ϕ519 Problems   Assigned 10/14/91. Due 10/21/91.

(21) Consider the hypergeometric ODE : $x(1-x)y'' + [\gamma - (1+\alpha+\beta)x]y' - \alpha\beta y = 0$, with α, β, γ = cnsts. As it stands, this ODE is not in a Sturm - Liouville form, i.e. not of form: $\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] + [q(x) + \lambda w(x)]y = 0$. By constructing an appropriate multiplicative function, put the hypergeometric ODE into Sturm-Liouville form, and identify the fcns $p(x)$, $q(x)$, $w(x)$, and the eigenvalues λ.

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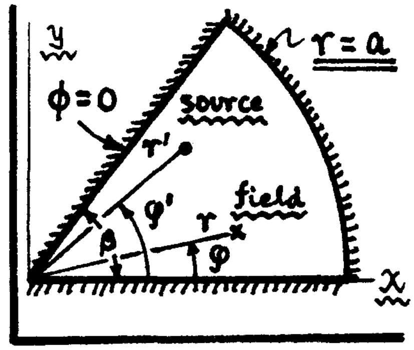
(22) [Jackson, Prob. (3.20)]. Use the result of problem (19) to show that the Green's function on an infinite domain for a 2D wedge of opening $\measuredangle\,\beta$ is:

$$G_\infty(r,\varphi; r',\varphi') = \sum_{n=1}^{\infty} \frac{4}{n}(r_</r_>)^p \sin p\varphi \sin p\varphi', \text{ where } p = \frac{n\pi}{\beta}$$

and $r_{\lessgtr} = \left\{ \begin{smallmatrix} \text{lesser} \\ \text{greater} \end{smallmatrix} \right\}$ of (r, r'). G_∞ applies for $0 \leq (r, r') \to \infty$. Now do Jackson's problem: find the Green's function when there is a bounding (conducting) surface at $r = a$ as shown.



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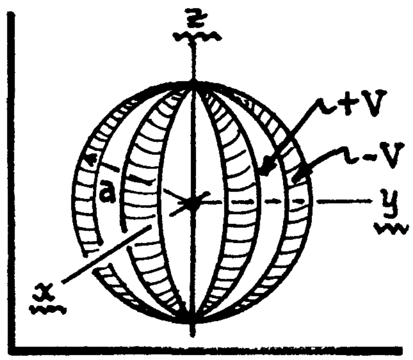
(23) [Jackson (3.4), 20 pts]. A hollow conducting sphere of radius a is divided into $2n$ segments by planes intersecting the z-axis at equal intervals of azimuth φ. The segments are kept at potentials $\pm V$, alternately. (A) Write the series for the potential inside the sphere for the general case of $2n$ segments. Use symmetry arguments to show which coefficients must vanish. For the non-vanishing terms, find the appropriate coefficients as integrals over $\cos\theta$ (θ = colatitude $\measuredangle$).



(B) For the case of $n = 1$, we have hemispheres at $\pm V$, resp. In this case, determine the potential $\phi(r \leq a, \theta, \varphi)$ explicitly up to and including all terms with $l = 3$. Then, by a coordinate transformation, verify that your value of ϕ reduces to the "well-known" result in Jk^n Eq. (3.36). (C) Comment on how you would react if someone electrified your beachball in this fashion.

ϕ519 Solutions

7

21 Put hypergeometric eqtn: $x(1-x)y'' + [\gamma - (1+\alpha+\beta)x]y' - \alpha\beta y$, into S-L form.

1) The hypergeometric eqtn is: $\mathcal{D}_H y = 0$, where the operator is:

$$\rightarrow \mathcal{D}_H = f\frac{d^2}{dx^2} + g\frac{d}{dx} + h \quad \text{w/} \quad f = x(1-x),\ h = -\alpha\beta,\quad g = \gamma - (1+\alpha+\beta)x. \qquad (1)$$

Since $g \neq f'$, then $\mathcal{D}_H$ is not self-adjoint as it stands. By the result of ϕ519 prob^m 15, $\mathcal{D}_H$ can be made self-adjoint by multiplication by $\mu(x)$, where

$$\rightarrow \mu(x) = \exp[I(x)],\quad I(x) = \int [(g-f')/f]\,dx. \qquad (2)$$

Note... an integration cnst for $I(x)$ is superfluous... it just gives an overall mult. cnst in μ.

2) Evaluate $\mu(x)$ to find...

$$I(x) = \int \frac{(\gamma-1) + (1-\alpha-\beta)x}{x(1-x)}\,dx = -(\gamma-1)\ln\left(\frac{1-x}{x}\right) - (1-\alpha-\beta)\ln(1-x)$$

or// $I(x) = (\gamma-1)\ln x + (\alpha+\beta-\gamma)\ln(1-x)$,

and// $$\boxed{\mu(x) = \exp[I(x)] = x^{\gamma-1}(1-x)^{\alpha+\beta-\gamma}}. \qquad (3)$$

Then: $\mu\mathcal{D}_H = (\mu f)\frac{d^2}{dx^2} + (\mu g)\frac{d}{dx} + (\mu h)$, is self-adjoint

3) Compare $\mu\mathcal{D}_H$ term-by-term with the S-L operator: $\mathcal{D}_{SL} = p\frac{d^2}{dx^2} + p'\frac{d}{dx} + (q + \lambda w)$, to find...

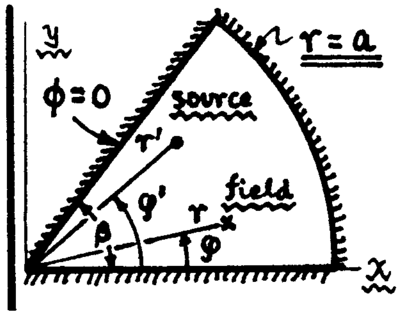
$$\boxed{\begin{array}{l} p(x) = \mu f = x^{\gamma}(1-x)^{1+\alpha+\beta-\gamma},\quad q(x) = 0, \\ \lambda = -\alpha\beta,\quad w(x) = \mu = x^{\gamma-1}(1-x)^{\alpha+\beta-\gamma}. \end{array}} \qquad (4)$$

The Sturm-Liouville form of the hypergeometric eqtn is thus...

$$\left[\!\left[\frac{d}{dx}\left[x(1-x)w(x)\frac{dy}{dx}\right] - \alpha\beta\, w(x) y = 0,\quad \text{w/}\ w(x) = x^{\gamma-1}(1-x)^{\alpha+\beta-\gamma} \right]\!\right] \qquad (5)$$

For the Legendre eqtn, e.g., $\gamma = 1$, $\alpha = -l$, $\beta = l+1 \Rightarrow w = 1$, and the eqtn reads: $\frac{d}{dx}\left[x(1-x)\frac{dy}{dx}\right] + l(l+1)y = 0$. Which is correct.

㉒ [Jackson #(3.20)]. Find G-fcn for a bounded 2D wedge.



1) By a simple notational change $(a,\alpha) \to (r',\varphi')$, the potential calculated in problem ⑲, for line charge λ at (r',φ'), is:

$$\to \phi(r,\varphi;r',\varphi') = \lambda \sum_{n=1}^{\infty} \frac{4}{n} \left\{ \begin{matrix} (r/r')^p, & r \leq r' \\ (r'/r)^p, & r' \leq r \end{matrix} \right\} \sin p\varphi \sin p\varphi', \quad (1)$$

where $\underline{p = n\pi/\beta}$. On setting $\lambda = 1$, and $\{\ \} = (r_</r_>)^p$, we immediately obtain

$$\to G_\infty(r,\varphi;r',\varphi') = \sum_{n=1}^{\infty} \frac{4}{n} [(r_</r_>)^p] \sin p\varphi \sin p\varphi', \quad (2)$$

which is the Green's fcn for an unbounded wedge, where $0 \leq (r,r') \to \infty$.

2) Let $r' = r_<$ and $r = r_>$ for sake of definiteness. The radial dependence of G_∞ of Eq. (2) is then★

$$[\ ] = A_p\left(\frac{r'}{r}\right)^p + B_p\left(\frac{r}{r'}\right)^p \quad \int A_p = 1 \ \& \ B_p = 0 \text{ gives } G_\infty. \quad (3)$$

$B_p = 0$ is chosen so that G_∞ does not diverge as $r \to \infty$. But when r is restricted to $r \leq a$ = finite, we need not choose $B_p = 0$. What we do need for the present problem, however, is that $G \equiv 0$ @ $r = a$. So, with $A_p = 1$, we need...

$$[\ ] = \left[\left(\frac{r'}{r}\right)^p + B_p\left(\frac{r}{r'}\right)^p\right]\Bigg|_{r=a} = \left(\frac{r'}{a}\right)^p + B_p\left(\frac{a}{r'}\right)^p = 0 \Rightarrow \underline{\underline{B_p = -\left(\frac{r'^2}{a^2}\right)^p}}$$

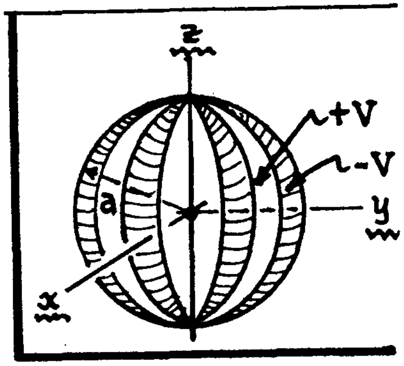
$$\xrightarrow{\text{and}} [\ ] = \left(\frac{r'}{r}\right)^p - \left(\frac{rr'}{a^2}\right)^p, \quad \text{on } 0 \leq r \leq a, \text{ w/ } r' < r. \quad (4)$$

3) With the radial dependence per Eq. (4), the G-fcn for the bounded wedge is:

$$\boxed{G(r,\varphi;r',\varphi') = \sum_{n=1}^{\infty} \frac{4}{n}\left[\left(\frac{r'}{r}\right)^p - \left(\frac{rr'}{a^2}\right)^p\right] \sin p\varphi \sin p\varphi'}, \quad p = \frac{n\pi}{\beta}. \quad (5)$$

This is for $r' < r$. When $r' > r$, interchange r' & r. This result is the same as that quoted by Jackson in Prob. (3.20). Note that the boundary contribution in $1/a$ vanishes as $a \to \infty$. So G of Eq. (5) also includes G_∞ of Eq. (2).

★ See Jackson Eq. (2.69) for general solution to the 2D problem in polar coordinates.



(23) [Jackson (3.4), 20 pts]. ϕ (inside) for electrified beachball.

(A) 1) There is no charge inside the sphere, so -- in the general solution of Jackson's Eq. (3.61) -- inverse powers of r do not contribute for $0 \leq r \leq a$, and the interior potential is of form:

$$\rightarrow \phi(r,\theta,\varphi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{+l} (r/a)^l A_{lm} Y_{lm}(\theta,\varphi), \quad r \leq a. \qquad (1)$$

We have put in a factor $1/a^l$ in an obvious place. On the sphere surface, we need:

$$\left[\!\left[\phi(r=a,\theta,\varphi) = \sum_{l,m} A_{lm} Y_{lm}(\theta,\varphi) = \begin{cases} +V, & 2k\cdot\frac{2\pi}{2n} \leq \varphi \leq (2k+1)\frac{2\pi}{2n}; \\ -V, & (2k+1)\frac{2\pi}{2n} \leq \varphi \leq (2k+2)\frac{2\pi}{2n}; \end{cases} \right\} = V(\varphi). \qquad (2)$$

Here $k = 0, 1, \ldots, n-1$ for $2n$ segments, and this B.C. determines the A_{lm} via

$$\rightarrow A_{lm}^* = \int_{4\pi} d\Omega\, Y_{lm}(\theta,\varphi) V(\varphi), \quad \text{[Jackson Eq. (3.58)]}. \qquad (3)$$

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2) In Eqs. (1)-(3), have $Y_{lm}(\theta,\varphi) = N_{lm} P_l^m(\cos\theta) e^{im\varphi}$, w/ $N_{lm} = \left[\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}\right]^{1/2}$.

Note that ϕ of Eq. (1) must be symmetric about the xy-plane, i.e. ϕ is unchanged by the reflection $\theta \rightarrow \pi - \theta$, or $\cos\theta \rightarrow (-)\cos\theta$, or $P_l^m(\mu) \rightarrow P_l^m(-\mu)$, w/ $\mu = \cos\theta$. Since $P_l^m(-\mu) = (-)^{l-|m|} P_l^m(\mu)$,¶ then the only Y_{lm}'s which can appear in ϕ are those with $l - |m| = $ even #. The case $l = 0 = m$ is ruled out also, because Y_{00} would give an unwanted cnst background term which did not change with φ. If we put Y_{lm} into Eq. (3), then...

$$\rightarrow A_{lm}^* = N_{lm} \int_{-1}^{+1} d\mu\, P_l^m(\mu) \int_0^{2\pi} d\varphi\, e^{im\varphi} V(\varphi)$$

$$= N_{lm} V \int_{-1}^{+1} d\mu\, P_l^m(\mu) \sum_{k=0}^{n-1} \left\{ \int_{2k\frac{\pi}{n}}^{(2k+1)\frac{\pi}{n}} d\varphi\, e^{im\varphi} - \int_{(2k+1)\frac{\pi}{n}}^{(2k+2)\frac{\pi}{n}} d\varphi\, e^{im\varphi} \right\}$$

this { } $= -(1-e^{im\frac{\pi}{n}})^2$

$$= N_{lm} \frac{V}{im} \int_{-1}^{+1} d\mu\, P_l^m(\mu) \sum_{k=0}^{n-1} e^{im\cdot 2k\frac{\pi}{n}} \left\{ (e^{im\frac{\pi}{n}} - 1) - (e^{im\frac{2\pi}{n}} - e^{im\frac{\pi}{n}}) \right\}$$

$$= N_{lm} \frac{iV}{m} (1 - e^{im\frac{\pi}{n}})^2 \int_{-1}^{+1} d\mu\, P_l^m(\mu) \sum_{k=0}^{n-1} \left(e^{2im\frac{\pi}{n}}\right)^k. \qquad (4)$$

The series is a geometric series which is easily summed to give ...

¶ $P_l^m(\mu) \propto (d/d\mu)^m P_l(\mu)$. When $\mu \rightarrow (-)\mu$, get sign change $(-)^l$ from P_l, $(-)^m$ from $(d/d\mu)^m$.

$$\rightarrow A^*_{lm} = N_{lm}\frac{iV}{m}\,S(m,n)\int_{-1}^{+1} d\mu\, P_l^m(\mu)\,,\quad \text{w/}\ S(m,n) = \left(1-e^{i\frac{m\pi}{n}}\right)^2\left[\frac{e^{2im\pi}-1}{e^{2i\frac{m}{n}\pi}-1}\right]. \qquad (5)$$

When m = arbitrary integer $\neq 0$, generally $S(m,n)\equiv 0$. But when $m \rightarrow$ multiple of n, say $m = pn+\epsilon$, where $p = 0, 1, 2, \ldots$ & $\epsilon\rightarrow 0$, it is easy to show...

$$\lim_{\epsilon\to 0} S(m=pn+\epsilon, n) = \left(1-e^{i\pi p}\right)^2[n] = \begin{cases} 0, & p \text{ even } (=0,2,4,\ldots) \\ 4n, & p \text{ odd } (=1,3,5,\ldots)\end{cases}$$

So $\boxed{A^*_{lm} = 4iV N_{lm}\,\frac{n}{m}\int_{-1}^{+1} d\mu\, P_l^m(\mu)}$ $m = 1n, 3n, 5n, \ldots;\ m\leq l;$ $l-m$ = even #. (and $\mu=\cos\theta$) (6)

(B) 3) For $n=1$ (two hemispheres), the only nonvanishing A^*_{lm} through $l=3$ are -- by the conditions in Eq.(6) -- A^*_{11}, A^*_{31} & A^*_{33}. With the $N_{lm}P_l^m(\cos\theta)$ entries on pp. 99-100 of Jackson, these are...

$$A^*_{11} = 4iV\left(-\sqrt{\frac{3}{8\pi}}\right)\int_0^\pi \sin^2\theta\, d\theta = -iV\sqrt{\frac{3\pi}{2}} = A_{1,-1};$$

$$A^*_{31} = 4iV\left(-\frac{1}{4}\sqrt{\frac{21}{4\pi}}\right)\int_0^\pi (5\cos^2\theta-1)\sin^2\theta\, d\theta = -iV\,\frac{\sqrt{21\pi}}{16} = A_{3,-1};$$

$$A^*_{33} = \frac{4iV}{3}\left(-\frac{1}{4}\sqrt{\frac{35}{4\pi}}\right)\int_0^\pi \sin^4\theta\, d\theta = -iV\,\frac{\sqrt{35\pi}}{16} = A_{3,-3}. \qquad (7)$$

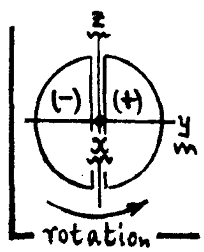
Form $\phi(r,\theta,\varphi) = \sum_{l,m}(r/a)^l A_{lm}Y_{lm}(\theta,\varphi)$ of Eq.(1) from results of Eq.(7), plus the Y_{lm} on Jackson's pp. 99-100. Then get the hemispheres' interior potential...

$$\phi(r\leq a,\theta,\varphi) = \frac{3}{2}\left(\frac{r}{a}\right)\sin\theta\sin\varphi + \frac{7}{64}\left(\frac{r}{a}\right)^3\sin\theta\left[3(5\cos^2\theta-1)\sin\varphi + {}\right.$$
$$\left. {} + 5\sin^2\theta\sin 3\varphi\right]; \qquad (8)$$

This is ϕ(inside), as required, up through terms with $l=3$. It can be made to look more palatable by use of the identity: $\sin 3\varphi = 3\sin\varphi - 4\sin^3\varphi$. Then:

$$\boxed{\phi(r\leq a,\theta,\varphi) = \frac{3}{2}\left(\frac{r}{a}\right)\sin\theta\sin\varphi - \frac{7}{16}\left(\frac{r}{a}\right)^3\sin\theta\sin\varphi\left[5\sin^2\theta\sin^2\varphi - 3\right]+\ldots} \qquad (9)$$

This ϕ describes the hemispheres oriented as shown at right. Upon rotation about the x-axis ($y\rightarrow +z$, $z\rightarrow(-)y$; i.e. $\varphi = \frac{\pi}{2}$ & $\theta\rightarrow\theta+\frac{\pi}{2}$), Eq.(9) yields:



$$\phi = \frac{3}{2}\left(\frac{r}{a}\right)P_1(\cos\theta) - \frac{7}{8}\left(\frac{r}{a}\right)^3 P_3(\cos\theta)+\ldots \quad \left\{\begin{array}{l} P_1(\mu)=\mu \\ P_3(\mu) = \frac{1}{2}(5\mu^3-3\mu)\end{array}\right., \text{ Jk}^{\text{n}}\ \text{Eq.}(3.36).$$