~~Kinetic Theory~~ (G.T.) ~~or Stat. Mech~~ RC

10. N atoms of a monatomic gas in a box of volume V have a Maxwell-Boltzmann velocity distribution

$$n_0(v) = \left(\frac{N}{V}\right)\left(\frac{m}{2\pi k_B T}\right)^{3/2} 4\pi \; v^2 e^{-mv^2/2k_BT}$$

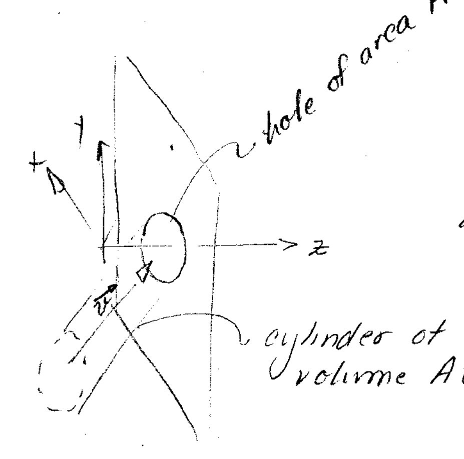
where $n_0(v)\,dv$ is the number density of atoms with speeds in the interval dv at v, T = absolute temperature, m = mass of each atom, and k_B = Boltzmann's constant. A small hole is made in the box, so that atoms can leak out.

a) Find an expression for the velocity distribution $n'(v)$ of escaping atoms - ie. the number (per unit time and unit surface area of the hole) escaping with speeds in the interval dv at v. Explain qualitatively why n' differs in functional form from n_0. ~~[note: different units. JH]~~

b) Find the rms velocity of escaping atoms, and compare it with the rms velocity of atoms inside the container. Based on your result, explain whether the remaining gas will become hotter or colder.

END

Soln:



In time dt, all atoms with velocity $\vec{v}$ will escape through the hole provided they lie in the slanted cylinder shown at left, with volume $A v_z\,dt$. Thus we need to multiply n_0 by $A v_z\,dt$ to get the number escaping in time dt with

$$\psi_n(k) = e^{ikx} u_k^{(n)}(x)$$

$$\nabla_x^2 \psi_n(k) = \frac{2m(V-E)}{\hbar^2} \psi_n(k)$$

$$e^{ikx}(\nabla_x^2 + 2ik\nabla_x - k^2) u_k^{(n)}$$

$$(\nabla_x^2 + 2ik\nabla_x) u_k^{(n)} = \left(k^2 - \frac{2mE}{\hbar^2}\right) u_k^{(n)}$$

with BC : $u_k^{(n)}(0) = u_k^{(n)}(a)$

Try plane wave solu: $u_k^{(n)} \propto e^{i\sigma x}$

B.C. $e^{i\sigma a} = 1$, $\sigma a = 2\pi n$, $n = 0, \pm 1, \pm 2 \cdots$

put into D.E.

$$\sigma_n^2 - 2\sigma_n k + k^2 = \frac{2m E(k, n)}{\hbar^2}$$

$$E(k,n) = \frac{\hbar^2}{2m}(\sigma_n + k)^2 = \frac{\hbar^2}{2m}\left(k + \frac{2\pi n}{a}\right)^2$$

$$u_k^{(n)} = \frac{1}{\sqrt{a}} e^{\frac{2\pi n}{a} x} \qquad n = 0, \pm 1, \pm 2$$

$$-\frac{\pi}{2} < k \leq \pi/a$$

$$\rightarrow \psi_n(k) = \frac{1}{\sqrt{Na}} e^{i(k + \frac{2\pi n}{a})x}$$

$n=1$, $k + \frac{2\pi n}{a} = \frac{\pi}{a}$ for $k = -\frac{\pi}{a}$

$= +\frac{3\pi}{a}$ " $k = +\frac{\pi}{a}$

$n=-1$ $k + \frac{2\pi n}{a} = -\frac{\pi}{a}$ for $k = +\frac{\pi}{a}$

$= -\frac{3\pi}{a}$ " $k = -\frac{\pi}{a}$