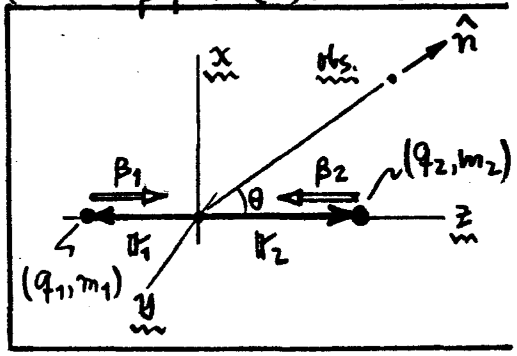
● Do Jackson's Problem (15.5), p. 734, parts (a) & (b). Skip part (c). This problem is blessedly nonrelativistic, and the point of it is to see what happens to the radiation spectrum when charge (q_1, m_1) scatters from a center which is not fixed, e.g. a charge (q_2, m_2) where it may be that $m_2 \sim m_1$. Let the particles collide along the z-axis as shown. Can you reduce the integral in part (b)?



● In class, we showed that the 1D wave equation: $u_{xx} - u_{\tau\tau} = 0$ (with x the space coordinate and $\tau = vt$ the time) could be solved in general by:

$u(x,\tau) = f(x-\tau) + g(x+\tau)$. The arbitrary functions f and g are usually fixed by initial and/or boundary conditions.

Show that for the initial value problem, where at time $\tau = 0$ both the amplitude $u(x,0) = u_0(x)$ and its derivative $u_\tau(x,0) = v_0(x)$ are specified [i.e. u_0 & v_0 are given functions of x], the 1D wave solution (on an ∞ domain) is:

$$\boxed{u(x,\tau) = \frac{1}{2}\left[u_0(x-\tau) + u_0(x+\tau)\right] + \frac{1}{2}\int_{x-\tau}^{x+\tau} v_0(\xi)\,d\xi\,.}$$

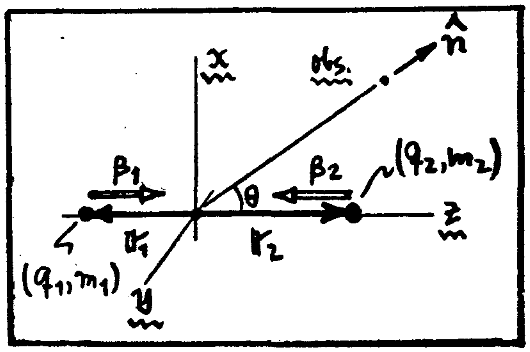
● A type of equation which often appears in wave propagation problems is of the form: $\boxed{P_{tt} + 2\beta P_t + \omega_0^2 P = \omega_p^2 E(\mathbf{r},t)}$, where β, ω_0 & ω_p are cnsts. We want a particular solution for $P(\mathbf{r},t)$ when the driving field $E(\mathbf{r},t)$ is arbitrary.

(A) Use Fourier Transforms: $P(\mathbf{r},t) \to \tilde{P}(\mathbf{r},\omega) = \int_{-\infty}^{\infty} P(\mathbf{r},t)\,e^{-i\omega t}\,dt$, etc., to show that $\tilde{P}$ and $\tilde{E}$ are related by: $\tilde{P} = \omega_p^2 \tilde{E} / [(\omega_0^2 - \omega^2) + 2i\beta\omega]$.

(B) Invert the transform of part (A) and show [Contour integration is easiest] that:

$P(\mathbf{r},t) = \int_0^{t \text{ (or } \infty)} K(\tau)\,E(\mathbf{r},t-\tau)\,d\tau$ is the desired particular solution. $K(\tau)$ is the "kernel" of the P_{tt} equation. Find $K(\tau)$ explicitly, and sketch $K(\tau)$ vs. τ. Interpret your result by clever commentary.

Prob^m (43) [Jk^n # (15.5)]. Coulomb collision for finite m's.



(A) **1.** Non-relativistic version of Jk^n Eq. (15.1) for q_1:

$$\frac{d^2 I_1}{d\omega\, d\Omega} = \frac{q_1^2}{4\pi^2 c}\left|\int_{\text{coll}^n} [\hat{n}\times(\hat{n}\times\dot{\boldsymbol{\beta}}_1)]\, e^{i\omega\{t - \frac{1}{c}\hat{n}\cdot\mathbf{r}_1(t)\}}\, dt\right|^2. \quad (1)$$

There is a similar expression for q_2. $\mathbf{r}_1$ is the position cd. relative to the system CM (center-of-mass), defined by

$$\mathbf{r}_1 = +(\mu/m_1)\,\mathbf{r} \qquad \text{w/}\ \ \mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2, \text{ relative cd.}$$
$$\mathbf{r}_2 = -(\mu/m_2)\,\mathbf{r} \qquad \mu = m_1 m_2/(m_1+m_2), \text{ reduced mass.} \quad (2)$$

From this def^n: $\boldsymbol{\beta}_1 = \frac{1}{c}\dot{\mathbf{r}}_1 = +(\mu/m_1 c)\,\dot{\mathbf{r}}$, $\boldsymbol{\beta}_2 = -(\mu/m_2 c)\,\dot{\mathbf{r}}$. Then for q_1...

$$\frac{d^2 I_1}{d\omega\, d\Omega} = \frac{\mu^2}{4\pi^2 c^3}\left|\frac{q_1}{m_1}\int_{\text{coll}^n} [\hat{n}\times(\hat{n}\times\ddot{\mathbf{r}})]\, e^{-i\omega t}\, e^{i\omega(\mu/m_1 c)\hat{n}\cdot\mathbf{r}(t)}\, dt\right|^2. \quad (3)$$

We've taken the complex conjugate of the integrand to produce the $e^{-i\omega t}$.

2. The combined radiation from q_1 & q_2 is coherent, i.e. it is the square of the sum of amplitudes as in Eq. (3), rather than the sum of the amplitudes squared. So...

$$\boxed{\frac{d^2 I}{d\omega\, d\Omega} = \frac{\mu^2}{4\pi^2 c^3}\left|\int_{\text{coll}^n} e^{-i\omega t}(\hat{n}\times\ddot{\mathbf{r}})\left[\left(\frac{q_1}{m_1}\right) e^{i\left(\frac{\omega\mu}{m_1 c}\right)\hat{n}\cdot\mathbf{r}(t)} \overset{①}{-} \left(\frac{q_2}{m_2}\right) e^{\overset{②}{-} i\left(\frac{\omega\mu}{m_2 c}\right)\hat{n}\cdot\mathbf{r}(t)}\right] dt\right|^2.} \quad (4)$$

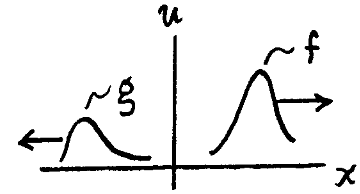
Minus sign ① follows from $\dot{\boldsymbol{\beta}}_2 = (-)(\mu/m_2 c)\,\ddot{\mathbf{r}}$; ② follows from $\mathbf{r}_2 = -(\mu/m_2)\,\mathbf{r}$.
We have used the fact that $[\hat{n}\times\ddot{\mathbf{r}}]^2 = [\hat{n}\times(\hat{n}\times\ddot{\mathbf{r}})]^2$ for $\ddot{\mathbf{r}}$ along z-axis.

(B) **3.** In low frequency limit, and for $q_1/m_1 = q_2/m_2$, Eq. (4) reduces to...

$$\left[\!\left[\frac{d^2 I}{d\omega\, d\Omega} = \frac{\omega^2}{4\pi^2 c^5}\left[\frac{q_1\mu^2}{m_1^2} + \frac{q_2\mu^2}{m_2^2}\right]^2 \left|\int_{\text{coll}^n} e^{-i\omega t}(\hat{n}\times\ddot{\mathbf{r}})(\hat{n}\cdot\mathbf{r})\, dt\right|^2,\right]\!\right] \quad (5)$$

straightforwardly. The usual dipole term (with just $\hat{n}\times\ddot{\mathbf{r}}$ in the integral) is not present. In fact the resulting integral has the structure for quadrupole radiation.

Prob^{m} (44) Complete D'Alembert's initial value solution to 1D wave equation.



1. $u_{xx} - u_{\tau\tau} = 0$ has general solution:

$$u(x,\tau) = f(x-\tau) + g(x+\tau). \qquad (1)$$

We suppose the initial values are fixed (at $\tau=0$):

$$u(x,0) = f(x) + g(x) = u_0(x), \qquad (2)$$

$$\left.\frac{\partial}{\partial\tau} u(x,\tau)\right|_{\tau=0} = -f'(x) + g'(x) = v_0(x); \qquad (3)$$

u_0 & v_0 are given fcns of x.

2. Integrate through Eq. (3) to get: $g(x) - f(x) = \int^x v_0(s)\,ds$, and combine this with Eq. (2): $g(x) + f(x) = u_0(x)$, to solve for $f(x)$ & $g(x)$...

$$f(x) = \tfrac{1}{2}\Big[u_0(x) - \int^x v_0(s)\,ds\Big],\quad g(x) = \tfrac{1}{2}\Big[u_0(x) + \int^x v_0(s)\,ds\Big]. \qquad (4)$$

The lower limit on the integral is arbitrary... call it a. Then, from (4)...

$$f(x-\tau) = \tfrac{1}{2}\Big[u_0(x-\tau) - \int_a^{x-\tau} v_0(s)\,ds\Big],$$

$$g(x+\tau) = \tfrac{1}{2}\Big[u_0(x+\tau) + \int_a^{x+\tau} v_0(s)\,ds\Big]; \qquad (5)$$

$$u(x,\tau) = f(x-\tau) + g(x+\tau) = \tfrac{1}{2}\big[u_0(x-\tau) + u_0(x+\tau)\big] +$$

$$+ \tfrac{1}{2}\Big(\int_a^{x+\tau} - \int_a^{x-\tau}\Big) v_0(s)\,ds. \qquad (6)$$

But $\Big(\int_a^{x+\tau} - \int_a^{x-\tau}\Big) \equiv \int_{x-\tau}^{x+\tau}$. Then, as advertised...

$$\boxed{u(x,\tau) = \tfrac{1}{2}\big[u_0(x-\tau) + u_0(x+\tau)\big] + \tfrac{1}{2}\int_{x-\tau}^{x+\tau} v_0(\xi)\,d\xi,} \qquad (7)$$

Prob$^{\underline{m}}$ (45) Solve the SHO polarization model: $\underline{\underline{P_{tt} + 2\beta P_t + \omega_0^2 P = \omega_p^2 E(x,t)}}$, for $\underline{\text{any}}$ E.

(a) For a Fourier Transform: $\tilde{F}(\omega) = \int_{-\infty}^{\infty} F(t)\, e^{-i\omega t}\, dt$, repeated partial integrations show that: $\int_{-\infty}^{\infty} [\partial^n F(t)/\partial t^n]\, e^{-i\omega t}\, dt = (i\omega)^n \tilde{F}(\omega)$, with $F(t=\pm\infty) \equiv 0$ assumed. Then the Fourier transformed P_{tt} yields immediately (with $\tilde{P}$ & $\tilde{E}$ the F.T.'s of P & E)...

$$-\omega^2 \tilde{P} + 2i\beta\omega \tilde{P} + \omega_0^2 \tilde{P} = \omega_p^2 \tilde{E} \;\rightarrow\; \boxed{\tilde{P}(x,\omega) = \omega_p^2\, \tilde{E}(x,\omega)/[(\omega_0^2 - \omega^2) + 2i\beta\omega].} \quad (1)$$

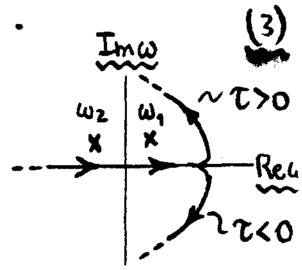
NOTE: x is just a spectator variable. Jk$^{\underline{n}}$ Eq. (7.50) is the monochromatic (fixed ω) version of this; he has chosen $\omega = (-)\,\omega\,[\text{us}]$, and $\gamma = 2\beta\,[\text{us}]$.

(b) The Fourier inverse of $\tilde{P}$ is the desired particular integral; it is...

$$\rightarrow P(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{P}(x,\omega)\, e^{i\omega t}\, d\omega = \frac{\omega_p^2}{2\pi}\int_{-\infty}^{\infty} [(\omega_0^2 - \omega^2) + 2i\beta\omega]^{-1}\, \tilde{E}(x,\omega)\, e^{i\omega t}\, d\omega. \quad (2)$$

Put in: $\tilde{E}(x,\omega) = \int_{-\infty}^{\infty} E(x,t')\, e^{-i\omega t'}\, dt'$, and rearrange terms to write...

$$\rightarrow P(x,t) = \int_0^{\infty} dt'\, K(t-t')\, E(x,t'), \quad \text{w/}\; \underline{K(\tau)} = \frac{\omega_p^2}{2\pi}\int_{-\infty}^{\infty} \frac{e^{i\omega\tau}\, d\omega}{(\omega_0^2 - \omega^2) + 2i\beta\omega}. \quad (3)$$



Evaluate $K(\tau)$ by contour integration. Note the integrand has two simple poles in the upper half ω-plane, @: $\omega^2 - 2i\beta\omega - \omega_0^2 = 0 \Rightarrow$ $\omega = \omega_{1,2} = \pm\omega_r + i\beta$, where $\underline{\omega_r} = \sqrt{\omega_0^2 - \beta^2}$ is the damped SHO resonant frequency. When $\tau < 0$, contour is closed in lower half-plane (why?); then $K(\tau<0) \equiv 0$. For $\tau > 0$, closure in the upper half-plane + Residue Theorem yields the results...

$$\boxed{K(\tau) = \frac{\omega_p^2}{\omega_r}\,\theta(\tau)\, e^{-\beta\tau} \sin\omega_r\tau}, \quad \xrightarrow{\text{and}} \underline{P(\mathbf{r},t) = \int_0^{\infty} d\tau\, K(\tau)\, E(\mathbf{r}, t-\tau)}. \quad (4)$$

∞ ~ if upper limit $= t$, then $E \equiv 0$ @ $t < 0$.

$\theta(\tau)$ is the unit step fcn: $\theta(\tau) = \begin{cases} 1, & \tau > 0 \\ 0, & \tau < 0 \end{cases}$. The spectator cd x is generalized to $\mathbf{r}$.

$K(\tau)$ vs τ is sketched at right; it is the system response to a δ-fcn impulse at $\tau = 0$. In the integral for the polarization P, what K "does" is to gather all elements of E in the past (i.e. $t - \tau < t$) which have excited oscillations, and combine them to form P at time t. $K(\tau<0) \equiv 0$ is required by causality.