Electrostatic B.V. Problems I

Following are elements of Jackson's Ch. 2 on "Boundary-Value Problems..."

1) We have seen that a complete solution to the potential problem $\nabla^2\phi = -4\pi\rho$ can be written in terms of a Green's fcn G as follows...

$$\rightarrow \phi(\mathbf{r}) = \underbrace{\int_V G(\mathbf{r},\mathbf{r}')\,\rho(\mathbf{r}')\,d^3x'}_{\text{part of }\phi\text{ generated by charges }\rho\text{ in V.}} + \underbrace{\frac{1}{4\pi}\oint_S \left[G(\mathbf{r},\mathbf{r}')\frac{\partial\phi}{\partial n'} - \phi(\mathbf{r}')\frac{\partial G}{\partial n'}\right] dS'}_{\text{part of }\phi\text{ generated by potentials/fields specified on surface(s) S enclosing V.}}, \quad (1)$$

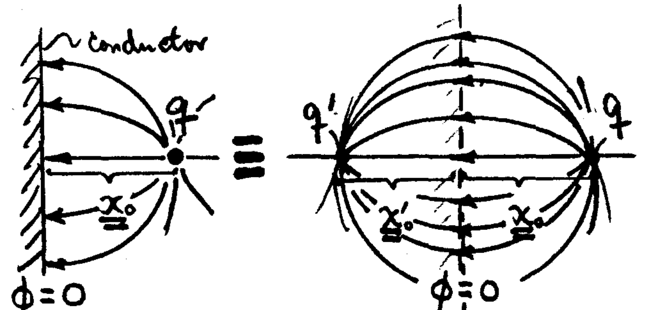
$$\text{where: } G(\mathbf{r},\mathbf{r}') = \frac{1}{|\mathbf{r}-\mathbf{r}'|} + \underbrace{F(\mathbf{r},\mathbf{r}')}_{\text{use for boundary conditions}}, \text{ such that } \begin{cases} \nabla_r^2 G = -4\pi\,\delta(\mathbf{r}-\mathbf{r}'), \\ \nabla_r^2 F = 0. \end{cases} \quad (2)$$

The (free) fcn F is available to invest G with whatever boundary conditions must be obeyed on S. The fact that F is a potential which obeys $\nabla^2 F = 0$ (Laplace Eqtn) suggests that the required boundary conditions can be supplied by "fictitious" charges which lie outside V. This degree of freedom is the basis for solutions to ϕ by the **METHOD OF IMAGES**.

$\boxed{\text{Ex}}$ Charge q outside plane conductor.

Interested in $\mathbb{E}$, ptl @ $x \geq 0$. Also want surface charge distribution induced on plane by presence of q, etc.

$$\begin{cases} V = \text{righthand half-space, } x \geq 0; \\ S = \text{the } \infty \text{ conducting plane, } x = 0.^{\star} \end{cases}$$



q at distance x_0 outside ∞ conducting plane ($\phi\equiv 0$). $\mathbb{E}$ lines are $\perp$ plane. ‖ Placement of $q' \equiv (-)q$ at $x_0' = x_0$ behind plane reproduces the B.C. $\phi = 0$.

Lefthand half-space is excluded, and there we can place an "image charge" q', whose potential obeys $\nabla^2 F = 0$ in V. Obviously $q' = (-)q$ placed at $x_0' = (-)x_0$ (behind plane) ensures plane $x=0$ has $\phi \equiv 0$.

★ S can be extended to enclose RH half-space V w/o affecting problem.

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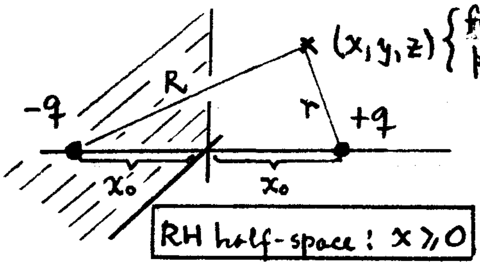
B.V. Probs I (cont'd)

Then, in RH half-space, the potential problem is...



$$\phi\left(\begin{smallmatrix}\text{field}\\ \text{point}\end{smallmatrix}\right) = \frac{q}{r} - \frac{q}{R} \quad \int^{\text{w/}} r = \sqrt{(x-x_0)^2+y^2+z^2},$$

$$\text{and: } \phi(x=0) \equiv 0. \qquad R = \sqrt{(x+x_0)^2+y^2+z^2}; \qquad (3)$$

This result applies only in V (i.e. $x \geq 0$).

From it, we can get $\mathbb{E}(x \geq 0) = -\nabla\phi$, etc. Will do this in Jk$^{\underline{n}}$ prob. (2-1).

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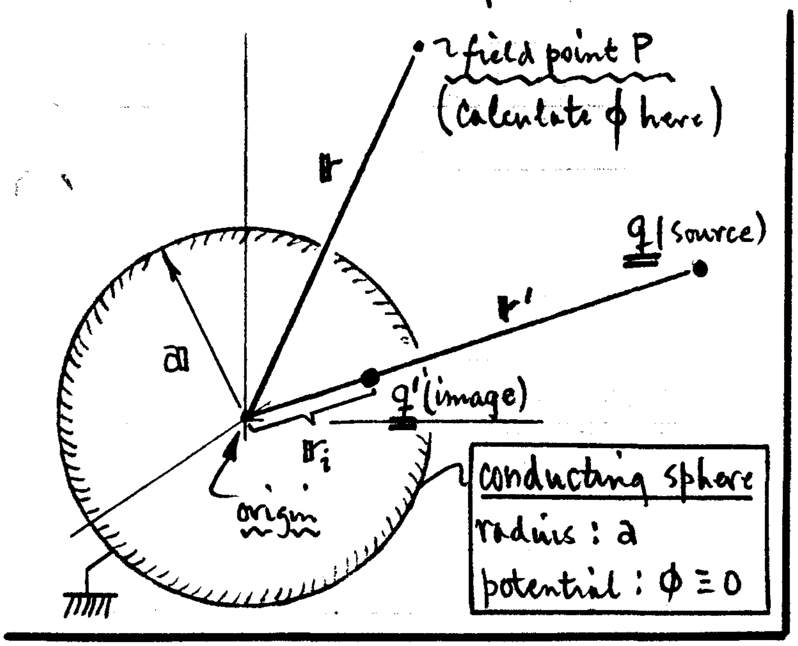
2) A more difficult example of the use of image charges is given in Jk$^{\underline{n}}$ Sec. 2-2... it is q in presence of a grounded conducting sphere.



For q outside sphere ($r' > a$), wish to calculate ϕ, $\mathbb{E}$, etc. Here have:

V = space outside sphere ($r' \geq a$);
S = sphere surface ($r' = a$).

S, plus sphere at ∞, "encloses" V. Put an image charge q' along $\mathbf{r}'$ and inside sphere -- this q' lies outside V, and so has $\nabla^2 F = 0$ in V. q and q' together will generate potential:

$$\longrightarrow \phi_P(\mathbf{r}) = \frac{q}{|\mathbf{r}-\mathbf{r}'|} + \frac{q'}{|\mathbf{r}-\mathbf{r}_i|}, \text{ at field pt. P.} \qquad (4)$$

The image size q' and its position $\mathbf{r}_i$ (with $r_i < a$) are now adjusted so that the boundary condition on S, viz $\phi \equiv 0$ on S is respected:

$$0 = \phi_P(\mathbf{r}=\mathbf{a}) = \frac{q}{|a\hat{n} - r'\hat{n}'|} + \frac{q'}{|a\hat{n} - r_i\hat{n}'|} \quad \int \begin{array}{l}\hat{n} = \text{unit vector along } \mathbf{r} \\ \hat{n}' = \;\text{"} \quad \text{"} \quad \text{"} \quad \mathbf{r}'\end{array}$$

$$0 = \frac{q/a}{|\hat{n} - (r'/a)\hat{n}'|} + \frac{q'/r_i}{|(a/r_i)\hat{n} - \hat{n}'|}, \quad \underline{\text{Satisfied for}} \begin{cases} q'/r_i = (-)q/a, \\ a/r_i = r'/a. \end{cases} \qquad (5)$$

So -- somewhat amazingly -- we get a solution for a single image charge q':

IMAGE CHARGE $\begin{cases} \text{position}: r_i = a^2/r', \leftarrow r_i = \text{geom. mean}\{a, r'\}; \\ \text{size}: q' = -(a/r')\,q. \end{cases}$ (6)

and
$$\boxed{\phi_P(\mathbf{r}) = \frac{q}{|\mathbf{r}-\mathbf{r}'|} - \frac{kq}{|\mathbf{r}-k^2\mathbf{r}'|}}$$
soln for ϕ for $r' \geq a$ (q outside), where: $k = (a/r') \leq 1$. (7)

From this solution for $\phi_P(\mathbf{r})$, can get several "interesting" quantities:

1. Surface charge density induced by q: $\sigma = \frac{1}{4\pi}\left(-\frac{\partial\phi}{\partial r}\right) \leftarrow$ See Jkⁿ Eq. (2.5);
2. Force between q & sphere (q'): $F_{qq'} = \frac{q^2}{a^2}\left(\frac{a^3}{r'^3}\right) \Big/ \left[1-(a^2/r'^2)\right]^2 \leftarrow$ attractive [Jkⁿ Eq. (2.6)]
3. Behavior inside sphere ($r' \leq a$)... need only change sign of sphere normal.

3) Jackson proceeds to do variations on the theme of a conducting sphere...

Conducting sphere problems:

Sec #	field source	sphere grounded	sphere insulated	surface charge	surface potential	Remarks
2.2	pt. q	yes	no	0, to begin	0	problem done above
2.3	"	no	yes	Q	$\frac{Q}{a}$, to begin	q & Q same sign ⇒ repulsion at large dist., but attractⁿ in close.
2.4	"	no	no	$Q = Va$ (to begin)	$\phi = V$, const	as with above example, ∃ an unstable point at $r' > a$.
2.5	uniform ext'l E_0	no	yes	0	0	Surface charge density $\}$ $\sigma = \frac{3}{4\pi} E_0 \cos\theta$ w/ $\theta = \measuredangle$ w.r.t. E_0

We will not pursue details here; you can read these Secs. for yourself. Only ~ new physics is in Sec. (2.3), where q (outside sphere) & Q (on sphere) of same sign ⇒ expected repulsion at $r' \gg a$, but (unexpected) attraction as $r' \sim a$. The force $F_{qq'}$ changes sign as $\infty > r' \to a$; there is an unstable pt. w/ $F_{qq'} = 0$.