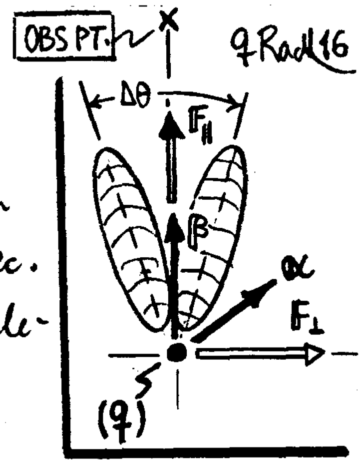
$\measuredangle$ distribution for ultrarelativistic q, at $\beta \to 1$.

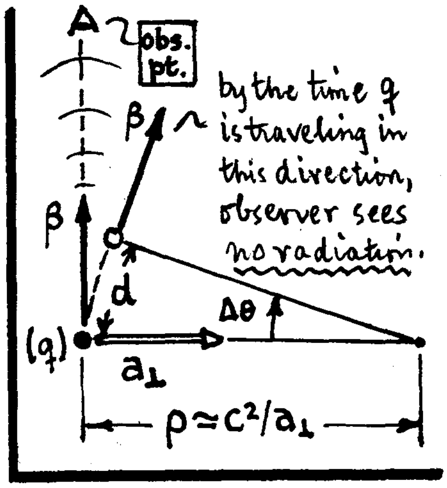


12) A semi-quantitative argument re the radiation emitted by an extremely relativistic q ($\beta \to 1$, $\gamma \to \infty$) is made in Jackson's Sec. 14.4. It is worth repeating here. q is acted upon by an accelerating force $\mathbb{F}$ w/ components $\mathbb{F}_\parallel$ & $\mathbb{F}_\perp$ w.r.t. its velocity β.

For accel^n α due to $\mathbb{F}$, know (from above linear/circular comparison, Eq. (49)) that Comparative radiation rates go as: $\underline{\underline{P_\parallel(\text{due to } \mathbb{F}_\parallel)/P_\perp(\text{due to } \mathbb{F}_\perp) = \frac{1}{\gamma^2} \to 0}}$, $\beta \to 1$.

So, for ultrarelativistic q, radiation loss P comes mainly from $\mathbb{F}_\perp$ & $\alpha_\perp$. As far as loss P is concerned, q moves in a circle whose instantaneous radius is ρ, such that: $a_\perp = v^2/\rho$, or: $\underline{\underline{\rho \simeq c^2/a_\perp}}$.

q's radiation is only seen in a forward direction, in cone of $\measuredangle$ $\underline{\underline{\Delta\theta \sim \frac{1}{\gamma} \to \text{small}}}$. (for $\Delta\theta$, see Jk^n Fig. (14.4), and Eq. (14.40)). So observer -- situated along q's instantaneous travel direction β, only sees q's radiation for a brief instant, before q turns away under the transverse acceleration $\alpha_\perp$.



During the time when observer sees rad^n, q travels a distance d along its "orbit" of radius ρ, so...

$$\rightarrow d = \rho\,\Delta\theta \simeq \rho/\gamma \Rightarrow \underline{\text{broadcast time}}\Big\}\ \Delta t = \frac{d}{v} \simeq \rho/v\gamma. \quad (50)$$

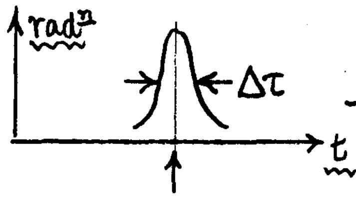
During this "broadcast", the leading edge of the radiation pulse travels distance $\underline{z = c\Delta t \simeq \rho/\beta\gamma}$ toward the observer. The pulse cuts off when q has moved up the z-axis by distance $d\cos\Delta\theta \simeq d$. So pulse length & duration are:

$$\underline{\text{length}}: \Delta z \simeq z - d \simeq \left(\frac{1}{\beta} - 1\right)\frac{\rho}{\gamma} \simeq \rho/2\gamma^3{}^{\bigstar}; \quad \underline{\text{duration}}: \Delta\tau = \frac{\Delta z}{c} \simeq \frac{\rho}{2c\gamma^3}. \quad (51)$$

★ Have used: $\gamma^2 = 1/(1-\beta^2) = 1/(1+\beta)(1-\beta) \simeq 1/2(1-\beta)$, as $\beta \to 1$. Then: $(1-\beta) \simeq 1/2\gamma^2$.

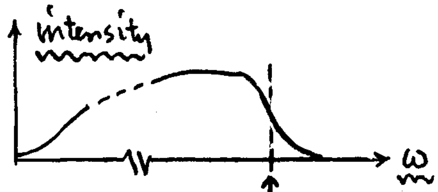
Frequency Spectrum of an ultrarelativistic q.

Now have the picture that as q moves by the observer, observer sees a short, sharp pulse of radiation of duration $\Delta\tau \simeq \rho/2c\gamma^3 \to 0$, so...



q approaching head-on here

this pulse has a frequency spectrum



cutoff frequency $\boxed{\omega_c \sim \frac{1}{\Delta\tau} \sim \left(\frac{c}{\rho}\right)\gamma^3}$ (52)

This is the **Key Result** for this analysis... an ultrarelativistic q radiates over a broad band of frequencies, from $\omega = 0$ up to a cutoff frequency ω_c [Eq.(52)], which can be arbitrarily large as $\beta \to 1$ and $\gamma \to$ large.

If q is actually moving in a synchrotron orbit of radius ρ, then

$$\omega_c \sim (E/mc^2)^3\, \omega_s \quad \begin{cases} E = \text{particle energy,} \\ \omega_s \simeq c/\rho, \text{ orbit freq.} \end{cases} \tag{53}$$

Specs on TANTULUS I (Wisconsin) electron storage synchrotron...

$E = 0.24\ \text{GeV} \Rightarrow \gamma = E/mc^2 = 470.$

$\rho = 0.64\ \text{m} \Rightarrow \omega_s = c/\rho = 4.69 \times 10^8\ \text{Hz}.$

(and CW beam current ~ 200mA)

$$\text{So, } \omega_c \sim \gamma^3 \omega_s \sim 5\times10^{16}\ \text{Hz} \longleftrightarrow \underline{30\,\text{eV photons}}. \tag{54}$$

Such synchrotron radiation is used in surface science experiments.

In Secs. 14.5 & 14.6, Jackson proceeds to analyse synchrotron radiation in detail, working out angular distributions, polarization of the radiation, and frequency distributions. As $\gamma \to$ large, the radiation is mainly confined to lie in the orbit plane, and is polarized in that plane (i.e. the radiation E-field lies in the orbit plane). The high frequency cutoff ($1/e \times$ peak) actually occurs at: $\underline{\omega_c \simeq 3\gamma^3(c/\rho)}$, so estimates in Eq.(54) are conservative.

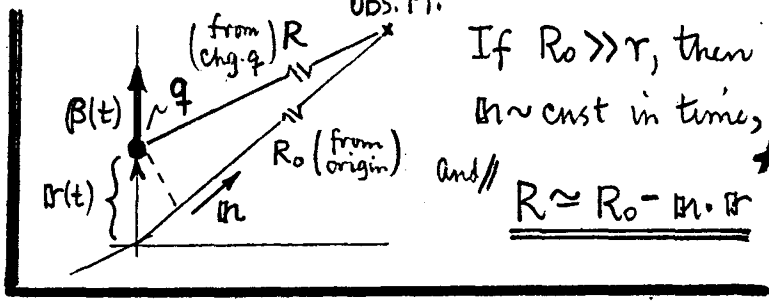
q in a synchrotron orbit. Geometry for the calculation.

Synchrotron Radiation [Jackson, Sec. (14.6)]

13) As an application of the general radiation formula [Jackson Eq. (14.67)]:

energy per unit solid ∡ $d\Omega$ and freq. interval $d\omega$ }
$$\boxed{\frac{d^2I}{d\Omega\, d\omega} \doteq \frac{q^2\omega^2}{4\pi^2 c}\left|\int_{-\infty}^{\infty}[\mathbf{n}\times(\mathbf{n}\times\boldsymbol{\beta}(t))]\,e^{i\omega[t-\frac{1}{c}\mathbf{n}\cdot\mathbf{r}(t)]}\,dt\right|^2,} \quad (1)$$

(for a fully relativistic point q in arbitrary motion at velocity $\boldsymbol{\beta}(t)$ [geometry at right]), we consider a charge in circular motion

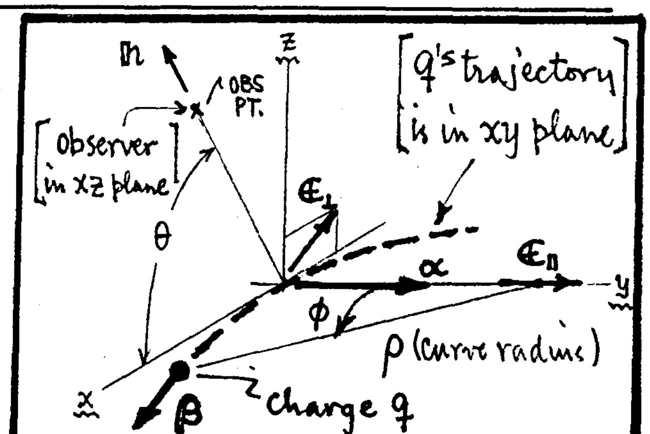


If $R_0 \gg r$, then $\mathbf{n} \sim$ cnst in time, and // $R \simeq R_0 - \mathbf{n}\cdot\mathbf{r}$ ★

(at least instantaneously -- i.e. q could be in a stable synchrotron orbit at radius ρ, or -- at a given instant -- its orbit could have radius-of-curvature ρ).

NOTE In the time integral in Eq. (1), the observer's unit vector $\mathbf{n}$ is cnst in time, for the "radiation zone approxn" we are using.★ In the integrand, however, it is still true that q's velocity $\boldsymbol{\beta}(t)$ can change arbitrarily, in both magnitude & direction. Also, in the phase factor, q's position $\mathbf{r}(t)$ is ~ arbitrary (except $r \ll R_0$). Since -- for a radiation event lasting Δt -- q's position change $\Delta r \sim c\Delta t$, both terms in the phase are comparable, and any further approximations would render Eq. (1) useless. All further simplifications in using Eq. (1) come from choosing "simple" trajectories $\mathbf{r}$ & $\boldsymbol{\beta}$.

14) A "simple" geometry -- for circular motion of q -- is shown at right [Jackson Fig. (14.9)]... q moves in a circular orbit of radius ρ in the xy plane; the observer is in the xz plane, oriented at latitude ∡ θ. As $\beta \to 1$, observer sees only a short flash of radiation



★ This is the only approxn made in getting to Eq. (1), i.e. the observation distance R_0 is large compared to the characteristic length r over which q radiates.

as q passes origin. We want to calculate $d^2I/d\omega d\Omega$ for this motion (for $\beta \to 1$), principally as a fcn of the observer's orientation θ [note: $\theta = 0 \Rightarrow$ observer is in the orbit plane; $\theta \neq 0 \Rightarrow$ observer is out of plane], and we want to keep track of the polarization of the radiation¶... i.e. does the emitted radiation have its $\mathbb{E}$-vector in or out of the orbit plane?

In the integrand in Eq. (1), have: $\mathbf{n} \times (\mathbf{n} \times \beta) = (\mathbf{n} \cdot \beta)\mathbf{n} - \beta$, which gives vector directions associated with the radiation fields. But $\beta = \beta(t)$ changes in time, so it is not suitable for keeping track of polarization. For this reason, the decomposition of $\mathbf{n} \times (\mathbf{n} \times \beta)$ is done in terms of two other fixed vectors:

$$\left[\begin{array}{l} \epsilon_{\parallel} = \text{unit vector along } y\text{-axis (in orbit plane, } \parallel \text{ inst. acceleration } \vec{\alpha}), \\ \epsilon_{\perp} = \mathbf{n} \times \epsilon_{\parallel}, \text{ unit vector } (\epsilon_{\perp} \text{ is out of orbit plane, } \parallel z\text{-axis when } \theta \to 0). \end{array}\right. \quad (2)$$

Straightforward vector conjuring then gives the integrand term...

$$\longrightarrow \mathbf{n} \times (\mathbf{n} \times \beta) = \beta\left[-\epsilon_{\parallel} \sin\phi + \epsilon_{\perp} \cos\phi \sin\theta\right], \quad (\phi = vt/\rho), \quad (3)$$

where $\phi = vt/\rho$ is the azimuthal ∡ traced by q's orbit near the origin. Note the time dependence of $\beta(t)$ is now neatly sequestered in ϕ. Note also that with fixed $\epsilon_{\parallel}$ (in orbit plane) and $\epsilon_{\perp}$ (out of orbit plane), we can keep track of the polarization of the contributions to $d^2I/d\omega d\Omega$.

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15) The other t-dept term in Eq. (1) is the phase. In the chosen geometry...

$$\longrightarrow \omega\left[t - \frac{1}{c}\mathbf{n} \cdot \mathbf{r}(t)\right] = \omega\left[t - \frac{1}{c}\rho \sin\phi \cos\theta\right], \quad (\phi = vt/\rho) \quad (4)$$

This is the end of the beginning... now we just need to put this phase in Eq. (1), together with $\mathbf{n} \times (\mathbf{n} \times \beta)$ of Eq. (3), to find the desired radiation intensity. Unfortunately, the resulting integral is not doable (i.e. not tabulated), and so

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¶ The polarization is important for experimental reasons (transition selection rules, etc.)

Sync Rad (cont'd). Frequency-angle distribution. Remarks.

We need some further approximations to get a compact result. What proves useful is to note -- in $\mathbf{n}\times(\mathbf{n}\times\boldsymbol{\beta})$ of Eq. (3) and the phase of Eq. (4) -- that (because of the headlight effect as $\beta\to 1$) no usefully large radiation will be detected for large θ (out of the orbit plane); then $\theta\to$ small, and the trig fcns in Eqs (3) & (4) are: $\sin\theta\simeq\theta$, $\cos\theta\simeq 1-\frac{1}{2}\theta^2$. Also, the $\measuredangle\,\phi = vt/\rho \to$ small, since the observed radiation pulse is short, and -- keeping terms to $\mathcal{O}(t^3)$ -- $\sin\phi\simeq\phi-\frac{1}{6}\phi^3$, $\cos\phi\simeq 1-\frac{1}{2}\phi^2$. Putting this all together, find...

$$\rightarrow \underline{\underline{\frac{d^2I}{d\omega\, d\Omega} \simeq \frac{q^2\omega^2}{4\pi^2 c}\left|-\epsilon_\parallel Z_\parallel(\omega)+\epsilon_\perp Z_\perp(\omega)\right|^2}};$$

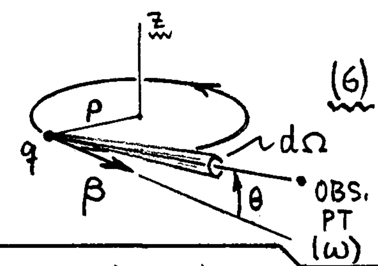
$$\left.\begin{aligned} \text{w/}\quad Z_\parallel(\omega) &= \frac{\rho}{c}(\theta^2+\gamma^{-2})\int_{-\infty}^{\infty} x\, e^{\frac{3}{2}i\xi\left(x+\frac{1}{3}x^3\right)}dx, \\ Z_\perp(\omega) &= \frac{\rho}{c}\,\theta\,(\theta^2+\gamma^{-2})\int_{-\infty}^{\infty} e^{\frac{3}{2}i\xi\left(x+\frac{1}{3}x^3\right)}dx. \end{aligned}\right\rangle \underline{\underline{\xi = \frac{\omega\rho}{3c}(\theta^2+\gamma^{-2})}}. \quad (5)$$

...after some arithmetic [N.B. $\mathcal{O}(t^3)$ terms are retained in the phase only]. The major approxns now are that $\theta\to$ small and $\beta\to 1$. But the integrals are now "well-known"... they give modified Bessel fcns $K_\nu(\xi)$ for $\nu=\frac{1}{3}$ and $\frac{2}{3}$. Finally, the detailed frequency distribution for "synchrotron radiation" is (for $\theta\to 0$ & $\gamma\gg 1$):

ickson 4.83)

$$\boxed{\frac{d^2I}{d\omega\, d\Omega} \simeq \frac{q^2}{3\pi^2 c}\left(\frac{\omega\rho}{c}\right)^2(\theta^2+\gamma^{-2})^2\Big[\underbrace{K^2_{\frac{2}{3}}(\xi)}_{\text{pol}^{\underline{3}}\text{n} \parallel \text{ orbit plane}}+\underbrace{\left(\frac{\theta^2}{\theta^2+\gamma^{-2}}\right)K^2_{\frac{1}{3}}(\xi)}_{\text{pol}^{\underline{3}}\text{n} \perp \text{ orbit plane}}\Big]}, \quad (6)$$

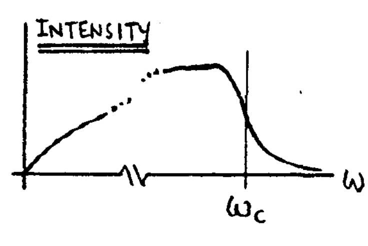
$$\text{w/}\quad \xi = \frac{\omega\rho}{3c}(\theta^2+\gamma^{-2}).$$



REMARKS

1. This formula ⇒ q's radiation is polarized mainly in orbit plane ($\parallel$ pol$^{\underline{3}}$n intensity is 7× $\perp$ pol$^{\underline{3}}$n intensity (over all ω)).
2. Since $K_\nu(\xi)\sim\sqrt{\pi/2\xi}\,e^{-\xi}$ as $\xi\to$ large, the radiation is negligible at "large" θ's (for ω fixed) or at "high" ω's (for θ fixed). The low freq. behavior follows from: $K_\nu(\xi)\simeq\frac{1}{2}\Gamma(\nu)\left[\frac{2}{\xi}\right]^\nu$, $\xi\to 0$.
3. Overall, q's radiation is ~ confined to the orbit plane, and broadcasts frequencies from $\omega=0$ up to a cutoff $\boxed{\omega\simeq\omega_c = 3\gamma^3 c/\rho}$. The asymptotic behaviors are (up to numerical factors):

$$\frac{d^2I}{d\omega\, d\Omega}\sim\frac{q^2}{c}(\omega\rho/c)^{\frac{2}{3}},\ \omega\to 0;\quad \frac{d^2I}{d\omega\, d\Omega}\sim\frac{q^2}{c}\gamma^2\left(\frac{\omega}{\omega_c}\right)e^{-\frac{2\omega}{\omega_c}},\ \omega\to\infty \Rightarrow$$



4. Freqs $\omega>\omega_c$ @ useful intensity only up to $\theta_c\simeq\frac{1}{\gamma}\sqrt{\frac{\omega_c}{3\omega}}$. Etc.