PROBLEMS

DEPARTMENT OF PHYSICS

2001 COMPREHENSIVE EXAM

September 2001

Answer each of the following questions, all of which carry equal weight. Begin your answer to each question on a new sheet of paper; solutions to different questions must not appear on the same sheet. Each sheet of paper must be labeled with your name and the problem number in the upper right hand corner. When more than one sheet is submitted for a problem, be sure the pages are ordered properly.

PHYSICAL CONSTANTS

Quantity	Symbol	Value
acceleration due to gravity	g	$9.8\ \mathrm{m\ s^{-2}}$
gravitational constant	G	$6.67 \times 10^{-11}\ \mathrm{N\ m^2\ kg^{-2}}$
permittivity of vacuum	ε_o	$8.85 \times 10^{-12}\ \mathrm{C^2 N^{-1} m^{-2}}$
permeability of vacuum	μ_o	$4\pi \times 10^{-7}\ \mathrm{N\ A^{-2}}$
speed of light in vacuum	c	$3.00 \times 10^{8}\ \mathrm{m\ s^{-1}}$
elementary charge	e	$1.602 \times 10^{-19}\ \mathrm{C}$
mass of electron	m_e	$9.11 \times 10^{-31}\ \mathrm{kg}$
mass of proton	m_p	$1.673 \times 10^{-27}\ \mathrm{kg}$
Planck constant	h	$6.63 \times 10^{-34}\ \mathrm{J\ s}$
Avogadro constant	N_A	$6.02 \times 10^{23}\ \mathrm{mol^{-1}}$
Boltzmann constant	k	$1.38 \times 10^{-23}\ \mathrm{J\ K^{-1}}$
molar gas constant	R	$8.31\ \mathrm{J\ mol^{-1}\ K^{-1}}$
standard atmospheric pressure		$1.013 \times 10^{5}\ \mathrm{Pa}$

SOLUTIONS

DEPARTMENT OF PHYSICS

2001 COMPREHENSIVE EXAM

September 2001
9am - noon

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1

In this problem you will analyze projectile motion in the presence of a resistive force that is directly proportional to the velocity. Let the proportionality constant be designated as k.

(a) Write down the equations of motion in the horizontal and vertical directions.

(b) Find the horizontal components of the projectile's velocity and position as a function of time. Assume that the projectile has an initial position (0,0), an initial speed v_o, and is launched at an angle θ with respect to the horizontal.

(c) Find the vertical components of the projectile's velocity and position as a function of time.

(d) Show that these results reduce to the expected results in the absence of any air resistance.

(e) On the same graph sketch trajectories (y vs. x) for the cases $k = 0$ and $k > 0$.

Classical Mechanics
2001 Comprehensive Exam
Larry D. Kirkpatrick

Solution

a) $$m\ddot{x} = -k\dot{x}$$
$$m\ddot{y} = -k\dot{y} - mg$$

b) We rewrite the equation of motion in terms of v_x

$$m\frac{dv_x}{dt} = -kv_x$$

$$\int_{v_{xo}}^{v_x} \frac{dv_x}{v_x} = -\frac{k}{m}\int_0^t dt$$

$$\ln v_x - \ln v_{xo} = -\frac{k}{m}t$$

$$v_x = v_{xo}e^{-kt/m} = v_o \cos\theta\, e^{-kt/m}$$

$$\int_0^x dx = v_o \cos\theta \int_0^t e^{-kt/m} dt$$

$$x = \frac{mv_o \cos\theta}{k}\left(1 - e^{-kt/m}\right)$$

c) $$m\frac{dv_y}{dt} = -kv_y - mg$$

$$\int_{v_{oy}}^{v_y} \frac{dv_y}{kv_y + mg} = -\int_0^t dt$$

$$\frac{m}{k}\ln\left(kv_y + mg\right) - \frac{m}{k}\ln\left(kv_{yo} + mg\right) = -t$$

$$v_y = -\frac{mg}{k} + \frac{kv_o \sin\theta + mg}{k}e^{-kt/m} = v_o \sin\theta e^{-kt/m} + \frac{mg}{k}\left(e^{-kt/m} - 1\right)$$

$$\int_0^y dy = \int_0^t \left(-\frac{mg}{k} + \frac{kv_o \sin\theta + mg}{k}e^{-kt/m}\right)dt$$

$$y = -\frac{mgt}{k} + \frac{mkv_o \sin\theta + m^2 g}{k^2}\left(1 - e^{-kt/m}\right)$$

d) To show that these results reduce to the expected results in the absence of air resistance, we need to expand the exponential in a power series.

$$e^{-kt/m} = 1 + \left(-\frac{k}{m}t\right) + \frac{1}{2!}\left(-\frac{k}{m}t\right)^2 + \ldots = 1 - \frac{k}{m}t + \frac{1}{2}\frac{k^2}{m^2}t^2 + \ldots$$

$$1 - e^{-kt/m} = \frac{k}{m}t - \frac{1}{2}\frac{k^2}{m^2}t^2 + \ldots$$

$$v_x = v_o \cos\theta\, e^{-kt/m} \to v_o \cos\theta$$

$$x = \frac{mv_o \cos\theta}{k}\left(1 - e^{-kt/m}\right) = \frac{mv_o \cos\theta}{k}\left(\frac{k}{m}t + \ldots\right) \to v_o t \cos\theta$$

$$v_y = -\frac{mg}{k} + \frac{kv_o \sin\theta + mg}{k} e^{-kt/m} = -\frac{mg}{k} + \frac{kv_o \sin\theta + mg}{k}\left(1 - \frac{k}{m}t + \ldots\right)$$

$$= \frac{-mg}{k} + v_o \sin\theta + \frac{mg}{k} - \frac{k}{m} v_o t \sin\theta - gt + \ldots \to v_o \sin\theta - gt$$

$$y = -\frac{mgt}{k} + \frac{mkv_o \sin\theta + m^2 g}{k^2}\left(1 - e^{-kt}\right) = -\frac{mgt}{k} + \frac{mkv_o \sin\theta + m^2 g}{k^2}\left(\frac{k}{m}t - \frac{k^2}{2m^2}t^2 + \ldots\right)$$

$$= \frac{-mgt}{k} + v_o t \sin\theta + \frac{mgt}{k} - \frac{1}{2}\frac{k}{m} v_o t^2 \sin\theta - \frac{1}{2} g t^2 + \ldots$$

$$\to v_o t \sin\theta - \frac{1}{2} g t^2$$

e) The curve for $k = 0$ is the normal parabolic curve for projectile motion. The curve for $k > 0$ begins with the same slope, but lies lower and does not extend as far. It is not symmetric, falling off sharply.

2

A uniform current density, $\mathbf{J} = J_0\hat{\mathbf{z}}$, flows through an infinitely long, straight wire of radius R. The wire has bulk resistivity ρ.

(a) What is the power dissipated per unit volume in the wire?

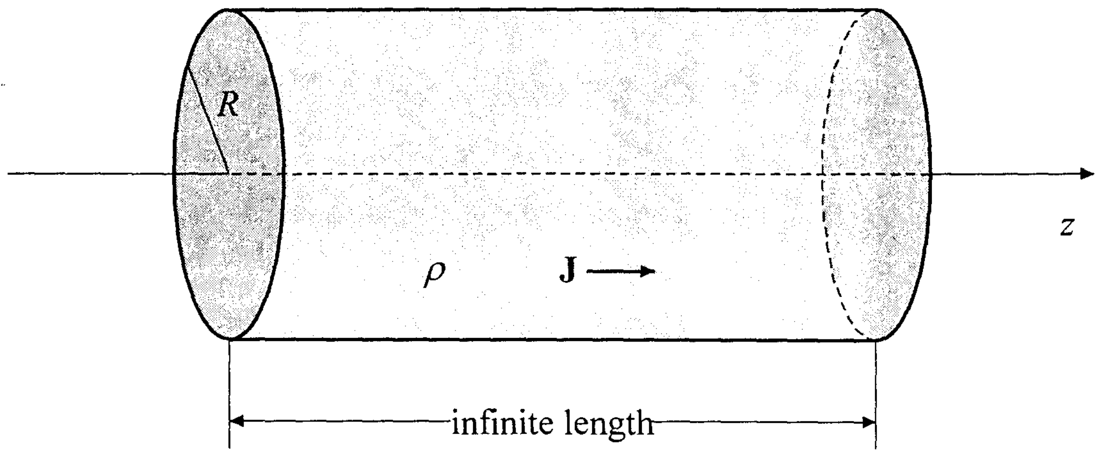
(b) Find the electric field **E** and magnetic field **B**, both inside and outside the wire.

(c) Calculate the vector quantity

$$\mathbf{S} \equiv \frac{1}{\mu_0}(\mathbf{E} \times \mathbf{B}),$$

both inside and outside the wire. Also calculate $\nabla \cdot \mathbf{S}$.

(d) What are the physical interpretations of **S** and $\nabla \cdot \mathbf{S}$?



CCK1 (E&M) **Solution**

1. The power per unit volume dissipated in the wire is given by

$$P = \mathbf{E}\cdot\mathbf{J}.$$

According to Ohm's law, the electric field in the resistor is

$$\mathbf{E} = \mathbf{J}\rho = J_0\rho\hat{\mathbf{z}}.$$

The power consumed per unit volume is:

$$P = J_0^{\,2}\rho.$$

2. The electric field inside the resistor is given in part 1, above. We note that the entire system is charge neutral, and so $\nabla\cdot\mathbf{E}=0$. There are no time dependencies, so $\nabla\times\mathbf{E}=0$. Since the derivatives of the electric field are zero, the constant uniform field inside the wire must extend to infinity outside the wire.

An amperian loop of radius r centered on the z-axis will contain a total current

$$I_{enc} = \begin{cases} \pi r^2 J_0, & r \le R \\ \pi R^2 J_0, & r > R \end{cases}$$

The magnetic field, which is due entirely to this current, is all in the positive azimuthal direction. According to Ampere's law,

$$2\pi r\mathbf{B} = \hat{\phi}\mu_0 I_{enc}.$$

So the magnetic field is:

$$\mathbf{B} = \hat{\phi}\mu_0 I_{enc} = \begin{cases} \hat{\phi}\frac{1}{2}\mu_0 J_0 r, & r\le R \\ \hat{\phi}\frac{1}{2}\mu_0 J_0 R^2/r, & r > R \end{cases}$$

3. The Poynting vector is

$$\mathbf{S} \equiv \frac{1}{\mu_0}(\mathbf{E}\times\mathbf{B}) = \begin{cases} -\hat{\mathbf{r}}\frac{1}{2}J_0^{\,2}\rho r, & r\le R \\ -\hat{\mathbf{r}}\frac{1}{2}J_0^{\,2}\rho R^2/r, & r > R \end{cases}$$

Since there is only a radial component, the divergence is merely:

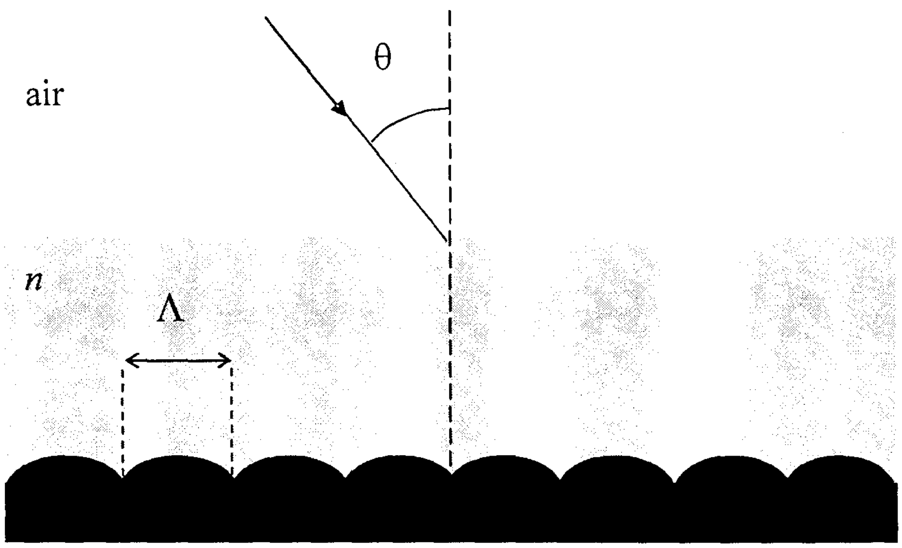
$$\nabla\cdot\mathbf{S} = \frac{1}{r}\frac{d}{dr}(rS) = \begin{cases} -J_0^{\,2}\rho, & r\le R \\ 0, & r > R \end{cases}$$

4. **S** is a vector field mapping the direction and magnitude of the flow of electromagnetic energy through space. The units are energy per time per area. The result shows that the energy of the electromagnetic field is flowing inward, converging on the resistive wire.

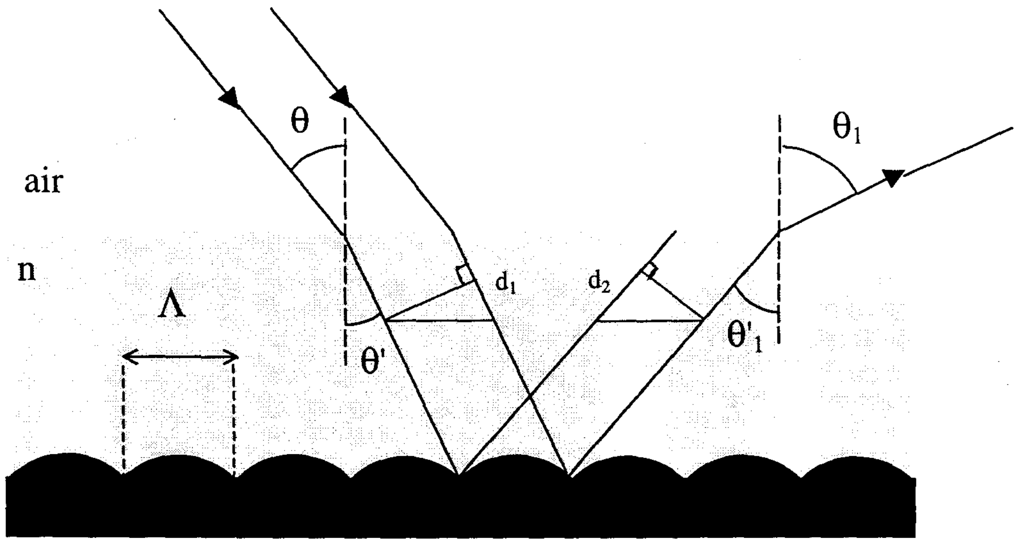
The divergence of **S** measures the local source or sink of electromagnetic energy (energy per volume per time). The resistor is a "sink" in the sense that electromagnetic energy flows inward to the resistor and is converted to another form (heat). Note that the power per unit volume dissipated in the resistor (part 1) is exactly equal to the divergence of S inside the wire.

3

A diffraction grating with a spatial groove period Λ is immersed in a transparent liquid with an index of refraction n. A monochromatic plane wave with wavelength λ (in air) is incident on the air-liquid interface at an angle θ (see Figure). Find the angle between the surface normal and the propagation direction in air of the first-order wave diffracted from the grating.



Solution



a) According to Snell's law, angles in air and in the liquid are related as:

$$\frac{\sin\theta}{\sin\theta'} = \frac{n}{1} \quad ; \quad \frac{\sin\theta_1}{\sin\theta_1'} = \frac{n}{1}$$

b) Optical pathlength difference between incoming and diffracted beams is

$$d_2 - d_1 = \Lambda \sin\theta_1' - \Lambda \sin\theta_1 = \Lambda(\sin\theta_1' - \sin\theta')$$

c) Condition for (constructive interference) a diffraction maximum is

$d_2 - d_1 = m \cdot \lambda'$, where $\lambda' = \frac{\lambda}{n}$ is wavelength in the medium and $m = 0, \pm 1, \pm 2, \ldots$

d) $$m\frac{\lambda}{n} = \Lambda\left(\frac{\sin\theta_1}{n} - \frac{\sin\theta}{n}\right) \quad \Big| \cdot n$$

$$\theta_1 = \operatorname{Arc\,Sin}\left(\frac{m\lambda}{\Lambda} + \sin\theta\right)$$

Diffraction angle does <u>not</u> depend on n!

A solid ball with radius R and mass M has moment of inertia $\left(\frac{2}{5}\right)MR^2$. The ball rolls without slipping along a rail that prevents transverse motion. The rail is formed into a shape described by $y = L(2(x/L)^2 - (x/L)^4)$, where $L >> R$, x is the horizontal distance along the ground, and y is the vertical distance above the ground. The ball experiences a constant downward acceleration g due to gravity.

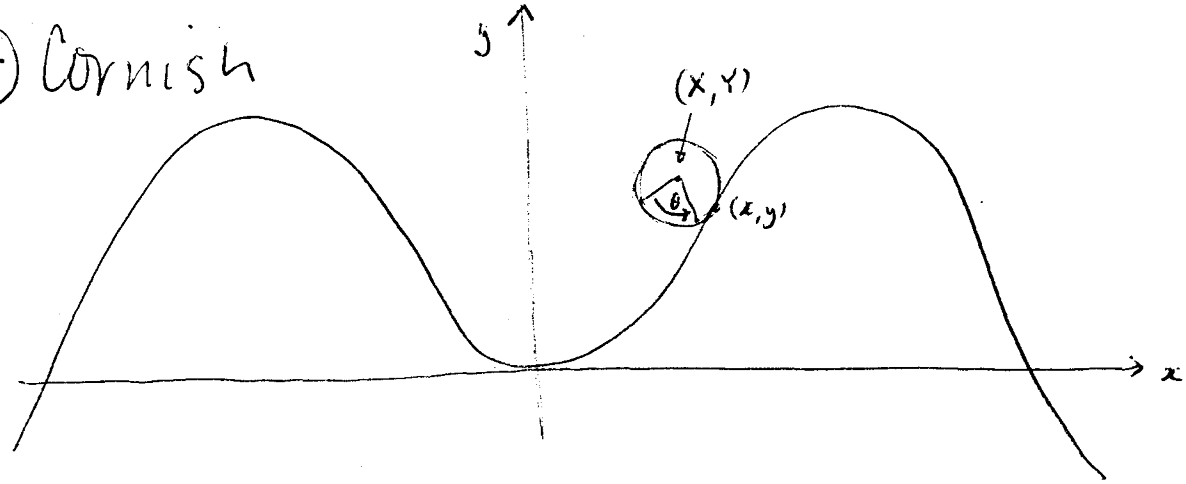
(a) Write down the Lagrangian for the ball in terms of appropriate generalized coordinate(s).

(b) Find all constants of the motion that follow directly from the Lagrangian.

(c) Write down the equation(s) of motion and locate any equilibrium positions.

(d) Solve the equations of motion for small oscillations about equilibrium.

④ Cornish



Denote position of com by X, Y and position of contact point by (x,y). Rotation about com by angle θ.

$$\Rightarrow \quad T = \tfrac{1}{2} M(\dot{X}^2 + \dot{Y}^2) + \tfrac{1}{2} I \dot{\theta}^2 \qquad I = \tfrac{2}{5} M R^2$$

$$V = MgY$$

X, Y related to x, y by

$$X = x - \frac{Ry'}{\sqrt{1+y'^2}}$$

$$Y = y + \frac{R}{\sqrt{1+y'^2}}$$

Here

$$y(x) = L\left(2\left(\tfrac{x}{L}\right)^2 - \left(\tfrac{x}{L}\right)^4\right)$$

$$y'(x) = 4\left(\tfrac{x}{L}\right)\left(1-\left(\tfrac{x}{L}\right)^2\right)$$

$$y''(x) = \tfrac{4}{L}\left(1 - 3\left(\tfrac{x}{L}\right)^2\right)$$

Constraint of rolling w/o slipping $\Rightarrow$ $R\,d\theta = ds = \sqrt{dx^2+dy}$

$$\Rightarrow \quad \dot{\theta} = \frac{\dot{x}}{R}\sqrt{1+y'^2}$$

Now, $\dot{X} = \dot{x}\left(1 - \frac{Ry''}{(1+y'^2)^{3/2}}\right)$ and $\dot{Y} = \dot{y}\left(1 - \frac{Ry''}{(1+y'^2)^{3/2}}\right)$

<u>But</u> $Ry'' \sim \frac{R}{L} \ll 1 \quad \Rightarrow \quad \dot{X} \simeq \dot{x}$ and $\dot{Y} \simeq \dot{y}$

Thus $T \simeq \tfrac{1}{2} M(\dot{x}^2 + \dot{y}^2) + \tfrac{1}{5} M \dot{x}^2 (1+y'^2) = \frac{7}{10} M \dot{x}^2 (1+y'^2)$

$\therefore \quad L = T - V \simeq \frac{7}{10} M \dot{x}^2 (1 + y'^2) - Mg\left(y + \frac{R}{\sqrt{1+y'^2}}\right)$

Since $\frac{\partial L}{\partial t} = 0$, the total energy $E = T + V$ is a constant of the motion.

<u>EOM</u> $\quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$

$$\Rightarrow \quad \frac{7M}{5}\left(\ddot{x}(1+y'^2) + \dot{x}^2 y' y''\right) + Mg y' = 0$$

Equilibrium $\quad \ddot{x} = \dot{x} = 0 \quad \Rightarrow \quad y' = 0$

$$\Rightarrow \quad x = 0 \quad \text{or} \quad x = \pm L$$

<u>Case 1</u> Near $x = 0$ let $x = \eta$, $\eta \ll L$

$$\Rightarrow \quad \frac{7}{5}\ddot{\eta} = -4\frac{g}{L}\eta$$

$\therefore$ Stable oscillations with frequency $\omega = \sqrt{\frac{20g}{7L}}$

<u>Case 2</u> Near $x = \pm L$, let $x = \pm L + \eta$, $\eta \ll L$

$$\Rightarrow \quad \frac{7}{5}\ddot{\eta} = +\frac{8g}{L}\eta$$

$\therefore$ Unstable equilibrium. η grows exponentially on an e-folding timescale of $\sqrt{\frac{40g}{7L}}$

5

In low-temperature condensed matter physics, ultra-low temperatures (from millikelvin to picokelvin) are achieved by magnetic cooling. This problem illustrates the principle of such a refrigerator. Consider an ideal two-state paramagnet composed of N distinguishable non-interacting spin-½ particles with each spin having a magnetic moment μ. The initial temperature of the system is T_o and the initial external magnetic field is B_o. Assume that the external magnetic field is increased isothermally until $B = \lambda B_o$, where $\lambda >> 1$. Answer the following questions assuming that all processes are reversible:

(a) Using an appropriate partition function, determine the entropy of the system under the conditions $T = T_o$ and $B = \lambda B_o$.

(b) Now assume that the paramagnetic system is thermally isolated from the environment and the magnetic field is reduced from λB_o to B_o. Determine the final temperature T of the system. Clearly explain how you arrived at your result.

(Hint: Recall that $S/k = \ell nZ + \beta E$ and $E = -\left(\dfrac{\partial \ell nZ}{\partial \beta}\right)_{V,N}$, where $\beta = 1/kT$)

Solution to Thermal Physics problem by Avci:

(a) For N identical and distinguishable particles the partition function can be written in terms of single spin partition function Z_1:

$$Z = Z_1^N$$

Recall that the possible energies of each spin in magnetic field B is given by $E_\pm = \pm \mu B$. Therefore, Z_1 is given by

$$Z_1 = e^{\beta\mu B} + e^{-\beta\mu B} = 2\,\text{Cosh}\,\beta\mu B\,, \quad \text{where} \quad \beta = \frac{1}{kT_0}$$

The entropy can be determined from the relation:

$$S = k\left(\ln Z - \beta\frac{\partial \ln Z}{\partial\beta}\right) = kN\left(\ln 2\,\text{Cosh}\,\beta\mu B - \beta\frac{\partial \ln Z_1}{\partial\beta}\right)$$

$$\frac{\partial \ln Z_1}{\partial\beta} = \mu B\,\frac{e^{\beta\mu B} - e^{-\beta\mu B}}{e^{\beta\mu B} + e^{-\beta\mu B}} = \mu B\,\text{Tanh}\,\beta\mu B$$

$$\Rightarrow \boxed{S = kN\left\{\ln(2\,\text{Cosh}\,x) - x\,\text{Tanh}\,x\right\}, \text{ where } x = \beta\mu B}$$

(b) Since the system is thermally isolated process of decrease in B takes place adiabatically. This means the entropy found in part (a) remains constant while B goes to B_0.

Therefore, $S_i = S_f$, but S is only a function of $x = \beta\mu B$, this means $x_i = x_f \Rightarrow \frac{\mu B}{kT_0} = \frac{\mu B_0}{kT}$

$$\Longrightarrow \boxed{T = \frac{B_0}{B}T_0 = \frac{T_0}{\lambda} \ll T_0}$$

—o—

6

You are given a spherical shell of radius a centered on the origin. The surface charge distribution is $\sigma_0 > 0$ for $z > 0$ and $-\sigma_0$ for $z < 0$.

(a) Calculate the dipole moment for this charge distribution.

(b) Using this result, write down the electric potential for points along the positive z-axis with $z >> a$.

(c) Calculate the electric potential for points along the positive z-axis for $z > a$. Show that this potential reduces to the one in (b) for $z >> a$.

(d) Calculate all of the quadrupole moments Q_{ij} for this charge distribution, where $Q_{ij} = \int \left(3x_i x_j - r^2\delta_{ij}\right)\rho(\vec{r})\,d^3r$

LDK 3/E&M-1

6

Electricity and Magnetism
2001 Comprehensive Exam
Larry D. Kirkpatrick

Solution

a) We begin by calculating the contribution to the dipole moment from the positive top-half of the sphere.

$$\mathbf{P}_+ = \int_0^{2\pi}\int_0^1 \sigma_o\left(\cos\theta\,\hat{\mathbf{z}} + \sin\theta\,\hat{\boldsymbol{\rho}}\right)a^3 d\varphi\, d\left(\cos\theta\right) = \pi\sigma_o a^3\hat{\mathbf{z}}$$

To get the contribution from the lower half, we reverse the sign of the charge and reverse the limits of integration. This gives the same result. Therefore,

$$\mathbf{P} = 2\pi\sigma_o a^3\hat{\mathbf{z}}$$

b) For large distances

$$\phi = \frac{1}{4\pi\varepsilon_o}\frac{\mathbf{P}\cdot\mathbf{r}}{r^3} = \frac{\sigma_o a^3}{2\varepsilon_o z^2}$$

c) Let $u = \cos\theta$.

$$\phi = \frac{\sigma_o a^2}{4\pi\varepsilon_o}\int_0^{2\pi} d\varphi\left(\int_0^1 \frac{du}{\sqrt{z^2+a^2-2azu}} - \int_{-1}^0 \frac{du}{\sqrt{z^2+a^2-2azu}}\right) = \frac{\sigma_o a}{\varepsilon_o z}\left(\sqrt{z^2+a^2} - z\right)$$

$$= \frac{\sigma_o a}{\varepsilon_o}\left[\left(1+\left(\frac{a}{z}\right)^2 - 1\right)\right] \approx \frac{\sigma_o a}{\varepsilon_o}\left[1+\frac{a^2}{2z^2} - 1\right] = \frac{\sigma_o a^3}{2\varepsilon_o z^2}$$

d) $$Q_{zz} = \sigma_o\int_0^{2\pi} d\varphi\left(\int_0^1\left(3z^2-a^2\right)du - \int_{-1}^0\left(3z^2-a^2\right)du\right) = 0$$

Therefore, $Q_{xx} = Q_{yy} = 0$. All off-diagonal elements are zero because of the axial symmetry.

Electricity and Magnetism
2001 Comprehensive Exam
Larry D. Kirkpatrick

Solution

a) We begin by calculating the contribution to the dipole moment from the positive top-half of the sphere.

$$\mathbf{p} = \int_S \sigma(\mathbf{r})\mathbf{r}\,dA$$

You can argue that the *x*- & *y*-components must be zero by symmetry, or you can show this explicitly.

$$\mathbf{p}_+ = \int_0^{2\pi}\int_0^1 \sigma_o\left(\cos\theta\,\hat{\mathbf{z}} + \sin\theta\,\hat{\boldsymbol{\rho}}\right)a^3 d\varphi\, d\left(\cos\theta\right)$$

The $\hat{\boldsymbol{\rho}}$ term vanishes because $\int_0^{2\pi}\hat{\boldsymbol{\rho}}\,d\varphi = 0$.

$$\mathbf{p}_+ = 2\pi\sigma_o a^3\hat{\mathbf{z}}\int_0^1 \cos\theta\, d\cos\theta = 2\pi\sigma_o a^3\hat{\mathbf{z}}\tfrac{1}{2}\cos^2\theta\Big|_0^1 = \pi\sigma_o a^3\hat{\mathbf{z}}$$

To get the contribution from the lower half, we reverse the sign of the charge and reverse the limits of integration. This gives the same result. Therefore,

$$\mathbf{p} = 2\pi\sigma_o a^3\hat{\mathbf{z}}$$

b) For large distances

$$\phi = \frac{1}{4\pi\varepsilon_o}\frac{\mathbf{p}\cdot\mathbf{r}}{r^3}$$

Along the $+z$-axis

$$\phi = \frac{\sigma_o a^3}{2\varepsilon_o z^2}$$

c) $$\phi = \frac{1}{4\pi\varepsilon_o}\int_S \frac{\sigma(\mathbf{r})}{R}\,dA$$

Let $u = \cos\theta$ and use the law of cosines to find R.

$$\phi = \frac{\sigma_o a^2}{4\pi\varepsilon_o}\int_0^{2\pi} d\varphi\left(\int_0^1 \frac{du}{\sqrt{z^2+a^2-2azu}} - \int_{-1}^0 \frac{du}{\sqrt{z^2+a^2-2azu}}\right)$$

$$\phi = \frac{\sigma_o a^2}{2\varepsilon_o} = \frac{\sigma_o a}{\varepsilon_o z}\left[\frac{2\sqrt{z^2+a^2-2azu}}{-2az}\Bigg|_0^1 - \frac{2\sqrt{z^2+a^2-2azu}}{-2az}\Bigg|_{-1}^0\right] = \frac{\sigma_o a}{\varepsilon_o z}\left(\sqrt{z^2+a^2} - z\right)$$

$$= \frac{\sigma_o a}{\varepsilon_o}\left[\left(1+\left(\frac{a}{z}\right)^2\right)^{1/2} - 1\right] \approx \frac{\sigma_o a}{\varepsilon_o}\left[1+\frac{a^2}{2z^2} - 1\right] = \frac{\sigma_o a^3}{2\varepsilon_o z^2}$$

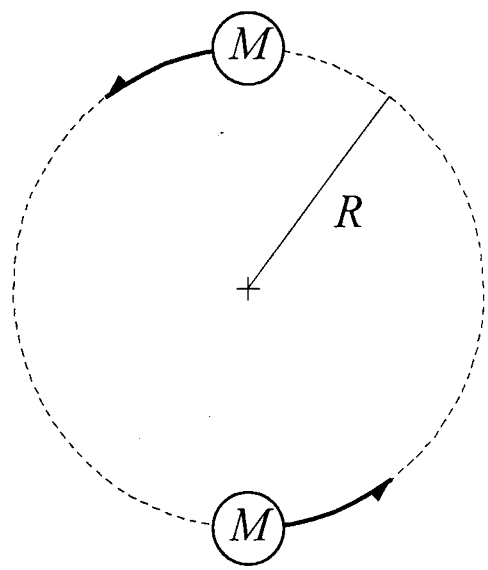
d) $$Q_{zz} = \sigma_o\int_0^{2\pi} d\varphi\left(\int_0^1\left(3z^2-a^2\right)du - \int_{-1}^0\left(3z^2-a^2\right)du\right) = 0$$

By symmetry $Q_{xx} = Q_{yy}$. Because $Q_{xx}+Q_{yy}+Q_{zz} = 0$, $Q_{xx} = Q_{yy} = 0$.
All off-diagonal elements are zero because of the axial symmetry. In fact, you can argue that this is a pure dipole and therefore all elements must be zero.

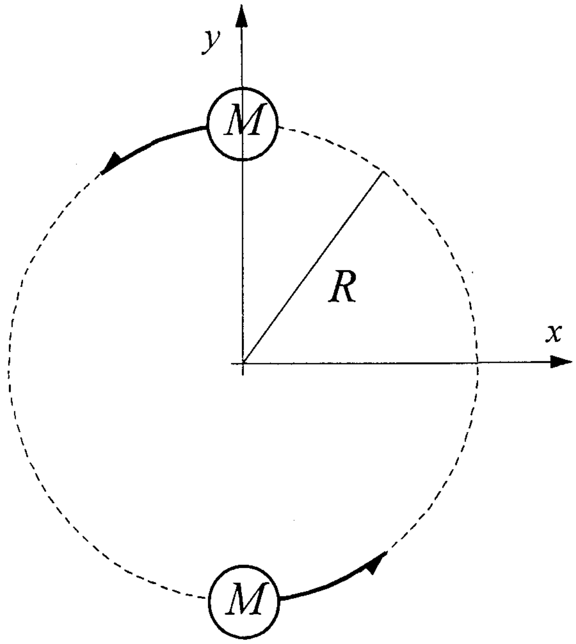
Two stars, each of identical mass M, orbit each other in a circular trajectory of radius R.

(a) What is the angular velocity ω of the binary system? You may assume $v/c \approx 0$.

(b) We wish to make a comparative observational study of these two stars from close range. Find the possible orbits that would keep a space probe at a *fixed* and *equal* distance from both stars.



CCK2 (Astrophysics) **Solution**



We are looking for orbits that are fixed in the reference frame of the two stars. For a stationary object (space probe) in that rotating frame, there is a gravitational force from each star as well as centrifugal force. We choose cartesian coordinates in the rotating frame, fixed with respect to the two masses as drawn above. The force is then:

$$\mathbf{F} = -\frac{GMm}{\left[x^2+(y+R)^2\right]^{3/2}}\begin{pmatrix} x \\ y+R \end{pmatrix} - \frac{GMm}{\left[x^2+(y-R)^2\right]^{3/2}}\begin{pmatrix} x \\ y-R \end{pmatrix} + m\omega^2\begin{pmatrix} x \\ y \end{pmatrix}.$$

The desired orbits are also equidistant from the two stars, which implies that $\boxed{y=0}$. By symmetry, this leads to zero net force in the y-axis, so a stationary orbit only needs to satisfy

$$\frac{F_x}{m} = -\frac{2GMx}{\left(x^2+R^2\right)^{3/2}} + \omega^2 x = 0.$$

For the x coordinate there are three solutions, $$x = \left\{0,\ \pm\sqrt{3}R\right\}.$$

8

The Hamiltonian of a two-dimensional simple harmonic oscillator is

$$\hat{H} = a_x^\dagger a_x + a_x^\dagger a_y + 1,$$

where the raising and lowering operator for x and y are

$$a_x^\dagger = \frac{1}{\sqrt{2}}\left(x - ip_x\right), \qquad a_x = \frac{1}{\sqrt{2}}\left(x + ip_x\right),$$

$$a_y^\dagger = \frac{1}{\sqrt{2}}\left(y - ip_y\right), \qquad a_y = \frac{1}{\sqrt{2}}\left(y + ip_y\right).$$

(a) Using basis states $\left|n_x n_y\right\rangle$, which are simultaneously eigenfunctions of each number operator,

$$N_x = a_x^\dagger a_x, \qquad N_y = a_y^\dagger a_y,$$

give all states belonging to the lowest three distinct energy levels of the system.

(b) Express the two-dimensional angular momentum operator

$$L = xp_y - yp_x$$

in terms of raising and lowering operators.

(c) Use the same basis function from (a) to write out the complete set of eigenstates for each of the **two** energy levels having definite angular momentum. Indicate the possible angular momenta for each energy level.

8

Solution

a. The eigenstates of each operator are

$$N_j|n_j\rangle = n_j|n_j\rangle \ . \tag{1}$$

Expressing the Hamiltonian in terms of the number operators

$$\hat{H} = N_x + N_y + 1 \ ,$$

it is immediately possible to write

$$\hat{H}|n_x n_y\rangle = (n_x + n_y + 1)|n_x n_y\rangle = E\,|n_x n_y\rangle \ .$$

Thus the eigenstates from the lowest three energy levels are

E	states
1	$\|00\rangle$
2	$\|01\rangle$, $\|10\rangle$
3	$\|02\rangle$, $\|11\rangle$, $\|20\rangle$

b. Re-writing the positions and momenta

$$x = \frac{1}{\sqrt{2}}(a_x^\dagger + a_x) \ , \quad p_x = \frac{i}{\sqrt{2}}(a_x^\dagger - a_x) \ ,$$

$$y = \frac{1}{\sqrt{2}}(a_y^\dagger + a_y) \ , \quad p_y = \frac{i}{\sqrt{2}}(a_y^\dagger - a_y) \ .$$

The angular momentum can be written

$$L = xp_y - yp_x = \frac{i}{2}(a_x^\dagger + a_x)(a_y^\dagger - a_y) - \frac{i}{2}(a_y^\dagger + a_y)(a_x^\dagger - a_x)$$

expanding the products and noting the a_x and a_y commute we get

$$L = i(a_y^\dagger a_x - a_x^\dagger a_y)$$

c. The raising and lowering operators work as follows:

$$a_j|n_j\rangle = \sqrt{n_j}\,|n_j - 1\rangle \ , \quad a_j^\dagger|n_j\rangle = \sqrt{n_j + 1}\,|n_j + 1\rangle \ .$$

(These relationship are derived by: $\Big|a_j|n_j\rangle\Big|^2 = \langle n_j|a_j^\dagger a_j|n_j\rangle = n_j$.) So the angular momentum operator will act on the states $|n_x n_y\rangle$

$$L|n_x n_y\rangle = i\sqrt{n_x(n_y+1)}\,|(n_x-1)(n_y+1)\rangle - i\sqrt{(n_x+1)n_y}\,|(n_x+1)(n_y-1)\rangle$$

The explicit actions on the states from the lowest two energy levels are

$$\begin{aligned} L|00\rangle &= 0 \\ L|01\rangle &= -i|10\rangle \\ L|10\rangle &= i|01\rangle \end{aligned}$$

Thus the lowest energy state $|00\rangle$ is an eigenstate of L with eigenvalue $\ell = 0$. The other two states are not eigenstates of L, but may be combined to make eigenstates

$$\begin{aligned} |\psi_1\rangle &= |10\rangle + i|01\rangle \ , \quad \ell = +1 \\ |\psi_{-1}\rangle &= |10\rangle - i|01\rangle \ , \quad \ell = -1 \end{aligned}$$

Thus $E = 2$ can have angular momentum $\ell = 1$ or $\ell = -1$.

9

A spin-1/2 system is initially in the state χ such that $-i/2$ is the amplitude for $S_z = \hbar/2$, and $(i\sqrt{3})/2$ is the amplitude for $S_z = -\hbar/2$. Suppose the system is subjected to a uniform, constant magnetic field B in the z-direction.

(a) Find the expectation value of S_x evaluated in the Schroedinger picture for $t > 0$.
(b) Find the expectation value of S_x evaluated in the Heisenberg picture for $t > 0$.

(Hint: Solve the Heisenberg equations of motion in (b).)

9

Solution

Given $\chi(t=0) = \begin{pmatrix} -i/2 \\ i\sqrt{3}/2 \end{pmatrix}\begin{matrix} + \\ - \end{matrix}$

Use $S_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{matrix} + \\ - \end{matrix}$; $S_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

and $H = \mu_B B \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{matrix} + \\ - \end{matrix}$; $\mu_B = \frac{|e|\hbar}{2mc}$

(a) In the S.P., the state vector carries the time dependence:

$$\chi(t) = e^{-i\frac{Ht}{\hbar}}\chi(0) = \begin{pmatrix} -\frac{i}{2} e^{-\frac{i\omega t}{2}} \\ i\frac{\sqrt{3}}{2} e^{\frac{i\omega t}{2}} \end{pmatrix}\begin{matrix} + \\ - \end{matrix}$$

where $\omega = \frac{|e|B}{mc}$

$$\langle S_x \rangle = \chi(t)^{\dagger} S_x \chi(t)$$

$$= \left(\frac{i}{2} e^{\frac{i\omega t}{2}}, -\frac{i\sqrt{3}}{2} e^{-\frac{i\omega t}{2}}\right)\frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} -\frac{i}{2} e^{-\frac{i\omega t}{2}} \\ \frac{i\sqrt{3}}{2} e^{\frac{i\omega t}{2}} \end{pmatrix}$$

$$= \frac{\hbar}{2}\left(-\frac{\sqrt{3}}{4} e^{i\omega t} - \frac{\sqrt{3}}{4} e^{-i\omega t}\right)$$

$$\boxed{\langle S_x \rangle = -\frac{\sqrt{3}}{4}\hbar\cos\omega t}$$

precession about B at the Larmor frequency ✓

(b) In the H.P., the operators carry the time dependence:

$$\dot{S}_x = -\frac{i}{\hbar}[S_x, H] = -\frac{i}{\hbar}\underbrace{[S_x, S_z]}_{-i\hbar S_y}\omega$$

$$\dot{S}_x = -\omega S_y$$

coupled d.e.'s

$$\dot{S}_y = -\frac{i}{\hbar}[S_y, H] = -\frac{i}{\hbar}\underbrace{[S_y, S_z]}_{i\hbar S_x}\omega$$

$$\dot{S}_y = \omega S_x$$

Solution: $S_x = S_x^0 \cos\omega t - S_y^0 \sin\omega t$

$S_y = S_x^0 \sin\omega t + S_y^0 \cos\omega t$

These reduce properly as $t \to 0$

$$\langle S_x \rangle = \langle S_x^0 \rangle \cos\omega t - \langle S_y^0 \rangle \sin\omega t$$

Now $\langle S_x^0 \rangle = \left(\frac{i}{2}, -\frac{i\sqrt{3}}{2}\right) \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -i/2 \\ i\sqrt{3}/2 \end{pmatrix} = -\frac{\sqrt{3}}{4}\hbar$

similarly, $\langle S_y^0 \rangle = 0$

$$\boxed{\langle S_x \rangle = -\frac{\sqrt{3}}{\hbar}\hbar \cos\omega t}$$

same as (a) ✓

Consider a *one-dimensional* metal of length L. Assume that there are N free electrons in the conductor and the system temperature is T. Answer the following questions:

(a) Explain, using the dispersion relation, the exact meaning of $D(\varepsilon)d\varepsilon$, where $D(\varepsilon)$ is the density of states and ε is the energy.

(b) Prove that $D(\varepsilon) = \frac{L}{\pi}(\frac{2m}{\hbar^2})^{\frac{1}{2}} \frac{1}{\sqrt{\varepsilon}}$ for the system described above.

(c) Determine the Fermi energy ε_F of the system in terms of N and L. The Fermi energy is the energy of the highest occupied level at zero temperature.

(d) Construct an integral-equation from which one can determine the chemical potential $\mu(T)$ of the metal at temperature T. Do not attempt to solve the equation.

10

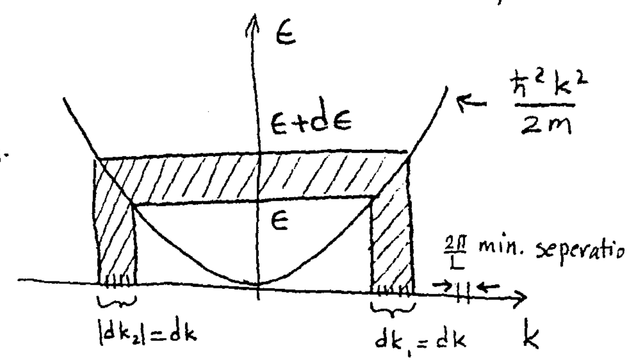
comp-01
RA2/CMP-1

Solution to Solid State problem by Avci

(a) Periodic boundary conditions impose that the minimum separation between two k-values is $\frac{2\pi}{L}$. This leads to $D(\epsilon)\,d\epsilon$ to mean 2 x number of k's that fall between the shaded regions on the k-axis, which is the number of states that fall between ϵ and $\epsilon + d\epsilon$ as shown below. The factor 2 is due to spin-degeneracy. Mathematically,

$$D(\epsilon)\,d\epsilon = \frac{2\,dk \times \overset{\leftarrow \text{spin}}{2}}{\left(\frac{2\pi}{L}\right)} = \frac{2L}{\pi}\,dk$$

Refer to the figure on the right for clarity.



(b) $$\epsilon = \frac{\hbar^2}{2m}k^2 \Rightarrow k = \left(\frac{2m}{\hbar^2}\right)^{1/2}\epsilon^{1/2}$$

$$dk = \frac{1}{2}\left(\frac{2m}{\hbar^2}\right)^{1/2}\epsilon^{-1/2}\,d\epsilon$$

Insert in $D(\epsilon)\,d\epsilon = \frac{2L}{\pi}\,dk \Rightarrow$ $\boxed{D(\epsilon) = \frac{L}{\pi}\left(\frac{2m}{\hbar^2}\right)^{1/2}\frac{1}{\sqrt{\epsilon}}}$

(c) $$N = \int_0^{\epsilon_F} D(\epsilon)\,d\epsilon = \int_0^{\epsilon_F}\frac{L}{\pi}\left(\frac{2m}{\hbar^2}\right)^{1/2}\epsilon^{-1/2}\,d\epsilon = \frac{2L}{\pi}\left(\frac{2m}{\hbar^2}\right)^{1/2}\epsilon_F^{1/2} \quad (T = 0\,K$$

$$\Rightarrow \epsilon_F = \underbrace{\left(\frac{N\pi}{2L}\right)^2}_{k_F^2}\frac{\hbar^2}{2m} \Rightarrow \boxed{\epsilon_F = \frac{\hbar^2 k_F^2}{2m},\ \text{where } k_F = \frac{\pi N}{2L}}$$

not asked

Numerical: $$\epsilon_F = \frac{(1.05 \times 10^{-27}\ \text{erg-s})^2}{2 \times 9.11 \times 10^{-28}\,g}\left(\frac{\pi}{2}\,\frac{10^8}{2\ \text{cm}}\right)^2 = 3.73 \times 10^{-12}\ \text{erg}$$

$$\epsilon_F = \frac{3.72 \times 10^{-12}\ \text{erg}}{1.6 \times 10^{-12}\ \text{erg/eV}} = \boxed{2.33\ \text{eV}}$$

(d) At finite temperature we must use the Fermi-Dirac distribution, $f(\epsilon)$, for determining the number of electrons, N, in the system. This leads to:

$$\boxed{N = \int_0^\infty f(\epsilon)\,D(\epsilon)\,d\epsilon}, \text{ where } f(\epsilon) = \frac{1}{e^{(\epsilon - \mu(T))/kT} + 1}$$, and $D(\epsilon)$ is given above. The chemical potential $\mu(T) \left(\approx \epsilon_F + \frac{\pi^2}{12}\frac{(kT)^2}{\epsilon_F}\right)$ can be determined by a number of ways from the integral equation shown above.

11

Consider the linear operator L acting on functions $y(x)$ defined on $-\infty < x < \infty$ and vanishing at $x \to \pm\infty$,

$$L = \frac{\partial^2}{\partial x^2} - 1.$$

(a) Write down the Green's function for L.

(b) Use the Green's function to solve the inhomogeneous equation

$$Ly = f(x) = \begin{cases} 1, & -a < x < a \\ 0, & \text{otherwise} \end{cases}$$

11

Solution

a. The Green's function $G(x)$ will solve the equation

$$LG = \delta(x) = \frac{d^2G}{dx^2} - G \ .$$

For the regions $x < 0$ and $x > 0$ this is the homogeneous equation

$$G''(x) - G(x) = 0 \ ,$$

for which the two independent solutions are e^x and e^{-x}. To satisfy the boundary conditions $G(x) \to 0$ at $x \to \pm\infty$, we have

$$G(x) = \begin{cases} Ae^x & , \quad x < 0 \\ Be^{-x} & , \quad x > 0 \end{cases}$$

To be continuous we must have $A = B$. Integrating equation (2) across $x = 0$ gives

$$1 = G'(0+) - G'(0-) = -2A \ ,$$

so $A = -1/2$. The complete Green's function is

$$G(x) = -\tfrac{1}{2}e^{-|x|} \ .$$

b. The Green's function can be used to solve any inhomogeneous problem $Ly = f(x)$

$$y(x) = \int_{-\infty}^{\infty} G(x - x')f(x')\,dx' \ .$$

For the RHS given in (1) this integral is

$$y(x) = \int_{-a}^{a} G(x - x')\,dx' = -\tfrac{1}{2}\int_{-a}^{a} e^{-|x-x'|}dx' \ .$$

The integral is easily done if $|x| > a$ so that $|x - x'| = \pm(x - x')$ over the entire range. In this case

$$y(x) = -\tfrac{1}{2}e^{\mp x}\int_{-a}^{a} e^{\pm x'}\,dx' = -\sinh(a)\,e^{\mp x} \quad , \quad \pm x > a.$$

When $|x| < a$ the integral is done in two pieces

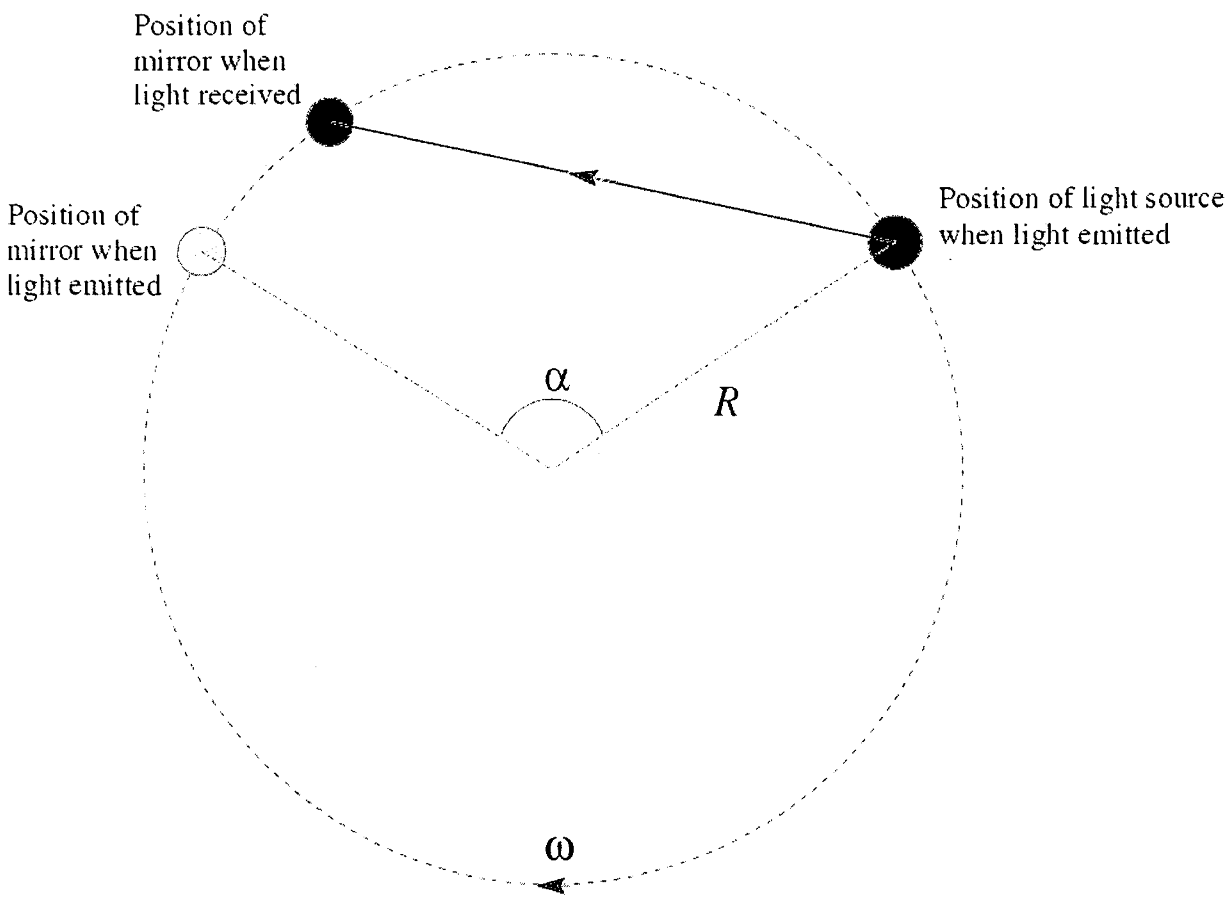
$$\begin{aligned} y(x) &= -\tfrac{1}{2}\int_{-a}^{x} e^{(x'-x)}dx' - \tfrac{1}{2}\int_{x}^{a} e^{(x-x')}dx' \\ &= \tfrac{1}{2}\left[-e^{-x}(e^x - e^{-a}) + e^x(e^{-a} - e^{-x})\right] \\ &= \tfrac{1}{2}(e^{-x-a} + e^{x-a} - 2) = e^{-a}\cosh(x) - 1 \end{aligned}$$

So the complete solution to equation (1) is

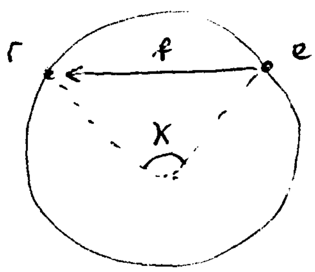
$$y(x) = \begin{cases} -\sinh(a)\,e^x & , \quad x < -a \\ e^{-a}\cosh(x) - 1 & , \quad -a < x < a \\ -\sinh(a)\,e^{-x} & , \quad a < x \end{cases}$$

In 1913, Sagnac developed a novel interferometer to study rotating frame effects. A set of mirrors was placed around a circle in such a way that light could pass clockwise and counter clockwise around the interferometer, with the two beams forming an interference pattern after completing a round trip. Consider the diagram below that shows part of the Sagnac interferometer. A monochromatic light source operating at frequency ν_0 emits a photon that travels to the mirror, where it is measured to have frequency ν_1. Both the light source and the mirror move at constant angular frequency ω on a circle of radius R. Find the redshift $z = (\nu_0 - \nu_1)/\nu_1$ in terms of R, ω, and the angle between the light source and mirror, α.

Note: The frequencies ν_0 and ν_1 are measured in the rest frame of the light source and mirror, respectively.



(12) Without loss of generality choose a coordinate system where the photon travels in the $-x$ direction



The angle between the receiving and emitting positions is denoted χ.

$\left(\alpha = \chi + \frac{2R}{c} \sin \chi/2 \right.$, but this is irrelevant$\left.\right)$

In the lab frame we have

Photon 4-momentum $\quad \vec{p} \xrightarrow[\text{lab}]{} \left(E/c , -\frac{E}{c}, 0, 0 \right)$

Emitter 4-velocity $\quad \vec{u}_e \xrightarrow[\text{lab}]{} \gamma (c, \underset{\sim}{v}_e)$

Receiver 4-velocity $\quad \vec{u}_r \xrightarrow[\text{lab}]{} \gamma (c, \underset{\sim}{v}_r)$

Where $\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$ and $v = \omega R$.

$\underset{\sim}{v} = v \cos(\omega t + \delta) \hat{i} + v \sin(\omega t + \delta) \hat{j}$, where $\underset{\sim}{v}_e$ and $\underset{\sim}{v}_r$ have $\delta_e = \delta_r + \alpha$.

Now, $\quad \nu_e = - \frac{\vec{u}_e \cdot \vec{p}}{h}$ and $\nu_r = - \frac{\vec{u}_r \cdot \vec{p}}{h}$

Thus $\quad z = \frac{\nu_e - \nu_r}{\nu_r} = \frac{(\vec{u}_e - \vec{u}_r) \cdot \vec{p}}{\vec{u}_r \cdot \vec{p}}$

where $\quad (\vec{u}_e - \vec{u}_r) \cdot \vec{p} = -(\gamma c - \gamma c) \frac{E}{c} + (v_{ex} - v_{rx}) p_x$

$$= \frac{E}{c} (v_{rx} - v_{ex})$$

But At time of emission and reception, the emitter and receiver have the same velocity in the x direction

$$\Rightarrow \boxed{z = 0}$$

13

A group of intrepid graduate students became obsessed with the sign asymmetry between the time-dependent terms in the Faraday and Ampere-Maxwell equations. First they tried switching the sign in front of the $\partial B/\partial t$ term in Faraday's law, but they quickly discovered this led to several physical inconsistencies. Undeterred, they tried switching the sign in front of the $\partial E/\partial t$ term in the Ampere-Maxwell equation, but again they discovered several physical inconsistencies. In each case, describe two of the physical inconsistencies they would have discovered. Support your conclusions mathematically.

(13) The changes the students make to Maxwell's equations break gauge invariance and Lorentz invariance.
By Noether's Theorem, this implies that quantities such as energy and charge will no longer be conserved.
Explicitly we have

Case a		Case b	
(1a)	$\nabla \cdot \underline{E} = 4\pi \rho$	(1b)	$\nabla \cdot \underline{E} = 4\pi \rho$
(2a)	$\nabla \times \underline{B} = \frac{4\pi}{c} \underline{J} + \frac{1}{c}\frac{\partial E}{\partial t}$	(2b)	$\nabla \times \underline{B} = \frac{4\pi}{c} \underline{J} - \frac{1}{c}\frac{\partial E}{\partial t}$
(3a)	$\nabla \times \underline{E} = \frac{1}{c}\frac{\partial B}{\partial t}$	(3b)	$\nabla \times \underline{E} = -\frac{1}{c}\frac{\partial B}{\partial t}$
(4a)	$\nabla \cdot \underline{B} = 0$	(4b)	$\nabla \cdot \underline{B} = 0$

Consider wave equation in vacuum.

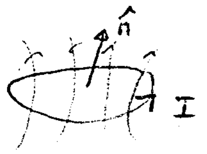
Have $\frac{1}{c}\frac{\partial}{\partial t}(2) + \nabla \times (3) = 0 \Rightarrow \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2} + \nabla^2 \underline{E} = 0$

same for cases $\underline{a}$ and $\underline{b}$, and similarly for $\underline{B}$ wave eqn.
Change in relative sign means that

$$\underline{E} = \underline{E}_0\, e^{\pm \omega t} e^{i \underline{k}\cdot \underline{r}} \quad \underline{\text{or}} \quad \underline{E} = \underline{E}_0\, e^{\pm kr} e^{i\omega t}$$

$\Rightarrow$ waves damp out and energy not conserved
(or grow).

Case a also messes up Lenz's law:



$$\mathcal{E} = \oint_c \underline{E} \cdot d\underline{\ell} \quad , \quad \underline{\Phi} = \int_s \underline{B} \cdot \hat{n}\, da$$

$$\Rightarrow \quad \mathcal{E} = \oint_c \underline{E} \cdot d\underline{\ell} = \int_s (\nabla \times \underline{E}) \cdot \hat{n}\, da = \frac{1}{c}\frac{\partial}{\partial t}\int_s \underline{B} \cdot \hat{n}\, da$$

$$= +\frac{1}{c}\frac{\partial \Phi}{\partial t}$$ $\Rightarrow$ get runaway current build up

Case b messes up charge conservation:

$$\frac{1}{c}\frac{\partial}{\partial t}(1b) - \nabla \cdot (2b) = 0 \Rightarrow \frac{\partial \rho}{\partial t} - \nabla \cdot \underline{J} = 0$$

A hydrogen-like atom has one unpaired electron in a p-state (all other electrons form a closed shell). The interaction between the electron's orbital motion and its spin is described by a spin-orbit coupling operator

$$\hat{H}_{SO} \propto \vec{\hat{L}} \cdot \vec{\hat{S}} = \hat{L}_x \hat{S}_x + \hat{L}_y \hat{S}_y + \hat{L}_z \hat{S}_z .$$

(a) Find the matrix form of $\hat{H}_{SO}$ in the uncoupled representation $|m_z\rangle |s_z\rangle$, where $\hat{L}_z |m_z\rangle = \hbar m_z |m_z\rangle$ and $\hat{S}_z |s_z\rangle = \hbar\, s_z |s_z\rangle$.

(b) Find the eigenvalues of $\hat{H}_{SO}$.

TABLE 9.4 Normalized relations between $\hat{L}_+$, $\hat{L}_-$, $\hat{L}_x$, $\hat{L}_y$ and the states $|lm\rangle$[a]

$$\hat{L}_z |lm\rangle = m\hbar |lm\rangle$$
$$\hat{L}_+ |lm\rangle = \hbar[(l - m)(l + m + 1)]^{1/2} |l, m + 1\rangle$$
$$\hat{L}_- |lm\rangle = \hbar[(l + m)(l - m + 1)]^{1/2} |l, m - 1\rangle$$
$$\hat{L}_x |lm\rangle = \frac{1}{2}\hbar[(l - m)(l + m + 1)]^{1/2} |l, m + 1\rangle + \frac{1}{2}\hbar[(l + m)(l - m + 1)]^{1/2} |l, m - 1\rangle$$
$$\hat{L}_y |lm\rangle = -\frac{1}{2}i\hbar[(l - m)(l + m + 1)]^{1/2} |l, m + 1\rangle + \frac{1}{2}i\hbar[(l + m)(l - m + 1)]^{1/2} |l, m - 1\rangle$$
$$\hat{L}_\pm |lm\rangle = \hbar[l(l + 1) - m(m \pm 1)]^{1/2} |l, m \pm 1\rangle$$

[a] These normalization relations also apply to the total angular momentum operators, $\hat{J}_\pm$, $\hat{J}_x$, $\hat{J}_y$, $\hat{J}_z$, and $\hat{J}^2$, where

$$\hat{J}^2 |jm_j\rangle = \hbar^2 j(j + 1) |jm_j\rangle$$
$$\hat{J}_z |jm_j\rangle = \hbar m_j |jm_j\rangle$$

(a) Rewrite the operator $\hat{\vec{L}}\cdot\hat{\vec{S}}$ in terms of ladder operators:

$$\hat{L}_x = \frac{1}{2}\left(\hat{L}_+ + \hat{L}_-\right)$$

$$\hat{L}_y = \frac{1}{2i}\left(\hat{L}_+ - \hat{L}_-\right)$$

$$\hat{S}_x = \frac{1}{2}\left(\hat{S}_+ + \hat{S}_-\right)$$

$$\hat{S}_y = \frac{1}{2i}\left(\hat{S}_+ - \hat{S}_-\right)$$

$$\hat{\vec{L}}\cdot\hat{\vec{S}} = \hat{L}_z\cdot\hat{S}_z + \hat{L}_x\hat{S}_x + \hat{L}_y\hat{S}_y$$

$$= \hat{L}_z\hat{S}_z + \frac{1}{4}\left(\cancel{\hat{L}_+\hat{S}_+} + \hat{L}_+\hat{S}_- + \hat{L}_-\hat{S}_+ + \cancel{\hat{L}_-\hat{S}_-}\right)$$
$$- \frac{1}{4}\left(\cancel{\hat{L}_+\hat{S}_+} - \hat{L}_+\hat{S}_- - \hat{L}_-\hat{S}_+ + \cancel{\hat{L}_-\hat{S}_-}\right)$$

$$= \hat{L}_z\hat{S}_z + \frac{1}{2}\hat{L}_+\hat{S}_- + \frac{1}{2}\hat{L}_-\hat{S}_+$$

For $\ell = 1$ there are six basis functions

$|1\ \tfrac{1}{2}\rangle, |1\ -\tfrac{1}{2}\rangle, |0\ \tfrac{1}{2}\rangle, |0\ -\tfrac{1}{2}\rangle, |-1\ \tfrac{1}{2}\rangle, |-1\ -\tfrac{1}{2}\rangle$

$$\hat{L}_+|m_z\ s_z\rangle = \hbar\sqrt{(1-m_z)(2+m_z)}\ |m_z+1, s_z\rangle$$

$$\hat{L}_-|m_z\ s_z\rangle = \hbar\sqrt{(1+m_z)(2-m_z)}\ |m_z-1, s_z\rangle$$

$$\hat{S}_+|m_z, s_z\rangle = \hbar\sqrt{(\tfrac{1}{2}-s_z)(\tfrac{3}{2}+m_z)}\ |m_z, s_z+1\rangle$$

$$\hat{S}_-|m_z, s_z\rangle = \hbar\sqrt{(\tfrac{1}{2}+s_z)(\tfrac{3}{2}-m_z)}\ |m_z, s_z-1\rangle$$

	$\hat{L}_z \hat{S}_z$	$\frac{1}{2}\hat{L}_+ \hat{S}_-$	$\frac{1}{2}\hat{L}_- \hat{S}_+$
$\lvert 1\ \frac{1}{2}\rangle$	$\hbar\cdot\frac{\hbar}{2}\lvert 1\ \frac{1}{2}\rangle$	0	0
$\lvert 1\ -\frac{1}{2}\rangle$	$\hbar(-\frac{\hbar}{2})\lvert 1\ -\frac{1}{2}\rangle$	0	$\frac{\hbar^2\sqrt{2}}{2}\lvert 0\ \frac{1}{2}\rangle$
$\lvert 0\ \frac{1}{2}\rangle$	0	$\frac{\hbar^2\sqrt{2}}{2}\lvert 1\ -\frac{1}{2}\rangle$	0
$\lvert 0\ -\frac{1}{2}\rangle$	0	0	$\frac{\hbar^2\sqrt{2}}{2}\lvert -1\ \frac{1}{2}\rangle$
$\lvert -1\ \frac{1}{2}\rangle$	$-\hbar\cdot\frac{\hbar}{2}\lvert -1\ \frac{1}{2}\rangle$	$\frac{\hbar^2\sqrt{2}}{2}\lvert 0\ -\frac{1}{2}\rangle$	0
$\lvert -1\ -\frac{1}{2}\rangle$	$-\hbar(-\frac{\hbar}{2})\lvert -1\ -\frac{1}{2}\rangle$	0	0

$$(\hat{\vec{L}}\cdot\hat{\vec{S}}) = \hbar^2 \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & \frac{\sqrt{2}}{2} & 0 & 0 & 0 \\ 0 & \frac{\sqrt{2}}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 0 & \frac{\sqrt{2}}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}$$

(b) We notice that there is mixing only between two pairs of states $\{|1\ -\tfrac{1}{2}\rangle, |0\ \tfrac{1}{2}\rangle\}$ and $\{|0\ -\tfrac{1}{2}\rangle, |-1\ \tfrac{1}{2}\rangle\}$
To diagonalize the matrix $(\hat{\vec{L}}\cdot\hat{\vec{S}})_{ij}$ we need only to diagonalize the submatrices

$$\begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{1}{2} \end{pmatrix}$$

$$\det\begin{pmatrix} -\frac{1}{2}-\lambda & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\lambda \end{pmatrix} = \lambda\left(\tfrac{1}{2}+\lambda\right) - \tfrac{1}{2} = 0$$

$$\lambda^2 + \tfrac{1}{2}\lambda - \tfrac{1}{2} = 0 \Rightarrow \lambda_\pm = -\tfrac{1}{4} \pm \sqrt{\tfrac{1}{16}+\tfrac{1}{2}}$$

$$\lambda_\pm = -\tfrac{1}{4} \pm \tfrac{3}{4}$$

Eigenvalues of $(\hat{\vec{L}}\cdot\hat{\vec{S}})_{ij}$ are:

$$\left\{\frac{\hbar^2}{2}, -\hbar^2, \frac{\hbar^2}{2}, -\hbar^2, \frac{\hbar^2}{2}, \frac{\hbar^2}{2}\right\}$$

15

A simple model of a self-gravitating body such as a star or planet, is the "water-moon". It is so called because its mass density ρ is constant independent of pressure. The body is in spherically symmetric equilibrium. The gravitational force from all material inside a radius r is balanced by an outward pressure gradient

$$\frac{dP}{dr} = -g(r)\rho \quad .$$

(a) What is the central pressure as a function of the total mass M and radius R?

(b) What is the gravitational binding energy of the water-moon? (In other words, how much work is required to separate all material to infinity?)

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Water moom – solution

a. Gravitational acceleration $g(r)$ is given by Newton's law

$$g(r) = \frac{G\,m(r)}{r^2} \quad ,$$

where $m(r)$, the mass inside radius r, can be considered to be concentrated at the center. For constant mass density ρ

$$m(r) = \rho \frac{4\pi}{3} r^3 \quad , \tag{1}$$

making the force balance equation

$$\frac{dP}{dr} = -\frac{4\pi G\,\rho^2}{3} r \quad .$$

This may be integrated from the center ($r = 0$) to yield the pressure.

$$P(r) - P(0) = -\frac{2\pi G\,\rho^2}{3} r^2 \quad .$$

The pressure at the surface $P(R) = 0$, since there is nothing outside it. Imposing this constraint yields an expression for the central pressure

$$P(0) = \frac{2\pi G\,\rho^2}{3} R^2 \quad .$$

To replace the density with the total mass $M = m(R)$ we use expression (1) evaluated there

$$\rho = \frac{3M}{4\pi R^3} \quad . \tag{2}$$

The central pressure is therefore

$$P(0) = \frac{3G\,M^2}{8\pi R^4} \quad . \tag{3}$$

b. The work required to lift a mass dm against a gravitational force $dm\,g(r)$ is

$$dW = dm \int_r^\infty g(r')\,dr' = dm\,G\,m(r) \int_r^\infty \frac{dr'}{(r')^2} = dm\,\frac{G\,m(r)}{r}$$

The mass in a shell of thickness dr is

$$dm = \rho\,dV = \rho\,4\pi r^2\,dr \quad .$$

Using this along with (1) gives an expression for the energy required to decrease the moon's radius by dr

$$dW = G\,\rho^2 \frac{16\pi^2}{3} r^4\,dr \quad .$$

The total energy to decrease the stars radius from $r = R$ to $r = 0$ is

$$W = G\,\rho^2 \frac{16\pi^2}{3} \int_0^R r^4\,dr = G\,\rho^2 \frac{16\pi^2}{15} R^5 \quad .$$

Using expression (2) gives the gravitational binding energy

$$W = \frac{3G\,M^2}{5R} \quad . \tag{4}$$

Alternative for part b

Writing the gravtational acceleration using a potential $\Psi(r)$

$$\mathbf{g} \;=\; -\nabla\Psi \;, \tag{5}$$

where the accelration from part (a) is

$$\mathbf{g}(r) \;=\; \begin{cases} -\frac{GMr}{R^3}\,\hat{\mathbf{r}} & , \quad r \le R \\ \\ -\frac{GM}{r^2}\,\hat{\mathbf{r}} & , \quad r > R \end{cases}$$

Integrating this from $\Psi = 0$ at $r \to \infty$ gives the potential

$$\Psi(r) \;=\; -\int_r^\infty g_r(r')\,dr' \;=\; \begin{cases} \frac{GM}{2R^3}(r^2 - R^2) - \frac{GM}{R} & , \quad r \le R \\ \\ -\frac{GM}{r} & , \quad r > R \end{cases} \tag{6}$$

A mass element dm at radius r has potential energy

$$dW \;=\; \Psi(r)\,dm \;\;.$$

(To see this note that the force on the element will be $\mathbf{F} = -\nabla V = dm\,\mathbf{g}$, where V is a potential energy function, which upon comparison with (5) is see to be $V = \Psi\,dm$). The potential energy of the entire collection is one-half the integral the potential energies of all mass elements,

$$W \;=\; \tfrac{1}{2}\int \Psi\,dm \;=\; \tfrac{1}{2}\int \Psi\,\rho\,d^3x \;\;.$$

Using expression (6) and integrating over the spherical moon gives a potential energy

$$\begin{aligned} W \;&=\; 2\pi\rho\int_0^R \Psi(r)\,r^2\,dr \;=\; 2\pi\rho\frac{GM}{2R}\int_0^R \left(\frac{r^2}{R^2} - 3\right)\,r^2\,dr \\ &=\; 2\pi\rho\frac{GM}{2R}\,R^3\left(\frac{1}{5} - 1\right) \;=\; -\,\frac{4\pi\rho R^3\,GM}{5R} \;=\; -\,\frac{3GM^2}{5R} \end{aligned}$$

To raise all of the matter out of its potential well we must add energy

$$E \;=\; -W \;=\; \frac{3GM^2}{5R} \;\;.$$

which is the gravitational binding energy.