DEPARTMENT OF PHYSICS

1994 COMPREHENSIVE EXAMINATION

Problem #1.

Consider a thin rod of length L and mass M having a mass distribution $\rho(r)$ (mass per unit length).

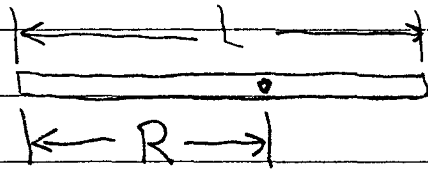
a) (2 points) Give expressions for the position of the center of mass R and the moment of inertia I of the rod with respect to rotations about an axis that is perpendicular to the rod and which passes through one end of the rod. Prove that $MR^2 \leq I$ for any mass distribution $\rho(r) \geq 0$.

b) (4 points) Assume now that this rod is placed with one end at rest on a horizontal surface in a uniform gravitational field. Assume that the point of contact between the rod and the surface may not slip horizontally, but that it is free to move vertically (thus breaking contact with the surface). The rod is free to rotate about this contact point. Assume the rod is placed initially at rest in a vertical position. Determine the equations of motion for the subsequent motion of the rod using suitable generalized coordinates. Find one first integral of the motion.

c) (4 points) The end of the rod and the point of contact on the horizontal surface may in principle separate as the rod falls. Determine under what conditions (e.g., on the mass distribution and/or the orientation of the rod) the point of contact of the rod will separate from the surface as it falls.

Solution to Classical Mechanics Problem (Lee Lindblom)

a) Let R denote the position of the center of mass



R is defined by the condition:

$$0 = \int_0^L (r - R)\,\rho(r)\,dr$$

Thus: $$R = \frac{1}{M}\int_0^L r\,\rho(r)\,dr$$

where $$M = \int_0^L \rho(r)\,dr.$$

The moment of inertia (for rotations about an axis $\perp$ to the rod through one end of the rod) is given by:

$$I = \int_0^L r^2 \rho(r)\,dr$$

Now $$MR^2 = \frac{1}{M}\left[\int_0^L r\,\rho(r)\,dr\right]^2$$

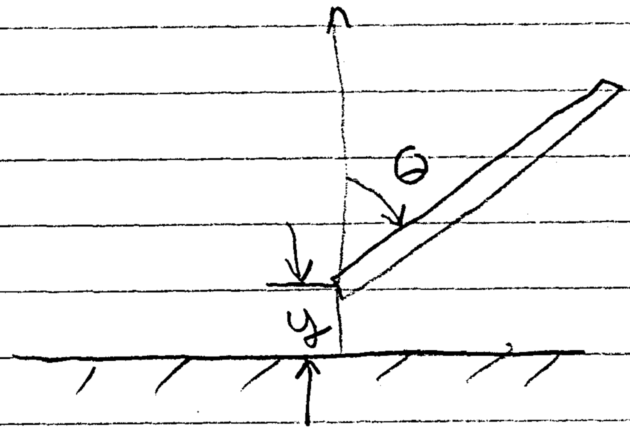
By the Schwarz inequality (for $\rho(r) \geq 0$):

$$\left[\int_0^L r\,\rho(r)\,dr\right]^2 \leq \left[\int_0^L \rho(r)\,dr\right]\left[\int_0^L r^2 \rho(r)\,dr\right]$$

Thus: $$MR^2 \leq I$$

(2)

b) Let y and Θ denote generalized coordinates that specify the vertical position of the end of the rod above the plane, and the orientation of the rod:



First determine the kinetic energy of the rod in terms of y and Θ: Let X and Y be the position of individual points in the rod.

$$X = r \sin\Theta$$
$$Y = y + r\cos\Theta$$

Where r is the position along the rod. Thus, the kinetic energy is:

$$T = \frac{1}{2}\int_0^L \rho(r)\left[\dot{X}^2 + \dot{Y}^2\right]dr$$
$$= \frac{1}{2}\int_0^L \rho(r)\left\{r^2\cos^2\Theta\,\dot{\Theta}^2 + \left[\dot{y} - r\dot{\Theta}\sin\Theta\right]^2\right\}dr$$

③

$$= \frac{1}{2}\int_0^L \rho(r)\left\{\dot{y}^2 + r^2\dot{\theta}^2 - 2r\dot{y}\dot{\theta}\sin\theta\right\}dr$$

$$= \frac{1}{2}M\dot{y}^2 + \frac{1}{2}I\dot{\theta}^2 - \dot{y}\dot{\theta}\sin\theta\int_0^L \rho(r)r\,dr$$

$$= \frac{1}{2}M\dot{y}^2 + \frac{1}{2}I\dot{\theta}^2 - MR\dot{y}\dot{\theta}\sin\theta$$

Next the potential energy is:

$$V = Mg(y + R\cos\theta)$$

Thus the Lagrangian is:

$$L = T - V = \frac{1}{2}M\dot{y}^2 + \frac{1}{2}I\dot{\theta}^2 - MR\dot{y}\dot{\theta}\sin\theta - Mg(y + R\cos\theta)$$

The equations of motion are:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) - \frac{\partial L}{\partial y} \equiv \lambda = M\frac{d}{dt}\left(\dot{y} - R\dot{\theta}\sin\theta\right) + Mg$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0 = \frac{d}{dt}\left(I\dot{\theta} - MR\dot{y}\sin\theta\right) - MgR\sin\theta$$

In addition there is a constraint $y = 0$ which applies until the generalized force λ changes sign. When λ is positive the horizontal surface pushes on the rod, but if λ becomes negative the surface would have to pull on the rod (but it is unable to do this).

When the constraint is satisfied $y = 0 = \dot{y} = \ddot{y}$.
Thus the equation of motion becomes:

$$I\ddot{\theta} - MgR\sin\theta = 0$$

And the generalized force is

$$\lambda = -MR\ddot{\theta}\sin\theta - MR\dot{\theta}^2\cos\theta + Mg$$

The first integral of the equation of motion is:

$$\frac{1}{2}\dot{\theta}^2 = MgR(1-\cos\theta)$$

c) Now evaluate the generalized force:

$$\lambda = -MR\ddot{\theta}\sin\theta - MR\dot{\theta}^2\cos\theta + Mg$$

$$= Mg\left\{-\frac{MR^2}{I}\sin^2\theta - \frac{2MR^2}{I}\cos\theta(1-\cos\theta) + 1\right\}$$

$$= Mg\left\{1 - \frac{MR^2}{I} + \frac{MR^2}{I}(3\cos^2\theta - 2\cos\theta)\right\}$$

This quantity will change sign when

$$0 = 1 - \frac{MR^2}{I} + \frac{MR^2}{I}(3\cos^2\theta - 2\cos\theta)$$

This equation will have roots for moments of inertia satisfying

$$MR^2 \leq I \leq \frac{4}{3}MR^2$$

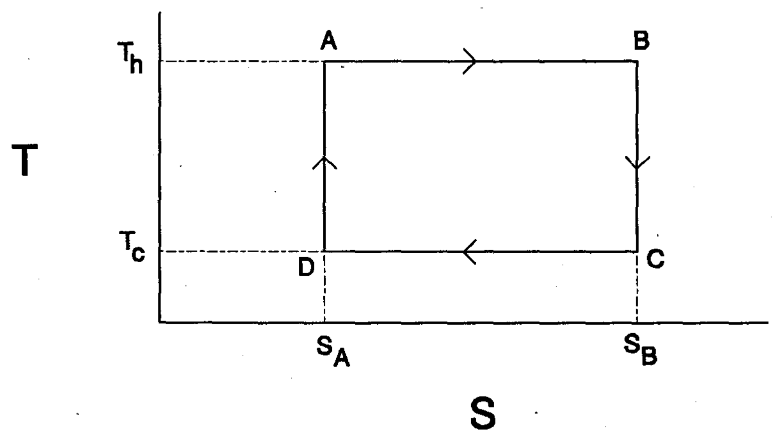
#2

Thermodynamics

A thermodynamic substance is described by the entropy function

$$S = A(UNV)^{1/3}$$

where U, N and V are the internal energy, particle number and volume of the substance, respectively. The substance is taken through the cycle ABCDA as shown in the S-T diagram below. Find the change in internal energy (ΔU), the heat absorbed (Q) and the work performed on the surroundings (W) for each process A→B, B→C, C→D, and D→A. Evaluate the thermodynamic efficiency of the closed cycle.



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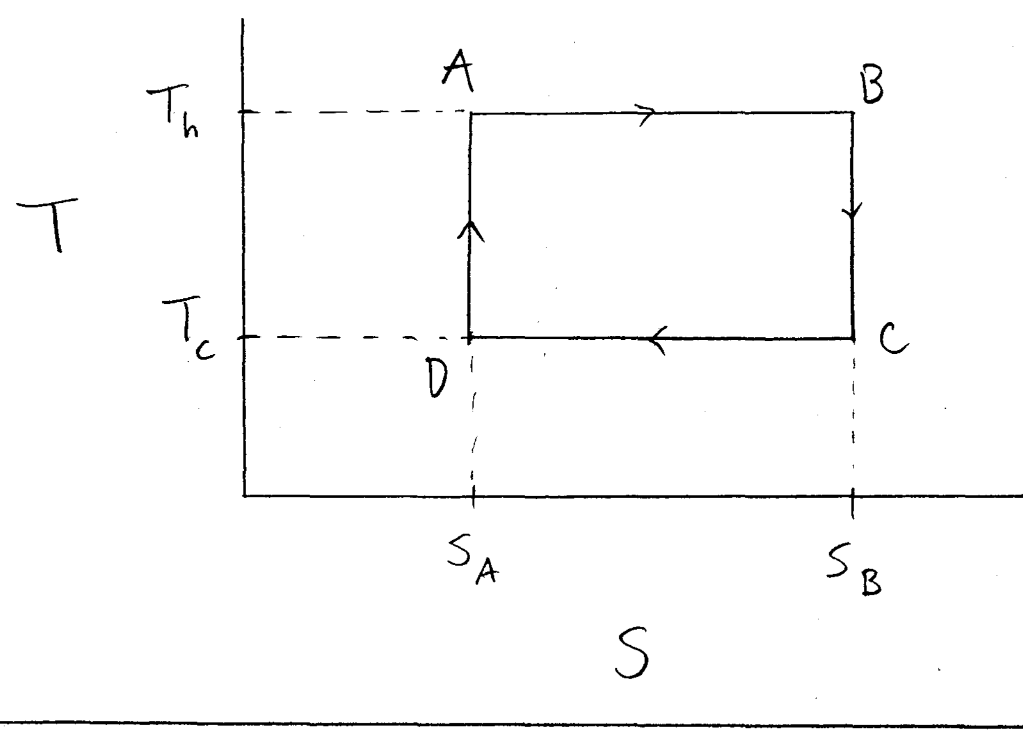


Solution:

First, relate U to S and T:

$$U = \frac{S^3}{A^3 N V}$$

$$T = \frac{\partial U}{\partial S} = \frac{3U}{S} \Rightarrow \boxed{U = \frac{1}{3} S T}$$

Using $\Delta U = \frac{1}{3}\Delta(ST) = \frac{1}{3} T\Delta S + \frac{1}{3} S \Delta T$

$Q = T \Delta S$

and $W = -\Delta U + Q$,

4 points

we find for each process (4 points)

AB :

$$\Delta U = \frac{1}{3}(S_B - S_A) T_h$$
$$Q = T_h (S_B - S_A)$$
$$W = \frac{2}{3} T_h (S_B - S_A)$$

BC :

$$\Delta U = \frac{1}{3} S_B (T_c - T_h)$$
$$Q = 0$$
$$W = -\frac{1}{3} S_B (T_c - T_h)$$

CD :

$$\Delta U = \frac{1}{3}(S_A - S_B) T_c$$
$$Q = T_c (S_A - S_B)$$
$$W = \frac{2}{3} T_c (S_A - S_B)$$

DA :

$$\Delta U = \frac{1}{3} S_A (T_h - T_c)$$
$$Q = 0$$
$$W = -\frac{1}{3} S_A (T_h - T_c)$$

Efficiency: $\varepsilon = \dfrac{\Sigma W}{Q_h}$

$$\Sigma W = \frac{2}{3} T_h (S_B - S_A) - \frac{1}{3}(T_c - T_h) S_B$$
$$+ \frac{2}{3} T_c (S_A - S_B) - \frac{1}{3}(T_h - T_c) S_A$$
$$= (T_h - T_c)(S_B - S_A) = \underset{(>0)}{\Delta T \Delta S} \quad \checkmark$$

$$Q_h = T_h (S_B - S_A)$$

$$\boxed{\varepsilon = \frac{T_h - T_c}{T_h} = 1 - \frac{T_c}{T_h}} = \text{Carnot efficiency}$$

(2 points)

Thus, the process is reversible

Math Physics

A stretched string of length L has the end $x = 0$ fastened down, and the end $x = L$ moves in prescribed manner:

$$y(L, t) = \cos \omega_0 t, \qquad t > 0.$$

Initially, the string is stretched in a straight line with unit displacement at the end $x = L$ and zero initial velocities everywhere. Find the displacement $y(x, t)$ as a function of x and t after $t = 0$.

S. TSURUTA

Problem 3: Solution

$y(0,t)=0, \quad y(L,t)=\cos\omega_0 t$ (1)

$y(x,0)=x/L$ (2)

Write the solution in the form

$y = u(x,t)+v(x)\cos\omega_0 t,$ (3)

where v takes care of the inhomogeneous boundary conditions. Both u and v obey the wave eqn:

$y'' - \frac{1}{c^2}\ddot{y}=0$ (4), where $y''=\frac{\partial^2 y}{\partial x^2}$ and $\ddot{y}=\frac{\partial^2 y}{\partial t^2}$.

$\therefore v''+\frac{1}{c^2}\omega_0^2 v=0. \quad \therefore v=A\sin\left(\frac{\omega_0 x}{c}\right)+B\cos\left(\frac{\omega_0}{c}x\right).$

We choose v to obey the boundary conditions.

$v(0)=0$, and $v(L)=1$. $\therefore B=0$ and $A=\csc\left(\frac{\omega_0}{c}L\right)$

$\therefore v=\sin\left(\frac{\omega_0}{c}x\right)/\sin\left(\frac{\omega_0}{c}L\right).$ (5)

Then, $u(x,t)$ obeys homogeneous conditions at the end points, and the initial condn $u(x,0)=\frac{x}{L}-\frac{\sin(\omega_0 x/c)}{\sin(\omega_0 L/c)}$. (6)

We expand u in a series of normal modes

$u(x,t)=\sum a_n \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi c}{L}t\right)$

$\therefore u(x,0)=\frac{x}{L}-\frac{\sin(\omega_0 x/c)}{\sin(\omega_0 L/c)}=\sum a_n \sin\left(\frac{n\pi x}{L}\right)$ from (6) & (7).

$$\therefore a_n=\frac{2}{L}\int_0^L\left(\frac{x}{L}-\frac{\sin(\omega_0 x/c)}{\sin(\omega_0 L/c)}\right)\sin\left(\frac{n\pi x}{L}\right)dx$$

$$\star \longrightarrow =\frac{2}{L}\left\{\left(-\frac{L}{n\pi}(-1)^n-\frac{1}{2\sin\frac{\omega_0}{c}L}\left[\frac{1-\cos\left(n\pi-\frac{\omega_0 L}{c}\right)}{\frac{n\pi}{L}-\frac{\omega_0}{c}}-\frac{1-\cos\left(n\pi+\frac{\omega_0 L}{c}\right)}{\frac{n\pi}{L}+\frac{\omega_0}{c}}\right]\right)\right.$$

$$=\frac{2}{L}\left\{\frac{(-1)^{n+1}}{n\pi}-\left[1+(-1)^{n+1}\cos\frac{\omega_0 L}{c}\right]\cdot\frac{\frac{\omega_0}{c}}{\sin\frac{\omega_0}{c}L\left(\frac{n^2\pi^2}{L^2}-\frac{\omega_0^2}{c^2}\right)}\right\}$$

$$\star \quad \frac{2}{L}\left\{\left(-\frac{L}{n\pi}\right)\frac{x}{L}\cos\frac{n\pi}{L}x\,\Big|_0^L+\frac{L}{n\pi}\frac{1}{L}\int_0^L\cos\frac{n\pi}{L}x\,dx\right.$$

$$\left.-\frac{1}{\sin\frac{\omega_0}{c}L}\frac{1}{2}\int_0^L\left[\sin\left(\frac{n\pi}{L}-\frac{\omega_0}{c}\right)x-\sin\left(\frac{n\pi}{L}+\frac{\omega_0}{c}\right)x\right]dx\right\}$$

at $t=0$, $u=\sum_{n=1}^{\infty} A_n \frac{n\pi c}{L} \sin\frac{n\pi}{L}x = 0$ $\therefore$ $A_n = 0$.

$$\therefore\ u=\sum_{n=1}^{\infty}\frac{2}{L}\left\{\frac{(-1)^{n+1}}{n\pi}-\frac{\frac{\omega_0}{c}\left[1+(-1)^{n+1}\cos\frac{\omega_0 L}{c}\right]}{\sin\frac{\omega_0}{c}L\left(\frac{n^2\pi^2}{L^2}-\frac{\omega_0^2}{c^2}\right)}\right\}\sin\frac{n\pi}{L}x\cos\frac{n\pi c}{L}t.$$

$\therefore$

$$\boxed{y=\sum_{n=1}^{\infty}\frac{2}{L}\left\{\frac{(-1)^{n+1}}{n\pi}-\frac{\frac{\omega_0}{c}\left[1+(-1)^{n+1}\cos\frac{\omega_0 L}{c}\right]}{\sin\frac{\omega_0}{c}L\left(\frac{n^2\pi^2}{L^2}-\frac{\omega_0^2}{c^2}\right)}\right\}\sin\frac{n\pi x}{L}\cos\frac{n\pi c}{L}t+\frac{\sin\frac{\omega_0}{c}x}{\sin\frac{\omega_0 L}{c}}\cos\omega_0 t}$$

Ans.

$(0\le x\le L,\ t\ge 0)$

Problem #4.

a) (3 points) Consider a homogeneous isotropic material with large conductivity (i.e., a good conductor). Show that the macroscopic magnetic field $\vec{B}$ and the macroscopic current density $\vec{J}$ satisfy the diffusion equations,

$$\nabla^2 \vec{B} = \kappa \frac{\partial \vec{B}}{\partial t}$$

and

$$\nabla^2 \vec{J} = \kappa \frac{\partial \vec{J}}{\partial t}.$$

in this material. Find an expression for κ in terms of the electromagnetic properties of this material.

b) (4 points) Assume that space is half filled (i.e., for $x \geq 0$) with a good conductor as in part a), while the other half (i.e., for $x < 0$) is vacuum. Assume that a plane symmetric electromagnetic wave propagates towards the conductor in the direction that is normal to its surface. Solve Maxwell's equations for the electromagnetic fields everywhere.

c) (3 points) Find the reflection coefficient for the solution found in b). Discuss the conservation of energy for this solution.

#4

L. LINDBLOM

Solution to Electromagnetism Problem:

a) The macroscopic Maxwell equations are

$$\vec{\nabla}\cdot\vec{D} = 4\pi\rho$$
$$\vec{\nabla}\times\vec{E} + \frac{1}{c}\frac{\partial\vec{B}}{\partial t} = 0$$
$$\vec{\nabla}\cdot\vec{B} = 0$$
$$\vec{\nabla}\times\vec{H} = \frac{4\pi}{c}\vec{J} + \frac{1}{c}\frac{\partial\vec{D}}{\partial t}$$

The material is assumed to be a good conductor, thus in its own rest frame $\rho = 0$ and $\vec{J} = \sigma\vec{E}$. Further since it is a good conductor we assume

$$\frac{\partial\vec{D}}{\partial t} \ll 4\pi\vec{J}$$

We assume the conductor is isotropic and homogeneous, thus,

$$\vec{B} = \mu\vec{H}$$
$$\vec{D} = \epsilon\vec{E}$$

where μ and ϵ are constants.

Now compute: $\vec{\nabla}\times(\vec{\nabla}\times\vec{H}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{H}) - \nabla^2\vec{H}$

$$= \frac{1}{\mu}\vec{\nabla}(\vec{\nabla}\cdot\vec{B}) - \frac{1}{\mu}\nabla^2\vec{B}$$
$$= -\frac{1}{\mu}\nabla^2\vec{B}$$

Thus from Maxwell's Eqns: $-\frac{1}{\mu}\nabla^2\vec{B} = \frac{4\pi}{c}\vec{\nabla}\times\vec{J} = \frac{4\pi\sigma}{c}\vec{\nabla}\times\vec{E} = -\frac{4\pi\sigma}{c^2}\frac{\partial\vec{B}}{\partial t}$

Thus $\boxed{\nabla^2 \vec{B} = \frac{4\pi\sigma\mu}{c^2} \frac{\partial \vec{B}}{\partial t}}$

And again from Maxwell's equations:

$$\frac{4\pi}{c} \frac{\partial \vec{J}}{\partial t} = \vec{\nabla} \times \left(\frac{\partial \vec{H}}{\partial t} \right) = \frac{1}{\mu} \vec{\nabla} \times \left(\frac{\partial \vec{B}}{\partial t} \right)$$
$$= -\frac{c}{\mu} \vec{\nabla} \times (\vec{\nabla} \times \vec{E})$$
$$= -\frac{c}{\mu} \vec{\nabla}(\vec{\nabla} \cdot \vec{E}) + \frac{c}{\mu} \nabla^2 \vec{E}$$
$$= \frac{c}{\mu} \nabla^2 \vec{E} = \frac{c}{\sigma\mu} \nabla^2 \vec{J}$$

Thus $\boxed{\nabla^2 \vec{J} = \frac{4\pi\sigma\mu}{c^2} \frac{\partial \vec{J}}{\partial t}}$ so $\boxed{K = \frac{4\pi\sigma\mu}{c^2}}$

b) In region I $(x < 0)$ we have a plane electromagnetic wave propagating in the $+x$ direction

$$\vec{E}_i \, e^{i\omega(t - x/c)}$$

where $\vec{E}_i$ is constant and $\perp$ to $\hat{x}$. There may also be a reflected wave traveling in the $-x$ direction. Thus the complete $\vec{E}$ field is:

$$\vec{E} = \vec{E}_i \, e^{i\omega(t - x/c)} + \vec{E}_r e^{i\omega(t + x/c)}$$

The $\vec{B}$ field is similarly:

$$\vec{B} = \hat{x} \times \vec{E}_i \, e^{i\omega(t - x/c)} - \hat{x} \times \vec{E}_r e^{i\omega(t + x/c)}$$

③

In region II $(x \geq 0)$ we have a conductor. Thus the $\vec{E}$ and $\vec{B}$ fields satisfy the diffusion eqn. found in a).

Let $\quad \vec{E} = \vec{E}_0 e^{kx + i\omega t}$

Then: $\quad k^2 - iK\omega = 0 \quad$ from the diffusion equation

$$k^2 = +i\omega K = +i\frac{4\pi\sigma\mu\omega}{c^2} = \frac{4\pi\sigma\mu\omega}{c^2} e^{+i\pi/2}$$

$$k = \pm\sqrt{\frac{4\pi\sigma\mu\omega}{c^2}}\, e^{+i\pi/4}$$

$$e^{i\pi/4} = \frac{\sqrt{2}}{2}(1+i)$$

So $\boxed{k = \pm(1+i)\sqrt{\frac{2\pi\sigma\mu\omega}{c^2}}}$

Thus the general solution in region II is

$$\vec{E} = \vec{E}_+ e^{kx + i\omega t} + \vec{E}_- e^{-kx + i\omega t}$$

The + solution diverges as $x \to +\infty$ so is unphysical, thus, the general solution is:

$$\vec{E} = \vec{E}_- e^{-kx + i\omega t}$$

Where $\vec{E}_-$ is a constant and $k = (1+i)\sqrt{\frac{2\pi\sigma\mu\omega}{c^2}}$

④

At the surface ($x=0$) of the conductor $\vec{E}$ and its first derivative must be continuous. Thus:

$$\vec{E}_i + \vec{E}_r = \vec{E}_-$$
$$-\frac{i\omega}{c}\vec{E}_i + \frac{i\omega}{c}\vec{E}_r = -k\vec{E}_-$$

Thus: $\vec{E}_i + \vec{E}_r = \vec{E}_- = \frac{i\omega}{kc}\vec{E}_i - \frac{i\omega}{kc}\vec{E}_r$

$$\left(1+\frac{i\omega}{kc}\right)\vec{E}_r = \left(-1+\frac{i\omega}{kc}\right)\vec{E}_i$$

$$\vec{E}_r = \frac{i\omega - kc}{i\omega + kc}\vec{E}_i$$

$$\vec{E}_- = \frac{2i\omega}{i\omega + kc}\vec{E}_i$$

This determines the electromagnetic field every where.

c) The reflection coefficient is simply the ratio of the magnitudes of the reflected to the incident waves.

$$R = \frac{|\vec{E}_r|}{|\vec{E}_i|} = \left|\frac{i\omega - kc}{i\omega + kc}\right|$$

We can express this more clearly by recalling that

$$k = (1+i)\delta \quad \text{where} \quad \delta = \sqrt{\frac{2\pi\sigma\mu\omega}{c^2}}$$

So $$R = \left|\frac{-\delta c + i(\omega - \delta c)}{\delta c + i(\omega + \delta c)}\right| = \left|\frac{[-\delta c + i(\omega - \delta c)][\delta c - i(\omega + \delta c)]}{[\delta c + i(\omega + \delta c)][\delta c - i(\omega + \delta c)]}\right|$$

(5)

So $R = \left| \dfrac{-\delta^2 c^2 + 2i\delta c\omega + \omega^2 - \delta^2 c^2}{\delta^2 c^2 + (\omega + \delta c)^2} \right|$

$$\boxed{R = \frac{\sqrt{(\omega^2 - 2\delta^2 c^2)^2 - 4\delta^2 c^2 \omega^2}}{\delta^2 c^2 + (\omega + \delta c)^2}}$$

Note that $\lim_{\omega \to \infty} R = 1$ and $\lim_{\omega \to 0} R = 1$

but $R \neq 1$ for other frequencies.

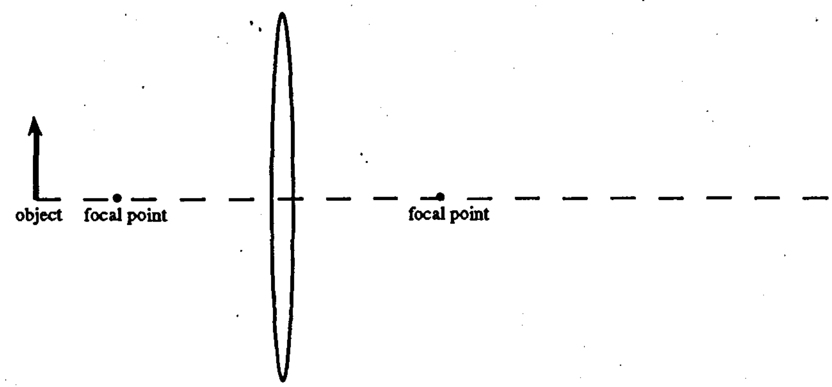
Although energy does not propagate to $x \to +\infty$, it is conserved dispite the fact that $R \neq 1$ (and hence not all of the incoming energy is reflected back to $x \to -\infty$) because some of the energy is absorbed by the conductor via ohmic heating.

Problem #5

This problem consists of two parts which are unrelated.

I. In the problems that follow, assume that the lenses are thin.

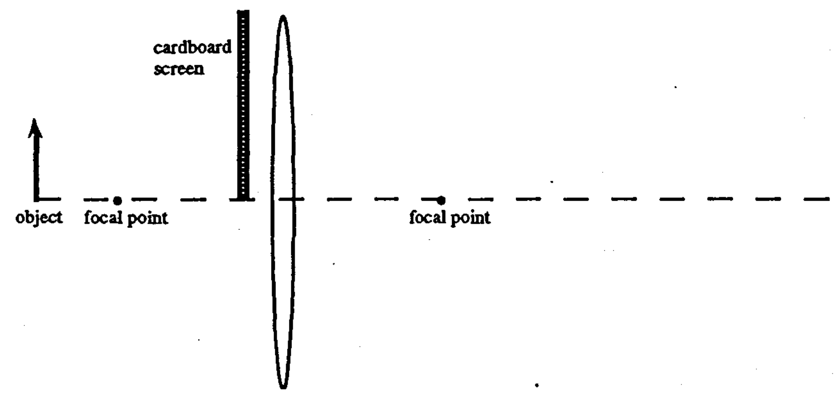
A. An object is placed 15 cm from a converging lens having focal length of 10 cm.



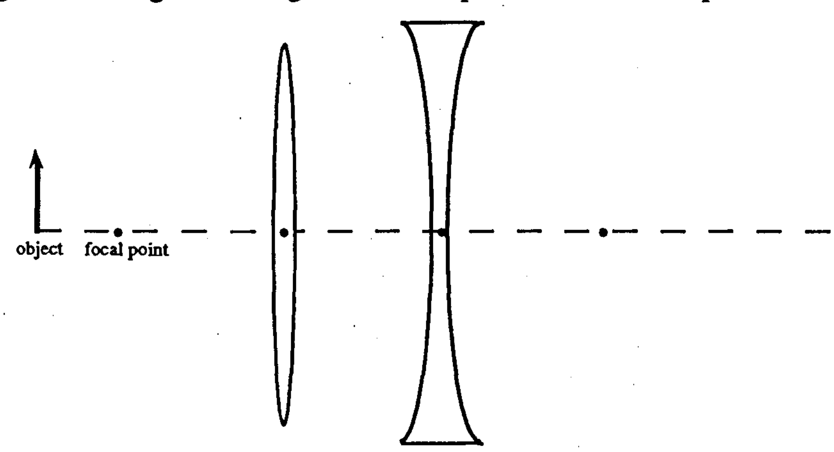
i. Calculate the position, orientation, and magnification of the image produced by this lens.

ii. Draw the appropriate ray diagram.

B. A cardboard sheet is used to cover the upper half of the lens (as shown below). Describe the effect this will have on the image.



C. A diverging lens having focal length -10 cm is placed at the focal point of the first.



i. Calculate the position, orientation, and magnification of the image produced by this system.

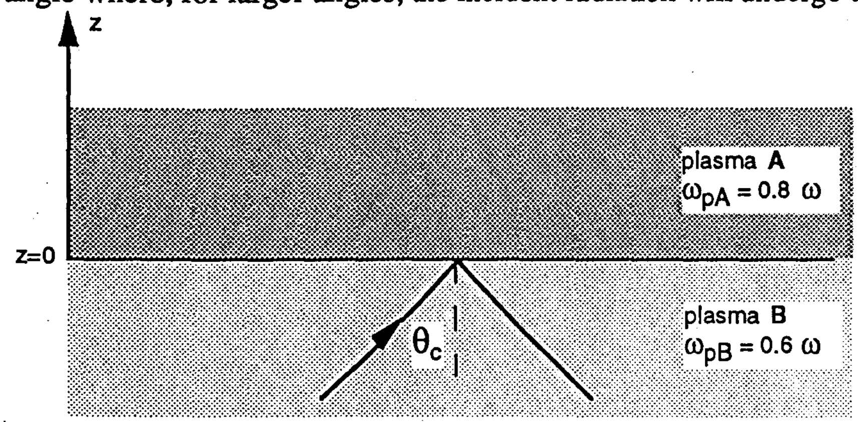
ii. Draw the appropriate ray diagram.

II. The dispersion relation for electromagnetic waves propagating in a plasma with no dc magnetic field is:

$$\omega^2 = k^2c^2 + \omega_p{}^2,$$

where $\omega_p{}^2$ is the square of the plasma frequency, and is proportional to the plasma density.

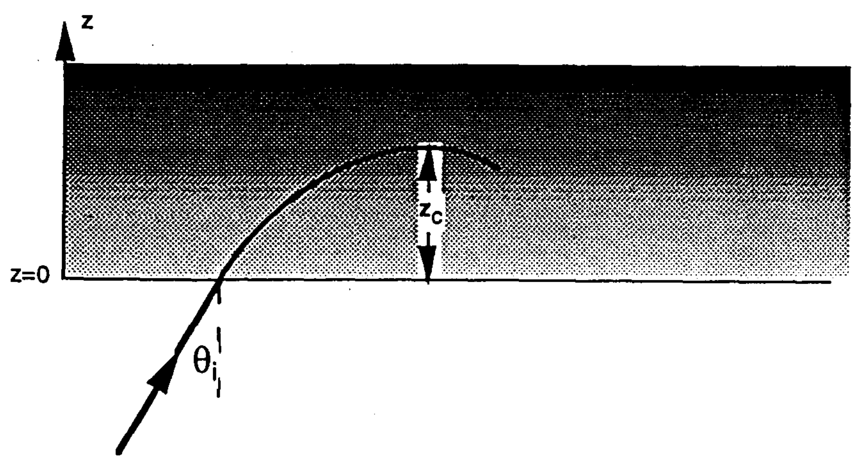
A. Two plasmas (A and B) with plasma frequencies $\omega_{pA} = 0.8\ \omega$ and $\omega_{pB} = 0.6\ \omega$ are separated by an ideal plane boundary at z=0 (as in the diagram below). An electromagnetic wave of frequency ω propagates in plasma **B** and impinges on the boundary between **A** and **B**. What is the critical angle where, for larger angles, the incident radiation will undergo total reflection?



B. Suppose now that the plasma density varies linearly with z in a region of space, such that:

$$\omega_p^2 = \begin{cases} 0, & z \leq 0 \\ \omega^2 \dfrac{z}{L}, & z > 0 \end{cases}$$

If an electromagnetic wave of frequency ω propagates from the region of space $z < 0$ and impinges on the plasma at z= 0 with an angle of incidence $\theta_i = 30^o$ (as shown below), find the distance z_c that the wave will propagate into the plasma before being reflected. You may assume that L is large compared to the wavelength of the wave.



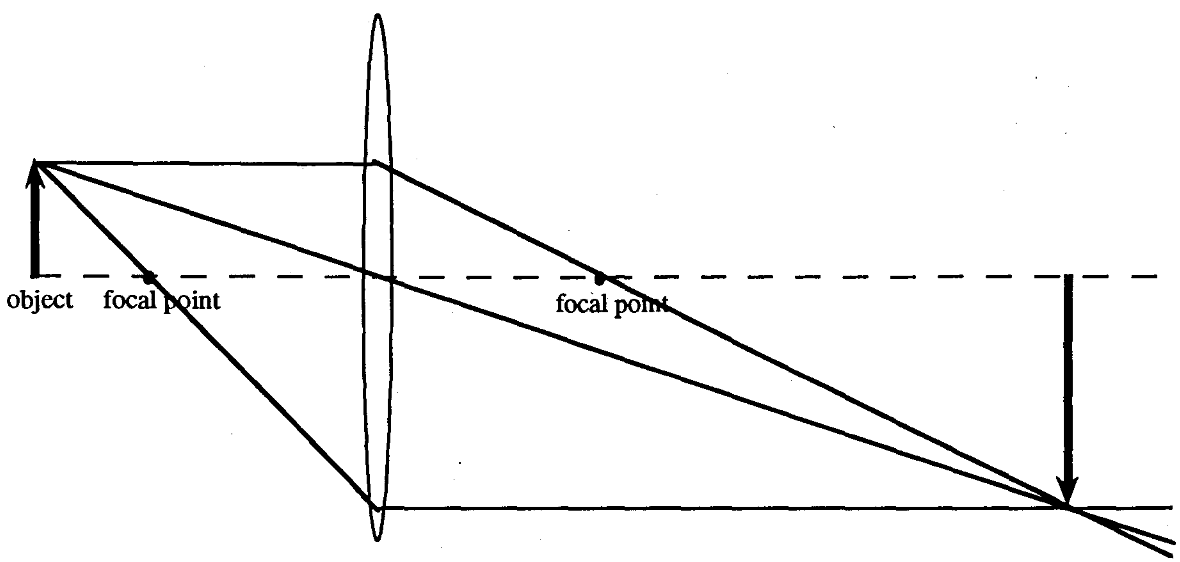
I. A. i. $\frac{1}{f} = \frac{1}{i} + \frac{1}{o}$

$$\left.\begin{array}{l} f = +10\,cm \\ o = +15\,cm \end{array}\right\} \quad i = +30\,cm$$

$$m = \frac{-i}{o} = -\frac{30\,cm}{15\,cm} = -2$$

The image is real, located 30cm to the right of the lens. It is magnified by a factor of 2, and is inverted (as indicated by a negative magnification).

ii. The ray diagram is:



B. The three principal rays are drawn to represent the convergence of all light which reflects off the tip of the object (arrow) and passes through the lens. Only the light passing from the object to the top half of the lens will be blocked. The image will therefore have the same size, position, and orientation as in part A. It will appear more dim because of the missing light.

C.i. The image of the first lens becomes the object of the second lens.

$$\frac{1}{f} = \frac{1}{i} + \frac{1}{o}$$

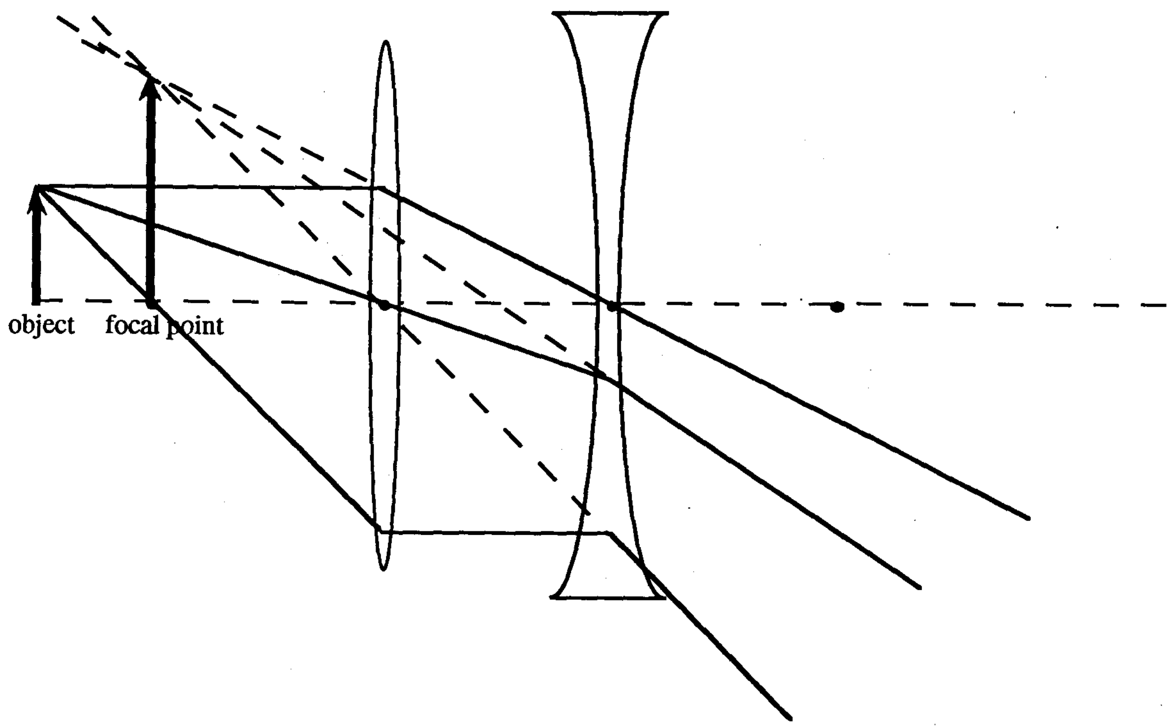
$$\left.\begin{array}{l} f = -10\text{cm} \\ o = -20\text{cm} \end{array}\right\} \quad i = -20\text{ cm}$$

The image is virtual, located 20 cm to the left of the second lens, placing it at the focal plane of the first lens.

$$m = -\frac{i}{o} = \frac{20\text{cm}}{-20\text{cm}} = -1$$

The image is the same size as the object of the second lens (the original image), and is inverted. Hence, it is twice the size of the original object and is not inverted.

ii. The ray diagram is:



II. A. By Snell's law, the tangential component of the propagation vector $\vec{k}$ must conserved across the boundary. Since the incident frequency must also equal the refracted frequency, we may write Snell's law as:

$$n_A \sin \Theta_A = n_B \sin \Theta_B$$

where $n = \frac{kc}{\omega}$. From the dispersion relation:

$$n_A = \left[1 - \frac{\omega_{pA}^2}{\omega^2}\right]^{1/2} = 0.6$$

$$n_B = \left[1 - \frac{\omega_{pB}^2}{\omega^2}\right]^{1/2} = 0.8$$

The critical angle satisfies $\Theta_A = \frac{\pi}{2}$.

$$\Theta_c = \sin^{-1}\left(\frac{0.6}{0.8}\right) = \underline{.85 \text{ rad}}$$

$$= 49°$$

B. If the index of refraction, n, varies by a negligible amount over one wavelength, then we may imagine the plasma to be stratified in infinitely thin layers.

By Snell's laws:

$$n \sin\theta = n_i \sin\theta_i = \sin\theta_i$$

all along the ray. The wave is reflected when $\theta = \pi/2$, or:

$$n = \sin\theta_i$$

From the dispersion relation:

$$\left[1 - \frac{\omega_{pc}^2}{\omega^2}\right]^{1/2} = \sin\theta_i$$

$$\left[1 - \frac{z_c}{L}\right]^{1/2} = \sin\theta_i$$

$$\boxed{\frac{z_c}{L} = 1 - \sin^2\theta_i = \frac{3}{4}}$$

#6

Ant War (Math Physics) - GT

Two ant colonies engage in warfare. The loss rate of red ants, dr/dt, is proportional to the number b of black ants, and vice versa:

$$dr/dt = -\alpha b$$
$$db/dt = -\gamma r$$

where α and γ are positive constants. Initially there are r_o red ants and b_o black ants. Suppose that $\alpha = 2\gamma$ (black ants are better fighters) but $r_o = 2b_o$ (there are more red ants to start with).

(a) Which colony wins the war (eliminates the other colony)?

(b) What fraction of the winning side is left at the end?

(c) How long does the war last?

(d) Now assume that the loss rates in the ant war are proportional to the number of encounters between red and black ants -- that is, proportional to the product rb. Now we have

$$dr/dt = -\alpha rb$$
$$db/dt = -\gamma rb$$

Assume again that $\alpha = 2\gamma$ and $r_o = 2b_o$, and describe the outcome of the ant war.

#6

Ant War - Math Physics - Solution

(a) $$\left.\begin{aligned}\dot r &= -\alpha b = -2\gamma b\\ \dot b &= -\gamma r\end{aligned}\right\} \;\&\; r_0 = 2b_0$$

note $\ddot r = 2\gamma^2 r \quad \& \quad \ddot b = 2\gamma^2 b$

so $r = Ae^{\sqrt{2}\gamma t} + Be^{-\sqrt{2}\gamma t}, \quad b = Ce^{\sqrt{2}\gamma t} + De^{-\sqrt{2}\gamma t}$

but

$\dot r = -2\gamma b$ so

$$\sqrt{2}\gamma\left[Ae^{\sqrt{2}\gamma t} - Be^{-\sqrt{2}\gamma t}\right] = -2\gamma\left[Ce^{\sqrt{2}\gamma t} + De^{-\sqrt{2}\gamma t}\right]$$

$$\Rightarrow \quad C = -\frac{A}{\sqrt{2}} \quad \text{and} \quad D = +\frac{B}{\sqrt{2}}$$

Further $r_0 = A + B = 2b_0 = 2(C+D)$

so

$$\left.\begin{aligned}A + B &= r_0\\ -\frac{A}{\sqrt{2}} + \frac{B}{\sqrt{2}} &= \frac{r_0}{2}\end{aligned}\right\} \Rightarrow \begin{aligned}A &= \tfrac{1}{2} r_0\left[1 - \tfrac{1}{\sqrt{2}}\right]\\ B &= \tfrac{1}{2} r_0\left[1 + \tfrac{1}{\sqrt{2}}\right]\end{aligned}$$

and

$$r = \frac{1}{2} r_0 \left\{ \left(1 - \frac{1}{\sqrt{2}}\right) e^{+\sqrt{2}\gamma t} + \left(1 + \frac{1}{\sqrt{2}}\right) e^{-\sqrt{2}\gamma t} \right\}$$

$$b = \frac{r_0}{2\sqrt{2}} \left\{ -\left(1 - \frac{1}{\sqrt{2}}\right) e^{+\sqrt{2}\gamma t} + \left(1 + \frac{1}{\sqrt{2}}\right) e^{-\sqrt{2}\gamma t} \right\}$$

Can either r or b equal zero?

$r = 0 \Rightarrow e^{2\sqrt{2}\gamma t} = -\dfrac{1 + \frac{1}{\sqrt{2}}}{1 - \frac{1}{\sqrt{2}}} < 0$ impossible for real t!

$b = 0 \Rightarrow e^{2\sqrt{2}\gamma t} = +\dfrac{1 + \frac{1}{\sqrt{2}}}{1 - \frac{1}{\sqrt{2}}}$ ok. so red wins!

(c) $$t = \frac{1}{2\sqrt{2}\gamma} \ln \frac{\sqrt{2}+1}{\sqrt{2}-1} = \frac{.623}{\gamma}$$

(b) At this time, $e^{\sqrt{2}\gamma t} = \sqrt{2}+1$ and

$$r = \frac{r_0}{2}\left\{ \frac{\sqrt{2}-1}{\sqrt{2}}(\sqrt{2}+1) + \frac{\sqrt{2}+1}{\sqrt{2}}\frac{1}{\sqrt{2}+1} \right\}$$

$$= \frac{r_0}{2}\left\{ \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \right\} = \frac{r_0}{\sqrt{2}}$$

so at the war's end $r/r_0 = \frac{1}{\sqrt{2}} = \underline{\underline{.707}}$

(d) $\left.\begin{aligned} \dot{r} &= -2\gamma r b \\ \dot{b} &= -\gamma r b \end{aligned}\right\}$ so $\dot{r} = 2\dot{b}$

$$r - r_0 = 2b - 2b_0$$

Since $r_0 = 2b_0$ this means $r = 2b$ always, so $b=0$ implies $r=0$ and vice-versa. Neither side wins!

(An exact solution can also be easily obtained at this point: Since $b = r/2$,

$$\dot{r} = -2\gamma r\left(\frac{r}{2}\right) = -\gamma r^2$$

$$\int_{r_0}^{r} \frac{dr}{r^2} = -\gamma\int_0^t dt$$

$$-\frac{1}{r} + \frac{1}{r_0} = -\gamma t \implies r = \frac{r_0}{1+\gamma r_0 t}, \quad b = \frac{1}{2}\frac{r_0}{1+\gamma r_0 t}$$

Thus $r, b \to 0$ only as $t \to \infty \implies$ The war goes on "forever"!)

#7

Solid State

A free electron is described by a wavefunction $\psi(x,y,z)$ that satisfies periodic boundary conditions:

$$\psi(x+L,\ y,\ z) = \psi(x,\ y,\ z)$$
$$\psi(x,\ y+L,\ z) = \psi(x,\ y,\ z)$$
$$\psi(x,\ y,\ z+L) = \psi(x,\ y,\ z)$$

(a) write the wavefunction and associated energy for all states that satisfy the boundary conditions.

(b) determine the density of states per unit volume of wavevector space.

(c) determine the density of states per unit energy.

(d) repeat (c) if the electron is confined to two dimensions.

(e) repeat (c) if the electron is confined to one dimension.

#7 Solid State J. Hemminger

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(c) determine the density of states per unit energy

(d) repeat (c) if the electron is confined to two dimensions

(e) repeat (c) if the electron is confined to one dimension

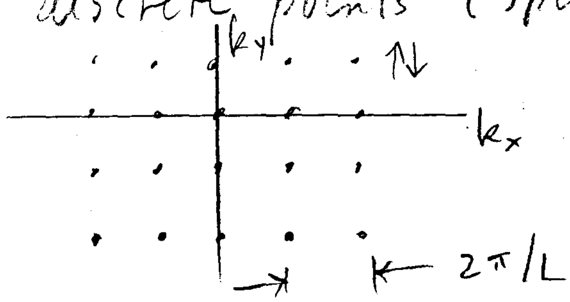
Sol'n

(a) $\psi_{\underline{k}} = e^{i\underline{k}\cdot\underline{x}}$, $\underline{k} = \frac{2\pi}{L}(n_x, n_y, n_z)$

n_x, n_y, n_z are integers

$$E_{\underline{k}} = \frac{\hbar^2 \underline{k}^2}{2m}$$

(b) In wavevector space, the allowed states are indicated by discrete points (spin degeneracy = 2)



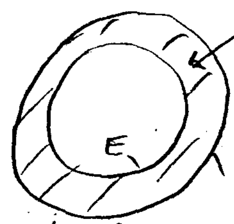
spin degeneracy

density of states $= \overset{\downarrow}{2} \times \dfrac{1}{(2\pi/L)^3}$

$$\boxed{\rho(k) = \frac{L^3}{4\pi^3}} = \frac{V}{4\pi^3}$$

(c) density of states per unit energy follows by considering a shell of energy width dE in k-space:

shell of volume $4\pi k^2 dk$



states in shell:

$$\rho(E)\, dE = \rho(k) \cdot 4\pi k^2 dk$$

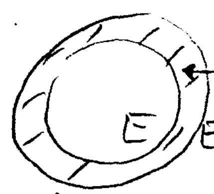
$$\rho(E) = \rho(k) \frac{4\pi k^2}{dE/dk}$$

$$= \frac{V}{4\pi^3} \cdot 4\pi k^2 \cdot \frac{m}{\hbar^2 k}$$

$$\boxed{\rho(E) = \frac{V}{\pi^2} \cdot \frac{m}{\hbar^2} \cdot \sqrt{\frac{2mE}{\hbar^2}}}$$

(d) In 2D, $\rho(k) = 2 \times \dfrac{1}{(2\pi/L)^2} = \dfrac{L^2}{2\pi^2} = \dfrac{A}{2\pi^2}$

The energy shell has area $2\pi k\, dk$:



states in shell:

$$\rho(E)\, dE = \rho(k)\, 2\pi k\, dk$$

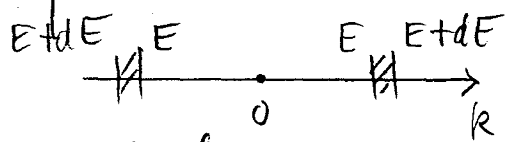
$$\rho(E) = \rho(k)\,\frac{2\pi k}{dE/dk}$$

$$= \frac{A}{2\pi^2}\cdot 2\pi k \cdot \frac{m}{\hbar^2 k}$$

$$\boxed{\rho(E) = \frac{A}{\pi}\cdot\frac{m}{\hbar^2} = \text{const}}$$

(e) In 1D, $\rho(k) = 2 \times \frac{1}{2\pi/L} = \frac{L}{\pi}$

The energy shell has width $2\,dk$:



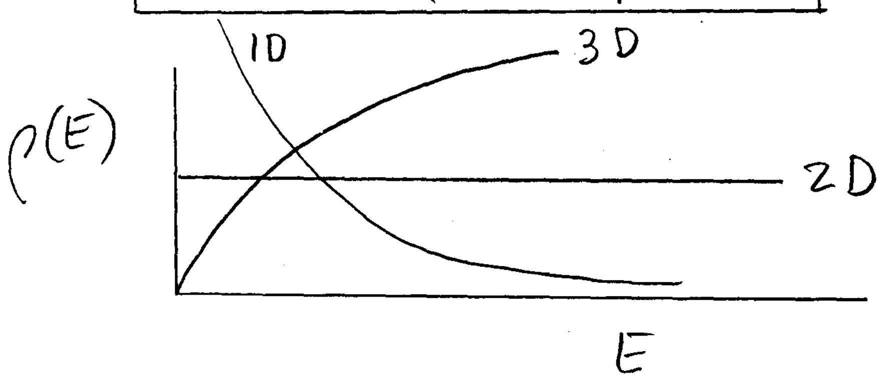
states in shell:

$$\rho(E)\,dE = \rho(k)\,2\,dk$$

$$\rho(E) = \rho(k)\cdot\frac{2}{dE/dk}$$

$$= \frac{L}{\pi}\cdot 2\cdot\frac{m}{\hbar^2 k}$$

$$\boxed{\rho(E) = \frac{2L}{\pi}\cdot\frac{m}{\hbar^2}\cdot\sqrt{\frac{\hbar^2}{2mE}}}$$



Problem #8.

Lee Lindblom

Pions can be produced during the collision of two protons via the following reaction:

$$p + p \rightarrow p + n + \pi^+$$

a) Assume that one proton is at rest in the "laboratory" frame of reference and undergoes a collision with a second proton moving with speed v_i with respect to this frame. Determine the minimum value of v_i needed to create a π^+ via the above reaction.

b) The π^+ decays with an average lifetime $\tau_{\pi^+} = 2.54 \times 10^{-8}$ s as measured in its own rest frame. Determine the average distance the π^+ created in the minimum energy collision considered in a) will travel in the laboratory frame (and thus the length of the track it would leave in a detector) before it decays.

For your information:

$$m_p c^2 = 938.211 \text{ Mev}$$
$$m_n c^2 = 939.505 \text{ Mev}$$
$$m_{\pi^+} c^2 = 139.6 \text{ Mev}$$
$$c = 2.9979 \times 10^{10} \text{cm/s}$$

#8 Solution to Special Relativity Problem: L. LINDBLOM

a) In the center of mass frame the minimum energy collision will leave p, n and π^+ at rest. In any other frame then, this collision will leave p, n and π^+ moving with common speed V_f.

Impose the conservation of 4-momentum in the rest frame of the target p:

$$m_p \gamma_i \begin{pmatrix} 1 \\ V_i \end{pmatrix} + m_p \begin{pmatrix} 1 \\ 0 \end{pmatrix} = (m_p + m_n + m_{\pi^+}) \gamma_f \begin{pmatrix} 1 \\ V_f \end{pmatrix}$$

where $\gamma_i = (1 - V_i^2/c^2)^{-1/2}$, $\gamma_f = (1 - V_f^2/c^2)^{-1/2}$

Thus: $$\gamma_f^2 = \frac{m_p^2 (1+\gamma_i)^2}{(m_p + m_n + m_{\pi^+})^2}$$

$$\gamma_f^2 V_f^2/c^2 = \frac{m_p^2 \gamma_i^2 V_i^2/c^2}{(m_p + m_n + m_{\pi^+})^2}$$

But $$\gamma_f^2 - \gamma_f^2 V_f^2/c^2 = \frac{1}{1 - V_f^2/c^2} - \frac{V_f^2/c^2}{1 - V_f^2/c^2} = 1$$

Thus: $$\frac{(1+\gamma_i)^2}{\left(1 + \frac{m_n}{m_p} + \frac{m_{\pi^+}}{m_p}\right)^2} - \frac{\gamma_i^2 V_i^2/c^2}{\left(1 + \frac{m_n}{m_p} + \frac{m_{\pi^+}}{m_p}\right)^2} = 1$$

So: $$1 + 2\gamma_i + \underbrace{\gamma_i^2 - \gamma_i^2 V_i^2/c^2}_{1} = 2(1+\gamma_i) = \left(1 + \frac{m_n}{m_p} + \frac{m_{\pi^+}}{m_p}\right)^2$$

(2)

So $\quad \gamma_i = \frac{1}{2}\left(1 + \frac{m_n}{m_p} + \frac{m_{\pi^+}}{m_p}\right)^2 - 1$

Plugging in numbers for the masses we get,

$$\gamma_i = 1.312$$

So: $\quad V_i/c = \sqrt{\frac{\gamma_i^2 - 1}{\gamma_i^2}} = 0.647$

$$\boxed{V_i = 0.647c = 1.94 \times 10^{10}\,\text{cm/sec}}$$

b) After the collision the π^+ will move a distance $\Delta x = V_f \Delta t$ in the time Δt measured in the laboratory frame. It thus moves from the spacetime point $(0,0)$ at collision to the point $(c\Delta t, \Delta x) = \Delta t(c, V_f)$. The distance between these points is the time experienced in the rest frame of the π^+: $\Delta\tau$. This is related to Δt using the special relativistic distance formula:

$$c^2\Delta\tau^2 = c^2\Delta t^2 - \Delta x^2 = c^2\Delta t^2(1 - V_f^2/c^2)$$

$$\Delta\tau = \frac{\Delta t}{\gamma_f}$$

This is of course just the usual Lorentz transformation (No deduction if the formula is reproduced from memory.)

(3)

Thus: $\Delta x = V_f \Delta t = \gamma_f V_f \Delta \tau$

Since: $\gamma_f V_f = \gamma_i V_i \left(1 + \frac{m_n}{m_p} + \frac{m_{\pi^+}}{m_p}\right)^{-2}$

So, $\gamma_f V_f = 0.184c$

And finally since $\Delta \tau = 2.54 \times 10^{-8}\,s$

$$\boxed{\Delta x = \gamma_f V_f \Delta \tau = 140\,cm}$$

#7

Quantum Mechanic

A hydrogen atom is in a $2p_{1/2}$ state with $m_j = 1/2$. Given the Clebsch-Gordan coefficients for $j_1 = 1$, $j_2 = 1/2$,

$$C_{\frac{3}{2}\frac{1}{2}}(1 - \frac{1}{2}) = \sqrt{\frac{1}{3}} \qquad C_{\frac{3}{2}\frac{1}{2}}(0\ \frac{1}{2}) = \sqrt{\frac{2}{3}}$$

$$C_{\frac{1}{2}\frac{1}{2}}(1 - \frac{1}{2}) = \sqrt{\frac{2}{3}} \qquad C_{\frac{1}{2}\frac{1}{2}}(0\ \frac{1}{2}) = -\sqrt{\frac{1}{3}}$$

evaluate the expectation values of S_x, S_z, L_x, L_z, J_x, and J_z.

#9 A hydrogen atom is in a $2p_{1/2}$ state with $m_j = 1/2$. Given the Clebsch-Gordan coefficients for $j_1 = 1$, $j_2 = 1/2$,

$$C_{\frac{3}{2}\frac{1}{2}}(1 -\tfrac{1}{2}) = \sqrt{\tfrac{1}{3}} \qquad C_{\frac{3}{2}\frac{1}{2}}(0\ \tfrac{1}{2}) = \sqrt{\tfrac{2}{3}}$$

$$C_{\frac{1}{2}\frac{1}{2}}(1 -\tfrac{1}{2}) = \sqrt{\tfrac{2}{3}} \qquad C_{\frac{1}{2}\frac{1}{2}}(0\ \tfrac{1}{2}) = -\sqrt{\tfrac{1}{3}},$$

evaluate the expectation values of S_x, S_z, L_x, L_z, J_x and J_z.

Solution:

The spinor for the specified state is

$$\psi_{\frac{1}{2}}^{\frac{1}{2}} = C_{\frac{1}{2}\frac{1}{2}}(1 -\tfrac{1}{2})\, R_{2p} Y_1^1 \beta + C_{\frac{1}{2}\frac{1}{2}}(0\ \tfrac{1}{2})\, R_{2p} Y_1^0 \alpha$$

where $\alpha \leftrightarrow S_z = \frac{\hbar}{2}$ and $\beta \leftrightarrow S_z = -\frac{\hbar}{2}$

$$\psi_{\frac{1}{2}}^{\frac{1}{2}} = \sqrt{\tfrac{2}{3}}\, R_{2p} Y_1^1 \beta - \sqrt{\tfrac{1}{3}}\, R_{2p} Y_1^0 \alpha$$

Note that $\displaystyle\int_0^\infty R_{2p}^2 r^2 dr = 1$

$$\int |Y_\ell^m|^2 d\Omega = 1$$

$$\langle \alpha | \alpha \rangle = 1$$

$$\langle \beta | \beta \rangle = 1$$

$$\langle \alpha | \beta \rangle = 0$$

Then $\langle S_x \rangle = \frac{2}{3}\int R_{2p}^2 r^2 dr \int |Y_1^1|^2 d\Omega \ \langle \beta | S_x | \beta \rangle$

$- \frac{\sqrt{2}}{3}\int R_{2p}^2 r^2 dr \int Y_1^{1*} Y_1^0 d\Omega \ \langle \beta | S_x | \alpha \rangle$

$- \frac{\sqrt{2}}{3}\int R_{2p}^2 r^2 dr \int Y_1^{0*} Y_1^1 d\Omega \ \langle \alpha | S_x | \beta \rangle$

$+ \frac{1}{3}\int R_{2p}^2 r^2 dr \int |Y_1^0|^2 d\Omega \ \langle \alpha | S_x | \alpha \rangle$

But $S_x = \frac{S_+ + S_-}{2}$ so $\langle \alpha | S_x | \alpha \rangle = \langle \beta | S_x | \beta \rangle = 0$

Also $\int Y_1^{1*} Y_1^0 d\Omega = \int Y_1^{0*} Y_1^1 d\Omega = 0$

Thus $\boxed{\langle S_x \rangle = 0}$

$\langle S_z \rangle$ can be computed in similar fashion, but it can also be evaluated essentially by inspection of the spinor Ψ:

$\langle S_z \rangle = \frac{2}{3}\left(-\frac{\hbar}{2}\right) + \frac{1}{3}\left(\frac{\hbar}{2}\right)$

$\boxed{\langle S_z \rangle = -\frac{\hbar}{6}}$

$\langle L_x \rangle = \frac{2}{3}\int R_{2p}^2 r^2 dr \int Y_1^{1*} L_x Y_1^1 d\Omega \ \langle \beta | \beta \rangle$

$- \frac{\sqrt{2}}{3}\int R_{2p}^2 r^2 dr \int Y_1^{1*} L_x Y_1^0 d\Omega \ \langle \beta | \alpha \rangle$

$- \frac{\sqrt{2}}{3}\int R_{2p}^2 r^2 dr \int Y_1^{0*} L_x Y_1^1 d\Omega \ \langle \alpha | \beta \rangle$

$+ \frac{1}{3}\int R_{2p}^2 r^2 dr \int Y_1^{0*} L_x Y_1^0 d\Omega \ \langle \alpha | \alpha \rangle$

But $L_x = \frac{L_+ + L_-}{2}$

so $\int Y_1^{1*} L_x Y_1^1 \, d\Omega = \int Y_1^{0*} L_x Y_1^0 \, d\Omega = 0$

Also $\langle \alpha | \beta \rangle = \langle \beta | \alpha \rangle = 0$

so $\boxed{\langle L_x \rangle = 0}$

$\langle L_z \rangle$ can be evaluated by inspection of ψ:

$\langle L_z \rangle = \frac{2}{3}(\hbar) + \frac{1}{3}(0)$

$\boxed{\langle L_z \rangle = \frac{2\hbar}{3}}$

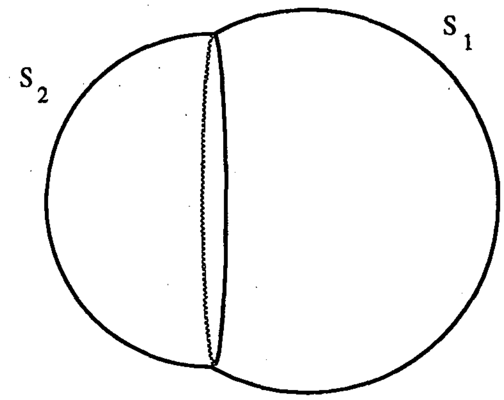
Finally,

$$\boxed{\begin{aligned} \langle J_x \rangle &= \langle L_x + S_x \rangle = 0 \\ \langle J_z \rangle &= \langle L_z + S_z \rangle = \frac{\hbar}{2} \end{aligned}}$$

$(m_j = \frac{1}{2})$ ✓

This problem consists of two parts which are unrelated.

I. The Gaussian surface shown at right consists of portions of two spherical surfaces of different size. Half of the surface of the smaller sphere of radius r=3m, and 3/4 of the surface of the larger sphere of radius $R=\sqrt{12}$ m, were used to form the composite surface.



A point charge of $+3.0 \times 10^{-9}$ C is fixed in place at the center of the larger sphere and a negative charge of -3.0×10^{-9} C is fixed in place at the center of the smaller sphere..

A. Calculate the electric flux through the surface S_1.

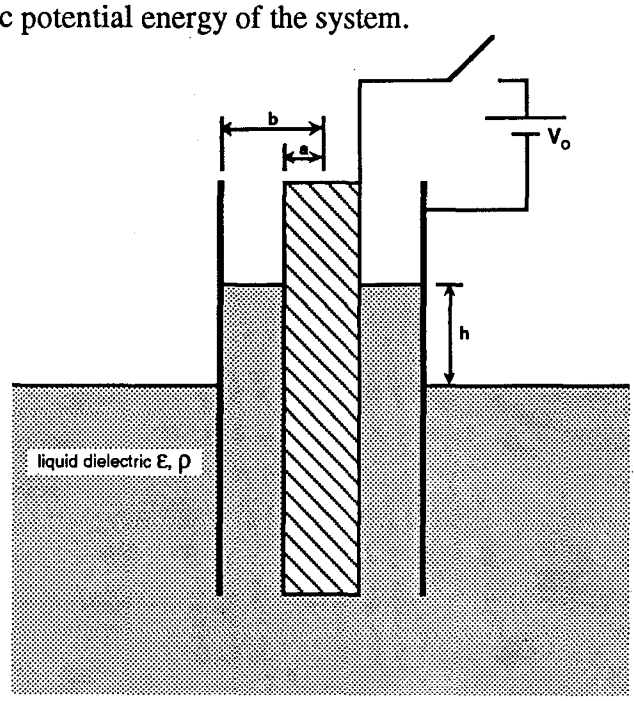
B. Calculate the electric flux through the surface S_2.

II. A cylindrical capacitor is composed of a long conducting rod of radius **a** and a long, coaxial conducting shell with inner radius **b**. One end of the system is immersed in a liquid with dielectric constant ε and density ρ, as shown in the diagram below. A voltage V_o is switched on across the capacitor. Assume that the capacitor is fixed in space and that no conduction current flows in the liquid. To what height **h** does the liquid dielectric rise between the conductors? Neglect surface tension.

You may recall from Jackson, page 162, that the electrical force exerted on a dielectric when electrostatic potential V is held fixed is given by:

$$\vec{F} = \vec{\nabla} U|_V$$

where U is the electric potential energy of the system.



I. A. By the Principal of Superposition, the total flux Φ_{total} through surface S_1 is just the sum of the flux due to the positive charge and the flux due to the negative charge:

$$\Phi_{total} = \Phi_{\substack{\text{due to}\\ +q}} + \Phi_{\substack{\text{due to}\\ -q}}$$

Consider first the positive charge. If S_1 were a complete sphere, then the flux due to $+q$ would be:

$$\Phi_{\substack{\text{through total}\\ \text{sphere}}} = \frac{+q}{\epsilon_0} = 340 \ \text{Nm}^2/\text{C}$$

Only 3/4 of the field lines, and therefore, only 3/4 of the flux passes through S_1.

$$\therefore \Phi_{\substack{\text{due to}\\ +q}} = \frac{3}{4}(340 \ \text{Nm}^2/\text{C}) = 255 \ \text{Nm}^2/\text{C}$$

A similar arguement yields:

$$\Phi_{\substack{\text{due to}\\ -q}} = \frac{1}{2}(-340 \ \text{Nm}^2/\text{C}) = -170 \ \text{Nm}^2/\text{C}$$

$$\Phi_{\substack{\text{total}\\ \text{through } S_1}} = 255 \ \text{Nm}^2/\text{C} - 170 \ \text{Nm}^2/\text{C} = \underline{85 \ \text{Nm}^2/\text{C}}$$

B. $$\Phi_{\text{through } S_1} + \Phi_{\text{through } S_2} = \frac{Q_{total}}{\epsilon_0} = 0$$

$$\Rightarrow \Phi_{\text{through } S_2} = -85 \ \text{Nm}^2/\text{C}$$

II. By Newton's 2nd Law, the electrical force exerted upward on the dielectric must be equal in magnitude to the gravitational force exerted downward. The gravitational force acting on the column of liquid is:

$$F_g = mg = \rho\pi(b^2-a^2)hg$$

To find the electrical force, we displace the column of liquid upward by an infintesimal amount δz. The change in electric potential energy is:

$$\delta U = \frac{1}{2}\int_{\substack{\text{inside}\\ \text{dielectric}}} \vec{D}\cdot\vec{E}\,d\tau \;-\; \frac{1}{2}\int_{\substack{\text{in free}\\ \text{space}}} \vec{D}_0\cdot\vec{E}_0\,d\tau$$

By Gauß's Law:

$$\vec{E}_0 = \frac{\lambda}{2\pi\epsilon_0 r}\,\hat{r}$$

where λ is linear charge density on inner conductor. The potential difference is:

$$\Delta V = \int_a^b \frac{\lambda}{2\pi\epsilon_0 r}\,dr = \frac{\lambda}{2\pi\epsilon_0}\ln(b/a) = V_0$$

If the potential difference across the capacitor is held fixed by the battery, then:

$$\vec{E} = \vec{E}_0 = \frac{V_0}{r \ln(b/a)} \hat{r}$$

Since ϵ is a scaler constant (class A dielectric),

$$\vec{D} = \epsilon \vec{E} = \epsilon \vec{E}_0$$

$$\vec{D}_0 = \epsilon_0 \vec{E}_0$$

$$\therefore \ \delta U = \frac{(\epsilon - \epsilon_0)}{2} \int \vec{E}_0 \cdot \vec{E}_0 \, d\tau$$

$$= \frac{(\epsilon - \epsilon_0)}{2} \int_a^b \frac{V_0^2}{r^2 [\ln(b/a)]^2} 2\pi \, \delta z \, r \, dr$$

$$= \frac{(\epsilon - \epsilon_0) \pi V_0^2 \delta z}{\ln(b/a)}$$

The electrical force on the dielectric is:

$$F_z = \left. \frac{\partial U}{\partial z} \right|_V = \frac{\delta U}{\delta z} = \frac{(\epsilon - \epsilon_0) \pi V_0^2}{\ln(b/a)}$$

$$F_z = F_g \implies \boxed{h = \frac{(\epsilon - \epsilon_0) V_0^2}{\rho g (b^2 - a^2) \ln(b/a)}}$$

#11

Quantum Mechanics

(a) A spin-1/2 particle is in a pure state, for which $\langle S_x\rangle=\langle S_z\rangle=\hbar/4$ and $\langle S_y\rangle < 0$. Determine the spinor for the particle.

(b) An ensemble of identical spin-1/2 particles is in a state described by a density matrix such that $\langle S_x\rangle=\langle S_z\rangle=-\langle S_y\rangle=\hbar/4$. Find the elements of the density matrix. Is the system in a pure state or a mixed state?

#11

Solution - Quantum - G. TUTHILL

(a)

Let $\chi = \begin{pmatrix}\alpha\\ \beta\end{pmatrix}$ w/ α real + positive, $\beta = |\beta| e^{i\phi}$

$$\langle S_x \rangle = \frac{\hbar}{4} = \frac{\hbar}{2}\, \overbrace{\alpha^* \beta^*} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \frac{\hbar}{2}(\alpha^*\beta + \beta^*\alpha)$$

$$= \hbar\, \alpha |\beta| \cos\phi$$

$$\langle S_z \rangle = \frac{\hbar}{4} = \frac{\hbar}{2}\left(|\alpha|^2 + |\beta|^2\right)$$

$$\langle S_y \rangle = -i\frac{\hbar}{2}(\alpha^*\beta - \beta^*\alpha) = \hbar\, \alpha |\beta| \sin\phi$$

We also have $|\alpha|^2 + |\beta|^2 = 1$ so

$$|\alpha|^2 = \frac{1 + \langle S_z \rangle \frac{2}{\hbar}}{2} = \frac{3}{4} \Rightarrow \alpha = \frac{\sqrt{3}}{2}$$

$$|\beta|^2 = \frac{1 - \langle S_z \rangle \frac{2}{\hbar}}{2} = \frac{1}{4} \Rightarrow |\beta| = \frac{1}{2}$$

$$\cos\phi = \frac{1}{4\alpha|\beta|} = \frac{1}{\sqrt{3}} \qquad \phi = \pm\cos^{-1}\frac{1}{\sqrt{3}}$$

This places ϕ in the 1^{st} or 4^{th} quadrant.

But $\langle S_y \rangle$ is negative, so ϕ must be in the 4^{th} quadrant:

$$\phi = -\cos^{-1}\frac{1}{\sqrt{3}} = -0.955 \text{ radians}$$

(b) Now $\hat{\rho} = \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix}$ w/ $\rho_{11} + \rho_{22} = 1$

$$\langle S_x \rangle = \frac{\hbar}{4} = Tr\begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix}\frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \frac{\hbar}{2}(\rho_{12} + \rho_{21})$$

$$\langle S_z \rangle = \frac{\hbar}{4} = \frac{\hbar}{2}(\rho_{11} - \rho_{22})$$

$$\langle S_y \rangle = -\frac{\hbar}{4} = \frac{i\hbar}{2}(\rho_{12} - \rho_{21})$$

so
$$\rho_{11} = \frac{1 + 1/2}{2} = \frac{3}{4}$$

$$\rho_{22} = \frac{1 - 1/2}{2} = \frac{1}{4}$$

$$\rho_{12} = \frac{1}{2}(1+i)$$

$$\rho_{21} = \frac{1}{2}(1-i)$$

$$\hat{\rho} = \begin{pmatrix} 3/4 & \frac{1}{2}(1+i) \\ \frac{1}{2}(1-i) & 1/4 \end{pmatrix}$$

Pure or mixed state? Find eigenvalues of $\hat{\rho}$:

$$\left(\frac{3}{4} - \lambda\right)\left(\frac{1}{4} - \lambda\right) - \frac{1}{2} = 0$$

is <u>not</u> satisfied by $\lambda = 0$ or 1

so this is a mixed ~~state~~.

#12

E&M:

A particle of mass m charge q, moves in a plane perpendicular to a uniform, static, magnetic induction B.

a) Show that the total power (energy radiated per unit time) is given by

$$P = \frac{2}{3} q^4 \frac{B^2}{M^2} \frac{1}{c^3} (\gamma^2 - 1),$$

where γ is the ratio of the particle's total energy to its rest energy, and q, m, and c are the electric charge, mass, and the velocity of light.

(Assume that the magnetic induction is in the $\hat{z}$ direction, while the initial velocity is perpendicular to the $\hat{z}$ direction.)

b) If at time t=0 the particle has a total energy $E_0 = \gamma_0 Mc^2$, show that it will have energy $E = \gamma Mc^2 < E_0$ at the time t, where

$$t = \frac{3}{2} \frac{M^3 c^5}{q^4 B^2} \left[\frac{1}{\gamma} - \frac{1}{\gamma_0} \right].$$

c) If the particle is initially nonrelativistic and has a kinetic energy E_0 at t=0, what is its kinetic energy at time t?

S. TSURUTA

Problem 12

e & m

Solution

(a) Take $\vec{B}$ along z axis, $\bar{B} = B\hat{k}$.

The acceleration is then

$$\dot{\vec{v}} = \vec{v} \times \vec{\omega}_B = \vec{v} \times \frac{q\vec{B}}{\gamma mc}. \quad (1)$$

We may solve for $\bar{v}$ to get

$$\bar{v} = v_0 \cos\omega_B t\,\hat{i} - v_0 \sin\omega_B t\,\hat{j} + v_z\hat{k}, \quad (2)$$

where v_0 is initial velocity in xy-plane (we may take this in x direction).
v_z is initial z-velocity.

Since from the Liénard's result (e.g. Jackson Eqn 14-26), the total instantaneous power radiated is obtained by

$$P = \frac{2}{3}\frac{q^2}{c^3}|\dot{\vec{v}}|^2 = \frac{2}{3}\frac{q^2}{c}\gamma^6\left[(\dot{\vec{\beta}})^2 - (\vec{\beta}\times\dot{\vec{\beta}})^2\right], \quad (3)$$

where $\bar{\beta} \equiv \bar{v}/c$,

$$P = \frac{2}{3}\frac{q^2}{c}\gamma^6\left[\left(\frac{\dot{\bar{v}}}{c}\right)^2 - \left(\frac{1}{c^2}\bar{v}\times\dot{\bar{v}}\right)^2\right] \quad (3')$$

But

$$(1) \rightarrow \bar{v}\times\dot{\bar{v}} = \bar{v}\times\left[\bar{v}\times\frac{q\bar{B}}{\gamma mc}\right] = \frac{q}{\gamma mc}\left[\bar{v}(\bar{v}\cdot\bar{B}) - \bar{B}v^2\right]$$

$$= \frac{q}{\gamma mc}\left[Bv_z\vec{v} - \hat{k}Bv^2\right],$$

$$(\bar{v}\times\dot{\bar{v}})^2 = \frac{q^2}{\gamma^2 m^2 c^2}\left[B^2v_z^2v^2 + B^2v^4 - 2B^2v^2v_z^2\right] = \frac{q^2B^2v^2}{\gamma^2m^2c^2}\left[v^2 - v_z^2\right].$$

$$(\dot{\bar{v}})^2 = \dot{\bar{v}}\cdot\dot{\bar{v}} = \frac{q^2}{\gamma^2m^2c^2}(\vec{v}\times\vec{B})\cdot(\bar{v}\times\bar{B})$$

$$= \frac{q^2}{\gamma^2m^2c^2}\bar{v}\cdot\left[\bar{v}(\bar{B}\cdot\bar{B}) - \bar{B}(\bar{v}\cdot\bar{B})\right] = \frac{q^2}{\gamma^2m^2c^2}\left(v^2B^2 - B^2v_z^2\right).$$

note: $\bar{l}\cdot\bar{m}\times\bar{n} = \bar{m}\times\bar{n}\cdot\bar{l} = \bar{m}\cdot\bar{n}\times\bar{l} = \bar{v}\cdot\left[(\bar{v}\times\bar{B})\times\bar{B}\right]$

$= \bar{v}\cdot\left[\bar{B}\times(\bar{v}\times\bar{B})\right]$

#12 (cont'd.)

e&m Soln.

$$\therefore P = \frac{2}{3}\frac{q^2}{c}\gamma^6\left[\frac{1}{c^2}(V^2B^2 - B^2V_z^2)\frac{q^2}{\gamma^2 m^2 c^2} - \frac{q^2}{\gamma^2 m^2 c^2}\frac{B^2V^2}{c^4}(V^2 - V_z^2)\right]$$

$$= \frac{2}{3}\frac{q^2}{c}\gamma^6\frac{1}{c^2}B^2(V^2 - V_z^2)\frac{q^2}{\gamma^2 m^2 c^2}\left[1 - \frac{V^2}{c^2}\right].$$

$$P = \frac{2}{3}q^4\frac{B^2\gamma^2}{m^2c^5}V^2, \quad \text{for } V_z = 0.$$

$$= \frac{2}{3}q^4\frac{B^2\gamma^2}{m^2c^5}\frac{c^2}{\gamma^2}(\gamma^2 - 1) = \boxed{\frac{2}{3}q^4\frac{B^2}{m^2}\frac{1}{c^3}(\gamma^2 - 1)} \quad \text{Ans.}$$

(b). $E(t) = mc^2\gamma(t)$. For $\gamma \gg 1$,

$P = \frac{dE}{dt} = -K\gamma^2$, where $K = \frac{2}{3}q^4\frac{B^2}{m^2}\frac{1}{c^3}$ from (a).

$\frac{d\gamma}{\gamma^2} = -\frac{K}{mc^2}dt$. Integrating,

get $t = \frac{mc^2}{K}\frac{1}{\gamma} + \alpha$, where $\alpha = \text{const}$.

at $t=0$, $\gamma = \gamma_0$, $\alpha = -\frac{mc^2}{K\gamma_0^2}$.

$$t = \frac{mc^2}{K}\left(\frac{1}{\gamma} - \frac{1}{\gamma_0}\right) = \frac{3}{2}\frac{m^3c^5}{q^4B^2}\left[\frac{1}{\gamma} - \frac{1}{\gamma_0}\right]. \quad // \text{QED}.$$

Since $t > 0$, $\frac{1}{\gamma} - \frac{1}{\gamma_0} > 0 \therefore \gamma < \gamma_0 \therefore E = \gamma mc^2 < \gamma_0 mc^2 = E_0$.

(c) non-rel.

Can write $P = \frac{2}{3}q^4\frac{B^2}{m^2c^5}\gamma^2V^2 \approx \frac{2}{3}q^4\frac{B^2}{m^2c^5}V^2$, from (a).

$P = \frac{dT}{dt} = -\frac{4}{3}\frac{q^4B^2}{m^3c^5}T = -K'T$, where $T \equiv$ kinetic energy,

So, $-\frac{dT}{K'T} = dt$, where $K' \equiv \frac{4}{3}\frac{q^4B^2}{m^3c^5}$.

Integrating, $t = -\frac{1}{K'}\ln T + \alpha'$, where $\alpha' = \text{const}$.

e & m: Soln.

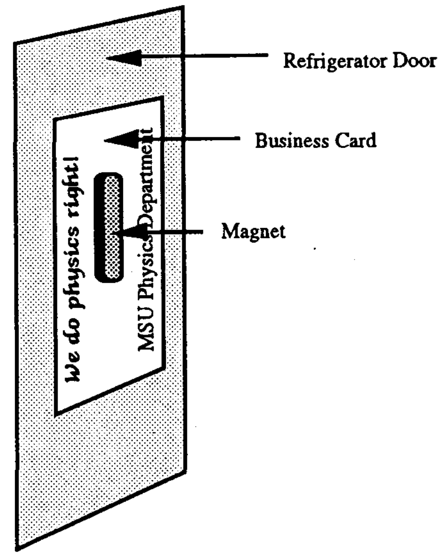
at $t=0$, $T=\mathcal{E}_0$; so $\alpha = \frac{1}{K'} \ln \mathcal{E}_0$. Thus,

$$t = \frac{1}{K'} \ln\left(\frac{\mathcal{E}_0}{T}\right).$$

$$\frac{\mathcal{E}_0}{T} = e^{K't}; \quad \frac{T}{\mathcal{E}_0} = e^{-K't}.$$

$$\therefore T = \mathcal{E}_0 e^{-K't} = \boxed{\mathcal{E}_0 \exp\left[-\frac{4}{3}\frac{q^4 B^2}{m^3 c^5} t\right]}$$ Ans.

Refrigerator Problems



A. A paper business card of mass 25g is held to the face of a refrigerator door (as shown to the right) by a magnet of mass 50g. The business card and magnet are at rest.

i. Sketch a free-body diagram for the *business card* and a separate free-body diagram for the *magnet.* In each case:

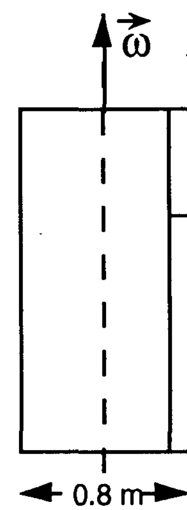
- draw vectors to indicate the forces that are exerted on the object,
- label and describe each force,
- identify the object exerting each force and the object on which that force is exerted.

The force vectors on your diagrams should be drawn consistent with any known relative magnitudes.

ii. For each of the forces shown in your diagram for the **magnet**, identify the corresponding force that completes the third-law (action-reaction) force pair.

B. The refrigerator now spins about a vertical axis through its geometric center, with constant angular velocity $\vec{\omega}$, as shown to the right. The business card and magnet are located at the center of the refrigerator door and they remain at rest relative to the door. At the instant shown, draw separate free-body diagrams for the business card and the magnet. Vector lengths on these diagrams should be consistent with the diagrams in part A and with the laws of physics.

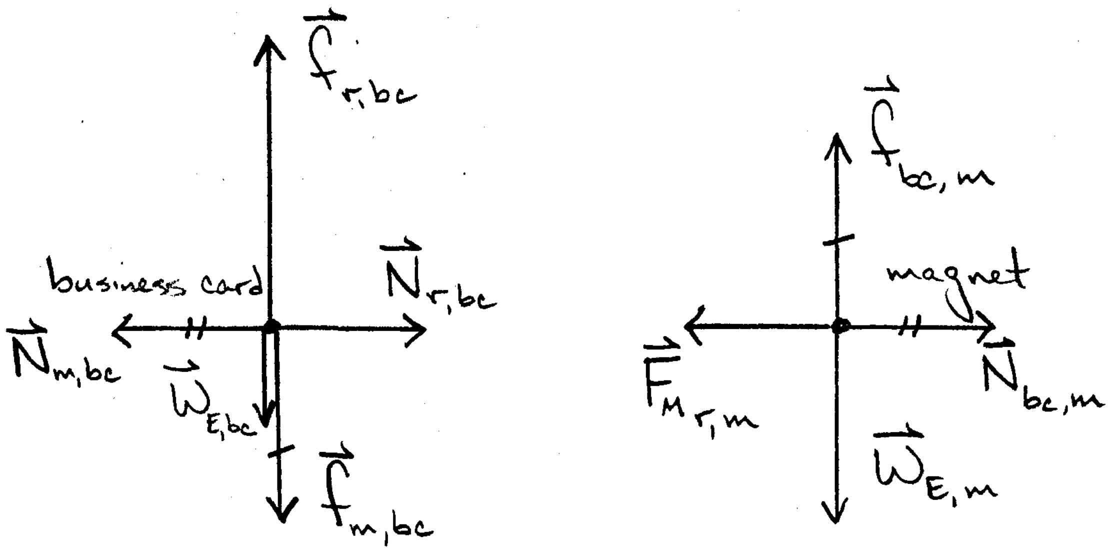
Explain the differences (if any) between these new diagrams and the diagrams drawn in part A. (For example, if a force increases in magnitude, explain why.)



C. The refrigerator is now drifting in space with some rockets attached. This Turbo-Fridge has moments of inertia $I_1=I_2=100\text{kg-m}^2$ and $I_3=40\text{kg-m}^2$ in a principal axis coordinate system. Initially, the Turbo-Fridge is not rotating. Assume that the center of mass is at the geometric center. Two of the rockets are mounted directly opposite one another, parallel and anti-parallel to the x_1 body axis, such that they cause a torque in the positive x_3 direction. These rockets each have a thrust of 500N and are 0.4m from the center of mass. They are fired for 2 sec and then turned off. Now rockets, which are at opposite ends of the Turbo-Fridge mounted parallel and anti-parallel to the body x_2 axis such that they produce a torque in the positive x_1 direction, are fired for $\Delta t={}^{10\pi}/_{16}\text{sec}$. These rockets each produce 200N of thrust and are 1.2m from the center of mass. Assume that the amount of material ejected by the rockets is small enough that the moments of inertia are not affected. Use Euler's Equations of Motion, $\tau_i=I_{ij}\dot{\omega}_j+\delta_{ijk}\omega_j I_{kl}\omega_l$, to find the components of angular velocity in the body set of axes for the final motion.

#13 Refrigerator Problems:

A. i.



key: $\vec{N}$ - normal force (contact)
$\vec{f}$ - frictional force (contact)
$\vec{W}$ - gravitational force (non-contact)
$\vec{F}_M$ - magnetic force (non-contact)
m - magnet
bc - business card
r - refrigerator

$\therefore$ $\vec{N}_{bc,m}$ is the normal force exerted by the business card on the magnet.

$W_{E,m} = 2W_{E,bc}$ given
$W_{E,m} = f_{bc,m}$ Newton's 2nd Law
$f_{bc,m} = f_{m,bc}$ Newton's 3rd Law
$f_{r,bc} = f_{m,bc} + W_{E,bc}$ Newton's 2nd

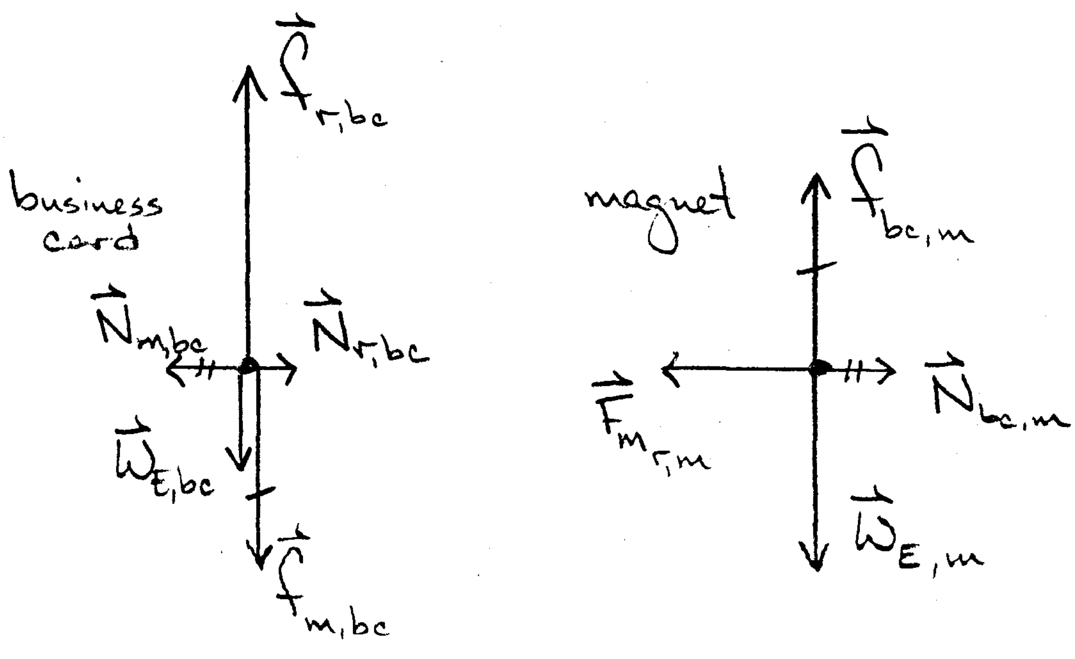
$N_{bc,m} = F_{M_{r,m}}$ N 2nd Law
$N_{m,bc} = N_{bc,m}$ N 3rd Law
$N_{r,bc} = N_{m,bc}$ N 2nd Law

(NB: We can not yet compare horizontal and vertical forces.)

A. ii.

Force	Companion Force	
$\vec{N}_{bc,m}$	$\vec{N}_{m,bc}$	shown with 2 tick marks
$\vec{f}_{bc,m}$	$\vec{f}_{m,bc}$	shown with 1 tick mark
$\vec{W}_{E,m}$	$\vec{W}_{m,E}$	grav. pull of magnet on the Earth
$\vec{F}_{M_{r,m}}$	$\vec{F}_{M_{m,r}}$	magnetic pull by magnet on the refrig.

B. The gravitational, frictional, and magnetic forces remain unchanged. The magnet has a centripetal acceleration, so $N_{bc,m} < F_{M_{r,m}}$. Hence, $N_{bc,m}$ must decrease (by Newton's 2nd Law). $N_{m,bc}$ decreases an equal amount (by N 3rd Law). Since the business card is accelerating as well, $N_{r,bc} < N_{m,bc}$.



Euler's Equations of motion are:

$$\tau_1 = I_1\dot{\omega}_1 + \omega_2\omega_3(I_3 - I_2)$$
$$\tau_2 = I_2\dot{\omega}_2 + \omega_3\omega_1(I_1 - I_3)$$
$$\tau_3 = I_3\dot{\omega}_3 + \omega_1\omega_2(I_2 - I_1)$$

During the first blast, $\tau_1 = \tau_2 = 0$, and

$$\tau_3 = 2(500\,N)(.4\,m) = 400\,N\text{-}m$$

Euler's Eqs. yield:

$$\omega_1 = \omega_2 = 0$$

$$\dot{\omega}_3 = \frac{\tau_3}{I_3} = 10\,s^{-2}$$

$$\therefore \quad \omega_3 = \dot{\omega}_3\,\Delta t = 20\,sec^{-1} \qquad \text{after } \Delta t = 2\,sec.$$

During the second blast:

$$\tau_1 = I_1\dot{\omega}_1 + \omega_2\omega_3(I_3 - I_1)$$
$$0 = I_2\dot{\omega}_2 + \omega_3\omega_1(I_1 - I_3)$$
$$0 = I_3\dot{\omega}_3$$

$$\therefore \dot{\omega}_3 = 0 \implies \omega_3 = 20\,sec^{-1}$$

Let $z \equiv \omega_1 + i\omega_2$

$$\Upsilon_1 = I_1 \dot{z} + i\omega_3(I_1 - I_3) z$$

or:

$$\dot{z} + i\alpha z = \frac{\Upsilon_1}{I_1} \quad , \text{ where } \alpha \equiv \frac{\omega_3(I_1 - I_3)}{I_1}$$

$$z = e^{-i\alpha t}\int \frac{\Upsilon_1}{I_1} e^{+i\alpha t} dt + C e^{-i\alpha t}$$

$$= -i\beta + C_1 e^{-i\alpha t} \quad , \text{ where } \beta \equiv \frac{\Upsilon_1}{\omega_3(I_1 - I_3)}$$

At $t=0$, $z=0$, so $C_1 = i\beta$.

$$z = i\beta(e^{-i\alpha t} - 1)$$

$$\omega_1(t) = \beta \sin(\alpha t)$$

$$\omega_2(t) = \beta\left[\cos(\alpha t) - 1\right]$$

For the given parameters:

$$\alpha = 12 \text{ sec}^{-1}$$

$$\beta = 0.4 \text{ sec}^{-1}$$

After $t = 10\pi/16$, $\alpha t = \frac{15\pi}{2}$, and

$$\omega_1 = -\beta = -0.4 \text{ sec}^{-1}$$

$$\omega_2 = -\beta = -0.4 \text{ sec}^{-1}$$

We now look at the torque-free solution for subsequent motion:

$$0 = I_1 \dot{\omega}_1 + \omega_2 \omega_3 (I_3 - I_1)$$
$$0 = I_1 \dot{\omega}_2 + \omega_3 \omega_1 (I_1 - I_3)$$
$$0 = I_3 \dot{\omega}_3$$

So: $\dot{\omega}_3 = 0 \implies \omega_3 = 20 \text{ sec}^{-1}$ forever.

$$\dot{z} + i\alpha z = 0$$
$$z = C_2 e^{-i\alpha t}$$

At $t = 0$, $z = -\beta(1+i) \implies C_2 = -\beta(1+i)$

$$\boxed{\begin{aligned} \omega_1(t) &= -\beta[\cos(\alpha t) + \sin(\alpha t)] \\ \omega_2(t) &= -\beta[\cos(\alpha t) - \sin(\alpha t)] \\ \omega_3(t) &= 20 \text{ sec}^{-1} = \text{constant} \end{aligned}}$$

#14

Quantum Mechanics

A particle of mass m is subject to a three dimensional delta-shell potential of the form

$$V = -\lambda\, \delta(r-a)$$

where r is the radial coordinate. λ and a are positive constants with the dimensions (energy·length) and (length), respectively.

(a) Find the minimum value of λ such that there is at least one bound state of the particle. (Hint: Consider the l=0 case.)

(b) A beam of particles of mass m and energy E is scattered from this potential, under the cond ition that $\lambda \gg \hbar^2/2ma$. What is the lowest value of E such that resonant s-wave scattering (i.e., a maximum of the s-wave scattering cross-section) occurs?

#14 Solution:

The Schrödinger equation

$$\left[-\frac{\hbar^2}{2m}\nabla^2 - \lambda\delta(r-a)\right]\psi = E\psi$$

becomes, for the $l=0$ case

$$\left(\frac{1}{r}\frac{d^2}{dr^2}r\right)\psi + \frac{2m\lambda}{\hbar^2}\delta(r-a)\psi + \frac{2mE}{\hbar^2}\psi = 0$$

where ψ depends on r alone. Making the usual substitution $\psi = u(r)/r$ we obtain

$$u'' + \frac{2m\lambda}{\hbar^2}\delta(r-a)u + \frac{2mE}{\hbar^2}u = 0$$

A bound state has $E<0$, so letting $E = -\frac{\hbar^2\kappa^2}{2m}$ we have

$$u'' - \kappa^2 u = 0 \text{ for } r \neq a$$

and

$$\begin{cases} u(a+\epsilon) = u(a-\epsilon) \\ u'(a+\epsilon) - u'(a-\epsilon) = -\frac{2m\lambda}{\hbar^2}u(a) \end{cases}$$

must also be satisfied.

Now for $r<a$ we have $u(r) = A\sinh\kappa r$ (since u/r must be finite as $r\to 0$) and for $r>a$, $u(r) = Be^{-\kappa r}$ (since $u\to 0$ as $r\to\infty$ for a bound state)

Thus

$$A\sinh\kappa a = Be^{-\kappa a}$$

and $$\kappa(-Be^{-\kappa a} - A\cosh\kappa a) = -\frac{2m\lambda}{\hbar^2}Be^{-\kappa a}$$

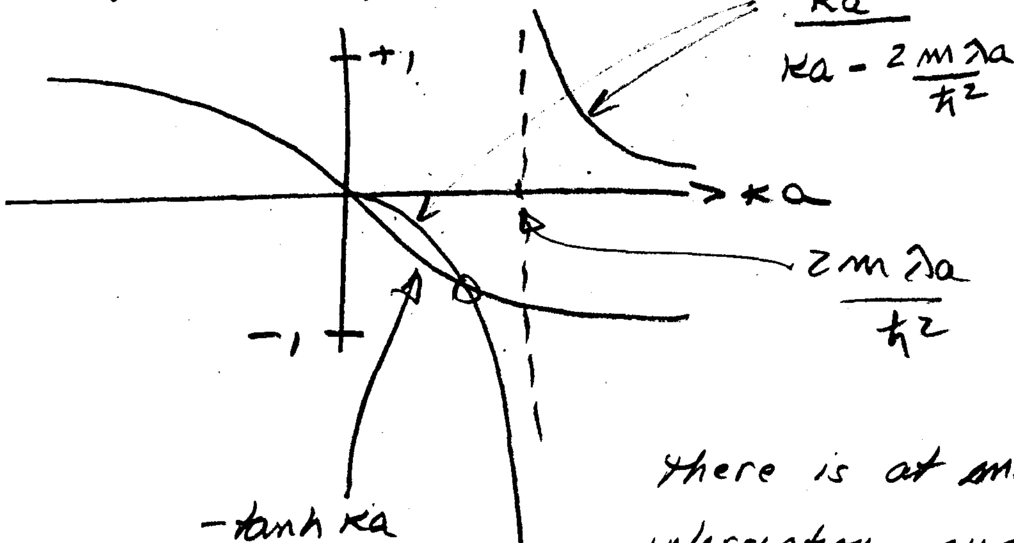
The 2^{nd} equation can be written

$$A\cosh\kappa A = -\kappa B\left(1 - \frac{2m\lambda}{\hbar^2\kappa}\right)e^{-\kappa a}$$

So

$$-\tanh\kappa a = \frac{\kappa a}{\kappa a - \frac{2m\lambda a}{\hbar^2}}$$

Graphing the right + left sides vs ka:



there is at most one intersection, and that exists provided the initial slope of the r.h.s. is less than one in magnitude

$$\text{ie}\quad \frac{d}{dx}\left(\frac{x}{x-C}\right)\Bigg|_{x=0} = \frac{d}{dx}\left(1+\frac{C}{x-c}\right)\Bigg|_{x=0} = -\frac{1}{C}$$

So we have to have $C > 1$

or $\lambda > \frac{\hbar^2}{2ma}$

(b) The s-wave scattering cross-section is proportional to $\sin^2\delta_0$, where δ_0 is the s-wave phase shift. Thus there will be a resonance when $\delta_0 = \frac{\pi}{2}$ (or $\frac{3\pi}{2}$ etc.)

For an s-wave soln

$$u = \begin{cases} A\sin kr & (r<a) \quad \text{w/ } E = \hbar^2k^2/2m \\ B\sin(kr+\delta_0) = B\cos kr & \text{for } \delta_0 = \frac{\pi}{2} \end{cases}$$

then $\quad A\sin ka = B\cos ka$

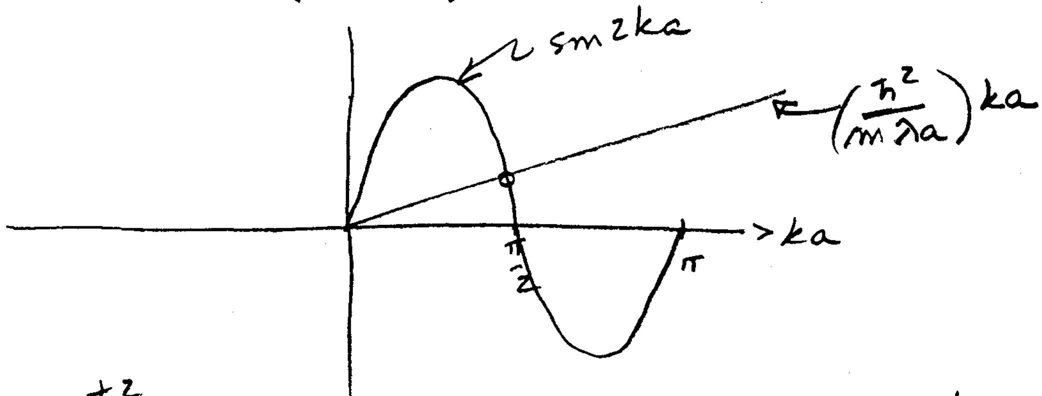
$$k(-B\sin ka - A\cos ka) = -\frac{2m\lambda}{\hbar^2}A\sin ka$$

or $\quad k(-\tan^2 ka \sin ka - \cos ka) = -\frac{2m\lambda}{\hbar^2}\sin ka$

$$\underbrace{\tan^2 ka + 1}_{=\frac{1}{\cos^2 ka}} = \frac{2m\lambda a}{\hbar^2 (ka)}\tan ka$$

or

$$\left(\frac{\hbar^2}{2m\lambda a}\right)(ka) = \sin ka \cos ka = \frac{1}{2}\sin 2ka$$



Since $\frac{\hbar^2}{m\lambda a} \ll 1$ the intersection is near $ka = \pi/2$:

$$\frac{\hbar^2}{m\lambda a}\left(\frac{\pi}{2} - \delta\right) = \sin 2\left(\frac{\pi}{2} - \delta\right) = \sin 2\delta$$
$$\cong 2\delta$$
$$\Rightarrow \delta \cong \frac{\pi}{2}\left(\frac{\hbar^2}{2m\lambda a}\right)$$

or $\quad ka \cong \frac{\pi}{2}\left(1 - \frac{\hbar^2}{2m\lambda a}\right)$

and $\quad E = \frac{\hbar^2 k^2}{2m} \cong \frac{\hbar^2}{2ma^2}\frac{\pi^2}{4}\left(1 - \frac{\hbar^2}{m\lambda a}\right)$

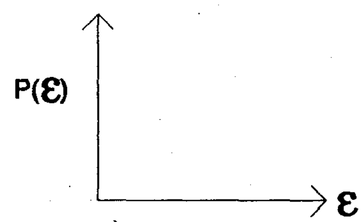
#15

Statistical Mechanics:

a) Find the density distribution of electrons in thermal equilibrium, $n_e(\vec{p})\,d^3\vec{p}$, as a function of the total number density n_e, momentum p, temperature T, and familiar constants such as the electron mass m_e and the Boltzman constant k. Assume that the electrons are non-relativistic and non-degenerate.

b) The occupation index $P(\varepsilon)$ is defined as $n(\varepsilon) / g(\varepsilon)$, where $n(\varepsilon)$ is the number density of electrons and $g(\varepsilon)$ is the number of possible particle states of energy ε. Sketch the distribution of the occupation index of electrons as a function of energy ε, when

(i) $\frac{\mu}{kT}$ is negative and large; (ii) $\frac{\mu}{kT}$ is positive and large, and; (iii) electrons are completely degenerate, where μ is the chemical potential



c) Find $n(\varepsilon)\,d\varepsilon$, the number density of non-relativistic electrons of energy ε in the range of $d\varepsilon$, in terms of μ, kT, m_e and h.

Problem 15 S. TSURUTA

Stat. mech. Solution

(a) Since the electrons are non-degenerate and in thermal equilibrium, they obey Maxwell-Boltzmann statistics. Hence

$n_e(\epsilon) = g(\epsilon)e^{-\alpha - \epsilon/kT}$, where $\boxed{\alpha = -\frac{\mu}{kT}}$

$n_e(\epsilon)d\epsilon = n_e(\bar{p})d\bar{p} = e^{-\alpha - \epsilon(p)/kT} g(p)dp.$

But $g(\bar{p})d\bar{p} = \frac{2}{h^3} 4\pi p^2 dp$ for electrons.

Since non-relativistic, $\epsilon(p) = \frac{p^2}{2m}$.

$\therefore n_e(\bar{p})d\bar{p} = e^{(-\alpha - p^2/2mkT)} \frac{8\pi}{h^3} p^2 dp$

$= A e^{-p^2/2mkT} 4\pi p^2 dp$

where $A \equiv e^{-\alpha} \frac{2}{h^3}$.

$\therefore n_e = \int n_e(\bar{p})d\bar{p} = \int_0^\infty A e^{-p^2/2m_e kT} 4\pi p^2 dp.$

Let $a \equiv \frac{1}{2m_e kT}$. Then,

$n_e = \int n_e(\bar{p})d\bar{p} = \int_0^\infty A e^{-p^2 a} p^2 dp\, 4\pi.$

Let $u = p^2 \rightarrow p = u^{1/2}$, $du = 2p\,dp \rightarrow dp = \frac{du}{2\sqrt{u}}$.

$\therefore \int n_e(\bar{p})d\bar{p} = n_e = 4\pi A \int_0^\infty e^{-au} u \frac{du}{2\sqrt{u}} = 2\pi A \int_0^\infty e^{-au} \sqrt{u}\, du.$

But $\int_0^\infty \sqrt{u}\, e^{-au} du = \frac{1}{2a}\sqrt{\frac{\pi}{a}}.$

#15 (contd.)
Stat Mech. Soln.

$$\therefore n_e = 2\pi A \int_0^\infty e^{-au} \sqrt{u}\, du = 2\pi A \frac{1}{2a}\sqrt{\frac{\pi}{a}} = A\left(\frac{\pi}{a}\right)^{3/2}.$$

$$\therefore A = n_e \left(\frac{a}{\pi}\right)^{3/2} = n_e \frac{1}{(2m_e kT)^{3/2}\, \pi^{3/2}}.$$

$$\therefore A = \frac{n_e}{(2\pi m_e kT)^{3/2}}.$$

$$\therefore \boxed{n_e(p)\,dp = \frac{n_e 4\pi p^2 dp}{(2\pi m_e kT)^{3/2}}\, e^{-\frac{p^2}{2m_e kT}}}$$

(b). For electrons which obey the Fermi-Dirac statistics,

$$P(\epsilon) = \frac{n(\epsilon)}{g(\epsilon)} = \frac{1}{e^{(\epsilon-\mu)/kT} + 1}.$$

(i). $\mu < 0$, $|\mu|/kT$ large. $\frac{\mu}{kT} = -\frac{|\mu|}{kT}$

$$e^{(\epsilon-\mu)/kT} + 1 \rightarrow e^{(|\mu|+\epsilon)/kT}$$

$$\therefore P(\epsilon) = e^{(-|\mu|-\epsilon)/kT}$$ Maxwell-Boltzman.

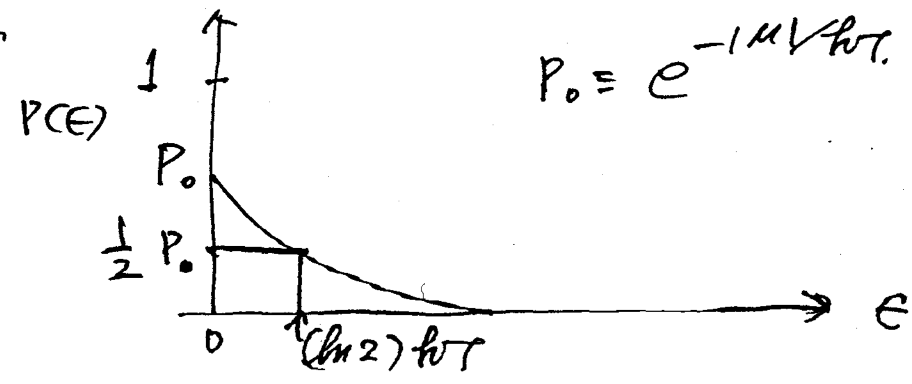
$\therefore$ when $\epsilon = 0$ $\quad P(0) = e^{-|\mu|/kT} = P_0 \ll 1$ for $\frac{|\mu|}{kT}$ large

$\epsilon = \infty \quad P(\infty) \doteq 0$

when $P(\epsilon) = \frac{1}{2}P_0 = e^{-|\mu|/kT - \epsilon/kT} = P_0 e^{-\epsilon/kT}$

$$e^{\epsilon/kT} = 2 \rightarrow \epsilon = (\ln 2) kT.$$

$\therefore$



$$P_0 = e^{-|\mu|/kT}.$$

#15 (cont'd) Soln.

Stat. Mech.

(b)(ii) For large μ/kT, >0

$$P(\epsilon) = \frac{1}{e^{(\epsilon-\mu)/kT}+1}$$

$\epsilon = 0$ $\quad P(0) = \dfrac{1}{\underset{>>}{e^{-\mu/kT}}+1} \to 1$ since $e^{-\frac{\mu}{kT}} << 1$ for large $\frac{\mu}{kT}$.

$\epsilon = \infty$ $\quad P(\infty) = \dfrac{1}{e^{(\infty-\mu)/kT}+1} \to e^{-\infty/kT} \to 0.$

$$P(\epsilon) = \frac{1}{2} = \frac{1}{e^{(-\mu+\epsilon)/kT}+1}$$

$\therefore 2 = e^{(-\mu+\epsilon)/kT}+1 \to e^{-\frac{\mu}{kT}} e^{\epsilon/kT} = 1$

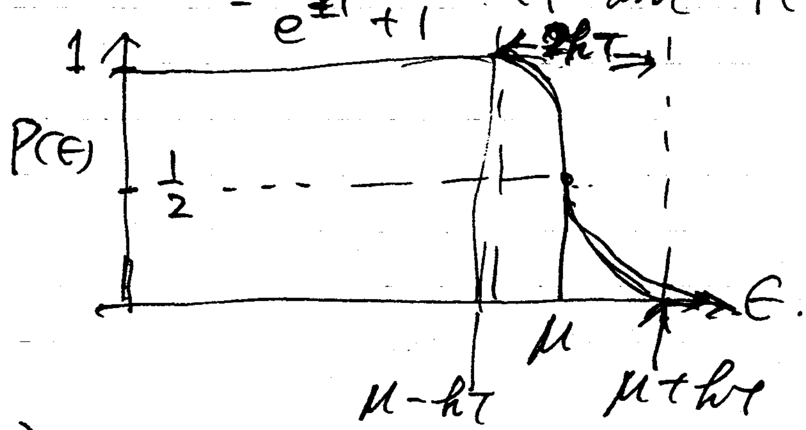
$\therefore \quad e^{\mu/kT} = e^{\epsilon/kT} \to \therefore \quad \epsilon = \mu.$

$1 > P(\epsilon) > 0$, when $\epsilon \approx \mu \pm kT$.

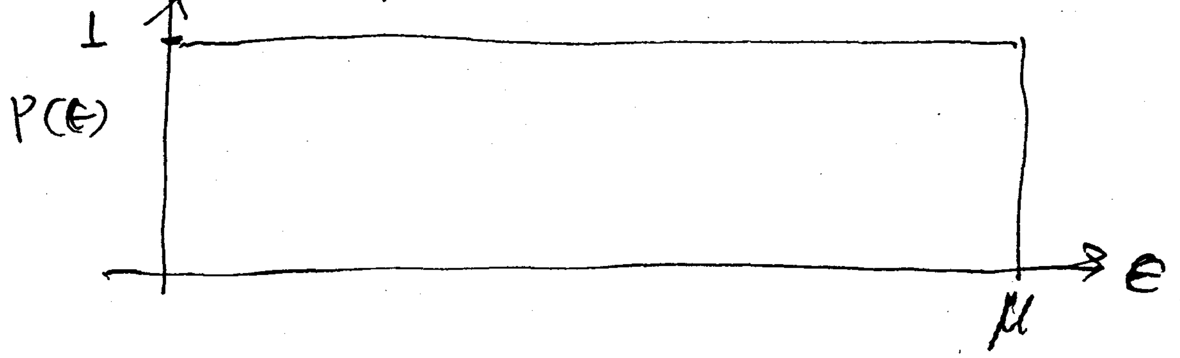
Since $P(\epsilon) = \dfrac{1}{e^{(-\mu+\mu\pm kT)/kT}+1}$

$= \dfrac{1}{e^{\pm 1}+1} < 1$ and $P(\epsilon) > 0$.

$\therefore$



(iii) Complete degeneracy means $\frac{\mu}{kT} \to \infty$ as $\frac{\mu}{kT} \to \infty$ $\quad \frac{kT}{\mu} \to 0$. So (ii) reduce to



#15 (conti.) Soln.

Stat. mech.

(c). $n(\epsilon) = P(\epsilon)\, g(\epsilon)$, where ①

$$P(\epsilon) = \frac{1}{e^{(\epsilon-\mu)/kT}+1} \quad ②.$$

and non-relativistic means that $\epsilon = \frac{p^2}{2m}$. $\therefore p = \sqrt{2m_e\epsilon}$.

$\therefore g(\epsilon)d\epsilon = \frac{8\pi}{h^3}(2m_e\epsilon)dp$.

$d\epsilon = \frac{2p\,dp}{2m_e} = \frac{p}{m_e}dp$. $\therefore dp = \frac{m_e}{p}d\epsilon = \frac{m_e}{\sqrt{2m_e\epsilon}}d\epsilon$.

$\therefore n(\epsilon)d\epsilon = g(\epsilon)P(\epsilon)d\epsilon = \frac{8\pi}{h^3}(2m_e^3)^{1/2}\epsilon^{1/2}d\epsilon\, P(\epsilon)$.

$\therefore n(\epsilon)d\epsilon = \frac{8\pi}{h^3}(2m_e^3)^{1/2}P(\epsilon)\sqrt{\epsilon}\,d\epsilon$.

$$= \frac{8\pi}{h^3}(2m_e^3)^{1/2}\frac{1}{e^{(\epsilon-\mu)/kT}+1}\sqrt{\epsilon}\,d\epsilon$$

// ans.