ϕ506 Problems

(13) The free-particle Klein-Gordon Eqtn is : $[\nabla^2 - \frac{1}{c^2}(\partial^2/\partial t^2) - (mc/\hbar)^2]\,\phi(\mathbf{r},t) = 0$, for the free motion of mass m in 3D. Verify (per CLASS NOTES, pp. Sch. 4-5) that upon substituting $\psi(\mathbf{r},t) = \phi(\mathbf{r},t)\exp[i(mc^2/\hbar)t]$, the KG Eqtn becomes (exactly) $\underline{[\nabla^2 - \frac{1}{c^2}(\partial^2/\partial t^2) + \frac{2im}{\hbar}(\partial/\partial t)]\,\psi(\mathbf{r},t) = 0}$. **(A)** Under what general condition on ψ can the term in $1/c^2$ be considered "small"? **(B)** Consider a planewave solution to the ψ eqtn : $\underline{\psi(\mathbf{r},t) = \exp[\frac{i}{\hbar}(\mathbf{p}\cdot\mathbf{r} - Kt)]}$, w/ $\mathbf{p}$ = m's cnst momentum, and K its (relativistic) kinetic energy ($K = E - mc^2$). What condition must K obey in order that the $1/c^2$ term is "small"? Then, what do you conclude re KG Eqtn?

---

(14) **(A)** For a Dirac delta function δ defined by : $\int_{-\infty}^{+\infty}\delta(x-x')f(x')\,dx' = f(x)$, establish the relation : $\underline{\delta(kx) = \delta(x)/|k|}$, for k any nonzero cnst. **(B)** Use the result in part (A) to show that if $g(x)$ is a fcn with one zero @ $x = x_0$, i.e. if $g(x_0) = 0$, then : $\underline{\delta[g(x)] = \delta(x-x_0)/|(dg/dx)_{x=x_0}|}$. HINT: expand $g(x)$ in a Taylor series about x_0. **(C)** Generalize the result in part (B) to $g(x)$ with n zeros x_i (i.e. $g(x_i) = 0$, $i = 1$ to n) : $\underline{\delta[g(x)] = \sum_{i=1}^{n}\delta(x-x_i)/|(dg/dx)_{x=x_i}|}$. What must be assumed about the derivatives $(dg/dx)_{x=x_i}$? **(D)** Use part (C) to show that when a & b are cnsts, and $b \neq a$: $\underline{\delta[(x-a)(x-b)] = \{\delta(x-a) + \delta(x-b)\}/|a-b|}$.

---

(15) **(A)** Start from the definition : $\underline{\langle x\rangle_0 = \int_\infty \psi^*(x,0)\{x\}\psi(x,0)\,dx}$, for the expectation value of position x in configuration space at time $t = 0$. Transform this integral to momentum space, and show that $\underline{x \to x_{op} = i\hbar\,\partial/\partial p}$ in that space. Is this operator equivalence independent of time? **(B)** To generalize the transformations between x & p spaces, begin with the "natural" definitions of $\langle x^n\rangle$ & $\langle p^n\rangle$ in their own spaces, and establish the equivalences : $\underline{x^n \to (i\hbar)^n\partial^n/\partial p^n}$, and : $\underline{p^n \to (-i\hbar)^n\partial^n/\partial x^n}$, upon transformation. **(C)** Use part (B) to show that for any fcns $f(x)$ & $F(p)$ expansible in power series : $\underline{f(x) \to f(i\hbar\frac{\partial}{\partial p})}$ & $\underline{F(p) \to F(-i\hbar\frac{\partial}{\partial x})}$, similarly.

ϕ506 Solutions

⑬ Explicit NR limit for KG Eqtn. Example of plane-waves.

(A) Substitution of $\phi = \psi e^{-i(mc^2/\hbar)t}$ into $[\nabla^2 - \frac{1}{c^2}\partial^2/\partial t^2 + (mc/\hbar)^2]\phi = 0$ indeed produces the (quoted) modified eqtn: $\underline{[\nabla^2 - \frac{1}{c^2}\partial^2/\partial t^2 + \frac{2im}{\hbar}\partial/\partial t]\psi = 0}$. If the term in $1/c^2$ is "small", if -- by comparison with the other t-dept term --

$$\rightarrow \left|\frac{1}{c^2}\left(\frac{\partial^2\psi}{\partial t^2}\right)\right| \ll \left|\frac{2im}{\hbar}\left(\frac{\partial\psi}{\partial t}\right)\right| \Rightarrow \underline{\underline{\left|\frac{1}{\dot\psi}\left(\frac{\partial\dot\psi}{\partial t}\right)\right| \ll mc^2/\hbar}}, \text{ w/ } \dot\psi = \partial\psi/\partial t. \qquad (1)$$

Under this condition, the 2nd term LHS in the ψ eqtn is ~negligible compared with the 3rd term LHS.

(B) For a planewave solution: $\psi = e^{(i/\hbar)[\mathbf{p}\cdot\mathbf{r} - Kt]}$, w/ $K = E - mc^2$ the relativistic K.E.*, we note: $\dot\psi = -i(K/\hbar)\psi$, and $\frac{\partial\dot\psi}{\partial t} = -(K/\hbar)^2\psi$. By putting these values into the general condition of Eq.(1), we find...

$$\rightarrow |-i(K/\hbar)| \ll mc^2/\hbar, \text{ i.e. } \underline{\underline{K \ll mc^2}}. \qquad (2)$$

Under this condition on KG planewaves, the single relativistic term in the KG eqtn for ψ, namely the term in $1/c^2$, is negligibly small. Now, the relativistic form of K is...

$$\rightarrow K = E - mc^2 = (\gamma - 1)mc^2, \text{ w/ } \gamma = \sqrt{1-(v/c)^2}, \qquad (3)$$

where v = m's velocity. So $K \ll mc^2 \Rightarrow (\gamma - 1) \ll 1$, or...

$$\rightarrow \gamma - 1 \simeq \frac{1}{2}(v/c)^2 \ll 1, \text{ i.e. } \underline{\underline{v \ll c}}. \qquad (4)$$

Thus m is moving slowly, at nonrelativistic velocities. Upon neglecting the $1/c^2$ term in the KG Eqtn under this condition, we obtain...

$$\boxed{\left[\nabla^2 + (2im/\hbar)\frac{\partial}{\partial t}\right]\psi = 0} \leftarrow \text{Schrödinger's free-particle wave eqtn (as NR limit of KG Eq. CLASS NOTES, Eq.(13), p. Sch.6)} \qquad (5)$$

---

* For this ψ, the original $\phi = e^{(i/\hbar)[\mathbf{p}\cdot\mathbf{r} - Et]}$, w/ E = relativistic <u>total</u> energy.

(14) Some algebraic facts about Dirac delta functions.

(A) Ref. CLASS NOTES, Eqs. (26), p. Sch. 11. From the def$^{\underline{n}}$ $\int_{-\infty}^{\infty}\delta(z)\,dz = 1$, evidently $\delta(z)$ is an even fcn of z, since the same result obtains for $z = \pm x$ (i.e. find $\int_{-\infty}^{\infty}\delta(\pm x)\,dx = 1$). So $\delta(-x) = \delta(x)$, and this means for a cnst k of either sign: $\delta(kx) = \delta(|k|x)$. Now, with $u = |k|x$, note that...

$$\rightarrow 1 = \int_{-\infty}^{\infty}\delta(u)\,du = \int_{-\infty}^{\infty}\delta(|k|x)\,d(|k|x) = |k|\int_{-\infty}^{\infty}\delta(kx)\,dx,$$

$$\text{i.e.,}\quad \underbrace{\int_{-\infty}^{\infty}[\delta(kx)]\,dx = 1/|k| = \int_{-\infty}^{\infty}\left[\frac{\delta(x)}{|k|}\right]dx}_{\text{identify}}, \quad \text{so}\quad \boxed{\delta(kx) = \delta(x)/|k|}. \quad \text{QED} \qquad (1)$$

(B) If $g(x)$ has a single zero @ $x = x_0$, then $\delta[g(x)]$ vanishes everywhere except in the neighborhood of x_0. In that neighborhood, g's Taylor series is:

$$\rightarrow g(x) = \cancelto{0}{g(x_0)} + (x-x_0)g'(x_0) + (x-x_0)\left[\tfrac{1}{2}(x-x_0)g''(x_0) + \ldots\right] \qquad (2)$$

The terms in the [ ] vanish faster than $(x-x_0)$ as $x \to x_0$, and so in that neighborhood: $g(x) = g'(x_0)(x-x_0)$, provided $g'(x_0) \neq 0$. Then

$$\rightarrow \boxed{\delta[g(x)] \rightarrow \delta[g'(x_0)(x-x_0)] = \delta(x-x_0)/|g'(x_0)|}. \quad \text{QED} \qquad (3)$$

The last step follows by use of Eq. (1).

(C) For $g(x)$ with N zeros, $g(x_i) = 0$ for $i = 1, 2, \ldots, N$, the calculation of part (B) can be repeated for each x_i, so: $\delta[g(x)] = \delta(x-x_i)/|g'(x_i)|$, as $x \to x_i$. As $x \to x_1 \to x_2 \to \cdots \to x_N$, such contributions accumulate independently, and so:

$$\boxed{\delta[g(x)] = \sum_{i=1}^{N}\delta(x-x_i)/|g'(x_i)|}. \quad \text{QED} \quad (4)$$

Each $g'(x_i)$ assumed nonzero.

(D) For $g(x) = (x-a)(x-b)$, zeros are at $x = a$ & $x = b$, w/ $g'(a) = (a-b)$, and $g'(b) = (b-a)$. Both g's are nonzero if $b \neq a$. Then, by use of Eq. (4) above...

$$\boxed{\delta[(x-a)(x-b)] = \frac{\delta(x-a)}{|a-b|} + \frac{\delta(x-b)}{|b-a|} = \frac{1}{|a-b|}[\delta(x-a) + \delta(x-b)]}. \quad \text{QED} \qquad (5)$$

(15) Show $x_{op} = i\hbar\,\partial/\partial p$ in momentum space. Generalize to $f(x) \to f(i\hbar\frac{\partial}{\partial p})$, etc.

(A) $\psi(x,0) \to \varphi(k) = \frac{1}{\sqrt{2\pi}} \int \psi(x,0)\, e^{-ikx}\, dx$, in momentum space, per CLASS NOTES, Eq. (24), p. Sch. 10. The $\int = \int_{-\infty}^{+\infty}$ is over an ∞ domain. At $t=0$, the position expectation value: $\langle x\rangle_0 = \int \psi^*(x,0)\{x\}\psi(x,0)\,dx$, can then be transformed as...

$$\rightarrow \langle x\rangle_0 = \int dx \left[\frac{1}{\sqrt{2\pi}}\int \varphi(k')\, e^{+ik'x}\, dk'\right]^* \{x\} \left[\frac{1}{\sqrt{2\pi}}\int \varphi(k)\, e^{+ikx}\, dk\right]$$

$$= \frac{1}{2\pi}\int dx \int dk'\, \varphi^*(k')\, e^{-ik'x} f(x), \text{ w/ } \underline{\underline{f(x) = \int dk\, \varphi(k)\, x\, e^{ikx}}}. \qquad (1)$$

But: $x e^{ikx} = -i\frac{\partial}{\partial k} e^{ikx}$, so: $f(x) = -i\int dk\,\varphi(k)\frac{\partial}{\partial k} e^{ikx}$. By a simple partial integration: $f(x) = +i\int dk\, e^{ikx}\frac{\partial}{\partial k}\varphi(k)$, for particles with finite momenta (i.e. $\varphi(k)\to 0$ as $|k|\to\infty$). Use this in Eq. (1) and rearrange integrals to get...

$$\langle x\rangle_0 = \int dk'\,\varphi^*(k')\int dk \left\{i\frac{\partial}{\partial k}\right\}\varphi(k)\cdot\underbrace{\left[\frac{1}{2\pi}\int dx\, e^{i(k-k')x}\right]}_{[\;] = \delta(k-k')}$$

... integrate over k' to get: $\langle x\rangle_0 = \int dk\,\varphi^*(k)\left\{i\frac{\partial}{\partial k}\right\}\varphi(k)$, i.e. $\underline{\underline{x \to i\frac{\partial}{\partial k}}}$. (2)

identify

The momentum $p = \hbar k$ (de Broglie); so we've shown: $\boxed{x \to i\hbar\,\partial/\partial p}$. **QED.**

Above derivation is OK at $t=0$. When $t>0$, $\psi(x,0)\to\psi(x,t)$, and the mean position is: $\langle x\rangle_t = \int \psi^*(x,t)\{x\}\psi(x,t)\,dx$. The packet representation of ψ is: $\psi(x,t) = \frac{1}{\sqrt{2\pi}}\int \varphi(k)\, e^{i(kx-\omega t)}\,dk$, w/ $\omega = \omega(k)$. Eq. (1) above becomes...

$$\rightarrow \langle x\rangle_t = \frac{1}{2\pi}\int dx\int dk'\,\varphi^*(k')\, e^{-i(k'x-\omega' t)} f(x,t), \qquad (3)$$

w/ $f(x,t) = \int dk\,\varphi(k)\, x\, e^{i(kx-\omega t)} = -i\int dk\,\varphi(k)\, e^{-i\omega t}\frac{\partial}{\partial k} e^{ikx}$

$$= +i\int dk\, e^{ikx}\frac{\partial}{\partial k}\left[\varphi(k)\, e^{-i\omega t}\right], \text{ by partial integration;}$$

$$\text{so} \rightarrow \langle x\rangle_t = \int dk'\,\varphi^*(k')\, e^{i\omega' t}\int dk\, e^{-i\omega t}\left\{i\frac{\partial}{\partial k} + t\left(\frac{\partial\omega}{\partial k}\right)\right\}\varphi(k)\cdot\underbrace{\left[\frac{1}{2\pi}\int dx\, e^{i(k-k')x}\right]}_{[\;] = \delta(k-k')} \qquad (4)$$

This is the counterpart of the first of Eqs. (2) above, for $t>0$. Integrating over k' in the delta fcn, we find that...

$$\underline{\underline{\langle x\rangle_t = \langle x\rangle_0 + \langle v_g\rangle t}}, \qquad (5)$$

w/ $\langle x\rangle_0 = \int dk\, \varphi^*(k)\left\{i\frac{\partial}{\partial k}\right\}\varphi(k)$ ← initial position per Eq. (2);

$\langle v_g\rangle = \int dk\, \varphi^*(k)\left\{\frac{\partial\omega}{\partial k}\right\}\varphi(k)$ ← exp. value of group velocity.

This result shows that the packet moves in time @ group velocity $v_g = \partial\omega/\partial k$. $x \to i\partial/\partial k$ operating on $\varphi(k)$ alone cannot specify this time dependence, **BUT** $i\partial/\partial k$ operating on the time-dependent momentum distribution (CLASS NOTES, Eq. (35), p. Sch. 15) $\varphi(k)e^{-i\omega(k)t}$ does the trick, since...

$$\rightarrow i\frac{\partial}{\partial k}\left[\varphi(k)e^{-i\omega t}\right] = e^{-i\omega t}\left[\underbrace{i\frac{\partial}{\partial k}}_{\text{initial position}} + \underbrace{t\left(\frac{\partial\omega}{\partial k}\right)}_{\text{group velocity motion}}\right]\varphi(k) \qquad (6)$$

In this sense, $\underline{x \to i\partial/\partial k}$ is a time-independent assignment, if -- in momentum space -- $i\partial/\partial k$ is allowed to operate on the appropriate amplitude.

(B) Have $x \to i\partial/\partial k$ in momentum space, and $k \to -i\partial/\partial x$ in position space (CLASS NOTES, Eq. (36), p. Sch. 15). Need only do analysis on x; k results follow by inversion.

$$\rightarrow \langle x^n\rangle_0 = \int dx\, \Psi^*(x,0)\{x^n\}\Psi(x,0) = \frac{1}{2\pi}\int dx \int dk'\varphi^*(k')e^{ik'x} f_n(x), \qquad (7)$$

w/ $f_n(x) = \int dk\, \varphi(k)\, x^n e^{ikx}$; Eq. (1) revisited. Use: $\underbrace{x^n e^{ikx} = \left(\frac{1}{i}\frac{\partial}{\partial k}\right)^n e^{ikx}}$, do $\underline{\underline{n}}$ partial integrations on f_n to show: $f_n(x) = \int dk\, e^{ikx}\left(i\frac{\partial}{\partial k}\right)^n\varphi(k)$. Put this result back into (7), segregate the $\delta(k-k')$ fcn as in Eq. (2), integrate over k', and find: $\langle x^n\rangle_0 = \int dk\, \varphi^*(k)\left\{\left(i\frac{\partial}{\partial k}\right)^n\right\}\varphi(k)$. Thus: $\underline{\underline{x^n \to (i\partial/\partial k)^n}}$, as required. In similar (reciprocal) fashion: $\underline{\underline{k^n \to (-i\partial/\partial x)^n}}$.

(C) For $f(x)$ in a power series: $f(x) = \sum_{n=0}^{\infty}\left[\frac{1}{n!}f^{(n)}(0)\right]x^n$, the [ ]'s are csts. In momentum space, $x^n \to (i\partial/\partial k)^n$, so: $\underline{f(x)} \to \sum_{n=0}^{\infty}\left[\frac{1}{n!}f^{(n)}(0)\right](i\partial/\partial k)^n = \underline{\underline{f(i\partial/\partial k)}}$, as required. Similarly (and reciprocally): $\underline{\underline{F(k) \to F(-i\partial/\partial x)}}$, in position space.