DEPARTMENT OF PHYSICS

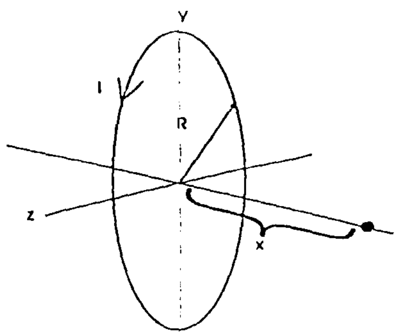
1995 COMPREHENSIVE EXAM

June 19, 20, 21, 1995

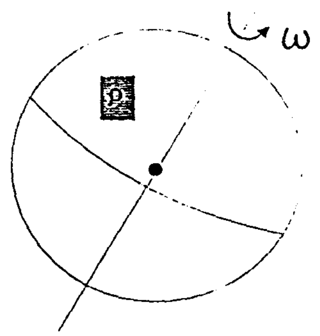
Answer each of the following questions, all of which carry equal weight. Begin your answer to to each question on a new sheet of paper; solutions to different questions must **not** appear on the same sheet. Each sheet of paper must be labeled with your name and the problem number in the upper right hand corner. When more than one sheet is submitted for a problem, be sure the pages are ordered properly.

1. Magnetic fields in the sun.

A. A circular conducting loop of radius R lies in the yz-plane with its center at the origin. If the loop carries current I, find the magnetic field generated at distance x along the axis of the loop.



B. A sphere of radius R has a uniform volume charge density ρ. Find the magnetic field at the center of the sphere when it rotates as a rigid body with angular velocity ω about an axis through its center.

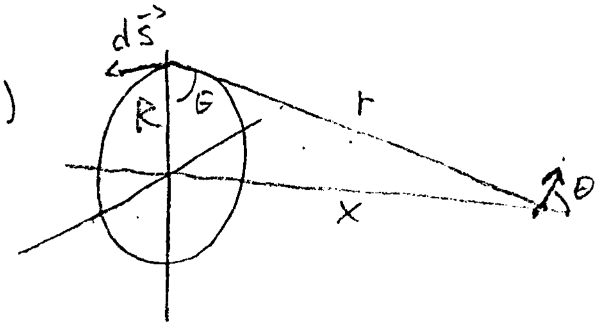


C. The Zeeman effect seen in the spectra of sunspots reveals the existence of fields as large as 0.4 tesla. Suppose that we model the source of this field in the sun as a spinning, flat disk of electrons, with the following properties: the disk radius is $R = 10^7$ m, its thickness is $\Delta x \ll R$, it rotates at 3×10^{-2} rad/sec, it is uniformly charged, and its center coincides with the center of the sun. Find the surface charge density $\sigma = \rho\Delta x$ on this disk required to produce a magnetic field of 0.4 tesla at its center.

Comp 95 Solution: Current Loop (Smith) #1

Prob #1

(a)



$$d\vec{B} = \frac{\mu_0 I}{4\pi} \frac{d\vec{s} \times \hat{r}}{r^2}$$

component along x is

$|d\vec{B}| \cos\theta$ $ds \perp \hat{r}$

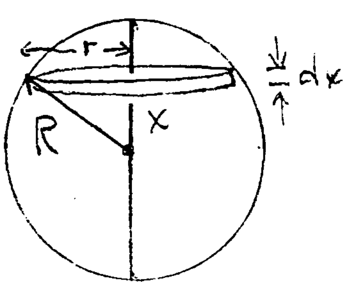
$$|d\vec{B}| = \frac{\mu_0 I}{4\pi} \frac{ds}{r^2} \cos\theta$$

$$B = \frac{\mu_0 I}{4\pi} \oint \frac{ds}{(x^2+R^2)} \frac{R}{\sqrt{x^2+R^2}}$$

$$B = \frac{\mu_0 I}{4\pi} \frac{R}{(x^2+R^2)^{3/2}} 2\pi R$$

$$\boxed{B_x = \frac{\mu_0 I R^2}{2(x^2+R^2)^{3/2}}}$$

(b) Consider the sphere as made up of rings of radius r, thickness dx, centered on the rotation axis



$$dB = \frac{\mu_0 I r^2}{2(x^2+r^2)^{3/2}}$$

where

$$I = \rho \frac{2\pi r\,dr\,dx}{T} = \rho\, 2\pi r\,dr\,dx \frac{\omega}{2\pi}$$

So

$$dB = \frac{\mu_0}{2} \frac{r^2}{(x^2+r^2)^{3/2}} \rho\, 2\pi r\,dr\,dx \frac{\omega}{2\pi}$$

$$= \frac{\mu_0}{2} \frac{\rho\omega r^3\,dr\,dx}{(x^2+r^2)^{3/2}}$$

$$B = \frac{\mu_0 \rho \omega}{2} \int_{-R}^{R} dx \int_0^{\sqrt{R^2-x^2}} \frac{r^3 dr}{(r^2+x^2)^{3/2}}$$

Let $v = r^2+x^2 \quad dv = 2r\,dr \quad r^2 = v - x^2$

$$B = \frac{\mu_0 \rho \omega}{4} \int_{-R}^{R} dx \int_{x^2}^{R^2} \frac{(v-x^2)\,dv}{v^{3/2}}$$

$$= \frac{\mu_0 \rho \omega}{\cancel{4}} \int_{-R}^{R} dx \left\{ 2v^{1/2} + \frac{x^2 2}{v^{1/2}} \right\}_{x^2}^{R^2}$$

$$= \frac{\mu_0 \rho \omega}{2} \int_{-R}^{R} dx \left(R + \frac{x^2}{R} - x + x \right)$$

$$= \frac{\mu_0 \rho \omega}{2} \left[Rx + \frac{x^3}{3R} \right]_{-R}^{R}$$

$$= \frac{\mu_0 \rho \omega}{2} \left[R^2 + \frac{1}{3}R^2 + R^2 + \frac{1}{3}R^2 \right]$$

$\boxed{B = 4/3\, \rho \mu_0 \omega R^2}$ at center

(c) From intermediate result above,

$B = \frac{\mu_0 \rho \omega}{2} dx\, R \qquad$ Let $\rho = ne$

Then

$$n\,dx = \frac{2B}{\mu_0 e \omega R} = \frac{2(0.4\,T)}{\left(4\pi \times 10^{-7} \frac{W}{A \cdot m}\right)\left(1.6 \times 10^{-19}\, C\right)\left(3 \times 10^{-2}\, \frac{1}{s}\right)(10$$

$= 1.3 \times 10^{19} / m^2$ about the same as atom density ...

2. The equation of motion of a magneto-elastic beam, subject to damping and driven by a sinusoidal force, is given by

$$\ddot{x} + \gamma\dot{x} - x + x^3 = \alpha \cos\omega t.$$

Here α, γ, ω are positive real constants, and x measures the displacement of the beam.

A. Write appropriate differential equations in the standard form for a dynamical system for $\alpha = 0$.

B. Determine the fixed points when $\alpha = 0$.

C. Classify the fixed points for $\alpha = 0$, and determine their stability in terms of the system parameters.

D. Repeat part A when $\alpha > 0$.

E. Discuss whether chaos is possible when $\alpha = 0$, and when $\alpha > 0$.

F. Find the time dependence of the volume of a ball of initial points centered at the origin of state space.

CM#1 : Hermanson

The equation of motion of a magneto-elastic beam subject to damping and a sinusoidal driving force is given by

$$\ddot{x} + \gamma \dot{x} - x + x^3 = \alpha \cos \omega t$$

where α, γ, and ω are positive real constants and x measures the beam's displacement.

(a) Write appropriate differential equations in the standard form for a dynamical system

(b) Determine the fixed points when $\alpha = 0$

(c) Classify the fixed points for $\alpha = 0$ and determine their stability in terms of the system parameters

(d) Repeat (a) when $\alpha > 0$

(e) Discuss whether chaos in possible when $\alpha = 0$ and when $\alpha > 0$

(f) Determine the time dependence of the volume of a ball of initial points centered at the origin of state space.

CM#1 Solution J. Hermanson

(a) Given $\alpha = 0$, write 1st order d.e.'s

$$\begin{cases} \dot{v} = x - x^3 - \gamma v & = f(v,x) \\ \dot{x} = v & = g(v,x) \end{cases}$$

(b) Fixed points require $f = 0 = g$

$\Rightarrow \quad (v,x) = (0,0)\,;\ (0,1)\,;\ (0,-1)$

(c) Stability analysis uses the eigenvalues of the Jacobian matrix

$$J = \begin{pmatrix} \frac{\partial f}{\partial v} & \frac{\partial f}{\partial x} \\ \frac{\partial g}{\partial v} & \frac{\partial g}{\partial x} \end{pmatrix} = \begin{pmatrix} -\gamma & 1-3x^2 \\ 1 & 0 \end{pmatrix}$$

(i) at $(0,0)$ $\quad J = \begin{pmatrix} -\gamma & 1 \\ 1 & 0 \end{pmatrix} \Rightarrow \lambda_{1,2} = -\frac{\gamma}{2} \pm \frac{\sqrt{\gamma^2+4}}{2}$

Thus $\lambda_1 > 0 \quad \lambda_2 < 0 \Rightarrow$ <u>saddle node</u>

(ii) at $(0,\pm 1)$ $\quad J = \begin{pmatrix} -\gamma & -2 \\ 1 & 0 \end{pmatrix} \Rightarrow \lambda_{1,2} = -\frac{\gamma}{2} \pm \frac{\sqrt{\gamma^2-8}}{2}$

– For $\gamma > 2\sqrt{2}$ $\quad \lambda_{1,2} = \text{real} < 0 \Rightarrow$ <u>attracting node</u>

– For $\gamma < 2\sqrt{2}$ $\quad \lambda_1 = \lambda_2^* =$ complex but $\mathrm{Re}\,\lambda_{1,2} < 0$

$\Rightarrow$ <u>Spiral node</u>

(d) when $\alpha > 0$, write 3 d.e.'s

$$\begin{cases} \dot{v} = x - x^3 - \gamma v + \alpha \cos\theta & = f(v x \theta) \\ \dot{x} = v & = g(v x \theta) \\ \dot{\theta} = \omega & = h(v x \theta) \end{cases}$$

(e) chaos cannot exist in 2D ($\alpha = 0$) but may exist in 3D ($\alpha > 0$)

(f) the volume of the ball satisfies

$$\frac{1}{V}\frac{dV}{dt} = \underline{\nabla} \cdot \underline{F} = \frac{\partial f}{\partial v} + \frac{\partial g}{\partial x} + \frac{\partial h}{\partial \theta} = -\gamma$$

independent of x

$$\frac{dV}{dt} = -\gamma V$$

$$\boxed{V = V_0 e^{-\gamma t}}$$

3. If, in the hydrogen atom, both electron and proton are point particles, then the Coulomb potential between them: $V(r) = -e^2/r$, extends all the way down to radial separation $r = 0$, where $V(r)$ is singular. You may assume the electron is point-like, but experiments show that the proton is an extended structure, with a change radius $R = 10^{-13}$ cm. Any such $R>0$ removes the Coulomb singularity at $r = 0$, and thereby shifts the hydrogen atom energy levels.

A. Assume the proton's charge is distributed uniformly over a thin spherical shell of radius R. How does this perturb the otherwise "pure" Coulomb potential $V(r)$ as $r \to 0$?

B. Using first-order perturbation theory, find the energy shift due to ΔV (of part A) in the hydrogen atom ground state. It helps to note that the scale length for the ground state wavefunction is $\gg R$.

C. What numerical fraction of the ground state binding energy is the shift you have calculated in part B? Is the shift positive or negative? Comment on why this makes sense.

φ Comp. Exam : June '95 (Robiscoe) 5/28/95

③ ✓ #3

QM -- A Perturbation Calculation

"If, in the hydrogen atom, both electron and proton are point particles, then the Coulomb potential between them : $V(r) = -e^2/r$, extends all the way down to radial separation $r=0$, where $V(r)$ is singular. You can assume the electron is point-like, but experiments show that the proton is an extended structure, with a charge radius $R = 10^{-13}$ cm. Any such $R > 0$ removes the Coulomb singularity at $r=0$, and thereby shifts the hydrogen atom energy levels.

A. Assume the proton's charge is distributed over a spherical shell of radius R. What perturbation ΔV does this cause on the otherwise "pure" Coulomb potential $V(r) = -e^2/r$?

B. Using first order perturbation theory, find the energy shift caused by ΔV (of part A) on the hydrogen atom ground state. HINT: the ground state wavefunction is : $\psi(r) \propto \exp(-r/a_0)$, with $a_0 \gg R$.

C. What fraction of the ground state binding energy is the shift you have calculated in part B ? Is the shift positive or negative ? Comment on why this makes sense."

SOLUTION

A. Inside a spherical shell of radius R, the electron-proton potential is constant, at a value of $-e^2/R$. The Coulomb potential thus changes as :

$$\rightarrow V(r) = (-)\frac{e^2}{r} \rightarrow (-)\frac{e^2}{R} = (-)\frac{e^2}{r} + \left(\frac{e^2}{r} - \frac{e^2}{R}\right), \quad 0 \leq r \leq R$$

i.e.// $V(r) \rightarrow V(r) + \Delta V$, w/ $\underline{\Delta V = (e^2/r) - (e^2/R)}$. (1)

(OVER)

B. By first-order perturbation theory, the energy shift caused by ΔV of part A is, in the ground state $\psi(r)$...

$$\rightarrow \Delta E = \int_\infty \psi^*(r)[\Delta V]\psi(r)\,dV = \int_0^R |\psi(r)|^2 \left[\frac{e^2}{r} - \frac{e^2}{R}\right] \cdot 4\pi r^2 dr$$

or// $$\Delta E = 4\pi e^2 \int_0^R |\psi(r)|^2 \left[r - \frac{r^2}{R}\right] dr. \qquad (2)$$

This assumes $\psi(r)$ is normalized. Next, since $\psi(r) \propto \exp(-r/a_0)$, with $a_0 \simeq 0.53 \times 10^{-8}$ cm., it is clear that $\psi(r)$ deviates by a negligibly small amount from $\psi(0)$ over the range $0 \leq r \leq R \simeq 10^{-13}$ cm. So, in Eq. (2), we evaluate $\psi(r) \simeq \psi(0)$, and take it outside the integral. Then...

$$\rightarrow \Delta E = 4\pi e^2 |\psi(0)|^2 \int_0^R \left[r - \frac{r^2}{R}\right] dr = \frac{2}{3}\pi e^2 |\psi(0)|^2 R^2. \qquad (3)$$

In fact, a properly normalized ground state wavefcn is: $\psi(r) = (1/\pi a_0^3)^{1/2} \exp(-r/a_0)$, so: $|\psi(0)|^2 = 1/\pi a_0^3$, and in (3)...

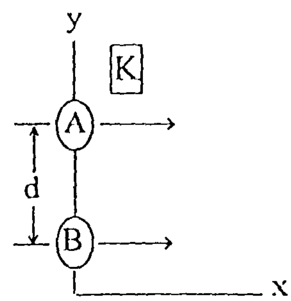
$$\Delta E = \frac{2}{3}(R/a_0)^2 \frac{e^2}{a_0}. \qquad (4)$$

C. The ground state binding energy is $B = e^2/2a_0$. So, fractionally...

$$\rightarrow \frac{\Delta E}{B} = \frac{4}{3}(R/a_0)^2 = \frac{4}{3}(10^{-13}/0.53 \times 10^{-8})^2 = 5 \times 10^{-10}. \qquad (5)$$

The shift is very small, about $\frac{1}{2}$ part per billion. ΔE in Eq. (4) is positive, which means that the effect slightly weakens the binding of the state. This makes sense, since the effect "removes" part of the Coulomb attraction between the electron & proton.

4. In reference frame K, two runners A and B are lined up at distance d apart on the y-axis for a race parallel to the x-axis. Two starters, one beside each runner, will fire their starting pistols at slightly different times in order to give an advantage to runner B, who is slower than A. The time delay is ΔT, in frame K.



A. For what range of time delays ΔT is there a frame K′, moving along the y-axis at velocity v, in which there is a <u>true</u> (not just apparent) advantage to B? Express your answer in terms of the given d, v, and c.

B. For the minimum ΔT_m which applies in part A, find the <u>explicit</u> Lorentz transformation from K to frame K′. As a fixed point, assume the slower runner B was at the origin ($x = 0$, $y = 0$) at his starting time, $t = 0$. Your K $\rightarrow$ K′ transform should not only include coordinates x, y and t, but should also incorporate d, ΔT_m, etc.

C. In K, assume B's velocity is V, while A runs at $V + \Delta V$. After the race has begun, find the positions (x', y') of both A and B as seen in the K′ frame. Your positions should be quoted in terms of v, ΔT, V, ΔV, d, and c, etc., and they should be given as a function of time t' for the K′ frame (not t for K frame).

D. For a given delay time ΔT at the start of the race in K, there is a maximum time T_M that the race can be run such that B wins. This translates to a time T'_M in K′. Find explicit forms for T_M and T'_M (in terms of V, ΔV, ΔT and $\Delta T'$) in the K and K′ frames. You should show that T_M and T'_M have the same forms, so that observers in K and K′ agree that B wins.

Comp. 95

Problem (#4)

Relativity: Ans

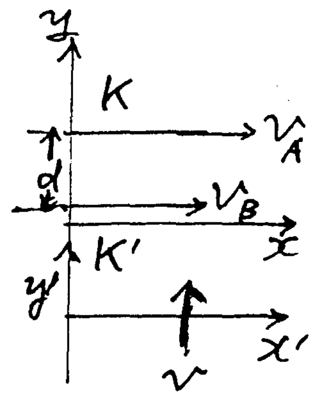


Fig. 1.

(a) K' moving along $\hat{y}$ axis of K with v.

• $\beta = v/c$; $\gamma = 1/\sqrt{1-\beta^2}$ } ①

$\Delta y = d$ in K.

• For K', the delay time recorded is:

$\Delta T' = \gamma(\Delta T - \frac{v}{c^2}\Delta y)$ ② from LT (Lorentz transformation).

But $\Delta y = y_A - y_B = d$. ①'

Let: $\begin{cases} \text{A's location} = x_A, y_A, \ \& \\ \text{B's} \quad \text{"} \quad = x_B, y_B. \end{cases}$

From ①' & ②

$$\Delta T' = \gamma(\Delta T - \frac{v}{c^2} d) = \gamma(\Delta T - \beta \frac{d}{c}) \quad ②'$$

• B sees a true advantage in K' if $\Delta T' > 0$. Then, from ②', get

$\boxed{\Delta T > vd/c^2}$ Ans. ③

(b). From LT, K → K' transforms as:

$t' = \gamma(t - \frac{v}{c^2} y)$; $y' = \gamma(y - vt)$; & $x' = x$ ④.

• From Eqn ③, $\Delta T_m = vd/c^2$ ⑤ So, for ΔT_m, we get

$$\left\{ \begin{array}{l} \underline{t' = \gamma(t - \Delta T_m y/d)}; \ \underline{y' = \gamma[y - \Delta T_m c^2 t/d]}, \\ \underline{x' = x}, \text{ where } \underline{\gamma = 1/\sqrt{1-(v/c)^2}} = \underline{1/\sqrt{1-(\Delta T_m c/d)^2}} \end{array} \right\} \text{Ans}$$

(c) $v_A = V + \Delta V$; $v_B = V$. given.

For B:

In K system: $x_B = V t_B$; $y_B = 0$ ⑤'

In K' " : $x'_B = x_B = V t_B$, but

$t'_B = \gamma(t_B - (V/c^2) y_B) = \gamma t_B$, since $y_B = 0$, from LT.

So; $x'_B = (V/\gamma) t'_B$; $y'_B = \gamma(y_B - V t_B) = -\gamma V t_B = -V t'_B$ from LT.

So $x'_B = (V/\gamma) t'_B$; $y'_B = -V t'_B$

Since $t_B = t_A = t$, we get

$\boxed{x'_B = (V/\gamma) t' \ ; \ y'_B = -V t'}$ ⑥a Ans.

Relativity (cont'd) Ans

(C) Cont'd.)

For A:

In K system: $x_A = (V+\Delta V)(t_A - \Delta T)$; $y_A = d$; for $t_A \geq \Delta T$ (5)''

In K' " : $x_A' = x_A = (V+\Delta V)(t_A - \Delta T)$; (7)

but $t_A' = \gamma(t_A - (V/c^2) y_A) = \gamma[t_A - (V/c^2) d]$, for $y_A = d$.

So $t_A' = \gamma(t_A - (V/c^2)d) \rightarrow t_A = (t_A'/\gamma) + (V/c^2)d$ (8)

So (8) → (7) & get:

$x_A' = (V+\Delta V)(t_A'/\gamma + Vd/c^2 - \Delta T)$ (9)

$y_A' = \gamma(y_A - V t_A) = \gamma(d - V t_A)$ (10)

(8) → (10) & get

$$y_A' = \gamma[d - (V/c)^2 d - (V/\gamma) t_A']$$
$$= \gamma[d(1-\beta^2) - (V/\gamma) t_A'] = \gamma[d/\gamma^2 - V t_A'/\gamma]$$
$$= d/\gamma - V t_A' \quad (11)$$

But since $t_A = t_B = t$, we get from (9) & (11) ⇒

$$\left\{ \boxed{\begin{array}{l} x_A' = (V+\Delta V)(t'/\gamma + Vd/c^2 - \Delta T); \\ y_A' = (d/\gamma - V t') \end{array}} \right\} \text{ ans } (6b)$$

(d) Let $\Delta x = x_A - x_B$. Then, if $\Delta x < 0$ at the end of the race at $t = t_m$, B will still win. So from (5)'' & (5)'

$x_B = Vt$; $x_A = (V+\Delta V)(t - \Delta T)$. So,

$\Delta x = x_A - x_B = (V+\Delta V)(t - \Delta T) = \Delta V(t - \Delta T) - V\Delta T < 0$, for $t < t_m$.

So, $\Delta V(t_m - \Delta T) - V\Delta T = 0$ at $t = t_m$.

$\rightarrow$ $t_m = (V\Delta T + \Delta V \Delta T)/\Delta V$,

or $\boxed{t_m = [(V+\Delta V)/\Delta V]\Delta T.}$ (12) Ans.

Similarly, $\Delta x' = x_A' - x_B' < 0$ in the K' system. From (6a) & (6b)

$\Delta x' = x_A' - x_B' = (V+\Delta V)(t'/\gamma + Vd/c^2 - \Delta T) - Vt'/\gamma < 0$ for $t' < t_m'$

at $t' = t_m'$; get $\Delta x' = 0 = (V+\Delta V)(t_m'/\gamma + Vd/c^2 - \Delta T) - V t_m'/\gamma$.

$\rightarrow$ $\Delta V t_m'/\gamma = (V+\Delta V)(\Delta T - Vd/c^2)$. (13)

But from (2)', $\rightarrow \Delta T' = \gamma(T - Vd/c^2)$ (14)

(14) → (13) & get $\Delta V t_m'/\gamma = (V+\Delta V)\Delta T'/\gamma$, or

$\boxed{t_m' = [(V+\Delta V)/\Delta V]\Delta T'}$ (15) Ans.

(12) & (15) are in the same form QED

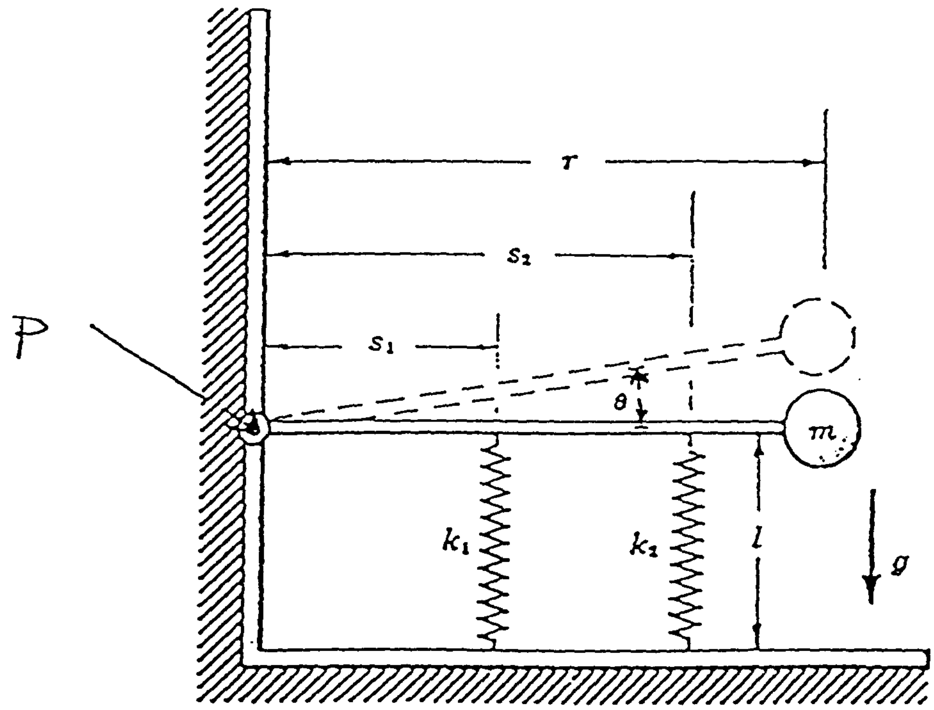
5. This problem consists of two parts which are unrelated.

I. A uniform cylinder of radius R has several turns of light string wrapped around it. The free end of the string is held in your hand and the cylinder is allowed to fall due to gravity, unwinding the string as it does. Let g be the gravitational acceleration, M be the mass of the cylinder, and assume that the string's mass is negligible. Assume that the string remains vertical throughout the cylinder's motion.

A. With what acceleration must you raise your hand in order to keep the height of the cylinder above the ground constant?

B. Find the tension in the string under these conditions.

II. A particle of mass m is attached to a massless rod which pivots at point P. The particle's center of mass is located a distance r from point P. The rod is supported by two springs, as shown below, with spring constants k_1 and k_2 and unstretched lengths L_1 and L_2, respectively.



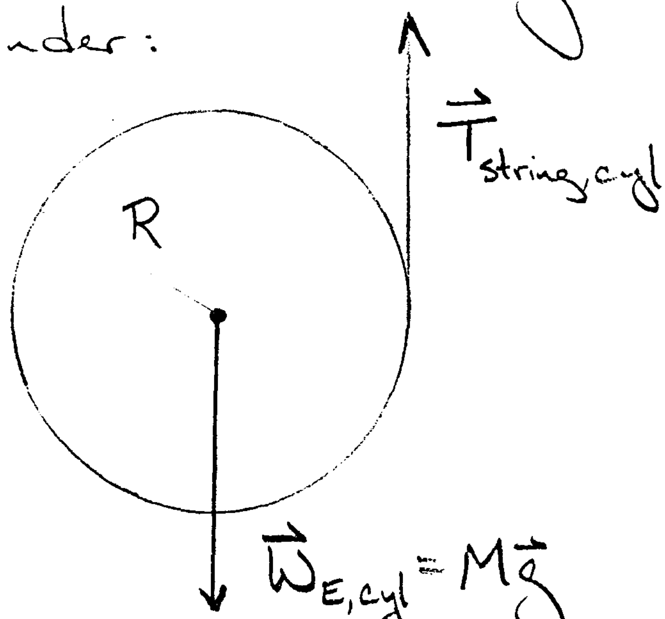
A. Using the angle θ as one generalized coordinate, write the Lagrangian for this system. Use the small angle approximation, retaining terms through quadratic order in θ, but discarding all higher order terms. (The rod is horizontal when $\theta = 0$.)

B. Determine the equation of motion for this system.

C. Assume that the springs have been adjusted so that the rod is in static equilibrium for $\theta = 0$. Find the frequency of small angle oscillations about this equilibrium point.

5. Solution (Part I)

A. Draw an extended free-body diagram for the cylinder:



By Newton's 2nd Law:

$$\Sigma \vec{F} = m\vec{a}_{cm}$$

$$\vec{T} + \vec{W} = m\vec{a}_{cm} \qquad \text{(Let up be + dir.)}$$

$$T - W = m a_{cm}^{\nearrow 0} = 0$$

$$T = W = Mg \qquad \text{(This is solution to part B)}$$

Find α about center of mass:

$$TR = I\alpha = \tfrac{1}{2}MR^2\alpha$$

$$\alpha = \frac{TR}{\tfrac{1}{2}MR^2} = \frac{2g}{R}$$

Rolling condition: $a_{hand} = R\alpha = 2g$

Solution (Part II)

A.
$$L = T - V$$
$$= \frac{1}{2} m (r\dot{\theta})^2 - \frac{1}{2} k_1 \xi_1^2 - \frac{1}{2} k_2 \xi_2^2 - mgr\theta$$

where:
$$\xi_1 = l + s_1\theta - L_1$$
$$\xi_2 = l + s_2\theta - L_2$$

[We have used the small angle approximation $\sin\theta \approx \tan\theta \approx \theta$]

B. The equation of motion is given by:
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = \frac{\partial L}{\partial \theta}$$

$$\boxed{mr^2\ddot{\theta} = -k_1 \xi_1 s_1 - k_2 \xi_2 s_2 - mgr}$$

C. At $\theta = 0$, $\ddot{\theta} = 0$, and the equation of motion yields:

#5

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$$0 = -k_1 s_1 (l - L_1) - k_2 s_2 (l - L_2) - mgr$$

Substituting this back into the equation of motion for general θ:

$$mr^2 \ddot{\theta} = -(k_1 s_1^2 + k_2 s_2^2)\theta$$

$$\ddot{\theta} = -\omega^2 \theta$$

where

$$\omega = \sqrt{\frac{k_1 s_1^2 + k_2 s_2^2}{mr^2}}$$

is the frequency of small angle oscillation.

6. An electron (mass m, charge -e) in a hydrogen atom moves in a circular orbit of radius r. Assume an infinitely heavy nucleus of charge +e, and treat this problem classically.

A. Find an expression for the electron's total orbital energy E in terms of r alone. Ultimately, the electron loses E by radiation.

B. In general, the power radiated by an accelerated electron, per unit solid angle, can be expressed as

$$\frac{dP}{d\Omega} = \frac{e^2}{4\pi c}\left[\frac{\vec{n} \times [(\vec{n} - \vec{\beta}) \times \vec{\alpha}]}{(1 - \vec{n} \cdot \vec{\beta})^3}\right]^2_{\text{ret}},$$

where $\vec{n}$ is a unit vector from source to observation point, $\vec{\beta} = v/c$ is the electron velocity in units of c, and $\vec{\alpha} = d\vec{\beta}/dt$. The subscript "ret" means that the righthand side of the expression is evaluated at the retarded time. In this problem, it is appropriate to find the total power P radiated by the electron in the nonrelativistic limit, as a function of its acceleration a and such constants as e and c. Find this version of P from the above, when $v \ll c$.

C. Assume that, per orbit, the electron radiates energy $\Delta E \ll |E|$. Find the radial dependence of the total radiated power P, from the results of part B. Here, your answer for P should depend on r alone, except for such constants as e and c.

D. Obtain a differential equation for the decrease in r as a function of time, due to radiation.

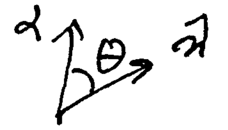
E. Calculate the elapsed time for the electron to spiral into the nucleus if it starts from $r = a_o$, where $a_o = 0.53 \times 10^{-8}$ cm is the Bohr radius. Find a number for this "collapse time". NOTE: the classical electron radius is $r_o = e^2/mc^2 = 2.82 \times 10^{-13}$ cm.

Problem (#6)

E & M: Ans.

(a) Centripetal force = Coulomb force. So
$mv^2/r = e^2/r^2$ (1) for electron in H atom.
So kinetic energy $KE = \frac{1}{2}mv^2 = \frac{1}{2}\frac{e^2}{r}$ (2) for $v \ll c$.
The potential energy for the orbiting electron is
$PE = -e^2/r$. (3)
So, from (2) & (3) total orbital energy is:
$E = KE + PE = \boxed{-e^2/(2r)}$ Ans (4)

(b) General case:
$dP/d\Omega = (e^2/4\pi c)\,\{\hat{n}\times[(\hat{n}-\bar{\beta})\times\bar{a}]/(1-\hat{n}\cdot\bar{\beta})^3\}^2$ (5) given.
In the non-relativistic limit, retarded time = t, $\beta \ll 1$ and so $1-\hat{n}\cdot\bar{\beta}\simeq 1$. Also, $\{\hat{n}\times[(\hat{n}-\bar{\beta})\times\bar{a}]\}^2$
$\simeq (\hat{n}\times(\hat{n}\times\bar{a}))^2 = a^2\sin^2\theta$, where θ is the angle between $\bar{a}$ and $\hat{n}$. So (5) →



$dP/d\Omega = (e^2/4\pi c^3)|a|^2\sin^2\theta$ and
$P = \int (dP/d\Omega)\,d\Omega = \int (e^2/4\pi c^3)\,|a^2|\sin^2\theta\,d\Omega$
$\boxed{P = \left(\frac{2e^2}{3c^2}\right)|\bar{a}|^2}$ Ans, (6)

(c) With the centripetal acceleration, $a = v^2/r$ (7)
The electron must radiate at a rate (6). So (7) → (6) →
$P = (2e^2/3c^2)(v^4/r^2)$, but from (1) $v^2 = e^2/mr$.
So $P = (2e^2/3c^3)(e^4/m^2r^4) = (2e^2/3c^3)(e^2/m)^2\,r^{-4}$
or $\boxed{P = \frac{2}{3}\left(\frac{e^6}{m^2c^3}\right)\frac{1}{r^4}}$ Ans (7)

(d) Equate the radiative loss rate P to the rate of loss of E. i.e. dE/dt (from (4)) = (7). Then
$dE/dt = d(-e^2/2r) = (-)(2/3)(e^6/m^2c^3)\,r^{-4}$.
→ get $\boxed{\frac{dr}{dt} = (-)\frac{4}{3}\left(\frac{e^4}{m^2c^3}\right)\frac{1}{r^2}}$ (8)

(e) The total time T for the electron to spiral down from the orbital radius $r = R$ to $r = 0$ is: (from ⑧)

$$T(\text{collapse}) = \int_{r=R}^{r=0} dt = -\frac{3}{4}\left(\frac{m^2c^3}{e^4}\right)\int_R^0 r^2 dr = \frac{1}{4}\left(\frac{m^2c^3}{e^4}\right)R^3.$$

$$\therefore \boxed{T(\text{collapse}) = \frac{1}{4}\left(\frac{r_0}{c}\right)\left(\frac{R}{r_0}\right)^3,} \quad \text{Ans.} \qquad (9)$$

where $r_0 = e^2/(mc^2) = 2.82\times10^{-13}$ cm (for electrons).
For electrons $R = a_0 = 0.53\,\text{Å} = 0.53\times10^{-8}$ cm.
Then, from eqn (9), we get
$T(\text{collapse}) = 1.6\times10^{-11}$ sec. Ans.

7. The specific heat (per unit volume) at constant pressure for a metal at temperature T can be represented approximately by

$C_s = aT^3$, in the superconducting state;
$C_n = bT^3 + \gamma T$, in the normal state.

a, b, and γ are constants. From the equivalent of the Clausius-Clapeyron equation for this transition, we can write

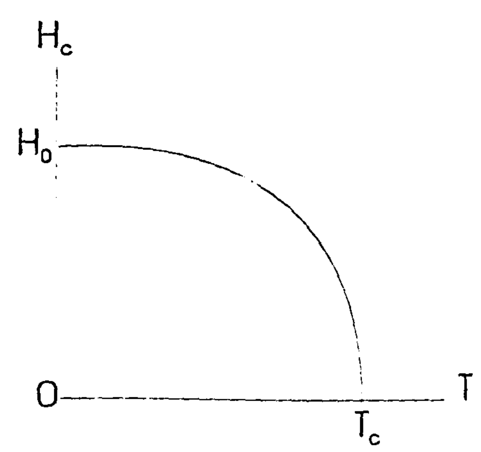
$$T(S_n - S_s) = nL = -\mu_o T H_c(\partial H_c/\partial T).$$

Here S_n and S_s are the entropies in the normal and superconducting states, L is the latent heat associated with the superconducting transition, H_c is the critical magnetic field that destroys superconductivity at temperature T, and μ_o is the permeability of free space. Note that the superconducting transition is a second-order transition in zero magnetic field -- i.e. there is no latent heat associated with the transition for zero field.

A. Derive an expression for T_c, the transition temperature in zero field. Your result for T_c should depend on the constants a, b, and γ.

B. Find the temperature at which the difference between the internal energies of the two states, in zero field, is a maximum.

C. Show that the critical magnetic field depends on temperature as: $H_c = (1-t^2)H_o$, where: $t = T/T_c$, and: $H_o = T_c\,(\gamma/2\mu_o)^{1/2}$. You may assume that $H_c = 0$ at $T = T_c$, and that $dH_c/dT = 0$ at $T = 0$, as shown in the figure below.



Solution

Supercond. Transition (Smith) $C_p = \frac{\partial U}{\partial T}\Big)_p = T\frac{\partial S}{\partial T}\Big)_p$ $dV=0$

CCMP 95 PROB #7

(a) Zero field $\Rightarrow$ ~~second~~ order $\Rightarrow$ $L_c = 0$

So

$$S_n = S_s$$

Use fact that $dS = C_p\, dT/T$ to get

Then $$\int \left(\frac{C_n}{T}\right) dT = \int \left(\frac{C_s}{T}\right) dT$$

$$\frac{aT_c^3}{3} + \text{const} = \frac{bT_c^3}{3} + \gamma T_c + \text{const}$$

Set arbitrary const $= 0$ to make $S \to 0$

Then

$$T_c^2 (a-b) = 3\gamma$$

$$\boxed{T_c = \left[\frac{3\gamma}{a-b}\right]^{1/2}}$$

(b) Again use $C_p = \partial U/\partial T$

Define $U_n - U_s = \Delta U$

Set $\frac{\partial \Delta U}{\partial T} = 0$ for max so $\frac{\partial U_n}{\partial T} = \frac{\partial U_s}{\partial T}$

so $C_{p_n} = C_{p_s}$

Then

$$aT^3 = bT^3 + \gamma T$$

$$T^2 = \gamma/(a-b) = T_c^2/3$$

$$\boxed{T = T_c/\sqrt{3}}$$

(c) Given $T(S_n - S_s) = nL_c$

$$= -\mu_0 T H_c \left(\frac{dH_c}{dT}\right)$$

Use $C_p = T \left.\frac{\partial S}{\partial T}\right)_p$

$$S_n - S_s = -\mu_0 H_c \left(\frac{dH_c}{dT}\right)$$

$$\frac{\partial S_n}{\partial T} - \frac{\partial S_s}{\partial T} = -\mu_0 \frac{\partial}{\partial T}\left(H_c \frac{\partial H_c}{\partial T}\right)$$

$$T\left(\frac{\partial S_n}{\partial T}\right)_p - T\left(\frac{\partial S_s}{\partial T}\right)_p = -\mu_0 T/_2 \frac{\partial^2}{\partial T^2}\left(H_c^2\right)$$

$$C_n - C_s = -\mu_0 T/_2 \frac{\partial^2}{\partial T^2} H_c^2$$

$$\frac{d^2}{dT^2}(H_c^2) = +\frac{2}{\mu_c}\left[(a-b)T^2 - \gamma\right]$$

↓ mostly math from hereon

Integrate $\frac{d}{dT}(H_c^2) = \frac{2}{\mu_0}\left[(a-b)\frac{T^3}{3} - \gamma T + \alpha\right]$

To find α note that $\frac{dH_c}{dT} = 0$ at $T=0$ but $H_c \neq 0$

So LHS above is

$$2H_c \frac{dH_c}{dT} = 0 = 2/\mu_c \, \alpha \qquad \therefore \alpha = 0$$

Then $H_c^2 = \frac{2}{\mu_0}\left[\frac{(a-b)}{3\cdot 4}T^4 - \gamma T^2/_2 + \beta\right]$

$T_c^2 = \frac{3\gamma}{a-b}$ $$= \frac{2}{\mu_0}\left[\frac{\gamma}{T_c^2}\frac{T^4}{4} - \gamma T^2/_2 + \beta\right]$$

At $T=T_c$ $H_c = 0$ so $\gamma/_4 T_c^2 - \gamma T_c^2/_2 + \beta = 0$

$$\beta = \gamma/_4 T_c^2$$

$$H_c^2 = \frac{2}{\mu_0}\left[\frac{\gamma}{T_c^2}\frac{T^4}{4} - \gamma T^2/2 + \gamma/4\, T_c^2\right]$$

$$= \frac{\gamma}{2\mu_0}T_c^2\left[\frac{T^4}{T_c^4} - 2\frac{T^2}{T_c^2} + 1\right]$$

Define $t = T/T_c < 1$ for $0 < T < T_c$

and $H_0 = T_c\sqrt{\gamma/2\mu_0}$

$$H_c^2 = H_0^2(t^4 - 2t^2 + 1)$$

$$H_c = \pm H_0(t^2 - 1)$$

but since $H_c > 0$ and $t < 1$ must take $-$ root

$$\boxed{H_c = H_0(1 - t^2)}$$

8. Consider the unit step-function: $\theta(\tau) = \begin{cases} 1, \tau > 0; \\ 0, \tau < 0. \end{cases}$ By doing an appropriate contour integral in the complex ω-plane

A. Show that an integral representation for the step-function is:

$$\theta(\tau) = (i/2\pi) \int_{-\infty}^{+\infty} \left[e^{-i\omega\tau} / (\omega + i\varepsilon)\right] d\omega, \text{ in } \lim \varepsilon \to 0+.$$

B. What is the result for the integral of part A when $\lim \varepsilon \to 0-$?

C. By differentiating your results in parts A and/or B, find an integral representation of the Dirac delta function, $\delta(\tau)$. [Recall: $\delta(\tau) = 0$ for all $\tau \neq 0$, but: $\int_{-\infty}^{+\infty} \delta(\tau) d\tau = 1$]. Does your representation for $\delta(\tau)$ depend on whether $\varepsilon \to 0+$ or $0-$?

ϕ Comp. Exam : June '95 (Robiscoe)

5/28/95

8 ✓ #8

MP -- Complex Variables

"By evaluating an appropriate contour integral in the complex ω-plane, show that

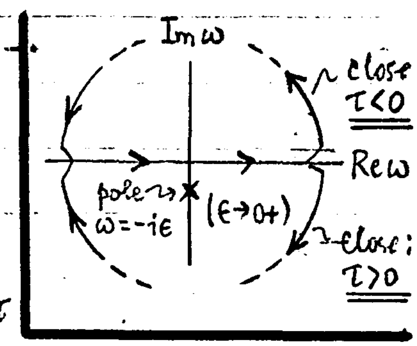
A. The unit step function : $\theta(\tau) = \begin{cases} 1, & \text{for } \tau > 0; \\ 0, & \text{for } \tau < 0; \end{cases}$ has representation :

$$\theta(\tau) = (i/2\pi) \int_{-\infty}^{\infty} [e^{-i\omega\tau}/(\omega + i\epsilon)]\, d\omega \,, \text{ in } \lim \epsilon \to 0+ .$$

B. What results for the integral of part A when $\lim \epsilon \to 0-$?

C. By an appropriate differentiation of your results, derive an integral representation of the Dirac delta function, $\delta(\tau)$. Recall : $\delta(\tau) \equiv 0$ for all $\tau \neq 0$, but $\int_{-\infty}^{+\infty} \delta(\tau)\, d\tau = 1$. Does your representation for $\delta(\tau)$ depend on whether $\epsilon \to 0+$ or $\epsilon \to 0-$?"

SOLUTION



A. For $\epsilon > 0$ (i.e. $\epsilon \to 0+$), the integrand has a simple pole @ $\omega = -i\epsilon$, on the (−)ve Im ω axis. Whatever the sign of ϵ, the contour (for the contour integral counterpart of $\theta(\tau)$) must be closed so that the factor $e^{-i\omega\tau}$ vanishes on a large semicircle. Since $e^{-i\omega\tau} = (e^{\tau \operatorname{Im}\omega})(e^{-i\tau \operatorname{Re}\omega}) \to 0$ when $\tau > 0$ & Im$\omega \to -\infty$, or $\tau < 0$ & Im$\omega \to +\infty$, then closure must be made in the upper half-plane for $\tau < 0$, and lower half-plane for $\tau > 0$.

For $\epsilon \to 0+$, the contour integral version of $\theta(\tau)$ is zero for $\tau < 0$, since closure in the upper half-plane captures no poles. When $\tau > 0$, closure in the lower half-plane captures the pole @ $\omega = -i\epsilon$, and so by the

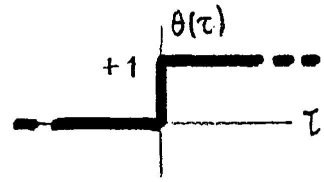
(OVER)

Residue Theorem...

$$\rightarrow \theta(\tau>0) = (-2\pi i)\,\frac{i}{2\pi} \lim_{\epsilon\to 0+} \mathrm{Res}\left\{ \frac{e^{-i\omega\tau}}{\omega+i\epsilon},\ \omega=-i\epsilon \right\} = \lim_{\epsilon\to 0+} e^{-i(-i\epsilon)\tau}$$

↑ (−) because of CW contour

or// $\theta(\tau>0) = \lim_{\epsilon\to 0+} e^{-\epsilon\tau} = 1$, and/ $\theta(\tau<0) = 0$.



(1)

So, indeed: $\theta(\tau) = (i/2\pi) \lim_{\epsilon\to 0+} \int_{-\infty}^{\infty} [e^{-i\omega\tau}/(\omega+i\epsilon)]\, d\omega$ is the step fcn.

B. If $\epsilon\to 0-$, the pole in the above ω-plane diagram moves to a position on the (+)ve $\mathrm{Im}\,\omega$ axis. Closure in the lower half-plane now gives a new fcn such that: $\tilde{\theta}(\tau>0)=0$. Closure in the upper half-plane produces: $\tilde{\theta}(\tau<0) = -1$ (by a calc^n similar to that in Eq. (1)). So...

$$\rightarrow \tilde{\theta}(\tau) = (i/2\pi) \lim_{\epsilon\to 0-} \int_{-\infty}^{+\infty} [e^{-i\omega\tau}/(\omega+i\epsilon)]\, d\omega = \theta(\tau) - 1 = \begin{cases} 0, & \tau>0; \\ -1, & \tau<0. \end{cases} \quad (2)$$

The extra (−) sign comes from the CCW contour for $\tau<0$.

C. By doing $d/d\tau$ on the integral of part A, we find...

$$\rightarrow \delta(\tau) = \frac{d}{d\tau}\left\{ \left(\frac{i}{2\pi}\right) \lim_{\epsilon\to 0+} \int_{-\infty}^{\infty} [e^{-i\omega\tau}/(\omega+i\epsilon)]\, d\omega \right\} = \frac{1}{2\pi} \lim_{\epsilon\to 0+} \int_{-\infty}^{\infty} \left(\frac{\omega}{\omega+i\epsilon}\right) e^{-i\omega\tau}\, d\omega$$

i.e.// $$\boxed{\delta(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega\tau}\, d\omega} \quad (3)$$

By the fcnal nature of $\theta(\tau)$ & $\tilde{\theta}(\tau)$, $\delta(\tau)$ is zero everywhere but <u>at</u> $\tau=0$, and it is <u>independent</u> of whether $\epsilon\to 0+$ or $\epsilon\to 0-$. Moreover...

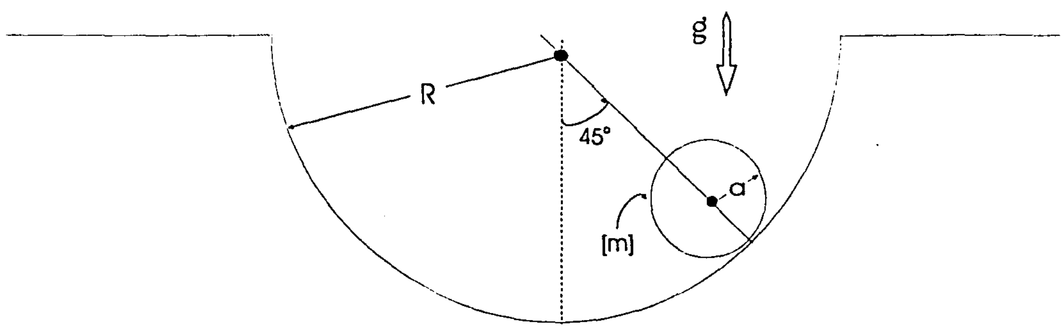
$$\rightarrow \int_{-\infty}^{\infty} \delta(\tau)\, d\tau = \lim_{T\to 0} \int_{-T}^{+T} d\tau \cdot \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega\tau}\, d\omega = \lim_{T\to 0} \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \int_{-T}^{+T} e^{-i\omega\tau}\, d\tau$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \left(\frac{\sin x}{x}\right) dx = 1, \text{ independent of } T. \quad (4)$$

So, $\delta(\tau)$ of Eq. (3) is fully qualified as a rep^n of the Dirac $\delta(\tau)$.

9. A thin uniform hoop of mass m and radius a rolls without slipping inside a hemispherical bowl of radius $R > a$, as shown below. The hoop's motion is constrained to lie in a vertical plane. Initially, the hoop is released from a position halfway between the bottom of the bowl and its upper edge, as indicated in the diagram.

Find the normal force exerted by the bowl on the hoop, when the hoop arrives at the bottom of the bowl.

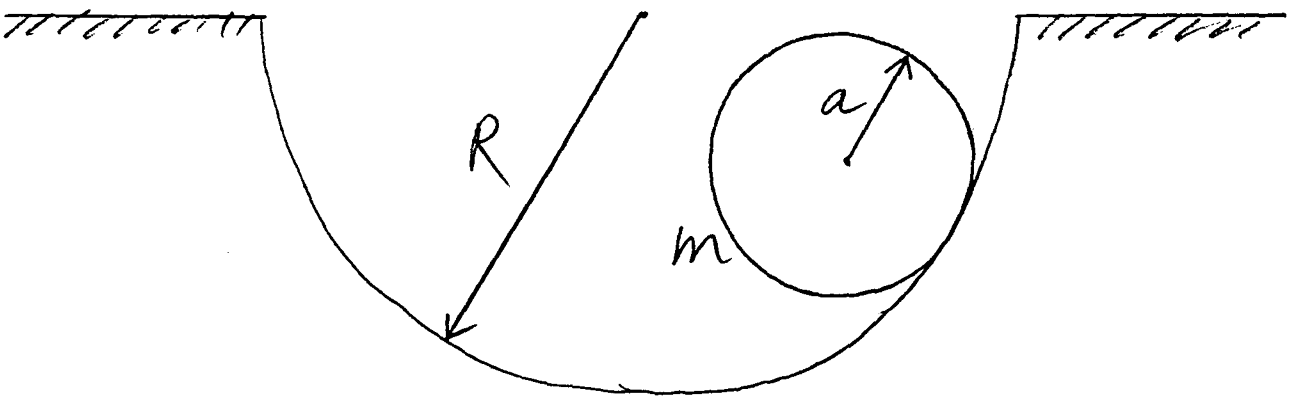


CM#2: Hermanson

A thin uniform hoop of mass m and radius a rolls without slipping within a hemi spherical bowl of radius R as shown below. The motion is constrained to lie in a vertical plane. The is released from a position halfway between the bottom of the bowl and its upper edge.

Find the normal force exerted by the bowl on the hoop when the hoop is at the bottom of the bow.

~~(b) Find the frequency of small oscillations near the bottom of the bowl~~

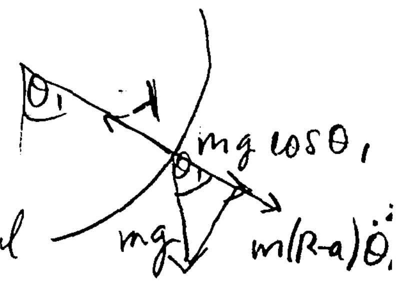


#9

The normal force λ:

(write Lagrange eqn's incorporating the constraints)

$$-m(R-a)\dot{\theta}_1^2 - mg\cos\theta_1 = \lambda$$



In words:

The normal force is the sum of the centrifugal force and the gravitational force projected in the radial direction; it is negative because it tends to <u>decrease</u> r

At the bottom, $\dot{\theta}_1^2$ is related to the kinetic energy

$$E = \frac{m}{2}(R-a)^2\dot{\theta}_1^2 + \frac{m}{2}a^2\dot{\theta}_2^2, \text{ but } \dot{\theta}_2 = \frac{R-a}{a}\dot{\theta}_1$$

$$= \frac{m}{2}(R-a)^2\dot{\theta}_1^2 + \frac{m}{2}(R-a)^2\dot{\theta}_1^2$$

$$= m(R-a)^2\dot{\theta}_1^2$$

$$= mg(R-a)(1-\cos\underbrace{\theta_0}_{\frac{\pi}{4}}) \quad \text{initial energy} = \text{P.E.}$$

$$= mg(R-a)\left(1-\frac{1}{\sqrt{2}}\right)$$

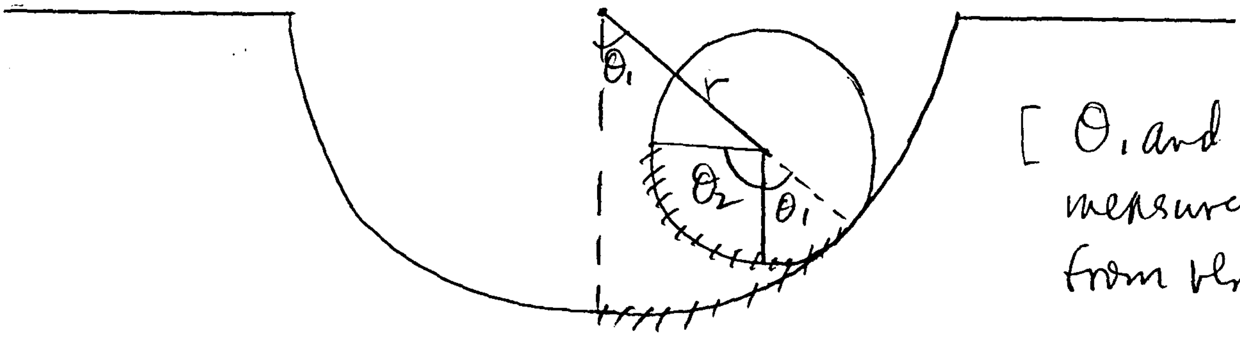
So the centrifugal force at the bottom is

$$m(R-a)\dot{\theta}_1^2 = mg\left(1-\frac{1}{\sqrt{2}}\right) > 0 \checkmark$$

adding the grav. force mg at the bottom,

$$\boxed{\lambda = -mg\left(2-\frac{1}{\sqrt{2}}\right)} \quad \text{directed inward (up)} \checkmark$$

CM #2 Solution J. Hermanson



[θ_1 and θ_2 measured from vertical]

Contraints: $r = R - a$ hoop keeps contact and

$R\theta_1 = a(\theta_1 + \theta_2)$ rolls without slipping

$$\boxed{\begin{aligned} f &= r - R + a = 0 \\ g &= (R-a)\theta_1 - a\theta_2 = 0 \end{aligned}}$$

$$T = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}_1^2\right) + \frac{m}{2}a^2\dot{\theta}_2^2$$

$$V = mg(R-a)(1 - \cos\theta_1)$$

$$L = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}_1^2 + a^2\dot{\theta}_2^2\right) + mg(R-a)\cos\theta_1 + \text{const}$$

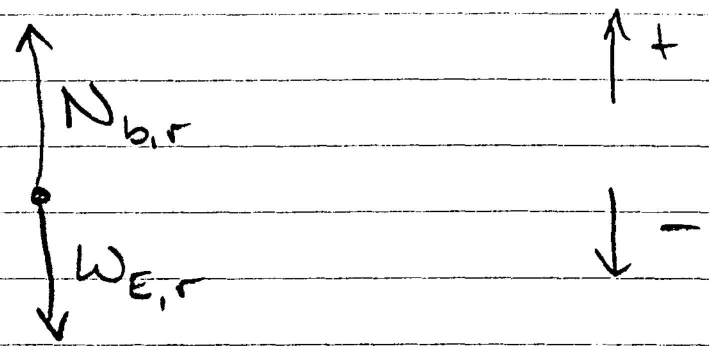
$$\left\{\begin{aligned} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} &= \lambda\frac{\partial f}{\partial r} + \lambda'\frac{\partial g}{\partial r} = \lambda = \text{normal force} \\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) - \frac{\partial L}{\partial \theta_1} &= \lambda\frac{\partial f}{\partial \theta_1} + \lambda'\frac{\partial g}{\partial \theta_1} = \lambda'(R-a) \\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) - \frac{\partial L}{\partial \theta_2} &= \lambda\frac{\partial f}{\partial \theta_2} + \lambda'\frac{\partial g}{\partial \theta_2} = -\lambda' a = \text{torque} \end{aligned}\right.$$

[rather than work with these equations directly I used the first one and energy conservation]

CM #2

Alternate solution

At bottom of the bowl the free-body-diagram for ring is:



$$\vec{F}_{net} = m\vec{a}_{cm}$$

Choose up to be positive

$$N - W = ma_c = m\frac{v_{cm}^2}{R}$$

$$N = W + m\frac{v_{cm}^2}{R}$$

$$= m\left[g + \frac{v_{cm}^2}{R}\right]$$ find v_{cm}

② #9

$$PE_i = KE_f$$

$\cos 45° = \frac{1}{\sqrt{2}}$

$$mgR\left(1-\frac{1}{\sqrt{2}}\right) = \frac{1}{2}mv_{cm}^2 + \frac{1}{2}I\omega^2$$

use rolling condition $v_{cm} = a\omega$

$$mgR\left(1-\frac{1}{\sqrt{2}}\right) = \frac{1}{2}mv_{cm}^2\left(1+\frac{I}{ma^2}\right)$$

for a ring: $I = ma^2$

$$\therefore \quad v_{cm}^2 = gR\left(1-\frac{1}{\sqrt{2}}\right)$$

$$N = m\left[g + g\left(1-\frac{1}{\sqrt{2}}\right)\right]$$

$$= mg\left(2-\frac{1}{\sqrt{2}}\right)$$ directed <u>up</u>

10. You are asked to discuss the propagation of electromagnetic radiation in metals, using the approximations described below.

Suppose the electrons in a metal can be treated as a uniform gas of density n. Electrons under the influence of an external electric field $\vec{E}$ experience a frictional force due to collisions with the metal ions that form the crystal lattice. Often the equation of motion for electrons in the presence of collisions and external fields is approximated by:

$$\frac{d\vec{p}}{dt} = -e\vec{E} - \frac{1}{\tau}\vec{p}, \tag{1}$$

where $1/\tau$ denotes the frequency of collisions between electrons and ions and $\vec{p}(t) = m\vec{v}(t)$ is the average electron momentum at time t, and -e and m are the charge and mass of the electron, respectively.

A. Give a brief explanation in words of the equation

$$\vec{J}(\vec{r},\omega) = \sigma(\omega)\vec{E}(\vec{r},\omega) \tag{2}$$

and then show that within the above model the electrical conductivity of metals is

$$\sigma(\omega) = \frac{\sigma_0}{1 - i\omega\tau}, \tag{3}$$

with

$$\sigma_0 = \frac{ne^2\tau}{m}. \tag{4}$$

Discuss the phase relationship between the electric field and the current density in the limits $\omega\tau \gg 1$ and $\omega\tau \ll 1$. [Note: We assume $\vec{E}(\vec{r},t) = \mathrm{Re}\left\{\vec{E}(\vec{r},\omega)\exp(-i\omega t)\right\}$.]

B. Derive from Maxwell's equations the following wave equation for the electric field in a metal

$$-\vec{\nabla}^2\vec{E} = \left(\frac{\omega}{c}\right)^2 \varepsilon(\omega)\vec{E} \tag{5}$$

with

$$\varepsilon(\omega) = 1 + \frac{4\pi i\sigma(\omega)}{\omega} \tag{6}$$

Here $\varepsilon(\omega)$ is the dielectric constant of the metal. Assume that the magnetic susceptibility is the same as in free space.

(continued on next page)

C. Under certain conditions on the frequency of the radiation, Eq. (5) will have propagating oscillatory solutions, i.e., the metal should become transparent. Provided that $\omega\tau >> 1$, show that a metal should be transparent for $\omega \geq \omega_p$, where

$$\omega_p^2 = \frac{4\pi n e^2}{m}. \tag{7}$$

ω_p is referred to as the plasma frequency of the metal.

D. From the above criterion, are metals reflecting or transmitting for optical radiation?

Specifically for the alkali metal Na, check whether the condition $\omega_p\tau >> 1$ can be satisfied and predict the wavelength λ below which Na should be transparent.

[Note: Typical values are $\omega_p \approx 10^{16} Hz$ or $\lambda_p = 2\pi c / \omega_p \approx 10^3 \dot{A}$. For Na: $\lambda_p = 2 \times 10^3 \dot{A}$ and $\tau \approx 3.2 \times 10^{-14}$ sec at 273 K. $1\dot{A} \equiv 10^{-8} cm$.]

10- Solution

A. $\vec{J}(\vec{r},\omega) = \sigma(\omega)\vec{E}(\vec{r},\omega)$

This is the generalized Ohm's Law in harmonic form. The current density is directly proportional to the electric field, where the constant of proportionality is the conductivity of the metal. This conductivity depends only on the frequency of the field.

To find $\sigma(\omega)$ we use Eq. (1) and assume a harmonic time dependence for all variables

$$\vec{p}(\vec{r},t) = Re\{\vec{p}(\vec{r},\omega)e^{-i\omega t}\}$$

$$\vec{J}(\vec{r},t) = Re\{\vec{J}(\vec{r},\omega)e^{-i\omega t}\} \quad \text{etc.}$$

Eq. (1) becomes:

$$-i\omega\vec{p}(\vec{r},\omega) = -e\vec{E}(\vec{r},\omega) - \frac{1}{\tau}\vec{p}(\vec{r},\omega)$$

Using Ohm's Law.

$$\vec{E}(\vec{r},\omega) = \frac{\vec{J}(\vec{r},\omega)}{\sigma(\omega)} = \frac{-en\vec{p}(\vec{r},\omega)}{m\sigma(\omega)}$$

We get:

$$-i\omega\vec{p} = \frac{ne^2}{m\sigma(\omega)}\vec{p} - \frac{1}{\tau}\vec{p}$$

Solving for $\sigma(\omega)$:

$$\boxed{\sigma(\omega) = \frac{ne^2\tau}{m}\frac{1}{1-i\omega\tau}}$$

In the limit $\omega\tau \gg 1$, $\sigma(\omega)$ is almost purely imaginary, and the current in the metal will lag the field by 90°.

In the limit $\omega\tau \ll 1$, $\sigma(\omega)$ is almost purely real, and the current will be in phase with the field.

B. In Gaussian units, Maxwell's Equations are:

$$\nabla\cdot\vec{B} = 0$$
$$\nabla\cdot\vec{D} = 4\pi\rho$$
$$\nabla\times\vec{H} = \frac{4\pi}{c}\vec{J} + \frac{1}{c}\frac{\partial\vec{D}}{\partial t}$$
$$\nabla\times\vec{E} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t}$$

where

$$\vec{D} = \vec{E} + 4\pi\cancelto{0}{\vec{P}} = \vec{E}$$
$$\vec{H} = \vec{B} - 4\pi\cancelto{0}{\vec{M}} = \vec{B}$$

In the absence of any dielectric or magnetic materials.

Given the harmonic time-dependence of the fields:

$$\frac{\partial \vec{B}}{\partial t} = -i\omega\vec{B}$$

$$\frac{\partial \vec{E}}{\partial t} = -i\omega\vec{E}$$

Faraday's Law becomes:

$$\nabla \times \vec{E} = \frac{i\omega}{c}\vec{B}$$

and Ampere's Law may be written:

$$\nabla \times \left(\frac{c}{i\omega}\nabla \times \vec{E}\right) = -\frac{i\omega}{c}\vec{E} + \frac{4\pi}{c}\vec{J}$$

Using Ohm's Law, we get:

$$\nabla \times (\nabla \times \vec{E}) = \frac{\omega^2}{c^2}\vec{E} + \frac{4\pi i\omega\sigma(\omega)}{c^2}\vec{E}$$

Use the vector identity $\nabla \times \nabla \times \vec{E} = \nabla(\nabla\cdot\vec{E}) - \nabla^2\vec{E}$ and the fact that the metal is neutral:

$$\nabla\cdot\vec{E} = 4\pi\rho = 4\pi e(n_i - n_e) = 0.$$

$$\therefore \boxed{\begin{aligned} -\nabla^2\vec{E} &= \frac{\omega^2}{c^2}\left[1 + \frac{4\pi i\sigma(\omega)}{\omega}\right]\vec{E} \\ &\equiv \frac{\omega^2}{c^2}\varepsilon(\omega)\vec{E} \end{aligned}}$$

C. Eq. (5) has oscillatory solutions when $\varepsilon(\omega) > 0$.

$$\varepsilon(\omega) = 1 + \frac{4\pi i \sigma(\omega)}{\omega} = 1 + \frac{4\pi i n e^2 \tau}{m\omega}\left(\frac{1}{1 - i\omega\tau}\right)$$

Assuming $\omega\tau \gg 1$,

$$\varepsilon(\omega) \approx 1 - \frac{4\pi n e^2}{m\omega^2} \geq 0$$

$$\therefore \quad \frac{4\pi n e^2}{m\omega^2} \leq 1$$

So for $\quad \omega^2 \geq \dfrac{4\pi n e^2}{m} \equiv \omega_p^2$

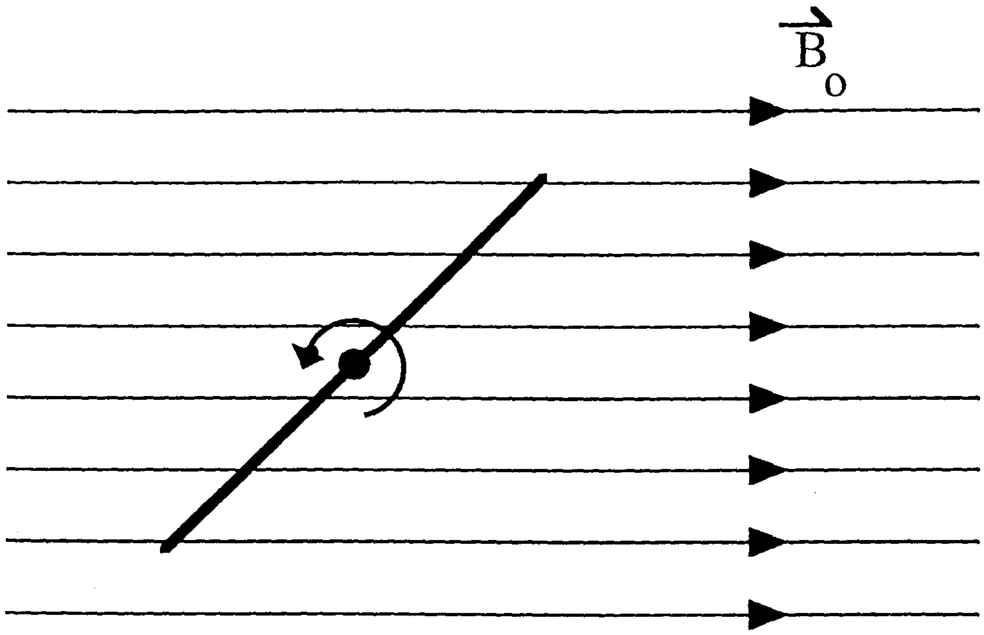
the metal will be transparent.

D. From observation, metals reflect optical radiation. For the example given:

$\omega_p \tau \approx 300 \gg 1$, so the condition of an imaginary $\sigma(\omega)$ is satisfied.
Na is transparent for $\omega \geq \omega_p$, or for $\lambda \leq \lambda_p = 2 \times 10^3$ Å $= 200$ nm
This λ is outside the range of optical radiation;

$$400\text{ nm} < \lambda < 700\text{ nm}$$

11. A thin circular wire ring is rotating about an axis perpendicular to a uniform magnetic field B_0, as shown below. At time $t = 0$ its angular velocity is ω_0. The ring has electrical conductivity σ and mass density ρ. The size of the ring is unknown.



side view of ring

A. Explain in words why the ring slows down.

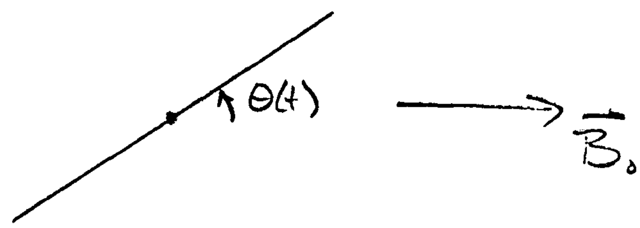
B. Derive an equation for the rate at which angular velocity ω changes in time. Solve this equation to find $\omega(t)$ in terms of the parameters σ, ρ, B_0, and ω_0. Do not assume a small change in velocity, but neglect self-inductance effects.

C. Use the results of part B to determine how many complete revolutions the ring will make before coming to rest?

Solution

11\. A. Magnetic flux linked by the ring is changing with time, inducing an EMF in the ring. The ring has finite conductivity so there will be energy lost to the system due to Joule heating. The initial energy of the ring is all kinetic ($\frac{1}{2} I \omega_0^2$), so a loss of energy requires the ring to slow.

B.



The magnetic flux through the ring is

$$\Phi_B = B_0 A_0 \sin \theta(t)$$

where $A_0 = \pi a^2$ is the area of the ring and a is the unknown radius.

Faraday's Law predicts an induced EMF:

$$\mathcal{E} = -\frac{1}{c} \frac{\partial \Phi_B}{\partial t} = -\frac{B_0 A_0 \omega}{c} \cos \theta(t)$$

The power loss is due to Joule heating:

$$\frac{d}{dt}(K.E.) = -\frac{\langle \mathcal{E}^2 \rangle}{R}$$

where R is the resistence in the ring, and the brackets $\langle\ \rangle$ indicate an average over a complete rotation.

$$I\omega\dot{\omega} = -\frac{B_0^2 A_0^2 \omega^2}{c^2 R} \langle \cos^2\theta(t) \rangle$$

$$= -\frac{B_0^2 A_0^2 \omega^2}{2c^2 R}$$

$$\dot{\omega} = -\frac{\omega}{\tau}$$

where

$$\tau = \frac{2 I c^2 R}{B_0^2 A_0^2}$$

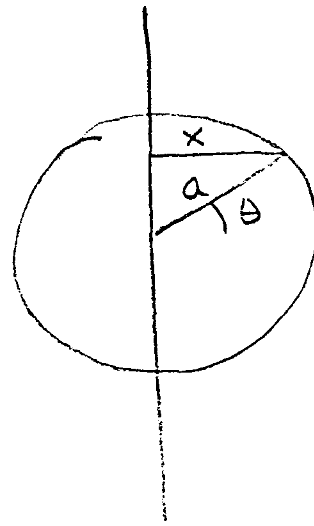
$$\int \frac{d\omega}{\omega} = -\int \frac{dt}{\tau}$$

$$\omega(t) = \omega_0 e^{-t/\tau}$$

It remains only to write τ in terms of parameters given.

$I = \frac{1}{2} M a^2$ where $M = \rho\, 2\pi a A'$ is the total mass of the ring, and A' is cross sectional area of wire.

If we don't remember the moment of inertia, we can derive it:



$dm = \rho A' a\, d\theta$

$x = a \cos\theta$

$$I = \int x^2\, dm$$

$$= \int_0^{2\pi} \rho A' a^3 \cos^2\theta\, d\theta$$

$$= \pi \rho A' a^3$$

$$= \frac{1}{2} M a^2$$

$$R = \frac{l}{\sigma A'} = \frac{2\pi a}{\sigma A'}$$

So

$$\tau = \frac{2 I c^2 R}{B_0^2 A_0^2} = \frac{2(\pi\rho A' a^3)\left(\frac{2\pi a}{\sigma A'}\right) c^2}{B_0^2 (\pi a^2)^2}$$

$$\boxed{\tau = \frac{4\rho c^2}{B_0^2 \sigma} \qquad \omega(t) = \omega_0 e^{-t/\tau}}$$

#11

(4)

C.

$$\omega(t) = \frac{d\theta(t)}{dt}$$

$$\int d\theta = \int \omega\, dt$$

When the ring comes to a stop, it will have turned through total angle $\theta = 2\pi n$, where n is the number of revolutions

$$2\pi n = \int_0^\infty \omega(t)\, dt = \int_0^\infty \omega_0 e^{-t/\tau} dt$$

$$= -\omega_0 \tau (0 - 1) = \omega_0 \tau$$

$$\boxed{\therefore \quad n = \frac{\omega_0 \tau}{2\pi} = \frac{2\omega_0 \rho c^2}{\pi B_0^2 \sigma}}$$

12. This problem has three parts, which may be completed in any order.

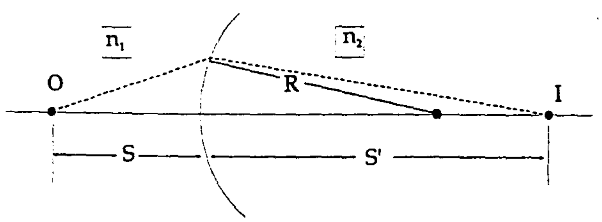
A. The refracting spherical surface shown below has radius R, and the medium to the left of the surface has refractive index n_1 while the medium to the right has index n_2. An object is placed on the axis at distance s to the left of the interface; this results in formation of an image at distance s′ on the axis to the right, as shown. Assume all angles relative to the optical axis are small (paraxial approximation). Show that

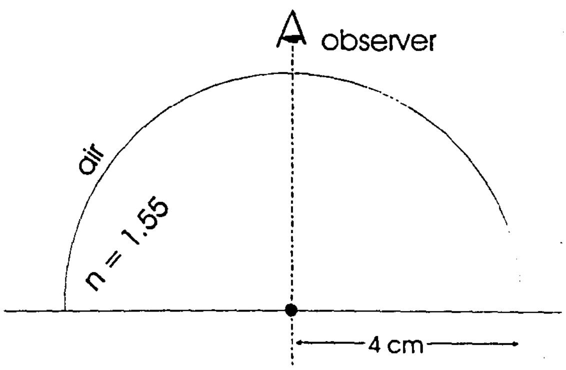
$$(n_1 / s) + (n_2 / s') = \frac{1}{R}(n_2 - n_1).$$

Explain any sign conventions you use.

B. Show that the magnification of the image relative to the object is: $M = -n_1 s'/n_2 s$.

C. Given the results of parts A and B, find the magnification and position of the image for the following situation. A glass hemisphere is used as a paperweight with its flat face resting on a stack of papers. The center of the hemisphere is directly over a letter "P" (as in Ph.D. Pass). The hemisphere has radius R = 4 cm, and is made of glass of refractive index n = 1.55. What does the letter "P" look like to the indicated observer? Draw a ray diagram consistent with your answer.

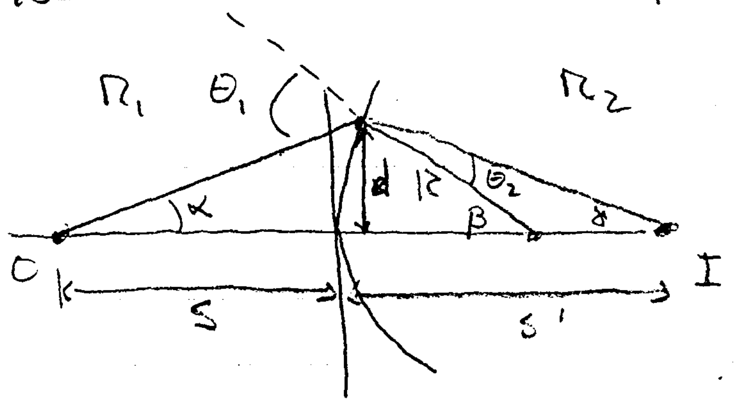




Comp 95
Prob #12

Solution: Spherical Lens (Smith)

a)



Relate s to s'. Use Snell's Law for refraction

$$n_1 \sin\theta_1 = n_2 \sin\theta_2$$

or for small angles $n_1 \theta_1 \approx n_2 \theta_2$

$$\tan\alpha \approx \alpha = d/s$$

$$\tan\gamma \approx \gamma = d/s'$$

$$\theta_1 = \alpha + \beta \qquad \text{exterior angles}$$

$$\beta = \gamma + \theta_2$$

So

$$n_1(\alpha + \beta) = n_2(\beta - \gamma)$$

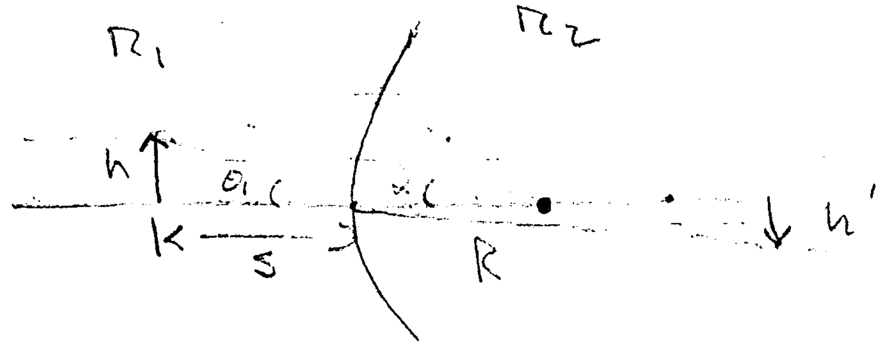
For small angles $\beta \approx \sin\beta = d/R$

So

$$n_1 \alpha + n_2 \gamma = (n_2 - n_1)\beta$$

$$\boxed{\frac{n_1}{s} + \frac{n_2}{s'} = \frac{(n_2 - n_1)}{R}}$$

(cancel out d



$$M = \frac{h'}{h} = \frac{(s'-R)\tan\alpha}{(s+R)\tan\alpha}$$ then use a α cancels $\tan\alpha = \frac{h}{s+R} = \frac{h'}{s'-R}$

or

$\tan\theta_1 = h/s$

$\tan\theta_2 = -h'/s'$ (note sense of h' is <0 if $h>0$ but $\theta_1, \theta_2 > 0$)

$$M = h'/h = -\frac{s'\tan\theta_2}{s\tan\theta_1}$$

$$\simeq -\frac{s'}{s}\frac{\sin\theta_2}{\sin\theta_1} = -\frac{s'}{s}\frac{n_1}{n_2}$$

c) $$\frac{n_1}{s} + \frac{n_2}{s'} = \frac{n_2 - n_1}{R}$$

$R<0$ now for concave interface

$$n_2/s' = \frac{n_2-n_1}{R} - \frac{n_1}{s}$$

$n_2 = 1$
$n_1 = 1.55$
~~So~~ $R = -4$ cm
$s = 4$ cm

$$\frac{1}{s'} = \frac{(1-1.55)}{-4\text{ cm}} - \frac{1.55}{4\text{ cm}}$$

$$1/s' = \frac{0.55 - 1.55}{4} = -1/4$$

$\boxed{s' = -4\text{ cm}}$

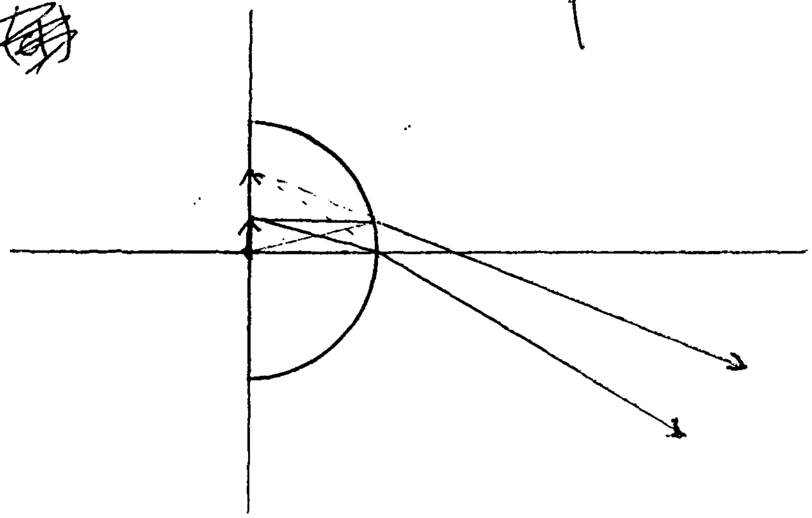
at interface also!

$$M = -\frac{s'}{s}\frac{n_1}{n_2} = -\frac{(-4\text{cm})}{(4\text{cm})}\frac{(1.55)}{1}$$

$$= \boxed{1.55}$$

#12

part c diagram



13. Consider a quantum mechanical system with total spin angular momentum $j = {}^3/_2$, and with no spatial degrees of freedom.

The system is in a state represented by the ket

$$|\psi\rangle = \frac{1}{2}\left|\frac{3}{2}\frac{3}{2}\right\rangle - \frac{1}{2}\left|\frac{3}{2}\frac{1}{2}\right\rangle + \frac{1}{2}\left|\frac{3}{2}-\frac{1}{2}\right\rangle - \frac{1}{2}\left|\frac{3}{2}-\frac{3}{2}\right\rangle.$$

Each of the kets $|jm\rangle$ on the righthand side are eigenstates of J^2 and J_z, with eigenvalues $j(j+1)\hbar^2$ and $m\hbar$, resp.

Find $\langle J_x\rangle$, $\langle J_y\rangle$, and $\langle J_z\rangle$ for the state $|\psi\rangle$.

QM #3 Hermanson

Consider a spin-$\frac{3}{2}$ system with no spatial degrees of freedom, in a quantum state represented by the ket

$$|\psi\rangle = \frac{1}{2}\left|\frac{3}{2}\,\frac{3}{2}\right\rangle - \frac{1}{2}\left|\frac{3}{2}\,\frac{1}{2}\right\rangle + \frac{1}{2}\left|\frac{3}{2}\,-\frac{1}{2}\right\rangle - \frac{1}{2}\left|\frac{3}{2}\,-\frac{3}{2}\right\rangle$$

where the kets $|j\,m\rangle$ are eigenstates of J^2 with eigenvalues $j(j+1)\hbar^2$ and of J_z with eigenvalues $m\hbar$.

Determine $\langle J_x\rangle$, $\langle J_y\rangle$ and $\langle J_z\rangle$ for this state.

QM#3 Solution J. Hermanson

Given $|\psi\rangle = \frac{1}{2}\left|\frac{3}{2}\,\frac{3}{2}\right\rangle - \frac{1}{2}\left|\frac{3}{2}\,\frac{1}{2}\right\rangle + \frac{1}{2}\left|\frac{3}{2}\,-\frac{1}{2}\right\rangle - \frac{1}{2}\left|\frac{3}{2}\,-\frac{3}{2}\right\rangle$, [normalized]

(a) Compute $\langle J_x\rangle$ and $\langle J_y\rangle$ from $\langle J_+\rangle$:

$$J_+ = J_x + iJ_y$$

real

$$\langle J_+\rangle = \langle J_x\rangle + i\langle J_y\rangle \quad \text{or} \quad \langle J_x\rangle = \mathrm{Re}\langle J_+\rangle$$
$$\langle J_y\rangle = \mathrm{Im}\langle J_+\rangle$$

Use the jm representation for J_+:

$$\begin{array}{cc} [J_+] = \hbar & \begin{pmatrix} 0 & \sqrt{3} & & \\ & 0 & 2 & \\ & & 0 & \sqrt{3} \\ & & & 0 \end{pmatrix} \\ m = & \quad \frac{3}{2} \quad \frac{1}{2} \quad -\frac{1}{2} \quad -\frac{3}{2} \end{array}$$

$$\left[\ \langle j\,m+1|\,J_+\,|j\,m\rangle = \hbar\sqrt{j(j+1)-m(m+1)}\ \right]$$

While, in the jm representation we also have

$$[\psi] = \frac{1}{2}\begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$$

Then
$$\langle J_+\rangle = \frac{\hbar}{4}\begin{pmatrix} 1 & -1 & 1 & -1 \end{pmatrix}\begin{pmatrix} 0 & \sqrt{3} & & \\ & 0 & 2 & \\ & & 0 & \sqrt{3} \\ & & & 0 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$$
$$= -\hbar\left(\frac{1+\sqrt{3}}{2}\right) = \text{real}$$

$$\boxed{\begin{aligned} \langle J_x\rangle &= \mathrm{Re}\langle J_+\rangle = -\frac{\hbar}{2}\left(1+\sqrt{3}\right) \\ \langle J_y\rangle &= \mathrm{Im}\langle J_+\rangle = 0 \end{aligned}}$$

#13

(b) Compute $\langle J_z \rangle$ in similar fashion:

$$[J_z] = \hbar \begin{pmatrix} \frac{3}{2} & & & \\ & \frac{1}{2} & & \\ & & -\frac{1}{2} & \\ & & & -\frac{3}{2} \end{pmatrix}$$

$$m = \quad \frac{3}{2} \quad \frac{1}{2} \quad -\frac{1}{2} \quad -\frac{3}{2}$$

$$\langle J_z \rangle = \frac{\hbar}{4} \begin{pmatrix} 1 & -1 & 1 & -1 \end{pmatrix} \begin{pmatrix} \frac{3}{2} & & & \\ & \frac{1}{2} & & \\ & & -\frac{1}{2} & \\ & & & -\frac{3}{2} \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$$

$$= \frac{\hbar}{4} \left(\frac{3}{2} + \frac{1}{2} - \frac{1}{2} - \frac{3}{2} \right)$$

$$\boxed{\langle J_z \rangle = 0}$$

Or, more simply, note that $|\psi\rangle$ is expanded in eigenstates of J_z, and use the coefficients squared as the probability of each J_z value:

$$\langle J_z \rangle = \frac{1}{4}\left(\frac{3}{2}\hbar\right) + \frac{1}{4}\left(\frac{1}{2}\hbar\right) + \frac{1}{4}\left(-\frac{1}{2}\hbar\right) + \frac{1}{4}\left(-\frac{3}{2}\hbar\right)$$

$$= 0 \quad \checkmark$$

14. Following are two problems in wave propagation for cylindrical geometries. Parts A and B are unrelated.

A. Outside an infinitely long cylinder of radius a, a potential function u(**r**,t) satisfies the wave equation $\nabla^2 u = \frac{1}{c^2}(\partial^2 u / \partial t^2)$. The cylinder is split along its length, and on its surface: $u = \begin{cases} +u_o \exp(-i\omega_o t), \text{ for } 0 < \phi < \pi; \\ -u_o \exp(-i\omega_o t), \text{ for } \pi < \phi < 2\pi; \end{cases}$ here u_o and ω_o are positive constants. Find u(**r**,t) everywhere outside the cylinder under the assumption that only outgoing waves are present at large distances.

B. A quantity u satisfies the wave equation (as quoted in Part A above) inside a long cylindrical pipe of radius a, with u = 0 on the walls of the pipe. Let the axis of the pipe be the z-axis, and suppose at z = 0 there is maintained an excitation of the form: $u = u_o \exp(-i\omega_o t)$, where u_o and ω_o are positive constants. Waves will travel along the pipe in various modes -- i.e. various transverse spatial distributions. Find the phase velocity of the fundamental mode as a function of the frequency ω_o, and interpret your result for small ω_o.

Problem #14
Math Phys.

Comp 1995
Answers

Part (a)

∘ In cylindrical coordinates an outgoing wave condition of the form $u = \sum_m C_m H_m^{(1)}(K\rho) \sin m\varphi \, e^{-i\omega_0 t}$ ①
where $K = \omega_0 / c$ ②, and we have chosen
$H_m^{(1)} = J_m + i Y_m$ ③ to ensure outgoing waves.

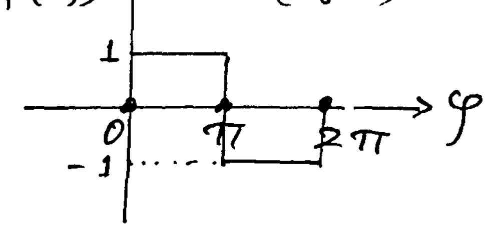
∘ The boundary condition at $\rho = a$ requires us to expand a square-wave in φ. (See Fig. 1)

∘ $f(\varphi) = \sum A_n \sin(n\varphi)$ ④, where

$$A_n = \frac{1}{\pi}\int_0^{2\pi} f(\varphi)\sin(n\varphi)\,d\varphi$$

$$= \frac{2}{\pi}\int_0^{\pi} \sin(n\varphi)\,d\varphi = \frac{2}{n\pi}\left[-\cos(n\varphi)\right]_0^{\pi}$$

(Fig. 1)



$\to \therefore A_n = \dfrac{4}{n\pi}$, } ⑤ where n odd; $A_n = 0$, n even

∘ Hence, matching at $\rho = a$, from ①, ④ & ⑤, →

$$u(a) = f(\varphi)\, u_0 \, e^{-i\omega_0 t}$$
$$= \sum A_m \sin(n\varphi)\, u_0 \, e^{-i\omega_0 t}$$
$$= \sum_{n \text{ odd}} \frac{4}{n\pi} \sin(n\varphi)\, u_0 \, e^{-i\omega_0 t}$$
$$= \sum_m C_m H_m^{(1)}(Ka) \sin(m\varphi)\, e^{-i\omega_0 t}$$

$\to \therefore C_m H_m^{(1)}(Ka) = \dfrac{4u_0}{m\pi} \Rightarrow C_m = \dfrac{4u_0}{m\pi H_m^{(1)}(Ka)}$ ⑥

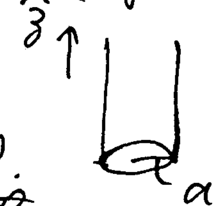
$\to \therefore$ ⑥ ⇒ ① & get $u(x,t) = \dfrac{4u_0}{\pi}\displaystyle\sum_m \left\{\frac{H_m^{(1)}(K\rho)}{m H_m^{(1)}(Ka)}\right\} \sin m\varphi \, e^{-i\omega_0 t}$

⑦ // ans.

Part (b)

∘ Cylindrical coordinates. $z = 0$, $u = u_0 e^{-i\omega_0 t}$.
$u = 0$ when $\rho = a$.

(Fig. 2)



Due to the cylindrical form, soln for the spacial part will become the following form:

$\nabla^2 u + k_0^2 u = 0$ ⑧, with $k_0 = \omega_0 / c$ ⑨.

∘ ⑧ is the Helmholtz eqn in cylindrical coordinates.
So, the general soln is of the form

$u = \sum_m A_m J_m\left(\sqrt{k_0^2 - \alpha^2}\,\rho\right) e^{i\alpha z - i\omega_0 t}$ ⑩

If the boundary condition at $\rho = a$ is to be obeyed, MP Ans. (contd.)

Part (b) (contd.)

we must have:

$J_m(x) = 0$, at $\rho = a$, when $x = \sqrt{k_0^2 - \alpha^2}\,\rho$.

So $x = \sqrt{k_0^2 - \alpha^2}\,\rho = \sqrt{k_0^2 - \alpha^2}\,a \to J_m(x) = 0$

• For the lowest mode, $\sqrt{k_0^2 - \alpha^2}\,a = 2.4$ (11)

$\rightarrow \quad \alpha^2 = k_0^2 - (2.4/a)^2 = \dfrac{\omega_0^2}{c^2} - \left(\dfrac{2.4}{a}\right)^2.$ (12)

The phase velocity down the pipe is then, from (12)

$$V_{ph} = \frac{dz}{dt} = \frac{\omega_0}{\alpha} = \frac{\omega_0 c}{\sqrt{\omega_0^2 - 2.4^2c^2/a^2}}$$

$$\rightarrow \quad V_{ph} = \frac{c}{\sqrt{1 - \left(\dfrac{2.4^2c^2}{a^2\omega_0^2}\right)}} \quad // \text{ Ans.}$$

(c) For small ω_0, the expression for α becomes imaginary. This means the wave is damped, and will not propagate for ω_0 less than $\omega_c = \dfrac{2.4c}{a}$ // ans.

(d) The cut-off frequency $= \omega_c$. Near $\omega = \omega_c$ the phase velocity becomes infinite.

15. In a particular quantum mechanical system, let Q be an operator representing some dynamical quantity, and suppose that Q does <u>not</u> depend explicitly on time. Consider the energy eigenstates of the system, described by wavefunctions Ψ_n that satisfy Schrödinger's eigenvalue equation: $H\Psi_n = E_n \Psi_n$, with H the system Hamiltonian and E_n the (real) eigenenergy of the $n^{\underline{th}}$ state.

A. In the state Ψ_n, show that Q is a constant of the motion, in an expectation value sense.

B. Find the eigenfunctions of Q, i.e. those functions ϕ_n such that: $Q\phi_n = q_n \phi_n$, where the q_n=cnsts.

$\boxed{\phi \text{ Comp. Exam : June '95}}$ (Robiscoe) 5/28/95

15 ✓#15

QM -- Operator Formalism

"In a quantum-mechanical system, the dynamical quantity Q does not depend explicitly on time. Q may or may not be an operator. Consider the nth stationary state of the system, as described by a wavefunction ψ_n that satisfies Schrödinger's eigenvalue equation: $H\psi_n = E_n\psi_n$, with H the system Hamiltonian and E_n the real eigen-energy of the nth state.

A. Show that Q is a constant of the motion, in an expectation value ~~sense~~.

B. What are the eigenfunctions of Q? Explain your choice."

Solution

A. By the QM equation-of-motion, w/ $\langle\ \rangle$ denotes an expⁿ value:

$$\rightarrow \frac{d}{dt}\langle Q\rangle = \frac{i}{\hbar}\langle [H,Q]\rangle + \cancelto{0}{\langle \partial Q/\partial t\rangle}. \tag{1}$$

$\partial Q/\partial t \equiv 0$ by hypothesis, since Q does not depend explicitly on time. $\langle Q\rangle$ will be a cnst of the motion, i.e. $\frac{d}{dt}\langle Q\rangle = 0$, only if Q commutes with the Hamiltonian, i.e. $[H,Q]=0$. In particular, we must show $\langle [H,Q]\rangle_n = 0$ for the nth stationary state.

In the nth state, the expectation value of the commutator is:

$$\rightarrow \langle [H,Q]\rangle_n = \langle n|HQ - QH|n\rangle$$

$$= \langle n|H(Q|n\rangle - \langle n|Q\,\overbrace{(H|n\rangle}^{=E_n|n\rangle}. \tag{2}$$

In the 2nd integral RHS, the cnst E_n comes out, i.e. $\langle n|Q(H|n\rangle = E_n\langle n|Q|n\rangle$. For the 1st integral RHS in (2), use the fact (OVER) that E_n real $\leftrightarrow$ H Hermitian (self-adjoint), so that

$$\rightarrow \langle n|H(Q|n\rangle = \langle Hn|Q|n\rangle = \langle E_n n|Q|n\rangle = E_n\langle n|Q|n\rangle. \quad (3)$$

The (real) cnst E_n can be removed from the integral once again. Now Eqs. (1)-(3) yield...

$$\rightarrow \frac{d}{dt}\langle Q\rangle_n = \frac{i}{\hbar}\langle[H,Q]\rangle_n = \frac{i}{\hbar}\{\underbrace{E_n\langle n|Q|n\rangle - E_n\langle n|Q|n\rangle}_{\text{cancel}}\}$$

i.e.// $\frac{d}{dt}\langle Q\rangle_n \equiv 0$, so// $\underline{\underline{\langle Q\rangle_n = \langle n|Q|n\rangle = \text{Cnst of motion.}}}$ (4)

This holds for all eigenstates $\psi_n = |n\rangle$ of the system.

B. When H & Q commute, they share the same eigenfcns, as we can see by noting: $\langle[H,Q]\rangle_n = 0$ implies...

$$\rightarrow H(Q|n\rangle) = Q(H|n\rangle) = Q(E_n|n\rangle) = E_n(Q|n\rangle), \quad (5)$$

...so $Q|n\rangle$ is an eigenfcn of H with eigenenergy E_n. Thus, the operation of Q on $|n\rangle$ does not change the eigenstate; at most:

$$\rightarrow Q|n\rangle = q_n|n\rangle, \quad q_n = \text{some \# characteristic of state } |n\rangle. \quad (6)$$

The $\{q_n\}$ are evidently the eigenvalues of Q, and--as advertised--the states $|n\rangle$ are simultaneously eigenfcns of both H and Q.