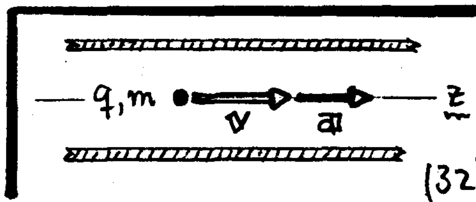
Radiation from relativistic q's in accelerators.

9) Now use Liénard's formula, Eq.(31), for several <u>applications</u> to the radiation from a relativistic accelerated single charge q.

LINEAR ACCELERATOR : $\mathbf{v} \parallel \mathbf{a} = \frac{d\mathbf{v}}{dt}$.



$\rightarrow$ So, $|\boldsymbol{\beta}\times\dot{\boldsymbol{\beta}}| \equiv 0$, in (31), and : $P_{\text{radn}} = \frac{2q^2}{3c^3}\left[\gamma^3 \frac{d\mathbf{v}}{dt}\right]^2$. (32)

Let motion be along z-axis, so : $\gamma^3(d\mathbf{v}/dt) = \hat{z}\,\gamma^3(dv/dt)$. Then note, by differentiation : $\frac{d}{dt}(\gamma v) = \gamma^3(dv/dt)$. So q's radiation rate can be written :

$$\boxed{P_{\text{radn}} = \frac{2}{3}(q^2/m^2c^3)\left[d\mathbf{p}/dt\right]^2} \quad \text{for straight-line motion, with: } \mathbf{p} = \gamma m\mathbf{v} \left(\begin{smallmatrix}\text{relativistic}\\ \text{3-momentum}\end{smallmatrix}\right),\ t = \text{lab time.} \tag{33}$$

Now recall work-energy theorem : $\frac{dE}{dt} = \mathbf{v}\cdot\frac{d\mathbf{p}}{dt}$ ‖ w/ $\mathbf{p} = \gamma m\mathbf{v}$, rel. 3-momentum; $E = \gamma mc^2$, rel. total energy.

Then, $\longrightarrow dE = d\mathbf{z}\cdot\frac{d\mathbf{p}}{dt}$, and : $\frac{dp}{dt} = \frac{dE}{dz}$, for straight-line motion. (34)

Using this in Eq.(33), we get the convenient expression...

$$\boxed{P_{\text{radn}} = \frac{2}{3}(q^2/m^2c^3)\left[dE/dz\right]^2} \quad \text{for motion in straight line.} \tag{35}$$

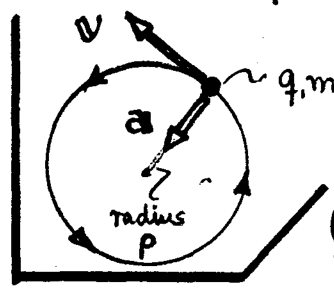
What is notable about this expression is that P_{radn} depends only on the <u>rate</u> (dE/dz) at which the external fields supply energy to q ; P_{radn} does <u>not</u> depend on E or v itself, so there are no γ's on the RHS to get large as $v\rightarrow c$.

We can look at the relative radiation loss/power in for a linear accelerator:

$$\rightarrow \mathcal{R}_{\text{loss}} = \frac{\text{power radiated}}{\text{power supplied}} = \frac{P_{\text{radn}}}{dE/dt} \underset{(v\to c)}{\sim} \frac{2}{3}\left(\frac{q}{mc^2}\right)^2 \frac{dE}{dz} = \frac{2}{3}\left(\frac{dE/mc^2}{dz/r_0}\right). \tag{36}$$

Here $r_0 = q^2/mc^2$ is the classical charge radius of (q,m); it is <u>very</u> small. Evidently $\mathcal{R}_{\text{loss}} \sim 1$ only when $dE \sim mc^2$ is supplied per length $dz \sim r_0$. For an electron, this requires supplying : $dE/dz \sim mc^2/r_0 \sim 2\times10^{12}$ MeV/cm ! Typical designs give : $dE/dz \sim 0.1$ MeV/cm. So <u>linear</u> accelerators are ~ radiationless.

Radiation loss for q in a circular accelerator.



CIRCULAR ACCELERATOR : $\mathbf{v} \perp \mathbf{a} = \frac{d\mathbf{v}}{dt}$.

$\xrightarrow{\text{So}}$ $|\beta \times \dot{\beta}| = \beta\dot{\beta}$, in (31), and: $P_{radn} = \frac{2q^2}{3c^3}\gamma^4[\mathbf{a}]^2$. (37)

For the circular motion: $|\mathbf{a}| = v^2/\rho$, for orbit radius ρ (assumed $\simeq$ cnst per rev).

So $\boxed{P_{radn} = \frac{2}{3}(q^2c/\rho^2)\gamma^4\beta^4}$ for circular motion, in orbit radius ρ, w/ velocity $v = \beta c$, $\gamma = 1/\sqrt{1-\beta^2}$. (38)

Unlike the linear accelerator, P_{radn}(circular) depends on particle's β, and in fact it $\to \infty$ as $v \to c$. Evidently the radiation dominates as $\beta \to 1$. Write:

$$\rightarrow P_{radn} = \frac{2q^2c}{3\rho^2}\beta^4\left[\frac{E}{mc^2}\right]^4 \rightarrow \frac{2q^2c}{3\rho^2}[E/mc^2]^4, \text{ as } \beta \to 1, \tag{39}$$

The energy loss per revolution (w/ $\Delta t = 2\pi\rho/v$, the orbit period) is:

$$\rightarrow \left[\!\left[\Delta\mathcal{E}_{loss}^{(circ)} = P_{radn}\cdot\Delta t = \frac{4\pi q^2}{3\rho}\beta^3[E/mc^2]^4 \longrightarrow \frac{4\pi q^2}{3\rho}[E/mc^2]^4.\right]\!\right] \tag{40}$$

Compare $\Delta\mathcal{E}_{loss}^{(circ)}$ with loss in a linear accelerator over same distance $\Delta z = 2\pi\rho$:

$$\Delta\mathcal{E}_{loss}^{(lin.)} = \frac{2}{3}\left(\frac{q^2}{m^2c^3}\right)\left[\frac{\Delta E}{2\pi\rho}\right]^2\cdot\frac{2\pi\rho}{v} = \frac{q^2}{3\pi\rho}\left(\frac{1}{\beta}\right)\left[\frac{\Delta E}{mc^2}\right]^2$$ ΔE is the energy supplied in $\Delta z = 2\pi\rho$

So

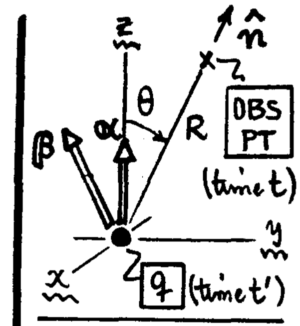
$$\frac{\text{circular loss}}{\text{linear loss}}\Bigg|_{\substack{(\text{over})\\ \Delta z = 2\pi\rho}} = \frac{\Delta\mathcal{E}_{loss}^{(circ.)}}{\Delta\mathcal{E}_{loss}^{(lin.)}} = 4\pi^2\beta^4\left(\frac{E}{mc^2}\right)^2\left(\frac{E}{\Delta E}\right)^2 \gg 1 \tag{41}$$ when: $E \gg \Delta E$, $E \gg mc^2$.

This ratio is independent of particles (no q, m here) and independent of the accelerator size (no ρ either). So it is an <u>intrinsic feature</u> of circular vs. linear accelerator design: <u>circular accelerators radiate enormously more than linear accelerators at high energies.</u>

Q/ 1. Why build circular accelerators (CERN, SSC) at all?

2. For a circular accelerator, let $\rho \to$ earth radius. What E_{max} could you get?

Angular Distribution of Radiation from a single q.



$\beta = \frac{\mathbf{v}}{c}$, $\alpha = \dot{\beta}$

10) We have calculated the total radiated power from an arbitrarily moving charge q -- cf. P(t') of Eq.(26), p. Rad 16. Now we will concentrate on the angular distribution of that radiation (how much radiation goes off into solid ∡ dΩ at the obsᵛn point). A bit later, we will look at the frequency spectrum of the radiation. Information on the distribution & spectrum is important for...

A. Synchrotron design & radiation shielding thereof.

B. Use of synchrotrons as sources of UV & X-rays.

C. Astrophysics -- in assessing mechanisms of cosmic catastrophes {Supernovas, Crab Nebula.

From Eq.(23), p. q Rad 8, we have the energy/unit time & area at the obsᵛn point:

$$\rightarrow [\mathbb{S}\cdot\hat{n}]_t = \left[\frac{q^2}{4\pi c R^2}\cdot\frac{(\hat{n}\times[(\hat{n}-\beta)\times\alpha])^2}{(1-\hat{n}\cdot\beta)^6}\right]_{t'} \quad \left\{\begin{array}{l}\text{evaluation @ retarded time:}\\ t' = t - \frac{1}{c}R(t'),\ \text{w/ } t = \text{obs. time}\end{array}\right. \tag{42}$$

The energy/area emitted by q during the interval $t_1' \leq t' \leq t_2'$ (in its own time), which reaches the observer during $t_1 \leq t \leq t_2$ ... w/ $t_i = t_i' + \frac{1}{c}R(t_i')$, for $i = 1$ & 2... is calculated by the observer to be...

$$\rightarrow \frac{\Delta E_{rad}}{\Delta A} = \int_{t_1}^{t_2}[\mathbb{S}\cdot\hat{n}]_t\,dt = \int_{t_1'}^{t_2'}\underbrace{[\mathbb{S}\cdot\hat{n}]_{t'}\left(\frac{dt}{dt'}\right)}_{\text{this integrand evaluated at } q\text{'s time } t'}dt'. \tag{43}$$

$\frac{dt}{dt'} = (1-\hat{n}\cdot\beta)|_{t'}$ [Eq(20a), p. [redacted]5]

The first form of the integral is clumsy, because it is generally difficult to find the times t_i when q undergoes some complicated motion $R(t_1') \rightarrow R(t_2')$. It is simpler to work in t'(particle) rather than t (obsᵛr). So we consider...

$$\left[\!\left[\begin{array}{l}\text{power per}\\ \text{unit solid}\measuredangle\end{array}\right\} \quad \frac{dP(t')}{d\Omega} = R^2[\mathbb{S}\cdot\hat{n}]_{t'}\left(\frac{dt}{dt'}\right) = R^2[\mathbb{S}\cdot\hat{n}]_{t'}(1-\hat{n}\cdot\beta)_{t'}. \tag{44}$$

This is the integrand of Eq.(43), and it differs from the dP(t')/dΩ calculated in Eq.(25) by the "headlight factor" $(1-\hat{n}\cdot\beta)_{t'}$. This factor -- when integrated

Radiated Power per Solid ∡. Linear Accelerator example.

over all times t' when q is actually radiating (i.e. accelerating), per Eq. (43) -- is just what is needed to convert the radiation signal from q's time t' to observer's time t. Put in $[\$\cdot\hat{n}]$ of Eq. (42) to get...

$$\boxed{\frac{dP}{d\Omega} = \frac{q^2}{4\pi c}\underbrace{(1-\hat{n}\cdot\beta)^{-5}}_{①}\,\underbrace{\left|\hat{n}\times[(\hat{n}-\beta)\times\alpha]\right|^2}_{②}} \tag{45}$$

both sides of this eqtn are evaluated at the retarded q-time t'.

We will drop the retarded time labelling at this point, noting that the ensuing calculations are all done in time t', and that when $(dP/d\Omega)$ in (45) is integrated against dt' (per Eq. (43)), it gives $dE/d\Omega$ in observer's time t.

Most of Jackson's Chap. 14 is devoted to an analysis of $\frac{dP}{d\Omega}$ in Eq. (45) above. It may be worth remarking that most of the relativity occurs in factor ① (a factor that $\rightarrow 1$ when $c\rightarrow\infty$), while factor ② contains directional information.

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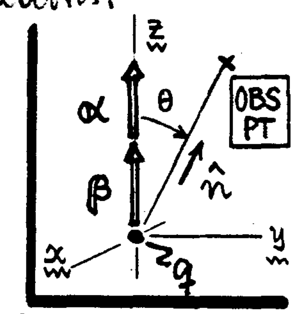
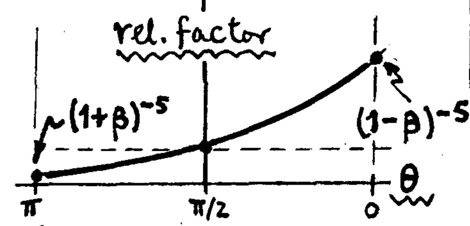
11) Now apply Eq. (45) for radiated power $\frac{dP}{d\Omega}$ to analyse ∡ distributions.



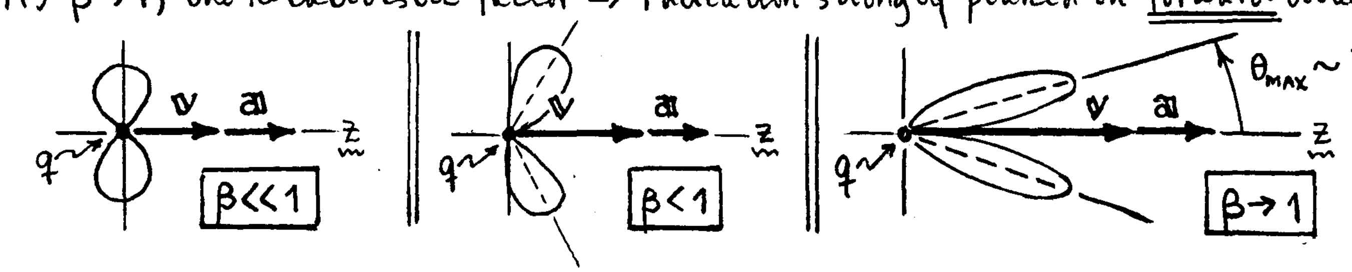
LINEAR ACCELERATOR : $\beta \parallel \alpha$ for q. Observer at ∡ θ.

Have: $\hat{n}\times[(\hat{n}-\beta)\times\alpha] = \hat{n}\,\alpha\cos\theta - \alpha$. Then...

$$\frac{dP}{d\Omega} = \underbrace{\left[\left(\frac{q^2a^2}{4\pi c^3}\right)\sin^2\theta\right]}_{\text{Larmor result}}\underbrace{\frac{1}{(1-\beta\cos\theta)^5}}_{\text{(relativistic factor)}} \tag{46}$$



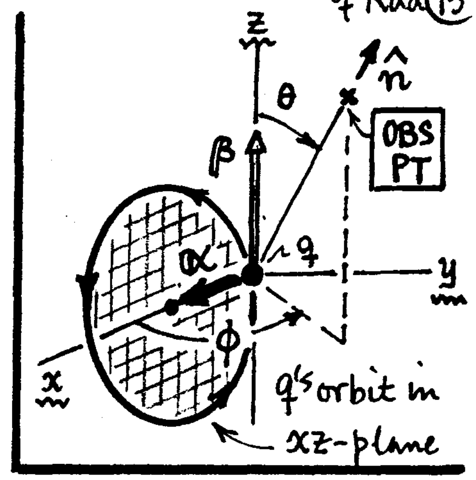
As $\beta\rightarrow 1$, the relativistic factor $\Rightarrow$ radiation strongly peaked in forward direction:



The total radiated power: $P = \int_{4\pi}(dP/d\Omega)\,d\Omega = \left(\frac{2q^2a^2}{3c^3}\right)\gamma^6$, gets large as $\beta\rightarrow 1$.

∡ distribution for a Circular Accelerator.

CIRCULAR ACCELERATOR : $\beta \perp \alpha$ for q.

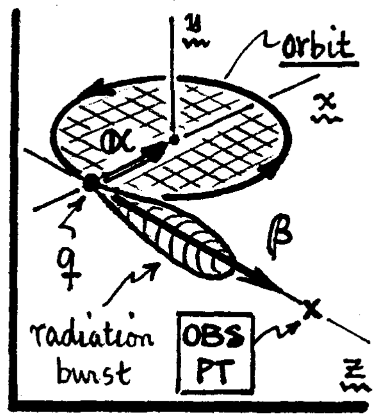


Choose cd. system shown : q's orbit is in the xz plane. Some algebra on $|\hat{n}\times[(\hat{n}-\beta)\times\alpha]|^2$ shows that...

$$\left[\!\left[\frac{dP}{d\Omega}=\frac{q^2a^2}{4\pi c^3}\frac{1}{(1-\beta\cos\theta)^3}\left[1-\frac{\sin^2\theta\cos^2\phi}{\gamma^2(1-\beta\cos\theta)^2}\right].\right.\right. \tag{47}$$

Compare w/ Eq. (46) for linear acceleration. We now have an extra ∡ dependence on ϕ, which specifies whether you are in orbit plane ($\phi=0$ or π) or not -- this extra term comes from the fact that β is no longer $\parallel \alpha$. We can compare the circular with the linear case in the yz plane ($\phi=\pi/2$)...

$$\rightarrow \left(\frac{dP}{d\Omega}\right)_{\substack{\text{circular,}\\ \text{yz-plane}}}=\left(\frac{q^2a^2}{4\pi c^3}\right)\frac{1}{(1-\beta\cos\theta)^3},\ \text{vs.}\ \left(\frac{dP}{d\Omega}\right)_{\substack{\text{linear,}\\ \text{yz-plane}}}=\left(\frac{q^2a^2}{4\pi c^3}\right)\frac{\sin^2\theta}{(1-\beta\cos\theta)^5}. \tag{48}$$



Both quantities show forward peaking (more pronounced for linear case), and differ principally in that the straight-ahead radiation, @ $\theta=0$, does not vanish for the circular case. An observer on the z-axis, as shown, sees periodic short bursts of radiation each time q completes an orbit. These bursts become more intense as $\beta\to1$.

Total radiated power : $P=\int_{4\pi}(dP/d\Omega)d\Omega=\left(\frac{2q^2a^2}{3c^3}\right)\gamma^4$, for circular acceleration. This seems to be less than the linear case : $P=\left(\frac{2q^2a^2}{3c^3}\right)\gamma^6$, as $\beta\to1$. BUT, it isn't... if you look at both P's under equivalent accelerating conditions, as Jackson remarks in his Eq. (14.47)...

$$\left[\!\left[\begin{array}{ll}\text{linear accel}^{\underline{n}}: & P_\parallel=(2q^2/3m^2c^3)|\mathbb{F}_\parallel|^2,\ \ \mathbb{F}_\parallel=(d\mathbb{p}/dt)_\parallel=\text{force applied}\parallel\text{motion},\\ \text{circular accel}^{\underline{n}}: & P_\perp=(2q^2/3m^2c^3)\gamma^2|\mathbb{F}_\perp|^2,\ \ \mathbb{F}_\perp=(d\mathbb{p}/dt)_\perp=\text{force applied}\perp\text{motion}.\end{array}\right.\right. \tag{49}$$

Under the same applied forces, i.e. $|\mathbb{F}_\perp|=|\mathbb{F}_\parallel|$, have : $\underline{P_\perp(\text{circular})=\gamma^2P_\parallel(\text{linear})}$, and we recover the fact (p. Rad 19) that circular accelerators radiate more.