(76) [20 pts]. In Sec. 12.9, Jackson quotes the Proca Lagrangian [Eq. (12.91)], and derives Proca's wave eqtn for a massive photon field [Eq. (12.93)]. Along the way, he claims that the "Lorentz gauge is now required for current conservation."

(A) Show why Jackson's claim about the Lorentz gauge is justified.

(B) Assume the initial choice for $\mathcal{L}_P$ is the Lorentz gauge: $\partial_\alpha A^\alpha = 0$, so the current J_α is conserved: $\partial_\alpha J^\alpha = 0$. Now do a gauge transformation: $A_\alpha \rightarrow A'_\alpha = A_\alpha + \partial_\alpha G$. What condition on the gauge fcn G is needed to maintain current conservation? (The theory is ~nonsense w/o conserved currents!). What gauge are you in now?

(C) With the current-conserving gauge transforms permitted in part (B), show the transformed Lagrange density is: $\underline{\mathcal{L}'_P = \mathcal{L}_P - \partial^\alpha U_\alpha}$, w/ U_α a vector field depending on J_α & A_α. Find U_α explicitly. Are $\mathcal{L}'_P$ & $\mathcal{L}_P$ "gauge equivalent" in the sense of probm (72)?

(D) Draw a conclusion regarding the gauge freedom of $\mathcal{L}_P$ (as prototype for massive vector fields).

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(77) [Jkn # 12.14, 20 pts]. Consider how a nonzero photon mass $\mu = m_\gamma c/\hbar$ alters the magnetic field of the earth. Use existing satellite data to establish an upper limit on m_γ. Do the problem as stated in Jackson, pp. 616-617.

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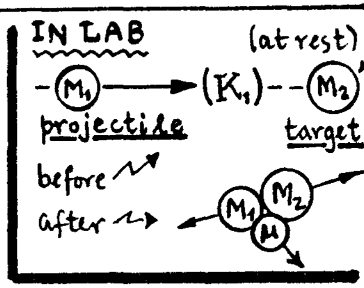
(78) Recall the results of probm (73), for the center-of-momentum description of a relativistic collision between a projectile of rest energy $M_1 = m_1 c^2$ and initial lab kinetic energy K_1, and a target of rest energy $M_2 = m_2 c^2$ initially at rest in lab. Suppose the $M_1 \rightarrow M_2$ collision leaves these particles intact and also produces a new particle of rest energy μ.



(A) The minimum CM energy at which μ is produced is: $E_{cm} = M_1 + M_2 + \mu$; say why. Use this E_{cm} to find the minimum lab energy or "threshold", K_1 (min.) for μ-production.

(B) At threshold, find the kinetic energy K_μ (min.) at which μ first appears in lab.

(C) What is K_1(min.) for proton-antiproton production via: $p + p \rightarrow p + p + (p + \bar{p})$? At what kinetic energy does the $\bar{p}$ first appear in lab?

(D) The "efficiency" of this μ-production process can be measured by $\epsilon = \mu / K_1$. Find ϵ, and consider the case where: M_2 (target) = M(proton), $\mu = 2M$ for a $p\bar{p}$ pair. Is it more efficient to use protons or electrons for M_1 (projectile)? What argues against using e's?

(76) [20 pts.]. Analyse gauge constraints on Proca Lagrangian.

(A) 1. The Proca Lagrangian is Jk^n Eq. (12.91):

$$\rightarrow \mathcal{L}_P = -\frac{1}{16\pi} F_{\alpha\beta} F^{\alpha\beta} - \frac{1}{c} J_\alpha A^\alpha + \frac{\mu^2}{8\pi} A_\alpha A^\alpha, \quad \mu = \frac{m_\gamma c}{\hbar} = \text{photon mass.} \qquad (1)$$

The eqtns-of-motion (i.e. $\partial^\beta[\partial\mathcal{L}_P/\partial(\partial^\beta A^\alpha)] = \partial\mathcal{L}_P/\partial A^\alpha$) are Eq. (12.92):

$$\rightarrow \partial^\beta F_{\beta\alpha} + \mu^2 A_\alpha = \frac{4\pi}{c} J_\alpha, \quad \text{w/ } F_{\beta\alpha} = \partial_\beta A_\alpha - \partial_\alpha A_\beta. \qquad (2)$$

To see how current J_α is conserved, operate through Eq. (2) by ∂^α, so that

$$\rightarrow \partial^\alpha \partial^\beta F_{\beta\alpha} + \mu^2 \partial^\alpha A_\alpha = \frac{4\pi}{c} \partial^\alpha J_\alpha. \qquad (3)$$

It is easy to show that with $F_{\beta\alpha}$ the antisymmetric field tensor defined in (2), the first term LHS in (3) vanishes: $\partial^\alpha\partial^\beta F_{\beta\alpha} \equiv 0$.★ Then (3) yields

$$\rightarrow \underline{\underline{\partial^\alpha J_\alpha = (\mu^2 c/4\pi)\,\partial^\alpha A_\alpha}}, \quad \text{so/ current is conserved: } \underline{\underline{\partial^\alpha J_\alpha = 0}}, \text{ only when } \underline{\underline{\partial^\alpha A_\alpha = 0}} \leftrightarrow \text{Lorentz gauge.} \qquad (4)$$

This justifies Jk^{n}'s claim that the Lorentz gauge is <u>required</u> for current consvn.

(B) 2. Under a gauge transform: $A_\alpha \rightarrow A'_\alpha = A_\alpha + \partial_\alpha G$, <u>the field tensor $F_{\beta\alpha}$ remains unchanged</u>. The field eqtns (2) become: $\partial^\beta F_{\beta\alpha} + \mu^2[A_\alpha + \partial_\alpha G] = \frac{4\pi}{c} J_\alpha$, and operation through this eqtn by ∂^α produces the counterpart of Eq. (4):

$$\rightarrow \partial^\alpha J_\alpha = (\mu^2 c/4\pi)[\partial^\alpha A_\alpha + (\partial^\alpha\partial_\alpha) G]. \qquad (5)$$

If the original gauge was Lorentz, then $\partial^\alpha A_\alpha = 0$. According to (5), current is conserved in the new gauge $(A_\alpha + \partial_\alpha G)$ only if the gauge function is restricted:

$$\rightarrow \underline{\underline{(\partial^\alpha\partial_\alpha) G = \left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right) G = 0}} \Rightarrow G = \text{free field scalar.} \qquad (6)$$

But then we are still <u>in</u> the Lorentz gauge (Jk^n p. 221), since $\partial^\alpha A'_\alpha = 0$ also. In any case, current <u>can</u> be conserved for $\mathcal{L}_P$, for gauge fcns G obeying Eq. (6).

★ $$\partial^\alpha\partial^\beta F_{\beta\alpha} \overset{\text{relabel indices}}{=} \partial^\beta\partial^\alpha F_{\alpha\beta} \overset{\text{change diff}^n\text{ order}}{=} +\partial^\alpha\partial^\beta F_{\alpha\beta} \overset{\text{use } F = \text{antisym.}}{=} -\partial^\alpha\partial^\beta F_{\beta\alpha}, \text{ so } \partial^\alpha\partial^\beta F_{\beta\alpha} \equiv 0.$$

3. When $A_\alpha \to A'_\alpha = A_\alpha + \partial_\alpha G$, the Proca Lagrangian in Eq. (1) becomes...

(C) $$\to \mathcal{L}'_p = -\frac{1}{16\pi} F_{\alpha\beta} F^{\alpha\beta} - \frac{1}{c} J_\alpha (A^\alpha + \partial^\alpha G) + \frac{\mu^2}{8\pi}(A_\alpha + \partial_\alpha G)(A^\alpha + \partial^\alpha G)$$

$$= \mathcal{L}_p - \frac{1}{c} J_\alpha \underbrace{\partial^\alpha G}_{4} + \frac{\mu^2}{8\pi}\left[A_\alpha \underbrace{\partial^\alpha G}_{3} + A^\alpha \underbrace{\partial_\alpha G}_{2} + \underbrace{(\partial_\alpha \partial^\alpha) G}_{1}\right]. \tag{7}$$

We've used the fact that $F_{\alpha\beta}$ is invariant under the gauge transform, and have gathered together the terms that form $\mathcal{L}_p$. Term 1 $\equiv 0$ for current conservation [Eq. (6) above], terms 2 & 3 combine to give $2A^\alpha \partial_\alpha G$, and term 4 can be written as: $J_\alpha \partial^\alpha G = \partial^\alpha (J_\alpha G) - (\cancel{\partial^\alpha J_\alpha})^{0} G$ (current conservation again). Then...

$$\to \mathcal{L}'_p = \mathcal{L}_p - \frac{1}{c}\partial^\alpha (J_\alpha G) + \frac{\mu^2}{4\pi} A_\alpha \partial^\alpha G. \tag{8}$$

For the third term on the RHS here, we can write

$$A_\alpha \partial^\alpha G = \partial^\alpha (A_\alpha G) - (\cancel{\partial^\alpha A_\alpha})^{0} G, \text{ by Lorentz gauge.} \tag{9}$$

So// $$\boxed{\mathcal{L}'_p = \mathcal{L}_p - \partial^\alpha\left[\left(\frac{1}{c} J_\alpha - \frac{\mu^2}{4\pi} A_\alpha\right) G\right]}; \ \alpha = 0,1,2,3. \tag{10}$$

4. Let $U_\alpha = \left(\frac{1}{c} J_\alpha - \frac{\mu^2}{4\pi} A_\alpha\right) G$, and integrate $\mathcal{L}'_p$ of Eq. (10) over a hypervolume with invariant volume element $d^4x = dx^0 dx^1 dx^2 dx^3$, with the $x^0 = ct$ cd. ranging from time t_1 to t_2 (fixed endpoints of the motion). Then the <u>action</u> is

$$\to \mathcal{A}'_p = \int_1^2 \mathcal{L}'_p \, d^4x = \mathcal{A}_p - \int_1^2 (\partial^\alpha U_\alpha)\, d^4x. \tag{11}$$

The $\alpha = 0$ term in the integral gives just: $\int d^3x \, U_0 \Big|_{t_1}^{t_2}$, fixed at the endpts; it contributes nothing to the variation $\delta\mathcal{A}'_p$. The $\alpha = 1,2,3$ terms can -- by Gauss' Theorem -- be transformed to integrals over hypersurfaces at ∞, where they vanish. Thus we get $\delta\mathcal{A}'_p = \delta\mathcal{A}_p = 0$ together, and $\mathcal{L}'_p$ & $\mathcal{L}_p$ of Eq. (10) <u>are</u> gauge equivalent... they will give the same eqtns-of-motion. We can state:

(D)

For current conservation (and gauge equivalence), $\mathcal{L}_p$ is totally restricted to the Lorentz gauge ($\partial^\alpha A_\alpha = 0$). Any further gauge freedom requires additional terms for $\mathcal{L}_p$.

(77) [Jk^{n} # 12.14, 20 pts]. Analyse changes in $\mathbb{B}$(earth) due to m(photon) > 0, per Proca.

(A) The steady-state Proca Eq. is : $(\nabla^2 - \mu^2) A^\alpha = -(4\pi/c) J^\alpha$. If we momentarily put $\mu = ik$, then the Green's fcn for this eqtn is : $G_k = \frac{1}{R} e^{\pm ikR}$ [by Jk^{n} Eq.(6.62)], w/ $R = |\mathbf{r} - \mathbf{r}'|$ the field pt - source pt separation. Now put $ik = \mu$, and choose the exponentially <u>damped</u> Green's fcn : $G_\mu = \frac{1}{R} e^{-\mu R}$, and solution :

$$\rightarrow A^\alpha(\mathbf{r}) = \int \frac{d^3x'}{R} e^{-\mu R} \cdot \frac{1}{c} J^\alpha(\mathbf{r}'), \quad \underline{\mu = m_\gamma c/\hbar = \text{photon Compton wavelength}}. \quad (1)$$

Suppose $\phi = 0$, and : $\mathbb{J} = c \nabla \times \mathbb{M}$, w/ $\mathbb{M} = \mathbb{m} f$ and $\mathbb{m}$ a cnst vector. Then...

$$\rightarrow \mathbb{A}(\mathbf{r}) = -\mathbb{m} \times \int d^3x' \left(\frac{e^{-\mu R}}{R}\right) \nabla' f(\mathbf{r}'), \quad (2)$$

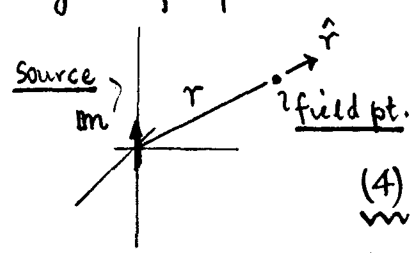
is the vector potential [we've used : $\nabla \times (\mathbb{m} f) = (\nabla f) \times \mathbb{m}$, for $\mathbb{m} = $ cnst]. Now, note that : $F(R) \nabla' f(\mathbf{r}') = \nabla'[F(R) f(\mathbf{r}')] - f(\mathbf{r}') \nabla' F(R)$, and $\nabla' F(R) = -\nabla F(R)$, for $F(R) = \frac{1}{R} e^{-\mu R}$ (or in fact for any fcn of $R = |\mathbf{r} - \mathbf{r}'|$). Then, with ∇ acting on the <u>field</u> pt. $\mathbf{r}$ (so that $-f(\mathbf{r}') \nabla' F(R) = \nabla[f(\mathbf{r}') F(R)]$) :

$$\boxed{\mathbb{A}(\mathbf{r}) = -\mathbb{m} \times \nabla \int d^3x' \left(\frac{e^{-\mu R}}{R}\right) f(\mathbf{r}')} - \underbrace{\cancel{\mathbb{m} \times \int d^3x' \, \nabla' \left[\left(\frac{e^{-\mu R}}{R}\right) f(\mathbf{r}')\right]}}_{\to 0} \quad (3)$$

The 1st term RHS in (3) gives the required $\mathbb{A}$. The 2nd term RHS integrates out to the values $[(e^{-\mu R}/R) f(\mathbf{r}')]_{\mathbf{r}' \to \infty}$, which vanish by assumption.

(B) For a point dipole at the origin, $f(\mathbf{r}') = \delta(\mathbf{r}')$, and Eq.(3) gives for ptl. $\mathbb{A}$...

$$\mathbb{A}(r) = -\mathbb{m} \times \nabla \left(\frac{e^{-\mu r}}{r}\right) = -(\mathbb{m} \times \hat{r}) \frac{\partial}{\partial r} \left(\frac{e^{-\mu r}}{r}\right)$$



or $\underline{\underline{\mathbb{A}(r) = \mathbb{G}(r) \times \hat{r}}}$, w/ $\mathbb{G}(r) = \mathbb{m}\, g(r)$, $\hat{r} = \mathbf{r}/r$, and : $\underline{\underline{g(r)}} = \frac{1}{r^2}(1 + \mu r) e^{-\mu r}$. (4)

Now we need : $\mathbb{B} = \nabla \times \mathbb{A} = \nabla \times [\mathbb{G}(r) \times \hat{r}]$. By the usual messy identity (inside Jackson's front cover)...

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$$\to \mathbf{B}(\mathbf{r}) = \overbrace{\mathbf{G}(\nabla\cdot\hat{r})}^{\textcircled{1}} - \overbrace{(\mathbf{G}\cdot\nabla)\hat{r}}^{\textcircled{2}} - \overbrace{\hat{r}(\nabla\cdot\mathbf{G})}^{\textcircled{3}} + \overbrace{(\hat{r}\cdot\nabla)\mathbf{G}}^{\textcircled{4}}. \qquad (5)$$

For term ①, we straightforwardly calculate: $\nabla\cdot\hat{r} = 2/r$. For the other terms in (5)...

$$[\textcircled{2}]_x = \left(G_x\frac{\partial}{\partial x} + G_y\frac{\partial}{\partial y} + G_z\frac{\partial}{\partial z}\right)\frac{x}{r}, \quad \text{and}\quad \frac{\partial}{\partial x}\left(\frac{x}{r}\right) = \frac{1}{r} - \frac{x^2}{r^3},\ \frac{\partial}{\partial y}\left(\frac{x}{r}\right) = -\frac{xy}{r^3}\ \text{(etc.)}$$

$$= \frac{G_x}{r} - \frac{x}{r^3}(\mathbf{r}\cdot\mathbf{G}) = \frac{1}{r}\left[\mathbf{G} - \hat{r}(\hat{r}\cdot\mathbf{G})\right]_x; \qquad (6A)$$

so $$\textcircled{1} - \textcircled{2} = \frac{1}{r}\left[\mathbf{G} + \hat{r}(\hat{r}\cdot\mathbf{G})\right] = \frac{g}{r}\left[\mathbf{m} + \hat{r}(\hat{r}\cdot\mathbf{m})\right]. \qquad (6B)$$

Also, for ③: $$\nabla\cdot\mathbf{G} = \nabla\cdot(\mathbf{m}g) = \mathbf{m}\cdot\nabla g + g\nabla\cdot\mathbf{m} = (\mathbf{m}\cdot\hat{r})\frac{\partial g}{\partial r}; \qquad (6C)$$

and $$[\textcircled{4}]_x = \left(\frac{x}{r}\frac{\partial}{\partial x} + \frac{y}{r}\frac{\partial}{\partial y} + \frac{z}{r}\frac{\partial}{\partial z}\right)m_x g = m_x\frac{\partial g}{\partial r} = \left(\mathbf{m}\frac{\partial g}{\partial r}\right)_x. \qquad (6D)$$

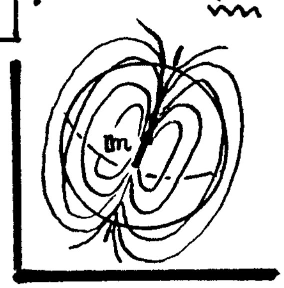
Combine 6B, C & D to form $\mathbf{B}$ of Eq. (5), with result...

$$\to \mathbf{B}(\mathbf{r}) = \left(\frac{g}{r} - \frac{\partial g}{\partial r}\right)\hat{r}(\mathbf{m}\cdot\hat{r}) + \left(\frac{g}{r} + \frac{\partial g}{\partial r}\right)\mathbf{m}, \qquad (7)$$

w/ $g(r) = \frac{1}{r^2}(1+\mu r)e^{-\mu r}$ [Eq. (4)], so $\frac{\partial g}{\partial r} = (-)\frac{1}{r^3}\left[2(1+\mu r) + \mu^2 r^2\right]e^{-\mu r}$.

Calculating the coefficients in g in Eq. (7), we find as required...

$$\boxed{\mathbf{B}(\mathbf{r}) = \left[3\hat{r}(\mathbf{m}\cdot\hat{r}) - \mathbf{m}\right]\left(1 + \mu r + \tfrac{1}{3}\mu^2 r^2\right)\frac{e^{-\mu r}}{r^3} - \frac{2}{3}\mu^2\,\mathbf{m}\,\frac{e^{-\mu r}}{r}}. \qquad (8)$$



(C) At the equator, $\mathbf{m}\cdot\hat{r} = 0$, and with R_e = earth radius, (8) gives:

$$\to B = -\frac{m e^{-x}}{R_e^3}\Big[\underbrace{(1 + x + \tfrac{1}{3}x^2)}_{\text{earth fld}} - \underbrace{\tfrac{2}{3}x^2}_{\text{added fld}}\Big], \quad \text{with: } x = \mu R_e. \qquad (9)$$

The ratio of added field to earth's field is $< 4\times10^{-3}$, i.e.

$$B(\text{added})/B(\text{earth}) = \tfrac{2}{3}x^2/(1 + x + \tfrac{1}{3}x^2) < 4\times10^{-3} \Rightarrow \underline{x = \mu R_e < 0.078}. \qquad (10)$$

This establishes the lower limit: $\mu^{-1} > R_e/0.078 = 12.8$ earth radii $= 81.7\times10^6$ m. In turn, the photon mass is: $\boxed{m_\gamma = \frac{\hbar}{c}/\mu^{-1} < \frac{\hbar/c}{8.17\times10^9\,\text{cm}} = 4.3\times10^{-48}\,\text{gm}}$

This agrees with (and justifies) Jackson's claim on p. 6, regarding m_γ. It is remarkable that you can weigh a photon with your compass!

ϕ520 Solutions

78 Establish relativistic threshold condition for $M_1+M_2 \to M_1+M_2+\mu$.

(A) μ will appear at a CM threshold: $E_{cm} = M_1+M_2+\mu$, where there is just enough energy to supply its rest mass, with no energy "wasted" in any relative motion of the particles. Then, since $E_{cm}^2 = M_1^2+M_2^2+2E_1M_2$, and $E_1 = K_1+M_1$,* have

$$(M_1+M_2+\mu)^2 = M_1^2+M_2^2+2(K_1+M_1)M_2, \text{ at } \mu \text{ threshold;}$$

so $\boxed{K_1(\min) = \frac{\mu}{M_2}\left(M_1+M_2+\frac{\mu}{2}\right)}$, min. K.E. of M_1 for μ-production. (1)

(B) At threshold, μ appears at rest in CM frame, which is moving at V_{cm} in lab. μ's kinetic energy will be: $K_\mu = (\Gamma_{cm}-1)\mu$, w/ CM dilation factor...

$$\rightarrow \Gamma_{cm} = (E_1+M_2)/E_{cm} = \frac{K_1+M_1+M_2}{\mu+M_1+M_2} \Rightarrow \Gamma_{cm}-1 = \frac{K_1-\mu}{\mu+M_1+M_2},$$

@ $K_1 = K_1(\min)$ of Eq.(1) above $\Big\}$ $\Gamma_{cm}-1 = \frac{\mu}{2M_2}\left(\frac{2M_1+\mu}{M_1+M_2+\mu}\right)$. (2)

so at threshold: $\boxed{K_\mu = \left(\frac{2M_1+\mu}{M_1+M_2+\mu}\right)\frac{\mu^2}{2M_2}}$ $\Big\{$ lab K.E. of μ at production threshold. (3)

(C) For: $p+p \to p+p+(p+\bar{p})$, have: $M_1(\text{projectile}) = M_2(\text{target}) = M(\text{proton mass}) = 938$ MeV, and $\mu = 2M$ for the $p\bar{p}$ pair produced. Eq.(1) $\Rightarrow$ $K_1(\min) = 6M = 5.63$ GeV, and Eq (3) $\Rightarrow$ $K_\mu = 2M$. Since K_μ is equally shared between the p & $\bar{p}$ produced, the $\bar{p}$ first appears at lab K.E.: $K(\bar{p}) = M = 938$ MeV.

(D) Using Eq.(1): $\epsilon = \mu/K_1(\min.) = M_2/(M_1+M_2+\frac{\mu}{2})$, and for $M_2 = \frac{\mu}{2} = M$, have:

$\boxed{\epsilon = M/(M_1+2M)}$ (4) For projectile $M_1 = M(\text{proton})$, have $\epsilon = 33\%$, while for $M_1 = m(\text{electron}) \ll M$, have $\epsilon \simeq 50\%$, so electrons are more efficient. What militates against using electrons is that they are ultrarelativistic here, so they suffer much greater radiation losses. The projectile γ is: $K_1 = (\gamma-1)M_1 \Rightarrow \gamma = 1+(K_1/M_1)$,

so $\gamma = 1+\frac{\mu}{M_1M_2}\left(M_1+M_2+\frac{\mu}{2}\right) = 1+2\left(1+\frac{2M}{M_1}\right)$, here $\Rightarrow$ $\gamma(\text{protons}) = 7$, $\gamma(\text{electrons}) = 7350$.

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* We are using results from prob^m 73. The "masses" are all rest energies: $M_1 = m_1c^2$, etc.