DEPARTMENT OF PHYSICS

PH. D. COMPREHENSIVE EXAMINATION

SEPTEMBER 24-25, 1984

DEPARTMENT of PHYSICS

PH.D. COMPREHENSIVE EXAMINATION

MONDAY, SEPT. 24, 1984, 9-12 AM

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Answer each of the following questions. Each question carries equal weight. Begin your answer to each question on a new sheet of paper; solutions to different questions must not appear on the same sheet of paper. Label each page of your exam as follows:

A. Your name in upper left-hand corner.

B. Problem number, and page number for that problem, in upper right hand corner.

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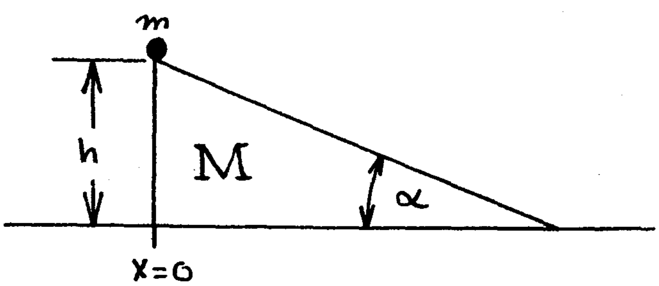
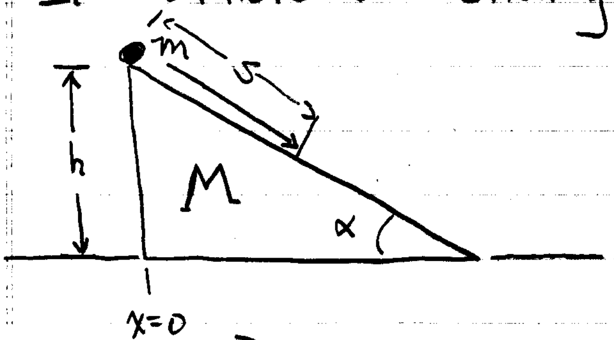
1. A point particle of negligible size, and mass m, slides without friction down the inclined plane shown below. The incline also slides without friction on the table and has mass M. The incline angle is α and the side height is h.



If the particle starts at the top of the incline at t=0, and the left edge of the incline is initially at X=o, as shown, at what point on the table does the particle actually hit the table?

Solution

1. Particle on sliding, inclined plane



Where does particle actually hit the table?

— " —

Describe position of incline as $\bar{X}$, and position of mass as: $x = \bar{X} + s\cos\alpha$

$y = h - s\sin\alpha$

where s is position of m along incline (note 2 degrees of freedom)

Lagrangian $L = T - V$

$\dot{x} = \dot{\bar{X}} + \dot{s}\cos\alpha$

$\dot{y} = -\dot{s}\sin\alpha$

$$T = \frac{1}{2}M\dot{\bar{X}}^2 + \frac{1}{2}m(\dot{x}^2+\dot{y}^2)$$

$$= \frac{1}{2}(M+m)\dot{\bar{X}}^2 + \frac{1}{2}m\dot{s}^2 + m\dot{\bar{X}}\dot{s}\cos\alpha$$

$$V = mg(h - s\sin\alpha)$$

$$L = T - V = \frac{1}{2}(M+m)\dot{\bar{X}}^2 + \frac{1}{2}m\dot{s}^2 + m\dot{\bar{X}}\dot{s}\cos\alpha - mg(h - s\sin\alpha)$$

$\bar{X}$ is cyclic (ignorable), so

$p_x = \text{constant} = (M+m)\dot{\bar{X}} + m\dot{s}\cos\alpha$

and

$$\boxed{\ddot{\bar{X}} = -\frac{m\cos\alpha}{(M+m)}\ddot{s}}$$

Now find $s(t)$: $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{s}}\right) - \frac{\partial L}{\partial s} = 0$

$$\frac{d}{dt}\left(m\dot{s} + m\dot{X}\cos\alpha\right) - mg\sin\alpha = 0$$

$$m\ddot{s} + m\cos\alpha\left(-\frac{m\cos\alpha}{(M+m)}\ddot{s}\right) - mg\sin\alpha = 0$$

$$\ddot{s} = \frac{g\sin\alpha}{1 - \frac{m\cos^2\alpha}{(M+m)}} = \frac{(M+m)g\sin\alpha}{M + m\sin^2\alpha}$$

$\ddot{s}$ = constant $= a_s$ so

$$s(t) = 1/2\, a_s t^2 + \cancel{s_0}^{\,0} + \cancel{v_{0s}}^{\,0} t$$

When m reaches bottom of incline, $s = h/\sin\alpha$
So, time to hit table is

$$t^2 = \frac{2h}{(\sin\alpha)a_s} = \frac{2h}{\sin\alpha}\frac{(M + m\sin^2\alpha)}{(M+m)g\sin\alpha}$$

Then, since no initial velocity for incline, left edge is at

$$X = \cancel{X_0}^{\,0} + 1/2\,\ddot{X}t^2 = -\frac{1}{\cancel{2}}\frac{m\cos\alpha}{(M+m)}\frac{\cancel{2}h}{\sin\alpha}$$

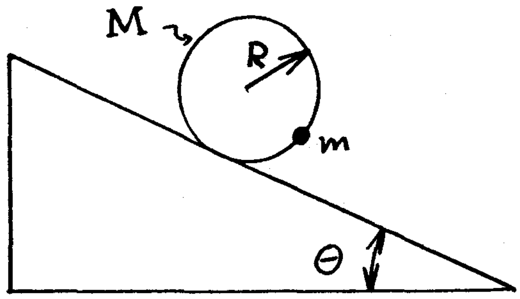
and particle hits table at $X + h/\tan\alpha$
↑ width of incline

or at

$$x' = -\frac{m\cos\alpha\, h}{\sin\alpha(M+m)} + \frac{h}{\tan\alpha}$$

$$\boxed{x' = h/\tan\alpha\left(\frac{M}{m+M}\right)}$$

2. A hollow cylinder of radius R and mass M is loaded at one point by a mass m. (That is, m is firmly attached to the cylinder.) The cylinder is on an inclined plane of angle θ.



Beyond what critical angle of the inclined plane θ_c will the cylinder roll down the hill regardless of its initial orientation? You may assume the cylinder rolls without slipping. For $\theta<\theta_c$, find the frequency of small rolling oscillations of the cylinder.

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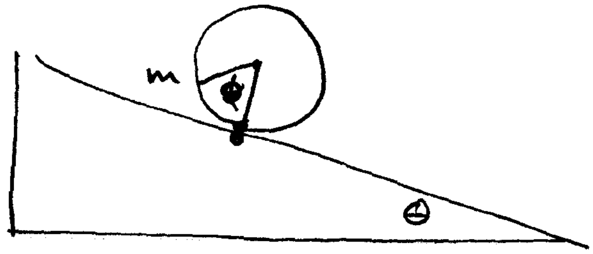
Solution:



ϕ is angle of roll relative to m being in contact with plane

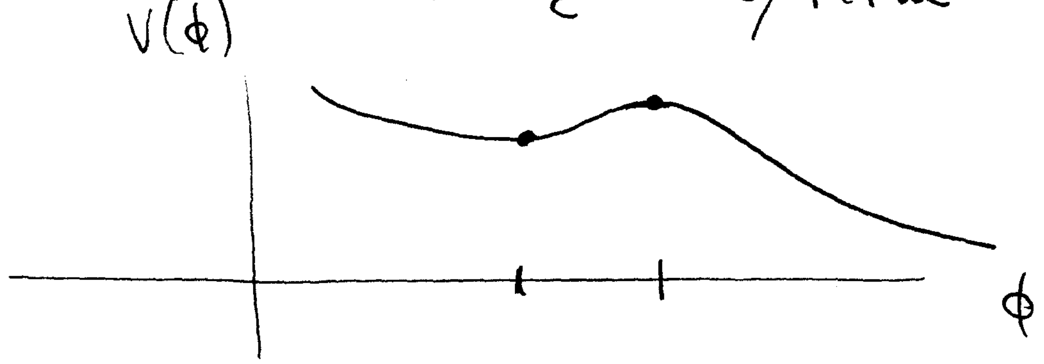
$$V(\phi) = -Mg\sin\theta R\phi$$

$$- mg\left[\sin\theta R\phi + R\left[\cos(\theta+\phi) - \cos\theta\right]\right]$$

$$\frac{\partial V(\phi)}{\partial \phi} = 0 = (M+m)gR\sin\theta + mgR\sin(\theta+\phi)$$

$$\sin(\theta+\phi) = \frac{M+m}{m}\sin\theta \le 1$$

$$\sin\theta_c = m/M+m$$



$$T = \frac{1}{2}MR^2\dot{\phi}^2 + \frac{1}{2}MR^2\dot{\phi}^2 + T_m$$

$$T_m = \frac{1}{2}mR^2\dot{\phi}^2\left[2-2\cos\phi\right]$$

$$T = \left[M + m(1-\cos\phi)\right]R^2\dot{\phi}^2$$

$$L = T - V \qquad \omega^2 = V''(\phi=\phi_0) \Big/ M(\phi=\phi_0)$$

$$V'' = mgR\cos(\theta+\phi_0) \qquad \sin(\theta+\phi_0) = \frac{(M+m)\sin\theta}{m}$$

$$\cancel{M} = 2\left[M + m(1-\cos\phi_0)\right]R^2$$

$$\boxed{\omega^2 = \frac{g}{R}\,\frac{m}{2(M+m(1-\cos\phi_0)}\cos(\theta+\phi_0)}$$

$$\text{with } \sin(\theta+\phi_0) = \frac{(M+m)\sin\theta}{m} = \frac{\sin\theta}{\sin\theta_c}$$

$$=$$

$$\omega^2 = \frac{g}{R}\,\frac{m}{2(M+m(1-\cos\phi_0))}\,\frac{\sqrt{\sin^2\theta_c - \sin^2\theta}}{\sin\theta_c}$$

$$= \frac{g}{R}\,\frac{M+m}{2(M+m(1-\cos\phi_0)}\sqrt{\sin^2\theta_c - \sin^2\theta}$$

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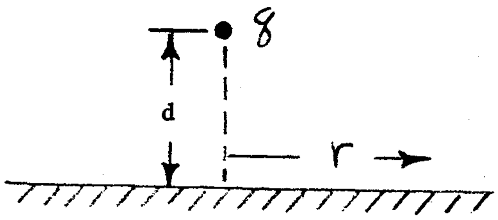
3. A point charge q is placed a distance d from an infinite conducting plate. Find the charge density at the surface of the plate as a function of the distance r from the point on the plate directly under the charged point.



(Hint: use the image charge method.)

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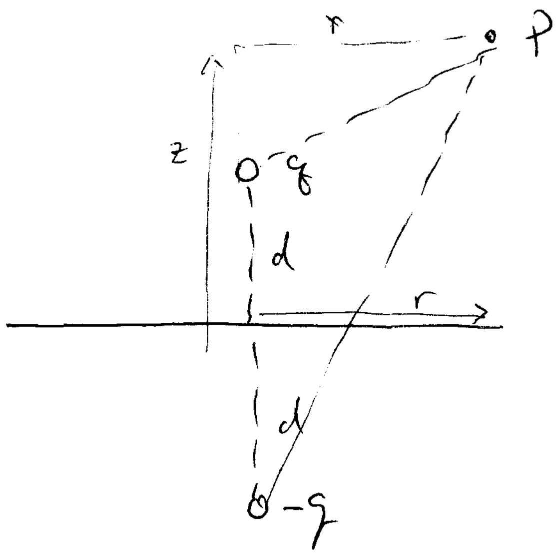
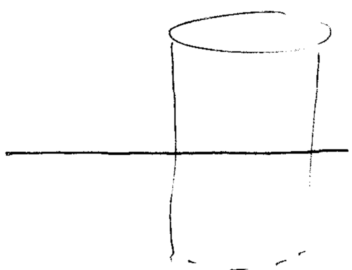


Using an image charge the single charge and plate can be replaced by a positive and a negative charge in free space. The potential at any point can then be calculated

$$V_P = \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{\sqrt{(z-d)^2+r^2}} + \frac{-q}{\sqrt{(z+d)^2+r^2}}\right]$$

From Gauss's law the electric field can be related to the charge density



$$\oint \vec{E}\cdot d\vec{S} = \frac{q}{\varepsilon_0}$$

$$EA = \frac{\sigma A}{\varepsilon_0}$$

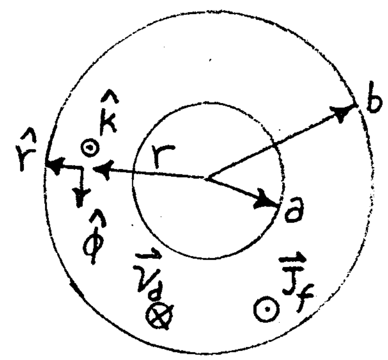
$$\therefore \varepsilon_0 E = \sigma$$

E is related to the potential as $E = -\left.\frac{\partial V}{\partial z}\right|_{z=0}$

$$\therefore \quad \sigma = \varepsilon_0\left(-\left.\frac{\partial V}{\partial z}\right|_{z=0}\right) = -\varepsilon_0 \frac{q}{4\pi\varepsilon_0}\left[\frac{(z-d)}{\left((z-d)^2+r^2\right)^{3/2}} - \frac{z+d}{\left((z+d)^2+r^2\right)^{3/2}}\right]_{z=0}$$

$$= \frac{-qd}{2\pi}\,\frac{1}{(d^2+r^2)^{3/2}}\,.$$

4. The hollow wire shown in cross section at right has a given uniform current density $\vec{J}_f$ and electron drift velocity $\vec{v}_d$. Find the magnitudes and directions of $\vec{B}$ and of the Hall field $\vec{E}_h$, for $a<r<b$. The unit vectors at an arbitrary point in the wire are labeled $\hat{r}$, $\hat{\phi}$ and $\hat{k}$.



From Ampère's Law,

$$2\pi r B = \mu_0 J_f \pi (r^2 - a^2) \qquad \boxed{\vec{B} = \frac{\mu_0 J_f (r^2 - a^2)}{2r} \hat{\phi}}$$

$$\vec{F}_r = -e(\vec{E}_h + \vec{v}_d \times \vec{B}) = 0 \quad \text{(no radial drift in equilibrium)}$$

$$\vec{E}_h = -\vec{v}_d \times \vec{B} = +v_d \hat{k} \times \frac{\mu_0 J_f (r^2 - a^2)}{2r} \hat{\phi} = \boxed{-\frac{\mu_0 v_d J_f (r^2 - a^2)}{2r} \hat{r}}$$

DEPARTMENT of PHYSICS

PH.D. COMPREHENSIVE EXAMINATION

MONDAY, SEPT. 24, 1984, 2-5 PM

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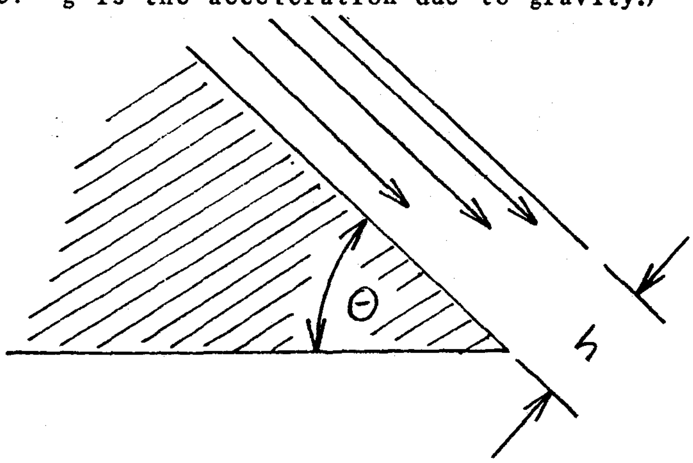
5. The Navier-Stokes equation for a viscous incompressible fluid is

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u}\cdot\vec{\nabla})\vec{u} = -\frac{1}{\rho}\vec{\nabla}p + \nabla^2\vec{u} + \vec{g}$$

Such a fluid undergoes steady parallel flow down an inclined plane under the influence of gravity. The thickness of the layer, from the plane to the free surface, is h. Find the velocity vector $\vec{u}$ everywhere in the fluid. (Note: $\vec{g}$ is the acceleration due to gravity.)

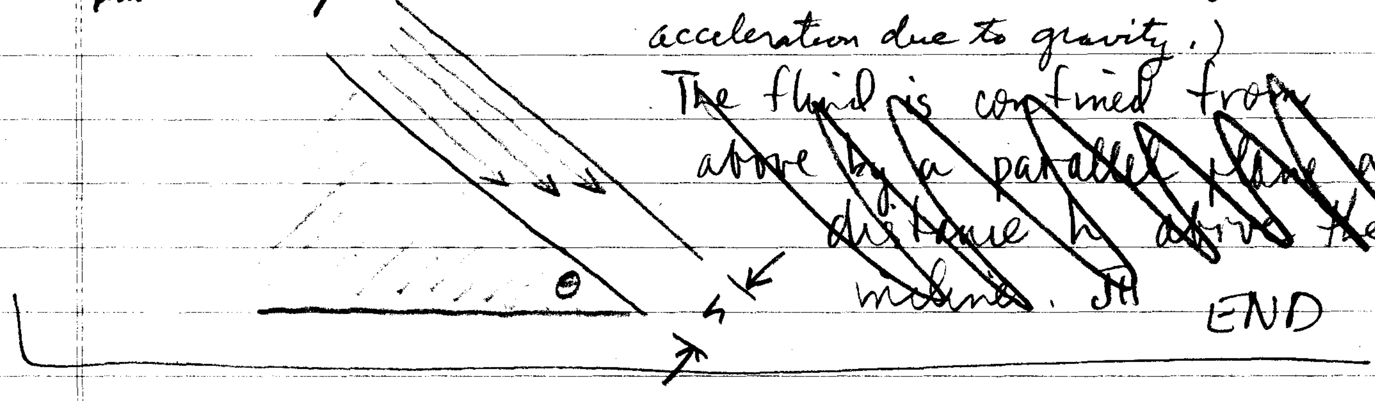


5. The Navier-Stokes equation for a viscous incompressible fluid is

$$\underbrace{\frac{\partial \underset{\sim}{u}}{\partial t}}_{\text{"you"}} + (\underset{\sim}{u} \cdot \nabla)\underset{\sim}{u} = -\frac{1}{\underbrace{\rho}_{\text{rho}}} \nabla p + \underbrace{\nu}_{\text{nu}} \nabla^2 \underset{\sim}{u} + \overset{\text{"gee"}}{\underset{\sim}{g}}$$

Such a fluid undergoes steady parallel flow down an inclined plane under the influence of gravity. Th[e] thickness of the layer, from the pla[ne] to the free surface is h. Find the velocity vector $\underset{\sim}{u}$ everywhere in the fluid. (Note: g is the acceleration due to gravity.)

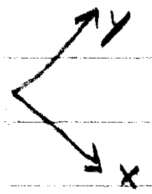
~~The fluid is confined from above by a parallel plane a distance h above the incline.~~ END



A. "steady" $\Rightarrow \partial u/\partial t = 0$

"parallel" $\Rightarrow (u\cdot\nabla)u = 0$

The remaining forces must balance. Choose axes inclined along the plane.



Then $p = p(y)$, $\tilde{u} = \tilde{e}_x u(y)$. The x-component of the N-S Eq. is

$$\nu u_{yy} = -g \sin\theta$$

$$u = A + By - \tfrac{1}{2}\left(\tfrac{g}{\nu}\sin\theta\right) y^2$$

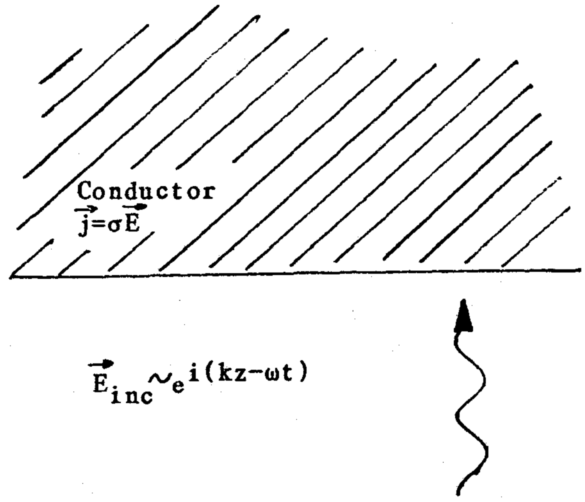
B.C.'s are ① $u = 0$ at $y = 0$
② $u_y = 0$ at $y = h$ $\Rightarrow$ $\boxed{u = -\frac{g\sin\theta}{2\nu}(y^2 - 2yh)}$

6. A plane electromagnetic wave is normally incident on a conductor of finite conductance σ.

a. Find the solution of Maxwell's Equations throughout space.

b. Calculate the fraction of power absorbed in the conductor as a function of frequency ω.

c. Calculate the penetration range of radiation into the conductor as a function of ω.



For part c, assume $\sigma \gg \omega\varepsilon$, where ε is the dielectric permittivity.

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Solution: $\nabla \times E = -\dot{\overline{B}}$

$$\nabla \times B = \dot{\overline{E}} + 4\pi \vec{J}$$
$$= \dot{\overline{E}} + 4\pi\sigma \overline{E}$$

so in conductor $\nabla^2 E = \ddot{E} + 4\pi\sigma \dot{E}$

In free space $\overline{E} = \hat{\epsilon}\left(e^{i(kz-\omega t)} + R\, e^{i(-kz-\omega t)}\right)$

In conductor $\overline{E} = \hat{\epsilon}\, T\, e^{i(k'z-\omega t)}$

$\overline{E}$ & $\overline{B}$ continuous at boundary

$$1+R = T$$
$$k(1-R) = k'T$$
$$k'(1+R) = k'T$$

$$(k')^2 = \omega^2 + 4\pi\sigma\omega i$$
$$= k^2 + 4\pi\sigma k i$$

units $c=1$

$$R = \frac{k-k'}{k+k'}$$

fraction of power absorbed is

$$f = 1 - RR^*$$

For $\sigma \gg \omega$ $\quad k' = e^{i\pi/4}\sqrt{4\pi\sigma k}$

$$= \frac{1+i}{\sqrt{2}}\sqrt{4\pi\sigma k}$$

In conductor wave goes as $e^{-\sqrt{2\pi\sigma k}\,x}$

Penetration depth $\ell \sim \dfrac{1}{\sqrt{2\pi\sigma k}}$

*   *   *   *   *   *   *   *   *   *   *   *

7. A system of three spin-1/2 particles is pairwise exchange-coupled according to the Ising-model Hamiltonian.

$$H = -JS_1^z S_2^z - j(S_1^z S_3^z + S_2^z S_3^z); \quad |J| >> |j|.$$

a) What must be the signs of J and j if the ground state is ferromagnetic?

b) Determine the eigenstates and their energies for arbitrary J and j.

c) Sketch the energy-level diagram for $J=2j>0$. Also for $J=-2j>0$. What is peculiar about the case $J<0$?

d) Sketch the Zeeman splitting of the ground state in the two cases where $J>0$. If the selection rules for an experiment are $\Delta M = \pm 1$, which case would have detectable transitions within the ground state at very low temperatures?

*   *   *   *   *   *   *   *   *   *   *   *

Note – the problem was revised. This solution was written for the original version.

a.) $\mathcal{H} = -J\, S_1^z S_2^z - j\,(S_1^z S_3^z + S_2^z S_3^z)$

makes "parallel" arrangement of spins lowest in energy.

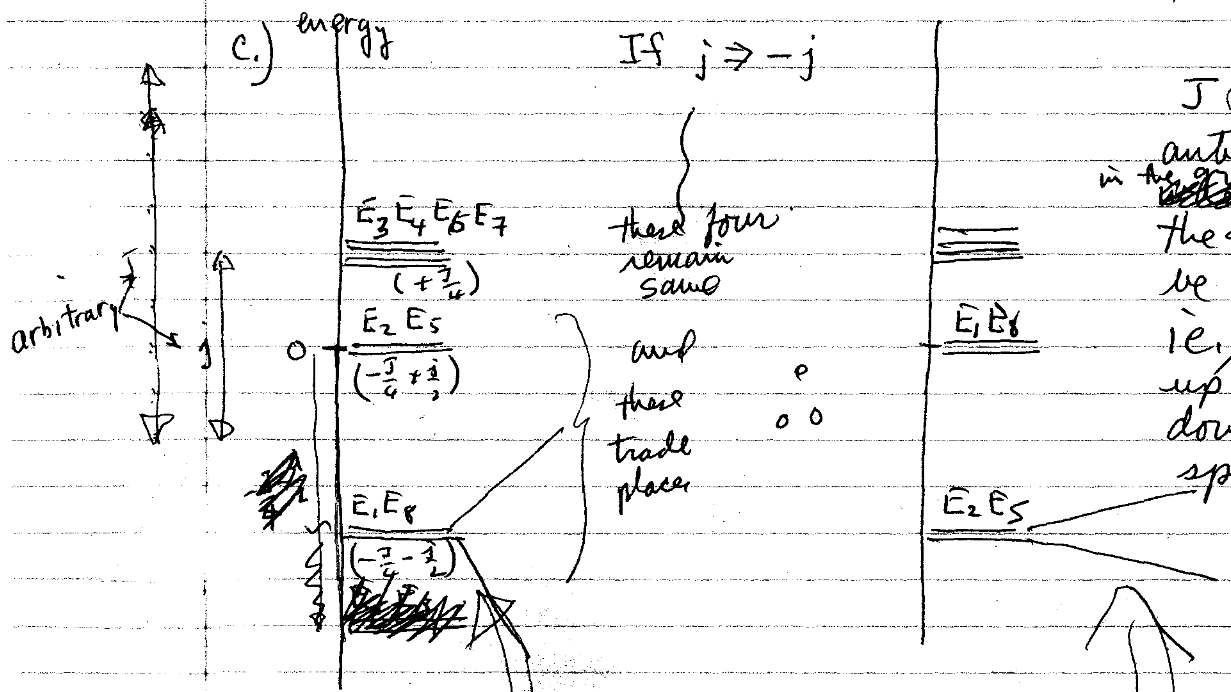
b.) States:

1. $\|{+}{+}{+}\rangle$	$E_1 = -\frac{J}{4} - \frac{j}{4} - \frac{j}{4}$	$= -\frac{J}{4} - \frac{j}{2}$,	$M_S = \frac{3}{2}$
2. $\|{+}{+}{-}\rangle$	$E_2 = -\frac{J}{4} + \frac{j}{4} + \frac{j}{4}$	$= -\frac{J}{4} + \frac{j}{2}$,	$M_S = \frac{1}{2}$
3. $\|{+}{-}{+}\rangle$	$E_3 =$	$= \frac{J}{4}$	$M_S = \frac{1}{2}$
4. $\|{-}{+}{+}\rangle$	$E_4 =$	$= \frac{J}{4}$	$M_S = \frac{1}{2}$
5. $\|{-}{-}{+}\rangle$	$E_5 = -\frac{J}{4} + \frac{j}{4} + \frac{j}{4}$	$= -\frac{J}{4} + \frac{j}{2}$	$M_S = -\frac{1}{2}$
6. $\|{-}{+}{-}\rangle$	$E_6 =$	$= \frac{J}{4}$	$M_S = -\frac{1}{2}$
7. $\|{+}{-}{-}\rangle$	$E_7 =$	$= \frac{J}{4}$	$M_S = -\frac{1}{2}$
8. $\|{-}{-}{-}\rangle$	$E_8 = -\frac{J}{4} - \frac{j}{4} - \frac{j}{4}$	$= -\frac{J}{4} - \frac{j}{2}$	$M_S = -\frac{3}{2}$

degen. degen. degen.

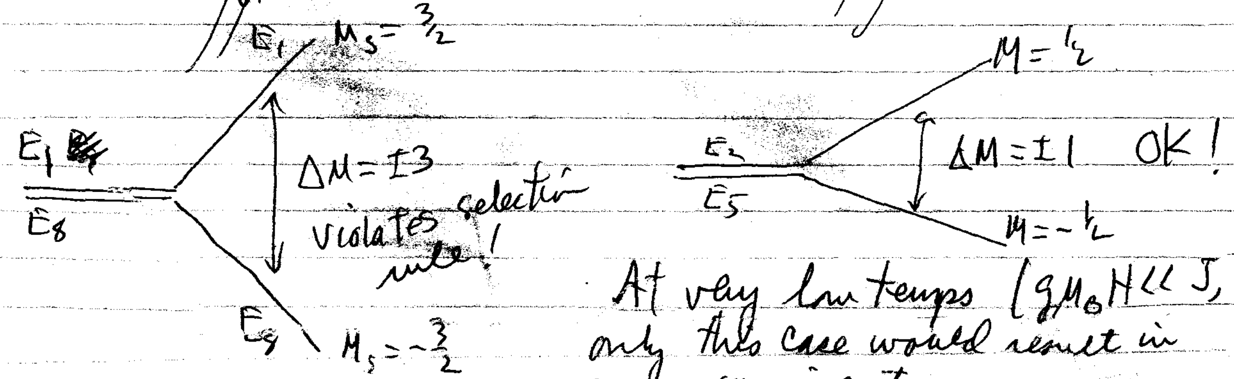
gd. st. has extra degen.

c.)



J cannot be antiferromagnetic ~~either~~ in the ground state because the system would be "frustrated", ie, if spin 1 is up and spin 2 is down, what could spin 3 do?

d.) Add $g\mu_B H M_s$



At very low temps ($g\mu_B H \ll J, j$) only this case would result in an experiment.

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8. An electron in the spinor state at t=0,

$$[\chi_o] = \begin{pmatrix} \cos\frac{\theta}{2} \\ \\ \sin\frac{\theta}{2} \end{pmatrix} e^{i\vec{k}\cdot\vec{r}} \qquad [\theta,\ \vec{k} = \text{constant}]$$

is subjected to a uniform magnetic field B in the z-direction, i.e. the quantization axis.

a) Interpret the state $[\chi_o]$.

b) Determine $[\chi]$ for $t>0$ and interpret it.

8. <u>QM Hermanson</u>

"kay" "are"

An electron in the Spinor state at $t=0$,

chi

$$[\chi_0] = \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \end{pmatrix} e^{i\underset{\sim}{k}\cdot\underset{\sim}{r}} \qquad [\theta, \underset{\sim}{k} = \text{constant}]$$

is subjected to a uniform magnetic field B in the z-direction, i.e. the quantization axis.

a) Interpret the state $[\chi_0]$.

END b) Determine $[\chi]$ for $t>0$ and interpret it.

Sol'n : a) Compute spin projections

$$\langle S_x\rangle = \left(\cos\frac{\theta}{2}, \sin\frac{\theta}{2}\right)\frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \end{pmatrix}$$

$$= \frac{\hbar}{2}\left(\cos\frac{\theta}{2}\sin\frac{\theta}{2} + \sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)$$

$$= \frac{\hbar}{2}\sin\theta$$

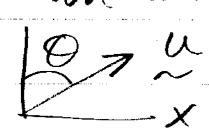
$$\langle S_y\rangle = \frac{\hbar}{2}\left(\cos\frac{\theta}{2}, \sin\frac{\theta}{2}\right)\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \end{pmatrix}$$

$$= 0$$

$$\langle S_z\rangle = \frac{\hbar}{2}\left(\cos\frac{\theta}{2}, \sin\frac{\theta}{2}\right)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \end{pmatrix}$$

$$= \frac{\hbar}{2}\cos\theta$$

The spin is aligned along an axis $\underset{\sim}{u}$ in the xz plane and moves along $\underset{\sim}{k}$.



b) when $B \neq 0$, $H = \frac{p^2}{2m} + H'$

$$H' = -M_z B$$

$$= -g\left(\frac{\mu_B}{\hbar}\right) S_z B \quad ; \quad \mu_B = \frac{q\hbar}{2m_e} < 0 \text{ for } e^-$$

$$= -\left(\frac{g}{2}\right) \frac{qB}{m_e} S_z$$

$$= \omega_0 S_z \quad ; \quad \omega_0 = -\left(\frac{g}{2}\right) \frac{qB}{m_e} > 0$$

Now $[\chi] = e^{-iHt/\hbar} \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \end{pmatrix} e^{i\underset{\sim}{k}\cdot\underset{\sim}{r}}$

$$= e^{i(\underset{\sim}{k}\cdot\underset{\sim}{r} - E_k t/\hbar)} e^{-i\omega_0\left(\frac{S_z}{\hbar}\right)t} \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \end{pmatrix} \begin{matrix} \leftarrow S_z = \frac{\hbar}{2} \\ \leftarrow S_z = -\frac{\hbar}{2} \end{matrix}$$

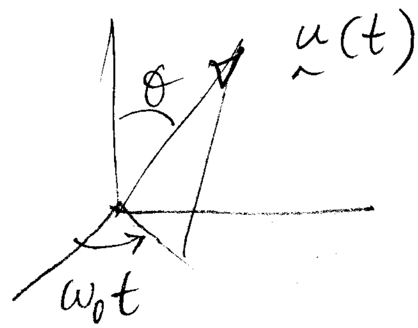
$$= e^{i\phi} \begin{pmatrix} \cos\frac{\theta}{2}\, e^{-i\omega_0 t/2} \\ \sin\frac{\theta}{2}\, e^{i\omega_0 t/2} \end{pmatrix} ; \quad \phi = \underset{\sim}{k}\cdot\underset{\sim}{r} - \frac{\hbar k^2 t}{2m}$$

And $\langle S_x \rangle = \frac{\hbar}{2} \sin\theta \cos\omega_0 t$

$$\langle S_y \rangle = \frac{\hbar}{2} \sin\theta \sin\omega_0 t$$

$$\langle S_z \rangle = \frac{\hbar}{2} \cos\theta$$

The spin precesses about B with frequency ω_0:



DEPARTMENT of PHYSICS

PH.D. COMPREHENSIVE EXAMINATION

TUESDAY, SEPT. 25, 1984, 9-12 AM

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9. In a crystal lattice of N sites, m Schottky defects are formed, each defect corresponding to one of the N original atoms being removed from the lattice and leaving a vacancy behind. An energy ε ($\varepsilon > 0$) is required to form one such defect.

a) Find the average number of Schottky defects when the crystal is at temperature T.

b) Calculate the contribution to the specific heat associated with defect creation.

c) Suppose a volume decrease δ is associated with the formation of each defect. Find the equilibrium volume at temperature T and pressure p, assuming the volume of the perfect crystal is V_o.

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~~Stat Mech / Sol. St.~~ - GT. ✓RC

9. In a crystal lattice of N uc sites, m lc Schottky defects are formed, each defect corresponding to one of the N original atoms being removed from the lattice and leaving a vacancy behind. An energy ϵ $(\epsilon > 0)$ is required to form one such defect. lc epsilon ε on spinwriter

a) Find the average number of Schottky defects when the crystal is at temperature T. uc

b) Calculate the contribution to the specific heat associated with defect creation.

c) Suppose a volume decrease δ (l.c. delta) ~~were~~ is associated with the formation of each defect. Find the equilibrium volume at temperature T uc and pressure p lc, assuming the volume of the perfect crystal is V_0. u.c vee

END

Soln: a) Calculate partition fcn., realizing that there are $\binom{N}{m}$ ways of picking the n atoms to be removed:

$$Z = \sum_{m=0}^{N} \binom{N}{m} e^{-n\epsilon/kT} = \left(1 + e^{-\epsilon/kT}\right)^N$$

$$\langle n \rangle = \frac{\partial}{\partial(-\epsilon/kT)} \ln Z = \frac{N e^{-\epsilon/kT}}{1 + e^{-\epsilon/kT}} = \frac{N}{1 + e^{\epsilon/kT}}$$

Or calculate $F = U - TS = n\epsilon_0 - T k_B \ln\binom{N}{n}$

$\langle n \rangle$ found from $\left.\frac{\partial F}{\partial n}\right|_{n=\langle n \rangle} = 0$, so $0 = \epsilon_0 - k_B T \ln \frac{N - \langle n \rangle}{\langle n \rangle}$

using Stirling's approx.

b) either write $F = -k_B T \ln Z = -Nk_B T \ln(1+e^{-\epsilon/kT})$

$$\text{+ use } C = T\frac{\partial S}{\partial T} = -T\frac{\partial^2 F}{\partial T^2}$$

or use $C = \frac{\partial U}{\partial T} = \frac{\partial}{\partial T}\langle n\rangle \epsilon$

$$= \frac{\partial}{\partial T}\left(\frac{N\epsilon}{1+e^{\epsilon/kT}}\right) = Nk_B\left(\frac{\epsilon}{kT}\right)^2 \frac{e^{\epsilon/kT}}{\left(1+e^{\epsilon/kT}\right)^2}$$

c) Now

$$Z = e^{-p\frac{V_0}{kT}}\left(1+e^{-\epsilon/kT + p\delta/kT}\right)^N$$

+

$$G = pV_0 - Nk_B T \ln\left(1+e^{-\epsilon/k_BT + p\delta/k_BT}\right).$$

(where, since the free energy is explicitly a fcn of p, we denote it G for Gibbs)

$$V = \frac{\partial G}{\partial p} = V_0 - \frac{N\delta}{1+e^{\epsilon/kT - p\delta/kT}}$$

10. N atoms of a monatomic gas in a box of volume V have a Maxwell Boltzmann velocity distribution

$$n_o(v) = \frac{N}{V}\left(\frac{m}{2\pi k_B T}\right)^{3/2} 4\pi\, v^2 e^{-mv^2/2k_BT}$$

where $n_o(v)\,dv$ is the number density of atoms with speeds in the interval dv at v, T=absolute temperature, M=mass of each atom, and k_B=Boltzmann's constant. A small hole is made in the box, so that atoms can leak out.

a) Find an expression for the velocity distribution $n'(v)$ of escaping atoms - i.e. the number (per unit time and unit surface area of the hole) escaping with speeds in the interval dv at v. Explain <u>qualitatively</u> why n' differs in functional form from n_o.

b) Find the rms velocity of escaping atoms, and compare it with the rms velocity of atoms inside the container. Based on your result, explain whether the remaining gas will become hotter or colder.

~~Kinetic Theory~~ (G.T.) ✓ RC

~~or Stat. Mech~~

10. N atoms of a monatomic gas in a box of volume V have a Maxwell-Boltzmann velocity distribution

$$n_0(v) = \left(\frac{N}{V}\right)\left(\frac{m}{2\pi k_B T}\right)^{3/2} 4\pi \; v^2 e^{-mv^2/2k_BT}$$

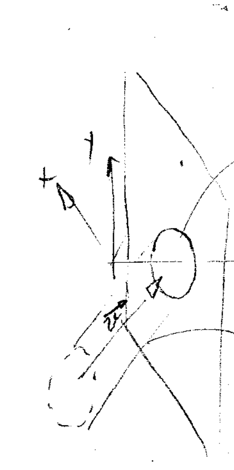
where $n_0(v)\,dv$ is the number density of atoms with speeds in the interval dv at v, T = absolute temperature, m = mass of each atom, and k_B = Boltzmann's constant. A small hole is made in the box, so that atoms can leak out.

a) Find an expression for the velocity distribution $n'(v)$ of escaping atoms - ie, the number (per unit time and unit surface area of the hole) escaping with speeds in the interval dv at v. Explain qualitatively why n' differs in functional form from n_0. ~~[note: different units! JH]~~

b) Find the rms velocity of escaping atoms, and compare it with the rms velocity of atoms inside the container. Based on your result, explain whether ~~will~~ the remaining gas will become hotter or colder.

END

Soln:



In time dt, all atoms with velocity $\vec{v}$ will escape through the hole provided they lie in the slanted cylinder shown at left, with volume $A v_z\,dt$. Thus we need to multiply n_0 by $A v_z\,dt$ to get the number escaping in time dt with vector velocity $\vec{v}$. We then integrate over all angles of $\vec{v}$, subject to the restriction $v_z > 0$. In polar coords, this amounts to $0 \le \varphi < 2\pi$, $0 \le \theta < \pi/2$.

$$n'(v)\, dt\, A = \int_0^{\pi} d\varphi \int_0^{\pi/2} d\theta (\sin\theta) v\cos\theta \quad n_0(v)\, A\, dt.$$

where the 1st factor of $\cos\theta$ comes from the transcription to polar coordinates, and the 2nd from v_z. Thus.

$$n'(v) = \frac{n_0(v)\, v}{4}$$

This differs from n_0 for the physical reason that faster-moving atoms strike the walls more often, and therefore are more likely to escape.

b) $$\langle v^2 \rangle_{esc} = \frac{\int_0^\infty n'(v)\, v^2\, dv}{\int_0^\infty n'(v)\, dv} = \frac{\int_0^\infty v^5 e^{-mv^2/2kT} dv}{\int_0^\infty v^3 e^{-mv^2/2kT} dv}$$

$$= \left(\frac{2kT}{m}\right) \frac{\int_0^\infty x^5 e^{-x^2} dx}{\int_0^\infty x^3 e^{-x^2} dx} = \left(\frac{2kT}{m}\right) \cdot 2 = \frac{4kT}{m}$$

so for escaping atoms $\boxed{v_{rms} = \sqrt{\frac{4kT}{m}}}$

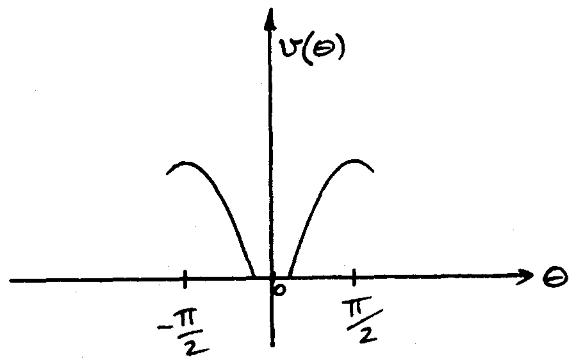
$$\langle v^2 \rangle_{cont.} = \frac{\int_0^\infty n_0(v)\, v^2\, dv}{\int_0^\infty n_0(v)\, dv} = \frac{\int_0^\infty v^4 e^{-mv^2/2kT} dv}{\int_0^\infty v^2 e^{-mv^2/2kT}} = \left(\frac{2kT}{m}\right) \frac{\int_0^\infty x^4 e^{-x^2} dx}{\int_0^\infty x^2 e^{-x^2} dx}$$

$$= \left(\frac{2kT}{m}\right) \cdot \frac{3}{2}$$

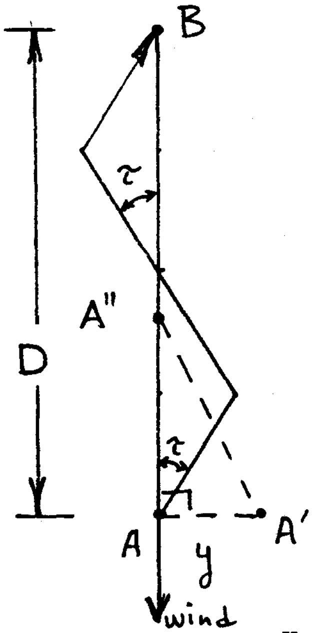
so for atoms in the container $\boxed{v_{rms} = \sqrt{\frac{3kT}{m}}}$

The slower atoms are left behind, and the container <u>cools</u>.

11. A sailboat sails at speed $v(\theta) = -\alpha+\beta \sin|\theta|$ when heading at an angle θ to the wind. $\alpha>0$, $\beta>0$

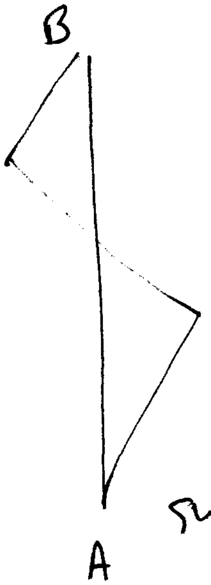


Find the optimum tacking angles τ in order to sail straight upwind at the fastest rate from A to B.



Starting from A' instead of A ($\frac{Y}{D} < \tan \tau$), show that the fastest way to get to B is still to tack at angles τ. (Hint: calculate time to sail from A' to A'' on line AB plus time to sail from A'' to B by optimum tacking, and minimize.)

Solution



$$t = \frac{D}{v(\theta)\cos\theta} \qquad \frac{dt}{d\theta} = 0$$

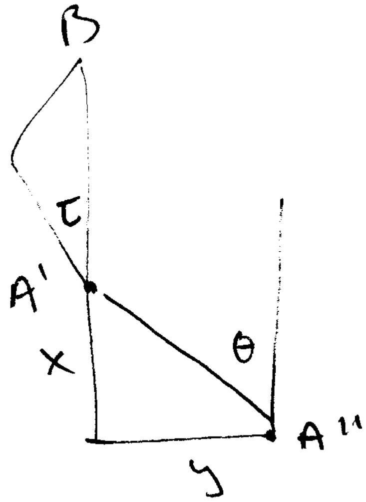
$$\frac{v'(\theta)}{v(\theta)} = \frac{\sin\theta}{\cos\theta} = \frac{\beta\cos\theta}{-\alpha+\beta\sin\theta}$$

$$\sin\theta\left[-\alpha+\beta\sin\theta\right] = \beta\left(1-\sin^2\theta\right)$$

$$2\beta\sin^2\theta - \alpha\sin\theta - \beta = 0$$

$$\sin^2\theta - \frac{\alpha}{2\beta}\sin\theta - \frac{1}{2} = 0$$

$$\boxed{\sin\tau = \frac{\alpha}{4\beta} + \sqrt{\frac{1}{2}+\left(\frac{\alpha}{4\beta}\right)^2}}$$



$$t = \frac{x}{v(\theta)\cos\theta} + \frac{D-x}{v(\tau)\cos\tau} \qquad \frac{y}{x} = \tan\theta$$

$$\frac{dt}{dx} = \left(\frac{1}{v(\theta)\cos\theta} - \frac{1}{v(\tau)\cos\tau}\right) - \frac{x}{v(\theta)\cos\theta}\frac{d}{d\theta}\left(v\cos\theta\right)\frac{d\theta}{dx}$$

$$= 0$$

$\theta = \tau$ fulfills $\frac{dt}{dx} = 0$!

12. ^3H nuclei collide with ^4He nuclei to produce ^6Li nuclei plus neutrons ^1n. Find the kinetic energy threshold for this reaction in the lab frame where the helium nuclei are the targets. Mass defects of the nuclei are:

$\Delta M(^3H) = 15.84\ Mev/c^2$

$\Delta M(^4He) = 3.61\ Mev/c^2$

$\Delta M(^1n) = 8.37\ Mev/c^2$

$\Delta M(^6Li) = 15.86\ Mev/c^2$

(The mass defect of a nucleus is:

$$\Delta M = M(A,Z) - A\mu,$$

where M(A,Z) is the actual mass of the nucleus, A is the number of baryons, and μ is the nuclear mass unit.)

Solution: Cons. of energy + momentum in Lab

$$\frac{P^2}{2M_1} + M_1c^2 + M_2c^2 = \frac{P^2}{2(M_3+M_4)} + (M_3+M_4)c^2$$

$$\frac{P^2}{2M_1}\left\{1 - \frac{M_1}{M_3+M_4}\right\} = (M_3+M_4-M_1-M_2)\,c^2$$

$$M_1/(M_3+M_4) \cong 3/7$$

$$\frac{P^2}{2M_1} = \frac{7}{4}\left(15.86 + 8.37 - 3.61 - 15.84\right)\ \text{MeV}$$

DEPARTMENT of PHYSICS

PH.D. COMPREHENSIVE EXAMINATION

TUESDAY, SEPT. 25, 1984, 2-5 PM

* * * * * * * * * * * *

Answer each of the following questions. Each question carries equal weight. Begin your answer to each question on a new sheet of paper; solutions to different questions must not appear on the same sheet of paper. Label each page of your exam as follows:

A. Your name in upper left-hand corner.

B. Problem number, and page number for that problem, in upper right hand corner.

* * * * * * * * * * * *

13. A rocket has engines which give it a constant acceleration of one g relative to its instantaneous inertial frame as measured by an accelerometer attached to the rocket. The rocket starts from rest near the earth. Ignore all gravitational effects.

Compute the proper time (τ) for the occupants of the rocket ship to travel the 30,000 light years from the earth to the center of the galaxy, assuming that they accelerate at one g for half the trip and decelerate at one g for the remaining half.

Suggestions:

Use the velocity and acceleration four-vectors. Note that $u^\alpha = dx^\alpha/d\tau$, $a^\alpha = du^\alpha/d\tau$, and that $g \approx 1$ year^{-1} in units where $c=1$. Also note that the four-velocity and four acceleration are perpendicular.

* * * * * * * * * * * *

Problem (1) solution

(a) Take the rocket's motion to be along the x-axis. Let t be Earth time and τ proper "ship" time. The initial condition of rest near the Earth is then that

$$u^x = 0 \quad \text{at } t=0 \text{ and/or } \tau = 0$$

We have the following equations for the four-velocity $\vec{u}$ and four-acceleration $\vec{a}$:

$$\vec{u}\cdot\vec{u} = -1 = -(u^t)^2 + (u^x)^2 \qquad \text{(normalization of four velocity)} \qquad (1)$$

$$\vec{u}\cdot\vec{a} = 0 = -a^t u^t + a^x u^x \qquad (\vec{a} \text{ orthogonal to } \vec{u}) \qquad (2)$$

$$\vec{a}\cdot\vec{a} = g^2 = -(a^t)^2 + (a^x)^2 \qquad \text{(proper acceleration is } g) \qquad (3)$$

(2) $\Rightarrow$ $\quad a^t = a^x \left(\dfrac{u^x}{u^t}\right)$ substituting this into (3), we get

$$g^2 = (a^x)^2\left[1 - \left(\frac{u^x}{u^t}\right)^2\right] = -\frac{(a^x)^2}{(u^t)^2}\left[-(u^t)^2 + (u^x)^2\right] \quad \text{now use Eq. (1)}$$

$$g^2 = (a^x)^2/(u^t)^2 \quad \text{or} \quad \boxed{a^x = g\,u^t} \qquad (4)$$

which further implies that $\quad \boxed{a^t = g\,u^x} \qquad (5)$

Now differentiate Eq. (4) with respect to proper time to get a differential equation for u^x:

$$\underbrace{\frac{da^x}{d\tau} = \frac{d^2u^x}{d\tau^2}}_{\text{by defn. of } \vec{a},\vec{u}} \underbrace{=}_{\text{by Eq. (4)}} g\,\frac{du^t}{d\tau} \underbrace{=}_{\text{by defn. } \vec{a} \,\&\, \vec{u}} g\,a^t \underbrace{=}_{\text{Equ. (5)}} g^2 u^x \qquad (6)$$

The solutions to this equ, $\dfrac{d^2u^x}{d\tau^2} = g^2 u^x$, are, obviously, just

$$\boxed{u^x = A\exp(g\tau) + B\exp(-g\tau)} \qquad (7)$$

Since the initial condition is rest at $\tau=0$, and $a^x = \dfrac{du^x}{d\tau} = g$ at $\tau = 0$, we must have $A = -B = 1$, so that

$$u^x = \sinh(g\tau) \qquad (8)$$

and, by Eq. (1),

$$u^t = \cosh(g\tau) \qquad (9)$$

To find $x(\tau)$, we integrate Eq. (8), subject to the initial condition that $x = 0$ at $\tau = 0$:

$$x(\tau) = g^{-1}\left[\cosh(g\tau) - 1\right] \quad . \qquad (10)$$

In units with $c = 1 = \frac{3\times10^{10}\,\text{cm}}{\text{sec}}$

$$g = 980\ \frac{\text{cm}}{\text{sec}^2} \cdot \frac{1\ \text{sec}}{3\times10^{10}\,\text{cm}} = 3.27\times10^{-8}\ \text{sec}^{-1} = \frac{1}{3.06\times10^{7}\ \text{sec}} \approx \frac{1}{\text{yr.}}$$

To get half-way to the galactic center requires $x = 15{,}000$ light years, so

$$15{,}000\ \text{years} = \frac{1}{1\ \text{yr}^{-1}}\left[\cosh(g\tau) - 1\right]$$

or

$$\cosh(g\tau) \approx 15{,}000$$

For such a large value, cosh is well approximated by $\frac{1}{2}\exp(g\tau)$, so

$$g\tau \approx \ln(30{,}000)$$

$$\tau \approx g^{-1}\ln(30{,}000) \approx \underline{10.3\ \text{years}}$$

The deceleration half of the trip is identical, so the total time is

$$\boxed{2\tau = 20.6\ \text{years}}$$

14. Find the quantum-mechanical eigenfunction $\psi_n(k)$ and energy bands $E_n(k)$ of a one-dimensional empty lattice $[V(x)=0]$ with lattice constant a; n and k are the band index and wave-vector. Illustrate your results with a sketch of the energy bands. Hint: Use Bloch's theorem to represent $\psi_n(k)$ in terms of its periodic part $u_n(k)$.

quantum mechanical psi l.c. kay

14. Find the eigenfunction $\psi_n(k)$ (n band index) "IN"
and draw the energy bands of a one dimensional
empty lattice with lattice constant a.
Empty lattice means $V(x)=0$. (Hint use the
wave equation ~~and~~ boundary conditions for $u_n(x)$,
the periodic part of the Bloch function). "you"

Omit

"aitch"

Find the quantum-mechanical eigenfunctions $\psi_n(k)$ and energy bands $E_n(k)$ of a one-dimensional empty lattice [$V(x)=0$] with lattice constant a; n and k are the band index and wave-vector. Hint: Use Bloch's theorem to represent $\psi_n(k)$ in terms of its periodic part $u_n(k)$.

$$\psi_n(k) = e^{ikx} u_k^{(n)}(x)$$

$$\nabla_x^2 \psi_n(k) = \frac{2m(V-E)}{\hbar^2} \psi_n(k)$$

$$e^{ikx}(\nabla_x^2 + 2ik\nabla_x - k^2)\, u_k^{(n)}$$

$$(\nabla_x^2 + 2ik\nabla_x)\, u_k^{(n)} = \left(k^2 - \frac{2mE}{\hbar^2}\right) u_k^{(n)}$$

with BC : $u_k^{(n)}(0) = u_k^{(n)}(a)$

Try plane wave solu' $u_k^{(n)} \propto e^{i\sigma x}$

BC. $e^{i\sigma a} = 1$, $\sigma a = 2\pi n$, $n = 0, \pm 1, \pm 2 \cdots$

put into D.E.

$$\sigma_n^2 - 2\sigma_n k + k^2 = \frac{2m E(k,n)}{\hbar^2}$$

$$E(k,n) = \frac{\hbar^2}{2m}(\sigma_n + k)^2 = \frac{\hbar^2}{2m}\left(k + \frac{2\pi n}{a}\right)^2$$

$u_k^{(n)} = \frac{1}{\sqrt{a}} e^{\frac{2\pi n}{a} x}$ $n = 0, \pm 1, \pm 2$

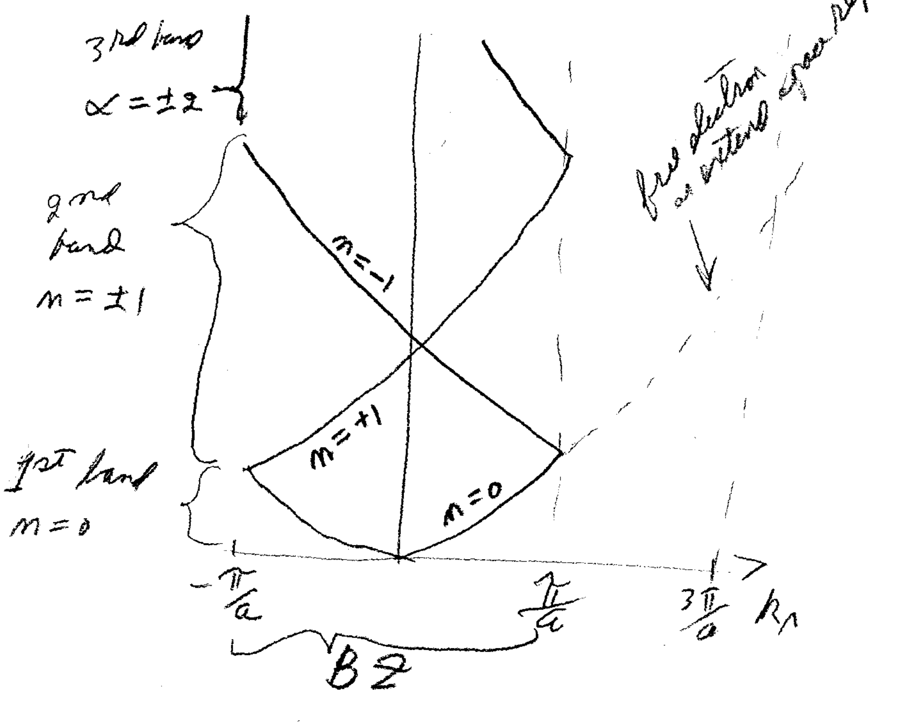
$\rightarrow \psi_n(k) = \frac{1}{\sqrt{Na}} e^{i(k + \frac{2\pi n}{a})x}$ $-\frac{\pi}{2} < k \leq \pi/a$

$n=1$, $k + \frac{2\pi n}{a} = \frac{\pi}{a}$ for $k = -\frac{\pi}{a}$

$= +\frac{3\pi}{a}$ " $k = +\frac{\pi}{a}$

$n=-1$ $k + \frac{2\pi n}{a} = -\frac{\pi}{a}$ for $k = +\frac{\pi}{a}$

$= -\frac{3\pi}{a}$ " $k = -\frac{\pi}{a}$



15. The mercury atom has the following energy levels expressed in terms of energy units $1/\lambda$.

$6s^2$	^1S_o	0
6s6p	^3P_o	37,645 cm^{-1}
6s6p	$^3P_1$	39,412 cm^{-1}
6s6p	$^3P_2$	44,043 cm^{-1}
6s7s	$^3S_1$	62,350 cm^{-1}

a) Explain the meaning of the spectroscopic notation above.

b) What transitions will occur between these energy levels in a gas discharge? Explain in moderate detail.

c) Briefly outline an experimental method for verifying the total angular momenta J assigned to the levels above.

a) $\underset{n}{\overset{\uparrow}{6}}s\ \underset{\ell}{\overset{\uparrow}{6p}}$ gives nature of indiv. electron wavefn
$\ell = 0, 1, 2, 3, \ldots$ called $s, p, d, f \ldots$

$^3P_0$: $^{2S+1}L_J$ gives nature of multielectron wavefn coupled to give $\vec{L} = \vec{\ell}_1 + \vec{\ell}_2$
$\vec{S} = \vec{s}_1 + \vec{s}_2$ and $\vec{J} = \vec{j}_1 + \vec{j}_2$

b) ①
$$\left.\begin{array}{l} \Delta S = 0 \\ |\Delta L| \le 1 \\ |\Delta J| \le 1 \\ |\Delta m_J| \le 1 \\ \text{parity change req'd} \end{array}\right\} \text{for electric dipole operator } e\vec{r}$$

derived on basis of $\rightarrow$ $\left\{\begin{array}{l} \text{operator } \vec{r} \text{ does not act on } \vec{S} \text{ space} \\ \text{operator } \vec{r} \text{ is a vector (rank 1) operator in } \vec{L} \text{ space} \\ \quad \Rightarrow \text{max change } \pm 1 \text{ or } 0 \end{array}\right.$

② Fluorescence will occur, since the atoms in the discharge are excited by electron collisions

Conclude transitions are allowed between the triplets only (by electric dipole). Furthermore must be $S \to P$ to have parity change

Hence $^3S_1 \to {}^3P_2,\ {}^3P_1,\ {}^3P_0$ only $\boxed{\therefore \text{ 3 lines}}$

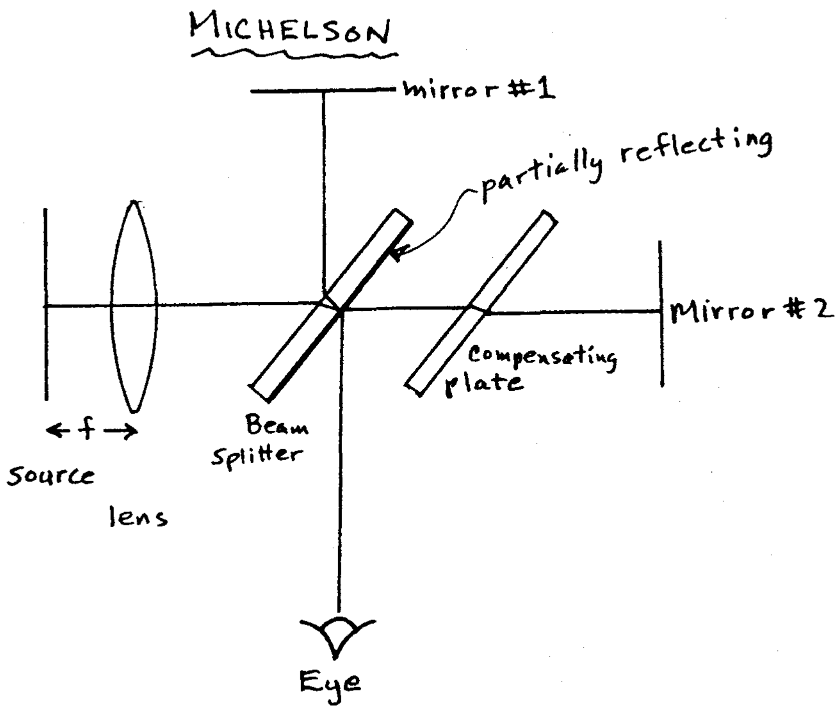
c) Zeeman effect : measure $\left\{\begin{array}{l} \#\text{ sublevels} \\ g_J \text{ factors} \end{array}\right.$

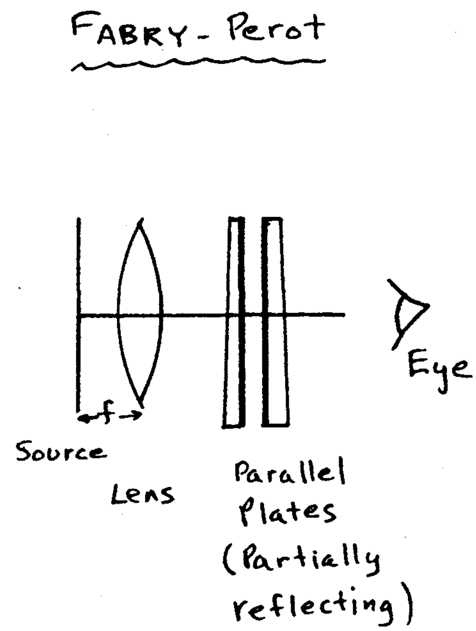
16. Consider the circular pattern of fringes resulting from a Michelson interferometer which is illuminated by an extended monochromatic source and which is viewed by eye. A schematic drawing is given below.

a) If the difference in distance between the beam splitter and the two mirrors is d=2 mm, find the order m of the central fringe for λ=500 nm and discuss whether it is bright or dark. (You may take these numbers to be exact.)

b) Find the angular radius of the 3rd dark fringe seen off-axis.

c) Describe the difference in the fringe pattern for a Fabry-Perot inteferometer (consisting of two parallel partially reflecting mirrors) <u>vs</u>. the Michelson interferometer and comment on their relative usefulness.





#16

a) path difference is $\Lambda = 2d\cos\theta$ (See optics texts)
Depending on whether or not there is a phase shift upon reflection, a given Λ value can give either constructive or destructive interference. For example, an uncoated glass beamsplitter would introduce a net phase difference of π. If we ignore this effect, $\Lambda = m\lambda$ is the condition for constructive interference:

$$\Lambda = 2d = m\lambda \qquad m = \frac{4\times10^{-3}\,m}{500\times10^{-9}\,m} = \underline{8000}$$

b) Increasing $\theta \Rightarrow \Lambda = 2d\cos\theta$ Decreases

$\therefore$ all off-axis parts of the pattern correspond to smaller m value

1^{st} dark ring	$m = 7999\frac{1}{2}$
2^{nd} " "	$m = 7998\frac{1}{2}$
3^{rd} " "	$m = 7997\frac{1}{2}$

so $\frac{\Lambda}{\lambda}\cos\theta = 7997.5$

$\cos\theta = \frac{7997.5}{8000}$ and $\theta = 1.43^\circ$

c) Fabry-Perot uses multiple reflections. Multiple beam interference gives much sharper interference than the two-beam case of the Michelson. Hence, Fabry-Perot is more useful for spectroscopy. The Michelson, with two $\perp$ arms, played a key role in confirming special relativity theory. Laser versions have now proven that space is isotropic to better than 2.5 parts in 10^{15}!