⑩ Consider the 2P states of a one-electron atom. Here, the orbital ∢ momentum $\vec{L}$ (eigenwerte $l=1$) and electron spin ∢ momentum $\vec{S}$ (eigenwerte $s=1/2$) couple to form $\vec{J}=\vec{L}+\vec{S}$, with j-values $3/2$ & $1/2$. By using the step-down operator J_-, and imposing orthonormality, explicitly do a Clebsch-Gordan transform$^{\underline{n}}$ from the uncoupled states $|lsm_lm_s\rangle$ to the coupled states $|lsjm_j\rangle$. Make a table of your results, i.e. each $|lsjm_j\rangle$ state in turn, as a linear combination of the $|lsm_lm_s\rangle$, with appropriate C-G coefficients.

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⑪ [15pts]. To generalize prob.⑩, let l have any value >0; then $j=l\pm\frac{1}{2}$. With $m_s=\pm\frac{1}{2}$ only, there are just two m_l values for a given m: viz. $m_l=m\mp\frac{1}{2}$ (here $m=m_j$). Let $\alpha=|s=\frac{1}{2},m_s=+\frac{1}{2}\rangle$ & $\beta=|s=\frac{1}{2},m_s=-\frac{1}{2}\rangle$ be the spin-up & spin-down eigenfcns. Then the eigenfcns of the coupled states have just two terms (suppress l & s, ad libitum):

$$\boxed{|n;lsjm\rangle = C_1(jm)|n;l,m_l=m-\tfrac{1}{2}\rangle\alpha + C_2(jm)|n;l,m_l=m+\tfrac{1}{2}\rangle\beta.}$$

The C-G transform in this case amounts to finding two pairs of constants C_k, one pair for each of $j=l\pm\frac{1}{2}$. By using the J_- operator, calculate the $C_k(jm)$ explicitly.

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⑫ In an atom where the orbital & spin ∢ momenta $\vec{L}$ & $\vec{S}$ couple to form $\vec{J}=\vec{L}+\vec{S}$, the magnetic moments $\vec{\mu}_L=-g_L\mu_0\vec{L}$ & $\vec{\mu}_S=-g_S\mu_0\vec{S}$ likewise couple to form a total $\vec{\mu}_J=\vec{\mu}_L+\vec{\mu}_S$. Use the Vector Model to show that (in an expectation-value sense): $\vec{\mu}_J=-g_J\mu_0\vec{J}$. Show that g_J -- which is called the Landé g-factor -- is given by:

$$\boxed{g_J=\left[\frac{j(j+1)+l(l+1)-s(s+1)}{2j(j+1)}\right]g_L+\left[\frac{j(j+1)+s(s+1)-l(l+1)}{2j(j+1)}\right]g_S.}$$

Calculate g_J-values for the hydrogen states $2P_{3/2}$, $2P_{1/2}$ & $2S_{1/2}$. What is the maximum observable μ_J in each state? If a weak magnetic field H were applied to this system how would the state energies vary with H? Draw a picture. [This is the Zeeman Effect].

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⑬ Consider the hydrogenic states $2S_{\frac{1}{2}}$ [the $m=\pm\frac{1}{2}$ levels are denoted α & β] and $2P_{3/2}$ [$m=+\frac{3}{2}, +\frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}$ levels denoted a, b, c, d]. Some of the m-levels are coupled by a Stark interaction $V=\vec{\mathcal{E}}\cdot\vec{r}$, w/ $\vec{r}$ = position and $\vec{\mathcal{E}}$ = cnst. Find the absolute value of all matrix elements $\mathcal{M}=|\langle 2S_{\frac{1}{2}}|V|2P_{\frac{3}{2}}\rangle|$ allowed between the six m-levels, up to a reduced matrix element R. If $\Gamma\propto\mathcal{M}^2$ is the transition rate induced by V, establish the equalities:

$$\boxed{\Gamma(\alpha\to b)=\Gamma(\beta\to c),\quad \Gamma(\alpha\to a)=\Gamma(\beta\to d)=3\Gamma(\alpha\to c)=3\Gamma(\beta\to b).}$$

⑩ Carry out a Clebsch-Gordan transform for hydrogen states $2P_{3/2}$ & $2P_{1/2}$.

1. Denote the wavefcn for the coupled repⁿ by : $|nlsjm\rangle = \psi_j^m$ ($m = m_j$ here), for the uncoupled space part : $|nlm_l\rangle = \phi_l^{m_l}$, and for the spin $\frac{1}{2}$ spinors : $|s=\frac{1}{2}, m_s = \pm\frac{1}{2}\rangle = \alpha, \beta$ (spin up & spin down, resp.). Start at the top of the ladder : $m = j = \frac{3}{2}$, where

$\rightarrow \underline{\underline{\psi_{3/2}^{3/2}}} = \phi_1^{+1}\alpha$. (1) (i.e. $|n=2, l=1, s=\frac{1}{2}, j=\frac{3}{2}, m=j\rangle = |n=2, l=1, m_l = l\rangle \otimes |s=\frac{1}{2}, m_s=\frac{1}{2}\rangle$)

is the only possibility. Operate with $J_- = L_- + S_-$, w/ $J_-\psi_j^m = \sqrt{j(j+1)-m(m-1)}\,\psi_j^{m-1}$, and likewise for L_- & S_- (note : $S_-\alpha = \beta$, and no more). Then...

$$J_-\psi_{3/2}^{3/2} = \sqrt{2j}\,\psi_{3/2}^{1/2} = (L_-\phi_1^{+1})\alpha + \phi_1^{+1}(S_-\alpha) = \sqrt{2l}\,\phi_1^0\alpha + \phi_1^{+1}\beta.$$

so/ $j = 3/2$, $l = 1$ $\Rightarrow$ $\underline{\underline{\psi_{3/2}^{1/2}}} = \sqrt{2/3}\,\phi_1^0\alpha + \sqrt{1/3}\,\phi_1^{+1}\beta$. (2)

Normalization is evidently preserved. Apply J_- twice more to get to $m = -j$...

$$J_-\psi_{3/2}^{1/2} = \sqrt{4}\,\psi_{3/2}^{-1/2} = \sqrt{\tfrac{2}{3}}\left[(L_-\phi_1^0)\alpha + \phi_1^0(S_-\alpha)\right] + \sqrt{\tfrac{1}{3}}(L_-\phi_1^{+1})\beta,$$

so/ $\underline{\underline{\psi_{3/2}^{-1/2}}} = \sqrt{1/3}\,\phi_1^{-1}\alpha + \sqrt{2/3}\,\phi_1^0\beta$. (3) $\underline{\underline{\psi_{3/2}^{-3/2}}} = \phi_1^{-1}\beta$ (obvious!) (4)

---

2. We now have all the coupled states for $2P_{3/2}$, i.e. the ψ_j^m for $j = 3/2$. The ψ_j^m for $2P_{1/2}$ must be orthogonal to the $2P_{3/2}$ states, so -- for the first one -- try

$\rightarrow \psi_{1/2}^{1/2} = A\phi_1^0\alpha + B\phi_1^{+1}\beta$, w/ $\langle\psi_{1/2}^{1/2}|\psi_{3/2}^{1/2}\rangle = 0 \Rightarrow A\sqrt{\frac{2}{3}} + B\sqrt{\frac{1}{3}} = 0$. (5)

This condition, together with the norm : $|A|^2 + |B|^2 = 1$, implies $A = \sqrt{\frac{1}{3}}$, $B = -\sqrt{\frac{2}{3}}$,

so/ $\underline{\underline{\psi_{1/2}^{1/2}}} = \sqrt{1/3}\,\phi_1^0\alpha - \sqrt{2/3}\,\phi_1^{+1}\beta$, (6) and/ $\underline{\underline{\psi_{1/2}^{-1/2}}} = \sqrt{2/3}\,\phi_1^{-1}\alpha - \sqrt{1/3}\,\phi_1^0\beta$. (7)

$\psi_{1/2}^{-1/2}$ is gotten by a $J_-\psi_{1/2}^{1/2}$ operation (as above). A table of C-G coefficients for the states $2P_{3/2}$ & $2P_{1/2}$ appears at right.

		ψ_j^m	$\phi_1^{+1}\alpha$	$\phi_1^0\alpha$	$\phi_1^{+1}\beta$	$\phi_1^{-1}\alpha$	$\phi_1^0\beta$	$\phi_1^{-1}\beta$
$2P_{3/2}$	$j = 3/2$	$m = +3/2$	1	~	~	~	~	~
		$+1/2$	~	$\sqrt{2/3}$	$\sqrt{1/3}$	~	~	~
		$-1/2$	~	~	~	$\sqrt{1/3}$	$\sqrt{2/3}$	
		$-3/2$	~	~	~	~	~	1
$2P_{1/2}$	$j = 1/2$	$m = +\frac{1}{2}$	~	$\sqrt{1/3}$	$-\sqrt{2/3}$	~	~	~
		$-\frac{1}{2}$	~	~	~	$\sqrt{2/3}$	$-\sqrt{1/3}$	~

(8)

ϕ507 Solutions

(1) [15 pts]. Derive Clebsch-Gordan coefficients, in general, for $s = 1/2$.

1. Denote the wavefcns for the coupled repⁿ and l-eigenfcn repⁿ by, resp...

$$|n; l, s, j, m\rangle = \psi_j^m \quad , \quad |n; l, m_l\rangle = \phi_l^{m_l}. \tag{1}$$

Start at the top of the ladder, $m = \max = j$, noting the only possibility is...

$$\psi_{j=l+\frac{1}{2}}^{m=j} = \phi_l^{+l}\alpha \quad , \quad \alpha = |s=\tfrac{1}{2}, \uparrow\rangle = \text{spin-up eigenfcn.} \tag{2}$$

Apply the step-down operator : $J^- = L^- + S^-$, where in general...

$$J^-|j,m\rangle = \sqrt{j(j+1) - m(m-1)}\;|j, m-1\rangle. \tag{3}$$

So// $J^-\psi_{j=l+\frac{1}{2}}^{m=j} = \sqrt{2j}\;\psi_{j=l+\frac{1}{2}}^{m=j-1}$, and// $(L^- + S^-)\phi_l^{+l}\alpha = \sqrt{2l}\;\phi_l^{l-1}\alpha + \phi_l^l\beta$,

$$\Rightarrow \psi_{j=l+\frac{1}{2}}^{m=j-1} = \sqrt{\frac{2l}{2l+1}}\;\phi_l^{l-1}\alpha + \sqrt{\frac{1}{2l+1}}\;\phi_l^l\beta \quad , \quad \beta = |s=\tfrac{1}{2}, \downarrow\rangle. \quad \left\{ \text{J- applied } \underline{\text{once}} \right. \tag{4}$$

---

2. This is for J^- applied <u>once</u>. Operate again with J^- to get...

$$J^-\psi_{j=l+\frac{1}{2}}^{m=j-1} = \sqrt{\frac{2l}{2l+1}}\left[(L^-\phi_l^{l-1})\alpha + \phi_l^{l-1}\beta\right] + \sqrt{\frac{1}{2l+1}}\left[(L^-\phi_l^l)\beta + 0\right],$$

or// $$\sqrt{2(2j-1)}\;\psi_{j=l+\frac{1}{2}}^{m=j-2} = \sqrt{\frac{2l}{2l+1}}\sqrt{2(2l-1)}\;\phi_l^{l-2}\alpha + 2\sqrt{\frac{2l}{2l+1}}\;\phi_l^{l-1}\beta$$

or// $$\psi_{j=l+\frac{1}{2}}^{m=j-2} = \sqrt{\frac{2l-1}{2l+1}}\;\phi_l^{l-2}\alpha + \sqrt{\frac{2}{2l+1}}\;\phi_l^{l-1}\beta \quad \left\{ J^- \text{ applied } \underline{\text{twice}} \right. \tag{5}$$

Another application of J^- gives...

$$\psi_{j=l+\frac{1}{2}}^{m=j-3} = \sqrt{\frac{2l-2}{2l+1}}\;\phi_l^{l-3}\alpha + \sqrt{\frac{3}{2l+1}}\;\phi_l^{l-2}\beta \quad \left\{ J^- \text{ applied } \underline{\text{thrice}} \right. \tag{6}$$

---

3. Comparing Eqs. (2), (4), (5) & (6), the obvious generalization is...

$$\psi_{j=l+\frac{1}{2}}^{m=j-k} = \sqrt{\frac{2l+1-k}{2l+1}}\;\phi_l^{l-k}\alpha + \sqrt{\frac{k}{2l+1}}\;\phi_l^{l-k+1}\beta. \tag{7}$$

But $m = j - k \Rightarrow k = l + \frac{1}{2} - m$. Then Eq. (7) can be written as...

$$\boxed{\psi^m_{j=l+\frac{1}{2}} = \left[\sqrt{(l+\tfrac{1}{2}+m)/(2l+1)}\right]\phi_l^{m-\frac{1}{2}}\,\alpha + \left[\sqrt{(l+\tfrac{1}{2}-m)/(2l+1)}\right]\phi_l^{m+\frac{1}{2}}\,\beta} \tag{8}$$

The [ ]'s here are the desired Clebsch-Gordan coefficients for $j = l + \frac{1}{2}$...

$$\longrightarrow \quad C_{1,2}(l+\tfrac{1}{2}, m) = \sqrt{(l+\tfrac{1}{2} \pm m)/(2l+1)}\,, \quad \text{for } m_l = m \mp \tfrac{1}{2}. \tag{9}$$

4. To get the $\psi^m_{j=l-\frac{1}{2}}$, we need only construct them $\perp$ $\psi^m_{j=l+\frac{1}{2}}$, and impose normalization. If the Clebsch-Gordan coefficients are $C_{1,2}(l-\frac{1}{2}, m)$, then

$$C_1(l-\tfrac{1}{2},m)\sqrt{\frac{l+\frac{1}{2}+m}{2l+1}} + C_2(l-\tfrac{1}{2},m)\sqrt{\frac{l+\frac{1}{2}-m}{2l+1}} = 0, \text{ orthogonality;}$$

$$|C_1(l-\tfrac{1}{2},m)|^2 + |C_2(l-\tfrac{1}{2},m)|^2 = 1, \text{ normalization.} \tag{10}$$

These conditions are satisfied by the desired C-G coefficients for $j = l - \frac{1}{2}$...

$$\longrightarrow \quad C_{1,2}(l-\tfrac{1}{2}, m) = \pm\sqrt{(l+\tfrac{1}{2} \mp m)/(2l+1)}\,, \quad \text{for } m_l = m \mp \tfrac{1}{2}. \tag{11}$$

Eqs. (9) & (11) give the general C-G coefficients for $s = \frac{1}{2}$ and arbitrary $j = l \pm \frac{1}{2}$.

5. For the $2P_{3/2}$ & $2P_{1/2}$ states in hydrogen, $l = 1$ and $j = 3/2$ & $1/2$. The above gives...

	m	ψ
$2P_{3/2}$	$+3/2$	$\phi_1^{+1}\alpha$
	$+1/2$	$(\sqrt{2/3})\,\phi_1^0\alpha + (\sqrt{1/3})\,\phi_1^{+1}\beta$
	$-1/2$	$(\sqrt{1/3})\,\phi_1^{-1}\alpha + (\sqrt{2/3})\,\phi_1^0\beta$
	$-3/2$	$\phi_1^{-1}\beta$
$2P_{1/2}$	$+1/2$	$(\sqrt{1/3})\,\phi_1^0\alpha - (\sqrt{2/3})\,\phi_1^{+1}\beta$
	$-1/2$	$(\sqrt{2/3})\,\phi_1^{-1}\alpha - (\sqrt{1/3})\,\phi_1^0\alpha$

One can see by inspection that these eigenfcns ψ are mutually orthogonal. This is true for general $j = l \pm \frac{1}{2}$, m eigenfcns constructed via the C-G transformation... the C-G transformation on an orthogonal set of product wavefcns will in turn produce an orthogonal set.

ϕ507 Solutions

(12) Derive the Landé g-factor and apply it to the 2P-2S levels in hydrogen.

1. By the Vector Model, we average **everything** in the direction of $\vec{J}=\vec{L}+\vec{S}$, so...

$$\langle\vec{\mu}_J\rangle_J = \langle\vec{\mu}_L+\vec{\mu}_S\rangle_J = [(\vec{\mu}_L+\vec{\mu}_S)\cdot\vec{J}/\vec{J}^2]\,\vec{J}$$

$$= (-)\mu_0\,[(g_L\,\vec{L}\cdot\vec{J} + g_S\,\vec{S}\cdot\vec{J})/\vec{J}^2]\,\vec{J} = -g_J\,\mu_0\,\vec{J},$$

where: $g_J = (\vec{L}\cdot\vec{J}/\vec{J}^2)\,g_L + (\vec{S}\cdot\vec{J}/\vec{J}^2)\,g_S \leftarrow$ Landé g-factor. (1)

But: $\vec{L}\cdot\vec{J} = \vec{L}^2+\vec{L}\cdot\vec{S}$, and: $\vec{S}\cdot\vec{J} = \vec{S}^2+\vec{L}\cdot\vec{S}$, with: $\vec{L}\cdot\vec{S} = \frac{1}{2}(\vec{J}^2-\vec{L}^2-\vec{S}^2)$.

Then... $\vec{L}\cdot\vec{J} = \frac{1}{2}(\vec{J}^2+\vec{L}^2-\vec{S}^2) = \frac{1}{2}[j(j+1)+l(l+1)-s(s+1)]$,

$\vec{S}\cdot\vec{J} = \frac{1}{2}(\vec{J}^2-\vec{L}^2+\vec{S}^2) = \frac{1}{2}[j(j+1)-l(l+1)+s(s+1)]$. (2)

The Landé g-factor is then given by -- as advertised:

$$\boxed{g_J = \left[\frac{j(j+1)+l(l+1)-s(s+1)}{2j(j+1)}\right]g_L + \left[\frac{j(j+1)-l(l+1)+s(s+1)}{2j(j+1)}\right]g_S}. \quad (3)$$

---

2. For the hydrogen 2P-2S levels, $s=\frac{1}{2}$ always. With $g_L=1$ & $g_S=2$, get...

$2P_{3/2}\left\{\begin{matrix} j=3/2\\ l=1\end{matrix}\right\}$ $g_J = [+\frac{2}{3}]\,g_L + [+\frac{1}{3}]\,g_S = \underline{\underline{\frac{4}{3}}}$; $\mu_J(\max) = g_J\,\mu_0 \times \frac{3}{2} = 2\mu_0$. (4)

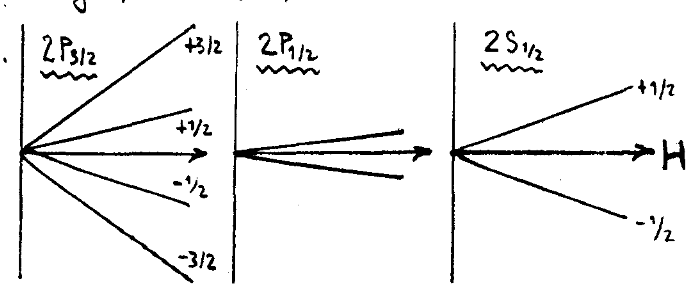
$2P_{1/2}\left\{\begin{matrix} j=1/2\\ l=1\end{matrix}\right\}$ $g_J = [+\frac{4}{3}]\,g_L + [-\frac{1}{3}]\,g_S = \underline{\underline{\frac{2}{3}}}$; $\mu_J(\max) = g_J\,\mu_0 \times \frac{1}{2} = \frac{1}{3}\mu_0$. (5)

$2S_{1/2}\left\{\begin{matrix} j=1/2\\ l=0\end{matrix}\right\}$ $g_J = [0]\,g_L + [+1]\,g_S = \underline{\underline{2}}$; $\mu_J(\max) = g_J\,\mu_0 \times \frac{1}{2} = \mu_0$. (6)

The $\mu_J(\max)$ values are just $\mu_J(\max) = g_J\,\mu_0 \times [m_J(\max) = J]$. In a weak magnetic fld H, get a linear Zeeman effect...

$$\mathcal{E}_J = -\vec{\mu}_J\cdot\vec{H} = g_J\,\mu_0 H\, m_J, \quad (7)$$

which governs the pictures (not to scale) ↗



ϕ507 Solutions

⑬ Calculate $2P_{\frac{3}{2}} \to 2S_{\frac{1}{2}}$ transition rates for coupling by $V = \vec{\mathcal{E}} \cdot \vec{r}$.

1. The transition rates $\Gamma \propto |M|^2$, where the transition matrix elt is...

$$M = \vec{\mathcal{E}} \cdot \underbrace{\langle \alpha j m |}_{2S_{1/2}} \vec{r} \underbrace{| \alpha' j' m' \rangle}_{2P_{3/2}} \quad \begin{cases} j = \frac{1}{2} \text{ for } 2S_{1/2},\ j' = \frac{3}{2} \text{ for } 2P_{3/2} \\ m \to m' = m,\ m \pm 1 \text{ only} \end{cases} \tag{1}$$

By the $\vec{T}$-vector formalism, M is calculable in terms of a reduced elt R...

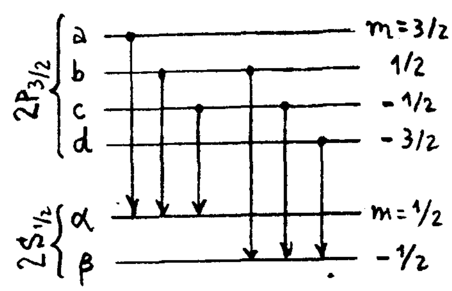
$$R = \langle \alpha j \| r \| \alpha' j' \rangle \leftarrow \text{radial matrix elt};$$

e.g.// $$M(m \to m' = m \pm 1) = \mp \frac{1}{2} (\mathcal{E}_x \pm i \mathcal{E}_y) R \sqrt{(j \pm m + 1)(j \pm m + 2)},$$

$$M(m \to m' = m) = \mathcal{E}_z R \sqrt{(j+1)^2 - m^2}. \tag{2}$$

---

2. There are six allowed transitions $2P_{3/2} \to 2S_{1/2}$ (all with $\Delta m = 0, \pm 1$), as shown in the diagram. Using Eq. (2), the transition matrix elts are...



$$\begin{cases} M(a \leftarrow \alpha) = -\frac{1}{2} (\mathcal{E}_x + i \mathcal{E}_y) R \sqrt{(2)(3)}, \\ M(b \leftarrow \alpha) = \frac{1}{2} \mathcal{E}_z R \sqrt{8} = \mathcal{E}_z R \sqrt{2} \\ M(c \leftarrow \alpha) = +\frac{1}{2} (\mathcal{E}_x - i \mathcal{E}_y) R \sqrt{(1)(2)}, \end{cases}$$

and// $$M(b \leftarrow \beta) = (-) M^*(c \leftarrow \alpha),\ M(c \leftarrow \beta) = M(b \leftarrow \alpha),\ M(d \leftarrow \beta) = (-) M^*(a \leftarrow \alpha). \tag{3}$$

---

3. If we call $\mathcal{E}_\perp^2 = \mathcal{E}_x^2 + \mathcal{E}_y^2$, $\mathcal{E}_\parallel^2 = \mathcal{E}_z^2$, then the $\Gamma = |M|^2$ values are...

$$\begin{cases} \Gamma(a \leftarrow \alpha) = \frac{3}{2} |R|^2 \mathcal{E}_\perp^2 = \Gamma(d \leftarrow \beta), \\ \Gamma(b \leftarrow \alpha) = 2 |R|^2 \mathcal{E}_\parallel^2 = \Gamma(c \leftarrow \beta), \\ \Gamma(c \leftarrow \alpha) = \frac{1}{2} |R|^2 \mathcal{E}_\perp^2 = \Gamma(b \leftarrow \beta); \end{cases}$$

so// $$\boxed{\begin{aligned} &\Gamma(a \leftarrow \alpha) = \Gamma(d \leftarrow \beta),\ [\text{via } \mathcal{E}_\perp] \\ &\qquad = 3\Gamma(c \leftarrow \alpha) = 3\Gamma(b \leftarrow \beta); \\ \text{and}//\ &\Gamma(b \leftarrow \alpha) = \Gamma(c \leftarrow \beta),\ [\text{via } \mathcal{E}_\parallel]. \end{aligned}} \tag{4}$$