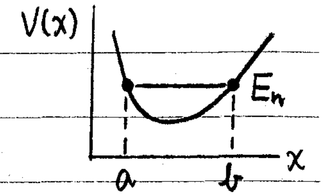
Exam Questions for Phys. 506 (Hom Test) 12/29/70

12/29/70 **1** "Consider a particle bound in an arbitrary attractive 1D potential $V(x)$ as shown. Show that the spacing between adjacent quantized energy levels E_n is given, in WKB approximation, by $\Delta E_n \simeq \hbar\omega_n$, where ω_n is the natural vibration frequency of the n^{th} level. (Hint: differentiate the WKB energy quantization condition with respect to n)."



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Level E_n found from: $\int_a^b \left[2m(E_n - V(x))\right]^{\frac{1}{2}} dx = (n+\tfrac{1}{2})\pi\hbar$

Differentiating w.r.t. n gives

$$\pi\hbar = \int_a^b \tfrac{1}{2}\left[2m(E_n - V(x))\right]^{-\frac{1}{2}} 2m\left(\frac{dE_n}{dn}\right) dx = \left(\frac{dE_n}{dn}\right)\int_a^b \frac{dx}{p_n(x)/m}$$

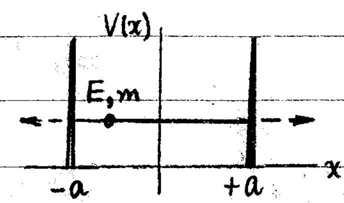
where: $p_n(x) = \left[2m(E_n - V(x))\right]^{\frac{1}{2}}$ is momentum in n^{th} level

But natural period is: $\tau = \frac{2\pi}{\omega} = 2\int_a^b \frac{dx}{p(x)/m}$

$\therefore \int_a^b \frac{dx}{p_n(x)/m} = \frac{\pi}{\omega_n}$, ω_n = natural freq. in n^{th} level

This $\Rightarrow \frac{dE_n}{dn} = \hbar\omega_n$, or $\Delta E_n \simeq \hbar\omega_n$ for adjacent levels.

1/5/71 **2** "A particle of mass m and energy E is trapped in a 1D box of length $2a$. The walls of the box (at $\pm a$) may be represented by δ-fcns of strength C, i.e. the potential is: $V(x) = C\left[\delta(x+a) + \delta(x-a)\right]$. Estimate the lifetime of the particle in the box; i.e., how long before it gets out?"



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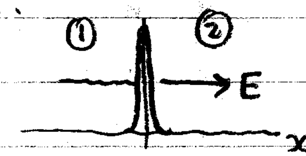
Decay rate : $\Gamma = \left(\frac{1}{\tau/2}\right) T$ $\begin{cases} \tau = \text{natural period inside box} \\ T = \text{trans. coeff. at one barrier} \end{cases}$

Particle is free inside box $\Rightarrow \tau = \frac{4a}{v}$, $v = \sqrt{2E/m}$

$\therefore \left(\frac{1}{\tau/2}\right) = \frac{1}{2a}\sqrt{\frac{2E}{m}}$

Must now calculate T for δ-fcn barrier...



$V = C\delta(x)$. Let $\hbar k = \sqrt{2mE}$.

$\psi_1 = e^{+ikx} + Ae^{-ikx}$, $\psi_2 = Be^{+ikx}$, Want $T = |B|^2$

ψ continuous at $x = 0$: $1 + A = B \leftarrow$ (I)

ψ' discontinuous at $x = 0$ ...

$$\psi_2'(0+) - \psi_1'(0-) = \frac{2mC}{\hbar^2}\psi_2(0)$$

or, $ik[B - (1-A)] = \frac{2mC}{\hbar^2} B \Rightarrow 1 - A = \left(1 - \frac{2mC}{ik\hbar^2}\right)B \leftarrow$ (II)

Add (I) & (II) $\Rightarrow 2 = \left(2 + i\frac{2mC}{\hbar^2 k}\right)B \Rightarrow B = 1\Big/\left(1 + i\frac{mC}{\hbar^2 k}\right)$

$\therefore T = |B|^2 = 1\Big/\left(1 + \frac{m^2C^2}{\hbar^2(\hbar k)^2}\right) = 1\Big/\left(1 + \frac{mC^2}{2E\hbar^2}\right)$

$\Rightarrow \Gamma = \frac{1}{2a}\sqrt{\frac{2E}{m}}\Big/\left(1 + \frac{mC^2}{2E\hbar^2}\right)$

Lifetime is : $\Delta t = \frac{1}{\Gamma} = 2a\sqrt{\frac{m}{2E}}\left(1 + \frac{mC^2}{2E\hbar^2}\right)$

1/27/71 ③ "A QM system in state $\psi(x)$ at time $t=0$ is subjected to an interaction H which generates two discrete eigenstates ϕ_n and eigenenergies E_n, with $E_2 - E_1 = \hbar\Omega$. The energy spectrum fcn of H is therefore discrete, with values

$$W_n = \left| \int \phi_n^*(x)\,\psi(x)\,dx \right|^2, \quad n = 1 \,\&\, 2.$$

Assume $\sum_{n=1}^{2} W_n = 1$ for convenience. Derive an expression for the probability of finding the state $\psi(x)$ at time $t>0$. What is the oscillation period between times of maximum probability?"

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Desired probability is : $P(t) = \left| \sum_E W(E)\, e^{-\frac{i}{\hbar}Et} \right|^2$

or $P(t) = \left| W_1 e^{-\frac{i}{\hbar}E_1 t} + W_2 e^{-\frac{i}{\hbar}E_2 t} \right|^2$

$$= \left| e^{-\frac{i}{\hbar}E_1 t}\left(W_1 + W_2 e^{-i\Omega t}\right) \right|^2$$

$$= \left(W_1 + W_2 e^{+i\Omega t}\right)\left(W_1 + W_2 e^{-i\Omega t}\right)$$

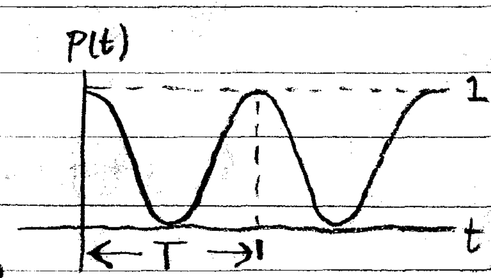
$$= W_1^2 + W_2^2 + W_1 W_2 \left(e^{i\Omega t} + e^{-i\Omega t}\right)$$

$$= W_1^2 + W_2^2 + 2W_1W_1 \cos\Omega t$$

$$= W_1^2 + W_2^2 + 2W_1W_2 - 2W_1W_2\underbrace{(1-\cos\Omega t)}_{2\sin^2\frac{1}{2}\Omega t}$$

$$= \underbrace{(W_1+W_2)^2}_{=1} - 4W_1W_2 \sin^2 \tfrac{1}{2}\Omega t$$

$\therefore\ P(t) = 1 - 4W_1W_2 \sin^2 \frac{1}{2}\Omega t$



Period T : $\frac{1}{2}\Omega T = \pi \Rightarrow T = 2\pi/\Omega$

2/3/71 **4** "Start from the definition of the S-matrix in the form

$$\psi_\alpha(x',t') = \sum_\beta S_{\beta\alpha}\,\phi_\beta(x',t'),$$

which describes the evolution of a free particle state $\phi_\alpha(x,t)$ in the distant past to the state $\psi_\alpha(x',t')$ in the distant future. Suppose the ϕ_β are orthonormal, and that the total interaction is at all times Hermitian. Then the normalization and orthogonalization of the ψ_α must be time-independent. Use this fact to show that the S-matrix is unitary, i.e.

$$S^\dagger S = 1, \quad \underline{\text{or}} \quad (S^\dagger S)_{ij} = \sum_\beta S^\dagger_{i\beta} S_{\beta j} = \sum_\beta S^*_{\beta i} S_{\beta j} = \delta_{ij}."$$

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Since the ψ normalization is t' indpt, we can write

$$\int dx'\,\psi_i^*(x',t')\,\psi_j(x',t') = \lim_{t'\to-\infty} \int dx'\,\psi_i^*(x',t')\,\psi_j(x',t')$$

$$= \int dx'\,\phi_i^*(x',t')\,\phi_j(x',t') = \delta_{ij}$$

Plugging in the expansions for ψ on the LHS, we get

$$\sum_{\beta,\gamma} S^*_{\beta i} S_{\gamma j} \underbrace{\int dx'\,\phi^*_\beta(x',t')\,\phi_\gamma(x',t')}_{\delta_{\beta\gamma}} = \delta_{ij}$$

or// $\sum_\beta S^*_{\beta i} S_{\beta j} = \delta_{ij}$. <u>QED</u>

An alternate, somewhat more physical, proof is the following...

$$\psi_\alpha(x',t') = \sum_\beta S_{\beta\alpha}\,\phi_\beta(x',t') \Rightarrow S_{\beta\alpha} = \int dx'\,\phi_\beta^*(x',t')\,\psi_\alpha(x',t')$$

Plugging the matrix element directly into the sum, we get

$$\sum_\beta S_{\beta i}^* S_{\beta j} = \sum_\beta \left(\int dx'\,\phi_\beta^*(x',t')\,\psi_i(x',t')\right)^* \left(\int dx\,\phi_\beta^*(x,t)\,\psi_j(x,t)\right)$$

$$= \int dx'\,\psi_i^*(x',t') \int dx \left[\sum_\beta \phi_\beta^*(x,t)\,\phi_\beta(x',t')\right] \psi_j(x,t)$$

But the $[\ ] = i G_0(x',t';x,t)$, the free particle propagator. So

$$\sum_\beta S_{\beta i}^* S_{\beta j} = \int dx'\,\psi_i^*(x',t')\, \underbrace{i \int dx\, G_0(x',t';x,t)\,\psi_j(x,t)}_{=\psi_j(x',t')}$$

$= \psi_j(x',t')$, since $G_0 \Rightarrow$ propagation without any interactions to distort ψ_j.

$$\therefore \sum_\beta S_{\beta i}^* S_{\beta j} = \int dx'\,\psi_i^*(x',t')\,\psi_j(x',t') \leftarrow$$ this is the orthonormalization integral for ψ, which can be assumed to be δ_{ij} as $t' \to -\infty$.

$$= \delta_{ij}\,,\ \text{QED}$$