DEPARTMENT OF PHYSICS

PH. D. COMPREHENSIVE EXAMINATION

SEPTEMBER 22-23, 1986

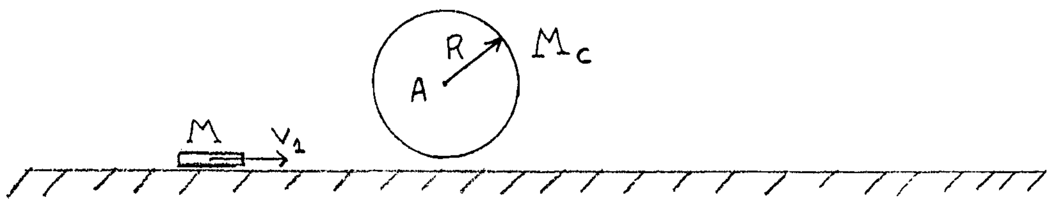
DEPARTMENT OF PHYSICS

Ph.D. COMPREHENSIVE EXAMINATION

MONDAY, SEPTEMBER 22, 1986, 9AM -12 PM

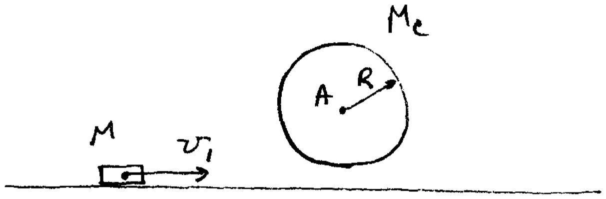
Answer each of the following questions, all of which carry equal weight. Begin your answer to each question on a new sheet of paper; solutions to different questions must not appear on the same sheet of paper. Each sheet of paper must be labeled with your name and the problem number in the upper right hand corner of the page. If more than one sheet is submitted for a problem, be sure the pages are ordered properly.

1. A cylinder of radius R and mass M_c is at rest, supported on a frictionless axis A. A block of mass M and initial velocity v_1 moves on a frictionless plane transverse to A and makes contact with the cylinder. The contact friction is large enough that no slipping occurs between the block and cylinder. The height of the block is negligible compared to R. Find the final velocity of the block, after it breaks contact with the cylinder. How does the final velocity depend on R?



Ph.D. Comps

[Mech] WKF



Edit statement

1. A cylinder of radius R and mass M_c is a rest, supported on a frictionless axis A. A block of mass M and initial velocity v_1 moves on a frictionless plane transverse to A and makes contact with the cylinder. ~~The height of the block is negligible compared to R.~~ The contact friction is large enough that no slipping occurs between the block and cylinder. The height of the block is negligible compared to R. ~~How does the final velocity of the block, after it ~~leaves the~~ breaks contact with the cylinder, depend on ~~the radius~~ R?~~

Find the final velocity of the block, after it breaks contact with the cylinder. How does the final velocity depend on R?

Initial linear momentum $\quad Mv_1$

Angular momentum at contact $\quad RMv_1$

Moment of inertia (Cyl.) $\quad I_c = \frac{1}{2} M_c R^2$

(Block) $\quad I_B = MR^2$

$$I = I_c + I_B = \frac{1}{2} M_c R^2 + MR^2$$

Angular momentum (final)

$$I\omega = \left(\frac{1}{2} M_c R^2 + MR^2\right)\left(\frac{v_{final}}{R}\right)$$

Cons. of ang. momentum.

$$\left(\frac{1}{2} M_c R^2 + MR^2\right)\left(\frac{v_f}{R}\right) = RMv_1$$

$$v_f = \frac{Mv_1}{\left(\frac{1}{2} M_c + M\right)} = \left(\frac{2M}{M_c + 2M}\right) v_1$$

No dependence on R.

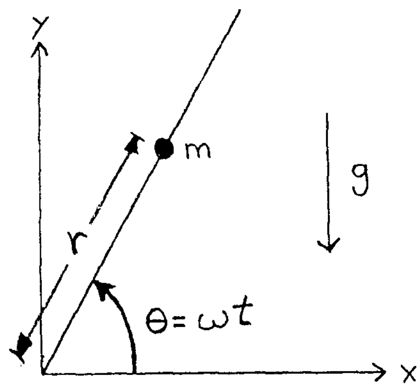
2. Consider a bead of mass m sliding on a straight wire without friction. The wire rotates at a constant angular speed ω in the vertical plane; gravity acts downward with acceleration g.

(a) Find the Lagrangian for the bead

(b) Solve the equation of motion to find r(t); assume that at t=0, r=R and dr/dt=0

(c) Find the Hamiltonian

(d) Is the Hamiltonian equal to the energy? Is it constant?



You may assume $R > g/\omega^2$

Ph.D Comps

Classical Mechanics Hiscock

2. Consider a bead of mass m sliding on a straight wire without friction. The wire rotates at a constant angular speed ω in the vertical plane; gravity acts downward with acceleration g.

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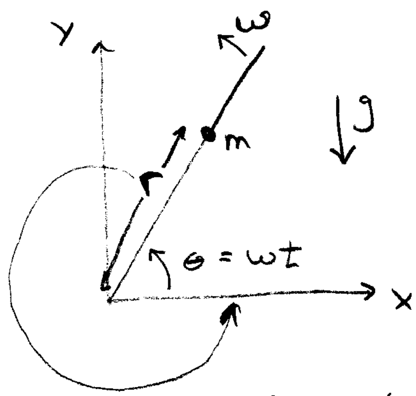


You may assume $R > g/\omega^2$

Solution:

(a) $\theta = \omega t$, $\dot\theta = \omega$ $\quad x = r\cos\theta = r\cos\omega t \quad y = r\sin\theta = r\sin\omega t$

$\dot x = \dot r\cos\omega t - r\omega\sin\omega t \qquad \dot y = \dot r\sin\omega t + r\omega\cos\omega t$

$T = \frac{1}{2}m(\dot x^2+\dot y^2) = \frac{1}{2}m(\dot r^2 + r^2\omega^2) \qquad V = mgr\sin\theta = mgr\sin(\omega t)$

$$\boxed{L = T - V = \frac{1}{2}m(\dot r^2 + r^2\omega^2) - mgr\sin(\omega t)}$$

(b) $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot r}\right) - \frac{\partial L}{\partial r} = 0$

$\frac{d}{dt}(m\dot r) - \left[mr\omega^2 - mg\sin(\omega t)\right] = 0$

$\rightarrow \quad \ddot r - r\omega^2 = -g\sin(\omega t)$

solution to inhomogeneous eq.: Let $r_1 = a\sin(\omega t)$; then

$\ddot r_1 - r_1\omega^2 = [-a\omega^2 - a\omega^2]\sin(\omega t) = -g\sin(\omega t)$ so $\boxed{a = \frac{g}{2\omega^2}}$

homogeneous solution:

$\ddot r_2 = \omega^2 r_2 \qquad r_2 = A\cosh(\omega t) + B\sinh(\omega t)$

$r(t) = r_1(t) + r_2(t) = A\cosh(\omega t) + B\sinh(\omega t) + \frac{g}{2\omega^2}\sin(\omega t)$

initial conditions

$r(0) = R = A$

$\dot r(0) = 0 = B\omega + \frac{g}{2\omega} \Rightarrow B = -g/2\omega^2$

so

$$\boxed{r(t) = R\cosh(\omega t) - \frac{g}{2\omega^2}\sinh(\omega t) + \frac{g}{2\omega^2}\sin(\omega t)}$$

may need a lower limit on ω, to keep $r(t)$ positive? -GT.

Yes! Thanks George!

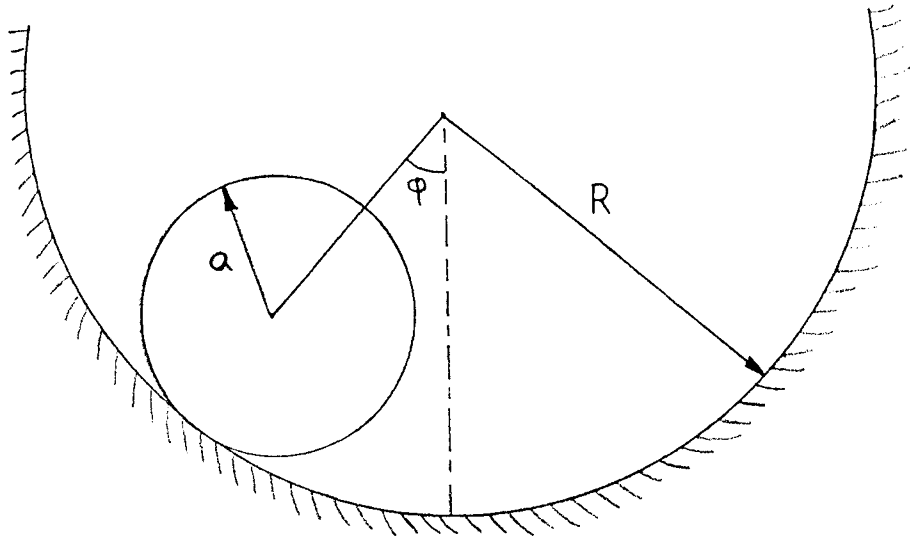
(c) Hamiltonian: $p_r = m\dot r$

$H = p_r\dot r - L = \boxed{\frac{p_r^2}{2m} - \frac{1}{2}mr^2\omega^2 + mgr\sin(\omega t)}$

(d) $H \neq T+V$, so $H \neq E$; $\boxed{\text{it is \underline{not} the energy}}$

$\boxed{\text{H is also not constant}}$; since $\frac{dH}{dt} = \frac{\partial H}{\partial t} \neq 0$

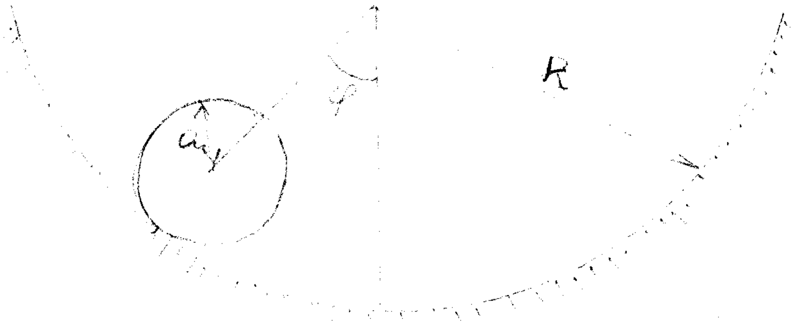
3. Find the kinetic energy of a homogeneous cylinder of radius a and mass M, rolling (without slipping) inside a cylindrical surface of radius R.



Note: You must calculate the moment of inertia about the axis of the cylinder.

Classical Mechanics (alternate problem) (A.E.)

3. Find the kinetic energy of a homogeneous cylinder of radius a and mass M, rolling (without ~~sliding~~ slipping) inside a cylindrical surface of radius R.



Note: You must calculate the moment of inertia about the axis of the cylinder.

Solution

The (magnitude of the) velocity of the center of mass of the cylinder is $(R-a)\,\dot{\varphi}(t) \equiv v$

The angular velocity Ω is related to v through the condition that the line of contact has zero velocity (it is an instantaneous axis of pure rotation due to the assumption of rolling without sliding). From the equation $\vec{v} + \vec{\Omega} \times \vec{r} = 0$ we get:

$$(R-a)\,\dot{\varphi}(t) + \Omega\, a = 0$$

$$\rightarrow \qquad \Omega(t) = -\frac{1}{a}(R-a)\,\dot{\varphi}(t)$$

The total kinetic energy consists of a translational piece and a rotational piece:

$$T = \frac{1}{2} M (R-a)^2 \dot{\varphi}^2(t) + \frac{1}{2} I_3 \frac{(R-a)^2}{a^2} \dot{\varphi}^2(t)$$

Noting that I_3, the moment of inertia about the axis of the cylinder, is given by

$$I_3 = \frac{1}{2} M a^2 \quad ,$$

we have that

$$T = \frac{3}{4} M (R-a)^2 \dot{\varphi}^2(t)$$

4. An outstanding problem in astrophysics is that the Sun only seems to be giving off about 1/3 the number of electron neutrinos that the best theoretical stellar models predict. A possible solution to this puzzle is suggested by particle physics: if neutrinos have mass, <u>and</u> if the eigenstates of the weak interactions (ν_e, the electron neutrino; ν_μ, the muon neutrino; and ν_τ, the tau neutrino) are not mass eigenstates, then neutrino oscillations will occur. A neutrino which starts out as an electron neutrino in the center of the Sun may have "oscillated" into a muon (or tau) neutrino by the time it reaches the neutrino detector here on Earth (which is only sensitive to electron neutrinos). For the purposes of this problem, ignore the existence of the tau neutrino (so the neutrino is only a two state system, $|\nu_e\rangle$ and $|\nu_\mu\rangle$) and special relativistic effects (i.e., use the Schrodinger equation, not the Dirac equation)

The weak interaction eigenstates, $|\nu_e\rangle$ and $|\nu_\mu\rangle$, may then be written in terms of the mass eigenstates $|\nu_1\rangle$ (mass m_1) and $|\nu_2\rangle$ (mass m_2) as follows:

$$|\nu_e\rangle = \cos\theta|\nu_1\rangle + \sin\theta|\nu_2\rangle$$

$$|\nu_\mu\rangle = -\sin\theta|\nu_1\rangle + \cos\theta|\nu_2\rangle$$

where θ is the (fixed) mixing angle.

(a) express the mass eigenstates $|\nu_1\rangle$, $|\nu_2\rangle$ in terms of the weak eigenstates $|\nu_e\rangle, |\nu_\mu\rangle$

(b) A neutrino is created at t=0, x=0 in the state $|\psi(t=0, x=0)\rangle = |\nu_e\rangle$ with definite energy E_o. Find the probability that this neutrino will be an electron neutrino at time t and location x; in other words, evaluate $|\langle\nu_e|\psi(t,x)\rangle|^2$.

Ph.D. Comp. B.H

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Solution

(a) $|\nu_1\rangle = \cos\theta\,|\nu_e\rangle - \sin\theta\,|\nu_\mu\rangle$
$|\nu_2\rangle = \sin\theta\,|\nu_e\rangle + \cos\theta\,|\nu_\mu\rangle$

(b) Since the states $|\nu_1\rangle$ $|\nu_2\rangle$ are mass eigenstates, their amplitudes will each evolve according to **the Schrodinger** equation:

Let

$$|\Psi(t,x)\rangle = \alpha_1(t,x)\cos\theta\,|\nu_1\rangle + \alpha_2(t,x)\sin\theta\,|\nu_2\rangle$$

initial condition: $t=x=0$ $\alpha_1 = \alpha_2 = 1$

Evolution:

$$i\hbar\frac{\partial\alpha_1}{\partial t} = -\frac{\hbar^2}{2m_1}\frac{\partial^2\alpha_1}{\partial x^2} \qquad i\hbar\frac{\partial\alpha_2}{\partial t} = -\frac{\hbar^2}{2m_2}\frac{\partial^2\alpha_2}{\partial x^2}$$

plane wave free particle solutions:

$$\alpha_1 = \alpha_1^0\exp(-i\omega_1 t + k_1 x) \qquad \alpha_2 = \alpha_2^0\exp(-i\omega_2 t + k_2 x)$$

initial condition $\alpha_1 = \alpha_2 = 1$ at $t=x=0 \Rightarrow \alpha_1^0 = \alpha_2^0 = 1$

Evolution eqn. becomes:

$$+\hbar\omega_1 = +\frac{\hbar^2 k_1^2}{2m_1} \qquad \hbar\omega_2 = \frac{\hbar^2 k_2^2}{2m_2}$$

state of definite energy $\Rightarrow E_0 = \dfrac{\hbar^2 k_1^2}{2m_1} = \dfrac{\hbar^2 k_2^2}{2m_2}$

$$\Rightarrow \omega_1 = \frac{E_0}{\hbar} = \omega_2 \quad \& \quad k_1 = \frac{\sqrt{2m_1E_0}}{\hbar} \qquad k_2 = \frac{\sqrt{2m_2E_0}}{\hbar}$$

so

$$|\Psi(t,x)\rangle = \left[e^{-i\frac{E_0 t}{\hbar}}\right]\left\{\cos\theta\exp\left(i\frac{\sqrt{2m_1E_0}}{\hbar}x\right)|\nu_1\rangle + \sin\theta\exp\left(i\frac{\sqrt{2m_2E_0}}{\hbar}x\right)|\nu_2\rangle\right\}$$

DEPARTMENT OF PHYSICS

Ph.D. COMPREHENSIVE EXAMINATION

MONDAY, SEPTEMBER 22, 1986, 2-5 PM

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1. Calculate the capacitance of a spherical capacitor of inner radius R_1, and outer radius R_2, with the space between the spheres filled with a dielectric varying as $\varepsilon = \varepsilon_1 + \varepsilon_2 \cos^2\theta$ and θ is the polar angle.

E+M | John Carlsten | PhD Comps

1. Calculate the capacitance of a spherical capacitor of inner radius R_1 and outer radius R_2, with the space between the spheres filled with a dielectric varying as $\epsilon = \epsilon_1 + \epsilon_2 \cos^2\theta$
and θ is the polar angle.

E+M solution

Apply a charge +Q to the inner sphere and −Q to the outer sphere. Now apply Gauss' Law between the spheres

$$\int \vec{D}\cdot d\vec{A} = 4\pi Q$$

$$= \int_0^\pi (\epsilon_1 + \epsilon_2 \cos^2\theta)\, E \quad 2\pi R^2 \sin\theta \, d\theta$$

$$= 2\pi E R^2 \left[-\epsilon_1 \cos\theta \Big|_0^\pi - \epsilon_2 \frac{\cos^3\theta}{3} \Big|_0^\pi \right]$$

$$= 2\pi E R^2 \left[2\epsilon_1 + \frac{2\epsilon_2}{3} \right]$$

$$= 4\pi E R^2 (\epsilon_1 + \epsilon_2/3)$$

so $E = \dfrac{Q}{R^2(\epsilon_1 + \epsilon_2/3)}$

NOTE: E is constant over θ + φ but D is not since Dielectric attracts charge, so Q is not evenly distributed.

Now we can calculate the potential difference between the plates

$$V = -\int_{R_2}^{R_1} \vec{E}\cdot d\vec{R} = -\int_{R_2}^{R_1} \frac{Q}{R^2(\epsilon_1 + \epsilon_2/3)}\, dR = \frac{Q}{(\epsilon_1 + \epsilon_2/3)} \frac{1}{R} \Big|_{R_2}^{R_1}$$

$$V = \frac{Q}{(\epsilon_1 + \epsilon_2/3)} \left(\frac{1}{R_1} - \frac{1}{R_2} \right)$$

Thus

$$C = \frac{Q}{V} = \frac{(\epsilon_1 + \epsilon_2/3)(R_1 R_2)}{(R_2 - R_1)}$$

2. A cylindrical column of liquid sodium is supported between two electrodes which impress a current I (uniformly) through the column. The self-interaction of the moving charges gives rise to a hydrostatic pressure within the column and a net force on the electrodes which terminate the structure. Find the force on the electrodes. How does it depend on R? Neglect end effects, viscosity, and gravity.

cylindrical Ph.D. ~~Comp~~ liquid sodium Wayne Ford (E&M) WKA

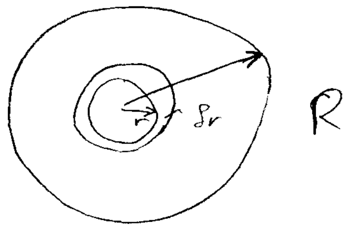
2\. A cylindrical column of ~~mercury~~ liquid sodium is supported between two electrodes which impress a current I ~~~~ (uniformly) through the column. The self-interaction of the moving charges gives rise to a ~~~~ hydrostatic pressure ~~and~~ within the column and a net force on the electrodes which terminate the structure. ~~[How does the force on the electrodes depend on the cylinder radius R? Neglect end effects, viscosity, and gravity.]~~

Find the force on the electrodes. How does it depend on R? Neglect end effects, viscosity, and gravity.

JED.

Interesting but -?

$B = B(r)$ by symmetry



$$\oint \vec{B}\cdot d\vec{l} = \mu\mu_0 \int_0^r J(r)\, da$$

$$J = \frac{I}{\pi R^2} \Rightarrow B_\phi (2\pi r) = \mu\mu_0 \frac{I}{\pi R^2} \pi r^2$$

$$B_\phi = \mu\mu_0 \frac{Ir}{2\pi R^2}$$ B_ϕ gives rise to radial force.

At r a radially inward force is created giving rise to a hydrostatic pressure

$$\delta p = \frac{B_\phi\, \delta I}{(2\pi r)} = \left(\mu\mu_0 \frac{Ir}{2\pi R^2}\right)\left(\frac{I}{\pi R^2} 2\pi \delta r\, r\right) \frac{1}{2\pi r}$$

$$\delta p = \mu\mu_0 \frac{I^2}{2\pi^2 R^4} r\, \delta r$$

Thus,
$$p(r) = \int_r^R \mu\mu_0 \frac{I^2}{2\pi^2 R^4} r\, dr = \frac{\mu\mu_0 I^2}{4\pi^2 R^4}(R^2 - r^2)$$

This pressure is translated to axial force on the ends

$$F = \int p(r)\, r\, dr\, d\phi = \frac{\mu\mu_0}{2\pi R^4} I^2 \int_0^R (R^2 r - r^3)\, dr$$

$$= \frac{\mu\mu_0}{2\pi R^4} I^2 \left(\frac{R^4}{2} - \frac{R^4}{4}\right) = \frac{\mu\mu_0}{8\pi} I^2$$

$\Rightarrow$ No dependence.

3. a) Obtain the electromagnetic field due to a charge e moving with uniform angular velocity ω on a circle of radius a. Assume that the speed v of particle is such that $v \ll c$, where c is the speed of light. The field is to be evaluated at distances $r \gg a$ from the center of the particle's orbit. Use the electric dipole approximation.

b) Obtain the power radiated per unit solid angle as a function of the polar angle θ, measured from the normal to the plane of the orbit.

Electromagnetism. (A.E.)

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b) Obtain the power radiated per unit solid angle as a function of the ~~azimuthal~~ polar angle θ, measured from the normal to the plane of the orbit.

Too long " JED

Solution :

Lienard - Wiechert potential :

$$\vec{A}(\vec{x}t) = \frac{1}{c}\int d^3x' \frac{1}{|\vec{x}-\vec{x}'|} \vec{J}(\vec{x}'; t - \frac{|\vec{x}-\vec{x}'|}{c})$$

For $r = |\underset{\sim}{x}| \gg a$:

$$\vec{A}(\vec{x}t) \cong \frac{1}{cr}\int d^3x' \vec{J}(\vec{x}'; t - \frac{|\vec{x}-\vec{x}'|}{c})$$

$$|\vec{x}-\vec{x}'| \cong r - \vec{x}'\cdot\hat{n} ,$$

where $\hat{n} = \frac{\vec{x}}{r}$ is the unit vector in the direction of the vector $\vec{x}$.

If $v \ll c$ we can take the delay to be the same for all the points of the orbit. Then :

$$\vec{A}(\vec{x}t) \cong \frac{1}{cr}\int d^3x' \vec{J}(\vec{x}'; t - \frac{r}{c})$$

Call : $t' = t - \frac{r}{c}$

$$\vec{A}(\vec{x}t) \cong \frac{1}{cr}\int d^3x' \vec{J}(\vec{x}'; t') = \frac{e}{cr}\vec{v}(t')$$

$$e\,\underset{\sim}{v}(t') = \dot{\underset{\sim}{p}}(t')$$

$\therefore$

$$\vec{A}(\vec{x}t) = \frac{1}{cr}\dot{\underset{\sim}{p}}(t)$$

In the electric dipole approximation:

$$\vec{B} = \frac{1}{c}\frac{\partial}{\partial t}\vec{A}\times\hat{n} = \frac{1}{c}\dot{\underset{\sim}{A}}\times\hat{n}$$

and

$$\vec{E} = \frac{1}{c}(\dot{\underset{\sim}{A}}\times\hat{n})\times\hat{n} = \vec{B}\times\hat{n}$$

Now:

$$\vec{v}(t') = \omega a\,(-\sin\omega t'\,\hat{e}_x + \cos\omega t'\,\hat{e}_y)$$

$$\hat{n} = \sin\theta\cos\varphi\,\hat{e}_x + \sin\theta\sin\varphi\,\hat{e}_y + \cos\theta\,\hat{e}_z$$

$$\dot{\underset{\sim}{A}} = \frac{e}{cr}\frac{\partial}{\partial t}\vec{v} =$$

$$= \frac{e}{cr}\,\omega a\,(-\omega\cos\omega t'\,\hat{e}_x - \omega\sin\omega t'\,\hat{e}_y)$$

$$= -\frac{e\omega^2 a}{cr}\,(\cos\omega t'\,\hat{e}_x + \sin\omega t'\,\hat{e}_y)$$

$$B_x = \frac{1}{c} (A_y n_z - \underbrace{A_z}_{=0} n_y) =$$

$$= \frac{1}{c} (-) \frac{e\omega^2 a}{cr} \sin\omega t' \cos\theta$$

$$B_y = \frac{1}{c} (\underbrace{A_z}_{=0} n_x - A_x n_z) =$$

$$= -\frac{1}{c} (-) \frac{e\omega^2 a}{cr} \cos\omega t' \cos\theta$$

$$B_z = \frac{1}{c} (A_x n_y - A_y n_x) =$$

$$= \frac{1}{c} (-) \frac{e\omega^2 a}{cr} \left[\cos\omega t' \sin\theta \sin\varphi - \sin\omega t' \sin\theta \cos\varphi \right]$$

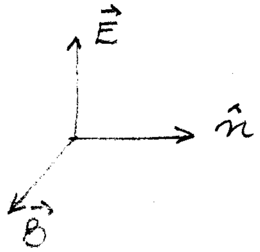
$$= - \frac{e\omega^2 a}{c^2 r} \sin\theta \underbrace{(\cos\omega t' \sin\varphi - \sin\omega t' \cos\varphi)}_{= \sin(\varphi - \omega t')}$$

$$= \frac{e\omega^2 a}{c^2 r} \sin\theta \sin(\omega t' - \varphi)$$

$\therefore$

$$\vec{B}(\vec{x}, t) = \frac{e\omega^2 a}{c^2 r} \left[- \cos\theta \sin\omega t' \, \hat{e}_x + \cos\theta \cos\omega t' \, \hat{e}_y + \sin\theta \sin(\omega t' - \varphi) \, \hat{e}_z \right]$$

Call: $\alpha = \dfrac{e\omega^2 a}{c^2 r}$



$$|\vec{E}| = |\vec{B}|$$

$$\vec{S} = \frac{c}{4\pi} \vec{E} \times \vec{B}$$

$$= \frac{c}{4\pi} |\vec{B}|^2 \hat{n}$$

$$|\vec{B}|^2 = \alpha^2 \left[\cos^2\theta \sin^2\omega t' + \cos^2\theta \cos^2\omega t' + \right.$$
$$\left. + \sin^2\theta \sin^2(\omega t' - \varphi) \right]$$

$\therefore$

$$|\vec{B}|^2 = \alpha^2 \left[\cos^2\theta + \sin^2\theta \sin^2(\omega t' - \varphi) \right]$$

$$\frac{1}{T}\int_0^T dt \, \vec{S}(t) = \frac{c}{4\pi}\alpha^2 \hat{n} \left[\cos^2\theta + \sin^2\theta \underbrace{\frac{1}{T}\int_0^T dt \, \sin^2(\omega t' - \varphi)}_{= \frac{1}{2}} \right]$$

$$\equiv \bar{\underset{\sim}{S}}$$

$\therefore$

$$\bar{\underset{\sim}{S}} = \frac{c}{4\pi}\alpha^2 \frac{1}{2} \left(2\cos^2\theta + \sin^2\theta \right) \hat{n}$$

$$= \frac{c}{8\pi}\alpha^2 \left(1 + \cos^2\theta \right) \hat{n}$$

Power radiated $\quad P = \int d\vec{A} \cdot \bar{\underset{\sim}{S}}$

$$\therefore \quad P = \int d\Omega \ \frac{c}{8\pi} \alpha^2 (1+\cos^2\theta) \ r^2$$

$$\therefore \quad \frac{dP}{d\Omega} = \frac{c\,\alpha^2 r^2}{8\pi} (1+\cos^2\theta)$$

$$= \frac{c}{8\pi} \frac{e^2\omega^4 a^2}{c^4 r^2} r^2 (1+\cos^2\theta)$$

$$\therefore \quad \frac{dP}{d\Omega} = \frac{e^2\omega^4 a^2}{8\pi c^3} (1+\cos^2\theta)$$

4. For a paramagnetic gas of N atoms cm^{-3} with L=0 and S=1/2:

a) Calculate the number of atoms per cubic cm in each level at temperature T and in field H.

b) Calculate the resultant magnetization.

c) With $N=10^{22}$ cm^{-3}, H = 25,000 Gauss and g=2, compute values for a) and b) above at T=300K and at T=4K.

Note: k (Boltzmann's constant) = 1.38×10^{-16} erg/K
$\mu_B = 9.274 \times 10^{-21}$ erg/Gauss.

~~(Solid State~~)
Stat Mech

Drumheller

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4. For a paramagnetic gas of N atoms cm^{-3} with $L=0$ and $S=\frac{1}{2}$:

a.) Calculate the number of atoms per cubic cm in each level at temperature T and in field H.

b.) Calculate the resultant magnetization.

c.) With $N = 10^{22}$ cm^{-3}, $H = 2.5\,T$, ~~and~~ $T = 300K$ and 4 K, and $g = 2$, compute values for a.) and b.) above.

---

include: note: k (Boltzmann's constant) $= 1.38 \times 10^{-16}$ erg / K
$\mu_B = 9.274 \times 10^{-21}$ erg / Gauss

Sol'n

For $L=0$, $S=\frac{1}{2} \Rightarrow J=\frac{1}{2}$ $\therefore M_J = \pm\frac{1}{2}$
Level population proportional to $e^{-E/kT}$ where

$$E = -\vec{\mu}_B \cdot \vec{H} = -\mu_B\, g\, M_J H = -\mu_B g(\pm\tfrac{1}{2})H$$

$$N_{+\frac{1}{2}} = C \exp(+g\mu_B H/2kT)$$
$$N_{-\frac{1}{2}} = C \exp(-g\mu_B H/2kT)$$

with $C\left[e(+\cdots) + e(-\cdots)\right] = N$

$\therefore$ $C = \frac{N}{\underbrace{e^{+h}+e^{-h}}}$ where $h = g\mu_B H/2kT$

Note: this is the partition fn Z

a./ $N_{+\frac{1}{2}} = \frac{N}{Z} e^{h}$ $\qquad N_{-\frac{1}{2}} = \frac{N}{Z} e^{-h}$

b./ $$M = \frac{g\mu_B}{2}\left(N_{+\frac{1}{2}} - N_{-\frac{1}{2}}\right) = \frac{g\mu_B N}{2}\frac{e^{h}-e^{-h}}{e^{h}+e^{-h}}$$

$$= \frac{g\mu_B N}{2}\tanh h = \frac{g\mu_B N}{2}\tanh\left(\frac{g\mu_B H}{kT}\right)$$

c./ $\exp\left(\frac{2\,(9.274(10)^{-21}\,\frac{erg}{G})\; 2.5(10)^4 G}{2(1.38(10)^{-16}\,\frac{erg}{K})\;\; 300\,K}\right) =$

$$N_{+\frac{1}{2}} = \frac{10^{22}\exp(5.6(10)^{-2})}{2\cosh(5.6(10)^{-2})} = =$$

$\exp(5.6(10)^{-2}) = 1.0056$

$= 5.000078\,(10)^{22}$

$M = 0.516$ emu/cm^3 at 300K

$M = 37.2$ emu/cm^3 at 4K

DEPARTMENT OF PHYSICS

Ph.D. COMPREHENSIVE EXAMINATION

TUESDAY, SEPTEMBER 23, 1986, 9AM -12 PM

Answer each of the following questions, all of which carry equal weight. Begin your answer to each question on a new sheet of paper; solutions to different questions must not appear on the same sheet of paper. Each sheet of paper must be labeled with your name and the problem number in the upper right hand corner of the page. If more than one sheet is submitted for a problem, be sure the pages are ordered properly.

1. A hydrogen atom is placed in a field such that the electron experiences a perturbing potential of the form

$$V = A(x+y)^2 \qquad\qquad (A = \text{const.})$$

a) Calculate the correction to the ground state energy, to 1st order in perturbation theory.

b) The n=2 hydrogen level is initially 4-fold degenerate (ignoring spin). Comment on the effect of V on this level. How many levels appear to 1st order in perturbation theory?

Normalized wavefunctions $\psi_{n\ell m}$: (a_o = Bohr radius)

$$\psi_{100} = (\pi a_o^3)^{-1/2} \exp(-r/a_o)$$

$$\psi_{200} = (8\pi a_o^3)^{-1/2} (1 - r/2a_o) \exp(-r/2a_o)$$

$$\psi_{210} = (32\pi a_o^3)^{-1/2} (r/a_o) \cos\theta \exp(-r/2a_o)$$

$$\psi_{21\pm1} = (64\pi a_o^3)^{-1/2} (r/ao) \sin\theta\, e^{\pm i\phi} \exp(-r/2a_o)$$

c) Suppose the perturbation has cylindrical symmetry: $V = A(x^2 + y^2)$. How would your answers to parts a) and b) change?

Ph.D Comps.

– Quantum Mech – (GT)

A hydrogen atom is placed in a field such that the electron experiences a perturbing potential of the form

$$V = A(x+y)^2 \qquad (A = \text{const.})$$

(a) Calculate the correction to the ground state energy, to 1st order in perturbation theory

(b) The $n=2$ hydrogen level is initially 4-fold degenerate (ignoring spin). Comment on the effect of V on this level. How many levels appear to 1st order in perturbation theory?

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$$\psi_{21\pm1} = (64\pi a_0^3)^{-1/2} (r/a_0) \sin\theta\, e^{\pm i\phi} \exp(-r/2a_0)$$

(c) Suppose the perturbation has cylindrical symmetry – $V = A(x^2+y^2)$. How would your answers to parts (a) + (b) change?

Soln: (a) $\Delta E_0 = \langle \phi_{100} | V | \phi_{100} \rangle = \dfrac{A}{\pi a_0^3} \displaystyle\int_0^\infty r^2 dr \int_0^{2\pi} d\phi \int_0^\pi d\theta \sin\theta \, e^{-\frac{2r}{a_0}} \underbrace{r^2 (x+y)^2}$

$r^2 \sin^2\theta (1 + \sin 2\phi)$

$$= \frac{A}{\pi a_0^3} 2\pi \underbrace{\int_0^\infty dr\, r^4 e^{-2r/a_0}}_{(a_0/2)^5\, 4!} \int_0^\pi d\theta \sin^3\theta$$

$$\int_{-1}^{+1} d(\cos\theta)(1-\cos^2\theta) = 4/3$$

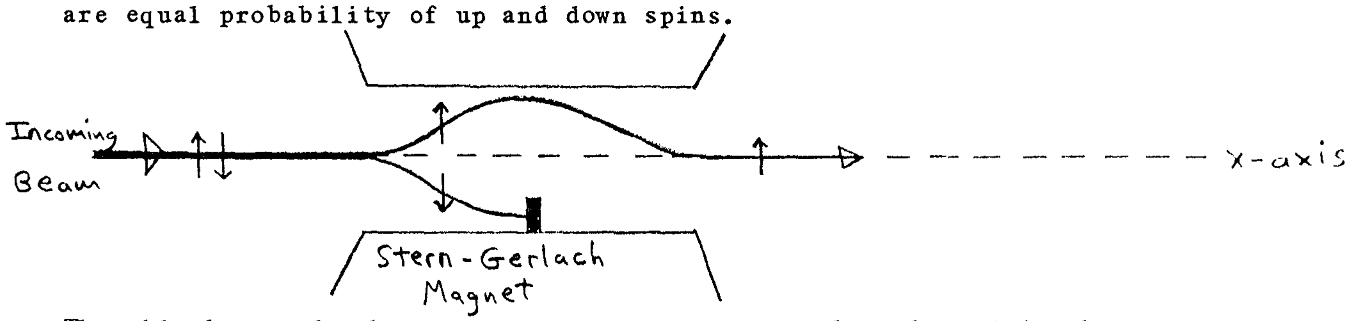
$$= \underline{2 a_0^2 A}$$

→

(b) Off-diagonal elements of V', in the $n=2$ subspace are all zero except those with m's differing by 2 — i.e. only $\langle \phi_{211} | V | \phi_{21-1} \rangle$. Among the diagonal elements, all are distinct except $\langle \phi_{211} | V | \phi_{211} \rangle = \langle \phi_{21-1} | V | \phi_{21-1} \rangle$. Thus the degeneracy is completely lifted in 1st order.

(c) Answer to part (a) is the same, since the $r^2 \sin\theta \sin 2\phi$ part of V never contributed to ΔE anyway. In part (b), <u>all</u> off diagonal elements of V vanish, and there are only 3 distinct diagonal elements, as before. Thus ~~the~~ $n=2$ is split into <u>3</u> levels, rather than 4.

2. Imagine a Stern–Gerlach magnet system that separates the up–spin from the down–spin of an incoming beam of particles in which initially there are equal probability of up and down spins.



Then block out the down–spins in some manner so that the exiting beam is up–spin only.

Next imagine a second identical Stern–Gerlach magnet in line (along the x-axis) but rotated about the x-axis by 60^o. In this new frame the up (or "z") axis will be rotated 60^o from the first stage. Again block the "down" spins (i.e., the "down" spins in the new frame). Finally a third stage is rotated back 60^o to be in the same orientation as the first.

In the third stage, calculate the probability that you will measure up–spins and down–spins.

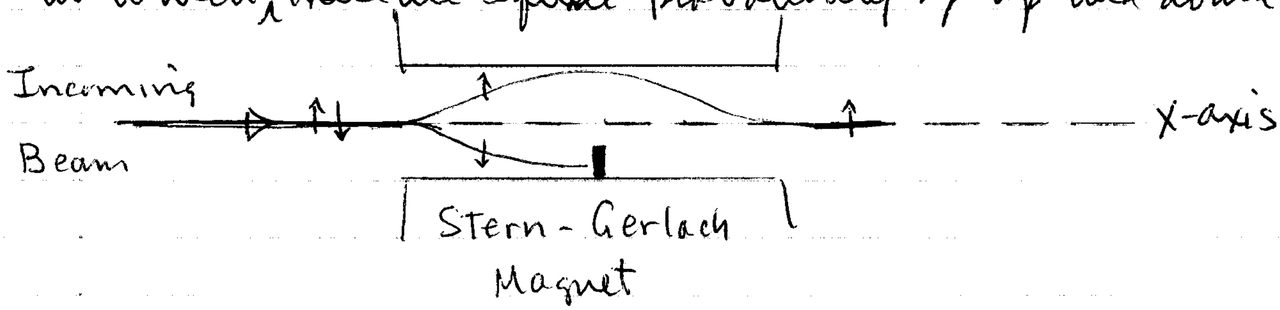
Hint: The appropriate unitary rotation operator is $U = \exp\left(-\frac{i}{2}\,\theta\,\hat{n}\cdot\vec{\sigma}\right)$ and can be written

$$U_R = \vec{I}\cos\frac{\theta}{2} - i\,\hat{n}\cdot\vec{\sigma}\sin\frac{\theta}{2}$$

where $\vec{I}$ is a unit $\begin{pmatrix}1&0\\0&1\end{pmatrix}$ matrix and $\vec{\sigma}$ are the Pauli matrices. $\hat{n}$ is a unit vector along the axis of rotation.

Quantum Ph.D. Comps Drumheller

2. Imagine a Stern-Gerlach magnet system that separates the up-spin from the down-spin of an incoming beam of particles in which initially there are equal probability of up and down spins.



Then block out the down-spins in some manner so that the exiting beam is up-spin only.

Next imagine a second identical Stern-Gerlach magnet in line (along the x-axis) but rotated about the x-axis by 60°. In this new frame the up (or "z") axis will be rotated 60° from the first stage. Again block the "down" spins (i.e., the "down" spins in the new frame). Finally a third stage is rotated back 60° to be in the same orientation as the first.

In the third stage, calculate the probability that you will measure up-spins and down-spins.

Hint: the appropriate unitary rotation operator is $U = \exp\left(-\frac{i}{2}\theta\, \hat{n}\cdot\vec{\sigma}\right)$ and can be written

$$U_R = \vec{1}\cos\frac{\theta}{2} - i\,\hat{n}\cdot\vec{\sigma}\,\sin\frac{\theta}{2}$$

$\begin{pmatrix}1&0\\0&1\end{pmatrix}$

where $\vec{1}$ is a unit matrix and $\vec{\sigma}$ are the Pauli matrices.
$\hat{n}$ is a unit vector along ~~the~~ axis of rotation.

$$\mathcal{U} = \exp\left(-\frac{i}{2}\theta\, \hat{n}\cdot\vec{\sigma}\right)$$

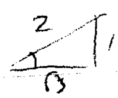
$$= 1 \cos\frac{\theta}{2} - i\,\hat{n}\cdot\vec{\sigma}\,\sin\frac{\theta}{2}$$

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad U_{R_x} = 1\cos\frac{\theta}{2} - i\,\sigma_x \sin\frac{\theta}{2}$$

$$U_{R_x} = \begin{vmatrix} \cos\frac{\theta}{2} & -i\sin\frac{\theta}{2} \\ -i\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{vmatrix}$$

This assumes the α and β before the first block ~~are~~ normalized so that $\alpha = \frac{1}{2}$ i.e. $\alpha^2 = \frac{1}{4}$.

$$\begin{pmatrix} c_1' \\ c_2' \end{pmatrix} = U_{R_x}\begin{pmatrix} \alpha \\ \beta_2 \end{pmatrix} \qquad \beta = 0$$



$\frac{\sqrt{3}}{2}$

...ming out of the first SG magnet is:

Spin "up" → c_1', Spin "down" → c_2' in new frame

$$\begin{pmatrix} c_1' \\ c_2' \end{pmatrix} = \begin{vmatrix} \cos\frac{\theta}{2} & -i\sin\frac{\theta}{2} \\ -i\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{vmatrix}\begin{pmatrix} \alpha \\ 0 \end{pmatrix} \xleftarrow{\theta} \text{Spin down blocked}$$

$\theta = 60°$

$\frac{\theta}{2} = 30°$

where c_1' is spin "up", c_2' is spin "down".

$$c_1' = \alpha\cos\frac{\theta}{2}$$

$$c_2' = -i\,\alpha\sin\frac{\theta}{2}$$

Probability

$$|c_1'|^2 = |\alpha|^2\cos^2\frac{\theta}{2} = \frac{1}{4}\cdot\frac{3}{4} = \frac{3}{16}$$

$$|c_2'|^2 = |\alpha|^2\sin^2\frac{\theta}{2} = \frac{1}{4}\cdot\frac{1}{4} = \frac{1}{16}$$

After 2nd stage

$$\begin{pmatrix} c_1'' \\ c_2'' \end{pmatrix} = \begin{pmatrix} \cos\left(-\frac{\theta}{2}\right) & -i\sin\left(-\frac{\theta}{2}\right) \\ -i\sin\left(-\frac{\theta}{2}\right) & \cos\left(-\frac{\theta}{2}\right) \end{pmatrix}\begin{pmatrix} c_1' \\ 0 \end{pmatrix} \xleftarrow{\theta} \text{Spin "down" (new frame) blocked}$$

if $\alpha = 1$ i.e. normalized after 1st block.

$$c_1'' = c_1'\cos\frac{\theta}{2}$$

$$c_2'' = c_1'\,|+i|\,\sin\frac{\theta}{2}$$

$$\boxed{|c_1''|^2 = \frac{3}{16}\cdot\frac{3}{4} = \frac{9}{64}} = \frac{9}{16}$$

$$\boxed{|c_2''|^2 = \frac{1}{16}\cdot\frac{1}{4} = \frac{1}{64}} = \frac{1}{16}$$

3. A hydrogen atom in its ground state is subjected to a time-dependent potential of the form:

$$V(\vec{x},t) = V_o \cos(k_o z - \omega t),$$

switched on adiabatically at $t = -\infty$. Using time-dependent perturbation theory, obtain the transition rate per unit solid angle for emission of the electron with momentum $\vec{p}$ in a direction defined by the angles θ and ϕ (referred to the z-axis).

Make the simplest possible approximation for the (properly normalized) final-state wave function. Comment on the validity or failure of your approximation.

Note: the wave function for the ground state of the hydrogen atom is

$$\psi_{100}(\vec{x}) = (\pi a_o^3)^{-1/2} \exp(-r/ao)$$

and its Fourier transform is

$$\psi_{100}(\underset{\sim}{k}) = 8\sqrt{\pi}\, a_o^{3/2} \frac{1}{(1 + k^2 a_o^2)^2}$$

Quantum Mechanics. (A.E.)

3. A hydrogen atom in its ground state is subjected to a time-dependent potential of the form:

$$V(\vec{x}t) = V_0 \cos(k_0 z - \omega t) \quad ,$$

switched on adiabatically at $t = -\infty$. Using time-dependent perturbation theory, obtain the transition rate per unit solid angle for emission of the electron with momentum $\vec{p}$ in a direction defined by the angles θ and φ (referred to the z-axis).

Make the simplest possible approximation for (properly normalized) final-state wave function. Comment on the validity or failure of your approximation.

~~Compare this problem briefly with the photoelectric effect.~~

Note: the wave function for the ground state of the hydrogen atom is

$$\psi_{100}(\vec{x}) = \frac{1}{\sqrt{\pi}} \frac{1}{a_0^{3/2}} e^{-r/a_0} \quad ,$$

and its Fourier transform is

$$\psi_{100}(\underset{\sim}{k}) = 8\sqrt{\pi}\, a_0^{3/2} \frac{1}{(1+k^2 a_0^2)^2} \quad .$$

~~Identify~~ a_0 ← this is identified in another q.m. problem! -GT

Solution:

$$\hat{V}(\vec{x}t) = \frac{V_0}{2}\left(e^{i(k_0 z - \omega t)} + e^{-i(k_0 z - \omega t)} \right)$$

The positive-frequency term ($\sim e^{-i\omega t}$) leads to absorption. (The other one leads to stimulated emission.) Thus we keep the first term only.

The transition rate for the atom to go from $|0\rangle$ to $|n\rangle$ is given by:

$$\Gamma_{0\to n} = \frac{2\pi}{\hbar} \left| \langle n | -e\frac{V_0}{2} e^{ik_0 \hat{z}} | 0 \rangle \right|^2 \delta(E_n - (E_0 + \hbar\omega)),$$

If the electron is ejected with momentum $\vec{p}$ in the direction (θ, φ) we must sum over the states subtended by the solid angle $d\Omega$. The true transition rate is then:

$$d\Gamma = \sum_{\substack{n \\ (\in d\Omega)}} \Gamma_{0\to n}$$

$$= \frac{2\pi e^2 \left| \langle n | V_0 e^{ik_0 \hat{z}} | 0 \rangle \right|^2}{\hbar \, 4} \rho(E_0 + \hbar\omega) \frac{d\Omega}{4\pi},$$

where $\rho(E)$ is the density of final states, or number of final states per unit energy.

The simplest approximation one can make about the final electron state is to set

$$\varphi_n(\underset{\sim}{x}) = \varphi_{\underset{\approx}{k}}(\underset{\sim}{x}) = \langle \underset{\sim}{x} | \underset{\sim}{k} \rangle = \frac{1}{L^{3/2}} e^{i\underset{\sim}{k}\cdot\underset{\sim}{x}} ,$$

i.e., we assume the electron to be entirely free (L is the side of our quantization box). This approximation is good at high energies; it breaks down near threshold.

of $\underset{\sim}{k}$-states in the volume d^3k is

$$\frac{L^3}{(2\pi)^3} d^3k = \frac{L^3}{(2\pi)^3} k^2 dk \, d\Omega$$

For free-electrons

$$E_k = \frac{\hbar^2 k^2}{2m} \rightarrow dE_k = \frac{\hbar^2}{m} k\,dk$$

$\therefore$ # of states in the interval $(E_k, E_k + dE_k)$, in the solid angle $d\Omega$ is given by

$$\frac{L^3}{2^{1/2}\, 4\pi^3} \frac{m^{3/2} E_k^{1/2} dE_k \, d\Omega}{\hbar^3}$$

$\therefore$ # of states per unit energy in $d\Omega$ is given by

$$\frac{L^3 m^{3/2} E_k^{1/2}}{2^{1/2} \pi^2 \hbar^3} \frac{d\Omega}{4\pi} = \rho(E_k) \frac{d\Omega}{4\pi}$$

Now:

$$\langle \underset{\sim}{k} | \, e^{i k_0 \hat{z}} \, | 0 \rangle =$$

$$= \int d^3x \, \frac{e^{-i \underset{\sim}{k} \cdot \underset{\sim}{x}}}{L^{3/2}} \, e^{i k_0 z} \, \psi_0(\underset{\sim}{x})$$

$$= \frac{1}{L^{3/2}} \int d^3x \, e^{-i (\underset{\sim}{k} - \underset{\sim}{k_0}) \cdot \underset{\sim}{x}} \, \psi_0(\underset{\sim}{x})$$

with $\underset{\sim}{k_0} = k_0 \hat{e}_z$

Then:

$$\langle \underset{\sim}{k} | \, e^{i k_0 \hat{z}} \, | 0 \rangle = \frac{1}{L^{3/2}} \, 8\pi^{1/2} a_0^{3/2} \, \frac{1}{\left[1 + | \underset{\sim}{k} - k_0 \hat{e}_z |^2 \right]^2}$$

Putting everything together:

$$d\Gamma = \frac{2\pi e^2}{\hbar} \, \frac{V_0^2}{4} \, \frac{1}{L^3} \, \frac{64 \pi a_0^3}{\left(1 + | \underset{\sim}{k} - k_0 \hat{e}_z |^2 \right)^4} \, \frac{L^3 m^{3/2} (E_0 + \hbar\omega)^{1/2}}{2^{1/2} \pi^2 \hbar^3} \times \frac{d\Omega}{4\pi}$$

Numerical factor =

$$= \frac{2\pi e^2}{\hbar} \; \frac{V_0^2}{4} \; \frac{1}{L^3} \; 64\pi a_0^3 \; \frac{L^3 \, m^{3/2}}{2^{1/2}\,\pi^2 \hbar^3} \; \frac{1}{4\pi} =$$

$$= \frac{128}{16\pi} \; \frac{1}{\sqrt{2}} \; \frac{e^2}{\hbar^4} \; V_0^2 \, a_0^3 \; m^{3/2} \qquad\qquad a_0 = \frac{\hbar^2}{m e^2}$$

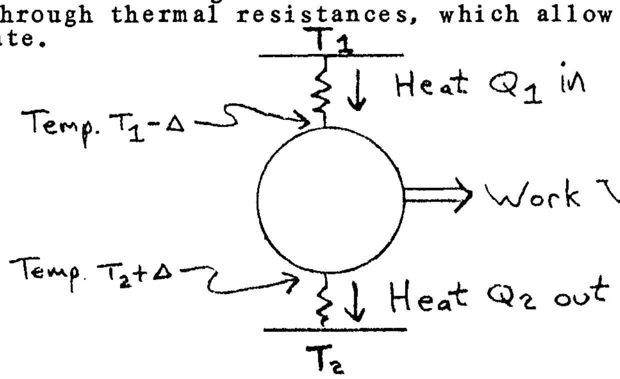
$$= \frac{8}{\sqrt{2}\,\pi} \; \frac{e^2}{\hbar^2 \hbar^2} \, m \; m^{1/2} \; V_0^2 \, a_0^3$$

$$= \frac{8}{\sqrt{2}\,\pi} \; \frac{m^{1/2}}{\hbar^2} \; V_0^2 \, a_0^2$$

$\therefore$

$$\frac{d\Gamma}{d\Omega} = \frac{8}{\sqrt{2}\,\pi} \; \frac{m^{1/2}}{\hbar^2} \; V_0^2 \, a_0^2 \; \frac{(E_0 + \hbar\omega)^{1/2}}{(1 + |\underset{\sim}{k} - k_0 \hat{e}_z|)^4}$$

4. A finite-speed Carnot engine is in contact with its heat reservoirs at T_1 and T_2 through thermal resistances, which allow the flow of heat at a finite rate.



$T_1 > T_2$

Assume:

1. The Carnot engine absorbs heat when it is at temperature $T_1-\Delta$, and rejects heat at $T_2+\Delta$ [note: $(T_1-\Delta) > (T_2+\Delta)$]

2. The total time for heat flow, for one cycle, is

$$t_1 + t_2 = \alpha \frac{Q_1}{\Delta} + \alpha \frac{Q_2}{\Delta}$$

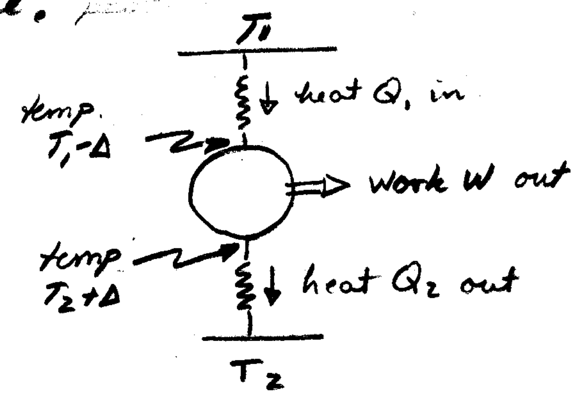
where α is a parameter which depends on thermal conductance.

3. The adiabatic portions of the Carnot cycle require no time.

For the case $T_2 = T_o$, $T_1 = 2T_o$ find the value of Δ which maximizes the engine's <u>power</u> output. What is the maximum power, and the efficiency, in this case?

– Thermo – (GT)

4\. A finite-speed Carnot engine is in contact with its ~~the~~ heat reservoirs at T_1 & T_2 through thermal resistances, which allow the flow of heat at a finite rate.



$T_1 > T_2$

Assume

① The Carnot engine absorbs heat when it is at temperature $T_1 - \Delta$, and rejects heat at $T_2 + \Delta$ $\left((T_1 - \Delta) > (T_2 + \Delta) \right)$

② The total time for heat flow, for one cycle, is

$$t_1 + t_2 = \alpha \frac{Q_1}{\Delta} + \alpha \frac{Q_2}{\Delta}$$

where α is a parameter which depends on thermal conductance.

③ The adiabatic portions of the Carnot cycle require no time.

For the case $T_2 = T_0$, $T_1 = 2T_0$ find the value of Δ ~~the~~ which maximizes the engine's power output. What is the maximum power, and the efficiency, ~~#~~ in this case?

Soln: The engine's efficiency is given by the usual Carnot expression

$$\eta = 1 - \frac{T_2 + \Delta}{T_1 - \Delta} = 1 - \frac{T_0 + \Delta}{2T_0 - \Delta} = \frac{T_0 - 2\Delta}{2T_0 - \Delta}$$

while Q_1 and Q_2 are related by

$$\frac{Q_1}{Q_2} = \frac{T_1 - \Delta}{T_2 + \Delta} = \frac{2T_0 - \Delta}{T_0 + \Delta}$$

→

The power is just $P = \frac{W}{t_1+t_2} = \frac{Q_1 - Q_2}{t_1+t_2}$

or:

$$P = \frac{Q_1\left(1 - \frac{T_0+\Delta}{2T_0-\Delta}\right)}{\frac{\alpha}{\Delta} Q_1\left(1 + \frac{T_0+\Delta}{2T_0-\Delta}\right)} = \frac{\Delta}{\alpha}\,\frac{T_0 - 2\Delta}{3T_0}$$

Maximizing w/ resp. to Δ gives:

$$\frac{dP}{d\Delta} = 0 = \frac{1}{\alpha}\,\frac{T_0-2\Delta}{3T_0} - \frac{\Delta}{\alpha}\,\frac{2}{3T_0} \Rightarrow T_0 - 2\Delta = 2\Delta$$

$$\boxed{\Delta = \frac{T_0}{4}}$$

Then $P_{max} = \frac{T_0/4}{\alpha} \cdot \frac{T_0 - \frac{T_0}{2}}{3T_0} = \frac{T_0}{24\alpha}.$

and $\eta = \frac{T_0 - 2\Delta}{2T_0 - \Delta} = \frac{1/2}{7/4} = \frac{2}{7}.$

DEPARTMENT OF PHYSICS

Ph.D. COMPREHENSIVE EXAMINATION

TUESDAY, SEPTEMBER 23, 1986, 2-5 PM

Answer each of the following questions, all of which carry equal weight. Begin your answer to each question on a new sheet of paper; solutions to different questions must <u>not</u> appear on the same sheet of paper. Each sheet of paper must be labeled with your name and the problem number in the upper right hand corner of the page. If more than one sheet is submitted for a problem, be sure the pages are ordered properly.

1. Evaluate the following integral

$$\int_{-\infty}^{+\infty} dz \ \frac{\exp(-i\lambda z)}{z^2 + z_o^2} \ \cos\,[(z^2 + z_o^2)^{1/2}]$$

for $\lambda > 1$ (z_o is a real number)

Mathematical Physics. (A.E.)

1. Evaluate the following integral

$$\int_{-\infty}^{+\infty} dz \; \frac{e^{-i\lambda z}}{z^2 + z_0^2} \cos\left(z^2 + z_0^2\right)^{1/2},$$

for $\lambda > 1$ (z_0 is a real number)

Students who simply forget about the branch cut will (unfortunately) get the correct answer. 4π.

Solution

On the imaginary axis : $z = i z_i$

$$e^{-i\lambda i z_i} = e^{\lambda z_i} \underset{z_i \to -\infty}{\longrightarrow} 0$$

$$\cos\left[(i z_i)^2 + z_0^2\right]^{1/2} = \cos\left[-z_i^2 + z_0^2\right]^{1/2} =$$

$$= \frac{1}{2}\left[e^{i(-z_i^2+z_0^2)^{1/2}} + e^{-i(-z_i^2+z_0^2)^{1/2}}\right]$$

for $z_i \to -\infty$:

$$\cos\left[-z_i^2 + z_0^2\right]^{1/2} \to \frac{1}{2}\left[e^{i(-i)z_i} + e^{-i(-i)z_i}\right]$$

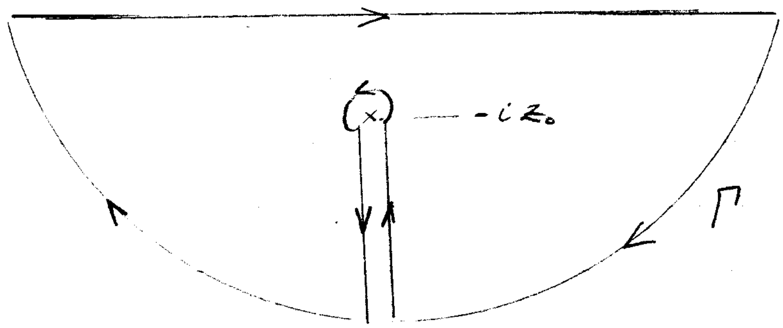
$$= \frac{1}{2}\left[e^{z_i} + e^{-z_i}\right]$$

Then:

$$e^{-i\lambda(i z_i)} \cos\left[(i z_i)^2 + z_0^2\right]^{1/2} \underset{z_i \to -\infty}{\longrightarrow} \frac{1}{2}\left[e^{(\lambda+1) z_i} + e^{(\lambda-1) z_i}\right]$$

and both terms decay exponentially for $\lambda > 1$.

We can then close the contour on the lower half plane. Call Γ the entire closed contour.



Cauchy's theorem:

$$\oint_{\Gamma} dz \quad \frac{e^{-i\lambda z} \cos(z^2+z_0^2)^{1/2}}{z^2+z_0^2} = 0$$

since there are no poles enclosed by Γ.

1. Integral around the branch point at $z=-iz_0$:

Set $\quad z = -iz_0 + \rho e^{i\theta} \qquad\qquad dz = i\rho e^{i\theta} d\theta$

$\rho \to 0$

$-\frac{3\pi}{2} \le \theta \le \frac{3\pi}{2}$

This contribution to the total $\oint_{\Gamma} dz$ is:

$$\int_{-\pi/2}^{3\pi/2} d\theta \; i\rho e^{i\theta} \; \frac{e^{-i\lambda(-iz_0+\rho e^{i\theta})} \cos[-2i\rho e^{i\theta} + \rho^2 e^{2i\theta}]^{1/2}}{\cancel{-z_0^2} - 2iz_0\rho e^{i\theta} + \rho^2 e^{2i\theta} \cancel{+z_0^2}}$$

$$\xrightarrow[\rho\to 0]{} \int_{-\pi/2}^{3\pi/2} d\theta \; \frac{e^{-\lambda z_0}}{-2z_0} = \frac{e^{-\lambda z_0}}{-2z_0}\left(\frac{3\pi}{2}+\frac{\pi}{2}\right) = \frac{2\pi}{-2z_0} e^{-\lambda z_0}$$

$$= -\frac{\pi}{z_0} e^{-\lambda z_0}$$

2. Integral along the branch line :

$$z = -iz_0 + \rho e^{i\theta} \qquad 0 \le \rho \le \infty$$

a) For the upward path : $\theta = -\frac{\pi}{2}$

$$e^{i\theta} = e^{-i\pi/2} = \cos\frac{\pi}{2} - i\sin\frac{\pi}{2} = -i$$

$$\therefore \quad z = -iz_0 - i\rho = -i(z_0 + \rho)$$

b) For the downward path : $\theta = \frac{3\pi}{2}$

$$e^{i\frac{3\pi}{2}} = \cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2} = -i$$

$$\therefore \quad z = -iz_0 - i\rho = -i(z_0 + \rho)$$

The only function in the integrand which depends on whether we are on the upward or downward path about the branch cut is the argument of the cosine, namely $[z^2 + z_0^2]^{1/2}$. However, for a given value of ρ on either side of the branch cut, the square root differs only in sign. Since $\cos -x = \cos x$, we have that the full integrand is the same for either path.

Thus the integrals over the upward and lower paths cancel each other out.

We conclude that

$$\oint_{\Gamma} dz \ldots \quad = 0 = \int_{-\infty}^{+\infty} dz \quad e^{-i\lambda z} \frac{\cos(z^2+z_0^2)^{1/2}}{z^2+z_0^2}$$

$$+ \quad (-) \frac{\pi}{z_0} e^{-\lambda z_0}$$

$$\Rightarrow \quad \int_{-\infty}^{+\infty} dz \, \frac{e^{-i\lambda z} \cos(z^2+z_0^2)^{1/2}}{z^2+z_0^2} = \frac{\pi}{z_0} e^{-\lambda z_0}$$

2. A particle with mass m and charge q is injected into a region with a uniform electric field $\vec{E} = E_0\hat{y}$. It is injected at t=0 with initial conditions $x = y = o$, $\vec{v} = v_0\hat{x}$. Find the shape of its trajectory for $t>0$; i.e., find $y(x)$.

<u>You may not assume that the particle's velocity is always small compared to the speed of light!</u>

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~~Electromagnetism~~

c. A particle with mass m and charge q is injected into a region with a uniform electric field $\vec{E} = E\hat{y}$. It is injected at $t=0$ with initial condition $x=y=0$ $\vec{V} = V_0 \hat{x}$. Find the shape of its trajectory for $t>0$; i.e., find $y(x)$.

You may not assume that the particle's velocity is always small compared to the speed of light!

Solution:

Lorentz Force Law: $m\dfrac{du^\mu}{d\tau} = q F^\mu{}_\nu u^\nu$ $\qquad u^\mu \equiv \dfrac{dx^\mu}{d\tau}$

$\tau \equiv$ proper time

Only nonzero components of $F^\mu{}_\nu$ are $F^t{}_y = F^y{}_t = E$

$\mu = t$

$$\frac{du^t}{d\tau} = \frac{q}{m} E u^y \quad (1)$$

$\mu = x$ $\quad \dfrac{du^x}{d\tau} = 0 \Rightarrow u^x = \text{constant} = \gamma_0 V_0 \qquad \gamma_0 \equiv (1-V_0^2)^{-\frac{1}{2}}$

$$\Rightarrow \boxed{x = \gamma_0 V_0 \tau} \quad \& \quad \tau = x/\gamma_0 V_0 \quad (2)$$

$\mu = y$ $\quad \dfrac{du^y}{d\tau} = \dfrac{q}{m} E u^t \quad (3)$

Differentiate (3), use (1) to eliminate u^t:

$$\frac{d^2u^y}{d\tau^2} = \left(\frac{qE}{m}\right)^2 u^y \Rightarrow u^y = A\exp\left(\frac{qE}{m}\tau\right) + B\exp\left(\frac{-qE}{m}\tau\right)$$

at $t=x=y=\tau=0$ $\quad u^y=0 \Rightarrow A=-B \Rightarrow u^y = A\sinh\left(\frac{qE}{m}\tau\right)$

now use (3) to fix A: at $\tau=0$ $\quad u^t = \gamma_0$

$$\Rightarrow A\frac{q\cancel{E}}{m} = \frac{q\cancel{E}}{m}\gamma_0 \Rightarrow A = \gamma_0$$

so

$$\boxed{u^y(\tau) = \gamma_0 \sinh\left(\frac{qE}{m}\tau\right)} \quad (4)$$

now integrate (4) to find

$$y(\tau) = y_0 + \frac{m\gamma_0}{qE} \cosh\left(\frac{qE}{m}\tau\right)$$

at $\tau = 0$ $y = 0$ $\Rightarrow$ $y_0 = -m\gamma_0/qE$. Finally, use (2) to replace τ by $x/\gamma_0 v_0$:

$$\boxed{y(x) = \frac{m\gamma_0}{qE}\left[\cosh\left(\frac{qE}{m\gamma_0 v_0}x\right) - 1\right]}$$

Probably a good problem – I ~~was~~ did have ~~had~~ difficulty with it.
JFD

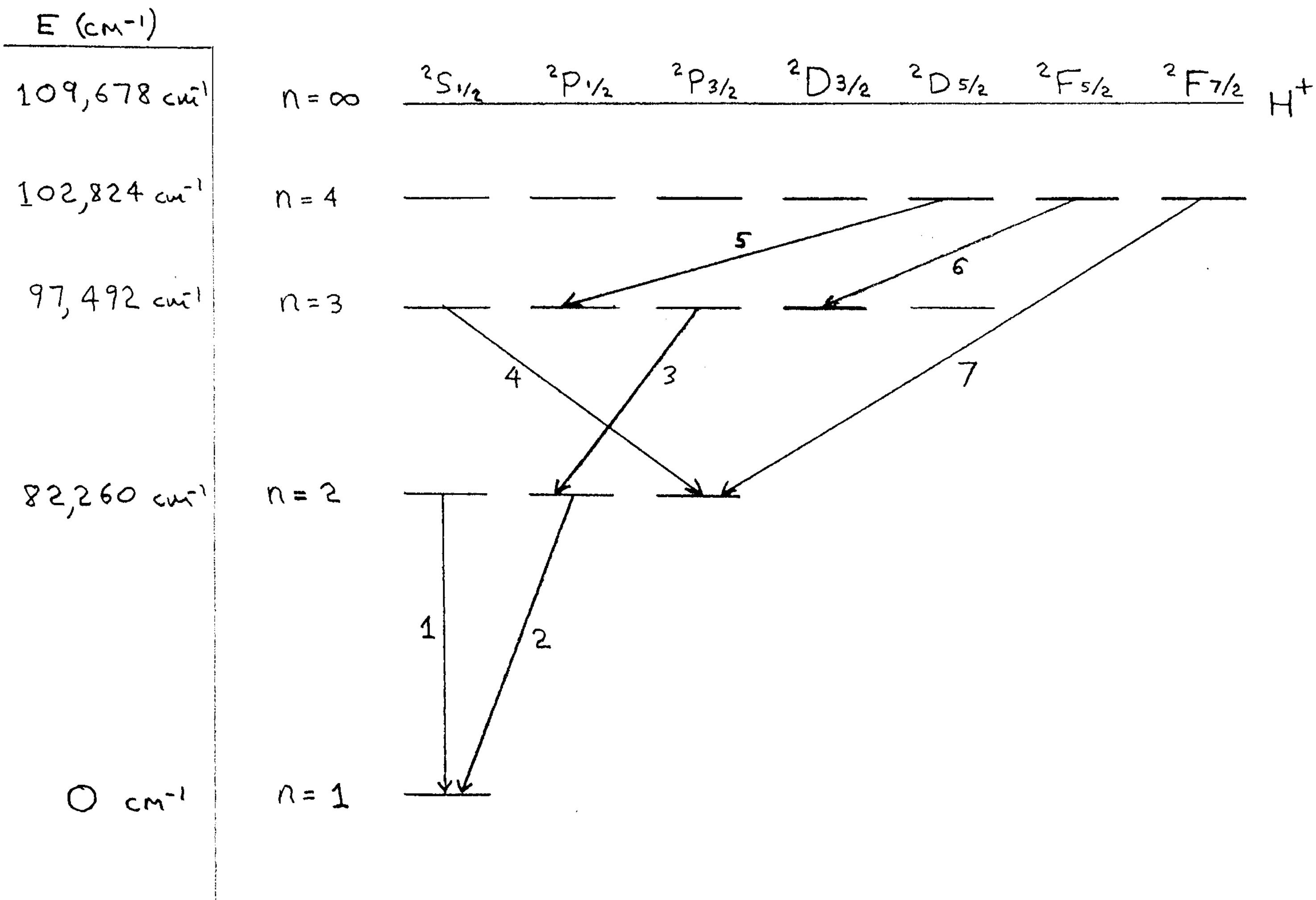
me, too – GT

3. The approximate term energies for hydrogen are shown below. Fine structure is ignored.

a) which transitions are allowed for one-photon, electric dipole transitions?

b) which transitions are allowed for two-photon, electric dipole transitions?

c) Assuming that transitions 1,3,6,7 are allowed, determine the wavelength (in Å) for the emissions. Also state whether these transitions will be observable to the eye.



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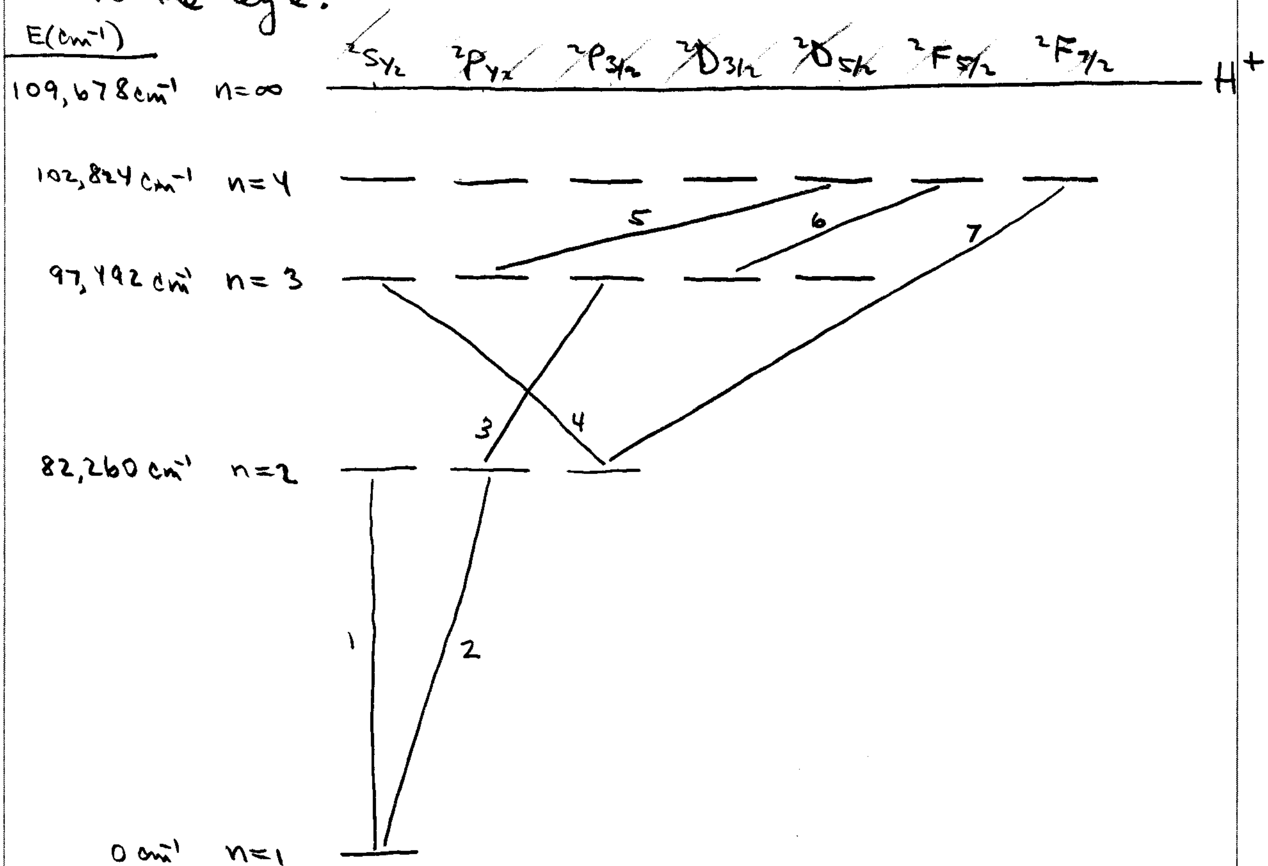


Hydrogen Energy Levels

Solution Atomic Physics

a) the selection rules for one-photon electric dipole transitions are $\Delta L = \pm 1$ $\Delta J = 0, \pm 1$

therefore transitions 2,4,6 are allowed.

b) For two-photon, electric dipole transitions

$\Delta L = 0, \pm 2$ and $\Delta J = 0, \pm 1, \pm 2$

therefore transitions 1,3,7 are allowed.

c)

transition	λ	Observable?
1	1216 Å	No, this is in the ultraviolet.
3	6565 Å	Yes, this is red
6	18,755 Å	No, this is in the infrared.
7	4863 Å	Yes, this is blue.

$$\lambda_{n,n'} = (E_n - E_{n'})^{-1} \qquad 1\text{Å} = 10^{-8}\ \text{cm}$$

Visible spectrum is ~ 7000 Å – 4000 Å

4. A polymer consists of strands of molecular moieties linked together more or less like a tangled chain. Some polymers such as polyacetylene can be pictured, at least locally, as having a one dimensional topography. For the purposes of discussing the electronic structure of such a one dimensional system, a tight binding Hamiltonian is often applied:

$$\hat{H} = \sum_{\ell=1}^{N} \varepsilon_\ell \ \hat{c}_\ell^\dagger \ \hat{c}_\ell - \sum_{\ell=1}^{N} t_{\ell+1,\ell}\left[\hat{c}_{\ell+1}^\dagger \ \hat{c}_\ell + \hat{c}_\ell^\dagger \ \hat{c}_{\ell+1}\right]$$

Here, $\hat{H}$ is the second quantized Hamiltonian for the electronic system in the single particle approximation. ε_ℓ is the energy of an electron at the site ℓ. $\hat{C}_\ell$ is the annihilation operator for an electron on site ℓ. $t_{\ell+1,\ell}$ is the real valued transfer integral which measures the ability of an electron to hop to a neighboring site. The polymer is viewed as having N sites, where a site can be considered as the individual moiety, or unit cell, building block of the polymer. The polymer has total length L. Determine and sketch the band structure of this Hamiltonian. To do this one should canonically transform the Hamiltonian to use Bloch states with periodic boundary conditions. Assume translational symmetry so that $\varepsilon_\ell = \varepsilon_o$, and $t_{\ell+1,\ell} = t_o$ for all ℓ. Finally, denote the Fermi energy position in your sketch assuming each unit cell contributes only one electron.

Ph.D. Comps. Wayne Ford (Sol. St.

WGF

4. A polymer consists of strands of molecular ~~moities~~ moieties linked together more or less like a ~~chain~~ tangled chain. Some polymers such as polyacetylene can be pictured, at least locally, as having a one dimensional topography. For the purposes of discussing the electronic structure of such a one dimensional system, a tight binding Hamiltonian is often applied:

$$\hat{H} = \sum_{l=1}^{N} \varepsilon_l \hat{c}_l^{+} \hat{c}_l - \sum_{l=1}^{N} t_{l+1,l} \left[\hat{c}_{l+1}^{+} \hat{c}_l + \hat{c}_l^{+} \hat{c}_{l+1} \right].$$

Here, $\hat{H}$ is the second quantized Hamiltonian for the electronic system in the single particle approximation, ε_l is the energy of an electron at the site l. $\hat{c}_l$ is the annihilation operator for an electron on site l. $t_{l+1,l}$ is the (real valued) transfer integral which measures the ability of an electron to hop to a neighboring site. The polymer is viewed as having N sites, where ~~sites~~ a site can be considered as the individual moiety, or unit cell, ~~of~~ building block of the polymer. The polymer has total length L.

Determine and sketch the band structure of this ~~system~~ Hamiltonian. To do this one should canonically transform the Hamiltonian to use Bloch states with ~~Assume~~ periodic boundary conditions. Assume translational symmetry so that $\varepsilon_l = \varepsilon_0$, ~~[illegible]~~ and $t_{l+1,l} = t_0$ for all l. Finally, denote the Fermi energy position in your sketch assuming ~~only one~~ each unit cell contributes ~~to the~~ only one electron.

$$\hat{c}_l = \frac{1}{\sqrt{N}} \sum_n \hat{a}_n e^{i2\pi n l/N}$$

$$\hat{c}_l^+ = \frac{1}{\sqrt{N}} \sum_n \hat{a}_n^+ e^{-i2\pi n l/N}$$

$$\hat{c}_l^+ c_l = \frac{1}{N} \sum_{nn'} \hat{a}_{n'}^+ \hat{a}_n e^{-i2\pi n' l/N} e^{i2\pi n l/N}$$

$$\sum_l \varepsilon_l \hat{c}_l^+ c_l = \frac{1}{N} \varepsilon_0 \sum_{nn'} \hat{a}_{n'}^+ \hat{a}_n \underbrace{\sum_l e^{i2\pi(n-n')l/N}}_{= N\delta_{nn'}}$$

$$= \varepsilon_0 \sum_n \hat{a}_n^+ \hat{a}_n$$

$$\hat{c}_{l+1}^+ \hat{c}_l = \frac{1}{N} \sum_{n'} \sum_n \hat{a}_{n'}^+ \hat{a}_n e^{-i2\pi n'(l+1)/N} e^{i2\pi n l/N}$$

$$\sum_l t_{l+1,l} \hat{c}_{l+1}^+ \hat{c}_l = \frac{1}{N} t_0 \sum_{n'} \sum_n \hat{a}_{n'}^+ \hat{a}_n \sum_l e^{i2\pi(n-n')l/N} e^{-i2\pi n'/N}$$

$$= t_0 \sum_n \hat{a}_n^+ \hat{a}_n e^{-i2\pi n/N}$$

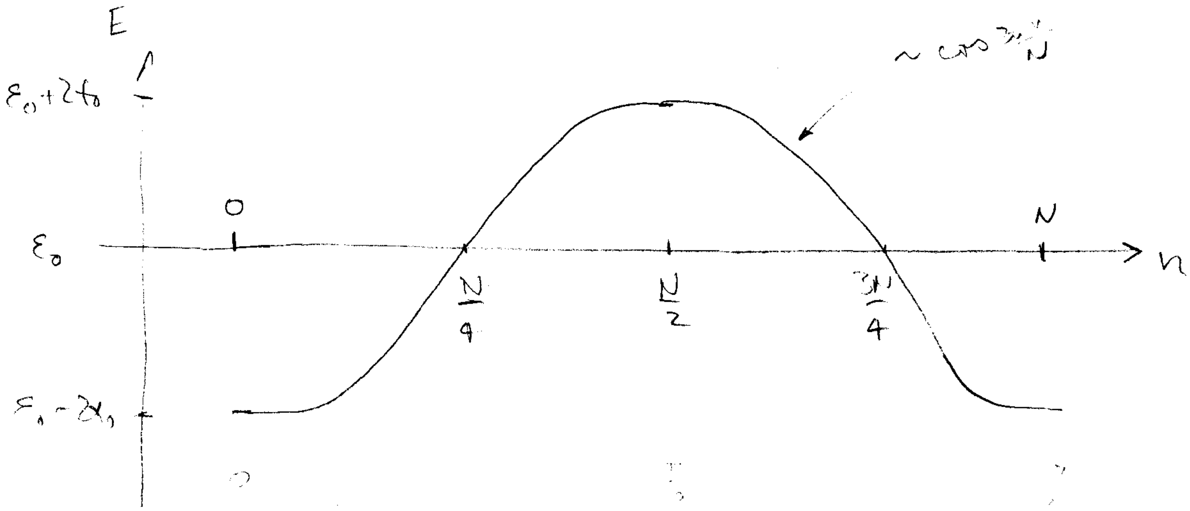
$$\{\ \} + \text{h.c.} = 2t_0 \sum_n \hat{a}_n^+ \hat{a}_n \cos 2\pi \frac{n}{N}$$

Thus

$$\hat{H} = \sum_{n=1}^{N} \left(\varepsilon_0 - 2t_0 \cos 2\pi \frac{n}{N} \right) \hat{a}_n^+ \hat{a}_n$$

The eigenstates have energies

$$\varepsilon_n = \varepsilon_0 - 2t_0 \cos 2\pi \frac{n}{N}$$



$\varepsilon_F = \varepsilon_0$ since there are $\frac{N}{2}$ levels below.

Let $k = \frac{2\pi n}{Na}$, Reduced zone scheme