ϕ520 Problems

(52) [Jk^n # (7.4)]. Analyse reflection of light at normal incidence from the surfaces of good and poor conductors. Do the problem as stated on Jackson's p. 328.

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(53) [Jk^n # (7.7)]. Find the relaxation time for any local charge accumulation in a good conductor [by model: $\sigma(\omega) = \sigma_0/(1-i\omega\tau)$]. Do the problem per Jackson's p. 329.

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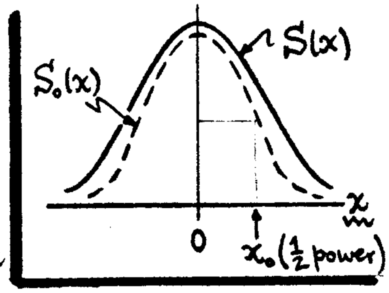
(54) [20 pts]. Extend the analysis in Jk^n Sec. 7.8 to include the group velocity dispersion factor $\alpha = \left(\frac{d^2\omega}{dk^2}\right)_0$. Begin by expanding $\omega(k)$ to $\theta(k^2)$. Show: $u(x,t) = e^{i\phi}[u(\xi,0) - \Delta u(\xi)]$, w/ $\phi = (k_0 v_g - \omega_0)t$, $\xi = x - v_g t$, per class notes. Assume $u(x,0)$ is real, and $|\alpha t| \ll 1/(\Delta k)^2$.

(A) Show: $\underline{\underline{|u(x,t)|^2 = [u(\xi,0)]^2\{1 + 2k_0\alpha t[u_x(\xi,0)/u(\xi,0)]\}}}$, to first order in αt.

(B) The term in αt in part (A) distorts the pulse intensity $|u|^2$. How does the pulsewidth change? To this end, consider a general pulse profile: $S(x) = S_0(x)[1 + \lambda f(x)]$, w/ $S_0(x)$ an unperturbed intensity, and $\lambda f(x)$ a "small" distortion. Let $S_0(0) = 1$, and define the (upper) half-power point of $S_0(x)$ by: $S_0(x_0) = \frac{1}{2}$. To first order in λ, show the corresponding half-power point of $S(x)$ lies at: $\underline{\underline{x = x_0 + \frac{\lambda}{2}[f(0) - f(x_0)]/S_0'(x)}}$, w/ $S_0' = dS_0/dx$.



(C) Apply (B) to (A): show, at early times, the half-power point of $|u(x,t)|^2$ lies at: $\underline{\underline{x(t) = x(0) - \alpha k_0 t}}$. So $|u|^2$ broadens or narrows, depending on $\alpha \lessgtr 0$. What happens as $t \to \infty$?

(D) Show that the pulse energy: $\underline{\underline{\mathcal{E} \propto \int_{-\infty}^{\infty} |u(x,t)|^2\, dx}}$, remains unchanged in this approx^n.

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(55) In class, we found the dispersion relation for an EM wave propagating in an ionosphere (density: $n \frac{\text{free } e\text{'s}}{\text{unit vol.}}$), along a magnetic field line (strength B_0): $\underline{\underline{kc = \omega[1 - \omega_p^2/\omega(\omega \mp \omega_B)]^{\frac{1}{2}}}}$, w/ $\omega_p = \sqrt{4\pi n e^2/m}$ = plasma freq., $\omega_B = eB_0/mc$ = cyclotron freq. Low freqs. ($\omega \ll \omega_B$ & ω_p) give "whistler waves."

(A) Show that the dispersion relation for whistlers is: $kc = \omega_p\sqrt{\omega/\omega_B}$.

(B) Calculate the group velocity v_g for whistlers. Find v_g numerically for whistlers at average frequency $\bar{\omega} = 10^4$ Hz, in the earth's ionosphere: $n = 10^5\ e\text{'s}/\text{cm}^3$, $B_0 = 0.3$ G.

(C) A whistler pulse starts out with frequencies in the range $(\omega - \Delta\omega)$ to ω, w/ $\Delta\omega \ll \omega$; it propagates over distance D. Calculate the time delay Δt between arrival of the high and low freq. components of the pulse. Find Δt numerically if: $\omega = 10^4$ Hz, $\Delta\omega = 10^3$ Hz, and the trip is in the earth's ionosphere (part B) between--roughly--N & S poles.

ϕ 520 Solutions

(53) [Jk^{n} (7.7)]. Find charge relaxation rate in a (Drude) medium, w/ $\sigma(\omega) = \sigma_0/(1-i\omega\tau)$.

The (wrong) way this problem is usually done goes as follows.* Combine $\nabla\cdot\mathbf{E} = 4\pi\rho$, $\nabla\cdot\mathbf{J} + \partial\rho/\partial t = 0$, and Ohm's Law: $\mathbf{J} = \sigma\mathbf{E}$, to obtain $\underline{\partial\rho/\partial t + 4\pi\sigma\rho = 0}$, an eqtn for decay of accumulated charge density ρ. If the conductivity $\underline{\sigma = \text{cnst}}$ in the medium, then the solution $\underline{\rho(t) = \rho(0)e^{-\Gamma t}}$, w/ $\underline{\Gamma = 4\pi\sigma}$, shows that ρ decays (relaxes) in a characteristic time $1/\Gamma$ (very quickly: $1/\Gamma \sim 10^{-19}$ sec., for Cu); the charge appears on the conductor surface. This calculation fails if $\sigma \neq$ cnst.

(A) When $\sigma = \sigma(\omega)$ is frequency dept., above solution needs repair. Write decay eqtn as:

$$\rightarrow \frac{\partial}{\partial t}\rho(t) + [4\pi\sigma(\omega)]\rho(t) = 0 \qquad (1)$$

$\sqrt{}$ r-dependence of ρ is suppressed (we are assuming a homog^{s} medium)

... do a Fourier Transform through Eq. (1): $\tilde{\rho}(\omega) = \int_{-\infty}^{\infty}\rho(t)\,e^{i\omega t}\,dt$ ...

$$\boxed{[4\pi\sigma(\omega) - i\omega]\,\tilde{\rho}(\omega) = 0}, \text{ as advertised.} \qquad (2)$$

(B) An "initial disturbance" is an accumulation: $\rho(t) = \begin{cases}\text{zero, for } t<0;\\ \text{nonzero, } t\geq 0;\end{cases}$, which generates some $\tilde{\rho}(\omega)$ in Eq. (2). Clearly, Eq. (2) does not fix $\tilde{\rho}(\omega)$ [nor should it], but it does require that for any oscillation at freq. ω: $4\pi\sigma(\omega) = i\omega$. For our model ...

$$\rightarrow \sigma(\omega) = \sigma_0/(1-i\omega\tau), \text{ and: } 4\pi\sigma(\omega) = i\omega \Rightarrow \tau\omega^2 + i\omega - 4\pi\sigma_0 = 0, \qquad (3)$$

So/ $\underline{\omega = -(i/2\tau) \pm \frac{1}{2\tau}\sqrt{16\pi\sigma_0\tau - 1}}$, w/ τ = collision time for conduction process.

Physics here is: the medium can't support arbitrary ω... the operative ω's are defined by τ. Write $\sigma_0 = \omega_p^2\tau/4\pi$ (w/ $\omega_p = \sqrt{4\pi n e^2/m}$ = plasma freq.). Then (3) reads...

$$\underline{i\omega = (1/2\tau) \pm i\omega_p\sqrt{1-(1/2\omega_p\tau)^2}}. \qquad (4)$$

The component of $\rho(t)$ at an allowed freq. ω [per Eq. (3)] now behaves as:

$$\boxed{\tilde{\rho}(\omega)e^{-i\omega t} \propto e^{-\lambda t}\,e^{\pm i\Omega_p t}} \qquad (5)$$

$\sqrt{}$ $\rho(t)$ relaxes at rate: $\lambda = 1/2\tau \ll \Gamma_0 = 4\pi\sigma_0$, while oscillating at freq.: $\Omega_p = \omega_p[1-(1/4\Gamma_0\tau)]^{\frac{1}{2}}$.

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* E.g. see P. Lorrain & D. Corson "EM Fields & Waves" (Freeman, 2nd ed, 1970), Eq (10.15).

54 [20 pts]. Analyse pulse distortion at early times by "group velocity dispersion" $\alpha = d^2\omega/dk^2$.

1. Start from the in-class result. Where $\phi = (k_0 v_g - \omega_0)t$, $\xi = (x - v_g t)$, the pulse is...

(A) $\rightarrow u(x,t) = e^{i\phi}\{u(\xi,0) - \Delta u(\xi)\}$, w/ $u(\xi,0) = \int_{-\infty}^{\infty} dk\, A(k)\, e^{ik\xi}$, (1)

We need to estimate the distortional part $\Delta u(\xi)$. $\boxed{\Delta u(\xi) = \int_{-\infty}^{\infty} dk\, A(k) e^{ik\xi}\left[1 - e^{-\frac{1}{2}i\alpha(k-k_0)^2 t}\right]}$. Expand the [ ] at early times : $\alpha t \ll 1/(\Delta k)^2 \sim (\Delta x)^2$. Then...

$$\Delta u(\xi) \simeq \tfrac{1}{2} i\alpha t \int_{-\infty}^{\infty} (k-k_0)^2 A(k)\, e^{ik\xi}\, dk \tag{2}$$

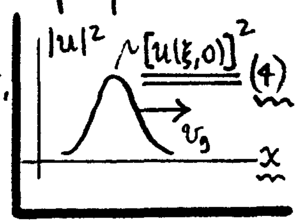
... use fact : $u(x,0) = \int A(k) e^{ikx} dk \longleftrightarrow \partial^n u/\partial x^n = \int (ik)^n A(k) e^{ikx} dk$...

So $\Delta u(\xi) \simeq \frac{1}{2} i\alpha t\left[k_0^2 u(\xi,0) + 2ik_0 u_x(\xi,0) - u_{xx}(\xi,0)\right]$, to $O(\alpha t)$;

and $\xrightarrow{}$ $u(x,t) \simeq e^{i\phi}\left\{\left[1 - \tfrac{1}{2} i k_0^2 \alpha t\right] u(\xi,0) + k_0\alpha t\, u_x(\xi,0) + \tfrac{1}{2} i\alpha t\, u_{xx}(\xi,0)\right\}$. (3)

2. Assume initial pulseform $u(x,0)$ is <u>real</u>. Calculate $|u|^2$ from Eq (3), keeping $O(\alpha)$ terms:

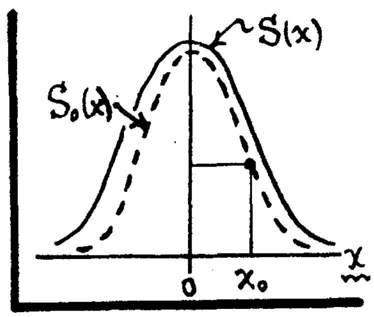
$$\boxed{|u(x,t)|^2 \simeq [u(\xi,0)]^2\left\{1 + (2k_0\alpha t)\left[u_x(\xi,0)/u(\xi,0)\right]\right\}} \quad \text{w/ } \xi = x - v_g t, \text{ for } \alpha t \ll (\Delta x)^2. \tag{4}$$



The factor in front shows that at early times $|u|^2$ propagates mainly as the undistorted pulse $[u(\xi,0)]^2$, at group velocity $v_g = (d\omega/dk)_0$. The term in α in the { } is the dispersive correction which distorts the pulse as t goes on.

(B) 3. Consider an intensity profile $S(x)$, which is max @ $x=0$ and decreases for $|x|>0$...

$\rightarrow S(x) = S_0(x)[1 + \lambda f(x)]$ — $S_0(x)$ is unperturbed profile, term in $\lambda \ll 1$ is the distortion. (5)



The unperturbed profile has norm $S_0(0) = 1$, and its HWHM lies at x_0 such that $S_0(x_0) = 1/2$. The distortion term in λ generally <u>shifts</u> the HWHM of $S(x)$ to a new pt. $x_0' = x_0 + \delta x$ such that :

$\rightarrow S(x_0') = \frac{1}{2} S(0)$, or $S_0(x_0')[1 + \lambda f(x_0')] = \frac{1}{2}[1 + \lambda f(0)]$. (6)

4. To estimate the shift $\delta x = x_0' - x_0$, expand both S_0 & f in Eq. (6) about x_0, to $O(\delta x)$:

$$\left[\tfrac{1}{2} + \delta x\, S_0'(x_0)\right]\left\{1 + \lambda\left[f(x_0) + \delta x\, f'(x_0)\right]\right\} \simeq \tfrac{1}{2}[1 + \lambda f(0)],$$

Sol $\rightarrow$ $\delta x \simeq \frac{\lambda}{2}[f(0) - f(x_0)] \Big/ \left\{S_0'(x_0) + \lambda\left[f(x_0) S_0'(x_0) + \tfrac{1}{2} f'(x_0)\right]\right\}$. (7)

But $\lambda \ll 1$, and to 1st order in λ we can <u>drop</u> the term in λ in the denom. of Eq.(7). So the shift $\delta x \simeq \frac{\lambda}{2}[f(0)-f(x_0)]/S_0'(x_0)$, to lowest order, and the HWHM of S lies at:

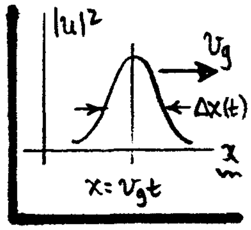
$$\boxed{x_0' = x_0 + \frac{\lambda}{2}[f(0)-f(x_0)]/S_0'(x_0)} \int \begin{array}{l} x_0 = \text{HWHM of } S_0(x); \\ x_0' = \text{HWHM of } S(x) = S_0(x)[1+\lambda f(x)]. \end{array} \quad (8)$$

This result is to $O(\lambda)$ only. The term in λ measures how much $S(x)$ is $\left\{\begin{array}{l}\text{broadened}\\ \text{narrowed}\end{array}\right\}$ w.r.t. $S_0(x)$.

(C) 5. To apply Eq. (8) to $|u(x,t)|^2$ of Eq. (4), comparison of (4) & (5) shows: $\lambda = 2k_0\alpha t$ (small by assumption), $S_0(x) = [u(\xi,0)]^2$ (assume $|u(\xi,0)|^2$ has HWHM $x_0 = \Delta x$), $f(x) = \frac{u_x(\xi,0)}{u(\xi,0)}$.

In Eq. (8), the peak is at $x=0$, and x_0 is the HWHM of S_0. Now, for $[u(\xi,0)]^2$, the peak is at $\xi = x - v_g t = 0$, and Δx is the HWHM. Choose an <u>initially</u> <u>symmetric</u> <u>pulse</u> [$u(x,0)$ = even fcn of x], so $u_x(0,0)=0$ and $f(0)=0$ in Eq. (8). The correction term in (8) is...

$$\rightarrow \frac{\lambda}{2}[0-f(x_0)]/S_0'(x_0) = -\frac{\lambda}{2}\left(\frac{\cancel{u_x(\xi_0,0)}}{u(\xi_0,0)}\right)\Big/ 2u(\xi_0,0)\cancel{u_x(\xi_0,0)} = -\frac{\lambda}{2}. \quad (9)$$



We've used $[u(\xi_0,0)]^2 = \frac{1}{2}$, with ξ_0 locating the HWHM point on the travelling pulse. The HWHM of $|u(x,t)|^2$ at early times $[\alpha t \ll (\Delta x)^2]$ is then

$$\Delta x' \simeq \Delta x - \frac{\lambda}{2} = \Delta x - k_0\alpha t, \text{ or } \boxed{\Delta x(t) \simeq \Delta x(0) - \alpha k_0 t} \int \begin{array}{l} \text{w/ } \alpha = (d^2\omega/dk^2)_0, \\ k_0 = \text{carrier wave\#}. \end{array} \quad (10)$$

The pulse broadens or narrows depending on whether the GVD factor $\alpha \lessgtr 0$. At later times, $[\alpha t \sim (\Delta x)^2]$, we <u>know</u> the pulse must disperse (broaden), so we expect the next correction to Eq. (10) [i.e. the term in $\lambda^2 \sim O(\alpha t)^2$] will be (+)ve, and so will <u>increase</u> $\Delta x(t)$. ¶

(D) 6. The pulse intensity in Eq. (4) can be written as...

$$\rightarrow |u(x,t)|^2 \simeq [u(\xi,0)]^2 + \lambda\, u(\xi,0)\, u_x(\xi,0), \text{ to 1st order in } \lambda = 2k_0\alpha t. \quad (11)$$

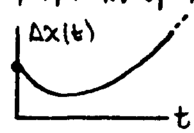
In the 2nd term RHS here, recognize $u u_x = \frac{1}{2}(\partial/\partial x)u^2$. The pulse energy to $O(\lambda)$ is...

$$\rightarrow \mathcal{E}(t) \propto \int_{-\infty}^{\infty} |u(x,t)|^2 dx \simeq \int_{-\infty}^{\infty} [u(\xi,0)]^2 d\xi + \frac{\lambda}{2}\int_{-\infty}^{\infty} d\xi \underbrace{\cancel{\frac{\partial}{\partial \xi}[u(\xi,0)]^2}}_{\to 0}. \quad (12)$$

The 1st term RHS is the initial energy; the 2nd vanishes (because $[u(\pm\infty,0)]^2 = 0$). The GVD correction does not change the total pulse energy; it just redistributes it.

¶ To $O(\lambda^2)$, can show: $\Delta x(t) \simeq \Delta x(0) - k_0\alpha t + (k_0\alpha t)^2 G\{u\}$, w/ $G\{u\} \simeq \frac{1}{2\sqrt{2}}\{-u_{xx}(0,0)/|u_x(x_0,0)|\}$

For initially symmetric pulses [per Eq. (9)], $u_{xx}(0,0) < 0$ and $G\{u\}$ is (+)ve, so the $O(\lambda^2)$ correction <u>is</u> (+)ve. For $\alpha > 0$, the HWHM $\Delta x(t)$ goes thru a min @ $k_0\alpha t \sim 1/2G$.



ϕ520 Solutions

(55) Some numerical work on "whistler waves".

A) At low freqs.: $\omega \ll \omega_B$ & ω_P, the dispⁿ relation is: $kc \simeq \omega[1 \pm \frac{\omega_P^2}{\omega\omega_B}]^{1/2}$, w/ ($\pm$) corresponding to ($\pm$) helicity. When $\omega \to 0$, k[(−)helicity] → imaginary, and the wave that propagates is (+) helicity only, w/ $k \simeq \omega[1+\frac{\omega_P^2}{\omega\omega_B}]^{1/2} = \omega_P[(\omega/\omega_B)+(\omega/\omega_P)^2]^{1/2}$. The leading term here is the desired dispersion relation, viz.

→ $kc = \omega_P\sqrt{(\omega/\omega_B)}$, for $\omega \to 0$ (i.e. $\omega \ll \omega_B$ & ω_P) and (+) helicity only. (1)

(B) From (1): $\omega = \omega_B[c^2k^2/\omega_P^2]$, so the group velocity is $v_g = \partial\omega/\partial k$, i.e.

$\underline{\underline{v_g = 2c\sqrt{(\omega_B\omega/\omega_P^2)}}}$ plasma frequency: $\omega_P = \sqrt{4\pi ne^2/m} = 0.056\sqrt{n}$, MHz ($n = \#/\text{cm}^3$); cyclotron freq.: $\omega_B = eB_0/mc = 17.6\,B_0$, MHz ($B_0$ in G). (2)

From these numerical values, we find the general v_g for whistlers...

$\underline{\underline{v_g = (14.2\times10^5\ \text{km/sec})\times\sqrt{\omega B_0/n}}}$ B_0 in G, ω in kHz and n in $\#/\text{cm}^3$. (3)

For the values given: $\bar{\omega} = 10$ kHz, $B_0 = 0.3$ G, $n = 10^5/\text{cm}^3$, we find:

$\boxed{v_g = 7.78\times10^3\ \text{km/sec} = 0.0259c}$, in earth's ionosphere, @ $\bar{\omega} = 10$ kHz. (4)

This EM wave moves relatively slowly, at < 3% light speed.

(C) Trip time at freq. ω is: $t(\omega) = D/v_g(\omega)$, so for an increment in ω...

→ $\frac{dt(\omega)}{t(\omega)} = -\frac{dv_g(\omega)}{v_g(\omega)} = -\frac{1}{2}\left(\frac{d\omega}{\omega}\right)$. (5) The (−) sign ⇒ $t(\omega)$ decreases with ω, so high freq. components arrive first.

For $d\omega \sim \Delta\omega \ll \omega$, the overall time delay between reception of ω & $(\omega - \Delta\omega)$ is//

$\underline{\underline{\Delta t(\omega) = t(\omega)\cdot\frac{1}{2}\left(\frac{\Delta\omega}{\omega}\right) = \frac{D}{2v_g(\omega)}\cdot\left(\frac{\Delta\omega}{\omega}\right)}}$. (6)

For numbers given: $D = 20{,}000$ km (N to S poles), $v_g(\omega) = 7800$ km/sec [Eq.(4)], $\Delta\omega/\omega = 1\,\text{kHz}/10\,\text{kHz} = 0.1$, we find:

$\boxed{\text{trip time: } t(\omega) = D/v_g(\omega) = 2.56\ \text{sec.};\ \text{delay: } \Delta t(\omega) = 128\ \text{msec.}}$ (7)