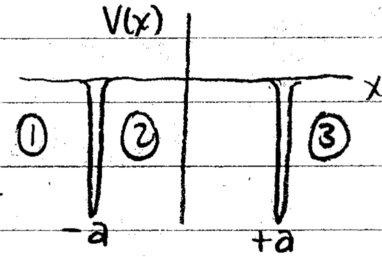
1/3/71 (36) For a single δ-fcn potential, see prob (2) on Phys. 505 final exam.

Get one bound state : $\mathcal{E} = -\hbar^2 k^2/2m = -\frac{1}{2} mC^2/\hbar^2$, $k = mC/\hbar^2$

For double δ-fcn well ...

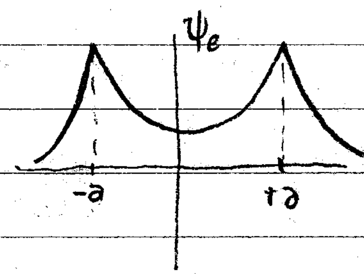


$$V(x) = -C\left[\delta(x+a) + \delta(x-a)\right]$$

Let $E = -\hbar^2 \kappa^2/2m$

$$\psi_1 = Ae^{+\kappa x}, \quad \psi_2 = Be^{+\kappa x} + Ce^{-\kappa x}, \quad \psi_3 = De^{-\kappa x} \quad \Big\{ \text{general solutions}$$

a) Even Parity Solutions



Choose: $D = +A$, $C = +B$ $\Rightarrow$ $\psi_1 = Ae^{+\kappa x}$, $\psi_2 = B(e^{+\kappa x} + e^{-\kappa x})$, $\psi_3 = Ae^{-\kappa x}$

Continuity in ψ : $Ae^{-\kappa a} = B(e^{+\kappa a} + e^{-\kappa a})$

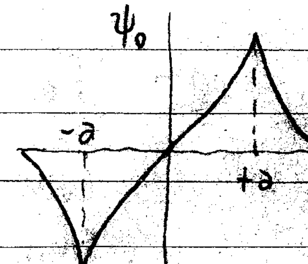
Discontinuity in ψ' ...

$$\psi_3'(a+) - \psi_2'(a-) = -\frac{2mC}{\hbar^2}\psi_3(a)$$

$$\Rightarrow \left(\frac{2k}{\kappa} - 1\right)Ae^{-\kappa a} = B(e^{+\kappa a} - e^{-\kappa a}), \quad k = mC/\hbar^2$$

$\therefore$ energy condition is : $\boxed{\tanh \kappa a = \frac{2k}{\kappa} - 1}$

Odd Parity Solutions



Choose: $D = -A$, $C = -B$ $\Rightarrow$ $\psi_1 = -De^{+\kappa x}$, $\psi_2 = B(e^{+\kappa x} - e^{-\kappa x})$, $\psi_3 = De^{-\kappa x}$

Continuity in ψ & discontinuity in ψ', as above, gives

energy condition : $\boxed{\operatorname{ctnh} \kappa a = \frac{2k}{\kappa} - 1}$, $k = mC/\hbar^2$

For $a \to \infty$ (separated wells), we have ...

even solns

$$\frac{2k}{\kappa} - 1 = \tanh \kappa a \underset{a\to\infty}{\simeq} 1 - 2e^{-2\kappa a} \quad , \ \kappa a \gg 1$$

$$\Rightarrow \kappa \simeq k/(1-e^{-2\kappa a}) \simeq k(1+\epsilon), \ \epsilon \simeq e^{-2ka} \ll 1$$

$$\therefore E_e = -\frac{\hbar^2 \kappa_e^2}{2m} \simeq (1+2\epsilon)\mathcal{E}, \quad \mathcal{E} = -\frac{\hbar^2 k^2}{2m} \left\{ \begin{array}{l} \text{energy of a} \\ \text{single well} \end{array} \right.$$

$$= -\tfrac{1}{2} mC^2/\hbar^2$$

odd solns

$$\frac{2k}{\kappa} - 1 = \operatorname{ctnh} \kappa a \simeq 1 + 2e^{-2\kappa a} \quad , \ \kappa a \gg 1$$

$$\Rightarrow \kappa \simeq k(1+e^{-2\kappa a})^{-1} \simeq k(1-\epsilon)$$

$$\therefore E_o = -\frac{\hbar^2 \kappa_o^2}{2m} \simeq (1-2\epsilon)\mathcal{E}$$

{ N.B. Both E_e & $E_o \to \mathcal{E}$ (single well) as $a \to \infty$, as you'd expect.

Note ψ_e is more tightly bound than ψ_o (as in H_2 molecule)
Energy splitting is

$$\Delta E = E_o - E_e = 4\epsilon|\mathcal{E}| \simeq (2mC^2/\hbar^2)\, e^{-2mCa/\hbar^2}$$

For $a \to 0$ (united wells), we have ...

even solns } $\frac{2k}{\kappa} - 1 = \tanh \kappa a \underset{\kappa a \ll 1}{\simeq} \kappa a$

For $a=0$, we have potential $V=-2C\delta(x)$, and $K=2k=2mC/\hbar^2$ which is what we'd expect for a single well of strength $2C$. So there is always a bound state for ψ_e. For $a\neq 0$, but small...

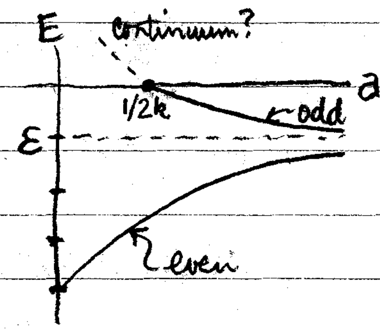
$$aK^2+K-2k=0 \Rightarrow K=\frac{1}{2a}\left[-1+\sqrt{1+8ka}\right] \quad \begin{cases}\text{choose + sign}\\ \text{for } K\geq 0\end{cases}$$

or, $K \simeq 2k(1-2ka)$

odd solns $\Big\}$ $\frac{2k}{K}-1=\coth Ka \underset{Ka\ll 1}{\simeq} 1/Ka \Rightarrow Ka\simeq 2ka-1$

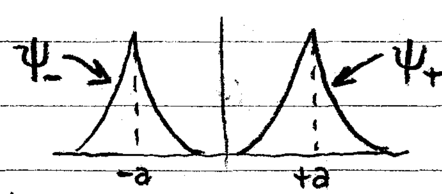
or, $K\simeq 2k-\frac{1}{a}$

$K>0$ only for $a>1/2k$. So there is no bound state for ψ_o for sufficiently small a. A rough sketch of the allowed energies is given at right.



b) A rough sketch of ψ_e & ψ_o is provided above. The states

$$\psi_\pm=\frac{1}{\sqrt{2}}(\psi_e\pm\psi_o)$$



obviously correspond to the particle being located at either the RH well (i.e. ψ_+) or LH well (i.e. ψ_-)

c) Note that

$$\psi_+ = \frac{1}{\sqrt{2}}(\psi_e + \psi_o) \qquad \Rightarrow \qquad \psi_e = \frac{1}{\sqrt{2}}(\psi_+ + \psi_-)$$
$$\psi_- = \frac{1}{\sqrt{2}}(\psi_e - \psi_o) \qquad\qquad \psi_o = \frac{1}{\sqrt{2}}(\psi_+ - \psi_-)$$

$$\therefore\ \Psi = \left(\psi_e e^{-\frac{i}{\hbar}E_e t} + \psi_o e^{-\frac{i}{\hbar}E_o t}\right)/\sqrt{2}$$
$$= \frac{1}{2}\psi_+\left(e^{-\frac{i}{\hbar}E_e t} + e^{-\frac{i}{\hbar}E_o t}\right) + \frac{1}{2}\psi_-\left(e^{-\frac{i}{\hbar}E_e t} - e^{-\frac{i}{\hbar}E_o t}\right)$$

Write $E_e = \left(\frac{E_o + E_e}{2}\right) - \left(\frac{E_o - E_e}{2}\right)$, $E_o = \left(\frac{E_o + E_e}{2}\right) + \left(\frac{E_o - E_e}{2}\right)$; define $\hbar\Omega = (E_o - E_e)/2$, $\bar{E} = (E_o + E_e)/2$

$$\therefore\ \Psi = \left(\psi_+ \cos\Omega t + i\,\psi_- \sin\Omega t\right) e^{-\frac{i}{\hbar}\bar{E}t}$$

or, $|\Psi|^2 = |\psi_+|^2\cos^2\Omega t + |\psi_-|^2\sin^2\Omega t$

Physical interpretation is that particle oscillates at freq. Ω between states ψ_+ & ψ_-, i.e. between localization first near one well then near the other one.

1/26/71 42 By defn : $\phi_\beta(\xi') = i\int dx\, G_0(\xi',\xi)\,\phi_\beta(\xi)$

Redone as prob. # 16, p507 (Jan '93)

For plane wave : $\phi_\beta(\xi) = \frac{1}{\sqrt{2\pi}} e^{i(k_\beta x - \omega_\beta t)}$, $\phi_\beta^*(\xi) = \frac{1}{\sqrt{2\pi}} e^{-i(k_\beta x - \omega_\beta t)}$.

But $\phi_\beta(-\xi) = \frac{1}{\sqrt{2\pi}} e^{i(k_\beta(-x) - \omega_\beta(-t))} = \frac{1}{\sqrt{2\pi}} e^{-i(k_\beta x - \omega t)} = \phi_\beta^*(+\xi)$

That this is always true can be seen from S. eq. for free particle

$$i\hbar \frac{\partial}{\partial t}\phi(x,t) = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\phi(x,t)$$

Complex conjugation $\Rightarrow$ $-i\hbar\frac{\partial}{\partial t}\phi^*(x,t) = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\phi^*(x,t)$

$(+x,+t)\to(-x,-t)$ $\Rightarrow$ $-i\hbar\frac{\partial}{\partial t}\phi(-x,-t) = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\phi(-x,-t)$

So $\phi(-x,-t)$ & $\phi^*(x,t)$ both satisfy same eqtn. They can at most differ by a phase factor, which we can choose $= 0$.

Propagation eqtn for $\phi_\beta(-\xi)$ is, by defn

$$\phi_\beta(-\xi') = i\int dx\, G_0(-\xi',-\xi)\,\phi_\beta(-\xi)$$

or// $\phi_\beta^*(\xi') = i\int dx\, \phi_\beta^*(\xi)\, G_0(-\xi',-\xi)$

But: $G_0(-\xi',-\xi) = -i\,\theta(-t'+t)\left(\frac{m/2\pi i\hbar}{-t'+t}\right)^{\frac{1}{2}} e^{+\frac{im}{2\hbar}|-x'+x|^2/(-t'+t)}$

$$= -i\,\theta(t-t')\left(\frac{m/2\pi i\hbar}{t-t'}\right)^{\frac{1}{2}} e^{+\frac{im}{2\hbar}|x-x'|^2/(t-t')} = G_0(+\xi,+\xi')$$

$\therefore$ $\phi_\beta^*(\xi') = i\int dx\, \phi_\beta^*(\xi)\, G_0(\xi,\xi')$ QED.

1/28/71 43 By comm. rules, have... 5

$$i\hbar L_x = L_y L_z - L_z L_y \quad \underline{\text{and}} \quad i\hbar L_y = L_z L_x - L_x L_z$$

Take exp values of both sides, remembering $L_z = m\hbar$ w.r.t. ψ, and L_z is Hermitian. Thus...

$$i\hbar \langle L_x \rangle = m\hbar(\langle L_y \rangle - \langle L_y \rangle) \equiv 0 \Rightarrow \langle L_x \rangle = 0$$
$$i\hbar \langle L_y \rangle = m\hbar(\langle L_x \rangle - \langle L_x \rangle) \equiv 0 \Rightarrow \langle L_y \rangle = 0$$

QED

1/28/71 44 $H\psi_1 = E\psi_1$ and $H\psi_2 = E\psi_2$. Suppose $[H, Q] = 0$.

Used as possible problem for Ph.D. qualifying exam - June 1992 -

Let $\phi = b_1\psi_1 + b_2\psi_2$ { Note: $H\phi = E\phi$, i.e. ϕ = eigenfcn of H. Now impose $Q\phi = q\phi$, i.e. ϕ also eigenfcn of Q

$$Q\phi = b_1 Q\psi_1 + b_2 Q\psi_2 = q(b_1\psi_1 + b_2\psi_2)$$

Operate first with $\langle\psi_1|\rangle$, then with $\langle\psi_2|\rangle$, assuming $\langle\psi_i|\psi_j\rangle = \delta_{ij}$

$$\boxed{\begin{aligned} (Q_{11}-q)b_1 + Q_{12}b_2 &= 0 \\ Q_{21}b_1 + (Q_{22}-q)b_2 &= 0 \end{aligned}}$$

Have soln only if...

$$\det\begin{pmatrix} Q_{11}-q & Q_{12} \\ Q_{21} & Q_{22}-q \end{pmatrix} = 0$$

$$\det = 0 \Rightarrow (q-Q_{11})(q-Q_{22}) - |Q_{12}|^2 = 0 \quad (\text{using } Q_{21} = Q_{12}^*)$$

$$\Rightarrow q = \tfrac{1}{2}\left[(Q_{11}+Q_{22}) \pm \sqrt{(Q_{11}-Q_{22})^2 + 4|Q_{12}|^2}\right]$$

Now use $Q_{22} = Q_{11}$ to get Q eigenvalues...

$$q_1 = Q_{11} + |Q_{12}|, \quad q_2 = Q_{22} - |Q_{12}|$$

Each q_i has a corresponding set $b_1^{(i)}$ & $b_2^{(i)}$ of b-coefficients.
Using $Q_{22} = Q_{11}$, $Q_{21} = Q_{12}^*$, and $Q_{12} = |Q_{12}| e^{-i\theta}$, have...

$\underline{q = q_1 = Q_{11} + |Q_{12}|}$

$$\begin{array}{lcl} (Q_{11} - q_1) b_1^{(1)} + Q_{12} b_2^{(1)} = 0 & & -|Q_{21}| b_1^{(1)} + Q_{12} b_2^{(2)} = 0 \\ Q_{12}^* b_1^{(1)} + (Q_{11} - q_1) b_2^{(1)} = 0 & \Rightarrow & Q_{12}^* b_1^{(1)} - |Q_{12}| b_2^{(2)} = 0 \end{array}$$

$$\therefore \; b_2^{(1)} = (|Q_{12}|/Q_{12}) b_1^{(1)} = (Q_{12}^*/|Q_{12}|) b_1^{(1)} = e^{i\theta} b_1^{(1)}$$

Impose normalization : $|b_1^{(1)}|^2 + |b_2^{(1)}|^2 = 1 \Rightarrow b_1^{(1)} = 1/\sqrt{2}$

$\therefore$ Eigenfcn for $q = q_1 = Q_{11} + |Q_{12}|$ $\Big\}$ $\phi_1 = \frac{1}{\sqrt{2}} (\psi_1 + e^{i\theta} \psi_2)$

$\underline{q = q_2 = Q_{11} - |Q_{12}|}$

Everything is the same here except $|Q_{12}|$ changes sign.

Get $b_2^{(2)} = -e^{i\theta} b_1^{(2)}$, and $b_1^{(2)} = 1/\sqrt{2}$

$\therefore$ Eigenfcn for $q = q_2 = Q_{11} - |Q_{12}|$ $\Big\}$ $\phi_2 = \frac{1}{\sqrt{2}} (\psi_1 - e^{i\theta} \psi_2)$

The ϕ_j are clearly orthonormal. To check the eigenvalues, note

$$\begin{aligned} \langle \phi_1 | Q | \phi_1 \rangle &= \tfrac{1}{2} \langle \psi_1 + e^{i\theta} \psi_2 | Q | \psi_1 + e^{i\theta} \psi_2 \rangle \\ &= \tfrac{1}{2} (Q_{11} + e^{i\theta} Q_{12} + e^{-i\theta} Q_{21} + Q_{22}) \quad \left\{ \begin{array}{l} Q_{12} = |Q_{12}| e^{-i\theta} \\ Q_{21} = |Q_{12}| e^{+i\theta} \end{array} \right. \\ &= \tfrac{1}{2} (2Q_{11} + 2|Q_{12}|) = Q_{11} + |Q_{12}| \end{aligned}$$

and similarly for $\langle \phi_2 | Q | \phi_2 \rangle$.

1/31/71 45 $\frac{d}{dt}P(t) = -\Gamma_0 \frac{1}{\tau}\int_{t-\tau}^{t} P(x)\,dx$

a) By mean value theorem (or something)

$$\lim_{\tau\to 0}\int_{t-\tau}^{t} P(x)\,dx = \lim_{\tau\to 0} P(t-\tfrac{1}{2}\tau)\,\tau$$

$$\therefore \lim_{\tau\to 0}\frac{d}{dt}P(t) = -\Gamma_0 \lim_{\tau\to 0} P(t-\tfrac{1}{2}\tau) = -\Gamma_0 P(t), \quad \text{QED}$$

b) Converting it to a 2ND order diff. eqtn, have

$$\frac{d^2}{dt^2}P(t) = -\frac{\Gamma_0}{\tau}\frac{d}{dt}\left[\int_{-\infty}^{t} P(x)\,dx - \int_{-\infty}^{t-\tau} P(x)\,dx\right]$$

$$= -\frac{\Gamma_0}{\tau}\left[P(t) - P(t-\tau)\right]$$

{ this is a "difference" diff eqtn. Obviously non-local in t. So what?

c) Assuming solution $P(t) = P_0 e^{-\Gamma t}$, get in above eqtn

$$-\Gamma e^{-\Gamma t} = -\frac{\Gamma_0}{\tau}\int_{t-\tau}^{t} e^{-\Gamma x}\,dx = -\frac{\Gamma_0}{\Gamma\tau}e^{-\Gamma t}(e^{\Gamma\tau}-1)$$

or $x^2 = x_0(e^{x}-1)$ $\begin{cases} x = \Gamma\tau \\ x_0 = \Gamma_0\tau \end{cases}$ i.e. $\Gamma^2 = \frac{\Gamma_0}{\tau}(e^{\Gamma\tau}-1)$

If $\tau \to 0$, expand exp to get

$x \simeq x_0(1 + \frac{1}{2}x + \frac{1}{6}x^2 + \dots)$

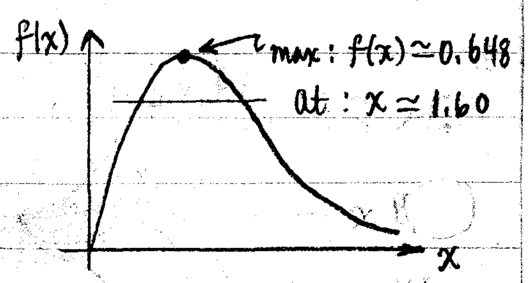
$\to$ 0^{th} order soln: $x \simeq x_0$

1^{ST} " " : $x \simeq x_0(1+\frac{1}{2}x_0)$

or $\Gamma \simeq (1+\frac{1}{2}\Gamma_0\tau)\Gamma_0 > \Gamma_0$

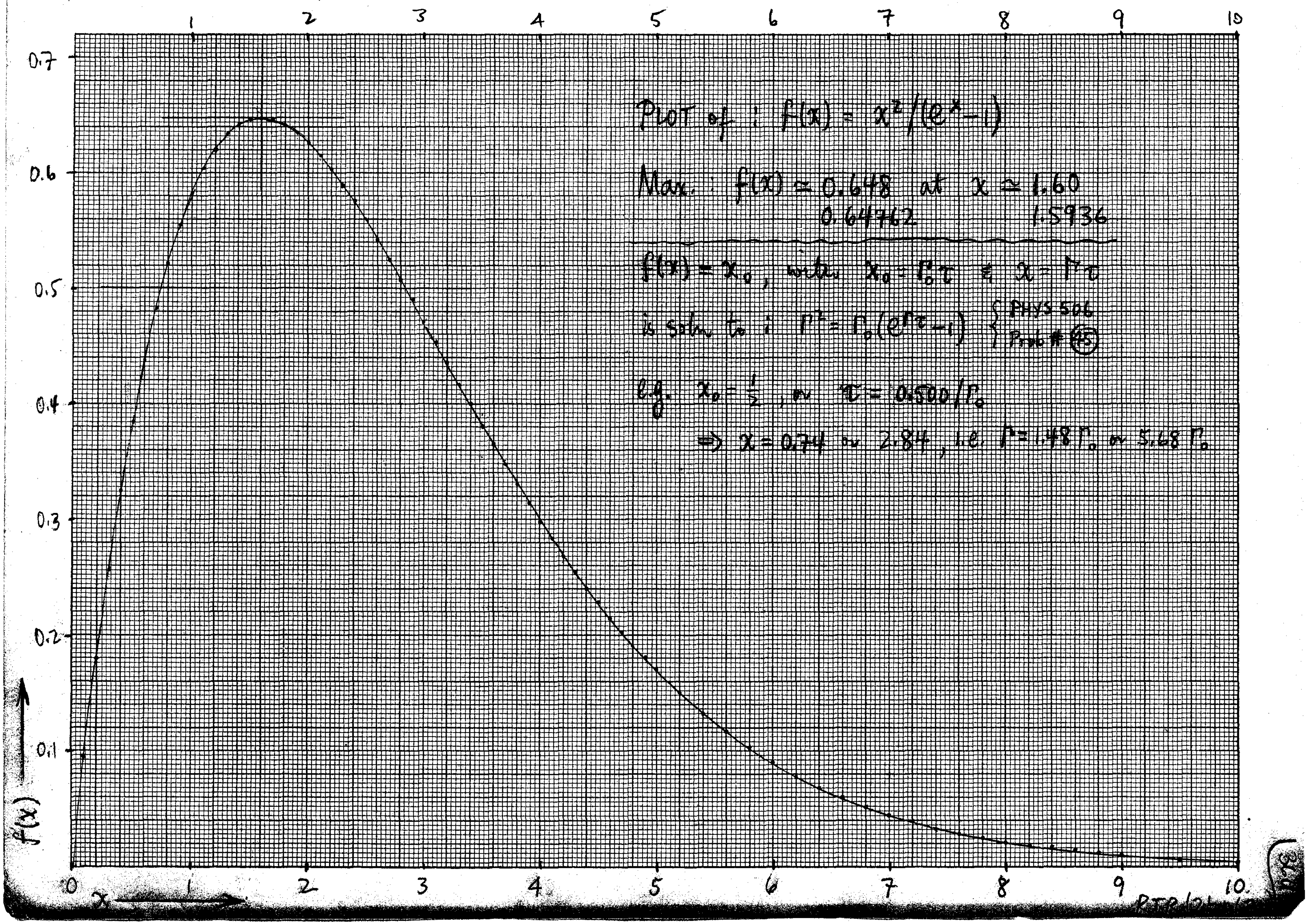
Define: $f(x) = x^2/(e^x-1)$.

Soln to above eqtn is: $f(x) = x_0$



E.g. $x_0 = \frac{1}{2} \Rightarrow \tau = 0.500/\Gamma_0 \Rightarrow \Gamma = \begin{matrix}1.48\,\Gamma_0 \\ 5.68\,\Gamma_0\end{matrix}$

Max permissible soln is for: $x_0 = 0.648 \Rightarrow \tau = 0.648/\Gamma_0 \Rightarrow \Gamma = 2.47\,\Gamma_0$


Plot of : $f(x) = x^2/(e^x - 1)$
Max. : $f(x) \simeq 0.648$ at $x \simeq 1.60$
0.64762 1.5936
$f(x) = x_0$, with $x_0 = \Gamma_0 \tau$ & $x = \Gamma\tau$
is soln to : $\Gamma^2 = \Gamma_0 (e^{\Gamma\tau} - 1)$
PHYS 506
Prob # 45
e.g. $x_0 = \frac{1}{2}$, or $\tau = 0.500/\Gamma_0$
$\Rightarrow x = 0.74$ or 2.84, i.e. $\Gamma = 1.48\,\Gamma_0$ or $5.68\,\Gamma_0$
f(x)
x
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
1
2
3
4
5
6
7
8
9
10

2/4/71 **46** This is stfwd arithmetic -- soln is indicated in Merzbacher, pp. 175-176.

2/4/71 **47** $T = e^{-\gamma}$. See Landau & Lifshitz, p. 174. Exact result for $l=0$ is

$$\gamma_0 = \frac{2}{\hbar}\int_{r_0}^{C/E}\left[2m\left(\frac{C}{r}-E\right)\right]^{\frac{1}{2}}dr = \frac{2C}{\hbar}\sqrt{\frac{2m}{E}}\left[\cos^{-1}\sqrt{\frac{E}{V(r_0)}} - \sqrt{\frac{E}{V(r_0)}\left(1-\frac{E}{V(r_0)}\right)}\right]$$

Let $V_0 = V(r_0)$ for convenience. Assume $E/V_0 \ll 1$. Expansion gives...

$$\gamma_0 \simeq \frac{2\pi C}{\hbar v}\left[1-\frac{4}{\pi}\sqrt{\frac{E}{V_0}}\left(1-\frac{1}{6}\frac{E}{V_0}+\theta\left(\left(\frac{E}{V_0}\right)^2\right)\right)\right] \simeq \frac{2\pi C}{\hbar v}\left[1-\frac{4}{\pi}\sqrt{\frac{E}{V_0}}\right]$$

~~Here we have neglected terms of relative order $(E/V_0)^{\frac{3}{2}}$ and higher.~~

If we include the centrifugal barrier $B_l(r) = l(l+1)\hbar^2/2mr^2$, then

m ← α-particle mass. $(6.646\times10^{-24}\,\mathrm{gm})$

$$\gamma_l = \frac{2}{\hbar}\int_{r_0}^{C/E}\left[2m\left(\frac{C}{r}+B_l(r)-E\right)\right]^{\frac{1}{2}}dr$$

For U^{238}, $r_0 \simeq 1.37\times10^{-13}\,\mathrm{cm}\times(238)^{\frac{1}{3}} = 8.49_4\times10^{-13}\,\mathrm{cm}$

~~Then, write $C = 2(Z-2)e^2$, $V_0 = C/r_0 = 30.5_2$ MeV~~ ($Z = 92$)

On the other hand, $B_l(r_0) = l(l+1)\dfrac{\hbar^2}{2mr_0^2} = l(l+1)\times 0.0724$ MeV

Note $\sigma = B_l(r_0)/V(r_0) = 0.00237\times l(l+1) \ll 1$ $\left\{\begin{array}{l}\text{even for } l=10\\ \sigma = 0.2609\end{array}\right.$

~~So $B_l(r)$ is never more than say 1% of $V(r)$, and vanishes quickly~~ as $r > r_0$. We can roughly approximate the integral by

$$\gamma_l = \frac{2}{\hbar}\int_{r_0}^{C/E}\left[2m\left(\frac{C}{r}-E'\right)\right]^{\frac{1}{2}}dr, \quad E' = E - B_l(r_0)$$

The desired result then follows from the above expansion for γ_0, with E there replaced by E'. Get

$$\gamma_\ell \simeq \frac{2\pi C}{\hbar v}\left[1-\frac{4}{\pi}\sqrt{\frac{E'}{V_0}}\right] \simeq \frac{2\pi C}{\hbar v}\left[1-\frac{4}{\pi}\sqrt{\frac{E}{V_0}}\left(1-\tfrac{1}{2}\sigma\right)\right]$$

where : $\sigma = B_\ell(r_0)/V(r_0)$.

We can ignore σ altogether in $U^{238} \to Th^{234} + \alpha$, especially since the gnd state of U^{238} is 0^+ (i.e. $\ell = 0$). Then

$E = \frac{1}{2}mv^2 = 4.25$ MeV for emitted $\alpha \Rightarrow v = 1.432 \times 10^9$ cm/sec $= 0.04775 \times c$

$$\therefore \frac{2\pi C}{\hbar v} = 2\pi\alpha\,\frac{2(Z-2)}{v/c} = 172.8_4$$

and// $$\frac{4}{\pi}\sqrt{E/V_0} = 0.4751_3$$

$\left.\right\}\ \gamma_0 \simeq 90.716$

Transmission coefficient is : $T = e^{-\gamma_0} = 4.005 \times 10^{-40}$

Lifetime : $\Delta t = \tau/T$, $\tau = 2r_0/v = 11.86 \times 10^{-22}$ sec

$\therefore \Delta t = 2.962 \times 10^{18}$ sec $= 93.93 \times 10^9$ years $\left\{\begin{array}{l}\text{this is for}\\ r_0 = 8.494\ \text{f}\end{array}\right.$

$\gamma_0 = 84.39$

This is about 20× too large. If we increase r_0 to $9.853\,\mathring{f}$ (see Halliday "Intro Nuc. Physics" p. 86), then $V_0 = 26.31$ MeV $\Rightarrow T = 2.236 \times 10^{-37}$, $\tau = 13.76 \times 10^{-22}$ sec $\Rightarrow \Delta t = 6.153 \times 10^{15}$ sec $=$ 0.1951×10^9 years, which is about 25 times too small. Clearly Δt is an extremely sensitive fcn of r_0.

from QM 505-6-7 notes... problem solutions

2/21/71 (1) a) From lecture of 2/8/71 (with $\hbar = 1$)

$$F(\vec{r}') = R(\delta\vec{\alpha})F(r), \quad R(\delta\vec{\alpha}) = 1 - i\,\delta\vec{\alpha}\cdot\vec{L} \quad \text{for } \infty\text{smal rotation}$$

For N sucessive ∞smal rotations about same axis, have

$$F(\vec{r}') = R^N(\delta\vec{\alpha})F(r) = (1 - i\,\delta\vec{\alpha}\cdot\vec{L})^N F(\vec{r})$$

Have rotated through $\vec{\alpha} = N\delta\vec{\alpha}$ by this time. So can write

$$F(\vec{r}') = \left(1 - i\,\frac{\vec{\alpha}\cdot\vec{L}}{N}\right)^N F(\vec{r})$$

Let $N \to \infty$ such that $\vec{\alpha} = $ const, but $\delta\vec{\alpha} = \vec{\alpha}/N \to \infty$smal. Then

$$F(\vec{r}') = \lim_{N\to\infty}\left(1 - i\,\frac{\vec{\alpha}\cdot\vec{L}}{N}\right)^N F(\vec{r}) = e^{-i\vec{\alpha}\cdot\vec{L}}\,F(\vec{r})$$

This is by def^{n} of the exponential: $e^x = \lim_{N\to\infty}\left(1 + \frac{x}{N}\right)^N$

Thus have; $R(\vec{\alpha}) = e^{-i\vec{\alpha}\cdot\vec{L}}$ for finite rotation. QED

b) For ∞smal translation $\delta\vec{\epsilon}$, have -- by Taylor's expansion

$$F(\vec{r}') = F(\vec{r} - \delta\vec{\epsilon}) = F(\vec{r}) - (\delta\vec{\epsilon}\cdot\vec{\nabla})F(\vec{r}) = T(\delta\vec{\epsilon})F(\vec{r})$$

where: $T(\delta\vec{\epsilon}) = 1 - \frac{i}{\hbar}\,\delta\vec{\epsilon}\cdot\vec{p}$ is ∞smal translation operator.

By same machinations as above, after N translations by $\delta\vec{\epsilon}$

$$T(N\delta\vec{\epsilon}) = (1 - i\,\delta\vec{\epsilon}\cdot\vec{p})^N, \quad \hbar = 1$$

or $T(\vec{\epsilon}) = \left(1 - i\,\frac{\vec{\epsilon}\cdot\vec{p}}{N}\right)^N$, for $\vec{\epsilon} = N\delta\vec{\epsilon}$

In limit that $N \to \infty$, while $\vec{\epsilon} = $ finite, have as desired

$$T(\vec{\epsilon}) = \lim_{N\to\infty}\left(1 - i\,\frac{\vec{\epsilon}\cdot\vec{p}}{N}\right)^N = e^{-i\vec{\epsilon}\cdot\vec{p}} \quad \text{QED}$$

2/21/71 (49) $\overline{L^2} = 3\overline{L_z^2} = 3\hbar^2\,\overline{m^2}$

m assumes $2l+1$ values $-l, -l+1, \ldots, +l$ with equal a priori probability. So

$$\overline{m^2} = \sum_{m=-l}^{m=+l} m^2 \Big/ (2l+1) = \frac{2}{2l+1}\sum_{m=1}^{l} m^2 = \frac{2}{2l+1} \times \frac{l}{6}(l+1)(2l+1)$$

$$\therefore\ \overline{m^2} = \frac{l}{3}(l+1), \quad \underline{\text{and}} \quad \overline{L^2} = \hbar^2 l(l+1). \quad \underline{\text{QED}}$$

2/21/71 (50) This is just a crummy problem. From PHYS 505 lecture 10/28/70 (#(12))

<u>Choose</u> : $v_0(x) = u_0(x) = x^0 = 1$. Let...

$$\langle v_0|v_0\rangle = \int_{-1}^{+1} 1\,dx = 2 = b_{00}, \quad \langle v_0|u_1\rangle = \int_{-1}^{+1} x\,dx = 0 = b_{01}$$

<u>Choose</u> : $v_1(x) = u_1(x) - \frac{b_{01}}{b_{00}} u_0(x) = x$. Note...

$\langle v_0|v_1\rangle = \int_{-1}^{+1} x\,dx = 0$. So v_1 & v_0 are orthogonal. Let...

$$\langle v_1|v_1\rangle = \int_{-1}^{+1} x^2\,dx = \frac{2}{3} = b_{11}, \quad \langle v_0|u_2\rangle = \int_{-1}^{+1} x^2\,dx = \frac{2}{3} = b_{02}$$

and $\langle v_1|u_2\rangle = \int_{-1}^{+1} x^3\,dx = 0 = b_{12}$.

<u>Choose</u> : $v_2(x) = u_2(x) - \frac{b_{12}}{b_{11}} v_1(x) - \frac{b_{02}}{b_{00}} v_0(x) = x^2 - \frac{1}{3}$. Note...

$$\langle v_0|v_2\rangle = \int_{-1}^{+1}(x^2 - \tfrac{1}{3})\,dx = 0, \quad \langle v_1|v_2\rangle = \int_{-1}^{+1} x(x^2 - \tfrac{1}{3})\,dx = 0$$

So v_2 is $\perp$ both v_0 & v_1. Let...

$$\langle v_2|v_2\rangle = \int_{-1}^{+1}(x^2 - \tfrac{1}{3})^2\,dx = \frac{8}{45} = b_{22}$$

$$b_{23} = \langle v_2|u_3\rangle = \int_{-1}^{+1} x^3(x^2 - \tfrac{1}{3})\,dx = 0$$

In general... $v_n(x) = u_n(x) - \sum_{k=0}^{n-1} \frac{b_{kn}}{b_{kk}} v_k(x)$ $\begin{cases} b_{kn} = \langle v_k|u_n\rangle \\ b_{kk} = \langle v_k|v_k\rangle \end{cases}$

$$b_{13} = \langle v_1 | u_3 \rangle = \int_{-1}^{+1} x^4 dx = \frac{2}{5}$$

$$b_{03} = \langle v_0 | u_3 \rangle = \int_{-1}^{+1} x^3 dx = 0$$

Choose: $v_3(x) = u_3(x) - \sum_{k=0}^{2} \frac{b_{kn}}{b_{kk}} v_k(x) = x^3 - \frac{3}{5} x$

At this pt, we have

$$\left.\begin{aligned} v_0(x) &= 1 = P_0^0(x) \\ v_1(x) &= x = P_1^0(x) \\ v_2(x) &= \tfrac{1}{3}(3x^2-1) = \tfrac{2}{3} P_2^0(x) \\ v_3(x) &= \tfrac{1}{5}(5x^3-3x) = \tfrac{2}{5} P_3^0(x) \end{aligned}\right\}$$

So, indeed, for the cases of $n=0$ to 3, the $v_n(x)$ are $\propto P_n^0(x)$, the Legendre polynomials.

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The proof for the general case can be done by induction. The Schmidt process gives after the $n+1^{\text{st}}$ step

$$v_{n+1}(x) = u_{n+1}(x) - \sum_{k=0}^{n} \frac{\langle v_k | u_{n} \rangle}{\langle v_k | v_k \rangle} v_k(x)$$

should be $(n+1)$ / RTR (2 Feb. 83)

Proposition that $v_k(x) \propto P_k(x)$ is true for $k=0$ (and $k=1,2,3$, as per above). Assume it is true for $k=n$, i.e.

$$v_k(x) = C_k P_k(x), \quad k=0 \text{ to } n \quad (C_k = \text{some cnst})$$

Now show it is true for $k=n+1$. We must show that

$$v_{n+1}(x) = u_{n+1}(x) - \sum_{k=0}^{n} \frac{\langle P_k | u_{n} \rangle}{\langle P_k | P_k \rangle} P_k(x)$$

$\to (n+1)$

is $\propto P_{n+1}(x)$. Now we know $\langle P_k | P_k \rangle = 2/(2k+1)$. So have

$$v_{n+1}(x) = x^{n+1} - \sum_{k=0}^{n} \frac{2k+1}{2} \langle P_k | \overset{n+1}{u_{n}} \rangle P_k(x)$$

Is this $\propto P_{n+1}(x)$? Contemplate expansion of x^{n+1} in terms of the $P_k(x)$ -- over $-1 \leq x \leq +1$. Must have

$$x^{n+1} = \sum_{k=0}^{n+1} B_k P_k(x) \quad \left\{ \begin{array}{l} \text{series truncates at } k = n+1 \text{ since} \\ P_k(x) \text{ is a polynomial of degree } k. \end{array} \right.$$

Expansion coefficients B_k are calculated as

$$B_\ell = \frac{2\ell+1}{2} \langle P_\ell | x^{n+1} \rangle$$

$$\therefore\ x^{n+1} = \sum_{k=0}^{n+1} \frac{2k+1}{2} \langle P_k | u_{n+1} \rangle P_k(x)$$

$$= \sum_{k=0}^{n} \frac{2k+1}{2} \langle P_k | u_{n+1} \rangle P_k(x) + \frac{2n+3}{2} \langle P_{n+1} | u_{n+1} \rangle P_{n+1}(x)$$

Plugging this into above, we have

$$v_{n+1}(x) = \left[\frac{2n+3}{2} \langle P_{n+1} | u_{n+1} \rangle \right] P_{n+1}(x)$$

The [ ] is just a (non-zero!) cnst. So indeed $v_{n+1} \propto P_{n+1}$, and the induction is complete.

N.B. This proof is due to A.B. Western.

506 : Note on Problem 50 — 28 Feb 72

We wish to show that the Schmidt-orthogonalized $v_n(x)$ are proportional to $P_n(x)$ for general n. The proof can be done by induction.

1) After the $n+1^{\text{ST}}$ step, the Schmidt process gives

$$v_{n+1}(x) = u_{n+1}(x) - \sum_{k=0}^{n} \frac{\langle v_k | u_n \rangle}{\langle v_k | v_k \rangle} v_k(x), \tag{1}$$

should be $(n+1)$ / RTR (2 Feb. '83)

where $u_n(x) = x^n$. The proposition that $v_k(x) \propto P_k(x)$ is true for $k=0$ (and $k=1,2,3$), as demonstrated in the problem. Assume the proposition is true for $k=n$, i.e. assume

$$v_k(x) = C_k P_k(x), \quad k = 0 \text{ to } n, \tag{2}$$

where C_k is some cnst. The induction will be complete if we can show from this that the proposition is true for $k=n+1$. That is, we must show that

$$v_{n+1}(x) = u_{n+1}(x) - \sum_{k=0}^{n} \frac{\langle P_k | u_n \rangle}{\langle P_k | P_k \rangle} P_k(x), \tag{3}$$

$(n+1)$

is in fact proportional to $P_{n+1}(x)$.

2) Now we know $\langle P_k | P_k \rangle = 2/(2k+1)$ from the normalization of the P_k. So eq. (3) reads

$$v_{n+1}(x) = x^{n+1} - \sum_{k=0}^{n} \left(\frac{2k+1}{2}\right) \langle P_k | u_n \rangle P_k(x) \tag{4}$$

$(n+1)$

We must show this is $\propto P_{n+1}(x)$. Contemplate the expansion of x^{n+1}

in terms of $P_k(x)$, over $-1 \leq x \leq +1$. This can be done because the P_k are a complete set on this interval. The series must be of the form

$$x^{n+1} = \sum_{k=0}^{n+1} B_k P_k(x) \tag{5}$$

The series truncates at $k=n+1$ because $P_k(x)$ is a polynomial of degree k in x. The expansion coefficients B_k are calculable as...

$$B_\ell = \frac{2\ell+1}{2} \langle P_\ell | x^{n+1} \rangle$$

$$\therefore x^{n+1} = \sum_{k=0}^{n+1} \frac{2k+1}{2} \langle P_k | u_{n+1} \rangle P_k(x)$$

$$= \sum_{k=0}^{n} \frac{2k+1}{2} \langle P_k | u_{n+1} \rangle P_k(x) + \frac{2n+3}{2} \langle P_{n+1} | u_{n+1} \rangle P_{n+1}(x) \tag{6}$$

We now plug this expression for x^{n+1} into eq.(4). We note that the 2ND term on the RHS of eq.(4) is cancelled by the 1ST term on the RHS here. The result of plugging in is

$$v_{n+1}(x) = \left[\frac{2n+3}{2} \langle P_{n+1} | u_{n+1} \rangle \right] P_{n+1}(x) \tag{7}$$

The [ ] here is just a non-zero constant. So indeed $v_{n+1} \propto P_{n+1}$, and the induction is complete. QED.

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This proof is due in part to A.B. Westerm.