ϕ506 Problems

NOTE: Problems are worth 10 points unless marked otherwise.

① (A) Convert Planck's Radiation Law from an energy distribution $U(\nu)$ over frequencies ν to a distribution $U(\lambda)$ over wavelengths λ by using: $U(\lambda)\,d\lambda = U(\nu)\,d\nu$.

(B) Show that $U(\lambda)$ vs. λ passes through a maximum at wavelength λ_m such that: $\lambda_m T = \text{cnst}$; this is Wien's Displacement Law. Evaluate the cnst in cgs units.

(C) The solar spectrum peaks at $\lambda_m \simeq 5000$ Å. Assuming the sun radiates as a blackbody, find the effective temperature of the sun's surface.

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② (A) Cesium has a work function of 1.8 eV. If a retarding potential of 5 V completely stops the photocurrent emitted from Cs, what is the wavelength of the most energetic incident photon?

(B) Light of wavelength 4000 Å is incident on Lithium. If the work function for Li is 2.1 eV, what is the speed of the fastest emitted photoelectron?

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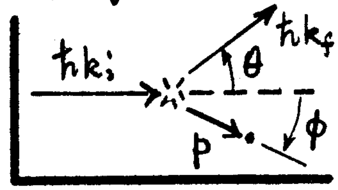
③ [15 pts]. Ref. class notes on Compton Effect, pp. Intro. 10–12, Eqs. (24)–(27).

(A) Carry out the algebra between Eqs. (26) and (27) -- i.e. show that the conservation laws for the photon-electron collision actually lead to Compton's formula for the scattered photon frequency: $\nu_f = \nu_i / [1 + (h\nu_i/mc^2)(1-\cos\theta)]$.



(B) For a given incident photon frequency ν_i and scattered photon $\measuredangle\,\theta$, find the kinetic energy K of the recoil electron. Note that K may be relativistic, so use $K = E - mc^2$, w/ $E =$ is the total energy of the electron.

(C) For given ν_i and θ, find the recoil $\measuredangle\,\phi$ for the scattered electron.

(D) Photons at energy 1.02 MeV undergo Compton scattering. Find the energy of the photons scattered at $\theta = 60°$, the kinetic energy of the recoil electrons, and the recoil $\measuredangle\,\phi$ for the scattered electrons.

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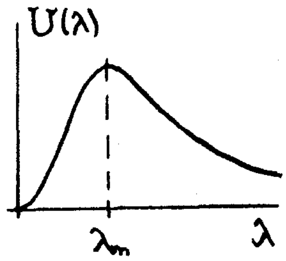
① Derive Wien's Displacement Law from the Planck Radiation Law.

(A) Since $\lambda\nu = c$, then $|d\nu/d\lambda| = c/\lambda^2$. The absolute value eliminates an unimportant (−) sign, which refers only to the direction of counting. Now since: $U(\nu) = (8\pi h/c^3)\,\nu^3/(e^{h\nu/kT}-1)$, CLASS NOTES p. Intro 6, Eq.(13), we have:

$$U(\lambda) = U(\nu)\,|d\nu/d\lambda|\Big|_{\nu=c/\lambda} = \underline{\underline{\frac{8\pi hc}{\lambda^5}\Big/\left[e^{\left(\frac{hc}{kT}\right)\frac{1}{\lambda}}-1\right]}}. \quad (1)$$



$U(\lambda)$ vs. λ has the general shape sketched at right -- it falls to zero in the far UV ($\lambda \to 0$) and far IR ($\lambda \to \infty$), while going through an absolute maxm at an intermediate wavelength λ_m.

(B) To find λ_m, first define a new variable: $\underline{x = (hc/kT)\frac{1}{\lambda}}$, and write (1) as:

$$\rightarrow U(\lambda) = C\,x^5/(e^x-1), \quad \text{w/ } C = 8\pi hc\,(kT/hc)^5 = \text{cnst}. \quad (2)$$

Since $\frac{d}{d\lambda} = (dx/d\lambda)\frac{d}{dx} = -(kT/hc)\,x^2\,\frac{d}{dx}$, then we can calculate from (2):

$$\rightarrow \frac{dU(\lambda)}{d\lambda} = -\left(\frac{kT}{hc}\right)Cx^2\frac{d}{dx}\left[x^5/(e^x-1)\right] = -\left(\frac{kT}{hc}\right)C\,\frac{x^6}{(e^x-1)^2}\left[5(e^x-1)-xe^x\right]. \quad (3)$$

$\lambda = \lambda_m$ when $dU/d\lambda = 0$, whence in (3): [ ] = 0, or...

$$\rightarrow (5-x)e^x = 5 \Rightarrow x \simeq 4.965, \text{ i.e. } \lambda_m \simeq (hc/kT)/4.965,$$

$$\text{or/ } \boxed{\lambda_m T = hc/4.965k = 0.290 \text{ cm-}^\circ\text{K}} \leftarrow \text{WIEN'S LAW} \quad (4)$$

The x-value is an approximate solution to the transcendental eqtn. The numerical value of the RHS cnst in (4) is obtained by putting in the CGS values $h = 6.63\times10^{-27}$, $c = 3.00\times10^{10}$, $k = 1.38\times10^{-16}$.

(C) For solar radiation, the maxm output is at $\lambda_m = 5000\,\text{Å} = 5\times10^{-5}$ cm. Then, if the sun radiates as a blackbody (a good approxn), Eq.(4) ⇒ surface temp...

$$\underline{\underline{T = (0.290 \text{ cm-}^\circ\text{K})/\lambda_m = 5800^\circ\text{K}}}. \quad (5)$$

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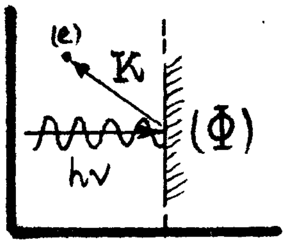

② Some numbers re Photo Electric Effect.

(A) The (max) kinetic energy of the ejected electron is...

$\rightarrow K = \mathcal{E} - \Phi$ $\int$ $\mathcal{E} = h\nu$ = incident photon energy, Φ = work function of metal. (1)

If these electrons are stopped by a retarding potential of 5V, then K_{max} = 5eV, and the max. photon energy must be -- with $\Phi = 1.8$ eV...

$\rightarrow \mathcal{E} = K + \Phi = 5.0 + 1.8 = 6.8$ eV. (2)

Since $\mathcal{E} = h\nu$, and $\nu = c/\lambda$, then $\lambda = hc/\mathcal{E}$ is the photon wavelength. On putting in h & c, a useful relation is

$\underline{\lambda = (12{,}400\,\text{Å})/\mathcal{E}}$ $\int$ $\mathcal{E}$ = photon energy in eV, λ = photon wavelength. (3)

For the photon in Eq (2) : $\underline{\underline{\lambda}} = (12{,}400)/6.8 = \underline{\underline{1824\,\text{Å}}}$. (4)

(B) The incident light, @ $\lambda = 4000\,\text{Å}$, has energy : $\mathcal{E} = \frac{12{,}400}{4000} = 3.1$ eV, by (3). The max. ejected electron K.E. is, by (1)

$\rightarrow K = \mathcal{E} - \Phi = 3.1 - 2.1 = 1.0$ eV, (5)

for a Li work function $\Phi = 2.1$ eV. Since the electron rest energy is $E_0 = mc^2 = 0.511$ MeV >> K, this electron is non-relativistic, and one may use the Newtonian $K = \frac{1}{2}mv^2$, w/ v = electron velocity. So...

$K = \frac{1}{2}mv^2 = 1.0\,\text{eV} = 1.6\times10^{-12}$ erg;

so $v = \sqrt{(2/m)\times 1.6\times10^{-12}}$, w/ $m = 9.11\times10^{-28}$ gm, for e,

or $\underline{\underline{v = 5.93\times10^{7}\ \frac{\text{cm}}{\text{sec}}}} = 0.00198\,c$. (6)

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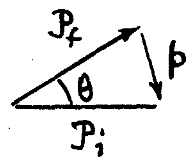
③ [15 pts]. Details of Compton scattering.

Adopt a shorthand notation: $\mathcal{P} = \hbar k$ = photon momentum, $K = E - mc^2$ = recoil electron kinetic energy. The conservation laws in CLASS NOTES, p. Intro. 12 are...
① $\mathcal{P}_i - \mathcal{P}_f = K/c$;
② $\mathcal{P}_i - \mathcal{P}_f \cos\theta = p\cos\phi$,
③ $\mathcal{P}_f \sin\theta = p\sin\phi$.

The photon energy is $\mathcal{E} = \mathcal{P}c = h\nu$, and K is related to the electron momentum p by $(pc)^2 = K^2 + 2mc^2 K$.

(A) Form ②^2 + ③^2 to show...



$$\rightarrow \mathcal{P}_i^2 - 2\mathcal{P}_i\mathcal{P}_f\cos\theta + \mathcal{P}_f^2 = p^2. \qquad (1)$$

Now ①^2 $\Rightarrow$ $\mathcal{P}_i^2 - 2\mathcal{P}_i\mathcal{P}_f + \mathcal{P}_f^2 = (K/c)^2$. Subtract this from Eq (1) to get...

$$\rightarrow 2\mathcal{P}_i\mathcal{P}_f(1-\cos\theta) = p^2 - (K/c)^2. \qquad (2)$$

By the p-K relation above: $p^2 - (K/c)^2 = 2mK = 2mc(\mathcal{P}_i - \mathcal{P}_f)$; the last relation follows from ①. Use of this in Eq. (2) produces...

$$\rightarrow \mathcal{P}_i\mathcal{P}_f(1-\cos\theta) = mc(\mathcal{P}_i - \mathcal{P}_f) \Rightarrow \frac{1}{mc}(1-\cos\theta) = \frac{1}{\mathcal{P}_f} - \frac{1}{\mathcal{P}_i}$$

w/ with $\mathcal{P} = h\nu/c$ ... $\boxed{\nu_f = \nu_i/[1+(h\nu_i/mc^2)(1-\cos\theta)]}$. **QED** (3)

(B) By ①: $K = c(\mathcal{P}_i - \mathcal{P}_f)$. Since $\mathcal{P} = h\nu/c$, have: $K = h(\nu_i - \nu_f)$, i.e. the energy gained by the electron = energy given up by the photon. Then, by by use of Eq. (3) for ν_f, we have immediately...

$$\boxed{K = h\nu_i\left\{1 - [1+(h\nu_i/mc^2)(1-\cos\theta)]^{-1}\right\} \leqslant h\nu_i}. \quad \textbf{QED} \qquad (4)$$

(C) To get the electron recoil $\measuredangle\phi$, divide ③ by ②, and use $\mathcal{P} = h\nu/c$. Then:

$$\rightarrow \tan\phi = \mathcal{P}_f\sin\theta/(\mathcal{P}_i - \mathcal{P}_f\cos\theta) = \left(\frac{\nu_f}{\nu_i}\right)\sin\theta/\left[1-\left(\frac{\nu_f}{\nu_i}\right)\cos\theta\right]. \qquad (5)$$

The ratio ν_f/ν_i is prescribed by Eq. (3). As a further shorthand, use

$r = h\nu_i/mc^2$; then: $\nu_f/\nu_i = 1/[1+r(1-\cos\theta)]$. Use this in (5) to get:

$$\rightarrow \tan\phi = \frac{\sin\theta}{1+r(1-\cos\theta)} \Big/ \left[1 - \frac{\cos\theta}{1+r(1-\cos\theta)}\right] = \sin\theta/(1+r)(1-\cos\theta)$$

w/ $\left.\begin{array}{l}\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\\ 1-\cos\theta = 2\sin^2\frac{\theta}{2}\end{array}\right\}$ $\boxed{\tan\phi = \frac{1}{1+r}\,\mathrm{ctn}\,\frac{\theta}{2}}$, $r = \frac{h\nu_i}{mc^2}$. (6)

(D) Compton scattering @ $\theta = 60°$, at incident energy...

$\rightarrow h\nu_i = 1.02\,\mathrm{MeV} \Rightarrow r = h\nu_i/mc^2 = 1.02/0.511 = 2$. (7)

So, scattered photon energy is -- from Eq. (3) above...

$$\left[\!\left[\, h\nu_f = h\nu_i/[1+r(1-\cos\theta)] = 1.02/[1+2(1-\tfrac{1}{2})] = \underline{\underline{0.511\,\mathrm{MeV}}}. \right.\right. \quad (8)$$

The recoil electron K.E. is the energy given up by the photon...

$$\left[\!\left[\, K = h\nu_i - h\nu_f = 1.02 - 0.511 = \underline{\underline{0.511\,\mathrm{MeV}}} \right.\right. \quad (9)$$

Both the scattered photon & recoil electron go off at the electron rest energy $mc^2 = 0.511$ MeV. The electron scattering ∡ is, from Eq. (6)...

$$\left[\!\left[\, \tan\phi = \frac{1}{1+r}\,\mathrm{ctn}\,\frac{\theta}{2}\Big|_{\substack{r=2\\ \theta=60°}} = \frac{1}{3}\times 1.7321 = 0.5774; \right.\right.$$

$$\text{so/}\ \underline{\underline{\phi = 30°}}. \quad (10)$$

This is a ~radical event: the photon is cut in half, and the electron is significantly agitated. However, everybody had a good time.