DEPARTMENT OF PHYSICS

M.S. COMPREHENSIVE/PH.D. QUALIFYING EXAMINATION

MONDAY, APRIL 1, 1991, 8 A.M.- 12 NOON

Answer each of the following questions, all of which carry equal weight. Begin your answer to each question on a new sheet of paper; solutions to different questions must not appear on the same sheet. Each sheet of paper must be labeled with your name and the problem number in the upper right hand corner. When more than one sheet is submitted for a problem, be sure the pages are ordered properly. [Problems 1-8]

CLASSICAL MECHANICS

1. A uniform cylinder and a uniform sphere, each of mass m and radius R are simultaneously released from rest from the top of an inclined plane of height h. They roll down the plane without slipping.

   a) Find the velocity of each object at the bottom of the plane.

   b) Which one reaches the bottom first?

#1

D. Lee
Mech.

Soln:

A) $\frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 = mgh$ for each object.

$v = R\omega$

$\therefore \frac{1}{2}mv^2 + \frac{1}{2}I\frac{v^2}{R^2} = mgh$

$V^2 = 2mgh \big/ \left(m + \frac{I}{R^2}\right)$

cylinder $I = \frac{1}{2}MR^2$ ←
sphere $I = \frac{2}{5}mR^2$ ←
] They may quote or derive

$V_c^2 = \dfrac{2mgh}{m + \frac{mR^2}{2R^2}} = \dfrac{4gh}{3} = 1.33gh$

$V_s^2 = \dfrac{2mg}{\left(m + \frac{2}{5}\frac{mR^2}{R^2}\right)} = \dfrac{10gh}{7} = 1.43gh$

B) Sphere wins since $V=0$ initially and $V_s > V_c$ at all points down plane.

CLASSICAL MECHANICS

2. A sounding rocket is fired vertically upward. Its rocket engine provides an upward acceleration of 2g for 100 seconds and then burns out.

   a) What is the maximum height reached by the rocket?

   b) What is the rocket's maximum speed? Where in its flight is the maximum speed achieved?

   c) What is the total time-of-flight (launch to impact)?

(Neglect air resistance.)

Divide flight into 3 parts:

(1) upward boost: Initial condition $x=0$ at $t=0$

$$a = 2g \cong 20\ m/sec^2$$

$$V(t) = 2gt$$

$$x(t) = gt^2$$

Final condition: $t = 100$ sec $V = 2000$ m/sec

$$x = 100{,}000\ m$$

(2) upward coast I.C.: $x = 100{,}000$ m $V = 2000$ m/sec $t = 100$ sec
$a = -g$

$$v(t) = 2000 - g(t-100)$$

$$x(t) = 100{,}000 + 2000(t-100) - \tfrac{1}{2}g(t-100)^2$$

Final condition $v(t_1) = 0$

$$2000 - 10(t_1 - 100) = 0 \qquad t_1 = 300\ sec$$

$$x(t_1) = 100{,}000 + 2000 \cdot 200 - 5 \cdot (200)^2$$

$$= 100{,}000 + 400{,}000 - 200{,}000 = 300{,}000\ m$$

(4) return to Earth:
I.C. ⇒ $t = 300$ sec $V = 0$ $x = 300{,}000$ m
$a = -g$

$$V_x = -g(t-300)$$

$$x(t) = 300{,}000 - \tfrac{1}{2}g(t-300)^2$$

Final condition $x = 0$ ⇒ $300{,}000 - 5(t_2 - 300)^2 = 0$

$(t_2 - 300\ sec)^2 = 60{,}000$ $\quad t_2 = 300\ sec + \sqrt{6} \cdot 100\ sec$

then $V_x = -10 \frac{m}{sec^2}(\sqrt{6} \cdot 100\ sec) = \sqrt{6} \cdot 1000\ m/sec \simeq 2445\ m/sec$

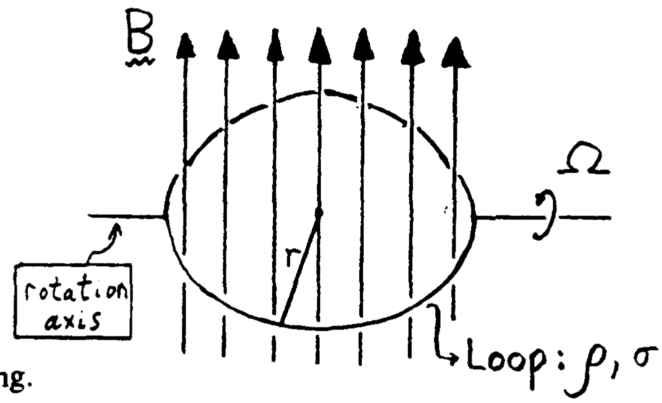
so:

(a) $\boxed{300\ km}$

(b) $\boxed{\simeq 2445\ m/sec;\ \text{at moment of impact}}$

(c) $\boxed{t \cong 545\ sec}$

ELECTROMAGNETISM



3. A circular metal loop of radius r, mass density ρ and conductivity σ is initially rotating about a horizontal axis at an angular velocity Ω as shown. At t=0, a uniform magnetic field B is turned on in the vertical direction, when the plane of the loop is horizontal. Assume the loop's rotational energy is dissipated by Joule heating.

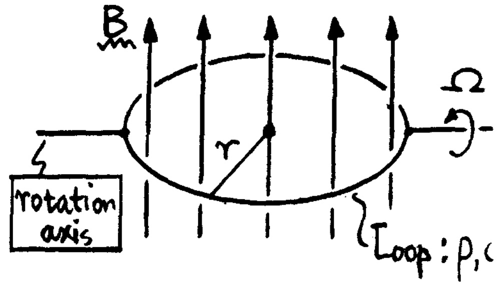
   a) Find the differential equation which governs the rate of decrease of the loop's rotational angular velocity $\omega(t)$.

   b) Assume B is weak enough so the loop rotates a number of times before stopping. Solve the equation in part a, approximately, for $\omega=\omega(t)$. NOTE: in effect, you should time-average the Joule loss.

   c) Find the (approximate) angle through which the loop rotates from t=0 until it stops. Show that this angle is independent of loop dimensions (i.e. independent of loop radius r and cross-sectional area A.

for/ MSU M.S. Qualifying Exam: Apr. 1991 (Robiscoe) #3 3/24/91

E&M: Rotating loop in a magnetic field.



3. "A circular metal loop of radius r, mass density ρ and conductivity σ is initially rotating about a horizontal axis at angular velocity Ω as shown. At time $t=0$, a uniform magnetic field B is turned on in the vertical direction, when the plane of the loop is horizontal. Assume the loop's rotational energy is dissipated by Joule heating.

A. Find the differential equation which governs the rate of decrease of the loop's rotational angular velocity ω.

B. Assume B is weak enough so the loop rotates a number of times before stopping. Solve the equation in part A, approximately, for $\omega = \omega(t)$. NOTE: in effect, you should time-average the Joule loss.

C. Find the (approximate) angle through which the loop rotates from $t=0$ until it stops. Show that this angle is independent of loop dimensions (i.e. independent of loop radius r and cross-sectional area A)."

A. Faraday's Law ⇒ emf around loop while rotating at ω is: $\mathcal{E} = -\frac{1}{c}\, d\phi/dt$, where magnetic flux: $\phi = \pi r^2 B \cos\omega t$. So the emf is

$\longrightarrow$ $\mathcal{E} = +\frac{1}{c}\pi r^2 B\omega \sin\omega t$, assuming $\omega \simeq$ cnst per rotation. (1)

If R is the resistance of the loop ($R = 2\pi r/\sigma A$), the Joule heating dissipates power $P = \mathcal{E}^2/R$. This is at the expense of the rotational kinetic energy $K = \frac{1}{2} I\omega^2$, where $I = \frac{1}{2} m r^2$ is the loop's moment-of-inertia about a diameter, and $m = 2\pi r A\rho$ is the loop mass.

(over)

ELECTROMAGNETISM

4. A particle of mass *m* and electric charge *q* is situated in an alternating electric field along the *x*- axis:

$$E_x = E_0\cos(\omega t) \qquad [E_0 \text{ is a constant}].$$

The particle also experiences an additional force proportional to the <u>third time derivative</u> of its x-position:

$$F_x = +\alpha(d^3x/dt^3) \qquad [\alpha \text{ constant}].$$

Find *x(t)* for the oscillation of the particle in the steady state. This model gives an approximate description of a charged particle which scatters radiation.

#4

Qual 90 Solution Hiscock

Math. Phys: electron w/ radiation reaction

Newton's 2nd:

$$m\ddot{x} = eE_0\cos(\omega t) + \alpha\dddot{x}$$

complexify $\cos(\omega t) \to e^{i\omega t}$

$$m\ddot{x} = eE_0 e^{i\omega t} + \alpha\dddot{x}$$

assume $x = x_0 e^{i\omega t}$

$\ddot{x} = -\omega^2 x_0 e^{i\omega t}$

$\dddot{x} = -i\omega^3 x_0 e^{i\omega t}$

$$\to \quad -\omega^2 m x_0 = eE_0 - i\alpha\omega^3 x_0$$

$$\to \quad x_0 = \left(\frac{eE_0}{\omega^2}\right)\frac{1}{i\alpha\omega - m} = -\left(\frac{eE_0}{\omega^2}\right)\frac{m + i\alpha\omega}{m^2+\alpha^2\omega^2}$$

$$m + i\alpha\omega = \sqrt{m^2+\alpha^2\omega^2}\,e^{i\phi} \quad \phi = \tan^{-1}\left(\frac{\alpha\omega}{m}\right)$$

so $$x_0 = -\left(\frac{eE_0}{\omega^2}\right)\frac{e^{i\phi}}{(m^2+\alpha^2\omega^2)^{1/2}}$$

then
$$x = -\frac{eE_0}{\omega^2}\frac{1}{(m^2+\alpha^2\omega^2)^{1/2}}e^{i(\omega t+\phi)}$$ Now take real part

$$\boxed{x = -\frac{eE_0}{\omega^2}\frac{1}{(m^2+\alpha^2\omega^2)^{1/2}}\cos(\omega t+\phi) \quad \text{where } \phi = \tan^{-1}\left(\frac{\alpha\omega}{m}\right)}$$

or $$x = -\frac{eE_0}{\omega^2}\frac{1}{(m^2+\alpha^2\omega^2)^{1/2}}\left\{\cos(\omega t)\cos(\phi) - \sin(\omega t)\sin(\phi)\right\}$$

$$\boxed{x = \left(\frac{eE_0}{\omega^2}\right)\frac{1}{m^2+\alpha^2\omega^2}\left\{-m\cos(\omega t) + \alpha\omega\sin(\omega t)\right\}}$$

5. The ground state wave function of the hydrogen atom electron is found to be

$$\psi_{100} = [\pi \alpha_0^3]^{-1/2}\, e^{-r/a_0}$$

where a_0 is the radius of the smallest Bohr orbit. What is the probability that the electron will be found <u>outside</u> of this Bohr radius?

Solution

#5

$$\int x^2 e^{ax}\,dx = \frac{x^2 e^{ax}}{a} - \frac{2}{a}\left(\frac{e^{ax}}{a^2}(ax-1)\right)$$

$$\int_{a_0}^{\infty} \cancel{x}^2 e^{-2r/a_0}\,d\cancel{x} = \left[\frac{\cancel{x}^2 e^{-2r/a_0}}{-\frac{2}{a_0}} - \frac{2}{-\frac{2}{a_0}}\left(\frac{e^{-2r/a_0}}{\left(-\frac{2}{a_0}\right)^2}\left(-\frac{2}{a_0}x-1\right)\right)\right]_{a_0}^{\infty}$$

$$= +\frac{a_0^3}{2}e^{-2} + \frac{a_0^3}{4}e^{-2}(+2+1)$$

$$= e^{-2}a_0^3\left(\frac{1}{2}+\frac{3}{4}\right) = a_0^3\,\frac{5}{4}\,e^{-2}$$

$$\psi(r) = \frac{1}{\sqrt{\pi a_0^3}}\, e^{-r/a_0}$$

Prob. density $= |\psi(r)|^2$

Prob. electron <u>outside</u> a_0: $\int_{a_0}^{\infty} |\psi(r)|^2\, r^2 dr\, d\Omega$

$$= \frac{4\pi}{\pi a_0^3}\int_{a_0}^{\infty} e^{-2r/a_0}\, r^2 dr = \frac{4}{a_0^3}\, a_0^3\, \frac{5}{4}\, e^{-2}$$

$$= 5e^{-2} = 0.677 \Rightarrow 68\%$$

CLASSICAL MECHANICS

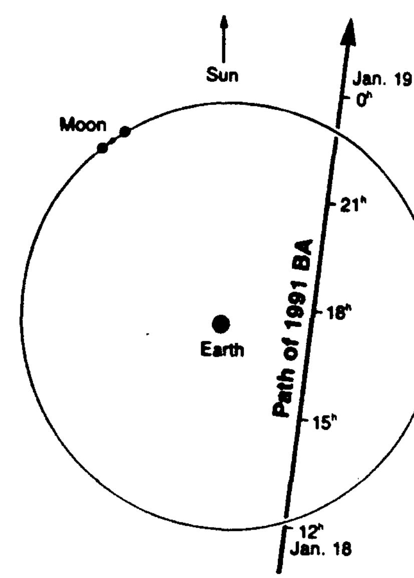
6. On January 18, 1991, the smallest asteroid ever detected (estimated diameter 9 meters) passed closer to the Earth than any other asteroid ever detected (closest approach 170,000 km - less than half the distance to the Moon), at a speed of roughly 16 km/sec (the kinetic energy is equivalent to about 1 Megaton of TNT). Consider the system consisting of an asteroid passing (or possibly impacting) the Earth on a hyperbolic orbit. You may treat the Earth as a fixed center of attraction in this problem Work in the plane defined by the asteroid's orbit and the Earth; use plane polar coordinates.

   a) What is the Lagrangian for the motion of the asteroid?

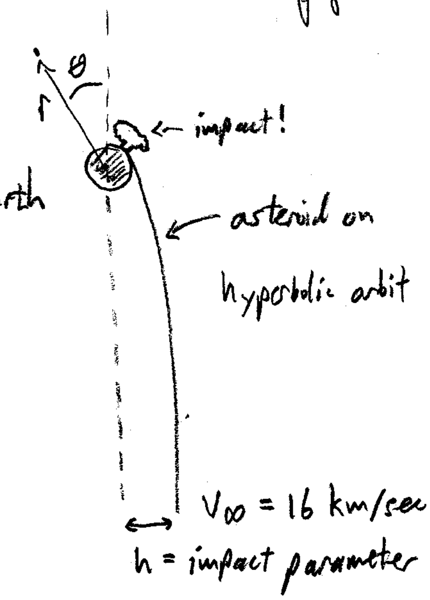
   b) Write out the Euler-Lagrange equations of motion for this system and find two first integrals of the equations of motion.

   c) Let the velocity of the asteroid at infinity be 16 km/sec. What is the total cross-section for the asteroid to impact the Earth? (i.e., what is the maximum impact parameter which yields a minimum radius equal to the Earth's radius?). Is this substantially larger than the physical size of the Earth? Comment.

Some useful numbers: $G = 6.67 \times 10^{-11}$ N m^2 kg^{-2}; $M_{Earth} = 6 \times 10^{24}$ kg;
$R_{Earth} = 6.4 \times 10^{6}$ m.



Asteroid Impact



(a)

$$T = \tfrac{1}{2} m \dot{r}^2 + \tfrac{1}{2} m r^2 \dot{\theta}^2$$

$$V = -\frac{GMm}{r}$$

$$\boxed{L = \tfrac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2) + \frac{GMm}{r}}$$

(b) E-L eqns:

$$\frac{\partial L}{\partial \dot{r}} = m\dot{r} \qquad \frac{\partial L}{\partial r} = m r \dot{\theta}^2 - \frac{GMm}{r^2}$$

$$\Rightarrow \boxed{m\ddot{r} - m r \dot{\theta}^2 + \frac{GMm}{r^2} = 0} \quad (1)$$

$$\frac{\partial L}{\partial \theta} = 0 \qquad \frac{\partial L}{\partial \dot{\theta}} = m r^2 \dot{\theta} \Rightarrow \boxed{(m r^2 \dot{\theta})^{\cdot} = 0} \quad (2)$$

first integrals: integrate (2) immediately:

$$\boxed{m r^2 \dot{\theta} = L} \quad (3) \text{ (angular momentum)}$$

Substitute this ↗ into (1), eliminating $\dot{\theta}$:

$$\dot{\theta} = L / m r^2$$

$$(1) \Rightarrow \quad m\ddot{r} - \frac{L^2}{m r^3} + \frac{GMm}{r^2} = 0 \qquad \text{multiply by } \dot{r} \text{ \& integrate:}$$

$$\boxed{\tfrac{1}{2} m \dot{r}^2 + \frac{L^2}{2 m r^2} - \frac{GMm}{r} = E} \quad (4) \text{ energy integral}$$

(c) Cross-section: $\sigma = \pi h^2$ where h = impact parameter

$L = m V_\infty h$; (4) may then be written as: $\{$also, $E = \tfrac{1}{2} m V_\infty^2\}$

$$\tfrac{1}{2} m \dot{r}^2 + \tfrac{1}{2} m V_\infty^2 \left(\frac{h}{r}\right)^2 - \frac{GMm}{r} = \tfrac{1}{2} m V_\infty^2 \quad (5)$$

now $\div$ (5) by m, set $r = R_{Earth}$ & $\dot{r} = 0$ (grazing impact) solve for h:

$$V_\infty^2 \frac{h^2}{R_E^2} - \frac{2GM}{R_E} = V_\infty^2$$

$$h^2 = \frac{R_E^2}{V_\infty^2}\left[V_\infty^2 + \frac{2GM}{R_E}\right]$$

$$\boxed{h = R_E\left[1 + \frac{2GM}{R_E V_\infty^2}\right]^{1/2}} \qquad \boxed{\sigma = \pi h^2 = \pi R_E^2\left[1 + \frac{2GM}{R_E V_\infty^2}\right]}$$

Now for numbers:

$$\frac{2GM}{R_E V_\infty^2} = \frac{2(6.67\times10^{-11})(6\times10^{24})}{(6.4\times10^6)(1.6\times10^4)^2} \simeq .489$$

so

$$\boxed{\begin{array}{ll} h \simeq 1.22\,R_E = 7.81\times10^6\,\mathrm{m} & \\ \sigma \simeq 1.489\,(\pi R_E^2) & h - R_E \simeq 1410\,\mathrm{km} \end{array}}$$

almost a 50% increase in cross section - significant, but note that for larger values of V_∞, h will be closer to R_E.

THERMODYNAMICS

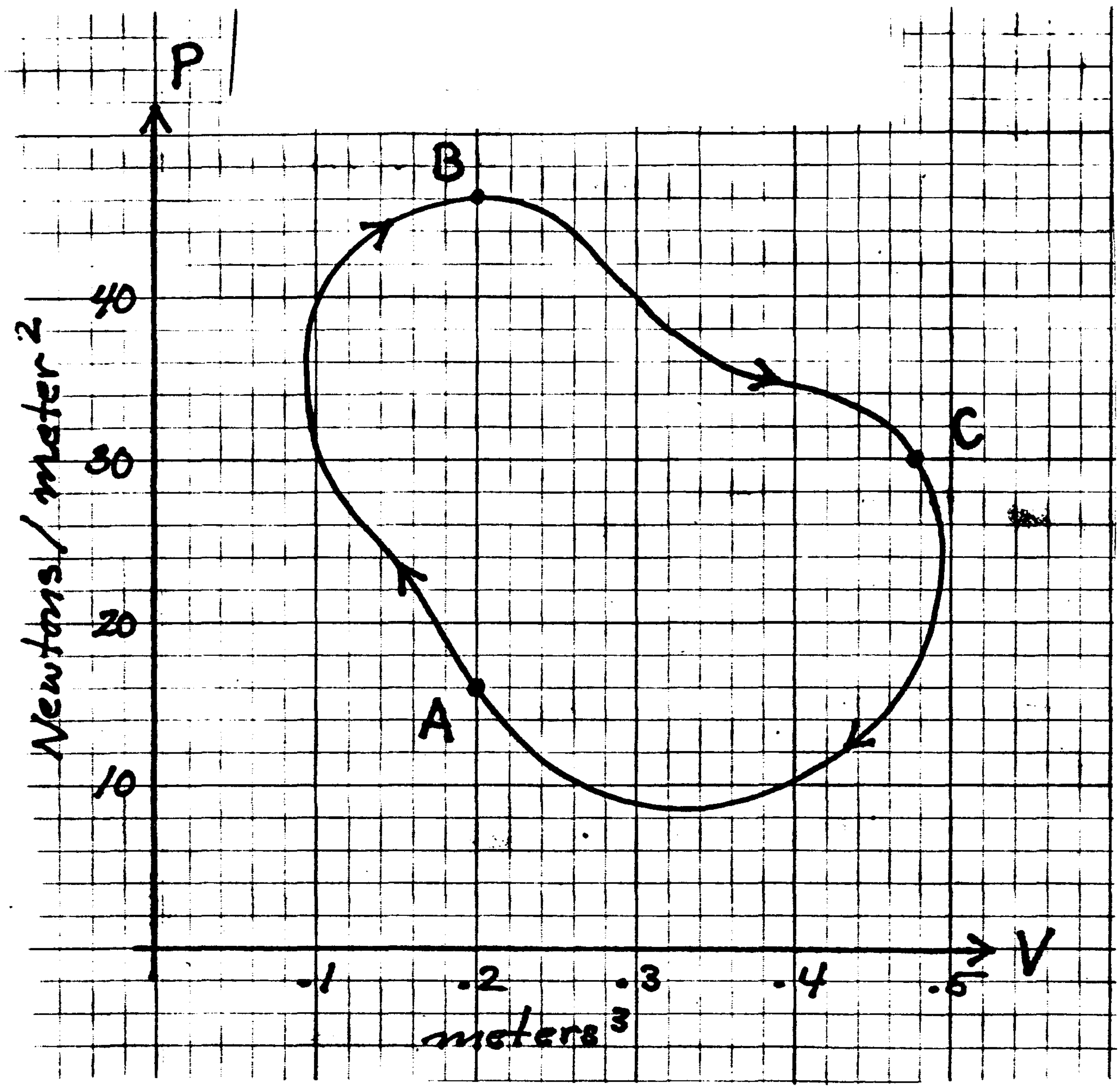
7. A small engine cycle is approximated by the cyclic process A→B→C→A as shown below. The heat input into the engine occurs during A→B and is roughly 16.0 Joules.

a) About how much work is done by the engine during a single cycle?

b) What is the efficiency of the engine?

c) Compare your answer in part b) with the maximum thermodynamic efficiency of a heat engine operating between the maximum temperature T_C and minimum T_A. For this, you may assume the system is composed of an ideal gas.

d) If the engine operates at 20 Hz, what is the approximate power output?



Answers: Counting squares, I find 216 little squares inscribed by the cycle and 269 little squares will in turn inscribe the cycle.

(a) $\frac{1}{2}(216+269) = 242.5$ squares in the cycle. Each has $\frac{1}{25}$ Joules, so 9.7 Joules of work are done per cycle.

(b) The efficiency is $\frac{9.7}{16.0} = .606$ or about 61 %.

(c) The maximum thermodynamic efficiency is $1-\frac{T_A}{T_C}$

$$= 1 - \frac{P_A V_A}{P_C V_C} = 1 - \frac{3.2}{14.4} = 1 - .222$$

or .777 or about 78%.

That means the cycle operates at about 78% of maximum thermal efficiency, since $\frac{.61}{.78} \approx .78$.

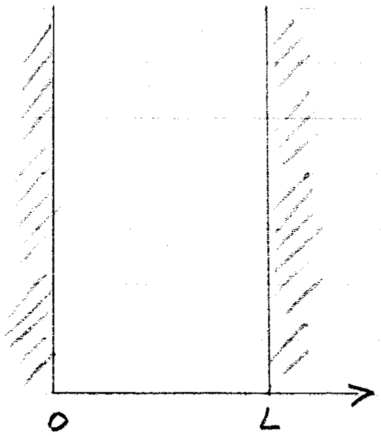
(d) Running at 20 Hz, the power output is just (20 Hz)(9.7 Joules) = 194 Watts.

(e) Quasi static processes consist of a succession of equilibrium states in which the system homogeneous and p and T are well defined. Non-quasistatic processes cannot be represented as curves in pV.

QUANTUM MECHANICS

8. A spinless particle of mass m is confined between two parallel, impenetrable walls separated by a distance L. The particle has its lowest possible energy.

One of the two plane walls is suddenly moved away from the other one, increasing the separation to 2L. What is the probability for the particle to end up in a state with the same energy as it originally had?



$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x) = E\,\psi(x)$$

$$\therefore \left(\frac{d^2}{dx^2} + k^2\right)\psi(x) = 0 \,,$$

where $k^2 = \dfrac{2mE}{\hbar^2}$

Boundary conditions:

$$\psi(x=0) = 0 = \psi(x=L)$$

$$\psi_n(x) = A \sin k_n x \,, \qquad k_n = \frac{n\pi}{L} \,, \text{ with } n = 1, 2, 3, \ldots$$

Normalization: $\langle \psi_n | \psi_n \rangle = 1 \Rightarrow A = \sqrt{\frac{2}{L}}$

$\therefore$

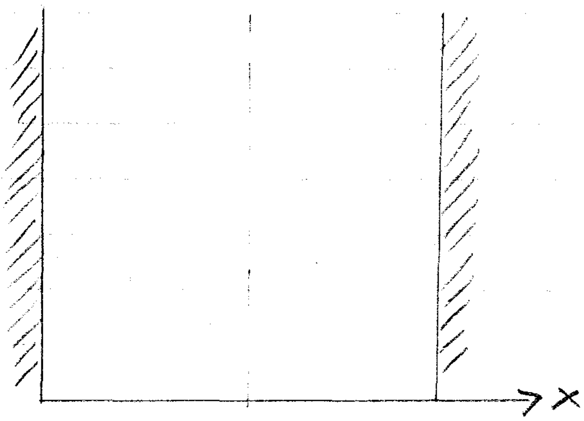
$$\psi_n(x) = \sqrt{\frac{2}{L}} \sin \frac{n\pi}{L} x \qquad n = 1, 2, 3, \ldots$$

$$E_n = \frac{\hbar^2}{2m}\left(\frac{n\pi}{L}\right)^2$$

Ground state: $n = 1$:

$$\psi_1(x) = \sqrt{\frac{2}{L}} \sin \frac{\pi}{L} x$$

$$E_1 = \frac{\hbar^2 \pi^2}{2mL^2}$$



The normalized energy eigen functions for the new well are

$$\phi_n(x) = \sqrt{\frac{2}{2L}} \sin \frac{n\pi}{2L} x$$

$n = 1, 2, 3, ..$

and the associated energy eigenvalues are

$$E'_n = \frac{\hbar^2}{2m}\left(\frac{n\pi}{2L}\right)^2$$

Clearly, $E'_{n=2} = E_1$

Now:

$$|\psi_1\rangle = \sum_n |\phi_n\rangle\langle\phi_n|\psi_1\rangle ,$$

and the probability for the particle to end up in a state with the same energy is

$$\mathcal{P}(E'_{n=2}) = |\langle\phi_{n=2}|\psi_1\rangle|^2$$

Now:

Note that $\psi_1(x)$ vanishes for $x > L$.

$$\langle\phi_{n=2}|\psi_1\rangle = \int dx\, \phi^*_{n=2}(x)\, \psi_1(x) =$$

$$= \sqrt{\frac{1}{L}}\sqrt{\frac{2}{L}} \int_0^L dx \sin\frac{\pi}{L}x \sin\frac{\pi}{L}x$$

$\therefore$

$$\langle \phi_{n=2} | \psi_1 \rangle = \frac{\sqrt{2}}{L} \times \frac{L}{2} = \frac{1}{\sqrt{2}}$$

$$\therefore \quad \mathcal{P}(E'_{n=2}) = \frac{1}{2}$$

DEPARTMENT OF PHYSICS

M.S. COMPREHENSIVE/PH.D. QUALIFYING EXAMINATION

MONDAY, APRIL 1, 1991, 1 P.M.- 5 P.M.

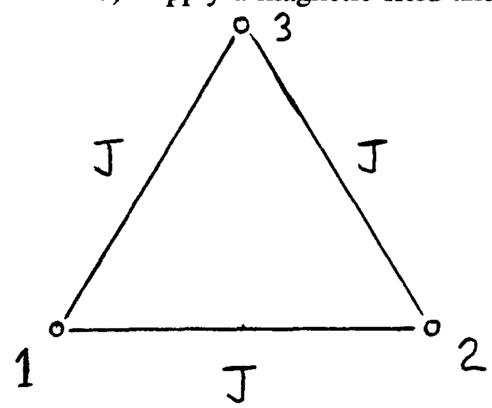
Answer each of the following questions, all of which carry equal weight. Begin your answer to each question on a new sheet of paper; solutions to different questions must not appear on the same sheet. Each sheet of paper must be labeled with your name and the problem number in the upper right hand corner. When more than one sheet is submitted for a problem, be sure the pages are ordered properly. [Problems 9-16]

QUANTUM MECHANICS

9. Consider a 3 atom solid, each atom of which has spin 1/2 coupled to each of the other atoms ferromagnetically with Heisenberg exchange energy J between each pair.

a) Write the Hamiltonian and find the ground state energy.

b) Apply a magnetic field and show the splitting of the ground state.



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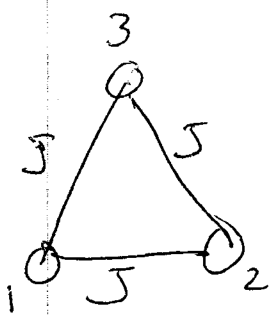


a/ $\boxed{\mathcal{H} = -J\vec{S}_1\cdot\vec{S}_2 - J\vec{S}_2\cdot\vec{S}_3 - J\vec{S}_3\cdot\vec{S}_1}$

Simplest method:

$$(\vec{S}_1+\vec{S}_2+\vec{S}_3)^2 = S_1^2+S_2^2+S_3^2 + 2\,\vec{S}_1\cdot\vec{S}_2 + 2\vec{S}_2\cdot\vec{S}_3 + 2\vec{S}_3\cdot\vec{S}_1$$

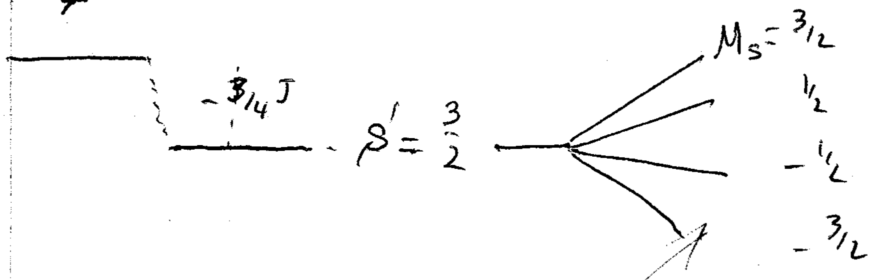
$S_1+S_2+S_3$
$S_1+S_2+S_3$
$S_1^2+S_1\cdot S_2+S_1\cdot S_3$
$S_1\cdot S_2 + \quad S_2\cdot S_3 + S_2^2$
$S_1\cdot S_3 \quad S_1\cdot 3 \quad + S_1^3$

$$\therefore \vec{S}_1\cdot\vec{S}_2+\vec{S}_2\cdot\vec{S}_3+\vec{S}_3\cdot\vec{S}_1 = \tfrac{1}{2}\left[(\vec{S}_1+\vec{S}_2+\vec{S}_3)^2-(S_1^2+S_2^2+S_3^2)\right]$$
$$= \tfrac{1}{2}\left[S'^2-(S_1^2+S_2^2+S_3^2)\right]$$

For ferromagn: $S' = \frac{3}{2}$ $\quad S_1=S_2=S_3=1/2$

$$\left[E_{3/2} = -\frac{J}{2}\left[S'(S'+1) - 3S_1(S_1+1)\right]\right. = -\frac{J}{2}\left[\frac{3}{2}\frac{5}{2}-3\frac{1}{2}\frac{3}{2}\right]$$

a./ $$= -\frac{J}{2}\left(\frac{15}{4}-\frac{9}{4}\right) = \frac{1}{2}\frac{3}{2} = \boxed{-\frac{3}{4}J} \qquad E = -\frac{3}{4}J$$



b./ $\mathcal{H}_z = g\mu_B H S_z$

$\boxed{E_{M_s} = g\mu_B H M_s}$

MATHEMATICAL PHYSICS

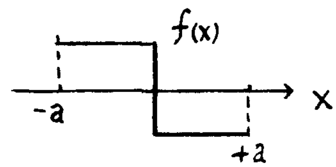
10. Consider the ordinary differential equation:

$$d^2y/dx^2 + \kappa f(x)y = 0, \quad \text{on } -a \le x \le +a;$$

κ = constant, and : $y(-a) = y(+a) = 0$.

a) Find the condition which determines the eigenvalue spectrum for κ if:

$$f(x) = \begin{cases} +1, \text{ for } -a \le x < 0; \\ -1, \text{ for } 0 < x \le a. \end{cases}$$



b) Sketch how the eigenvalues in part (a) might be found graphically. What are the approximate values of the κ's when $|\kappa| \to$ large?

for/ MSU M.S. Qualifying Exam: Apr. 1991 $\#10$ 3/24/91

Math: SHO equation with a twist.

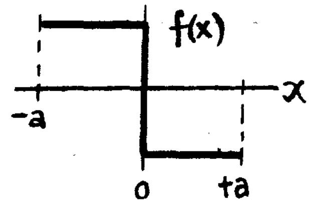
10. Consider the ordinary differential equation:

$$d^2y/dx^2 + \kappa f(x)\, y = 0, \quad \text{on} \quad -a \leq x \leq +a;$$

$$\kappa = \text{constant}, \quad \text{and}: \quad y(-a) = y(+a) = 0.$$

A. Find the condition which determines the eigenvalue spectrum for κ if:

$$f(x) = \begin{cases} +1, & \text{for } -a \leq x < 0; \\ -1, & \text{for } 0 < x \leq a. \end{cases}$$



B. Sketch how the eigenvalues in part A might be found graphically. What are the approximate values of the κ's when $|\kappa| \to$ large?

A. Let $\kappa = k^2$ for convenience. Solutions to $y'' \pm k^2 y = 0$ are $\sin kx$ & $\cos kx$ for $x<0$, and $\sinh kx$ & $\cosh kx$ for $x>0$. With the given boundary conditions $y(-a)=0$ & $y(+a)=0$, it is obvious that the required y's are

$$\begin{cases} y(x) = A \sin k(a+x), & -a \leq x < 0; \\ y(x) = B \sinh k(a-x), & 0 < x \leq a; \end{cases} \qquad (1)$$

where A & B are constants. $y(x)$ should be "smooth" at $x=0$, i.e. both $y(x)$ & $y'(x)$ should be continuous. Thus...

$$\begin{cases} y(0) \text{ cont}^s \Rightarrow A \sin ka = B \sinh ka, \\ y'(0) \text{ cont}^s \Rightarrow Ak \cos ka = -Bk \cosh ka. \end{cases} \qquad (2)$$

(over)

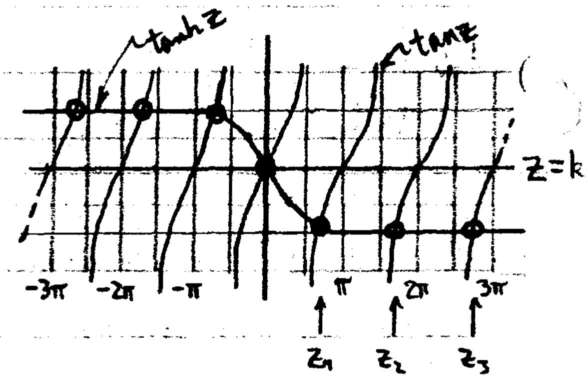
Dividing the first of Eqs. (2) by the second, we get...

$$\boxed{\tan ka = -\tanh ka} \qquad (3)$$

This equation has a set of discrete solutions k_n (only), and it generates the eigenvalues $K_n = k_n^2$ for the quoted problem.

NOTE If k is a <u>real</u> solution to Eq. (3), then $p = ik$ is also a solution since: $\tan pa = -\tanh pa$, just regenerates Eq. (3) [because $\tan(ix) = i\tanh x$, and $\tanh(ix) = i\tan x$]. Thus, real solutions k_n to Eq. (3) will generate both $K_n = +k_n^2$ and $K_n = (-)k_n^2$ classes of eigenvalues.

B. Let $z = ka$. A graph of the functions in Eq. (3) is sketched at right. The circled intersection points are solutions to Eq. (3), and they give eigenvalues via



$$z_n = a\sqrt{K_n} \Rightarrow K_n = (z_n/a)^2. \qquad (4)$$

The eigenvalue at $z_0 = 0$ can be discarded as trivial -- $K_0 = 0$ in the differential equation requires $y(x) \equiv 0$ on the interval.

Evidently, after $z > \pi$ or so, Eq. (3) goes over to...

$$\tan z \simeq -1 \Rightarrow z \simeq (n - \tfrac{1}{4})\pi, \quad n = 1, 2, 3, \ldots$$

so//
$$\boxed{K_n = \pm\left[\left(n - \frac{1}{4}\right)\frac{\pi}{a}\right]^2}, \quad n = 1, 2, 3, \ldots \text{ for "large" } K_n. \qquad (5)$$

In a sense, the (+)ve part of $f(x)$ generates the (+) K_n-values, while the (-)ve part of $f(x)$ gives the (-) K_n-values.

SPECIAL RELATIVITY

11. A photon (γ), when scattered off an electron (e^-), produces an electron- positron pair:

$$\gamma + e^- \rightarrow 2e^- + e^+ \quad .$$

Find the minimum photon energy, in the rest frame of the electron, for this reaction to occur.

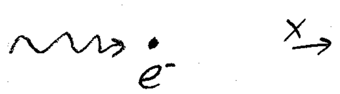
A photon (γ) scattered by an electron (e^-), may if energetic enough, produce an electron-positron pair as follows:

$$\gamma + e^- \to 2e^- + e^+ .$$

Find the minimum photon energy, in the rest frame of the initial electron, for this reaction to occur.

Solution: Use conservation of 4-momentum; $c = 1$

Before:



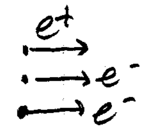
$p_\gamma^\mu = (E_\gamma, p_\gamma, 0, 0)$

$p_e^\mu = (m_e, 0, 0, 0)$

photon massless $\Rightarrow$

$E_\gamma = p_\gamma \quad (0)$

After:



At threshold, the 3 particles will move off together (this is obvious if one thinks of the process in the CM frame).

$p_3^\mu = (E_3, p_3, 0, 0)$

Normalization:

$m_3^2 = (3m_e)^2 = E_3^2 - p_3^2 \qquad (1)$

Conserve 4-momentum:

energy: (before = after)

$$E_\gamma + m_e = E_3 \qquad (2)$$

momentum

$$p_\gamma = p_3 \qquad (3)$$

Square (2): $E_\gamma^2 + 2E_\gamma m_e + m_e^2 = E_3^2$ ← replace using (1)

$E_\gamma^2 + 2E_\gamma m_e + m_e^2 = 9m_e^2 + p_3^2$ ← replace using (3) & (0)

$\cancel{E_\gamma^2} + 2E_\gamma m_e + \cancel{m_e^2} = \cancel{9}\,8\,m_e^2 + \cancel{E_\gamma^2}$

$\Rightarrow$

$$2E_\gamma m_e = 8m_e^2 \to \boxed{E_\gamma = 4m_e} \boxed{\simeq 2.044\ \text{MeV}} \quad \checkmark$$

ELECTROMAGNETISM

12. Calculate the magnetic field on the axis of an ideal, but short solenoid, of N turns and length ℓ.

E & M

#12

12. Calculate the magnetic field on the axis of an ideal, but short solenoid

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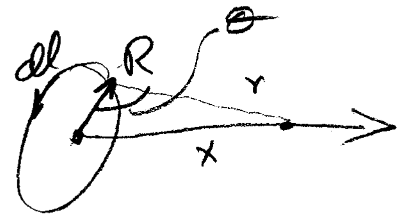
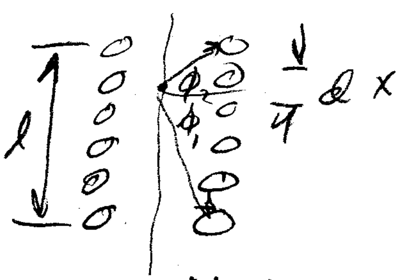
solution 1st do a circular loop



$$dB = \frac{\mu_0 I}{4\pi} \frac{dl \times \vec{r}}{r^2} = \frac{\mu_0 I}{4\pi} \frac{dl}{x^2+R^2}$$

$$\therefore B_x = \frac{\mu_0 I}{4\pi} \frac{\cos\theta}{x^2+R^2} \int dl = \frac{\mu_0 I R^2}{2(x^2+R^2)^{3/2}}$$

then add them for a solenoid



N turns

$$dB = \frac{\mu_0 R^2}{2(x^2+R^2)^{3/2}} I \left(\frac{N}{l}\right) dx$$

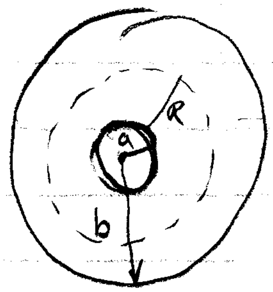
a way to integrate is let $x = R\tan\phi$

$$\rightarrow B = \frac{\mu_0 N I}{2l}(\sin\phi_2 - \sin\phi_1)$$

13. A coaxial cable consists of two thin coaxial conducting cylinders. The radius of the inner cylinder is a and the radius of the outer cylinder is b. Find the capacitance per unit length of the cable.

#13 DLee E&M

Soln:



assume equal but opposite charge distributions of λ coul/m on each cylinder.

Apply Gauss' law to find E in between $\epsilon_0 \int E \cdot dA = q_{net}$

$$\epsilon_0 2\pi R E L = \lambda L \Rightarrow E = \frac{\lambda}{2\pi\epsilon_0 R} \qquad a \leq R \leq b$$

$$V_{ab} = -\int \vec{E} \cdot d\vec{l} = -\frac{\lambda}{2\pi\epsilon_0} \int_a^b \frac{dR}{R} = -\frac{\lambda}{2\pi\epsilon_0} \ln R \Big|_a^b = -\frac{\lambda}{2\pi\epsilon_0} \ln\frac{b}{a}$$

$$C = \frac{Q}{V} = \frac{\cancel{\lambda L}}{\frac{\cancel{\lambda L}}{2\pi\epsilon_0} \ln\frac{b}{a}} = \frac{2\pi\epsilon_0}{\ln b/a}$$

SOLID STATE

14. The thermodynamics of magnetic systems may be studied by replacing V (volume) by $-M$ (negative of the magnetization) and P (pressure) by H (the applied magnetic field) in the usual thermodynamic functions.

a) Use the first law of thermodynamics and derive a simple expression for the heat capacity of a magnetic system at constant magnetic field in terms of M, H, and T.

b) Assume that the system behaves according to the Curie law with constant internal energy ($dU=0$) and calculate the specific heat at constant H.

#14

First Law:
$$\boxed{dU = dQ - PdV}$$

$$dQ = dU + PdV \qquad V \Rightarrow -M \qquad P \Rightarrow H$$

$$dQ = dU - HdM \qquad dU = 0$$

Definition of Heat Cap:
$$\boxed{C_H = \left(\frac{dQ}{dT}\right)_H} = -H\left(\frac{\partial M}{\partial T}\right)_H$$

Curie Law:
$$\boxed{M = \frac{C}{T}H} \qquad C = \text{Curie Constant}$$

$$\left(\frac{\partial M}{\partial T}\right)_H = -\frac{CH}{T^2}$$

$$C_H = \frac{CH^2}{T^2}$$

MATHEMATICAL PHYSICS

15. a) Prove that the trace of an operator is independent of the representation used to evaluate it.

b) Prove the following equation:

$$\mathrm{Det}[e^{i\alpha A}] = e^{i\lambda\alpha \mathrm{Tr}[A]}$$

for an arbitrary operator A. You must identify the value of λ as part of your proof.

a) In a certain basis spanned by the vectors $\{|u_i\rangle\}$ we have that

$$\mathrm{Tr}\,\hat{A} = \sum_i \langle u_i|\hat{A}|u_i\rangle = \sum_i A_{ii}$$

Consider a new basis $\{|v_j\rangle\}$. We evaluate:

$$\sum_i \langle v_i|\hat{A}|v_i\rangle =$$

$$= \sum_i \sum_{jk} \langle v_i|u_j\rangle\langle u_j|\hat{A}|u_k\rangle\langle u_k|v_i\rangle$$

$$= \sum_{jk} A_{jk} \sum_i \langle u_k|v_i\rangle\langle v_i|u_j\rangle$$

$$= \sum_{jk} A_{jk} \underbrace{\langle u_k|u_j\rangle}_{=\delta_{jk}}$$

$$= \sum_j A_{jj}$$

$$\therefore \sum_i \langle v_i|\hat{A}|v_i\rangle = \sum_i \langle u_i|\hat{A}|u_i\rangle$$

Q.E.D.

b) The matrix which represents the operator $e^{i\alpha\hat{A}}$ in the basis of the eigenvectors of $\hat{A}$ is of the form

$$e^{i\alpha A} = \begin{pmatrix} e^{i\alpha a_1} & 0 & 0 & \cdots & 0 \\ 0 & e^{i\alpha a_2} & 0 & \cdots & 0 \\ \vdots & & & & \\ 0 & 0 & \cdots & & e^{i\alpha a_n} \end{pmatrix}$$

where we have denoted by $\{a_i\}$ $i=1,2,\ldots n$, the eigenvalues of $\hat{A}$. The above equation follows immediately once we note that

$$e^{i\alpha\hat{A}} \,|a_i\rangle = e^{i\alpha a_i} \,|a_i\rangle ,$$

where $|a_i\rangle$ denotes the eigenvector that corresponds to the eigenvalue a_i.

We then have that

$$\begin{aligned} \mathrm{Det}\; e^{i\alpha\hat{A}} &= \prod_{i=1}^{n} e^{i\alpha a_i} \\ &= e^{i\alpha \sum_{i=1}^{n} a_i} \\ &= e^{i\alpha \,\mathrm{Tr}\,\hat{A}} . \end{aligned}$$

Note that the value of the determinant does not depend on the representation used to evaluate it — So $\lambda = 1$.

QUANTUM MECHANICS

16. Suppose that at $t = 0$ a spin 1/2 particle is found to be in the "spin up" state along the y - axis of an arbitrarily chosen coordinate system. The particle is in the presence of a static, uniform, magnetic field of strength B, oriented along the x - direction.

a) At a later time T we measure that z - component of the spin. What is the probability for finding the spin to be "up"? Interpret your result physically.

b) Assume that the previous measurement is not performed. Instead, at $t = T$ we suddenly (i.e. instantaneously) reverse the magnetic field. Is the state at $t = 2T$ the same as that for $t = 0$ (spin up along the y - axis) or is it "spin down" (along the y-axis)?

Recall:

$$|\pm\rangle_x = \frac{1}{\sqrt{2}} \left(|+\rangle_z \pm |-\rangle_z \right)$$

$$|\pm\rangle_y = \frac{1}{\sqrt{2}} \left(|+\rangle_z \pm i |-\rangle_z \right)$$

#16
pg 1 of 2

a) $\hat{H} = -\gamma \hat{\underset{\sim}{S}} \cdot \underset{\sim}{B} = \omega_0 \hat{S}_x$,
where
$$\omega_0 = -\gamma B \; .$$

We require the amplitude ${}_z\langle +| \, e^{-\frac{i}{\hbar}\omega_0 \hat{S}_x T} \, |+\rangle_y$.
Now:
$$|+\rangle_z = \frac{1}{\sqrt{2}} \left(|+\rangle_x + |-\rangle_x \right)$$

$\therefore$

$$ {}_z\langle +| \, e^{-\frac{i}{\hbar}\omega_0 \hat{S}_x T} \, |+\rangle_y = \frac{1}{\sqrt{2}} \, {}_x\langle +| \, e^{-\frac{i}{\hbar}\omega_0 \hat{S}_x T} \, |+\rangle_y + \frac{1}{\sqrt{2}} \, {}_x\langle -| \, e^{-\frac{i}{\hbar}\omega_0 \hat{S}_x T} \, |+\rangle_y $$

$$ = \frac{1}{\sqrt{2}} \left[e^{-\frac{i}{2}\omega_0 T} \, {}_x\langle +|+\rangle_y + e^{\frac{i}{2}\omega_0 T} \, {}_x\langle -|+\rangle_y \right] $$

But:

$$ {}_x\langle +|+\rangle_y = \frac{1}{2}(1+i) \quad ; \quad {}_x\langle -|+\rangle_y = \frac{1}{2}(1-i) $$

$\therefore$

$$ {}_z\langle +| \, e^{-\frac{i}{2}\omega_0 \hat{S}_x T} \, |+\rangle_y = \frac{1}{2\sqrt{2}} \left[e^{-\frac{i}{2}\omega_0 T}(1+i) + e^{\frac{i}{2}\omega_0 T}(1-i) \right] $$

$$ = \frac{1}{2\sqrt{2}} \left[\left(e^{\frac{i}{2}\omega_0 T} + e^{-\frac{i}{2}\omega_0 T} \right) - i \left(e^{\frac{i}{2}\omega_0 T} - e^{-\frac{i}{2}\omega_0 T} \right) \right] $$

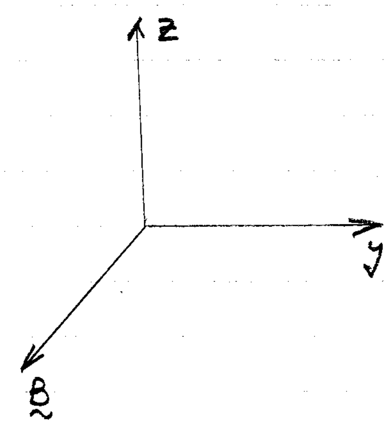
$\therefore$

$$ {}_z\langle +| \, e^{-\frac{i}{2}\omega_0 \hat{S}_x T} \, |+\rangle_y = \frac{1}{\sqrt{2}} \left(\cos \frac{\omega_0}{2} T + \sin \frac{\omega_0}{2} T \right) $$

Thus, the probability to find the spin "up" along the z-axis is

$$\mathcal{P} = \frac{1}{2}\left(1 + 2\sin\frac{\omega_0}{2}T\cos\frac{\omega_0}{2}T\right)$$

$$= \frac{1}{2}\left(1 + \sin\omega_0 T\right)$$



For $\omega_0 T = 0$: $\mathcal{P} = 1/2$
" $\omega_0 T = \pi/2$: $\mathcal{P} = 1$
" $\omega_0 T = \pi$: $\mathcal{P} = 1/2$
" $\omega_0 T = 3\pi/2$: $\mathcal{P} = 0$

Clearly, our solution corresponds to the precession of the spin state in the plane normal to the magnetic field.

b) $$|\psi(T)\rangle = e^{-\frac{i}{\hbar}\omega_0 \hat{S}_x T}|+\rangle_y$$

$$|\psi(2T)\rangle = e^{+\frac{i}{\hbar}\omega_0 \hat{S}_x T}|\psi(T)\rangle$$

$$= e^{\frac{i}{\hbar}\omega_0 \hat{S}_x T}e^{-\frac{i}{\hbar}\omega_0 \hat{S}_x T}|+\rangle_y$$

$$= |+\rangle_y \quad !$$

Then, conservation of energy demands: $\frac{dK}{dt} = -P$, or...

$$\frac{d}{dt}\left[\frac{1}{2}\left(\frac{1}{2}mr^2\right)\omega^2\right] = -\mathcal{E}^2/R \qquad \text{later...} \quad \tau = \frac{4\rho c^2}{\sigma B^2} \qquad \text{... (2)}$$

or// $$\frac{1}{2}mr^2\omega\frac{d\omega}{dt} = -\frac{1}{R}(\pi r^2 B/c)^2\,\omega^2 \sin^2\omega t$$

or// $$\boxed{\frac{d\omega}{dt} = -\frac{\omega}{\tau}\left[2\sin^2\omega t\right]}, \quad \text{w/ } \tau = mRc^2/(\pi r B)^2. \qquad (3)$$

B. If the loop rotates several times before stopping, a time-average of Eq. (3) yields $\langle 2\sin^2\omega t\rangle = 1$ on the RHS, so that approximately

$$\frac{d\omega}{dt} \simeq -\frac{\omega}{\tau} \Rightarrow \boxed{\omega(t) \simeq \Omega e^{-t/\tau}}, \quad \tau \text{ given in Eq. (3).} \qquad (4)$$

Same effect produced by time-averaging the Joule loss $\frac{\mathcal{E}^2}{R}$ in Eq. (2).

C. Turning ∡ between $t=0$ & loop stop is

$$\theta = \int_0^\infty \omega(t)\,dt = \Omega\tau. \qquad (5)$$

Now τ involves $\left\{\begin{matrix} m = 2\pi r A\rho \\ R = 2\pi r/\sigma A \end{matrix}\right\} mR = 4\pi^2 r^2\rho/\sigma$, i.e.

$$\left.\begin{matrix} \tau = mRc^2/(\pi r B)^2 = 4\rho c^2/\sigma B^2 \\ \text{So}// \ \boxed{\theta = \Omega\tau = 4\Omega\rho c^2/\sigma B^2} \end{matrix}\right\} \text{indpt of loop dimensions} \qquad (6)$$