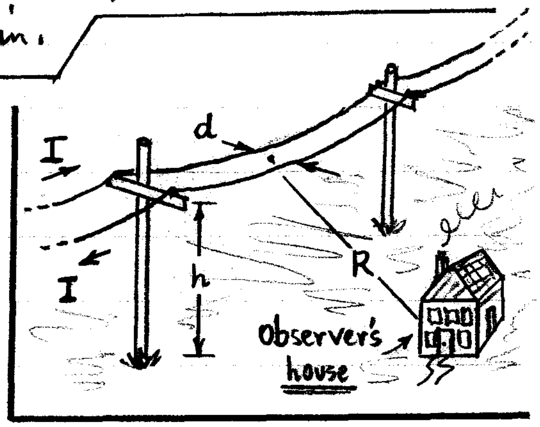
ϕ520 Final Exam (in class, 3 hrs) — Wed., 20 Mar. 1991

This exam is open-book, open notes, and is worth 150 points. There are 2 pages to this exam, and 5 problems with point-values as marked. In your solutions, box the answer when appropriate, number pages consecutively, put your name on p. 1, and staple pages together before handing them in.

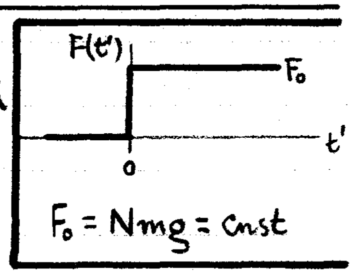


① [30 pts]. In class, we briefly discussed the phenomenon of "ELF radiation", i.e. the broadcast of Extra Low Frequency EM waves (such as those from power lines at 60 Hz), which may constitute a health hazard. Consider the situation as sketched: an observer's house is at distance $R \sim 100\,m$ from power lines which are at height $h \sim 10\,m$; the lines are distance $d \sim 1\,m$ apart, and carry current $I \sim 100\,A$ at $60\,Hz$, phased so that the currents are (instantaneously) in opposite directions. If these power lines are a major feeder, I may be delivered at voltages $\sim 7200\,V$.

A. Show that any radiation fields, as such, are entirely negligible in this system.

B. The observer is not exposed to any EM radiation. What fields can he see?

C. Estimate the size of the fields in part B, for the given geometry and for $\left\{ \begin{array}{l} I = 100\,A, \\ V = 7.2\,kV. \end{array} \right.$



② [30 pts]. Jackson illustrates the "pre-acceleration" of a charged particle (e, m) in Fig. (17.1), p. 798. For an external force:

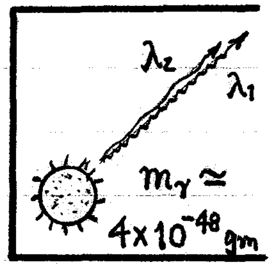
$\longrightarrow F(t') = 0$ for $t' < 0$; $F(t') = F_0 = \text{cnst}$, for $t' \geq 0$;

he shows that the particle begins to move at $t' < 0$, before the force is turned on. Consider the 1D problem, with (e, m) initially at rest (at $t' = -\infty$) somewhere on the x-axis, and the above $F(t')$ acting along that axis.

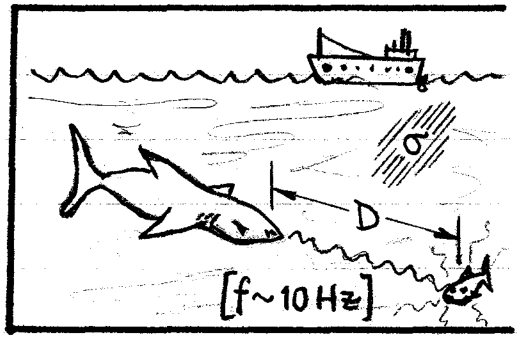
A. Find the distance Δx that (e, m) moves during its pre-acceleration period.

B. If $F_0/m = Ng$, where g = gravitational acceleration, find that value of N needed to give $\Delta x \sim 10^{-13}\,cm$ (movement across a nuclear diameter) in part A.

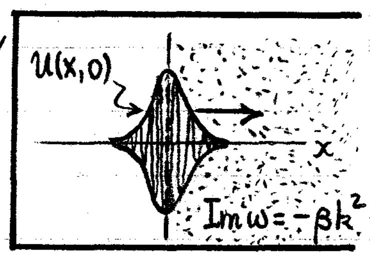
C. If F_0 is Coulombic, what charge separation is needed for the acceleration in part B?



③ [30 pts]. Suppose the photon has a small mass, about the size quoted by Jackson on p. 6 (as an experimental upper limit): $m_\gamma \simeq 4\times10^{-48}$ gm. Consider two photons emitted from a star at the same time and in the same direction: photon #1 is in the ultraviolet at wavelength $\lambda_1 = 1000$ Å, while photon #2 is in the infrared at $\lambda_2 = 10{,}000$ Å. How far will the photons have traveled by the time #2 falls behind #1 by its own wavelength (i.e. by λ_2?).



④ [30 pts]. It has been claimed that sharks can find their prey by detecting weak, low-frequency EM signals "broadcast" by the prey. Suppose these signals are at frequency $f \sim 10$ Hz, and the shark can sense them down to 1/1000 of their original broadcast strength. Given that $\sigma = 4.3$ (ohm-m)$^{-1}$ is the conductivity of seawater [note MKS units], calculate the maximum distance D at which the shark can sense its prey.



⑤ [30 pts.]. At time $t=0$, an EM pulse, at carrier wave# k_0 and width Δx, enters a medium with an absorptive component specified by: $\mathrm{Im}\,\omega(k) = -\beta k^2$, where ω = frequency, k = wave #, and β = cnst > 0. The initial pulse has amplitude: $u(x,0) \propto e^{ik_0 x}\, e^{-(x/\Delta x)^2}$. It will help you to know: $\int_{-\infty}^{\infty} e^{-p^2x^2 \pm qx}\,dx = (\sqrt{\pi}/p)\, e^{q^2/4p^2}$, for $\mathrm{Re}\,p > 0$.

A. Normalize $u(x,0)$ to carry a given energy $\mathcal{E}$, i.e. $\int_{-\infty}^{\infty} |u(x,0)|^2 dx = \mathcal{E}$.

B. Find the amplitude $A(k)$ such that $u(x,0) = \int_{-\infty}^{\infty} A(k)\, e^{ikx}\, dk$.

C. Find the pulse energy at $t>0$; i.e. $W(t) = \int_{-\infty}^{\infty} |u(x,t)|^2 dx$. [HINT: $W(t)$ is most conveniently calculated in terms of $|A(k)|^2$ & $\mathrm{Im}\,\omega(k)$]. Sketch $W(t)$ vs. t and discuss. How does $W(t)$ behave as $t \to$ large?

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⑥ [extra credit]. Design an electron.

ϕ 520 Final Solutions (1991)

① [30 pts]. Analyse ELF "radiation".

A. At $\omega = 2\pi f$, $f = 60$ Hz, wavelength is: $\lambda = c/f = \frac{3\times10^{10}}{60} = 5000$ km.

We are in the static zone (Jk^{n} p.392), where: $d\binom{\text{system}}{\text{size}} \ll R\binom{\text{obs}^{\text{n}}}{\text{distance}} \ll \lambda\binom{\text{wave}}{\text{length}}$.

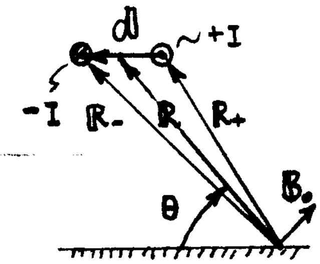
For the field of a single charge [Jk^{n} Eq.(14.14), for non-relativistic motion]:

$$\rightarrow \mathbb{E} = \underbrace{e\left[\frac{\hat{n}}{R^2}\right]}_{\text{static fld}} + \underbrace{\frac{e}{c}\left[\frac{\hat{n}\times(\hat{n}\times\dot{\boldsymbol{\beta}})}{R}\right]}_{\text{radiation fld}} \Rightarrow \left|\frac{E(\text{rad}^{\text{n}})}{E(\text{static})}\right| \sim \left|\frac{e\dot{\beta}}{cR}\Big/\frac{e}{R^2}\right| = \left|\frac{R\dot{\beta}}{c}\right|. \quad (1)$$

Since $\dot{\beta} = \omega\beta$ for the current motion: $\left|\frac{E(\text{rad}^{\text{n}})}{E(\text{static})}\right| \sim 2\pi\beta\frac{R}{\lambda}$. But $\frac{R}{\lambda} = 2\times10^{-5}$, and also $\beta \ll 1$ ($\beta\sim10^{-5}$ perhaps), so $E(\text{rad}^{\text{n}})$ is entirely negligible. The observer (i.e. victim) will at most "see" the static fields.

B: As noted above, the observer can at most see "static" $\mathbb{E}$ & $\mathbb{B}$ fields -- which oscillate harmonically at $e^{-i\omega t}$. The $\mathbb{E}$-fld vanishes because the system is overall charge neutral. That leaves the $\mathbb{B}$-fld... observer will see: $\underline{\mathbb{B}(R,t) = \mathbb{B}_0(R)\,e^{-i\omega t}}$, where $\mathbb{B}_0(R)$ is generated by the wires.

C. $\mathbb{B}_0$ is generated by the two-wire system as shown. Its magnitude at the observer is [cf Jk^{n} Eq.(5.6), p.171]:



$$B_0 = \frac{2I}{c}\left(\frac{1}{R_+} - \frac{1}{R_-}\right), \quad \text{w/}\ \mathbb{R}_\pm = \mathbb{R} \mp \tfrac{1}{2}\mathbf{d},\quad d \ll R. \quad (2)$$

So/

$$R_\pm = \left(R^2 \mp \mathbb{R}\cdot\mathbf{d} + \tfrac{1}{4}d^2\right)^{\frac{1}{2}} = R\left(1 \mp \frac{d}{R}\cos\theta + \frac{1}{4}\underbrace{\cancel{\frac{d^2}{R^2}}}_{\text{negligible}}\right)^{\frac{1}{2}}$$

or/

$$R_\pm \simeq R\left(1 \mp \frac{1}{2}\frac{d}{R}\cos\theta\right), \text{ to 1}^{\text{st}} \text{ order in } \frac{d}{R}. \quad (3)$$

and/ $\boxed{B_0(R) \simeq \frac{2I}{c}\left(\frac{d\cos\theta}{R^2}\right), \text{ likewise}}$, and: $B_0 < \frac{2I}{c}(d/R^2)$

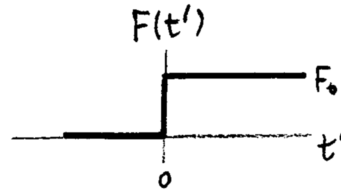
$I = 100$ A [MKS] $\leftrightarrow$ $I = 3\times10^{11}$ statA [CGS, Jk^{n} p.820]. Then for $d = 1$m, $R = 100$m, get:

$\underline{B_0 < (2\times3\times10^{11}/3\times10^{10})(1/10^6) = 20\times10^{-6}\text{ Gauss}}$, less than $10^{-5}\times B_{\text{earth}}$.

ϕ 520 Final Solutions (Cont'd)

② [30pts]. Find charge motion due to "pre-acceleration".

A. The motion follows Jackson's Eq. (17.51) in 1D:



$$m\dot{v}(t) = \int_0^\infty e^{-s} F(t+\tau s)\, ds \qquad (1)$$

If $t \geq 0 \Rightarrow F = F_0$ in the integrand, motion is $\dot{v} = F_0/m$. When $t = -T$ is negative, F in integrand does not kick in until $-T + \tau s \geq 0 \Rightarrow s \geq T/\tau$,

So// $t = -T < 0 \Rightarrow m\dot{v} = \int_{T/\tau}^\infty e^{-s} F_0\, ds = F_0 e^{-T/\tau}$

i.e.// $\underline{\underline{m\dot{v}(t) = F_0 e^{t/\tau}}}$, when $t < 0$. (2)

Integrate this eqtn twice, from reference time $t_0 = -\infty$, when particle is assumed at rest $[v(t_0) = 0]$ at some position $x(t_0)$, up to $t < 0$. Then

$$\rightarrow v(t) = (F_0\tau/m)\, e^{t/\tau} \;;\; x(t) = x(t_0) + (F_0\tau^2/m)\, e^{t/\tau},\quad t \leq 0. \qquad (3)$$

Distance moved during preacceleration period: $\boxed{\Delta x = x(0) - x(t_0) = \frac{F_0\tau^2}{m}}$. (4)

B. If $F_0 = Nmg$, then: $\Delta x = Ng\tau^2$, or $N = \Delta x/g\tau^2$ for a given Δx.

#'s// $\left.\begin{array}{l} \Delta x = 10^{-13}\text{ cm (nuclear diam.)} \\ \tau = 6.3\times10^{-24}\text{ sec (Jk}^{\text{n}}\text{ p.782)} \\ g = 980\text{ cm/sec}^2 \end{array}\right\} N = \frac{10^{-13}}{980\times(6.3\times10^{-24})^2} = \underline{\underline{2.57\times10^{30}}}. \quad (5)$

This sort of N is not found on most carnival rides.

C. If two e's are separated by distance r, then $F_0 = e^2/r^2$. It is "natural" to write: $r = nr_0$, where $r_0 = \frac{e^2}{mc^2}$ and n = numerical factor. Then:

$$\rightarrow a = F_0/m = \frac{1}{m}\frac{e^2}{(nr_0)^2} = \frac{c^2}{n^2 r_0} = \frac{1}{n^2}\frac{(3\times10^{10})^2}{2.82\times10^{-13}} = \frac{1}{n^2}\times 3.26\times10^{30}\, g. \qquad (6)$$

is the Coulombic acceleration for an electron at $r = nr_0$. a matches Ng of part B when $n \simeq 1.13$, i.e. the electrons would be "intermingled".

③ [30 pts.]. Yet another if-the-photon-had-a-mass problem.

1) As discussed in Jk$^{\underline{n}}$ Sec. (12.9), if the photon had a mass m_γ, then it would obey a free-space dispersion relation [Jk$^{\underline{n}}$ Eq. (12.95)]...

$$\omega^2 = k^2c^2 + \mu^2c^2, \quad \mu = m_\gamma c/\hbar, \quad \text{w/} \quad \underline{\underline{\omega = \sqrt{k^2c^2+\omega_p^2}}}, \tag{1}$$

where: $\omega_p = m_\gamma c^2/\hbar = \left(\frac{m_\gamma}{m_e}\right)\frac{c}{\hbar/m_e c} = \left(\frac{m_\gamma}{m_e}\right)\left(\frac{e^2}{\hbar c}\right)\frac{c}{r_0}$ $\int e^2/\hbar c \simeq 1/137$, $r_0 = 2.82\times10^{-13}$ cm.

Let $r = \frac{m_\gamma}{m_e} \simeq \frac{1}{2}\times10^{-20}$. So: $\omega_p = r\times\frac{1}{137}\times\frac{3\times10^{10}}{2.82\times10^{-13}} = r\times0.78\times10^{21}$ Hz. (2)

$\omega_p \sim 4$ Hz is very small, because $r = m_\gamma/m_e$ is so small. Anyway, the photon will move with a group velocity

$$\rightarrow v_g = \frac{\partial\omega}{\partial k} = \frac{kc^2}{\sqrt{k^2c^2+\omega_p^2}} = \frac{c}{\omega}\sqrt{\omega^2-\omega_p^2} \simeq c\left[1-\frac{1}{2}\frac{\omega_p^2}{\omega^2}\right], \quad \omega_p \ll \omega. \tag{3}$$

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2) Photons at frequencies ω_1 and $\omega_2 < \omega_1$, after having traveled for time T, will have moved apart by distance...

$$\rightarrow d = T\Delta v_g = \frac{cT}{2}\omega_p^2\left(\frac{1}{\omega_2^2}-\frac{1}{\omega_1^2}\right) = \frac{cT}{2}\left(\frac{\omega_p}{\omega_2}\right)^2\left[1-\left(\frac{\lambda_1}{\lambda_2}\right)^2\right], \tag{4}$$

where $\lambda = 2\pi c/\omega$ is the wavelength. In our case $(\lambda_1/\lambda_2)^2 = 0.01$ is negligible, and we want $d = \lambda_2 = 10^4$ Å. Since $\omega_2 = \frac{2\pi c}{\lambda_2} = 2\pi\times3\times10^{14}$ Hz...

$$\lambda_2 = \frac{cT}{2}\left(\frac{\omega_p}{\omega_2}\right)^2 \Rightarrow T = \frac{2\lambda_2}{c}\left(\frac{\omega_2}{\omega_p}\right)^2 = \frac{2\times10^{-4}}{3\times10^{10}}\left(\frac{6\pi\times10^{14}}{r\times0.78\times10^{21}}\right)^2$$

$$\rightarrow \text{w/} \; T = \frac{1}{r^2}\times3.89\times10^{-26}\text{ sec} = 20.1\times10^{14}\text{ sec} = \underline{\underline{6.37\times10^7\text{ years}}}. \tag{5}$$

This is the travel time required for $d = \lambda_2$ separation (use $r = \frac{4\times10^{-48}}{9.1\times10^{-28}}$, from Jk$^{\underline{n}}$ p. 6). The required travel distance is:

$$\boxed{D = cT = 63.7\times10^6 \text{ light years}} \quad \left(\begin{array}{c}\sim 600\times \text{ as large as} \\ \text{Milky Way galaxy}\end{array}\right). \tag{6}$$

ϕ 520 Final Solutions (cont'd)

⟨4⟩[30 pts].

1) By Jackson's Eqs. (7.72) & (7.77), the EM field amplitudes will be attenuated by a factor $e^{-D/\delta}$, after propagating distance D in a medium with penetration depth $\underline{\delta = c/\sqrt{2\pi\mu\sigma\omega}}$, w/ σ = conductivity, $\omega = 2\pi f$ = frequency. If the prey broadcasts unit signal strength, the shark receives an EM Signal strength $= (e^{-D/\delta})^2$, which must exceed $1/N$, $N = 10^3$, by the conditions of the problem. Thus, the shark senses its prey if

$$(e^{-D/\delta})^2 \geq \frac{1}{N} \Rightarrow \underline{\underline{D \leq \frac{\delta}{2}\ln N = (c/4\pi\sqrt{\sigma f})\ln N}}. \quad (1)$$

2) Eq. (1) is in CGS, so we need $\sigma_{CGS} = 9\times10^9\,\sigma_{MKS}$, Hz [Jk^{n}, p. 820]. Then, with f the broadcast frequency in Hz, maximum prey detection distance is

$$D_{MAX} = (3\times10^{10}/4\pi\sqrt{9\times10^9\sigma_{MKS} f})\ln N = (\underbrace{2.52\times10^4\,\text{cm}}_{=252\,\text{m}}/\sqrt{f\sigma_{MKS}})\ln N.$$

σ_{MKS} is now the conductivity in MKS units of $(\text{ohm-m})^{-1}$. (2)

3) For the problem at hand...

$$\left.\begin{array}{l}\text{attenuation factor}: N = 10^3\\ \text{broadcast freq}: f = 10\,\text{Hz}\\ \text{Seawater cond.}: \sigma = 4.3\ \text{MKS}\end{array}\right\} \boxed{D_{max} = (252\,\text{m})\frac{\ln 10^3}{\sqrt{10\times4.3}} = 265\,\text{m}} \quad (3)$$

The shark can sense its prey in a volume : $\frac{4\pi}{3}D_{max}^3 = 78\times10^6$ cu. meters, which contains about 21×10^9 gal. of water. The chances are that you cannot hide from a shark in your backyard swimming pool.

$$\int_{-\infty}^{\infty} e^{-p^2x^2 \pm qx}\,dx = \frac{\sqrt{\pi}}{p}\, e^{q^2/4p^2}, \quad \mathrm{Re}\,p \neq 0.$$

⑤ [30 pts]. Analyse energy transport of a Gaussian wave-packet.

A. Write the pulse as: $u(x,0) = N e^{ik_0x} e^{-(x/\Delta x)^2}$. Its normalization follows:

$$\int_{-\infty}^{\infty} |u(x,0)|^2 dx = N^2 \int_{-\infty}^{\infty} e^{-2(x/\Delta x)^2} dx = N^2 \Delta x \sqrt{\frac{\pi}{2}} = \mathcal{E} \Rightarrow N^2 = \sqrt{\frac{2}{\pi}}\,\frac{\mathcal{E}}{\Delta x}. \quad (1)$$

B. As found in class (Xerox notes of 3/5/91), the Fourier amplitude of u is:

$$A(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} u(x,0)\, e^{-ikx}\, dx = (N\Delta x/2\sqrt{\pi})\, e^{-\frac{1}{4}[(k-k_0)\Delta x]^2}. \quad (2)$$

C. In the class notes cited, we found the pulse energy transported was

$$\rightarrow W(t) = 2\pi \int_{-\infty}^{\infty} dk\, |A(k)|^2 e^{[2\,\mathrm{Im}\,\omega(k)]t}. \quad (3)$$

In this problem, we are given $\mathrm{Im}\,\omega(k) = -\beta k^2$. Using $A(k)$ of Eq. (2)...

$$W(t) = 2\pi \left(\frac{N^2(\Delta x)^2}{4\pi}\right) \int_{-\infty}^{\infty} dk\, e^{-\frac{1}{2}[(k-k_0)\Delta x]^2} e^{-2\beta k^2 t}$$

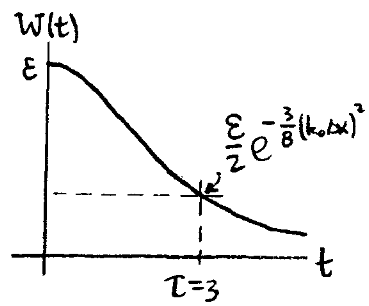
$$= \frac{1}{2}\sqrt{\frac{2}{\pi}}(\mathcal{E}\Delta x)\, e^{-\frac{1}{2}(k_0\Delta x)^2} \int_{-\infty}^{\infty} dk\, e^{-[\frac{1}{2}(\Delta x)^2 + 2\beta t]k^2 + [k_0(\Delta x)^2]k}$$

or//

$$W(t) = \frac{(\mathcal{E}\Delta x)\, e^{-\frac{1}{2}(k_0\Delta x)^2}}{\sqrt{(\Delta x)^2 + 4\beta t}}\, e^{\frac{1}{2}(k_0\Delta x)^2 (\Delta x)^2/[(\Delta x)^2 + 4\beta t]}. \quad (4)$$

Define the dimensionless time: $\tau = (4\beta/(\Delta x)^2)\,t$. With a bit of algebra:

$$\boxed{W(t) = \frac{\mathcal{E}}{\sqrt{1+\tau}}\, e^{-\frac{1}{2}(k_0\Delta x)^2 \tau/(1+\tau)}} \quad (5)$$



If $\beta > 0$, the pulse rapidly attenuates on a time scale $\Delta t \sim (\Delta x)^2/4\beta$, reaching an asymptotic value $W(t) \sim (\mathcal{E}/\sqrt{\tau})\, e^{-\frac{1}{2}(k_0\Delta x)^2}$ in a few times τ. Evidently, at a given τ, a broad pulse is more severely attenuated than a narrow one. This is the reverse of dispersion, where the broad pulse would "survive" longer.