Physics 506 Final Exam — 16 Mar. 1972

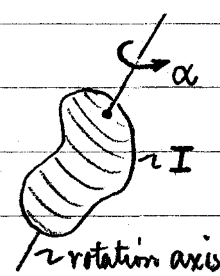
25 pts. ① In a certain QM system, it is found that the eigenfcn $\psi(x)$ of energy E is translationally invariant, i.e. if $\psi(x)$ is a solution to $H\psi = E\psi$, then so is $\psi(x+\Delta x)$, where Δx is an arbitrary translation of the coordinate origin. Shew, as a result of this, that the system's linear momentum operator p must commute with the Hamiltonian H, i.e. $[H,p]=0$, so that p is a constant of the motion, which in turn means ψ must describe a free particle.

Prob. # 1
ϕ507 Final
(May 1992)

25 pts. ② The wavefcn describing the motion of a free particle starting from (x,t) and moving to (x',t') obviously depends only on the differences between initial and final coordinates. Consequently, the free particle propagator G_0 is at most a fcn of $x'-x$ and $t'-t$. A full Fourier integral representation of G_0 must then be of the form (in 1D)

$$G_0(x'-x,t'-t) = \frac{1}{(2\pi)^2}\int_{-\infty}^{+\infty} dk \int_{-\infty}^{+\infty} d\omega\, g(k,\omega)\, e^{ik(x'-x)}\, e^{-i\omega(t'-t)}.$$

Derive an expression for $g(k,\omega)$, which is known as the free particle propagator in momentum space, by taking account of the fact that G_0 obeys the point-source Schrödinger equation.



50 pts. ③ a) A "plane rotator" is a rigid body constrained to rotate (with arbitrary angular momentum) about a fixed axis in space. The rotation can be specified by choosing a point on the body and giving its azimuthal ∡ α w.r.t. the rotation axis. Suppose the body has moment of inertia I about the rotation axis. For the QM plane rotator, solve the time-independent Schrodinger equation for the allowed energies E_m and normalized eigenfcns $\psi_m(\alpha)$ of the rotation. What is the degeneracy of the state with energy E_m?

b) Suppose that, at time $t=0$, the plane rotator of part a) above is in a state specified by the wavefcn $\Psi(\alpha,0) = C\sin^2\alpha$, where C is a normalizing constant. What will be the wavefcn $\Psi(\alpha,t)$ of this state for times $t>0$?

25 pts. ④ The expectation value of $1/r^2$ in the state $|nlm\rangle$ of a hydrogen-like atom (potential : $V(r) = -Ze^2/r$) is calculated to be

Prob. #④ ϕ507 Final (May 1992)

$$\langle 1/r^2\rangle = \langle nlm|\tfrac{1}{r^2}|nlm\rangle = \left(\tfrac{Z}{a_0}\right)^2/n^3(l+\tfrac{1}{2}), \quad a_0 = \hbar^2/me^2.$$

Use this to show that $\langle 1/r^3\rangle$ in the same state is given by

$$\langle 1/r^3\rangle = \langle nlm|\tfrac{1}{r^3}|nlm\rangle = \left(\tfrac{Z}{a_0}\right)^3/n^3 l(l+\tfrac{1}{2})(l+1).$$

Hint : do not use explicit wavefcns. Instead, cleverly look at the equation of motion for an electron in orbit.

25 pts. ⑤ Let $\vec{A} = (A_x, A_y, A_z)$ be a general QM vector operator, and consider the quantity $\vec{I} = \psi^\dagger\vec{A}\psi$, which is the integrand of the expectation value of $\vec{A}$ in the state ψ. Under a rotation of the coordinate system by an infinitesimal angle about any one of the coordinate axes, there are two equivalent ways to describe how $\vec{I}$, and hence $\langle\vec{A}\rangle$, transforms. Either ψ is transformed, leaving $\vec{A}$ unchanged (i.e. $\psi\to\psi'$, so that $\vec{I}\to\vec{I}' = \psi'^\dagger\vec{A}\psi'$), or $\vec{A}$ is transformed, leaving ψ unchanged (i.e. $\vec{A}\to\vec{A}'$, so $\vec{I}\to\vec{I}' = \psi^\dagger\vec{A}'\psi$). By equating the two equivalent forms for the transformed $\vec{I}$, derive a commutation relation between $\vec{A}$ and $\vec{J}$, where $\vec{J}$ is the total system angular momentum operator. Hint : work with one component of $\vec{I}$ at a time. Check your result for $\vec{A}=\vec{J}$. Do not make any misteaks.

25 pts. ⑥ A hydrogen-like atom (potential : $V(r) = -Ze^2/r$) is in its ground state, with total energy given by the usual Bohr formula. Calculate (i.e. get a number for) the probability that the electron will be found at a distance from the nucleus greater than its energy would permit from a classical standpoint.

Prob. #④
ϕ507 MIDTERM
(Spr. 92)

25 pts. ⑦ An operator F depends on the position vector $\vec{x}$ and particle momentum $\vec{p}$ only through the combinations $\vec{x}^2$, $\vec{p}^2$ and $\vec{x}\cdot\vec{p}$; that is, considered as a function of $\vec{x}$ and $\vec{p}$, $F = F(\vec{x}^2, \vec{p}^2, \vec{x}\cdot\vec{p})$ only. Let $\vec{L}$ be the system orbital angular momentum operator, and denote the eigenstates $Y_{lm}(\vartheta,\varphi)$ of $\vec{L}^2$ and L_z by $|lm\rangle$.

a) Calculate the commutator bracket $[\vec{L}, F]$.

b) State all that can be said about the matrix elements $\langle l'm'|F|lm\rangle$.

① [20 pts.]. Analyse consequences of translational invariance in a QM system.

(A) 1. We are given that:

$$\mathcal{H}\, u_n(x) = E_n\, u_n(x), \quad \text{and} \quad \mathcal{H}\, u_n(x+\Delta x) = E_n\, u_n(x+\Delta x). \tag{1}$$

Suppose $\Delta x \to$ infinitesimal, and expand $u_n(x+\Delta x)$ by Taylor series...

$$u_n(x+\Delta x) = u_n(x) + \Delta x \left(\frac{\partial u_n}{\partial x}\right)\Big|_{\Delta x=0} + \cdots \leftarrow \frac{\partial}{\partial x} = \frac{i}{\hbar}\, p \text{ (operator)}$$

so, $$u_n(x+\Delta x) = u_n(x) + \frac{i\Delta x}{\hbar}\, p\, u_n(x) + \cdots \tag{2}$$

The second of Eqs. (1) then gives...

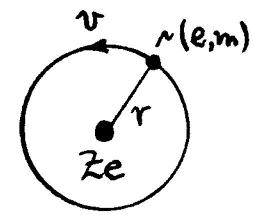
$$\mathcal{H}\left[u_n(x) + \frac{i\Delta x}{\hbar}\, p\, u_n(x) + \cdots\right] = E_n\left[u_n(x) + \frac{i\Delta x}{\hbar}\, p\, u_n(x) + \cdots\right] \tag{3}$$

terms cancel

so, $$\cancel{\left(\frac{i\Delta x}{\hbar}\right)} \mathcal{H}\, p\, u_n(x) = \cancel{\left(\frac{i\Delta x}{\hbar}\right)}\, p\, \underbrace{E_n\, u_n(x)}_{= \mathcal{H}\, u_n(x)}$$ since E_n commutes with p.

i.e., $$[\mathcal{H} p - p\mathcal{H}]\, u_n(x) = 0,$$ so, $$\underline{\underline{[\mathcal{H}, p] = 0, \text{ as required.}}} \tag{4}$$

④ [25 pts.]. For H-like atom, manufacture $\langle 1/r^3\rangle$ from $\langle 1/r^2\rangle$.



1. The equation of the electron orbit at r, viz...

$$\rightarrow \quad mv^2/r = Ze^2/r^2, \tag{1}$$

can be written in terms of the orbital ∡ momentum $L = mvr$ as:

$$\rightarrow \quad L^2/r^3 = Zme^2/r^2. \tag{2}$$

Quantum-mechanically, Eq. (2) will hold in an expectation-value sense (by Ehrenfest's Theorem: Sakurai, p. 87) and so in the state $|nlm\rangle$

$$\rightarrow \quad \langle nlm|\frac{L^2}{r^3}|nlm\rangle = \hbar^2 \frac{Z}{a_0}\langle nlm|\frac{1}{r^2}|nlm\rangle, \quad a_0 = \hbar^2/me^2. \tag{3}$$

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2. In Eq. (3), L^2 is an <u>operator</u>, which operates on the ∡ cds of $|nlm\rangle$, and which has the eigenvalue $l(l+1)\hbar^2$ in that state. Then (3) reads...

$$\rightarrow \quad l(l+1)\langle nlm|\frac{1}{r^3}|nlm\rangle = \frac{Z}{a_0}\langle nlm|\frac{1}{r^2}|nlm\rangle$$

So, $$\langle nlm|\frac{1}{r^3}|nlm\rangle = \frac{Z/a_0}{l(l+1)}\langle nlm|\frac{1}{r^2}|nlm\rangle$$

$$= \underline{\underline{(Z/a_0)^3/n^3\, l(l+1)(l+\tfrac{1}{2})}}, \tag{4}$$

as required.
