ϕ 519 Problems { Set #③ : Probs. 7-10. Assigned 10/7/88; due 10/14/88.

3

P4

[redacted] A "plasma" (a uniformly ionized gas, electrically neutral overall) flows at velocity $\mathbb{V}$ w.r.t. the lab frame, in which the observed fields are $\mathbb{E}$ & $\mathbb{B}$. The plasma has an electrical conductivity $\sigma = $ cnst, so that a current density: $\mathbb{J} = \sigma(\mathbb{E} + \frac{\mathbb{V}}{c} \times \mathbb{B})$ flows in the plasma (as seen in lab). Show that, in lab, the magnetic field changes as

(7) (A) $\boxed{\partial \mathbb{B}/\partial t = \nabla \times (\mathbb{V} \times \mathbb{B}) + (c^2/4\pi\sigma)\left[\nabla^2 \mathbb{B} - \frac{1}{c^2}(\partial^2 \mathbb{B}/\partial t^2)\right]}$.

(3) (B) Find a reduced form of the eqtn in part (A) which holds when $\nabla \cdot \mathbb{J} = 0$ is imposed (i.e. there is no net current flow out of the plasma). [redacted]: see Jk^{n} Eq. (10.10).

(5) (C) Using the reduced eqtn of part (B), show that for slow-moving plasmas, the interior $\mathbb{B}$ fields decay as time goes on. Calculate the decay constant.

(5) (D) At the other extreme, consider a highly conducting plasma ($\sigma \to \infty$), moving at $\mathbb{V} \neq 0$. Show that the magnetic flux through any loop moving with the plasma does <u>not</u> change with time -- the $\mathbb{B}$ field is "frozen" in the plasma.

[redacted] Lyttleton & Bondi [Proc. Roy. Soc. (London) <u>A252</u>, 313 (1959)] suggested that the expansion of the universe might be due to matter carrying a net electric charge. Consider a spherically symmetric universe (centered on the Big Bang locus) containing un-ionized hydrogen atoms of uniform density n atoms/unit vol. Assume the proton & electron charges are slightly different, i.e.: $|e_p/e_e| = 1+\beta$, w/ $|\beta| \ll 1$, but $\beta \neq 0$.

(A) Find the minimum value β_m of β at which this universe begins expanding.

(B) Assume the density n remains constant due to continuous creation of matter (sic). Show, for $\beta > \beta_m$, the repulsive force on an atom is $\propto r$, its radial distance from the center of the universe. Consequently, show: (1) the atom's radial velocity $v_r \propto r$, (2) this universe expands exponentially in time.

(C) Show that $v_r = r/T$, where T is the time required for expansion by factor e. If $T \sim 10^{10}$ yr. ($\sim$ age of universe), and the observed average density $n \sim 6 \frac{\text{atoms}}{\text{m}^3}$, calculate the size of β needed to "explain" the expansion of the universe.

ϕ519 Problems

● Start from the Maxwell Eqs. for an uncharged ($\rho=0$) material medium:

$$\boxed{\nabla\cdot\mathbb{D}=0,\ \nabla\cdot\mathbb{B}=0,\ \nabla\times\mathbb{E}=-\frac{1}{c}(\partial\mathbb{B}/\partial t),\ \nabla\times\mathbb{H}=\frac{1}{c}(\partial\mathbb{D}/\partial t)+\frac{4\pi}{c}\mathbb{J}.}$$

Assume the medium is conducting: $\mathbb{J}=\sigma\mathbb{E}$, polarizable: $\mathbb{D}=\mathbb{E}+4\pi\mathbb{P}$, and magnetizable: $\mathbb{B}=\mathbb{H}+4\pi\mathbb{M}$. Derive the wave eqtn for $\mathbb{E}$, in the form: all terms in (and operations on) $\mathbb{E}$ = all source (driving) terms in $\mathbb{P}$ & $\mathbb{M}$. [redacted] Which of the source terms is likely to dominate in a typical case.

● Read enough of Jackson's Sec. (6.2) to convince yourself that his Eq. (6-12) is correct. Using this, do Jackson's problem 6.1(a), p. 261, viz: Show that for current-carrying elements in empty space, the total magnetic field energy: $\boxed{W_m=\frac{1}{2c^2}\int d^3r\int d^3r'\,\frac{\mathbb{J}(\mathbf{r})\cdot\mathbb{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}}$.

Confused by the loud drums, Roy is flushed into the net.

⑦ Analyse how a moving plasma affects magnetic fields.

(7pts) A. If $\vec{E}'$ is the electric field at a point fixed in the plasma, then Faraday's Law in this moving medium reads [see Jackson's Eq. (6.6)]...

10/1/84

$$-\frac{1}{c}\frac{\partial \vec{B}}{\partial t} = \vec{\nabla}\times\left(\vec{E}' - \frac{1}{c}\vec{v}\times\vec{B}\right), \text{ or } \frac{\partial \vec{B}}{\partial t} = \nabla\times(\vec{v}\times\vec{B}) - c\vec{\nabla}\times\vec{E}'. \quad (1)$$

The current density $\vec{J}$ in the plasma is generated by $\vec{E}'$, i.e. $\vec{J} = \sigma\vec{E}'$, so...

$$\partial\vec{B}/\partial t = \vec{\nabla}\times(\vec{v}\times\vec{B}) - \frac{c}{\sigma}\vec{\nabla}\times\vec{J}. \quad (2)$$

Now $\vec{J}$ is related to $\vec{B}$ and $\vec{E}$ via Ampere's Law...

$$\vec{\nabla}\times\vec{B} = \frac{4\pi}{c}\vec{J} + \frac{1}{c}(\partial\vec{E}/\partial t) \Rightarrow \vec{\nabla}\times\vec{J} = \frac{c}{4\pi}\vec{\nabla}\times\left[\vec{\nabla}\times\vec{B} - \frac{1}{c}\frac{\partial\vec{E}}{\partial t}\right]. \quad (3)$$

For the first term here: $\vec{\nabla}\times(\vec{\nabla}\times\vec{B}) = \vec{\nabla}(\underbrace{\cancel{\vec{\nabla}\cdot\vec{B}}}_{0}) - \nabla^2\vec{B}$, and in the second term: $\vec{\nabla}\times(\partial\vec{E}/\partial t) = \frac{\partial}{\partial t}(\vec{\nabla}\times\vec{E}) = -\frac{1}{c}(\partial^2\vec{B}/\partial t^2)$, by use of Faraday's Law (in lab). Thus, we have for curl $\vec{J}$...

$$\vec{\nabla}\times\vec{J} = -\frac{c}{4\pi}\left[\nabla^2\vec{B} - \frac{1}{c^2}(\partial^2\vec{B}/\partial t^2)\right]. \quad (4)$$

Use of this in Eq. (2) gives the desired eqtn of motion for $\vec{B}$...

$$\boxed{\partial\vec{B}/\partial t = \vec{\nabla}\times(\vec{v}\times\vec{B}) + \frac{c^2}{4\pi\sigma}\left[\nabla^2\vec{B} - \frac{1}{c^2}(\partial^2\vec{B}/\partial t^2)\right].} \quad (5)$$

(3pts) B. From Ampere's Law, Eq (3): $\vec{\nabla}\cdot(\vec{\nabla}\times\vec{B}) \equiv 0 = \frac{4\pi}{c}\vec{\nabla}\cdot\vec{J} + \frac{1}{c}\vec{\nabla}\cdot(\partial\vec{E}/\partial t)$, so that $\vec{\nabla}\cdot\vec{J} = 0$ implies neglect of the displacement current $\propto \partial\vec{E}/\partial t$. Throwing out this term in Eqs. (3) & (4) is equivalent to neglecting $\partial^2\vec{B}/\partial t^2$, so Eq. (5) becomes

$$\boxed{\partial\vec{B}/\partial t = \nabla\times(\vec{v}\times\vec{B}) + \frac{c^2}{4\pi\sigma}\nabla^2\vec{B}} \quad (6)$$

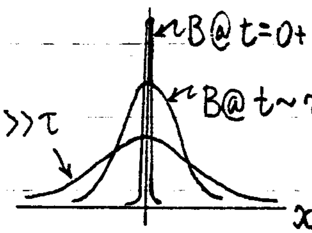
~ Required form for $\vec{\nabla}\cdot\vec{J} = 0$.

C. For slow-moving plasmas, $\vec{v} \to 0$, and Eq. (6) reads approximately...

(5pts) $$\nabla^2 \vec{B} = \frac{1}{\kappa}(\partial \vec{B}/\partial t), \quad \kappa = c^2/4\pi\sigma. \tag{7}$$

This is in the form of a diffusion eqtn (see Mathews & Walker "Math. Methods of Physics" p. 218). In 1D, the prototype solution -- for a B-field initially concentrated at the origin -- is given by (M & W, p. 243)

$$B(x,t) \propto (1/\sqrt{4\pi\kappa t})\, e^{-x^2/4\kappa t}. \tag{8}$$



If $x \sim D$ is a characteristic dimension over which B is appreciable, then B will die away in a characteristic time...

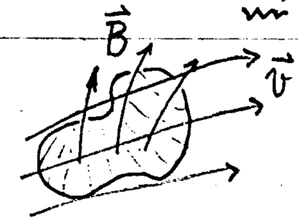
$$\underline{\underline{t \sim \tau = D^2/4\kappa = \pi\sigma D^2/c^2}} \leftarrow \text{diffusion time.} \tag{9}$$

D. For $\vec{v} \neq 0$, but $\sigma \to \infty$, Eq. (6) is approximately...

(5pts) $$\partial \vec{B}/\partial t = \vec{\nabla} \times (\vec{v} \times \vec{B}), \quad \text{or} \quad \frac{\partial \vec{B}}{\partial t} - \vec{\nabla} \times (\vec{v} \times \vec{B}) = 0. \tag{10}$$

This last construction is recognizable as the "convective derivative" of $\vec{B}$, as described in the footnote on Jackson's p. 212...

$$\frac{\partial \vec{B}}{\partial t} - \vec{\nabla} \times (\vec{v} \times \vec{B}) = \left(\frac{\partial}{\partial t} + \vec{v} \cdot \vec{\nabla}\right)\vec{B} = \frac{d\vec{B}}{dt} \tag{11}$$



This, together with Eq. (10), says that in the plasma: $d\vec{B}/dt = 0$, so that for any loop within the plasma, the magnetic flux: $\underline{\Phi = \int \vec{B} \cdot d\vec{S}, \text{ is constant}}$ ... there can be no EMF's induced in the plasma because the ∞ conductivity immediately reacts to cancel them. The $\vec{B}$ lines are thus "frozen" in (i.e. carried along by the plasma).

9/30/84

(5) Explore how e-p charge imbalance might explain expansion of universe.

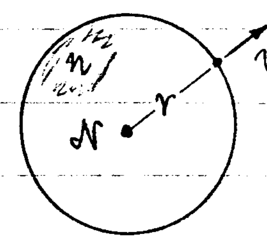
A. The excess charge $q = \beta e$ on each atom (assumed to be mass M) can -- at atom-atom separation r -- cause a net repulsion...

$$q, M \quad \odot \overset{r}{\longrightarrow} \odot \xrightarrow{} f_r \quad \Big\} \; f_r = \frac{k(\beta e)^2}{r^2} - G\frac{M^2}{r^2} = \frac{ke^2}{r^2}(\beta^2 - \beta_m^2), \quad \beta_m = \sqrt{\frac{GM^2}{ke^2}},$$

where: $k = 1/4\pi\epsilon_0 = 9\times 10^{9}$ MKS, $e = 1.6\times 10^{-19}$ C
grav. const $\}$ $G = 6.67\times 10^{-11}$ MKS, $M = 1.67\times 10^{-27}$ kgm $\Big\}$ $\underline{\underline{\beta_m = 0.90\times 10^{-18}}}$.

$\beta \geq \beta_m$ ensures $f_r \geq 0$, i.e. actual repulsion, so this $\Rightarrow$ the universe expands.

B. f_r is an inverse square law, so Gauss' Law applies, and an atom at radial distance r experiences a repulsive force as though all the atoms in the r-sphere were concentrated at its center; the atoms outside the r-sphere don't count. This $\Rightarrow$ a net force F_r on atom at r...



$$\mathcal{N} = \begin{matrix}\#\text{ atoms in}\\ r\text{-sphere}\end{matrix} = \left(\frac{4\pi}{3}r^3\right)n \Rightarrow \boxed{F_r = \mathcal{N}f_r = Kr}, \quad \underline{\underline{K}} = \frac{4\pi n}{3}ke^2(\beta^2 - \beta_m^2).$$

The eqtn of motion for the radial velocity $v_r = dr/dt$ is...

$$M\frac{dv_r}{dt} = F_r, \;\overset{\text{or}}{/\!/}\; Mv_r\frac{dv_r}{dr} = Kr \Rightarrow \underline{\underline{v_r = \Omega r}}, \text{ where: } \Omega = \sqrt{\frac{K}{M}}.$$

So $v_r \propto r$ as advertised. Since: $dr/dt = \Omega r$, another trivial integration yields

$\underline{\underline{r(t) = r(0)\exp(\Omega t)}}$, and: Δt req'd for e× expansion is: $T = \frac{1}{\Omega} = \sqrt{\frac{M}{K}}$.

C. For a given T, need $K = M/T^2$. Numerically, for $T = 10^{10}$ yrs, this yields...

$(\beta^2 - \beta_m^2)n = 1.74\times 10^{-35}$, or: $\beta^2 = \beta_m^2 + 2.90\times 10^{-36}$, when $n = 6$ atoms/m^3.

Then: $\underline{\beta = 1.92\times 10^{-18}} \simeq 2\beta_m$ charge imbalance is "all" that is required.

11/8/84

● (10 pts). Derive wave eqtn for $\vec{E}$ in an uncharged, polarizable/magnetizable med.^{m}

1) Insert $\vec{D}=\vec{E}+4\pi\vec{P}$, $\vec{H}=\vec{B}-4\pi\vec{M}$, $\vec{J}=\sigma\vec{E}$ into given Maxwell set, so....

① $\vec{\nabla}\cdot\vec{E}=-4\pi\,\vec{\nabla}\cdot\vec{P}$, ② $\vec{\nabla}\cdot\vec{B}=0$,

③ $\nabla\times\vec{E}=-\frac{1}{c}\,\partial\vec{B}/\partial t$, ④ $\vec{\nabla}\times\vec{B}=4\pi\vec{\nabla}\times\vec{M}+\frac{4\pi}{c}\sigma\vec{E}+\frac{1}{c}\frac{\partial}{\partial t}(\vec{E}+4\pi\vec{P})$.

Take $\vec{\nabla}\times$ Eq. ③ and use identity: $\vec{\nabla}\times(\vec{\nabla}\times\vec{E})=\vec{\nabla}(\vec{\nabla}\cdot\vec{E})-\nabla^2\vec{E}$. Then...

$$\vec{\nabla}\underbrace{(\vec{\nabla}\cdot\vec{E})}_{\text{use ①}}-\nabla^2\vec{E}=-\frac{1}{c}\frac{\partial}{\partial t}\underbrace{(\vec{\nabla}\times\vec{B})}_{\text{use ④}};$$

$$\Rightarrow 4\pi\,\vec{\nabla}(\vec{\nabla}\cdot\vec{P})+\nabla^2\vec{E}=\frac{4\pi}{c}\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{M})+\left(\frac{4\pi\sigma}{c^2}\right)\frac{\partial\vec{E}}{\partial t}+\frac{1}{c^2}\frac{\partial^2}{\partial t^2}(\vec{E}+4\pi\vec{P}).$$

Rearrange terms to get desired wave eqtn: ① ② ③

$$\boxed{\left(\nabla^2-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{E}-\left(\frac{4\pi\sigma}{c^2}\right)\frac{\partial\vec{E}}{\partial t}=\frac{4\pi}{c^2}\left(\frac{\partial^2\vec{P}}{\partial t^2}\right)-4\pi\vec{\nabla}(\vec{\nabla}\cdot\vec{P})+\frac{4\pi}{c}\vec{\nabla}\times\left(\frac{\partial\vec{M}}{\partial t}\right)}.$$

2) Microscopically, the polarization: $P\sim nea_0$, where $n=\#$ atoms/vol. and $a_0=\frac{\hbar^2}{me^2}$ is the Bohr radius, while the magnetization: $M\sim n\mu_0$, $\mu_0=e\hbar/2mc$ is the Bohr magneton. Then, nominally: $M/P\sim\mu_0/ea_0=e^2/2\hbar c\sim 1/274\ll 1$, and when electric polarization is not forbidden by some exotic selection rule in the medium, it will dominate the magnetization. Term ③ above is thus (usually) negligible.

The relative sizes of terms ①&② depend on the medium. If P varies appreciably over some characteristic distance D, then term ② $\sim P/D^2$. If we are propagating waves at freq. ω, then term ① $\sim\frac{\omega^2}{c^2}P\sim k^2P$ (where $k\sim\frac{\omega}{c}$ is the wave #). In this case, have: term ②/term ① $\sim 1/k^2D^2\sim(\lambda/D)^2$, where λ = wavelength. For isotropic media & optical wavelengths, $D\gg\lambda$, and term ① usually dominates.

10/7/84

Calculate magnetic energy of system of currents in empty space.

1. Jackson's arithmetic up through Eq. (6-12) is correct, so that the increment of work done on current $\vec{J}$ by a change $\delta\vec{A}$ in the vector potential is

$$\delta W = \frac{1}{c}\int d\tau\, \vec{J}\cdot\delta\vec{A}, \quad d\tau = d^3x. \tag{1}$$

2. Now suppose the $\delta\vec{A}$ here is caused by a current change at some distant location. By the usual relation between $\vec{A}$ & its $\vec{J}$, have...

$$\delta\vec{A}(\vec{x}) = \frac{1}{c}\int \frac{d\tau'}{R}\,\delta\vec{J}(\vec{x}'), \quad R = |\vec{x}-\vec{x}'|, \quad d\tau' = d^3x'. \tag{2}$$

3. In Eq. (1), the integrand coordinates are $\vec{x}$. Put Eq. (2) into (1) to get...

$$\delta W = \frac{1}{c^2}\int d\tau \int \frac{d\tau'}{R}\,\vec{J}(\vec{x})\cdot\delta\vec{J}(\vec{x}') \tag{3}$$

The primed & unprimed coordinates can be interchanged. Since R is unaffected by this, then all that happens to this expression for δW is that the cds $\vec{x}$ & $\vec{x}'$ of $\vec{J}$ & $\delta\vec{J}$ are interchanged. Thus we can write...

$$\delta W = \frac{1}{2c^2}\int d\tau \int \frac{d\tau'}{R}\left[\vec{J}(\vec{x})\cdot\delta\vec{J}(\vec{x}') + \vec{J}(\vec{x}')\cdot\delta\vec{J}(\vec{x})\right],$$

or// $$\delta W = \frac{1}{2c^2}\int d\tau \int \frac{d\tau'}{R}\,\delta\left[\vec{J}(\vec{x})\cdot\vec{J}(\vec{x}')\right]. \tag{4}$$

The δ inside signifies current-change, not variation in R e.g. So it can be taken outside the integral. Then we get the desired energy

$$\boxed{W = \frac{1}{2c^2}\int d\tau \int d\tau'\, \frac{\vec{J}(\vec{x})\cdot\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|}} \tag{5}$$