DEPARTMENT OF PHYSICS

1997 COMPREHENSIVE EXAM

Answer each of the following questions, all of which carry equal weight. Begin your answer to each question on a new sheet of paper; solutions to different questions must not appear on the same sheet. Each sheet of paper must be labeled with your name and the problem number in the upper right hand corner. When more than one sheet is submitted for a problem, be sure the pages are ordered properly.

PHYSICAL CONSTANTS

Quantity	Symbol	Value
acceleration due to gravity	g	$9.8\ \text{m s}^{-2}$
gravitational constant	G	$6.67 \times 10^{-11}\ \text{N m}^2\ \text{kg}^2$
permittivity of vacuum	ε_o	$8.85 \times 10^{-12}\ \text{C}^2\ \text{N}^{-1}\ \text{m}^{-2}$
permeability of vacuum	μ_o	$4\pi \times 10^{-7}\ \text{N A}^{-2}$
speed of light in vacuum	c	$3.00 \times 10^{8}\ \text{m s}^{-1}$
elementary charge	e	$1.602 \times 10^{-19}\ \text{C}$
mass of electron	m_e	$9.11 \times 10^{-31}\ \text{kg}$
mass of proton	m_p	$1.673 \times 10^{-27}\ \text{kg}$
Planck constant	h	$6.63 \times 10^{-34}\ \text{J s}$
Avogadro constant	N_A	$6.02 \times 10^{23}\ \text{mol}^{-1}$
Boltzmann constant	k	$1.38 \times 10^{-23}\ \text{J K}^{-1}$
molar gas constant	R	$8.31\ \text{J mol}^{-1}\ \text{K}^{-1}$
standard atmospheric pressure		$1.013 \times 10^{5}\ \text{Pa}$

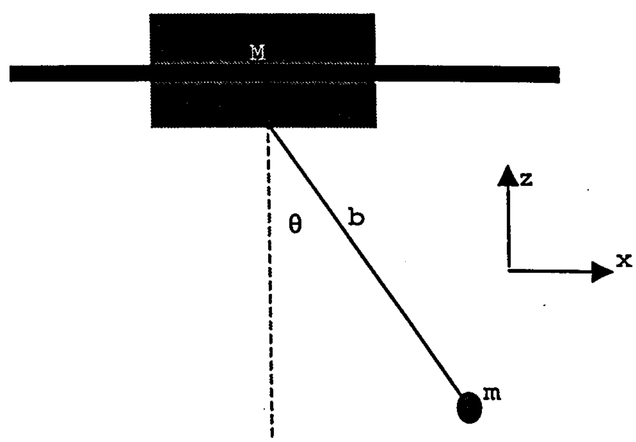
QUESTION #1

A block of mass M moves horizontally along a smooth rail without friction. A pendulum of length b and mass m hangs from the block. In addition to the motion along the rail, the pendulum can also move perpendicular to the rail.

a) Find the Lagrangian for the system for small angular displacements. (Keep only quadratic terms.)

b) What are the three characteristic frequencies for the system? (If you cannot do the math in part a or your answers don't seem reasonable, say so and make your best guesses at the frequencies.)

c) Describe the normal modes.



L KIRKPATRICK

#1

a) If you recognize that the motion in and out of the page is a normal mode, you realize that you can reduce the problem to a two-dimensional one.

Then $U = mgy = -mgb(1-\cos\theta) \approx \frac{1}{2} mgb\theta^2$

$$T = \tfrac{1}{2} M\dot{X}^2 + \tfrac{1}{2} m(\dot{x}^2 + \dot{y}^2)$$
$$= \tfrac{1}{2} M\dot{X}^2 + \tfrac{1}{2} m(\dot{x}^2 + 2b\dot{x}\dot{\theta}\cos\theta + b^2\dot{\theta}^2)$$
$$\approx \tfrac{1}{2}(M+m)\dot{X}^2 + \tfrac{1}{2} b^2\dot{\theta}^2 + mb\dot{x}\dot{\theta}$$

$$x = X + b\sin\theta$$
$$y = b - b\cos\theta$$
$$\dot{x} = \dot{X} + b\ddot{\theta}\cos\theta$$
$$\dot{y} = b\dot{\theta}\sin\theta$$

b) Therefore, the characteristic determinant is

$$\begin{vmatrix} -w^2(M+m) & -w^2 mb \\ -w^2 mb & mgb - w^2 mb^2 \end{vmatrix} = 0$$

with solutions $w_1 = 0$ $\quad w_2 = \sqrt{\frac{g}{b}\left(\frac{M+m}{M}\right)}$

$w_3 = \sqrt{\frac{g}{b}}$ for the motion in and out of the page

c) w_1 corresponds to a translation of the entire system w/ a constant velocity.

w_2 corresponds to the antisymmetric mode with the two masses oscillating out of phase.

w_3 corresponds to simple pendulum mode in and out of the page.

In three dimensions, we cannot use spherical coordinates because the coordinate must be zero at the equilibrium position. Let φ be the angle w/ the vertical measured normal to the page.

$$U = mgz \cong -mgb(1-\cos\theta + 1 - \cos\varphi)$$
$$\cong \tfrac{1}{2}mgb(\theta^2+\varphi^2)$$
$$T = \tfrac{1}{2}M\dot{X}^2 + \tfrac{1}{2}m(\dot{x}^2+\dot{y}^2+\dot{z}^2)$$

$$x = X + b\sin\theta \qquad \dot{x} = \dot{X} + b\dot{\theta}\cos\theta \approx \dot{X} + b\dot{\theta}$$
$$y = b\sin\varphi \qquad \dot{y} = b\dot{\varphi}\cos\varphi \approx b\dot{\varphi}$$
$$z = b(1-\cos\theta) + b(1-\cos\varphi) \qquad \dot{z} = -b\dot{\theta}\sin\theta - b\dot{\varphi}\sin\varphi \approx 0$$

$$\Rightarrow L \cong \tfrac{1}{2}\left[(M+m)\dot{X}^2 + 2mb\dot{X}\dot{\theta} + mb^2\dot{\theta}^2 + b^2\dot{\varphi}^2\right] - \tfrac{1}{2}mgb(\theta^2+\varphi^2)$$

$$\begin{vmatrix} -(M+m)w^2 & -mbw^2 & 0 \\ -mbw^2 & mgb - mb^2w^2 & 0 \\ 0 & 0 & mgb - mb^2w^2 \end{vmatrix} = 0$$

This determinant has the same solutions as on the first page.

QUESTION #2

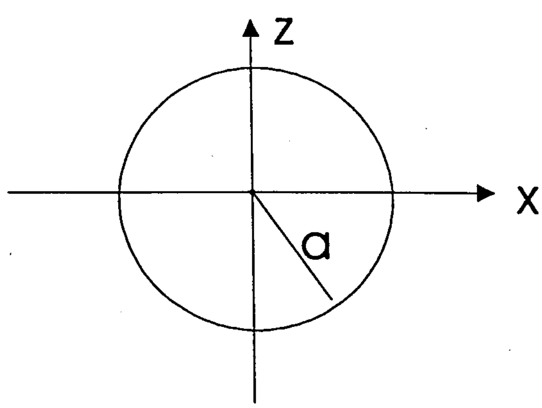
A sphere of radius a centered on the origin has a uniform permanent magnetization $\mathbf{M} = M\hat{\mathbf{z}}$. The sphere is cut into two hemispheres by the xy-plane, and the hemispheres are separated infinitesimally (not enough to alter the field within the material or for $r > a$).

a) What is the magnetic induction $\mathbf{B}$ for $r > a$?

b) What is the magnetic induction $\mathbf{B}$ in the gap between the two hemispheres?

c) Use the stress tensor to calculate the force between the hemispheres.

Hint: You may want to use the fact that $\int T_{ij}\, dA_j \to 0 \; as \; R \to \infty$, where R is the radius of the surface of integration.



S. TSURUTA

#2

Comp. Exam/97

E & M

A sphere of radius a centered on the origin has a uniform permanent magnetization $\bar{M} = M\hat{z}$. The sphere is cut into two hemispheres by the x-y plane, and the hemispheres are separated infinitesimally (not enough to alter the field within the material or for $r>a$). (See Fig.)

(a) What is the magnetic induction, $\bar{B}$, for $r>a$?

(b) What is the magnetic induction, $\bar{B}$, in the gap between the two hemispheres?

(c) Use the stress tensor to calculate the force between the hemispheres.

Hint: you may want to use the fact that $\int T_{ij}\, dA_j \to 0$ as $R \to \infty$, where R is the radius of the surface of integration.

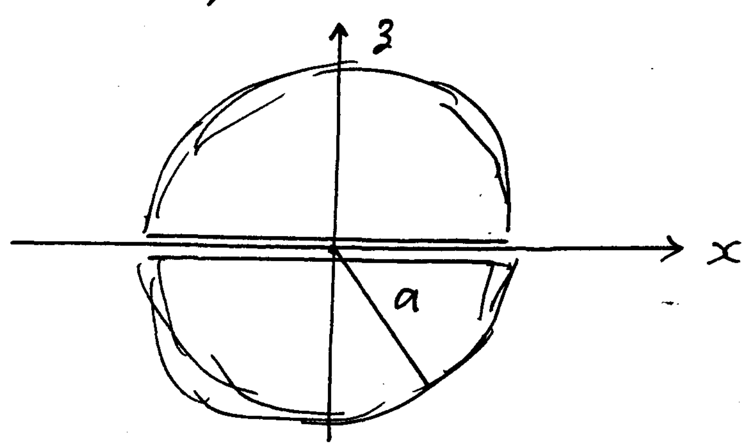


Fig.

e&m (conti.)

Ans.:

(a) $\Phi = m\frac{\cos\theta}{r^2}$; $m = \frac{4\pi}{3}Ma^3$ (e.g. J. 5.106)

So $\bar{B} = -\bar{\nabla}\Phi = \boxed{\frac{m}{r^3}[2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}]}$ Ans.

(b) On either side of the gap, just inside the hemisphere,

Ans. $\boxed{\bar{B} = \frac{8\pi}{3}\bar{M} = \frac{8\pi}{3}M\hat{z}}$ (e.g. J.5.105)

Since $B_\perp$ must be continuous, $\bar{B}$ must have the same value within the gap.

(c). By symmetry, $\bar{F}$ can only be in the $\hat{z}$ direction.

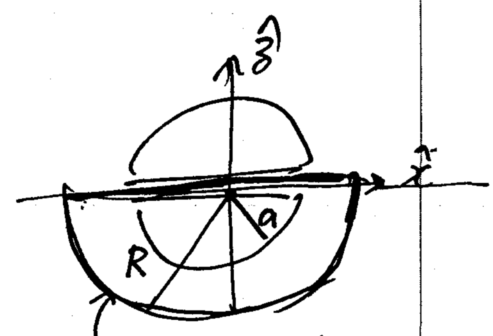
So, $\bar{F} = F_z\hat{z}$ &

$$F_z = \int T_{zz}\,dA_z + \int T_{zx}\,dA_x + \int T_{zy}\,dA_y.$$

$T_{ij} = \frac{1}{4\pi}\{B_iB_j - \frac{1}{2}\delta_{ij}\bar{B}\cdot\bar{B}\}$

Since $B \propto \frac{1}{R^3}$ $T \propto \frac{1}{R^6}$,

but the curved part of $dA \propto R^2$, there is no contribution from it.



let dA be this closed surface & let $R \to \infty$.

Then, $F_z = \int_{\text{(flat part of } dA)} T_{zz}\,dA_z$

for $r<a$, (within gap),

$$T_{zz} = \frac{1}{4\pi}\{B_z^2 - \frac{B^2}{2}\} = \frac{1}{8\pi}\left(\frac{8\pi}{3}M\right)^2 = \frac{8\pi M^2}{9}.$$

for $r>a$ (at $\theta = \pi/2$),

$$T_{zz} = \frac{1}{4\pi}\left(B_z^2 - \frac{B^2}{2}\right) = \frac{1}{8\pi}\left(\frac{4\pi a^3}{3}\frac{M}{r^3}\right)^2 = \frac{2\pi M^2 a^6}{9 r^6}$$

So, $$F_z = \int_0^a \frac{8\pi}{9} M^2 r\,dr\,d\theta + \int_a^\infty \frac{2\pi M^2 a^6}{9 r^6} r\,dr\,d\theta$$

$$= \frac{16\pi^2 M^2}{9}\frac{a^2}{2} + \frac{4\pi^2 M^2 a^6}{9}\frac{1}{4a^4} = \pi^2 M^2 a^2.$$

So. Ans $\boxed{\bar{F} = \pi^2 M^2 a^2 \hat{z}}$

QUESTION #3

A one-dimensional simple harmonic oscillator is subject to the perturbation

$$V(x,t) = Ax\delta(t); \qquad A = \sqrt{m\hbar\omega_0}\,,$$

where m and ω_0 are the mass and natural frequency of the oscillator respectively. Suppose the oscillator is in its ground state at $t = -\infty$.

After the perturbation, evaluate the expectation values of position, momentum, and energy to first order in A.

Hints:

$$x = \frac{\Delta}{\sqrt{2}}\left(a + a^{\dagger}\right), \quad \Delta = \sqrt{\frac{\hbar}{m\omega_0}}$$

$$p = \frac{-i\hbar}{\sqrt{2}\Delta}\left(a - a^{\dagger}\right)$$

$$|\alpha\, t\rangle = \sum_n c_n e^{-i\omega_n t}|n\rangle\,.$$

#3 Quantum Mechanics J. HERMANSON

A one-dimensional simple harmonic oscillator is subject to the perturbation

$$V(x,t) = A\,x\,\delta(t)\ ;\ A = \sqrt{m\hbar\omega_0},$$

where m and ω_0 are the mass and natural frequency of the oscillator.
Suppose the oscillator is in its ground state at $t=-\infty$. After the perturbation, evaluate the expectation values of position, momentum, and energy to first order in A.

<u>Hints:</u> $x = \frac{\Delta}{\sqrt{2}}(a+a^\dagger)\ ;\ \Delta = \sqrt{\frac{\hbar}{m\omega_0}}$

$$p = \frac{-i\hbar}{\sqrt{2}\,\Delta}(a-a^\dagger)$$

$$|\alpha t\rangle = \sum_n c_n e^{-i\omega_n t}|n\rangle$$

Solution (ω_0 = natural frequency)

We look for the coefficients c_n in the expansion

$$|\alpha t\rangle = c_0 e^{-i\frac{\omega_0 t}{2}}|0\rangle + c_1 e^{-i\frac{3}{2}\omega_0 t}|1\rangle + \cdots$$

where, to 1st order in A,

$$c_n(t) = -\frac{i}{\hbar}\int_{-\infty}^{t} V_{n0}(t')\, e^{i\omega_{n0}t'}\, dt'$$

[Initially, $c_0 = 1$, $c_n = 0$ for $n > 0$]

$$V_{n0} = \langle n | A x\, \delta(t) | 0\rangle$$
$$= A\, x_{n0}\, \delta(t)$$

$$\boxed{c_n = -\frac{i}{\hbar} A\, x_{n0}} \quad ; t > 0$$

Now $x = \frac{\Delta}{\sqrt{2}}(a + a^\dagger)$; $\Delta = \sqrt{\frac{\hbar}{m\omega_0}}$

$$x_{10} = \frac{\Delta}{\sqrt{2}}\langle 1 | \cancel{a} + a^\dagger | 0\rangle$$
$$= \frac{\Delta}{\sqrt{2}} \quad ; \quad \boxed{\text{all other } x_{n0} = 0}$$

So $c_1 = -\frac{i}{\hbar} A \frac{\Delta}{\sqrt{2}} = -\frac{i}{\hbar}\sqrt{m\hbar\omega_0}\cdot\sqrt{\frac{\hbar}{2m\omega_0}} = -\frac{i}{\sqrt{2}}$

To lowest order, then,

$$\boxed{|\alpha t\rangle = e^{-i\frac{\omega_0 t}{2}}|0\rangle - \frac{i}{\sqrt{2}}\, e^{-i\frac{3\omega_0 t}{2}}|1\rangle}$$

after the perturbation, $t > 0$

Then $|\alpha t\rangle = |0\rangle - \frac{i}{\sqrt{2}} e^{-i\omega_0 t} |1\rangle$,

apart from factor $e^{-\frac{i\omega_0 t}{2}}$
which can be discarded below.

$$\langle x\rangle = \left\{\langle 0| + \frac{i}{\sqrt{2}} e^{i\omega_0 t}\langle 1|\right\} x \left\{|0\rangle - \frac{i}{\sqrt{2}} e^{-i\omega_0 t}|1\rangle\right\}$$

$$= \frac{i}{\sqrt{2}} e^{i\omega_0 t} x_{10} - \frac{i}{\sqrt{2}} e^{-i\omega_0 t} x_{01}$$

But $x_{10} = x_{01} = \frac{\Delta}{\sqrt{2}}$

$$\langle x\rangle = \frac{i\Delta}{2}\left(e^{i\omega_0 t} - e^{-i\omega_0 t}\right)$$

$$= \frac{\Delta}{-2i}\left(\quad \right)$$

$$\boxed{\langle x\rangle = -\Delta \sin\omega_0 t} = -\sqrt{\frac{\hbar}{m\omega_0}} \sin\omega_0 t$$

From Ehrenfest's theorem,

$$\langle p\rangle = \frac{d}{dt} m\langle x\rangle$$

$$= -m\omega_0 \Delta \cos\omega_0 t$$

$$\boxed{\langle p\rangle = -\sqrt{m\hbar\omega_0} \cos\omega_0 t}$$

$$\langle E\rangle = \langle \alpha t|H|\alpha t\rangle$$

$$= \left\{\langle 0| + \frac{i}{\sqrt{2}} e^{i\omega_0 t}\langle 1|\right\} H \left\{|0\rangle - \frac{i}{\sqrt{2}} e^{-i\omega_0 t}|1\rangle\right\}$$

$$= \langle 0|H|0\rangle + \frac{1}{2}\langle 1|H|1\rangle$$

$$= \frac{\hbar\omega_0}{2} + \frac{1}{2}\frac{3\hbar\omega_0}{2}$$

$$\boxed{\langle E\rangle = \frac{5}{4}\hbar\omega_0}$$ time-independent ✓

QUESTION #4

A set of basis functions $\{u_n(x)\}$, n=1, 2,..., ∞, is defined on an interval $a \le x \le b$. The $u_n(x)$ are in general complex, and – over the interval – the set is "complete" and "orthonormal".

Be certain to define any terms or symbols you use in answering the questions that follow.

a) Show that $\{u_n(x)\}$ is a "linearly independent" set of functions.

b) Let Λ be the linear operator that generates the u_n and eigenvalue λ_n via:

$$\Lambda u_n - \lambda_n u_n = 0.$$

Now consider the inhomogenous problem on $a \le x \le b$

$$\Lambda v(x) - \lambda v(x) = f(x) \,, \qquad \lambda = \text{constant}$$

Show that a solution to this problem can be expressed as

$$v(x) = \int_a^b G(x,x')\, f(x')dx',$$

where: $G(x,x') = \sum_n u_n(x) u_n^*(x')/(\lambda_n - \lambda).$

The asterisk * means "complex conjugate".

D. ROBISCOE

Robiscoe : Math ϕ #④ [Aug. 97] 06/18/97

"A set of basis functions $\{u_n(x)\}$, $n=1,2,\ldots,\infty$, is defined on an interval $a \leq x \leq b$. The $u_n(x)$ are in general complex, and -- over the interval -- the set is "complete" and "orthonormal."

Be certain to define any terms or symbols you use in answering the questions that follow."

(A) Write an equation expressing the fact that the $u_n(x)$ are "orthonormal"

(B) Write an equation expressing the fact that the $u_n(x)$ are "complete".

(C) Show that $\{u_n(x)\}$ is a "linearly independent" set of functions.

(D) Let Λ be the linear operator that generates the u_n and eigenvalues λ_n via : $\Lambda u_n - \lambda_n u_n = 0$. Now consider the inhomogeneous problem

$$\Lambda v(x) - \lambda v(x) = f(x), \text{ on } a \leq x \leq b.$$

Show that a solution to this problem can be expressed as

$$v(x) = \int_a^b G(x,x')\, f(x')\, dx',$$

where : $G(x,x') = \sum_n u_n(x)\, u_n^*(x') / (\lambda_n - \lambda)$.

The asterisk $*$ means "complex conjugate".

NOTE: parts (A) & (B) could be deleted for sake of brevity.

Solution...

(A) $\underline{\underline{\int_a^b u_m^*(x)\, u_n(x)\, dx = \delta_{mn}}}$ ✓ $*$ ⇒ complex conjugate, δ_{mn} = Kronecker delta $= \begin{cases} 1, & m=n; \\ 0, & m \neq n. \end{cases}$

(B) $\underline{\underline{\sum_{n=1}^{\infty} u_n(x)\, u_n^*(x') = \delta(x-x')}}$ ✓ $\delta(\xi)$ = Dirac delta: $\int_-^+ \delta(\xi)\, d\xi = 1$.

(over)

Robiscoe: Math ϕ (cont'd)

(C) A set of fcns $\{w_n(x)\}$ is linearly independent if, for any set of constants $\{c_n\}$, the condition $\sum_n c_n w_n(x) = 0 \Rightarrow$ all $c_n \equiv 0$. In our case, if $\sum_n c_n u_n(x) = 0$, then -- by operating through w/ $\int dx\, u_m^*(x)$

$$\rightarrow \int_a^b dx\, u_m^*(x) \cdot \left[\sum_n c_n u_n(x) = 0\right] \Rightarrow \sum_n c_n \underbrace{\int_a^b u_m^*(x) u_n(x) dx}_{\delta_{mn}} = 0;$$

so/ $\sum_n c_n \delta_{mn} = c_m = 0$. All the $c_n \equiv 0$, and $\{u_n(x)\}$ is lin. indpt.

(D) Since $\{u_n\}$ is complete on $a \leq x \leq b$, can expand: $v(x) = \sum_n \alpha_n u_n(x)$, and: $f(x) = \sum_n \beta_n u_n(x)$, w/ $\{\alpha_n\}$ & $\{\beta_n\}$ = cnsts. The eqtn is then...

$$\rightarrow \sum_n \alpha_n (\lambda_n - \lambda) u_n(x) = \sum_n \beta_n u_n(x) \Rightarrow \alpha_n = \beta_n / (\lambda_n - \lambda).$$

This last step follows from the linear independence of the $\{u_n(x)\}$. Now, from the orthonormality condition, we can find $\{\beta_n\}$, as...

$$\rightarrow \int_a^b dx\, u_n^*(x) \cdot \left[f(x) = \sum_p \beta_p u_p(x)\right] \Rightarrow \int_a^b u_n^*(x) f(x) dx = \sum_n \beta_p \delta_{np}$$

so/ $\beta_n = \int_a^b u_n^*(x') f(x') dx'$

and/ $\alpha_n = \dfrac{1}{\lambda_n - \lambda} \int_a^b u_n^*(x') f(x') dx'$

With this result, $v = \sum_n \alpha_n u_n$ can be written, as desired

$$\rightarrow v(x) = \sum_n \frac{u_n(x)}{\lambda_n - \lambda} \int_a^b u_n^*(x') f(x') dx' = \int_a^b G(x, x') f(x') dx'$$

w/ $G(x, x') = \sum_n u_n(x) u_n^*(x') / (\lambda_n - \lambda)$.

QUESTION #5

Most (90%) primary cosmic rays are high energy protons. The highest energy particles ever observed have been cosmic rays with energies of order 10^{14} MeV. Such high energy particles are rare, and are not observed directly, but are of great interest since their energies greatly exceed what can be obtained in Earthly accelerators (e.g., Fermilab's maximum energy is about 10^6 MeV).

An upper limit to cosmic ray energies may exist due to the Cosmic Background Radiation (CBR). A high energy cosmic ray proton (p) could interact with a CBR photon (γ) to produce a neutral pi meson (π^0), thereby lowering the energy of the proton:

$$\gamma + p \rightarrow p + \pi^0$$

The threshold energy for this reaction may provide a cutoff in the high-energy cosmic ray spectrum.

a) Calculate the threshold energy of the proton for it to undergo this reaction if γ represents a CBR photon of temperature 3 K. Assume the collision to be head-on; take the photon energy to be kT; $m_p = 940$ MeV; $m_\pi = 135$ MeV.

b) Comment on the relation of your answer to the present upper limit on observed cosmic ray energies–is it higher, lower, about the same? Should we expect to observe substantially higher energy cosmic rays than 10^{14} MeV?

Possible useful conversion factor: 1 eV is equivalent to a temperature of 11,600 K.

B. Hiscock

#5

Special Relativity / Cosmic Ray Problem Solution

Conservation of 4-momentum gives

$$\vec{p}_\gamma + \vec{p}_p = \vec{p}_{p'} + \vec{p}_\pi \qquad (p' = \text{proton coming out of reaction})$$

The length of the total 4-momentum before and after must be the same:

$$(\vec{p}_\gamma + \vec{p}_p)^2 = (\vec{p}_{p'} + \vec{p}_\pi)^2 \qquad (1)$$

In the "lab" frame (where $E_\gamma \sim 3K = 2.6\times10^{-10}\,MeV$)

$$\vec{p}_\gamma = (E_\gamma, \vec{p}_\gamma) \qquad \vec{p}_p = (E_p, \vec{p}_p)$$

In the C.M. frame at threshold the proton and pion are at rest as a "lump" (any additional relative motion would require greater energy)

$$\vec{p}_{p'} + \vec{p}_\pi = (m_p + m_\pi, 0)$$

Substituting into Eq. (1) we find

$$|\vec{p}_\gamma|^2 + 2\,\vec{p}_\gamma \cdot \vec{p}_p + |\vec{p}_p|^2 = -(m_p + m_\pi)^2 \qquad (2)$$

note $|\vec{p}_\gamma|^2 = 0 \quad |\vec{p}_p|^2 = -m_p^2$

$$\vec{p}_\gamma \cdot \vec{p}_p = -E_\gamma E_p + \vec{p}_\gamma \cdot \vec{p}_p$$

head on collision $\Rightarrow$ $\vec{p}_\gamma \cdot \vec{p}_p = -p_\gamma p_p$

and since the photon is massless, $p_\gamma = E_\gamma$, so (2) becomes:

$$-2E_\gamma E_p - 2E_\gamma p_p = -(m_p + m_\pi)^2 + m_p^2$$

$p_p = (E_p^2 - m_p^2)^{\frac{1}{2}}$ so:

$$E_p + (E_p^2 - m_p^2)^{1/2} = \frac{2m_p m_\pi + m_\pi^2}{2E_\gamma}$$

$$= \frac{2\cdot 940\cdot 135 + (135)^2}{2\cdot 2.6\times 10^{-10}} \text{ MeV}$$

$$= 5.2\times 10^{14} \text{ MeV}$$

Since $E_p \gg m_p$, we can take $(E_p^2 - m_p^2)^{1/2} \approx E_p$, so we find

$$\boxed{E_p = 2.6\times 10^{14} \text{ MeV}}$$

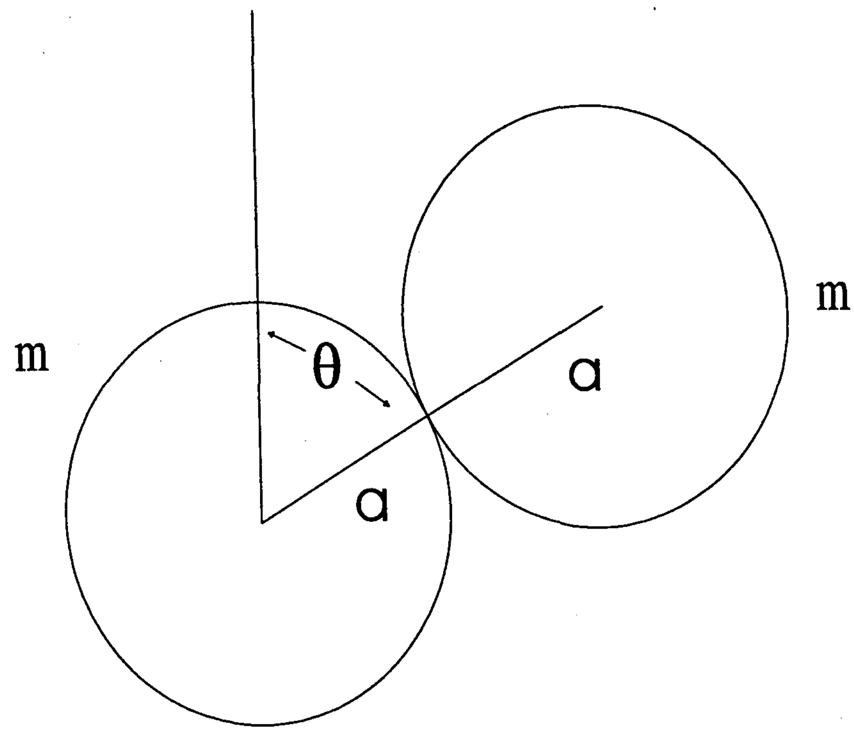
which is of the same order as the energy of the most energetic cosmic rays yet observed.

Hence, we may already be seeing the highest energy cosmic rays. We should not expect to observe cosmic rays with substantially higher energies.

QUESTION #6

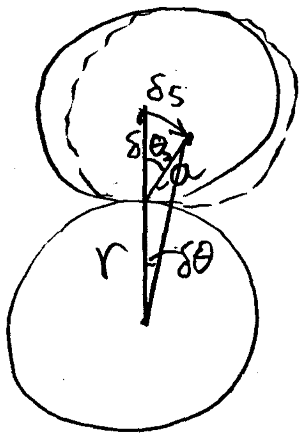
A uniform, circular hoop of mass *m* and radius *a* rolls without shipping on an identical hoop which is stationary. Both hoops are confined to a vertical plane. Motion is initiated by placing the moving hoop at the very top of the stationary hoop, and giving it kinetic energy *mga*.

At what angle or point of contact do the hoops separate?



#6 Solution:

The angle of rotation θ_2 of the moving hoop is related to θ by the condition of no slipping:



$r = 2a$

The c.m. displacement is

$$\delta s = r\,\delta\theta = a\,\delta\theta_2$$

Integrating,

$$r\theta = a\theta_2$$

$$\boxed{\theta_2 = \frac{r}{a}\theta}$$

Now the lift-off occurs when the centrifugal force

$$F_{out} = mr\dot{\theta}^2$$

is exactly balanced by the centripetal force

$$F_{in} = mg\cos\theta$$

$$\Rightarrow \quad mr\dot{\theta}^2 = mg\cos\theta$$

To relate $\dot{\theta}$ and θ, use energy conservation. The total energy at $t=0$ is

$$T+V = mga + mgr = \boxed{\frac{3}{2}mgr = E}$$

This also the total energy at any later time.

The kinetic energy is given in terms of $\dot{\theta}$:

$$T = \frac{m}{2} r^2 \dot{\theta}^2 + \frac{I}{2} \dot{\theta}_2^2 \quad ; \quad I = ma^2 \text{ for hoop}$$

$$= \frac{m}{2} r^2 \dot{\theta}^2 + \frac{ma^2}{2} \left(\frac{r}{a} \dot{\theta}\right)^2$$

$$= mr^2 \dot{\theta}^2$$

$$= E - V = \frac{3}{2} mgr - mgr\cos\theta \qquad \underline{\text{energy conservation}}$$

$$mr^2 \dot{\theta}^2 = mgr\left(\frac{3}{2} - \cos\theta\right)$$

or $\dot{\theta}^2 = \frac{g}{r}\left(\frac{3}{2} - \cos\theta\right)$

Thus

$$F_{out} = mr\dot{\theta}^2 = mg\left(\frac{3}{2} - \cos\theta\right)$$

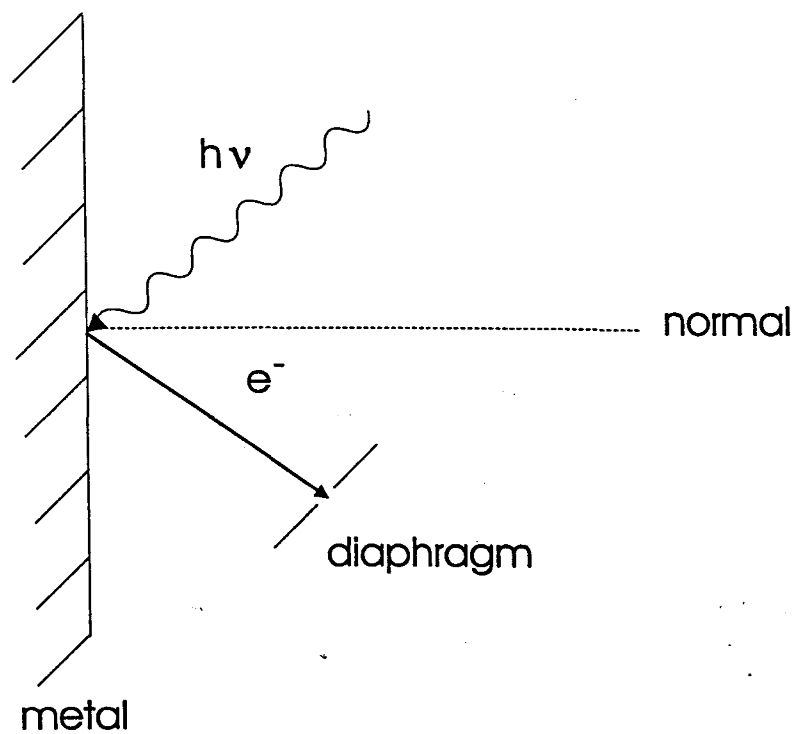
$$= F_{in} = mg\cos\theta$$

when $\cos\theta_0 = \frac{3}{4} \Rightarrow \boxed{\theta_0 = 41.41^\circ}$

QUESTION #7

The work function ϕ of a metal is defined as the smallest energy required to remove an electron and place it at infinity at rest. An experimentalist wishes to measure the work function of a metal with the photoelectric effect but does not have a spectrometer to measure the kinetic energy of the ejected electrons, which are assumed to be excited from the Fermi level of the metal. Instead he places a diaphragm with a small slit perpendicular to the path of the ejected electrons and measures the intensity of the electrons that pass through the slit as a function of the angle. The monochromatic radiation has a frequency of 1.45×10^{15} Hz, the angle from the normal to the diaphragm of the first destructive interference spot (no intensity) is 1.01×10^{-3} radians, and the slit width is 1.25×10^{-6} m.

Determine ϕ in units of electron volts.



R. Avci

#7 Solution:

$\phi = h\nu - K$ (Einstein's photoelectric equation)

$K = \frac{p^2}{2m}$; and $p = \frac{h}{\lambda}$ (λ = de Broglie wavelength of electron)

$a \sin\theta = \lambda$ (First minima of the single-slit diffraction)

where $a = 1.25 \times 10^{-6}$ m (slit width)

$$\lambda \simeq a\,\theta(\text{rad}) \simeq 1.25\times10^{-6}\,\text{m} \times 1.01\times10^{-3}\,\text{rad} \simeq \boxed{1.26\times10^{-9}\,\text{m}}$$

$$K = \frac{p^2}{2m} = \frac{h^2}{2m\lambda^2} = \frac{(6.63\times10^{-34}\,\text{J.s})^2}{2\times9.11\times10^{-31}\,\text{kg}\times(1.26\times10^{-9}\,\text{m})^2} = \boxed{1.51\times10^{-19}\,\text{J}}$$

$$\phi = h\nu - K = 6.63\times10^{-34}\,\text{J.s}\times1.45\times10^{15}\,\text{s}^{-1} - K \;(\text{J})$$

$$\phi = 9.61\times10^{-19} - 1.51\times10^{-19} = 8.10\times10^{-19}\,\text{J} = \boxed{5.06\,\text{eV}}$$

This is a typical work function.

QUESTION #8

Consider a neutral helium atom. Ignore the Coulomb interaction between the electrons and the spin-orbit interaction.

a) Determine the electronic energy levels for $n = 1$ and $n = 2$.

b) Write the allowable (that is, consistent with the Pauli principle) two-electron wave functions, including spin, for excited helium in the 1s2p configigruation. Give your answers in terms of radial functions $R_{1s}(r)$ and $R_{2p}(r)$, spherical harmonics Y_ℓ^m, and spin functions α and β.

c) Describe, in qualitative terms, the effect of the interelectronic Coulomb interaction on the three energy levels in (a) and (b).

#8

J. HERMANSON

Solution

(a) Ignoring $\frac{e^2}{r_{12}}$, each electron moves in a potential $-\frac{2e^2}{r}$, and so the energy levels are hydrogenic but with $Z=2$:

$$E_n = -\frac{mZ^2e^4}{2\hbar^2 n^2} = -\frac{Z^2}{n^2} \times 13.6\,eV$$

$$\boxed{\begin{aligned} E_1 &= -4 \times 13.6 = -54.4\,eV \\ E_2 &= -\tfrac{4}{4} \times 13.6 = -13.6\,eV \end{aligned}}$$

$\Rightarrow$ for a single e^-. For both e, $E_1 = 2(-54.4)\,eV = -108.8\,eV$.

(b) The 1s 2p excited state has four spin states:

	State	S	S_z	Exchange Symmetry
SINGLET	$\frac{\alpha\beta-\beta\alpha}{\sqrt{2}}$	0	0	odd
TRIPLETS	$\alpha\alpha$	1	1	even
	$\frac{\alpha\beta+\beta\alpha}{\sqrt{2}}$	1	0	even
	$\beta\beta$	1	-1	even

It also has three spatial states,

$$R_{1s}^{(1)} Y_0^0{}^{(1)} R_{2p}^{(2)} Y_1^m{}^{(2)}, \quad m = 0, \pm 1,$$

which must be <u>symmetrized for singlets</u> and <u>antisymmetrized for triplets</u> (Pauli principle)

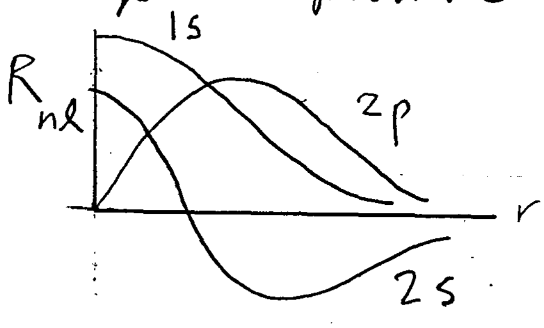
The three singlet WFs for $1s2p$ are (with $m = 0, \pm 1$)

$$\frac{R_{1s}(1)Y_0^0(1)R_{2p}(2)Y_1^m(2) + R_{1s}(2)Y_0^0(2)R_{2p}(1)Y_1^m(1)}{\sqrt{2}} \left(\frac{\alpha\beta - \beta\alpha}{\sqrt{2}}\right)$$

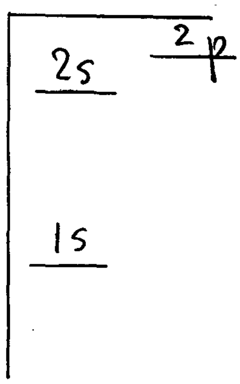
For the triplets, there are 9 WFs ($S_z = 0, \pm 1$; $m = 0, \pm$)

$$\frac{R_{1s}(1)Y_0^0(1)R_{2p}(2)Y_1^m(2) - R_{1s}(2)Y_0^0(2)R_{2p}(1)Y_1^m(1)}{\sqrt{2}} \begin{cases} \alpha\alpha, & S_z = 1 \\ \dfrac{\alpha\beta + \beta\alpha}{\sqrt{2}}, & S_z = 0 \\ \beta\beta, & S_z = -1 \end{cases}$$

(c) Coulomb interaction $\frac{e^2}{r_{12}}$ raises the energy of each level due to screening of the nucleus. But 2s levels are lower than 2p because the former have larger amplitude near the nucleus



E_{nl}



For the two-electron WFs (part (b)), triplets lie lower than singlets (Hund's Rule)

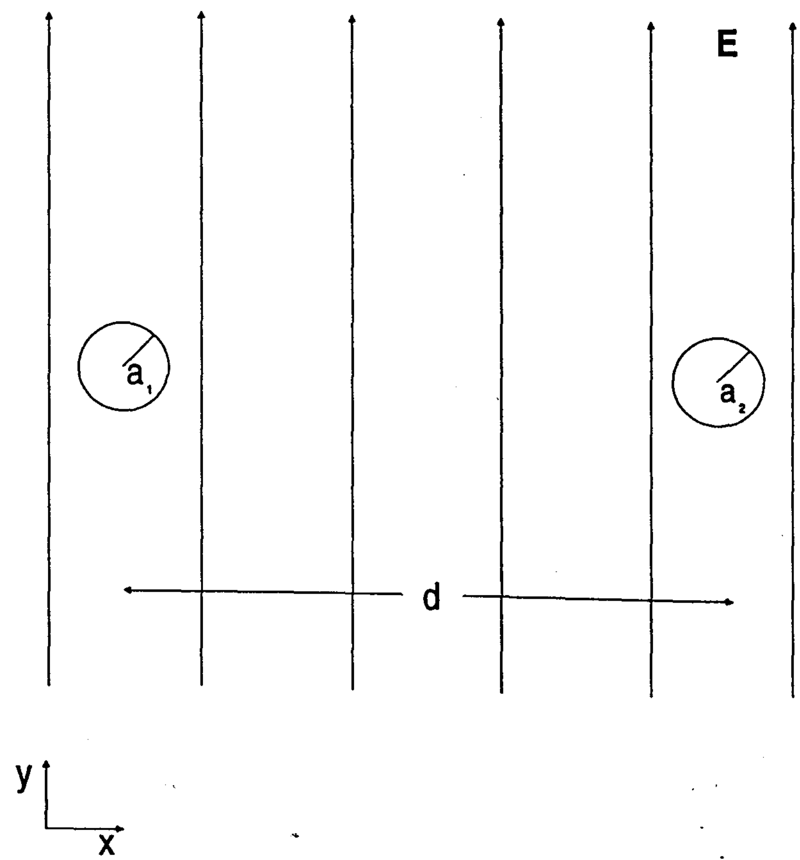
QUESTION #9

A perfectly conducting sphere of radius a is placed into a uniform electric field $\mathbf{E} = E_0\,\mathbf{z}$.

a) Find the electrostatic potential for $r \geq a$ with the sphere in place.

b) What is the electric dipole moment $\mathbf{p}$ of the sphere?

c) Two spheres of radii a_1 and a_2 are separated by a horizontal distance $d >> a_1, a_2$. Find the dipole moments of each sphere resulting from a vertical external field $\mathbf{E} = E_0\mathbf{y}$. Do not find the potential. You may neglect terms $O(a^4/d^4)$ and higher.



DANA LONGCOPE

Question # 9.

(a) Outside of the sphere there are no charges so the potential Φ must satisfy $\nabla^2\Phi = 0$. Placing the origin at the center of the sphere we may assume axisymmetry, $\partial/\partial\phi = 0$. In spherical coordinates Laplace's equation takes the form

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right) = 0 \ . \tag{1}$$

To match the external field we demand

$$\Phi(r\,\theta) \to -E_0\,z = -E_0\,r\cos\theta \quad , \qquad r\to\infty \ . \tag{2}$$

Since the sphere is a perfect conductor it is at a single potential, say $\Phi(r,\theta) = 0$.

A solution of Laplace's equation can be expressed using multipoles, however, based on the boundary conditions we can restrict ourselves to only the dipole term:

$$\Phi(r,\theta) = R(r)\cos\theta \ .$$

Placing this in eq. (1) gives the equation for the radial function

$$R'' + 2\frac{1}{r}R' - \frac{2R}{r^2} = 0 \ . \tag{3}$$

The general solution of this equation is

$$R(r) = A\,r + \frac{B}{r^2} \ . \tag{4}$$

Matching boundary condition eq. (2) gives $A = -E_0$. Requiring $\Phi(a,\theta) = 0$ gives $B = E_0\,a^3$. The final result is therefore

$$\Phi(r,\theta) = E_0\left(\frac{a^3}{r^2} - r\right)\cos\theta \ . \tag{5}$$

(b) The first term in this expression is the electric field due to the polarization of the conducting sphere, the second is the external field. The coefficient of $r^{-2}\cos\theta$ is the dipole moment of the sphere. In vector form this is

$$\mathbf{p} = E_0 a^3\,\hat{\mathbf{z}} \ , \tag{6}$$

the sense of the dipole is parallel to the field.

(c) To lowest order each of the spheres will develop a dipole moment from the external field, just as in the problem above. The dipole field from one sphere will, however, change the field at the other sphere affecting its dipole moment. At sphere 1, the combination of external field and field of sphere 2 is

$$E_{y1} = E_0 - \frac{p_{y2}}{d^3} \ , \tag{7}$$

where the second term arises from the dipole p_{y2} on sphere 2. The dipole moment of sphere 2 is simply proportional to the field there (see part [a])

$$p_{y2} = a_2^3 E_{y2} \ . \tag{8}$$

#9 cont.

Combining these expressions and writing a similar one for E_{y2} gives the coupled system

$$\begin{bmatrix} 1 & a_2^3/d^3 \\ a_1^3/d^3 & 1 \end{bmatrix} \cdot \begin{bmatrix} E_{y1} \\ E_{y2} \end{bmatrix} = \begin{bmatrix} E_0 \\ E_0 \end{bmatrix} \tag{9}$$

A bit of algebra yields the solution

$$\begin{bmatrix} E_{y1} \\ E_{y2} \end{bmatrix} = \frac{E_0}{1 - a_1^3 a_2^3/d^6} \begin{bmatrix} 1 - a_2^3/d^3 \\ 1 - a_1^3/d^3 \end{bmatrix} . \tag{10}$$

The denominator can be simplified by neglecting $a_1^3 a_2^3/d^6$. This gives the dipole moments

$$\mathbf{p}_1 = a_1^3(1 - a_2^3/d^3)E_0\,\hat{\mathbf{y}} \quad , \quad \mathbf{p}_2 = a_2^3(1 - a_1^3/d^3)E_0\,\hat{\mathbf{y}} \ . \tag{11}$$

Each sphere has a slightly reduced dipole moment due to their interaction.

QUESTION #10

A light ray travels in the xy-plane between points A & B, whose coordinates are (-1, 1) and (+1, 1), respectively. The intervening medium has index of refraction n such that: $n(y) = \sqrt{ky}$, where k is a positive constant and y is the y-coordinate. By Fermat's principle, the ray will travel along a path such that the optical path-length

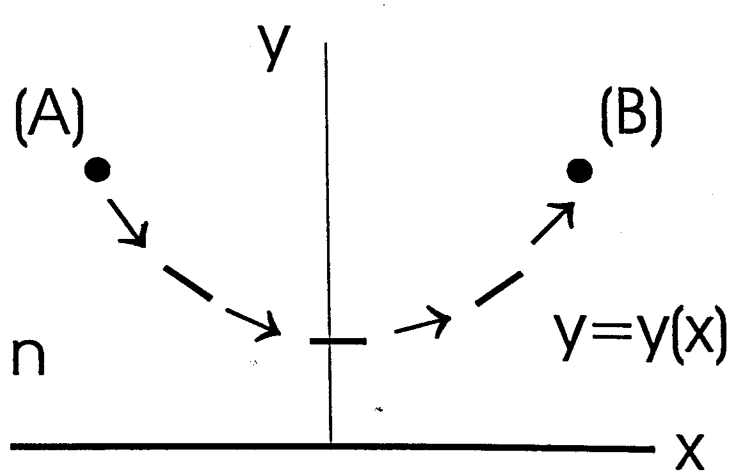
$$P = \int_A^B n ds, \quad ds = \text{element of length along } y = y(x),$$

is an extremum.

a) Convert P to the form: $P = \int_A^B J(y, y')\,dx$, and impose the extremum condition by means of the appropriate form of the Euler-Langrange equation.

b) Solve the Euler-Lagrange equation of part (a), and find the general form of the light ray path between <u>any</u> two points in the medium (for which $y > 0$).

c) Find $y = y(x)$ explicitly for the points A[-1, 1] and B[+1, 1].



Question #10

a)
$$P = \int_A^B n\, ds = \int_A^B \underbrace{\sqrt{k}\,\sqrt{y}\,\sqrt{1+(y')^2}}_{L(y,\,y')}\, dx$$

$$\delta P = 0 \quad \Rightarrow \quad \frac{d}{dx}\frac{\partial L}{\partial y'} - \frac{\partial L}{\partial y} = 0$$

b)
$$\frac{\partial L}{\partial y} = \sqrt{k} \quad \frac{\sqrt{1+(y')^2}}{2\sqrt{y}}$$

$$\frac{\partial L}{\partial y'} = \sqrt{k}\,\frac{\sqrt{y}\; y'}{\sqrt{1+(y')^2}}$$

$$\frac{d}{dx}\frac{\partial L}{\partial y'} = \frac{\left(\frac{1}{2}\frac{1}{\sqrt{y}}(y')^2 + \sqrt{y}\, y''\right)\sqrt{1+(y')^2} - \sqrt{y}\, y' \frac{1}{2}\frac{2y'y''}{\sqrt{1+(y')^2}}}{\left(\sqrt{1+(y')^2}\right)^2} \cdot \sqrt{k}$$

$$= \left[\frac{(y')^2}{2\sqrt{y}\sqrt{1+(y')^2}} + \frac{\sqrt{y}\, y''}{\sqrt{1+(y')^2}} - \frac{\sqrt{y}(y')^2 y''}{(1+(y')^2)^{3/2}}\right]\sqrt{k}$$

$$\boxed{\text{IT } \underline{\underline{\text{DOES}}} \text{ SIMPLIFY!}}$$

Lagrange equation:

$$\frac{y'^2(1+y'^2)+2yy''(1+y'^2)-2yy''y'^2-(1+y'^2)^2}{2\sqrt{y}\,(1+y'^2)^{3/2}}=0$$

$$(\cancel{y'^2}-1-\cancel{y'^2})(1+y'^2)+2yy''(1+\cancel{y'^2}-\cancel{y'^2})=0$$

$$-1-y'^2+2yy''=0$$

$$2yy''-y'^2-1=0$$

$$\frac{d}{dx}(2yy''-y'^2-1)=2\cancel{y'y''}+2yy'''-2\cancel{y'y''}=0$$

$$y'y'''=0 \quad\rightarrow\quad y'''=0$$

$$y''=C_1$$

$$y'=C_1x+C_2$$

$$\underline{y=C_1x^2+C_2x+C_3}$$

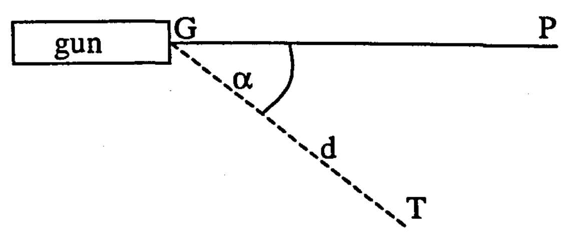
QUESTION #11

An electron gun accelerates electrons through a potential difference V and emits them along the direction GP, as shown in the figure. (Assume that the electrons are non-relativistic.) We want the electrons to hit the target T located a distance d from the gun and at an angle α relative to GP.

Find the strength of a uniform magnetic field B required for each of the following situations:

a) the field is perpendicular to the plane defined by GP and GT

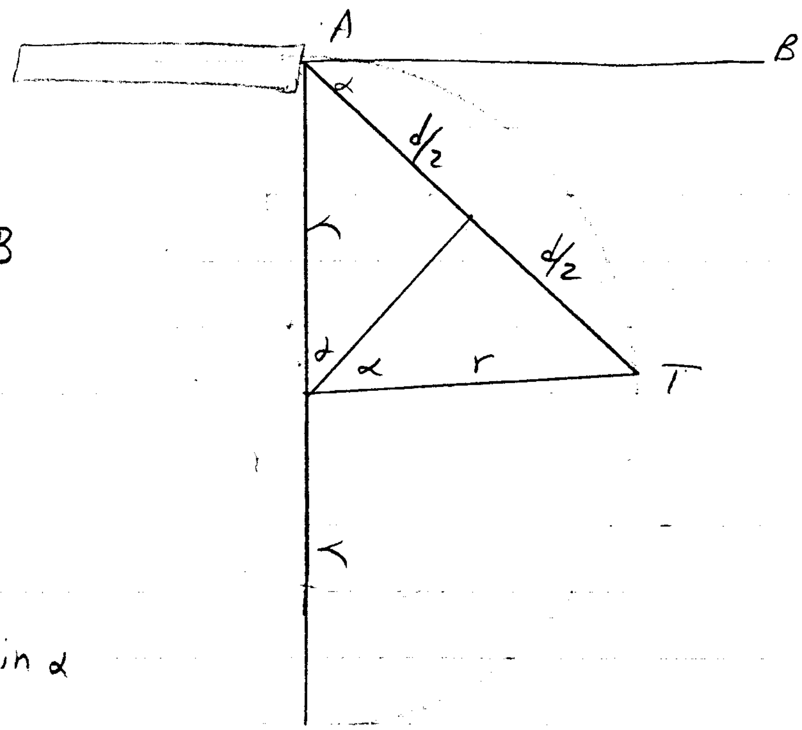
b) the field is parallel to GT



L. KIRKPATRICK

#11

a)



$$\frac{mv^2}{r} = qvB$$

$$\Rightarrow B = \frac{mv}{qr}$$

From figure

$$\frac{d}{2} = r\sin\alpha$$

Also $\frac{1}{2}mv^2 = qV$

$$\Rightarrow v = \sqrt{\frac{2qV}{m}}$$

$$B = \frac{m}{q}\sqrt{\frac{2qV}{m}}\,\frac{2\sin\alpha}{d} = \frac{2\sin\alpha}{d}\sqrt{\frac{2mV}{q}}$$ inward

b) Look at components of v parallel (∥) and perpendicular (⊥) to AT

$$v_{\parallel} = v\cos\alpha \qquad v_{\perp} = v\sin\alpha$$

⇒ The path will be a helix starting at A and passing through T; the line AT is along the edge of the helix parallel to the axis of the helix.

(cont).

The time required to reach T is given by

$$t_{\parallel} = \frac{d}{v\cos\alpha}$$

During this time the electron must make an integral number of orbits. n.

$$t_{\perp} = n\frac{2\pi r}{v\sin\alpha}$$

But $r = \dfrac{mv\sin\alpha}{qB}$

$$\Rightarrow \quad t_{\perp} = n\frac{2\pi m}{qB}$$

$t_{\perp} = t_{\parallel}$

$$\Rightarrow \quad \frac{d}{v\cos\alpha} = n\frac{2\pi m}{qB}$$

$$\text{or} \quad B = \frac{2\pi n v\cos\alpha}{qd} = \frac{2\pi n\cos\alpha}{d}\sqrt{\frac{2mV}{q}}$$

QUESTION #12

Possibly useful properties of the Airy function:

- $Ai''(x) = xAi(x)$
- $Ai(x) > 0 \; for \; x > 0$
- $Ai(x) = 0 \; at \; x = a_s = -2.338, -4.088, -5.552$
- $Ai''(x) \to 0 \; as \; x \to$ infinity

A rectangular box has a 5 mm x 5 mm horizontal section. It has a bottom but extends upward quite a ways.

a) Find the eigenfunctions of an electron confined to the bottom of the box by gravity.

b) Draw an energy diagram with the four lowest energy states, listing their degeneracy.

DANA LONGCOPE

Question # 12.

(a) The potential is $V(z) = mgz$ so the Schrödinger's equation is

$$\left(-\frac{\hbar^2}{2m}\nabla^2 + mgz\right)\psi \;=\; E\psi \;. \tag{1}$$

This can be solved by separation of variables

$$\psi(x,y,z) = f(z)\sin(p\pi x/L)\sin(q\pi y/L)$$

where $L = 5$mm and $p, q = 1, 2, \ldots$. Placing this in Schrödingder's equation gives

$$-f''(z) + \left[\frac{2m^2g}{\hbar^2}z \;+\; \frac{\pi^2(p^2+q^2)}{L^2} \;-\; \frac{2mE}{\hbar^2}\right] f(z) \;=\; 0 \;.$$

This can be put into the form

$$f''(u) - (u-u_0)f(u) \;=\; 0 \tag{2}$$

By introducing the scaled variable

$$z \;=\; \left(\frac{\hbar^2}{2m^2g}\right)^{1/3} u \;\equiv\; \lambda u \;, \tag{3}$$

where $\lambda = 1.4$ mm. u_0 is given by

$$u_0 \;=\; \frac{2mE\lambda^2}{\hbar^2} \;-\; \pi^2(p^2+q^2)\frac{\lambda^2}{L^2} \tag{4}$$

The solution to (2) is

$$f(u) = \mathrm{Ai}(u-u_0) \;. \tag{5}$$

The boundary condition is $\psi(z{=}0) = f(u{=}0) = 0$. This implies that

$$\mathrm{Ai}(-u_0) = 0 \;.$$

which implies that u_0 must take on the values

$$u_0 = -a_s = 2.338, 4.088, 5.552 \cdots. \tag{6}$$

(b) Replacing (6) into (4) yields the energy levels

$$E_{pqs} = \frac{\hbar^2}{2m\lambda^2}\left[(p^2+q^2)\pi^2\frac{\lambda^2}{L^2} + |a_s|\right] \;. \tag{7}$$

The leading coefficient can be evaluated

$$E_0 = \frac{\hbar^2}{2m\lambda^2} = (\tfrac{1}{2}\hbar^2g^2m)^{1/3} = 7.9\times10^{-26}\,\mathrm{erg} = 4.9\times10^{-14}\,\mathrm{eV} \;.$$

Finally, noting that $\pi^2\lambda^2/L^2 = 0.774$ in this case we can evaluate some energy levels

(p,q)	s	E	degen.
(1,1)	1	$3.89\,E_0$	2
(1,1)	2	$5.64\,E_0$	2
(1,2)	1	$6.21\,E_0$	4
(1,1)	3	$7.10\,E_0$	2
(1,2)	2	$7.96\,E_0$	4
(2,2)	1	$8.53\,E_0$	2

QUESTION #13

Consider a single spin-½ particle as a binary model paramagnetic system in which the spin has a magnetic moment μ and is in the presence of an external magnetic field $\mathbf{B} = B\hat{\mathbf{z}}$, where $\hat{\mathbf{z}}$ is the unit vector along the z-direction. Assume that the binary system is in thermal equilibrium with its surroundings at the fundamental temperature $\tau = k_B T$, where k_B is the Boltzmann constant and T is the Kelvin temperature. Answer the following questions:

a) In an appropriate reference frame the energy levels of this system can be represented by 0 (zero) and ε. For each energy level describe the orientation of the spin relative to **B**, and determine the energy splitting ε.

b) Recall that $Z = e^{-F/\tau}$ and $dF = -\sigma d\tau - p dV + \mu dN$, where Z is the partition function, F is the Helmholtz free energy, σ is the entropy, and μ is the chemical potential. Determine Z for the single spin described above. Now generalize the result to N noninteracting spins and write down the partition function Z for the N-spin-½ system and explain briefly how you arrive at the result.

c) In your own words, what is the definition of the entropy of a system? Determine the entropy σ of the N-spin system in terms of N, τ, and B. Determine the value of σ for $\lim_{\tau\to 0}$ and in the $\lim_{\tau\to\infty}$, keeping B constant. Do the results make sense? Explain. Similarly, discuss the limits of σ for $\lim_{B\to 0}$ and $\lim_{B\to\infty}$, keeping τ constant. Do the results make sense? Explain.

d) If we increase B reversibly from $B = 0$ to $B > 0$ at τ = constant (isothermally), do you expect σ to increase or to decrease? Explain. Now if we insulate the spin system from the surroundings and reduce the magnetic field from $B > 0$ to $B = 0$ reversibly and adiabatically (σ = constant), do you expect the temperature of the spin system to go up or down? Explain. (This is the basis of cooling/heating by adiabatic demagnetization.)

R. AVCI

#13 Solution:

(a)

state 2 ———————— ϵ $\uparrow \vec{B} \downarrow \vec{s}_z$ is opposite to $\vec{B}$

state 1 ———————— 0 $\uparrow \vec{B} \uparrow \vec{s}_z$ is parallel to $\vec{B}$

Interaction energy: $U = -\vec{\mu}\vec{B} = \begin{cases} +\frac{\mu B}{2} \\ -\frac{\mu B}{2} \end{cases}$

$\epsilon = \Delta U = \frac{\mu B}{2} - \left(-\frac{\mu B}{2}\right) = \boxed{\mu B}$ (energy splitting)

(b) $$Z = \sum_{\text{states}} e^{-\frac{\epsilon_n}{\tau}}, \quad \text{where} \quad \tau = k_B T$$

For a single spin system:

$$Z_1 = e^{-\frac{0}{\tau}} + e^{-\frac{\epsilon}{\tau}} = 1 + e^{-\epsilon/\tau} = \boxed{1 + e^{-\frac{\mu B}{\tau}}}$$

For N-spins which are independent (non-interacting) and distinguishable:

$$Z = Z_1^N = \boxed{\left(1 + e^{-\frac{\mu B}{\tau}}\right)^N}$$

(instead of $\frac{Z_1^N}{N!}$ for indistinguishable particles)

(c) Entropy is defined as $\ln g$ where g is the number of accessible states that system can be in under the specified constraints such as constant τ or constant total energy. For example, for a single spin-$\frac{1}{2}$ and for $B=0$, $\tau \neq 0$ $g=2$ ($\uparrow$ or $\downarrow$) this means $\sigma = \ln 2$, or $\tau = 0$, $B \neq 0$ $g=1$ ($\uparrow$) this means $\sigma = \ln 1 = 0$. Now let us determine σ:

From $dF = -\sigma\, d\tau - p\, dV + \mu\, dN$

$$\sigma = -\left(\frac{\partial F}{\partial \tau}\right)_{V,N} \quad ; \quad F = -\tau \ln Z$$

or $$F = -\tau N \ln\left(1 + e^{-\mu B/\tau}\right)$$

$$\sigma = -\left(\frac{\partial F}{\partial \tau}\right)_{V,N} = N \ln\left(1 + e^{-\mu B/\tau}\right) + \frac{N \cancel{\tau} \frac{\mu B}{\tau^{\cancel{2}}} e^{-\frac{\mu B}{\tau}}}{1 + e^{-\mu B/\tau}}$$

$$\Rightarrow \boxed{\sigma = N \left\{ \ln\left(1 + e^{-\frac{\mu B}{\tau}}\right) + \frac{(\mu B/\tau)}{e^{\mu B/\tau} + 1} \right\}}$$

Limits:

$$\lim_{\substack{\tau\to 0\\ B\neq 0}} \sigma = N(\ln 1 + 0) = 0$$ (all spins are lined up with $\vec{B}$)

$$\lim_{\substack{\tau\to \infty\\ B\neq 0}} \sigma = N\ln(1+1) + 0 = N\ln 2$$ (most random case)

$$\lim_{\substack{B\to 0\\ \tau\neq 0}} \sigma = N\ln 2 + 0 = N\ln 2$$ (most random case)

$$\lim_{\substack{B\to \infty\\ \tau\neq 0}} \sigma = N\ln 1 + 0 = 0$$ (all spins are lined up with $\vec{B}$)

These limits all make sense because, as indicated above spins are either all lined up with magnetic field hence there is only one accessible state : $\underbrace{\uparrow\uparrow\uparrow \cdots \uparrow}_{N}$ which means

$g=1$ or $\sigma = \ln 1 = 0$, or they will be in most random configuration which gives $g = 2^N$ accessible states, or $\sigma = N\ln 2$.

(d) As illustrated above when B is increased while keeping $\tau = \text{const.}$ σ will decrease because of ordering of spins. When we insulate the system, no heat will flow in or out of the system. If one notices $\sigma = f\left(\frac{B}{\tau}\right)$, which means if $\sigma = \text{const.}$, $\frac{B}{\tau}$ must be constant, or when $B\to 0$, $\tau \to 0$ to keep $\frac{B}{\tau} = \text{const.}$ This means spin temperature will approach to absolute zero. In practice spin temperatures of 10^{-6} K can be achieved by this method. — o —

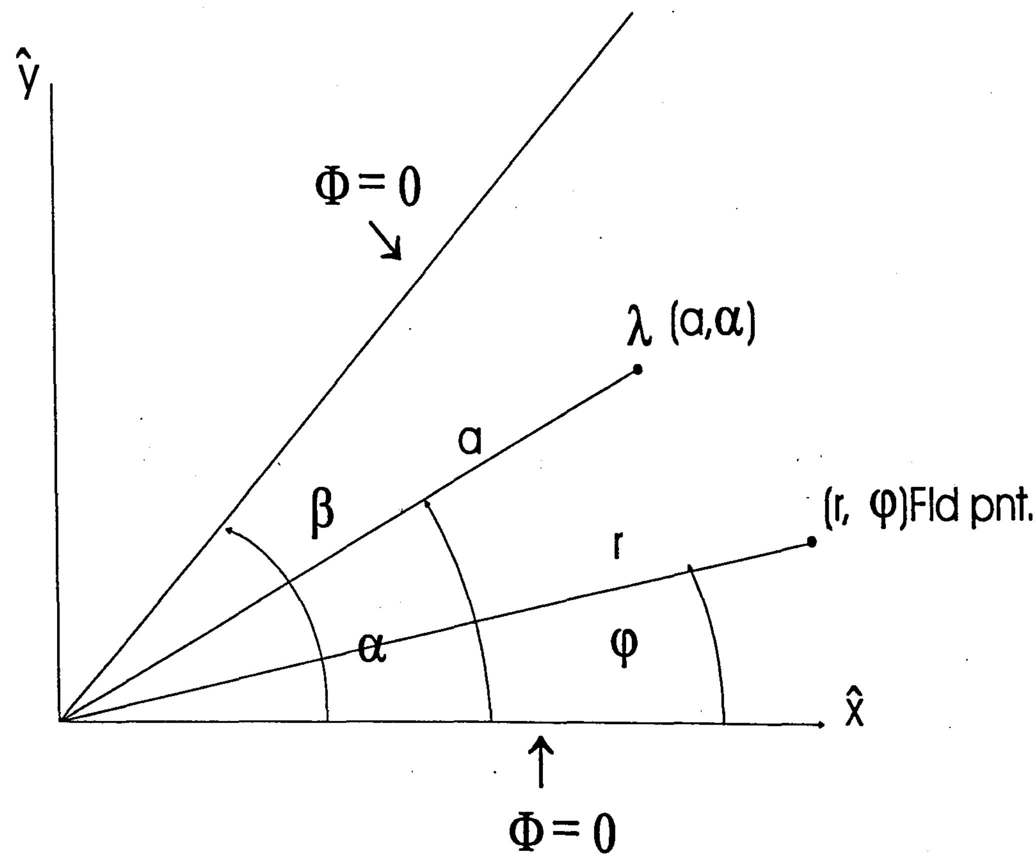
QUESTION #14

Two conducting planes intersect at angle β and each plane is held at potential $\Phi = 0$. At a point with polar coordinates (a, α) inside the wedge, there is a line charge parallel to the $\hat{\mathbf{z}}$ axis carrying charge /length λ = constant.

a) For field points (r, φ), write expressions for potential Φ_1 $(r < a, \varphi)$ and Φ_2 $(r > a, \varphi)$ valid over $0 \le \varphi \le \beta$. Require $\Phi_1 \to 0$ as $r \to 0$, and $\Phi_2 \to 0$ as $r \to \infty$. Note that at $r = a$, $\Phi_2 = \Phi_1$.

b) Account for the surface discontinuity at $(r = a, \varphi = \alpha)$ by a singular surface charge density σ on the cylinder $r = a$. Relate σ to the electric field and hence to derivatives of Φ. Use this relation to find the unknown expansion coefficients for Φ_1 and Φ_2 of part (a).

c) Find the charge density σ_p on the plates ($\varphi = 0$ and $\varphi = \beta$) for $r < a$.



14 Math Phys. (MP)

Two conducting planes intersect at angle β, and each plane is held at potential $\Phi = 0$. At a point with polar coordinates (a, α) inside the wedge, there is a line charge ($\parallel \hat{z}$ axis) carrying charge/length λ = constant. (See Fig.).

(a) For field points (r, φ), write expressions for potential Φ_1 $(r<a, \varphi)$ and Φ_2 $(r>a, \varphi)$ valid over $0 \leq \varphi \leq \beta$. Require $\Phi_1 \to 0$ as $r \to 0$, and $\Phi_2 \to 0$ as $r \to \infty$. Note that at $r=a$, $\Phi_2 = \Phi_1$.

(b) Account for the surface discontinuity at $(r=a, \varphi=\alpha)$ by a singular surface charge density σ on the cylinder $r=a$. Relate σ to ~~find~~ field and hence to derivatives of Φ. Use this relation to find the unknown expansion coefficients in Φ_1 and Φ_2 of Part (a).

(c) Find the charge density σ_p on the plates ($\varphi = 0$ and $\varphi = \beta$) at $r < a$.

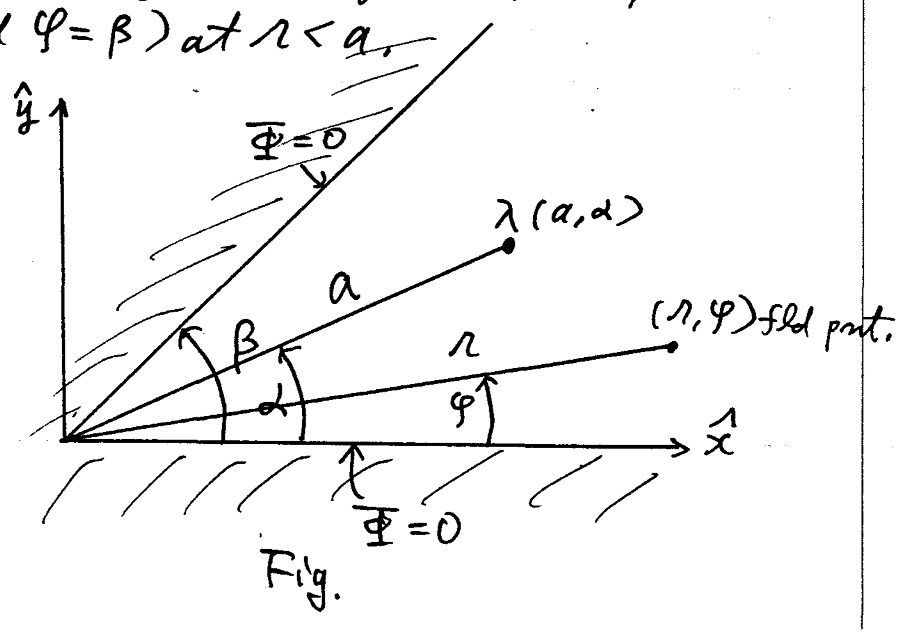


Fig.

MP (   ) cont'.

Ans.:

(a) For $r<a$:

$$\nabla^2\Phi_1 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \Phi_1}{\partial r}\right) + \frac{1}{r^2}\left(\frac{\partial^2 \Phi_1}{\partial \varphi^2}\right) = 0 \quad ①$$

$$\Phi_1(r,\varphi) = R(r)\Psi(\varphi) \quad ②$$

$$R = a r^{\nu} + b r^{-\nu};\ \Psi_\nu(\varphi) = A\cos\nu\varphi + B\sin\nu\varphi;\ \nu \neq 0 \quad ③$$

For B.C. (boundary conditions), $\Phi_1 = 0$ at $\varphi = 0$ & $\varphi = \beta$ & $r \to 0$ $\Phi_1 \to 0$,

Then, $$\boxed{\Phi_1(r,\varphi) = \sum_{n=1}^{\infty} a_n r^{n\pi/\beta}\sin\frac{n\pi}{\beta}\varphi} \quad ④$$ Ans.

For $r > a$:

B.C. $\Phi_2 \to 0$ as $r \to \infty$, $\Phi_2 = 0$ at $\varphi = 0$ & $\varphi = \beta$.

So $$\Phi_2(r,\varphi) = \sum_{n=1}^{\infty} b_n r^{-n\pi/\beta}\sin\frac{n\pi}{\beta}\varphi \quad ⑤$$

at $r = a$, $\Phi_1 = \Phi_2$. So $\Phi_1(a,\varphi) = \Phi_2(a,\varphi) \to$ ④ & ⑤

$$\to \sum_{n=1}^{\infty} a_n a^{n\pi/\beta}\sin\frac{n\pi}{\beta} = \sum_{n=1}^{\infty} b_n a^{-n\pi/\beta}\sin\frac{n\pi}{\beta}$$

$\to$ get $b_n = a_n a^{2n\pi/\beta}$ ⑥.

⑥ → ⑤ & get: $$\boxed{\Phi_2(r,\varphi) = \sum_{n=1}^{\infty} a_n a^{\frac{2n\pi}{\beta}} r^{-\frac{n\pi}{\beta}}\sin\frac{n\pi}{\beta}\varphi} \quad ⑦$$ Ans.

(b) $$E_2 - E_1 = -\left.\frac{\partial \Phi_2}{\partial r}\right|_{r=a} + \left.\frac{\partial \Phi_1}{\partial r}\right|_{r=a} = 4\pi\sigma = 4\pi\lambda\,\delta(\varphi-\alpha) \quad ⑧$$

$$\to \sum_{n=1}^{\infty}\Big[a_n\left(\frac{n\pi}{\beta}\right) a^{2n\pi/\beta} a^{-\frac{n\pi}{\beta}-1}\sin\frac{n\pi\varphi}{\beta}$$

$$+ a_n\left(\frac{n\pi}{\beta}\right) a^{\frac{n\pi}{\beta}-1}\sin\frac{n\pi\varphi}{\beta}\Big] = 4\pi\lambda\,\delta(\varphi-\alpha).$$

$$\to \sum_{n=1}^{\infty} a_n\left(\frac{2n\pi}{\beta}\right) a^{\frac{n\pi}{\beta}-1}\sin\left(\frac{n\pi\varphi}{\beta}\right) = \frac{4\pi\lambda}{a}\,\delta(\varphi-\alpha).$$

MP(   ) Contd.

14-3

So $\frac{a}{4\pi\lambda} a_n \left(\frac{2n\pi}{\beta}\right) a^{\frac{n\pi}{\beta}-1} = \frac{2}{\beta}\int_0^\beta \delta(\varphi-\alpha)\sin\frac{n\pi\varphi}{\beta}\,d\varphi$

$$= \frac{2}{\beta}\sin\left(\frac{n\pi\alpha}{\beta}\right).$$

So $\boxed{a_n = \frac{4\lambda}{n} a^{-\frac{n\pi}{\beta}} \sin\left(\frac{n\pi}{\beta}\alpha\right)}$ (9) Ans//

(c). $\sigma(\varphi=0) = \frac{1}{4\pi} E_\varphi\Big|_{\varphi=0} = -\frac{1}{4\pi}\frac{1}{r}\frac{\partial\Phi_1}{\partial\varphi}\Big|_{\varphi=0}$ (10)

$$= -\frac{1}{4\pi}\frac{1}{r}\frac{\partial}{\partial\varphi}\left[\sum_{n=1}^{\infty} a_n r^{n\pi/\beta}\sin\frac{n\pi}{\beta}\varphi\right]\Bigg|_{\varphi=0}$$

$$= -\frac{1}{4\pi}\frac{1}{r}\sum_{n=1}^{\infty}\frac{n\pi}{\beta} a_n r^{n\pi/\beta}\cos\frac{n\pi}{\beta}\varphi\Big|_{\varphi=0}$$

$$= -\frac{1}{4\pi}\sum_{n=1}^{\infty}\frac{n\pi}{\beta} a_n r^{\frac{n\pi}{\beta}-1} \quad (11)$$

with a_n given by (9).

So, $\boxed{\sigma(\varphi=\beta) = \frac{1}{4\pi}\sum_{n=1}^{\infty}(-1)^n\frac{n\pi}{\beta} a_n r^{\frac{n\pi}{\beta}-1} \text{ with } a_n \text{ given by (9)}}$ (12) Ans//

Well written!

S/

QUESTION #15

Potassium (K) is an alkali metal with $Z = 19$.

a) What is the electron configuration of the ground state?

b) What are the L, S, and J quantum numbers for this state?

c) Discuss the Zeeman splitting for the ground state.

d) The normal Zeeman effect assumes $S = 0$. Discuss the normal Zeeman transitions between the first excited state and the ground state.

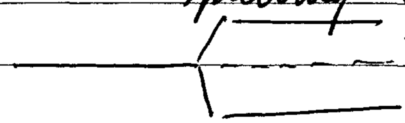
#15 Atomic Physics Sol'n

J. Drumheller

a. $Z=19$; $1s^2\ 2s^2\ 2p^6\ 3s^2\ 3p^6\ \underline{4s^1}$

b. ∴ $L=0$ $S=1/2$ $J=1/2$ $\left({}^2S_{1/2}\right)$

The only Zeeman splitting would be $\pm M_s$ g-factor



$\Delta E = \vec{\mu}\cdot\vec{B} = 2\underbrace{\frac{e\hbar}{2mc}}_{\mu_B}\underbrace{\left(\frac{1}{2}\right)}_{M_s}\underbrace{B}_{B}$

d. The first excited state is, most likely, $4p^1$

∴ $L=1$ $S=1/2$ ($J=\frac{3}{2}$ although not important)

$\left({}^2P_{3/2}\right)$

$\Delta E = \frac{e\hbar}{2mc}B$