Solution to problem 1

1. In the absence of uniform charge distribution on the insulating cylindrical surface, potential V satisfies Laplace's equation, $\nabla^2 V = 0$. Because of the cylindrical symmetry the most appropriate form of Laplace's equation would be the one expressed in the cylindrical coordinates— r, θ and z. Furthermore, because it is an infinitely long cylinder the potential must be a function of r and θ only. The corresponding Laplace's equation is simply given by $\frac{1}{2}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2}=0$ with the boundary conditions that (a) at $r \to \infty$ $V(r,\theta) \to -E_o r\cos\theta$ because $\vec{E} = E_o\hat{x} = -\frac{\partial V}{\partial x}\hat{x}$, where $x = r\cos\theta$; (b) the parallel (to the cylindrical surface) component of the electric field, $\vec{E}$, is continuous, which implies that the potential is finite in all regions and is continuous across the dielectric boundary; and (c) the perpendicular component of $\vec{D} = \varepsilon\vec{E}$ or $-\varepsilon\frac{\partial V}{\partial r}$ is continuous across the dielectric boundary at $r = r_o$. The most general solution of Laplace's equation in a given region can be obtained using the separation of variables and is given by the cylindrical harmonics as

$$V(r,\theta) = a^{\pm} + b^{\pm}\ln r + \sum_{n=1}^{\infty} r^n(c_n^{\pm}\cos n\theta + d_n^{\pm}\sin n\theta) + r^{-n}(g_n^{\pm}\cos n\theta + h_n^{\pm}\sin n\theta)$$

where $\pm$ refers to the regions where $+$ indicates $r > r_o$ and $-$ $r < r_o$. The Uniqueness Theorem ensures that the solution that satisfies all the boundary conditions is the only solution. Let us apply the boundary conditions to these piece-by-piece solutions:

(a) At $r \to \infty$ the $V(r,\theta) \to -E_o r\cos\theta$ implies that

$a^+ = 0,\ b^+ = 0,\ c^+_{\ 1} = -E_o,\ c^+_{\ n} = 0$ for $n \neq 1$ and finally, $d^+_{\ n} = 0$ for all *n*.

(b) $V(r,\theta)$ is finite for $r < r_o$. This implies that $b^- = 0$ and $V(r,\theta)$ is continuous across the dielectric boundary, which simply implies that at $r = r_o$ $V^- = V^+$. By equating the coefficients of $\cos n\theta$ and $\sin n\theta$ where $n \geq 1$ we obtain the following equations:

$a^- = 0,\ r_o c_1^- = -E_o r_o + g_1^+ r_o^{-1},\ r_o^n c_n^- = r_o^{-n} g_n^+$ for $n \geq 2$ and $r_o^n d_n^- = r_o^{-n} h_n^+$ for $n \geq 1$.

(c) Similarly, the boundary condition at $r = r_o$, $\varepsilon \dfrac{\partial V^-}{\partial r} = \varepsilon_o \dfrac{\partial V^+}{\partial r}$, yields

$\varepsilon c_1^- = -\varepsilon_o E_o - \varepsilon_o g_1^+ r_o^{-2}$, $\varepsilon n r_o^{n-1} d_n^- = -\varepsilon_o n r_o^{-n-1} h_n^+$ for $n \geq 2$ and

$\varepsilon n r_o^{n-1} d_n^- = -\varepsilon_o n r_o^{-n-1} h_n^+$ for $n \geq 1$

Solving the coupled equations obtained from the boundary conditions expressed in parts (b) and (c) above immediately yields

$c_n^- = g_n^+ = 0$ for $n \geq 2$; $d_n^- = h_n^+ = 0$ for $n \geq 1$;

$c_1^- = -\dfrac{2\varepsilon_o E_o}{(\varepsilon + \varepsilon_o)}$; and $g_1^+ = -\left(\dfrac{\varepsilon - \varepsilon_o}{\varepsilon + \varepsilon_o}\right) E_o r_o^2$

In the absence of any surface charge on the dielectric cylinder

$V^-(r,\theta) = r c_1^- \cos\theta = -\dfrac{2\varepsilon_o E_o}{(\varepsilon + \varepsilon_o)} r \cos\theta$ for $r < r_o$, and

$V^+(r,\theta) = -E_o r \cos\theta + r^{-1} g_1^+ = -E_o r \cos\theta - \left(\dfrac{\varepsilon - \varepsilon_o}{\varepsilon + \varepsilon_o}\right) E_o r_o^2 r^{-1} \cos\theta$ for $r > r_o$

2. Now we introduce the uniform charge on the surface of the dielectric cylinder. Here we can simply use the *principle of superposition* because the electric field will not alter the charge distribution over an insulating surface. This would not have worked if this was a metallic cylinder. Therefore, we simply add electric potential V_σ obtained from uniform charge density σ over the surface of the insulating dielectric. This potential can easily be found using Gauss' theorem, which yields

   $V_\sigma^- = 0$ (no free charge inside the cylinder) for $r < r_o$ and

   $V_\sigma^+ = -\dfrac{\sigma r_o}{\varepsilon_o} \ln\left(\dfrac{r}{r_o}\right)$ (note that $V_\sigma^- = V_\sigma^+ = 0$ at $r = r_o$ as it should) for $r > r_o$

   By adding this to the potential distribution we obtain above the net electrostatic potential distribution in all space can be determined as

   $V_{net}^- = V^- + V_\sigma^- = r c_1^- \cos\theta = -\dfrac{2\varepsilon_o E_o}{(\varepsilon + \varepsilon_o)} r \cos\theta$ for $r < r_o$, and

   $V_{net}^+ = V^+ + V_\sigma^+ = -\dfrac{\sigma r_o}{\varepsilon_o} \ln\left(\dfrac{r}{r_o}\right) - E_o r \cos\theta - \left(\dfrac{\varepsilon - \varepsilon_o}{\varepsilon + \varepsilon_o}\right) E_o r_o^2 r^{-1} \cos\theta$ for $r > r_o$

Solution #2

The potential energy associated w/ the dipole is

$$U = -\vec{p}\cdot\vec{E} = +eEx \qquad \text{since } \vec{p} = -(-e)\times\hat{x}$$

So if the e^- is moved in $+x$ direction, U increases.

The Hamiltonian

$$H = \frac{-\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}m\omega^2x^2 + eEx$$

$$= \frac{-\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}m\omega^2\left(x + \frac{eE}{m\omega^2}\right)^2 - \frac{e^2E^2}{2m\omega^2}$$

So the oscillation is now about the point

$x = \frac{-eE}{m\omega^2}$ and the energy will be shifted down

by $\Delta E = \frac{-e^2E^2}{2m\omega^2}$

If $\psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega x^2}{2\hbar}}$ w/o E

$\psi_0'(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar}\left(x + \frac{eE}{m\omega^2}\right)^2}$ will be the new w.f.

So the Schrödinger equation looks like ...

$$\left[\frac{-\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}m\omega^2\left(x + \frac{eE}{m\omega^2}\right)^2 - \frac{e^2E^2}{2m\omega^2}\right]\psi_0' = E_0\psi_0'$$

after churning through

$$\left(\frac{\hbar\omega}{2} - \frac{e^2E^2}{2m\omega^2}\right)\psi_0' = E_0'\psi_0'$$

So the new ground state energy is

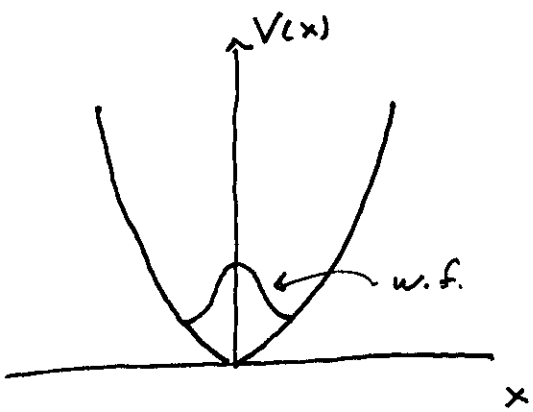
$$E_0' = \frac{\hbar\omega}{2} - \frac{e^2E^2}{2m\omega^2}$$ shifted downward, as we thought.

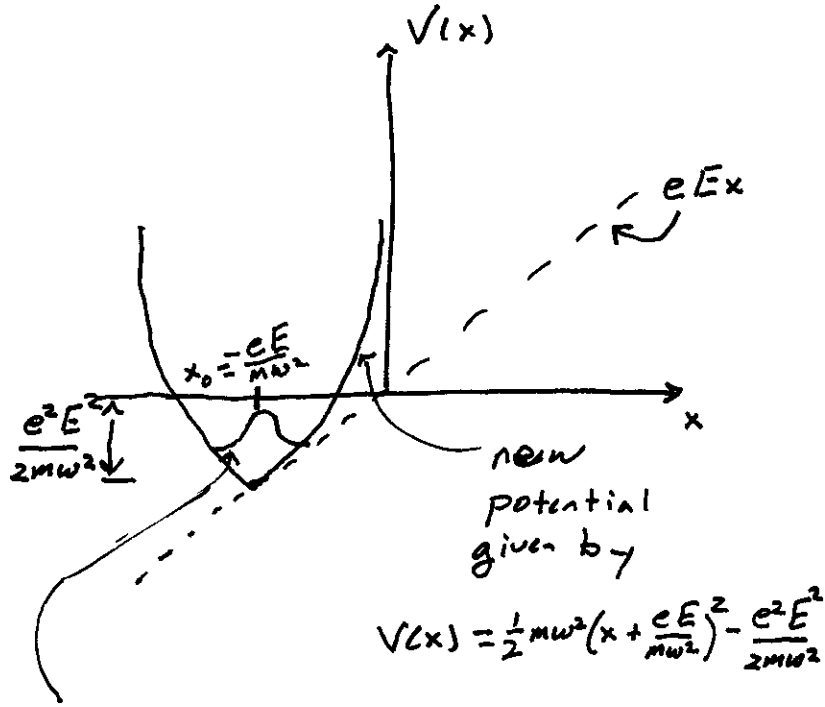
The w.f. can be expressed in terms of $\vec{p}$

$$U = -\vec{p}\cdot\vec{E} = exE \qquad \text{so } \vec{p} = -ex\hat{\imath}$$

$$\psi_0' = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar}\left(x - \frac{\vec{p}\cdot\vec{E}}{m\omega^2 x}\right)}$$

For the drawings





wave function same as to left, but shifted

3.

a. Denote by x the position of the bar, and by θ_1 and θ_2 the angles of the left and right pendular from vertical. The position and velocity of the left bob (bob 1) is

$$\begin{aligned} x_1 = x + L\sin\theta_1 \quad &, \quad y_1 = -L\cos\theta_1 \\ \dot{x}_1 = \dot{x} + L\cos\theta_1\,\dot{\theta}_1 \quad &, \quad \dot{y}_1 = L\sin\theta_1\,\dot{\theta}_1 \end{aligned}$$

Its kinetic energy is therefore

$$T_1 \;=\; \tfrac{1}{2}m(\dot{x}_1^2 + \dot{y}_1^2) \;=\; \tfrac{1}{2}m\dot{x}^2 + mL\cos\theta_1\,\dot{x}\,\dot{\theta}_1 + \tfrac{1}{2}mL^2\dot{\theta}_1^2$$

The kinetic energy of bob 2 is found by changing the subscript. The kinetic energy of the pipe is $T_{\rm p} = 2m\dot{x}^2$. Finally the potential energy is

$$V \;=\; mgy_1 + mgy_2 \;=\; -mgL\cos\theta_1 - mgL\cos\theta_2 \quad . \tag{1}$$

The Lagrangian is therefore

$$\begin{aligned} L \;=\;& 3m\dot{x}^2 + mL\cos\theta_1\,\dot{x}\,\dot{\theta}_1 + mL\cos\theta_2\,\dot{x}\,\dot{\theta}_2 \\ &+ \tfrac{1}{2}mL^2\dot{\theta}_1^2 + \tfrac{1}{2}mL^2\dot{\theta}_2^2 + mgL\cos\theta_1 + mgL\cos\theta_2 \end{aligned} \tag{2}$$

b. Equilibrium is $\dot{x} = 0$ and $\theta_1 = \theta_2 = 0$. Linearizing about this gives kinetic and potential energies

$$T \;=\; 3m\dot{x}^2 + mL\,\dot{x}\,\dot{\theta}_1 + \tfrac{1}{2}mL^2\dot{\theta}_1^2 + mL\,\dot{x}\,\dot{\theta}_2 + \tfrac{1}{2}mL^2\dot{\theta}_2^2 \;=\; \tfrac{1}{2}[\dot{x},\,\dot{\theta}_1,\,\dot{\theta}_2\,]\cdot\underline{\underline{M}}\cdot\begin{bmatrix}\dot{x}\\ \dot{\theta}_1\\ \dot{\theta}_2\end{bmatrix}$$

$$V \;=\; -2mgL + \tfrac{1}{2}mgL\theta_1^2 + \tfrac{1}{2}mgl\theta_2^2 \;=\; -2mgL + \tfrac{1}{2}[x,\,\theta_1,\,\theta_2\,]\cdot\underline{\underline{V}}\cdot\begin{bmatrix}x\\ \theta_1\\ \theta_2\end{bmatrix}$$

with the matrices

$$\underline{\underline{M}} \;=\; \begin{bmatrix} 6m & mL & mL \\ mL & mL^2 & 0 \\ mL & 0 & mL^2 \end{bmatrix} \quad , \quad \underline{\underline{V}} \;=\; \begin{bmatrix} 0 & 0 & 0 \\ 0 & mgL & 0 \\ 0 & 0 & mgL \end{bmatrix} \tag{3}$$

The eigenfrequencies must satisfy the determinental equation

$$\det(\omega^2\underline{\underline{M}} - \underline{\underline{V}}) \;=\; 0 \;=\; m^3\begin{vmatrix} 6\omega^2 & L\omega^2 & L\omega^2 \\ L\omega^2 & L^2\omega^2 - gL & 0 \\ L\omega^2 & 0 & L^2\omega^2 - gL \end{vmatrix} \tag{4}$$

Taking the determinant

$$\begin{aligned} 0 \;&=\; 6L^4\,\omega^2(\omega^2 - g/L)(\omega^2 - g/L) \;-\; 2L^4\omega^4(\omega^2 - g/L) \\ &=\; L^4\,\omega^2(\omega^2 - g/L)\,(4\omega^2 - 6g/L) \end{aligned} \tag{5}$$

The eigenfrequnecies are evidently

$$\omega_1 \;=\; 0 \;\;,\;\; \omega_2 \;=\; \sqrt{\frac{g}{L}} \;\;,\;\; \omega_3 \;=\; \sqrt{\frac{3g}{2L}} \;\;. \tag{6}$$

The first two correspond to translation and anti-symmetric oscillation whose eigenvectors

$$\vec{v}^{(1)} \;=\; \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \;\;,\;\; \vec{v}^{(2)} \;=\; \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \tag{7}$$

can be easily verified. The third must satisfy the equation

$$\begin{bmatrix} 9/L & 3/2 & 3/2 \\ 3/2 & L/2 & 0 \\ 3/2 & 0 & L/2 \end{bmatrix} \cdot \begin{bmatrix} v_1^{(3)} \\ v_2^{(3)} \\ v_3^{(3)} \end{bmatrix} \;=\; 0$$

whose solution is

$$\vec{v}^{(3)} \;=\; \begin{bmatrix} L \\ -3 \\ -3 \end{bmatrix} \tag{8}$$

c. The general solution for a configuration initially in equilibrium is

$$\begin{bmatrix} x(t) \\ \theta_1(t) \\ \theta_2(t) \end{bmatrix} \;=\; A_1\, t\, \vec{v}^{(1)} + A_2 \sin(\omega_2 t)\vec{v}^{(2)} + A_3 \sin(\omega_3 t)\vec{v}^{(3)} \;\;. \tag{9}$$

Where A_1, A_2 and A_3 are constants. The initial velocity is

$$\begin{bmatrix} \dot{x}(0) \\ \dot{\theta}_1(0) \\ \dot{\theta}_2(0) \end{bmatrix} \;=\; \begin{bmatrix} 0 \\ v_0/L \\ 0 \end{bmatrix} \;=\; A_1\, \vec{v}^{(1)} + A_2\omega_2\vec{v}^{(2)} + A_3\omega_3\vec{v}^{(3)}$$

The top and bottom rows gives

$$0 \;=\; A_1 + A_3 L\omega_3 \quad\Longrightarrow\quad A_1 \;=\; -\,A_3 L\omega_3 \;\;. \tag{10}$$

$$0 \;=\; -\,A_2\omega_2 \;-\; 3A_3\omega_3 \quad\Longrightarrow\quad A_2 \;=\; -\,3A_3\frac{\omega_3}{\omega_2} \;\;. \tag{11}$$

The middle row is then

$$\frac{v_0}{L} \;=\; \omega_2 A_2 - 3\omega_3 A_3 \;=\; -\,6\omega_3 A_3 \quad\Longrightarrow\quad A_3 \;=\; -\,\frac{v_0}{6\omega_3 L} \;=\; -\,\frac{v_0}{3\sqrt{6}\,\sqrt{gL}} \tag{12}$$

Substituting this into eq. (10) yeilds

$$A_1 \;=\; L\omega_3\frac{v_0}{3\sqrt{6}\,\sqrt{gL}} \;=\; \frac{v_0}{6} \tag{13}$$

Placing these two expression into the top row of eq. (9)

$$\boxed{x(t) \;=\; \frac{1}{6}\,v_0 t \;-\; \frac{v_0}{3\sqrt{6}}\sqrt{\frac{L}{g}}\,\sin\left(\sqrt{\frac{3g}{2L}}\,t\right)} \tag{14}$$

4) Solution

(a) The wavefunctions for eigenstates are given by the spherical harmonics $Y_{\ell\ell_z}(\theta,\phi)$, times the spinor part. The energies depend only on $\ell = 0, 1, 2, \ldots$ quantum number, from the eigenvalues of these harmonics:

$$\boldsymbol{\nabla}^2_{\theta,\phi} Y_{\ell\ell_z}(\theta,\phi) = -\ell(\ell+1) Y_{\ell\ell_z}(\theta,\phi) \qquad \Rightarrow \qquad \boxed{E_\ell = \frac{\hbar^2 \ell(\ell+1)}{2mR^2}}$$

with degeneracies given by orbital degeneracy, $\ell_z = -\ell, -\ell+1, \ldots, \ell-1, \ell$, times two, for spin projections: $\boxed{2(2\ell+1)}$.

(b) Wave function $\propto \cos\theta$ describes $\ell = 1,\ \ell_z = 0$ orbital state. The energy of this state is

$$\boxed{E_1 = \frac{\hbar^2}{mR^2}}$$

and this level is $\boxed{\text{6-fold}}$ degenerate. Dropping $\ell = 1$ and $s = 1/2$ in each ket $|\,\ell, \ell_z; s, s_z\,\rangle$ we indicate only $|\,\ell_z, s_z\,\rangle$:

$$|\,+1, +\frac{1}{2}\,\rangle \qquad |\,+1, -\frac{1}{2}\,\rangle \qquad |\,0, +\frac{1}{2}\,\rangle \qquad |\,0, -\frac{1}{2}\,\rangle \qquad |\,-1, +\frac{1}{2}\,\rangle \qquad |\,-1, -\frac{1}{2}\,\rangle$$

(c) With spin-orbit interaction, spin and orbital angular momenta no longer conserved separately, but the total angular momentum is. We then use classification of new states in terms of orbital ℓ, spin s, and total angular momentum j, and projection j_z of total angular momentum on the z-axis:

$$|\,\ell, s, j, j_z\,\rangle$$

Introduce $\hat{\mathbf{J}} = \hat{\mathbf{L}} + \hat{\mathbf{S}}$ - operator of total angular momentum; $\hat{\mathbf{J}}^2$ has eigenvalues $\hbar^2 j(j+1)$ where possible values of $\boxed{j = \ell \pm \frac{1}{2}}$. Exercise (not required on exam): check that these are indeed good quantum numbers by showing that $\hat{\mathbf{L}}^2$, $\hat{\mathbf{S}}^2$, $\hat{\mathbf{J}}$ are all commute with the Hamiltonian $\hat{\mathcal{H}}$, but $[\hat{\mathcal{H}}, \hat{\mathbf{L}}] \neq 0$ and $[\hat{\mathcal{H}}, \hat{\mathbf{S}}] \neq 0$ (use general angular momentum commutation rule $[\hat{J}_i, \hat{J}_k] = \epsilon_{ikm}\hat{J}_m$).

We can use it to express the dot-product

$$\hat{\mathbf{L}} \cdot \hat{\mathbf{S}} = \frac{1}{2}\left[\hat{\mathbf{J}}^2 - \hat{\mathbf{L}}^2 - \hat{\mathbf{S}}^2\right]$$

that gives energy shifts to E_1 states

$$\boxed{\Delta E^j_\ell = \frac{A\hbar^2}{2}\left[j(j+1) - \ell(\ell+1) - s(s+1)\right]} \qquad \text{with degeneracies given by } j_z \text{ values} \qquad \boxed{2j+1}$$

which for $\ell = 1$, $s = 1/2$ results in

$$\Delta E^{\frac{3}{2}}_1 = \frac{A\hbar^2}{2}\left[\frac{3}{2}\frac{5}{2} - 2 - \frac{1}{2}\frac{3}{2}\right] = \frac{A\hbar^2}{2} \qquad \text{degeneracy} \quad 4$$

$$\Delta E^{\frac{1}{2}}_1 = \frac{A\hbar^2}{2}\left[\frac{1}{2}\frac{3}{2} - 2 - \frac{1}{2}\frac{3}{2}\right] = -A\hbar^2 \qquad \text{degeneracy} \quad 2$$

(d) To use perturbation theory for degenerate states, one employs the secular equation, $\boxed{det(V_{n'n} - \Delta E \delta_{n'n}) = 0}$ with the matrix elements taken between 6 original degenerate states with $\ell = 1$ $s = \frac{1}{2}$:

$$V_{n'n} = \langle\, \ell'_z, s'_z \,|\hat{V}|\, \ell_z, s_z \,\rangle = A\langle\, \ell'_z, s'_z \,|\hat{\mathbf{L}} \cdot \hat{\mathbf{S}}|\, \ell_z, s_z \,\rangle = A\langle\, \ell'_z, s'_z \,|\frac{\hat{L}_+\hat{S}_- + \hat{L}_-\hat{S}_+}{2} + \hat{L}_z\hat{S}_z|\, \ell_z, s_z \,\rangle$$

The matrix has lots of zeros that are coming from the orthogonal nature of $|\, \ell_z, s_z \,\rangle$ states. For example, $\langle\, \ell'_z, s'_z \,|\hat{L}_+\hat{S}_-|\, \ell_z, s_z \,\rangle$ is non-zero only if $\ell'_z = \ell_z + 1$ AND $s'_z = s_z - 1$. Similar considerations tell us that non-vanishing matrix elements is only possible when conservation $\boxed{\ell'_z + s'_z = \ell_z + s_z}$ occurs, and all other elements are zero by symmetry (bold red in table below), so the interaction matrix $\hat{V}/A\hbar^2$ is obtained by computing several remaining elements and using symmetry:

	$\vert +1,+\frac{1}{2}\rangle$	$\vert +1,-\frac{1}{2}\rangle$	$\vert 0,+\frac{1}{2}\rangle$	$\vert 0,-\frac{1}{2}\rangle$	$\vert -1,+\frac{1}{2}\rangle$	$\vert -1,-\frac{1}{2}\rangle$
$\langle +1,+\frac{1}{2}\vert$	$\frac{1}{2}$	**0**	**0**	**0**	**0**	**0**
$\langle +1,-\frac{1}{2}\vert$	**0**	$-\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	**0**	**0**	**0**
$\langle 0,+\frac{1}{2}\vert$	**0**	$\frac{1}{\sqrt{2}}$	0	**0**	**0**	**0**
$\langle 0,-\frac{1}{2}\vert$	**0**	**0**	**0**	0	$\frac{1}{\sqrt{2}}$	**0**
$\langle -1,+\frac{1}{2}\vert$	**0**	**0**	**0**	$\frac{1}{\sqrt{2}}$	$-\frac{1}{2}$	**0**
$\langle -1,-\frac{1}{2}\vert$	**0**	**0**	**0**	**0**	**0**	$\frac{1}{2}$

and the secular equation is easily obtained from this block-diagonal matrix,

$$\left(\frac{A\hbar^2}{2} - \Delta E\right)^2 \left(-\Delta E\left[-\frac{A\hbar^2}{2} - \Delta E\right] - \frac{A^2\hbar^4}{2}\right)^2 = \left(\Delta E - \frac{A\hbar^2}{2}\right)^4 \left(\Delta E + A\hbar^2\right)^2 = 0$$

that gives eigenvalues and degeneracies for new energy states:

$$\Delta E = \frac{A\hbar^2}{2} \qquad \text{with degeneracy} \quad 4$$

and

$$\Delta E = -A\hbar^2 \qquad \text{with degeneracy} \quad 2$$

Solution to *Problem 5*

a. There are two key assumptions of quantum statistical mechanics: (1) in an isolated system (i.e., the internal energy, U, is constant) all the accessible states, Ω, have the same probability, which is given by $P=\frac{1}{\Omega}$, and (2) in the limit where the number of particles goes to infinity ($N\to\infty$) the time average of the physical quantities over a long period of time is the same as the ensemble average of the same quantities (this is sometimes known as the Ergodic Theorem). In principle the two assumptions are equivalent.

b. Quantities p,V,N cannot be independent variables of a fundamental equation. This is because p and V are conjugate to each other. We can have either p or V at a given time but not both. Conversion from one independent variable to its conjugate is done through Legendre transformation using the Euler relation, $U=TS-pV+\mu N$. For example, while S,V,N are independent variables for internal energy $U(S,V,N)$, quantities T,p,N are the independent variables for the Gibbs free energy: $G(T,p,N)=U-TS+pV$.

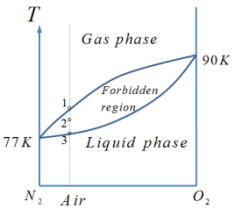
c. ^4He is a boson and obeys the Bose-Einstein statistics. Near absolute zero ($T\approx 1K$) the chemical potential, $\mu=-\frac{1}{\beta N}$, is negative and much smaller than the first excited state energy (i.e. $|\mu|<<\varepsilon_1$). Therefore, if we compare the occupation number of the ground state, N_o, to that of the first excited state, N_1, we notice that $N_o=\frac{1}{e^{\beta(\varepsilon_o-\mu)}-1}\approx-\frac{kT}{\mu}$ and $N_1=\frac{1}{e^{\beta(\varepsilon_1-\mu)}-1}\approx\frac{kT}{\varepsilon_1}$; hence $\frac{N_o}{N_1}=-\frac{\varepsilon_1}{\mu}\approx 10^8$. This simply says that relative to the first excited state the majority of the ^4He are in the ground state. *Some extra considerations:* We must be careful about assessing what fraction of the total particles are in the ground state: Above we only considered the occupation number of one of three first excited states. In order to assess the total number of particles occupying all the excited states we have to integrate the occupation numbers over all the excited states taking into account the density of the states. The total number of particles occupying all the excited states, N_e, is given by $N_e=\int_0^\infty N_\varepsilon D(\varepsilon)d\varepsilon$ where

$N_\varepsilon = \frac{1}{e^{\beta(\varepsilon-\mu)}-1}$ and $D(\varepsilon)$ is the density of the states at ε. The result of this integration, which is readily available in the textbooks, is $N_e = 2.6(mkT/2\pi h^2)^{3/2}V$. Taking into account that $N_o = N(1-N_e/N)$ one can easily show that $N_o = N(1-(T/T_E)^{3/2})$; here $T_E = (2\pi\hbar^2/M)(n/2.6)^{3/2} \approx 3\,K$ for ^4He, where $n = N/V$ is the particle density of the liquid ^4He. This means that at $T = 1\,K$ $N_o \approx 0.81N$, or only $\sim 81\%$ of the total particles are in the ground state.

d. The quantum concentration is $n_Q \approx \frac{1}{\lambda^3} = \left(\frac{p}{h}\right)^3 \sim m^{3/2}$, where $p^2/2m \approx 3kT/2$ is used; therefore, $\frac{n_Q^e}{n_Q^n} = \left(\frac{m_e}{m_n}\right)^{3/2} \approx 1.3\times10^{-5}$. This means neutrons can be packed much more densely than electrons. Neutron stars are super-dense fermions (neutrons). Their density is around nuclear densities. Packing electrons and protons at such densities is energetically very costly because of Coulomb repulsion; hence, the $p^+ + e^- \to n^o + \nu_e$ reaction combines *p*'s and *e*'s into highly packed *n*'s to form the core of a neutron star.

e. Quantities p, T, N are the independent variables of the Gibbs free energy: $G(T,V,\mu) = U - TS + pV$, and $dG = -SdT + VdP + \mu dN$. Therefore, $\frac{\partial^2 G}{\partial T \partial p} = \frac{\partial^2 G}{\partial p \partial T} \to \frac{\partial}{\partial T}\left(\frac{\partial G}{\partial p}\right)_{T,N} = \frac{\partial}{\partial p}\left(\frac{\partial G}{\partial T}\right)_{p,N}$. Since $\left(\frac{\partial G}{\partial p}\right)_{T,N} = V$ and $\left(\frac{\partial G}{\partial T}\right)_{p,N} = -S$, we obtain the answer: $\left(\frac{\partial V}{\partial T}\right)_{p,N} = -\left(\frac{\partial S}{\partial p}\right)_{T,N}$.

f. The entropy of mixing is the key reason for this. The Gibbs free energy of a mixture is determined by the interactions among the constituents of the mixture and the entropy of mixing. The phase of the system assumes the state that minimizes the Gibbs free energy. The phase diagram suggests that air remains in the gas phase until the temperature of the mixture reaches point 1, well below the liquid transition temperature of O_2. In the forbidden region, such as point 2, part of the mixture stays in the gas phase rich in N_2 and the other part becomes liquid rich in O_2. Constituents of these phases are determined by the lever rule. At point 3, at $\sim 78\,K$, all the air becomes



liquid. The entropy of the mixture keeps the air in the gas phase way below the liquid transition temperature of pure O_2.

g. The first law of thermodynamics, $dU = \cancel{dQ} + \cancel{dW}$, asserts that during a free expansion of an isolated system $dU = 0$ because neither is heat given to the system nor work done on the system. For a monotonic ideal gas all the internal energy of the *ideal* gas is due to its kinetic energy, K. Symbolically we can represent this as $U_{ideal} = K$. On the other hand, for a *van der Waals* gas the potential energy $P \approx -\frac{aN^2}{V}$ due to attraction among the N gas atoms (or molecules) contributes to the internal energy. This can be symbolically represented as $U_{vdW} = K + P$. One can imagine that when the gas expands the attraction between the van der Waals gas atoms decreases, which causes an increase in the potential energy, P (i.e. $P_i \approx -\frac{aN^2}{V_i} < P_f = -\frac{aN^2}{V_f}$ because $V_f > V_i$). In order for the total energy, $U = K + P$, to be conserved the kinetic energy of the gas atoms (molecules) must decrease. This can be achieved by the temperature of the van der Waals gas being reduced. That is to say, the attractive forces among the monotonic gas atoms slow the velocities of the freely expanding gas atoms in such a way that $dU = dK + dP = 0$.

6.

(a) *Consider first the case of a single shell of radius $r = b$, which has $\vec{B}_{\rm in} = (4\pi IN/c)\hat{z}$ and $\vec{B}_{\rm out} = 0$. The EMF around a circle of radius r is $\oint \vec{E}\cdot d\vec{r} = 2\pi r E_\theta = -\dot{\Phi}/c$, with $\Phi = \pi B_{\rm in} r^2$ for $r < b$ and $\Phi = \pi B_{\rm in} b^2$ for $r > b$. This gives $\vec{E}_{\rm in} = -(2\pi \dot{I} N r/c^2)\hat{e}_\theta$ and $\vec{E}_{\rm out} = -[2\pi \dot{I} N b^2/(rc^2)]\hat{e}_\theta$.*

The two-shell solution can be obtained by superposition since electromagnetism is a linear theory. In the region $r < a$, we have

$$\vec{B} = 0\,, \tag{1}$$

$$\vec{E} = 0\,. \tag{2}$$

In the region $a \le r \le b$, we have

$$\vec{B} = \left(\frac{4\pi IN}{c}\right)\hat{e}_z = \left(\frac{4\pi I_0 N}{c}\right)\cos\omega t\;\hat{e}_z\,, \tag{3}$$

$$\vec{E} = \left(\frac{2\pi \dot{I} N}{c^2}\right)\left(\frac{a^2}{r} - r\right)\hat{e}_\theta = \left(\frac{2\pi I_0 \omega N}{c^2}\right)\sin\omega t\left(\frac{a^2}{r} - r\right)\hat{e}_\theta\,. \tag{4}$$

In the region $r > b$, we have

$$\vec{B} = 0 \tag{5}$$

$$\vec{E} = -\left(\frac{2\pi \dot{I} N}{c^2}\right)\frac{b^2 - a^2}{r}\hat{e}_\theta = -\left(\frac{2\pi I_0 \omega N}{c^2}\right)\sin\omega t\frac{b^2 - a^2}{r}\hat{e}_\theta\,. \tag{6}$$

(b) The magnetic field energy is $U_{\rm field} = (8\pi)^{-1}\int B^2 dV$, so per unit length of the cylinder we have

$$\frac{U_{\rm field}}{L} = \left(b^2 - a^2\right)\frac{B_{\rm in}^2}{8} = 2\pi^2 I^2 N^2 \frac{b^2 - a^2}{c^2}\,. \tag{7}$$

Writing this as $1/2\mathcal{L}I^2$, with $\mathcal{L}$ the self-inductance per unit length, we find that

$$\mathcal{L} = 4\pi^2 N^2 \frac{b^2 - a^2}{c^2}\,. \tag{8}$$

(c) The Poynting vector is $\vec{S} = [c/(4\pi)]\vec{E}\times\vec{B}$, which is radial and vanishes at the $r = a$ surface. Integrating $\vec{S}$ over the surface just inside $r = b$ gives

$$\int_{\partial V}\vec{S}\cdot\hat{n}\frac{dA}{L} = -4\pi^2 I\dot{I}N^2\frac{b^2 - a^2}{c^2}\,, \tag{9}$$

where $\hat{n} = \hat{e}_r$ is radially outward. This agrees with

$$\frac{dU_{\rm field}}{dt} + \int_{\partial V}\vec{S}\cdot\hat{n}\,dA = 0\,. \tag{10}$$

(d). The Poynting vector is also the linear momentum flux density. Thus, the angular momentum flux density is $\dot{j} = \vec{r}\times\vec{S}$. But $\vec{S}$ is purely radial, so this cross product vanishes. Thus, there is no angular momentum supported by these fields.

7) Solution

(a) Energy of electon in the cavity is determined from Schrödinger equation

$$\boxed{-\frac{\hbar^2}{2m}\boldsymbol{\nabla}^2\psi(\mathbf{r}) = E\psi(\mathbf{r})} \qquad \text{with boundary condition} \qquad \boxed{\psi(r=R)=0},$$

which is solved in spherical coordinates by the separation of variables method using wavefunction of the form

$$\boxed{\psi(r,\theta,\phi) = R_{n\ell}(r)Y_{\ell m}(\theta,\phi)}.$$

Using properties of spherical harmonics $Y_{\ell m}(\theta,\phi)$ we eliminate the angular part of the Laplacian and get equation for the radial part, after defining the wavenumber k through $E = \hbar^2k^2/2m$:

$$\frac{1}{r^2}\frac{d}{dr}\left[r^2\frac{d}{dr}R_{n\ell}(r)\right] + \left(k^2 - \frac{\ell(\ell+1)}{r^2}\right)R_{n\ell}(r) = 0\,.$$

Solutions of this equation are spherical Bessel functions

$$x = kr \qquad \Rightarrow \qquad \frac{d^2}{dx^2}R_{n\ell} + \frac{2}{x}\frac{d}{dx}R_{n\ell} + \left(1 - \frac{\ell(\ell+1)}{x^2}\right)R_{n\ell} = 0$$

and we select only functions, finite in the $r \to 0$ limit:

$$\boxed{R_{n\ell}(r) \propto j_\ell(kr)}.$$

Spectrum of allowed energies is determined from the boundary condition

$$j_\ell(kR) = 0 \qquad \Rightarrow \qquad \boxed{k = \frac{a_{\ell n}}{R}} \qquad \text{where lowest zeros are} \quad a_{01} = \pi = 3.142 \quad a_{11} = 4.493 \quad a_{21} = 5.763$$

for orbital numbers $\ell = 0, 1, 2$ and $n = 1$. The lowest energy state is given by $\ell = 0, n = 1$ zero $a_{01} = \pi$ that corresponds to spherically symmetric wavefunction.

$$\boxed{E_{\ell,n} = \frac{\hbar^2 a_{\ell n}^2}{2mR^2}} \qquad \text{with} \qquad E_{min} = \frac{\hbar^2\pi^2}{2mR^2}$$

(b) The pressure in radial direction is

$$\boxed{p_r = \frac{F_r}{4\pi R^2} = -\frac{1}{4\pi R^2}\frac{\partial E_{min}(R)}{\partial R} = \frac{\hbar^2\pi}{4mR^5}}$$

- positive, as is expected from electron bouncing off the walls.

(c) It is balanced by the surface tension pressure

$$\boxed{\frac{\hbar^2\pi}{4mR^5} = \frac{2\sigma}{R}} \qquad \Rightarrow \qquad R = \left[\frac{\hbar^2\pi}{8m\sigma}\right]^{1/4} = \left[\frac{1.1\ 10^{-68} * \pi}{8 * 9.1\ 10^{-31} * 1.5\ 10^{-4}}\right]^{1/4} \approx \boxed{2.4\ 10^{-9}\, m}$$

about 2.4 nm.

8.

(a) *Recall that the reduced mass is $\mu = m_1 m_2/m$, with $m = m_1 + m_2$ the total mass. Recall also that central circular motion has*

$$F = \frac{G m_1 m_2}{r_0^2} = \mu \omega^2 r_0 \,, \tag{1}$$

where r_0 is here (and henceforth) the orbital separation and where the second equality comes from centripetal acceleration. [The problem stated that r_0 was the radius of the orbit instead of the orbital separation, so credit was given whether the student used r_0 or $2r_0$ in their solutions]. Solving for r_0 we find

$$\omega^2 = \frac{Gm}{r_0^3} \,, \tag{2}$$

and thus

$$r_0^3 = \frac{Gm\tau^2}{4\pi^2} \,. \tag{3}$$

(b) *For the Earth-Sun system, $m_1 + m_2 \sim M_\odot$. By comparison, here r_0 is 4 times bigger and m is 2.5 times bigger, so τ is $(4^3/2.5)^{1/2} = 5.06$ yrs ~ 5 yrs. Alternatively, you can plug in the numbers given to find*

$$\omega^2 = \frac{m}{r_0^3} \sim \frac{2.5 \cdot 1.476 \text{ km}}{(4 \cdot 8 \cdot 60 \text{ sec})^3} \sim 1.7 \times 10^{-15} \text{ sec}^{-2} \,, \tag{4}$$

which then leads to $\tau = 2\pi/\omega \sim 1.6 \times 10^8$ sec, which is about 5 years. We have here used that 1 AU is about 8.3 light minutes.

(c) *This is a special case of central, non-circular motion. You must realize that when the aliens push the bottom, the bodies are still r_0 apart from each other (since they were previously in a circular orbit) and their radial velocities are zero. Using energy conservation, we then have that*

$$\frac{1}{2}\mu \dot{r}^2 - \frac{G m_1 m_2}{r} = E = -\frac{G m_1 m_2}{r_0} \,, \tag{5}$$

where note that the $\dot{\theta}$ term in the kinetic energy is absent since the aliens have stopped the stars from going in a circle. Solving this differential equation, we find

$$t = \frac{r_0^{3/2}}{\sqrt{2G(m_1+m_2)}} \int_0^1 \frac{du}{\sqrt{u^{-1}-1}} = \frac{\tau}{2\pi\sqrt{2}} \frac{\pi}{2} = \frac{\tau}{4\sqrt{2}} = \frac{5}{4\sqrt{2}} \text{ yrs} \sim 0.884 \text{ yrs} \,, \tag{6}$$

where we used that $r = r_0 \cos\theta = r_0 u$.

(d) *The toy removed energy and angular momentum, both of which should be conserved. The only way to make the stars lose so much energy and angular momentum would be to make them emit a giant amount of gravitational waves or of photons.*

(e) *The impulse acts infinitesimally, so the only way to make the stars collide is for the magnitude of the impulse to be so large that it makes the momenta of the stars equal to each other (in magnitude and direction) in the center of mass frame. Any impulse different than this will force the stars into a new orbit (eccentric orbit) that will not lead to a collision. In the center of mass frame, the momentum of, say, star 2 is*

$$p_2 = m_2 v_2 = m_2 \left(\frac{m_1}{m}\right) v = \frac{m_1 m_2}{\sqrt{m r_0}} \,, \tag{7}$$

where in the last equality we used that $v = (m/r_0)^{1/2}$. Since in the center of mass frame $\vec{p}_1 = -\vec{p}_2$, to make $\vec{p}_1 = +\vec{p}_2$ you must impart an impulse of $\Delta\vec{p} = 2p_2$ to star 2, which after plugging in numbers is

$$\Delta\vec{p} = -2\vec{p}_2 = -\frac{3}{\sqrt{2.5}} \frac{M_\odot^{3/2}}{r_0^{1/2}} \hat{p}_2 \sim -1.9 \frac{M_\odot^{3/2}}{r_0^{1/2}} \hat{p}_2 \,. \tag{8}$$

We note that this result is symmetric under particle exchange, i.e. under $m_1 \leftrightarrow m_2$, and thus, it doesn't matter which star you impart this impulse to, i.e. the magnitude will be the same, but the direction will be opposite. Once the stars have equal momentum they just fall to each other in exactly the same way as with the newer Romulan toy. Thus, the amount of time it takes the stars to collide is the same as in part (c).

(f) *The energy of a circular orbit is $E = -Gm_1m_2/(2r_0)$, so if r_0 is a function of time, then*

$$\frac{dE}{dt} = \frac{G\mu m}{r_0^2}\dot{r}_0 = -\frac{32}{5}\frac{G^4}{c^5}\frac{m^3\mu^2}{r_0^5}\,, \tag{9}$$

where the last equality comes from the rate of change of energy due to gravitational wave emission. Solving for r_0, we find

$$\dot{r}_0 = -\frac{64}{5}\frac{G^3}{c^5}\frac{m^2\mu}{r_0^3}\,, \tag{10}$$

$$\int_i dr_0 r_0^3 = -\int_i dt \frac{64}{5}\frac{G^3}{c^5}m^2\mu\,, \tag{11}$$

$$r_0(t) = \left[r_i^4 - \frac{256}{5}\frac{G^3}{c^5}m^2\mu\, t\right]^{1/4}\,. \tag{12}$$

where we have assumed that $r_0(t=0) = r_i$ as our initial condition. We can find the amount of time it takes the bodies to coalesce by setting the left-hand side to zero, which then yields

$$t_{\rm collision} = \frac{5}{256}\frac{c^5}{G^3}\frac{r_i^4}{m^2\mu}\,. \tag{13}$$

Plugging in numbers, we find that $t_{\rm collision} \sim 8 \times 10^{18}$ yrs, which is 500 million times longer than the age of the Universe!

Problem 9 Solution

The partition function for this system is formally given by

$$Z = \sum_{J=0}^{\infty} (2J+1) \exp\left\{-\beta \frac{\hbar^2}{2I} J(J+1)\right\}. \qquad \beta = \frac{1}{kT}$$

Using it, we can find the heat capacity. Z is found by

$$Z = \int_0^{\infty} (2J+1) \exp\left\{-\beta \frac{\hbar^2}{2I} J(J+1)\right\} dJ$$

for a continuum i.e. $kT \gg \frac{\hbar^2}{2I}$

let $u = -\frac{\beta \hbar^2}{2I} J(J+1)$ $\qquad du = -\frac{\beta \hbar^2}{2I}(2J+1)$

so

$$Z = \frac{2I}{\beta\hbar^2} \int_{-\infty}^{0} e^u du = \frac{2I}{\beta\hbar^2}$$

From thermodynamics $\qquad F = -kT \ln Z$ $\qquad$ Helmholtz function

want $\left(\frac{\partial U}{\partial T}\right)_V = C_V$

so

$$dU = TdS - PdV \quad \Rightarrow \quad \left(\frac{\partial U}{\partial T}\right)_V = T\left(\frac{\partial S}{\partial T}\right)_V$$

$$dF = dU - TdS - SdT = -PdV - SdT$$

so $\quad S = -\left(\frac{\partial F}{\partial T}\right)_V$ $\quad$ and now we have

$$C_V = -T\left(\frac{\partial^2 F}{\partial T^2}\right)_V = k \quad \text{after a bit of work.}$$

$C_v = k$ is the solution if $kT >> \frac{\hbar^2}{2I}$

For $kT << \frac{\hbar^2}{2I}$, many terms in Z can be neglected.

Let's consider only the first two

$$Z = 1 + 3e^{-\frac{B\hbar^2}{I}}$$

$$C_v = kB^2 \frac{\partial^2}{\partial B^2} \ln Z = kB^2 \frac{\frac{3\hbar^4}{I^2} e^{-\frac{B\hbar^2}{I}}}{\left(1 + 3e^{-\frac{B\hbar^2}{I}}\right)^2}$$

$$\approx kB^2 \frac{3\hbar^2}{I^2} \left(1 - 6e^{-\frac{B\hbar^2}{I}}\right) e^{-\frac{B\hbar^2}{I}}$$

$$\approx 3kB^2 \left(\frac{\hbar}{I}\right)^2 e^{-\frac{B\hbar^2}{I}}$$

So at high T, $C_v = k$ as we expect from the equipartition theorem.

At low T $C_v \to 0$ since only the lowest quantum states are accessible.

10.

a. Introduce a perturbation $y(x) \to y(x) + \delta y(x)$ into the potential, expand to linear order, and integrate by parts twice

$$\begin{aligned}
\delta V &= \int_0^L \left[\frac{\alpha}{12} \frac{d^2y}{dx^2} \frac{d^2(\delta y)}{dx^2} + g\lambda(x)\,\delta y(x) \right] dx \\
&= \int_0^L \left[-\frac{\alpha}{12} \frac{d^3y}{dx^3} \frac{d(\delta y)}{dx} + g\lambda(x)\,\delta y(x) \right] dx \\
&= \int_0^L \left[\frac{\alpha}{12} \frac{d^4y}{dx^4} + g\lambda(x) \right] \delta y(x)\,dx \quad , \qquad (2)
\end{aligned}$$

after using the boundary conditions (1) to eliminate all surface terms. To be a minimum $\delta V = 0$ for every possible perturbation, $\delta y(x)$. This means the factor of the final expression in square brackets must vanish

$$\boxed{\frac{d^4y}{dx^4} = -\frac{12g}{\alpha}\,\lambda(x) \quad . \qquad (3)}$$

b. For a uniform mass density $\lambda(x) = \lambda_0$ the general solution to eq. (3) is

$$y(x) = -\frac{g\lambda_0}{2\alpha}\,x^4 + Ax^3 + Bx^2 + Cx + D \quad , \qquad (4)$$

for unknown constants A, B, C, and D. Applying the first two boundary conditions

$$y(0) = D = 0 \quad , \quad y'(0) = C = 0 \quad ,$$

leaving only A and B. The next two yield the conditions

$$y(L) = -\frac{g\lambda_0}{2\alpha}\,L^4 + AL^3 + BL^2 = 0 \qquad (5)$$

$$y'(L) = -\frac{2g\lambda_0}{\alpha}\,L^3 + 3AL^2 + 2BL = 0 \qquad (6)$$

This yields

$$B = \frac{g\lambda_0}{2\alpha}\,L^2 - A\,L \quad , \qquad (7)$$

and

$$-\frac{2g\lambda_0}{\alpha}\,L^3 + 3AL^2 + \underbrace{\frac{g\lambda_0}{\alpha}\,L^3 - 2AL^2}_{2BL} = -\frac{g\lambda_0}{\alpha}\,L^3 + AL^2 = 0 \quad . \qquad (8)$$

For which the solution is

$$A = \frac{g\lambda_0}{\alpha}\,L \quad , \quad B = -\frac{g\lambda_0}{2\alpha}\,L^2 \quad . \qquad (9)$$

Placing all those in the general solution (4) yields

$$\boxed{y(x) = -\frac{g\lambda_0}{2\alpha}\left(x^4 - 2Lx^2 + L^2x^2\right) = -\frac{g\lambda_0}{2\alpha}\,x^2(x-L)^2 \quad . \qquad (10)}$$

c. We can superpose the solution to b. with a solution to

$$\frac{d^4y}{dx^4} \;=\; -\frac{12mg}{\alpha}\,\delta(x-L/2) \quad . \tag{11}$$

The general solution to this is

$$y(x) \;=\; \begin{cases} Ax^3+Bx^2 & , \quad x<L/2 \\ A'(x-L)^3+B'(x-L)^2 & , \quad x>L/2 \end{cases} \tag{12}$$

where A, B, A', and B' are unknown constants and we have taken inspiration from part b. to satisfy all boundary conditions. The remaining conditions involve the behavior of the solution across $x=L/2$. The first three conditions are that $y(x)$, $y'(x)$ and $y''(x)$ must all be continuous

$$A(L/2)^3+B(L/2)^2 \;=\; -A'(L/2)^3+B'(L/2)^2 \tag{13}$$

$$3A(L/2)^2+2B(L/2) \;=\; 3A'(L/2)^2-2B'(L/2) \tag{14}$$

$$6A(L/2)+2B \;=\; -6A'(L/2)+2B' \quad . \tag{15}$$

The final expression yields

$$B \;=\; -3(A'+A)(L/2)+B' \tag{16}$$

whose substitution into (13) gives

$$(A+A')(L/2)^3-3(A+A')(L/2)^3+B'(L/2)^2 \;=\; B'(L/2)^2 \quad .$$

whose only solution is $A'=-A$. Placing this into eq. (16) yields $B=B'$. Using these two in eq. (14) gives

$$6A(L/2)^2 \;=\; -4B(L/2) \quad\Longrightarrow\quad B \;=\; -(3/4)\,AL \quad .$$

Finally, integrating (11) across $x=L/2$ yields

$$\left.\frac{d^3y}{dx^3}\right|_{x=L/2+0} - \left.\frac{d^3y}{dx^3}\right|_{x=L/2-0} \;=\; 6(A'-A) \;=\; -\frac{12mg}{\alpha} \quad . \tag{17}$$

This yields the solutions

$$A \;=\; \frac{mg}{\alpha} \quad , \quad A' \;=\; -\frac{mg}{\alpha} \quad , \quad B \;=\; B' \;=\; -\frac{3mg}{4\alpha} \quad . \tag{18}$$

Using these in eq. (12) and adding the solution from b. yields the complete solution

$$y(x) \;=\; \begin{cases} -\dfrac{g\lambda_0}{2\alpha}\,x^2(x-L)^2 \;+\; \dfrac{mg}{4\alpha}x^2(4x-3L) & , \quad x<L/2 \\ -\dfrac{g\lambda_0}{2\alpha}\,x^2(x-L)^2 \;+\; \dfrac{mg}{4\alpha}(x-L)^2(L-4x) & , \quad x>L/2 \end{cases} \tag{19}$$