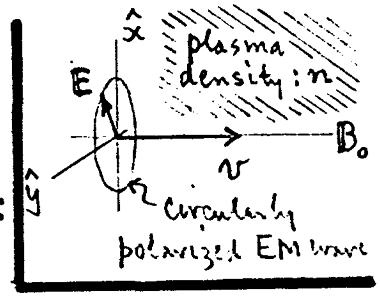
ASIDE Ionospheric Propagation.

Propagation of EM Waves in Earth's Ionosphere [Jkn Sec. 7.6].



1) As an interesting example of how a plasma affects propagation of an EM wave, we study Jkn Sec. 7.6. By choice, the wave is circularly polarized in the xy-plane and at freq. ω:

$\rightarrow \mathbb{E}(t) = \mathbb{E}\, e^{-i\omega t}$, w/ $\mathbb{E} = (\hat{\epsilon}_x \pm i\,\hat{\epsilon}_y)E$ & $E = \text{cnst.}$ (A1)

(upper sign: left circ. pol$^{\underline{n}}$; lower sign: right circ. pol$^{\underline{n}}$)

This wave moves in a plasma of $\underline{n}$ electrons/cm^3, along a magnetic field line $\underline{\mathbb{B}_0}$ lying along the z-axis. The wave will polarize the medium and thus be affected by the electron motion -- which proceeds according to...

$\rightarrow$ $$m\ddot{\mathbf{r}} = -e\left[\mathbb{E}(t) + \frac{1}{c}\dot{\mathbf{r}} \times \mathbb{B}_0\right],$$ for a single (free) electron $[-e, m]$, (A2)

to lowest order approx$^{\underline{n}}$. We want to solve Eq. (A2) to find the dynamic response of the medium -- i.e. the dielectric cnst $\epsilon(\omega)$. Then $\epsilon(\omega) \Rightarrow$ wave features.

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2) Rewrite Eq. (A2), putting in $\mathbb{E}(t)$ from (A1) and also $\mathbb{B}_0 = B_0\hat{\epsilon}_z$...

$$\ddot{\mathbf{r}} - \omega_B\, \hat{\epsilon}_z \times \dot{\mathbf{r}} = -\frac{e}{m}\mathbb{E}\, e^{-i\omega t}, \quad \underline{\omega_B} = \frac{eB_0}{mc} = \text{cyclotron frequency.} \quad \text{(A3)}$$

For a steady-state solution, $\mathbf{r}(t)$ will synchronize with $\mathbb{E}(t)$. So we try...

$$\left[\begin{array}{l} \mathbf{r}(t) = (x\,\hat{\epsilon}_x + iy\,\hat{\epsilon}_y)e^{-i\omega t}, \text{ w/ amplitudes } x \,\&\, y \text{ independent of time;} \\ \Rightarrow \dot{\mathbf{r}} = -i\omega\mathbf{r},\ \ddot{\mathbf{r}} = -\omega^2\mathbf{r};\ \ \hat{\epsilon}_z \times \dot{\mathbf{r}} = -i\omega(x\,\hat{\epsilon}_y - iy\,\hat{\epsilon}_x)e^{-i\omega t} \end{array}\right\} \text{(A4)}$$

Put all this into Eq. (A3), and put $\mathbb{E} = (\hat{\epsilon}_x \pm i\hat{\epsilon}_y)E$. Rearrange terms to get:

$$\omega[x\hat{\epsilon}_x + iy\,\hat{\epsilon}_y] - i\omega_B[x\,\hat{\epsilon}_y - iy\,\hat{\epsilon}_x] = \frac{eE}{m\omega}[\hat{\epsilon}_x \pm i\,\hat{\epsilon}_y]$$

$$\begin{pmatrix} \omega & -\omega_B \\ -\omega_B & \omega \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} u \\ \pm u \end{pmatrix}, \quad \text{where: } \underline{u} = \frac{eE}{m\omega}$$ ✓ u has units of velocity.

$\Rightarrow$ Solution: $x = u/(\omega \mp \omega_B)$, $y = \pm u/(\omega \mp \omega_B)$. (A5)

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3) The steady-state motion of the electrons, as induced by the $\mathbf{E}$-wave, is:

$$\mathbf{r} = (x\hat{\epsilon}_x + iy\hat{\epsilon}_y)e^{-i\omega t} = \frac{eE/m\omega}{\omega \mp \omega_B}(\hat{\epsilon}_x \pm i\hat{\epsilon}_y)e^{-i\omega t}$$

or $\boxed{\mathbf{r}(t) = \left(\frac{e/m\omega}{\omega \mp \omega_B}\right)\mathbf{E}(t)}$; ω = EM wave freq., $\underline{\omega_B} = eB_0/mc$ = cyclotron frequency; and $\mp$ in denom. $\leftrightarrow$ $\pm$ helicity for $\mathbf{E}$. (A6)

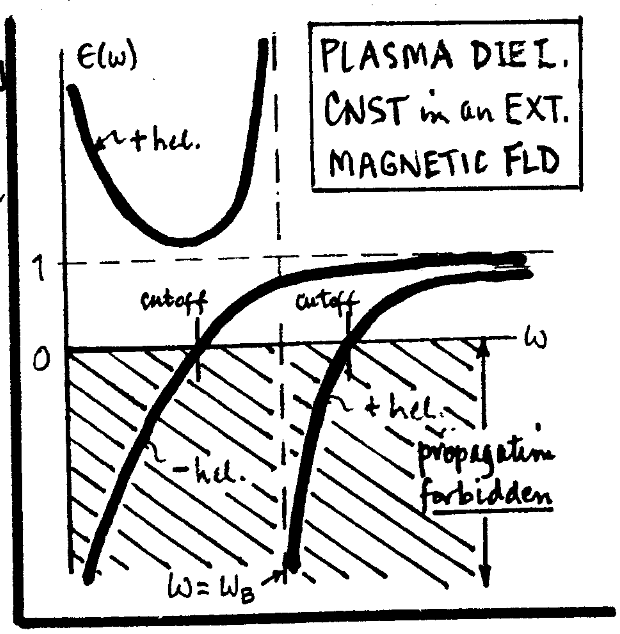
Then, if the plasma has n (free) e's/cm^3, $\mathbf{E}(t)$ polarizes the plasma, as:

$$\rightarrow \mathbf{P}(t) = -ne\,\mathbf{r}(t) = \underbrace{-\left(\frac{ne^2/m\omega}{\omega \mp \omega_B}\right)}\mathbf{E}(t) \qquad (A7)$$

$\equiv$ polarizability: $\alpha = -\frac{1}{4\pi}\omega_p^2/\omega(\omega \mp \omega_B)$, w/ $\underline{\omega_p} = \sqrt{\frac{4\pi ne^2}{m}}$ = plasma frequency

From this, we find the dielectric cnst for the plasma:

$$\boxed{\epsilon(\omega) = 1 + 4\pi\alpha = 1 - \frac{\omega_p^2}{\omega(\omega \mp \omega_B)}} \quad \mp \leftrightarrow \pm \text{ helicity} \qquad (A8)$$



$\epsilon(\omega)$ vs. ω is sketched at right. Evidently, the $\pm$ helicity waves propagate in very different ways. For the size of things, note...

PLASMA FREQ. } $\omega_p = 0.056\sqrt{n}$, MHz ($n = \frac{\#e\text{'s}}{\text{cm}^3}$). (A9)

earth's ionosphere: $n = 10^{4-6}/\text{cm}^3 \Rightarrow \omega_p = 6\text{-}60$ MHz.

CYCLOTRON FREQ. } $\omega_B = 17.6\,B_0$, MHz (B_0 in Gs). (A10)

earth's magnetosphere: $B_0 \sim 0.1-0.5$ Gs $\Rightarrow \omega_B \sim 1.8-8.8$ MHz.

The "interesting" structure in $\epsilon(\omega)$ is at low freqs., i.e. @ $0 < \omega \sim \omega_B < 10$ MHz.

When $\epsilon(\omega) < 0$, there is no propagation (as noted) since the wave # k is imaginary:

$$\rightarrow k = \omega/v_{phase} = \frac{\omega}{c}\sqrt{\epsilon(\omega)} \;;\; \text{need } \epsilon(\omega) > 0 \text{ for propagation}. \qquad (A11)$$

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The condition $\epsilon(\omega)>0$ limits the frequency ranges in which the $\pm$ helicity waves can propagate in this magnetized plasma. Per above $\epsilon(\omega)$ sketch:

+ HELICITY: no propagation in range...

$$\omega_B \leqslant \omega \leqslant \frac{1}{2}\left[\omega_B+\sqrt{\omega_B^2+4\omega_p^2}\,\right] \xrightarrow{\sim\text{cutoff}} \begin{cases}\left[1+(\omega_p^2/\omega_B^2)\right]\omega_B, & \text{for } \omega_p \ll \omega_B,\\ \left[1+\frac{1}{2}(\omega_B/\omega_p)\right]\omega_p, & \text{for } \omega_p \gg \omega_B;\end{cases} \quad \text{(A12)}$$

− HELICITY: no propagation in range...

$$0 \leqslant \omega \leqslant \frac{1}{2}\left[-\omega_B+\sqrt{\omega_B^2+4\omega_p^2}\,\right] \xrightarrow{\sim\text{cutoff}} \begin{array}{ll}(\omega_p/\omega_B)\,\omega_p, & \text{when: } \omega_p \ll \omega_B,\\ \left[1-\frac{1}{2}(\omega_B/\omega_p)\right]\omega_p, & \text{for } \omega_p \gg \omega_B.\end{array} \quad \text{(A13)}$$

REMARKS on propagation in a magnetized plasma.

1. The EM wave is reflected when $\epsilon(\omega)=0$. From (A8), this happens when:

$$\left[\begin{array}{l}\epsilon(\omega)=0 \Rightarrow \omega^2 \mp \omega_B\omega - \omega_p^2 = 0,\\ \text{or } \underline{\omega = \frac{1}{2}\left[\pm\omega_B + \sqrt{\omega_B^2+4\omega_p^2}\,\right]}\end{array}\right. \quad \text{(A14)}$$

here $\pm \leftrightarrow$ cutoffs noted in (A12) & (A13); ($\pm$ here refers to $\pm$ helicity).

The fact that: $\omega^2=\omega_p^2 \pm \omega_B\omega$, and $\omega_p^2 = 4\pi n e^2/m \Rightarrow$ that a +ve helicity wave reflects from a lower density ionospheric layer (smaller n) than does a (−) helicity wave.

2. Analyse group (transport) velocity at "high freqs" $\omega \gg \omega_B$. From (A8)...

$$\rightarrow kc = \omega\sqrt{\epsilon(\omega)} = \omega\left[1-\frac{\omega_p^2}{\omega(\omega\mp\omega_B)}\right]^{1/2} \simeq \omega\left[1-\frac{\omega_p^2}{\omega^2}\left(1\pm\frac{\omega_B}{\omega}\right)\right]^{\frac{1}{2}}, \text{ for } \omega \gg \omega_B;$$

$$\text{So } k^2c^2 \simeq \omega^2-\omega_p^2\left(1\pm\frac{\omega_B}{\omega}\right), \text{ or } \underline{\underline{\omega^2 \simeq k^2c^2+\omega_p^2\left(1\pm\frac{\omega_B}{\omega}\right)}}. \quad \text{(A15)}$$

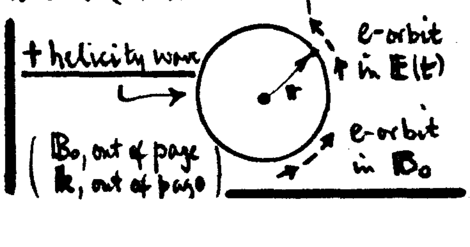
Do $\partial/\partial k$ on Eq. (A15) and solve for group velocity $v_g = \partial\omega/\partial k$. Then...

$$2\omega v_g \simeq 2kc^2 \pm \omega_p^2\omega_B \frac{\partial}{\partial k}(1/\omega) = \text{etc}... \quad \text{(A16)}$$

$$... \Rightarrow \underline{\underline{v_g^2/c^2 \simeq 1-(\omega_p^2/\omega^2) \mp 2\epsilon\left(\frac{\omega_p^2}{\omega^2}\right)\left[1-\frac{\omega_p^2}{2\omega^2}\right]}}, \text{ to 1st order in } \epsilon = \frac{\omega_B}{\omega}.$$

CONCLUSION: The (+) helicity waves travel slower than the (−) helicity waves.

3. The marked distinctions between $\pm$ helicities waves relate to the cyclotron resonance possible for (+) helicities.



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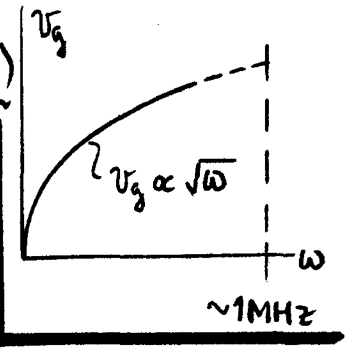
4) At low frequencies only the (+) helicity wave can propagate [sketch, p A2], and:

$\underline{\omega_B \gg \omega \to 0}$

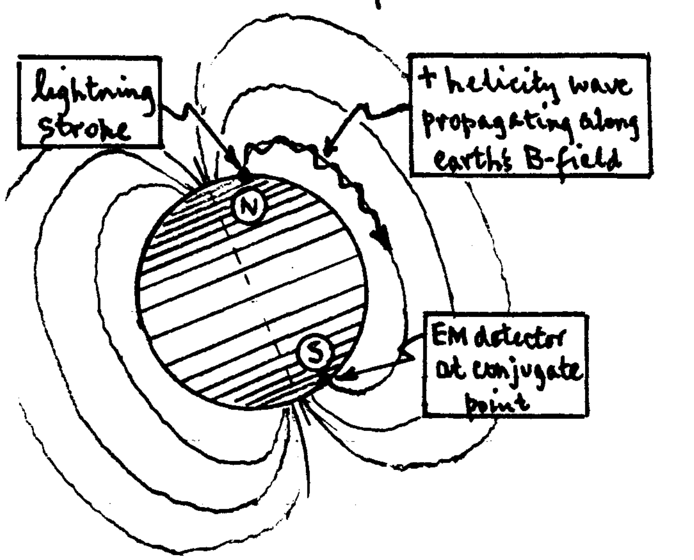
so $$kc = \omega\left[1-\frac{\omega_p^2}{\omega(\omega-\omega_B)}\right]^{1/2} \simeq \omega_p\sqrt{\frac{\omega}{\omega_B}}\left[1+\frac{1}{2}\frac{\omega}{\omega_B}\left(1+\frac{\omega_B^2}{\omega_p^2}\right)+\mathcal{O}\left(\frac{\omega}{\omega_B}\right)^2+\dots\right]$$

$\Rightarrow$ GROUP VELOCITY : $$\boxed{v_g = \frac{\partial\omega}{\partial k} = c\sqrt{(4\omega_B/\omega_p^2)\,\omega}} \qquad \text{(A17)}$$



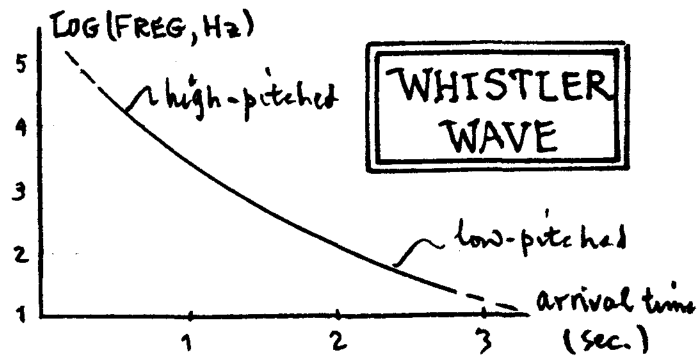
This transport velocity is highly dispersive... an EM pulse with an initially broad spectrum of frequencies (say $0 \lesssim \omega \lesssim 2\pi \times 1$ MHz, or so) will become very much spread out in time & space during propagation. And, from (A17), the low frequency components of the pulse markedly trail the high frequency components.

This behavior explains the mildly amusing occurrence of "Whistler waves"...



Wave propagates along B-field line (more or less) because of coupling to electron cyclotron resonance.

Arrival of signal at point Ⓢ looks like:



The wave is called a "whistler" because it sounds like a long, drawn-out whistle, sliding monotonically downward in frequency, from $\sim 10^5$ Hz to ~ 10 Hz.

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