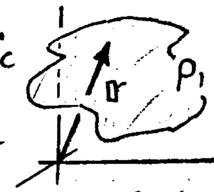
Φ 519 Problems { Set #1 : Probs. 1-3.
Assigned 23 Sept 88; due 30 Sept 88.



[redacted]: Problems are graded at 10 pts. each, unless indicated otherwise.

1 Consider a region of otherwise empty space where there are electric charge & current densities $\rho$ & $\mathbb{J}$; both $\rho$ & $\mathbb{J}$ are general fcns of position $\mathbb{r}$ & time $t$. Wanted: the 4 Maxwell Eqtns specifying the electric & magnetic fields $\mathbb{E}$ & $\mathbb{B}$ in the region. You should already "know" Maxwell's Eqtns (from previous courses), so this problem is just bookkeeping. Refer to (and cite) whatever sources you use, up to and including Jackson.




(A) Write the Maxwell Eqtns for the above region, in cgs units only, in both their differential and integral forms. Specify any constants which appear.

(B) Give a brief description (25 words or less!) of the physical basis of each of the Maxwell Eqtns in part (A). You can start by naming them.

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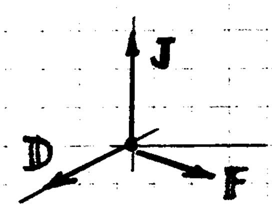
2 Suppose $\mathbb{F}$ is an unknown vector, about which we do know:

$$\rightarrow \mathbb{D}\cdot\mathbb{F} = \rho, \quad \mathbb{D}\times\mathbb{F} = \mathbb{J},$$




where $\mathbb{D}$, $\rho$ and $\mathbb{J}$ are all known quantities. Solve for $\mathbb{F}$ in terms of $\mathbb{D}$, $\rho$, and $\mathbb{J}$. Your solution should be a vector eqtn for $\mathbb{F}$ (not involving angles, etc.). If $\rho$ & $\mathbb{J}$ are identified as charge & current densities, discuss analogies to the solution of the system: $\nabla\cdot\mathbb{F} = \rho$, $\nabla\times\mathbb{F} = \mathbb{J}$.

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3 Show that any vector field $\mathbb{F}(\mathbb{r})$ can be decomposed into transverse (T) and longitudinal (L) parts: $\mathbb{F} = \mathbb{F}_T + \mathbb{F}_L$, such that: $\nabla\cdot\mathbb{F}_T = 0$, $\nabla\times\mathbb{F}_L = 0$, and also:

$$\boxed{\mathbb{F}_T(\mathbb{r}) = \frac{1}{4\pi}\nabla\times\left\{\nabla\times\int_\infty \mathbb{F}(\mathbb{r}')\frac{d^3r'}{|\mathbb{r}-\mathbb{r}'|}\right\}, \quad \mathbb{F}_L(\mathbb{r}) = (-)\frac{1}{4\pi}\nabla\left\{\int_\infty [\nabla'\cdot\mathbb{F}(\mathbb{r}')]\frac{d^3r'}{|\mathbb{r}-\mathbb{r}'|}\right\}}$$

The integrals are over all space. HINT: use $\nabla^2(1/|\mathbb{r}-\mathbb{r}'|) = (-)4\pi\,\delta(\mathbb{r}-\mathbb{r}')$.

[redacted]: Jackson uses this decomposition in his Eqs. (6-47) → (6-50), p. 222.

[redacted] Set #1: Probs 1-3.
Assigned 9/23/88; due 9/30/88.

List Maxwell's Eqtns of E & M, in cgs units, and describe.

9/24/88

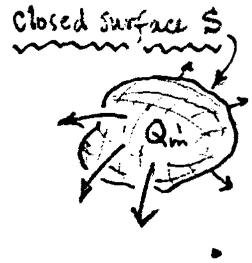
1) A concise statement of the cgs Maxwell Eqs. appears in Jackson Eq. (6.28). In "otherwise empty space", the fields are related by: $\mathbb{D} = \epsilon\,\mathbb{E}$, $\mathbb{B} = \mu\,\mathbb{H}$, and $\epsilon = 1$, $\mu = 1$ in the cgs system. The densities $\rho$ & $\mathbb{J}$ are the "true" densities -- there are no polarization or magnetization charges. If $\underline{\underline{c}}$ = light velocity (empty space):

| # | Differential Form | Integral Form | Name of Law | Remarks |
|---|---|---|---|---|
| 1 | $\nabla\cdot\mathbb{E} = 4\pi\rho$ | $\oint_S \mathbb{E}\cdot d\mathbb{s} = 4\pi Q_{in}$ | Gauss | $Q_{in} = \int \rho\, dv$ (inside $S$) |
| 2 | $\nabla\cdot\mathbb{B} = 0$ | $\oint_S \mathbb{B}\cdot d\mathbb{s} = 0$ | (Dirac) | — |
| 3 | $c\nabla\times\mathbb{E} = -\dot{\mathbb{B}}$ | $c\oint_L \mathbb{E}\cdot d\mathbb{l} = (-)\,\dot{\Phi}_m$ | Faraday | $\Phi_m = \int \mathbb{B}\cdot d\mathbb{s}$ (inside $L$) |
| 4 | $c\nabla\times\mathbb{B} = 4\pi\mathbb{J} + \dot{\mathbb{E}}$ | $c\oint_L \mathbb{B}\cdot d\mathbb{l} = 4\pi I_{in} + \dot{\Phi}_e$ | Maxwell-Ampere | $I_{in} = \int \mathbb{J}\cdot d\mathbb{s}$; $\Phi_e = \int \mathbb{E}\cdot d\mathbb{s}$ |

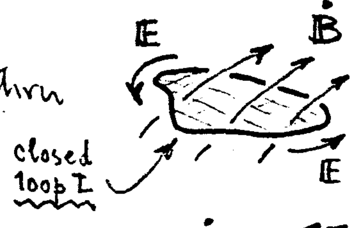
The "$\cdot$" signifies $\partial/\partial t$ (not $d/dt$). The differential forms hold locally (in any osmal neighborhood) at the spacetime pt. $(\mathbb{r}, t)$. The integral forms hold globally, over extended geometries as noted below.

2) ① & ② relate $\mathbb{E}$ & $\mathbb{B}$ to their possible scalar (monopole) sources:

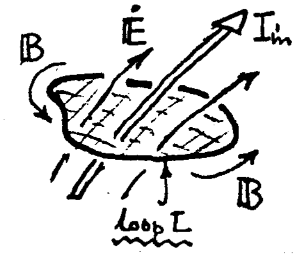
① ⇒ net $\mathbb{E}$-flux through a closed surface $S$ is proportional to the electric charge $Q_{in}$ within; ② ⇒ no magnetic monopoles.




③ is the law of induction: a changing magnetic flux $\Phi_m$ thru a closed loop $L$ generates an emf in the loop.



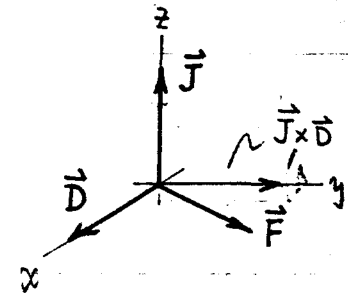

④ implies that the magnetic field around loop $L$ can be generated in two ways: by the enclosed current $I_{in}$ (per Ampere), and/or by a changing electric flux $\dot{\Phi}_e$ (Maxwell).



# 519 Suppl. Problem Solutions

9/24/84

1 Solve the system: $\vec{D}\cdot\vec{F}=\rho$, $\vec{D}\times\vec{F}=\vec{J}$, for unknown $\vec{F}$ ($\vec{D}$, $\rho$ & $\vec{J}$ known).




1) Label the axes $xyz$ as shown: $x$-axis along $\vec{D}$, $z$-axis along $\vec{J}$, which must be $\perp$ the $xy$ plane that contains both $\vec{D}$ & $\vec{F}$ (this is because $\vec{D}\times\vec{F}=\vec{J}$). Notice that $\vec{J}\times\vec{D}$ lies along the $y$-axis. Then $\vec{F}$, lying in the $xy$ plane, must be a linear combination of the form...

$$\vec{F}=\alpha\vec{D}+\beta\,\vec{J}\times\vec{D}\,,\quad \alpha\ \&\ \beta = \text{coefficients to be found.}$$

2) But: $\rho=\vec{D}\cdot\vec{F}=\alpha D^2+\beta\underbrace{\vec{D}\cdot(\vec{J}\times\vec{D})}_{0}=\alpha D^2$, so: $\alpha=\rho/D^2$.

And: $\vec{J}=\vec{D}\times\vec{F}=\alpha\underbrace{\vec{D}\times\vec{D}}_{0}+\beta\,\vec{D}\times(\vec{J}\times\vec{D})=\beta\,[\vec{J}D^2-\vec{D}\underbrace{(\vec{D}\cdot\vec{J})}_{0\ (\text{obvious})}]$,

So// $\beta=1/D^2$, and overall the desired vector $\vec{F}$ is...

$$\boxed{\vec{F}=\frac{1}{D^2}\left[\vec{D}\rho-\vec{D}\times\vec{J}\right]}\ \checkmark\quad \text{where: } \rho=\vec{D}\cdot\vec{F},\ \ \vec{J}=\vec{D}\times\vec{F}$$

3) If we replace $\vec{D}$ by the symbol $\vec{\nabla}$, have: $\vec{F}=(1/\nabla^2)[\vec{\nabla}\rho-\vec{\nabla}\times\vec{J}]$. Then, if $\nabla^2$ is a differential operator, $1/\nabla^2$ (the inverse operator) must be some kind of integral operator (in fact it is). This suggests that when $\vec{F}$ varies throughout space, the solution of the system: $\vec{\nabla}\cdot\vec{F}=\rho$, $\vec{\nabla}\times\vec{F}=\vec{J}$, will look like...

$$\vec{F}\sim\vec{\nabla}\int\rho\cdot\text{something}\cdot d\tau-\vec{\nabla}\times\int\vec{J}\cdot\text{something}\cdot d\tau\,.$$

In fact this turns out to be the case, as we know from our Vector Calculus Theorem.

# $\phi$519 Prob. Solutions

9/25/94

**S** Show: $\vec{F} = \vec{F}_T + \vec{F}_L$, where: $\vec{F}_T = \frac{1}{4\pi}\vec{\nabla}\times\left[\vec{\nabla}\times\int_\infty \vec{F}(\vec{x}')\frac{d^3x'}{|\vec{x}-\vec{x}'|}\right]$,

and $\vec{\nabla}\cdot\vec{F}_T = 0$, $\vec{\nabla}\times\vec{F}_L = 0$. $\quad \vec{F}_L = (-)\frac{1}{4\pi}\vec{\nabla}\left[\int_\infty \vec{\nabla}'\cdot\vec{F}(\vec{x}')\frac{d^3x'}{|\vec{x}-\vec{x}'|}\right]$

1) Since $\vec{F}_T \propto \vec{\nabla}\times$(vector), then $\vec{\nabla}\cdot\vec{F}_T = 0$ is immediate; also $\vec{\nabla}\times\vec{F}_L = 0$ follows immediately from $\vec{F}_L \propto \vec{\nabla}$(vector). As for the details, compute directly...

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{V}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{V}) - \nabla^2\vec{V}, \text{ by usual vector operator identity,}$$

so// $$\vec{F}_T = \frac{1}{4\pi}\left[\vec{\nabla}\left(\vec{\nabla}\cdot\int_\infty\frac{\vec{F}(\vec{x}')\,d^3x'}{|\vec{x}-\vec{x}'|}\right) - \nabla^2\int_\infty\frac{\vec{F}(\vec{x}')\,d^3x'}{|\vec{x}-\vec{x}'|}\right].$$

2) The $\vec{\nabla}$ operates on $\vec{x}$ cds, not $\vec{x}'$. Take $\nabla^2$ inside $2^{ND}$ integral and note that: $\nabla^2(1/|\vec{x}-\vec{x}'|) = -4\pi\,\delta(\vec{x}-\vec{x}')$, so: $\nabla^2\int_\infty\frac{\vec{F}(\vec{x}')\,d^3x'}{|\vec{x}-\vec{x}'|} = -4\pi\,\vec{F}(\vec{x})$. Inside the $1^{ST}$ integral, note that: $\vec{\nabla}\cdot(\vec{F}/|\vec{x}-\vec{x}'|) = \vec{F}\cdot\vec{\nabla}(1/|\vec{x}-\vec{x}'|)$, since $\vec{\nabla}\cdot\vec{F}(\vec{x}')\equiv 0$. Then, use: $\vec{\nabla}(1/|\vec{x}-\vec{x}'|) = -\vec{\nabla}'(1/|\vec{x}-\vec{x}'|)$, and write...

$$\vec{F}_T = (-)\frac{1}{4\pi}\vec{\nabla}\left(\int_\infty d^3x'\,\vec{F}(\vec{x}')\,\vec{\nabla}'\frac{1}{|\vec{x}-\vec{x}'|}\right) + \vec{F}(\vec{x}).$$

3) For the remaining integral, integrate by parts: $\int_\infty dx'\,F\frac{\partial}{\partial x'}\psi = F\psi\Big|_\infty - \int_\infty dx'\,\psi\frac{\partial F}{\partial x'}$, and claim the integrated part (boundary term) vanishes at $\infty$. Then...

$$\vec{F}_T = +\frac{1}{4\pi}\vec{\nabla}\left(\int\frac{d^3x'}{|\vec{x}-\vec{x}'|}\nabla'\cdot\vec{F}(\vec{x}')\right) + \vec{F}, \quad \text{w}/\!/ \ \vec{F} = \vec{F}_T + \vec{F}_L.$$

The $1^{ST}$ term RHS is just $(-)\vec{F}_L$, as specified above. Thus, we have an identity which obeys the relation $\vec{F}_T + \vec{F}_L = \vec{F}$, with the integral assignments as given above. **QED**