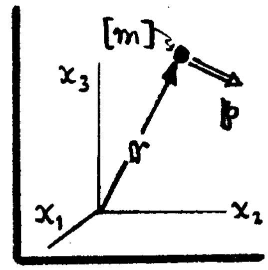
ϕ506 Problems

(20) For general QM operators A, B and C, establish the commutator identities:

(1) $[AB, C] = A[B, C] + [A, C]B$;

(2) $[A, BC] = B[A, C] + [A, B]C$.



(21) Classically, angular momentum $\mathbb{L}$ is defined in terms of the position $\mathbf{r}$ and linear momentum $\mathbf{p}$ of a particle by $\mathbb{L} = \mathbf{r} \times \mathbf{p}$. In rectangular coordinates $\mathbf{r} = (x_1, x_2, x_3)$, the components of $\mathbb{L}$ are: $L_1 = x_2 p_3 - x_3 p_2$, etc. In QM, the x_α & p_β (w/ $\alpha, \beta = 1, 2, 3$) are linear Hermitian operators which obey $[x_\alpha, p_\beta] = i\hbar\delta_{\alpha\beta}$. Use these facts to show that $\mathbb{L}$ is <u>also</u> a linear Hermitian operator, and is therefore acceptable as the QM counterpart of ∡ momentum. Then, prove the commutation rules:

(1) $[L_\alpha, x_\beta] = i\hbar\, x_\gamma$;

(2) $[L_\alpha, p_\beta] = i\hbar\, p_\gamma$;

(3) $[L_\alpha, L_\beta] = i\hbar\, L_\gamma$.

Here, $\alpha\beta\gamma$ is any even permutation of the indices 123 (i.e. $\alpha\beta\gamma = 123, 231$, or 312).

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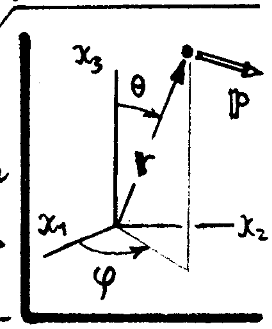
(22) For the QM ∡ momentum operator $\mathbb{L}_{op} = \mathbf{r}_{op} \times \mathbf{p}_{op}$ as defined in problem (21), establish the torque equation: $\frac{d}{dt}\langle \mathbb{L} \rangle = \langle \mathbf{r} \times \mathbb{F} \rangle$, for $\mathbb{F}$ an external force. In the QM version here, the $\langle\ \rangle$ mean an expectation value, of course.



(23) In problem (21), the QM ∡ momentum $\mathbb{L} = \mathbf{r} \times \mathbf{p}$ was defined in a rectangular coordinate system, w/ $\mathbf{r} = (x_1, x_2, x_3)$ and $\mathbf{p} = -i\hbar(\partial/\partial x_1, \partial/\partial x_2, \partial/\partial x_3)$. Transform to spherical polar cds: $(x_1, x_2, x_3) \rightarrow r(\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$, w/ $r = (x_1^2 + x_2^2 + x_3^2)^{1/2}$ the radial distance, θ the colatitude ∡ (from x_3 axis), φ the azimuth ∡ (around x_3 axis).

(A) Show that in spherical cds, the operator $L_3 = -i\hbar\,\partial/\partial\varphi$.

(B) Calculate the commutator $[L_3, \varphi]$. Then, establish the angular uncertainty relation: $\Delta L_3 \Delta\varphi \geq \frac{1}{2}\hbar$.

ϕ506 Solutions

(20) Establish commutator identities: $[AB,C] = A[B,C] + [A,C]B$, etc.

It is easiest just to expand the RHS of the identity, and show that it is equivalent to the LHS.

1) For (1)...

$$\rightarrow \text{RHS} = A[B,C] + [A,C]B = A(BC - CB) + (AC - CA)B$$

$$= ABC - \underbrace{\cancel{ACB} + \cancel{ACB}}_{\text{cancel}} - CAB = (AB)C - C(AB). \tag{1}$$

The grouping in the last step is allowable because multiplication of operators is <u>associative</u>, if not commutative: $ABC = (AB)C = A(BC)$, so long as the left-to-right order remains intact. Then in (1)...

$$\rightarrow \text{RHS} = (AB)C - C(AB) = [AB, C] = \text{LHS}. \quad \underline{\textbf{QED}} \tag{2}$$

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2) For (2)...

$$\text{RHS} = B[A,C] + [A,B]C = B(AC - CA) + (AB - BA)C$$

$$= \underbrace{\cancel{BAC} - BCA + ABC - \cancel{BAC}}_{\text{cancel}} = -(BC)A + A(BC)$$

$$= A(BC) - (BC)A = [A, BC] = \text{LHS}. \quad \underline{\textbf{QED}} \tag{3}$$

This identity was used, in effect, in problem (4) of the MidTerm Exam, with $A = V$, and $B = C = p$, so: $[V, p^2] = p[V,p] + [V,p]p$.

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(21) Develop QM version of ∡ momentum $\mathbb{L} = \mathbb{r} \times \mathbb{p}$.

1. The components of $\mathbb{L}$ are: $L_\alpha = x_\beta p_\gamma - x_\gamma p_\beta$, w/ $\alpha\beta\gamma = \widehat{123}$ (cyclic perm$^{\text{ns}}$s). The L_α, considered as operators (with $p_\gamma = -i\hbar\,\partial/\partial x_\gamma$, etc.) will be linear because the x_β & p_γ are both linear, and the product of two linear operators is also linear.* For Hermitian character, with both $\mathbb{r}$ & $\mathbb{p}$ Hermitian, have: $L_\alpha^\dagger = p_\gamma^\dagger x_\beta^\dagger - p_\beta^\dagger x_\gamma^\dagger = p_\gamma x_\beta - p_\beta x_\gamma$. But, since $\beta \neq \gamma$, both $[x_\beta, p_\gamma]$ & $[x_\gamma, p_\beta] = 0$, so these last two products commute: $p_\gamma x_\beta = x_\beta p_\gamma$, and $p_\beta x_\gamma = x_\gamma p_\beta$. Then: $L_\alpha^\dagger = x_\beta p_\gamma - x_\gamma p_\beta = L_\alpha$; so L_α is Hermitian.

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2. Use the commutator identities from problem (20).

(1) $$[L_\alpha, x_\beta] = [x_\beta p_\gamma, x_\beta] - [x_\gamma p_\beta, x_\beta]$$

$$= x_\beta \underbrace{[p_\gamma, x_\beta]}_{0,\ \text{for } \gamma\neq\beta} + \underbrace{[x_\beta, x_\beta]}_{0,\ \mathbb{r}\text{ comps commute}} p_\gamma - x_\gamma \underbrace{[p_\beta, x_\beta]}_{-i\hbar,\ \gamma=\beta} - \underbrace{[x_\gamma, x_\beta]}_{0,\ \mathbb{r}\text{ comps commute}} p_\beta$$

or/ $[L_\alpha, x_\beta] = i\hbar x_\gamma$. **QED** (1)

(2) $$[L_\alpha, p_\beta] = [x_\beta p_\gamma, p_\beta] - [x_\gamma p_\beta, p_\beta]$$

$$= x_\beta [\cancel{p_\gamma, p_\beta}]^{0} + \underbrace{[x_\beta, p_\beta]}_{i\hbar} p_\gamma - x_\gamma [\cancel{p_\beta, p_\beta}]^{0} - [\cancel{x_\gamma, p_\beta}]^{0\,(\gamma\neq\beta)} p_\beta$$

$[L_\alpha, p_\beta] = i\hbar p_\gamma$. **QED** (2)

(3) $$[L_\alpha, L_\beta] = [L_\alpha, x_\gamma p_\alpha - x_\alpha p_\gamma] = [L_\alpha, x_\gamma p_\alpha] - [L_\alpha, x_\alpha p_\gamma]$$

respect the cyclic perm$^{\text{n}}$

$$= x_\gamma [\cancel{L_\alpha, p_\alpha}]^{0} + \underbrace{[L_\alpha, x_\gamma]}_{-i\hbar x_\beta\ [\text{from (1)}]} p_\alpha - x_\alpha \underbrace{[L_\alpha, p_\gamma]}_{-i\hbar p_\beta\ [\text{from (2)}]} - [\cancel{L_\alpha, x_\alpha}]^{0} p_\gamma$$

or/ $[L_\alpha, L_\beta] = i\hbar (x_\alpha p_\beta - x_\beta p_\alpha) = i\hbar L_\gamma$, w/ $\alpha\beta\gamma = \widehat{123}$. (3)

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* A & B linear → for c = cnst: $(AB)(c\psi) = A[B(c\psi)] = A[c(B\psi)] = c(AB)\psi$.

(22) Establish torque eqtn: $(d/dt)\langle \mathbb{L}\rangle = \langle \mathbb{r}\times\mathbb{F}\rangle$, for QM $\measuredangle$ momentum.

1. With $L_\alpha = x_\beta p_\gamma - x_\gamma p_\beta$, $\alpha\beta\gamma = \overrightarrow{123}$, the QM Eqtn-of-Motion is...

$$\rightarrow \frac{d}{dt}\langle L_\alpha\rangle = \langle \partial L_\alpha/\partial t\rangle + \frac{i}{\hbar}\langle[\mathcal{H}, L_\alpha]\rangle. \tag{1}$$

The $\partial L_\alpha/\partial t$ term involves integrals over $\partial x/\partial t$ & $\partial p/\partial t$. But x & $p \propto \partial/\partial x$ are integration variables, so these integrals vanish, and $\langle \partial L_\alpha/\partial t\rangle = 0$. Then...

$$\rightarrow \frac{d}{dt}L_\alpha = \frac{i}{\hbar}[\mathcal{H}, L_\alpha], \text{ in an expectation value sense.} \tag{2}$$

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2. Evidently, we need the commutator $[\mathcal{H}, L_\alpha]$. Use the identities of problem (20), and results from CLASS NOTES, p. Prop.(15), Eqs.(11)...

$$\rightarrow [\mathcal{H}, L_\alpha] = [\mathcal{H}, x_\beta p_\gamma] - [\mathcal{H}, x_\gamma p_\beta]$$

$$= x_\beta \underbrace{[\mathcal{H}, p_\gamma]}_{+i\hbar\left(\frac{\partial\mathcal{H}}{\partial x_\gamma}\right)} + \underbrace{[\mathcal{H}, x_\beta]}_{-i\hbar\left(\frac{\partial\mathcal{H}}{\partial p_\beta}\right)} p_\gamma - x_\gamma \underbrace{[\mathcal{H}, p_\beta]}_{+i\hbar\left(\frac{\partial\mathcal{H}}{\partial x_\beta}\right)} - \underbrace{[\mathcal{H}, x_\gamma]}_{-i\hbar\left(\frac{\partial\mathcal{H}}{\partial p_\gamma}\right)} p_\beta$$

$$\left[\!\left[\frac{i}{\hbar}[\mathcal{H}, L_\alpha] = x_\beta\left(-\frac{\partial\mathcal{H}}{\partial x_\gamma}\right) - x_\gamma\left(-\frac{\partial\mathcal{H}}{\partial x_\beta}\right) + \left(\frac{\partial\mathcal{H}}{\partial p_\beta}\right)p_\gamma - \left(\frac{\partial\mathcal{H}}{\partial p_\gamma}\right)p_\beta.\right.\right. \tag{3}$$

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3. In (3), put $\mathcal{H} = \frac{1}{2m}\mathbb{p}^2 + V$ for the Hamiltonian (V could be time-dept).

The space derivatives $\partial\mathcal{H}/\partial x_\alpha = \partial V/\partial x_\alpha = -F_\alpha$ are the force components. So:

$$\rightarrow \frac{i}{\hbar}[\mathcal{H}, L_\alpha] = \underbrace{(x_\beta F_\gamma - x_\gamma F_\beta)}_{(\mathbb{r}\times\mathbb{F})_\alpha} + \frac{1}{m}\underbrace{(p_\beta p_\gamma - p_\gamma p_\beta)}_{\text{zero}}. \tag{4}$$

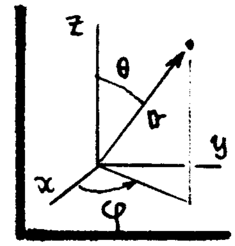
Use of this result in Eq. (2) allows us to write...

$$\underline{\underline{\frac{d}{dt}L_\alpha = \frac{i}{\hbar}[\mathcal{H}, L_\alpha] = (\mathbb{r}\times\mathbb{F})_\alpha}}, \text{ i.e. } \boxed{\frac{d}{dt}\langle\mathbb{L}\rangle = \langle\mathbb{r}\times\mathbb{F}\rangle}. \tag{5}$$

So, Ehrenfest's Theorem works for $\measuredangle$ momentum $\mathbb{L} = \mathbb{r}\times\mathbb{p}$.

ϕ506 Solutions

㉓ Transform ∡ momentum operator $\mathbb{L} = \mathbf{r}\times\mathbf{p}$ from rectangular to spherical cds.



1. The transformation and its inverse are...

(A)
$$\begin{cases} x = r\sin\theta\cos\varphi,\ y = r\sin\theta\sin\varphi,\ z = r\cos\theta; \\ r = (x^2+y^2+z^2)^{1/2},\ \theta = \cos^{-1}(z/r),\ \varphi = \tan^{-1}(y/x). \end{cases} \quad (1)$$

From the 2nd set of eqtns, we can calculate certain key derivatives...

$$\begin{cases} \dfrac{\partial r}{\partial x_i} = \dfrac{x_i}{r},\ x_i \leftrightarrow (x,y,z);\ \dfrac{\partial\theta}{\partial x} = \dfrac{xz}{r^3\sin\theta},\ \dfrac{\partial\theta}{\partial y} = \dfrac{yz}{r^3\sin\theta}; \\ \text{and}/\!/\ \tan\varphi = y/x \Rightarrow (\sec^2\varphi)\dfrac{\partial\varphi}{\partial x_i} = \dfrac{\partial}{\partial x_i}\left(\dfrac{y}{x}\right). \end{cases} \quad (2)$$

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2. Now we want to transform $L_z = xp_y - yp_x = i\hbar\left(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\right)$ to (r,θ,φ) cds. By the chain rule for differentiation...

$$\rightarrow \frac{\partial}{\partial x} = \left(\frac{\partial r}{\partial x}\right)\frac{\partial}{\partial r} + \left(\frac{\partial\theta}{\partial x}\right)\frac{\partial}{\partial\theta} + \left(\frac{\partial\varphi}{\partial x}\right)\frac{\partial}{\partial\varphi}, \text{ and similarly for } \frac{\partial}{\partial y};$$

$$\left.\begin{aligned} &\text{so}/\!/\ y\frac{\partial}{\partial x} = \left(\frac{yx}{r}\right)\frac{\partial}{\partial r} + \left(\frac{yxz}{r^3\sin\theta}\right)\frac{\partial}{\partial\theta} + \left(\frac{-y^2}{x^2\sec^2\varphi}\right)\frac{\partial}{\partial\varphi}, \\ &\text{and}/\!/\ x\frac{\partial}{\partial y} = \left(\frac{xy}{r}\right)\frac{\partial}{\partial r} + \left(\frac{xyz}{r^3\sin\theta}\right)\frac{\partial}{\partial\theta} + \left(\frac{1}{\sec^2\varphi}\right)\frac{\partial}{\partial\varphi}. \end{aligned}\right\} \quad (3)$$

In the <u>difference</u> between these last two operators, only the terms in $\frac{\partial}{\partial\varphi}$ survive...

$$\rightarrow y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} = \underbrace{-\frac{1}{\sec^2\varphi}\left(\frac{y^2}{x^2}+1\right)}_{=\cos^2\varphi(\tan^2\varphi+1) = \sin^2\varphi+\cos^2\varphi = 1}\frac{\partial}{\partial\varphi} = -\partial/\partial\varphi;$$

So/\!/
$$\boxed{L_z = i\hbar\left(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\right) = -i\hbar(\partial/\partial\varphi)}. \quad \textbf{QED}^{¶} \quad (4)$$

(B) **3.** With $L_z = -i\hbar\,\partial/\partial\varphi$, the commutator $[L_z, \varphi] = -i\hbar$. Then, by Heisenberg's version of the uncertainty relations: $\Delta L_z\,\Delta\varphi \geq \frac{1}{2}|\langle[L_z,\varphi]\rangle| = \hbar/2$.

¶ The other components of $\mathbb{L}$, L_x & L_y, are not needed here. For the record, they look like:

$$L_x = i\hbar\left(\sin\varphi\frac{\partial}{\partial\theta} + \text{ctn}\,\theta\cos\varphi\frac{\partial}{\partial\varphi}\right),\quad L_y = i\hbar\left(-\cos\varphi\frac{\partial}{\partial\theta} + \text{ctn}\,\theta\sin\varphi\frac{\partial}{\partial\varphi}\right).$$