DEPARTMENT OF PHYSICS

2000 COMPREHENSIVE EXAM
SECTION 1

Monday, 28 August 2000
9am - noon

Answer each of the following questions, all of which carry equal weight. Begin your answer to each question on a new sheet of paper; solutions to different questions must not appear on the same sheet. Each sheet of paper must be labeled with your name and the problem number in the upper right hand corner. When more than one sheet is submitted for a problem, be sure the pages are ordered properly.

PHYSICAL CONSTANTS

Quantity	Symbol	Value
acceleration due to gravity	g	$9.8\ \mathrm{m\ s^{-2}}$
gravitational constant	G	$6.67 \times 10^{-11}\ \mathrm{N\ m^2\ kg^{-2}}$
permittivity of vacuum	ε_o	$8.85 \times 10^{-12}\ \mathrm{C^2\ N^{-1}\ m^{-2}}$
permeability of vacuum	μ_o	$4\pi \times 10^{-7}\ \mathrm{N\ A^{-2}}$
speed of light in vacuum	c	$3.00 \times 10^{8}\ \mathrm{m\ s^{-1}}$
elementary charge	e	$1.602 \times 10^{-19}\ \mathrm{C}$
mass of electron	m_e	$9.11 \times 10^{-31}\ \mathrm{kg}$
mass of proton	m_p	$1.673 \times 10^{-27}\ \mathrm{kg}$
Planck constant	h	$6.63 \times 10^{-34}\ \mathrm{J\ s}$
Avogadro constant	N_A	$6.02 \times 10^{23}\ \mathrm{mol^{-1}}$
Boltzmann constant	k	$1.38 \times 10^{-23}\ \mathrm{J\ K^{-1}}$
molar gas constant	R	$8.31\ \mathrm{J\ mol^{-1}\ K^{-1}}$
standard atmospheric pressure		$1.013 \times 10^{5}\ \mathrm{Pa}$

2000 COMPS

Aug. 28, 29, 30, 2000

1.

A physicist falls through a hole in the ice. The rescuers cannot walk on the ice because the ice is too thin. Luckily the physicist has a small bag of sand (mass m) in a coat pocket. The physicist ties a massless (but very strong) fishing line to the bag of sand and throws the bag toward the rescuers. The bag is thrown with speed v_0 at an angle θ to the horizontal. Assume that the bag of sand is thrown from the level of the ice and that the ice has a constant coefficient of kinetic friction μ. Assume further that the string exerts no force on the bag of sand and that the bag of sand does not dent the ice upon impact.

a) How far does the bag travel horizontally?

b) For what angle is this distance of travel a maximum?

1

Solution

LK

a) The distance x_p traveled during the projectile part of the path is given by

$$x_t = \frac{2v_o \sin\theta \cos\theta}{g}$$

During the impact with the ice, the bag experiences a vertical impulse to stop the vertical motion of the bag.

$$I_v = \int_0^{\Delta t} N dt = \Delta p_v = m v_o \sin\theta$$

This results in a horizontal impulse that slows the horizontal component of the bag's velocity.

$$I_h = \int_0^{\Delta t} f dt = \mu \int_0^{\Delta t} N dt = \mu m v_o \sin\theta = m \Delta v_h$$

If we assume that Δt is small, the bag will not slide very far during the impact. However, the bag's horizontal velocity at the beginning of the sliding part of the motion is now

$$v_x = v_o \left(\cos\theta - \mu \sin\theta\right)$$

Therefore, the bag slides a distance

$$x_s = \frac{v_x^2}{2a} = \frac{v_o^2 \left(\cos\theta - \mu \sin\theta\right)^2}{2\mu g}$$

This gives a total distance of horizontal travel of

$$R = \frac{v_o^2}{g}\left[2\sin\theta\cos\theta + \frac{\left(\cos\theta - \mu\sin\theta\right)^2}{2\mu}\right]$$

$$= \frac{v_o^2}{2\mu g}\left(\cos^2\theta + 2\mu\sin\theta\cos\theta + \mu^2\sin^2\theta\right)$$

b) We can find the maximum by taking the derivative with respect to θ and setting it equal to zero.

$$\frac{dR}{d\theta} = 0 = 2\cos\theta\left(-\sin\theta\right) + 2\mu\left(\cos^2\theta - \sin^2\theta\right) + 2\mu\sin\theta\cos\theta$$

or

$$\left(\mu^2 - 1\right)\cos\theta\sin\theta + \mu\left(\cos^2\theta - \sin^2\theta\right) = 0$$

Using some trig substitutions, we can simplify this

$$\left(1-\mu^2\right)\tfrac{1}{2}\sin 2\theta = \mu \cos 2\theta$$

which gives

$$\tan 2\theta = \frac{2\mu}{1-\mu^2}$$

2.

An electromagnetic wave of the form

$$\mathbf{E}(\mathbf{r},t) = \mathrm{Re}\left[\mathbf{E}_0\, e^{i\mathbf{k}\cdot\mathbf{r}}\, e^{-i\omega t}\right]$$

is present in a semi-infinite, uniform conductor in the half space $z > 0$. In the conducting medium, the following quantities are real constants: $\mathbf{E}_0$, ω, conductivity σ, magnetic permeability μ, and dielectric constant ε. Assume that $\sigma >> \omega\varepsilon$ and that $\mathbf{k}$ is oriented along the z–axis.

a) Determine an expression for $k = k_1 + ik_2$.

b) Evaluate $\mathbf{E}(\mathbf{r},t)$ for $z > 0$ using your expression for k.

SOLUTION

E&M Problem J. Hermanson

(a) For a conductor with $\vec{J} = \sigma \vec{E}$, in cgs units,

$$\left\{ \begin{array}{ll} \nabla \cdot E = 0 & ; \quad \nabla \times E = -\frac{\mu}{c}\frac{\partial H}{\partial t} \\ \nabla \cdot B = 0 & ; \quad \nabla \times H = \frac{\epsilon}{c}\frac{\partial E}{\partial t} + \frac{4\pi\sigma}{c}\vec{E} \end{array} \right\}$$

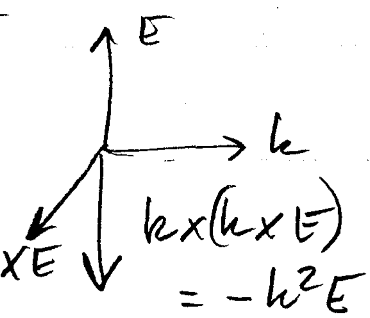
Given $E \sim e^{i(k\cdot r - \omega t)}$, $\vec{k}\cdot\vec{E} = 0$ and

$\nabla \times E$: $\quad ik \times E = i\frac{\mu\omega}{c} H \longrightarrow H = \frac{c}{\mu\omega} k \times E$

$\nabla \times H$: $\quad ik \times H = -i\frac{\epsilon\omega}{c} E + \frac{4\pi\sigma}{c} E$

or $\quad k \times \frac{c}{\mu\omega}(k \times E) = -\left(\frac{\epsilon\omega}{c} + i\frac{4\pi\sigma}{c}\right) E$

Now $k \times (k \times E) = -k^2 E$:



so $\quad -\frac{c}{\mu\omega} k^2 E = \left(-\frac{\epsilon\omega}{c} + i\frac{4\pi\sigma}{c}\right) E$

$$\left[k^2 - \mu\epsilon\frac{\omega^2}{c^2} - i\frac{4\pi\mu\omega\sigma}{c^2}\right] E = 0$$

For a nontrivial solution $E \neq 0$, and

$$k^2 = \mu\epsilon\frac{\omega^2}{c^2}\left(1 + i\frac{4\pi\sigma}{\omega\epsilon}\right)$$

But $\sigma \gg \omega\epsilon$, so

$$k^2 \simeq i\frac{4\pi\sigma\mu\omega}{c^2}$$

or $\boxed{k = \frac{1+i}{\sqrt{2}}\sqrt{\frac{4\pi\sigma\mu\omega}{c^2}} = (1+i)\sqrt{\frac{2\pi\sigma\mu\omega}{c^2}}}$

$= (1+i) K$ (over)

(b) Then $\vec{E} = \mathrm{Re}\left[\vec{E}_0 \, e^{i\vec{k}\cdot\vec{r}} e^{-i\omega t}\right]$

$$e^{i\vec{k}\cdot\vec{r}} = e^{ikz}$$

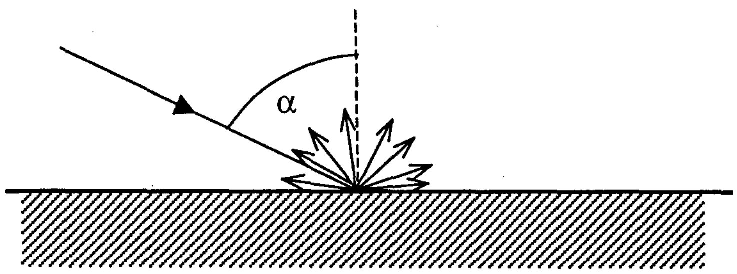
But $k = (1+i)K$, $K = \sqrt{\frac{2\pi\sigma\mu\omega}{c^2}} = \delta^{-1}$

$$e^{ikz} = e^{iKz} e^{-Kz}$$

$$\boxed{\begin{array}{c} \vec{E} = \vec{E}_0 \, e^{-Kz} \cos(Kz - \omega t) \\ \text{where } K = \sqrt{\frac{2\pi\sigma\mu\omega}{c^2}} \end{array}}$$

3.

Write an expression for the pressure exerted by a monochromatic plane wave of frequency ν and intensity I, incident at angle α on a surface in vacuum (see Figure). Assume isotropic scattering within the half-space above the surface.



1. $$I = \frac{N \cdot h\nu}{S \cdot t}$$

N - number of photons in a volume $V = S \cdot c \cdot t$
S - cross-section of the beam

Each (incident) photon carries amount of momentum perpendicular to the surface

$$p_i = \hbar k_\perp = \frac{h\nu}{c} \cos\alpha$$

Each scattered photon carries a momentum perpendicular to the surface

$$p_s = \frac{h\nu}{c} \cos\theta \qquad \left(0 < \theta < \frac{\pi}{2} \right)$$

Average momentum carried by a scattered photon:

$$\bar{p}_s = \int_0^{\pi/2} \frac{h\nu}{c} \cos\theta \sin\theta \, d\theta = \frac{h\nu}{2c}$$

Total momentum change in the beam of photons in time t

$$\Delta p = N(p_i + \bar{p}_s) = N \frac{h\nu}{c} \left(\cos\alpha + \frac{1}{2} \right)$$

Pressure exerted on the surface

$$P = \frac{\Delta p}{t} \cdot \frac{\cos\alpha}{S} = \frac{N \cdot \frac{h\nu}{c} \left(\cos\alpha + \frac{1}{2} \right) \cos\alpha}{t \cdot S}$$

$$= \frac{I \cdot \cancel{S} \cdot \cancel{t} \cdot \frac{\cancel{h\nu}}{c} \left(\cos\alpha + \frac{1}{2} \right) \cos\alpha}{\cancel{t} \cdot \cancel{S} \cdot \cancel{h\nu}} = \underline{\underline{\frac{1}{c} I \left(\cos\alpha + \frac{1}{2} \right) \cos\alpha}}$$

4.

A one-dimensional simple harmonic oscillator with a mass m and natural frequency ω is subject to the perturbation $V = A\,x\,\delta(t)$, where A = real constant. To first order in A, evaluate the expectation value of x for $t > 0$ if the oscillator is in its ground state for $t < 0$.

Useful relations:

$$x = \frac{\Delta}{\sqrt{2}}\left(a + a^\dagger\right); \quad \Delta = \sqrt{\frac{\hbar}{m\omega}}$$

$$p = \frac{-i\hbar}{\sqrt{2}\Delta}\left(a - a^\dagger\right)$$

$$a^\dagger|0\rangle = |1\rangle$$

Solution:

Find the coefficients in the expansion

$$|\psi(t)\rangle = \sum_n c_n e^{-i\omega_n t} |n\rangle \quad \text{or}$$

$$|\psi(t)\rangle = c_0 e^{-i\omega t/2} |0\rangle + c_1 e^{-i\frac{3}{2}\omega t} |1\rangle + \cdots$$

c_0 ← initial value = 1

To first order,

$$c_n = -\frac{i}{\hbar} \int_{-\infty}^{t} V_{n0}(t') e^{i\omega_{n0} t'} dt', \quad n > 0$$

where $\omega_{n0} = n\omega_0$ and

$$V_{n0} = \langle n | A x \delta(t) | 0 \rangle$$

$$= A x_{n0} \delta(t)$$

But $x_{n0} = \langle n | \frac{\Delta}{\sqrt{2}} (\cancel{a} + a^\dagger) | 0 \rangle$

$$= \frac{\Delta}{\sqrt{2}} \delta_{n1}$$

$$c_1 = -\frac{i}{\hbar} \int_{-\infty}^{t} A \frac{\Delta}{\sqrt{2}} \delta(t') e^{i\omega_{n0} t'} dt'$$

$$= -\frac{iA\Delta}{\sqrt{2}\hbar}, \quad t > 0$$

$c_0 = 1$ (initial condition)

$c_n = 0$ for $n > 1$

Now $|\psi(t)\rangle = e^{-i\omega t/2}|0\rangle + c_1 e^{-i\frac{3}{2}\omega t}|1\rangle$

$$\langle x\rangle = \left\{\langle 0| e^{i\omega t/2} + c_1^* e^{i\frac{3}{2}\omega t}\langle 1|\right\} x$$

$$\left\{|0\rangle e^{-i\omega t/2} + c_1 e^{-i\frac{3}{2}\omega t}|1\rangle\right\}$$

$$= c_1\langle 0|x|1\rangle e^{-i\omega t} + c_1^*\langle 1|x|0\rangle e^{i\omega t}$$

$$= c_1 x_{01} e^{-i\omega t} + c_1^* x_{10} e^{i\omega t}$$

where $x_{01} = x_{10} = \frac{\Delta}{\sqrt{2}}$ from previous page

$$\langle x\rangle = \frac{\Delta}{\sqrt{2}}\left(c_1 e^{-i\omega t} + c_1^* e^{i\omega t}\right)$$

$$= \frac{\Delta}{\sqrt{2}}\,\frac{-iA\Delta}{\sqrt{2}\hbar}\left(e^{-i\omega t} - e^{i\omega t}\right)$$

$$= -\frac{A\Delta^2}{\hbar}\sin\omega t$$

$$\boxed{\langle x\rangle = -\frac{A}{m\omega}\sin\omega t}$$

5.

A non-relativistic electron moving in one dimension is subject to the periodic potential

$$V(z) = V_0 \cos\left(\frac{2\pi}{a} z\right),$$

where V_0 and a are positive, real constants. As implied by Bloch's Theorem, the probability density $|\Psi|^2$ of the electron is also periodic.

Two wavefunctions which satisfy Bloch's Theorem are given by

$$\Psi_1 = \sqrt{\frac{2}{Na}} \cos\left(\frac{\pi}{a} z\right)$$

$$\Psi_2 = \sqrt{\frac{2}{Na}} \sin\left(\frac{\pi}{a} z\right)$$

where N is a large positive integer.

a) Find the expectation value of the energy for Ψ_1 and Ψ_2.

b) Sketch $|\Psi|^2$ for each state and use your sketch to explain the ordering of the energies in part (a).

SSP JHermanson

(a) For $\psi_1 = \sqrt{\frac{2}{L}} \cos \frac{\pi}{a} z$; $L \equiv Na$

$$E_1 = \langle \psi_1 | -\frac{\hbar^2}{2m}\frac{d^2}{dz^2} + V(z) | \psi_1 \rangle$$

$$\equiv KE + PE$$

$$KE = -\frac{\hbar^2}{2m} \int dz \left(\frac{2}{L}\right) \cos\left(\frac{\pi}{a} z\right) \frac{d^2}{dz^2} \cos\left(\frac{\pi}{a} z\right)$$

integrate over length $L = Na$

$$= +\frac{\hbar^2}{2m}\left(\frac{2}{L}\right)\left(\frac{\pi}{a}\right)^2 \underbrace{\int dz \cos^2\left(\frac{\pi}{a} z\right)}_{\frac{1}{2}\cdot L}$$

$$\boxed{KE = \frac{\hbar^2}{2a}\left(\frac{\pi}{a}\right)^2}$$

$$PE = \int dz \cos\left(\frac{\pi}{a} z\right) V_0 \cos\left(\frac{2\pi}{a} z\right) \cos\left(\frac{\pi}{a} z\right)$$

$$= V_0 \cdot \frac{2}{L} \int dz \cos^2\left(\frac{\pi}{a} z\right) \cos\left(\frac{2\pi}{a} z\right)$$

$$= V_0 \cdot \frac{2}{L} \cdot \frac{1}{2} \int dz \left(1 + \cos\left(\frac{2\pi}{a} z\right)\right) \cos\left(\frac{2\pi}{a} z\right)$$

$$= V_0 \cdot \frac{2}{L} \cdot \frac{1}{2} \underbrace{\int dz \cos^2\left(\frac{2\pi}{a} z\right)^2}_{\frac{1}{2}\cdot L}$$

$$\boxed{PE = \frac{V_0}{2}}$$

Thus $\boxed{E_1 = \frac{\hbar^2}{2m}\left(\frac{\pi}{a}\right)^2 + \frac{V_0}{2}}$

For $\psi_2 = \sqrt{\frac{2}{L}} \sin\left(\frac{\pi}{a} z\right)$

$$KE = -\frac{\hbar^2}{2m} \cdot \frac{2}{L} \int dz \sin\left(\frac{\pi}{a} z\right) \frac{d^2}{dz^2} \sin\left(\frac{\pi}{a} z\right)$$

$$= \frac{\hbar^2}{2m}\left(\frac{\pi}{a}\right)^2 \text{, same as for } \psi_1$$

$$PE = V_0 \cdot \frac{2}{L} \int dz \sin^2\left(\frac{\pi}{a} z\right) \cos\left(\frac{2\pi}{a} z\right)$$

$$= V_0 \cdot \frac{2}{L} \cdot \frac{1}{2} \int dz \left(1 - \cos\left(\frac{2\pi}{a} z\right)\right) \cos\left(\frac{2\pi}{a} z\right)$$

$$= V_0 \cdot \frac{2}{L} \cdot \frac{1}{2} \underbrace{\int dz\, (-) \cos^2\left(\frac{2\pi}{a} z\right)}_{-\frac{1}{2} L}$$

$$= -\frac{V_0}{2}$$

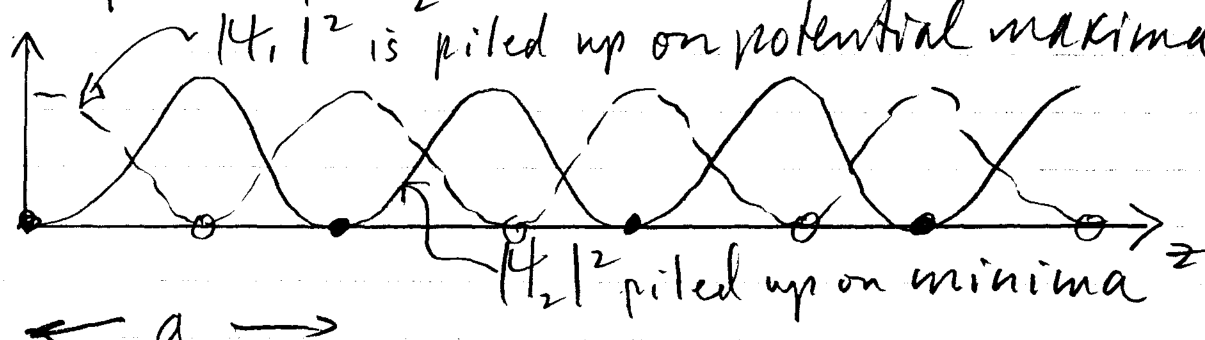
$$\boxed{E_2 = \frac{\hbar^2}{2m}\left(\frac{\pi}{a}\right)^2 - \frac{V_0}{2}}$$

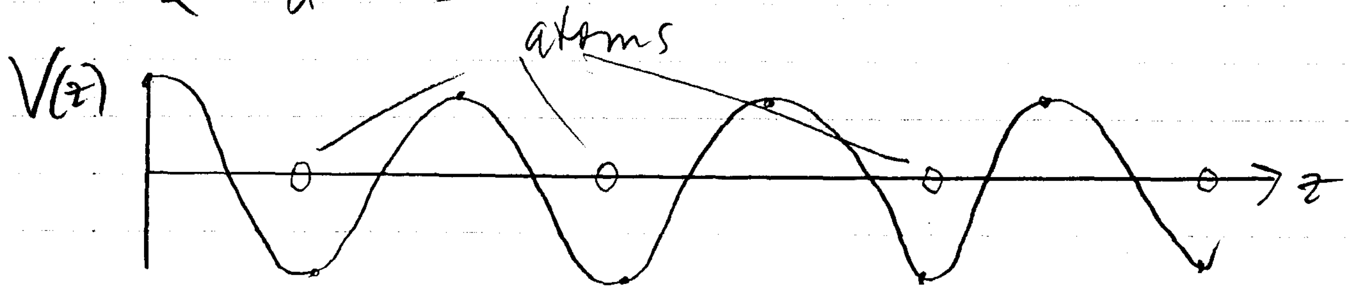
E_1 $\left(\cos \frac{\pi}{a} z\right)$

V_0

E_2 $\left(\sin \frac{\pi}{a} z\right)$

(b) Why is $E_1 > E_2$?





6.

A motor boat on open water must get from point A at $(x, y) = (0, 0)$ to point B at $(L, 0)$ against a current in a minimum amount of time. The water exerts a drag force on the boat of

$$\mathbf{F}_w = b(\mathbf{v}_c - \mathbf{v}_b) \qquad b > 0, \text{ constant,}$$

where $\mathbf{v}_b$ is the velocity of the boat and $\mathbf{v}_c$ is the velocity of the current. The current is described by

$$\mathbf{v}_c = -\hat{\mathbf{x}} g_0 e^{y/L}$$

where g_0 is a positive constant. The motor exerts a force on the boat of

$$\mathbf{F}_b = bc\hat{\mathbf{s}} \qquad c > 0, \text{ constant}$$

where $\hat{\mathbf{s}}$ is a unit vector in the direction the boat is pointed. The boat has negligible mass, so its equation of motion is

$$\mathbf{F}_b + \mathbf{F}_w = 0.$$

Find the trajectory $y(x)$ of minimum time. To simplify the analysis, you may work in the limit of a powerful motor, $c >> g_0$; in this limit $\left|\frac{dy}{dx}\right|$ will be $<< 1$ for all x. How far must the boat veer from a straight course?

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Math Phys. B. Link

Solution: The time required for the journey from A to B is

$$T = \int_0^L dx\, \frac{\sqrt{1+y'^2}}{v_b} \cong \int_0^L dx \left(\frac{1+y'^2}{c^2 + g(y)^2 - 2cg(y)/\sqrt{1+y'^2}} \right)^{1/2},$$

where the boat speed v_b follows from the kinematics. For $c \gg g_0$, the optimal solution will be close to $y = 0$ for all x and $|y'|$ will $\ll 1$. In this limit,

$$T \simeq \frac{1}{c} \int_0^L dx \left(\sqrt{1+y'^2} + \frac{1}{c} g(y) \right).$$

The Euler-Lagrange equation for this integrand gives the extremal solution

$$\frac{d}{dx}\left(\frac{y'}{\sqrt{1+y'^2}} \right) - \frac{1}{c}\frac{\partial g}{\partial y} = 0.$$

Or, since $|y'| \ll 1$,

$$y'' - \frac{1}{c}\frac{\partial g}{\partial y} = 0.$$

Near $y = 0$, $\partial g/\partial y \simeq g_0/l$. The solution with the boundary conditions $y(0) = y(L) = 0$ is

$$y(x) = \frac{g_0}{2cl} x(x - L).$$

7.

Consider a system with two degrees of freedom with the following Lagrangian

$$L(x, \dot{x}, \dot{y}) = \tfrac{1}{2}\dot{x}^2 - x\ln|\dot{y}+x| ,$$

where $x > 0$ but y is unconstrained.

a) Give explicit expressions for any and all independent constants of the motion.

b) Solve for the motion near equilibrium.

c) At $t = 0$ the particle is given the velocity $\dot{x} = \delta$, $\dot{y} = \varepsilon$ at the equilibrium position. The constants δ and ε are arbitrarily small. Find the subsequent motion.

7 CM - DANA LONGCOPE

Solution

(a) The Lagrangian is cyclic in y, so one constant is the y-momentum

$$\boxed{p_y = \frac{\partial L}{\partial \dot{y}} = -\frac{x}{\dot{y}+x} \quad . \qquad (1)}$$

1

The Lagrangian is formally independant of time ($\partial L/\partial t = 0$) so the Hamiltonian is also a constant. The explicit form of this is

$$\boxed{H = p_x\dot{x} + p_y\dot{y} - L = \tfrac{1}{2}\dot{x}^2 - \frac{x\dot{y}}{\dot{y}+x} + x\ln|\dot{y}+x| \quad . \qquad (2)}$$

1

$$= \tfrac{1}{2}p_x^2 - x(p_y+1) + x\ln\left|\frac{x}{p_y}\right|$$

(b) From eq. (1) we find

$$\dot{y} = -(1+p_y^{-1})x \quad . \qquad (3)$$

The second EOM is

$$\ddot{x} = \frac{\partial L}{\partial x} = -\ln|\dot{y}+x| - \frac{x}{\dot{y}+x} \quad . \qquad (4)$$

The equilibrium must have $\dot{x} = \dot{y} = \ddot{x} = 0$. Using this in eq. (4) gives

$$-\ln|x_0| - 1 = 0 \quad , \qquad (5)$$

or $x_0 = e^{-1}$. Placing this in eq. (3) gives the requirement $p_y = -1$ for equilibirum. Because it is a cyclic variable the y-position is irrelevant for equilibrium. The equilibrium position istherefore

$$\boxed{x = e^{-1} \quad , \quad y = y_0 \quad . \qquad (6)}$$

1

Consider small perturbations to each variable

$$x(t) = e^{-1} + \xi(t) \quad , \quad y(t) = y_0 + \eta(t) \quad .$$

Also, the y-momentum will be perturbed by a small amount $p_y = -1 + p_y'$. Replacing these in eqs. (3) and (4)

$$\dot{\eta} = -p_y' e^{-1} = \text{const.} \quad . \qquad (7)$$

$$\ddot{\xi} = -e(\dot{\eta}+\xi) - e\xi + e(\dot{\eta}+\xi) = -e\xi \quad . \qquad (8)$$

The explicit solutions for small oscillations are

$$\boxed{\begin{aligned} \xi(t) &= \xi(0)\cos(e^{1/2}t) + e^{-1/2}\dot{\xi}(0)\sin(e^{1/2}t) \quad , & (9) \\ \eta(t) &= \eta(0) + \dot{\eta}(0)\,t \quad . & (10) \end{aligned}}$$

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(c) The perturbation can be written as $\xi(0) = \eta(0) = 0$ and $\dot{\xi}(0) = \delta$ and $\dot{\eta}(0) = \epsilon$. Using these in the expressions above yields

$$\boxed{\begin{aligned} x(t) &= e^{-1} + \delta e^{1/2}\sin(e^{1/2}t) \quad , & (11) \\ y(t) &= y_0 + \epsilon t \quad . & (12) \end{aligned}}$$

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8.

Far from an oscillating electric dipole, the magnetic and electric fields are given by

$$\mathbf{B} = \frac{\omega^2}{c^2}(\hat{\mathbf{r}}\times\mathbf{d})\frac{\cos(kr-\omega t)}{r} \text{ and } \mathbf{E} = \mathbf{B}\times\hat{\mathbf{r}}.$$

Here k and ω denote the wavenumber and angular frequency, $\mathbf{r}$ is the outward radial vector from the center of the dipole, and $\mathbf{d}$ is the dipole moment. Calculate the (instantaneous) energy flux and angular-momentum density flux vectors. Use these to calculate the average power and angular momentum radiated by the dipole. Give a physical explanation for why no angular momentum is carried off by the radiation.

Electromagnetism Solution 8.

$$\vec{S} = \frac{c}{4\pi}(\vec{E}\times\vec{B}) \quad , \quad \vec{J} = \frac{1}{4\pi c}\,\vec{r}\times(\vec{E}\times\vec{B})$$

$$\vec{E}\times\vec{B} = (\vec{B}\times\hat{r})\times\vec{B} = (\vec{B}\cdot\vec{B})\hat{r} - (\vec{B}\cdot\hat{r})\vec{B}$$

but $\quad \vec{B}\cdot\hat{r} \propto (\hat{r}\times\vec{d})\cdot\hat{r} = 0$

$$\therefore \quad \vec{S} = \frac{c}{4\pi}(\vec{B}\cdot\vec{B})\hat{r} = \frac{c}{4\pi}\frac{d^2\omega^4}{c^4}\frac{\cos^2(kr-\omega t)}{r^2}\sin^2\theta$$

where θ is the angle between $\vec{r}$ and $\vec{d}$.

$$\vec{J} = \frac{1}{4\pi c}(\vec{r}\times\hat{r})(\vec{B}\cdot\vec{B}) = 0$$

$\Rightarrow$ no angular momentum flux

$\Rightarrow$ no angular momentum radiated by the dipole

Reason: System is axisymmetric $\Rightarrow$ angular momentum is conserved.

Power radiated

$$P = \left\langle \int_{S^2} (\vec{S}\cdot\hat{r})\, r^2\sin\theta\, d\theta\, d\phi \right\rangle$$

$$\langle A\rangle = \frac{1}{T}\int A\,dt \quad , \qquad \langle\cos^2(kR-\omega t)\rangle = 1/2$$

$$P = \frac{c}{8\pi}\frac{\omega^4 d^2}{c^4}\int_{S^2}\sin^3\theta\, d\theta\, d\phi$$

$$= \frac{c}{4}\frac{\omega^4 d^2}{c^4}\int_0^\pi \sin^3\theta\, d\theta = \frac{\omega^4 d^2}{3c^3}$$

9.

This problem is concerned with the spin states of a hydrogen atom in a magnetic field **B**. The Hamiltonian that describes the interaction of the proton and electron spins with the magnetic field and with each other is

$$\mathbf{H} = -\mu_e \sigma^e \cdot \mathbf{B} - \mu_p \sigma^p \cdot \mathbf{B} + \alpha\, \sigma^e \cdot \sigma^p,$$

where σ^e and σ^p are Pauli matrices for the electron and the proton, μ_e is the Bohr magneton, μ_p is the nuclear magneton and α is a positive constant. Calculate the spin-dependent energy level splittings in a very strong magnetic field (but neglecting relativistic effects). State the inequality that must be satisfied by the magnetic field

Useful identity:

$$\sigma^e \cdot \sigma^p = 2\mathbf{P}_{\mathrm{exch}} - 1,$$

where $\mathbf{P}_{\mathrm{exch}}$ is the "spin exchange operator"; it exchanges the electron and proton spin states.

9.

QM - BL (Cornish) Sol'n

Solution to question C

We are told that B is very large. This suggests that we should treat

$$\widehat{H}_0 = -\mu_e \sigma^e \cdot \mathbf{B} - \mu_p \sigma^p \cdot \mathbf{B}$$

as the unperturbed Hamiltonian, and

2

$$\widehat{H}_i = \alpha\, \sigma^e \cdot \sigma^p$$

as a perturbation. This split will be valid so long as

$$|<\psi|\,\widehat{H}_0\,|\psi>| \quad \gg \quad |<\psi|\,\widehat{H}_i\,|\psi>|\,, \tag{1}$$

where $|\psi>$ is an eigenstate of $\widehat{H}_0$. The eigenstates of the spin operators σ are $|\uparrow>$ and $|\downarrow>$, so the eigenstate of $\widehat{H}_0$ are given by the tensor products

2

$$|++>=|\uparrow>_e|\uparrow>_p\,, \quad |+->=|\uparrow>_e|\downarrow>_p\,,$$
$$|-+>=|\downarrow>_e|\uparrow>_p\,, \quad |-->=|\downarrow>_e|\downarrow>_p\,.$$

The interaction operator can be written as

$$\begin{aligned} \sigma^e \cdot \sigma^p &= \sigma^e_x \sigma^p_x + \sigma^e_y \sigma^p_y + \sigma^e_z \sigma^p_z \\ &= 2\left(\sigma^e_+ \sigma^p_- + \sigma^e_- \sigma^p_+\right) + \sigma^e_z \sigma^p_z \\ &= 2\widehat{P} - \widehat{1} \end{aligned}$$

where $\widehat{P} = \sigma^e_+ \sigma^p_- + \sigma^e_- \sigma^p_+ + \frac{1}{2}\sigma^e_z \sigma^p_z(\widehat{1} + \sigma^e_z \sigma^p_z)$ is the spin exchange operator: $\widehat{P}\,|ab>=|ba>$. Without loss of generality, we can take $\mathbf{B}$ along the z axis so that *e.g.*

5

$$\widehat{H}_0|++>=-(\mu_e+\mu_p)B|++>\,, \quad \widehat{H}_i|++>=\alpha|++>$$
$$\widehat{H}_0|+->=-(\mu_e-\mu_p)B|+->\,, \quad \widehat{H}_i|+->=2\alpha|-+>-\alpha|+->$$

Thus, the energy levels are given by

$$E = \alpha \pm (\mu_e + \mu_p)B \quad \text{and} \quad E = -\alpha \pm (\mu_e - \mu_p)B$$

The inequality (1) is satisfied so long as $|\mu_e B \pm \mu_p B| \gg \alpha$, which is equivalent to demanding that $\underbrace{\mu_e B \gg \alpha}_{1}$ since $\mu_e \gg \mu_p$.

#10 ASTRO B. LINK

As the Solar System was forming, it was filled with dust grains of various sizes and similar densities in nearly circular orbits about the young Sun. About 5 billion years ago, the Sun "suddenly" switched on to its present luminosity (power output). Radiation pressure exerted by the solar radiation field on the dust then began to accelerate some grains out of the Solar System. Determine the critical grain mass below which grains are driven out of the Solar System. Make any assumptions you need about the properties of grains, but please describe and justify these assumptions.

10. 0 $\boxed{\text{STS}}$ Astroϕ - BL

Astrophysics. As the Solar System was forming, it was filled with ~~dust~~ grains of various sizes and similar densities in nearly circular orbits about ~~the young~~ Sun. ~~Each grain of mass m had about the same angular momentum per unit mass L/m.~~ About 5 billion years ago, the Sun "suddenly" switched on to its ~~present luminosity~~ (power output). Radiation pressure exerted by the solar radiation ~~field on~~ the dust then began to accelerate some grains out of the Solar System. Determine ~~for which grains this happened~~. Make any assumptions you need about the properties of the grains, but please describe and justify these assumptions.

the critical grain mass below which grains are driven out of the Solar System.

Solution: The equations of motion for a dust grain of mass m are

$$\begin{aligned} m\ddot{r} - \frac{mv_\phi^2}{r} &= F_g + F_{\rm rad} \\ mrv_\phi &= L = \text{constant}, \end{aligned}$$

where F_g and $F_{\rm rad}$ are the gravitational and radiation forces. Assume that:

- grains are perfect absorbers of solar radiation and reemit absorbed radiation isotropically.

$$\frac{\pi r_g^2}{4\pi r^2} = \frac{1}{4}\left(\frac{r_g}{r}\right)^2$$

- grains are spheres, all of the same density ρ.

The radiation force is then $F_{\rm rad} = (L_\odot/4c)(r_g/r)^2$, where $L_\odot$ is the luminosity of the Sun and r_g is the grain radius. Then

$$\ddot{r} = \frac{1}{r^2}\alpha + \frac{(L/m)^2}{r^3},$$

$$\frac{v_\phi^2}{r} = \frac{(L/m)^2}{r^3}$$

where $M_\odot$ is the mass of the Sun and

$$\alpha \equiv \frac{L_\odot}{4c}\frac{r_g^2}{m} - GM_\odot.$$

If $\alpha < 0$, $\ddot{r}$ will be negative beyond some r, and the orbit is stable. If $\alpha > 0$, however, $\ddot{r} > 0$ for all r. This is the case when

$$m^{1/3} < \frac{L_\odot}{4GM_\odot c}\left(\frac{3}{4\pi\rho}\right)^{2/3}$$

Hence, grains below a certain mass are driven out of the Solar System by radiation pressure.

11.

Consider two electrons interacting with a spherically–symmetric 3–dimensional potential $V(r)$. Solutions for the Schrödinger equation

$$\frac{-\hbar^2}{2m}\nabla^2\Psi_k(\mathbf{r})+V(r)\Psi_k(\mathbf{r})=E_k\Psi_k(\mathbf{r})$$

are known and are given by one-electron eigenfunctions Ψ_k and eigenenergies E_k. Find approximate eigenfunctions and energies for the two-electron system if the interaction between the electrons H_{12} may be considered a perturbation. Assume that the spectrum $\{E_k\}$ is non-degenerate. *Hint*: Set up the two-electron Schrödinger equation for the trial wavefunction

$$\Psi_j(\mathbf{r}_1,\mathbf{r}_2)=a\Psi_k(\mathbf{r}_1)\Psi_l(\mathbf{r}_2)+b\Psi_l(\mathbf{r}_1)\Psi_k(\mathbf{r}_2).$$

You may leave integrals in symbolic form.

Assume that at time $t = 0$ the state of the two-electron system is given by a product of two one-electron eigenfunctions, $\Psi=\Psi_k(\mathbf{r}_1)\Psi_l(\mathbf{r}_2)$, where electron 1 is in state k and electron 2 is in state l. Find the time interval Δt after which the system will be in the state $\Psi=\Psi_l(\mathbf{r}_1)\Psi_k(\mathbf{r}_2)$, that is, the electrons have exchanged states.

11. QM - Rehane
p. 2

Solution

(a) The known eigenfunctions of one-electron problem (i= 1,2) satisfy the equation:

$$\hat{H}_i\Psi_k(\vec{r}_i) = -\frac{\hbar^2}{2m}\Delta_i\Psi_k(\vec{r}_i) + V(r)\Psi_k(\vec{r}_i) = E_k\Psi_k(\vec{r}_i).$$

We are looking for an approximate solution of the equation:

$$(\hat{H}_1 + \hat{H}_2 + \hat{H}_{12})\Psi_j(\vec{r}_1,\vec{r}_2) = E_j\Psi_j(\vec{r}_1,\vec{r}_2),$$

where $\hat{H}_{12} = \hat{H}_{21}$ is considered as perturbation. The eigenfunctions of unperturbed system are given by direct product of one-electron functions:

$$(\hat{H}_1 + \hat{H}_2)\Psi_k(\vec{r}_1)\Psi_l(\vec{r}_2) = (E_k + E_l)\Psi_k(\vec{r}_1)\Psi_l(\vec{r}_2).$$

However, unperturbed eigenfunction, $\Psi_l(\vec{r}_1)\Psi_k(\vec{r}_2)$,also corresponds to the same energy, E_k+E_l. This means that all unperturbed eigenfunctions are at least double degenerate. According to perturbation theory of degenerate eigenfunctions, we are looking for perturbed eigenfuncions in the form of linear superposition:

$$\Psi_j(\vec{r}_1,\vec{r}_2) = a\Psi_k(\vec{r}_1)\Psi_l(\vec{r}_2) + b\Psi_l(\vec{r}_1)\Psi_k(\vec{r}_2).$$

Substituting this into equation

$$(\hat{H}_1 + \hat{H}_2 + \hat{H}_{12} - E_k - E_l - \varepsilon)[a\Psi_k(\vec{r}_1)\Psi_l(\vec{r}_2) + b\Psi_l(\vec{r}_1)\Psi_k(\vec{r}_2)] = 0$$

$$(\hat{H}_{12} - \varepsilon)[a\Psi_k(\vec{r}_1)\Psi_l(\vec{r}_2) + b\Psi_l(\vec{r}_1)\Psi_k(\vec{r}_2)] = 0$$

By multiplying the last equation from left with $\Psi_k^*(\vec{r}_1)\Psi_l^*(\vec{r}_2)$ and $\Psi_l^*(\vec{r}_1)\Psi_k^*(\vec{r}_2)$ we get:

$$(C_{11} - \varepsilon)a + C_{12}b = 0$$

$$C_{21}a + (C_{22} - \varepsilon)b = 0$$

where

$$C_{11} = \iint\Psi_k^*(\vec{r}_1)\Psi_l^*(\vec{r}_2)\hat{H}_{12}\Psi_k(\vec{r}_1)\Psi_l(\vec{r}_2)d\tau_1 d\tau_2;$$

$$C_{22} = \iint\Psi_l^*(\vec{r}_1)\Psi_k^*(\vec{r}_2)\hat{H}_{12}\Psi_l(\vec{r}_1)\Psi_k(\vec{r}_2)d\tau_1 d\tau_2;$$

$$C_{12} = \iint\Psi_k^*(\vec{r}_1)\Psi_l^*(\vec{r}_2)\hat{H}_{12}\Psi_l(\vec{r}_1)\Psi_k(\vec{r}_2)d\tau_1 d\tau_2;$$

$$C_{21} = \iint\Psi_l^*(\vec{r}_1)\Psi_k^*(\vec{r}_2)\hat{H}_{12}\Psi_k(\vec{r}_1)\Psi_l(\vec{r}_2)d\tau_1 d\tau_2;$$

Due to symmetry of the above integrals with respect to exchange of electrons 1 and 2, C_{11}=C_{22} and C_{12}=C_{21}. This gives two solutions:

<u>First solution</u>: $\varepsilon_1 = C_{11} + C_{12}$ gives $a = b$, and the normalized eigenfunction is:

$$\Psi_j^1(\vec{r}_1,\vec{r}_2) = \frac{1}{\sqrt{2}}[\Psi_k(\vec{r}_1)\Psi_l(\vec{r}_2) + \Psi_l(\vec{r}_1)\Psi_k(\vec{r}_2)],$$

and the first eigenenergy is $E_1 = E_k + E_l + C_{11} + C_{12}$.

<u>Second solution</u>: $\varepsilon_2 = C_{11} - C_{12}$ gives $a = -b$, and the normalized eigenfunction is:

$$\Psi_j^2(\vec{r}_1,\vec{r}_2) = \frac{1}{\sqrt{2}}[\Psi_k(\vec{r}_1)\Psi_l(\vec{r}_2) - \Psi_l(\vec{r}_1)\Psi_k(\vec{r}_2)],$$

and the second eigenenergy is $E_2 = E_k + E_l + C_{11} - C_{12}$.

(b) The time development is obtained by presenting the state function as a linear superposition of energy eigenfunctions, and by multiplying each eigenfunction with corresponding complex exponential factor. In our case at time $t = 0$ the initial state function can be presented as linear superposition of two eigenfunctions:

$$\Psi(\vec{r}_1,\vec{r}_2,t=0) = \Psi_k(\vec{r}_1)\Psi_l(\vec{r}_2) = \frac{1}{\sqrt{2}}[\Psi^1_j(\vec{r}_1,\vec{r}_2) + \Psi^2_j(\vec{r}_1,\vec{r}_2)]$$

The time development of the state is given by:

$$\Psi(\vec{r}_1,\vec{r}_2,t) = \frac{1}{\sqrt{2}}[\Psi^1_j(\vec{r}_1,\vec{r}_2)e^{-i\omega_1 t} + \Psi^2_j(\vec{r}_1,\vec{r}_2)e^{-i\omega_2 t}]$$

$$= \frac{1}{\sqrt{2}}e^{-i\omega_1 t}[\Psi^1_j(\vec{r}_1,\vec{r}_2) + \Psi^2_j(\vec{r}_1,\vec{r}_2)e^{-i(\omega_2-\omega_1)t}]$$

where $\omega_i = \frac{E_i}{\hbar}$ $(i=1,2)$.The target state function is:

$$\Psi_l(\vec{r}_1)\Psi_k(\vec{r}_2) = \frac{1}{\sqrt{2}}[\Psi^1_j(\vec{r}_1,\vec{r}_2) - \Psi^2_j(\vec{r}_1,\vec{r}_2)].$$

The initial state evolves into the target state function (disregarding the unessential pure phase factor), if the complex exponential within the brackets becomes equal to -1:

$$e^{-i(\omega_2-\omega_1)\Delta t} = -1$$

$$\Rightarrow (\omega_2 - \omega_1)\Delta t = \pm\pi$$

$$\Rightarrow \Delta t = \left|\frac{\pi\hbar}{2C_{12}}\right|$$

12.

Enrico Fermi, an Italian-American physicist, designed the first atomic reactor, which went critical in 1942 under the football bleachers at the University of Chicago. Fermi's tremendous memory and powerful ability to simplify made him somewhat of an oracle of science. It is often reported that he calculated the yield of the first fission bomb by dropping bits of paper as the shock wave went by. As a result of this and other such calculations, "back-of-the-envelope" calculations have become known as Fermi problems. The idea is to simplify the problem and calculate an answer to one or two significant figures with the intent of obtaining an answer that is valid to better than an order of magnitude.

In working the following Fermi problems, be sure that you explain your assumptions and simplifications in *words*.

a) What area of photovoltaic cells would be required to supply the United States with its electrical energy needs?

b) When you take a single breath, how many of the nitrogen molecules in your lungs were once in the lungs of Aristotle?

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12.

Specialized Question for 2000 Comprehensive Exam
Larry Kirkpatrick

Enrico Fermi, an Italian-American physicist, designed the first atomic reactor, which went critical in 1942 under the football bleachers at the University of Chicago. Fermi's tremendous memory and powerful ability to simplify made him somewhat of an oracle of science. It is often reported that he calculated the yield of the first fission bomb by dropping bits of paper as the shock wave went by. As a result of this and other such calculations, "back-of-the-envelope" calculations have become known as Fermi problems. The idea is to simply the problem and calculate an answer to one or two significant figures with the intent of obtaining an answer that is valid to better than an order of magnitude.

In working the following Fermi problems, be sure that you explain your assumptions and simplifications *in words*.

a) What area of photovoltaic cells would be required to supply the United States with its electrical energy needs?

b) When you take a single breath, how many of the oxygen molecules in your lungs were once in the lungs of Aristotle?

Solutions

a) The solar constant at ground level is about 1 kW/m^2. Let's assume that our solar cells are in Arizona. Taking into account the length of the day and the varying angle of the sunlight, we can probably use 33% of the day. Let's assume a conversion efficiency of 20%. Therefore, our power output per unit area is

$$\frac{1000\text{ W}}{\text{m}^2}\times\frac{1}{3}\times\frac{1}{5}=67\text{ W/m}^2$$

US usage of electrical energy is 3 trillion kWh per year. This is an average power of

$$\frac{3\times10^{15}\text{ kWh}}{1\text{ y}}\times\frac{1\text{ year}}{52\text{ weeks}}\times\frac{1\text{ week}}{7\text{ days}}\times\frac{1\text{ day}}{24\text{ hours}}=3.4\times10^{11}\text{ W}$$

Therefore, the area required is

$$\frac{3.4\times10^{11}\text{ W}}{67\text{ W/m}^2}=5\times10^9\text{ m}^2=5000\text{ km}^2$$

Note that this is about 2000 square miles or about 1.8% of the land area of Arizona.

b) Let's begin by estimating the volume of the atmosphere. The Earth has a radius of 6400 km. The atmosphere has a half-height of 6 km, so

$$V=4\pi\left(6.4\times10^6\text{ m}\right)^2\left(6\times10^3\text{ m}\right)=2\times10^{18}\text{ m}^3=2\times10^{21}\text{ L}$$

Special - Kirkpatrick
p. 2

Aristotle lived for 62 years or 2×10^9 s. Assume one breath every two seconds, 1 liter per breath, each molecule was in his lung an average of two times, and none of these molecules get incorporated into rocks or absorbed in the ocean. Then the fraction of the nitrogen in the atmosphere he had in his lungs was

$$\frac{\frac{1}{2}\times 10^9\ \text{L}}{3\times 10^{21}\ \text{L}} = 1.6\times 10^{-13}$$

Your one breath of 1 L contains 2.7×10^{22} molecules of which four-fifths are nitrogen. Therefore, the average number of oxygen molecules that were once in Aristotle's lungs is

$$\frac{4}{5}\left(2.7\times 10^{22}\right)\left(1.6\times 10^{-13}\right) = 3.5\times 10^9$$

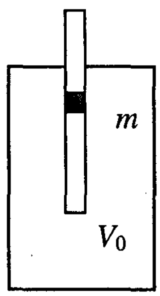
13.

An ideal gas is contained in a large volume V_0. As shown in the figure, the top has a pipe in which a cylindrical plug of area A and mass m is free to move up and down without friction. No gas molecules can escape from the volume. The gas is isolated from the surroundings and no heat can go in or out. The equilibrium pressure p_0 is slightly higher than atmospheric pressure because of the weight of the plug. When the plug is displaced from its equilibrium position by dz, it is observed to execute simple harmonic motion. Assume quasi-static equilibrium conditions during the oscillations of the plug.

a) Determine the frequency of oscillation in terms of the known equilibrium quantities A, m, V_0, n, R, and γ.

b) Describe accurately and briefly what happens to the temperature while the plug is executing simple harmonic motion.

c) Determine the relation between dz and dT, the variation in temperature.



13.

Avci-Thermo

An ideal gas (air) is contained in a large volume V_0. Fitted to the volume is a glass tube of cross-sectional area A in which a cylindrical plug of mass m is free to move up and down without friction, preventing any gas molecules from escaping from the volume. The content of volume is isolated from the outside; hence no heat can go in or out. The equilibrium pressure p_0 is slightly higher than the atmospheric pressure because of the weight of the ball. If the ball is displaced from its equilibrium position by dz, it is observed to execute a simple harmonic motion. Assuming the state of the gas is a quasi-static equilibrium during the oscillations of the piston

(a) determine the frequency of the oscillations in terms of known equilibrium quantities A, m, V_0, T_0, n, R and γ, and

(b) describe accurately and briefly what happens to the temperature while the plug is executing a simple harmonic motion, and determine the relation between dz and dT, the variation in temperature.

SOLUTION

(a) Process is adiabatic.

$$PV^\gamma = P_0 V_0^\gamma$$

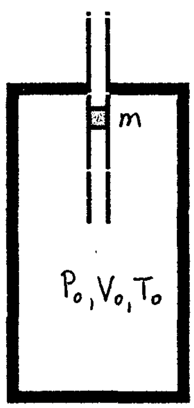
For a given displacement dz:

$$(P_0+dP)(V_0+dV)^\gamma = P_0 V_0^\gamma$$

$$\Rightarrow \cancel{P_0 V_0^\gamma}\left(1+\frac{dP}{P_0}\right)\left(1+\frac{dV}{V_0}\right)^\gamma = \cancel{P_0 V_0^\gamma}$$

$$\Rightarrow \left(1+\frac{dP}{P_0}\right)\left(1+\gamma\frac{dV}{V_0}\right) \approx 1$$

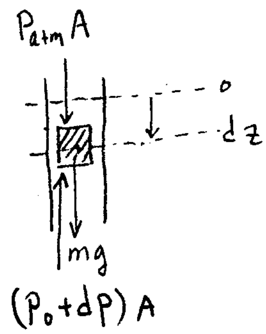
$$\Rightarrow \boxed{\frac{dP}{P_0} + \gamma\frac{dV}{V_0} = 0} \quad \text{---- (1)}$$



Force diagram:

Atm. pressure

$$\underbrace{P_{atm}A + mg = P_0 A}_{\text{Equilibrium}}$$



$\Rightarrow dP\,A =$ unbalanced force.

set $dP = \frac{F}{A}$, $dV = A\,dz$

This equation is of the form: $F = -kx$

where $k = \frac{\gamma P_0 A^2}{V_0}$, $x = dz$

Therefore, $\omega^2 = \frac{k}{m} = \frac{\gamma P_0 A^2}{m V_0} = \frac{A^2}{V_0^2}\frac{\gamma n R T_0}{m}$

$$\Rightarrow \boxed{\omega = \frac{A}{V_0}\left(\frac{\gamma n R T_0}{m}\right)^{1/2}}$$

Avci-Thermo cont'd P.2

(b) The temperature will vary sinosoidally as the plug oscillates back and forth.

Ideal gas law:

$$\frac{PV}{T} = \frac{P_0V_0}{T_0} \Rightarrow \cancel{\frac{P_0V_0}{T_0}}\frac{\left(1+\frac{dP}{P_0}\right)\left(1+\frac{dV}{V_0}\right)}{\left(1+\frac{dT}{T_0}\right)} = \cancel{\frac{P_0V_0}{T_0}}$$

$$\Rightarrow \boxed{\frac{dP}{P_0} + \frac{dV}{V_0} = \frac{dT}{T_0}} \quad (2)$$

Subtract equation (1) from equation (2):

$$\Rightarrow \frac{dV}{V_0}(1-\gamma) = \frac{dT}{T_0}, \qquad \text{set } dV = A\,dz$$

$$\Rightarrow \boxed{dT = \frac{AT_0}{V_0}(1-\gamma)\,dz}$$

— o —

14.

Consider the Sturm-Liouville problem for eigenfunction $\Psi(x)$ and eigenvalue λ

$$\frac{d^2\Psi}{dx^2} = -\frac{\lambda}{1-x^2}\Psi, \quad -1 \le x \le 1$$

subject to the boundary condition $\Psi(-1) = \Psi(1) = 0$.

a) Show that there is an orthogonality property of the form

$$\int_{-1}^{1} w(x)\Psi_i(x)\Psi_j(x)dx = 0 \quad \text{if} \quad \lambda_i \ne \lambda_j$$

for some weighting function $w(x)$.

b) Find a series solution. For which choices of λ will the series converge at $x = \pm 1$?

c) Give the first two eigenfunctions and eigenvalues explicitly.

14. E Mathϕ - DL

MP2

2000 Comps – Mathematical Physics

Consider the Sturm-Liouville problem for eigenfunction $\psi(x)$ and eigenvalue λ

$$\frac{d^2\psi}{dx^2} = -\frac{\lambda}{1-x^2}\,\psi \quad , \quad -1 \le x \le 1 \tag{1}$$

subject to the boundary condition $\psi(-1) = \psi(1) = 0$.

(a) **Show** an orthogonality property of the form

$$\int_{-1}^{1} w(x)\,\psi_i(x)\psi_j(x)\,dx \;=\; 0 \quad \text{if} \quad \lambda_i \neq \lambda_j$$

for some weighting function $w(x)$.

(b) Find a series solution. For which choices of λ will the series converge at $x = \pm 1$?

(c) Give the first two eigenfunctions and eignevalues explicitly.

Solution

(a) Writing the S-L equations for $\psi_i(x)$ and $\psi_j(x)$

$$\frac{d^2\psi_i}{dx^2} = -\frac{\lambda_i}{1-x^2}\,\psi_i \tag{2}$$

$$\frac{d^2\psi_j}{dx^2} = -\frac{\lambda_j}{1-x^2}\,\psi_j \tag{3}$$

Multiplying by $\psi_j(x)$ and $\psi_i(x)$ respectively, subtracting the equations and integrating gives

$$\int_{-1}^{1}\left[\psi_j\frac{d^2\psi_i}{dx^2} - \psi_i\frac{d^2\psi_j}{dx^2}\right]dx \;=\; (\lambda_j-\lambda_i)\int_{-1}^{1}\frac{\psi_i(x)\psi_j(x)}{1-x^2}\,dx \tag{4}$$

Integrating by parts we find

$$0 \;=\; (\lambda_j-\lambda_i)\int_{-1}^{1}\frac{\psi_i(x)\psi_j(x)}{1-x^2}\,dx \tag{5}$$

so in cases where $\lambda_i \neq \lambda_j$ we find

$$\boxed{\int_{-1}^{1}\frac{\psi_i(x)\psi_j(x)}{1-x^2}\,dx \;=\; 0 \quad \text{if} \quad \lambda_i \neq \lambda_j \ .} \tag{6}$$

3

(b) Proposing a solution of the form

$$\psi(x) \;=\; \sum_{n=0}^{\infty} c_n\,x^n \ . \tag{7}$$

gives the equation

$$\sum_{n=0}^{\infty}[(n+2)(n+1)\,c_{n+2} - n(n-1)c_n + \lambda\,c_n]\,x^n \;=\; 0 \tag{8}$$

The coefficients must satisfy the relation

$$\boxed{c_{n+2} \;=\; \frac{n(n-1)-\lambda}{(n+2)(n+1)}\, c_n \;.} \tag{9}$$

3

Convergence of series (7) requires that

$$\lim_{n\to\infty} \frac{c_{n+2}x^{n+2}}{c_n x^n} \;=\; x^2 \;<\; 1 \;. \tag{10}$$

Thus for convergence at $x = \pm 1$ we must have $c_{\ell+2} = 0$ for some ℓ. This means that

$$\boxed{\lambda \;=\; \ell(\ell-1) \quad , \quad \ell \;=\; 1,\, 2,\, 3,\, \ldots} \tag{11}$$

1

(c) The value $\ell = 1$ ($\lambda = 0$) gives the general solution

$$\psi_1(x) \;=\; c_0 + c_1\, x \tag{12}$$

for which the boundary conditions cannot both be satisfied.

For $\ell = 2$ ($\lambda = 2$) we get

$$c_2 \;=\; -\frac{\ell(\ell-1)}{2} c_0 \;=\; -\,c_0 \tag{13}$$

$$c_3 \;=\; -\frac{\ell(\ell-1)}{6} c_1 \;=\; -\,\tfrac{1}{3} c_1 \tag{14}$$

Which leads to the general form

$$\psi_2(x) \;=\; c_0\,(1-x^2) \;+\; c_1\,(x - \tfrac{1}{3}x^3) \;\;. \tag{15}$$

Since $\psi(\pm 1) = \pm 2c_1/3$ we must choose $c_1 = 0$ to satisfy the boundary conditions:

$$\boxed{\psi_2(x) \;=\; 1 - x^2 \quad , \quad \lambda_2 \;=\; 2} \tag{16}$$

2

Finally, $\ell = 3$ ($\lambda = 6$) gives

$$c_2 \;=\; -\frac{\ell(\ell-1)}{2} c_0 \;=\; -\,3c_0 \tag{17}$$

$$c_3 \;=\; -\frac{\ell(\ell-1)}{6} c_1 \;=\; -\,c_1 \tag{18}$$

$$c_4 \;=\; \frac{2-\ell(\ell-1)}{12} c_2 \;=\; -\,\tfrac{1}{3} c_2 \;=\; c_0 \tag{19}$$

(20)

This leads to the general form

$$\psi_3(x) \;=\; c_0\,(1 - 3x^2 + x^4) \;+\; c_1\,(x - x^3) \;\;. \tag{21}$$

Since $\psi(\pm 1) = -c_0$ we must have $c_0 = 0$ and

$$\boxed{\psi_3(x) \;=\; x - x^3 \quad , \quad \lambda_3 \;=\; 6} \tag{22}$$

1

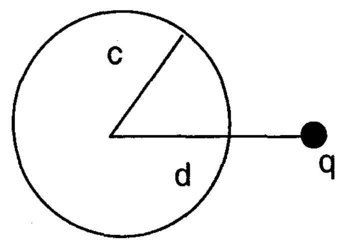
Background note — The solution to equation (1) is

$$\psi_\ell(x) \;=\; (1-x^2)\frac{dP_{\ell-1}}{dx} \;\;, \tag{23}$$

where $P_\ell(x)$ is the Legendre polynomial.

15.

A charge q is placed a distance d from the center of a grounded metal ball with radius $c < d$. What is the force acting on charge q?



15. J E/M - LDK
p. 1
EM3

Electricity and Magnetism Problem for the 2000 Comprehensive Exam
Larry Kirkpatrick

A charge q is placed a distance d from the center of a grounded metal ball with radius $c < d$. What is the force acting on charge q?

Solution

Let's choose an image charge Q located a distance $D < c$. As shown in the figure, let's denote the distances from the real charge and the image charge to a point on the surface of the ball as r and R, respectively. The electrostatic potential at this point is given by

$$V = \frac{kq}{r} + \frac{kQ}{R},$$

where

$$r^2 = d^2 - 2dc\cos\theta + c^2$$

and

$$R^2 = D^2 - 2Dc\cos\theta + c^2.$$

Since the electrostatic potential is zero at all points on the surface of the ball,

$$Qr = -qR.$$

We now square both sides of this relationship, plug in the values of r and R, and group terms in powers of $\cos\theta$ on each side to obtain

$$Q^2\left(d^2 + c^2\right) - 2dcQ^2\cos\theta = q^2\left(D^2 + c^2\right) - 2Dcq^2\cos\theta.$$

Because this relationship must be valid for all values of $\cos\theta$, the coefficients of each power of $\cos\theta$ must be equal on the two sides of this equation. This yields

$$Q^2 d = q^2 D$$

and

$$Q^2\left(d^2 + c^2\right) = q^2\left(D^2 + c^2\right).$$

We now solve both equations for the ratio Q^2/q^2 and equate them to obtain

$$\frac{D}{d} = \frac{D^2 + c^2}{d^2 + c^2}.$$

This is a quadratic equation in D, which has two roots

$$D = d, \frac{c^2}{d}.$$

The first root corresponds to the case where the real charge and the image charge are superimposed and the potential is zero everywhere. We are interested in the second root for which

$$Q = -\frac{D}{c}q = -\frac{c}{d}q.$$

(Note that the location of the image is not the same as that for an object outside a spherical mirror.)

Using this image charge and image distance, the attractive force on the charge outside a grounded conducting sphere is

$$F = kq^2\left(\frac{cd}{\left(d^2 - c^2\right)^2}\right).$$

It is interesting to check the limiting cases. As the charge approaches the surface of the sphere, *d* approaches *c* and force becomes very large. And as the charge is moved to very large distances, the force decreases to zero. Both of these behaviors are expected and agree with the case for the infinite conducting plane.