Covariant nature of Continuum Lagrange Eqtns.

Lagrangian Density for Fields & Sources [Jackson Sec. 12.8].

14) We remark that the continuum Lagrange Eqtns [Eq. (14), p. L&H 12] resemble a 4-divergence. They can be written in the form:

$$\rightarrow \frac{\partial}{\partial x^\mu}\left(\partial\mathcal{L}/\partial\xi_{x^\mu}\right) = \frac{\partial\mathcal{L}}{\partial\xi} \quad \Bigg\| \begin{array}{l} \text{sum on } \mu=0,1,2,3, \text{ w/ } \underline{\underline{x^0}}=ct, \\ \text{and: } \xi_{x^\mu} = \partial\xi/\partial x^\mu. \end{array} \tag{15}$$

... let : $\underline{\underline{\partial^\mu}} = \left(\frac{\partial}{\partial x^0}, (-)\nabla\right)$, contravariant del [Jk^n Eq. (11.76)]...

... define : $\underline{\underline{\mathcal{G}^\mu}} = \partial\mathcal{L}/\partial\xi_{x^\mu} = \partial\mathcal{L}/\partial(\partial^\mu\xi)$

So// Eq (15) is : $\boxed{\partial^\mu \mathcal{G}_\mu = \partial\mathcal{L}/\partial\xi}$, for one degree-of-freedom, ξ. (16)

From this, we can see that the continuum eqtns-of-motion are manifestly covariant if $\mathcal{G}_\mu$ is a 4-vector, since then $\partial^\mu\mathcal{G}_\mu$ is a 4-divergence -- which is automatically a Lorentz scalar (see class notes, p. SRT 24). All this is true if both the Lagrange density $\mathcal{L}$ and the generalized continuum cd. ξ are Lorentz scalars (for one degree-of-freedom).¶

But $\mathcal{L}$ = Lorentz scalar is just what we want to make the action invariant:

$$\left[\left[A = \int_{t_1}^{t_2} \iiint_{\text{space}} \mathcal{L}\, \underbrace{dx\,dy\,dz\,dt}_{\text{4-volume element}} \right.\right. \quad \begin{array}{l} dx\,dy\,dz\,dt \rightarrow dx'dy'dz'dt', \text{ for motion} \parallel z \\ = dx\,dy\left(\frac{dz}{\gamma}\right)(\gamma dt) \equiv dx\,dy\,dz\,dt \end{array} \tag{17}$$

Since the 4-volume element is Lorentz invariant, then A is an invariant Lorentz scalar if $\mathcal{L}$ is.

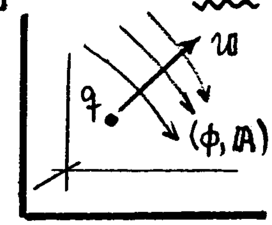
We use these facts to guide us in constructing a density $\mathcal{L}$ for particles plus fields: $\mathcal{L}$ should be a Lorentz scalar, and the ξ's $\rightarrow \xi^{(\nu)}$ 4-vectors.¶

¶ For more than one degree-of-freedom, e.g. for freedom in both t and **r**, the single cd ξ should $\rightarrow \xi^{(\nu)}$, a 4-vector. Then $\mathcal{G}_\mu \rightarrow \mathcal{G}_\mu^\nu$, a mixed 4-tensor, etc.

Lagrange Density $\mathcal{L}_{int}$ for Fields & Sources.

15) For a system of particles + fields: $\mathcal{L}_{total} = \mathcal{L}_{particles} + \mathcal{L}_{fields} + \mathcal{L}_{interaction}$. $\mathcal{L}_{particles}$ represents the motion of the particles in the absence of everything; $\mathcal{L}_{fields}$ likewise describes the field "motions" w/o anything else present; $\mathcal{L}_{interaction}$ covers the coupling of particles to fields -- e.g. how the sources ρ & $\mathbf{J}$ generate the fields. If we are not interested in the motions of the particles (i.e. we consider ρ & $\mathbf{J}$ as given), then we can just drop $\mathcal{L}_{particles}$ and focus on constructing:

$$\left[\!\left[\mathcal{L}_{EM} = \mathcal{L}_{fields} + \mathcal{L}_{int.} \longleftrightarrow \text{should give Maxwell Equations.}\right]\!\right] \tag{18}$$



$\mathcal{L}_{int.}$ is not hard to get. We already know that for a single q in potentials $(\phi, \mathbf{A})$, the particle-like L contains [see Eq.(9), p. L&H 3]:

$$\rightarrow L_{int.} = -q\phi + \frac{1}{c}q\,\mathbf{u}\cdot\mathbf{A}, \text{ w/ } \gamma L_{int} \text{ a Lorentz scalar.} \tag{19}$$

Pass to continuous limit of n = large # of q's per unit 3-volume, such that ¶

$$(qc, q\mathbf{u}) \rightarrow \gamma(nqc, nq\mathbf{u}) = (\rho c, \mathbf{J}), \text{ 4-current;}$$

$$\text{so/ } L_{int}(\text{single } q) \rightarrow \boxed{\mathcal{L}_{int.}(\rho \,\&\, \mathbf{J}) = -\rho\phi + \frac{1}{c}\mathbf{J}\cdot\mathbf{A} = -\frac{1}{c}J_\alpha A^\alpha} \tag{20}$$

$\mathcal{L}_{int}$ "derived" in this way is a manifest Lorentz scalar, since both the current J_α and potential A^α are qualified 4-vectors, so $J_\alpha A^\alpha$ is an invariant.

Note that $\mathcal{L}_{int}$ has units of an energy/unit volume, so $\mathcal{L}_{fields}$ in Eq.(18) must likewise be an energy/vol. We are thus looking for an $\mathcal{L}_{fields}$ which is quadratic in $\mathbf{E}$ and/or $\mathbf{B}$, and is at the same time a Lorentz scalar.

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16) Poynting's field energy density: $u = \frac{1}{8\pi}(E^2 + B^2)$, is quadratic in the fields, but it transforms as the time-like component of a 4-vector*, so it does not

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* field 4-momentum is: $c P^\mu_{fld} = \int T^{\mu\nu} d\sigma_\nu = \left(\int u\, d^3x, \frac{1}{c}\int \mathbf{S}\, d^3x\right)$ $\int$ $T^{\mu\nu}$ = stress tensor, $d\sigma_\nu$ = 4D hypersurface elt.

¶ The γ appears because the volume element contracts: $dV_0 \binom{q\text{'s}}{\text{frame}} \rightarrow \frac{1}{\gamma} dV \binom{\text{observer's}}{\text{frame}}$.

qualify for $\mathcal{L}_{fields}$. We need some other quadratic form, e.g. some Lorentz scalar associated with the Maxwell field tensor $F^{\alpha\beta}$. One such is:

$$F_{\alpha\beta}F^{\alpha\beta} = \mathrm{Tr}\left[(g_{\alpha\lambda}F^{\lambda\epsilon}g_{\epsilon\beta})F^{\alpha\beta}\right] = (-)\mathrm{Tr}\left[(g_{\alpha\lambda}F^{\lambda\epsilon})(g_{\epsilon\beta}F^{\beta\alpha})\right]$$

$$= -2(E^2 - B^2) \leftarrow \text{a Lorentz scalar}^{\star} \ (\text{since Tr} = \text{inv. under } \Lambda). \qquad (21)$$

This invariant is also a true scalar, since it does not change sign under the parity operation. Another invariant is (from Prob. 70): $F_{\alpha\beta}\mathcal{F}^{\alpha\beta} = 4\,\mathbf{E}\cdot\mathbf{B}$, but this quantity is a pseudoscalar, which does change sign under parity (when $\mathbf{E}\rightarrow(-)\mathbf{E}$ and $\mathbf{B}\rightarrow(+)\mathbf{B}$). On these grounds, we discard $\mathbf{E}\cdot\mathbf{B}$ for $\mathcal{L}_{fields}$.

Any multiple of of $F_{\alpha\beta}F^{\alpha\beta}$ can be used for $\mathcal{L}_{fields}$. The clever choice is

$$\boxed{\mathcal{L}_{fields} = -\frac{1}{16\pi}F_{\alpha\beta}F^{\alpha\beta} = +\frac{1}{8\pi}(E^2 - B^2)}, \qquad (22)$$

because this choice ultimately leads to the usual form of Maxwell's Equations.

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17) Combine (22) & (20) to find $\mathcal{L}_{EM}$ for the fields $\mathbf{E}$ & $\mathbf{B}$ plus sources ρ & $\mathbf{J}$:

$$\mathcal{L}_{EM} = \mathcal{L}_{fields} + \mathcal{L}_{int.} = (-)\frac{1}{16\pi}F_{\alpha\beta}F^{\alpha\beta} - \frac{1}{c}J_\alpha A^\alpha \leftarrow \text{Jk}^{\text{n}} \text{ Eq}(12.85)$$

or

$$\boxed{\mathcal{L}_{EM} = \frac{1}{8\pi}(E^2 - B^2) - \rho\phi + \frac{1}{c}\mathbf{J}\cdot\mathbf{A}}. \qquad (23)$$

To go from here to the Lagrange eqtns-of-motion, we must specify the "generalized coordinates" for $\mathcal{L}_{EM}$. Those coordinates cannot be the 6 field components of $\mathbf{E}$ & $\mathbf{B}$, since they are not mutually independent. In fact, the fields have

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*That $(E^2 - B^2)$ is a Lorentz invariant can be seen from the field transf$^{\text{n}}$ eqtns directly, viz:

$$\text{Jk}^{\text{n}} \text{ Eqs. (11-148)} \left\{\begin{array}{l|l} E'_z = E_z & B'_z = B_z \\ E'_x = \gamma(E_x - \beta B_y) & B'_x = \gamma(B_x + \beta E_y) \\ E'_y = \gamma(E_y + \beta B_x) & B'_y = \gamma(B_y - \beta E_x) \end{array}\right\} \Rightarrow \mathbf{E}'^2 - \mathbf{B}'^2 = \mathbf{E}^2 - \mathbf{B}^2 \ (\text{invariant})$$

by direct substitution & calculation.

just four degrees-of-freedom, as represented by the potential $A^\alpha = (\phi, \mathbb{A})$

and// $\longrightarrow \mathbb{E} = -\nabla\phi - \frac{1}{c}(\partial\mathbb{A}/\partial t)$, $\mathbb{B} = \nabla \times \mathbb{A}$. (24)

So we match the # degrees-of-freedom of the Maxwell field by assigning the 4-vector components $(\phi, \mathbb{A})$ to the continuum coordinates $\xi^{(\nu)}$ in the Lagrange eqtn-of-motion, viz.: $\partial^\mu[\partial\mathcal{L}/\partial(\partial^\mu\xi^{(\nu)})] = \partial\mathcal{L}/\partial\xi^{(\nu)}$; this is Eq. (16), p. I&H 13 for independent coordinates $\xi^{(\nu)}$, $\nu = 0,1,2,3$. Then, also, the fact that $\xi^{(\nu)} = A^\nu$ is a 4-vector means that the $\mathcal{L}_{EM}$ eqtns-of-motion will be Lorentz covariant (footnote, p. I&H 13). If, in fact, the $\mathcal{L}_{EM}$ eqtns-of-motion are the Maxwell Equations, we know (and require) already that they are Lorentz covariant. A double check!

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18) Now, with $\xi^{(\nu)} = (\phi, \mathbb{A})$, we want to show $\mathcal{L}_{EM}$ of Eq. (23) gives the "right" eqtns-of-motion, namely the Maxwell Equations for $\mathbb{E}$ & $\mathbb{B}$. Our system is:

$$\left[\!\left[\mathcal{L}_{EM} = \frac{1}{8\pi}(E^2 - B^2) - \rho\phi + \frac{1}{c}\mathbb{J}\cdot\mathbb{A}, \ \text{w/}\ \frac{\partial}{\partial t}\left(\frac{\partial\mathcal{L}_{EM}}{\partial\xi_t^{(\nu)}}\right) + \underbrace{\frac{\partial}{\partial x_k}}_{\text{sum on } k}\left(\frac{\partial\mathcal{L}_{EM}}{\partial\xi_{x_k}^{(\nu)}}\right) = \frac{\partial\mathcal{L}_{EM}}{\partial\xi^{(\nu)}}.\right]\!\right] \quad (25)$$

for $\nu = 0$ cd., i.e. $\xi^{(0)} = \phi$. Have: $\partial\mathcal{L}_{EM}/\partial\phi = -\rho$, $\partial\mathcal{L}_{EM}/\partial\phi_t = 0$, so...

$$-\rho = \frac{\partial}{\partial x_k}\left(\frac{\partial\mathcal{L}_{EM}}{\partial\phi_{x_k}}\right) = -\frac{\partial}{\partial x_k}\left(\frac{\partial\mathcal{L}_{EM}}{\partial E_k}\right) = -\frac{1}{4\pi}\underbrace{\frac{\partial}{\partial x_k}E_k}_{\text{div}\,\mathbb{E}}, \ \text{w/}\ \underbrace{\boxed{\nabla\cdot\mathbb{E} = 4\pi\rho}}_{\text{Gauss' Law}} \quad (26)$$

use $\phi_{x_k} = \partial\phi/\partial x_k = -E_k$

for $\nu = 1$ cd., i.e. $\xi^{(1)} = A_1$. Have: $\frac{\partial\mathcal{L}_{EM}}{\partial A_1} = \frac{1}{c}J_1$, $\frac{\partial\mathcal{L}_{EM}}{\partial A_{1t}} = -\frac{\partial\mathcal{L}_{EM}}{c\,\partial E_1} = -\frac{E_1}{4\pi c}$, so:

$$\frac{1}{c}J_1 = -\frac{\partial}{\partial t}\left(\frac{E_1}{4\pi c}\right) + \frac{\partial}{\partial x_k}\left(\frac{\partial\mathcal{L}_{EM}}{\partial(\partial A_1/\partial x_k)}\right) \longleftarrow \text{use: } \mathbb{B} = \left(\frac{\partial A_3}{\partial x_2} - \frac{\partial A_2}{\partial x_3}, \frac{\partial A_1}{\partial x_3} - \frac{\partial A_3}{\partial x_1}, \frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2}\right)$$

$$\text{So}\ \frac{1}{c}J_1 + \frac{1}{4\pi c}\left(\frac{\partial E_1}{\partial t}\right) = \frac{\partial}{\partial x_2}\left(\frac{\partial\mathcal{L}_{EM}}{\partial(\partial A_1/\partial x_2)}\right) + \frac{\partial}{\partial x_3}\left(\frac{\partial\mathcal{L}_{EM}}{\partial(\partial A_1/\partial x_3)}\right) = \frac{1}{4\pi}\left(\frac{\partial B_3}{\partial x_2} - \frac{\partial B_2}{\partial x_3}\right)$$

$$= -\frac{\partial\mathcal{L}_{EM}}{\partial B_3} = +\frac{B_3}{4\pi} \qquad\qquad = +\frac{\partial\mathcal{L}_{EM}}{\partial B_2} = -\frac{B_2}{4\pi}$$

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