6) While solving the potential problem $\nabla^2\phi = (-)4\pi\rho$ in terms of eigenfcns & eigenvalues may <u>seem</u> like an entirely different procedure than a solution by Green's fcns, there is a connection between the methods... viz. the <u>Green's fcn can be expressed in terms of the eigenfcns</u>. Per Jk$^{\underline{n}}$ Sec. (3.12), consider solving the PDE...¶

$$\rightarrow \nabla^2\psi(\mathbf{r}) + [\lambda + q(\mathbf{r})]\,\psi(\mathbf{r}) = 0 \quad \int \text{a 3D S-L eqtn, w/ } p(\mathbf{r})=1,\ w(\mathbf{r})=1. \tag{24}$$

Suppose the B.C. force eigenvalues λ_n & eigenfcns $\psi_n(\mathbf{r})$. Then the $\{\psi_n\}$ are an orthonormal set, etc. We have, in analogy with S-L theory:

$$\left[\!\left[\lambda \rightarrow \lambda_n, \underline{\text{real}} \text{ eigenvalues}; \ \psi \rightarrow \psi_n \text{ such that: } \int \psi_m^* \psi_n \, d^3x = \delta_{mn}. \right.\right. \tag{25}$$

The Green's Fcn for this problem is the solution to the point-source eqtn:

$$\rightarrow \nabla_r^2 G_\mu(\mathbf{r},\mathbf{r}') + [\mu + q(\mathbf{r})]\,G_\mu(\mathbf{r},\mathbf{r}') = -4\pi\,\delta(\mathbf{r}-\mathbf{r}'). \tag{26}$$

$\mu \neq$ any of the λ_n, because the RHS source term pushes μ off. Suppose, however, that G_μ in Eq. (26) satisfies <u>same</u> B.C. as the eigenfcns $\{\psi_n\}$ -- e.g. G_μ vanishes as $r \rightarrow \infty$, is finite at $r=0$, etc. Then try expansion...

$$G_\mu(\mathbf{r},\mathbf{r}') = \sum_m c_m(\mathbf{r}')\,\psi_m(\mathbf{r}), \quad G_\mu \text{ with same B.C. as } \{\psi_m(\mathbf{r})\}; \tag{27}$$

... put this into Eq. (26) and use $\nabla_r^2 \psi_m = -[\lambda_m + q(\mathbf{r})]\,\psi_m$...

Soy
$$\sum_m (\mu - \lambda_m)\,c_m(\mathbf{r}')\,\psi_m(\mathbf{r}) = -4\pi\,\delta(\mathbf{r}-\mathbf{r}'); \tag{28}$$

... multiply both sides by $\psi_n^*(\mathbf{r})$, integrate: $\int d^3x$, use orthogonality...

Soy
$$c_n(\mathbf{r}') = 4\pi\,\frac{\psi_n^*(\mathbf{r}')}{\lambda_n - \mu}, \quad \text{and/} \quad \boxed{G_\mu(\mathbf{r},\mathbf{r}') = 4\pi \sum_n \frac{\psi_n^*(\mathbf{r}')\,\psi_n(\mathbf{r})}{\lambda_n - \mu}}. \tag{29}$$

Indeed G_μ is related to the $\{\psi_n\}$. Compare w/ CLOSURE: $\delta(\mathbf{r}-\mathbf{r}') = \sum_n \psi_n^*(\mathbf{r}')\,\psi_n(\mathbf{r})$.

¶ This is Schrödinger's Eqtn (time indpt): $\nabla^2\psi + \frac{2m}{\hbar^2}[E - V(\mathbf{r})]\,\psi = 0 \ \begin{cases} \lambda = (2m/\hbar^2)E \\ q = -(2m/\hbar^2)V. \end{cases}$

7) The reduction of a PDE in n variables (like $\nabla^2\phi = 0$ in 3D) to n ODE's which can be solved by means of special functions [such as those generated by the generic Sturm-Liouville ODE] is called the SEPARATION OF VARIABLES. What you do goes as follows...

1. Suppose $\phi = \phi(x, y, \ldots)$, and $\nabla^2\phi = 0$. [$x, y, \ldots \leftrightarrow n$ coordinates]
2. Try a solution for ϕ as a product of separate fcns:

$$\phi(x,y,\ldots) = f_1(\alpha, x) \cdot f_2(\beta, y) \cdot \ldots$$

NOTE: $x, y, \ldots$ need not be rectangular coordinates. Could be spherical, etc.

$\alpha, \beta, \ldots$ are (adjustable) separation cnsts which "facilitate" the method.

3. Put $\phi = f_1 f_2 \cdots$ into $\nabla^2\phi = 0$. See if you can isolate terms so that:

$$\nabla^2\phi = 0 \Rightarrow \mathcal{D}_1 f_1(\alpha, x) = 0,\ \mathcal{D}_2 f_2(\beta, y) = 0, \ldots \Big\{ n \text{ ODE's, + some eq.: } F(\alpha, \beta, \ldots) = 0.$$

4. If these ODE's are solvable, then the general solution for ϕ is:

$$\phi(x, y, \ldots) = \sum_{\alpha, \beta, \ldots} C_{\alpha\beta\ldots}\, f_1(\alpha, x)\, f_2(\beta, y) \cdots, \text{ by superposition.} \quad (30)$$

REMARKS

A. Step 3 is the crucial one. It is not clear that the ODE for x will not have some y-dept term in it, and vice-versa--i.e. that the x dependent terms can be isolated, or truly separated. In general, the possibility of separation depends on properties of the coordinate system in which ∇^2 is written. Curvilinear coordinates make separation "harder."

B. The choice of coordinates for ∇^2 is dictated by the symmetry of the ϕ problem at hand: rectangular cds (x, y, z) for ϕ[boxes], cylindrical cds (ρ, ϕ, z) for ϕ[flagpoles], spherical cds (r, θ, ϕ) for ϕ[planets], prolate spheroidal cds (u, v, ϕ) for ϕ[cigars], etc. Wrong choice of coordinate symmetry makes imposing B.C. ~ impossible.

C. The B.C. on ϕ will most often require the cnsts $\alpha, \beta, \ldots$ be "quantized".

8) One place where separation-of-variables works beautifully is for $\nabla^2\phi=0$ in rectangular symmetry. Jackson does the problem in his Sec. (2.9). The coordinate system is rectangular cds (x, y, z), and we have...

$$\phi=\phi(x,y,z), \text{ w/ } \nabla^2\phi=\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\phi=0.$$

try// $\phi = U(x)V(y)W(z)$. NOTE: $\dfrac{\partial^2\phi}{\partial x^2}=VW\dfrac{d^2U}{dx^2}$, etc.

and/

$$\frac{1}{\phi}\nabla^2\phi=0 \Rightarrow \underbrace{\overset{\text{fcn}(x)\text{ only}}{\frac{1}{U}\left(\frac{d^2U}{dx^2}\right)}}_{\equiv\text{ cnst, }(-)\alpha^2}+\underbrace{\overset{\text{fcn}(y)\text{ only}}{\frac{1}{V}\left(\frac{d^2V}{dy^2}\right)}}_{\equiv\text{ cnst, }(-)\beta^2}+\underbrace{\overset{\text{fcn}(z)\text{ only}}{\frac{1}{W}\left(\frac{d^2W}{dz^2}\right)}}_{\equiv\text{ cnst, }\gamma^2}=0. \qquad (31)$$

The x, y, z variations are isolated (no cross-terms), and we have separated the PDE into 3 easily solvable ODE's, viz.

$$\left[\begin{array}{l}
U''/U=-\alpha^2, \text{ or: } U''+\alpha^2U=0 \Rightarrow U(x)\propto e^{\pm i\alpha x} \int \text{or } \sin\alpha x \text{ \& } \cos\alpha x\\
V''/V=-\beta^2, \text{ or: } V''+\beta^2V=0 \Rightarrow V(y)\propto e^{\pm i\beta y} \int \text{or } \sin\beta y \text{ \& } \cos\beta y\\
W''/W=\gamma^2, \text{ or } W''-\gamma^2W=0 \Rightarrow W(z)\propto e^{\pm\gamma z} \int \text{or } \sinh\gamma z \text{ \& } \cosh\gamma z\\
\underline{\text{with}}\ \underline{\text{restriction}}:\quad \gamma^2=\alpha^2+\beta^2.
\end{array}\right. \qquad (32)$$

The superposition of products UVW gives the general solution to $\nabla^2\phi=0$:

$$\rightarrow \phi(x,y,z)=\sum_{\alpha,\beta}C_{\alpha\beta}\left\{\begin{matrix}\sin\\ \cos\end{matrix}(\alpha x)\right\}\left\{\begin{matrix}\sin\\ \cos\end{matrix}(\beta y)\right\}\left\{\begin{matrix}\sinh\\ \cosh\end{matrix}\left(\sqrt{\alpha^2+\beta^2}\,z\right)\right\}. \qquad (33)$$

B.C. for ϕ-values on planes x, y or $z=$ cnst are easily applied, and the sin and/or cos alternatives are fixed by such B.C. The separation cnsts α & β are generally quantized by the B.C. (analogous to QM ψ obeying B.C., which generates quantized energies). As well, the expansion coefficients $C_{\alpha\beta}$ are calculable from some ambient ϕ.