = 50\ \ m.$$

Problem #15

A spherical bubble floats in air whose ambient pressure is p_0. The bubble consists of thin, massless, impermeable material which is stretchy but has *no surface tension.* Consequently its internal pressure is also p_0 while in its initial equilirbium, with radius R_0. Then an electric charge Q is deposited uniformly onto the bubble's surface, causing its radius to change many, many fold. What is the radius of the bubble once its internal temperature has equilibrated with the external temperature? State clearly whether the bubble has *expanded* (to a large multiple of R_0) or *contracted* (to a small fraction of R_0). It might help in calculating the force to differentiate the electrostatic energy,

$$W = \frac{\epsilon_0}{2} \int |\mathbf{E}|^2 \, d\mathbf{x} \ ,$$

with respect to the balloon's radius.

#15

The electric field outside the balloon is

$$E_r = \frac{Q}{4\pi\epsilon_0}\frac{1}{r^2} \quad , \quad r > R \ .$$

The total energy of the electric field is

$$W_E = \frac{\epsilon_0}{2}4\pi\int_R^\infty E_r^2\, r^2 dr = \frac{Q^2}{8\pi\epsilon_0}\frac{1}{R}$$

Differentiating this w.r.t. R gives

$$\frac{\partial W_E}{\partial R} = -\frac{Q^2}{8\pi\epsilon_0}\frac{1}{R^2}$$

meaning there is a net *outward* force. This means the balloon will *expand* when it is charged. Dividing by the surface area, $4\pi R^2$, gives an effective pressure due to the mutual repulsion of the charge

$$p_E = \frac{Q^2}{8\pi\epsilon_0}\frac{1}{4\pi R^4}$$

The balloon is impermeable so the internal mass is conserved. When it has a radius R it therefore has an internal mass density

$$\rho_i = \rho_0\left(\frac{R_0}{R}\right)^3 \ .$$

When the internal and external temperatures match the pressure difference across the surface will be

$$p_0 - p_i = p_0\left(1 - \frac{R_0^3}{R^3}\right) \ ,$$

directed inward (if its is positive — i.e. $R > R_0$). This inward pressure must balance the outward pressure of the charge leading to the requirement

$$p_0\left(1 - \frac{R_0^3}{R^3}\right) = \frac{Q^2}{32\pi^2\epsilon_0}\frac{1}{R^4}$$

This can be cast as an equation for the equilibrium radius R:

$$R^4 - R_0^3 R - \frac{Q^2}{32\pi^2\epsilon_0 p_0} = 0 \ .$$

Since th balloon has expanded many, many fold ($R_0 \ll R$) we can neglect the middle term to get the equilibrium radius

$$R \simeq \sqrt{\frac{|Q|}{4\pi\sqrt{2\epsilon_0 p_0}}} \ .$$