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① In a QM system with Hamiltonian $\mathcal{H}$, let the eigenfcns & eigenenergies be ψ_n & E_n, so: $\mathcal{H}\psi_n = E_n\psi_n$. To approximate the ground state energy E_0, suppose you use a trial fcn: $\underline{\psi = \psi_0 + \lambda\phi}$, w/ ψ_0 = actual ground state eigenfcn, λ = small (real) parameter, and ϕ = an arbitrary fcn with the expansion: $\phi = \sum_n c_n\psi_n$. Show that if the approximate (variational) energy $E(\lambda) = \langle\psi|\mathcal{H}|\psi\rangle / \langle\psi|\psi\rangle$ is expanded in a power series in λ, viz.: $\underline{E(\lambda) = E_0 + \lambda\mathcal{E}_1 + \lambda^2\mathcal{E}_2 + \lambda^3\mathcal{E}_3 + \cdots}$, then $\mathcal{E}_1 = 0$, while $\mathcal{E}_2$ is the positive quantity: $\boxed{\mathcal{E}_2 = \sum_n |c_n|^2(E_n - E_0)}$. NOTE: this result $\Rightarrow$ that any perturbation on $\mathcal{H}$ which shifts $\psi_0 \to \psi_0 + \lambda\phi$ by a term 1st order in some small parameter λ, can only shift the ground state energy $E_0 \to E_0 + \lambda^2\mathcal{E}_2$ by a term 2nd order in λ.

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② On p.2 of "Notes on the WKB Method", it is claimed that any 2nd order homogeneous ODE of the form: $\underline{y'' + f(x)y' + g(x)y = 0}$, can be cast into the WKB form: $\underline{\psi'' + k^2(x)\psi = 0}$, if $\underline{\psi(x)} = y(x)\exp\left[+\frac{1}{2}\int_a^x f(\xi)\,d\xi\right]$, and $\underline{k(x)} = \pm\left\{g(x) - \frac{1}{2}\left[f'(x) + \frac{1}{2}f^2(x)\right]\right\}^{1/2}$. Verify this claim by substituting $y(x) = \psi(x)u(x)$ into the original ODE and then choosing $u(x)$ judiciously. Why is the lower limit a in the integral $\int_a^x f(\xi)\,d\xi$ essentially arbitrary?

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③ Bessel's ODE is: $\underline{y'' + \frac{1}{x}y' + \left(1 - \frac{\nu^2}{x^2}\right)y = 0}$, w/ ν = real cnst. Find an approximate solution for the Bessel fcn $y = J_\nu(x)$ by the WKB method. Then find an asymptotic form for $J_\nu(x)$ as $x \to$ "large" (i.e. $x \gg |\nu|$). You may assume $|\nu| \gg 1/2$.

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④ This exercise is connected with the WKB "turning point" problem. (A) Show -- by substitution -- that a solution to: $\underline{y''(\xi) + \alpha\xi^n y(\xi) = 0}$, w/ α & n = cnsts and $\xi \geq 0$, is given by: $\underline{y(\xi) = A\sqrt{\xi}\,J_\nu(\zeta)}$, w/ A = cnst, $\nu = 1/(n+2)$, $\zeta = \left(\frac{2\sqrt{\alpha}}{n+2}\right)\xi^{\frac{1}{2}(n+2)}$, and $J_\nu(\zeta)$ is the Bessel fcn of order ν. (B) Assume an asymptotic form: $y(\xi) \sim \xi^{-k}e^{-a\xi^l}$, as $\xi \to \infty$. By properly choosing the cnsts k, l & a, show that as $\xi \to \infty$ this form satisfies the ODE: $y''(\xi) + \alpha\xi^n y(\xi) = \frac{n}{4}\left(\frac{n}{4} + 1\right)\xi^{-2}y(\xi) \to 0$.

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(10) Let the Hamiltonian $\mathcal{H} = \mathcal{H}_0 + V$, where $\mathcal{H}_0$ is a free-particle Hamiltonian, and V accounts for all interactions. Let ξ be the space-time point (x,t). The Schrödinger Eq. is: $\underline{(i\hbar \frac{\partial}{\partial t'} - \mathcal{H}_0')\psi(\xi') = \hbar\rho(\xi')}$, w/ $\rho(\xi') = \frac{1}{\hbar}V(\xi')\psi(\xi')$. ρ acts as a source fcn for the otherwise free propagation of ψ. Now let $G_0(\xi',\xi)$ be the free-particle Green's fcn which satisfies the point-source eqtn: $\underline{(i\hbar\frac{\partial}{\partial t'} - \mathcal{H}_0')G_0(\xi',\xi) = \hbar\delta(\xi'-\xi)}$. Show that the general solution to the S. Eqtn is: $\underline{\underline{\psi(\xi') = \psi_0(\xi') + \int G_0(\xi',\xi)\rho(\xi)d\xi}}$, where ψ_0 is a free-particle wavefcn. This justifies the claim in class notes, p. IF7.

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(11) Set $\hbar=1$. Consider a Schrödinger system with known eigenfcns $u_n(\mathbf{r})$ and eigenvalues ω_n [generated as usual by $\mathcal{H}u_n = \omega_n u_n$]. In class, we claimed the Green's fcn for this system was: $\underline{G(\mathbf{r},t;\mathbf{r}_0,t_0) = -i\,\theta(t-t_0)\sum_n u_n^*(\mathbf{r}_0)u_n(\mathbf{r})e^{-i\omega_n(t-t_0)}}$, w/ θ = unit step fcn [see Eq (A5) of class notes, p. IF7]. Verify this claim by showing that G actually obeys: $\underline{\underline{[i(\partial/\partial t) - \mathcal{H}]G = \delta(\mathbf{r}-\mathbf{r}_0)\delta(t-t_0)}}$, per Eq.(15), p. IF6.

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(12) A free particle in 1D has mean momentum k_0, and initially is localized in space to $\Delta x \sim \delta$; its wavefcn at $t=0$ is: $\underline{\psi(x,0) = Ae^{ik_0x}e^{-x^2/2\delta^2}}$. Adjust the cnst A so that $\int_{-\infty}^{\infty}|\psi(x,0)|^2dx = 1$. Now, by integrating $\psi(x,0)$ over the free-particle propagator K_0 [Eq.(19), class notes, p. IF8], show that $t>0$, ψ has evolved to:

$$\boxed{\psi(x,t) = \frac{A}{\sqrt{1+i\tau}}e^{i(k_0x-\omega_0t)}e^{-(1-i\tau)(x-v_0t)^2/2\delta^2(1+\tau^2)}},$$

w/ $\tau = \hbar t/m\delta^2$, $v_0 = \hbar k_0/m$, $\omega_0 = \hbar k_0^2/2m$.

Interpret the motion of ψ physically (e.g. draw pictures). What happens if $\delta \to 0$?

---

(13) This tidbit of complex variable arcana will be used in problem (15). By evaluating an appropriate contour integral in the complex ω-plane, show that the unit step fcn can be represented by: $\underline{\theta(\tau) = \lim_{\epsilon\to 0+}\left(\frac{i}{2\pi}\right)\int_{-\infty}^{\infty}\frac{e^{-i\omega\tau}d\omega}{\omega+i\epsilon}} = \begin{cases}1, & \text{for } \tau>0\\ 0, & \text{for } \tau<0\end{cases}$. If $\epsilon\to 0-$, rather than $0+$, show that the integral generates $\theta(\tau)-1$, the often popular out-of-step fcn. From the form for $\theta(\tau)$, what is the integral for the Dirac delta, $\delta(\tau)$?

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(17) A QM system consists of two particles, w/ masses m_1 & m_2. Express the operators for total momentum $\hat{\mathbb{P}} = \hat{\mathbb{p}}_1 + \hat{\mathbb{p}}_2$, and total ∡ momentum $\hat{\mathbb{L}} = \hat{\mathbb{l}}_1 + \hat{\mathbb{l}}_2$, in terms of the relative cd. $\mathbb{r} = \mathbb{r}_1 - \mathbb{r}_2$ and center-of-mass cd. $\mathbb{R} = (m_1\mathbb{r}_1 + m_2\mathbb{r}_2)/(m_1+m_2)$. Show that the K.E. part of the Hamiltonian, viz. $\hat{K} = \frac{1}{2m_1}\hat{\mathbb{p}}_1^2 + \frac{1}{2m_2}\hat{\mathbb{p}}_2^2$, can be put in the form: $\underline{\underline{\hat{K} = -(\hbar^2/2M)\nabla_R^2 - (\hbar^2/2\mu)\nabla_r^2}}$, w/ $M = (m_1+m_2)$ & $\mu = m_1 m_2/(m_1+m_2)$.

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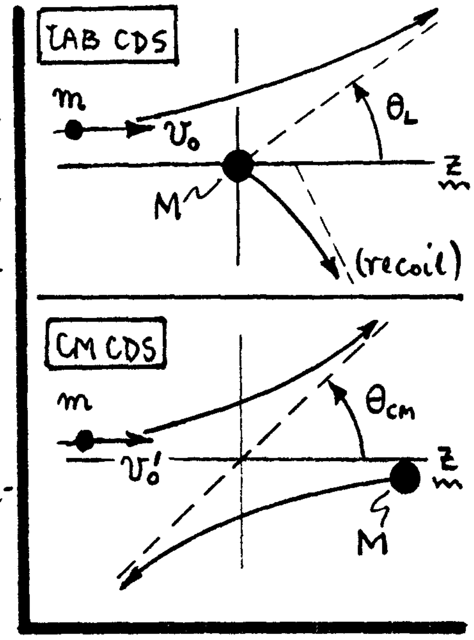
(18) [15 pts]. Most 2-body scattering events [w/ m (projectile) incident on M (target)] are described in terms of the scattering ∡ θ_{cm} in the <u>center-of-mass</u> (CM) <u>system</u>. When m/M is finite, θ_{cm} is generally $\neq \theta_L$, the actual scattering ∡ of m in the <u>lab</u> (L) <u>system</u>, because of M's recoil. Here we wish to relate θ_L to θ_{cm} for a classical <u>elastic</u> scattering event. Assume that M is initially at rest on the z-axis in lab, and m is incident at velocity $v_0 \parallel$ z-axis. Assume axial symmetry.



(A) After finding the CM velocity w.r.t. lab, and m's final velocity in CM (for an elastic event), show that: $\boxed{\tan\theta_L = \sin\theta_{cm}/[\cos\theta_{cm} + (m/M)]}$, is the required relation. Evidently $\theta_L \simeq \theta_{cm}$ when $m \ll M$. What is the relation when $m = M$?

(B) If $\frac{d\sigma}{d\Omega}$ is the differential scattering cross-section (# particles m scattered into solid ∡ $d\Omega = 2\pi\sin\theta\, d\theta$, per $d\Omega$), show: $\underline{\underline{(d\sigma/d\Omega)_L = (d\sigma/d\Omega)_{cm} \cdot \frac{d\cos\theta_{cm}}{d\cos\theta_L}}}$. What does this relation reduce to when $m = M$? What is the maxm. θ_L in this case?

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(19) [15 pts]. Use the Born approx^n to find the total cross section for an elastic scattering by a spherical well: $\underline{V(r) = (-)V_0,\ r<a;\ V(r) \equiv 0,\ r>a}$. <u>NOTE</u>: it is handy to verify and use Eq. (31), p. ScT 13, of class notes -- following from Eq. (14), p. ScT 7, for elastic & spherically symmetric events. Find limiting forms for $\sigma(k)$ for low energies [$ka \ll 1$] and high energies [$ka \gg 1$].

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(20) In prob$^{\underline{m}}$ (19), the Born Approxn (BA) provided cross-sections for scattering from a spherical well: $\underline{V(r) = (-)V_0,\ r<a;\ V(r) \equiv 0,\ r>a}$. Evaluate the validity of the BA in this case, per class notes p. ScT 10, Eq. (22). Show that the BA can hold down to ~ zero incident energy, if the well is shallow enough. Discuss the shallowness condition on V_0 w.r.t. formation of possible bound states in the well.

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(21) [15 pts.]. Using the Born Approxn, find both the differential and total scattering cross-sections for the central potentials: (A) $\underline{V(r) = V_0 e^{-\alpha r}}$, (B) $\underline{V(r) = V_0 e^{-\alpha^2 r^2}}$, w/ α & V_0 = cnst. Now, with the range parameter α held the same for each $V(r)$, adjust the amplitudes V_0 so that each potential has the same "volume", i.e. so that: $\underline{\int_0^\infty V(r)\cdot 4\pi r^2 dr = \Lambda = \text{cnst}}$. Finally, with this adjustment, intercompare and comment on the results for the $V(r)$'s in (A) & (B).

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(22) [15 pts]. Consider the scattering of an electron from a stationary charge distribution $\rho(\mathbf{r})$ which generates a potential ϕ per Poisson's eqtn: $\nabla^2\phi = -4\pi\rho$.

(A) The scattering potential is $V = -e\phi$. Show that in Born Approx$^{\underline{n}}$, the differential cross-section is: $\boxed{\frac{d\sigma}{d\Omega} = \left|\left(\frac{2me}{\hbar^2 q^2}\right)\int \rho(\mathbf{r})\, e^{i\mathbf{q}\cdot\mathbf{r}}\, d^3x\right|^2}$ w/ $\mathbf{q} = \mathbf{k}_{\text{before}} - \mathbf{k}_{\text{after}}$.

(B) Let $\rho(\mathbf{r})$ be due to an atomic ion w/ nucleus of charge Ze and N electrons distributed per their wavefcns ψ_k, i.e. $\underline{\rho(\mathbf{r}) = Ze\,\delta(\mathbf{r}) - e\sum_{k=1}^{N}|\psi_k(r)|^2}$. The atom is randomly oriented, so only the radial dependence of ψ_k is kept; the norm is $\int_0^\infty |\psi_k(r)\cdot 4\pi r^2 dr = 1$. Show that $d\sigma/d\Omega$ of part (A) can be written: $\boxed{d\sigma/d\Omega = (4/a_0^2 q^4)\,|Z - F(q)|^2}$, w/ $a_0 = \hbar^2/me^2$. $F(q)$ is the "form factor" for the atomic electrons. Find $F(q)$ and reduce it to a radial integral.

(C) Evaluate $F(q)$ for the single electron in the ground state of the H-atom [i.e. for $\underline{\psi(r) = (1/\sqrt{\pi a_0^3})\, e^{-r/a_0}}$]. Then, write down the cross-section $(d\sigma/d\Omega)$, and compare it with Sakurai's result ["Modern QM" (Addison Wesley, 1985), p. 448].

(23) In problem (22), you showed that for an electron scattering from a charge distribution $\rho(\mathbf{r})$, the transform of the scattering potential important for the Born Approx^n was: $\tilde{V}(\mathbf{q}) = -(4\pi e/q^2)\int \rho(\mathbf{r})\, e^{i\mathbf{q}\cdot\mathbf{r}}\, d^3x$, w/ $\mathbf{q} = \mathbf{k}(\text{before}) - \mathbf{k}(\text{after})$, the momentum transfer.

[10 pts.] (A) Put: $\rho(\mathbf{r}) = e\,\delta(\mathbf{r}) - e|\psi(r)|^2$, for e-scattering from a neutral H-atom, with the bound electron in a spherically symmetric eigenstate $\psi(r)$. By inverting the transform $\tilde{V}(\mathbf{q})$, first show that the actual scattering potential can be written as: $V(\mathbf{r}) = -e^2\left[\frac{1}{r} - \int \frac{d^3x'}{|\mathbf{r}-\mathbf{r}'|}|\psi(r')|^2\right]$. Interpret. Then, for the H-atom ground state: $\psi(r') = (1/\sqrt{\pi a_0^3})\, e^{-r'/a_0}$, w/ $a_0 = \hbar^2/me^2$, find $V(\mathbf{r})$. Show: $V(r) = -\frac{e^2}{a_0}\left(1+\frac{1}{\rho}\right)e^{-2\rho}$, w/ $\rho = r/a_0$. HINT: use the expansion for $1/|\mathbf{r}-\mathbf{r}'|$ per Jackson's Eq. (3.38).

[10 pts.] (B) For the V(r) in part (A), evaluate the Born Approx^n "validity criterion" (see class notes: Eq. (22), p. ScT 10). It is convenient to use the dimensionless energy parameter $\lambda = k^2 a_0^2 = E/E_H$ (E = incident electron energy, $E_H = e^2/2a_0$ = H-atom ionization). Show that the Born Approx^n fails at low energies, $\lambda \to 0$. Estimate a lower bound for λ, above which the Born Approx^n ~ OK.

[10 pts.] (C) Assume Sakurai's version of the differential cross-section for e→ H-atom scattering (as quoted in prob. (22)) is correct: $\frac{d\sigma}{d\Omega} = (4a_0^2/Q^4)\left[1 - 16/(Q^2+4)^2\right]^2$, w/ $Q = qa_0$, & $q = 2k\sin\frac{\theta}{2}$, θ = scattering ∡. From this, find the total cross-section $\sigma(\lambda)$. [HINT: develop and use the relation: $d\Omega = (2\pi/k^2a_0^2)\,Q\,dQ$]. Find limiting forms for σ for low & high energies. Compare with: $\sigma(\lambda\to 0) = (30\pm 5)\pi a_0^2$, measured; $\sigma(\lambda \gg 1) \simeq 7\pi/3k^2$, per Landau & Lifshitz "QM" (Addison Wesley, 2nd ed. 1965), p. 535. Comments?

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(24) Consider scattering of an electron from a screened Coulomb potential: $V(r) = -(Ze^2/r)\,e^{-\alpha r}$, by means of partial wave analysis. Using Eq. (32), p. PW 9 of class notes, show that the l^{th} partial wave phase shift $\delta_l(k)$ is given by: $\tan\delta_l(k) = \frac{Z}{ka_0}\,Q_l\left(1+\frac{\alpha^2}{2k^2}\right)$, w/ $a_0 = \hbar^2/me^2$, $k = \sqrt{2mE/\hbar^2}$, and $Q_l(z)$ = Legendre fcn of 2nd kind. Write $\tan\delta_0(k)$ explicitly, and find its limit for $k\to\infty$ & $\alpha > 0$. What happens to the analysis when $\alpha \to 0$?

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(25) [15 pts.] Phase shift analysis for hard-core scattering (class notes, pp. PW 10-12) requires knowing the radial wavefcn logarithmic derivative $R'_{k\ell}(a)/R_{k\ell}(a)$ at the cutoff $r=a$ of the scattering potential. Consider the dimensionless logarithmic derivative: $\beta_\ell(k) = a\,R'_{k\ell}(a)/R_{k\ell}(a)$. We wish to find how $\beta_\ell(k)$ depends on energy.

(A) For a given ∡ momentum ℓ, consider two closely spaced energies: E_1 & $E_2 = E_1 + \Delta E$. If $R_1 = R_{k_1\ell}(r)$ & $R_2 = R_{k_2\ell}(r)$ are the corresponding interior radial wavefcns, show that: $\frac{d}{dr}\left[r^2(R_1R_2' - R_2R_1')\right] + \frac{2m}{\hbar^2}(E_2-E_1)r^2R_1R_2 = 0$. [HINT: write the radial eqtns for R_1 & R_2 from Eq.(26), p. PW 8. Recall: $k^2 = 2mE/\hbar^2$. Then, think Green].

(B) Integrate the identity in part (A) over $0 \leq r \leq a$ to find an expression for $\Delta\beta_\ell/\Delta E = [\beta_\ell(k_2) - \beta_\ell(k_1)]/(E_2 - E_1)$. Pass to the limit $k_2 \to k_1 = k$ to derive the expression: $d\beta_\ell(k)/dE = -(2m/\hbar^2 a)\int_0^a [rR_{k\ell}(r)/R_{k\ell}(a)]^2 dr$. So, how does $\beta_\ell(k)$ vary w/ E?

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(26) (A) Find the exchange splitting of an energy level in a system of two electrons, by regarding the e-e interaction $V(\mathbf{r}_1 - \mathbf{r}_2)$ as a perturbation on the main electron binding terms. Use appropriate symmetrized wavefcns for the electrons.

(B) Show that the exchange-dependent terms in part (A) can be represented w.r.t. non-symmetrized electron spin states (i.e. product states) as eigenvalues of the exchange operator: $V_{ex} = -\frac{K}{2}(1 + 4\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)$. Here, K is the exchange integral from part (A), and $\boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_2$ are the (dimensionless) spin operators for electrons # 1 & 2.

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(27) [15 pts.]. Employ $a_0 = \frac{\hbar^2}{me^2}$ (Bohr radius) as a length unit, and $E_0 = \frac{e^2}{a_0}$ (2× H atom ionization) as an energy unit. Using the Thomas-Fermi model, estimate the average size of the following quantities [what's interesting is the scaling with Z]: (A) distance of an electron from the nucleus, (B) Coulomb interaction energy between two electrons, (C) kinetic energy of an electron, (D) energy needed to ionize the atom completely, (E) velocity of an electron in the atom, (F) angular momentum of an electron, (G) radial quantum number of an electron.

(28) [15 pts]. In the Thomas-Fermi atom, the total e-e repulsion energy can be written: $E_{ee} = \frac{1}{2}\int U_e(r)\,n(r)\,dV$, w/ $U_e(r)$ = potential energy of one e in the presence of the the other e's, and $n(r)$ = number density of electron states [n replaces the ρ in Eq. (39), p. ip 17, class notes]. In turn: $U_e(r) = U(r) + \frac{Ze^2}{r}$, w/ $U(r)$ the full T-F potential [$U(r)$ is (-) ve]; the nuclear attraction $(-Ze^2/r)$ has been subtracted out.

(A) Show that: $E_{ee} = -\frac{1}{2}E_{en} - \frac{5}{6}K_e$, w/ E_{en} is the total electron-nuclear binding energy, and K_e is the total electronic kinetic energy.

(B) Use the Virial Theorem for the Coulomb field to show: $\boxed{E_{ee} = -\frac{1}{7}E_{en}}$. Thus, in the T-F atom, the e-e repulsion is relatively small compared to the binding.

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(29) [25 pts]. In the ground state of a two-electron atom, both e's are in 1s orbitals (w/ one spin up and the other down ⇒ spin singlet). If we neglect magnetic interactions, the system Hamiltonian is: $\mathcal{H} = -\frac{\hbar^2}{2m}(\nabla_1^2 + \nabla_2^2) - Ze^2\left(\frac{1}{r_1} + \frac{1}{r_2}\right) + e^2/r_{12}$, w/ Z the nuclear charge and r_{12} the relative cd. If we neglect the e-e repulsion term, the problem reduces to each e moving separately in a Coulomb field $-Ze^2/r$; then for each e the energy is $-\frac{1}{2}Z^2e^2/a_0$ ($a_0 = \hbar^2/me^2$ = Bohr radius), and $\phi(r) = \frac{1}{\sqrt{\pi}}(Z/a_0)^{\frac{3}{2}}e^{-Zr/a_0}$ is its wavefunction. The overall system energy & wavefcn would be: $E_0 = -Z^2e^2/a_0$ & $\Psi_0(1,2) = \phi(r_1)\phi(r_2)$. Here we want to improve on this zeroth order approximation by a variational calculation which accounts for the e-e repulsion term.

(A) Use a trial wavefcn of form: $\Psi(1,2) = \mathcal{N}\exp\left[-\frac{\zeta}{a_0}(r_1 + r_2)\right]$, w/ ζ a variational parameter. Adjust the norm cnst $\mathcal{N}$ so that: $\int d^3x_1 \int d^3x_2 \,|\Psi(1,2)|^2 = 1$.

(B) Calculate the energy: $E(\zeta) = \langle\Psi|\mathcal{H}|\Psi\rangle$. The K.E. & P.E. integrals here are easy. Deal with the e-e repulsion term by expanding $1/r_{12}$ via the Spherical Harmonic Addition Theorem [e.g. Jackson, Sec. 3.6]. Show: $\int d^3x_1 \int d^3x_2 \,|\Psi|^2/r_{12} = \frac{5}{8}\zeta/a_0$.

(C) Minimize $E(\zeta)$ w.r.t. ζ and find E_{min}. Approximate the first ionization potential of the atom by: $I(Z) = -E_{min} - \frac{1}{2}Z^2e^2/a_0$. Calculate $I(Z)$ in Rydbergs (i.e. energy units of $e^2/2a_0 = 13.6$ eV), and compare with the experimental values given in the box.

I(Z) (Ryd.)	H^-	He	Li^+	Be^{2+}	B^{3+}
	0.055	1.807	5.559	11.311	19.063

(D) Do your $I(Z)$'s agree w/ R.T. Robiscoe, Am. J. Phys. 43, 538 (1975)? Why not?

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(37) [15 pts]. The Dirac gamma matrices obey: $\underline{\{\gamma_\mu,\gamma_\nu\} = \gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2\delta_{\mu\nu}}$, w/ μ & $\nu = 1,2,3,4$. Assume that the γ's are Hermitian, and show that their eigenvalues are at most ± 1. Next, show that each γ_μ has zero trace ($\mathrm{Tr}\,\gamma_\mu = 0$)... what does this imply about <u>(a)</u> the number of $+1$ vs. -1 eigenvalues? <u>(b)</u> the allowed rank of the γ's? What is the <u>lowest</u> rank of a set of four γ's which satisfy $\{\gamma_\mu,\gamma_\nu\} = 2\delta_{\mu\nu}$?

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(38) In an external EM field with 4-potential $A_\mu = (\mathbb{A}, i\phi)$, the Dirac Eqtn for a particle of charge q & mass m is: $\underline{[\partial/\partial x_\mu - i(q/\hbar c)A_\mu]\gamma_\mu\psi + (mc/\hbar)\psi = 0}$. Show that for the adjoint wavefcn $\bar\psi = \psi^\dagger\gamma_4$, the eqtn is: $\underline{[\partial/\partial x_\mu + i(q/\hbar c)A_\mu]\bar\psi\gamma_\mu - (mc/\hbar)\bar\psi = 0}$. Now, in the usual fashion, derive a continuity equation $\partial J_\mu/\partial x_\mu = 0$, for a Dirac probability 4-current in the presence of A_μ (HINT: multiply the adjoint eqtn on the right by ψ, the original eqtn on the left by $\bar\psi$, and combine). How does the present form for J_μ compare with the current for a free particle?

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(39) Start from the Dirac Eqtn in an external EM potential A_μ as stated in problem #38. Confirm that the eqtn can be written in Hamiltonian form as:

$$\underline{\underline{i\hbar\,\partial\psi/\partial t = \mathcal{H}\psi}}, \quad \text{w/} \quad \underline{\underline{\mathcal{H}(q,\mathbb{p}) = q\phi + \beta mc^2 + c\boldsymbol{\alpha}\cdot(\mathbb{p} - \tfrac{q}{c}\mathbb{A})}}.$$

By treating $\mathbb{p}$ as a real eigenvalue, show that if ψ satisfies the eqtn for operator $\mathcal{H}(+q,+\mathbb{p})$, then the charge conjugate wavefcn $\psi_c = \gamma_2\psi^*$ satisfies the eqtn for operator $\mathcal{H}(-q,-\mathbb{p})$. Thus, charge conjugation reverses the sign of q <u>and</u> $\mathbb{p}$.

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(40) [15 pts]. For a free Dirac particle, we have suggested that $v_k = c\alpha_k$ is the velocity operator. If so, we could define a momentum operator by: $\underline{p_k = mv_k = imc\beta\gamma_k}$ (we've used $\alpha_k = i\beta\gamma_k$). The value of p_k in state ψ is then the expectation value of $imc\bar\psi\gamma_k\psi$, w/ $\bar\psi = \psi^\dagger\beta$. Calculate this quantity for the charge conjugate wavefcn $\psi_c = \gamma_2\psi^*$. Can you show: $imc\bar\psi_c\gamma_k\psi_c = (-)\,imc\bar\psi\gamma_k\psi$, as one would expect for this quantity? Comment on: <u>(a)</u> what works for a definition of $\langle p_k\rangle$? <u>(b)</u> can $v_k = c\alpha_k$ be a constant of the motion for a free particle?
