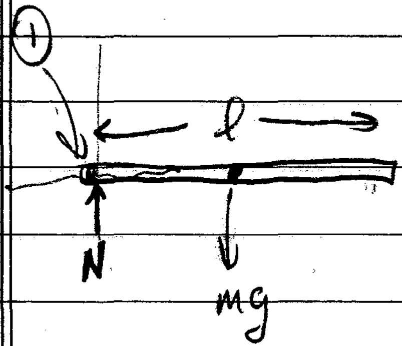
Mechanics

1. A horizontal uniform stick of length l and mass m is supported at each end on knife edges. At some instant in time, the right hand support is removed. Find the initial acceleration of the center of mass of the stick and the force which the left hand knife edge exerts at that instant.



$I_{O} = \frac{1}{3} m \ell^2$

$mg - N = ma$

$mg \frac{\ell}{2} = I\alpha$

$a = \frac{\ell}{2}\alpha \Rightarrow \alpha = \frac{2a}{\ell}$

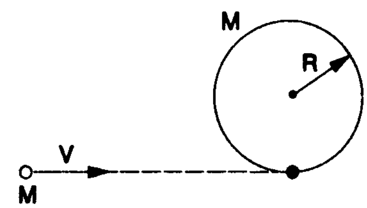
So $\cancel{m} g \frac{\ell}{2} = \frac{1}{3} \cancel{m} \ell^{\cancel{2}} \frac{2a}{\cancel{\ell}}$

$a = \frac{3}{4} g$

$N = mg - \frac{3}{4} mg = \frac{mg}{4}$

Mechanics

2. A thin circular wooden hoop of mass M and radius R sits on a horizontal frictionless plane. A bullet, also of mass M, moving with horizontal velocity V, strikes the hoop tangentially and becomes embedded in it as shown in the figure below. Calculate:



(a) the center of mass velocity

(b) the angular momentum of the system about the center of mass

(c) the angular velocity of the hoop

(d) the kinetic energy of the system before the collision

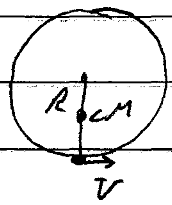
(e) the kinetic energy of the system after the collision

Qual 90 Solution Hiscock

Bullet & Hoop

(a) $\vec{V}_{CM} = \frac{M}{2M}\underbrace{\vec{V}_{bullet}}_{=V} + \frac{M}{2M}\cancelto{0}{\vec{V}_{hoop}} \Rightarrow \boxed{|\vec{V}_{CM}| = \frac{1}{2}V}$

(b) at the instant before the bullet strikes the hoop, only object in motion is the bullet; distance from CM is $R/2$:



$\Rightarrow \boxed{L = MVR/2}$

(c) After the collision: $I^{CM}_{TOTAL} = I_{bullet} + \underbrace{I^{CM\ of\ hoop}}_{\text{about center of loop}} + \underbrace{M(R/2)^2}_{\text{correction for new location of axis; new CM}}$

$$I^{CM}_{TOT} = M\left(\frac{R}{2}\right)^2 + MR^2 + M\left(\frac{R}{2}\right)^2$$

$$I^{TOT}_{CM} = \frac{3}{2}MR^2 \rightarrow L = I\omega \qquad \omega = L/I$$

$$\omega = \frac{\frac{\cancel{M}V\cancel{R}}{\cancel{2}}}{\frac{3}{\cancel{2}}\cancel{M}R^{\cancel{2}}} = \boxed{\frac{V}{3R} = \omega}$$

(d) before collision $\boxed{KE = \frac{1}{2}MV^2}$ (hoop is at rest)

(e) after collision: both translational & rotational motion

$$KE_{Trans} = \frac{1}{2}(2M)(V_{CM})^2 = M\left(\frac{1}{2}V\right)^2 = \frac{1}{4}MV^2$$

$$KE_{ROT} = \frac{1}{2}I\omega^2 = \frac{1}{2}\left(\frac{3}{2}MR^2\right)\left(\frac{V}{3R}\right)^2 = \frac{M}{12}V^2 \qquad \boxed{KE_{TOT} = \left(\frac{1}{4}+\frac{1}{12}\right)MV^2 = \frac{1}{3}MV^2}$$

Mechanics

3. According to the Yukawa theory of nuclear forces, the attractive force between a neutron and a proton has the potential:

$$V(r) = \frac{-Ke^{-\alpha r}}{r} \qquad (K, \alpha > 0)$$

Consider the motion of a neutron subject to this central potential.

(a) What is the Lagrangian for the system (use spherical coordinates)?

(b) What are the canonical momenta? Label each momentum as conserved or not conserved, and show how you determined this.

(c) Find the energy, E, and the angular momentum, L, of a circular orbit of radius a, as functions of K, α, a, and m, the neutron mass.

Qual 90 Solution Hirzeck

Yukawa Potential

$$V(r) = \frac{-Ke^{-\alpha r}}{r}$$

(a) Lagrangian in spherical coordinates

$$T = \tfrac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\phi}^2\right)$$

$$L = T - V$$

$$\boxed{L = \tfrac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\phi}^2\right) + \frac{Ke^{-\alpha r}}{r}}$$

(b) Momenta: $p_i = \partial L/\partial \dot{q}_i$

p_i conserved if $\partial L/\partial q_i = 0$

$p_r = m\dot{r}$ not conserved $\partial L/\partial r \neq 0$

$p_\theta = mr^2\dot{\theta}$ not conserved $\partial L/\partial\theta \neq 0$

$p_\phi = mr^2\sin^2\theta\,\dot{\phi}$ conserved $\partial L/\partial\phi = 0$

(c) circular orbit (choose orbit to lie in equatorial plane)

effective potential $\tilde{V}(r) = V(r) + L^2/(2mr^2)$

$\frac{\partial \tilde{V}}{\partial r} = 0$ for circular orbit

$$\tilde{V}(r) = \frac{-Ke^{-\alpha r}}{r} + \frac{L^2}{2mr^2}$$

$$\frac{\partial\tilde{V}}{\partial r} = \frac{\alpha Ke^{-\alpha r}}{r} + \frac{Ke^{-\alpha r}}{r^2} - \frac{L^2}{mr^3} = 0$$ solve for L at $r=a$

$$L^2 = ma^3 \left\{ \frac{Ke^{-\alpha a}}{a^2}(\alpha a+1) \right\} = maKe^{-\alpha a}(\alpha a+1)$$

$$\boxed{L = \left[maKe^{-\alpha a}(\alpha a+1) \right]^{1/2}}$$

could also work directly from E-L equations

Energy:

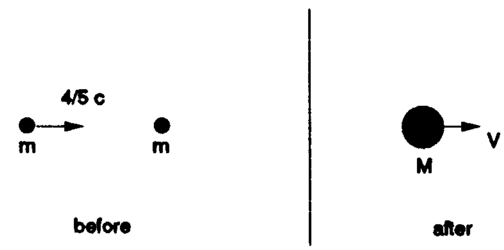
$E = \frac{1}{2}m\dot{r}^2 + \tilde{V}(r)$ $\quad \dot{r} = 0$ in circular orbit

$$\Rightarrow E = \tilde{V}(a) = \frac{-Ke^{-\alpha a}}{a} + \frac{\cancel{m}aKe^{-\alpha a}(\alpha a+1)}{2\cancel{m}a^2}$$

$$\boxed{E = \frac{Ke^{-\alpha a}}{2a}(\alpha a-1)}$$

Special Relativity

4. A particle of rest mass m moving at a speed $v = (4/5)c$ collides with a similar particle (also mass m) at rest to form a moving composite particle:



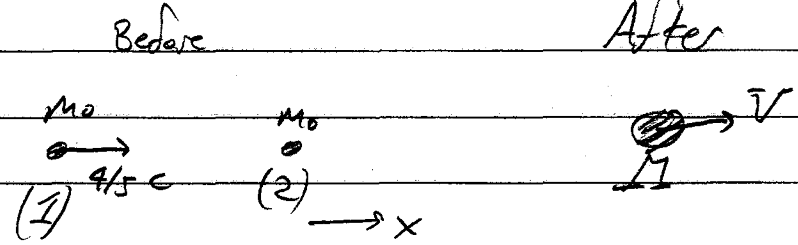
(a) What is the speed of the composite particle?

(b) What is the rest mass of the composite particle?

Qual 90 Solution Hiscock

Relativistic Inelastic Collision

$\boxed{\text{Work in units with } c=1}$



set up coordinates as shown above; label particles before collision as (1) & (2)

Then:

$$\vec{p}_1 = (E, p_x, 0, 0) = (m_0\gamma,\ m_0\gamma v^x, 0, 0)$$

$$\vec{p}_2 = (m_0, 0, 0, 0) \qquad \text{here } v^x = 4/5 \text{ so } \gamma = \frac{1}{\sqrt{1-\left(\frac{4}{5}\right)^2}} = \frac{5}{3}$$

$$\Rightarrow \vec{p}_1 = \left(\tfrac{5}{3}m_0,\ \tfrac{4}{3}m_0,\ 0, 0\right)$$

$$\underset{\uparrow \text{ conservation of 4-momentum}}{\vec{p}_1 + \vec{p}_2 = \vec{p}_{after}} = \left(\tfrac{8}{3}m_0,\ \tfrac{4}{3}m_0,\ 0,0\right) = (M\gamma',\ M\gamma' V, 0, 0)$$

(a) $V = \text{velocity after} = \dfrac{p^x_{after}}{p^t_{after}} = \dfrac{\frac{4}{3}m_0}{\frac{8}{3}m_0} = \boxed{\tfrac{1}{2}c = V}$

(b) if $V_{after} = \frac{1}{2}$, then $\gamma' = \dfrac{1}{\sqrt{1-\left(\frac{1}{2}\right)^2}} = \dfrac{2}{\sqrt{3}}$

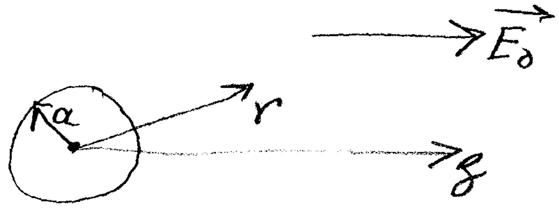
$$\Rightarrow p^t = M\gamma' = \tfrac{2}{\sqrt{3}}M = \tfrac{8}{3}m_0$$

$$\Rightarrow \boxed{M = \tfrac{4}{\sqrt{3}}m_0}$$

Electromagnetism

5. Given a conducting sphere of radius a which has been placed in a uniform electric field $E_o\hat{z}$, find the electric potential for this configuration. (You may use the following: $P_o = 1, \quad P_1 = \cos\theta, \quad P_2 = \frac{3}{2}\cos^2\theta - \frac{1}{2}$.)

a solution



$\nabla^2 u(r,\theta) = 0$

has general solution (no ϕ dependence)

$$u(r,\theta) = \sum_0^\infty B_n P_n r^{-(n+1)} + \sum_0^\infty A_n P_n r^n$$

B.C. a) $\vec{E}(r,\theta)_{r\to\infty} = \vec{E}_0 = E_0 \hat{z}$, $u(r,\theta)|_{r\to\infty} = -E_0 z + c$

$= -E_0 r\cos\theta + c$

b) $u(r=a,\theta) = u_0$

from a) for large $r \Rightarrow A_2, A_3, \cdots$ zero $A_n = 0 \quad n \geq 2$

$\therefore u(r,\theta) = \cdots B_1 P_1 r^{-2} + B_0 P_0 r^{-1} + A_0 + A_1 P_1 r$

as $r \to \infty$ have $-E_0 r\cos\theta \Rightarrow A_1 = -E_0$

from b) $u(a,\theta) = u_0 = \ldots B_1 \cos\theta\, a^{-2} + B_0 a^{-1} + A_0 - E_0 a\cos\theta$

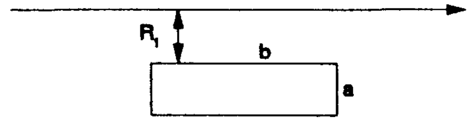
each P_n is independent

$\therefore A_0 = u_0$, $B_0 = 0$, $B_1 = E_0 a^3$ + for $n \geq 2$ $B_n = 0$

So. $u(r,\theta) = u_0 - E_0 r\cos\theta + \dfrac{E_0 a^3 \cos\theta}{r^2}$

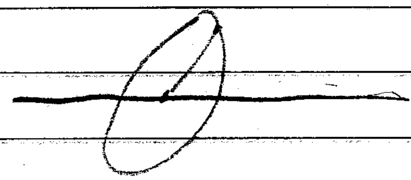
Electromagnetism

6.



(a) A long straight wire carries a current of I amperes to the right. Find the magnetic field due to the current everywhere outside the wire.

(b) A flat rectangular coil of wire of length b and width a is located at a distance of R_I from the wire as shown. Find the direction and magnitude of the induced EMF in the coil if the current in the wire is reduced at a rate of dI/dt.



Amperes Law

$$2\pi R B = \mu_0 I \Rightarrow B = \frac{\mu_0 I}{2\pi R}$$

$$\Phi_B = \int_{r=R}^{R+a} B \cdot dA = \frac{\mu_0 I}{2\pi} b \int_{R}^{r+a} \frac{dR}{r} = \frac{\mu_0 I b}{2\pi} \ln \frac{R+a}{R}$$

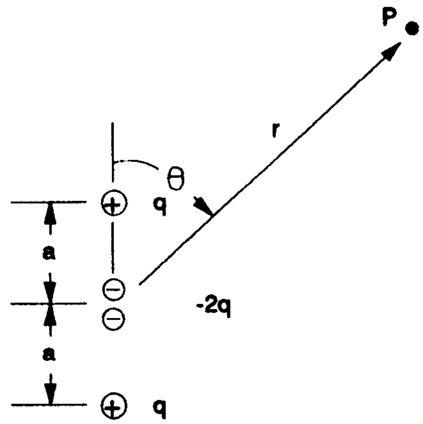
$$\mathcal{E} = \frac{\mu_0 b}{2\pi} \ln \frac{R+a}{a} \left(\frac{dI}{dt}\right)$$ clockwise

$$\frac{2I}{\cancel{r} c} = \frac{\mu_0 \cancel{I}}{2\pi \cancel{r}}$$

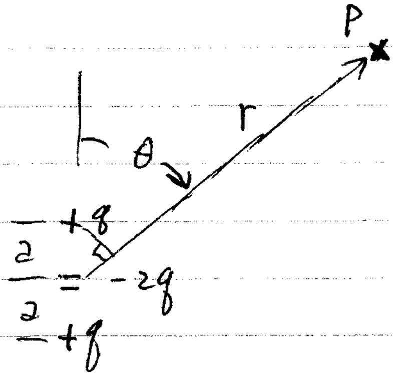
$$c = \frac{4\pi}{\mu_0}$$

Electromagnetism

7. Find an expression for the potential V from the linear electric quadrupole shown, valid far away where $r >> a$.



Find V by binomial expansion:



$$V = -\frac{2q}{4\pi\epsilon_0 r} + \frac{q}{4\pi\epsilon_0\left[(r - a\cos\theta)^2 + a^2\sin^2\theta\right]^{\frac{1}{2}}} + \frac{q}{4\pi\epsilon_0\left[(r + a\cos\theta)^2 + a^2\sin^2\theta\right]^{\frac{1}{2}}}$$

$$V = \frac{q}{4\pi\epsilon_0 r}\left[-2 + \frac{1}{\sqrt{1 - \frac{2a}{r}\cos\theta + \frac{a^2}{r^2}}} + \frac{1}{\sqrt{1 + \frac{2a}{r}\cos\theta + \frac{a^2}{r^2}}}\right]$$

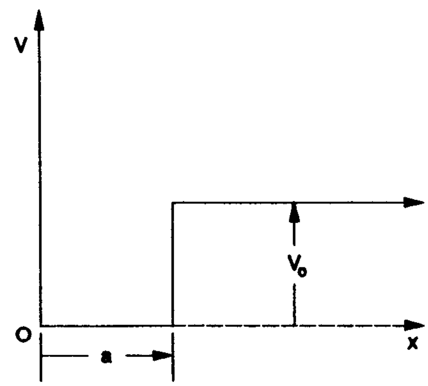
$$(1+x)^n = 1 + nx + \frac{n(n-1)x^2}{2} + \cdots, \quad n = -\tfrac{1}{2}$$

$$V = \frac{q}{4\pi\epsilon_0 r}\left[-2 + 1 - \frac{1}{2}\left(-\frac{2a}{r}\cos\theta + \frac{a^2}{r^2}\right) + \frac{3}{8}\left(\frac{4a^2}{r^2}\cos^2\theta + \cdots\right) + 1 - \frac{1}{2}\left(+\frac{2a}{r}\cos\theta + \frac{a^2}{r^2}\right) + \frac{3}{8}\left(\frac{4a^2}{r^2}\cos^2\theta + \cdots\right)\right]$$

$$= \frac{q}{4\pi\epsilon_0 r}\left[3\frac{a^2}{r^2}\cos^2\theta - \frac{a^2}{r^2}\right] = \boxed{\frac{qa^2}{4\pi\epsilon_0 r^3}(3\cos^2\theta - 1)}$$

Quantum Mechanics

8. A particle of mass m is placed in the semi-infinite square well as shown.



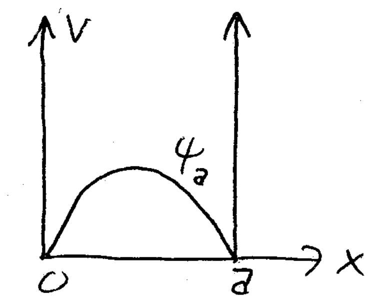
   (a) Find the ground state energy E_0 for $V_0 \rightarrow \infty$. Sketch the form of the wavefunction for this case.

   (b) Now consider the case where V_0 is large but finite ($V_0 >> E_0$). Sketch the form of the wavefunction in this case.

   (c) Find the magnitude and sign of the lowest-order correction ε to the ground state energy for the case described in (b). (That is, approximate the ground state energy by $E \approx E_0 + \varepsilon$; find ε for $\infty > V_0 \gg E_0$).

#8 Solution

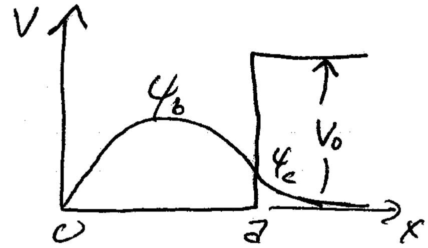
(a) $-\dfrac{\hbar^2}{2m}\dfrac{\partial^2\psi}{\partial x^2} + V\psi = E\psi$ (Schr. eq.)



$\psi_a = \psi_0 \sin\dfrac{\pi x}{a}$

$-\dfrac{\hbar^2}{2m}\left(-\dfrac{\pi^2}{a^2}\right) + 0 = \boxed{E_0 = \dfrac{\pi^2\hbar^2}{2ma^2}}$

(b)



(c) $\left.\begin{aligned}\psi_b &= \psi_{bo}\sin\left(\frac{\pi}{a}-\delta\right)x \\ \psi_c &= \psi_{co}\, e^{-\gamma x}\end{aligned}\right\}$ Solve for $\rho \equiv \psi_{co}/\psi_{bo}$, δ, γ, ε by matching amplitude & slope at $x=a$,

and solving Schrödinger equation in both regions for $E_0+\varepsilon$. From Schr. eq.,

Inside: $+\dfrac{\hbar^2}{2m}\left(\dfrac{\pi}{a}-\delta\right)^2 \simeq E_0 - \dfrac{\pi\hbar^2\delta}{ma} \simeq E_0+\varepsilon$ (1)

Outside: $-\dfrac{\hbar^2}{2m}\gamma^2 = -V_0 + E_0 + \varepsilon \simeq -V_0$ so $\gamma \simeq \dfrac{\sqrt{2mV_0}}{\hbar}$ (2)

Match ψ's: $\psi_{bo}\sin\left(\dfrac{\pi}{a}-\delta\right)a = \psi_{bo}\sin\delta a \simeq \psi_{bo}\,\delta a$ (3)

$\simeq \psi_{co}e^{-\gamma a} \simeq \psi_{co}\, e^{-\sqrt{2mV_0}\,a/\hbar}$

Match $(\partial\psi/\partial x)$: $-\psi_{bo}\dfrac{\pi}{a} \simeq -\gamma\psi_{co}e^{-\gamma a}$ (4)

$= -\dfrac{\sqrt{2mV_0}}{\hbar}\psi_{co}\, e^{-\frac{\sqrt{2mV_0}\,a}{\hbar}}$

(4) gives $\rho = \dfrac{\psi_{co}}{\psi_{bo}} = \dfrac{\pi\hbar\, e^{\sqrt{2mV_0}\,a/\hbar}}{a\sqrt{2mV_0}}$

(3) gives $\delta = \dfrac{\psi_{co}\, e^{-\sqrt{2mV_0}\,a/h}}{\psi_{bo}\, a} = \dfrac{\pi\hbar}{a^2\sqrt{2mV_0}}$

$\dfrac{\varepsilon}{E_0} = -\sqrt{\dfrac{2\hbar^2}{ma^2V_0}}$

(1) gives $\varepsilon = -\dfrac{\pi\hbar^2}{ma}\delta = -\dfrac{\pi\hbar^2}{ma}\dfrac{\pi\hbar}{a^2\sqrt{2mV_0}} = \boxed{\dfrac{-\pi^2\hbar^3}{ma^3\sqrt{2mV_0}}}$

Quantum Mechanics

9. The Gaussian wave packet $\Psi(x, t)$ is built out of plane waves according to the spectral function $a(k) = (C\alpha/\sqrt{\pi})e^{-\alpha^2 k^2}$ where C and α are constants, and k is the wave vector. Calculate $\Psi(x, t)$ for this packet and determine an expression for the width of the packet as a function of time, $\Delta x(t)$.

Spreading of a Gaussian Packet

Given $a(k) = \frac{C\alpha}{\sqrt{\pi}} e^{-\alpha^2 k^2}$

and
$$\Psi(x,t) = \int_{-\infty}^{\infty} a(k)\, e^{i(kx-\omega(k)t)}\, dk$$
✓

For free particle $\hbar\omega = E = \frac{\hbar^2 k^2}{2m}$ so $\omega = \left(\frac{\hbar}{2m}\right)k^2$

✓ 3

and
$$\Psi(x,t) = \int_{-\infty}^{\infty} \frac{C\alpha}{\sqrt{\pi}} e^{ikx}\, e^{-\alpha^2 k^2}\, e^{-i\left(\frac{\hbar t}{2m}\right)k^2}\, dk$$

Now pull trick of completing the square

$$ikx - \left(\alpha^2 + i\left(\frac{\hbar t}{2m}\right)\right)k^2 \quad ; \quad \text{Let } \beta^2 = \alpha^2 + i\left(\frac{\hbar t}{2m}\right)$$

Write
$$ikx - \beta^2 k^2 = -\left(\beta k \mp ix/2\beta\right)^2 - x^2/4\beta^2$$

So
$$\Psi(x,t) = \frac{C\alpha}{\sqrt{\pi}} \frac{1}{\beta} \int_{-\infty}^{\infty} e^{-z^2} e^{-x^2/4\beta^2}\, dz$$

where $z = (\beta k - ix/2\beta)$

$$\Psi(x,t) = \frac{C\alpha}{\beta\sqrt{\pi}} e^{-x^2/4\beta^2} \underbrace{\int_{-\infty}^{\infty} e^{-z^2} dz}_{=\sqrt{\pi}}$$

✓

So $$\boxed{\Psi(x,t) = \frac{C\alpha}{\left[\alpha^2 + i\left(\frac{\hbar t}{2m}\right)\right]^{1/2}} e^{-x^2/4\beta^2}}$$ ✓ 2

The complex denominator can be rewritten as

$$\Psi(x,t) = \frac{C\alpha\left[\alpha^2 - i\left(\frac{\hbar t}{2m}\right)\right]^{1/2}}{\left[\alpha^4 + \left(\frac{\hbar t}{2m}\right)^2\right]^{1/2}} e^{-x^2/4\beta^2}$$

So amplitude decreases in time.

The argument in exponent is

$$e^{-\frac{x^2}{4(\alpha^4 + (\hbar t/2m)^2)}(\alpha^2 - i(\hbar t/2m))}$$

and have oscillatory part with Gaussian envelope.

$$e^{\left(-\frac{x^2\alpha^2}{4(\alpha^4 + (\hbar t/2m)^2)}\right)}$$

which has a width based on $e^{-x^2/2(\Delta x)^2}$ of

$$(\Delta x)^2 = \frac{2}{\alpha^2}\left(\alpha^4 + (\hbar t/2m)^2\right)$$

$$(\Delta x) = \frac{\sqrt{2}}{\alpha}\sqrt{\alpha^4 + (\hbar t/2m)^2}$$ ✓ 2

OR

$$\psi(x,t) = \frac{C\alpha}{\left[\alpha^2 + i\left(\frac{\hbar t}{2m}\right)\right]^{1/2}} e^{-x^2/4\beta^2} \quad \text{where } \beta^2 = \alpha^2 + i\left(\frac{\hbar t}{2m}\right)$$

$$|\psi|^2 = \frac{C^2\alpha^2}{\left[\alpha^4 + \left(\frac{\hbar t}{2m}\right)^2\right]^{1/2}} e^{-\frac{x^2}{4\left(\alpha^2 + i\left(\frac{\hbar t}{2m}\right)\right)} - \frac{x^2}{4\left(\alpha^2 - i\left(\frac{\hbar t}{2m}\right)\right)}}$$

$$\exp \Rightarrow \quad -\frac{x^2}{4}\left(\frac{2\alpha^2}{(\alpha^4 + (\hbar t/2m)^2}\right)$$

$$|\psi|^2 = \frac{C^2\alpha^2}{\left[\alpha^4 + (\hbar t/2m)^2\right]^{1/2}} e^{-\frac{x^2\alpha^2}{2(\alpha^4 + (\hbar t/2m)^2)}}$$

Then $$|\psi| = \frac{C\alpha}{\left[\alpha^4 + (\hbar t/2m)^2\right]^{1/4}} e^{-\frac{x^2\alpha^2}{(\alpha^4 + (\hbar t/2m)^2)}}$$

Width for $|\psi|^2$, based on Gaussian envelope of width Δx according to $e^{-x^2/2(\Delta x)^2}$ is

$$\boxed{\Delta x^2 = \frac{(\alpha^4 + (\hbar t/2m)^2)}{\alpha^2}}$$

Quantum Mechanics

10. Two spin-1 particles are coupled by the Hamiltonian

$$H = -J\, S_1 \bullet S_2$$

where J is a positive constant.

(a) Find the energy eigenvalues and eigenfunctions for this system.

(b) Suppose the system is perturbed by adding to the Hamiltonian a term H':

$$H' = -D\left\{ (S_1^z)^2 + (S_2^z)^2 \right\}$$

where $0 < D << J$. Find the first order splitting of the ground state energy.

Helpful quantities may be the spin raising and lowering operators:

$$S_\pm |\, s\, , m > \;= \sqrt{s(s+1) - m(m \pm 1)}\;\; |\, s\, , m \pm 1 >$$

Quantum Mechanics

Two spin-1 particles are coupled by the Hamiltonian

$$H = -J\ S_1 \cdot S_2$$

where J is a positive constant.
(a) Find the energy eigenvalues and eigenfunctions for this system.
(b) Suppose the system is perturbed by adding to the Hamiltonian a term H':

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Helpful quantities may be the spin raising and lowering operators:

$$S_\pm |s\ m\rangle = \sqrt{s(s+1) - m(m\pm1)}\ |s\ m\pm1\rangle$$

Soln

(a) $\vec{S}_1\cdot\vec{S}_2 = \frac{1}{2}(\vec{S}_1+\vec{S}_2)^2 - 2\hbar^2$ for $s_1 = 1$

for $S_{tot} = 2$: (5-fold deg. state; $|s\,m\rangle = |22\rangle, |21\rangle$, etc.)

$$E = -J\left(\frac{6\hbar^2}{2} - 2\hbar^2\right) = \underline{-J\hbar^2}$$

for $S_{tot} = 1$: (3-fold deg; $|s\,m\rangle = |11\rangle, |10\rangle, |1\,-1\rangle$)

$$E = -J\left(\frac{2\hbar^2}{2} - 2\hbar^2\right) = \underline{+J\hbar^2}$$

for $S_{tot} = 0$: (nondeg. $|s\,m\rangle = |00\rangle$)

$$E = -J(0 - 2\hbar^2) = \underline{+2J\hbar^2}$$

(b) Ground state (unperturbed) is the $s=2$ multiplet and is degenerate; we will have to diagonalize H' in this subspace. Expand $|s\,m\rangle$ in $|1m_1 1m_2\rangle$ states, starting from $|22\rangle = |1111\rangle$ & using $S_- = S_{1-} + S_{2-}$.

$$|21\rangle = \frac{1}{\sqrt{2}}\left(|1011\rangle + |1110\rangle\right)$$

$$|20\rangle = \frac{1}{\sqrt{6}}\left(|1\,-1\,11\rangle + |11\,1\,-1\rangle + 2|1010\rangle\right)$$

$|2\,-1\rangle$ etc.

→

then $H'|22\rangle = -2D\hbar^2|22\rangle$

$$H'|21\rangle = -D\hbar^2|21\rangle$$

$$H'|20\rangle = -D\hbar^2\frac{2}{\sqrt{6}}\left(|1-1\,11\rangle + |11\,1-1\rangle\right)$$

$$H'|2-1\rangle = -D\hbar^2|2-1\rangle$$

$$H'|2-2\rangle = -2D\hbar^2|2-2\rangle$$

So H' is diagonal in this subspace, & the 1st order corrections are

$-2D\hbar^2$ (doubly degen)

$-D\hbar^2$ (" ")

$-\frac{2}{3}D\hbar^2$

Mathematical Physics

11. An object of mass m falls from rest in a location where the gravitational acceleration is g, and in a medium where it is subject to a retarding force $-\alpha v^2$ (where α is a constant and v the velocity).

   (a) What is the object's terminal velocity v_t?

   (b) From dimensional analysis, estimate a characteristic distance (in terms of the given parameters) over which the terminal velocity is attained.

   (c) Find the velocity v as a function of the distance x which the object falls.

9γ

Math Physics

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(c) Find the velocity v as a function of the distance x which the object falls.

Soln

(a) $m\ddot{x} = mg - \alpha v^2$

when $\ddot{x} = 0$ $\quad v = \sqrt{\frac{mg}{\alpha}} = v_t$

(b) Dimensions (M = mass, L = length, T = time)

$[\alpha] = ML^{-1}$, $[g] = LT^{-2}$, $[m] = M$

– g doesn't enter, because it is the only parameter with time in its dimension. Construct a length from α & m:

$$\lambda \sim m/\alpha$$

(c) $m\dot{v} = mg - \alpha v^2$

Now we want $v(x)$, not $v(t)$, so write $\dot{v} = \frac{dv}{dx}\dot{x} = v\frac{dv}{dx}$
$= \frac{1}{2}\frac{d}{dx}v^2$

$$\frac{d(v^2)}{dx} + \frac{2\alpha}{m}v^2 = 2g$$

$$e^{\frac{2\alpha}{m}x}\frac{dv^2}{dx} + \frac{d}{dx}\left(e^{\frac{2\alpha}{m}x}\right)v^2 = 2g\,e^{2\alpha x/m}$$

$$e^{\frac{2\alpha x}{m}}v^2 = 2g\frac{m}{2\alpha}\left(e^{\frac{2\alpha x}{m}} - 1\right)$$

$$v = \sqrt{\frac{mg}{\alpha}\left(1 - e^{-\frac{2\alpha x}{m}}\right)}$$

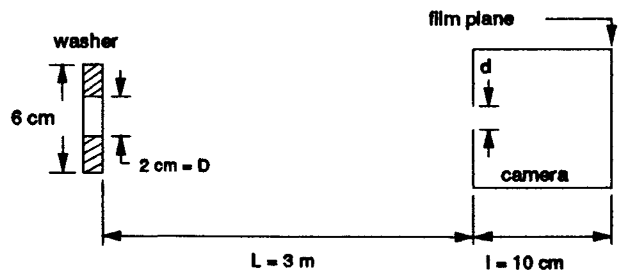
N.B. One can also integrate directly to find $v(t)$: $v(t) = \sqrt{\frac{mg}{\alpha}}\;\frac{e^{2t\sqrt{\frac{\alpha g}{m}}} - 1}{e^{2t\sqrt{\alpha g/m}} + 1}$

Optics

12. You want to photograph a black washer, 6 cm in diameter with a 2 cm diameter hole, using a pinhole camera (which has no lens) having a 10 cm distance from the round pinhole to the film plane. The washer is 3 m from the camera.

    (a) Explain what happens to the image quality, and why it happens, as the pinhole diameter *d* is varied over a wide range.

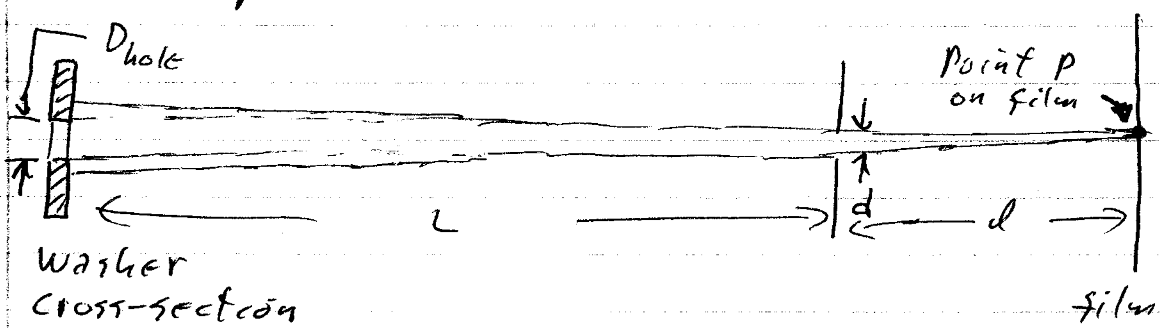
    (b) Over what approximate range of pinhole diameter *d* will the image be sharp enough to show that the washer is an object with a hole through it? (No complicated calculation expected.)



Not to Scale

As d becomes large, the image blurs because a point on the film sees a large portion of the washer so its image cannot have sharp edges.

As d becomes small, diffraction spreads out what on the ray theory would be a point of light at the film, and again the image blurs.

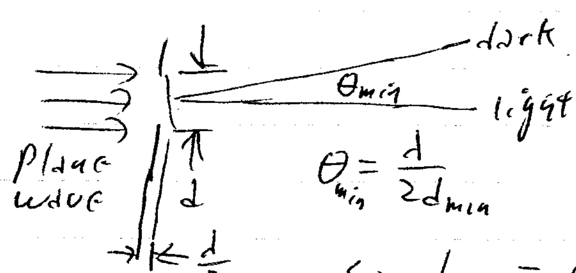


Roughly, when $\frac{d}{l} > \frac{2D}{L}$, hole can't be seen.

$$d_{max} \simeq \frac{2 \times 2\,cm}{3000\,cm} \times 10\,cm = \frac{40}{3000}\,cm = \boxed{0.013\,cm}$$

Hole image on screen has $D_i = \frac{l}{L} D_{hole} = \frac{10 \times 2\,cm}{3000}$

$= 0.0067\,cm$



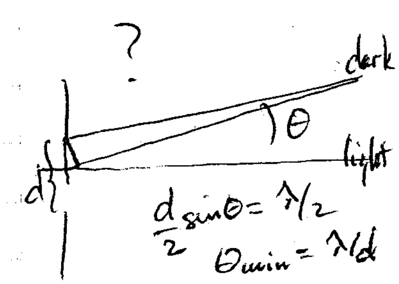
$\theta_{min} = \frac{d}{2 d_{min}}$

Diffraction spot size is $\sim 2\theta_{min} l = \frac{\lambda l}{d_{min}} \left(= \frac{l D}{L} \text{ to blur hole image} \right)$

so $d_{min} = \frac{\lambda L}{D} = \frac{5 \times 10^{-5}\,cm\ 3 \times 10^{3}\,cm}{2\,cm}$ $\times 2$?

$\boxed{d_{min} \simeq 0.0075\,cm}$

?



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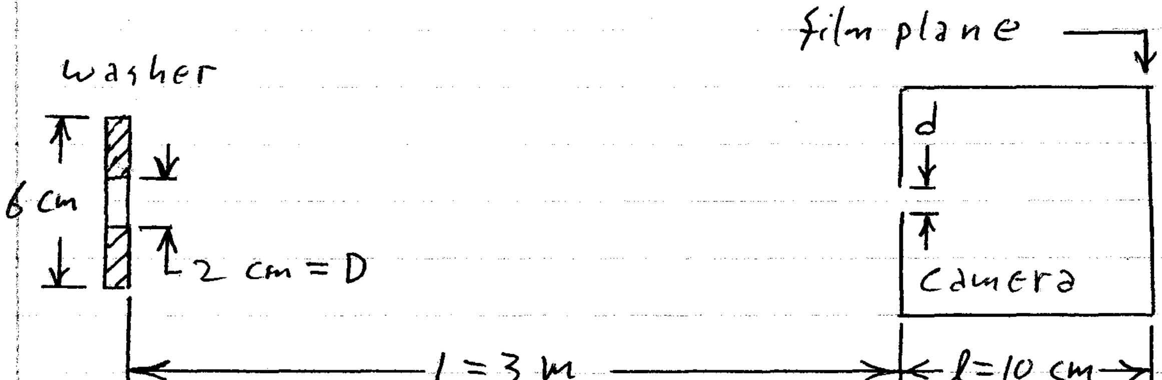
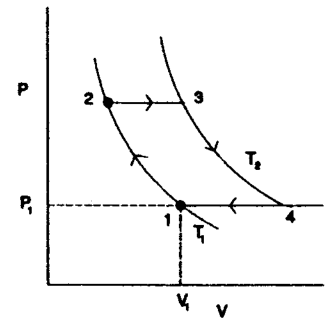


Not to scale

Thermodynamics

13. n moles of an ideal gas are compressed and expanded according to the PV diagram shown.



Find the net work done by the gas as the system starts from 1, goes around the cycle, and returns to 1.

Suppose T_1 = 100 K and T_2 = 300 K, $n = 2$, $V_1 = 8.314 \times 10^{-3}$ m^3, and $V_2 = V_1/2$. Note: $R = 8.31$ J/mole•K

$P_1 = 2 \times 10^5$ Pa $\quad V_1 = 8.314 \times 10^{-3} \quad T_1 = 100$

$P_2 = 4 \times 10^5$ Pa $\quad V_2 = 4.155 \times 10^{-3} \quad T_2 = 100$

$P_3 = 4 \times 10^5$ Pa $\quad V_3 = 12.46 \times 10^{-3} \quad T_2 = 300$

$P_4 = 2 \times 10^5$ Pa $\quad V_4 = 24.9 \times 10^{-3} \quad T_2 = 300$

$W_{12} = 2(8.314)100 \ln \frac{4.155}{8.314}$

$= 2(8.314)(100)(-.693) = -1152$ J

$W_{23} = (4 \times 10^5)(12.45 - 4.155) \times 10^{-3} = 33.18 \times 10^2 = 3318$ J

$W_{34} = 2(8.314)(300) \ln \frac{24.9}{12.46} = 2(8.314)(100)(.693) = 3456$

$W_{41} = (2 \times 10^5)(8.314 - 24.9) \times 10^{-3} = 33.18 \times 10^2 = -3318$ J

$W_{net} = 2304$ J

$pV = nRT$ $\qquad W = \int p\,dV = nRT\int \frac{dV}{V} = nRT_1 \ln \frac{V_2}{V_1}$

From (1 to 2)

From 2 to 3 $\qquad W = p_2 (V_3 - V_2)$

From 3 to 4 $\qquad W = nRT_2 \ln \frac{V_4}{V_3}$

From 4 to 1 $\qquad W = p_1 (V_1 - V_4)$

$P_1 = \frac{2 (\cancel{8.314})\, 100}{\cancel{8.314}\, 10^{-3}} = 2 \times 10^5 \text{ Pa}$

$P_2 = 2P_1$, $V_4 = 3V_1$, $V_3 = 3V_2$ chug chug.

Statistical Mechanics

14. Consider a system of three particles, each of which can have energy 0, ε_0, $2\varepsilon_0$, $3\varepsilon_0$, *etc.*

    (a) Suppose the three particles have fixed total energy $4\varepsilon_0$. Calculate the entropy of the system if:
    (i) the particles are distinguishable (*i.e.*, classical)
    (ii) the particles are identical fermions (disregard spin)
    (iii) the particles are identical bosons (disregard spin)

    (b) If the particles are identical bosons, and the system is maintained at temperature T, find the ratio P_1/P_2, where P_n is the probability that the toal system contains energy $n\varepsilon_0$.

Statistical Mechanics

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(iii) " " " " bosons

(b) If the particles are identical bosons, and the system is maintained at temperature T, find the ratio P_1/P_2, where P_n is the probability that the total system contains energy $n\varepsilon_0$.

Soln

(a) Use $S = k_B \ln g$ where g = # of microstates; evaluate g by combinatorics or 'brute force':

(i) $(\epsilon_1, \epsilon_2, \epsilon_3) = (4,0,0)$ + perm. $\to$ 3 states.
$(3,1,0)$ + " $\to$ 6 "
$(2,1,1)$ + " $\to$ 3 "
$(2,2,0)$ + " $\to$ 3 "

$g = 15 \Rightarrow S = k_B \ln 15$

(ii) no 2 particles can have same energy - only allowable microstate is $(3,1,0) \Rightarrow S = k_B \ln(1) = 0$
& permutations don't give new states

(iii) microstates are $(4,0,0), (3,1,0), (2,1,1), (2,2,0) \Rightarrow g = 4$

& $S = k_B \ln(4)$

(b) For bosons there are 2 microstates w/ $n=2$ (namely $(0,1,1)$ and $(0,0,2)$), and one microstate w/ $n=1$ (" $(0,0,1)$)

So
$$\frac{P_2}{P_1} = \frac{2e^{-2\epsilon_0/kT}}{e^{-\epsilon_0/kT}} = 2e^{-\epsilon_0/kT}$$

Atomic Physics

15. (a) Apply Hund's rules to find the ground state angular momentum for the following elements (with configurations of the outer shell electrons shown):

    (i) Cu ($4s^1 3d^{10}$)
    (ii) Ni ($4s^2 3d^8$)

    (b) Explain why the total electronic spin of the helium atom in the ground state is zero. What is the total spin in the 1st excited state?

9a

Atomic Physics/Quantum Mechanics

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(ii) Ni $(4s^2 3d^8)$

(b) Explain why the total electronic spin of the helium atom in the ground state is zero. What is the total spin in the 1st excited state?

Soln:

(a) (i) d shell is complete, so we have $s = 1/2$, $l = 0$ and so $J = 1/2 \Rightarrow {}^2S_{1/2}$ no rules needed!

$^{2S+1}L_J$

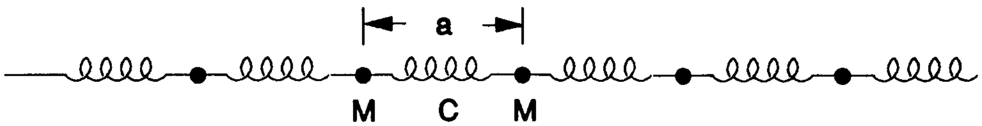
(ii) 2 vacancies in the d shell so $S = 1$ (largest s) and $l = 3$. (largest compat. w/ $s=1$). Then $J = |l+s| = 4$ since shell is more than $\frac{1}{2}$ full.

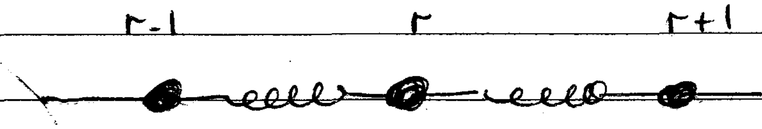
$\Rightarrow {}^3F_4$

(b) In the ground state the 2 electrons have the same spatial (1s) wavefunction (ie, the spatial state is symmetric), and so the spin state is antisymmetric; $S_{tot} = 0$. In the 1st excited state the spatial state may be symmetric or antisymmetric, but Coulomb repulsion between the electrons raises the energy of the symmetric state compared to the antisymmetric. Thus the antisymmetric spatial state and symmetric $(S=1)$ spin state is the 1st excited state

<u>Solid State Physics</u>

16. Consider the linear monatomic lattice shown below, with atoms of mass M separated at equilibrium by a distance a. Consider the interactions between neighboring atoms to follow Hooke's law, with restoring force constant C, and consider only nearest neighbor interactions. Derive an expression for the dispersion curve of this lattice, *i.e.*, an expression for the oscillation frequency as a function of wave number, $\omega(k)$, and draw a sketch of $\omega(k)$.





Consider displacement of atom r, u_r

Force on r is $+C(u_{r+1}-u_r)$ 1

and $-C(u_r-u_{r-1})$ 1

Write $M\dfrac{d^2u}{dt^2} = \sum \text{forces} = C(u_{r+1}-2u_r+u_{r-1})$ 3

Assume form for displacement $u_r = Ae^{i(kx-\omega t)}$ 1

then

$$-M\omega^2 u_r = C(u_{r+1}-2u_r+u_{r-1})$$ 1

Define x for each atom as $x = ra$ 1

then

$$-M\omega^2 u_r = CA\left(e^{ik(r+1)a} - 2e^{ikra} + e^{ik(r-1)a}\right)e^{-i\omega t}$$

$$= C\left(e^{ika} - 2 + e^{-ika}\right)u_r$$

$$\omega^2 = -C/M\,(2\cos ka - 2)$$

$$\boxed{\omega^2 = 2C/M\,(1-\cos ka)}$$ 2

or

$$= 2C/M\; 2\sin^2\frac{ka}{2}$$

$$\omega = (4C/M)^{1/2}\sin\frac{ka}{2}$$