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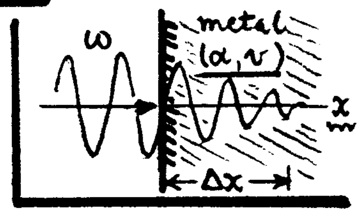
1992 ϕ520 Final Exam (in class, 3hrs.) Sat. 9 May 1992

This exam is open-book, open notes, and is worth 240 points total. There are six problems on 2 pages, with point-values as marked. For each problem, put a $\boxed{\text{box}}$ around your answer. Number your solution pages consecutively, write your name on page 1, and staple the pages together before handing them in.



① [40 pts.]. An EM plane wave at frequency ω strikes a metal surface at normal incidence, penetrates, and propagates inside the metal via a 1D wave eqtn: $u_{xx} - \alpha u_t - (1/v^2)\, u_{tt} = 0$.
Here u is any component of the wave's $\mathbb{E}$-field, α is a constant (at low ω) proportional to the metal's conductivity, and v is the wave velocity inside the metal. If α is "large", find the characteristic depth Δx to which the wave propagates before becoming "extinct" for all practical purposes.

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② [40 pts.]. Consider a relativistic particle (mass m, charge q) in external EM fields. An alternative Lagrange formalism treats the particle's 4-position x^α and 4-velocity u^α as generalized coordinates, so Hamilton's Principle yields the Euler-Lagrange eqtns:

$$\boxed{\frac{d}{d\tau}(\partial L/\partial u^\alpha) = \partial_\alpha L}$$

W/ Lagrangian L = Lorentz scalar, τ = particle proper time, and: $\partial_\alpha = \partial/\partial x^\alpha = (\partial/\partial x^0, \nabla)$, covariant del.

(A) Show that for (m, q) coupled to fields described by a 4-potential $A^\beta = (\phi, \mathbb{A})$, the Lagrangian: $L = \frac{1}{2} m u_\alpha u^\alpha + (q/c)\, u_\beta A^\beta$, gives the correct eqtn-of-motion for q.

(B) Find the canonical momenta P_α for the Lagrangian L of part (A). Show that the Hamiltonian $\mathcal{H}$ in this formulation is a Lorentz scalar, and find its value. How could this $\mathcal{H}$ be used in a quantum-mechanical context?

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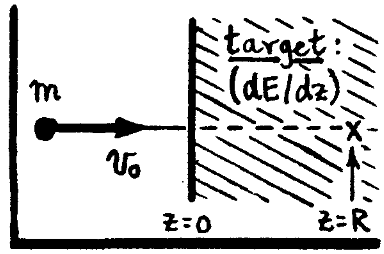
③ [40 pts.]. Show that it is not possible for an isolated free electron to emit or absorb a single photon. HINT: Analyse consequences of the conservation of (relativistic) momentum.

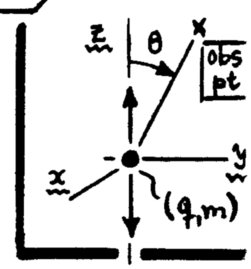
(next page)



④ [40 pts.]. A relativistic particle of mass m moves along the z-axis, initially at velocity v_0. It slams into a target whose surface is located in the plane $z=0$, penetrates the target, and travels in a straight line to a point $z=R$, where it stops. During $z=0 \rightarrow R$, the particle loses energy at a (lab) rate: $dE/dz = (c/v)^2 f_0$, where v is its instantaneous velocity, and f_0 is a constant characteristic of the target material.

(A) R is called the "range" of the particle. Calculate R for the given conditions.

(B) If K_0 is the particle's initial kinetic energy, show that $R \propto K_0$ for relativistic particles, but $R \propto K_0^2$ in the nonrelativistic limit.



⑤ [40 pts.]. Consider a particle (mass m, charge q) which executes a 1D simple harmonic motion along the z-axis; its position as a fcn of time is: $z(t) = R\cos\omega_0 t$, w/ R & ω_0 both = cnst.

(A) If q's motion is nonrelativistic, find the radiated power per unit solid ∡, $dP/d\Omega$, and the total radiated power, P. For the angular distribution, use the ∡ θ shown.

(B) What frequency spectrum does q broadcast? Find time-averaged values of $dP/d\Omega$ & P.

(C) Discuss semi-quantitatively how this analysis changes when q moves relativistically.

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⑥ [40 pts.]. Your TV set employs an electron beam at energy $\simeq$ 25 keV and current $I_0 \sim$ 1 mA which is stopped in a phosphor coating on the inside of the screen to form an image. Assume the phosphor thickness is $\delta \simeq 10^{-4}$ cm, and that the beam stops in distance δ by a uniform deceleration. (A) Find the frequency spectrum of the radiation produced at the screen. Estimate the highest frequency of concern in the spectrum. Is this radiation dangerous? (B) Find the ratio: (total radiation energy produced) / (total beam energy supplied), during the beam "stopping". What beam parameters would you adjust to keep this ratio as small as possible?

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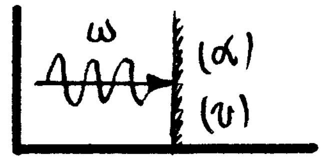
⟨1⟩ [40 pts.] EM wave propagation in a metal.



1. The planewave $u(x,t) = e^{i(kx-\omega t)}$ propagates in the metal according to $u_{xx} - \alpha u_t - (1/v^2)u_{tt} = 0$. By direct substitution...

$$\rightarrow -k^2 + i\alpha\omega + (\omega^2/v^2) = 0, \text{ ''} \quad k = \frac{\omega}{v}\sqrt{1 + i(\alpha v^2/\omega)}. \tag{1}$$

The (+) ve square root is chosen so that $k \geqslant 0$ when $\alpha \to 0$; this means the rightward traveling wave continues to the right.

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2. If $\alpha \to$ "large" (and ω is not too big), write k in Eq. (1) as...

$$k = \frac{\omega}{v}\left[i\left(\frac{\alpha v^2}{\omega}\right)\right]^{\frac{1}{2}}\sqrt{1 - i(\omega/\alpha v^2)} \simeq \sqrt{i}\,(\alpha\omega)^{\frac{1}{2}}\left[1 - \tfrac{1}{2}i(\omega/\alpha v^2)\right]. \tag{2}$$

$$\ldots \text{ but } \sqrt{i} = (e^{i\frac{\pi}{2}})^{\frac{1}{2}} = e^{i(\pi/4)} = \frac{1}{\sqrt{2}}(1+i)\ldots$$

so// $k \simeq \sqrt{\dfrac{\alpha\omega}{2}}\,(1+i)\left[1 - \tfrac{1}{2}i(\omega/\alpha v^2)\right]$, for $\alpha \to$ large,†

$$\text{or //} \quad \left[\!\left[\, k = k_R + i k_I \;\;\int\; \begin{aligned} k_R &= \sqrt{\frac{\alpha\omega}{2}}\left[1 + \tfrac{1}{2}(\omega/\alpha v^2)\right], \\ k_I &= \sqrt{\frac{\alpha\omega}{2}}\left[1 - \tfrac{1}{2}(\omega/\alpha v^2)\right]. \end{aligned} \,\right]\!\right] \tag{3}$$

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3. Put k of Eq. (3) into the planewave (in the metal) to get

$$\rightarrow u(x,t) = \left[e^{-k_I x}\right] e^{i(k_R x - \omega t)}. \tag{4}$$

The factor in front attenuates to ~ negligible values at distances Δx such that $k_I\Delta x \sim 1$. The characteristic penetration depth is thus

$$\boxed{\Delta x \sim 1/k_I = \sqrt{2/\alpha\omega}\left[1 + \tfrac{1}{2}(\omega/\alpha v^2)\right]}. \tag{5}$$

With $\alpha = 4\pi\mu\sigma/c^2$, it is easy to show $\Delta x \equiv \delta$, Jackson's "skin depth" of Eq. (7.77).

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† From class notes (2/12/91): $\alpha = 4\pi\mu\sigma/c^2$. In Eq. (3), $\alpha \to$ "large" means $\alpha v^2 \gg \omega$. With $v = c/\sqrt{\mu\epsilon}$, this translates to: $4\pi\sigma \gg \epsilon\omega$, as a condition on conductivity σ.

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② [40pts.] Work out $(q,m) \leftrightarrow$ field coupling via optional Lagrange formalism.

(A) $L = \frac{1}{2} m u_\alpha u^\alpha + \frac{q}{c} u_\beta A^\beta$, into $\frac{d}{d\tau}(\partial L/\partial u^\alpha) = \partial_\alpha L$ gives...

$$\rightarrow \frac{d}{d\tau}\left(m u_\alpha + \frac{q}{c} A_\alpha\right) = \frac{q}{c}(\partial_\alpha A_\beta) u^\beta, \tag{1}$$

where we have used $G_\sigma H^\sigma = H_\sigma G^\sigma$ for 4-vectors G & H. The 1st term on the LHS is the Minkowski force: $\frac{d}{d\tau}(m u_\alpha) = dp_\alpha/d\tau = f_\alpha$. For the 2nd term LHS, use the Chain Rule: $\frac{d}{d\tau} = (\partial x^\beta/\partial\tau)\frac{\partial}{\partial x^\beta} = (\partial_\beta) u^\beta$. Then write

$$f_\alpha + \frac{q}{c}(\partial_\beta A_\alpha) u^\beta = \frac{q}{c}(\partial_\alpha A_\beta) u^\beta,$$

or// $$f_\alpha = \frac{q}{c}(\partial_\alpha A_\beta - \partial_\beta A_\alpha) u^\beta. \tag{2}$$

Again use $G_\sigma H^\sigma = H_\sigma G^\sigma$ on the β-sum, and change the covariant index α to contravariant [see Jkn Eq. (11.75)]. Then...

$$\boxed{f^\alpha = \frac{d}{d\tau}(m u^\alpha) = \frac{q}{c} u_\beta F^{\alpha\beta}}, \quad F^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha \int \text{field tensor Jkn (11.136)} \tag{3}$$

This is the correct covariant form of the Lorentz force law (Jkn Eq. (11.144)).

(B) The canonical momenta are: $P_\alpha = \partial L/\partial u^\alpha = m u_\alpha + (q/c) A_\alpha$, and the Hamiltonian is [see Jkn Sec. (12.1)]

(cancel)

$$\mathcal{H} = P_\alpha u^\alpha - L = \left(m u_\alpha + \cancel{\frac{q}{c} A_\alpha}\right) u^\alpha - \left(\frac{1}{2} m u_\alpha u^\alpha + \cancel{\frac{q}{c} u_\beta A^\beta}\right)$$

or// $$\mathcal{H} = \frac{1}{2} m u_\alpha u^\alpha = -\frac{1}{2} mc^2 \int \text{this is a Lorentz scalar, as required.} \tag{4}$$

We have used: $u_\alpha u^\alpha = -c^2$, for the 4-velocity. If $\mathcal{H}$ were to be used in a QM formalism, we would write it in terms of the <u>canonical</u> momenta P_α, for which: $m u_\alpha = P_\alpha - \frac{q}{c} A_\alpha$, so that: $\mathcal{H} = \frac{1}{2m}(P_\alpha - \frac{q}{c} A_\alpha)(P^\alpha - \frac{q}{c} A^\alpha)$. We would then impose the QM condition: $P_\alpha = -i\hbar\,\partial_\alpha$. See Jkn Eq. (12.29).

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③ [40 pts.] Show that an isolated electron cannot emit / absorb a single photon.

1) If the electron emitted a photon of 4-momentum p_γ, overall 4-momentum would be conserved, so we would have:

$$\rightarrow p_1 = p_\gamma + p_2, \tag{1}$$

where p_1 & p_2 are the electron momenta before & after the emission. We shall now show that $p_\gamma \equiv 0$ under plausible assumptions, so the assumed photon doesn't exist. The absorption case follows similarly [let $t \rightarrow (-)t$].

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2) Since the photon is massless, then $p_\gamma \cdot p_\gamma = -(m_\gamma c)^2 = 0$. Thus:

$$\rightarrow 0 = p_\gamma \cdot p_\gamma = (p_1 - p_2)\cdot(p_1 - p_2) = p_1 \cdot p_1 + p_2 \cdot p_2 - 2 p_1 \cdot p_2. \tag{2}$$

But: $p_1 \cdot p_1 = p_2 \cdot p_2 = -(m_e c)^2$, for the electron of mass m_e. Then...

$$\rightarrow p_1 \cdot p_2 = -(m_e c)^2, \tag{3}$$

by Eq. (2), for <u>any</u> initial & final electron momenta p_1 & p_2.

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3) Choose the plausible case:

$$\begin{cases} p_1 = m_e(c, 0, 0, 0) \leftarrow \text{electron initially at rest,} \\ p_2 = m_e \gamma (c, v, 0, 0) \leftarrow \text{electron recoiling at } v \ \left(\gamma = 1/\sqrt{1-\beta^2}\right); \end{cases}$$

so, $$p_1 \cdot p_2 = -\gamma (m_e c)^2. \tag{4}$$

Eqs (3) & (4) are consistent only if $\gamma = 1$, i.e. $v \equiv 0$, which means $p_2 \equiv p_1$. Then $\underline{p_\gamma = p_1 - p_2 \equiv 0}$, as advertised, and the isolated electron cannot emit a photon while conserving momentum.

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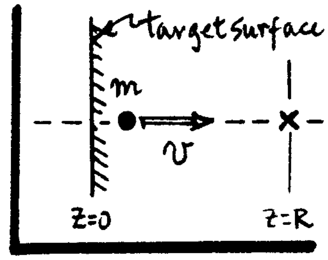
④ [40 pts]. Calculate range of particle stopping @ $dE/dz = (C/v)^2 f_0$.



1. For particle motion in a straight line we have the relation: $dp/dt = dE/dz$, from the relativistic work-energy theorem (see class notes, p. Rad 18); here $p = \gamma m v$, $E = \gamma m c^2$, and z & t are lab coordinates. Since the given dE/dz is an energy loss, m's eqtn-of-motion inside the target is

$$\rightarrow \frac{d}{dt}(\gamma m v) = -(c/v)^2 f_0, \quad \text{w/}\ \gamma = 1/\sqrt{1-\beta^2},\ \beta = v/c \tag{1}$$

$$\ldots \text{use: } \frac{d}{dt}(\gamma\beta) = \gamma^3 \frac{d\beta}{dt} = \gamma^3\left(\frac{dz}{dt}\right)\frac{d\beta}{dz} = c\gamma^3\beta\frac{d\beta}{dz} \ldots$$

$$\text{so/ } mc^2\gamma^3\beta\frac{d\beta}{dz} = -f_0/\beta^2, \quad \text{or/ } \int\frac{\beta^3 d\beta}{(\sqrt{1-\beta^2})^3} = -(f_0/mc^2)\int dz. \tag{2}$$

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2. The integral over β in Eq.(2) is tabulated: $\int\beta^3 d\beta/(\sqrt{1-\beta^2})^3 = \gamma + 1/\gamma$ [see e.g. Dwight # (323.03)], and so Eq. (2) yields β as a fcn of z...

$$\underline{\underline{\left(\gamma + \frac{1}{\gamma}\right) = \left(\gamma + \frac{1}{\gamma}\right)_0 - \frac{f_0 z}{mc^2}}} \quad \int^{\text{w/}} \gamma = 1/\sqrt{1-\beta^2}, \quad \&/ \left(\gamma + \frac{1}{\gamma}\right)_0 = \left(\gamma + \frac{1}{\gamma}\right)\Big|_{\beta = \beta_0 = v_0/c}. \tag{3}$$

The range $z = R$ is reached when $\beta \to 0$, so on the LHS of Eq.(3), $\gamma \to 1$. Then

$$\left[\!\!\left[\ \frac{f_0 R}{mc^2} = \left(\gamma + \frac{1}{\gamma}\right)_0 - 2 = (\gamma_0 - 1)\left[1 - \frac{1}{\gamma_0}\right],\ \gamma_0 = 1/\sqrt{1-\beta_0^2}.\ \right]\!\!\right] \tag{4}$$

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3. The initial particle K.E. is: $K_0 = (\gamma_0 - 1)mc^2$. So $\gamma_0 = 1 + (K_0/mc^2)$, and (4) $\Rightarrow$

$$f_0 R = K_0\left[1 - \frac{1}{\gamma_0}\right], \text{ or: } \boxed{R = \frac{K_0}{f_0}\left[\frac{(K_0/mc^2)}{1 + (K_0/mc^2)}\right]}. \tag{5}$$

So/ highly relativistic ($K_0 \gg mc^2$) $\Rightarrow R \simeq \frac{K_0}{f_0}\left[1 - (mc^2/K_0) + \cdots\right]$,

~non-relativistic ($K_0 \ll mc^2$) $\Rightarrow R \simeq \frac{K_0}{f_0}\left(\frac{K_0}{mc^2}\right)\left[1 - (K_0/mc^2) + \cdots\right]$. (6)

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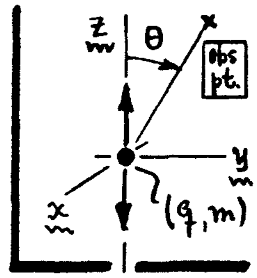

⑤ [40 pts]. Analyse radiation from a simple harmonic oscillator.

(A) $z(t) = R\cos\omega_0 t \Rightarrow$ acceleration: $a(t) = \ddot{z}(t) = -\omega_0^2 R\cos\omega_0 t$.

Nonrelativistic $\Rightarrow$ use Larmor radiation formulas [Jackson Eqs. (14.21) & (14.22)]. With θ the colatitude $\measuredangle$ shown in the sketch...

radiated power per solid $\measuredangle$ } $\boxed{\dfrac{dP}{d\Omega} = \dfrac{q^2}{4\pi c^3}|a|^2\sin^2\theta = \dfrac{(q\omega_0^2 R)^2}{4\pi c^3}\sin^2\theta\,[\cos^2\omega_0 t].}$ (1)

total radiated power } $\boxed{P = \dfrac{2}{3}(q^2/c^3)|a|^2 = \dfrac{2}{3c^3}(q\omega_0^2 R)^2[\cos^2\omega_0 t].}$ (2)

(B) Since $\cos^2 x = \frac{1}{2}[1+\cos 2x]$, then both $dP/d\Omega$ & P of part (A) have a time variation which goes as $[1+\cos 2\omega_0 t]$. A distant observer therefore sees a frequency spectrum which consists of the <u>single</u> <u>frequency</u> $\underline{\underline{\omega = 2\omega_0}}$.

Since the average value of $\cos^2\omega_0 t$ (over a few cycles) is $1/2$, then the time-averaged values of $dP/d\Omega$ & P (at the observer, and at frequency $2\omega_0$) are:

$$\rightarrow \langle dP/d\Omega\rangle = \frac{(q\omega_0^2 R)^2}{8\pi c^3}\sin^2\theta \quad,\quad \langle P\rangle = \frac{1}{3c^3}(q\omega_0^2 R)^2. \tag{3}$$

The radiation goes as $\omega_0^4 \propto 1/\lambda^4$, so short wavelengths radiate very strongly.

(C) For q moving relativistically, the radiated power per solid $\measuredangle$ should be calculated according to Jackson's Eq. (14.38). The calculation is done in retarded time t', and for linear motion ($\dot{\beta} \parallel \beta$): $\underline{\underline{dP(t')/d\Omega = \frac{q^2}{4\pi c^3}|\hat{n}\times(\hat{n}\times a)|^2/(1-\hat{n}\cdot\beta)^5}}$

For q's time t' on the RHS, it is still true that $a(t') = -\omega_0^2 R\cos\omega_0 t'$, and $\beta = \hat{z}\,\frac{1}{c}(dz/dt') = -\hat{z}\,\beta_0\sin\omega_0 t'$, w/ $\underline{\beta_0 = \omega_0 R/c}$. Also $|\hat{n}\times(\hat{n}\times a)|^2 = a^2\sin^2\theta$. So:

$$\rightarrow \frac{dP(t')}{d\Omega} = \frac{(q\omega_0^2 R)^2}{4\pi c^3}\sin^2\theta\,[\cos^2\omega_0 t']/(1+\beta_0\cos\theta\sin\omega_0 t')^5. \tag{4}$$

Compare w/ $dP/d\Omega$ of Eq. (1). The big change is the appearance of the "headlight" factor in the denominator, which changes the spectrum & time-averaging significantly.

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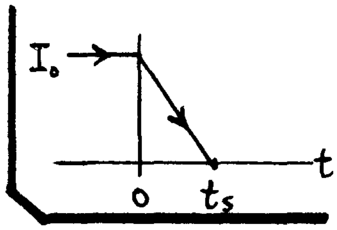
⑥ [40 pts.]. Consider a TV set as a source of radiation.

1. As it stops in the phosphor, of thickness δ, the scanning current I will radiate in the manner treated in problem (80)... the radiated energy per solid $\measuredangle$ is

$$\rightarrow \frac{d\mathcal{E}}{d\Omega} = \int_{-\infty}^{\infty} \sigma(\omega)\, d\omega, \quad \sigma(\omega) = \left(\frac{\sin^2\theta}{8\pi^2 c^3}\right) \delta^2 \left| \int_{-\infty}^{\infty} \dot{I}(t) e^{-i\omega t} dt \right|^2. \quad (1)$$

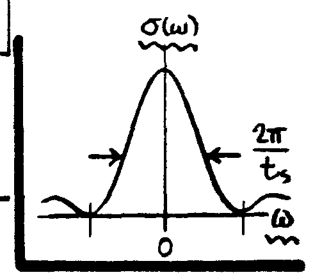


2. $I(t)$ drops from I_0 to zero in distance δ. If we assume a constant deceleration a in the phosphor, then: $a = v_0^2/2\delta$, and: $\delta = \frac{1}{2} a t_s^2$, for the stop time t_s. Hence: $t_s = 2\delta/v_0$, and in (1), $\dot{I} = -I_0/t_s$.

The integral in Eq. (1) is...

$$\rightarrow \left| \int_{-\infty}^{\infty} \dot{I}(t) e^{-i\omega t} dt \right|^2 = \left(\frac{I_0}{t_s}\right)^2 \left| \int_0^{t_s} e^{-i\omega t} dt \right|^2 = I_0^2 \left[\frac{\sin(\omega t_s/2)}{(\omega t_s/2)}\right]^2$$

so// spectral function $\Big\}$ $$\boxed{\sigma(\omega) = \frac{(I_0\delta)^2}{8\pi^2 c^3} \sin^2\theta \left[\frac{\sin(\omega t_s/2)}{(\omega t_s/2)}\right]^2} \quad (2)$$



3. The radiation is appreciable only up to $\omega t_s/2 \simeq \pi$, i.e.

$$f_{max} \simeq 1/t_s = \frac{v_0}{2\delta} = \frac{c}{2\delta}\sqrt{2K_0/mc^2}$$ $\int$ beam energy: $K_0 = 25$ keV, phosphor thickness $\delta = 10^{-4}$ cm, $mc^2 = 511$ keV.

so// $f_{MAX} \simeq 5\times10^{13}$ Hz $\leftrightarrow \lambda_{MIN} = c/f_{MAX} = 60{,}000$ Å (far IR, ~0.2 eV). (3)

These radiated wavelengths are biologically harmless; they mainly cause heating. Shorter wavelengths are present in $\sigma(\omega)$, but their intensity is a negligibly small.

4. For I_0 stopped in t_s, as above, total radiated energy is...

$$\rightarrow \mathcal{E}_{rad} = \int_{4\pi} d\Omega \int_{-\infty}^{\infty} \sigma(\omega)\, d\omega = \frac{(I_0\delta)^2}{8\pi^2 c^3} \int_{\theta=0}^{\theta=\pi} \sin^2\theta \cdot 2\pi \sin\theta\, d\theta \overbrace{\int_{-\infty}^{\infty} \frac{\sin^2(\omega t_s/2)}{(\omega t_s/2)^2} d\omega}^{=2\pi/t_s}$$

$\boxed{\mathcal{E}_{rad} = \frac{2}{3c^3}(I_0\delta)^2/t_s}$ (4). Total beam energy expended is: $\mathcal{E}_{beam} = I_0 V_0 t_s$, so the ratio is: $\mathcal{E}_{rad}/\mathcal{E}_{beam} = \frac{2}{3c^3}(I_0/V_0)(\delta/t_s)^2 = \frac{1}{6c^3}(I_0/V_0) v_0^2$. But the beam energy is $\frac{1}{2}mv_0^2 = eV_0$, so finally the ratio just depends on the beam current:

$\boxed{\mathcal{E}_{rad}/\mathcal{E}_{beam} = \frac{1}{3}(e/mc^3)\, I_0}$ (5). Beware of bright TV sets!