Maxwell Equations: Wave-like Solutions [Ref. Jackson Secs. (6.5) & (6.6)]

We have by now reduced Maxwell's Eqs. in a linear medium to the solution of two inhomogeneous wave equations...

$$\nabla\cdot\mathbb{E} = \left(\frac{4\pi}{\epsilon}\right)\rho, \quad \nabla\times\mathbb{E} = -\frac{1}{c}\partial\mathbb{B}/\partial t, \quad \nabla\cdot\mathbb{B} = 0, \quad \nabla\times\mathbb{B} = \frac{4\pi\mu}{c}\mathbb{J} + \frac{\mu\epsilon}{c}\frac{\partial\mathbb{E}}{\partial t};$$

use potentials...

$$\mathbb{B} = \nabla\times\mathbb{A}, \quad \mathbb{E} = -\nabla\phi - \frac{1}{c}\frac{\partial\mathbb{A}}{\partial t};$$

... and Lorentz Gauge

$$\nabla\cdot\mathbb{A} + \frac{\mu\epsilon}{c}\frac{\partial\phi}{\partial t} = 0$$

$$\boxed{\nabla^2\phi - \frac{1}{v^2}\frac{\partial^2\phi}{\partial t^2} = -\frac{4\pi}{\epsilon}\rho, \quad \nabla^2\mathbb{A} - \frac{1}{v^2}\frac{\partial^2\mathbb{A}}{\partial t^2} = -\frac{4\pi\mu}{c}\mathbb{J};}$$

where: $v = c/\sqrt{\mu\epsilon}$. (1)

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1) To proceed, evidently we should look at solutions to the generic wave eqtn...

$$\boxed{\nabla^2\psi - \frac{1}{v^2}\left(\partial^2\psi/\partial t^2\right) = -4\pi f(\mathbf{r},t)} \qquad (2)$$

$\psi = \psi(\mathbf{r},t)$ is the wave amplitude, $f(\mathbf{r},t)$ is a known source function.

REMARK Plane wave solutions.

The homogeneous eqtn is: $\left(\nabla^2 - \frac{1}{v^2}\frac{\partial^2}{\partial t^2}\right)\psi = 0$. Try plane wave: $\psi = Ae^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$.

So $\left(-k^2 + \frac{\omega^2}{v^2}\right)\psi = 0$ ... works if $\omega = \pm kv = \pm kc/\sqrt{\mu\epsilon}$ — relation: $\omega = \omega(k)$ is a "dispersion relation".

If μ & ϵ are indpt. of ω, then $\begin{cases}\text{wave phase velocity}: \omega/k = \pm v \\ \text{wave group velocity}: \partial\omega/\partial k = \pm v\end{cases}$ both cnst (wave shows zero dispersion)

If μ & ϵ depend on freq. ω, v(phase) & v(group) $\propto \omega$, and propagating wave distorts.

We first assume simplest case: $\omega = \text{cnst}\times k$, with no wave distortion/dispersion.

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2) With no dispersion [v(phase) & v(group) = cnst], Fourier analysis of Eq. (2) is useful:

transform ↓ inverse →

$$\tilde{\Psi}(\mathbf{r},\omega) = \int_{-\infty}^{\infty}\psi(\mathbf{r},t)\,e^{i\omega t}\,dt \leftrightarrow \psi(\mathbf{r},t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{\Psi}(\mathbf{r},\omega)\,e^{-i\omega t}\,d\omega,$$

$$\mathcal{F}(\mathbf{r},\omega) = \int_{-\infty}^{\infty} f(\mathbf{r},t)\,e^{i\omega t}\,dt \leftrightarrow f(\mathbf{r},t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathcal{F}(\mathbf{r},\omega)\,e^{-i\omega t}\,d\omega. \qquad (3)$$

This formulation anticipates problems that are unbounded in time. Plug these in (2):

ME Wave Solutions (cont'd)

$$\left(\nabla^2 - \frac{1}{v^2}\frac{\partial^2}{\partial t^2}\right)\psi = -4\pi f \Rightarrow \boxed{(\nabla^2 + k^2)\,\tilde{\psi}(\mathbf{r},\omega) = -4\pi \tilde{f}(\mathbf{r},\omega)}, \qquad (4)$$

where $k = \frac{\omega}{v}$ is linear in ω. The transformed wave eqtn in $\tilde{\psi}$ & $\tilde{f}$ is called an "inhomogeneous Helmholtz equation"; evidently it includes the Poisson eqtn $[\nabla^2\tilde{\psi} = -4\pi\tilde{f}$, for $k=0]$ as a special case. What's important about the Fourier transform is that it has reduced the t-variation to a "spectator" variable ω.

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3) PDE's of the Helmholtz (and allied) type can be solved by using <u>Green's functions</u>.

$$\text{Want } \tilde{\psi} \text{ in}: \; (\nabla^2 + k^2)\,\tilde{\psi}(\mathbf{r},\omega) = -4\pi\tilde{f}(\mathbf{r},\omega), \quad ①$$

$$\underline{\text{Define}}\; G_k \text{ by}: \; (\nabla^2 + k^2)\,G_k(\mathbf{r},\mathbf{r}') = -4\pi\,\delta(\mathbf{r}-\mathbf{r}'). \quad ② \qquad (5)$$

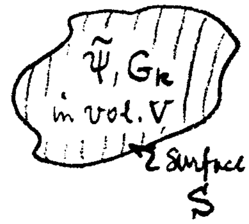
assume we can solve for $G_k(\mathbf{r},\mathbf{r}')$

Connect these equations by Green's identity...

$$\tilde{\psi}\nabla^2 G_k - G_k\nabla^2\tilde{\psi} = \nabla\cdot(\tilde{\psi}\nabla G_k - G_k\nabla\tilde{\psi})$$

integrate $\int_V d^3x$, use Divergence Thm

$$\int_V \underbrace{(\tilde{\psi}\nabla^2 G_k - G_k\nabla^2\tilde{\psi})}\, d^3x = \oint_S (\tilde{\psi}\nabla G_k - G_k\nabla\tilde{\psi})\cdot d\mathbf{S}$$



integrand $= 4\pi[G_k\tilde{f} - \tilde{\psi}\,\delta(\mathbf{r}-\mathbf{r}')]$ obtained by forming qty $\tilde{\psi}\cdot$Eq② $- G_k\cdot$Eq①

So//

$$4\pi\left[\int_V G_k\tilde{f}\,d^3x - \tilde{\psi}(\mathbf{r}',\omega)\right] = -\oint_S (G_k\nabla\tilde{\psi} - \tilde{\psi}\nabla G_k)\cdot d\mathbf{S}. \qquad (6)$$

Interchange labels $\mathbf{r}$ & $\mathbf{r}'$ [note that $G_k(\mathbf{r}',\mathbf{r}) = G_k(\mathbf{r},\mathbf{r}')$], rearrange terms to get

$$\boxed{\tilde{\psi}(\mathbf{r},\omega) = \int_V G_k(\mathbf{r},\mathbf{r}')\,\tilde{f}(\mathbf{r}',\omega)\,d^3x' + \frac{1}{4\pi}\oint_S (G_k\nabla'\tilde{\psi} - \tilde{\psi}\nabla' G_k)\cdot d\mathbf{S}'} \qquad (7)$$

Except for the "spectator" variable ω (& $k = \omega/v$), this solution is the <u>same</u> as for Poisson's eqtn [see Jkⁿ Eq. (1.42)]. It gives a particular solⁿ to $(\nabla^2 + k^2)\tilde{\psi} = -4\pi\tilde{f}$, provided G_k satisfies $(\nabla^2 + k^2)G_k = -4\pi\,\delta(\mathbf{r}-\mathbf{r}')$. The $\int_V$ term accounts for $\tilde{\psi}$ generated in V by the source $\tilde{f}$; the $\oint_S$ term provides freedom for B.C. on surface.

ME Wave Solutions (cont'd)

REMARKS on soln: $\tilde{\psi} = \int_V G_k \tilde{f}\, d^3x' + \frac{1}{4\pi}\oint_S (G_k \nabla'\tilde{\psi} - \tilde{\psi}\nabla' G_k)\cdot d\mathbb{S}'$.

1. Any sol$^{\underline{n}}$ $\tilde{\psi}_0$ to homogeneous eq$^{\underline{n}}$ $(\nabla^2 + k^2)\tilde{\psi}_0 = 0$ can be added to this $\tilde{\psi}$.

2. The surface term can be adjusted to meet B.C. on surface S enclosing volume V...

Dirichlet conditions $\}$ $\tilde{\psi}$ given on S $\Rightarrow$ construct $G_k(\text{on } S) \equiv 0$; then:

$$\text{surface term} \rightarrow (-)\frac{1}{4\pi}\oint_S (\tilde{\psi}\nabla' G_k)\cdot d\mathbb{S}; \tag{8}$$

Neumann conditions $\}$ $\nabla\tilde{\psi}$ given on S $\Rightarrow$ construct $\nabla G_k(\text{on } S) \equiv 0$; then:

$$\text{surface term} \rightarrow (+)\frac{1}{4\pi}\oint_S (G_k \nabla'\tilde{\psi})\cdot d\mathbb{S}. \tag{9}$$

Evidently the actual functional form of $G_k(\mathbf{r}, \mathbf{r}')$ depends on the B.C. required.

3. Sometimes the region of interest is an ∞ domain, i.e. the surface S is at ∞ -- where by definition $\tilde{\psi}$ & $\nabla\tilde{\psi}$ both vanish. Then our solution-to-date looks like:

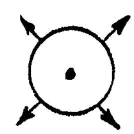
$$\left[\begin{array}{l} \tilde{\psi}(\mathbf{r},\omega) = \int_\infty G_{k\infty}(\mathbf{r},\mathbf{r}')\,\tilde{f}(\mathbf{r}',\omega)\, d^3x', \\ \text{with: } (\nabla^2 + k^2) G_{k\infty} = -4\pi\,\delta(\mathbf{r}-\mathbf{r}'). \end{array}\right\| \begin{array}{l} \text{for } \infty \text{ domain, with} \\ G_{k\infty} \rightarrow 0 \text{ as } |\mathbf{r}-\mathbf{r}'| \rightarrow \infty. \end{array} \tag{10}$$

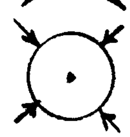
4. Jackson shows how to solve the $G_{k\infty}$ problem in his Eqs. (6.58)-(6.62). Results:

$$\boxed{G_{k\infty}(\mathbf{r},\mathbf{r}') = \frac{1}{R}e^{\pm ikR}}, \text{ w/ } R = |\mathbf{r}-\mathbf{r}'|, \text{ on } \infty \text{ domain}. \tag{11}$$

This can be verified by a plug-in, if you use $\nabla^2(1/R) = -4\pi\,\delta(\mathbf{r}-\mathbf{r}')$.

The results here have a geometrical interpretation...

(+) sign $\leftrightarrow$ <u>outgoing</u> spherical wave from point source at origin:  $\frac{e^{+ikR}}{R}$

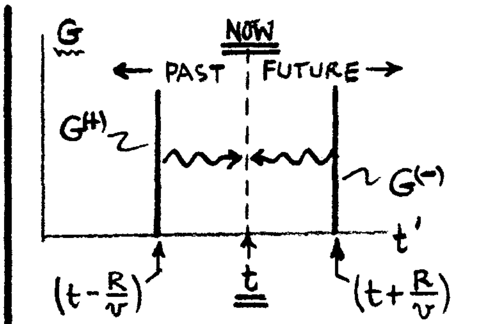
(−) sign $\leftrightarrow$ <u>incoming</u> " " to " " " "  $\frac{e^{-ikR}}{R}$

The wave amplitude $\psi(\mathbf{r}, t)$ [on an ∞ domain] can now be obtained by using $G_{k\infty}$ of Eq. (11) in $\tilde{\psi}(\mathbf{r},\omega)$ of Eq. (10), and then inverting the Fourier transforms $\tilde{\psi}$ and $\tilde{f}$. Jackson shows how this is done in his Eqs. (6.63)-(6.66).

ME Wave Solutions (cont'd)

4) With the procedure just noted, the resulting solution to the inhomogeneous wave eqtn: $\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\psi = -4\pi f(\mathbf{r},t)$, on an ∞ domain may be quoted as:

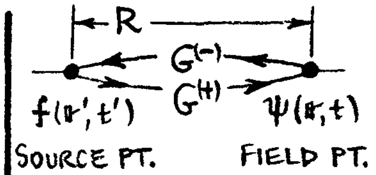
$$\boxed{\psi(\mathbf{r},t) = \int_\infty d^3x' \int_{-\infty}^{\infty} dt'\, G^{(\pm)}(\mathbf{r},t;\mathbf{r}',t')\, f(\mathbf{r}',t'), \quad \text{w/}\ G^{(\pm)} = \frac{1}{R}\delta\left(t' - \left[t \mp \frac{R}{v}\right]\right) \int R = |\mathbf{r}-\mathbf{r}'|,\ v = c/\sqrt{\mu\epsilon}.} \tag{12}$$



Signal (from f at time t') propagates to observation pt (ψ at time t) from the past by $G^{(+)}$, or from future by $G^{(-)}$.

[this is the un-numbered eqtn on Jkⁿ p. 225]. The δ-fcn in $G^{(\pm)}$ is really present, and it has a Novel Feature: the signal from f at time t' can arrive to form the disturbance ψ at time t, at: $\boxed{t = t' \pm (R/v)}$, i.e. $f \Rightarrow \psi$ from either the past or the future. There is no mathematical distinction between past & future (since wave eqtn is quadratic in x & t).

Time-ordering aside, we at least have a rational result for the finite propagation velocity v in the theory. The source & field points are (causally) connected only if: $R = v|t-t'|$.



Signal velocity = v

Details of the two solutions for ψ go as follows...

① $G^{(+)}$ solution: f signal at t' reaches ψ pt. at: $t = t' + \frac{R}{v} > t'$ {signal at t is from past.

$\uparrow$ $t' = t - \frac{R}{v} = t_{ret}$, is the "retarded time"; $G^{(+)}$ is called "retarded" Greens fcn;

and// $$\boxed{\psi(\mathbf{r},t) = \Psi_{inc}(\mathbf{r},t) + \int_\infty \frac{d^3x'}{R} f(\mathbf{r}', t_{ret})} \tag{13}$$ $\int$ Ψ_{inc} = solution to homogeneous wave eqtn as $t \to (-)\infty$ [starts in past].

② $G^{(-)}$ solution: f signal at t' reaches ψ pt. at: $t = t' - \frac{R}{v} < t'$ {signal at t is from future.

$\uparrow$ $t' = t + \frac{R}{v} = t_{adv.}$, is the "advanced time"; $G^{(-)}$ is called "advanced" Green's fcn;

and// $$\boxed{\psi(\mathbf{r},t) = \Psi_{out}(\mathbf{r},t) + \int_\infty \frac{d^3x'}{R} f(\mathbf{r}', t_{adv})} \tag{14}$$ $\int$ Ψ_{out} = solution to homogeneous wave eqtn as $t \to (+)\infty$ [starts in future].

The choice of $G^{(\pm)}$ for the ψ solution is dictated by whether we want the source integral to contribute $\sim$ zero at very early [$G^{(+)}$] or very late [$G^{(-)}$] times t.

ME Wave Solutions (cont'd)

5) SUMMARY A Complete Solution to Maxwell's Equations.

For a linear medium ($\mathbb{D}=\epsilon\mathbb{E}$, $\mathbb{B}=\mu\mathbb{H}$) ...

Lorentz gauge: $\nabla\cdot\mathbb{A}+\frac{\mu\epsilon}{c}\left(\frac{\partial\phi}{\partial t}\right)=0$

$$\left.\begin{aligned}&\nabla\cdot\begin{pmatrix}\mathbb{E}\\ \mathbb{B}\end{pmatrix}=\frac{4\pi}{\epsilon}\begin{pmatrix}\rho\\ 0\end{pmatrix},\\ &\nabla\times\begin{pmatrix}\mathbb{E}\\ \mathbb{B}\end{pmatrix}=\frac{4\pi\mu}{c}\begin{pmatrix}0\\ \mathbb{J}\end{pmatrix}+\frac{1}{c}\begin{pmatrix}-\partial\mathbb{B}/\partial t\\ \mu\epsilon\,\partial\mathbb{E}/\partial t\end{pmatrix}\end{aligned}\right\}\left(\nabla^2-\frac{1}{v^2}\frac{\partial^2}{\partial t^2}\right)\begin{pmatrix}\phi\\ \mathbb{A}\end{pmatrix}=-4\pi\begin{pmatrix}\rho/\epsilon\\ \frac{\mu}{c}\mathbb{J}\end{pmatrix},$$

where: $v=c/\sqrt{\mu\epsilon}$;

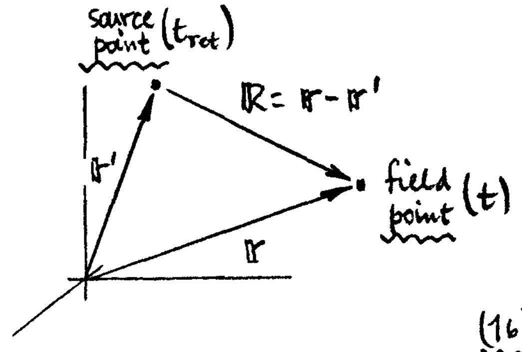
$$\text{w/}\quad \mathbb{B}=\nabla\times\mathbb{A},\quad \mathbb{E}=-\nabla\phi-\frac{1}{c}(\partial\mathbb{A}/\partial t). \tag{15}$$

Solutions on an ∞ domain ...

$$\phi(\mathbb{r},t)=\phi_0(\mathbb{r},t)+\frac{1}{\epsilon}\int_{\text{sources}}\frac{d^3x'}{R}\,\rho(\mathbb{r}',t_{ret}),$$

$$\mathbb{A}(\mathbb{r},t)=\mathbb{A}_0(\mathbb{r},t)+\frac{\mu}{c}\int\frac{d^3x'}{R}\,\mathbb{J}(\mathbb{r}',t_{ret});$$

$$\text{w/}\quad t_{ret}=t-\frac{1}{v}R(t_{ret}),\quad R=|\mathbb{r}-\mathbb{r}'|. \tag{16}$$



Here ϕ_0 & $\mathbb{A}_0$ are solutions to the homogeneous eqtns: $\left(\nabla^2-\frac{1}{v^2}\frac{\partial^2}{\partial t^2}\right)\begin{pmatrix}\phi_0\\ \mathbb{A}_0\end{pmatrix}=0$. We have chosen the retarded solutions for ϕ & $\mathbb{A}$, per convention.

REMARKS

1. The only apparent effect of adding the time-derivative terms (in $\dot{\mathbb{B}}$ & $\dot{\mathbb{E}}$) to the static Maxwell Eqtns [see Eqs. (1) above] is that in the integrals for ϕ & $\mathbb{A}$ -- i.e. the integrals $\int\frac{d^3x'}{R}(\rho\ \&\ \mathbb{J})$ -- <u>the time $t\to t_{ret}=t-(R/v)$</u>. At first glance, this is an unexpectedly simple way to include the $\dot{\mathbb{B}}$ & $\dot{\mathbb{E}}$ terms ... it just complicates the integrals a bit. <u>BUT</u>, the fact that the source pt - field pt distance R now becomes an explicit function of the time difference $(t-t_{ret})$ will cause grief, later.

2. The solutions of Eq. (16) hold <u>only</u> on an ∞ domain (i.e. the only B.C. are that the fields $\mathbb{E}$ & $\mathbb{B}$ and potentials ϕ & $\mathbb{A}$ vanish at ∞). Problems which require B.C. on a finite domain are much more complicated, but are solvable w/ suitable surface terms.