1992 Φ520 Problems Assigned: 4/20. Due: 4/27/92. file P25

(25)

(79) [20 pts]. In Sec. 12.9, Jackson quotes the Proca Lagrangian $\mathcal{L}_p$ [Eq. (12.91)], and derives Proca's wave eqtn for a massive photon field [Eq. (12.93)]. Along the way, he claims the "Lorentz gauge is now required for current conservation."

(A) Show why Jackson's claim about the Lorentz gauge is justified.

(B) Assume the initial choice for $\mathcal{L}_p$ is the Lorentz gauge: $\partial^\alpha A_\alpha = 0$, so the current J_α is conserved: $\partial^\alpha J_\alpha = 0$. Now consider a gauge transform: $A_\alpha \rightarrow A'_\alpha = A_\alpha + \partial_\alpha G$. What condition on the gauge fcn G is needed to maintain current conservation? (The theory is nonsense w/o conserved currents!). What gauge are you in now?

(C) With the current-conserving gauge transforms allowed in part (B), show that the transformed Lagrangian is: $\mathcal{L}'_p = \mathcal{L}_p - \partial^\alpha U_\alpha$, where U_α is a vector field that depends on J_α & A_α. Find U_α explicitly. Are $\mathcal{L}'_p$ & $\mathcal{L}_p$ "gauge equivalent" in the sense of problem (75)?

(D) Make a statement regarding the gauge freedom of $\mathcal{L}_p$ (as a massive vector field $\mathcal{L}$).

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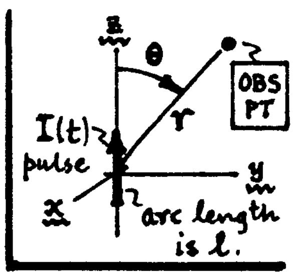
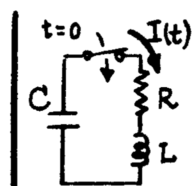
(80) [20 pts]. Consider a 1D arc discharge along the z-axis: a current pulse $I(t)$ begins at time $t=0$ and flows along a path of length ℓ. An observer at position (r, θ), w/ $r \gg \ell$, detects the arc's radiation.



(A) Start from the arc's Poynting vector S derived in class (NOTES: p. Rad 7, Eq (18)]. The energy/unit time radiated into solid ∡ $d\Omega$ is $r^2 S\, d\Omega$, so: $d\mathcal{E}/d\Omega = \int_{-\infty}^{\infty} r^2 S\, dt$, is the total energy radiated per unit solid ∡. Convert this to a frequency integral: $d\mathcal{E}/d\Omega = \int_{-\infty}^{\infty} \sigma(\omega)\, d\omega$ (Parseval's Theorem). Show the spectrum fcn is

$$\boxed{\sigma(\omega) = \left(\frac{\sin^2\theta}{8\pi^2 c^3}\right)\ell^2\omega^2 \left|\int_0^\infty I(t)\, e^{-i\omega t}\, dt\right|^2}$$

(B) A model for $I(t)$ is the discharge of a capacitor C (switched on at $t=0$, w/ initial voltage V_0) through a series resistor R & inductor L. Then (for the overdamped case): $\underline{I(t) = (V_0/L\Gamma)\, e^{-\gamma t} \sinh \Gamma t}$, w/ $\gamma = \frac{R}{2L}$, $\Gamma = \sqrt{\gamma^2 - (1/LC)}$.



Sketch this $I(t)$ pulse vs. t, and roughly estimate the pulse risetime & duration.

(C) Calculate the arc frequency spectrum $\sigma(\omega)$ for the $I(t)$ model of part (B). Sketch $\sigma(\omega)$ vs. ω. Over what frequency range is the arc radiation appreciable?

(D) Find the total energy radiated by the arc. Compare it with $\int I^2 R\, dt$ = discharge energy.

HINT: See R. Robiscoe & Z. Sui, J. Appl. Phys. 64, 4364 (Nov. 1988).

ϕ520 Solutions

(79) [20 pts.]. Analyse gauge constraints on Proca Lagrangian.

(A) 1. The Proca Lagrangian is Jk$^{\underline{n}}$ Eq. (12.91):

$$\rightarrow \mathcal{L}_p = -\frac{1}{16\pi} F_{\alpha\beta} F^{\alpha\beta} - \frac{1}{c} J_\alpha A^\alpha + \frac{\mu^2}{8\pi} A_\alpha A^\alpha, \quad \mu = \frac{m_\gamma c}{\hbar} = \text{photon mass}. \tag{1}$$

The eqtns-of-motion (i.e. $\partial^\beta[\partial\mathcal{L}_p/\partial(\partial^\beta A^\alpha)] = \partial\mathcal{L}_p/\partial A^\alpha$) are Eq. (12.92):

$$\rightarrow \partial^\beta F_{\beta\alpha} + \mu^2 A_\alpha = \frac{4\pi}{c} J_\alpha, \quad \text{w/ } F_{\beta\alpha} = \partial_\beta A_\alpha - \partial_\alpha A_\beta. \tag{2}$$

To see how current J_α is conserved, operate through Eq. (2) by ∂^α, so that

$$\rightarrow \partial^\alpha \partial^\beta F_{\beta\alpha} + \mu^2 \partial^\alpha A_\alpha = \frac{4\pi}{c} \partial^\alpha J_\alpha. \tag{3}$$

It is easy to show that with $F_{\beta\alpha}$ the antisymmetric field tensor defined in (2), the first term LHS in (3) vanishes: $\partial^\alpha\partial^\beta F_{\beta\alpha} \equiv 0$.* Then (3) yields

$$\rightarrow \underline{\underline{\partial^\alpha J_\alpha = (\mu^2 c/4\pi)\, \partial^\alpha A_\alpha}}, \quad \text{so current is conserved: } \underline{\underline{\partial^\alpha J_\alpha = 0}}, \text{ only when } \underline{\underline{\partial^\alpha A_\alpha = 0}} \leftrightarrow \text{Lorentz gauge}. \tag{4}$$

This justifies Jk$^{\underline{n}}$'s claim that the Lorentz gauge is <u>required</u> for current cons$^{\underline{n}}$.

(B) 2. Under a gauge transform: $A_\alpha \rightarrow A'_\alpha = A_\alpha + \partial_\alpha G$, <u>the field tensor $F_{\beta\alpha}$ remains unchanged</u>. The field eqtns (2) become: $\partial^\beta F_{\beta\alpha} + \mu^2[A_\alpha + \partial_\alpha G] = \frac{4\pi}{c} J_\alpha$, and operation through this eqtn by ∂^α produces the counterpart of Eq. (4):

$$\rightarrow \partial^\alpha J_\alpha = (\mu^2 c/4\pi)[\partial^\alpha A_\alpha + (\partial^\alpha \partial_\alpha) G]. \tag{5}$$

If the original gauge was Lorentz, then $\partial^\alpha A_\alpha = 0$. According to (5), current is conserved in the new gauge ($A_\alpha + \partial_\alpha G$) only if the gauge function is restricted:

$$\rightarrow \underline{\underline{(\partial^\alpha \partial_\alpha) G = \left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right) G = 0}} \Rightarrow G = \text{free field scalar}. \tag{6}$$

But then we are still <u>in</u> the Lorentz gauge (Jk$^{\underline{n}}$ p. 221), since $\partial^\alpha A'_\alpha = 0$ also. In any case, current <u>can</u> be conserved for $\mathcal{L}_p$, for gauge fcns G obeying Eq. (6).

$$\star \quad \partial^\alpha\partial^\beta F_{\beta\alpha} \overset{\text{relabel indices}}{=} \partial^\beta\partial^\alpha F_{\alpha\beta} \overset{\text{change diff}^{\underline{n}}\text{ order}}{=} +\partial^\alpha\partial^\beta F_{\alpha\beta} \overset{\text{use } F = \text{antisym.}}{=} -\partial^\alpha\partial^\beta F_{\beta\alpha}, \text{ so } \partial^\alpha\partial^\beta F_{\beta\alpha} \equiv 0.$$

3. When $A_\alpha \to A'_\alpha = A_\alpha + \partial_\alpha G$, the Proca Lagrangian in Eq. (1) becomes...

(C) $$\to \mathcal{L}'_p = -\frac{1}{16\pi} F_{\alpha\beta} F^{\alpha\beta} - \frac{1}{c} J_\alpha (A^\alpha + \partial^\alpha G) + \frac{\mu^2}{8\pi} (A_\alpha + \partial_\alpha G)(A^\alpha + \partial^\alpha G)$$

$$= \mathcal{L}_p - \frac{1}{c} J_\alpha \overset{④}{\partial^\alpha G} + \frac{\mu^2}{8\pi} \left[A_\alpha \overset{③}{\partial^\alpha G} + A^\alpha \overset{②}{\partial_\alpha G} + \overset{①}{(\partial_\alpha \partial^\alpha) G} \right]. \tag{7}$$

We've used the fact that $F_{\alpha\beta}$ is invariant under the gauge transform, and have gathered together the terms that form $\mathcal{L}_p$. Term ① $\equiv 0$ for current conservation [Eq. (6) above], terms ② & ③ combine to give $2A^\alpha \partial_\alpha G$, and term ④ can be written as: $J_\alpha \partial^\alpha G = \partial^\alpha (J_\alpha G) - (\cancelto{0}{\partial^\alpha J_\alpha}) G$ (current conservation again). Then...

$$\to \mathcal{L}'_p = \mathcal{L}_p - \frac{1}{c} \partial^\alpha (J_\alpha G) + \frac{\mu^2}{4\pi} A_\alpha \partial^\alpha G. \tag{8}$$

For the third term on the RHS here, we can write

$$A_\alpha \partial^\alpha G = \partial^\alpha (A_\alpha G) - (\cancelto{0}{\partial^\alpha A_\alpha}) G, \text{ by Lorentz gauge.} \tag{9}$$

So//
$$\boxed{\mathcal{L}'_p = \mathcal{L}_p - \partial^\alpha \left[\left(\frac{1}{c} J_\alpha - \frac{\mu^2}{4\pi} A_\alpha \right) G \right]}\ ;\ \alpha = 0, 1, 2, 3. \tag{10}$$

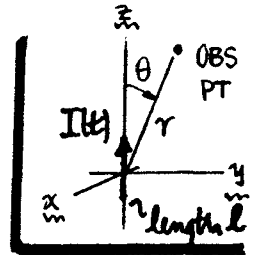
4. Let $U_\alpha = \left(\frac{1}{c} J_\alpha - \frac{\mu^2}{4\pi} A_\alpha \right) G$, and integrate $\mathcal{L}'_p$ of Eq. (10) over a hypervolume with invariant volume element $d^4x = dx^0 dx^1 dx^2 dx^3$, with the $x^0 = ct$ cd. ranging from time t_1 to t_2 (fixed endpoints of the motion). Then the action is

$$\to \mathcal{A}'_p = \int_1^2 \mathcal{L}'_p \, d^4x = \mathcal{A}_p - \int_1^2 (\partial^\alpha U_\alpha) \, d^4x. \tag{11}$$

The $\alpha = 0$ term in the integral gives just: $\int d^3x \, U_0 \Big|_{t_1}^{t_2}$, fixed at the endpts; it contributes nothing to the variation $\delta \mathcal{A}'_p$. The $\alpha = 1, 2, 3$ terms can -- by Gauss' Theorem -- be transformed to integrals over hypersurfaces at ∞, where they vanish. Thus we get $\delta \mathcal{A}'_p = \delta \mathcal{A}_p = 0$ together, and $\mathcal{L}'_p$ & $\mathcal{L}_p$ of Eq. (10) <u>are</u> gauge equivalent... they will give the same eqtns-of-motion. We can state:

(D)
For current conservation (and gauge equivalence), $\mathcal{L}_p$ is totally restricted to the Lorentz gauge ($\partial^\alpha A_\alpha = 0$). Any further gauge freedom requires additional terms for $\mathcal{L}_p$.

30) [20 pts]. Analyse freq. spectrum for a 1D arc discharge.



(A) Poynting vector as derived in class [class notes, p. Rad 7, Eq. (18)]★:

$$\rightarrow S(r,t) = \left(\frac{\sin^2\theta}{4\pi r^2}\right)\frac{1}{c^3}\left[\dot{I}(t')\,l\right]^2 \leftarrow \text{energy/unit time \& area at observer.} \quad (1)$$

The energy/unit time & solid ∡ is $r^2 S$, and if we integrate this over all t, we get

$$\rightarrow \frac{d\mathcal{E}}{d\Omega} = \int_{-\infty}^{\infty} r^2 S(r,t)\,dt = \left(\frac{\sin^2\theta}{4\pi c^3}\right) l^2 \int_{-\infty}^{\infty}\left[\dot{I}(t')\right]^2 dt' \leftarrow \text{radiated energy per unit solid ∡.} \quad (2)$$

The integral can be converted to an integration over a freq. variable ω by means of Parseval's Theorem for Fourier Integrals†, which states...

$$\rightarrow \int_{-\infty}^{\infty} |F(t)|^2 dt = \int_{-\infty}^{\infty} |f(\omega)|^2 d\omega, \text{ "if" } f(\omega) = (1/\sqrt{2\pi})\int_{-\infty}^{\infty} F(t)\,e^{-i\omega t}\,dt. \quad (3)$$

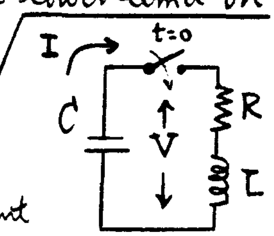
We identify $F(t)$ in Eq. (3) with $\dot{I}(t)$ in Eq. (2), so we can write...

$$\rightarrow \frac{d\mathcal{E}}{d\Omega} = \int_{-\infty}^{\infty}\sigma(\omega)\,d\omega, \quad \sigma(\omega) = \left(\frac{\sin^2\theta}{4\pi c^3}\right)\frac{l^2}{2\pi}\left|\int_{-\infty}^{\infty}\dot{I}(t)\,e^{-i\omega t}\,dt\right|^2. \quad (4)$$

$\sigma(\omega)$ is evidently the radiated energy per unit solid ∡, per unit frequency; it is what Jackson calls the frequency-angle spectrum. Thus we have, as desired...

$$\boxed{\frac{d^2 I}{d\omega\, d\Omega} = \sigma(\omega) = \left(\frac{\sin^2\theta}{8\pi^2 c^3}\right) l^2\omega^2\left|\int_0^{\infty} I(t)\,e^{-i\omega t}\,dt\right|^2.} \quad (5)$$

Two details in going Eq. (4) → (5): the {F.T. of $\dot{I}(t)$} → $i\omega$ {F.T. of $I(t)$}, by partial integration (for any $I(t)$ which vanishes as $t \to \pm\infty$); and the lower limit on the integral is put to zero [for $I(t)$'s which vanish @ $t<0$].



(B) For the passive CRL cct described, the cct eqtns are: $I = -C\dot{V}$, $V = RI + L\dot{I}$. It is easily verified that the solution for the current

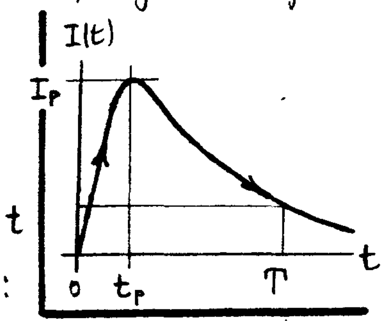
† G. Arfken "Math Methods for Physicists" (Academic Press, 3rd ed, 1985), Eq. (15.55).

★ Since we integrate over all times, the distinction between t & $t' = t - \frac{r}{c}$ is unimportant.

NOTE: most of the material in parts (A)-(C) is worked out in R. Robiscoe & Z. Sui, J. Appl. Phys. 64, 4364 (1988).

I which results from the initial conditions: $V(0)=V_0$, $I(0)=0$, is just the given:

$$\rightarrow I(t)=\frac{V_0}{L\Gamma}e^{-\gamma t}\sinh\Gamma t \quad \int \gamma = R/2L, \text{ and } \underline{\Gamma=\sqrt{\gamma^2-(1/LC)}}. \qquad (6)$$



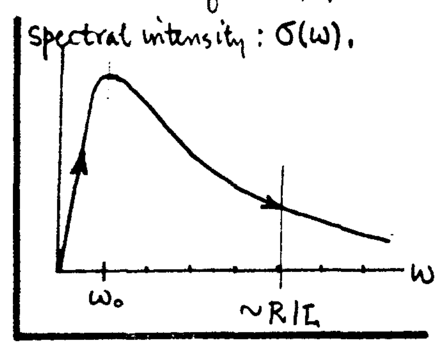
I(t) shows no oscillations so long as Γ is real, i.e. so long as $CR^2/L>4$; this is the condition for overdamping. The small-t behavior is: $I(t)\sim(V_0/2L)t$, while as $t\to$ large we have: $I(t)\sim(V_0/2L\Gamma)e^{-(\gamma-\Gamma)t}$. Roughly speaking: I(t) goes through a peak ($I_p\sim V_0/R$) at time $\boxed{t_p\sim L/R}$ (which is the "risetime"), then falls off exponentially in a characteristic time (i.e. "duration") $\boxed{T\sim RC}$. The overall behavior is sketched.

(C) For $I(t)=(V_0/L\Gamma)e^{-\gamma t}\sinh\Gamma t$, the spectrum of Eq. (5) requires the F.T.:

$$\rightarrow \int_0^\infty I(t)e^{-i\omega t}d\omega=\frac{V_0}{L\Gamma}\int_0^\infty e^{-\gamma t}\left(\frac{e^{\Gamma t}-e^{-\Gamma t}}{2}\right)e^{-i\omega t}dt=\frac{V_0/L}{(\omega_0^2-\omega^2)+2i\gamma\omega}, \qquad (7)$$

where $\omega_0=1/\sqrt{LC}$ is the cct natural frequency. Taking the absolute square, find...

$$\boxed{\sigma(\omega)=\left(\frac{\sin^2\theta}{8\pi^2c^3}\right)\frac{\ell^2V_0^2}{R^2}\left[\frac{4\gamma^2\omega^2}{(\omega_0^2-\omega^2)^2+4\gamma^2\omega^2}\right].} \qquad (8)$$



As a fcn of ω, the spectrum peaks @ $\omega=\omega_0$, then falls off slowly [as $\sim(\omega_0/\omega)^2$]. Beyond $\omega=\omega_0$, the spectrum does not fall to $1/n$ of its peak value until $\omega\sim 2\sqrt{n}(R/L)$; if $n=10$ (i.e. if $d^2I/d\omega d\Omega$ is detectable out to 10% of its peak value), then spectrum freq. range is $0\leq\omega\lesssim 7(R/L)$.

(D) From Eq. (2), with $d\Omega=2\pi\sin\theta\,d\theta$, the total arc radiation energy is...

$$\rightarrow \mathcal{E}_{rad}=\int_{4\pi}d\Omega\int_{-\infty}^{\infty}r^2S\,dt=\frac{\ell^2}{2c^3}\int_0^\pi\sin^3\theta\,d\theta\int_{-\infty}^{\infty}[\dot{I}(t)]^2dt=\frac{\ell^2}{2c^3}\cdot\frac{4}{3}\cdot\int_0^\infty[\dot{I}(t)]^2dt. \qquad (9)$$

To get MKS units, the RHS must be multiplied by $(1/4\pi\epsilon_0)$. Then, for the pulse: $I(t)=(V_0/L\Gamma)e^{-\gamma t}\sinh\Gamma t$, calculate: $\int_0^\infty\dot{I}^2dt=\frac{1}{4\gamma}(V_0/L)^2$, so that

use: $c=1/\sqrt{\mu_0\epsilon_0}$ (MKS)

$$\rightarrow \underline{\underline{\mathcal{E}_{rad}=\left(\frac{1}{4\pi\epsilon_0}\right)\cdot\frac{2\ell^2}{3c^3}\cdot\frac{1}{4\gamma}(V_0/L)^2}}=\ldots, \text{ "/} \boxed{\mathcal{E}_{rad}=\frac{1}{12\pi}\frac{Z_0}{R}\left(\frac{\omega_0\ell}{c}\right)^2CV_0^2}. \qquad (10)$$

Here $Z_0=\sqrt{\mu_0/\epsilon_0}=377\,\Omega$, and $\omega_0=1/\sqrt{LC}$. The total discharge energy $\mathcal{E}_{dis}=\int_0^\infty RI^2dt=\frac{1}{2}CV_0^2$ (clearly), so: $\underline{\mathcal{E}_{rad}/\mathcal{E}_{dis}=(Z_0/6\pi R)(\omega_0\ell/c)^2}$. This ratio $<10^{-6}$, typically.