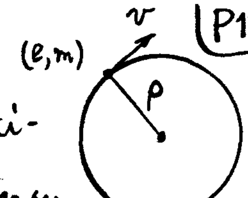
ϕ520 Problems (#34, 35, 36) { assigned : 1/24/91 ; due : 1/31/91 } **Set # ②** P12



Consider a large synchrotron which maintains a beam of highly relativistic protons [charge e, rest energy: $\mathcal{E}_0 = mc^2 = 938$ MeV] at total energy $\mathcal{E}$ in orbit at radius ρ. This machine supplies energy to the beam at a constant rate (in lab) of dU/dz, $\frac{\text{MeV}}{\text{meter}}$, per proton, and it has magnets generating large enough B fields to hold the protons in orbit for any "reasonable" $\mathcal{E}$ [see Jk$^{\underline{n}}$ Sec. 12.3]. Assume the limit on $\mathcal{E}$ is imposed by radiation losses alone.

(A) Find the limiting value of $\gamma = \mathcal{E}/mc^2$ under these circumstances.

(B) If $dU/dz = 10$ MeV/m and $\rho = 15$ km ($\sim$ SSC specs), calculate a <u>number</u> for γ.

(C) What magnetic field B is required for the orbit? Will this scheme work?

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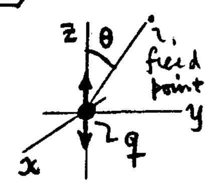
(35) [Jackson Prob. (14.5)]. Charge q oscillates along the z-axis according to $z(t') = R\cos\omega_0 t'$, R & ω_0 = cnsts. The motion is <u>relativistic</u>.



(A) Show that the instantaneous power radiated per unit solid ∡ is:

$$\longrightarrow dP(t')/d\Omega = \frac{q^2 c\beta^4}{4\pi R^2}\left[\sin^2\theta\cos^2\omega_0 t'\right]\Big/\left[1+\beta\cos\theta\sin\omega_0 t'\right]^5, \quad \text{w/ } \beta = \omega_0 R/c.$$

(B) Do a time average to show that the average radiated power/solid ∡ is:

$$\boxed{\langle dP/d\Omega\rangle = \frac{q^2 c\beta^4}{32\pi R^2}\left\{\left[4+\beta^2\cos^2\theta\right]\sin^2\theta\right\}\Big/\left(1-\beta^2\cos^2\theta\right)^{7/2}.}$$

[redacted]: Gradshteyn & Ryzhik, Sec. (3.66) applies.

(C) Sketch the angular distribution of $\langle dP/d\Omega\rangle$ for the relativistic case in part (B), and also for the nonrelativistic limit. In which case would your radio work better?

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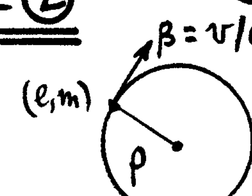
(36) [Jackson Prob. (14.10)]. By Bohr's Correspondence Principle for atomic radiation: for large principal quantum # n, the classical power radiated during transition $n \to n-1$ is $(\hbar\omega_0)\cdot\frac{1}{\tau}$, where ω_0 is the emitted frequency and τ is the transition lifetime. Consider a hydrogenlike Bohr atom, ignore relativity, and let $\alpha = \frac{e^2}{\hbar c} \simeq \frac{1}{137}$ (fine structure constant).

(A) For the transition $n \to n-1$, the reciprocal lifetime is called the "transition probability": $\Gamma_n = 1/\tau$. Show that: $\Gamma_n = \Gamma_1/n^5$, where: $\Gamma_1 = \frac{2}{3}Z^4\alpha^5(mc^2/\hbar)$.

(B) For hydrogen ($Z=1$), compare the quasi-classical result for the lifetime $\tau_n = 1/\Gamma_n$ with measured values: $\tau(2P\to1S) = 1.6$ ns, $\tau(4F\to3D) = 73$ ns, $\tau(6H\to5G) = 610$ ns.



(34) Radiation limit to synchrotron energy.

(A) From class notes, or Jkⁿ Eq. (14.46), each proton radiates at power level...

$$P_{rad} = \frac{2}{3}(e^2c/\rho^2)\beta^4\gamma^4, \quad \gamma = \mathcal{E}/mc^2 \quad (\mathcal{E} = \gamma mc^2 = \text{total energy}). \qquad (1)$$

This is relativistically correct, and it's what is seen in lab. The radiation energy loss during one orbit period $\Delta t = \frac{2\pi\rho}{\beta c}$ is $P_{rad}\Delta t$, and it must be less than the energy supplied during that circuit, viz $(dU/dz)\times 2\pi\rho$. So

$$P_{rad}\Delta t < \left(\frac{dU}{dz}\right)\cdot 2\pi\rho \Rightarrow \underline{\gamma^4 < \frac{3}{2}\left(\frac{dU}{dz}\right)\frac{\rho^2}{e^2}\cdot\cancel{\frac{1}{\beta^3}}}^{\,1\ \text{(highly relativistic)}}. \qquad (2)$$

(B) For numbers for the γ limit in Eq. (2), let the units of $\left(\frac{dU}{dz}\right)$ be $\frac{\text{MeV}}{\text{m}}$ and measure ρ in units of km. Then...

$$\gamma^4 < 1.043\times 10^{21}\,\rho^2(dU/dz), \text{ or } \underline{\gamma < 1.797\times 10^5\,[\rho^2(dU/dz)]^{\frac{1}{4}}}. \qquad (3)$$

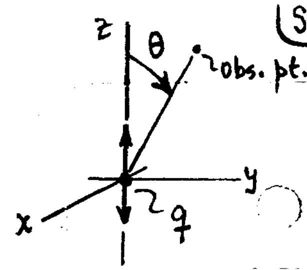
(ρ in km, dU/dz in MeV/m)

If $dU/dz = 10\,\frac{\text{MeV}}{\text{m}}$ and $\rho = 15$ km, then: $\boxed{\gamma < 1.24\times 10^6}$. The corresponding proton energy is $\mathcal{E} = \gamma\,\mathcal{E}_0 = 1160$ TeV, which is very robust. But this "limiting" energy is $\sim 10^3\times$ max. design energy for the SSC. So something else fails before the radⁿ limit is reached on this machine.

(C) The B-field needed to maintain the orbit is found from Jkⁿ (12.39):

$$\omega_B = \frac{v}{\rho} = \frac{eB}{\gamma mc} \Rightarrow B = \gamma\cancel{\beta}^{\,1}\frac{mc^2}{e\rho} = 31.3\,\gamma/\rho \quad \text{B is in Gauss, for } \rho \text{ in km}. \qquad (4)$$

If $\rho = 15$ km, then $B = 2.09\,\gamma$, Gauss. The beam magnets are capable of supplying (perhaps) $B \simeq 20{,}000$ G [this a big field for earthlings], and so the proton orbit can be held in place only up to $\gamma \sim 10^4$ (i.e. 10 TeV). <u>We cannot yet build a radiation-limited synchrotron.</u>



ϕ 520 Prob. Solutions

(35) [Jackson (14.5)] Analyse radiation from relativistic SHO.

(a) From Jn Eq. (14.38) : $\frac{dP(t')}{d\Omega} = \frac{q^2}{4\pi c}\frac{1}{D^5}\left|\hat{n}\times[(\hat{n}-\vec{\beta})\times\vec{\alpha}]\right|^2$, where $\vec{\alpha} = \dot{\vec{\beta}}$, and $D = (1-\vec{\beta}\cdot\hat{n})$. In this case of linear motion, $\vec{\alpha}$ is $\parallel \vec{\beta}$, and so we have...

$$\left|\hat{n}\times[(\hat{n}-\vec{\beta})\times\vec{\alpha}]\right|^2 = \left|\hat{n}\times(\hat{n}\times\vec{\alpha})\right|^2 = \alpha^2\sin^2\theta,$$

$$\xrightarrow{\text{so}} dP(t')/d\Omega = \frac{q^2}{4\pi c}\alpha^2\sin^2\theta/(1-\beta\cos\theta)^5, \quad t' = q\text{'s proper time}.$$

But: $z(t') = R\cos\omega_0 t' \Rightarrow \beta = \frac{1}{c}\frac{d}{dt'}z(t') = -\left(\frac{\omega_0 R}{c}\right)\sin\omega_0 t'$. Set $\beta_0 = \frac{\omega_0 R}{c}$.

Then: $\beta = -\beta_0\sin\omega_0 t'$, and $\alpha = \frac{d\beta}{dt'} = -\omega_0\beta_0\cos\omega_0 t'$. Thus...

$$\boxed{dP(t')/d\Omega = \frac{q^2 c}{4\pi R^2}\beta_0^4\sin^2\theta\cos^2\omega_0 t'/(1+\beta_0\cos\theta\sin\omega_0 t')^5}, \quad \beta_0 = \frac{\omega_0 R}{c},$$

is the required instantaneous radiation rate.

(b) The average power radiated per cycle will be (with $T = 2\pi/\omega_0$ = period)

$$\langle dP/d\Omega\rangle = \frac{1}{T}\int_0^T [dP(t')/d\Omega]\,dt' = \frac{q^2 c}{4\pi R^2}\beta_0^4\sin^2\theta\cdot\underbrace{\frac{1}{2\pi}\int_0^{2\pi}\frac{\cos^2 x\,dx}{(1+b\sin x)^5}}_{J}$$

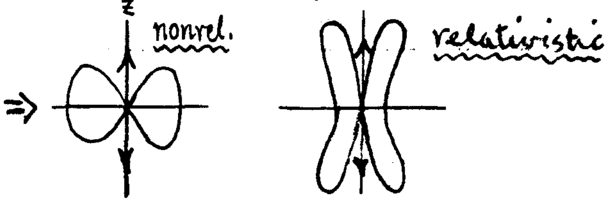
where: $b = \beta_0\cos\theta < 1$. The integral is*...

$$2\pi J = \int_{-\pi/2}^{3\pi/2}\frac{\cos^2 x\,dx}{(1+b\sin x)^5} = 2\int_0^{\pi}\frac{\sin^2 y\,dy}{(1+b\cos y)^5}, \quad \text{so } J = \frac{1}{b(1-b^2)^2}P_3^{-1}\left(1/\sqrt{1-b^2}\right)$$

$$\text{But: } P_3^{-1}(z) = \frac{2!}{4!}P_3^1(z) = \frac{1}{8}\sqrt{z^2-1}\,(5z^2-1) \Rightarrow J = \frac{1}{8}\frac{4+b^2}{(1-b^2)^{7/2}}$$

And so

$$\boxed{\left\langle\frac{dP}{d\Omega}\right\rangle = \frac{q^2 c\,\beta_0^4}{32\pi R^2}\left[\frac{4+\beta_0^2\cos^2\theta}{(1-\beta_0^2\cos^2\theta)^{7/2}}\right]\sin^2\theta} \Rightarrow$$

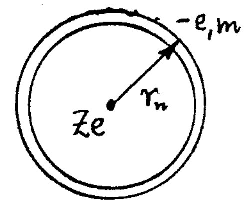


* Use Gradshteyn & Ryzhik, p. 384 #(3.664.4), in the form...

$$\int_0^{\pi}\frac{\sin^{2\nu}x\,dx}{(1+b\cos x)^{\mu}} = \frac{2^{\nu}\sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})}{b^{\nu}(1-b^2)^{\frac{1}{2}(\mu-\nu)}}P^{-\nu}_{\mu-\nu-1}\left(1/\sqrt{1-b^2}\right).$$

ϕ 520 Prob. Solutions

(36) [Jackson (14.10)]. Calculate semi-classical transition probability for H.



(a) Bohr claims radiated power: $P_n = \Gamma_n \Delta E(n \to n-1)$, so we want

$$\underline{\underline{\Gamma_n = P_n / \Delta E(n \to n-1)}} \quad \begin{cases} P_n = \text{radiated power, a lá Larmor;} \\ \Delta E(n \to n-1) = \text{transition energy, a lá Bohr.} \end{cases}$$

Assume circular orbits of radii r_n. Bohr model gives...

orbit energy: $E_n = -\frac{1}{2}(Z\alpha)^2 mc^2/n^2 \Rightarrow \underline{\underline{\Delta E(n \to n-1)}} \underset{n \gg 1}{\simeq} \dfrac{(Z\alpha)^2 mc^2}{n^3}$;

orbit radius: $r_n = \dfrac{n^2 a_0}{Z}$, $a_0 = \dfrac{\hbar^2}{me^2}$ = Bohr radius. Note: $\alpha = \dfrac{e^2}{\hbar c}$.

The electron's centripetal acceln in the n^{th} orbit is then calculable as

$$a_n = -\frac{1}{m} Ze^2/r_n^2 = -Z^3 e^6 m / n^4 \hbar^4 = -Z^3 e^2 \alpha^2 mc^2 / n^4 \hbar^2;$$

So// Larmor power: $\underline{\underline{P_n}} = \dfrac{2}{3}\dfrac{e^2}{c^3}|a_n|^2 = \dfrac{2}{3}\dfrac{Z^6 \alpha^7 (mc^2)^2}{n^8 \hbar}$,

and// Transition Prob$^{\underline{y}}$: $\Gamma_n = P_n / \Delta E(n \to n-1) = \frac{2}{3} Z^4 \alpha^5 mc^2 / n^5 \hbar$;

i.e.// $\boxed{\Gamma_n = \Gamma_1 / n^5, \quad \Gamma_1 = \frac{2}{3} Z^4 \alpha^5 mc^2/\hbar}$.

(b) The lifetime for $n \to (n-1)$ is: $\boxed{\tau_n = \Gamma_n^{-1} = n^5 \tau_1}$, where:

$$\tau_1 = 1/\Gamma_1 = \frac{1}{Z^4} \cdot \frac{3}{2} \hbar / \alpha^5 mc^2 = \frac{1}{Z^4} \cdot 9.32 \times 10^{-11} \text{ sec.}$$

For $Z = 1$, we calculate the #s at right. As expected, agreement with known values improves as $n \to$ larger.

transition	n	(this calcn) τ_n, nsec	(known) τ_n, nsec	ratio
$2p \to 1s$	2	3.0	1.6	1.88
$4f \to 3d$	4	95	73	1.30
$6h \to 5g$	6	725	610	1.19