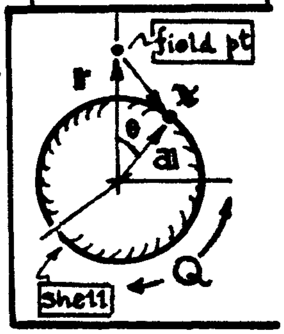
ϕ 519 Problems Assigned 9/6/91. Due 9/13/91.

④ A photon at ∡ freq. ω and wave# $k=\frac{2\pi}{\lambda}$ has mass $m_\gamma>0$. Use the relativistic mass-energy eqtn plus Einstein relations to show: $\boxed{\omega^2 = k^2c^2+\omega_0^2}$ (c = light speed & ω_0 = cnst). Find ω_0 numerically for $m_\gamma = 4\times10^{-48}$ gm (Jkⁿ, p. 6). Calculate the photon's group velocity $v_g=\frac{\partial\omega}{\partial k}$, and sketch v_g vs ω over $0\leq\omega\to\infty$. At what wavelength λ does v_g fall below $0.9999\,c$?

⑤ [25 pts]. What if Coulomb's Law were not inverse square? For point charges q & q' separated by distance r, suppose the force were: $\mathbf{F}=[qq'f(r)]\hat{r}$, where $f(r)$ decreases with r and $f(\infty)=0$, but f is otherwise arbitrary. $\mathbf{F}$ is radial, so we can still define a potential: $V(x)=q\int_x^\infty f(\xi)\,d\xi$, at distance x from q, and -- for an assembly of charges dq_i: $V=\sum_i dq_i\int_{x_i}^\infty f(\xi)\,d\xi$. Consider a uniformly charged conducting spherical shell of radius a and surface charge density $\sigma = Q/4\pi a^2$; we wish to find its potential. Let the field point be at $\mathbf{r}$ on the z-axis; treat $r>a$ (outside shell) and $r<a$ (inside shell) separately. With x the distance from field point to shell charge element, the potential at r is: $V(r)=\iint_{\text{shell}}\sigma\left\{\int_x^\infty f(\xi)\,d\xi\right\}a^2\sin\theta\,d\theta\,d\phi$. Check this form for good sense.



(5) (A) Do the ϕ integration for $V(r)$. Then show: $\boxed{V(r)=(Q/2ar)\int_L^U\left\{\int_x^\infty f(\xi)\,d\xi\right\}x\,dx}$; use the law of cosines. The limits U & L depend on $r \lessgtr a$. Find U & L for both $r\gtrless a$.

(5) (B) Put $f(\xi)=(1/\xi^2)\,h(\xi)$ in part (A). If $h(\xi)$ = cnst, show that the shell looks like a point charge outside ($r>a$), and inside $V(r<a)$ is everywhere cnst. Conversely, if $h(\xi)\neq$ cnst, neither of these "well-known" inverse-square-law results holds.

(7) (C) Suppose the Coulomb departure is: $h(\xi)=(\lambda/\xi)^\delta$, with λ a scale length and the exponent $|\delta|\ll1$. Show: $\boxed{V(r)\simeq(Q/2ar)\left\{(U-L)-\left[U\ln\left(\frac{U}{\lambda}\right)-L\ln\left(\frac{L}{\lambda}\right)\right]\delta\right\}}$, to 1st order in δ. Analyse: $\Delta V(r)=V(r)|_{\delta=0}-V(r)|_{\delta\neq0}$: find limiting forms for $\Delta V(r)$ when $r\ll a$, $r=a$, and $r\gg a$. Sketch a graph of $\Delta V(r)$ vs. r over $0\leq r\to\infty$.

(8) (D) Inside the shell, write: $V(r)\simeq\frac{Q}{a}[1-g(\rho)\delta]$, with $\rho=\frac{r}{a}$. Find $g(\rho)$. Suppose you had a shell of radius a = 50 cm, charged to 10 kV potential, and you detect ΔV's to an accuracy of ±1 μV over $\Delta r\sim1$ cm. What limits can you put on δ and/or λ?

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4 Analyse dispersion relation for a massive photon.

1) For any "particle" of mass m, momentum p, and total energy E, the relativistic mass-energy eqtn prescribes: $E^2 = p^2c^2 + m^2c^4$, c = light speed. For a photon (ω, k), the Einstein relations are: $(E, p) = \hbar(\omega, k)$, where $\hbar = 1.0546 \times 10^{-27}$ erg-sec is Planck's cnst. A simple plug-in yields

$$\rightarrow \omega^2 = k^2c^2 + \omega_0^2, \quad \omega_0 = m_\gamma c^2/\hbar = \text{cnst}, \qquad (1)$$

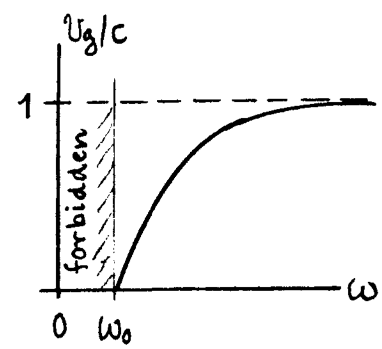
as desired. ω_0 is a very small frequency; numerically (from Jk$^{\underline{h}}$ p. 6)

$$\omega_0 = c\Big/\left(\frac{\hbar}{m_\gamma c}\right) \lesssim 3\times 10^{10}\,\frac{\text{cm}}{\text{sec}}\Big/10^{10}\,\text{cm} = \underline{3\,\text{Hz}}, \text{ about } 2\pi\times\frac{1}{2}\frac{\text{cycle}}{\text{sec}}. \qquad (2)$$

2) Since: $\omega = \sqrt{k^2c^2 + \omega_0^2}$ (and $kc = \sqrt{\omega^2 - \omega_0^2}$), some simple arithmetic gives

$$\left.\begin{matrix}\text{group}\\ \text{velocity}\end{matrix}\right\} v_g = \frac{\partial\omega}{\partial k} = kc^2\Big/\sqrt{k^2c^2+\omega_0^2} = c\sqrt{1-(\omega_0^2/\omega^2)},$$

$$\xrightarrow{\text{or}} \frac{v_g}{c} = \sqrt{1-(\omega_0^2/\omega^2)}. \qquad (3)$$



Notice that $v_g < c$ at any finite freq. ω, and in fact v_g falls to zero at $\omega = \omega_0$. At freqs. $\omega < \omega_0$, v_g is imaginary, and the photon does not propogate; $0 \leqslant \omega < \omega_0$ is a forbidden region of frequencies for the massive photon. Requested graph is shown.

3) From Eq. (3): $\frac{v_g}{c} = 1 - \epsilon$, when: $\omega_0^2/\omega^2 = 2\epsilon - \epsilon^2 \simeq 2\epsilon$ (for ϵ small). Thus v_g falls a fraction ϵ below c when the freq. ω reaches

$$\rightarrow \omega_\epsilon \simeq \omega_0/\sqrt{2\epsilon}, \quad \text{for } v_g = (1-\epsilon)c. \qquad (4)$$

Given $v_g = 0.9999c$, $\epsilon = 10^{-4}$, and Eq. (4) gives $\omega_\epsilon \simeq 70.7\,\omega_0$. With Eq. (2) the linear freq. is $\nu_\epsilon = \omega_\epsilon/2\pi = 33.8$ Hz. The associated wavelength is: $\underline{\lambda_\epsilon = c/\nu_\epsilon = 8890\text{ km}}$. This is ~ distance from Calcutta to the North Pole. Such a wave would be hard to measure!

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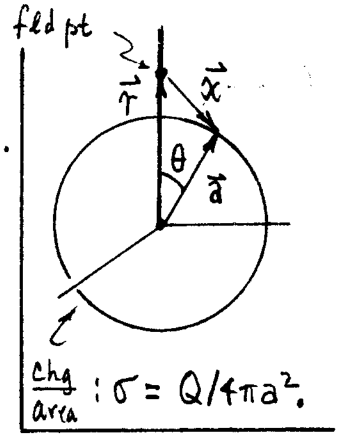


(25 pts) Find Shell potential for non-Coulombic force law.

A. In the given form for $V(r)$, do the ϕ integration $(0 \le \phi \le 2\pi)$, and put in $\sigma = Q/4\pi a^2$. Then...

5 pts

$$V(r) = \frac{Q}{2}\int_0^{\pi}\left(\int_x^{\infty} f(\xi)\,d\xi\right)\sin\theta\,d\theta. \tag{1}$$

Here, $x = x(\theta)$, via ; $x^2 = a^2 + r^2 - 2ar\cos\theta$. Since both a & r are fixed during the integration over the shell, in fact : $x\,dx = ar\sin\theta\,d\theta$ (differential form of law of cosines). Using this in Eq. (1), have immediately...

$$\boxed{V(r) = \frac{Q}{2ar}\int_L^U\left\{\int_x^{\infty} f(\xi)\,d\xi\right\}x\,dx} \quad \begin{cases} U, L = r \pm a, & \text{for } r > a \text{ (outside)}, \\ U, L = a \pm r, & \text{for } r < a \text{ (inside)}. \end{cases} \tag{2}$$

The limits U & L are the max & min values of x corresponding to the range $\pi \geqslant \theta \geqslant 0$.

B. Insert $f(\xi) = \frac{1}{\xi^2}h(\xi)$ in Eq. (2), and partial integrate...

5 pts

$$\int_x^{\infty} f(\xi)\,d\xi = \frac{1}{x}h(x) + \int_x^{\infty}\frac{d\xi}{\xi}h'(\xi),$$

So// $$V(r) = \frac{Q}{2ar}\int_L^U\left[h(x) + x\int_x^{\infty}\frac{d\xi}{\xi}h'(\xi)\right]dx. \tag{3}$$

Suppose $h(\xi) = C$, cnst, so that $h'(\xi) \equiv 0$. Then the potential is...

$$V(r) = \frac{QC}{2ar}\int_L^U dx = \frac{QC}{2ar}(U - L) = \begin{cases} QC/r, & r > a \text{ (outside)}, \\ QC/a, & r < a \text{ (inside)}. \end{cases} \tag{4}$$

Of course we would choose $C = 1$, in cgs units. Then $V = Q/r$ = point charge potential, everywhere outside, while $V = Q/a$ = cnst, everywhere inside shell. These are the characteristic features of an inverse-square law force.

C. If $h(\xi) = (\lambda/\xi)^{\delta} \Rightarrow f(\xi) = \lambda^{\delta}/\xi^{2+\delta}$, two trivial integrations give...

7pts $$\int_x^{\infty} f(\xi)\, d\xi = \frac{1}{1+\delta}\,\frac{1}{x}\left(\frac{\lambda}{x}\right)^{\delta},$$

and// $$V(r) = \frac{Q}{2ar}\left(\frac{1}{1-\delta^2}\right)\left[\left(\frac{\lambda}{U}\right)^{\delta} U - \left(\frac{\lambda}{L}\right)^{\delta} L\right]. \quad (5)$$

For $|\delta| \ll 1$: $N^{\delta} = e^{\delta \ln N} \simeq 1 + \delta \ln N$, to 1^{st} order in δ (and useful even if N varies over many orders of magnitude). Then, to $O(\delta)$, as advertised...

$$\boxed{V(r) \simeq \frac{Q}{2ar}\left\{(U-L) - \delta\left[U \ln(U/\lambda) - L \ln(L/\lambda)\right]\right\}} \quad (6)$$

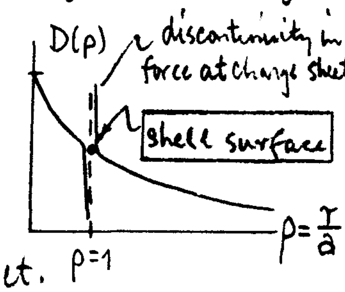
The "departure fcn": $\Delta V(r) = V(r)\big|_{\delta=0} - V(r)\big|_{\delta\neq 0}$ is easily found to be...

$$\Delta V(r) = \frac{Q\delta}{2a} D(r), \quad D(r) = \frac{1}{r}\left[U \ln(U/\lambda) - L \ln(L/\lambda)\right], \int \begin{matrix} U, L = r \pm a, \text{outside;} \\ U, L = a \pm r, \text{inside.} \end{matrix}$$

i.e.// $$D(\rho) = \begin{cases} \left(1+\frac{1}{\rho}\right)\ln\left(\frac{\rho+1}{n}\right) - \left(1-\frac{1}{\rho}\right)\ln\left(\frac{\rho-1}{n}\right), & \underline{\rho} = r/a > 1; \quad (7) \\ \left(\frac{1}{\rho}+1\right)\ln\left(\frac{1+\rho}{n}\right) - \left(\frac{1}{\rho}-1\right)\ln\left(\frac{1-\rho}{n}\right), & \rho = r/a < 1; \text{ and } \underline{n} = \frac{\lambda}{a}. \end{cases}$$

Using: $\lim_{x\to 0} x \ln x = 0$, and $\ln(1\pm\epsilon) \simeq \pm\epsilon$ as $\epsilon \to 0$, we straightforwardly get:

$$\begin{cases} D(\rho) \simeq 2\left[\ln(1/n) + 1\right] - \frac{1}{3}\rho^2, & \text{for } \rho \ll 1; \quad (8) \\ D(\rho) \simeq \frac{2}{\rho}\left[\ln(\rho/n) + 1\right], & \rho \gg 1; \quad D(\rho=1) = 2\ln\left(\frac{2}{n}\right). \end{cases}$$



$D(\rho)$ decreases with ρ, with usual $D'(\rho)$ discontinuity at the sheet.

8pts **D.** With reference to Eq. (7) above, and inside the shell, we can write...

$$V(r) \simeq \frac{Q}{a}\left[1 - \delta g(\rho)\right], \quad \underline{g(\rho)} = \frac{1}{2} D(\rho) = \frac{1}{2}\left[\left(\frac{1}{\rho}+1\right)\ln\left(\frac{1+\rho}{n}\right) - \left(\frac{1}{\rho}-1\right)\ln\left(\frac{1-\rho}{n}\right)\right], \quad (9)$$

where: $\rho = r/a \leq 1$, and: $n = \lambda/a$. If we measure potential differences ΔV over

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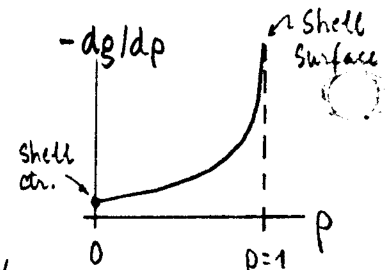
distances Δr which are "small" relative to the shell radius a, then we can determine the ratio...

$$\Delta V/\Delta r \simeq -\frac{Q\delta}{a}(\Delta g/\Delta r) \simeq -\frac{V_s\,\delta}{a}(dg/d\rho), \tag{10}$$

where: $V_s = Q/a$, is the potential at the shell surface. Notice that this measurement is indpt of the scale distance λ, since the derivative -- calculated via

$$g(\rho) = \frac{1}{2}\left[\frac{1}{\rho}\ln\left(\frac{1+\rho}{1-\rho}\right) + \ln\left(\frac{1-\rho^2}{\lambda^2}\right)\right], \quad 0 \leq \rho \leq 1;$$

is//
$$\frac{dg}{d\rho} = \frac{1}{\rho}\left[1 - \frac{1}{2\rho}\ln\left(\frac{1+\rho}{1-\rho}\right)\right] \simeq \begin{cases} -\rho/3, \text{ as } \rho \to 0, \\ \ln\sqrt{1-\rho}, \text{ as } \rho \to 1. \end{cases} \Big\rangle \underline{\text{indpt. of } \lambda}. \tag{11}$$



$dg/d\rho \to (-)\infty$ as $\rho \to 1-$ reflects the discontinuity in the force as we pass through the charge sheet. But here of course we are working over finite intervals Δr, so we cannot pass to $\rho = 1$; in fact, we can get down to a "fineness" of perhaps $\rho = 0.99$, so the max value of $-(dg/d\rho)$ is $-(dg/d\rho)|_{\rho=0.99} = 1.69$. At the other end, $\rho \to 0$, we can go to $\rho = 0.01$ (i.e. resolve $\Delta r \simeq 1/2$ cm out of $a = 50$ cm), so the min. is $-(dg/d\rho)|_{\rho=0.01} = 0.01/3$. Now solve Eq. (10) for δ, to get...

$$\rightarrow \delta \simeq \left(\frac{\Delta V}{V_s}\right)\frac{a}{\Delta r}\Big/\left[-(dg/d\rho)\right]. \tag{12}$$

We see that for max. sensitivity, we should work near the sphere center, where $-(dg/d\rho)$ is smallest. With $V_s = 10$ KV, $\Delta V = \pm 1\,\mu V$, $a = 50$ cm and $\Delta r = 1$ cm, and the $-(dg/d\rho)|_{\rho=0.01} = 0.01/3$ estimate from above, this gives $\underline{\delta \simeq \pm 1.5 \times 10^{-10}}$ as a limit of sensitivity for this expt. Plimpton & Lawton (Phys. Rev. 50, 1066 (1936)) did about this well, ~50 years ago.