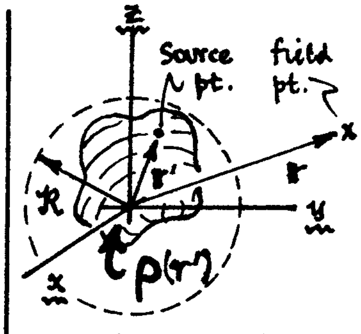
Multipoles & Dielectrics

Some topics from Jackson, Chap. 4.



source inside: $r' < R$,
field outside: $r > R$.

1) We have some powerful tools at our disposal to discuss the electrostatic potential ϕ in a general way. Suppose ϕ is generated (on an ∞ domain) by arbitrary distrib^{n} ρ. Then

$$\longrightarrow \phi(\mathbf{r}) = \int_\infty \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\, d^3x' \quad \int \text{solution for } \infty \text{ domain (Helmholtz' Thm)} \qquad (1)$$

... suppose $r >$ any r' at which $\rho(\mathbf{r}') \neq 0$ ... and use ...

use of the Addition Thm for Spherical Harmonics

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \sum_{l=0}^{\infty} \sum_{m=-l}^{+l} \frac{4\pi}{2l+1} \frac{1}{r}\left(\frac{r'}{r}\right)^l Y^*_{lm}(\theta',\varphi')\, Y_{lm}(\theta,\varphi) \leftarrow \text{Jk}^{n} \text{ Eq. (3.70) (or Eq. (26) p. II BV9)} \qquad (2)$$

So,

$$\boxed{\begin{aligned} \phi(\mathbf{r}) &= \sum_{l,m} \frac{4\pi}{2l+1} \underbrace{\left[q_{lm}\right]}_{\text{source}} \underbrace{\left(\frac{1}{r^{l+1}} Y_{lm}(\theta,\varphi)\right)}_{\text{field}}, \\ \text{where: } q_{lm} &= \int_\infty r'^{\,l}\, Y^*_{lm}(\theta',\varphi')\, \rho(\mathbf{r}')\, d^3x'. \end{aligned}}$$

... must have $r > R = \max r'$. The q_{lm} are called the "multipole moments" of the distrib^{n} ρ.. (3)

This representation neatly separates the field geometry, represented by the variation $r^{-(l+1)} Y_{lm}(\theta,\varphi)$ in the sum, from the source geometry, which enters as the coefficients q_{lm}. The so-called "multipole moments" q_{lm} are indpt of the field pt. location (so long as $r > \max r'$); they are an intrinsic property of the distrib^{n} ρ itself.

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2) The q_{lm} are of sufficient importance to merit names. E.g. (see listing of Y_{lm} on Jk^{n} pp. 99-100)...

$$q_{00} = \int_\infty Y^*_{00}\, \rho(\mathbf{r}')\, d^3x' = (1/\sqrt{4\pi})\, q, \quad q = \int_\infty \rho\, d^3x' = \text{total charge} = \text{"monopole moment"};$$

$$q_{10} = \int_\infty r'\, Y^*_{10}\, \rho(\mathbf{r}')\, d^3x' = (3/\sqrt{4\pi})\, p_z, \quad p_z = \int_\infty z' \rho\, d^3x' = \left(\text{"dipole moment"}\right)_z;$$

$z' = r'\cos\theta'$

[next page]

$$q_{11} = \int_\infty r' Y_{11}^* \rho(\mathbf{r}') dx' = -\sqrt{\frac{3}{8\pi}} \int_\infty r' [\sin\theta' e^{-i\varphi'}] \rho\, d^3x'$$

$$= -\sqrt{\frac{3}{4\pi}} \frac{1}{\sqrt{2}} \int_\infty (x' - i y') \rho\, d^3x' = -\sqrt{\frac{3}{4\pi}} \left(\frac{p_x - i p_y}{\sqrt{2}}\right) \quad \left\{ p_{x,y} = \int_\infty (x', y') \rho\, d^3x' = \underline{(\text{dipole moment})_{x,y}} \right. \qquad (4)$$

Note: $\underline{q_{l,-m} = (-)^m q_{lm}^*}$ ¶. For the $l=1$ case, we thus have...

$$\left. \begin{aligned} & q(l=1;\ m=+1,-1,0) = \sqrt{\frac{3}{4\pi}} \left(-\frac{p_x - i p_y}{\sqrt{2}},\ +\frac{p_x + i p_y}{\sqrt{2}},\ p_z \right) \\ & \text{where: } \left[\!\left[\mathbf{p} = \int \mathbf{r}' \rho(\mathbf{r}') d^3x' = (p_x, p_y, p_z) \leftarrow \text{dipole moment } \underline{\text{vector}} \right]\!\right] \end{aligned} \right\} \qquad (5)$$

$\boxed{\text{NOTE}}$ Gather terms through $l=1$...

$$\phi(\mathbf{r}) = 4\pi \frac{1}{r} \underbrace{q_{00} Y_{00}}_{= q/4\pi} + \frac{4\pi}{3} \frac{1}{r^2} \underbrace{\left(q_{11} Y_{11} + q_{1,-1} Y_{1,-1} + q_{10} Y_{10} \right)} + \cdots \qquad (6)$$

$$= \sqrt{\frac{3}{4\pi}} \left\{ \left(\frac{p_x - i p_y}{\sqrt{2}}\right) \sqrt{\frac{3}{8\pi}} \sin\theta\, e^{i\varphi} + \left(\frac{p_x + i p_y}{\sqrt{2}}\right) \sqrt{\frac{3}{8\pi}} \sin\theta\, e^{-i\varphi} + p_z \sqrt{\frac{3}{4\pi}} \cos\theta \right\}$$

$$= \frac{3}{4\pi} \left\{ (\sin\theta) \operatorname{Re}[(p_x - i p_y) e^{i\varphi}] + p_z \cos\theta \right\}$$

$$= \frac{3}{4\pi} \left\{ p_x \sin\theta \cos\varphi + p_y \sin\theta \sin\varphi + p_z \cos\theta \right\} = \frac{3}{4\pi} \mathbf{p} \cdot \hat{r}, \qquad (7)$$

So
$$\boxed{\phi(\mathbf{r}) = \frac{1}{r} \overbrace{q}^{\text{monopole}} + \frac{1}{r^2} \overbrace{\mathbf{p} \cdot \hat{n}}^{\text{dipole}} + O\left(\frac{1}{r^3}\right)}, \quad \hat{n} = \frac{\mathbf{r}}{r} \ \Big\{ \text{unit vector along } \mathbf{r}. \qquad (8)$$

In general, the $(l+1)$st term [w/ $l = 0, 1, 2, \ldots$] in the series will fall off with distance as $1/r^{l+1}$, and will have $(2l+1)$ indpt values of q_{lm} to find [need $q_{l0}, q_{l1}, \ldots, q_{ll}$; then $q_{l,-m} = (-)^m q_{lm}^*$]. Evidently the q_{lm}'s become successively more complicated.

¶ By def^{n} in Eq. (3): $q_{l,-m} = \int_\infty r'^l Y_{l,-m}^*(\theta', \varphi') \rho(\mathbf{r}') d^3x'$, and by Jk$^{\text{n}}$ Eq. (3.54): $Y_{l,-m}^* = (-)^m Y_{lm}$, so have: $q_{l,-m} = (-)^m \int_\infty r'^l Y_{lm}(\theta', \varphi') \rho(\mathbf{r}') d^3x' = (-)^m q_{lm}^*$, for $\rho \in$ real.

Multipoles & Dielectrics (cont'd)

3) In Eqs. (4.6) & (4.9), Jackson shows how to get the next term in Eq. (8):

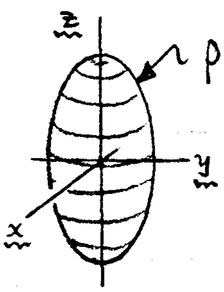
$$\left[\begin{array}{l} \Theta\left(\frac{1}{r^3}\right) = \frac{1}{2r^3}\sum_{i,j=1}^{3} Q_{ij}\, n_i n_j \;, \quad \hat{n} = \hat{r} = \frac{\mathbf{r}}{r} \text{ unit vector along } \mathbf{r}\,; \\ \text{w/ } Q_{ij} = \int_\infty (3x_i' x_j' - r'^2 \delta_{ij})\rho\, d^3x' \leftarrow \text{quadrupole moment } \underline{\text{tensor}} \end{array}\right\} \quad (9)$$

Drop the primes in the integral here and display the tensor...

$$\rightarrow (Q_{ij}) = \int \begin{pmatrix} (2x^2-y^2-z^2) & 3xy & 3xz \\ 3xy & (2y^2-z^2-x^2) & 3yz \\ 3xz & 3yz & (2z^2-x^2-y^2) \end{pmatrix} \rho(\mathbf{r})\, d^3x = \underset{\sim}{Q}. \quad (10)$$

$\boxed{\text{NOTE}}$: $Q_{ji} = Q_{ij}$, $\underset{\sim}{Q}$ is symmetric; $\operatorname{Tr}\underset{\sim}{Q} = \sum_{i=1}^{3} Q_{ii} = 0$, $\underset{\sim}{Q}$ is traceless; ‖ $\underset{\sim}{Q}$ has at most 5 indpt elements Q_{ij}; this is consistent w/ $(2l+1)|_{l=2}$. (11)

Often $\underset{\sim}{Q}$ can be simplified for ~ symmetric distributions ρ. E.g.



$\rho(\mathbf{r})$ w/ $\left\{\begin{array}{l}\text{cylindrical symmetry about } z\text{-axis} \\ \text{reflection symmetry in } xy \text{ plane}\end{array}\right.$ $\Rightarrow \langle xy\rangle = \langle yz\rangle = \langle xz\rangle = 0.$

$$\left.\begin{array}{l}\text{Also, } x \,\&\, y \text{ axes equivalent} \Rightarrow Q_{11} = Q_{22}, \\ \text{and } Q_{33} = 2\int (z^2-x^2)\rho\, d^3x = Q, \text{ etc.}\end{array}\right\} \text{ so } \underset{\sim}{Q} = \begin{pmatrix} -\frac{Q}{2} & & 0 \\ & -\frac{Q}{2} & \\ 0 & & Q \end{pmatrix}. \quad (12)$$

Quadrupoles like this are of interest in nuclear physics.

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The succeeding terms of the $\phi(\mathbf{r})$ expansion in Eq. (3) are cited. One rarely goes past the quadrupole term.

l-value	multipole name	falloff rate for ϕ	# indpt q_{lm}'s	Character of $\{q_{lm}\}$
0	monopole	$1/r$	1	scalar (charge q)
1	dipole	$1/r^2$	3	vector (dipole $\mathbf{p}$)
2	quadrupole	$1/r^3$	5	matrix ($\underset{\sim}{Q}$ above)
3	octupole	$1/r^4$	7	3-tensor
4	hexadecapole	$1/r^5$	9	4-tensor
5	?	$1/r^6$	11	5-tensor