Desirability of Lagrange & Hamilton forms of E&M

- Relativistic EM Lagrangian & Hamiltonian [ref. Jackson, Ch. 12]

In classical (particle) mechanics, the formulation of the theory in terms of a Lagrangian L or Hamiltonian H has a number of advantages. It ...

(a) emphasizes the central role of conservation laws in (particle) dynamics;
(b) allows dealing with scalar (energy) eqtns-of-motion rather than vector ($\mathbf{p}, \mathbf{F}$) eqtns;
(c) facilitates passage to QM (via $E \to i\hbar \frac{\partial}{\partial t}$, $\mathbf{p} \to -i\hbar\nabla$) from the Hamiltonian form.

For these reasons, it is useful to look at the Lagrangian-Hamiltonian formulation of electrodynamics. This adds nothing essentially new to the theory. We just recast it.

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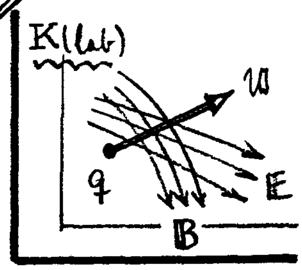
1) The eqtns-of-motion for charge q moving at velocity $\mathbf{u}$ in (external) fields $\mathbb{E}$ & $\mathbb{B}$¶ are already known... they are the Lorentz law:



$$\rightarrow \frac{d\mathbf{p}}{dt} = q\left(\mathbb{E} + \frac{1}{c}\mathbf{u}\times\mathbb{B}\right), \quad \frac{d\mathcal{E}}{dt} = q\mathbf{u}\cdot\mathbb{E}$$

w/ t = time in $\underset{\sim}{K}$(lab);

$\mathbf{p} = \gamma m \mathbf{u}$, $\mathcal{E} = \gamma m c^2$, $\gamma = 1/\sqrt{1-(u/c)^2}$. (1)

Or, manifestly covariantly...

$$\boxed{\frac{d}{d\tau}(m U^\alpha) = \frac{q}{c} F^{\alpha\beta} U_\beta}$$

w/ $d\tau = dt\sqrt{1-(u/c)^2}$, proper time for (q, m); (2)

$U^\alpha = \gamma(c, \mathbf{u})$, 4-velocity; $F^{\alpha\beta}$ = EM field tensor.

Any properly formulated Lagrangian/Hamiltonian theory must reproduce Eqs. (2).

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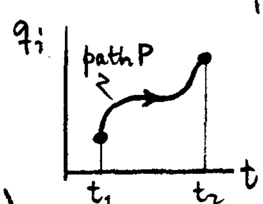
2) A Lagrangian formulation proceeds via "Hamilton's Principle". Nonrelativistically...

if ACTION: $A(P) = \int_{t_1}^{t_2} L(q_i, \dot{q}_i, t)\, dt$,

w/ $L = K\left(\substack{\text{kinetic}\\ \text{energy}}\right) - V\left(\substack{\text{potential}\\ \text{energy}}\right)$ = fcn of { generalized cds $q_i(t)$, " velocities $\dot{q}_i(t)$, & time t (endpts fixed);



then

$$\delta A(P) = \delta \int_{t_1}^{t_2} L\, dt = 0, \text{ for an allowed path } P \Rightarrow \boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \frac{\partial L}{\partial q_i}}$$

EULER-LAGRANGE EQTNS (3)

¶ We ignore the $\mathbb{E}$ & $\mathbb{B}$ fields due to q itself, i.e. we do not include radiation.

3) We will assume Hamilton's Principle: $\delta\int_{t_1}^{t_2} L\,dt = 0$, for an allowed path, holds true relativistically, if we can define a Lorentz invariant action $A = \int_{t_1}^{t_2} L\,dt$; We need A = Lorentz invariant to ensure covariant eqtns-of-motion. In fact, constructing a Lorentz invariant A for a free particle is ~ easy. Start by inserting the particle proper time $d\tau = dt/\gamma$ in the standard defn for A...

$$\rightarrow A = \int_{t_1}^{t_2} L\,dt = \int_{\tau_1}^{\tau_2} (\gamma L)\,d\tau \quad \Big/ \text{ w/ } \gamma = 1/\sqrt{1-(u/c)^2}, \text{ for free particle (of mass } m) \text{ moving @ } u. \tag{4}$$

Since $d\tau$ is a Lorentz scalar, then A is a Lorentz scalar if γL is (by construction).

Now free-particle motion can at most depend on: rest-mass m, 4-position x^α, and 4-velocity u^α (there is no acceleration, etc.). But L cannot depend on x^α -- since m is free, all positions x^α must be equivalent. So we take...

$$\gamma L = -m u_\alpha u^\alpha = -mc^2 \leftarrow \text{obvious Lorentz scalar,}$$

i.e.

$$\boxed{L_{free} = -mc^2\sqrt{1-(u/c)^2}} \quad \Big/ \text{ NOTE: } L_{free} \simeq \tfrac{1}{2}mu^2 - mc^2 + \cdots, \text{ for } u \ll c. \tag{5}$$

Want: $L_{free} \simeq$ Newton K.E. + cnst in nonrel. limit (this is the reason for choice of (−) sign out front).

This $L_{free} \Rightarrow$ eqtn-of-motion...

$$\frac{d}{dt}\left(\frac{\partial L_{free}}{\partial u_i}\right) = \frac{\partial L_{free}}{\partial x_i} = 0, \quad \text{or} \quad \underline{\underline{\frac{d}{dt}(\gamma m \mathbf{u}) = 0}}. \tag{6}$$

In fact this is the correct free particle eqtn-of-motion. So above L_{free} appears OK

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4) Motion of q in external $\mathbb{E}$ & $\mathbb{B}$ must perforce lead to a more complicated L. Since the fields are specified by the 4-potential $A^\alpha = (\phi, \mathbb{A})$, and in fact we are looking for a potential term to "add on" to the K.E.-like L_{free}, try

$$\rightarrow \gamma L = \gamma L_{free} - (q/c)\, u_\alpha A^\alpha \quad \Big/ \; u_\alpha = \gamma(c, -\mathbf{u}), \text{ covariant 4-velocity; } A^\alpha = (\phi, \mathbb{A}), \text{ contravariant 4-potential.} \tag{7}$$

γL constructed this way is an evident Lorentz scalar, and it at least has the correct form at nonrelativistic velocities $u \ll c$. We can see this by writing:

L_{EM} for rel$^{\underline{v}}$c q ⇒ H_{EM} for rel$^{\underline{v}}$c q

$$\left[\begin{array}{l} L = L_{free} - \frac{q}{c}(c, -\mathbf{u})\cdot(\phi, \mathbb{A}) = L_{free} - q\phi + q\frac{\mathbf{u}}{c}\cdot\mathbb{A}. \\ \text{so}/\!/ \quad \underline{L \to L_{free} - q\phi, \ @\, u \ll c} \ldots \text{ this is correct nonrel. form.} \end{array}\right. \tag{8}$$

This gives us good reason to suppose that this L is the correct relativistic form,

i.e.// $\boxed{L_{EM} = -mc^2\sqrt{1-(u/c)^2} - q\phi + \frac{q}{c}\mathbf{u}\cdot\mathbb{A}}$, RELATIVISTIC LAGRANGIAN for q in EXTERNAL $(\phi, \mathbb{A})$. (9)

Should describe the relativistic motion of q in the external EM pot$^{\underline{ls}}$ $(\phi, \mathbb{A})$. One can write this relation much more compactly as

$$\rightarrow \underline{\underline{\gamma L_{EM} = (-)\, u_\alpha P^\alpha}}, \quad \text{w/} \quad P^\alpha = m u^\alpha + \frac{q}{c}A^\alpha \left\{\begin{array}{l}\text{"canonical"} \\ \text{momentum}\end{array}\right. . \tag{10}$$

We will use this from later.

But the acid test for L of Eq.(9) is this: do the Lagrange eqtns, viz $\frac{d}{dt}(\partial L/\partial u_i) = \partial L/\partial x_i$, actually give the Lorentz force law [Eqs (1) or (2)] for (relativistic) q in $\mathbb{E}$ & $\mathbb{B}$? The answer is <u>YES</u>... details are left to a problem (see Prob. ⑪).

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5) The step from a system Lagrangian L to a Hamiltonian H is <u>prescribed</u>, via...

$$\rightarrow H = \mathbb{P}\cdot\mathbf{u} - L, \quad \text{w/} \quad P_k = \partial L/\partial u_k \quad \int \begin{array}{l} P_k \text{, defined this way, is called the} \\ \text{momentum "conjugate" to cd. } x_k. \end{array} \tag{11}$$

For the EM Lagrangian L in Eq.(9), the conjugate momenta are...

$$\rightarrow P_k = \frac{\partial}{\partial u_k}\left[-mc^2\sqrt{1-u^2/c^2} - q\phi + \frac{q}{c}\mathbf{u}\cdot\mathbb{A}\right] = \gamma m u_k + \frac{q}{c}A_k$$

ordinary 3-vectors

i.e.// $\underline{\underline{\mathbb{P} = \mathbb{p} + \frac{q}{c}\mathbb{A}}}$ $\int$ w/ $\mathbb{p} = \gamma m\mathbf{u}$ (ordinary relativistic 3-momentum of m). term in $(q/c)\mathbb{A}$ <u>modifies</u> $\mathbb{p}$ due to presence of fields. (12)

Now, put this conjugate $\mathbb{P}$ -- together with L of Eq.(9) -- into H of Eq.(11)...

Explicit form of H_{EM} for relativistic q.

$$\rightarrow H_{EM} = (\mathbb{P} - \tfrac{q}{c}\mathbb{A})\cdot \mathbf{u} + q\phi + mc^2/\gamma \;, \;\text{w/}\; \gamma = 1/\sqrt{1-u^2/c^2}. \qquad (13)$$

Want to eliminate explicit appearance of $\mathbf{u}$ in H_{EM}. To that end, define

$$\Pi = \mathbb{P} - \tfrac{q}{c}\mathbb{A}\;, \;\text{so/}\; \Pi = \mathbf{p} = \gamma m \mathbf{u} \quad \text{fact that } \Pi = \mathbf{p}\text{(particle) here is just fortuitous}$$

$$\text{so/}\; \mathbf{u} = \frac{1}{\gamma m}\Pi\;, \;\text{and/}\; \Pi^2 = (\gamma m)^2 u^2 = (\gamma m)^2 \underbrace{c^2\left(1-\frac{1}{\gamma^2}\right)}_{\text{this is just } u^2}$$

$$\Rightarrow \sqrt{\Pi^2 + (mc)^2} = \gamma m c \qquad (14)$$

Then: $\mathbf{u} = c\,\Pi/\sqrt{\Pi^2+(mc)^2}$ (this is just old $\mathbf{u} = c^2\mathbf{p}/\mathcal{E}$ relation [Eq.(18) p. SRT 15]), and when this relation is used in Eq. (13), we can rewrite $\mathcal{H}_{EM}$ as...

$$H_{EM} = \frac{c\,\Pi^2}{\sqrt{\Pi^2+(mc)^2}} + q\phi + \frac{c(mc)^2}{\sqrt{\Pi^2+(mc)^2}} = c\sqrt{\Pi^2+(mc)^2} + q\phi\;,$$

or/

$$\boxed{H_{EM} = \sqrt{(c\mathbb{P} - q\mathbb{A})^2 + (mc^2)^2} + q\phi} \qquad (15)$$

RELATIVISTIC HAMILTONIAN for q in EXTERNAL $(\phi, \mathbb{A})$.

REMARKS

1. This H_{EM}, together with Hamilton's eqtns-of-motion $\left(\dot{x}_k = \frac{\partial H}{\partial P_k}\;,\; \dot{P}_k = -\frac{\partial H}{\partial x_k}\right)$ in fact regurgitate the Lorentz force law (see Prob. 71).

2. H_{EM} is not a Lorentz scalar, but γH_{EM} is covariant (like an energy). See this by...

$$(H_{EM} - q\phi)^2 = c^2[\Pi^2 + (mc)^2]\;, \;\text{i.e./}\; \left(\tfrac{1}{c}(H_{EM} - q\phi)\right)^2 - \Pi^2 = (mc)^2 \qquad (16)$$

$$\text{or/}\; \underline{\underline{\Pi_\alpha \Pi^\alpha = (mc)^2}}\;, \;\text{if/}\; \Pi^\alpha = \left(\tfrac{1}{c}(H_{EM} - q\phi), \Pi\right) = \left(\tfrac{1}{c}H_{EM}, \mathbb{P}\right) - \tfrac{q}{c}\underbrace{(\phi, \mathbb{A})}_{\text{4-potential}}.$$

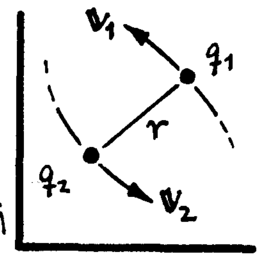
Π^α is evidently a 4-vector... then so is: $P^\alpha = (\tfrac{1}{c}H_{EM}, \mathbb{P}) = \Pi^\alpha + \tfrac{q}{c}A^\alpha$. Thus H_{EM} transforms like the timelike component of a 4-vector (conjugate) momentum.

3. Neither L_{EM} [Eq. (9)] nor H_{EM} [Eq. (15)] is gauge-invariant. But a gauge transform on $(\phi, \mathbb{A})$ alters neither eqtns-of-mot . All L_{EM} & H_{EM} are "gauge-equivalent".

4. For QM, prescription for H_{EM} [Eq. (15)] is $\mathbb{P} \rightarrow -i\hbar\nabla$. What does $H_{EM}\psi = \mathcal{E}\psi$ mean?

Darwin Interaction [Jackson, Sec. 12.7]

6) L_{EM} [Eq. (9)] & H_{EM} [Eq. (15)] govern the motion of a single particle (q,m) moving relativistically (i.e. at any allowed velocity $u<c$) in external fields $\mathbb{E}$ & $\mathbb{B}$, as specified by external potentials $(\phi, \mathbb{A})$.



But other "interesting" interacting (and simple) EM systems may have different configurations -- for example, two charges q_1 & q_2 interacting via each other's fields, with no external $\mathbb{E}$ and/or $\mathbb{B}$ present...

This configuration is of interest in atoms, and the Lagrangian L describing it will be different from the above L_{EM}. Nonrelativistically, we know:

$$\left.\begin{array}{l}\text{nonrelativistic}\\ \text{interaction potential}\end{array}\right\} V(r) = \frac{q_1 q_2}{r} \Rightarrow \underline{\underline{L_{int.}^{(nr)}}} = -V(r) = -\frac{q_1 q_2}{r} \int \begin{array}{l}\text{good for:}\\ v_1 \,\&\, v_2 \ll c.\end{array} \qquad (17)$$

Here we just focus on the interactive part of L; the full L will contain L_{int} plus contributions $L_{1,free}$ & $L_{2,free}$ from Eq. (5), p. L&H 2. L_{int} contains all the mutual potentials which q_1 & q_2 see because of their proximity.

What we want to do is calculate relativistic corrections to $L_{int}^{(nr)}$ of Eq. (17); we expect: $L_{int}^{(nr)} \rightarrow L_{int} = L_{int}^{(nr)} + \text{fcn}(v_{1,2}/c)$. The corrections are due to the fact that the relative motion of q_1 & q_2 generates $\mathbb{B}$-fields which alter the motion. The $\mathbb{B}$-fields are order (v/c) relative to the main $\mathbb{E}$-field interaction, and the q's interact with them at relative strength $\propto \frac{v}{c} B \sim (v/c)^2$. So we expect the corrections to be $\sim (v_{1,2}/c)^2$ relative to $L_{int}^{(nr)}$.

→ There is an immediate problem, however. $L_{int} = L_{int}(x_i; \dot{x}_i; t)$ is supposed to be a function of the instantaneous positions x_i and velocities $\dot{x}_i$ of all the particles q_i. But we know that the fields/potentials generated by q_2 at q_1 at time t are in fact functions of the retarded time $t_{ret} = t - (r/c)$ at q_2. So a single Lagrangian system of the q_1-q_2 system makes sense only if we can neglect retarded time corrections. This turns out to be possible to $O(v/c)^2$, but no higher. Point is: we can at best hope for an approximate L_{int}, good to $O(v/c)^2$ at best.

Derivation of Darwin Lagrangian

7) For q_2's effect on q_1, Lint must look like the potential interaction which appears in L_{Em} of Eq. (9), p. L&H 3, viz...

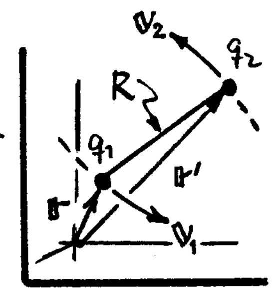
$$\rightarrow \boxed{L_{int} = -q_1\phi_{12} + q_1\left(\frac{\mathbb{v}_1}{c}\right)\cdot \mathbb{A}_{12}} \quad \int \phi_{12} \,\&\, \mathbb{A}_{12} \text{ are the scalar \& vector potentials generated at } q_1 \,(@\, t) \text{ by } q_2\,(@\, t-(r/c)). \qquad (18)$$

Since various choices of gauge for ϕ & $\mathbb{A}$ will give L_{int} forms that are "gauge equivalent" (see Prob. 92), meaning the L_{int}'s give the same eqtns-of-motion, then we have a gauge freedom. We shall choose the "Coulomb Gauge,"★ where $\nabla\cdot\mathbb{A}=0$ and:

$$\rightarrow \phi_{12}(\mathbb{r},t) = \int \frac{d^3x'}{R}\rho_2(\mathbb{r}',t), \quad \text{w/ } R = |\mathbb{r}-\mathbb{r}'| \quad \int \phi_{12} \text{ is "instantaneous" Coulomb potential, w/ no } t_{ret} \text{ corrections.} \qquad (19)$$

Big advantage here is that ϕ_{12} has no retarded time corrections; all t_{ret} dependence is thrown into $\mathbb{A}_{12}$. In this gauge, $\mathbb{A}_{12}$ is given by★

$$\rightarrow \mathbb{A}_{12}(\mathbb{r},t) = \frac{1}{c}\int \frac{d^3x'}{R}\mathbb{J}_{2T}(\mathbb{r}',t_{ret}), \quad \text{w/ } t_{ret} = t - \frac{1}{c}R(t_{ret}). \qquad (20)$$



$\mathbb{J}_{2T}$ is the transverse component of the current density due to the motion of q_2 relative to q_1. We only need $\mathbb{A}_{12}$ to $O(v_2/c)$ in order to get L_{int} of Eq. (18) correct to $O(v/c)^2$ [since factor $\frac{v_1}{c}$ is built-in]. But $\mathbb{A}_{12}$ of Eq. (20) is already $O(v_2/c)$, even without any t_{ret} corrections. So, per previous remarks, we can neglect t_{ret} altogether (i.e. set $t_{ret}\rightarrow t$), and still get L_{int} of Eq. correct to $O(v/c)^2$.

To get $\mathbb{A}_{12}$ of Eq. (20), we need to calculate $\mathbb{J}_{2T}(\mathbb{r}',t)$. Recalling Prob. 40...

$t_{ret}=t$, to our approxn.

$$\rightarrow \mathbb{J}_{2T}(\mathbb{r}',t) = \frac{1}{4\pi}\nabla'\times\left[\nabla'\times\int\frac{d^3x''}{|\mathbb{r}'-\mathbb{r}''|}\mathbb{J}(\mathbb{r}'',t)\right] \leftarrow \text{set } \mathbb{J}(\mathbb{r}'',t) = q_2\mathbb{v}_2\,\delta(\mathbb{r}''-\mathbb{r}_2)$$

$$= \frac{q_2}{4\pi}\nabla'\times\left[\nabla'\times\left(\frac{\mathbb{v}_2}{|\mathbb{r}'-\mathbb{r}_2|}\right)\right], \quad \mathbb{r}_2 \,\&\, \mathbb{v}_2 \text{ are fcns of } t. \qquad (21)$$

Now, for a reduction of this expression, expand the triple product...

★ ϕ519 Class Notes, p. ME 13... or Jackson Sec. 6.5, pp. 220-223.

Darwin Lagrangian

$$\rightarrow \mathbf{J}_{2T}(\mathbf{r}',t) = \frac{q_2}{4\pi}\left\{\nabla'\left[\nabla\cdot\left(\frac{\mathbf{v}_2}{|\mathbf{r}'-\mathbf{r}_2|}\right)\right] - \underbrace{\nabla'^2\left(\frac{\mathbf{v}_2}{|\mathbf{r}'-\mathbf{r}_2|}\right)}_{\text{this term} \Rightarrow \text{ a } \delta\text{-fcn}}\right\}$$

$\mathbf{v}_2$ = cnst for integration over $\mathbf{r}'$ in Eq. (20), w/ parameter t "frozen".

$$= q_2\mathbf{v}_2\,\delta(\mathbf{r}'-\mathbf{r}_2) + \frac{q_2}{4\pi}\nabla'\left[\mathbf{v}_2\cdot\nabla'\left(\frac{1}{|\mathbf{r}'-\mathbf{r}_2|}\right)\right]$$

$$= q_2\mathbf{v}_2\,\delta(\mathbf{r}'-\mathbf{r}_2) - \frac{q_2}{4\pi}\nabla'\left[\frac{\mathbf{v}_2\cdot(\mathbf{r}'-\mathbf{r}_2)}{|\mathbf{r}'-\mathbf{r}_2|^3}\right]. \tag{22}$$

Then...

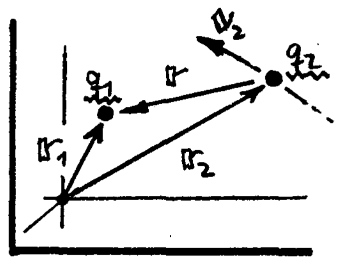
$$\mathbf{A}_{12}(\mathbf{r}_1,t) \simeq \frac{1}{c}\int\frac{d^3x'}{|\mathbf{r}_1-\mathbf{r}'|}\mathbf{J}_{2T}(\mathbf{r}',t) = \frac{q_2\mathbf{v}_2}{cr} - \frac{q_2}{4\pi c}\int\frac{d^3y}{|\mathbf{y}-\mathbf{r}|}\nabla_y\left(\frac{\mathbf{v}_2\cdot\mathbf{y}}{y^3}\right), \tag{23}$$

w/ $\mathbf{y} = \mathbf{r}'-\mathbf{r}_2$, and $\underline{\underline{\mathbf{r}}} = (\mathbf{r}_1-\mathbf{r}_2)$ is separation of q_1 & q_2 [replaces $\mathbf{R}$ in Eq. (19)].

The remaining integral in (23) is "straightforward" (partial-integrate, convert $\nabla_y(1/|\mathbf{y}-\mathbf{r}|)$ to $(-)\nabla_r(1/|\mathbf{y}-\mathbf{r}|)$, integrate & differentiate, etc). Finally...

$$\underline{\underline{\mathbf{A}_{12}(\mathbf{r},t) \simeq \frac{q_2}{2cr}\left[\mathbf{v}_2 + \frac{\mathbf{r}(\mathbf{v}_2\cdot\mathbf{r})}{r^2}\right]}} \tag{24}$$

(instantaneous) $\mathbf{A}$ generated <u>at</u> q_1 <u>by</u> q_2, to $O(v_2/c)$.



Now we can form L_{int} of Eq. (18) correct to $O(v/c)^2$...

$$\boxed{L_{int} = -q_1\left[\phi_{12} - \left(\frac{\mathbf{v}_1}{c}\right)\cdot\mathbf{A}_{12}\right] = -\frac{q_1 q_2}{r}\left\{1 - \frac{1}{2c^2}\left[(\mathbf{v}_1\cdot\mathbf{v}_2) + \frac{1}{r^2}(\mathbf{v}_1\cdot\mathbf{r})(\mathbf{v}_2\cdot\mathbf{r})\right]\right\}} \tag{25}$$

[We have put $\phi_{12} = \frac{q_2}{r}$, from Eq. (19)]. This L_{int} is known as the "**<u>Darwin Lagrangian</u>**". It identifies $O(v/c)^2$ corrections to the Coulomb interaction, as:

$$V^{(nr)}(r) = \frac{q_1 q_2}{r} \rightarrow \underline{\underline{V(r) = \frac{q_1 q_2}{r}\left\{1 - \underbrace{\frac{1}{2}\left[\boldsymbol{\beta}_1\cdot\boldsymbol{\beta}_2 + (\boldsymbol{\beta}_1\cdot\mathbf{r})(\boldsymbol{\beta}_2\cdot\mathbf{r})\right]}_{O(\beta^2)}\right\}}}. \tag{26}$$

REMARKS

1. Corrected $V(r)$ is symmetric in the subscripts 1 & 2, so $V(q_2 \text{ on } q_1) \equiv V(q_1 \text{ on } q_2)$. $V(r)$ can best be written in center-of-mass coordinates.

2. In QM (roughly speaking), the β_k are replaced by <u>operators</u> $\left[\beta_k \rightarrow -\frac{i\hbar}{m_k c}\nabla_k\right]$, and the correction term in (26) is called the "Breit Interaction." So long as $\beta \ll 1$, it $\Rightarrow$ easiest way to calculate $O(\beta^2)$ corrections to two-electron atoms.