1992 ϕ520 Problems   Assigned: 2/28. Due: 3/6/92.   P20 file   (20)

(63) [Jk^{n} #(7.14), part (a) only]. For a frequency-dependent dielectric constant $\epsilon(\omega)$, with: $\underline{\mathrm{Im}\,\epsilon(\omega) = \lambda[\theta(\omega-\omega_1) - \theta(\omega-\omega_2)]}$, w/ λ = cnst, θ = unit step fcn, and ordering $\omega_2 > \omega_1 > 0$, use the Kramers-Kronig relations to find $\mathrm{Re}\,\epsilon(\omega)$ at all $\omega \geqslant 0$. Sketch both $\mathrm{Im}\,\epsilon(\omega)$ & $\mathrm{Re}\,\epsilon(\omega)$ vs. ω. Compare your results for this model of $\epsilon(\omega)$ with the generic curves in Jk^{n} Fig. 7.8.

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(64) [Jk^{n} #(11.2)]. Show explicitly that two successive Lorentz Transformations in the same direction (at velocity β_1, followed by β_2) are equivalent to a single LT @: $\underline{\underline{\beta = (\beta_1+\beta_2)/(1+\beta_1\beta_2)}}$, w/ $\beta = v/c$. This is relativistic velocity addition.

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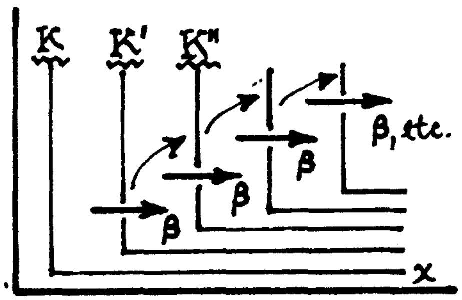
(65) Initially, K′ is moving at velocity $\beta_1 = \beta$ (w/ $0<\beta<1$) down the x-axis of reference system K. To boost her velocity, K′ boards a system K″ moving by her at relative velocity β. By velocity addition, K′ is now moving w.r.t. K @ $\beta_2 = 2\beta/(1+\beta^2)$. K′ continues the process, each time boarding a new system $K^{(m)}$ moving by her at β. Show that after $(n-1)$ such boosts, the K′ velocity relative to K is: $\beta_n = (1-\epsilon^n)/(1+\epsilon^n)$, w/ $0<\epsilon<1$. Find ϵ in terms of β. Can K′ get to $v=c$ by a finite number of finite accelerations?



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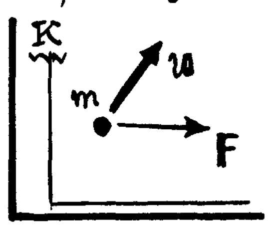
(66) Newton's $\mathbb{F} = m\mathbb{a}$ can be replaced in SRT by $\tilde{F} = m\tilde{a}$. The Minkowski 4-force $\tilde{F}$ is defined in terms of m's 4-momentum $\tilde{p} = (\mathcal{E}/c, \mathbb{p})$ [w/ $\mathcal{E} = \gamma mc^2$, $\mathbb{p} = \gamma m\mathbb{u}$, and $\gamma = 1/\sqrt{1-u^2/c^2}$] by: $\tilde{F} = d\tilde{p}/d\tau$, $d\tau = \frac{1}{\gamma}dt$ = particle proper time (and dt is observer's time in a reference frame K). The 4-acceleration $\tilde{a}$ is defined by the 4-velocity $\tilde{u} = \gamma(c, \mathbb{u})$ as: $\tilde{a} = d\tilde{u}/d\tau$.



(A) Show that $\tilde{F}\cdot\tilde{u} \equiv 0$, for the motion of m.

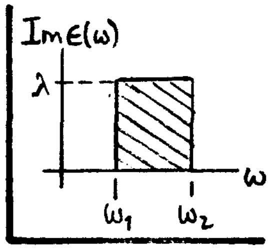
(B) From part (A), establish the relativistic work-energy theorem: $\underline{\underline{\mathbb{F}\cdot\mathbb{u} = d\mathcal{E}/dt}}$, w/ $\mathcal{E} = \gamma mc^2 = \mathcal{E}(u)$. $\mathbb{F} = d\mathbb{p}/dt$ (w/ $\mathbb{p} = \gamma m\mathbb{u}$) is the Newtonian force observed by K.

(63) [Jk^n # (7.14)(a)]. $\mathrm{Im}\,\epsilon(\omega) = \lambda[\theta(\omega-\omega_1) - \theta(\omega-\omega_2)] \Rightarrow \mathrm{Re}\,\epsilon(\omega) = ?$

1. From Jk^n Eq. (7.120), the prescription for $\mathrm{Re}\,\epsilon(\omega)$ is...

$$\rightarrow \frac{\pi}{2}[\mathrm{Re}\,\epsilon(\omega) - 1] = \mathcal{P}\int_0^\infty \frac{x\,\mathrm{Im}\,\epsilon(x)}{x^2-\omega^2}dx = \lambda\,\mathcal{P}\int_0^\infty \frac{x[\theta(x-\omega_1)-\theta(x-\omega_2)]}{x^2-\omega^2}dx,$$

or $$\frac{\pi}{2\lambda}[\mathrm{Re}\,\epsilon(\omega) - 1] = \mathcal{P}\int_{\omega_1}^{\omega_2}\frac{x\,dx}{x^2-\omega^2} = \frac{1}{2}\mathcal{P}\int_{\omega_1^2}^{\omega_2^2}\frac{dy}{y-\omega^2},\quad y = x^2. \quad (1)$$



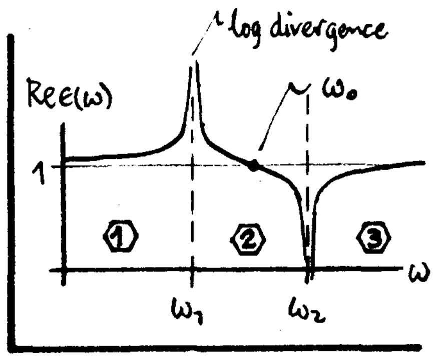
2. Suppose the running frequency $\omega \neq \omega_1$ or ω_2. Then Eq. (1) gives...

$$\rightarrow \frac{\pi}{\lambda}[\mathrm{Re}\,\epsilon(\omega) - 1] = \int_{\omega_1^2}^{\omega_2^2} d\ln(y-\omega^2) = \ln\left|\frac{\omega_2^2-\omega^2}{\omega_1^2-\omega^2}\right| \quad (2)$$

$$0 \leq \omega < \omega_1:\ \mathrm{Re}\,\epsilon(\omega) = 1 + \frac{\lambda}{\pi}\ln\left(\frac{\omega_2^2-\omega^2}{\omega_1^2-\omega^2}\right), \quad (3a)$$

$$\omega_1 < \omega < \omega_2:\ \mathrm{Re}\,\epsilon(\omega) = 1 + \frac{\lambda}{\pi}\ln\left(\frac{\omega_2^2-\omega^2}{\omega^2-\omega_1^2}\right), \quad (3b)\ \bigstar$$

$$\omega_2 < \omega:\ \mathrm{Re}\,\epsilon(\omega) = 1 + \frac{\lambda}{\pi}\ln\left(\frac{\omega^2-\omega_2^2}{\omega^2-\omega_1^2}\right). \quad (3c)$$



Note that in region ②, the $\ln \rightarrow 0$ @ $\omega_0 = \sqrt{(\omega_1^2+\omega_2^2)/2}$, and in region ③ $\mathrm{Re}\,\epsilon(\omega) \rightarrow 1$ as $\omega \rightarrow \infty$ (standard behavior).

3. At the boundary freqs. ω_1 & ω_2...

region ①, $\omega = \omega_1 - \delta$: $\mathrm{Re}\,\epsilon(\omega) \simeq 1 + \frac{\lambda}{\pi}\ln\left(\frac{\omega_2^2-\omega_1^2}{2\omega_1\delta-\delta^2}\right) \rightarrow +\infty$, as $\delta \rightarrow 0$

region ②, $\omega = \omega_1 + \delta$: $\mathrm{Re}\,\epsilon(\omega) \simeq 1 + \frac{\lambda}{\pi}\ln\left(\frac{\omega_2^2-\omega_1^2}{2\omega_1\delta+\delta^2}\right) \rightarrow +\infty$, as $\delta \rightarrow 0$

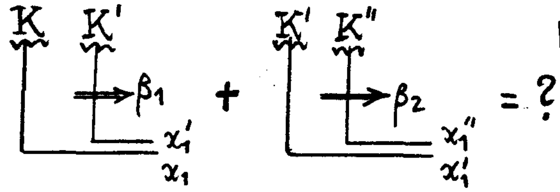
$\mathrm{Re}\,\epsilon(\omega)$ diverges $\sim \ln\frac{1}{|\delta|} \rightarrow \infty$ when approaching ω_1 from above or below.

Similarly, as $\omega \rightarrow \omega_2\pm$, $\mathrm{Re}\,\epsilon(\omega) \simeq 1 + \frac{\lambda}{\pi}\ln\left(\frac{2\omega_2\delta}{\omega_2^2-\omega_1^2}\right) \rightarrow (-)\infty$. There is no apparent (4) way to remove these divergences [due to ∞ derivatives $(d/d\omega)\,\mathrm{Im}\,\epsilon(\omega)$ @ $\omega = \omega_1$ & ω_2, normally forbidden in a theory of analytic fcns]. So they must stay in above sketch.

4. $\mathrm{Im}\,\epsilon(\omega)$ & $\mathrm{Re}\,\epsilon(\omega)$ above resemble Jk^s Fig. 7.8. But the $\omega_{1,2}$ problems $\Rightarrow$ this model is poor.

$\bigstar$ For $\omega_1 < \omega < \omega_2$, Eq. (3b) is defined by the $\mathcal{P}$, i.e. $\int_{\omega_1^2}^{\omega_2^2}\frac{dy}{y-\omega^2} = \lim_{\delta\to 0}\left\{\int_{\omega_1^2}^{(\omega-\delta)^2} + \int_{(\omega+\delta)^2}^{\omega_2^2}\right\}\frac{dy}{y-\omega^2} =$

$$= \lim_{\delta\to 0}\left\{\ln\left(\frac{\omega_2^2-\omega^2}{\omega^2-\omega_1^2}\right) + \ln\left(\frac{\omega^2-(\omega-\delta)^2}{(\omega+\delta)^2-\omega^2}\right)\right\} = \ln[(\omega_2^2-\omega^2)/(\omega^2-\omega_1^2)],\ \text{as stated.}$$



(64) Find equivalent velocity for two successive Lorentz transf$^{\underline{n}}$s [Jk$^{\underline{n}}$ #(11.2)].

1) Two successive Lorentz transf$^{\underline{n}}$s (along x_1-axis) yield, with $x_0 = ct$...

$$\underline{\underline{K \to K'(\beta_1)}} \begin{cases} x_0' = \gamma_1(x_0 - \beta_1 x_1), \\ x_1' = \gamma_1(x_1 - \beta x_0); \end{cases} \Bigg\| \quad \underline{\underline{K' \to K''(\beta_2)}} \begin{cases} x_0'' = \gamma_2(x_0' - \beta_2 x_1'), \\ x_1'' = \gamma_2(x_1' - \beta_2 x_0'). \end{cases} \quad (1)$$

2) Plug the x_0' & x_1' values from the $K \to K'$ transform into the x_0'' & x_1'' eqtns to get,

$$x_0'' = (1+\beta_1\beta_2)\gamma_1\gamma_2\left[x_0 - \left(\frac{\beta_1+\beta_2}{1+\beta_1\beta_2}\right)x_1\right],$$

$$x_1'' = (1+\beta_1\beta_2)\gamma_1\gamma_2\left[x_1 - \left(\frac{\beta_1+\beta_2}{1+\beta_1\beta_2}\right)x_0\right]. \quad (2)$$

3) Now work out the algebraic identity...

$$(1+\beta_1\beta_2)\gamma_1\gamma_2 = \left[\frac{(1+\beta_1\beta_2)^2}{\underbrace{(1-\beta_1^2)(1-\beta_2^2)}_{= 1+(\beta_1\beta_2)^2-(\beta_1^2+\beta_2^2) \,=\, 1+2\beta_1\beta_2+(\beta_1\beta_2)^2-(\beta_1^2+2\beta_1\beta_2+\beta_2^2)}}\right]^{\frac{1}{2}} = \left[\frac{(1+\beta_1\beta_2)^2}{(1+\beta_1\beta_2)^2-(\beta_1+\beta_2)^2}\right]^{\frac{1}{2}} = \left[\frac{1}{1-\left(\frac{\beta_1+\beta_2}{1+\beta_1\beta_2}\right)^2}\right]^{\frac{1}{2}},$$

i.e., $(1+\beta_1\beta_2)\gamma_1\gamma_2 = \gamma$, where: $\gamma = \frac{1}{\sqrt{1-\beta^2}}$, and: $\beta = (\beta_1+\beta_2)/(1+\beta_1\beta_2)$. (3)

4) The overall $K \to K''$ transf$^{\underline{n}}$ of Eq. (2) can now be written as...

$$x_0'' = \gamma(x_0 - \beta x_1), \quad x_1'' = \gamma(x_1 - \beta x_0),$$

with: $\boxed{\beta = (\beta_1+\beta_2)/(1+\beta_1\beta_2)}$, and: $\gamma = 1/\sqrt{1-\beta^2}$. (4)

This is a standard Lorentz transf$^{\underline{n}}$ for $K \to K''(\beta)$, at the advertised value of β. Velocities do <u>not</u> add linearly, as per Galileo. While K' <u>thinks</u> he boosts his velocity by β_2 in boarding K'', he only gets: $\beta - \beta_1 = \left(\frac{1-\beta_1^2}{1+\beta_1\beta_2}\right)\beta_2 < \beta_2$, w.r.t. K.

ϕ 520 Prob. Solutions

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Calculate aggregate velocity for a succession of Lorentz transf$^{\underline{n}}$s at β.

1) After $(n-1)$ boardings, the K–K' relative velocity will be...

$$\beta_n = \frac{\beta + \beta_{n-1}}{1+\beta\beta_{n-1}} \;;\; n \geqslant 1, \text{ and: } \beta_0 = 0,\ \beta_1 = \beta,\ \beta_2 = \frac{2\beta}{1+\beta^2},\ \text{etc} \tag{1}$$

In principle, this can be iterated for β_n in terms of $\beta_1 = \beta$. The first few terms are...

$$\beta_1 = \beta,\ \beta_2 = \frac{2\beta}{1+\beta^2},\ \beta_3 = \frac{3\beta+\beta^3}{1+3\beta^2},\ \beta_4 = \frac{4\beta+4\beta^3}{1+6\beta^2+\beta^4}, \tag{2}$$

$$\beta_5 = \frac{5\beta+10\beta^3+\beta^5}{1+10\beta^2+5\beta^4},\ \beta_6 = \frac{6\beta+20\beta^3+6\beta^5}{1+15\beta^2+15\beta^4+\beta^6},\ \beta_7 = \frac{7\beta+35\beta^3+21\beta^5+\beta^7}{1+21\beta^2+35\beta^4+7\beta^6},\ \text{etc.}$$

2) Consulting a table of binomial coefficients, it is apparent these results follow from:

$$\beta_n = \sum_{k=0}^{[n/2]} \binom{n}{2k+1}\beta^{2k+1} \Big/ \sum_{k=0}^{[n/2]} \binom{n}{2k}\beta^{2k} \quad \begin{cases} [n/2] = \text{greatest integer in } n/2, \\ \binom{n}{m} = n!/m!(n-m)!\,. \end{cases} \tag{3}$$

Indeed, this Ansatz satisfies the recursion relation in Eq. (1). Next, note that:

$$(1+\beta)^n = \sum_{m=0}^{n}\binom{n}{m}\beta^m = \binom{n}{0}\beta^0 + \binom{n}{1}\beta^1 + \binom{n}{2}\beta^2 + \binom{n}{3}\beta^3 + \cdots + \binom{n}{n}\beta^n$$

$$= \mathcal{D} + \mathcal{N},\ \text{where: } \mathcal{D} = \sum_{k=0}^{[n/2]}\binom{n}{2k}\beta^{2k},\ \text{and } \mathcal{N} = \sum_{k=0}^{[n/2]}\binom{n}{2k+1}\beta^{2k+1}. \tag{4}$$

Similarly...

$$(1-\beta)^n = \mathcal{D} - \mathcal{N} \quad \text{so} \quad \begin{cases} \mathcal{D} = \frac{1}{2}[(1+\beta)^n + (1-\beta)^n], \\ \mathcal{N} = \frac{1}{2}[(1+\beta)^n - (1-\beta)^n]. \end{cases} \tag{5}$$

3) With these identities, we can form the aggregate velocity of Eq. (3)...

$$\longrightarrow \beta_n = \mathcal{N}/\mathcal{D} = (1-\epsilon^n)/(1+\epsilon^n),\ \text{with: } \boxed{\epsilon = (1-\beta)/(1+\beta)}. \tag{6}$$

With $0<\beta<1 \Rightarrow 0<\epsilon<1$, ... so ... $0<\beta_n<1$ for any finite # of accelerations. For $n \to$ large: $\beta_n \simeq 1 - 2\epsilon^n$; we can at most approach $v = c$ from below; $v \leq c$ always.

ϕ520 Solutions

(66) With Minkowski force: $\tilde{F} = d\tilde{p}/d\tau$, show Work-Energy Thm: $\mathbb{F}\cdot\mathbb{u} = \frac{d}{dt}(\gamma mc^2)$.

A. 1) The Minkowski version of $\mathbb{F} = m\mathbb{a}$ is -- as stated...

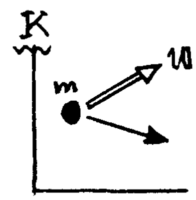
$$\rightarrow \tilde{F} = \frac{d\tilde{p}}{d\tau} = m\tilde{a}. \qquad (1)$$

Since $\tilde{a} = d\tilde{u}/d\tau$, we can form the scalar product of interest...

$$\rightarrow \tilde{F}\cdot\tilde{u} = m\left(\frac{d\tilde{u}}{d\tau}\right)\cdot\tilde{u} = \frac{m}{2}\frac{d}{d\tau}(\tilde{u}\cdot\tilde{u}) = \frac{m}{2}\frac{d}{d\tau}(c^2) = \underline{\underline{0}}. \qquad (2)$$

This works because $\tilde{u}$ has the invariant length: $\tilde{u}^2 = c^2 = $ cnst. So $\tilde{F} \perp \tilde{u}$.

B. 2) In $\tilde{F} = d\tilde{p}/d\tau$, put $\tilde{p} = (\mathcal{E}/c, \mathbb{p})$ [$\mathcal{E} = \gamma mc^2$, $\mathbb{p} = \gamma m\mathbb{u}$, w/ $\gamma = 1/\sqrt{1-u^2/c^2}$], and introduce observer K time dt by $d\tau = (1/\gamma)\,dt$. Then...



$$\rightarrow \tilde{F} = \gamma\frac{d}{dt}(\mathcal{E}/c, \mathbb{p}) = \gamma\left(\frac{1}{c}\frac{d\mathcal{E}}{dt}, \mathbb{F}\right), \qquad (3)$$

w/ $\mathbb{F} = d\mathbb{p}/dt$ the (relativized) version of the Newton 3-force on m seen by K.

The 4-velocity of m is, explicitly...

$$\rightarrow \tilde{u} = \gamma(c, \mathbb{u}), \text{ w/ } \mathbb{u} = \text{3-velocity seen by K, and } \gamma = 1/\sqrt{1-u^2/c^2}, \qquad (4)$$

... and the 4-vector scalar product in Eq. (2) is ...

$$\rightarrow \tilde{F}\cdot\tilde{u} = \gamma\left(\frac{1}{c}\frac{d\mathcal{E}}{dt}, \mathbb{F}\right)\cdot\gamma(c, \mathbb{u}) = \gamma^2\left[\frac{d\mathcal{E}}{dt} - \mathbb{F}\cdot\mathbb{u}\right]. \qquad (5)$$

But, by Eq. (2), $\tilde{F}\cdot\tilde{u} = 0$, and so the [ ] $\equiv 0$ here. Thus...

$$\boxed{\mathbb{F}\cdot\mathbb{u} = \frac{d\mathcal{E}}{dt}}, \text{ w/ } \left.\begin{array}{l}\mathbb{F} = \frac{d}{dt}(\gamma m\mathbb{u}) \\ \mathcal{E} = \gamma mc^2\end{array}\right\} \text{as observed in K.} \qquad (6)$$

This is the relativistic version of the work-energy theorem. The total energy $\mathcal{E}$ can be replaced by the relativistic kinetic energy $K = (\gamma-1)mc^2$.