● QM Observability and Heisenberg's Uncertainty Principle.

By now we have completed the first 3 topics listed in Eq. (1), p. Prop. 1, as properties of Schrödinger's wave mechanics. Here we treat the fourth topic, viz. the uncertainty relations. Although these relations are central to the QM realization of wave-particle duality, they do not bear directly on wave mechanics as such -- instead, they govern what is meant by "observability" in the theory, i.e. when is it possible to observe a precise number for some QM dynamical variable? With the machinery at our disposal, it is now possible to clearly define what we mean by QM "observability", and what role the uncertainty relations play here -- by and by, we will also define those relations more clearly. Both points are important to the structure of QM theory, if not directly to the details of wave mechanics.

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1) When doing QM wavepackets, we noted the (approximate) uncertainty★ relations for position & momentum, and energy & time [Eq. (27), p. Duality 12]:

$$\Delta x \Delta p \sim \hbar, \quad \Delta E \Delta t \sim \hbar. \tag{1}$$

These relations mean that x & p, and E & t cannot simultaneously be specified to arbitrary precision -- i.e. these pairs of variables cannot be "observed" as each having distinct values at a given moment. By now, we

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★ What does "uncertainty" mean here? $\Delta x \Delta p \sim \hbar$ implies that if the position x of a wave-particle is measured to lie in the range $x \pm \Delta x$ (i.e. x is "uncertain" to $\pm \Delta x$), then that wave-particle will automatically exhibit a momentum in the range $p \pm \Delta p$, where the momentum uncertainty is at least as large as $\Delta p \sim \hbar/\Delta x$. Any attempt to decrease the position uncertainty Δx results in an increase in the momentum uncertainty Δp, and vice-versa.

know that the operators in Eq. (1) do not commute; in particular...

$\rightarrow$ w/ $p = -i\hbar\frac{\partial}{\partial x}$: $[x,p] = i\hbar$; w/ $E = i\hbar\frac{\partial}{\partial t}$: $[E,t] = i\hbar$. (2)

This suggests that an uncertainty relation (i.e. impossibility of simultaneous observability of some pair of QM variables) is related to the commutator. For general operators A & B, it appears that...

$\rightarrow \Delta A\,\Delta B \sim |\langle [A,B] \rangle|$, $\langle\ \rangle \Rightarrow$ expectation value. (3)

Both x & p, and E & t, fit this Ansatz. In fact, Eq. (3) turns out to be true. But to make clear what is meant here, we need to define the terms "uncertainty" and "observability" more carefully.

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2) To define "uncertainty", we take a cue from probability theory, viz.

For a probability distribution $p(x)$, w/ $\int p(x)\,dx = 1$, define:★

MEAN VALUE of $A(x)$: $\overline{A} = \int A(x)\,p(x)\,dx$;

MEAN SQUARE VALUE: $\overline{A^2} = \int A^2(x)\,p(x)\,dx$;

ROOT MEAN SQUARE (RMS) DEVIATION ΔA, defined by:

$(\Delta A)^2 = \overline{A^2} - (\overline{A})^2 = \int (A-\overline{A})^2\,p(x)\,dx$. Note: $(\Delta A)^2 \geqslant 0$. (4)

In an analogous manner, for a QM operator A & wavefcn ψ, we define:

$\overline{A} = \int \psi^*(x)\,A(x)\,\psi(x)\,dx = \langle A \rangle$; $\overline{A^2} = \langle A^2 \rangle$;

UNCERTAINTY ΔA : $(\Delta A)^2 = \langle A^2 \rangle - \langle A \rangle^2 = \langle (A - \langle A \rangle)^2 \rangle$. (5)

★ x is the independent variable, something measurable as the outcome of an event. E.g. x could be the # heads that occur when you toss 100 coins in the air at the same time. $p(x)$ is the probability of x actually occurring. The integrals $\int$ over $p(x)$ span the available range of x... for the coins, $0 \leqslant x \leqslant 100$.

Definition(s) of QM Observability.

Now for QM "observability". We adopt the reasonable point-of-view that a definite and unique value a can be observed each time the dynamical variable represented by operator A is measured, only if the uncertainty ΔA in Eq. (5) vanishes... i.e. $a = \langle A \rangle$ is a certain number. This means:

QM OBSERVABILITY I : in state ψ, operator A has a definite value a iff : $\langle f(A)\rangle = \langle \psi | f(A)\psi\rangle = f(a)$, for all fcns f. (6)

In fact, this requirement $\Rightarrow$ the uncertainty ΔA must vanish, as...

$$\rightarrow \langle A\rangle = a,\quad \langle A^2\rangle = a^2 = \langle A\rangle^2,\quad \text{so}/\!/\ (\Delta A)^2 = \langle (A-a)^2\rangle = 0. \tag{7}$$

Then--if A is a Hermitian operator--$(\Delta A)^2 = 0$ means...

$$\langle \psi | (A-a)(A-a)\psi\rangle \overset{\text{Hermitian}}{=} \langle (A-a)\psi | (A-a)\psi\rangle = \int |(A-a)\psi|^2 dx = 0;$$

$$\text{so}/\!/\ \underline{\underline{A\psi = a\psi}},\ ^{\text{w}/\!/}\ a = \text{a number} \Rightarrow \underline{\underline{\psi \text{ is an eigenfcn of } A}}. \tag{8}$$

So we have a second & equivalent definition of QM "observability", viz.

QM OBSERVABILITY II : in state ψ, operator A has a definite value a, with uncertainty $\Delta A = 0$, if and only if : $A\psi = a\psi$, i.e. ψ is an eigenstate of A which corresponds to the eigenvalue a (a number). (9)

In particular, this means that a QM system can have definite energies E, if it is specified by wavefcns ψ that are eigenstates of the Hamiltonian $\mathcal{H}$, via solutions to Schrödinger's Eqtn : $\mathcal{H}\psi = E\psi$.

$\boxed{\text{ASIDE}}$ Simultaneous observability of QM operators A & B.

When can two operators A & B yield certain values a & b in state ψ? I.e.

$$\rightarrow \underline{A\psi = a\psi},\ \text{and : } \underline{B\psi = b\psi},\ ^{\text{w}/}\ a\ \&\ b = \text{eigenvalues (numbers)}. \tag{10}$$

When are A & B simultaneously observable? Heisenberg's Relation.

If "simultaneous observability" means (10) holds, then A & B commute, as...

$$\left.\begin{aligned} B(A\psi) &= B(a\psi) = a(B\psi) = (ab)\psi; \\ A(B\psi) &= A(b\psi) = b(A\psi) = (ba)\psi; \end{aligned}\right\} (AB - BA)\psi = [A, B]\psi = 0$$

so $A \to a$ & $B \to b$ are "simultaneously observable" iff $[A, B] \equiv 0$. (11)

As a contrapositive of this proposition, we have: if A & B do not commute, then they are not simultaneously observable. So, when $[A, B] \neq 0$, then $(\Delta A)(\Delta B) \neq 0$. This explains "why" x & p, and E & t, neither pair of which commute, are not simultaneously observable.