Φ 520 Problems { Set # 1 ; Probs. 36-38. Assigned 1/13/89; due 1/20/89.

NOTE! : Problems are graded at 10 pts. each, unless indicated otherwise.

[illegible] [Jackson Prob. (14.2)]. Using the Larmor formulas for the nonrelativistic motion of a point charge q, find the time-averaged quantities : $\langle dP/d\Omega\rangle$ = power radiated per unit solid angle, and $\langle P\rangle$ = total power radiated, when q moves as follows...

(A)... along the z-axis with instantaneous position : $z(t) = R\cos\omega_0 t$ (R & ω_0 = cnsts);

(B)... in a circle of radius R in the xy plane, at $\measuredangle$ frequency ω_0 (R & ω_0 = cnsts).

In each case, sketch the angular distribution of the radiation. Is there a significant difference in $\langle P\rangle$ for the linear motion vs. the circular motion?

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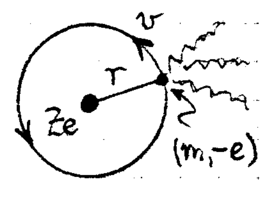
37 [Jackson Prob. (14.3)]. A nonrelativistic particle of mass m, charge ze, and initial kinetic energy K collides head-on with a fixed central force field. The interaction is repulsive, and is specified by a potential $V(r)$: $V(r)$ increases as the separation r decreases, and $V(r) > K$ for all $r < r_0$ (so r_0 = "closest distance of approach").

(A) Show that the total energy radiated by ze during this encounter is ... $\boxed{\Delta W = \frac{4}{3c}\left(\frac{ze}{mc}\right)^2\sqrt{\frac{m}{2}}\int_{r_0}^{\infty}\left|\frac{dV}{dr}\right|^2\frac{dr}{\sqrt{V(r_0)-V(r)}}}$.

(B) Let the potential be Coulombic : $V(r) = zZe^2/r$. If v_0 is the velocity of ze at infinity, show the radiated energy is : $\boxed{\Delta W = \frac{16}{45}(z/Z)(v_0/c)^3 K \ll K.}$

[illegible] An electron (mass m, charge $(-)e$), in a hydrogenlike atom (stationary nucleus of charge Ze), moves in a circular orbit of radius r. Treat the system classically, and assume the electron velocity $v \ll c$.



(A) Find an expression for the electron's total orbit energy E in terms of r alone.

(B) Assume the electron radiates energy $\Delta E \ll |E|$, per orbit. Find the radiated power P in terms of r alone. Equate P to the rate of loss of orbital energy, to obtain a differential eqtn for the decrease in orbit radius r due to radiation.

(C) Calculate the elapsed time for the electron to spiral into the nucleus if it starts from $r = a_0$. Set $Z=1$, $a_0 = 0.53\,\text{Å}$ (Bohr radius). Calculate a number for the collapse time.

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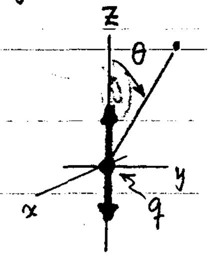
● [Jackson (14.2)] Use Larmor formulas for radiation from SHO & from cyclotron orbit.



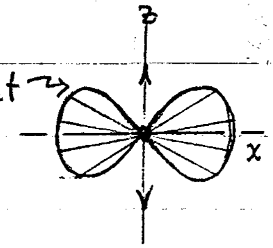
1/20/85

(a) SHO : $z(t) = R\cos\omega_0 t \Rightarrow$ accelⁿ : $a(t) = \ddot{z}(t) = -\omega_0^2 R\cos\omega_0 t$.

So// Jn Eq. (14.21) $\Rightarrow \quad \dfrac{dP}{d\Omega} = \dfrac{q^2}{4\pi c^3}|a|^2\sin^2\theta = \dfrac{(q\omega_0^2 R)^2}{4\pi c^3}[\cos^2\omega_0 t]\sin^2\theta$

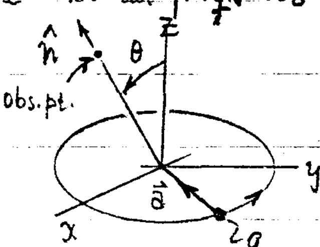
Time average (over many cycles): $\boxed{\left\langle\dfrac{dP}{d\Omega}\right\rangle = \left[\dfrac{(q\omega_0^2 R)^2}{8\pi c^3}\right]\sin^2\theta}$, since $\langle\cos^2\omega_0 t\rangle = \frac{1}{2}$.

The ∡ lar distribution of $\langle dP/d\Omega\rangle$ is the familiar $\sin^2\theta$, as at right →



From Jn Eq. (14.22), the Larmor version of the total radiated power is// $P = \dfrac{2q^2}{3c^3}|a|^2 = \dfrac{2}{3c^3}(q\omega_0^2 R)^2[\cos^2\omega_0 t] \rightarrow \boxed{\langle P\rangle = \dfrac{(q\omega_0^2 R)^2}{3c^3}}$

Notice that the radiation increases as the 4th power of the natural freq. ω_0



(b) Cyclotron Orbit : q's position in xy-plane : $\vec{r} = R(\cos\omega_0 t, \sin\omega_0 t, 0)$.

$\Rightarrow$ accelⁿ : $\vec{a} = \ddot{\vec{r}} = -a(\cos\omega_0 t, \sin\omega_0 t, 0)$, w/ $a = \omega_0^2 R$.

Since problem has cylindrical symmetry about z-axis after time averaging, can locate obs. pt. in xz-plane without loss of generality. Then, unit vector to obs pt is:

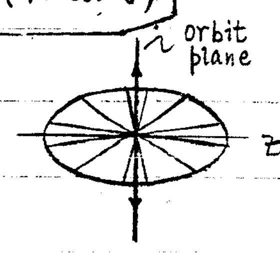
$\hat{n} = (\sin\theta, 0, \cos\theta) \Rightarrow \hat{n}\cdot\vec{a} = -a\cos\omega_0 t\sin\theta = a\cos\Theta$, $\Theta = \measuredangle(\hat{n}, \vec{a})$;

So// $\cos\Theta = -\cos\omega_0 t\sin\theta$, and// $\sin^2\Theta = 1 - [\cos^2\omega_0 t]\sin^2\theta$.

θ is the usual colatitude ∡ as shown. The time-averaged power/solid ∡ is

$$\boxed{\left\langle\frac{dP}{d\Omega}\right\rangle = \frac{q^2}{4\pi c^3}|a|^2\langle\sin^2\Theta\rangle = \frac{(q\omega_0^2 R)^2}{4\pi c^3}\left[1 - \frac{1}{2}\sin^2\theta\right] = \frac{(q\omega_0^2 R)^2}{8\pi c^3}(1+\cos^2\theta)}$$

The ∡ lar distribution is shown at right. Total radiated power is:



$$\boxed{\langle P\rangle = \int_0^\pi \langle dP/d\Omega\rangle \cdot 2\pi\sin\theta\, d\theta = 2\cdot\frac{(q\omega_0^2 R)^2}{3c^3}}$$

Note: twice as much $\langle P\rangle$ for part (b) vs (a).

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20 Jan. 85

37 [Jackson (14.3)]. Calculate rad^{n} energy loss for particle (m, ze) during collision.

(a) Total energy radiated: $\Delta W = \int_{-\infty}^{\infty} P\,dt = \frac{2}{3}\frac{(ze)^2}{m^2c^3}\int_{-\infty}^{\infty} |dp/dt|^2 dt$, via Larmor.

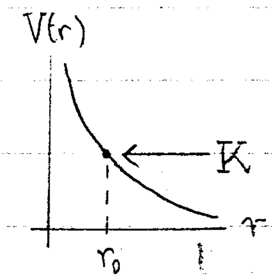
But Newton II $\Rightarrow$ $dp/dt = -dV/dr$. And total energy $E = K + V = V(r_0)$.

This last statement is true if the loss $\Delta W \ll E$. We assume this to be true.

Then plug in $|dp/dt|^2 = |dV/dr|^2$, and change integration variables $dt \to dr$, via

$$dt = \frac{dr}{v}, \quad v = \sqrt{\frac{2K}{m}} \Rightarrow dt = \sqrt{\frac{m}{2}}\, dr \Big/ [V(r_0) - V(r)]^{\frac{1}{2}};$$

Sol$_{/\!/}$
$$\boxed{\Delta W = 2 \cdot \frac{2}{3}\frac{(ze)^2}{m^2c^3}\sqrt{\frac{m}{2}}\int_{r_0}^{\infty}\left|\frac{dV}{dr}\right|^2 dr \Big/ [V(r_0) - V(r)]^{1/2}}$$



(b) Coulomb potential: $V(r) = zZe^2/r$, and: $V(r_0) = K$ (at $r = \infty$),

or $\frac{zZe^2}{r_0} = \frac{1}{2}mv_0^2 \Rightarrow \underline{\underline{r_0}} = 2zZe^2/mv_0^2$, is closest approach.

Also: $|dV/dr|^2 = (zZe^2)^2/r^4$. The above formula for ΔW is then...

$$\Delta W = \left\{\frac{4}{3c}\left(\frac{ze}{mc}\right)^2\sqrt{\frac{m}{2}}\,(zZe^2)^{\frac{3}{2}}\right\}\int_{r_0}^{\infty}\frac{dr}{r^4}\Big/\left[\frac{1}{r_0} - \frac{1}{r}\right]^{1/2}$$

$\int$ change variables to $y = \frac{r_0}{r} \Rightarrow dr = -\frac{r_0}{y^2}dy$

$$= \{\ \} \cdot \frac{\sqrt{r_0}}{r_0^3}\int_0^1 \frac{y^2\,dy}{\sqrt{1-y}} = \{\ \} \cdot \frac{1}{r_0^{5/2}} \cdot \underbrace{2\sqrt{1-y}\left[1 - \frac{2}{3}(1-y) + \frac{1}{5}(1-y^2)\right]\Big|_{y=1}^{y=0}}_{16/15}$$

i.e.$_{/\!/}$
$$\Delta W = \frac{4}{3c}\left(\frac{ze}{mc}\right)^2\sqrt{\frac{m}{2}}\,(zZe^2)^{\frac{3}{2}} \cdot \left(\frac{mv_0^2}{2zZe^2}\right)^{\frac{5}{2}} \cdot \frac{16}{15}$$

or$_{/\!/}$
$$\boxed{\Delta W = \frac{8}{45}\frac{z\,m v_0^5}{Z c^3} = \frac{16}{45}\frac{z}{Z}\left(\frac{v_0}{c}\right)^3 K}$$, where $K = \frac{1}{2}mv_0^2$ = incident K.E.

Notice that for the nonrelativistic case, $v_0 \ll c$, and the radiative loss $\Delta W \ll K$. This justifies the assumption made in part (a) that $E = K + V \simeq$ const.

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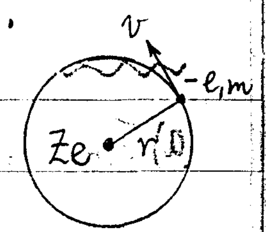
20 Jan. 1985

Calculate radiative time-of-collapse for the classical atom.



(a) Centripetal force = Coulomb force $\Rightarrow \frac{mv^2}{r} = \frac{Ze^2}{r^2}$, which gives:

K.E.: $K = \frac{1}{2}mv^2 = \frac{Ze^2}{2r}$.

The P.E. for the orbiting electron is: $V = -Ze^2/r$, so the total orbit energy is

$E = K + V = -Ze^2/2r$.

(b) With the centripetal accel^n: $a = v^2/r$, the electron radiates energy at rate

$$P = \frac{2}{3}\frac{e^2}{c^3}|a|^2 = \frac{2}{3}\frac{e^2}{c^3}\left|\frac{Ze^2}{mr^2}\right|^2 = \frac{2}{3}\frac{e^2(Ze^2)^2}{m^2c^3}\frac{1}{r^4},$$

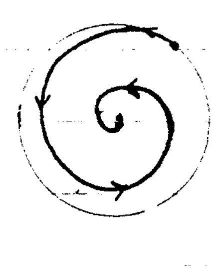
where we've used $v^2/r = Ze^2/mr^2$, from part (a).

Equate the radiative loss P to rate of loss of E(orbit) from part (a):

$$\frac{d}{dt}\left(-\frac{Ze^2}{2r}\right) = -\frac{2}{3}\frac{e^2(Ze^2)^2}{m^2c^3}\frac{1}{r^4} \Rightarrow \frac{dr}{dt} = -\frac{4}{3}\left(\frac{Ze^4}{m^2c^3}\right)\frac{1}{r^2}.$$

Note this can be written: $\frac{1}{c}(dr/dt) = (-)\frac{4Z}{3}(r_0/r)^2$, where $r_0 = \frac{e^2}{mc^2} = 2.8\times10^{-13}$ cm is the classical electron radius. Thus the radial velocity of the electron is $\ll c$ until it essentially hits the nucleus, and the radial shrinkage is **small** until the same point. The **total** time for the electron to spiral down from R to zero is

(c) $$T(\text{collapse}) = \int_{r=R}^{r=0} dt = \frac{3}{4}\left(\frac{m^2c^3}{Ze^4}\right)\int_0^R r^2\,dr = \frac{1}{4Z}\left(\frac{m^2c^3}{e^4}\right)R^3$$



or// $$\boxed{T(\text{collapse}) = \frac{1}{4Z}\frac{r_0}{c}\left(\frac{R}{r_0}\right)^3}, \quad r_0 = \frac{e^2}{mc^2} = 2.82\times10^{-13}\text{ cm}.$$

For $Z=1$ & $R = a_0 = 0.53\times10^{-8}$ cm, have: $R/a_0 = 1.88\times10^4$. The #s then give:

$T(\text{collapse}) = \frac{1}{4}\,\frac{2.82}{3}\times10^{-23}(1.88)^3\times10^{12} = \underline{1.6\times10^{-11}\text{ sec.}}$ Life is short!