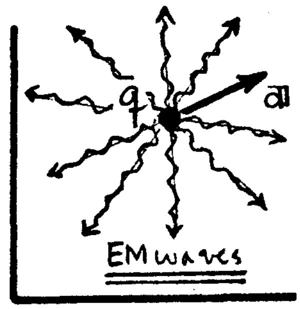
Simple Radiating Systems [Jk$^{\underline{n}}$ Secs. 9.1 & 9.2]



Any accelerated charge q "radiates", i.e. it acts as a source of EM waves which radiate away from q and carry off energy & momentum. The $\mathbb{E}$ & $\mathbb{B}$ fields in this radiation have new characteristics -- they fall off with distance R from q as $1/R$, vs. the $1/R^2$ falloff for static fields from point charges, and as $R\to\infty$ the radiation $\mathbb{E}$ & $\mathbb{B}$ fields combine to form outgoing transverse EM waves. It turns out to be ~ difficult to treat radiation from an arbitrarily moving single q (since q radiates over a broad band of frequencies ω -- see Jk$^{\underline{n}}$ Ch. 14). It is easier to treat the case where the q's are moving cooperatively, say in a narrow frequency band characteristic of an AC current.

For narrow band frequencies, or better yet a single frequency ω, we are thinking in terms of a Fourier rep$^{\underline{n}}$ for -- say -- a current source of radiation:

$$\rightarrow \mathbb{J}(\mathbb{r},t) = \int_{-\infty}^{\infty} \tilde{\mathbb{J}}(\mathbb{r},\omega)\, e^{-i\omega t}\, d\omega \Rightarrow \mathbb{J} \text{ is a large collection of SHO's @ freqs } \omega, \text{ ampls. } \tilde{\mathbb{J}}, \text{ time dep} \propto e^{-i\omega t}. \quad (1)$$

It would be useful to find out how the individual SHO's $\tilde{\mathbb{J}}e^{-i\omega t}$ radiate. Then, later, we can add up the individual effects (in principle) to get the Big Story.¶ So we will look at radiating systems where the sources have harmonic t-dependence:

$$\boxed{\rho(\mathbb{r},t) = \tilde{\rho}(\mathbb{r},\omega)e^{-i\omega t}, \quad \mathbb{J}(\mathbb{r},t) = \tilde{\mathbb{J}}(\mathbb{r},\omega)e^{-i\omega t}.} \quad (2)$$

Jackson writes $\rho(\mathbb{r}) \leftarrow \tilde{\rho}(\mathbb{r},\omega)$ and $\mathbb{J}(\mathbb{r}) \leftarrow \tilde{\mathbb{J}}(\mathbb{r},\omega)$, suppressing the ω-dependence.

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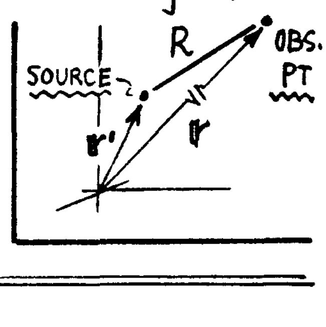
1) Our starting point is the solution to the wave eq$^{\underline{tn}}$ for the 3-vector potential $\mathbb{A}$ in terms of its source $\mathbb{J}$. In a non-medium ($\epsilon=1$, $\mu=1$, EMP velocity $= c$):

$$\left[\begin{array}{l} \mathbb{A}(\mathbb{r},t) = \frac{1}{c}\int d^3x' \int dt' \, \frac{1}{R}\, \mathbb{J}(\mathbb{r}',t')\, \delta(t'-t_R), \\ \text{w/ } R = |\mathbb{r}-\mathbb{r}'|, \text{ and } t_R = t - \frac{1}{c}R(t_R), \text{ retarded time.} \end{array}\right] \quad (3)$$



¶ Of course, this superposition idea holds only in linear systems.

Simp Rad (cont'd) Monochromatic Fields & Potentials.

This form for $\mathbb{A}$ is a transliteration of our previous solution to the wave eqtn for $\mathbb{A}$ [class notes on Maxwell's Eqtns, p. ME 18; or Jackson, Secs. 6.6 and 12.11]. The δ-fcn in the integrand $\Rightarrow$ the RHS is evaluated at the retarded time t_R. If, in Eq. (3), we put $\mathbb{J} = \tilde{\mathbb{J}}\, e^{-i\omega t}$, per Eq. (2), and then integrate over t'...

$$\rightarrow \mathbb{A}(\mathbf{r},t) = \tilde{\mathbb{A}}(\mathbf{r},\omega)\, e^{-i\omega t}, \quad \text{w/} \left[\tilde{\mathbb{A}}(\mathbf{r},\omega) = \frac{1}{c}\int d^3x' \underbrace{\left(\frac{e^{ikR}}{R}\right)}_{\text{a Green's fcn!}} \tilde{\mathbb{J}}(\mathbf{r}',\omega),\right] \tag{4}$$

where: $\underline{k = \omega/c}$, is the wave # of radiation at freq. ω.

So $\mathbb{A}$ is monochromatic if $\mathbb{J}$ is[†]; for monochromatic sources, all the potentials and fields have the same simple harmonic time dependence $e^{-i\omega t}$. Then we can afford to work with just the amplitudes $\tilde{\mathbb{A}}$ etc. The field ampl's are*

$$\tilde{\mathbb{B}} = \nabla\times\tilde{\mathbb{A}}, \quad \tilde{\mathbb{E}} = \frac{i}{k}\nabla\times\tilde{\mathbb{B}} - \cancel{(4\pi i/\omega)\tilde{\mathbb{J}}} \rightarrow 0, \text{ outside source, i.e. when: } |\mathbf{r}(\text{obs.pt})| \gg |\mathbf{r}'(\text{sourcept})|. \tag{5}$$

The monochromatic radiation problem is thus reduced to finding $\tilde{\mathbb{A}}$ in Eq. (4).*

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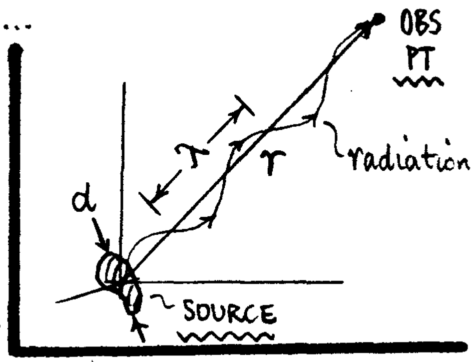
2) Evaluation of $\tilde{\mathbb{A}}$ in Eq. (4) leads naturally to a discussion of distance scales. There are 3 characteristic lengths in the problem, viz...

$$\left[\begin{array}{l} d: \text{size of source (extent of } |\Delta\mathbf{r}'(\text{sourcept})|), \\ \lambda = 2\pi/k: \text{wavelength of emitted radiation}, \\ r: \text{distance to observation point.} \end{array}\right\} \tag{6}$$



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* The fields $\tilde{\mathbb{B}}$ & $\tilde{\mathbb{E}}$ in Eq. (5) are entirely determined by $\tilde{\mathbb{A}}$; we don't need the scalar ptl. $\tilde{\Phi}$. This is because flds with $e^{-i\omega t}$ time dependence can't be monopolar. See Jk[n], p. 394.

* Expression for $\tilde{\mathbb{E}}$ is just Ampere's Law: $\nabla\times\mathbb{B} = \frac{1}{c}(\partial\mathbb{E}/\partial t) + \frac{4\pi}{c}\mathbb{J}$, after putting in the $(\mathbb{B},\mathbb{E},\mathbb{J}) = (\tilde{\mathbb{B}},\tilde{\mathbb{E}},\tilde{\mathbb{J}})\,e^{-i\omega t}$, doing $\frac{\partial}{\partial t}$, cancelling $e^{-i\omega t}$, and setting $\omega = kc$.

† Likewise, for the monochromatic density ρ in Eq. (2), the scalar potential is:

$$\Phi(\mathbf{r},t) = \tilde{\Phi}(\mathbf{r},\omega)\, e^{-i\omega t}, \quad \tilde{\Phi}(\mathbf{r},\omega) = \int d^3x' \frac{e^{ikR}}{R}\tilde{\rho}(\mathbf{r},\omega)$$

Φ is monochromatic if ρ is. No dispersion!

Simp Rad (cont'd) Radiation Distance Scales.

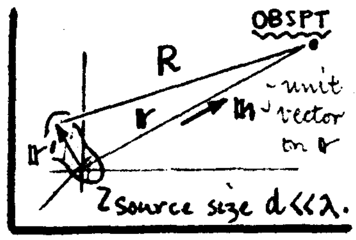
We (almost always) impose: $\underline{d \ll \lambda}$, which is the nonrelativistic case[&] and which fits cases of practical interest (radiation @ $\lambda \sim 5000$ Å from an atom of $d \sim 1$ Å; radiation @ $\lambda \sim 300$ m (1 MHz) from a radio antenna of $d \sim 30$ m). Then there are 3 orderings between λ & r which define zones where the fields behave in very different ways...

① NEAR (static) ZONE .... $d \ll r \ll \lambda \rightarrow$ fields ~ Coulombic;

② INTERMEDIATE (induction) ZONE .. $d \ll r \sim \lambda \rightarrow$ mixed $\binom{\text{Coulomb}}{+\text{EMP}}$ case; (7)

③ FAR (radiation) ZONE .... $d \ll \lambda \ll r \rightarrow$ true radiation fields (transverse planewaves)

The reason why the comparative size of λ & r is important seen by looking at $\tilde{\mathbb{A}}$:



$$\rightarrow R = |\mathbf{r} - \mathbf{r}'| \simeq r - \mathbf{n}\cdot\mathbf{r}', \quad \text{if } d \ll r \left(\begin{smallmatrix}\text{RAD'N}\\ \text{ZONE}\end{smallmatrix}\right)$$

So/ $$\tilde{\mathbb{A}}(\mathbf{r},\omega) = \frac{1}{c}\int d^3x' \, \tilde{\mathbb{J}}(\mathbf{r}',\omega)\, \frac{e^{ikR}}{R}$$

$$\simeq \left(\frac{e^{ikr}}{cr}\right)\int d^3x' \, \tilde{\mathbb{J}}(\mathbf{r}',\omega)\, e^{-\overbrace{(ik\mathbf{n}\cdot\mathbf{r}')}^{\sim \mathcal{O}(2\pi \frac{d}{\lambda}) \ll 1}} + \underbrace{\cancel{\mathcal{O}(d/r)}}_{\text{neglect}}$$

or/ $$\boxed{\tilde{\mathbb{A}}(\mathbf{r},\omega) \simeq \left(\frac{e^{ikr}}{cr}\right)\sum_{m=0}^{\infty}\frac{(-ik)^m}{m!}\int d^3x' \, \tilde{\mathbb{J}}(\mathbf{r}',\omega)\,[\mathbf{n}\cdot\mathbf{r}']^m.} \quad \left(\begin{smallmatrix}\text{RAD'N}\\ \text{ZONE}\end{smallmatrix}\right) \qquad (8)$$

Such an expansion is accurate only if $d \ll \lambda$, and $d \ll r$, with $\mathcal{O}(d/r)$ negligible. Since the $\mathcal{O}(d/\lambda)$ terms are kept [the m^{th} term in the sum is $\sim (d/\lambda)^m$], then r is the largest length present, and the appropriate size ordering for Eq. (8) is: $\underline{d \ll \lambda \ll r \rightarrow \infty}$. This is the RADIATION ZONE ③ in Eq. (7) above.

In what follows, we shall fool around a bit with the radiation zone $\tilde{\mathbb{A}}$ of Eq. (8). We leave the static & induction zones to the City Planning Commission.

& The velocity of the charges moving over length d @ $e^{-i\omega t}$ is $v \sim \omega d$. Then $v \ll c \Rightarrow d \ll c/\omega \sim \lambda$. So $d \ll \lambda$ is the nonrelativistic case, per claim.