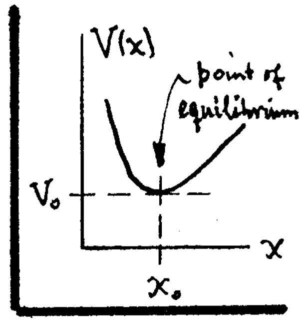
Schrödinger's Problem for a 1D Simple Harmonic Oscillator

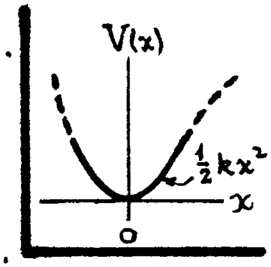
C. Simple Harmonic Oscillator [Davydov, Sec. 26].



1. For any attractive potential $V(x)$ which shows a minimum (or equilibrium pt) at some pt. x_0, do a Taylor expansion:

$$\rightarrow V(x) = V(x_0) + \cancel{V'(x_0)}^{\,0}(x-x_0) + \frac{1}{2}V''(x_0)(x-x_0)^2 + \ldots \quad (1)$$

near x_0. Now at x_0, have $V'(x_0) = 0$ [i.e. the force is zero there], and have $V''(x_0) > 0$ [$V(x)$ is concave upward]. Shift the zero of coordinates so that $x_0 = 0$ and $V(x_0) = V_0 = 0$. Then the leading term of $V(x)$ is...



$$V(x) = \frac{1}{2}kx^2, \text{ near } x=0, \text{ w/ } k = V''(0) > 0, \text{ spring cnst.} \quad (2)$$

This form of $V(x)$ corresponds to a Hooke's Law force: $F = -\frac{\partial V}{\partial x} = -kx$; that is why k is called a "spring cnst". Classically, a particle of mass m will execute a simple harmonic oscillation (SHO) about $x=0$ at the characteristic frequency: $\omega = \sqrt{k/m}$ (so long as the oscillation amplitude is not too large). Since $k = m\omega^2$, V in Eq. (2) is sometimes written: $V(x) = \frac{1}{2}m\omega^2x^2$.

The approximate form $V = \frac{1}{2}kx^2$, for small oscillations of m about an equilibrium pt. in an essentially arbitrary (but smoothly varying) attractive potential, has a ~ universal application -- this is the way bound states begin in most potential wells. The classical eqtn-of-motion, viz.

$$\rightarrow m\ddot{x} = -kx, \text{ w/ } \ddot{x} + \omega^2 x = 0, \quad (3)$$

also has many applications; it is applied whenever the system of interest can be thought of as being connected by springs. There are successful theories of the vibrations of diatomic molecules, crystal vibrations (phonons), etc. based on this simple idea. So now we want the QM -- i.e. a solution to:

$$\boxed{\frac{d^2\psi}{dx^2} + \frac{2m}{\hbar^2}\left(E - \frac{1}{2}m\omega^2x^2\right)\psi = 0} \quad \checkmark \text{ Schrödinger's Eqtn: for a 1D SHO @ freq. } \omega. \quad (4)$$

Simple Harmonic Oscillator (cont'd)

2. It is convenient to write Schrödinger's Eqtn, Eq.(4) in terms of dimensionless variables ξ (position) and λ (energy), as defined by...

$$\underline{\xi} = \left(\sqrt{m\omega/\hbar}\right)x \;,\quad \underline{\lambda} = 2E/\hbar\omega \;\Rightarrow\; \boxed{\frac{d^2\psi}{d\xi^2} + (\lambda - \xi^2)\psi = 0.} \tag{5}$$

As $|\xi| \to \infty$, the eqtn behaves asymptotically as...

$$\rightarrow \frac{d^2\psi}{d\xi^2} - \xi^2\psi \simeq 0 \Rightarrow \underline{\underline{\psi(\xi) \simeq \exp\left(\pm\tfrac{1}{2}\xi^2\right), \text{ as } |\xi| \to \infty.}} \tag{6}$$

$\psi \sim e^{+\frac{1}{2}\xi^2}$ is ruled out because ψ can never be infinite (esp. at $|\xi| = \infty$). To pick off the $\psi \sim e^{-\frac{1}{2}\xi^2}$, we look for a solution of the form...

$$\rightarrow \psi(\xi) = u(\xi)\exp\left(-\tfrac{1}{2}\xi^2\right), \tag{7}$$

Where $u(\xi)$ is a polynomial in ξ to be found. Limiting $u(\xi)$ to a combination of powers of ξ, say ξ^n at most, ensures $\psi(\xi) \to 0$ as $|\xi| \to \infty$. Put (7) into (5) to get:

$$\boxed{\frac{d^2u}{d\xi^2} - 2\xi\frac{du}{d\xi} + (\lambda - 1)u = 0} \tag{8}$$

Hermite's Differential Eqtn (a confluent hypergeometric ODE)

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3. The easiest way to "solve" Eq.(8) is to recognize it as a confluent hypergeometric ODE, i.e. an ODE of the form...*

$$\rightarrow z\frac{d^2u}{dz^2} + (c - z)\frac{du}{dz} - au = 0 \tag{9}$$

w/ $\underline{\underline{z = \tfrac{1}{2}\xi^2}}$, $\underline{\underline{2a = -(\lambda - 1)}}$, $\underline{\underline{c = 1/2}}$. (regular singularity @ $z=0$, essential singularity @ $z=\infty$)

Such an equation has a power series solution of the form...

$$\rightarrow u(\xi) = F(a,c;z) = \frac{\Gamma(c)}{\Gamma(a)}\sum_{k=0}^{\infty}\left[\frac{\Gamma(a+k)}{\Gamma(c+k)\Gamma(k+1)}\right]z^k, \quad \text{w/ } z = \tfrac{1}{2}\xi^2; \tag{10}$$

... and a second solution for $u(\xi)$ is $z^{1-c}F(a-c+1, 2-c; z)$; this solution is an independent solution to the ODE when $c \neq$ integer (as is true, here). The F's are called "confluent hypergeometric series", and they are "polynomials" of ∞ degree unless the index $a = -n$ is some (−)ve integer, with

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* Davydov gives a concise account of this ODE in his Appendix D, p. 619.

Simple Harmonic Oscillator (cont'd)

$n = 0, 1, 2, 3, \ldots$ ¶. Since, in $\psi(\xi) = u(\xi)\exp(-\frac{1}{2}\xi^2)$, we need $u(\xi)$ = polynomial of finite degree (to keep ψ from diverging as $|\xi| \to \infty$) we impose the condition that in fact $\underline{\underline{a = -n}}$. This quantizes the SHO energies, as...

$$\rightarrow a = -\tfrac{1}{2}(\lambda - 1) = -n \Rightarrow \lambda = 2E/\hbar\omega = 2n+1;$$

by// $\boxed{E_n = (n + \tfrac{1}{2})\hbar\omega}$, w/ $n = 0, 1, 2, \ldots$ ← allowed SHO energies. (11)

For each n-value, there is an eigenfcn solution for $u(\xi)$. It is a polynomial of $n^{\underline{th}}$ degree, called a "Hermite polynomial" $H_n(\xi)$, and defined by...

$$H_n(\xi) \propto F(-n, \tfrac{1}{2}; \tfrac{1}{2}\xi^2),$$

by// $H_n(\xi) = (-)^n e^{\xi^2} \cdot (d/d\xi)^n e^{-\xi^2}$, $n = 0, 1, 2, \ldots$ (12)

The normalized wavefcns for the stationary states of the SHO are then

$\underline{\underline{\psi_n(\xi) = \mathcal{N}_n H_n(x) \exp(-\tfrac{1}{2}\xi^2)}}$, w/ $\mathcal{N}_n = 1/(2^n n! \sqrt{\pi})^{1/2}$. (13)

The treatment of the SHO problem via the large number of tabulated results available for the confluent hypergeometric ODE is very compact... and similar compact treatments can be done for many other QM problems (and just ODE's in general). This is a persuasive argument for learning the intricacies of hypergeometric & confluent hypergeometric ODE's, if you do not know them already.

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4. A less elegant way of solving Hermite's Eqtn (8) is by Frobenius' Method, viz.// → put power series: $\underline{u(\xi) = \sum_{n=0}^{\infty} a_n \xi^n}$, into: $\underline{u'' - 2\xi u' + (\lambda - 1)u = 0}$; solve for $\{a_n\}$. (14)

¶ When $a = -n$, the series coefficient in Eq. (10): $\frac{\Gamma(a+k)}{\Gamma(a)} = a(a+1)\cdots(a+k-1)$, will vanish for all $k > n$; the power series truncates at the term $z^k|_{k=n}$.

Solution of Hermite's ODE by Method of Frobenius.

By carrying out the operation in Eq.(14), we find $u'' - 2\xi u' + (\lambda-1)u = 0$ requires: $\sum_{n=0}^{\infty}$ [fcn of n & a_n] $\xi^n = 0$, and then by setting the [ ] = 0, we obtain

$$\rightarrow a_{n+2} = \left[\frac{2n+1-\lambda}{(n+1)(n+2)}\right] a_n, \quad \text{for } n = 0,1,2,3,\ldots \tag{15}$$

This is the "recursion relation" for the series coefficients that is specific to Hermite's ODE. Since we are solving a 2nd order ODE, there are two arbitrary consts in the solution; in (15), we can choose these to be a_0 & a_1. Then we generate two classes of solutions, of $\pm$ parity, as follows:

I. $a_0 \neq 0,\ a_1 = 0 \Rightarrow \begin{cases} a_2 = \frac{1}{2}(1-\lambda)a_0, \quad a_4 = \frac{1}{12}(5-\lambda)a_2, \text{ etc.}; \\ a_1 = a_3 = a_5 = \cdots = 0; \end{cases}$

$\psi(-\xi) = (+)\,\psi(+\xi)$

so $u(\xi) = \sum_{n=0}^{\infty} a_{2n}\xi^{2n}$ & $\psi(\xi) = u(\xi)e^{-\frac{1}{2}\xi^2}$, have (+) parity. (16A)

II. $a_0 = 0,\ a_1 \neq 0 \Rightarrow \begin{cases} a_2 = a_4 = a_6 = \cdots = 0; \\ a_3 = \frac{1}{6}(3-\lambda)a_1, \quad a_5 = \frac{1}{20}(7-\lambda)a_3, \text{ etc.} \end{cases}$

$\psi(-\xi) = (-)\,\psi(+\xi)$

so $u(\xi) = \sum_{n=0}^{\infty} a_{2n+1}\xi^{2n+1}$ & $\psi(\xi) = u(\xi)e^{-\frac{1}{2}\xi^2}$, have (−) parity. (16B)

Class I & II solutions, of $\pm$ parity, are chosen to emulate the reflection symmetry of the SHO potential $V(x) = \frac{1}{2}kx^2$, w/ $V(-x) = V(x)$. We did this also for the rectangular potential well (see CLASS NOTES, pp. Sol$^{\underline{n}}$s 1-2, Eq.(4)).

An essential requirement for the series solutions $u(\xi) = \sum_{k=0}^{\infty} a_k \xi^k$ as generated above are that they do not $\rightarrow \infty$ as fast as $\exp(+\frac{1}{2}\xi^2)$, when $|\xi| \rightarrow \infty$; otherwise $\psi(\xi) = u(\xi)e^{-\frac{1}{2}\xi^2}$ will diverge. A way to ensure that this is true is to make the $u(\xi)$'s = polynomials of finite degree. Look at (15):

$$a_{n+2}/a_n = [(2n+1)-\lambda]/(n+1)(n+2). \tag{17}$$

Evidently, if we impose that the parameter $\lambda = 2E/\hbar\omega$ [from Eq.(5)] is

Remarks on Quantized SHO to date.

Such that $\underline{\underline{\lambda = 2n+1}}$, then in (17) we have $a_{n+2} = 0$, and also all the subsequent a_N vanish for $N > n+2$. The series $u(\xi) = \sum_{k=0}^{\infty} a_k \xi^k \to \sum_{k=0}^{k=n} a_k \xi^k$ becomes a polynomial of finite degree n, and as $|\xi| \to \infty$, the wavefcn $\psi \sim \xi^n e^{-\frac{1}{2}\xi^2}$ properly vanishes. This condition quantizes the SHO energies, as...

$$\lambda = 2n+1 \Longrightarrow \boxed{E_n = (n+\tfrac{1}{2})\hbar\omega}, \text{ w/ } n = 0, 1, 2, \ldots \tag{18}$$

This result confirms the energies quoted in Eq. (11), p. Sol$^{\underline{n}}$s 12 preceding.

$\boxed{\text{REMARKS}}$ On the quantized SHO.

1. As for the rectangular potential well (pp. Sol$^{\underline{n}}$s 1-4), the system quantization results from boundary conditions imposed on ψ; here $\psi(\xi) \to 0$ as $|\xi| \to \infty$.

2. The condition $\lambda = 2n+1$ in (18) is a way of ensuring $\psi = u e^{-\frac{1}{2}\xi^2}$ vanishes at ∞. One can show it is the only way. Any choice w/ $\lambda \neq 2n+1$ will give an ∞ series for $u(\xi) = \sum_k a_k \xi^k$, and it can be shown that this series diverges faster than $e^{+\frac{1}{2}\xi^2}$ as $|\xi| \to \infty$. So $\lambda \neq 2n+1 \Rightarrow \psi = u e^{-\frac{1}{2}\xi^2}$ diverges.

3. The choice $\lambda = 2n+1$ in the series for $u(\xi)$ allows us to write...

$$\rightarrow u(\xi) = H_n(\xi) = \sum_{k=0}^{n} a_k \xi^k, \text{ w/ } a_{k+2} = \frac{2(k-n)}{(k+1)(k+2)} a_k \left\{ \begin{matrix} \text{Hermite} \\ \text{polynomials} \end{matrix} \right., \tag{19}$$

as particular solutions to Hermite's ODE, Eq. (8). More on these, later.

4. The energies in Eq. (18) bear on Planck's Hypothesis (p. Intro. 4, Eq. (10)). He could (and did) picture the atomic oscillators that emitted radiation from the walls of his BB cavity as obeying a SHO eqtn: $\underline{\ddot{\phi} + \omega^2 \phi = 0}$, w/ ϕ = vibration amplitude, and ω = oscillation freq. Then, by solving this eqtn for ϕ, and insisting ϕ be finite, he would get: $E_n = (n+\frac{1}{2})\hbar\omega$, as above. Planck used just the $n\hbar\omega$ part of E_n, discarding the "zero-point" energy $\frac{1}{2}\hbar\omega$. But the $\frac{1}{2}\hbar\omega$ is needed for consistency w/ uncertainty relations.

5. We shall now study the SHO eigenfcns $\psi_n(\xi) = N_n H_n(\xi) e^{-\frac{1}{2}\xi^2}$ in detail. We do this not only because the SHO is an important QM system, but also because the analysis that follows is typical of what can be done for other "interesting" QM wavefcns, and we should look at the details for at least one example. What we shall do here is:

(A) Find explicit forms for the Hermite polynomials.
(B) Show that $H_n' = 2n H_{n-1}$. Then find the "generating fcn" for the $H_n(\xi)$.
(C) Establish Rodrigues' Formula: $H_n(\xi) = (-)^n e^{\xi^2} (d/d\xi)^n e^{-\xi^2}$.
(D) Show orthogonality & find normalization for the SHO $\psi_n(\xi)$'s.

When we are done, we will be able to show (among other things) how the QM SHO emulates a classical oscillator when $n \to \infty$ (Correspondence Principle works!).

(A) Find Hermite polynomials explicitly.

→ By Eq. (19) above: $H_n(\xi) = \sum_{k=0}^{n} a_k \xi^k$, w/ $a_{k+2} = \frac{2(k-n)}{(k+1)(k+2)} a_k$. (20)

By repeating the recursion from a_N down to a_0 or a_1, one finds...

$$a_N = \left[\frac{2^N}{N!}(-)^{\frac{N}{2}}\left(\frac{N}{2}\right)!\right] a_0 \;\checkmark \text{ for } N \text{ even}; \quad a_N = \left[\frac{2^{N-1}}{N!}(-)^{\frac{N-1}{2}}\left(\frac{N-1}{2}\right)!\right] a_1 \;\checkmark \text{ for } N \text{ odd} \tag{21}$$

The cnsts a_0 & a_1 are still free. It is customary to choose them so that the highest order term in the series for H_n has a coefficient $a_n = 2^n$. Thus:

$$\left.\begin{aligned} &\underline{n \text{ even}}: \text{choose}: a_0 = (-)^{\frac{n}{2}} n!/\left(\tfrac{n}{2}\right)! \Rightarrow a_n = 2^n; \\ &\underline{n \text{ odd}}: \text{Choose}: a_1 = 2\cdot(-)^{\frac{n-1}{2}} n!/\left(\tfrac{n-1}{2}\right)! \Rightarrow a_n = 2^n. \end{aligned}\right\} \tag{22}$$

The resulting Hermite polynomials are then of the form (for n = both even & odd):

$$\rightarrow \boxed{H_n(\xi) = (2\xi)^n - \frac{n(n-1)}{1!}(2\xi)^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2!}(2\xi)^{n-4} - \dots} \tag{23}$$

It is easy to see that for n = odd: $H_n(0) = 0$; the last term in the series $\propto \xi$.

On the other hand, when n = even : $H_n(0) = a_0 = (-)^{\frac{n}{2}} n!/(\frac{n}{2})!$. The first few polynomials that follow from (23) are...

$$\rightarrow H_0(\xi) = 1,\; H_1(\xi) = 2\xi,\; H_2(\xi) = 4\xi^2 - 2,\; H_3(\xi) = 8\xi^3 - 12\xi, \ldots \quad (24)$$

(B) Show $H_n' = 2n H_{n-1}$. Find generating fcn for the $H_n(\xi)$.

First show the identity. The $H_n(\xi)$ satisfy the ODE: $\frac{d^2}{d\xi} H_n - 2\xi \frac{d}{d\xi} H_n + 2n H_n = 0$. If we operate through this eqtn by $d/d\xi$, we can write...

$$\rightarrow \frac{d^2}{d\xi^2} H_n' - 2\xi \frac{d}{d\xi} H_n' + 2(n-1) H_n' = 0, \text{ w/ } H_n' = dH_n/d\xi. \quad (25)$$

But Eq. (25) is also satisfied by H_{n-1}. Then: $H_n' = C H_{n-1}$, where C = cnst. To fix C, look at H_n' & H_{n-1} as $\xi \to$ large. The leading terms are...

$$\xi \to \infty : H_n' \simeq 2^n n \xi^{n-1}, \text{ and : } H_{n-1} \simeq 2^{n-1}\xi^{n-1} = \frac{1}{2n} H_n'$$

$$\text{so// } \underline{\underline{H_n' = C H_{n-1}, \text{ with } C = 2n.}} \quad (26)$$

This relation is useful in finding the "generating fcn" $g(s,\xi)$ for the Hermite polynomials. By definition, $g(s,\xi)$ is that fcn which -- expanded in a Taylor Series in s -- has the $H_n(\xi)$ as expansion coefficients, i.e.

$$\rightarrow g(s,\xi) = \sum_{n=0}^{\infty} \frac{s^n}{n!} H_n(\xi). \quad (27)$$

To find an explicit form for g (whose uses will become apparent), differentiate (27) by $\partial/\partial\xi$, and use (26)...

$$\rightarrow \partial g/\partial\xi = \sum_{n=0}^{\infty} \frac{s^n}{n!} H_n'(\xi) = \sum_{n=1}^{\infty} \frac{s^n}{n!} \cdot 2n H_{n-1}(\xi) = 2s \underbrace{\sum_{n=1}^{\infty} \frac{s^{n-1}}{(n-1)!} H_{n-1}(\xi)}_{g(s,\xi),\text{ again}}$$

$$\text{so// } \partial g/\partial\xi = 2sg, \text{ and// } \underline{\underline{g(s,\xi) = g(s,0)\, e^{2s\xi}}} \quad (28)$$

But $g(s,0)$ can be evaluated from the defining eqtn (27), as...

$$\rightarrow g(s,0) = \sum_{k=0}^{\infty} \frac{s^{2k}}{(2k)!} H_{2k}(0) = \sum_{k=0}^{\infty} (-)^k \frac{s^{2k}}{k!} = e^{-s^2}. \quad (29)$$

Rodrigues' Formula for H_n. Orthogonality & Normalization of ψ_n.

At this point, we have an expression for the generating fcn g, i.e...

$$\rightarrow g(s,\xi) = \sum_{n=0}^{\infty} \frac{s^n}{n!} H_n(\xi) = e^{-s^2+2s\xi} = e^{\xi^2-(s-\xi)^2}. \tag{30}$$

This shows $g(0,\xi) = H_0(\xi) = 1$, so the norm is correct, per Eq. (24).

(C) Establish Rodrigues' Formula: $H_n(\xi) = (-)^n e^{\xi^2} (d/d\xi)^n e^{-\xi^2}$.

The Taylor Series for the generating fcn in Eq. (30) can also be written:

$$g(s,\xi) = \sum_{n=0}^{\infty} \frac{s^n}{n!} \left[\left(\frac{\partial}{\partial s}\right)^n g(s,\xi) \right]\Bigg|_{s=0}, \quad \text{so: } H_n(\xi) = \left[\left(\frac{\partial}{\partial s}\right)^n g(s,\xi) \right]\Bigg|_{s=0}$$

i.e. $\rightarrow$

$$H_n(\xi) = e^{\xi^2} (\partial/\partial s)^n e^{-(s-\xi)^2}\Big|_{s=0} \tag{31}$$

It is clear why g is called a "generating fcn"... the $H_n(\xi)$ are generated by g and its derivatives. To simplify (31), note that we are differentiating a fcn of $(s-\xi)$ only, so that the operation $\partial/\partial s$ is equivalent to $-\partial/\partial\xi$, and $(\partial/\partial s)^n = (-)^n (\partial/\partial\xi)^n$. Then we have...

$$\boxed{H_n(\xi) = (-)^n e^{\xi^2} (\partial/\partial\xi)^n e^{-\xi^2}}. \tag{32}$$

This is Rodrigues' formula, as desired.

(D) Orthogonality & normalization for the SHO eigenstates $\psi_n(x)$.

Write the SHO eigenfcns as

$$\psi_n(x) = N_n e^{-\frac{1}{2}\xi^2} H_n(\xi), \quad \text{w/ } \xi = \sqrt{m\omega/\hbar}\, x. \tag{33}$$

We want to demonstrate orthogonality, i.e. $\int_{-\infty}^{\infty} \psi_n^*(x)\psi_m(x)\,dx = 0$, when $m \neq n$. And we want to normalize the ψ_n, i.e. find the cnst N_n such that $\int_{-\infty}^{\infty} |\psi_n(x)|^2 dx = 1$. Both tasks are accomplished by considering the following integral over generating fcns:

$$\rightarrow J(s,t) = \int_{-\infty}^{\infty} g(s,\xi)\, g(t,\xi)\, e^{-\xi^2} d\xi. \tag{34}$$

Orthogonality & Normalization of the SHO ψ_n's (cont'd)

In detail, (34) reads...

$$\rightarrow \mathcal{J}(s,t) = \int_{-\infty}^{\infty} \underbrace{e^{-s^2+2s\xi}\, e^{-t^2+2t\xi}}_{g \text{ form in Eq. (30)}}\, e^{-\xi^2} d\xi = \overbrace{\sum_{m,n=0}^{\infty} \frac{s^n}{n!}\cdot\frac{t^m}{m!}}^{\text{def}^{\underline{n}} \text{ of } g \text{ in Eq. (27)}} \int_{-\infty}^{\infty} e^{-\xi^2} H_m(\xi) H_n(\xi)\, d\xi \quad (35)$$

The LHS integral is tabulated...

$$\rightarrow \mathcal{J}(s,t) = e^{-(s^2+t^2)} \int_{-\infty}^{\infty} e^{-\xi^2 + 2(s+t)\xi} d\xi = e^{-(s^2+t^2)} \cdot \sqrt{\pi}\, e^{(s+t)^2} = \sqrt{\pi}\, e^{2st}. \quad (36)$$

Expand this result in a power series and rewrite (35) as...

$$\left[\!\left[\sum_{n=0}^{\infty} \frac{s^n}{n!} (\sqrt{\pi}\, 2^n) t^n = \sum_{n=0}^{\infty} \frac{s^n}{n!} \left\{ \sum_{m=0}^{\infty} \underbrace{\left(\frac{1}{m!} \int_{-\infty}^{\infty} e^{-\xi^2} H_m(\xi) H_n(\xi)\, d\xi \right)}_{\text{call this integral } I_{mn}} t^m \right\}. \right. \quad (37)$$

For (37) to be an identity, must have...

A $I_{mn} \equiv 0$ for $m \neq n$, so: $\int_{-\infty}^{\infty} e^{-\xi^2} H_m(\xi) H_n(\xi)\, d\xi \equiv 0$, when $m \neq n$.

For: $\psi_k(x) = N_k e^{-\frac{1}{2}\xi^2} H_k(\xi)$, have: $\boxed{\int_{-\infty}^{+\infty} \psi_m^*(x)\, \psi_n(x)\, dx = 0,\ m \neq n.}$ (38)

B $I_{nn} = \sqrt{\pi}\, 2^n$, for $m = n \Rightarrow \int_{-\infty}^{\infty} e^{-\xi^2} H_n^2(\xi)\, d\xi = \sqrt{\pi}\, 2^n \cdot n!$;

...then// $\int_{-\infty}^{\infty} |\psi_n(x)|^2 dx = \frac{|N_n|^2}{\sqrt{m\omega/\hbar}} \int_{-\infty}^{\infty} e^{-\xi^2} H_n^2(\xi)\, d\xi = 1$,

...requires// $\boxed{N_n = \left(\sqrt{m\omega/\hbar} \,/\, \sqrt{\pi}\cdot 2^n n!\right)^{1/2}}$, norm$^{\underline{z}}$n cnst. (39)

With these results, we can write down the <u>orthonormal</u> <u>SHO</u> <u>eigenfcns</u>:

$$\boxed{\begin{array}{ll} \psi_n(x) = \left(\dfrac{\alpha/\sqrt{\pi}}{2^n n!}\right)^{1/2} e^{-\frac{1}{2}\alpha^2 x^2} H_n(\alpha x), & \text{w}/\!/\ \alpha = \sqrt{m\omega/\hbar} \\ & \uparrow \alpha \text{ has dimensions} \\ \text{obeying: } \int_{-\infty}^{+\infty} \psi_k^*(x)\, \psi_n(x)\, dx = \delta_{kn}. & \text{of a wave \# (inverse length)} \end{array}} \quad (40)$$

The ground state wavefcn (state with $n=0$ & energy $E_0 = \frac{1}{2}\hbar\omega$) is

ground state $\Big\}$ $\underline{\underline{\psi_0(x) = (\alpha/\sqrt{\pi})^{1/2} e^{-\frac{1}{2}\alpha^2 x^2}}}$. (41)

This is a Gaussian distribution of spatial extent $\Delta x \sim \frac{1}{\alpha} = \sqrt{\hbar/m\omega}$.