Liénard-Wiechert Potentials

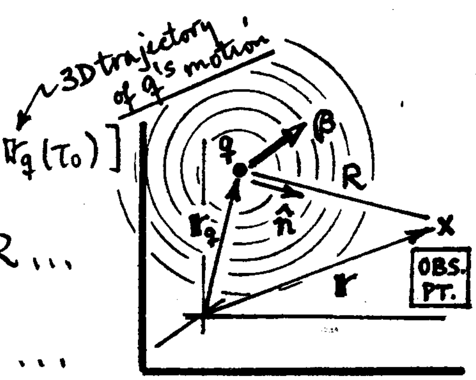
$$\left.\begin{array}{l}\boxed{A^{\alpha}(x) = q\,u^{\alpha}(\tau)/\{u^{\beta}(\tau)[x-r(\tau)]_{\beta}\}\Big|_{\tau=\tau_0}} \int \text{Jackson Eq. (14.6)} \\ \text{w/ } \tau_0 \text{ defined by: } [x-r(\tau_0)]^2 = 0 \;\&\; t\left[\begin{smallmatrix}\text{field}\\\text{point}\end{smallmatrix}\right] > t'(\tau_0)\left[\begin{smallmatrix}\text{source}\\\text{point}\end{smallmatrix}\right],\end{array}\right\} \quad (17)$$

These are the famous Liénard-Wiechert potentials $A^{\alpha} = (\phi, \mathbb{A})$ which specify the fields at an observation point x generated by the arbitrary motion of a point charge q along a trajectory $r(\tau)$ at (4-velocity) $u^{\alpha}(\tau)$.

REMARKS on Lienard Wiechert Potentials, Eq. (17).

1. We have now completed step one of the program outlined on p. qRad 1 -- Eq. (17) gives A^{α} generated by the arbitrary motion of a single point charge q.

2. If $\beta(\tau) = \frac{1}{c}\mathbb{v}(\tau)$ for q's motion @ velocity $\mathbb{v}$, then $u^{\alpha}(\tau) = \gamma c(1, \beta)$ in Eq. (17). The denominator is...



$$u^{\beta}(\tau)[x-r(\tau)]_{\beta}\Big|_{\tau=\tau_0} = u_0[x_0 - r_0(\tau_0)] - \mathbb{u}\cdot[\mathbb{r} - \mathbb{r}_q(\tau_0)]$$

... light-cone condition $\Rightarrow [x_0 - r_0(\tau_0)] = |\mathbb{r} - \mathbb{r}_q(\tau)| = R$ ...

... define: $\hat{n}$ = unit vector along $(\mathbb{r} - \mathbb{r}_q)$ $\left\{\begin{array}{l}\text{from } q \text{ to}\\ \text{obs. pt.}\end{array}\right.$ ...

$$\xrightarrow{\text{Soy}} u^{\beta}[x-r]_{\beta}\Big|_{\tau=\tau_0} = \gamma c R - \gamma c\,\beta\cdot\hat{n}\,R = \gamma c R(1-\hat{n}\cdot\beta)\Big|_{\tau=\tau_0}. \quad (18)$$

3. Then Eq. (17) for $A^{\alpha} = (\phi, \mathbb{A})$ assumes the more instructive form...

$$\boxed{\phi(\mathbb{r},t) = \left[\frac{q}{R}\cdot\frac{1}{(1-\hat{n}\cdot\beta)}\right]_{ret}, \quad \mathbb{A}(\mathbb{r},t) = \left[\frac{q\beta}{R}\cdot\frac{1}{(1-\hat{n}\cdot\beta)}\right]_{ret}}. \quad (19)$$

Subscript "ret" $\Rightarrow$ RHS's here are evaluated at the retarded time: $t' = t - \frac{1}{c}R(t')$

Evidently, when $\beta \to 0$, Eqs. (19) reduce to the nonrelativistic results, viz. $\phi = q/R$ (Coulomb) & $\mathbb{A} = q\mathbb{v}/cR$ (Biot-Savart). The fully covariant treatment has done two things: (A) re-emphasize retarded time, (B) introduce the denominator factor $(1-\hat{n}\cdot\beta)$. Now we can use ϕ & $\mathbb{A}$ of (19) to calculate the fields due to q.

$\mathbb{E}$ & $\mathbb{B}$ fields generated by single q in arbitrary motion.

6) The fields for q's (arbitrary) motion can be calculated from $(\phi, \mathbb{A})$ in (19) via usual: $\mathbb{E} = -\nabla\phi - \frac{1}{c}(\partial\mathbb{A}/\partial t)$, $\mathbb{B} = \nabla \times \mathbb{A}$, or they can be calculated covariantly from Eq. (17) via: $F^{\alpha\beta} = (\partial^\alpha A^\beta - \partial^\beta \partial^\alpha)$. Jackson does the latter in his Eqs. (14.9)-(14.11). If we do the former (direct) calculation, we need to know:

... for separation: $R(t') = c(t - t')$, w/ t' = retarded time ...

$$\frac{\partial R}{\partial t} = c\left(1 - \frac{\partial t'}{\partial t}\right) = \left(\frac{\partial R}{\partial t'}\right)\left(\frac{\partial t'}{\partial t}\right) = (-)\,\hat{n}\cdot\mathbb{v}(t')\,\frac{\partial t'}{\partial t},$$

So// $$\frac{\partial t'}{\partial t} = 1/(1 - \hat{n}\cdot\beta)\Big|_{ret} \qquad (20a)$$

[in (20), we found $\partial R/\partial t'$ by differentiating the identity $R^2 = \mathbb{R}\cdot\mathbb{R}$, then setting $\partial\mathbb{R}/\partial t' = -\mathbb{v}(t')$, and $\hat{n} = \mathbb{R}/R$ (all @ t')]. We also need the operation...

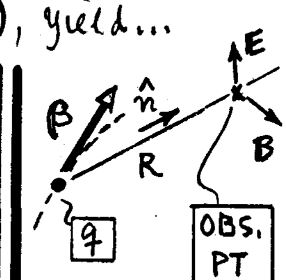
$$\nabla t' = -\frac{1}{c}\nabla R(t') = -\frac{1}{c}\left[\frac{\mathbb{R}}{R} + \left(\frac{\partial R}{\partial t'}\right)\nabla t'\right],$$

So// $$\nabla t' = -\hat{n}/c(1 - \hat{n}\cdot\beta)\Big|_{ret.} \qquad (20b)$$

Then $\mathbb{E} = -\nabla\phi - \frac{1}{c}(\partial\mathbb{A}/\partial t)$, $\mathbb{B} = \nabla\times\mathbb{A}$, from ϕ & $\mathbb{A}$ of Eqs. (19), yield...

Jk^n Eqs. 14.13-.14) (21)

$$\boxed{\begin{aligned}\mathbb{B}(\mathbb{r},t) &= [\hat{n}\times\mathbb{E}]_{ret}, \qquad \text{② \{rad^n field.\}} \qquad \text{① \{Coul. field.\}}\\ \mathbb{E}(\mathbb{r},t) &= \left[\frac{q}{cR}\left(\frac{\hat{n}\times[(\hat{n}-\beta)\times\alpha]}{(1-\hat{n}\cdot\beta)^3}\right)\right]_{ret} + \left[\frac{q}{R^2}\left(\frac{\hat{n}-\beta}{\gamma^2(1-\hat{n}\cdot\beta)^3}\right)\right]_{ret}\end{aligned}}$$



w/ Subscript "ret" ⇒ evaluation at retarded time: $\underline{t' = t - \frac{1}{c}R(t')}$ $\begin{cases} t = \text{lab time } \underline{at} \text{ obs. pt.} \\ t' = \text{lab time } \underline{at}\ q. \end{cases}$

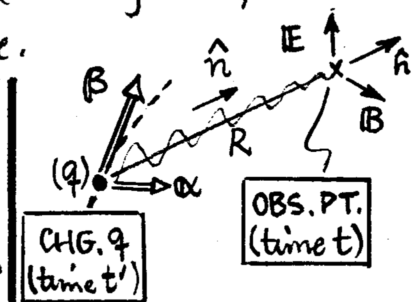
&// acceleration: $\alpha = \dot{\beta} = \frac{d}{dt'}\beta(t')$. This is α_{ret}, at source q.

[term ① = relativistic version of q's Coulomb field ($\sim \frac{q}{R^2}$). It is ~ static, with energy density $\propto E^2 \sim 1/R^4$, so $\int E^2 d(\text{vol}) \to 0$ as $R\to\infty$. This ⇒ no energy radiated to ∞.

term ② = something new... this field $E \propto 1/R$, with energy density $\propto E^2 \sim 1/R^2$, so $\int E^2 d(\text{vol.}) > 0$ as $R\to\infty$. This field can carry off energy; it is q's radiation field.

7) If we are only interested in the radiation produced by q (at large R)*, we need only deal with the fields which go as $1/R$ in Eq. (21), i.e.

$$\boxed{\mathbb{E}(\mathbf{r},t) = \frac{q}{c}\left[\frac{\hat{n}\times[(\hat{n}-\boldsymbol{\beta})\times\boldsymbol{\alpha}]}{(1-\hat{n}\cdot\boldsymbol{\beta})^3 R}\right]_{ret},\quad \mathbb{B}(\mathbf{r},t) = [n\times\mathbb{E}(\mathbf{r},t)]_{ret}} \tag{22}$$



retarded time $\}\ t' = t - \frac{1}{c}R(t')$

These are q's "radiation fields", and -- since they are both $\perp$ to the propagation direction $\hat{n}$, and $\perp$ each other -- they are a transverse wave propagating to the obsᵛn point.

The energy radiated per unit time & area to the obsᵛn pt. is measured by the Poynting vector $\mathbb{S} = (c/4\pi)\,\mathbb{E}\times\mathbb{B}$; in this case...

$$\rightarrow \mathbb{S}_{rad} = \frac{c}{4\pi}[\mathbb{E}\times(\hat{n}\times\mathbb{E})]_{ret} = \frac{c}{4\pi}[\,|\mathbb{E}|^2\hat{n} - (\underbrace{\cancel{\mathbb{E}\cdot\hat{n}}}_{0\ \text{(transverse wave)}})\mathbb{E}\,]_{ret}$$

So

$$\boxed{\mathbb{S}_{rad} = \frac{c}{4\pi}[\,|\mathbb{E}|^2\hat{n}\,]_{ret} = \left(\frac{q^2}{4\pi R^2 c}\left[\frac{\hat{n}\times[(\hat{n}-\boldsymbol{\beta})\times\boldsymbol{\alpha}]}{(1-\hat{n}\cdot\boldsymbol{\beta})^3}\right]^2\hat{n}\right)_{ret}}. \tag{23}$$

$\boxed{\textbf{REMARKS}}$ on $\mathbb{S}_{rad}$ of Eq. (23).

1. $\mathbb{S}_{rad}$ vanishes if q's acceleration $\alpha\to 0$. In order to radiate, q must accelerate.

2. If $c\to\infty$ (nonrelativistic limit), then $t'\simeq t$, and (23) is approximately

$$\rightarrow \mathbb{S}_{rad} = \frac{q^2}{4\pi R^2 c^3}\left[\hat{n}\times\left(\hat{n}\times\frac{d\mathbf{v}}{dt}\right)\right]^2\hat{n}, \quad \text{for } c\to\infty. \tag{24}$$

In a universe where $c=\infty$, there would be no radiation. The radiation we "see" (radio waves, light itself) that transports energy depends on lightspeed c = finite #.

3. In $\mathbb{S}_{rad}$ of Eq. (23), there is a bewildering amount of ∡lar dependence... the numerator involves the relative ∡'s between $\hat{n}$ & β, $\hat{n}$ & α, and β & α. The denom. $(1-\beta\cos\vartheta)$, $\vartheta = ∡(\hat{n}\,\&\,\beta)$ can → small when $\beta\parallel\hat{n}$ and $\beta\to 1$.

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* "large" R here means : $(q/cR)\alpha \gg q/R^2$, in Eq. (21). So we want: $\frac{R}{c}\alpha \gg 1$. Now $\alpha = \Delta\beta/\Delta t \sim 1/\Delta t$ (if $\Delta\beta\sim 1$ in time Δt), so: R