1. The Physics Dept. at MSU has just purchased a new piece of lecture demonstration equipment, an infinite frictionless plane manufactured by Infinite Plains, Inc. of Kansas. To test the infinite plane, a physics professor releases a puck with an initial velocity of 100 m/sec heading due North (the physics professor is quite strong). The infinite plane is horizontal here in Bozeman. The puck has a mass of 1 kg.

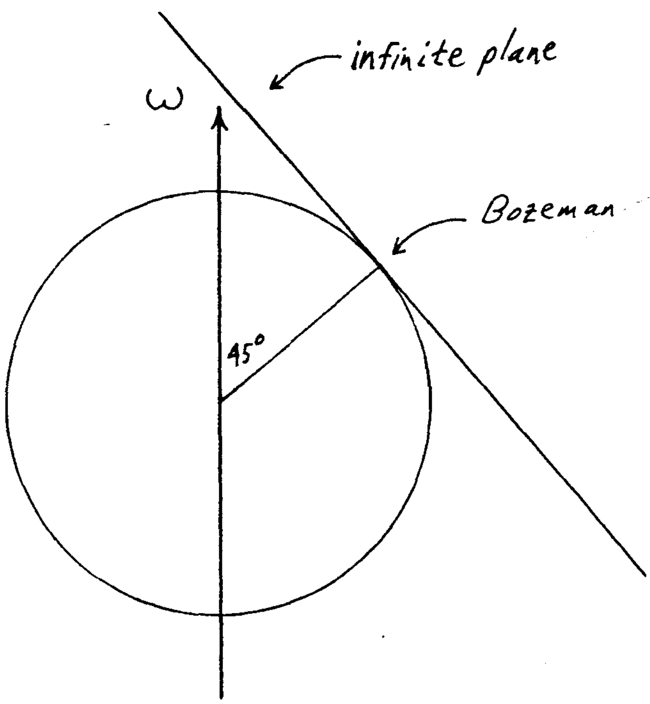
(10%) (a) As the professor watches the puck, he notices that it:
- (1) follows a straight line path
- (2) is deflected to the left
- (3) is deflected to the right
- (4) heads for Kansas, while singing "There's no place like home"

(Indicate correct description on your answer sheet.)

(35%) (b) Calculate the magnitude of the coriolis force on the puck by assuming that it has a negligible effect on the puck's motion.

(35%) (c) Using the force from (b), calculate how far the puck will be deflected from a straight line trajectory (in meters) after it has traveled 10 km.

(20%) (d) How much work has the Coriolis force done on the puck in 10 km of travel?



Note: $\omega_{Earth} = 7.27 \times 10^{-5}$ rad/sec; latitude of Bozeman = 45 degrees

Solution

(a) (iii) deflected to the right

(b) $\vec{F} = -2m\left(\vec{\omega} \times \frac{d\vec{r}}{dt}\right)$ $\quad |\vec{F}| = 2m\omega \frac{dr}{dt} \sin\theta$

$\theta = 45^\circ$ = angle between $\vec{\omega}$ & $\frac{d\hat{r}}{dt}$

$$|\vec{F}| = 2\,(1\text{kg}) \cdot \left(7.27 \times 10^{-5}\,\frac{\text{rad}}{\text{sec}}\right)\left(100\,\frac{\text{m}}{\text{sec}}\right)\left(\frac{1}{\sqrt{2}}\right)$$

$$\boxed{|\vec{F}| \simeq 1.03 \times 10^{-2}\,N}$$

(c) puck takes $t = \dfrac{10\text{ km}}{100\,\frac{\text{m}}{\text{sec}}} = 100\text{ sec}$ to travel 10 km $\quad |\vec{F}_{cor}|$ is constant

$\Rightarrow \vec{a}_{cor}$ constant $= 1.03 \times 10^{-2}\,\text{m/sec}^2$ eastward

Deflection $x = \frac{1}{2} a t^2 = \frac{1}{2}\left(1.03 \times 10^{-2}\,\frac{\text{m}}{\text{sec}^2}\right) 10^4\,\text{sec}^2 = \boxed{51.5\text{ m}}$

(d) work done:

$$W = \int_0^{100\text{ sec}} \vec{F} \cdot \vec{v}\, dt = \int_0^{100} -m\left(\vec{\omega} \times \frac{d\vec{r}}{dt}\right) \cdot \frac{d\vec{r}}{dt}\, dt = \boxed{0}$$

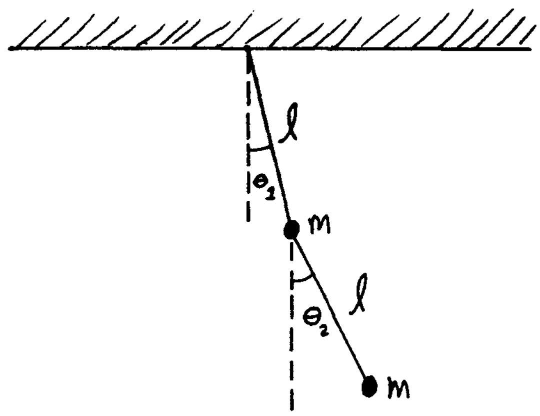
since $\vec{A} \cdot (\vec{A} \times \vec{B}) = 0$

the Coriolis force does no work!

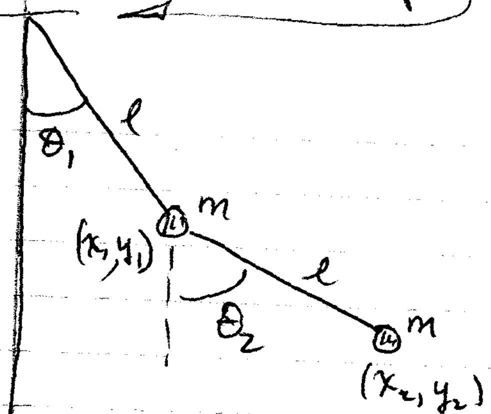
2. (a) Write the Lagrangian for the double pendulum in terms of θ_1 and θ_2 generalized coordinates.

(b) Find the equations of motion.

(c) Assume small oscillations and find the normal modes.



let $V=0$ at top.



$x_1 = l\sin\theta_1 \qquad x_2 = l(\sin\theta_1 + \sin\theta_2)$

$y_1 = l\cos\theta_1 \qquad y_2 = l(\cos\theta_1 + \cos\theta_2)$

$T = \frac{1}{2} m(\dot{x}_1^2 + \dot{x}_2^2 + \dot{y}_1^2 + \dot{y}_2^2)$

$\dot{x}_1 = l\cos\theta_1\dot{\theta}_1 \qquad \dot{x}_2 = l(\cos\theta_1\dot{\theta}_1 + \cos\theta_2\dot{\theta}_2)$

$\dot{y}_1 = -l\sin\theta_1\dot{\theta}_1 \qquad \dot{y}_2 = -l(\sin\theta_1\dot{\theta}_1 + \sin\theta_2\dot{\theta}_2)$

$$T = \frac{1}{2}ml^2\Big[(\cos^2\theta_1+\sin^2\theta_1)\dot{\theta}_1^2 + (\cos^2\theta_1+\sin^2\theta_1)\dot{\theta}_1^2 + (\cos^2\theta_2+\sin^2\theta_2)\dot{\theta}_2^2 + 2(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2)\dot{\theta}_1\dot{\theta}_2\Big]$$

$$= \frac{1}{2}ml^2\left[2\dot{\theta}_1^2 + \dot{\theta}_2^2 + 2\cos(\theta_1-\theta_2)\dot{\theta}_1\dot{\theta}_2\right]$$

a./ $V = -mgy_1 - mgy_2 = -mg[2\cos\theta_1 + \cos\theta_2]$

$$\boxed{L = \frac{ml^2}{2}\left[2\dot{\theta}_1^2 + \dot{\theta}_2^2 + 2\cos(\theta_1-\theta_2)\dot{\theta}_1\dot{\theta}_2\right] + mlg(2\cos\theta_1 + \cos\theta_2)}$$

$$\frac{d}{dt}\frac{\partial L}{\partial\dot{\theta}_1} - \frac{\partial L}{\partial\theta_1} = \frac{d}{dt}\left[ml^2 2\dot{\theta}_1 + ml^2\cos(\theta_1-\theta_2)\dot{\theta}_2\right] + ml^2\sin(\theta_1-\theta_2)\dot{\theta}_1\dot{\theta}_2$$

$-2mgl\sin\theta_1$

$$= ml^2\left[2\ddot{\theta}_1 + \cos(\theta_1-\theta_2)\ddot{\theta}_2 + \sin(\theta_1-\theta_2)\dot{\theta}_2(\cancel{\dot{\theta}_1}-\dot{\theta}_2) + \cancel{\dot{\theta}_1\dot{\theta}_2\sin(\theta_1-\theta_2)}\right] - 2mlg\sin\theta_1$$

b./

$$\boxed{= ml^2\left[2\ddot{\theta}_1 + \cos(\theta_1-\theta_2)\ddot{\theta}_2 + \sin(\theta_1-\theta_2)\dot{\theta}_2^2\right] - 2mg\sin\theta_1 = 0}$$

$$\frac{d}{dt}\frac{\partial L}{\partial\dot{\theta}_2} - \frac{\partial L}{\partial\theta_2} = \boxed{ml^2\left[\ddot{\theta}_2 + \cos(\theta_1-\theta_2)\ddot{\theta}_1 - \sin(\theta_1-\theta_2)\dot{\theta}_1^2\right] - mg\sin\theta_2 = 0}$$

Small angle $\cos(\theta_1-\theta_2) \approx 1 \quad \sin(\theta_1-\theta_2) \approx (\theta_1-\theta_2) \quad (\theta_1-\theta_2)\dot{\theta}^2 \approx 0$.

$$2\ddot{\theta}_1 l - \ddot{\theta}_2 l = 2g\theta_1$$
$$\ddot{\theta}_2 l + \ddot{\theta}_1 l = g\theta_2$$

$$\boxed{\begin{aligned} 2\ddot{\theta}_1 - \ddot{\theta}_2 &= 2g/l\,\theta_1 \\ +\ddot{\theta}_1 + \ddot{\theta}_2 &= g/l\,\theta_2 \end{aligned}}$$

let $\theta_1 = a_1 e^{i\omega t}$ $\quad \theta_2 = a_2 e^{i\omega t}$

$$-2\omega^2 a_1 + \omega^2 a_2 = 2g/l\ a_1$$
$$\mp\omega^2 a_1 \mp \omega^2 a_2 = g/l\ a_2$$

normal modes:

$$\begin{vmatrix} \left(2\omega^2 + \frac{2g}{l}\right)a_1 & -\omega^2 a_2 \\ \left(\omega^2 + \cancel{}\right)a_1 & +\left(\omega^2 + g/l\right)a_2 \end{vmatrix} = 0$$

$$\cancel{-2\omega^4 - \frac{2g}{l}\omega^2 + \omega^4 - \omega^2 \frac{g}{l}} = 0$$

$$2\omega^4 + 4\frac{g}{l}\omega^2 - \omega^4 + 2\left(\frac{g}{l}\right)^2 = 0$$

$$\omega^4 + 4\frac{g}{l}\omega^2 + 2\left(\frac{g}{l}\right)^2 = 0$$

$$\omega^2 = \frac{-4\frac{g}{l} \pm \sqrt{16\left(\frac{g}{l}\right)^2 - 8\left(\frac{g}{l}\right)^2}}{2} = -2\frac{g}{l} \pm \sqrt{2\left(\frac{g}{l}\right)^2}$$

$$\boxed{\omega^2 = \pm\left(\frac{g}{l}\left(-2 \pm \sqrt{2}\right)\right)}$$

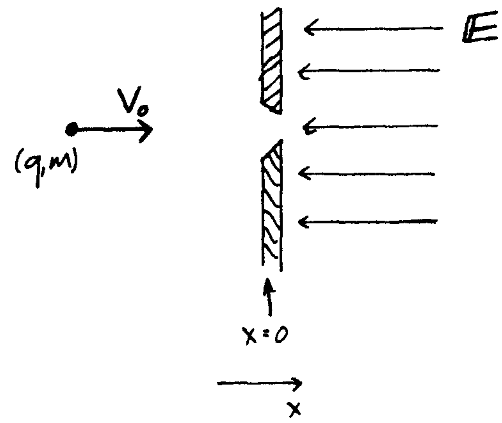
c/ frequencies:

$$\boxed{f_1 = \frac{1}{2\pi}\sqrt{\frac{g}{l}(2+\sqrt{2})} \qquad f_2 = \frac{1}{2\pi}\sqrt{\frac{g}{l}(2-\sqrt{2})}}$$

3. A particle of charge q and mass m travels at <u>relativistic</u> velocity v_0 along the x-axis of the lab frame. At $x=0$, q passes through a small hole in one plate of a capacitor (fixed in lab) and encounters a constant electric field $\boldsymbol{E} = -E\hat{x}$ which opposes its motion.

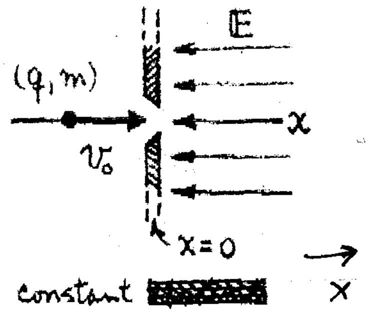
(a) Find the distance s (in lab) which q travels to the right of the plate before it <u>stops</u>. Do this part of the problem relativistically, but ignore radiation by q.

(b) For a <u>rough</u> estimate of radiation effects, assume q's motion is nonrelativistic. Compare the radiated energy to the initial kinetic energy. Is this ratio large or small?



for MSU Ph.D. Comprehensive Exam : Sept. 1990 — 8/27/90

PROBLEM



A particle of charge q and mass m travels at relativistic velocity v_0 along the x-axis of the lab frame. At $x=0$, q passes through a small hole in one plate of a capacitor (fixed in lab) and encounters a constant electric field $\mathbb{E} = -E\hat{x}$ which opposes its motion.

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(B) For a rough estimate of radiation effects, assume q's motion is nonrelativistic. Compare the radiated energy to the initial kinetic energy. Is this ratio large or small?*

SOLUTION

(A) Use relativistic work-energy relation : $\mathbb{F}\cdot\mathbf{v} = \frac{d}{dt}(\gamma mc^2)$, $\mathbf{F}$ = lab force on m.

For 1D motion in $\mathbb{F} = -q\mathbb{E}$, this translates to...

$$-qE\frac{dx}{dt} = mc^2\frac{d\gamma}{dt}, \quad \text{w/} \quad \int dx = -(mc^2/qE)\int d\gamma .$$

Integrate from $(x=0, \gamma = 1/\sqrt{1-(v_0/c)^2})$ at entry, to $(x=s, \gamma=1)$ at stop, to get :

$$\boxed{s = -(mc^2/qE)\int_{\gamma_0}^{1} d\gamma = \frac{mc^2}{qE}(\gamma_0 - 1)}, \quad \gamma_0 = 1/\sqrt{1-(v_0/c)^2}.$$

(B) Use nonrel^{c} Larmor formula for radiated power : $P = \frac{2}{3}(q^2/c^3)|\mathbf{a}|^2$, where $|\mathbf{a}|$ is q's deceleration ; $|\mathbf{a}| = qE/m$ = cnst, in this case. q's time-to-stop is $T = v_0/|\mathbf{a}|$ (for cnst deceleration) so the total radiated energy is $\mathcal{E} = PT$. Desired ratio is :

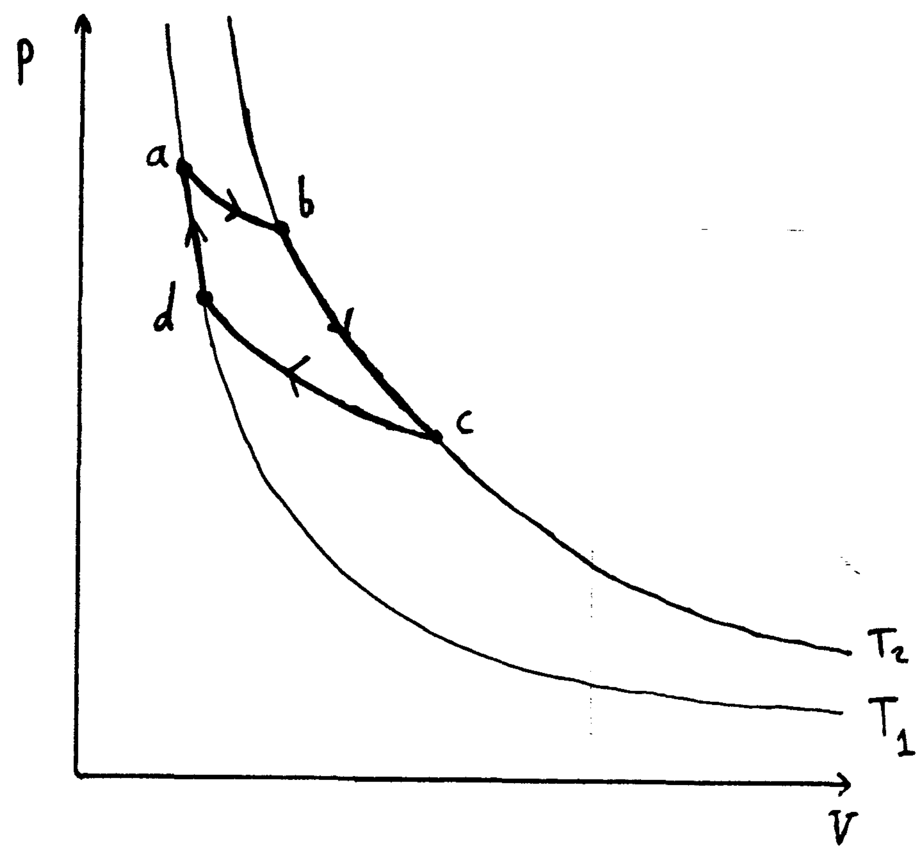
$$\boxed{\frac{\text{radiated energy}}{\text{kinetic energy}} = \frac{PT}{\frac{1}{2}mv_0^2} = \frac{4}{3}\cdot\frac{(q^2/c^3)|\mathbf{a}|^2}{mv_0^2}\cdot\frac{v_0}{|\mathbf{a}|} = \dots = \frac{4}{3}\left(\frac{E}{q/r_0^2}\right)\frac{1}{v_0/c}},$$

here $r_0 = q^2/mc^2$ is q's classical radius. Even for $v_0/c \to$ small, this ratio is very small, since -- with $r_0 \sim 10^{-13}$ cm -- the field $\frac{q}{r_0^2} \sim 10^{16}\,\frac{\text{statV}}{\text{cm}}$ is $\ggg$ any laboratory E.

4. A Carnot engine uses n moles of an ideal gas and operates between isotherms T_1 and T_2 as shown.

(a) Derive an expression for the efficiency of the Carnot engine.

(b) Prove that no other engine is more efficient than a Carnot engine.



Problem 4

Solution:

A) $W_{ab} = nC_v(T_2 - T_1)$ (since $\Delta Q = 0$ along the adiabatic and $W = \Delta U = nC_v \Delta T$)

?

$W_{bc} = nRT_2 \ln \frac{V_c}{V_b}$

$W_{cd} = nC_v(T_1 - T_2)$

$W_{da} = nRT_1 \ln \frac{V_a}{V_d}$

$$\text{Efficiency} = \frac{W_{ab} + W_{bc} + W_{cd} + W_{da}}{Q_{in}}$$

In isothermal expansion, $\Delta U = 0 \Rightarrow Q_{bc} = W_{bc}$

$$\eta = \frac{\cancel{nC_v(T_2 - T_1)} + nRT_2 \ln \frac{V_c}{V_b} + \cancel{nC_v(T_1 - T_2)} + nRT_1 \ln \frac{V_a}{V_d}}{nRT_2 \ln \frac{V_c}{V_b}}$$

For adiabatic: $T_1 V_a^{\gamma-1} = T_2 V_b^{\gamma-1}$, $T_1 V_d^{\gamma-1} = T_2 V_c^{\gamma-1}$ $\Rightarrow \frac{V_a}{V_d} = \frac{V_b}{V_c}$

$$\eta = \frac{T_2 \ln \frac{V_c}{V_b} - T_1 \ln \frac{V_c}{V_b}}{T_2 \ln \frac{V_c}{V_b}} = \frac{T_2 - T_1}{T_2}$$

B) Suppose one exists. Since Carnot engine is reversible, reverse it and use the more efficient one to drive the Carnot refrigerator. Contradiction of 2nd law

5. Consider an atom made up of an electron and a singly-charged ($Z=1$) triton (^3H). Initially the system is in its ground state. Suppose that somehow the nuclear charge suddenly increases by one unit. (By suddenly we mean that this occurs so fast that the electron wave function has no time to readjust during the charge-increase process -- which may be due to emission of an electron and antineutrino). By this process the nucleus turns into a helium nucleus ($Z=2$) of mass 3 (*i.e.*, ^3He). Obtain the probability for the system to be found in the ground state of the resulting helium ion.

The wave function for the ground state of a hydrogenic atom is given by

$$\psi_{n=1,\, \ell=0,\, m=0}(\vec{x}) = \frac{1}{\sqrt{\pi}} \left(\frac{Z}{a_0}\right)^{3/2} \exp(-Zr/a_0)$$

where a_0 is the Bohr radius.

Solution:

Since the electron wave function does not change discontinuously, it remains in the ground state of ^3H for a short while, before it "leaks" into an eigenstate of ^3He. Thus all we require is the overlap between the initial-state wave function with that for the ground state of ^3He.

$$^3H : \quad \psi(r) = \frac{1}{\sqrt{\pi}} \frac{1}{a_0^{3/2}} e^{-r/a_0}$$

$$^3He : \quad \psi(r) = \frac{1}{\sqrt{\pi}} \left(\frac{2}{a_0}\right)^{3/2} e^{-2r/a_0}$$

The amplitude for the process of interest is:

$$A = \langle \psi(^3He) \mid \psi(^3H) \rangle =$$

$$= \int d^3x \; \psi^*_{^3He}(\vec{x}) \; \psi_{^3H}(\vec{x}) =$$

$$= 4 \left(\frac{2}{a_0^2}\right)^{3/2} \left(\frac{a_0}{3}\right)^3 \int_0^\infty dx \; x^2 e^{-x}$$

$$= 16 \frac{\sqrt{2}}{27}$$

Thus the probability is

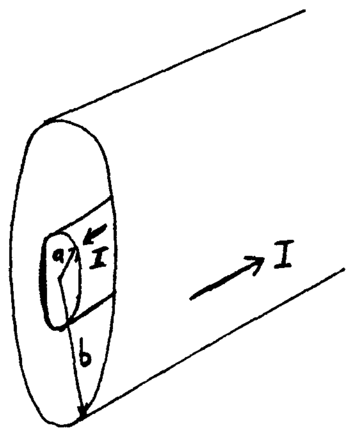
$$P = |A|^2 = \left(\frac{16}{27}\right)^2 \times 2 \cong 0.70$$

$$\Rightarrow \sim 70\% \text{ "chance"}$$

6. A coaxial cable has an inner wire of radius a and an outer metal sheath of radius b, as shown below. A current I flows down the inner wire and returns through the outer sheath.

(a) Calculate the self-inductance per unit length of the cable assuming that the current in the inner wire flows on the surface of the inner wire.

(b) Calculate the self-inductance per unit length if the current is uniformly distributed throughout the central wire. Compare and contrast your answers under these two assumptions.

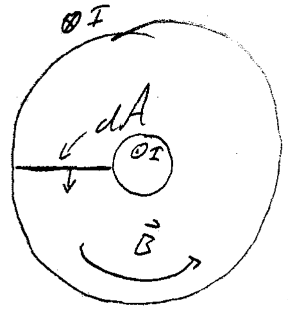


Solution:

(a) if current is all on the surface of the inner wire, then by Ampere's law,

$$\vec{B} = 0 \quad r < a$$

$$B = \frac{\mu_0 I}{2\pi r} \quad b > r > a$$



Take surface shown at left to compute ϕ

$$\phi = \int \vec{B} \cdot d\vec{A} = \int_0^{\ell} \int_{r=a}^{r=b} \frac{\mu_0 I}{2\pi r}\, dr\, dz$$

$$= \frac{\mu_0 I}{2\pi} \ln(b/a)\, \ell$$

$\vec{B}$, $d\vec{A}$ parallel

$$\mathcal{E} = -N \frac{d\phi}{dt} \qquad N = 1, \text{ so}$$

$$\mathcal{E} = -\frac{\mu_0 \ell}{2\pi} \ln(b/a) \frac{dI}{dt} = -L \frac{dI}{dt}$$

inductance/unit $= \boxed{L/\ell = \frac{\mu_0}{2\pi} \ln(b/a)}$

(b) Now assume I uniformly distributed in interior

$b > r > a \qquad B = \mu_0 I / 2\pi r$ still

but for $r < a$, Ampere's law tells us:

$$B = \frac{\mu_0\, i(\text{inside})}{2\pi r} = \frac{\mu_0 I r^2}{2\pi r a^2} = \frac{\mu_0 I r}{2\pi a^2}$$

Now surface for computing ϕ must reach $r = 0$ since $B \neq 0$ from $r = a$ to $r = b$

$$\phi = \int_0^{\ell} \left\{ \int_0^a \frac{\mu_0 I r}{2\pi a^2}\, dr + \int_a^b \frac{\mu_0 I}{2\pi r}\, dr \right\} dz$$

$$= \ell \left\{ \frac{\mu_0 I}{4\pi} + \frac{\mu_0 I}{2\pi} \ln(b/a) \right\}$$

new flux for $r < a$

same old flux from $b > r > a$

$$\frac{\partial \phi}{\partial t} = -\mathcal{E} = \frac{\mu_0 \ell}{2\pi}\left[\frac{1}{2} + \ln(b/a)\right]\frac{dI}{dt}$$

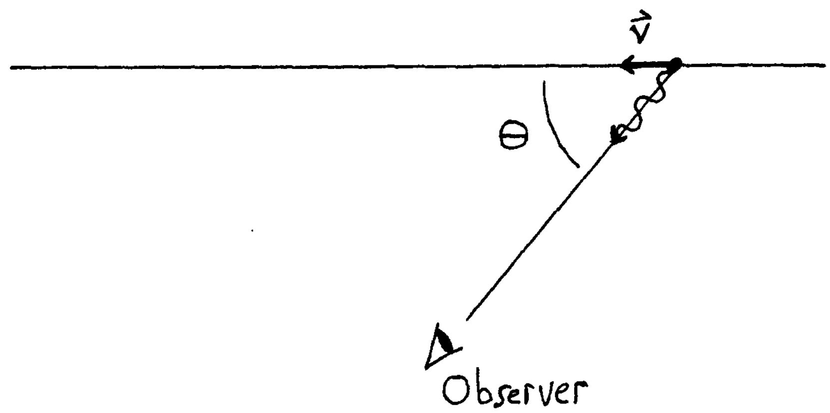
$$\Rightarrow \text{inductance/unit length} = \boxed{\frac{L}{\ell} = \frac{\mu_0}{2\pi}\left[\frac{1}{2} + \ln(b/a)\right]}$$

Comparing answer to part (a) shows that what is assumed about the location of the current can be very important; how important depends on the ratio of $\ln(b/a)$ to $1/2$!

7. A source of light moves with velocity $\vec{v}$ past an observer (see diagram). The light is emitted with frequency ν_0 in the rest frame of the source.

(a) Find the frequency measured by the observer as a function of $|\vec{v}|$ and θ.

(b) Is there any angle θ for which the observer sees no blueshift or redshift, so that $\nu_{observed}=\nu_0$? If so, find an expression for θ as a function of $|\vec{v}|$.

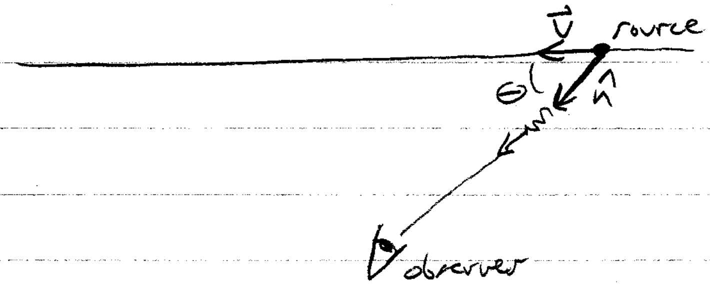


Solution: There are three four-velocities to consider: in the rest frame of the observer, they have components:

$$U_{obs} = (1, \vec{0})$$

$$U_{source} = (\gamma, \gamma\vec{\beta}) \qquad \vec{\beta} = \vec{V}/c, \quad \gamma = (1-\vec{\beta}\cdot\vec{\beta})^{-\frac{1}{2}}$$

four momentum $p_{light} = E(1, \hat{n})$ where $\hat{n}$ is the unit vector pointing along the photon's path from the source to the observer



The energy measured by an observer with four-velocity u is $-u\cdot p_{light}$

so: $$E_{emitted} = -U_{source}\cdot p_{light} = \gamma E(1-\vec{\beta}\cdot\hat{n})$$

but we know $E_{emitted} = h\nu_0 \Rightarrow E = h\nu_0/\gamma(1-\vec{\beta}\cdot\hat{n})$

The energy measured by the observer is then

$$E_{obs} = h\nu_{observed} = -U_{obs}\cdot p_{light} = E$$

$$\cancel{h}\nu_{obs} = \frac{\cancel{h}\nu_0}{\gamma(1-\vec{\beta}\cdot\hat{n})}$$

so

$$\boxed{\nu_{observed} = \frac{\nu_0}{\gamma(1-\vec{\beta}\cdot\hat{n})}}$$

now, $\vec{\beta}\cdot\hat{n} = \frac{|\vec{V}|}{c}\cos\theta$

so

$$\boxed{\nu_{observed} = \frac{\left[1-\left(\frac{V}{c}\right)^2\right]^{1/2}}{\left[1-\frac{V}{c}\cos\theta\right]}\nu_0}$$

(b) can $\nu_{observed} = \nu_0$? $\quad \left[1-\left(\frac{V}{c}\right)^2\right]^{1/2} \stackrel{?}{=} 1-\frac{V}{c}\cos\theta$

$$\cos\theta = \frac{c}{V}\left\{1-\left[1-\left(\frac{V}{c}\right)^2\right]^{1/2}\right\}$$

$\boxed{\text{Yes}}$ since $\{\}$ is always < 1, so there exists an angle $\rightarrow$ $\boxed{\theta_0 \equiv \cos^{-1}\left\{\frac{c}{V}\left[1-\left[1-\left(\frac{V}{c}\right)^2\right]^{1/2}\right]\right\}}$

8. (a) Consider the interesting case of the Fe^{2+} ion in a crystal field. It has four unpaired electrons coupled to give it an orbital angular momentum, which we will ignore (it is largely quenched in the crystal field) and a spin fivefold degenerate $S=2$. If the crystal field is axial, the Hamiltonian representing it can be written as DS_z^2 (D is a constant) so as to give an interaction with the spin system. Calculate the new energy levels of the ion in this axial field to first order perturbation.

(b) Further, if the symmetry of the crystal field is slightly lower, such as orthorhombic, the additional perturbation Hamiltonian can be written as $E(S_x^2-S_y^2)$. (E is a constant.) Calculate the energy levels in this field. (Do not consider normalization.)

(c) The unperturbed states are $|S,M_S\rangle$, *i.e.*, $|2,2\rangle$, $|2,1\rangle$, $|2,0\rangle$, $|2,-1|$, $|2,-2\rangle$. What are the approximate states in the orthorhombic crystal field? (Do not consider normalization.)

Ph.D. Comps Drumheller prob. 8

Quantum Mechanics

a./ Consider the interesting case of the Fe^{2+} ion in a crystal field. It has four unpaired electrons coupled to give it an orbital angular momentum, which we will ignore (it is largely quenched in the crystal field) (five fold degenerate) and a spin $S = 2$. ~~The~~ If the crystal field is axial, the Hamiltonian representing it can be written as DS_z^2 (D is a constant) so as to give an interaction with the spin system. Calculate the new energy levels of the ion in this axial field to first order perturbation.

b./ Further, if the symmetry of the crystal field is slightly lower, such as ~~tetragonal~~ orthorhombic, the ~~new~~ additional perturbation Hamiltonian ~~is~~ can be written as $E(S_x^2 - S_y^2)$ (E is a constant). Calculate ~~to second order, if necessary~~, the energy levels in this field. (Do not consider normalization)

c./ The unperturbed states are $|S, M_S\rangle$ (M or μ?), i.e., $|2,2\rangle$, $|2,1\rangle$, $|2,0\rangle$, $|2,-1\rangle$, $|2,-2\rangle$. What are the approximate states ~~[illegible]~~ in ~~[illegible]~~ the orthorhombic ~~tetragonal~~ crystal field? (Do not consider normalization)

~~too much?~~ omit

~~d./ Discuss what kind of crystal field symmetry term (in terms of spin operators) would have to be present [illegible] for a $\Delta M = \pm 1$ transition to be able to occur between the $|+2\rangle$ and $|-2\rangle$ levels.~~

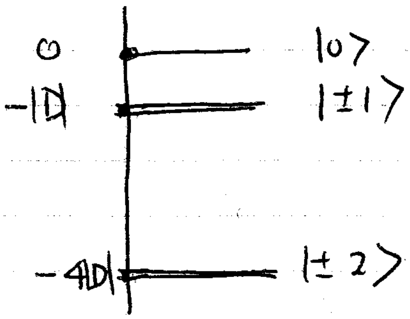
$S = 2 \qquad M_s = 2, 1, 0, -1, -2 \qquad |S, M_s\rangle \Rightarrow |M_s\rangle$

matrix elements

$$D S_z^2 |\pm 2\rangle = 4D |\pm 2\rangle \qquad \mathcal{H}' = D S_z^2$$

$$D S_z^2 |\pm 1\rangle = D |\pm 1\rangle$$

$$D S_z^2 |0\rangle = 0$$



$$S_x = \frac{1}{2}(S_+ + S_-) \qquad S_y = \frac{1}{2i}(S_+ - S_-)$$

$$S_\pm |M\rangle = \sqrt{S(S+1) - M(M \pm 1)}$$

$$\quad 2 \; 3 \qquad -2 \; 1$$

Tetragonal symmetry:

$$\mathcal{H}'' = E(S_x^2 - S_y^2) = \frac{1}{4} E \left[S_+^2 + S_-^2 + S_+S_- + S_-S_+ + S_+^2 + S_-^2 - S_+S_- - S_-S_+ \right]$$

$$= \frac{1}{2} E (S_+^2 + S_-^2)$$

Matrix elements

$$\frac{E}{2}(S_+^2 + S_-^2)\,|2\rangle = \frac{E}{2}\, 2\, S_-|1\rangle = E\sqrt{6}\,|0\rangle$$

$$\frac{E}{2}(\quad " \quad)\,|1\rangle = \frac{E}{2}\sqrt{6}\, S_-|0\rangle = \frac{E}{2}\, 6\, |-1\rangle = 3E|-1\rangle$$

$$\frac{E}{2}(\quad " \quad)\,|0\rangle = E\sqrt{6}\,|2\rangle + E\sqrt{6}\,|-2\rangle$$

$$\frac{E}{2}(\quad " \quad)\,|-1\rangle = 3E\,|+1\rangle$$

$$\frac{E}{2}(\quad " \quad)\,|-2\rangle = \sqrt{6}E\,|0\rangle$$

Energy levels. $\quad E_m = E_m^{(0)} + \sum \frac{|\langle k|\mathcal{H}''|m\rangle|^2}{E_m^{(0)} - E_k^{(0)}}$

$$E_2 = E_2^{(0)} + \frac{6E^2}{-4D} = E_2^{(0)} - \frac{3}{2}\frac{E^2}{D}$$

$$E_{-2} = E_{-2}^{(0)} + \;" \; = E_{-2}^{(0)} - \frac{3}{2}\frac{E^2}{D}$$

} curiously these are not split, they are moved

$$E_1 = E_1^{(0)} + \frac{9E^2}{0} \quad \therefore \text{ degenerate perturbation theory}$$

E_{-1} = same $\longrightarrow$ " " "

} curiously these are split by degenerate p.th.

$$E_0 = E_0^{0} + \frac{6E^2}{4D} + \frac{6E^2}{4D} = E_0 + \frac{3E^2}{D}$$

Note this one is moved by the appropriate amt.

For $|\pm 1\rangle$ degeneracy, choose new states $|+\rangle = \frac{1}{\sqrt{2}}\left(|1\rangle + |-1\rangle\right)$ and $|-\rangle = \frac{1}{2}\left(|1\rangle - |-1\rangle\right)$

Then $\frac{E}{2}\left(S_+^2+S_-^2\right)\left|\frac{|1\rangle+|-1\rangle}{\sqrt{2}}\right\rangle = \frac{E}{2}\left|\frac{3\cancel{E}|-1\rangle+3\cancel{E}|+1\rangle}{\sqrt{2}}\right\rangle$

$$= 3E\left|\frac{|1\rangle+|-1\rangle}{\sqrt{2}}\right\rangle$$

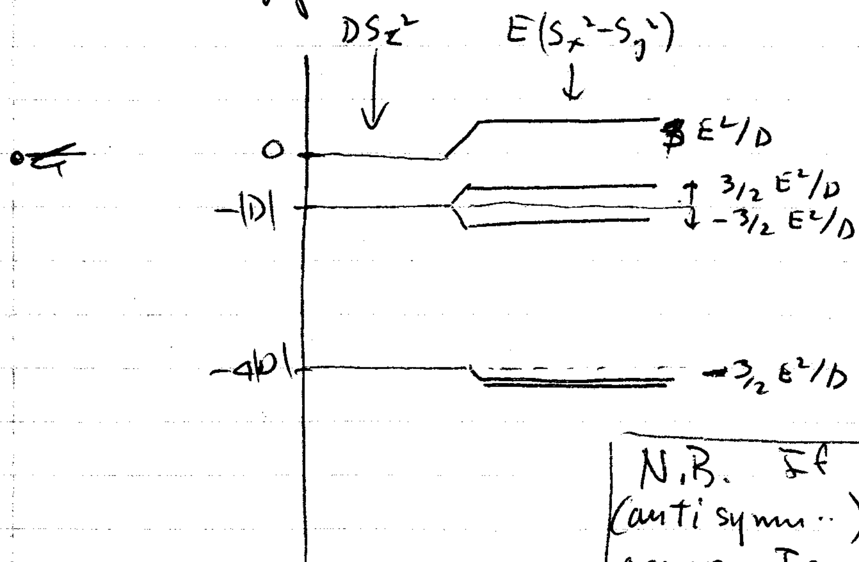
i.e., $\frac{E}{2}\left(S_+^2+S_-^2\right)|+\rangle = 3E|+\rangle$

Also $\frac{E}{2}\left(S_+^2+S_-^2\right)\left|\frac{|1\rangle-|-1\rangle}{\sqrt{2}}\right\rangle = \frac{E}{2}\left|3\frac{|-1\rangle-|1\rangle}{\sqrt{2}}\right\rangle$

$$= -3E\left|\frac{|1\rangle-|-1\rangle}{\sqrt{2}}\right\rangle$$

or $\frac{E}{2}\left(S_+^2+S_-^2\right)|-\rangle = -3E|-\rangle$

New energy levels are



Ψ's

$|0\rangle = |0\rangle^0 + \frac{\sqrt{6}}{4}\frac{E}{D}|2\rangle + \frac{\sqrt{6}}{4}\frac{E}{D}|-2\rangle$

$|+\rangle$

$|-\rangle$

$|\pm 2\rangle = |2\rangle^0 + \frac{\sqrt{6}}{4}\frac{E}{D}|0\rangle$

N.B. If the $|\pm 2\rangle$ are symmetrized (antisymm..), then a splitting does occur. I.e., if the 3X3 matrix $|\pm 2\rangle, |0\rangle$ is diagonalized the $\frac{(|2\rangle+|-2\rangle)}{\sqrt{2}}$ state moves and the $\frac{(|2\rangle-|-2\rangle)}{\sqrt{2}}$ doesn't.

$$|2\rangle = |2\rangle^0 + \frac{\sqrt{6}E}{4D}|0\rangle$$

$|-2\rangle =$ same

$$|0\rangle = |0\rangle^0 + \frac{\sqrt{6}E}{4D}|2\rangle + \frac{\sqrt{6}E}{4D}|-2\rangle$$

9. (a) Solve the spin-precession problem for a spin-1/2 particle working entirely in the Heisenberg picture. Interpret your answer physically.

(b) Suppose that at $t=0$ the spin state corresponds to "spin-up" for the x-component of the spin. (We take the external magnetic field to be directed along the z-axis.) Using your result to *a*, obtain the mean value of the measurements of the x-, y-, and z- components of the spin, performed at a later time t.

Quantum Mechanics

Solve the spin-precession problem for a spin-1/2 particle working entirely in the Heisenberg picture. Interpret your answer physically.

Solution.

The coupling Hamiltonian is:

$$\hat{H} = -\hat{\underset{\sim}{M}}.\underset{\sim}{B} = -\gamma\,\hat{\underset{\sim}{S}}.\underset{\sim}{B}$$

where γ is the gyromagnetic ratio. We are at liberty to pick the z-axis along the direction of the B-field, with the result that

$$\hat{H} = -\gamma B \hat{S}_z = \omega_0 \hat{S}_z ,$$

where $\omega_0 = -\gamma B$.

The Heisenberg equation of motion for an arbitrary observable $\hat{A}$ is (for a time-independent $\hat{H}$)

$$i\hbar \frac{d}{dt}\hat{A}(t) = [\hat{A}(t), \hat{H}]$$

Then:

$$\begin{aligned} i\hbar \frac{d}{dt}\hat{S}_x(t) &= [\hat{S}_x(t), \omega_0 \hat{S}_z(t)] \\ &= \omega_0 [\hat{S}_x(t), \hat{S}_z(t)] = -i\hbar\omega_0 \hat{S}_y(t), \end{aligned} \quad (1)$$

where we have made use of the angular momentum equal-time commutation relation

$$[\hat{S}_\alpha , \hat{S}_\beta] = \epsilon_{\alpha\beta\gamma} \hat{S}_\gamma .$$

Similarly,

$$i\hbar \frac{d}{dt} \hat{S}_y(t) = [\hat{S}_y(t), \omega_0 \hat{S}_z(t)] =$$

$$= \omega_0 [\hat{S}_y(t), \hat{S}_z(t)] = i\hbar \omega_0 \hat{S}_x(t) \tag{2}$$

and

$$i\hbar \frac{d}{dt} \hat{S}_z(t) = [\hat{S}_z(t), \omega_0 \hat{S}_z(t)] = 0 . \tag{3}$$

We then have that

$$\frac{d}{dt} \hat{S}_x(t) = - \omega_0 \hat{S}_y(t) \tag{1'}$$

$$\frac{d}{dt} \hat{S}_y(t) = \omega_0 \hat{S}_x(t) \tag{2'}$$

$$\frac{d}{dt} \hat{S}_z(t) = 0 \tag{3'}$$

$$(1') \Rightarrow \quad \frac{d^2}{dt^2}\hat{S}_x(t) = -\omega_0 \frac{d}{dt}\hat{S}_y(t)$$

$$(2') \Rightarrow \quad = -\omega_0^2 \hat{S}_x(t)$$

$\therefore$

$$\left(\frac{d^2}{dt^2} + \omega_0^2\right)\hat{S}_x(t) = 0 \qquad (4)$$

Similarly:

$$(2') \Rightarrow \quad \frac{d^2}{dt^2}\hat{S}_y(t) = \omega_0 \frac{d}{dt}\hat{S}_x(t)$$

$$(1') \Rightarrow \quad = -\omega_0^2 \hat{S}_y(t)$$

$\therefore$

$$\left(\frac{d^2}{dt^2} + \omega_0^2\right)\hat{S}_y(t) = 0 \qquad (5)$$

We then have that:

$$\hat{S}_x(t) = \hat{S}_x(0)\cos\omega t - \hat{S}_y(0)\sin\omega t$$

$$\hat{S}_y(t) = \hat{S}_y(0)\cos\omega t + \hat{S}_x(0)\sin\omega t$$

$$\hat{S}_z(t) = \hat{S}_z(0)$$

Our solution clearly corresponds to "spin precession" about the direction of the magnetic field.

b) $t=0$: $|\psi\rangle = |+\rangle_x$

In the Heisenberg picture the state vector is time-independent.

$\therefore$

$$\langle \hat{S}_x(t)\rangle = {}_x\langle +|\hat{S}_x(t)|+\rangle_x =$$

$$= \cos\omega t \; {}_x\langle +|\hat{S}_x(0)|+\rangle_x - \sin\omega t \; \underbrace{{}_x\langle +|\hat{S}_y(0)|+\rangle_x}_{=0}$$

$$= \frac{\hbar}{2}\cos\omega t$$

$$\langle \hat{S}_y(t)\rangle = {}_x\langle +|\hat{S}_y(t)|+\rangle_x$$

$$= \sin\omega t \; {}_x\langle +|\hat{S}_x(0)|+\rangle_x$$

$$= \frac{\hbar}{2}\sin\omega t$$

$$\langle \hat{S}_z(t)\rangle = {}_x\langle +|\hat{S}_z(0)|+\rangle_x$$

$$= 0$$

10. Assume the expansion of the universe is due to matter carrying a net electronic charge. Consider a spherically symmetric universe containing un-ionized hydrogen atoms of uniform density n atoms/unit volume. Assume the proton and electron charges are slightly different, *i.e.* $|e(\text{proton})/e(\text{electron})| = 1 + \beta$,, with $|\beta| << 1$ but β non-zero.

(a) Find the minimum value β_m of β at which this universe begins expanding.

(b) Assume the density n remains constant due to continuous creation of matter. For $\beta > \beta_m$, show that the repulsive force on an atom is proportional to r, its radial distance from the center of the universe. Show the atom's radial velocity $\propto r$, also.

(c) Show that this universe expands exponentially in time.

(d) Write the atom's radial velocity as $V_r = r/T$, with T the time required for expansion by factor e. If $T \sim 10^{10}$ years (age of universe) and the observed average density $n \sim 6$ atoms/m^3, find the size of β needed to "explain" the cosmic expansion.

NOTE: $e = 1.6 \times 10^{-19}$ coul, $M(\text{hydrogen}) = 1.67 \times 10^{-27}$ kg

$1/4\pi\epsilon_0 = 9 \times 10^9$ (MKS), $G = 6.67 \times 10^{-11}$ (MKS)

PROBLEM

"Assume the expansion of the universe is due to matter carrying a net electric charge. Consider a spherically symmetric universe containing un-ionized hydrogen atoms of uniform density n atoms/unit volume. Assume the proton and electron charges are slightly different, i.e. $|e(\text{proton})/e(\text{electron})| = 1+\beta$, with $|\beta| \ll 1$ but β non-zero.

(A) Find the minimum value β_m of β at which this universe begins expanding.

(B) Assume the density n remains constant due to continuous creation of matter. For $\beta > \beta_m$, show that the repulsive force on an atom is proportional to r, its radial distance from the center of the universe. Show the atom's radial velocity $\propto r$, also.

(C) Show that this universe expands exponentially in time.

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NOTE: $e = 1.6\times10^{-19}$ coul., $M(\text{hydrogen}) = 1.67\times10^{-27}$ kgm

$1/4\pi\epsilon_0 = 9\times10^{9}$ MKS, $G = 6.67\times10^{-11}$ MKS.

SOLUTION

(A) Excess charge βe on each atom can -- at separation r -- cause a net repulsion...

$$\rightarrow f_r = \overset{\text{e'static}}{k}\frac{(\beta e)^2}{r^2} - \overset{\text{gravitn}}{G}\frac{M^2}{r^2} = \frac{ke^2}{r^2}(\beta^2-\beta_m^2), \quad \beta_m = \sqrt{\frac{GM^2}{ke^2}}$$

w/ $k = 1/4\pi\epsilon_0 = 9\times10^9$ MKS, etc. Numerically: $\beta_m \simeq 0.90\times10^{-18}$, and for $\beta > \beta_m$ this universe expands.

(B) f_r is an inverse square law force, so Gauss' Law applies. Net force on an atom at surface of a sphere of radius r (anchored at center of universe) is: $F_r = N f_r$, where N = # atoms inside = $\left(\frac{4\pi}{3}r^3\right)n$, n = cnst. So we get F_r proportional to r, as...

$$\boxed{F_r = Kr, \text{ with: } K = \frac{4\pi}{3}ke^2(\beta^2-\beta_m^2).}$$

(next page)

part (B) cont'd

For above radial repulsive force F_r, eqtn-of-motion for radial velocity v_r is...

$$M\frac{dv_r}{dt} = F_r, \text{ or } Mv_r\frac{dv_r}{dr} = Kr \Rightarrow \boxed{v_r = \Omega r,\quad \Omega = \sqrt{K/M}}$$

So $v_r \propto r$, as advertised.

(C) $v_r = \frac{dr}{dt} = \Omega r \Rightarrow r(t) = r(0)\,e^{\Omega t}$, an exponential expansion in time.

(D) $v_r = r/T$, $T = 1/\Omega$ is the e-expansion time. But $\Omega = \sqrt{K/M}$, so the K-value needed for a given T is : $K = M/T^2$. For $T = 10^{10}$ yrs, this requires...

$$(\beta^2 - \beta_m^2)\,\eta = 1.74\times10^{-35},\ \text{ or } \beta^2 = \beta_m^2 + 2.90\times10^{-36},\ \text{ for } n = 6\ \frac{\text{atoms}}{\text{m}^2}$$

Then $\underline{\beta = 1.92\times10^{-18} \simeq 2\beta_m}$ fractional charge imbalance is all that's needed.

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Solution

1. Ideal gas law: $PV = NkT \Rightarrow N = \frac{PV}{kT}$

$$N = \frac{5\times10^{-10} \times 10^{-3} \times 10^{5}\ N/m^2 \times 10^{-6}\ m^3}{1.38\times10^{-23}\ J/K \times 296\ K} = \boxed{1.22\times10^{7}\ \text{molecules}/cm^3}$$

2. Partition Theorem: $\frac{1}{2} m v_{rms}^2 = \frac{3}{2} kT \Rightarrow v_{rms} = \sqrt{\frac{3kT}{m_{Ar}}}$

$$v_{rms} = \sqrt{\frac{3\times1.38\times10^{-23}\ J/K \times 296\ K}{6.64\times10^{-26}\ kg}} \simeq \underline{430\ m/s}$$

$$t \simeq \frac{\ell}{v_{rms}} = \frac{1\ m}{430\ m/s} = 2.33\times10^{-3}\ sec \simeq \boxed{2.33\ msec.}$$

3. Newton's 2^{nd} Law: $r\,\Delta P_n = F_n$

Normal force on the 1 cm² area of the surface.

change in momentum per strike in normal direction

rate of strike on 1 cm² area

$$\Rightarrow r^{(*)} = \frac{F_n}{\Delta P_n} = \frac{P\times A}{2 m v_n} \simeq \frac{P\times A}{2m\frac{v_{rms}}{\sqrt{3}}} = \frac{PA}{2\sqrt{kTm_{H_2}}} \quad (**)$$

$$\Rightarrow r = \frac{5\times10^{-10}\times10^{-3}\times10^{5}\ N/m^2\times10^{-4}\ m^2}{2\sqrt{1.38\times10^{-23}\ J/K \times 296 \times 3.35\times10^{-27}\ kg}} = \underline{6.76\times10^{11}\ \text{strikes per sec. per cm}^2}$$

$$n\,(\text{surface}) = \left(10^{23}\right)^{2/3} = 2.15\times10^{15}\ \text{atoms}/cm^2$$

$$\frac{n}{r} = \frac{2.15\times10^{15}\ \text{atoms}/cm^2}{6.76\times10^{11}\ H_2/cm^2\text{-sec}} = \boxed{3.18\times10^{3}\ sec = 53\ \text{minutes}}$$

NOTES:

(*) a simpler approach could be: $r \simeq \frac{N}{3} v_n \simeq \frac{N v_{rms}}{3\sqrt{3}} = \underbrace{\frac{1.22\times10^{7}}{3}}_{(cm^{-3})} \underbrace{\sqrt{\frac{kT}{m_{H_2}}}\times10^{2}}_{cm/sec} = 4.49\times10^{11}$ strikes per sec per cm²

(**) $r_{exact} = \frac{P\times A}{\sqrt{2\pi mkT}} \simeq 5.39\times10^{11}$ strikes per sec per cm².

12. Calculate the specific heat of a two-level quantum system in contact with a temperature bath at temperature T.

Ph.D. Comps

Drumheller prob. 12

Statistical Mechanics

Calculate the specific heat of a two-level quantum system in contact with a temperature bath at temperature T.

ε ————

0 ————

Partition fn:

$$Z = \sum_i e^{-\varepsilon_i/kT} = 1 + e^{-\varepsilon/kT}$$

Energy:

$$U = \langle E \rangle = \frac{1}{Z} \sum_i \varepsilon_i e^{-\varepsilon_i/kT} = \frac{\varepsilon e^{-\varepsilon/kT}}{1 + e^{-\varepsilon/kT}}$$

Heat Cap. (const. volume)

$$C_V = \frac{1}{k}\frac{\partial U}{\partial T} = \frac{\left(1+e^{-\varepsilon/kT}\right)\varepsilon\left(\frac{\varepsilon}{k}\frac{1}{T^2}\right) - \varepsilon e^{-\varepsilon/kT}\left(\frac{\varepsilon}{kT^2}\right)}{\left(1+e^{-\varepsilon/kT}\right)^2}$$

$$= \frac{\left(\frac{\varepsilon^2}{kT^2}\right) e^{-\varepsilon/kT}}{\left(1+e^{-\varepsilon/kT}\right)^2}$$

which can be written:

$$= \frac{k}{\cancel{k}}\left(\frac{\varepsilon}{kT}\right)^2 \frac{e^{-\varepsilon/kT}}{e^{-2\varepsilon/kT}\left(1+e^{\varepsilon/kT}\right)^2}$$

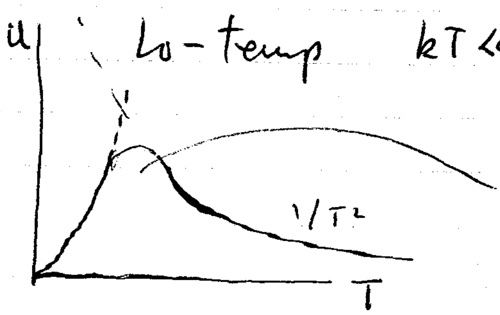
$$= \frac{k}{\cancel{k}}\left(\frac{\varepsilon}{kT}\right)^2 \frac{e^{\varepsilon/kT}}{\left(1+e^{\varepsilon/kT}\right)^2}$$

Hi-temp: $kT \gg \varepsilon$

$$C_V \cong k\left(\frac{\varepsilon}{kT}\right)^2 \frac{1}{2^2} = k\left(\frac{\varepsilon}{2kT}\right)^2 \sim \frac{1}{T^2}$$

Lo-temp $kT \ll \varepsilon$

$$C_V \cong k\left(\frac{\varepsilon}{kT}\right)^2 e^{-\varepsilon/kT}$$



called Schottky anomaly

13. Obtain an asymptotic expansion for the function

$$1 - erf(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty dt\, e^{-t^2} = erfc(x)$$

Keep at least two terms.

Hint: Integration by parts.

Mathematical Physics

Obtain an asymptotic expansion for the function

$$1 - \mathrm{erf}\,x = \frac{2}{\sqrt{\pi}} \int_x^\infty dt\; e^{-t^2} = \mathrm{erfc}\,x$$

Keep at least two terms.

Hint: Integration by parts. ~~(Do we give this?)~~ yes.

Solution:

$$\mathrm{erfc}\,x = \frac{2}{\sqrt{\pi}} \int_x^\infty dt\; e^{-t^2}$$

Set: $u = \frac{1}{t}$ $\quad v = -\frac{1}{2} e^{-t^2}$

$du = -\frac{dt}{t^2}$ $\quad dv = t e^{-t^2} dt$

$\therefore$

$$\mathrm{erfc}\,x = \frac{2}{\sqrt{\pi}} \left\{ -\frac{1}{2t} e^{-t^2} \Big|_x^\infty - \frac{1}{2} \int_x^\infty dt\, \frac{1}{t^2} e^{-t^2} \right\}$$

$$= \frac{2}{\sqrt{\pi}} \left\{ \frac{e^{-x^2}}{2x} - \frac{1}{2} \int_x^\infty t e^{-t^2} \frac{dt}{t^3} \right\}$$

$u = \frac{1}{t^3}$ $\quad v = -\frac{1}{2} e^{-t^2}$

$du = -3/t^4$

$\therefore$

$$\operatorname{erfc} x = \frac{2}{\sqrt{\pi}} \left\{ \frac{e^{-x^2}}{2x} - \frac{1}{2}\left[-\frac{e^{-t^2}}{2t^3}\Big|_x^\infty - \frac{3}{2}\int_x^\infty dt\, t\, e^{-t^2}\frac{1}{t^5} \right] \right\}$$

$\therefore$

$$\operatorname{erfc} x = \frac{2}{\sqrt{\pi}} \left\{ \frac{e^{-x^2}}{2x} - \frac{1}{4}\frac{e^{-x^2}}{x^3} + \dots \right\}$$

Proceeding further:

$$\operatorname{erfc} x = \frac{2}{\sqrt{\pi}} e^{-x^2} \left\{ \frac{1}{2x} - \frac{1}{2^2}\frac{1}{x^3} + \frac{3}{2^3 x^5} - \dots \right\}$$

$$= \frac{1}{\sqrt{\pi}} \frac{e^{-x^2}}{x} \left\{ 1 - \frac{1}{2}\frac{1}{x^2} + \dots \right\}$$

14. The proton is a bound state of two "up" (charge=+2/3) and one "down" (charge= -1/3) quarks. The neutron is a bound state of one up and two down quarks. All quarks are spin 1/2 particles.

The wave function of a spin-up proton may be written as:

$(18)^{-1/2}\{2[u(\uparrow)u(\uparrow)d(\downarrow)]-[u(\uparrow)u(\downarrow)d(\uparrow)]-[u(\downarrow)\,u(\uparrow)d(\uparrow)]+$ permutations on ordering of u,u,d$\}$.

The wave function of the neutron may be written in the same form, interchanging u's and d's.

(a) Calculate the magnetic moment of the proton, μ_p, in terms of the magnetic moments of the up and down quarks (μ_u, μ_d).

(b) Repeat (a) for the neutron.

(c) Calculate the ratio of the magnetic moment of the neutron to the magnetic moment of the proton. Assume the masses of the up and down quarks are equal. (The current experimental value for this ratio is -0.68497945 ± 0.00000058.)

Note: The Bohr magneton is $\mu = \frac{q\hbar}{2m}$ for a particle of mass m and charge q.

a) magnetic moment of proton $= \langle p\uparrow|(\mu_1+\mu_2+\mu_3)_z|p\uparrow\rangle$ where $|p\uparrow\rangle$ is the spin up wave function of the proton and $(\mu_i)_z$ are the z-components of the magnetic moment operators

First term in $|p\uparrow\rangle$ gives:

$$\left(\frac{4}{18}\right)\langle u(\uparrow)\, u(\uparrow)\, d(\downarrow)\,|(\mu_1+\mu_2+\mu_3)_z|\,u(\uparrow)\, u(\uparrow)\, d(\downarrow)\rangle$$

$$= \frac{4}{18}\langle u(\uparrow)\, u(\uparrow)\, d(\downarrow)\,|\,\underbrace{\mu_u+\mu_u-\mu_d}_{\text{eigenvalues now, not operators}}\,|\,u(\uparrow)\, u(\uparrow)\, d(\downarrow)\rangle$$

$= \frac{2}{9}(2\mu_u-\mu_d)$ where μ_u is the mag. moment of the up quark, μ_d is the mag moment of the down quark

2nd term gives:

$$\frac{1}{18}(\mu_u-\mu_u+\mu_d) = \frac{1}{18}\mu_d$$

3rd term gives

$$\frac{1}{18}(-\mu_u+\mu_u+\mu_d) = \frac{1}{18}\mu_d$$

Permuted terms give same values twice again (as d quark is moved around), so:

$$\mu_p = 3\left[\frac{2}{9}(2\mu_u-\mu_d)+\frac{1}{9}\mu_d\right] = \boxed{\frac{4}{3}\mu_u-\frac{1}{3}\mu_d}$$

b) Neutron calculation goes the same way, except $d \leftrightarrow u$, so

$$\boxed{\mu_n = \frac{4}{3}\mu_d-\frac{1}{3}\mu_u}$$

(c) Finally, for any spin 1/2 particle,

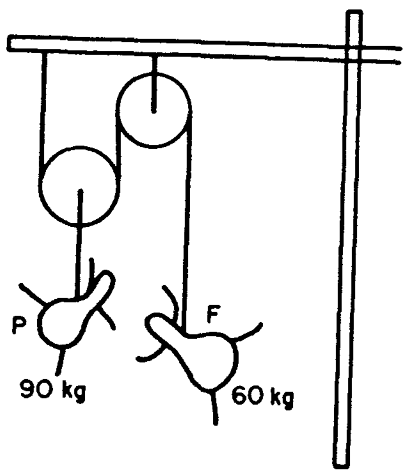
$$\mu = \frac{q\hbar}{2mc} \quad \text{so} \quad \mu_u = \frac{e\hbar}{3m_u c} \quad \mu_d = -\frac{e\hbar}{6m_d c}$$

$$\frac{\mu_n}{\mu_p} = \frac{\frac{1}{3}\left(\frac{e\hbar}{c}\right)\left[4\frac{-1}{6m_d} - \frac{1}{3m_u}\right]}{\frac{1}{3}\left(\frac{e\hbar}{c}\right)\left[4\frac{1}{3m_u} - \frac{-1}{6m_d}\right]}$$

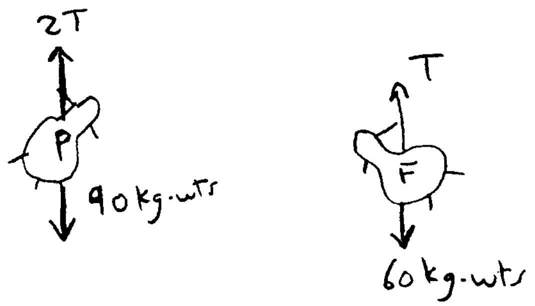
$$\boxed{\frac{\mu_n}{\mu_p} = -\left[\frac{4m_u + 2m_d}{8m_d + m_u}\right]}$$

if $m_u = m_d$, $\quad \dfrac{\mu_n}{\mu_p} = -\dfrac{6}{9} = \boxed{-\dfrac{2}{3}}$ fairly good comparison with expt. value

15. None of the identical boats on the Martian canals are quite able to support the combined load of both Paula and Fred, two affectionate Martians who refuse to go in separate boats. An enterprising boatman collects their fare by rigging them from the mast as shown in the figure, using the massless, frictionless, ropes and pulleys characteristic of Martian construction. The boatman ferries them accross before they hit either the mast or the deck. How much load does he save, in kg-wts?



(4) Draw free body diagrams for Paolo & Francesca



constraint: when F drops by Δx, P ascends only $\Delta x/2$, so $a_F = 2a_P$

use $F = ma$ for each:

Paolo:

$$(90\,\text{kg})\,a_P = 2T - 90\,\text{kg-wts} \qquad (1)$$

Francesca:

$$(60\,\text{kg})(2a_P) = 60\,\text{kg-wts} - T \qquad (2)$$

we want to find T, so we must eliminate a_P from Eqs. (1) & (2)

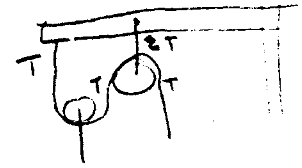
Take $4\cdot(1) - 3\cdot(2)$

$$0 = 8T - 360\,\text{kg-wts} - 180\,\text{kg-wts} + 3T$$

$$11T = 540\,\text{kg-wts}$$

$$T = \frac{540}{11}\,\text{kg-wts} \approx 49.09\,\text{kg-wts}$$

From the way the rope & pulleys are attached to the boom, the gondola has to support a weight of 3T:



so boom supports $3T = 147.27$ kg-wts

combined weight of Paolo & Francesca is 150 kg-wts, so Giuseppe saves

$\boxed{2.73\ \text{kg-wts}}$ of load

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Answer (a)

$$n(t+1) = \begin{cases} n(t)+1 & \text{with prob. } p \\ n(t)-1 & \text{with prob. } 1-p \end{cases}$$

So, $\langle n(t+1)\rangle = \langle n(t)\rangle$

$$+ (+1)p + (-1)(1-p)$$

$$= \langle n(t)\rangle + (2p-1),$$

Thus $\boxed{\langle n(t)\rangle = (2p-1)t}$

Answer (b) It is elementary that

$$\langle (n(t) - \langle n(t)\rangle)^2\rangle =$$

$$\langle n^2(t)\rangle - \langle n(t)\rangle^2$$ (proof below, see appendix)

Thus, compute $\langle n^2(t)\rangle$ in the same way as in part (a)

$$n^2(t+1) = \begin{cases} (n(t)+1)^2 & \text{prob. } p \\ (n(t)-1)^2 & \text{prob. } 1-p \end{cases}$$

$$= \begin{cases} n^2(t) + 2n(t) + 1 & \text{prob. } p \\ n^2(t) - 2n(t) + 1 & \text{prob. } 1-p \end{cases}$$

So $\langle n^2(t+1)\rangle = \langle n^2(t)\rangle + 1$

$$+ 2\left[\langle n(t)\rangle p - \langle n(t)\rangle (1-p)\right]$$

$$= \langle n^2(t)\rangle + 2(2p-1)\langle n(t)\rangle + 1$$

Thus, since $\sum_{t'=1}^{t} t' = \frac{1}{2} t(t+1)$,

$$\langle n^2(t)\rangle = 2(2p-1)^2 \tfrac{1}{2} t(t+1) + t$$

$$= (2p-1)^2 (t+1)t + t\,,$$

Thus, since $\langle n(t)\rangle^2 = (2p-1)^2 t^2$

$$\sigma^2 = (2p-1)^2 t + t = 4p(1-p)t$$

$$\therefore \boxed{\sigma(t) = 2\sqrt{p(1-p)t}}$$

End

(Note that $p = \frac{1}{2} \to \langle n\rangle = 0,\ \sigma = \sqrt{t}$

Appendix: proof that

$$\langle (n - \langle n \rangle)^2 \rangle = \langle n^2 \rangle - \langle n \rangle^2$$

Start: $\langle (n - \langle n \rangle)^2 \rangle =$

$$\langle n^2 - 2\langle n \rangle n + \langle n \rangle^2 \rangle$$

$$= \langle n^2 \rangle - 2\langle n \rangle^2 + \langle n \rangle^2$$

$$= \langle n^2 \rangle - \langle n \rangle^2 \qquad \text{end.}$$

This proof not required. Students should simply know this fact, but if necessary it can then be derived.