Adjoint Spinor & Standard Repⁿ for Continuity Eqtn. Uniqueness.

Now we define a quantity recurrent in the theory...

$$\boxed{\text{ADJOINT SPINOR}: \bar{\Psi} = \psi^\dagger\beta} \quad \text{if}/\!/ \ \psi = \begin{pmatrix}\varphi\\ \chi\end{pmatrix}, \text{ then}/\!/ \ \bar{\Psi} = (\varphi, -\chi). \quad (27)$$

In these terms we can write the current & density, and continuity eqtn, as...

$$\rightarrow J_k = ic\bar{\Psi}\gamma_k\psi, \quad \rho = \bar{\Psi}\gamma_4\psi, \text{ and}: \ \frac{\partial J_k}{\partial x_k} + ic\frac{\partial\rho}{\partial x_4} = 0, \ \text{w}/\!/\ x_4 = ict. \quad (28)$$

Clearly it makes sense to define a 4-vector current J_μ as...

$$\left.\begin{aligned} &\text{Dirac 4-current}: \ \underline{J_\mu = ic\bar{\Psi}\gamma_\mu\psi} \quad [\text{i.e.}/\!/\ J_\mu = (c\psi^\dagger\alpha_k\psi, ic\psi^\dagger\psi)] \\ &\text{so}/\!/\ \text{Dirac continuity eqtn is}: \ \boxed{\partial J_\mu/\partial x_\mu = 0}. \end{aligned}\right\} \quad (29)$$

Later we will show that J_μ in fact properly transforms as a Lorentz 4-vector, so that Dirac's continuity equation is manifestly (Lorentz) covariant.

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6) The "standard representation" we have given for the Dirac Eqtn, viz. [Eq. (22)];

$$\left(\gamma_\mu\frac{\partial}{\partial x_\mu} + \frac{mc}{\hbar}\right)\psi = 0, \ \text{w}/\!/\ \gamma_k = \begin{pmatrix}0 & -i\sigma_k\\ i\sigma_k & 0\end{pmatrix} \ \& \ \gamma_4 = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \ \text{STANDARD REPRESENTATION} \quad (30)$$

is not unique. If we transform the 4-spinor ψ by a unitary transformation S:

$$\rightarrow \psi \rightarrow \psi' = S\psi, \quad S \text{ a unitary transf}^{\underline{n}}: \ S^{-1} = S^\dagger, \quad (31)$$

then the Dirac Eqtn becomes...

$$\left(\gamma_\mu\frac{\partial}{\partial x_\mu} + \frac{mc}{\hbar}\right)S^{-1}\psi' = 0 \ \leftarrow \text{multiply on left by } S\ldots$$

$$\text{or}/\!/\ \left(\gamma'_\mu\frac{\partial}{\partial x_\mu} + \frac{mc}{\hbar}\right)\psi' = 0, \ \text{w}/\!/\ \gamma'_\mu = S\gamma_\mu S^{-1}, \ (\text{for}: \psi' = S\psi). \quad (32)$$

The γ'_μ obey the same anticommutation rule as the γ_μ: if $\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}$, then: $\{\gamma'_\mu, \gamma'_\nu\} = 2\delta_{\mu\nu}$. Consequently, there is no way to distinguish between the Dirac Eqtns for γ_μ & ψ and γ'_μ & ψ'. As advertised, the "standard representation" of Eq. (30) is not unique.

(next page)

Pauli's Theorem. General features of solutions for ± charge q.

That transf$^{\text{ns}}$ S actually exist to do the steps in Eqs. (30)→(32) is assured by Pauli's Fundamental Theorem, which states

"Given two sets of 4×4 matrices satisfying the rules $\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}$, $\{\gamma'_\mu, \gamma'_\nu\} = 2\delta_{\mu\nu}$, there always exists a non-singular 4×4 matrix S such that $S\gamma_\mu S^{-1} = \gamma'_\mu$. S is unique up to a multiplicative constant." (33)

For a proof, see Sakurai's "Advanced QM", App. C, p. 308.

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7) Before looking at explicit solutions to the Dirac Eqtn, we shall examine some general features which apply to ± charges and to ± energies. Curiously, these seemingly independent signs are linked in the Dirac Eqtn.

Begin by writing the Dirac Eqtn in an external field $A_\mu = (\mathbb{A}, i\phi)$. As usual:

In $A_\mu = (\mathbb{A}, i\phi)$: $p_\mu = -i\hbar\frac{\partial}{\partial x_\mu} \rightarrow p_\mu - (q/c)A_\mu$ ← here, p_μ is the momentum conjugate to 4-position x_μ *

... i.e. $\partial/\partial x_\mu \rightarrow \partial/\partial x_\mu - (iq/\hbar c)A_\mu$, for a particle of charge q. (34)

So// $$\boxed{\left[\gamma_\mu\left(\frac{\partial}{\partial x_\mu} - \frac{iq}{\hbar c}A_\mu\right) + \frac{mc}{\hbar}\right]\psi = 0}$$ ← DIRAC EQ. in extl. field A_μ. (35)

Next, write the Dirac Eqtn for the opposite sign of charge: $q \rightarrow (-)q$. The $(-)q$ obeys:

$$\rightarrow \left[\gamma_\mu\left(\frac{\partial}{\partial x_\mu} + \frac{iq}{\hbar c}A_\mu\right) + \frac{mc}{\hbar}\right]\psi_c = 0,$$ ψ_c = solution for $q \rightarrow (-)q$. (36)

QUESTION: How is $\psi_c(-q)$ related to $\psi(q)$?

Take the complex conjugate of the original eqtn, Eq. (35), noting that $x_k^* = x_k$, $x_4^* = -x_4$, and $A_k^* = A_k$, $A_4^* = -A_4$. Then (35) yields...

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* The EM Hamiltonian, in these terms, is: $\mathcal{H}_{EM} = \sqrt{(cp_k - qA_k)^2 + (mc^2)^2} + q\phi$; see Jackson, Sec. 12.1a. For this $\mathcal{H}_{EM}$, Hamilton's Eqs-of-Motion ⇒ Lorentz force law.

Relation between solutions $\psi(q)$ & $\psi_c(-q)$. Charge-conjugation.

$$\rightarrow \left[\gamma_k^*\left(\frac{\partial}{\partial x_k}+\frac{iq}{\hbar c}A_k\right)-\gamma_4^*\left(\frac{\partial}{\partial x_4}+\frac{iq}{\hbar c}A_4\right)+\frac{mc}{\hbar}\right]\psi^*=0;$$

or// $$\underline{\left[\gamma'_\mu\left(\frac{\partial}{\partial x_\mu}+\frac{iq}{\hbar c}A_\mu\right)+\frac{mc}{\hbar}\right]\psi^*=0}, \quad \text{w/ } \gamma'_\mu=(\gamma_k^*,-\gamma_4^*). \tag{37}$$

This conjugate version of (35) for $\psi(q)$ now resembles (36) for $\psi_c(-q)$. In fact, since the γ'_μ obey the same anticommutation rule as the γ_μ (i.e. $\{\gamma_\mu,\gamma_\nu\}=2\delta_{\mu\nu}$ $\Rightarrow \{\gamma'_\mu,\gamma'_\nu\}=2\delta_{\mu\nu}$), then by Pauli's Theorem [Eq. (33)], there exists a non-singular matrix S_c such that...

$$\rightarrow \gamma'_\mu = S_c\gamma_\mu S_c^{-1}, \text{ i.e. } \gamma_k^* = S_c\gamma_k S_c^{-1}, \text{ and: } \gamma_4^* = -S_c\gamma_4 S_c^{-1}. \tag{38}$$

With this, the conjugate eqtn (37) becomes the $(-)q$ eqtn (36), as follows...

$$\left[S_c\gamma_\mu S_c^{-1}\left(\frac{\partial}{\partial x_\mu}+\frac{iq}{\hbar c}A_\mu\right)+\frac{mc}{\hbar}\right]\psi^*=0 \leftarrow \text{mult. on left by } S_c^{-1}$$

So// $$\boxed{\left[\gamma_\mu\left(\frac{\partial}{\partial x_\mu}+\frac{iq}{\hbar c}A_\mu\right)+\frac{mc}{\hbar}\right]\psi_c=0}, \quad \text{w/ } \underline{\underline{\psi_c = S_c^{-1}\psi^*}}. \tag{39}$$

The solutions $\psi(q)$ & $\psi_c(-q)$ for $\pm q$ (in the same external field A_μ) are thus related -- but in a nontrivial way, through the matrix S_c as defined in (38).

Find S_c. In the standard representation (only!), S_c is defined by Eqs. (38), i.e.

$$\text{for}\begin{cases}\gamma_k=\begin{pmatrix}0 & -i\sigma_k\\ i\sigma_k & 0\end{pmatrix}\\[2ex] \gamma_4=\begin{pmatrix}1&0\\0&-1\end{pmatrix}\end{cases} \quad \left\| \quad \begin{aligned} S_c\gamma_1S_c^{-1}&=\gamma_1^*=-\gamma_1,\\ S_c\gamma_2S_c^{-1}&=\gamma_2^*=+\gamma_2,\\ S_c\gamma_3S_c^{-1}&=\gamma_3^*=-\gamma_3,\\ S_c\gamma_4S_c^{-1}&=-\gamma_4^*=-\gamma_4.\end{aligned}\right.$$

And, the γ_μ must satisfy: $\underline{\gamma_\mu\gamma_\nu+\gamma_\nu\gamma_\mu=2\delta_{\mu\nu}}$.

By inspection (sic)¶: $$\boxed{S_c=\gamma_2}=\begin{pmatrix}0&-i\sigma_2\\ i\sigma_2&0\end{pmatrix}=\begin{pmatrix}0&0&0&-1\\0&0&1&0\\0&1&0&0\\-1&0&0&0\end{pmatrix}, \quad S_c^{-1}=\gamma_2. \tag{40}$$

So// [If ψ is a solution for $+q$ in the field A_μ, then $\underline{\psi_c=\gamma_2\psi^*}$ is a solution for $-q$ in A_μ. The operation $q\rightarrow(-)q$ is called "charge conjugation", and ψ_c is the wavefcn that is "charge-conjugate" to ψ.] (41)

¶ Simple... $S_c=\gamma_2$ satisfies all four defining equations in (40). It is unique up to a phase.

Charge conjugation as it affects energy & momentum. Antiparticles.

Now we note that if ψ & $\psi_c = \gamma_2 \psi^*$ are charge-conjugate wavefcns, then they must have energy eigenvalues of opposite sign. Proof is simple...

$$\left[\begin{array}{l} \text{If } \psi \text{ is an eigenfcn of energy } E: \; i\hbar \frac{\partial \psi}{\partial t} = (+E)\psi, \\ \text{then}/\!/ \quad \gamma_2 \times \left\{ i\hbar \frac{\partial \psi}{\partial t} = E\psi \right\}^* \Rightarrow i\hbar \frac{\partial}{\partial t} \gamma_2 \psi^* = (-E)\gamma_2 \psi^*. \\ \text{So: } \psi_c = \gamma_2 \psi^* \text{ is an eigenfcn of energy } (-)E. \end{array}\right\} \tag{42}$$

We shall show in a short while (by looking at free particle solutions) that charge conjugation <u>also</u> reverses the sign of the particle's 3-momentum, i.e., $\mathbf{p} \to (-)\mathbf{p}$, under charge conjugation.

In summary, in Dirac theory the simple operation $q \to (-)q$ has the result:

> If ψ describes a particle with $(+q, +E, +\mathbf{p})$ in an external field A_μ, then the charge-conjugate wavefcn $\psi_c = \gamma_2 \psi^*$ describes an "anti-particle" with $(-q, -E, -\mathbf{p})$ in the same field. (43)

The ψ_c particle is called an "antiparticle" because it is the mirror-image (opposite-image) of the original particle. For bookkeeping purposes, we note...

$$\psi = \begin{bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{bmatrix} \Rightarrow \psi_c = \gamma_2 \psi^* = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \psi_1^* \\ \psi_2^* \\ \psi_3^* \\ \psi_4^* \end{bmatrix} = \begin{bmatrix} -\psi_4^* \\ \psi_3^* \\ \psi_2^* \\ -\psi_1^* \end{bmatrix}. \tag{44}$$

So the charge-conjugation operation literally stands ψ on its head, besides introducing the $*$'s and signs top & bottom.