7) EVALUATION of INFINITE SERIES.

While the tests just described may tell you that a series converges, they do not specify a value for the series. To (numerically) evaluate a convergent series, we rely mainly on values of already known series -- most often Taylor series. E.g..

Ex. Know: $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots = \sum_{n=0}^{\infty} x^n/n!$, as Taylor series for e^x.

So// $S = \sum_{n=0}^{\infty} (1/n!) = (e^x)|_{x=1} = e = 2.71828\ldots$

$$\tilde{S} = \sum_{n=0}^{\infty} (-)^n/n! = (e^x)|_{x=-1} = e^{-1} = 0.36788\ldots \qquad (22)$$

Related series can be manufactured from a known series by multiplying by a factor and differentiating. For example...

Ex. Consider: $F_k(x) = x^k e^x = \sum_{n=0}^{\infty} x^{n+k}/n!$, $k = \text{cnst} \geq 0$.

Differentiate: $\frac{d}{dx} F_k(x) = \sum_{n=0}^{\infty} \frac{n+k}{n!} x^{n+k-1}$.

$$\text{Evaluate @ } x=1: \sum_{n=0}^{\infty} (n+k)/n! = \frac{d}{dx}(x^k e^x)\Big|_{x=1} = (1+k)e. \qquad (23)$$

The major challenge here is to be familiar with enough "standard" Taylor series so as to recognize the Taylor series structure present in the desired $S = \sum_n a_n$. Some "standard" series are listed in M & W, p. 48. Some results are listed in Gradshteyn & Ryzhik, pp. 1-11.

Sometimes you can convert: numerical series → power series → tabulated integral. No kidding!

Ex. Evaluate: $S_k = \sum_{n=1}^{\infty} 1/n(n+k)$, $k = \text{cnst} > (-)1$.

$$1. \text{ Let: } f(x) = \sum_{n=1}^{\infty} x^n/n(n+k). \text{ We want } f(1) = S_k. \qquad (24)$$

Except for the factor $(n+k)$ in the denominator, the series for $f(x)$ is the same as: $(-)\ln(1-x) = \sum_{n=1}^{\infty} x^n/n$. So (per Eq.(23)), multiply by x^k and differentiate...

$$x^k f(x) = \sum_{n=1}^{\infty} x^{n+k}/n(n+k) \leftarrow \text{note: } x^k f(x)\big|_{x=0} = 0. \tag{25}$$

so// $\frac{d}{dx}[x^k f(x)] = \sum_{n=1}^{\infty} \frac{1}{n} x^{n+k-1} = x^{k-1} \sum_{n=1}^{\infty} \frac{x^n}{n} = (-) x^{k-1} \ln(1-x)$,

and//

$$\rightarrow x^k f(x) = (-) \int x^{k-1} \ln(1-x)\, dx + \text{cnst}. \tag{26}$$

2. Fix the constant in Eq. (26) by noting [per Eq. (25)]: $x^k f(x)\big|_{x=0} = 0$. Then...

$$\rightarrow x^k f(x) = (-) \int_0^x y^{k-1} \ln(1-y)\, dy = (-) \int_{1-x}^{1} (1-u)^{k-1} \ln u\, du. \tag{27}$$

Now evaluate (27) @ $x=1$ to get $S = f(1)$ as an integral...

$$\boxed{S_k = \sum_{n=1}^{\infty} 1/n(n+k) = (-) \int_0^1 (1-u)^{k-1} \ln u\, du}. \tag{28}$$

In fact, this integral is tabulated -- see Gradshteyn & Ryzhik # (4.253.1), p. 538. The result -- in terms of a known fcn Ψ [Euler's Psi Fcn] -- is:

$$\sum_{n=1}^{\infty} 1/n(n+k) = \frac{1}{k}\left\{\frac{1}{k} + [\Psi(k) - \Psi(1)]\right\}, \text{ for } \operatorname{Re} k > 0. \tag{29}$$

Numerical values of $\Psi(k)$ are tabulated -- e.g. see Abramovitz & Stegun, Ch. 6. Eq. (29) $\Rightarrow$ that for $k=1$: $\sum_{n=1}^{\infty} 1/n(n+1) = 1$. You should be able to verify this by evaluating $S_1 = (-)\int_0^1 \ln u\, du$ from Eq. (28).

8) GAUSS' SUMMATION FORMULA.

Sometimes a difficult series can be summed if it can be transformed (somehow) to a simpler form. One way of doing this is to exploit the mapping between a given fcn $f(x)$ and its Fourier transform $F(k)$, thereby transforming a series of $f(x=n)$ values [n = integer] into a series of $F(k=\nu)$ values [ν = integer].

1. Suppose we want: $S(\alpha) = \sum_{n=0}^{\infty} f(\alpha n)$, w/ $f = f(x)$ a given fcn, α = cnst, $\{n\}$ = integers. (30)

2. Assume $f(x)$ has a Fourier transform $F(k)$ and the usual inverse...

$$\rightarrow F(k) = \int_{-\infty}^{\infty} f(x) e^{ikx}\, dx, \quad f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(k) e^{-ikx}\, dk. \tag{31}$$

3. Now the idea is to plug the Fourier integral for $f(x)$ into $S(\alpha)=\sum_{n=0}^{\infty} f(\alpha n)$...

So// $\rightarrow S(\alpha)=\sum_{n=0}^{\infty} f(\alpha n)=\frac{1}{2\pi}\int_{-\infty}^{\infty} F(k)\left\{\sum_{n=0}^{\infty} e^{-ik\alpha n}\right\} dk,$ (32)

... evaluate the sum over $e^{-ik\alpha n}$ (a geometric series), and do the k-integral. The result is a sum over F-values, which we quote here w/o proof, viz...

$$\boxed{S(\alpha)=\sum_{n=0}^{\infty} f(\alpha n)=\frac{1}{\alpha}\sum_{\nu=-\infty}^{\nu=+\infty} F\left(\frac{2\pi}{\alpha}\nu\right)}. \quad (33)$$

We will do the proof later--it involves contour integration. This result does not sum S; it just transforms S into a new and (hopefully) simpler form.

Ex. Evaluate: $S=\sum_{n=0}^{\infty} 1/(1+n^2)$.

Here: $f(x)=1/(1+x^2)$ & $\alpha=1$, so: $S=\sum_{n=0}^{\infty} f(n)$. The Fourier transform is...

$$F(k)=\int_{-\infty}^{\infty}\frac{e^{ikx}}{1+x^2}dx=2\int_0^{\infty}\frac{\cos kx}{1+x^2}dx=\pi e^{-|k|}, \text{ from tables;}$$

so// $S=\frac{1}{\alpha}\sum_{\nu=-\infty}^{\nu=+\infty} F\left(\frac{2\pi}{\alpha}\nu\right)=\pi\sum_{\nu=-\infty}^{\nu=+\infty} e^{-2\pi|\nu|}=\pi\left(1+2\sum_{\nu=1}^{\infty} e^{-2\pi\nu}\right)$

This simple geometric series sums to: $\sum_{\nu=1}^{\infty} e^{-2\pi\nu}=e^{-2\pi}/(1-e^{-2\pi})$. A bit of arithmetic then gives S as a hyperbolic fcn...

$$S=\sum_{n=0}^{\infty} 1/(1+n^2)=\pi\,\mathrm{cth}\,\pi=3.15335\ldots \quad (34)$$

9) UNIFORM CONVERGENCE.

In dealing with functional series (e.g. Taylor series or power series), where the terms depend on some variable x as well as the summand n, e.g. ...

$$\rightarrow S(x)=\sum_{n=1}^{\infty} u_n(x)=\lim_{N\to\infty} S_N(x), \text{ w/ } S_N(x)=\sum_{n=1}^{N} u_n(x), \quad (35)$$

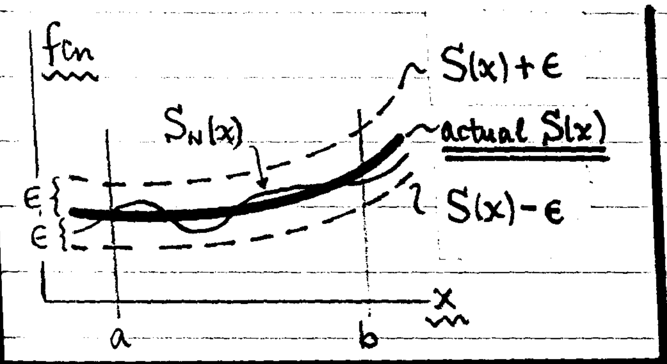
a new question arises -- i.e. how do the partial sums $S_N(x)$ depend on N and x? $S(x)$ may converge for some values of x, and not others. A more general definition of convergence than in Eq. (2) is needed; it is provided by...

Def$^{\underline{n}}$ of Uniform Convergence. M-Test & Abel's Test.

Uniform Convergence

"If, for any $\epsilon > 0$, there exists a number $\mathcal{N}$, independent of x in $a \leq x \leq b$, such that: $|S(x) - S_N(x)| < \epsilon$ for all $N \geq \mathcal{N}$, then the series $S(x) = \sum_{n=1}^{\infty} u_n(x)$ is said to be "uniformly convergent" in $a \leq x \leq b$." (36)

Pictorially, uniform convergence means...



$S_N(x)$ is always bounded on the interval, i.e. $S(x) - \epsilon \leq S_N(x) \leq S(x) + \epsilon$, where ϵ is arbitrarily small (so long as N is big enough). Evidently, the bounding cannot work if $S(x)$ is discontinuous (or ∞) in $a \leq x \leq b$.

Uniformly convergent series have 3 particularly useful properties...

(a) If each fcn $u_n(x)$ is a continuous fcn of x, then so is $S(x) = \sum_n u_n(x)$.

(b) The series may be integrated term-by-term: $\int_a^b S(x)\,dx = \sum_n \int_a^b u_n(x)\,dx$.

(c) The series is differentiable term-by-term: $\frac{d}{dx} S(x) = \sum_n \frac{d}{dx} u_n(x)$, provided:

1. $u_n(x)$ and $\frac{d}{dx} u_n(x)$ are both continuous in $a \leq x \leq b$;
2. $\sum_n \frac{d}{dx} u_n(x)$ is also uniformly convergent in $a \leq x \leq b$. (37)

We quote these properties w/o proof. Note how restrictive (c) is -- uniform convergence is delicate enough so that differentiation may destroy it.

10) WEIERSTRASS M-TEST; ABEL'S TEST.

There are two tests commonly used to establish uniform convergence. First is...

Weierstrass M-Test

"If $\sum_{n=1}^{\infty} M_n$ is a convergent series of positive numbers M_n, such that $M_n \geq |u_n(x)|$ for all x in $a \leq x \leq b$, then $\sum_{n=1}^{\infty} u_n(x)$ is uniformly convergent in that interval." (38)

This is a sort of comparison test (see Eq. (10)). The proof is ~ simple...

Proof of M-Test. Abel's Test. Application to Power Series.

proof

Since $\sum_n M_n$ converges to some number M, then $\lim_{N\to\infty}(M - \sum_{n=1}^{N} M_n) = 0$. This means that for any $\epsilon > 0$, there is some N such that: $\sum_{n=N+1}^{\infty} M_n < \epsilon$. Then, since by assumption $|u_n(x)| \leq M_n$ for all x in $[a,b]$, we have also: $\sum_{n=N+1}^{\infty} |u_n(x)| < \epsilon$. Thus:

$$|S(x) - S_N(x)| = \left|\sum_{n=N+1}^{\infty} u_n(x)\right| \leq \sum_{n=N+1}^{\infty} |u_n(x)| < \epsilon, \tag{39}$$

and (by definition) $S(x)$ is uniformly convergent in $a \leq x \leq b$.

REMARK Uniform convergence vs. absolute convergence.

Since absolute values $|u_n(x)|$ are used in the M-Test, then $\sum_n u_n(x)$ is absolutely convergent [see Eq(9)] and uniformly convergent in $a \leq x \leq b$. It is well to note that these two types of convergence are normally independent. E.g.

$$\sum_{n=1}^{\infty} (-)^{n+1} \frac{x^n}{n} = \ln(1+x), \text{ in } 0 \leq x \leq 1, \text{ converges } \begin{cases} \text{uniformly, but} \\ \text{not absolutely.} \end{cases}$$

$$\sum_{n=0}^{\infty} (1-x)x^n = \begin{cases} 1, & 0 \leq x < 1; \\ 0, & x = 1; \end{cases} \text{ in } 0 \leq x \leq 1, \text{ converges } \begin{cases} \text{absolutely, but not} \\ \text{uniformly (because of discontinuity @ } x=1\text{).} \end{cases} \tag{40}$$

The second common test for uniform convergence is Abel's (state w/o proof):

Abel's Test

"If $u_n(x) = a_n f_n(x)$, with $\sum_n a_n = A$ a convergent series, and the $f_n(x)$ are monotonic and bounded in $a \leq x \leq b$ [i.e. $f_{n+1}(x) \leq f_n(x)$ and $0 \leq f_n(x) \leq M$ over the interval], then the series $\sum_n u_n(x)$ converges uniformly in $a \leq x \leq b$." (41)

11) POWER SERIES.

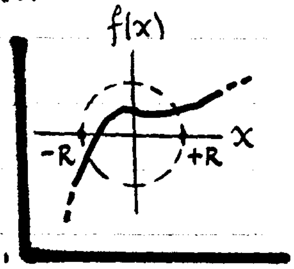
We have already used power series of the form $f(x) = \sum_{n=0}^{\infty} a_n x^n$ to solve ODEs; they are evidently very handy. They are also very well-behaved series, and -- on the basis of the above discussion of uniform convergence -- it is possible to establish the following facts about power series ...

Properties of Power Series.

Facts about $f(x) = \sum_{n=0}^{\infty} a_n x^n$.

1. Convergence. If, by the Ratio Test [Eq. (14)], one finds that...

→ $\lim_{n\to\infty} |a_{n+1}/a_n| = \frac{1}{R}$, then// $f(x)$ converges in $-R < x < +R$



R is called the "radius of convergence". $f(x)$ may converge or diverge at the endpoints $\pm R$; these require special attention.

2. Uniform & Absolute Convergence. Suppose $f(x)$ is known to converge in $-R < x < +R$. Then there is an interior interval $-r \leq x \leq +r$, w/ $0 < r < R$, such that $f(x)$ is uniformly and absolutely convergent for $|x| \leq r$. (This claim can be proved by the M-Test, with the ingenious choice $M_n = |a_n| r^n$).

3. Continuity. Each term $u_n(x) = a_n x^n$ in $f(x)$ is a continuous fcn of x in the region of uniform convergence $|x| \leq r$, so $f(x)$ is also continuous in $|x| \leq r$.

4. Differention & Integration. With $u_n(x) = a_n x^n$ continuous, and $f(x) = \sum_n u_n(x)$ uniformly convergent in $|x| \leq r$, the differentiated or integrated series is again a power series of continuous fcns with the same radius of convergence as the original one -- this is so because the new factors introduced by differentiation or integration do not change the outcome of the above Ratio Test. Thus a power series $f(x) = \sum_n a_n x^n$ can be differentiated or integrated an arbitrary number of times (within its convergence radius r).

5. Uniqueness. The series $f(x) = \sum_n a_n x^n$ uniquely represents $f(x)$ within the radius of convergence $|x| \leq r$ (i.e. the coefficients $\{a_n\}$ are unique).

oof

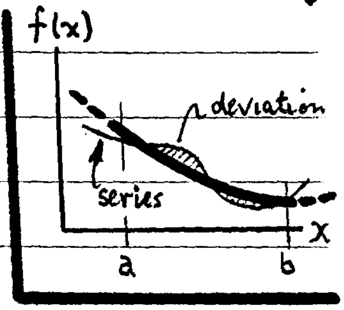
Suppose $f = \sum_n a_n x^n$ were not unique, i.e. there is another series $f = \sum_n b_n x^n$ in the same interval, so: $\sum_n a_n x^n = \sum_n b_n x^n$ in $|x| \leq r$. Set $x = 0$; then $b_0 = a_0$. Now, differentiate the series, so: $\sum_n n a_n x^{n-1} = \sum_n n b_n x^{n-1}$, and again set $x = 0$; then $b_1 = a_1$. Repeat, set $x = 0 \Rightarrow \ldots$ all $b_n = a_n$, and $f = \sum_n a_n x^n$ is unique.

12) CONVERGENCE IN THE MEAN.

It is possible to represent a fcn $f(x)$ by functional series $\sum_n a_n u_n(x)$ where the fcns $u_n(x)$ are more elaborate than simple powers x^n. For example, if $u_n(x) = \sin nx$, then $f(x) = \sum_n a_n \sin nx$ is a "Fourier Series" for $f(x)$, and the immediate question is: when does this series actually converge to $f(x)$? ¶ A way of dealing with such question follows

"If, on $a \leq x \leq b$, $f(x)$ is represented by a functional series: $f(x) = \sum_{n=0}^{N} a_n u_n(x)$, with $N \to \infty$, then $\sum_n a_n u_n(x)$ "converges in the mean" to $f(x)$ if:

$$\lim_{N\to\infty} D_N = 0, \text{ w/ } D_N(a_0, a_1, \ldots, a_N) = \int_a^b \Big[f(x) - \sum_{n=0}^{N} a_n u_n(x)\Big]^2 dx. \text{"} \qquad (42)$$



D_N is the mean (integrated) square deviation of $\sum_{n=0}^{N} a_n u_n(x)$ from $f(x)$ at a given N; for a given set of fcns $\{u_n(x)\}$, and given $f(x)$, evidently D_N depends on the choice of coefficients $a_0, a_1, \ldots, a_N$. The criterion in (42) specifies convergence if the shaded areas vanish (on average) when $N \to \infty$.

The best choice of $\{a_n\}$, to render D_N minimum at any given N, is found by:

$$\to \partial D_N / \partial a_\ell = (-)2\int_a^b \Big(f - \sum_n a_n u_n\Big) u_\ell \, dx = 0, \text{ for } \ell = 0 \text{ to } N; \qquad (43)$$

this makes $\Delta D_N = \sum_{\ell=0}^{N} (\partial D_N / \partial a_\ell) \Delta a_\ell = 0$, so D_N is stationary. Eq. (43)

¶ For Fourier Series: $f(x) = \sum_n (a_n \sin nx + b_n \cos nx)$, the answer is Dirichlet's Thm. If $f(x)$ $\boxed{1}$ is periodic in x, w/ period 2π, $\boxed{2}$ is single-valued in $-\pi \leq x \leq +\pi$, $\boxed{3}$ has a finite # of maxima & minima, and a finite # of discontinuities on the interval, and if $\boxed{4}$ $\int_{-\pi}^{\pi} |f(x)|\,dx$ exists, then the series converges to $f(x)$ at all points where $f(x)$ is continuous. At a point x_0 where $f(x)$ is discontinuous, the series converges to the midpoint value $\lim_{\epsilon\to 0} \frac{1}{2}[f(x_0+\epsilon) + f(x_0-\epsilon)]$.

then prescribes that the $\{a_n\}$ be chosen so that...

$$\rightarrow \sum_{n=0}^{N} U_{\ell n} a_n = \int_a^b f(x) u_\ell(x) dx \quad \begin{cases} \ell = 0 \text{ to } N, \text{ and} \\ U_{\ell n} = \int_a^b u_\ell(x) u_n(x) dx. \end{cases} \tag{44}$$

This eqtn can be solved (in principle) for the "best" $\{a_n\}$. An enormous simplication occurs if the $\{u_n(x)\}$ are a set of "orthogonal fcns", i.e. if

$$\left[U_{n\ell} = \int_a^b u_\ell u_n dx = \delta_{n\ell} = \begin{cases} 1, \text{ when } n=\ell \\ 0, \text{ otherwise.} \end{cases} \right\| \text{so} \quad a_\ell = \int_a^b f(x) u_\ell(x) dx. \tag{45}$$

On these grounds, sets of orthogonal expansion fcns $\{u_n(x)\}$ are prized.

13) BERNOULLI NUMBERS.

For a power series $f(x) = \sum_{n=0}^{\infty} a_n x^n$, the $\{a_n\}$ are fixed uniquely by the character of $f(x)$... $f(x)$ "generates" the $\{a_n\}$. We see this by writing a Taylor series

$$\rightarrow f(x) = \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} \frac{1}{n!} [f^{(n)}(0)] x^n \Rightarrow \underline{\underline{a_n = \frac{1}{n!} f^{(n)}(0)}}. \tag{46}$$

One set of $\{a_n\}$ that occurs frequently (in various guises) in the power series expansions of elementary fcns is the Bernoulli Numbers $\{B_n\}$. The B_n appear most directly in the expansion of the fcn*...

$$\left[f(x) = x/(e^x - 1) = \sum_{n=0}^{\infty} \frac{1}{n!} B_n x^n \leftarrow \text{the } \{B_n\} \text{ are Bernoulli \#s} \right.$$

$$\text{so} \quad B_n = f^{(n)}(0) = \left[\frac{d^n}{dx^n} \left(\frac{x}{e^x - 1} \right) \right]_{x=0} \Rightarrow B_0 = 1,\ B_1 = -\frac{1}{2}, \text{ etc.} \tag{47}$$

Since taking an infinite # of derivatives is ~tedious, we find the B_n by a clever method outlined in M&W Sec. 2-2. To wit...

$$\left[\text{Symbolically, let: } B_n \leftrightarrow (B)^n, \ \text{i.e. } B \text{ to } n^{\text{th}} \text{ power} \Rightarrow B_n; \right.$$

$$\text{so} \quad x/(e^x - 1) = \sum_{n=0}^{\infty} \frac{1}{n!} (Bx)^n = e^{Bx}, \tag{48}$$

* Such fcns are used in the Planck Radiation Law, Bose-Einstein statistics, etc.

The B_n are distinguishable in this way, as are the $(B)^n$. Now work on Eq.(48):

$\rightarrow x = (e^x - 1)e^{Bx} = e^{(B+1)x} - e^{Bx}$ ∫ expand each exponential in a power series;

so// $x = \sum_{n=0}^{\infty} \frac{1}{n!}\left[(B+1)^n - (B)^n\right] x^n$ ∫ identify like powers of x on LHS & RHS of eqtn;

and// $$\boxed{(B+1)^n - (B)^n = 0}\text{, except for } \begin{cases} n=0, \text{ w/ } B^0 \leftrightarrow B_0 = 1, \\ n=1, \text{ w/ } B^1 \leftrightarrow B_1 = -\frac{1}{2}. \end{cases} \quad (49)$$

For $n \geq 2$, this relation gives $B^{n-1} \leftrightarrow B_{n-1}$ in terms of $B_{n-2}, B_{n-3}, \ldots, B_1$.

e.g.// $(B+1)^3 - (B)^3 = 3(B)^2 + 3(B)^1 + 1 = 0 \leftrightarrow 3B_2 + 3B_1 + 1 = 0,$

$$\Rightarrow B_2 = -\tfrac{1}{3}(3B_1 + 1) = +1/6. \quad (50)$$

Repeating the procedure, we find...

$$\begin{cases} B_0 = 1,\ B_2 = +\frac{1}{6},\ B_4 = -\frac{1}{30},\ B_6 = +\frac{1}{42},\ B_8 = -\frac{1}{30}, \ldots; \\ B_1 = -\frac{1}{2},\ B_{2n+1} \equiv 0 \text{ for all } n \geq 1. \end{cases} \quad (51)$$

Tables exist... see Abramovitz & Stegun, Chap. 23 (B_0 to B_{60}).*

Since exponentials are involved in the def^n of the B_n, and also appear in the def^{n}s of trig fcns (both circular & hyperbolic), it is not surprising to see the B_n occur in many of the power series for trig fcns, e.g.

$$\begin{cases} \text{ctn}\, x = (e^{2ix}+1)/(e^{2ix}-1) = \frac{1}{x}\sum_{n=0}^{\infty} \frac{(-)^n}{(2n)!} B_{2n} \cdot (2x)^{2n}, \text{ for } |x| < \pi; \\ \tanh x = (e^{2x}-1)/(e^{2x}+1) = \sum_{n=1}^{\infty}\left[2^{2n}(2^{2n}-1)/(2n)!\right] B_{2n}\, x^{2n-1}, \ |x| < \frac{\pi}{2}. \end{cases} \quad (52)$$

14) TRANSFORMATION of SERIES.

Transforming a power series for $f(x)$ to a new independent variable is sometimes useful for improving the series rate-of-convergence, etc. This method does not

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* Later, by means of complex variable theory, we will show that: $B_{2n} = (-)^{n+1}\left[\frac{2(2n)!}{(2\pi)^{2n}}\right]\zeta(2n)$, where ζ is the Riemann Zeta Fcn, and $n \geq 1$ (recall $\zeta(p) = \sum_{k=1}^{\infty}(1/k)^p$).

Transformation of Series (cont'd)

Sum the series, but instead converts it to a friendlier form. As follows.

1. Let: $g(x) = \sum_{n=0}^{\infty} b_n x^n$, be a <u>known</u> power series [i.e. the fcn $g(x)$ is known].

Let: $f(x)$ be a series-to-be-transformed, and suppose it is written as:

$$f(x) = (b_0 c_0) + (b_1 c_1)x + (b_2 c_2)x^2 + \ldots = \sum_{n=0}^{\infty} b_n c_n x^n, \quad (53)$$

w/ the $\{b_n\}$ characterizing $g(x)$, and the $\{c_n\}$ additional known factors.

NOTE: any $f(x) = \sum_n a_n x^n$ can be written this way, with $c_n = a_n/b_n$.

2. The idea of transforming the series for $f(x)$ is to replace each (known) b_n by a derivative of $g(x)$, and so obtain $f(x)$ as a series in $x^n \cdot$ factors $g^{(n)}(x)$.

E.g.// $g(x) = b_0 + b_1 x + b_2 x^2 + \ldots \Rightarrow b_0 = g(x) - b_1 x - b_2 x^2 - b_3 x^3 - \ldots$

... so: $f(x) = c_0[g(x) - b_1 x - b_2 x^2 - \ldots] + c_1 b_1 x + c_2 b_2 x^2 + \ldots$

$$= c_0 g(x) + (c_1 - c_0) b_1 x + (c_2 - c_0) b_2 x^2 + \ldots \quad (54)$$

Now, just as b_0 has been replaced by $g(x)$, b_1 can be replaced by $g'(x)$, per...

$$g'(x) = b_1 + 2b_2 x + 3b_3 x^2 + \ldots \Rightarrow b_1 = g'(x) - 2b_2 x - 3b_3 x^2 - \ldots$$

... so: $f(x) = c_0 g(x) + (c_1 - c_0) x g'(x) + (c_2 - 2c_1 + c_0) b_2 x^2 + \ldots \quad (55)$

3. Repeat the procedure: replace b_2 by $g''(x) - [\text{stuff}]$, b_3 by $g'''(x) - [\text{stuff}]$, etc. The result is a transformed series for $f(x)$, viz...

$$\boxed{f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} A_n x^n g^{(n)}(x)} \quad \text{w/ } A_n = \sum_{\mu=0}^{n} (-)^{n-\mu} \binom{n}{\mu} c_\mu, \quad \binom{n}{\mu} = n!/\mu!(n-\mu)! = \text{Binomial Coefficient} \quad (56)$$

The x-dependence is quite different: $x^n g^{(n)}(x)$ vs. the x^n we had in Eq. (53). On pp. 54-55, M & W give an example: transforming $f(x) = \sum_{n=0}^{\infty} (-)^n x^n/(n+1)$, via $g(x) = \frac{1}{1+x} = \sum_{n=0}^{\infty} (-)^n x^n$, to: $f(x) = \frac{1}{1+x} \sum_{n=0}^{\infty} \frac{1}{n+1} \left(\frac{x}{1+x}\right)^n$, for improved convergence.

Operations with Power Series.

15) OPERATIONS WITH POWER SERIES

1. Power series $f(x) = \sum_n a_n x^n$ and $g(x) = \sum_n b_n x^n$ can be added, subtracted, multiplied or divided -- within their (mutual) radius of convergence -- to form new and convergent power series. For example, for multiplication...

$$\rightarrow \left(\sum_{n=0}^{\infty} a_n x^n\right)\left(\sum_{n=0}^{\infty} b_n x^n\right) = \sum_{n=0}^{\infty} c_n x^n, \quad w/\ c_n = \sum_{k=0}^{n} a_k b_{n-k}. \quad \int \text{G\&R, p.15 \#(0.316)}. \qquad (57)$$

2. Other operations may be of interest, such as $\left(\sum_{n=0}^{\infty} a_n x^n\right)^{-1}$, and often only the first few terms of the resultant series are needed -- e.g. to find what happens when $x \to 0$. The following table is useful (Abramovitz & Stegun, p. 15)

from "Handbook of Math. Functions" ed. Abramovitz & Stegun (Dover, 1973)

Operations With Series

Let $s_1 = 1 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \ldots$
$s_2 = 1 + b_1x + b_2x^2 + b_3x^3 + b_4x^4 + \ldots$
$s_3 = 1 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + \ldots$

	Operation	c_1	c_2	c_3	c_4
3.6.16	$s_3 = s_1^{-1}$	$-a_1$	$a_1^2 - a_2$	$2a_1a_2 - a_3 - a_1^3$	$2a_1a_3 - 3a_1^2a_2 - a_4 + a_2^2 + a_1^4$
3.6.17	$s_3 = s_1^{-2}$	$-2a_1$	$3a_1^2 - 2a_2$	$6a_1a_2 - 2a_3 - 4a_1^3$	$6a_1a_3 + 3a_2^2 - 2a_4 - 12a_1^2a_2 + 5a_1^4$
3.6.18	$s_3 = s_1^{1/2}$	$\frac{1}{2}a_1$	$\frac{1}{2}a_2 - \frac{1}{8}a_1^2$	$\frac{1}{2}a_3 - \frac{1}{4}a_1a_2 + \frac{1}{16}a_1^3$	$\frac{1}{2}a_4 - \frac{1}{4}a_1a_3 - \frac{1}{8}a_2^2 + \frac{3}{16}a_1^2a_2 - \frac{5}{128}a_1^4$
3.6.19	$s_3 = s_1^{-1/2}$	$-\frac{1}{2}a_1$	$\frac{3}{8}a_1^2 - \frac{1}{2}a_2$	$\frac{3}{4}a_1a_2 - \frac{1}{2}a_3 - \frac{5}{16}a_1^3$	$\frac{3}{4}a_1a_3 + \frac{3}{8}a_2^2 - \frac{1}{2}a_4 - \frac{15}{16}a_1^2a_2 + \frac{35}{128}a_1^4$
3.6.20	$s_3 = s_1^n$	na_1	$\frac{1}{2}(n-1)c_1a_1 + na_2$ *	$c_1a_2(n-1) + \frac{1}{6}c_1a_1^2(n-1)(n-2) + na_3$ *	$na_4 + c_1a_3(n-1) + \frac{1}{2}n(n-1)a_2^2 + \frac{1}{2}(n-1)(n-2)c_1a_1a_2 + \frac{1}{24}(n-1)(n-2)(n-3)c_1a_1^3$
3.6.21	$s_3 = s_1s_2$	$a_1 + b_1$	$b_2 + a_1b_1 + a_2$	$b_3 + a_1b_2 + a_2b_1 + a_3$	$b_4 + a_1b_3 + a_2b_2 + a_3b_1 + a_4$
3.6.22	$s_3 = s_1/s_2$	$a_1 - b_1$	$a_2 - (b_1c_1 + b_2)$	$a_3 - (b_1c_2 + b_2c_1 + b_3)$	$a_4 - (b_1c_3 + b_2c_2 + b_3c_1 + b_4)$
3.6.23	$s_3 = \exp(s_1 - 1)$	a_1	$a_2 + \frac{1}{2}a_1^2$	$a_3 + a_1a_2 + \frac{1}{6}a_1^3$	$a_4 + a_1a_3 + \frac{1}{2}a_2^2 + \frac{1}{2}a_2a_1^2 + \frac{1}{24}a_1^4$
3.6.24	$s_3 = 1 + \ln s_1$	a_1	$a_2 - \frac{1}{2}a_1c_1$	$a_3 - \frac{1}{3}(a_2c_1 + 2a_1c_2)$	$a_4 - \frac{1}{4}(a_3c_1 + 2a_2c_2 + 3a_1c_3)$ *

(58)

3. Finally, suppose we want to invert a series -- i.e. given $x(y) = \sum_{n=1}^{\infty} a_n y^n$, we want to find: $y(x) = \sum_{n=1}^{\infty} b_n x^n$. We will later show, by complex variables, that

$$\boxed{y(x) = \sum_{n=1}^{\infty} \frac{1}{n!} \left\{ \left(\frac{d}{d\xi}\right)^{n-1} [\xi / x(\xi)]^n \right\}\Big|_{\xi=0} x^n} \quad (59)$$

is the required series. As of now, we expect from complex variables Gauss' Sum Formula [Eq. (33)], Bernoulli #s [footnote p.∞16], Inversion Formula [Eq. (59)].