Reflection & Transmission Coefficients for the Rect$^{\underline{r}}$ Barrier.

4. As noted: $T = |E/A|^2$ & $R = |B/A|^2$ are the transmitted & reflected fractions for the incident wave. As such, they should obey: $T + R = 1$, in order that probability be conserved (i.e. we don't lose part of m). Let us check this out...

$$\rightarrow \frac{|E|^2}{|A|^2} + \frac{|B|^2}{|A|^2} = \left(1 + \tfrac{1}{4}\mu^2 \sinh^2\xi\right)\frac{|E|^2}{|A|^2}, \quad \text{w/ } \underline{\xi = 2\kappa a} \tag{12}$$

... but: $|E|^2/|A|^2 = 1/\left(\cosh^2\xi + \tfrac{1}{4}\lambda^2\sinh^2\xi\right)$...

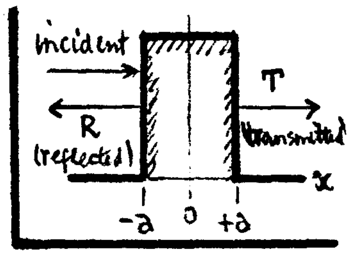
... use: $\cosh^2 = 1 + \sinh^2$, and note that...

$$\ldots 4 + \lambda^2 = 4 + \left(\frac{\kappa^2}{k^2} - 2 + \frac{k^2}{\kappa^2}\right) = \left(\frac{\kappa}{k} + \frac{k}{\kappa}\right)^2 = \mu^2 \Rightarrow 1 + \frac{\lambda^2}{4} = \frac{\mu^2}{4}\ldots$$

$$\xrightarrow{\text{so}} |E|^2/|A|^2 = 1/\left[1 + \left(1 + \tfrac{1}{4}\lambda^2\right)\sinh^2\xi\right] = 1/\left[1 + \tfrac{1}{4}\mu^2\sinh^2\xi\right]. \tag{13}$$

When this result is used in (12), we have immediate conservation of particles...

$$\rightarrow \frac{|E|^2 + |B|^2}{|A|^2} = 1, \quad \text{w/ } \underset{\text{incident}}{|A|^2} = \underset{\text{reflected}}{|B|^2} + \underset{\text{transmitted}}{|E|^2}, \quad \text{i.e. } \underline{\underline{T + R = 1}}. \tag{14}$$



Remarkable! m does get through the barrier... anything that isn't reflected at $x = -a$ is transmitted past $x = +a$. Classically, this transmission is impossible; classically, we would have (for $E < V_0$): $R = 1$, $T = 0$. But, QMly we have $T > 0$, i.e.

$$\left.\begin{array}{l}\text{transmission}\\ \text{coefficient}\end{array}\right\} \boxed{T = \frac{\text{transmitted intensity}}{\text{incident intensity}} = \frac{|E|^2}{|A|^2} = 1/\left(1 + \tfrac{1}{4}\mu^2\sinh^2 2\kappa a\right)}. \tag{15}$$

For a high & wide barrier: $\kappa a \gg 1 \Rightarrow \sinh 2\kappa a \simeq \frac{1}{2}e^{2\kappa a} \gg 1$, and (15) gives:

$$\rightarrow \underline{\underline{T \simeq \frac{4}{\mu^2}/\sinh^2 2\kappa a}} = 16\, e^{-4\kappa a}\underbrace{\left(\frac{\kappa k}{\kappa^2 + k^2}\right)^2}_{\to\ \frac{E}{V_0}\left(1 - \frac{E}{V_0}\right) \sim \text{ small, for } V_0 \gg E}, \quad \text{w/ } \kappa = \left[\frac{2m}{\hbar^2}(V_0 - E)\right]^{1/2} \tag{16}$$

Generally T is small, but non-zero. In any case, the reflection coefficient (fraction of m's reflected) is: $R = 1 - T$.

QM Measurements in Classically Forbidden Regions.

REMARKS On barrier penetration.

(1) To go from $x<-a$ to $x>+a$, m must pass through the region $-a<x<+a$, where $V_0>E$ and m's K.E. $=E-V_0$ is (−)ve. Indeed, we can calculate:

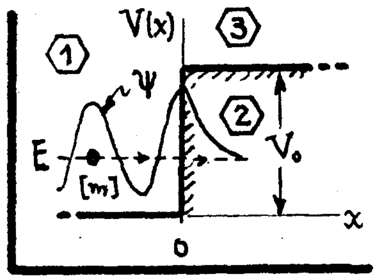
$$C=\frac{1}{2}\left(1-\frac{ik}{\kappa}\right)E\,e^{\kappa a+ika},\quad D=\frac{1}{2}\left(1+\frac{ik}{\kappa}\right)E\,e^{-\kappa a+ika}\ \int\text{inside barrier}$$

so $\rightarrow$ $$\underline{\frac{|C|^2+|D|^2}{|A|^2}=\frac{1}{2}\left(1+\frac{k^2}{\kappa^2}\right)T\cosh 2\kappa a}\rightarrow(4E/V_0)\,e^{-2\kappa a},\ \text{when }\kappa a\gg 1. \tag{17}$$

So, m has a nonvanishing presence inside the barrier, a classically forbidden region. But $|C|^2$ & $|D|^2$ do not enter into the probability conservation eqtn, Eq. (14). This $\Rightarrow$ m is never actually found in $-a<x<+a$!

---

(2) To explain this peculiarity, we invoke the uncertainty principle to show that any attempt to locate m in a classically inaccessible region -- such as region ② inside the potential step shown in the sketch -- will in fact perturb m just enough to boost it into a classically allowed region. For the potential step, the act of measurement boosts m from ② to ③. Argue as follows...



In region ② ($x>0$): $\psi(x)\propto e^{-\kappa x}$, w/ $\kappa=\left[\frac{2m}{\hbar^2}(V_0-E)\right]^{1/2}$.

To find m in ②, need to localize its position to within $\Delta x\sim 1/\kappa$.

This localization generates momentum components: $\Delta p\sim\hbar/\Delta x\sim\hbar\kappa$.

so the measurement perturbs m's energy by: $\Delta E\sim\frac{(\Delta p)^2}{2m}\sim\frac{\hbar^2\kappa^2}{2m}=V_0-E$.

and m's new energy is: $E+\Delta E\sim V_0$ $\Rightarrow$ <u>m is boosted from ② to ③</u>. (18)

Indeed, although $|\psi|^2>0$ in the forbidden region ②, m can never physically be located there. That is why $|C|^2$ & $|D|^2$ did not figure in the particle conservation eqtn. Generally, the uncertainty principle "protects" QM theory from allowing direct measurement of classically impossible situations.