1) solution

The grav. pot'l energy of the rod is found by integrating along its axis, $x \in (-L/2,\, L/2)$

$$\begin{aligned} V(\theta) &= = -GM\int\frac{dm}{r} = -GM\int_{-L/2}^{L/2}\frac{(m/L)\,dx}{\sqrt{R^2+x^2-2xR\cos\theta}} \\ &= -\frac{GMm}{LR}\int_{-L/2}^{L/2}dx\left(1+\frac{x^2}{R^2}-\frac{2x}{R}\cos\theta\right)^{-1/2} \\ &= -\frac{GMm}{LR}\int_{-L/2}^{L/2}\left[1-\frac{1}{2}\left(\frac{x^2}{R^2}-\frac{2x}{R}\cos\theta\right)+\frac{3}{8}\left(\frac{x^2}{R^2}-\frac{2x}{R}\cos\theta\right)^2+\cdots\right] \\ &= -\frac{GMm}{LR}\int_{-L/2}^{L/2}\left[1+\frac{x}{R}\cos\theta+\frac{x^2}{R^2}\left(-\frac{1}{2}+\frac{3}{2}\cos^2\theta\right)+\cdots\right] \\ &= -\frac{GMm}{R}-\frac{1}{12}\frac{GMmL^2}{R^3}\left(-\frac{1}{2}+\frac{3}{2}\cos^2\theta\right)+\cdots \end{aligned} \tag{2}$$

a. This potential can be used to compute the torque on the rod

$$N(\theta) = -\frac{\partial V}{\partial\theta} = -\frac{1}{4}\frac{GMmL^2}{R^3}\sin\theta\,\cos\theta \ . \tag{3}$$

This vanishes at $\theta = 0$ where $\sin\theta = 0$ and $\theta = \pi/2$ where $\cos\theta = 0$. The former is an absolute minimum of the potential energy.

The stable equilibrium is with the rod's axis along the radial diection: $\theta = 0$.

b. For small departures from the stable equilibrium θ is small and

$$N(\theta) \simeq -\frac{1}{4}\frac{GMmL^2}{R^3}\theta \ . \tag{4}$$

This is a physical pendulum whose equation of motion is

$$I\,\ddot\theta = N(\theta) \simeq -\frac{1}{4}\frac{GMmL^2}{R^3}\theta \ . \tag{5}$$

where the moment of inertia for a thin uniform rod is $I = mL^2/12$. This gives an equation

$$\ddot\theta = -\frac{3GM}{R^3}\theta \ . \tag{6}$$

This is a simple harmonic oscillator with frequency

$$\omega = \sqrt{\frac{3GM}{R^3}} = \sqrt{3}\,\Omega \ , \tag{7}$$

where $\Omega = \sqrt{GM/R^3}$ is the Keplerian orbital frequency.

An approximate method & its limitations

It is tempting to avoid the integral by replacing each of the bar's halves with a point mass. The obvious tack is for $m/2$ to be located $L/4$ from the bar's midpoint. The gravitational potential of each point is then

$$\begin{aligned} V_{1,2} &= -\frac{GM(m/2)}{r_{1,2}} = -GM(m/2)\left[R^2+(L/4)^2 \pm 2R(L/4)\cos\theta\right]^{-1/2} \\ &= -\frac{GMm}{2R}\left[1 \pm \tfrac{1}{2}(L/R)\cos\theta + (L/4R)^2\right]^{-1/2} \\ &= -\frac{GMm}{2R}\left[1 \mp \frac{L}{4R}\cos\theta - \frac{L^2}{32R^2} + \frac{3}{8}\left(\frac{L}{2R}\right)^2\cos^2\theta + \cdots\right] \quad , \end{aligned} \tag{8}$$

after expanding $r_{1,2}^{-1}$. The potential of the pair is

$$\tilde{V}(\theta) = V_1 + V_2 = -\frac{GMm}{R} + \frac{GMmL^2}{32\,R^3}(1-3\cos^2\theta) + \cdots \quad . \tag{9}$$

The torque

$$\tilde{N}(\theta) = -\frac{\partial \tilde{V}}{\partial\theta} = -\frac{3}{16}\frac{GMmL^2}{R^3}\sin\theta\,\cos\theta \tag{10}$$

is smaller than eq. (3) by a factor 3/4. This is a reminder that the replacement of a solid body by a point mass is valid only to lowest order: the *net force*.

The moment of inertia for the two point masses is

$$\tilde{I} = 2\,\frac{m}{2}\left(\frac{L}{4}\right)^2 = \frac{mL^2}{16} \quad , \tag{11}$$

is *also* smaller than the bar by a factor of 3/4. This means, quite remarkably, that the pair of point masses oscillates at exactly the same frequency as the uniform bar. If, however, one tried computed the torque using the point mass approximation and *then* use the moment of inertia of the uniform bar, one would obtain an incorrect frequency — too small by a factor $\sqrt{3}/2$.

Both torque and the moment of inertia are related to the bar's second moment — this is why they $\sim L^2$. It *is* possible to use two point masses, each of mass $m/2$, to accurately reproduce both, but they must be placed $L/\sqrt{12}$ from the mid-point of the bar, **not** at $L/4$ as tried above. The moment of inertia will then exactly match that of the bar and the potential energy of each point is

$$\begin{aligned} V_{1,2} &= -GM(m/2)\left[R^2+(L/\sqrt{12})^2 \pm 2R(L/\sqrt{12})\cos\theta\right]^{-1/2} \\ &= -\frac{GMm}{2R}\left[1 \mp \frac{L}{\sqrt{3}R}\cos\theta - \frac{L^2}{24R^2} + \frac{3}{8}\left(\frac{L}{\sqrt{3}R}\right)^2\cos^2\theta + \cdots\right] \\ &= -\frac{GMm}{2R}\left[1 \mp \frac{L}{\sqrt{3}R}\cos\theta - \frac{L^2}{24R^2}\left(1-3\cos^2\theta\right) + \cdots\right] \quad . \end{aligned} \tag{12}$$

The total potential energy

$$\tilde{V}(\theta) = V_1 + V_2 = -\frac{GMm}{R} + \frac{GMmL^2}{24\,R^3}(1-3\cos^2\theta) + \cdots \quad , \tag{13}$$

then agrees with the solid bar to $\mathcal{O}(L^2/R^2)$ — it will naturally disagree at the next order.

2) Solution

(a) In canonical ensemble, one fixes number of bound particles N_b. The partition function for the bound molecules involves just one (multiply degenerate) energy state

$$\boxed{Z_b = \underbrace{\frac{N_r!}{N_b!(N_r-N_b)!}}_{\text{degeneracy factor}} e^{-(-\varepsilon_0 N_b)/T}}$$

The free energy, and the chemical potential

$$F_b = -T\ln Z_b = -\varepsilon_0 N_b - T\ln\frac{N_r!}{N_b!(N_r-N_b)!} \approx -\varepsilon_0 N_b - T\left[N_r\ln N_r - N_b\ln N_b - (N_r-N_b)\ln(N_r-N_b)\right]$$

$$\boxed{\mu_b = \frac{\partial F_b}{\partial N_b} = -\varepsilon_0 - T\ln\left(\frac{N_r}{N_b}-1\right)}$$

and for the fraction of occupied sites in equilibrium ($\mu_b = \mu$) we have,

$$\boxed{\frac{N_b}{N_r} = \frac{1}{1+e^{-\beta(\varepsilon_0+\mu_b)}}\Big|_{\mu_b=\mu}}$$

Check: this quantity is always less than unity, as expected, and approaches 1 in the limit of large binding energy. In thermodynamic limit ($N_r >> 1$) the canonical and grand canonical ensembles give the same result. [1]

(b) To have 99% of receptors occupied we need

$$e^{-\beta(\varepsilon_0+\mu)} = \frac{v}{\lambda_T^3} Z_{int} e^{-\varepsilon_0/T} < 0.01 \qquad \Rightarrow \qquad N > 100\frac{V}{\lambda_T^3} Z_{int} e^{-\varepsilon_0/T}$$

For the given mass of antibiotic molecule $M = 400 m_p$ at room temperature $\lambda_T \sim (1/2) * 10^{-11} m$, volume $V = 10(0.1m)^3$, and $\exp(-12000/300) = \exp(-40) \approx \exp(-2.3*17) = 10^{-17}$. The number of antibiotic molecules and their net mass are:

$$\boxed{N > 100 * (10\cdot 10^{-3}) * (2^3\cdot 10^{33}) * 10^4 * 10^{-17} = 8\cdot 10^{20}}$$

$$mass = N * M = (8\cdot 10^{20}) * (400 * 2\cdot 10^{-27}\, kg) \approx 6.5\cdot 10^{-4}\, kg \boxed{= 650\, mg}$$

[1] In the grand canonical picture, the number of occupied sites is not fixed, but the chemical potential of adsorbed particles is given by that of the "particle bath", μ. Grand partition function $\mathcal{Z}_b = \sum_{N_b=0}^{N_r} \frac{N_r!}{N_b!(N_r-N_b)!} e^{-\beta(-\varepsilon_0 N_b - \mu N_b)} = \left(1+e^{\beta(\varepsilon_0+\mu)}\right)^{N_r}$. $\Phi = -T\ln\mathcal{Z}_b$ and the *average* number of occupied receptors is $N_b = -\dfrac{\partial\Phi}{\partial\mu} = \frac{N_r}{1+e^{-\beta(\varepsilon_0+\mu)}}$.

3) Solution

Since we will be looking at the two-slit diffraction pattern, we need to find the de Broglie wavelength of the C_{60} molecule. Thus we start by finding the momentum p of the C_{60} molecules of mass M at the temperature T using the equipartitian theorem.

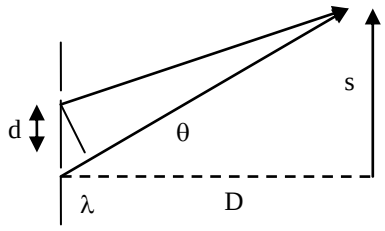
$$KE = \frac{p^2}{2M} = \frac{3}{2}kT \qquad \rightarrow p = \sqrt{3MkT}$$

Now we can find the de Broglie wavelength λ using Planck's constant h

$$\lambda = \frac{h}{p} = \frac{h}{\sqrt{3MkT}}$$

To find the angle off the direction of the collimated beam of the first maximum of the two-slit diffraction pattern, we draw a quick sketch and note that the two beams must differ in length by one de Broglie wavelength. Thus we get

$$\sin\theta = \frac{\lambda}{d}$$



We can now find the distance to the first maximum in the two-slit diffraction pattern using

$$s = D\tan\theta \approx D\sin\theta \approx \frac{D\lambda}{d}$$

$$\boxed{s \approx \frac{Dh}{d\sqrt{3MkT}}}$$

where we have used the fact that the angle is small since D>>d.

b) Now let's put in some numbers to find out how large an effect we can expect for the first maximum in the two-slit diffraction of C_{60} buckyballs. For D= 1m, T=1000K, and d=100nm. We also have to find the mass M of the C_{60} buckyball.

To find M we use the mass of 12amu for the carbon atom and we use the mass of the proton from the cover of the exam as an approximation for the amu.

$$M = (60)(12\text{amu})\left(\sim 1.67\text{x}10^{-27}\,\text{kg/amu}\right) \approx 1\text{x}10^{-24}\,\text{kg}$$

Now we can find s using

$$s \approx \frac{Dh}{d\sqrt{3MkT}} = \frac{(1\text{m})\left(6.6\text{x}10^{-34}\,\text{J}\cdot\text{s}\right)}{\left(100\text{x}10^{-9}\,\text{m}\right)\sqrt{3\left(10^{-24}\,\text{kg}\right)\left(1.4\text{x}10^{-23}\,\text{J}\cdot\text{K}^{-1}\right)(1000\text{K})}}$$

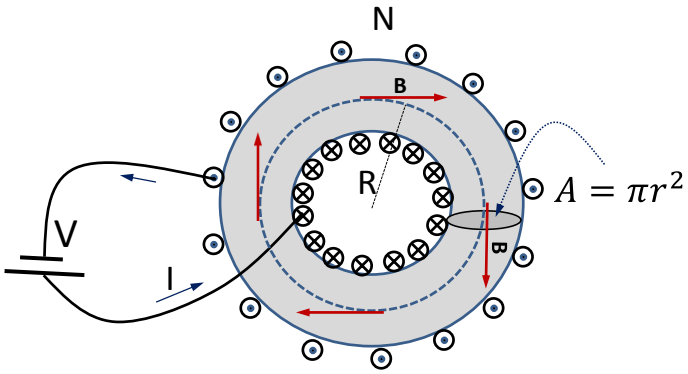
$$\approx \frac{\left(6.6\text{x}10^{-27}\,(\text{kg})(\text{m})^2(\text{s})^{-1}\right)}{\sqrt{4\text{x}10^{-44}\,(\text{kg})^2(\text{m})^2(\text{s})^{-2}}} \approx \frac{\left(6.6\text{x}10^{-27}\,(\text{kg})(\text{m})^2(\text{s})^{-1}\right)}{2\text{x}10^{-22}\,(\text{kg})(\text{m})(\text{s})^{-1}}$$

$$\boxed{s \approx 3\text{x}10^{-5}\,\text{m} \approx 30\mu\text{m}}$$

This distance is experimentally feasible to measure and indeed the two-slit diffraction using C_{60} buckyballs was seen in 1999 (see Arndt et al "Wave-Particle Duality of C_{60} Molecules", Nature Vol 40, pg 680, 14 Oct 1999)

4) Solution

1. As suggested, we consider only the work done against the magnetic attraction force in order to open up a gap of separation h across the cut in the iron ring. Since $h \ll r$, the field leakage across the gap is minimal and almost all of the magnetic flux is contained within the gap. Since the cross section of the iron ring is assumed to be uniform (because $R \gg r$) the magnitude of the magnetic field inside the ring will be approximately constant and perpendicular to the cross section of the iron ring (see figure below). Let us assume that the attractive force between the poles is $\vec{F}$. We need to apply a force equal but opposite to $\vec{F}$ in order to open up a narrow gap across the cut while we maintain a constant B in the ring and in the gap (note that $\vec{B}$ continues across the gap). The work required to open this gap is simply $W = F\,h$. All the energy expended to open up the gap is stored as field energy within the volume of the gap because the total energy stored in the volume $V = 2\pi RA$ of the ring remains constant. It is important to note here that the current has to be increased from the initial value to maintain the constant B in the coils. The force across the gap depends not on how the gap is opened, but only on the final state of the electromagnet. In our case the magnetic field in the final state is the same as that in the initial state before the air gap is opened up. The field energy stored in the magnetic field within the gap is entirely due to work done by -$\vec{F}$. Considering that magnetic field energy density *u* is given by $u = \dfrac{B^2}{2\mu_o}$, the total energy stored in volume $A\,h$ is equal to the work done against the magnetic force to open up the gap. Therefore,



$W_B = F_B h = u_B A h \rightarrow F_B = u_B A = \dfrac{B^2}{2\mu_o} A$, where subscript B indicates B is kept constant, and the magnetic field in the gap can be determined by Ampere's circuital law: $\oint \vec{H}.d\vec{l} = NI$.

Remember that the normal component of $\vec{B}$ continues across the gap. Using $\vec{H} = \dfrac{\vec{B}}{\mu}$ we obtain

$B\left(\dfrac{2\pi R}{\mu}\right) = NI \rightarrow B = \dfrac{\mu N I}{2\pi R}$ where μ is the permeability of iron, R is the mean diameter of the ring and current I is the initial current in the coils before the gap is opened. It is important to emphasize here that when we open up the gap in the ring the circumference of the ring is increased by *h*, i.e. $\oint dl = 2\pi R + h$. Since B is kept constant, this implies that the final current is increased, $I_B > I$. We can easily determine I_B from $\oint \vec{H}.d\vec{l} = NI$. Using $\vec{H} = \dfrac{\vec{B}}{\mu}$ we obtain

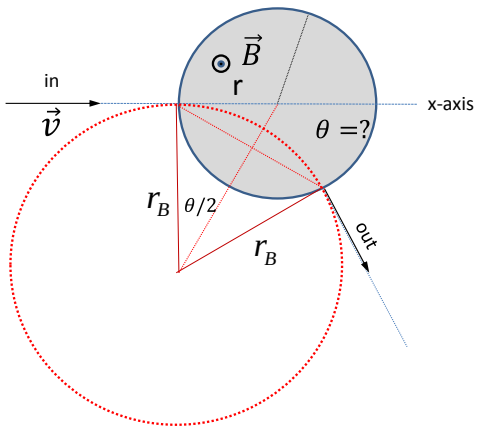
$B\left(\frac{2\pi R}{\mu}+\frac{h}{\mu_o}\right)=NI_B \rightarrow B=\frac{\mu\mu_o N I_B}{\mu_o 2\pi R+\mu h}$ where μ is the permeability of iron, μ_o is the permeability of free space and *R* is the mean diameter of the ring before the gap is opened up. This yields $\frac{I_B}{I}=\frac{\mu_o 2\pi R+\mu h}{\mu_o 2\pi R}=1+\frac{\mu}{\mu_o}\frac{h}{2\pi R}$; considering that $\mu \gg \mu_o$, this is not a negligible increase.

2. We notice, however, that if instead of the magnetic field we insist on keeping the current I in the coils constant while opening up the gap, the magnetic field in the coil will be reduced. As demonstrated above we can determine the final magnetic field B_I in the gap from Ampere's circuital law:

$\oint \vec{H}_I.d\vec{l}=NI$, which yields $B_I=\frac{\mu\mu_o NI}{\mu_o 2\pi R+\mu h}$, where subscript I indicates that the current I is kept constant. The attraction force between the poles is a function only of the magnetic field in the gap, which means $F_I=u_I\,A=\frac{B_I^{\,2}}{2\mu_o}A$. If we compare this force with the force F_B when the B field is kept constant we find $\frac{F_B}{F_I}=\frac{B^2}{B_I^{\,2}}=\left(\frac{\mu_o 2\pi R+\mu h}{\mu_o 2\pi R}\right)^2=\left(1+\frac{\mu}{\mu_o}\frac{h}{2\pi R}\right)^2$, which shows that F_I is substantially smaller than F_B. Therefore, work required to open up the gap is also less and can be found by integrating $W_I=\int_0^h F_I\,dh=\int_0^h A\frac{B_I^{\,2}}{2\mu_o}dh=Ah\frac{BB_I}{2\mu_o}$, where $B=\frac{\mu NI}{2\pi R}$ and

$B_I=\frac{\mu\mu_o NI}{\mu_o 2\pi R+\mu h}$.

3. An electron entering the gap region will encounter magnetic force $\vec{F}=-e\vec{v}\times\vec{B}$. We apply Newton's law of motion: $m_e a=m_e\frac{v^2}{r_B}=evB$, where $a=\frac{v^2}{r}$ is the centripetal acceleration of the electron. This suggests that the trajectory of an electron in the gap region is a circle with a radius of curvature r_B given by $r_B=\frac{m_e v}{eB}$. From the schematic on the right: $\tan(\theta/2)=\frac{r}{r_B}=\frac{reB}{m_e v}$.



5) Solution

(a) Centripetal force mv^2/r = Coulomb force Ze^2/r^2 (1)

Kinetic energy K = (1/2) $mv^2 = Ze^2/2r$ from (1); while

Potential energy $V = -Ze^2/r$. (2)

So, from (2) → Total energy $E = K + V = = -Ze^2/2r$. (3)

Using the centripetal acceleration **a** = v^2/r and the above relations in (1) and (2), we can eliminate **a** in the electron's radiating power equation $P = (2e^2/3c^{3)} \ |\mathbf{a}|^2$

Then, $P = [2e^2(Ze^2)^2/3m^2c^3]\ r^{-4}$ (4). Ans

Now equating P in (4) to energy loss rate dE/dt using E from (3), you obtain:

$dr/dt = -\frac{4}{3}(ze^2/m^2c^{3)}\ r^{-2}$. Ans

(b) $P_{rad} = \frac{2}{3}\frac{e^2c}{\rho^2}\ \beta^4\ \gamma^4$. Relativistic energy $E = \gamma\ mc^2$ → $\gamma = E/\ mc^2$. (1)

Orbital period $\Delta t = 2\pi\rho/\beta c$ (2)

Radiation energy loss/orbital period $= P_{rad}\Delta t$. (3)

Energy supplied during this period is $(dU/dz)2\pi\rho$ (4)

(3) = (4) and using (1) to eliminate E → gives the limiting γ as

$$\gamma = \left\{ \frac{3}{2}\frac{dU}{dz}\ \frac{\rho^2}{e^2}\ \frac{1}{\beta^3} \right\}^{1/4}$$ Ans

6) solution

The solution to eq. (1) can be written as a sum of eigenfunctions,

$$N(\mathbf{x},t) \;=\; \sum_{n=0}^{\infty} A_n\, e^{\gamma_n t}\psi_n(\mathbf{x}) \quad , \tag{2}$$

each of which must satisfy the eigenfunction equation

$$\nabla^2\psi_n \;=\; -\,\lambda_n\psi_n(\mathbf{x}) \quad , \quad 0 \le \lambda_0 \le \lambda_1 \le \cdots \tag{3}$$

along with the homogeneous boundary conditions $\psi_n(\mathbf{x}) = 0$ on the surface. Since eigenfunctions are orthogonal the growth rate of eigenfunction n is

$$\gamma_n \;=\; q \;-\; D\lambda_n \quad , \tag{4}$$

so the smallest eigenvalue, λ_0, leads to the fastest growth. If $\lambda_0 < q/D$ then $\gamma_0 > 0$ and at least one term in eq. (2) will diverge — therefore the solution will diverge and the Plutonium piece is *super-crititcal*. On the other hand, if $\lambda_0 > q/D$ then $\gamma_0 < 0$ and the other γ_n will be even more negative. Thus every term in eq. (2) decays over time and the piece of Plutonium is *subcritical*. The critical piece of Plutonium is defined by the condition $\lambda_0 = q/D$. It remains to find the minimum eigenvalue for the three configurations. The eigenfunction with the minimum eigenvalue will have *no nodes* within the Plutonium.

a. The eigenfunction with minimum eigenvalue for a $L \times L \times L$ cube is

$$\psi_0(x,y,z) \;=\; \sin(\pi x/L)\,\sin(\pi y/L)\,\sin(\pi z/L) \quad , \quad \lambda_0 = 3\pi^2/L^2 \quad . \tag{5}$$

Setting this equal to q/D yields the linear dimension of the critical cube

$$L_{\rm cr} \;=\; \sqrt{3}\pi\sqrt{D/q} \quad . \tag{6}$$

The critical volume is

$$V_{\rm cr} \;=\; L_{\rm cr}^3 \;=\; 3^{3/2}\pi^3(D/q)^{3/2} \quad . \tag{7}$$

b. An eigenfunction confined to a sphere of radius R must satisfy the equation

$$\nabla^2\psi_0 \;=\; \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\psi_0}{dr}\right) \;=\; \psi_0'' + 2\psi_0' \;=\; -\,\lambda_0\psi_0(r) \quad . \tag{8}$$

The solutiuon is the Spherical Bessel function of order zero

$$\psi_0(r) \;=\; j_0(\pi r/R) \;=\; \frac{\sin(\pi r/R)}{r} \quad , \quad \lambda_0 \;=\; \frac{\pi^2}{R^2} \quad . \tag{9}$$

Setting the eigenvalue equal to q/D yields the radius of the critical sphere

$$R_{\rm cr} \;=\; \pi\sqrt{D/q} \quad . \tag{10}$$

The critical volume is

$$V_{\rm cr} \;=\; \frac{4\pi}{3}R_{\rm cr}^3 \;=\; \frac{4\pi^4}{3}(D/q)^{3/2} \quad . \tag{11}$$

7) Solution

a) The kinetic energy of the wedge is easy to do in terms of S:

$$x = S \rightarrow \dot{x} = \dot{S} \rightarrow T_{M_1} = \frac{1}{2} M_1 \dot{S}^2$$

The kinetic energy of the block is a bit more involved as there is a cross term:

$$x = S + z\cos\theta \rightarrow \dot{x} = \dot{S} + \dot{z}\cos\theta$$

$$y = -z\sin\theta \rightarrow \dot{y} = -\dot{z}\sin\theta$$

Using these equations we can get the kinetic energy of the block

$$T_{M_2} = \frac{1}{2} M_2 \left(\dot{x}^2 + \dot{y}^2 \right) = \frac{1}{2} M_2 \left[\left(\dot{S} + \dot{z}\cos\theta \right)^2 + \left(-\dot{z}\sin\theta \right)^2 \right]$$

$$= \frac{1}{2} M_2 \dot{S}^2 + \frac{1}{2} M_2 \dot{z}^2 + M_2 \dot{S}\dot{z}\cos\theta$$

Thus the total kinetic energy is:

$$T = T_{M_1} + T_{M_2} = \frac{1}{2}\left(M_1 + M_2 \right)\dot{S}^2 + \frac{1}{2} M_2 \dot{z}^2 + M_2 \dot{S}\dot{z}\cos\theta$$

Now we find the potential energy V. Since the wedge only moves horizontally, we only have to consider the potential energy of the block. Both the stretch of the spring and the change of height in gravity are important. Thus we get:

$$V = \frac{1}{2} k z^2 - M_2 g z \sin\theta$$

Thus the Lagrangian is:

$$\boxed{L = T - V = \frac{1}{2}\left(M_1 + M_2 \right)\dot{S}^2 + \frac{1}{2} M_2 \dot{z}^2 + M_2 \dot{S}\dot{z}\cos\theta - \frac{1}{2} k z^2 + M_2 g z \sin\theta}$$

b) Now we find the Lagrange equation for S.

$$\frac{\partial L}{\partial S} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{S}} \right) = 0$$

$$\frac{\partial L}{\partial S} = 0 \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{S}} \right) = \frac{d}{dt}\left[\left(M_1 + M_2 \right)\dot{S} + M_2 \dot{z}\cos\theta \right] = \left(M_1 + M_2 \right)\ddot{S} + M_2 \ddot{z}\cos\theta$$

Thus our Lagrange equation for S becomes:

$$\boxed{\left(M_1 + M_2 \right)\ddot{S} + M_2 \ddot{z}\cos\theta = 0}$$

c) Now we find the Lagrange equation for z.

$$\frac{\partial L}{\partial S} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{S}} \right) = 0$$

$$\frac{\partial L}{\partial z} = -kz + M_2 g \sin\theta \qquad -\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}} \right) = -\frac{d}{dt}\left(M_2 \dot{z} + M_2 \dot{S}\cos\theta \right) = -M_2 \ddot{z} - M_2 \ddot{S}\cos\theta$$

Thus our Lagrange equation for z becomes:

$$\boxed{M_2 \ddot{z} + M_2 \ddot{S}\cos\theta + kz = M_2 g \sin\theta}$$

d) Before finding the normal frequencies, let's get rid of the constant term in the Lagrange equation for z using:

$$u = z - \frac{M_2 g \sin\theta}{k}$$

Using this, our two equations become:

$$(M_1 + M_2)\ddot{S} + M_2\ddot{u}\cos\theta = 0$$

$$M_2\ddot{u} + M_2\ddot{S}\cos\theta + ku = 0$$

Now to find the two normal frequencies of oscillation, we assume

$$u = u_o\cos(\omega t) \qquad\qquad S = S_o\cos(\omega t)$$

Putting these into our two equations we get:

$$-\omega^2(M_1 + M_2)S_o - \omega^2 M_2 u_o\cos\theta = 0$$

$$-\omega^2 M_2 u_o - \omega^2 M_2 S_o\cos\theta + ku_o = 0$$

Grouping terms we get:

$$-\omega^2(M_1 + M_2)S_o - \omega^2 M_2\cos\theta u_o = 0$$

$$-\omega^2 M_2\cos\theta S_o + (-\omega^2 M_2 + k)u_o = 0$$

To solve for the two normal modes, we need to find the determinant and set to zero.

$$\begin{vmatrix} -\omega^2(M_1 + M_2) & -\omega^2 M_2\cos\theta \\ -\omega^2 M_2\cos\theta & (-\omega^2 M_2 + k) \end{vmatrix} = 0$$

Multiplying out the determinant, we get:

$$\omega^2(M_1 + M_2)(-\omega^2 M_2 + k) + (\omega^2 M_2\cos\theta)^2 = 0$$

$$\rightarrow \omega^2(M_1 + M_2)k - \omega^4(M_1 + M_2)M_2 + \omega^4 M_2^2\cos^2\theta = 0$$

$$\rightarrow \omega^2\left\{\omega^2 M_2\left[(M_1 + M_2) - M_2\cos^2\theta\right] - (M_1 + M_2)k\right\} = 0$$

Thus our two normal frequencies are:

$$\boxed{\begin{aligned} &\omega^2 = 0 \\ &\omega^2 = \frac{k}{M_2}\left[\frac{M_1 + M_2}{M_1 + M_2\sin^2\theta}\right] \end{aligned}}$$

8) Solution

We need to find the first eigenvalue and eigenfunction to the Schrödinger equation, describing motion of an electron above the liquid,

$$\left(-\frac{\hbar^2}{2m}\frac{d^2}{dz^2}-\frac{\alpha}{z}\right)\psi(z)=E\psi(z)$$

Right away, from $\hbar^2/2mz_0^2 \sim \alpha/z_0$, we can figure out the typical length scale, $\boxed{z_0=\hbar^2/2m\alpha}$, and for the energy scale, $\boxed{E_0 \sim \alpha/z_0 = 2m\alpha^2/\hbar^2}$.

More precisely, for the bound states that decay for $z\to\infty$ and satisfy the boundary condition on the surface $\psi(0)=0$, we look for solution in the form

$$\psi(z)=\phi(z)e^{-kz} \qquad \text{with series representation for } \phi(z): \qquad \phi(z)=\sum_{n=1}^{\infty}\phi_n z^n\,.$$

Equation for $\phi(z)$ is

$$\left(-\frac{\hbar^2}{2m}\frac{d^2}{dz^2}+\frac{\hbar^2 k}{m}\frac{d}{dz}-\frac{\hbar^2k^2}{2m}-\frac{\alpha}{z}\right)\phi(z)=E\phi(z)$$

To find all energy levels, we get recursive equations on coefficients ϕ_n, and terminate the series at term n to get state E_n, $\psi_n(z)$, i.e. require vanishing coefficients starting from $\phi_{n+1}=0$.

For the ground state we need to terminate the series after the first term, i.e. take $\phi(z)=Cz$, which gives

$$\left(\frac{\hbar^2k}{m}-\alpha\right)C+\left(-\frac{\hbar^2k^2}{2m}-E\right)Cz=0 \qquad \Rightarrow \qquad \boxed{k=\frac{m\alpha}{\hbar^2} \qquad E_{GS}=-\frac{m\alpha^2}{2\hbar^2}=-\frac{E_0}{4}}$$

The wave function in the ground state is $\boxed{\psi_{GS}(z)=Cze^{-kz}}$ where C is a normalization constant. For the height of the surface electron layer above He:

$$\langle z\rangle=\frac{\int_0^\infty dz\, z^3e^{-2kz}}{\int_0^\infty dz\, z^2e^{-2kz}}=\frac{1}{2k}\frac{\int_0^\infty dx\, x^3e^{-x}}{\int_0^\infty dx\, x^2e^{-x}}=\frac{1}{2k}\frac{3!}{2!}=\frac{3}{2k} \qquad \boxed{\langle z\rangle=\frac{3\hbar^2}{2m\alpha}=3z_0}$$

N.B. The general solution for the energy states $n=1,2,3,\ldots$ is

$$E_n=-\frac{\hbar^2k_n^2}{2m} \quad \text{with} \quad k_n=\frac{m\alpha}{\hbar^2}\frac{1}{n} \qquad \Rightarrow \qquad E_n=-\frac{m\alpha^2}{2\hbar^2}\frac{1}{n^2}$$

9) Solution

(a)For the CRL circuit given, the circuit equations are:

$I = -C\, dV/dt$ (1); and $V = RI + L\, dI/dt$ (2).

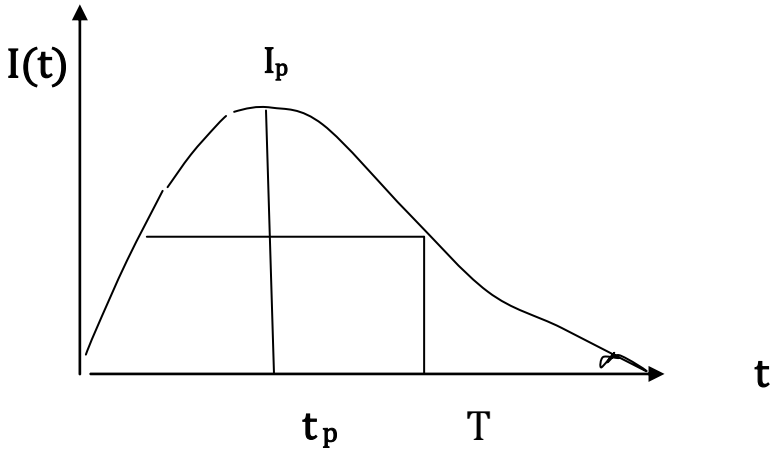
Then solving these equations with the initial conditions $V(0) = V_0$ and $I(0) = 0$, we obtain:

$I(t) = (V_0/L\Gamma)\ \exp(-\gamma t) \sinh(\Gamma t)$, where $\gamma = \frac{R}{2L}$ and $\Gamma = \sqrt{\gamma^2 - (\frac{1}{LC})}$. (3)

I(t) shows no oscillations so long as Γ is real, i.e., $\Gamma > 0$, which gives $CR^2/L > 4$ from (3). This is the condition for overdamping

(b) I(t) vs t sketch is given below. For small t, I(t) behaves as $I(t) \simeq (V_0/2L)t$, while $I(t) \simeq (V_0/2L\Gamma) \exp[-(\gamma-\Gamma)t]$ as $t \rightarrow \infty$. I(t) goes through peak at $I_p \simeq V_0/R$ at time $t_p \simeq L/R$.

Rise time $t_p \simeq L/R$; Duration $T \simeq RC$, and Peak amplitude $I_p \simeq V_0/R$ Ans



(c)

$\sigma(\omega) = \frac{sin^2\theta}{8\pi^2c^2}\ell^2\ \omega^2 \left| \int_0^\infty I(t)\, exp(-i\omega t)\ dt \right|^2$ (1) given.

Use $I(t) = (V_0/L\Gamma) \exp(-\gamma t) \sinh(\Gamma t)$ from (a) in $\int_0^\infty I(t)\, exp(-i\omega t)\, dt$ and carry out the integration, and use it in (1). The result is:

$$\sigma(\omega) = \frac{sin^2\theta}{8\pi^2c^3} (\ell^2 V_0^2/R^2) \{4\gamma^2\omega^2 / [(\omega_0^2 - \omega^2)^2 + 4\gamma^2\omega^2]\},$$ Ans

where $\omega_0 = 1/\sqrt{LC}$, and $\gamma = \frac{R}{2L}$.

10) Solution

a. It was known even in the late 19th century that a charged particle subjected to acceleration will emit electromagnetic radiation, Larmor radiation, and lose energy in the process. Classical mechanics does not help to explain why the electron does not fall into the nucleus when it is in a steady state orbital around the nucleus of an atom. This is because it is a purely quantum mechanical phenomenon: if the electrons are moving in the steady state quantum orbitals of an atom they do NOT emit electromagnetic radiation and lose energy even though they are moving around the nucleus with acceleration. Their locations in space are NOT known precisely and are determined by the probability density obtained from their wave function. They behave both as waves and as particles. Furthermore, the principle of uncertainty, a fundamental property of nature, will prevent an electron from being localized on the nucleus of an atom because of the huge uncertainties imposed on the momentum of the electron. On the other hand, the electron will absorb or emit radiation if it is taken out of its steady state quantum orbit, such as when it is forced to move around a circle in a synchrotron ring or when it is stimulated or energized to make a transition from one quantum state to another available state.

b. The common factor explaining all these observations is the scattering (changing the direction of travel) of light by air molecules or by small airborne particles in the atmosphere, whose sizes are much smaller than the wavelength of the visible light. This is called Rayleigh scattering and it is strongly dependent on the wavelength ($I \sim \frac{I_o}{\lambda^4}$) of the light being scattered but not too strongly dependent on the angle of scattering . This means shorter-wavelength light (i.e. blue) will be scattered in all directions, creating a diffuse blue sky radiation while long-wavelength light, such as red, will penetrate deeper into the sky without scattering. Therefore, an observer looking away from the sun will see a diffused blue light filling the sky. On the other hand, if the observer is in space outside the atmosphere, there will be no medium to scatter the light back to the observer and all he/she will see is a deep dark space. Rayleigh scattering also explains why the sun looks yellow in the atmosphere: the blue component of the light is mostly removed by Rayleigh scattering, and the remaining components arriving directly from the sun look yellow. In outer space, however, there is no atmosphere to scatter any component of the sunlight; hence the sun will look white, an equal mixture of all the colors. Similarly, at sunrise and sunset light travels a longer distance in the atmosphere and closer to the surface of the earth: only the long wavelengths (shades of red) will make it to the observer without scattering; the shorter wavelengths of light will be scattered off the direct path, causing the sun to look red. Furthermore, if there is dust in the air close to the surface of the earth whose sizes are much smaller than the visible wavelength of the

light, that too will contribute to the Rayleigh scattering and to the redness of the colors arriving directly from the sun.

c. The total power emitted by the sun is $P = 4\pi R^2 \times 1369$, where R is the distance from the surface of the earth to the center of the sun. Hence, we can determine R from $R = 8.3 \times 60 \times 3x10^8 + 7 \times 10^8 = 1.5x10^{11}$ m. This yields a radiative power emission of $P = 4\pi R^2 \times 1369 = 3.88 \times 10^{26} W$. This energy constitutes only 70% of the total loss, which means the total rate of energy loss by the sun is $P = 5.54 \times 10^{26} J/s$. Besides electromagnetic radiation the sun also loses energy because of solar flares (solar wind) ejecting mostly protons and electrons, neutrino emission and the radiation of other particles, such as axions. All of this energy is produced by fusion reactions at the center of the sun converting four hydrogen atoms into a single helium atom, and the mass lost in the process is the source of the sun's energy through Einstein's $E = \Delta m\, c^2$ relationship. Therefore, the rate of solar mass loss due to all emissions is $\frac{dm}{dt} = \frac{P}{c^2} = 6.15 \times 10^9 kg/s$. Even though this is a huge mass, it is nevertheless a very small fraction of the sun's total mass. We have to contrast this loss with the total energy in terms of the mass can be extracted from hyrogens at the centeral region of the sun where fusion reactions can take place and determine the life expectancy of the sun. The total fuel mass, m_f, can easily be calculated from $m_f = 2 \times 10^{30} \times 0.75 \times 0.007 \times 0.1 = 1.05 \times 10^{27} kg$. This means the life expectancy, τ, of the sun is $\tau = 1.05 \times 10^{27} / 6.15 \times 10^9 = 1.71 \times 10^{17} s$. Considering one year has about $1y \approx 3.16 \times 10^7 s$ this gives a life expectancy of the sun of $\tau \approx 5.41 \times 10^9 years = 5.41\, billion\ years$. Thus, human civilization has some time to think what to do about this doomsday scenario.

11) Solution

Let $y' \equiv \frac{dy}{dx}$; $y'' \equiv \frac{d^2y}{dx^2}$.

(a) $y\frac{d^2y}{dx^2} - (\frac{dy}{dx})^2 - 6xy^2 = 0$ (1)

Divide (1) by y^2 → $(y\,y'' - (y')^2)/y^2 = 6x$ (2).

But $d(y'/y)/dx = (y\,y'' - (y')^2)/y^2$ (3).

(3) → (2) and $d(y'/y)/dx = 6x$, and so separable.

So, integrating both sides, and obtain $y'/y = 6\int x\,dx = 3x^2 + A$, with A=constant.

Integrate again and get $\int (y'/y)\,dx = \ln y = \int (3x^2 + A)\,dx = x^3 + Ax + B$,
with B = another constant.

So, $y = \exp(x^3 + Ax + B)$ Ans.

(b)

$\frac{d^2y}{dx^2} + f(x)\frac{dy}{dx} + g(x)\,y = 0$ (1)

$y_1 = y_0$ (2); $y_2 = (y_0 u)$ (3);

and $u(x) = C\int .y_0^{-2} \exp[-\int f(x)dx]\,dx$ (4)

are given.

$y_1 = y_0$ and $y_2 = (y_0\,u)$ are linearly independent if Det $\begin{matrix} y1 & y2 \\ \frac{dy1}{dx} & \frac{dy2}{dx} \end{matrix} = y_1\,y_2' - y_2\,y_1' \neq 0$.

Now, calculate $(y_0\,u)'$ by using (4) for u and differentiating $(y_0 u)$.
Then, calculate $y_0\,(y_0\,u)' - (y_0\,u)\,y_0'$.

The result becomes:

$y_1\,y_2' - y_2\,y_1' = y_0\,(y_0\,u)' - (y_0\,u)\,y_0' = C \exp[-\int f(x)dx \neq 0$ Ans

(because it cannot be 0 even when f(x)=0).

12) Solution

Since the current exists only on the xy-plane, we have $\boldsymbol{\nabla}\times\mathbf{H}=0$ everywhere else, and we introduce scalar magnetic potential $\boxed{\mathbf{H}=\boldsymbol{\nabla}\varphi(x,z)}$. From the boundary condition on $\mathbf{H}$

$$H_x(x,z=0^+)-H_x(x,z=0^-)=\frac{4\pi}{c}j_y(x)\qquad\Rightarrow\qquad\boxed{\frac{\partial\varphi}{\partial x}\big|_{z=0^+}-\frac{\partial\varphi}{\partial x}\big|_{z=0^-}=\frac{4\pi j_0}{c}\cos kx}$$

The potential outside the xy plane satisfies equation $\boldsymbol{\nabla}\cdot\mathbf{B}=0\qquad\Rightarrow\boxed{\boldsymbol{\nabla}^2\varphi=0}$. To satidfy the boundary condition on the plane, we look for potential in the form $\boxed{\varphi(x,z)=\varphi(z)\sin kx}$. Equation for z-dependent part is obtained by substituting this form into $(\partial_x^2+\partial_z^2)\varphi(x,z)=0$

$$\left(\frac{d^2}{dz^2}-k^2\right)\varphi(z)=0\qquad\Rightarrow\qquad\varphi_>(z)=A_>e^{-kz}\qquad\varphi_<(z)=A_<e^{+kz}$$

where we assumed that only decaying solutions exist on the right ($>$) and left ($<$) sides of the plane.

The boundary condition for the normal component of $\mathbf{B}$-field gives,

$$\mu\frac{\partial\varphi_<}{\partial z}=\frac{\partial\varphi_>}{\partial z}\qquad\Rightarrow\qquad\boxed{\mu A_<=-A_>}.$$

and using it in the $\mathbf{H}$-field boundary condition we determine the values of A's:

$$\frac{\partial\varphi}{\partial x}\big|_{z=0^+}-\frac{\partial\varphi}{\partial x}\big|_{z=0^-}=(A_>-A_<)\,k\cos kx=\frac{4\pi j_0}{c}\cos kx\qquad\Rightarrow\qquad\boxed{A_>=\frac{4\pi j_0}{\left(1+\frac{1}{\mu}\right)ck}}$$

The field components are obtained from the potential (e.g. $z>0$)

$$\varphi_>(x,z)=\frac{4\pi j_0}{(1+\mu^{-1})ck}\sin kx\;e^{-kz}\qquad\Rightarrow$$

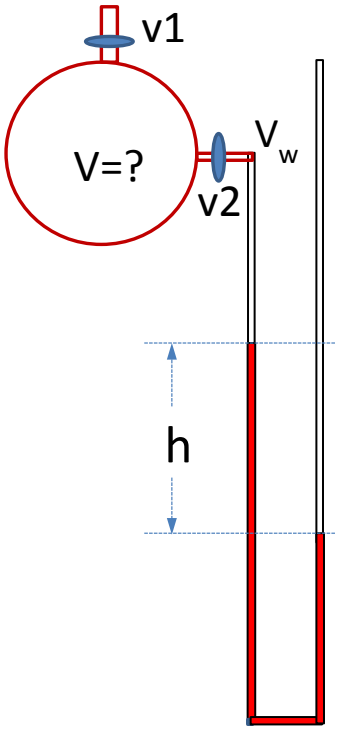
$$\boxed{H_x=\frac{\partial\varphi}{\partial x}=\frac{4\pi j_0}{(1+\mu^{-1})c}\cos kx\;e^{-kz}\qquad\&\qquad H_z=\frac{\partial\varphi}{\partial z}=-\frac{4\pi j_0}{(1+\mu^{-1})c}\sin kx\;e^{-kz}}$$

The field exponentially decays as we go away from the surface, on the scale of the oscillation wavelength $\lambda\sim 1/k$.

In the limit $k\to 0$, $\mu=1$ we have no z-component and $H_x=2\pi j_0/c$ - result for the uniform current density.

13) Solution

After the thermal and the mechanical equilibriums are reached we can apply the ideal gas law for two different situations:



1. $P_i = kT\frac{N_i}{V}$ =1.2 atm. Pressure in volume V when valves v1 and v2 are both closed.

2. $P_1 = kT\frac{N_1}{V} = kT\frac{N_2}{V_w} = P_o - \rho_{H_2O}gh$. Pressure in volume *V* and in volume $V_w = \pi r^2 h'$ after valve v2 is opened and the mechanical and thermal equilibriums have been reached. V_w is the volume confined between valve v2 and the surface of the water column in the left arm of the monometer. In the above equation *h'= 7 cm*, *r=0.3 cm*, P_o *=1 atm*, ρ_{H_2O} = 1 g/cm^3 and *h=70 cm*.

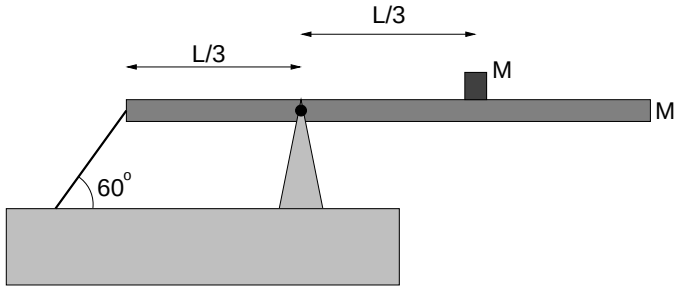
Furthermore, $N_i = N_1 + N_2$; this last equation can be rewritten as:

$\frac{P_i V}{kT} = \frac{P_1 V}{kT} + \frac{P_1 V_w}{kT} \rightarrow V = \frac{V_w}{\frac{P_i}{P_1} - 1}$. By plugging in the numerical values we obtain:

$$V = \frac{V_w}{\left(\frac{P_i}{P_1} - 1\right)} = \frac{\pi \times (0.3)^2 \times 7}{\left(\frac{1.2 \times 76 \times 13.6}{76 \times 13.6 - 70} - 1\right)} = 6.9\ cm^3$$

14) solution

a. The torques about the pivot must cancel in equilibrium. The weight force of the beam, Mg, is applied at its midpoint, which is located $(L/2)-(L/3) = L/6$ from the pivot. The torque balance condition is therefore



$$0 \;=\; N \;=\; T\sin(60^\circ)\frac{L}{3} - Mg\frac{L}{3} - Mg\frac{L}{6} \;=\; T\sin(60^\circ)\frac{L}{3} - Mg\frac{L}{2} \quad .$$

The tension is therefore

$$T \;=\; \frac{3Mg}{2\sin(60^\circ)} \;=\; \sqrt{3}Mg \quad . \tag{2}$$

The net forces on the beam must also balance

$$0 \;=\; \mathbf{F}_{\rm p,be} + \mathbf{F}_T - 2Mg\hat{\mathbf{z}} \;=\; \mathbf{F}_{\rm p,be} - T\cos(60^\circ)\,\hat{\mathbf{x}} - T\sin(60^\circ)\,\hat{\mathbf{z}} - 2Mg\hat{\mathbf{z}} \quad ,$$

where $\hat{\mathbf{x}}$ is to the right. This gives the force on the pivot

$$\mathbf{F}_{\rm p,be} \;=\; \sqrt{3}Mg\cos(60^\circ)\,\hat{\mathbf{x}} \;+\; Mg[\sqrt{3}\sin(60^\circ)+2]\,\hat{\mathbf{z}} \;=\; \frac{\sqrt{3}}{2}Mg\,\hat{\mathbf{x}} \;+\; \frac{7}{2}Mg\,\hat{\mathbf{z}} \tag{3}$$

b. Without the rope the torque on the beam-block system is

$$N \;=\; -\,Mg\frac{L}{2} \quad . \tag{4}$$

The rotational inertia of the beam about its pivot is

$$I_{\rm be} \;=\; \frac{ML^2}{12} \;+\; M\left(\frac{L}{6}\right)^2 \;=\; \frac{ML^2}{9} \quad , \tag{5}$$

using the parallel axis theorem. The rotational inertia of the beam-block system is

$$I_{\rm bl+be} \;=\; I_{\rm be} \;+\; M\left(\frac{L}{3}\right)^2 \;=\; \frac{2ML^2}{9} \quad . \tag{6}$$

The angular acceleration of the system is therefore

$$\alpha \;=\; \frac{N}{I_{\rm bl+be}} \;=\; -\,\frac{9}{4}\,\frac{g}{L} \quad , \tag{7}$$

The acceleration of the block is

$$a_{\rm bl} \;=\; \frac{L}{3}\alpha \;=\; -\,\frac{3}{4}\,g \quad .$$

The net force on the block is thus $-(3/4)Mg$. The weight force is $-Mg$ so the normal force from the beam is $+Mg/4$. By Newton's third law the normal force of the block on the beam is

$$\mathbf{F}_{\rm bl,be} \;=\; -\,\frac{Mg}{4}\,\hat{\mathbf{z}} \quad . \tag{8}$$

15) Solution

a) To find the first order correction of the **energy** of the harmonic oscillator states, we use:

$$E_n^{(1)} = \langle n|H'|n\rangle = \varepsilon\langle n|\left(\frac{\hbar}{m_e\omega} - x^2\right)|n\rangle = -\varepsilon\langle n|x^2|n\rangle + \frac{\varepsilon\hbar}{m_e\omega}\langle n|n\rangle$$

$$= -\frac{\varepsilon\hbar}{2m_e\omega}\langle n|\left(a_+ + a_-\right)^2|n\rangle + \frac{\varepsilon\hbar}{m_e\omega}$$

$$= -\frac{\varepsilon\hbar}{2m_e\omega}\langle n|\left(a_+a_+ + a_-a_+ + a_+a_- + a_-a_-\right)|n\rangle + \frac{\varepsilon\hbar}{m_e\omega}$$

$$= -\frac{\varepsilon\hbar}{2m_e\omega}\left[\langle n|a_+a_+|n\rangle + \langle n|a_-a_+|n\rangle + \langle n|a_+a_-|n\rangle + \langle n|a_-a_-|n\rangle\right] + \frac{\varepsilon\hbar}{m_e\omega}$$

In this case, only terms with the same number of raising and lowering operators will be non-zero. Thus:

$$E_n^{(1)} = -\frac{\varepsilon\hbar}{2m_e\omega}\left[\langle n|a_-a_+|n\rangle + \langle n|a_+a_-|n\rangle\right] + \frac{\varepsilon\hbar}{m_e\omega}$$

$$= -\frac{\varepsilon\hbar}{2m_e\omega}\left[\sqrt{n+1}\langle n|a_-|n+1\rangle + \sqrt{n}\langle n|a_+|n-1\rangle\right] + \frac{\varepsilon\hbar}{m_e\omega}$$

$$= -\frac{\varepsilon\hbar}{2m_e\omega}\left[(n+1)\langle n|n\rangle + n\langle n|n\rangle\right] + \frac{\varepsilon\hbar}{m_e\omega} = -\frac{\varepsilon\hbar(2n+1)}{2m_e\omega} + \frac{2\varepsilon\hbar}{2m_e\omega}$$

$$\boxed{E_n^{(1)} = \frac{\varepsilon\hbar}{2m_e\omega}[1-2n]}$$

Note that the ground state (n=0) energy gets shifted up as you might expect from the positive bump in the middle of the harmonic oscillator potential. Note however that this form of the potential perturbation actually lowers the potential a bit for large x so that higher states (n>0) actually lower in energy.

b) To find the first order correction to the ground state (n=0) **wave function** for the harmonic oscillator, we use:

$$\psi_n^{(1)} = \sum_{m\neq n}\frac{\langle\psi_m^{(0)}|H'|\psi_n^{(0)}\rangle}{E_n^{(0)} - E_m^{(0)}}\psi_m^{(0)} \rightarrow \psi_0^{(1)} = \sum_{m\neq 0}\frac{\langle m|H'|0\rangle}{E_0^{(0)} - E_m^{(0)}}|m\rangle = \sum_{m\neq 0}\frac{\langle m|\varepsilon\left(\frac{\hbar}{m_e\omega} - x^2\right)|0\rangle}{\hbar\omega\left[\left(0+\frac{1}{2}\right) - \left(m+\frac{1}{2}\right)\right]}|m\rangle$$

$$\psi_0^{(1)} = -\frac{\varepsilon}{\hbar\omega}\sum_{m\neq 0}\frac{\langle m|\left(\frac{\hbar}{m_e\omega} - x^2\right)|0\rangle}{m}|m\rangle$$

$$= -\frac{\varepsilon}{\hbar\omega}\sum_{m\neq 0}\frac{\langle m|\left(\frac{\hbar}{m_e\omega}\right)|0\rangle}{m}|m\rangle + \frac{\varepsilon}{\hbar\omega}\sum_{m\neq 0}\frac{\langle m|\left(x^2\right)|0\rangle}{m}|m\rangle$$

$$= -\frac{\varepsilon}{\hbar\omega}\sum_{m\neq 0}\left(\frac{\hbar}{m_e\omega}\right)\frac{\langle m|0\rangle}{m}|m\rangle + \frac{\varepsilon}{\hbar\omega}\left(\frac{\hbar}{2m\omega}\right)\sum_{m\neq 0}\frac{\langle m|\left(a_+ + a_-\right)^2|0\rangle}{m}|m\rangle$$

$$= -\frac{\varepsilon}{\hbar\omega}\sum_{m\neq 0}\left(\frac{\hbar}{m_e\omega}\right)\frac{\delta_{m0}}{m}|m\rangle + \frac{\varepsilon}{\hbar\omega}\left(\frac{\hbar}{2m\omega}\right)\sum_{m\neq 0}\frac{\langle m|a_+a_+ + a_+a_- + a_-a_+ + a_-a_-|0\rangle}{m}|m\rangle$$

$$= 0 + \frac{\varepsilon}{\hbar\omega}\left(\frac{\hbar}{2m_e\omega}\right)\sum_{m\neq 0}\frac{\langle m|a_+a_+ + a_-a_-|0\rangle}{m}|m\rangle$$

Note that we dropped the a_+a_- and a_-a_+ terms since $m \neq 0$. In addition, the a_-a_- will not contribute since $a_-|0\rangle = 0$. Thus we get:

$$\psi_0^{(1)} = \left(\frac{\varepsilon}{2m_e\omega^2}\right)\sum_{m\neq 0}\frac{\langle m|a_+a_+|0\rangle}{m}|m\rangle = \left(\frac{\varepsilon}{2m_e\omega^2}\right)\sum_{m\neq 0}\frac{\sqrt{1}\sqrt{2}\langle m|2\rangle}{m}|m\rangle$$

$$= \left(\frac{\varepsilon}{\sqrt{2}m_e\omega^2}\right)\sum_{m\neq 0}\frac{\delta_{m2}}{m}|m\rangle = \left(\frac{\varepsilon}{\sqrt{2}m_e\omega^2}\right)\left(\frac{1}{2}\right)|2\rangle$$

$$\boxed{\psi_0^{(1)} = \left(\frac{\varepsilon}{2\sqrt{2}m_e\omega^2}\right)\psi_2^{(0)}}$$

Thus the first order correction to the ground state wave function is to subtract a small amount of the n=2 state. Note that this makes sense since we expect the wave function to be symmetric since the perturbation is symmetric with x and the n=2 state is symmetric.

c) Now we do a plot of the initial wave function ψ_0, ψ_2 and the sum. Note that the resulting ground state wave function now has a dip in the center due to the positive perturbation in the center of the well.