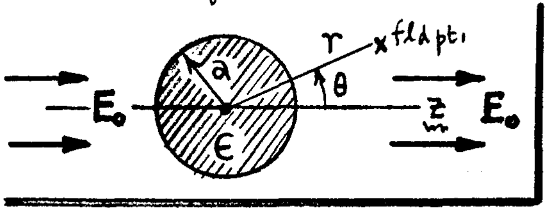
8) A more elaborate example of how $\epsilon > 1$ affects things is given in Jackson Sec. 4.4, Eqs. (4.48)-(4.58). It is the problem of a dielectric sphere placed in a uniform field E_0. Unlike the conducting sphere case (treated in Jk^{n} Sec. 2.5), the field <u>penetrates</u> to the interior of the sphere, and we need to find the potential $\phi(r \leq a)$ as well as $\phi(r \geq a)$.



With rotational symmetry about the z-axis, we want to solve $\nabla^2\phi = 0$ in the spherical cds r & θ. The solutions are of the well-known forms...

$$\rightarrow \phi_{in}(r \leq a) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta) \; ; \; \phi_{out}(r \geq a) = \sum_{l=0}^{\infty} \left[B_l r^l + C_l r^{-(l+1)} \right] P_l(\cos\theta). \quad (36)$$

<u>B.C.</u> $B_l = \begin{cases} -E_0, \text{ for } l = 1; \\ 0, \text{ otherwise;} \end{cases}$ so as $r \to \infty$, $\phi_{out} \to -E_0 r\cos\theta$, and $E_z = -\frac{\partial \phi_{out}}{\partial z} \to E_0$.

★ On the plane, $z = 0$, have $R_2 = R_1 = \sqrt{r^2 + d^2} = R$, and $\phi(z=0) = \frac{q}{R}\left(\frac{2}{\epsilon_1 + \epsilon_2}\right)$.

Multipoles & Dielectrics (cont'd)

What is new and exciting is that the B.C. on the sphere surface are different

at $r = a$ (sphere surface)

tangential comp. of $\mathbb{E}$ conserved : $-\frac{1}{a}\left(\frac{\partial\phi_{out}}{\partial\theta}\right) = -\frac{1}{a}\left(\frac{\partial\phi_{in}}{\partial\theta}\right)$;

normal comp. of $\mathbb{D}$ is conserved : $-\frac{\partial\phi_{out}}{\partial r} = -\epsilon\,\frac{\partial\phi_{in}}{\partial r}$. (37)

assumptions : no real surface charge at $r=a$; ϵ(inside) $= \epsilon$, ϵ(outside) $= 1$.

When these B.C. are imposed on the generic solutions in (36), Jk$^{\underline{o}}$ finds

$$\boxed{\phi_{in} = -\left(\frac{3}{\epsilon+2}\right)E_0\underbrace{r\cos\theta}_{z=r\cos\theta}, \quad \phi_{out} = -\left[1-\left(\frac{\epsilon-1}{\epsilon+2}\right)\frac{a^3}{r^3}\right]E_0\underbrace{r\cos\theta}_{z=r\cos\theta}} \quad \text{Jk}^{\underline{n}}\ (4.54) \qquad (38)$$

REMARKS

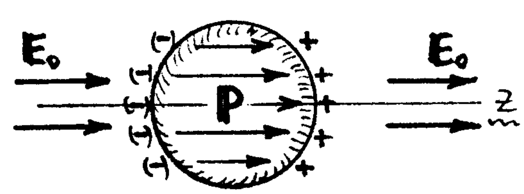
1. Compare above results with those for a conducting sphere in an external E_0 :

Jackson Eq. (2.14), p. 61
Conducting sphere in E_0 } $\phi_{in} \equiv 0$, $\phi_{out} = -\left[1-(a/r)^3\right]E_0 r\cos\theta$. (39)

Eqs. (38) & (39) are the same when $\epsilon\to\infty$, which is appropriate to a metal.

2. $E_{in} = -\partial\phi_{in}/\partial z = \left(\frac{3}{\epsilon+2}\right)E_0$ } ∥ applied E_0 and reduced ;

$E_{out} = E_0 + \left(\text{field of a dipole of strength } p = \left(\frac{\epsilon-1}{\epsilon+2}\right)a^3E_0\right)$ ← Changes due to polarization $\mathbb{P}$



Polarization field inside the sphere : $E_{induced} = E_{in} - E_0 = -\left(\frac{\epsilon-1}{\epsilon+2}\right)E_0$.

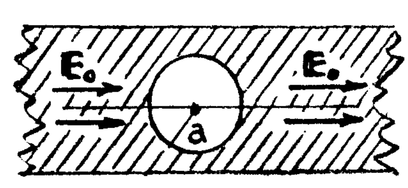
3. Actual polarization of the sphere is...

dipole moment per unit volume } $\mathbb{P} = \frac{1}{(4\pi/3)a^3}\,p\,\hat{z} = \left[\frac{3}{4\pi}\left(\frac{\epsilon-1}{\epsilon+2}\right)\right]\mathbb{E}_0$ (40)

the [ ] = α, polarizability

The induced surface density is : $\sigma_{pol} = \frac{1}{4\pi}\cdot(4\pi\,\mathbb{P}\cdot\hat{n}) = \frac{3}{4\pi}\left(\frac{\epsilon-1}{\epsilon+2}\right)E_0\cos\theta$. (41)

4. The important problem of the fields around and in a spherical cavity in a dielectric where there is a cnst field $\mathbb{E}_0$ is solved by sending $\epsilon\to 1/\epsilon$ in above eqtns.



9) We skip Jackson's Secs. 4.5 & 4.6 on molecular polarizability.

We close Chap. 4 with some discussion of the field energy in dielectric media (i.e. Jackson Sec. 4.7). In free space (no polarization $\mathbb{P}$), what we did was:

FREE SPACE ($\rho = \rho_{real}$, $\phi = \phi_{self}$) $\Big\}$ $W = \frac{1}{2}\int_\infty d^3x\, \phi(\mathbf{r})\,\rho(\mathbf{r}) \overset{\nabla\cdot\mathbb{E}=4\pi\rho}{=} \frac{1}{8\pi}\int_\infty d^3x\,(\underbrace{\phi\,\nabla\cdot\mathbb{E}}_{\nabla\cdot(\phi\mathbb{E})-(\nabla\phi)\cdot\mathbb{E}})$;

Convert $\nabla\cdot(\phi\mathbb{E})$ term to integral $\oint_\infty(\phi\mathbb{E})\cdot d\$ \to 0$; use $\nabla\phi = -\mathbb{E}$ $\Big\}$ $\boxed{W = \frac{1}{8\pi}\int_\infty(\mathbb{E}\cdot\mathbb{E})\,d^3x}$ energy-of-assembly in free space. (42)

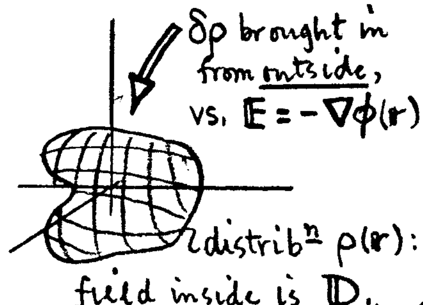
How is W modified by presence of polarizable medium: $\mathbb{E}\to\mathbb{D}=\epsilon\mathbb{E}$, $\phi\to\frac{1}{\epsilon}\phi$, $\rho\to\rho_{free}-\nabla\cdot\mathbb{P}$? Not clear that form $\frac{1}{2}\int d^3x\,\phi\rho$ is useful (or even comprehensible). So we review the way W can be derived microscopically.

DIELECTRIC ($\rho = \rho_{real}$, $\phi = \phi_{self}$) $\Big\}$ $\delta W = \int_\infty d^3x\,\phi(\mathbf{r})\,\delta\rho(\mathbf{r})$;

use: $\delta\rho = (1/4\pi)\,\nabla\cdot(\delta\mathbb{D})$ (accounts for Δ(polarization))

use: $\mathbb{E} = -\nabla\phi$ (outside ρ) $\Big\}$ $\boxed{\delta W = \frac{1}{4\pi}\int_\infty d^3x\,\mathbb{E}\cdot\delta\mathbb{D}}$

have discarded surface term.



(43)

The formal integral of this last expression is ...

$$W = \frac{1}{4\pi}\int_\infty d^3x \int_0^{D} \mathbb{E}\cdot\delta\mathbb{D}, \text{ for distrib}^{n}\text{ buildup } |\mathbb{D}| = 0\to D. \quad (44)$$

Suppose medium is nonisotropic but linear $\Big\{$ $\mathbb{D} = \underset{\sim}{\epsilon}\,\mathbb{E}$, i.e. $D_i = \epsilon_{ij}E_j$ (summation convention) and $\epsilon_{ij} \neq$ fcn of the field

So $W = \frac{1}{4\pi}\int_\infty d^3x\int_0^D E_i\,\epsilon_{ij}\,\delta E_j = \frac{1}{4\pi}\int_\infty d^3x\cdot\frac{1}{2}(E_i\,\epsilon_{ij}\,E_j)_{final}$

i.e., $\boxed{W = \frac{1}{8\pi}\int_\infty d^3x\,(\mathbb{E}\cdot\mathbb{D})}$, for linear media (w/ $\epsilon_{ji} = \epsilon_{ij}$). (45)

For nonlinear media ($\underset{\sim}{\epsilon}$ = fcn of $\mathbb{E}$), must calculate $W = \frac{1}{4\pi}\int_\infty d^3x\int_0^D \mathbb{E}\cdot\delta\mathbb{D}$.