Helmholtz' Theorem*

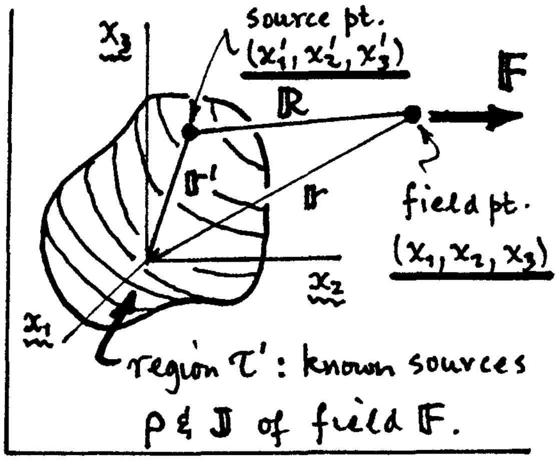
1) This theorem from vector calculus will show how:

A. a vector field $\mathbb{F}(x_i)$ is locally related to its sources $\left\{\begin{array}{l}\text{charge density } \rho(x_i),\\ \text{current} \;\;\text{"}\;\; \mathbb{J}(x_i);\end{array}\right.$

B. the sources ρ & $\mathbb{J}$ generate global potential fcns $\left\{\begin{array}{l}\text{scalar potential } \phi(x_i) \leftrightarrow \rho,\\ \text{vector} \;\;\text{"}\;\; \mathbb{A}(x_i) \leftrightarrow \mathbb{J};\end{array}\right.$

C. ϕ & $\mathbb{A}$ specify $\mathbb{F}$ globally; so we go from local to global rep$^{\underline{n}}$ of $\mathbb{F}$ via ρ & $\mathbb{J}$.†

Utility: Maxwell's Eqs. in fact locally (differentially) relate the electric & magnetic fields $\mathbb{E}$ & $\mathbb{B}$ to electric sources ρ & $\mathbb{J}$. We would like global (integral) rep$^{\underline{s}}$s of $\mathbb{E}$ & $\mathbb{B}$ in terms of ρ & $\mathbb{J}$. Since we will not consider time-dependence here, our rep$^{\underline{s}}$s (solutions) will be useful for electro- & magnetostatics.



THEOREM "Let $\mathbb{F}(x_i)$ be a vector field. Suppose in some region τ' of space the sources generating $\mathbb{F}$ are known, i.e. in τ' there is a "charge" density $\rho(x_i')$ and "current" density $\mathbb{J}(x_i')$ such that

$$\rightarrow \nabla\cdot\mathbb{F} = \rho(x_i), \quad \nabla\times\mathbb{F} = \mathbb{J}(x_i) \Big\} \text{for } x_i = x_i' \text{ in region } \tau'. \qquad (1)$$

Then a general solution to these differential eqtns for $\mathbb{F}$ can be written everywhere in space in terms of potentials ϕ & $\mathbb{A}$ as follows:

$$\mathbb{F}(x_i) = -\nabla\phi(x_i) + \nabla\times\mathbb{A}(x_i),$$

$$\text{w/ } \underline{\text{scalar potential}}\Big\}\; \phi(x_i) = \frac{1}{4\pi}\int_{\text{sources}} \frac{d\tau'}{R}\rho(x_i'), \quad \underline{\text{vector potential}}\Big\}\; \mathbb{A}(x_i) = \frac{1}{4\pi}\int_{\text{sources}} \frac{d\tau'}{R}\mathbb{J}(x_i'),$$

$$\text{where: } R = \Big[\sum_{i=1}^{3}(x_i - x_i')^2\Big]^{1/2} \Big\} \text{distance between field pt. } (x_i) \text{ \& source pt. } (x_i'). \qquad (2)$$

The integrals are over the entire source region τ' where ρ & $\mathbb{J}$ are non-zero. The source densities ρ & $\mathbb{J}$ are assumed to vanish at ∞."

† $(x_i) = (x_1, x_2, x_3) = (x, y, z)$ in 3D space; local $\leftrightarrow$ differential; global $\leftrightarrow$ integral.

* Ref: G. Arfken "Math. Methods for Physicists" (Academic, 3rd ed., 1985), pp. 78-83.

Helmholtz (cont'd)

REMARKS

1. Solution for $\mathbb{F}$ is general, but <u>not</u> unique. If ψ is a scalar field, and

$\longrightarrow \phi \to \phi + \psi$, so// $\mathbb{F} \to [-\nabla\phi + \nabla\times\mathbb{A}] + \nabla\psi$,

then//
$$\nabla\cdot\mathbb{F} \to [-\nabla^2\phi + \underbrace{\cancel{\nabla\cdot(\nabla\times\mathbb{A})}}_{\text{div curl}=0}] + \nabla^2\psi$$

$$\nabla\times\mathbb{F} \to [-\underbrace{\cancel{\nabla\times(\nabla\phi)}}_{\text{curl grad}=0} + \nabla\times(\nabla\times\mathbb{A})] + \underbrace{\cancel{\nabla\times(\nabla\psi)}}_{\text{ditto}}$$

$\mathbb{F}$ is unchanged by $\phi \to \phi + \psi$ so long as $\nabla^2\psi = 0$ (Laplace Equation) (3)

So $\mathbb{F}$ is "uniquely" defined by ϕ & $\mathbb{A}$ up to $\nabla\psi$, where $\nabla^2\psi = 0$.

2. But, a <u>given</u> ρ & $\mathbb{J}$, which vanish at ∞, specify $\mathbb{F}$ uniquely.

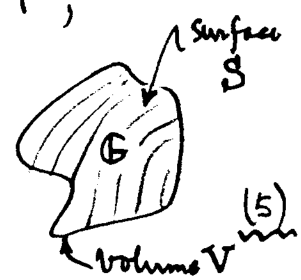
Suppose solns $\mathbb{F}_1$ & $\mathbb{F}_2$ result from given ρ & $\mathbb{J}$ ‖ $\nabla\cdot\mathbb{F}_i = \rho$, $\nabla\times\mathbb{F}_i = \mathbb{J}$ $(i=1,2) \Rightarrow \mathbb{G} = \mathbb{F}_1 - \mathbb{F}_2$ obeys $\left\{\boxed{\nabla\cdot\mathbb{G} = 0,\ \nabla\times\mathbb{G} = 0.}\right.$ (4)

We'll now show that $\mathbb{G} \equiv 0$, so $\mathbb{F}_2 \equiv \mathbb{F}_1$, and soln for $\mathbb{F}$ is unique.

Since $\nabla\times\mathbb{G} = 0$, then $\mathbb{G} = \nabla\Gamma$ [and $\nabla\times(\nabla\Gamma) = 0$, automatic]. The first of Eqs. (4) then requires $\nabla^2\Gamma = 0$ for the scalar fcn Γ. By vector identity:

$$\nabla\cdot(\Gamma\nabla\Gamma) = (\nabla\Gamma)\cdot(\nabla\Gamma) + \Gamma\nabla\cdot(\nabla\Gamma) = |\mathbb{G}|^2 + \Gamma\cancelto{0}{\nabla^2\Gamma},$$

and//
$$\int_V \nabla\cdot(\Gamma\nabla\Gamma)\,dV \overset{\text{Divergence Thm}}{=} \oint_S (\Gamma\nabla\Gamma)\cdot d\mathbb{S} = \int_V |\mathbb{G}|^2\,dV$$



So// $\longrightarrow \int_V |\mathbb{G}|^2\,dV = \oint_S \Gamma\,\mathbb{G}\cdot d\mathbb{S}$, Surface S enclosing vol. V. (5)

Let $S \to \infty$. There, ρ & $\mathbb{J}$ vanish, and so must the $\mathbb{F}_i$. Then $\mathbb{G} \to 0$ as $S \to \infty$.

So//
$$\int_{\text{all space}} |\mathbb{G}|^2\,dV = \oint_{S\to\infty} \Gamma\,\mathbb{G}\cdot d\mathbb{S} = 0, \quad \text{i.e.}// \int_{\text{all space}} |\mathbb{G}|^2\,dV = 0. \qquad (6)$$

But $|\mathbb{G}|^2 \geq 0$ (+ve definite), so $\int_{\text{all space}} |\mathbb{G}|^2\,dV = 0$ must $\Rightarrow |\mathbb{G}|^2 \equiv 0$. Then $\mathbb{G} = \mathbb{F}_1 - \mathbb{F}_2 = 0$, everywhere, and $\mathbb{F}_2 = \mathbb{F}_1$ is unique for the given ρ & $\mathbb{J}$. <u>QED.</u>

SUMMARY: Given ϕ & $\mathbb{A} \Rightarrow \mathbb{F}$ unique up to $\nabla\psi$; given ρ & $\mathbb{J} \Rightarrow \mathbb{F}$ <u>is</u> unique.

2) **PROOF** Need to show, for $\mathbb{F} = -\nabla\phi + \nabla\times\mathbb{A}$, that $\left\{\begin{array}{l}\nabla\cdot\mathbb{F} = \rho \\ \nabla\times\mathbb{F} = \mathbb{J}\end{array}\right\rangle$ for ϕ & $\mathbb{A}$, Eq. (2)

$$\nabla\cdot\mathbb{F} = -\nabla\cdot(\nabla\phi) + \cancel{\nabla\cdot(\nabla\times\mathbb{A})}^{\,0,\ \text{div. curl} \equiv 0} = -\nabla^2\phi\,;$$

So/ must show $\nabla^2\phi = -\rho$, for scalar potential in Eq. (2). (7)

Calculate: $\nabla^2\phi = \frac{1}{4\pi}\,\nabla^2_{\text{fld}}\int_{\text{sources}} \frac{d\tau'}{R}\,\rho(x_i') = \frac{1}{4\pi}\int_{\text{sources}} d\tau'\,\rho(x_i')\,\nabla^2_{\text{fld}}\left(\frac{1}{R}\right)$. (8)

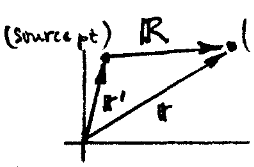
operates on field pts x_i, not source pts x_i'.

Here $R = [\sum_i (x_i - x_i')^2]^{1/2}$, as cited. The operation $\nabla^2_{\text{fld}}(1/R)$ is magical, as:

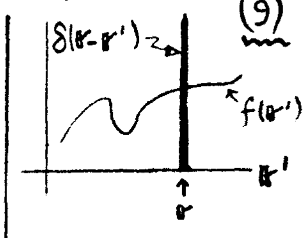
$\rightarrow \nabla^2_{\text{fld}}\left(\frac{1}{R}\right) = -4\pi\,\delta(\mathbb{R})$

$\delta(\mathbb{R})$, called 3D Dirac delta fcn, has properties: (9)

1. $\delta(\mathbb{R}) \equiv 0$, when $\mathbb{r}' \neq \mathbb{r}$;
2. $\delta(\mathbb{R}) \to \infty$, when $\mathbb{r}' \to \mathbb{r}$;
3. $\int_{\text{all space}} f(\mathbb{r}')\,\delta(\mathbb{r}-\mathbb{r}')\,d^3x_i' = f(\mathbb{r})$.



$\mathbb{R} = \mathbb{r} - \mathbb{r}'$



$\delta(\mathbb{r}-\mathbb{r}')$ is a fcnal "projection operator" ... it projects out $f(\mathbb{r}')$ value at $\mathbb{r}' = \mathbb{r}$.

If this δ-fcn assignment is true, then from (8) we have ...

$$\nabla^2_{\text{fld}}\phi = \frac{1}{\cancel{4\pi}}\int_{\text{all sources}} d\tau'\,\rho(x_i')\,[-\cancel{4\pi}\,\delta(\mathbb{R})] = -\rho(x_i),$$

So/ $\rightarrow \boxed{\nabla_{\text{fld}}\cdot\mathbb{F}(x_i) = -\nabla^2_{\text{fld}}\phi(x_i) = +\rho(x_i)}$, as advertised. QED. (10)

ASIDE Show that $\nabla^2_{\text{fld}}(1/R) = -4\pi\,\delta(\mathbb{R})$, per Eq. (9).

With $R = [\sum_i (x_i - x_i')^2]^{1/2}$, straightforwardly calculate ...

$$\nabla^2(1/R) = \nabla\cdot[\nabla(1/R)] = -\nabla\cdot(\mathbb{R}/R^3) = -\sum_i \frac{\partial}{\partial x_i}\left(\frac{x_i - x_i'}{R^3}\right)$$

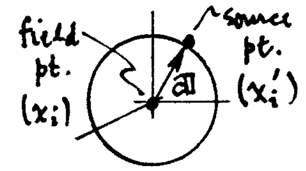
$$= \sum_i \left[\frac{3}{R^5}(x_i - x_i')^2 - \frac{1}{R^3}\right]. \qquad (11)$$

Each term here $\to \infty$ as $R \to 0$ $\left[\begin{array}{l}\text{source pt. } (x_i') \\ \text{field pt } (x_i)\end{array}\right\rangle$ coalesce$\Big]$. But when $R \neq 0$ ...

$$\nabla^2(1/R) = \frac{3}{R^5}\left[\sum_i (x_i - x_i')^2\right] - \sum_i \frac{1}{R^3} = \frac{3}{R^5}\cdot R^2 - 3\times\frac{1}{R^3} = 0, \quad R \neq 0$$

So/ $\rightarrow \nabla^2(1/R) = \left\{\begin{array}{l} 0, \text{ when } R \neq 0 \ (\mathbb{r} \neq \mathbb{r}') \\ \infty, \text{ when } R = 0 \ (\mathbb{r} = \mathbb{r}') \end{array}\right\rangle$ 1st two properties for $\delta(\mathbb{r}-\mathbb{r}')$. (12)

This shows that in an integral like Eq. (8) for $\nabla^2\phi$, containing $\nabla^2(1/R)$, we need only consider integrating over an ∞smal neighborhood about $R\to 0$ [i.e. when source (x_i') & field (x_i) pts coalesce]. Choose this to be a small sphere of radius $\mathbf{a}\to 0$ about (x_i). Then:



∞smal spherical vol. V, radius $\mathbf{a}$ about (x_i); V encl. by surface σ.

$$\rightarrow \nabla^2_{fld}\,\phi(x_i) = \frac{1}{4\pi}\int_{sources} d\tau'\,\rho(x_i')\left[\nabla^2_{fld}(1/R)\right] \quad \text{use: } \nabla\left(\tfrac{1}{R}\right) = -\frac{\mathbb{R}}{R^3}, \text{ put } \rho(x_i')\to\rho(x_i);$$

$$= -\frac{1}{4\pi}\rho(x_i)\int_V d\tau'\left[\nabla_{source}\cdot(\mathbb{R}/R^3)\right] \quad \text{use Divergence Thm: } \int_V \to \oint_\sigma;$$

$$= -\frac{1}{4\pi}\rho(x_i)\oint(\mathbf{a}/a^3)\cdot d\sigma \quad \text{on sphere } \sigma: \mathbf{a} = a\hat{a}, \text{ and } a = \text{cnst},$$

$$\ldots \text{and: } d\sigma = \hat{a}\,a^2\sin\theta\,d\theta\,d\phi;$$

$$= -\frac{1}{4\pi}\rho(x_i)\underbrace{\oint_\sigma \sin\theta\,d\theta\,d\phi}_{4\pi} \tag{13}$$

$$\boxed{\nabla^2_{fld}\,\phi(x_i) = -\rho(x_i)} \quad \begin{cases} \text{Justifies Eq. (10): } \nabla_{fld}\cdot\mathbb{F}(x_i) = \rho(x_i); \\ \text{Shows: } \nabla^2\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right) = -4\pi\,\delta(\mathbf{r}-\mathbf{r}'). \end{cases} \tag{14}$$

END ASIDE

3) Have shown that for $\mathbb{F} = -\nabla\phi + \nabla\times\mathbb{A}$, indeed $\nabla\cdot\mathbb{F} = \rho$, for $\phi = \frac{1}{4\pi}\int\frac{d\tau'}{R}\rho$. Proof of Helmholtz Thm complete if we can show $\nabla\times\mathbb{F} = \mathbb{J}$, for $\mathbb{A}$ as defined in (2).

$$\rightarrow \nabla\times\mathbb{F} = -\cancelto{0,\ \text{curl grad}\equiv 0}{\nabla\times(\nabla\phi)} + \underbrace{\nabla\times(\nabla\times\mathbb{A})}_{\text{identity}} = \underbrace{\nabla(\nabla\cdot\mathbb{A}) - \nabla^2\mathbb{A}}_{} \tag{15}$$

With: $\mathbb{A}(x_i) = \frac{1}{4\pi}\int_{sources}\frac{d\tau'}{R}\mathbb{J}(x_i')$, and $\nabla = \nabla_{fld}$ operating only on fld pts (x_i):

$$\nabla\times\mathbb{F} = \frac{1}{4\pi}\int_{sources}(\mathbb{J}\cdot\nabla)\nabla\left(\frac{1}{R}\right)d\tau' - \frac{1}{4\pi}\int_{sources}\mathbb{J}\underbrace{\left[\nabla^2(1/R)\right]}_{=-4\pi\delta(\mathbf{r}-\mathbf{r}')}d\tau'$$

this integral gives just $\mathbb{J}(\mathbf{r})$

i.e.

$$\nabla_{fld}\times\mathbb{F}(x_i) = \mathbb{I} + \mathbb{J}(x_i),$$

w/

$$\mathbb{I} = \frac{1}{4\pi}\int_{sources} d\tau'\,(\mathbb{J}\cdot\nabla)\nabla\left(\frac{1}{R}\right). \tag{16}$$

The proof is complete if we can show that integral $\mathbb{I} = 0$.

4) To show $\mathbb{I} = 0$, first note that for $R = [\sum_i (x_i - x_i')^2]^{1/2}$...

$$\left[\begin{array}{l} \nabla = (\partial/\partial x_i), \text{ field pt. del} \Rightarrow \nabla(1/R) = -\mathbb{R}/R^3 \\ \nabla' = (\partial/\partial x_i'), \text{ source pt. del} \Rightarrow \nabla'(1/R) = +\mathbb{R}/R^3 \end{array}\right\} \text{so } \nabla = (-)\nabla', \text{ w.r.t. } \frac{1}{R};$$

then $\mathbb{I} = \int d\tau' (\mathbb{J}\cdot\nabla)\nabla\left(\frac{1}{R}\right) = +\int d\tau' (\mathbb{J}\cdot\nabla')\nabla'\left(\frac{1}{R}\right)$,

$$\xrightarrow{\text{or}} I_k = \int d\tau' \left[\left(\sum_i J_i \frac{\partial}{\partial x_i'}\right)\frac{\partial}{\partial x_k'}\left(\frac{1}{R}\right)\right], \text{ for } k^{\text{th}} \text{ component.} \quad (17)$$

The messy [ ] in (17) can be written as a divergence, as follows...

$$\frac{\partial}{\partial x_i'}\left[J_i \frac{\partial}{\partial x_k'}\left(\frac{1}{R}\right)\right] = \cancelto{0}{\left(\frac{\partial J_i}{\partial x_i'}\right)}\frac{\partial}{\partial x_k'}\left(\frac{1}{R}\right) + \underbrace{\left[\left(J_i \frac{\partial}{\partial x_i'}\right)\frac{\partial}{\partial x_k'}\left(\frac{1}{R}\right)\right]}_{\text{this term is summand in Eq. (17)}}$$

... sum on i ... this $\Rightarrow \nabla'\cdot\mathbb{J} = 0$, since $\mathbb{J} = \nabla\times\mathbb{F}$.

$$\nabla'\cdot\left[\mathbb{J}\frac{\partial}{\partial x_k'}\left(\frac{1}{R}\right)\right] = 0 + \underbrace{\left[\sum_i \left(J_i \frac{\partial}{\partial x_i'}\right)\frac{\partial}{\partial x_k'}\left(\frac{1}{R}\right)\right]}_{\text{integrand in Eq. (17)}} \quad (18)$$

Put this result into Eq. (17), extend $\int d\tau'$ to all space, and use Div. Thm...

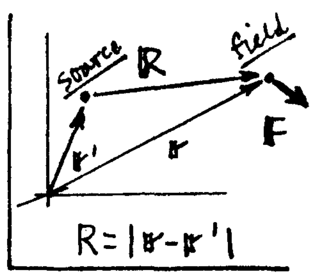
$$\rightarrow I_k = \int_{\text{all space}} d\tau' \, \nabla'\cdot\left[\mathbb{J}\frac{\partial}{\partial x_k'}\left(\frac{1}{R}\right)\right] \underset{\text{Div. Thm}}{=} \oint_{\sigma'\to\infty} d\sigma'\cdot\left[\mathbb{J}\frac{\partial}{\partial x_k'}\left(\frac{1}{R}\right)\right] \underset{\mathbb{J}\text{ vanishes at }\infty.}{=} 0. \quad (19)$$

So $\mathbb{I}$ in Eq. (16) vanishes, component-by-component, and we have shown:

$$\rightarrow \boxed{\nabla_{\text{fld}}\times\mathbb{F}(x_i) = \nabla_{\text{fld}}\times(\nabla_{\text{fld}}\times\mathbb{A}(x_i)) = \mathbb{J}(x_i)}, \text{ as advertised. } \underline{\text{QED.}} \quad (20)$$

We now have completed proof of Helmholtz' Theorem, viz:

$$\left[\begin{array}{l}\text{if } \nabla\cdot\mathbb{F} = \rho, \\ \nabla\times\mathbb{F} = \mathbb{J}\end{array}\right\} \text{then } \mathbb{F} = -\nabla\phi + \nabla\times\mathbb{A}, \quad \text{w/ } \begin{array}{l}\phi(\mathbf{r}) = \frac{1}{4\pi}\int\frac{d\tau'}{R}\rho(\mathbf{r}'), \\ \mathbb{A}(\mathbf{r}) = \frac{1}{4\pi}\int\frac{d\tau'}{R}\mathbb{J}(\mathbf{r}').\end{array}$$



PROVISOS: (1) ρ & $\mathbb{J}$ are indpt of time, and must vanish at ∞;
(2) ϕ is unique up to ψ, and $\mathbb{A}$ up to $\nabla\psi$, such that $\nabla^2\psi = 0$.