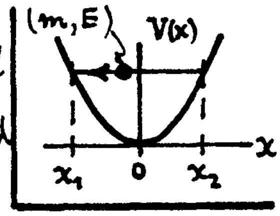
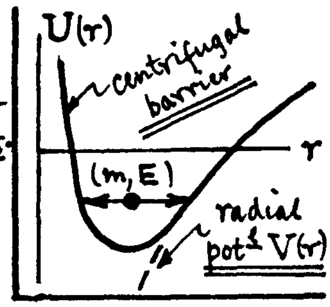
1992

ϕ507 Problems   Assigned: 3/2. Due: 3/9/92.   #7 file (7)

(25) In "Notes on the WKB Method", pp. 7–10, we solved the WKB problem $\ddot{v}+\Omega^2 v=0$ by transforming variables: $t\to s=\int\Omega(t)\,dt$, $v\to u=v\sqrt{\Omega}$, so the diff. eq. is $u''+[1+b(s)]u=0$, with $b(s)$ defined in Eq.(20) of Notes. For $b(s)=0$, we get the zeroth-order (WKB) solution: $u(s)\simeq u_0(s)=Ae^{+is}+Be^{-is}$. We then iterated to get: $u_1\simeq u_0+\int_0^s u_0 K\,d\sigma$, with K defined in Eq.(27). After $n+1$ iterations: $u_{n+1}=u_n+\int_0^s u_n K\,d\sigma$. Write out u_{n+1} <u>explicitly</u> as a series of $(n+2)$ terms, in successively higher "powers" of $b(s)$. Show that: $u_{n+1}(s)=u_0(s)+\sum_{k=1}^{n+1}\binom{n+1}{k}\int_0^s d\sigma_1\int_0^{\sigma_1}d\sigma_2\cdots\int_0^{\sigma_{k-1}}d\sigma_k\,u_0(\sigma_k)\,\mathcal{K}(\sigma_k,\dots,\sigma_1,s)$. Identify $\mathcal{K}$.



(26) A QM particle of mass m and energy E moves in a 1D SHO, w/ potential $V(x)=\frac{1}{2}m\omega^2x^2$, where ω = SHO natural frequency. Use Bohr-Sommerfeld quantization [i.e. $\int_{x_1}^{x_2}k(x)\,dx=(n+\frac{1}{2})\pi$, w/ $n=0,1,2,\dots$] to find the eigenenergies E_n for this motion. How does your result compare with the known E_n (SHO)?



(27) [30 pts]. For a QM particle (mass m, energy E) moving in >1D, and in an attractive radial pot$^{\underline{l}}$ $V(r)$, the <u>effective</u> potential $U(r)=V(r)+\frac{\mu^2\hbar^2}{2mr^2}$. The term in $\frac{1}{r^2}$ is the "centrifugal barrier", present because of m's rotational K.E. μ is a quantum # related to m's $\measuredangle$lar momentum [in 3D: $\mu^2=l(l+1)$, w/ $l=0,1,2,\dots$; in 2D: $\mu^2=m^2-\frac{1}{4}$, w/ $m=\pm1,\pm2,\dots$]. Here, just take $\mu^2\geqslant0$.

(A) Let length r_0 be the "size" of $U(r)$, and define a dimensionless variable: $x=r/r_0$. Write $V(r)=V_0 f(x)$, V_0 = cnst & $f(x)$ arbitrary. Show the Bohr-Sommerfeld condition becomes:

$$\boxed{\int_{x_1}^{x_2}\sqrt{\epsilon-[\sigma f(x)+\mu^2/x^2]}\,dx=(n+\tfrac{1}{2})\pi}$$

w/ x_1 & x_2 = solutions to: $\sigma f(x)+\mu^2/x^2=\epsilon$. Specify ϵ & σ in terms of $m, r_0, \hbar, E$ & V_0.

(B) Specialize to $f(x)=\ln x$ [log potentials are use to model quark confinement -- see Quigg & Rossner, Phys. Lett. 71B, 153 (1977)]. Sketch $U(x)$ vs. x, and find the minimum, x_0. Expand $U(x)$ about x_0, find the effective "spring constant" near x_0, and calculate the quantized energies $E_{n\mu}$ of a quark trapped near x_0: You have a SHO here. Why?

(C) For large vibrations: $\sigma\gg\mu^2$. Evaluate the above integral to find how $E_{n\mu}$ varies w/ n.

(D) For large rotations: $\mu^2\gg\sigma$. Find $E_{n\mu}$, approximately, to see how it varies w/ μ & n.

(25) Iterate the Neumann series for $u_{n+1}(s)$ [from p. 10 of "Notes on WKB Method"].

1) Start from the $\underline{m=1^{st}}$ iteration [Eq. (27) of "Notes on the WKB Method]: $u_{n+1}(s) = u_n(s) + \int_0^s d\sigma_1\, u_n(\sigma_1) K(\sigma_1, s)$, and insert: $u_n(x) = u_{n-1}(x) + \int_0^x d\sigma_2 u_{n-1}(\sigma_2) K(\sigma_2, x)$. So:

$$\rightarrow u_{n+1}(s) = u_{n-1}(s) + 2\int_0^s d\sigma_1\, u_{n-1}(\sigma_1) K(\sigma_1, s) + \int_0^s d\sigma_1 \int_0^{\sigma_1} d\sigma_2\, u_{n-1}(\sigma_2) K(\sigma_2, \sigma_1) K(\sigma_1, s). \quad (1)$$

This is the $\underline{m=2^{nd}}$ iteration. Put $u_{n-1}(x) = u_{n-2}(x) + \int_0^x d\sigma_3\, u_{n-2}(\sigma_3) K(\sigma_3, x)$ into Eq. (1) and again collect like terms to find for the $\underline{m=3^{rd}}$ iteration...

$$\rightarrow u_{n+1}(s) = u_{n-2}(s) + 3\int_0^s d\sigma_1\, u_{n-2}(\sigma_1) K(\sigma_1, s) + 3\int_0^s d\sigma_1 \int_0^{\sigma_1} d\sigma_2\, u_{n-2}(\sigma_2) K(\sigma_2, \sigma_1) K(\sigma_1, s) +$$

etc.

$$+ \int_0^s d\sigma_1 \int_0^{\sigma_1} d\sigma_2 \int_0^{\sigma_2} d\sigma_3\, u_{n-2}(\sigma_3) K(\sigma_3, \sigma_2) K(\sigma_2, \sigma_1) K(\sigma_1, s). \quad (2)$$

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2) In the $m=1$ iteration above, there are 2 terms, with numerical coefficients [1,1]. For $m=2$ in Eq. (1), we got 3 terms, with coefficients [1,2,1], and for $m=3$ in Eq. (2), we got 4 terms, with coefficients [1,3,3,1]. These sets are the binomial coefficients $\binom{m}{k} = m!/k!(m-k)!$, with m = iteration order #, and $k = 0, 1, \ldots, m$. After the m^{th} such operation as in Eq. (2) above, we will have the series...

$$u_{n+1}(s) = u_{n+1-m}(s) + \sum_{k=1}^{m} \binom{m}{k} \underbrace{\int_0^s d\sigma_1 \int_0^{\sigma_1} d\sigma_2 \cdots \int_0^{\sigma_{k-1}} d\sigma_k}_{k \text{ integrations}}\, u_{n+1-m}(\sigma_k)\, \underbrace{K^{(k)}}_{k \text{ factors of } K}(\sigma_k, \ldots, \sigma_1, s),$$

where: $\underline{K^{(k)}(\sigma_k, \ldots, \sigma_1, s) = K(\sigma_k, \sigma_{k-1}) K(\sigma_{k-1}, \sigma_{k-2}) \cdots K(\sigma_2, \sigma_1) K(\sigma_1, s)}$. (3)

Since $K(x,y) = b(x)\sin(x-y)$, then $K^{(k)}$ is of order $(b)^k$ in the small factor b.

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3) The iteration in Eq. (3) can be done a maximum of $m = n+1$ times. Then...

$$\boxed{u_{n+1}(s) = u_0(s) + \sum_{k=1}^{n+1} \binom{n+1}{k} \int_0^s d\sigma_1 \int_0^{\sigma_1} d\sigma_2 \cdots \int_0^{\sigma_{k-1}} d\sigma_k\, u_0(s)\, K^{(k)}(\sigma_k, \ldots, \sigma_1, s).} \quad (4)$$

This allows expressing $u_{n+1}(s)$ in terms of the WKB approximate for $u_0(s)$, with correction terms of order $K, (K)^2, \ldots, (K)^{n+1}$. Note that in Eq. (4), $n = 0, 1, 2, \ldots, \infty$.

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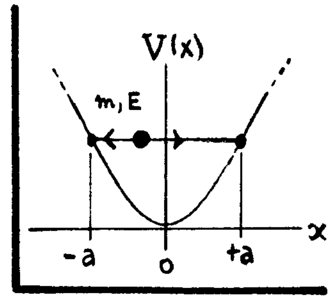
(25) Quantization of the SHO via Bohr-Sommerfeld rule [by way of WKB apprxn].

1) With $V(x) = \frac{1}{2}m\omega^2 x^2$, the QM version of the WKB interior phase integral is

$$\int_{x_1}^{x_2} k(x)dx = \int_{x_1}^{x_2} \left[\frac{2m}{\hbar^2}\left(E - \frac{1}{2}m\omega^2 x^2\right)\right]^{1/2} dx = \left(n+\frac{1}{2}\right)\pi, \quad (1)$$



with $n = 0, 1, 2, \ldots$, and $x_{1,2}$ "turning points"... i.e. points at which $E = V(x) = \frac{1}{2}m\omega^2 x^2$. Define these to be at $x = \pm a$...

$\longrightarrow$ $E = \frac{1}{2}m\omega^2 a^2 \longleftrightarrow$ turning points at $x_1 = (-)a$, $x_2 = +a$. (2)

2) Eq. (1), the Bohr-Sommerfeld quantization, now amounts to...

$$\frac{m\omega}{\hbar}\int_{-a}^{+a}(a^2 - x^2)^{1/2}dx = \left(n+\frac{1}{2}\right)\pi \quad (3)$$

... but: $\int_{-a}^{+a}(a^2-x^2)^{1/2}dx = \frac{1}{2}\left[x(a^2-x^2)^{1/2} + a^2\sin^{-1}\left(\frac{x}{a}\right)\right]\Big|_{x=-a}^{x=+a}$

$$= \frac{1}{2}a^2\left[\sin^{-1}(+1) - \sin^{-1}(-1)\right] = \frac{1}{2}a^2\pi$$

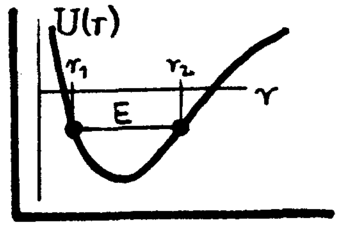
So, $\frac{m\omega}{\hbar}\cdot\frac{1}{2}a^2\pi = \left(n+\frac{1}{2}\right)\pi$, or, $\underline{\underline{\frac{1}{2}m\omega a^2 = \left(n+\frac{1}{2}\right)\hbar}}$. (4)

3) By def$^{\underline{n}}$ of a, in Eq. (2), we see that $\frac{1}{2}m\omega a^2 = E$ in Eq. (4). So the quantized energies of the SHO, via Bohr-Sommerfeld (á la WKB) are...

$$\boxed{E_n = \left(n+\frac{1}{2}\right)\hbar\omega, \quad n = 0, 1, 2, \ldots}. \quad (5)$$

These are the <u>exact</u> energies of a QM SHO (consult any telephone book, or QM directory, etc.). WKB (Bohr-Sommerfeld) quantization is usually a ~ good approxn, but not always this good. This is the only instance -- that I know of -- where the WKB energies agree <u>exactly</u> with the QM result.

7) (30 pts). Quark confinement: m in $V(r) = V_0 \ln(r/r_0)$. Analyse via Bohr-Sommerfeld.



(A) 1) The Bohr-Sommerfeld phase integral, namely...

$$\rightarrow \int_{r_1}^{r_2} \left[(2m/\hbar^2)\{E - U(r)\}\right]^{1/2} dr, \quad U(r) = V(r) + \frac{\mu^2\hbar^2}{2mr^2}, \quad (1)$$

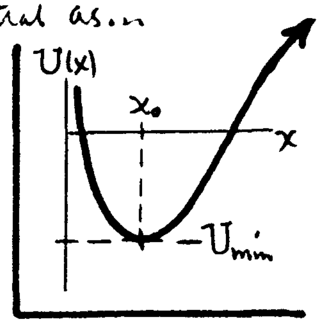
(for the radial problem, effectively 1D) can be written in terms of new variables...

$$\left[\begin{array}{l} \underline{x} = r/r_0, \text{ dimensionless length } \& \ V(r) = V_0 f(x), \\ \underline{\epsilon} = (2mr_0^2/\hbar^2)E \leftarrow \text{dim}^{\text{n}}\text{less energy}, \ \underline{\sigma} = (2mr_0^2/\hbar^2)V_0 \leftarrow \text{dim}^{\text{n}}\text{less ptl strength} \end{array}\right\} \quad (2)$$

So// Bohr-Sommerfeld Condition $\Big\}$ $$\boxed{\int_{x_1}^{x_2} \sqrt{\epsilon - [\sigma f(x) + (\mu^2/x^2)]}\, dx = (n + \tfrac{1}{2})\pi}, \quad (3)$$

Where: $n = 0, 1, 2, \ldots$ and turning pts x_1 & x_2 are solutions to $\sigma f(x) + (\mu^2/x^2) = \epsilon$.

(B) 2) For a logarithmic potential: $f(x) = \ln x$, write the effective potential as...



$$\rightarrow U(x) = U_0\left[\sigma \ln x + (\mu^2/x^2)\right], \quad U_0 = \hbar^2/2mr_0^2. \quad (4)$$

For the rotational quantum # $\mu \neq 0$, find min in $U(x)$ when...

$$\left[U'(x) = U_0\left[\frac{\sigma}{x} - \frac{2\mu^2}{x^3}\right] = 0 \Rightarrow x = \underline{\underline{x_0 = \sqrt{2\mu^2/\sigma}}}, \quad (5)\right.$$

and// $$U_{min} = U(x_0) = \tfrac{1}{2}\sigma U_0\left[\ln(2\mu^2/\sigma) + 1\right]. \quad (6)$$

For a particle near the bottom of the well (i.e. $x \sim x_0$ & $U \sim U_{min}$), eff. ptl is

$$\left[U(x) = U(x_0) + \overset{0,\ \text{by def}^{\text{n}}}{\cancel{U'(x_0)}}(x - x_0) + \tfrac{1}{2}U''(x_0)(x - x_0)^2 + \ldots \right.$$

an// $$U(x) \simeq U_{min} + \tfrac{1}{2}\kappa(x - x_0)^2, \quad \kappa = U''(x_0) = \text{dim}^{\text{n}}\text{less spring cnst} \quad (7)$$

Physical spring cnst is: $k = \kappa/r_0^2$ (since $x = \frac{r}{r_0}$). Calculating this, we find

$$\kappa = U''(x_0) = U_0\sigma^2/\mu^2, \quad \text{so//} \quad \underline{\underline{k = U_0\sigma^2/\mu^2 r_0^2}} \leftarrow \text{eff. spring cnst near } x_0. \quad (8)$$

In this approxn, the mass m moves in a SHO ptl [Eq.(7)] near the bottom of the well.

<u>By the Taylor expansion in Eq.(7), **all** wells are SHO ptls close enough to their minms.</u>

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7) (cont'd) The SHO motion of Eqs. (7) & (8) will have a natural frequency...

$$\rightarrow \omega = \sqrt{k/m} = \sqrt{2}\,(V_0/\hbar\mu), \text{ near } x = x_0, \tag{9}$$

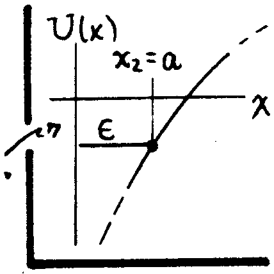
and the energies of this motion will be quantized by the usual SHO formula (prob. 26):

$$\boxed{E_{n\mu} = (n+\tfrac{1}{2})\hbar\omega = \sqrt{2}\,(n+\tfrac{1}{2})V_0/\mu} \quad \begin{cases}\text{with: } n=0,1,2,\dots;\ \mu \neq 0 \\ (\text{and near } x = x_0).\end{cases} \tag{10}$$

NOTE: $E_{n\mu}$ depends on both the vibrational quantum # n and rotational quantum # μ.

(C) 3) To see how $E_{n\mu}$ depends on n for vibrations big enough to invalidate the SHO approxn of part (B), suppose $\sigma \gg \mu^2 \rightarrow 0$ (ignore rotation altogether) and approx. Eq. (3) by...

$$\rightarrow \int_{x_1}^{x_2} (\epsilon - \sigma \ln x)^{1/2}\, dx \simeq (n+\tfrac{1}{2})\pi, \text{ for } \sigma \gg \mu^2 \rightarrow 0. \tag{11}$$



Ignoring rotation ⇒ ignoring centrifugal barrier, so $U(x)$ looks like:
Turning pts are at $x_1 = 0$ & $x_2 = a$, where a can be found from...

$$\rightarrow \sigma \ln a = \epsilon \Rightarrow a = \exp(\epsilon/\sigma) \tag{12}$$

change variables to $y = a/x$

Soln: Bohr-Sommerfeld ⇒ $\int_0^a [\sigma \ln(a/x)]^{\frac{1}{2}} \simeq (n+\tfrac{1}{2})\pi$,

$a\sqrt{\sigma}\,J \simeq (n+\tfrac{1}{2})\pi$, where: $J = \int_1^\infty \frac{dy}{y^2}\sqrt{\ln y} = \frac{1}{2}\sqrt{\pi}$ — G&R, p. 550 #(4.272.5);

put in $a = e^{\epsilon/\sigma}$ ... solve for ϵ... } $\boxed{E_n \simeq V_0 \ln[(n+\tfrac{1}{2})\pi / J\sqrt{\sigma}]}$ — for $\sigma \gg \mu^2 \rightarrow 0$ (i.e. large n) (13)

J here is just a number (indpt of system parameters); it does not matter if you show $J = \frac{\sqrt{\pi}}{2}$. Important point is that for large vibrations, the energies $E_n \propto$ logarithmically with n.

(D) 4) The case $\sigma \ll \mu^2$ is harder. We can't set $\sigma = 0$, because this would discard the binding part of $U(x)$, and m would go free. For $\mu \neq 0$ and $\sigma \neq 0$ change variables as above...

$\ln a = \epsilon/\sigma \Rightarrow \underline{a} = e^{\epsilon/\sigma}$, $\underline{y} = a/x$; $\underline{\lambda} = \mu^2/\sigma a^2$;

Soln: Eq. (3) ⇒ $$\boxed{\int_{y_2}^{y_1} \frac{dy}{y^2}[\ln y - \lambda y^2]^{1/2} = (n+\tfrac{1}{2})\pi / a\sqrt{\sigma}.} \tag{14}$$

This is the exact B-S condition. $y_{1,2} = \frac{a}{x_{1,2}}$ are inverse turning pts.

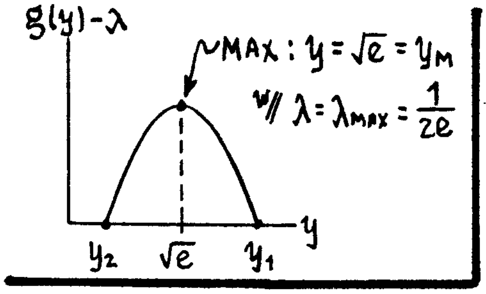
Case solved in Eq (12) was $\lambda = 0$, $x_{1,2} = 0, a \Rightarrow y_{1,2} = \infty, 1$. Now $\lambda \neq 0$. (next page)

(27) (cont'd). With $\lambda \neq 0$, the integral in Eq. (14) is difficult -- e.g. not tabulated. We will approximate the integral, assuming $\mu^2 \gg \sigma$ for large rotations. First write (14) as...

$$\rightarrow \int_{y_2}^{y_1} \frac{dy}{y}\left[g(y)-\lambda\right]^{1/2} = \frac{\sqrt{\pi}}{2}(\lambda/\lambda_0)^{\frac{1}{2}}, \quad \text{w/ } g(y) = \ln y/y^2, \; \sqrt{\lambda_0} = \mu/(2n+1)\sqrt{\pi}. \tag{15}$$

We want to solve this eqtn for $\lambda = \lambda(\epsilon)$. Quantization results from $\epsilon = \epsilon(\lambda(\text{fcn } n))$.

Note that λ_0 is the λ-value when $\mu \to 0$ [from Eq. (13): $a\sqrt{\sigma} \to (2n+1)\sqrt{\pi}$, for $\sigma \gg \mu^2 \to 0$, so $\lambda = \mu/a\sqrt{\sigma} \to \mu/(2n+1)\sqrt{\pi} = \lambda_0$, when $\mu \ll n$]. Anyway, we need the turning points $y_{1,2}$ in Eq. (15), i.e. solns to: $g(y)-\lambda = 0$. We graph $[g(y)-\lambda]$ vs. y at left, and note that since $g(y)$ is MAX at $y_M = \sqrt{e}$, with $g(y_M) = 1/2e$, then -- if the system energies are real -- $\lambda = (\mu^2/\sigma)e^{-2\epsilon/\sigma}$ must be bounded: $0 \leq \lambda \leq \lambda_{MAX} = 1/2e$. Then: $\infty \geq \epsilon \geq \frac{\sigma}{2}\left[\ln(2\mu^2/\sigma)+1\right]$, for energies.



Next, when $\mu^2/\sigma \gg 1$ (for large rotations), so that $\lambda \sim \lambda_{MAX}$, the turning pts $y_{1,2}$ both approach the midpt $y = y_M = \sqrt{e}$. Then we expand $g(y)$ about $y = y_M$, as...

$$g(y) = g(y_M) + \cancelto{0}{g'(y_M)}(y-y_M) + \frac{1}{2}g''(y_M)(y-y_M)^2 + \cdots$$

So, $$g(y)-\lambda \simeq (\lambda_M-\lambda) - \frac{1}{e^2}(y-y_M)^2, \quad \text{near } y = y_M = \sqrt{e}, \text{ for } \lambda_M = g(y_M) = 1/2e. \tag{16}$$

and, $$g(y)-\lambda = 0 \Rightarrow \text{turning points @ } \underline{\underline{y_{1,2} \simeq y_M \pm e\sqrt{\lambda_M-\lambda}}}. \tag{17}$$

Put these approxns (for $[g(y)-y]$, $y_{1,2}$) into Eq. (15), and change variables to $x = \frac{1}{e}(y-y_M)$. Then, for $\mu^2 \gg \sigma$, Bohr-Sommerfeld condition is

$$\rightarrow \int_{-q}^{+q}\left[\sqrt{q^2-x^2}/(p+x)\right]dx \simeq \frac{\sqrt{\pi}}{2}(\lambda/\lambda_0)^{1/2}, \quad q = \sqrt{\lambda_M-\lambda},\; p = 1/\sqrt{e}, \text{ and: } \lambda_M = 1/2e. \tag{18}$$

The integral in Eq. (18) is tedious, but doable -- see G&R, p. 89 #(2.282). Result is

$$\rightarrow \pi p + (q^2-p^2)\pi/\sqrt{\lambda_M+\lambda} \simeq \frac{\sqrt{\pi}}{2}(\lambda/\lambda_0)^{1/2}. \tag{19}$$

Standard accounting procedures allow rewriting this equation as

$\rightarrow \sqrt{2\lambda_M} - \sqrt{\lambda_M + \lambda} \simeq \rho\sqrt{\lambda}$, w/ $\rho = 1/2\sqrt{\pi\lambda_0} = \frac{1}{\mu}(n+\frac{1}{2})$ } ratio of vibrational to rotational Q#. (20)

The ratio ρ here is "small", since we doing the case μ (rot$^{\underline{l}}$) >> n (vib$^{\underline{l}}$). The solution to Eq. (20) to 1st order in ρ, and 1st order in $\lambda = (\mu^2/\sigma)\, e^{-2\epsilon/\sigma}$ is...

$\rightarrow \lambda \simeq \lambda_M(1 - 4\rho/\sqrt{2})$, $\lambda_M = 1/2e$ & ρ in Eq. (20). (21)

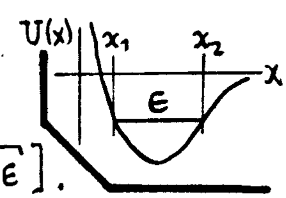
Since $\lambda = \lambda(\epsilon)$, this gives the system energies: $\epsilon = \frac{\sigma}{2}\ln(\mu^2/\sigma\lambda)$, or...

$\rightarrow \epsilon \simeq \frac{\sigma}{2}\left\{\left[\ln(2\mu^2/\sigma) + 1\right] - \ln\left(1 - \frac{4\rho}{\sqrt{2}}\right)\right\}$ } put in ϵ & σ of Eq. (2); expand 2nd ln ~ (−) $4\rho/\sqrt{2}$...

Sol$^{\underline{n}}$ $\boxed{E_{n\mu} \simeq \frac{V_0}{2}\left[\ln(2\mu^2/\sigma) + 1\right] + E_{n\mu}(\mathrm{SHO})}$ } for $\mu^2 >> \sigma$ (μ(rot$^{\underline{n}}$) >> n(vib$^{\underline{n}}$)) and $E_{n\mu}(\mathrm{SHO}) = \sqrt{2}(n+\frac{1}{2})\frac{V_0}{\mu}$. (22)

These are the required system energies for large rotations. Again, the main term $\propto$ logarithmically with the quantum #, as it did in Eq. (13). Notice that now the SHO energies $E_{n\mu}(\mathrm{SHO})$ of Eq. (10) appear as a perturbation on the larger rotational energies. In either case $\left(\begin{smallmatrix} n \rightarrow \text{large} \\ \mu \rightarrow \text{large}\end{smallmatrix}\right)$, all states $E_{n\mu}$ are bound.

NOT ASSIGNED, but out of curiosity, how does the BS rule work for hydrogen? Set up problem as follows... choose length unit $r_0 = \hbar^2/Zme^2 = a_0$ [Bohr radius]. Then system energy is: $E = (\hbar^2/2mr_0^2)\,\epsilon = \left[\frac{1}{2}(Z\alpha)^2 mc^2\right]\epsilon$, $\alpha = e^2/\hbar c \sim \frac{1}{137}$ {fine structure cnst}, and Coulomb potential: $V(r) = -Ze^2/r = V_0 f(x)$, w/ $V_0 = (Z\alpha)^2 mc^2$ & $f(x) = -1/x$, for $x = r/r_0$. The cnst $\sigma = (2mr_0^2/\hbar^2)V_0 = 2$, $\mu^2 = l(l+1)$, and Eq. (3) reads...



$\rightarrow \int_{x_1}^{x_2}\sqrt{\epsilon + (2/x) - (\mu^2/x^2)}\, dx = (n+\frac{1}{2})\pi$, $x_{1,2}$ sol$^{\underline{n}}$s to $\epsilon + \frac{2}{x} - \frac{\mu^2}{x^2} = 0$

$y = \frac{1}{x} \Rightarrow \int_{y_2}^{y_1}\frac{dy}{y^2}\sqrt{\epsilon + 2y - \mu^2 y^2} = (n+\frac{1}{2})\pi$, w/ $y_{1,2} = \frac{1}{\mu^2}\left[1 \pm \sqrt{1+\mu^2\epsilon}\right]$.

$y_2 \geq 0 \Rightarrow$ bound-state energies are (−)ve: $\epsilon = (-)\,\omega_n$. The BS integral is then...

$\rightarrow \int_{y_2}^{y_1}\frac{dy}{y^2}\sqrt{-\omega_n + 2y - \mu^2 y^2} = (n+\frac{1}{2})\pi$, w/ $y_{1,2} = \frac{1}{\mu^2}\left[1 \pm \sqrt{1 - \omega_n\mu^2}\right]$. } The integral is tabulated.