1. Solution

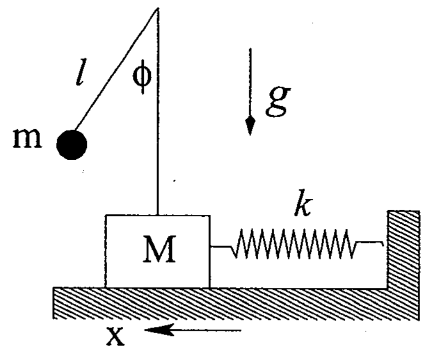
(a) By symmetry $V(r) = c/r$ for some constant c. Using $V(a) - V(b) = V$ we have $c = Vab/(b-a)$. Using $\mathbf{E} = -\nabla V$ we have $\mathbf{E} = \hat{\mathbf{r}} Vab/(b-a)r^2$.

(b) By symmetry $\mathbf{B} = B_\phi \hat{\phi}$. Drawing a circular Amperian loop at constant θ, we have $-2\pi r \sin\theta B_\phi = I$.

(c) The Poynting flux is $\mathbf{S} = \mathbf{E} \times \mathbf{B} = \hat{\theta} VIab/(2\pi r^3 \sin\theta(b-a))$. The flux through a surface with $\theta =$ constant is given by $\int \mathbf{S} \cdot d\mathbf{A} = \int_a^b S_\theta 2\pi r \sin\theta dr = VI$.

Answer of problem 2

Introduce normal coordinates, X, for deviations of M from equilibrium, and ϕ - angle of the pendulum with the vertical.



(a) The velocities are: for M it is $\dot{X}$, for m it is

$$v_x = \dot{x} = \frac{d}{dt}(X + l\sin\phi) = \dot{X} + l\cos\phi\,\dot{\phi}$$
$$v_y = \dot{y} = \frac{d}{dt}(-l\cos\phi) = l\sin\phi\,\dot{\phi} \qquad (1)$$

The Lagrangian

$$\mathcal{L} = \frac{M\dot{X}^2}{2} + \frac{m(\dot{X} + l\cos\phi\,\dot{\phi})^2 + m(l\sin\phi\,\dot{\phi})^2}{2} - \frac{kX^2}{2} - mgl(1-\cos\phi)$$

(b) For small X and ϕ we neglect terms of order higher than $O(\phi^2, X^2)$. The Lagrangian becomes,

$$\mathcal{L} = \frac{M\dot{X}^2}{2} + \frac{m(\dot{X} + l\dot{\phi})^2}{2} - \frac{kX^2}{2} - \frac{mgl\phi^2}{2}$$

The Euler-Lagrange equation

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{q}} - \frac{\partial\mathcal{L}}{\partial q} = 0$$

gives equations of motion for $q = (X, Y = l\phi)$,

$$(M+m)\ddot{X} + m\ddot{Y} + kX = 0$$
$$ml(\ddot{X} + \ddot{Y}) + mgY = 0$$

Introduce pendulum and spring frequencies:

$$\omega_p^2 = \frac{g}{l} \qquad\qquad \omega_s^2 = \frac{k}{M}$$

and search for collective mode solution $(X(t), Y(t)) = (X, Y)e^{i\omega t}$,

$$\begin{pmatrix} -(1+\frac{m}{M})\omega^2 + \omega_s^2 & -\omega^2\frac{m}{M} \\ -\omega^2 & -\omega^2 + \omega_p^2 \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} = 0$$

that gives frequencies of normal modes

$$\omega^2 = \frac{\omega_s^2 + \omega_p^2(1+m/M)}{2} \pm \sqrt{\left[\frac{\omega_s^2 + \omega_p^2(1+m/M)}{2}\right]^2 - \omega_s^2\omega_p^2}$$

with relative amplitudes proportional to

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} \omega_p^2 - \omega^2 \\ \omega^2 \end{pmatrix}_{\omega=\omega_\pm}$$

The limiting cases: $k = 0$ - no spring, $\omega_s^2 = 0$ and normal frequencies are

$$\omega^2 = 0 \qquad \omega^2 = \omega_p^2(1 + \frac{m}{M})$$

the last mode corresponding to "breathing" oscillations.

$k \to \infty$ - very rigid spring, almost immovable M. For this case $\omega_s^2 \gg \omega_p^2$ and the normal modes are

$$\omega_+^2 = \omega_s^2 \qquad \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \qquad \text{and} \qquad \omega_-^2 = \omega_p^2 \qquad \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

First one - pendulum mass is not moving relative to the walls and floor, and M performs fast oscillations with ω_s. Second mode - only pendulum moving with its natural frequency, M is not affected. In this case the two oscillators effectively decouple.

$l \to 0$ - very short pendulum, as if mass m was attached directly to mass M. Expanding the solution for $\omega_p^2 \gg \omega_s^2$, one gets

$$\omega_+^2 = \omega_p^2(1 + \frac{m}{M}) \qquad \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} -m/M \\ 1 \end{pmatrix} \qquad \text{and}$$

$$\omega_-^2 = \frac{\omega_s^2}{1 + m/M} = \frac{k}{m + M} \qquad \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

First mode resembles the "breathing" oscillations as if the spring was not there, and the other mode indeed corresponds to oscillations of two rigidly connected masses on a spring.

3. Solution

a) To find the two energies , we use Schrodinger's equation for the stationary states.

$$H\chi_+ = E_+\chi_+ \qquad\qquad H\chi_- = E_-\chi_-$$

$$-\frac{\gamma B_0\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = E_+\begin{pmatrix}1\\0\end{pmatrix} \qquad\qquad -\frac{\gamma B_0\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = E_-\begin{pmatrix}0\\1\end{pmatrix}$$

$$-\frac{\gamma B_0\hbar}{2}\begin{pmatrix}1\\0\end{pmatrix} = E_+\begin{pmatrix}1\\0\end{pmatrix} \qquad\qquad +\frac{\gamma B_0\hbar}{2}\begin{pmatrix}0\\1\end{pmatrix} = E_-\begin{pmatrix}0\\1\end{pmatrix}$$

$$\boxed{E_+ = -\frac{\gamma B_0\hbar}{2}} \qquad\qquad \boxed{E_- = +\frac{\gamma B_0\hbar}{2}}$$

b) We can get the time dependent state by simply inserting the energy exponentials into each term of the superposition state.

$$\boxed{\chi(t) = \frac{1}{\sqrt{2}}\begin{pmatrix}e^{-iE_+t/\hbar}\\ e^{+iE_+t/\hbar}\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}e^{+i\gamma B_0t/2}\\ e^{-i\gamma B_0t/2}\end{pmatrix}}$$

c) We can get this expectation value the usual way by putting the operator between the superposition state and it's hermitian conjugate. Writing all terms with matrices we get:

$$\langle S_x\rangle = \chi^\dagger S_x\chi = \frac{1}{\sqrt{2}}\begin{pmatrix}e^{-i\omega t/2} & e^{+i\omega t/2}\end{pmatrix}\frac{\hbar}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}e^{+i\omega t/2}\\ e^{-i\omega t/2}\end{pmatrix}$$

$$= \frac{\hbar}{2}\frac{1}{2}\begin{pmatrix}e^{-i\omega t/2} & e^{+i\omega t/2}\end{pmatrix}\begin{pmatrix}e^{-i\omega t/2}\\ e^{+i\omega t/2}\end{pmatrix} = \frac{\hbar}{2}\frac{1}{2}\left(e^{-i\omega t} + e^{+i\omega t}\right)$$

$$\boxed{\langle S_x\rangle = \frac{\hbar}{2}\cos(\omega t)}$$

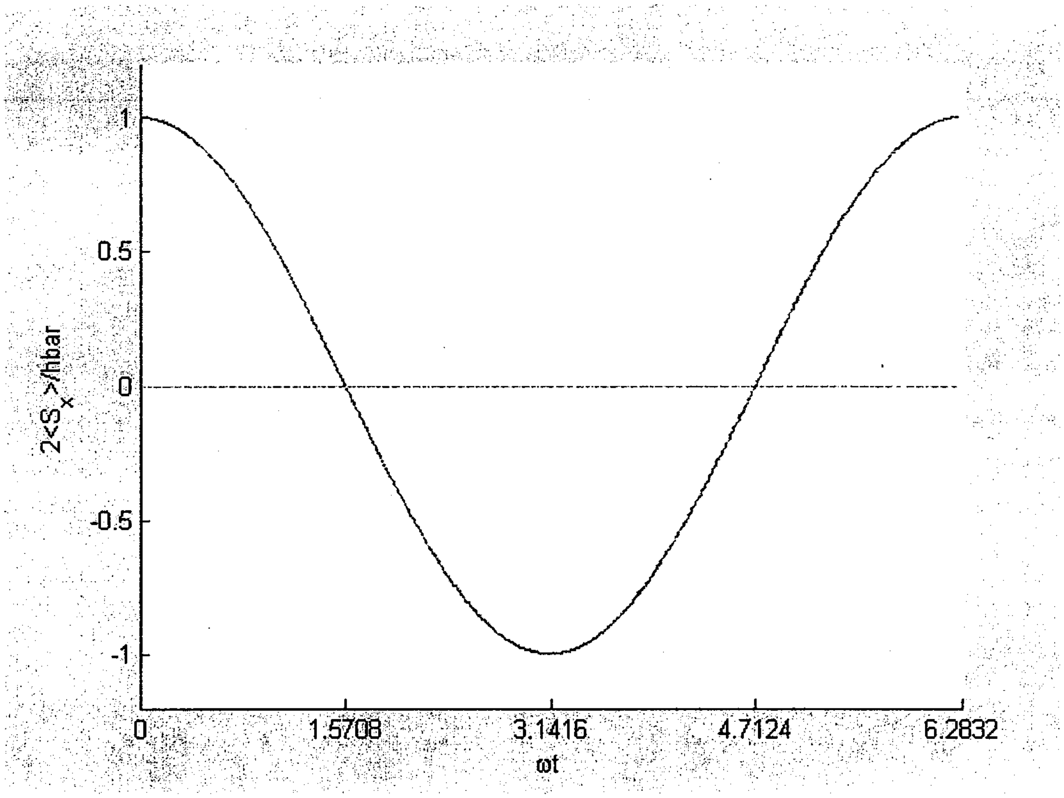
d) To get this probability, we first find the coefficient of the spinor along the +x direction.

$$c_+^{(x)} = \chi_+^{(x)\dagger}\chi(t) = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}e^{+i\omega t/2}\\ e^{-i\omega t/2}\end{pmatrix} = \frac{1}{2}\left(e^{+i\omega t/2} + e^{-i\omega t/2}\right) = \cos\left(\frac{\omega t}{2}\right)$$

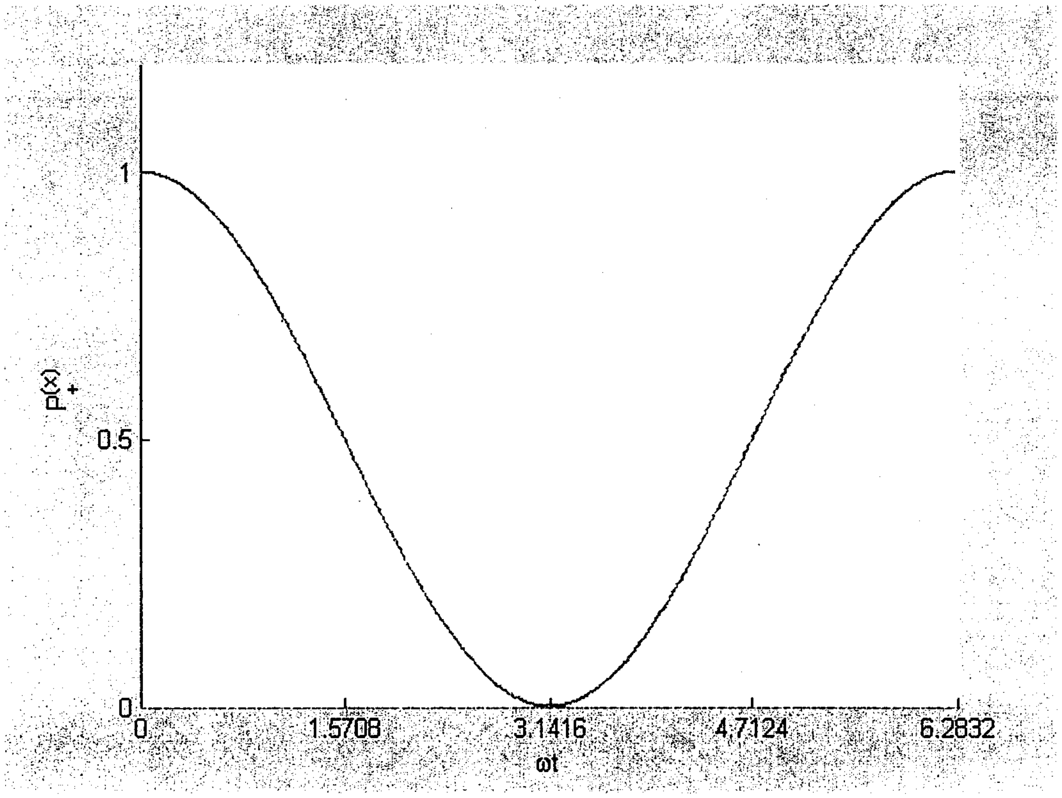
Using this coefficient we can find the probability that the spin will be along the +x direction:

$$\boxed{P_+^{(x)} = \left|c_+^{(x)}\right|^2 = \cos^2\left(\frac{\omega t}{2}\right)}$$

e) First we plot the expectation value $\langle S_x\rangle$ using: $\langle S_x\rangle = \frac{\hbar}{2}\cos(\omega t)$



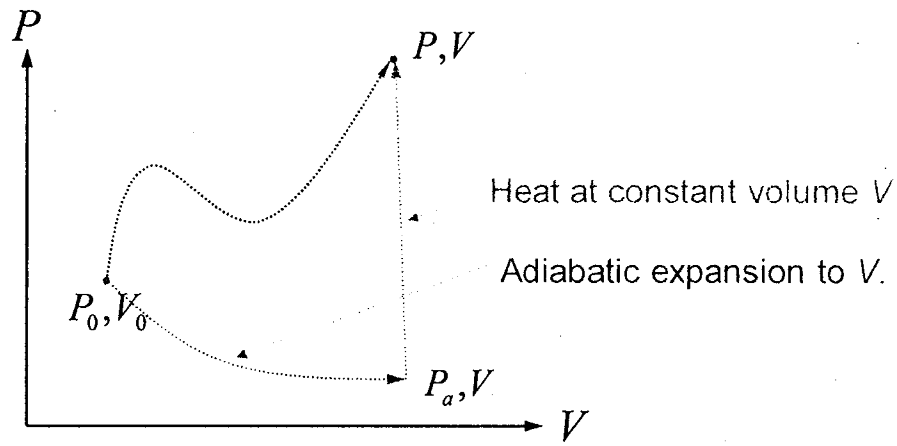
Then we plot the probability that the spin will be along the +x direction: $P_+^{(x)} = \cos^2\left(\frac{\omega t}{2}\right)$



We see that at $\omega t = \frac{\pi}{2}$ the expectation value $\langle S_x \rangle$ is zero since at this time the expectation value of the spin vector $\langle \vec{S} \rangle$ is pointing along the y axis as the spin precesses around the magnetic field. However at this time, $\omega t = \frac{\pi}{2}$, the probability $P_+^{(x)}$ of measuring $\frac{+\hbar}{2}$ along the +x direction is not zero but is equal to ½ . This is due to the non-zero uncertainty $\sigma_{S_x} = \frac{\hbar}{2}$. Thus if we measure the spin along in the x direction, we have a 50% chance of collapsing the spin wave function to $\frac{+\hbar}{2}$ (along the +x direction) and a 50% chance of collapsing the spin wave function to $\frac{-\hbar}{2}$ (along the -x direction). Thus we see that the plots are related but represent different aspects of the spin in the x direction.

4. Solution

For a reversible process a change in entropy depends only on the beginning and end points of the process. It is independent of the path that takes the gas from the initial point to the final point. This is because entropy, like the total energy of the system, is a state function. Thus, the change in entropy, ΔS, is simply given by $\Delta S = S(P,V) - S(P_0,V_0)$. Therefore, the problem can be solved by choosing a simple path, along which we can calculate the change in the entropy easily. For example, one can expand the gas at constant pressure from P_0,V_0 until it reaches P_0,V and then increase the pressure from P_0,V at constant volume until we reach the state P,V. The easiest way to solve this problem, shown in the figure below, is first to expand the gas adiabatically and reversibly from P_0,V_0 to P_a,V and then follow this by increasing the pressure at constant volume from P_a,V until we reach the final state, P,V.

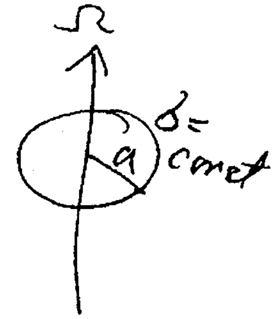


Because the expansion from P_0,V_0 to P_a,V is adiabatic there will be no change in the entropy of the gas. The only change in the entropy will take place during the heating process, when the pressure of the gas is increased from P_a to P at constant volume V. For an adiabatic process $P_0V_0^\gamma = P_aV^\gamma$. This immediately suggests $P_a = P_0(V_0/V)^\gamma$. Therefore, the change in the entropy can be written as $\Delta S = S(P,V) - S(P_0,V_0) = S(P,V) - S(P_a,V)$. This, then, can easily be calculated for an n mole ideal gas by integrating

$\Delta S = \int_{P_a,V}^{P,V} dS = \int_{P_a,V}^{P,V} \delta Q/T = \int_{T_a}^{T} nC_V dT/T = nC_V \ell n(T/T_a)$. For an ideal gas $P_aV/T_a = PV/T$, or $T/T_a = P/P_a$. Combining this with $P_a = P_0(V_0/V)^\gamma$ and inserting into $\Delta S = nC_V \ell n(T/T_a)$, we obtain $\Delta S = nC_V \ell n(P/P_0(V_0/V)^\gamma)$. By rearranging the last equation we arrive at the final expression: $\Delta S = nC_V \ell n PV^\gamma - nC_V \ell n P_0V_0^\gamma$, where $\gamma = C_p/C_v$.

#5

Comp/10 Key



1 $\boxed{\text{Special Topic}}$ by S. Tsuruta

(a) $\nabla^2 \phi_m = 0$ ①

$\phi_{in} = \sum_l A_l r^l P_l(\cos\theta)$ ②a

$\phi_{out} = \sum_l B_l r^{-(l+1)} P_l(\cos\theta)$ ②b

(b) $-\bar{B} = \hat{r}\dfrac{d\phi}{dr} + \hat{\theta}\dfrac{1}{r}\dfrac{\partial\phi}{\partial\theta}$ ③

So $\bar{B}_{in} = -\sum_l \left\{ A_l l r^{l-1} P_l(\cos\theta)\hat{r} + A_l r^{l-1} P_l'(\cos\theta)\hat{\theta} \right\}$ ④a

$\bar{B}_{out} = -\sum_l \left\{ B_l(-l-1) r^{-l-2} P_l(\cos\theta)\hat{r} + B_l r^{-l-2} P_l'(\cos\theta) \right\}$ ④b

Boundary condn. at $r=a$: $B_r^{in} = B_r^{out}$ ⑤a

So $\hat{r}\times(\bar{B}_{out} - \bar{B}_{in}) = \frac{4\pi}{c}\bar{K}$

$\rightarrow B_\theta^{out} - B_\theta^{in} = \frac{4\pi}{c} K = \frac{4\pi}{c}\sigma\Omega a \sin\theta$ (*) ⑤b

⑤a $\rightarrow l A_l a^{l-1} = -(l+1) B_l a^{-l-2}$ ⑥a

⑤b $\rightarrow (B_l a^{-l-2} - A_l a^{l-1}) = \frac{4\pi\sigma\Omega}{c} a\,\delta_{l1}$

$\rightarrow \boxed{A_1 = -\frac{8\pi}{3}\frac{\sigma\Omega a}{c},\; B_1 = \frac{4\pi}{3}\frac{\sigma\Omega a^4}{c}}$ ⑥

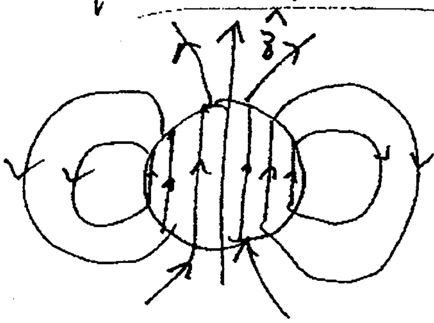
ans

(c) ④a ⑥ $\rightarrow \bar{B}_{in} = \frac{8\pi}{3}\frac{\sigma\Omega a}{c}[\hat{r}\cos\theta - \hat{\theta}\sin\theta] = \boxed{\frac{8\pi}{3}\frac{\sigma\Omega a}{c}\hat{z}}$

ans

④b ⑥ $\rightarrow \bar{B}_{out} = \boxed{\frac{4\pi}{3}\frac{\sigma\Omega a^4}{c r^3}[\hat{r}\,2\cos\theta - \hat{\theta}\sin\theta]}$

ans



ans //

6. Solution

a) To find the Lagrangian L, we first find the kinetic energy T for each mass. We use the Cartesian coordinate method here.

For M_1:

$$x = S \to \dot{x} = \dot{S} \to T_{M_1} = \frac{1}{2}M_1\dot{x}^2 = \frac{1}{2}M_1\dot{S}^2$$

For M_2:

$$x = S + R\sin\theta \to \dot{x} = \dot{S} + R\dot{\theta}\cos\theta \qquad y = -R\cos\theta \to \dot{y} = R\dot{\theta}\sin\theta$$

$$T_{M_2} = \frac{1}{2}M_2\dot{x}^2 + \frac{1}{2}M_2\dot{y}^2 = \frac{1}{2}M_2\left(\dot{S} + R\dot{\theta}\cos\theta\right)^2 + \frac{1}{2}M_2\left(R\dot{\theta}\sin\theta\right)^2$$

$$= \frac{1}{2}M_2\dot{S}^2 + \frac{1}{2}M_2R^2\dot{\theta}^2 + M_2\dot{S}R\dot{\theta}\cos\theta$$

Thus the total kinetic energy T is:

$$T = \frac{1}{2}(M_1 + M_2)\dot{S}^2 + \frac{1}{2}M_2R^2\dot{\theta}^2 + M_2\dot{S}R\dot{\theta}\cos\theta$$

For the potential energy V, we ignore M_1 since it stays on the horizontal level. For M_2 we have:

$$V = M_2g(R + y) = M_2g(R - R\cos\theta) = M_2gR(1 - \cos\theta)$$

Thus the Lagrangian L=T-V becomes:

$$\boxed{L = \frac{1}{2}(M_1 + M_2)\dot{S}^2 + \frac{1}{2}M_2R^2\dot{\theta}^2 + M_2\dot{S}R\dot{\theta}\cos\theta - M_2gR(1 - \cos\theta)}$$

b) Now to find Lagrange's equation for θ we get:

$$\frac{\partial L}{\partial \theta} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = 0 \qquad \frac{\partial L}{\partial \theta} = -M_2\dot{S}R\dot{\theta}\sin\theta - M_2gR\sin\theta$$

$$-\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = -\frac{d}{dt}\left(M_2R^2\dot{\theta} + M_2\dot{S}R\cos\theta\right) = -M_2R^2\ddot{\theta} - M_2\ddot{S}R\cos\theta + M_2\dot{S}R\dot{\theta}\sin\theta$$

Thus, with two terms canceling, we get:

$$M_2gR\sin\theta + M_2R^2\ddot{\theta} + M_2\ddot{S}R\cos\theta = 0 \quad \to \boxed{\ddot{\theta} = -\frac{g}{R}\sin\theta - \frac{\ddot{S}}{R}\cos\theta}$$

c) Now to find equilibrium angle during the acceleration, we let:

$$S = \frac{1}{2}at^2 \to \ddot{S} = a \quad \to \quad \ddot{\theta} = -\frac{g}{R}\sin\theta - \frac{a}{R}\cos\theta$$

Also at equilibrium during the acceleration,

$$\ddot{\theta} = 0 \to \frac{g}{R}\sin\theta_0 = -\frac{a}{R}\cos\theta_0 \to \tan\theta_0 = -\frac{a}{g}$$

Thus we get the equilibrium angle θ_0:

$$\boxed{\theta_0 = -\tan^{-1}\left(\frac{a}{g}\right)}$$

d) Now to find the frequency of small oscillations of the angle during the acceleration, we go back to our equation in b).

$$\ddot{\theta} = -\frac{g}{R}\sin\theta - \frac{a}{R}\cos\theta$$

Then we use the small angle α

$$\alpha = \theta - \theta_o \rightarrow \ddot{\alpha} = \ddot{\theta}$$

$$\begin{aligned}\ddot{\alpha} &= -\frac{g}{R}\sin(\alpha + \theta_o) - \frac{a}{R}\cos(\alpha + \theta_o) \\ &= -\frac{1}{R}\left\{g\left[\sin(\alpha + \theta_o)\right] + a\left[\cos(\alpha + \theta_o)\right]\right\} \\ &= -\frac{1}{R}\left\{g\left[\sin(\alpha)\cos(\theta_o) + \cos(\alpha)\sin(\theta_o)\right] + a\left[\cos(\alpha)\cos(\theta_o) - \sin(\alpha)\sin(\theta_o)\right]\right\}\end{aligned}$$

Now let's use the small angle approximation

$$\sin\alpha \approx \alpha \qquad \cos\alpha \approx 1$$

$$\begin{aligned}\ddot{\alpha} &= -\frac{1}{R}\left\{g\left[\alpha\cos(\theta_o) + \sin(\theta_o)\right] + a\left[\cos(\theta_o) - \alpha\sin(\theta_o)\right]\right\} \\ &= -\frac{1}{R}\left\{\alpha\left[g\cos(\theta_o) - a\sin(\theta_o)\right] + \left[g\sin(\theta_o) + a\cos(\theta_o)\right]\right\}\end{aligned}$$

Note that the last term in the right square brackets is zero from the results of part c) for the equilibrium angle θ_o. Thus:

$$\ddot{\alpha} = -\frac{1}{R}\left[g\cos(\theta_o) - a\sin(\theta_o)\right]\alpha$$

We can see that this is the classic harmonic oscillator equation with a oscillation frequency ω:

$$\boxed{\omega = \sqrt{\frac{g\cos(\theta_o) - a\sin(\theta_o)}{R}}}$$

Extra comment: Using some trig identities, we can write this in a simpler form:

$$\omega = \sqrt{\frac{g'}{R}} \qquad \text{with} \qquad g' = \sqrt{g^2 + a^2}$$

which we also expect from what we know about non-inertial frames.

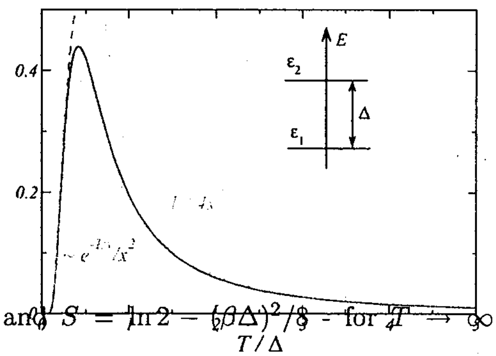
Answer of problem 7

To calculate thermodynamic properties we start with partition function, the sum over all possible configurations Γ with corresponding statistical weights. For a single two-level system, $\beta = 1/T$

$$Z = \sum_{\Gamma} e^{-\beta E_\Gamma} = e^{-\varepsilon_1/T} + e^{-\varepsilon_2/T} = e^{-\varepsilon_1/T}(1 + e^{-\beta\Delta})$$

Free energy,

$$\boxed{F = -T\ln Z = \varepsilon_1 - T\ln(1 + e^{-\beta\Delta})}$$



Entropy,

$$\boxed{S = -\frac{\partial F}{\partial T} = \ln(1 + e^{-\beta\Delta}) + \frac{\Delta}{T}\frac{1}{1 + e^{\beta\Delta}}}$$

at low temperatures $\beta\Delta \gg 1$ and $S \to \ln 1 = 0$ - only one state is available; at high temperatures $\beta\Delta \ll 1$ and $S = \ln 2 - (\beta\Delta)^2/8$ - for $T \to \infty$ approaching configuration with two equally populated states. Heat capacity,

$$\boxed{C(T,\Delta) = T\frac{\partial S}{\partial T} = \frac{\Delta^2}{T^2}\frac{e^{\Delta/T}}{(e^{\Delta/T}+1)^2}}$$

has exponential behavior at low temperatures, $\Delta/T \gg 1$, $C \sim (\Delta^2/T^2)\,e^{-\Delta/T}$, and inverse-$T$ behavior at high T, $\Delta/T \ll 1$, $C \sim \Delta^2/4T^2$. Profile of this function is shown in the figure and it is called Schottky barrier.

To calculate the heat capacity of glass, we integrate over the distribution of two-level systems,

$$C(T) = \int_0^\infty P(\Delta)C(T,\Delta)\,d\Delta$$

For energies $\Delta > \Lambda$ the $C(T,\Delta)$ is exponentially small (we are interested in low $T/\Lambda \ll 1$), so we can neglect the contribution from high energies, and use $P(\Delta) = P_0$ in this integral. We have,

$$C = P_0\int_0^\infty \frac{\Delta^2}{T^2}\frac{e^{\Delta/T}}{(e^{\Delta/T}+1)^2}\,d\Delta = P_0 T\int_0^\infty \frac{x^2e^x}{(e^x+1)^2}\,dx \quad , \qquad \text{with substitution } x = \Delta/T\,.$$

The integral gives just a numerical constant, whereas the temperature behavior is $C_{glass} \sim T$ - linear in temperature.

To evaluate the integral we perform integration by parts and then use series expansion of the fraction with exponent,

$$\int_0^\infty \frac{x^2e^x}{(e^x+1)^2}\,dx = -\int_0^\infty x^2\,d\frac{1}{e^x+1} = 2\int_0^\infty \frac{x}{e^x+1}\,dx = 2\int_0^\infty x\sum_{n=0}^\infty(-1)^n e^{-(n+1)x}\,dx = 2\sum_{n=0}^\infty\frac{(-1)^n}{(n+1)^2}\int_0^\infty ue^{-u}\,du$$

The integral here is equal to 1 (single integration by parts), while the sum is related to the Riemann-zeta function,

$$= 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2} = 2\left(\sum_{n=1}^{odd}\frac{1}{n^2} - \sum_{n=2}^{even}\frac{1}{n^2}\right) = 2\left(\sum_{n=1}^{\infty}\frac{1}{n^2} - 2\sum_{n=2}^{even}\frac{1}{n^2}\right) = 2\left(\sum_{n=1}^{\infty}\frac{1}{n^2} - 2\sum_{k=1}^{\infty}\frac{1}{(2k)^2}\right) = 2\sum_{n=1}^{\infty}\frac{1}{n^2}\left(1 - 2\frac{1}{4}\right)$$

$$= 2\zeta(2)\frac{1}{2} = \zeta(2) = \frac{\pi^2}{6} \qquad \Rightarrow \qquad \boxed{C = \frac{\pi^2}{6}P_0T}$$

8. Solution

Taking the Laplace transform we have

$$s^2L(x) - \frac{4}{5} + 4sL(x) + 8L(x) = \frac{4}{s^2+16}.$$

Solving for $L(x)$ we find

$$L(x) = \frac{1}{20}\frac{(s+2)+20}{(s+2)^2+4} - \frac{1}{20}\frac{s+2}{s^2+16}.$$

Looking up the inverse transforms in the CRC we get

$$x(t) = \frac{1}{40}\left(2e^{-2t}\cos(2t) - 2\cos(4t) - \sin(4t) + 20e^{-2t}\sin(2t)\right).$$

9. Solution

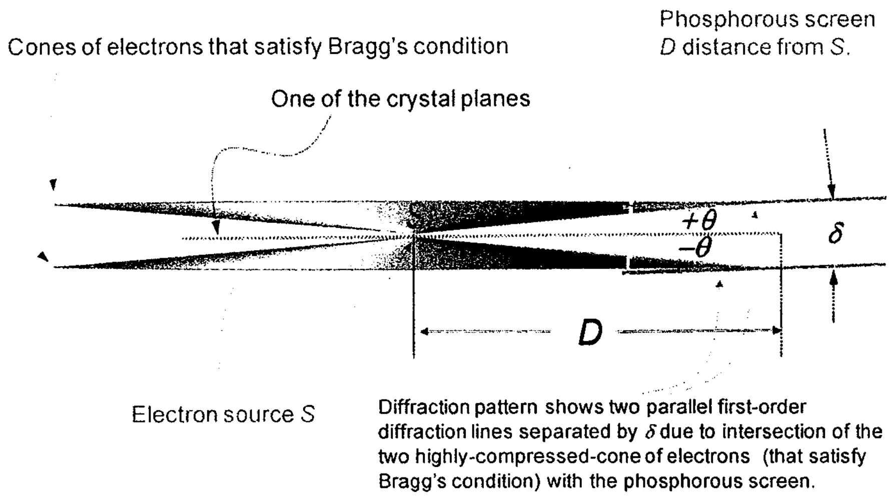
a. Typical lattice spacings are about *3-4* Å. The value of λ can be calculated from the De Broglie wavelength of the electron using $\lambda = h/p$, where momentum p can be calculated from $E = p^2 / 2m_e$. This yields $\lambda = h / p = h / \sqrt{2m_e E}$. By plugging in the numbers we obtain:

$$\lambda = \frac{6.63\times10^{-34} J.s}{\sqrt{2\times9.31\times10^{-31} kg\times2\times10^{4} eV\times1.6\times10^{-19} J/eV}}\times10^{10} \mathring{A}/m = 0.086\mathring{A},$$

which is much smaller than the typical lattice spacing of ~*3-4* Å.

b. Refer to the figure below for clarity. Only those electrons incident at angle θ that satisfy Bragg's diffraction condition, $2dSin\theta = n\lambda$ (where $n=1$ for the first-order diffraction), will come out of the crystal without being absorbed. Notice that because $\lambda << d$ the angle θ is expected to be a very small. Therefore, the diffracted electrons will fall into two low aspect-ratio cones subtended at the source *S*, corresponding to those electrons emitted at $\pm\theta$ incidence angles relative to the crystal planes, as shown in the figure below. The diffraction pattern on the screen will show two parallel lines, corresponding to the intersections of the two flat (low-aspect ratio) cones with the phosphorous screen. These patterns are known as Kikuchi lines or Kikuchi patters and are used commonly used in electron microscopy to identify crystal phase and orientation with submicron spatial resolution.

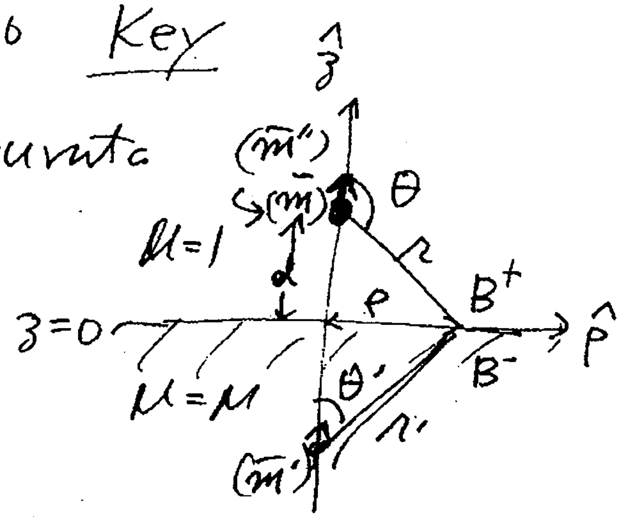
c. Because $\lambda << d$, Bragg's condition suggests that $Sin\theta = \lambda / 2d << 1$, which results in $Sin\theta \simeq \theta = \lambda / 2d$. This angle can easily be determined from the geometry shown in the figure below, where $\theta \simeq (\delta / 2) / D = \delta / 2D$. This immediately yields the lattice spacing, $d = \lambda R / \delta = (h / \sqrt{2m_e E}) \times (R / \delta)$. If we plug in the numbers: $d = 0.086\mathring{A} \times 20cm / 0.5cm = 3.44\mathring{A}$


Cones of electrons that satisfy Bragg's condition
Phosphorous screen
D distance from S.
One of the crystal planes
+θ
−θ
δ
D
Electron source S
Diffraction pattern shows two parallel first-order diffraction lines separated by δ due to intersection of the two highly-compressed-cone of electrons (that satisfy Bragg's condition) with the phosphorous screen.

#10

Comp/10 Key

1 Math Physics S. Tsuruta



(a) $\mu=\mu$ for $z<0$

$\mu=1$ for $z>0$

Find B

$\bar{B}^+ = \bar{B}\ (z>0)$;

$\bar{B}^- = B\ (z<0)$

$$\bar{B}^+ = \frac{m(3\hat{r}\cos\theta - \hat{z})}{r^3} + \frac{m'(3\hat{r}'\cos\theta' - \hat{z}')}{r'^3} \quad (1)$$

$r = r'$; $\theta' = \pi - \theta$ (1)

$$\bar{B}^- = \frac{\mu m''(3\hat{r}\cos\theta - \hat{z})}{r^3} = \mu \bar{H}^- \quad (2)$$

$\hat{r} = \hat{\rho}\sin\theta + \hat{z}\cos\theta'$ (3a)

$\hat{r}' = \hat{\rho}\sin\theta' + \hat{z}\cos\theta' = \hat{\rho}\sin\theta - \hat{z}\cos\theta$ (3b)

(i') (3b) → (1)

$$\bar{B}^+ = m\,\frac{3\sin\theta\cos\theta\,\hat{\rho} + (3\cos^2\theta - 1)\hat{z}}{r^3} + m'\,\frac{(-3\sin\theta\cos\theta)\hat{\rho}' + (3\cos^2\theta - 1)\hat{z}'}{r^3} \quad (4a)$$

// Ans

(i) (3a) → (2)

$$\mu H^- = B^- = \frac{\mu m''\ 3\sin\theta\cos\theta\,\hat{\rho} + (3\cos^2\theta - 1)\hat{z}}{r^3} \quad (4b)$$

// Ans

(b), Boundary condition $\underbrace{}_{B_\perp \text{ conti} \Rightarrow}$ $B^+_{\hat{n}} = B^+_{\hat{z}} \Rightarrow m + m' = \mu m''$ (5a)

$H_\parallel$ conti $\Rightarrow H^+_\rho \to$ (4) $m - m' = m''$ (5b)

(5a) (5b) → $\boxed{m'' = m\,2/(1+\mu) \quad m' = m\left(\frac{\mu-1}{\mu+1}\right)}$ Ans

(c) As $\mu \to 1$ $m' = 0$ & $m'' = m$ & $B^+ = 2m(3\cos^2\theta - 1)/r^3\,\hat{z}$

$\mu \to \infty$ $m' \to m$ & $m'' = 0$ & $B^- = 0$

So Field are ⊥ to plane. $\uparrow\ \uparrow\ \bar{B}$ field ($\mu \to \infty$) // Ans

11. Solution

a. Newton's law of motion dictates that the net force on mass m equals the net acceleration times the mass: The net force in the horizontal frictionless plane of motion is entirely due to the spring's restoring force and is given by $k(r - x_0)$, where k is the spring constant and x_0 is the distance from the center of the circle to mass m when the spring is relaxed (not stretched). Once the mass is spinning without any friction it will undergo two distinct accelerations along the radial direction. One of them is due to centripetal acceleration, which is equal to v^2 / r. The second one is due to the spring vibrations, which is given by d^2r / dt^2. Furthermore, while the centripetal acceleration is in the same direction as the spring force and always points towards the center, the vibrational acceleration has an oscillatory behavior. It changes direction along the radial direction. All of this can be put into a single equation by noting that the restoring force always opposes the extension or compression of the spring: $k(r - x_0) = m\, v^2 / r - m\, d^2r / dt^2$. The only quantity that is conserved during the action of the radial impulse would be the angular momentum of the system where no torque is applied to the system during the impulse. Therefore, the angular momentum of the system is preserved after the impulse, which can be expressed as: $mr_0v_0 = mvr$ or $r_0v_0 = vr$, where r_0 and v_0 are the radius of rotation and the speed of rotation $v_0 = \omega_0 r_0$, respectively, before the impulse.

b. In order to find the vibration frequency of the spring while the mass is spinning in the frictionless plane we introduce the vibration amplitude, $u = r - r_0$, where $|u| << r_0$ is oscillatory and can be positive or negative. Taking into account the conservation of angular momentum the equation of motion can be rewritten as
$k(r - x_0) = m\, v^2r^2 / r^3 - m\, d^2r / dt^2 \Rightarrow k(r - x_0) = m\, v_0^2 r_0^2 / (r_0 + u)^3 - m\, d^2r / dt^2$. This equation can be further simplified as
$k(r - x_0) = m\, v_0^2 / r_0(1 + u / r_0)^3 - m\, d^2r / dt^2 \Rightarrow k(u + r_0 - x_0) \simeq (m\, v_0^2 / r_0)(1 - 3u / r_0) - m\, d^2u / dt^2$
using the approximation $1/(1 + u / r_0)^3 \simeq (1 - 3(u / r_0))$. The last equation can be written as $d^2u / dt^2 + (k / m + 3\omega_0^2)u = -(k / m)(r_0 - x_0) + \omega_0^2 r_0$ using the relation $v_0 = \omega_0 r_0$. Noticing that before the application of impulse Newton's equation of motion implies that $k(r_0 - x_0) = m\, v_0^2 / r_0 = m\omega_0^2 r_0$. By inserting this in the last equation for u we obtain: $d^2u / dt^2 + (k / m + 3\omega_0^2)u = 0$ The last equation implies that the solution is of the form: $u = u_0 Sin(\omega t + \phi)$, where $\omega = 2\pi\nu = \sqrt{(k / m + 3\omega_0^2)}$.
The values of u_0 and ϕ can be determined from the initial conditions when necessary.

12. Solution

a) Wien's law - black body spectrum peaks at a wavelength $\lambda = b/T$ where $b = 2.9 \times 10^{-3}$ m K. Thus we have $\lambda = 1.07$ mm and an energy of 1.2×10^{-3} eV.

b) The threshold energy will be a minimum for head on collisions. At threshold, the pion and the proton will be at rest in the COM frame.

c) Method 1 (Undergraduate level approach)

Writing $\mathbf{k} = -k\hat{z}$ for the photon 3-momentum, $\mathbf{p} = p\hat{z}$ for the initial proton 3-momentum (we are looking at a head on collision), $\mathbf{q} = q\hat{z}$ for the final proton momentum and $\mathbf{g} = g\hat{z}$ for the pion momentum, we have, by momentum conservation $p - k = q + g$. At threshold the pion and proton will be at rest in the COM frame (which has velocity v in the frame we are using), so we can write $g = m_\pi \gamma v$ and $q = m_p \gamma v$ so that $(p-k)^2 = (m_\pi + m_p)^2(\gamma^2 - 1)$. Conservation of energy gives $E_p + k = E_g + E_q = \gamma(m_\pi + m_p)$. Squaring the energy equation and combining with the momentum equation yields

$$E_p + p = \frac{(m_\pi + m_p)^2 - m_p^2}{2k},.$$

Since $k \ll m_\pi, m_p$, we see that p must be large, so to a good approximation $E_p \simeq p$ and we have our final result.

Method 2 (Graduate level approach)

Write $\vec{k} \rightarrow (k, 0, 0, -k)$ for the components of the photon 4-momentum with $k = 1.2 \times 10^{-3}$ eV. Write $\vec{p} \rightarrow (E_p, 0, 0, p)$ for the components of the proton 4-momentum, and recall that $\vec{p} \cdot \vec{p} = -m_p^2$ (Using units where $c = 1$, and all quantities are in eV). The threshold corresponds to when the proton and pion end up with zero velocity in the center of momentum frame. The COM frame is defined by $\vec{P} = \vec{k} + \vec{p} \rightarrow_{\mathrm{COM}} (M, 0, 0, 0)$. At threshold we know that $M = m_p + m_\pi$. Using $\vec{P} \cdot \vec{P} = -M^2$ we get

$$p = \frac{M^2 - m_p^2}{4k} - \frac{m_p^2 k}{M^2 - m_p^2}.$$

If you are on the ball you will already have reasoned that $p \gg m_p$ and used a short-cut to get to a very good approximation to the above exact result. Plugging in the numbers yields $p = 5.85 \times 10^{19}$ eV.

d) Same as part b) but with $M = M_\Delta$. Get $p = 1.4 \times 10^{20}$ eV.

e) Using $m = 50$ grams and $v = 50$ m/s, get 62.5 J $= 3.9 \times 10^{20}$ eV - comparable to the energy of the cosmic rays.

f) At threshold the pion and the proton are at rest in the COM frame. In other words, they are co-moving with the COM frame, which has 4-velocity $\vec{P}/M$. Thus, the scattered proton has 4-momentum $\vec{q} = m_p\vec{P}/M$. Since the rest mass is negligible and the photon energy is small in the CMB rest frame, the fractional change in the energy is just $1 - m_p/M \approx 0.23$.

g) Dimensonal analysis gets you this one even if you don't remember any basic scattering theory! $L = 1/n\sigma = 4 \times 10^{21}\,\mathrm{m} = 0.13$ Mpc. Since about 20% of the proton's energy is lost in each scattering event, it doesn't take too many to stop them, so the stopping distance is less than a Mpc (a more detailed calculation taking into account the smaller cross section off the Delta resonance yields 10's of Mpc).

13. Solution

Use Laplace's equation $\nabla^2\phi = 0$ to solve this. Since Laplace's equation is separable in the rectangular coordinates, the general solution of Laplace's equation in xyz coordinates can be written as

$$\phi(x,y,z) = \sum\left(a_1e^{\alpha x} + a_2e^{-\alpha x}\right)\left(b_1e^{\beta y} + b_2e^{-\beta y}\right)\left(c_1e^{\gamma z} + c_2e^{-\gamma z}\right)$$

with $\alpha^2 + \beta^2 + \gamma^2 = 0$.

Since there is no z dependence for the boundaries, we expect γ=0. Thus we will have $\alpha^2 = -\beta^2$ for the coefficients for x and y. Also from the symmetry of the boundary conditions we expect that ϕ will be of the form:

$$\phi \sim \sum \cosh(\sim x)\cos(\sim y)$$

Thus we will let α be real and then we will have β=iα, so that β is pure imaginary. Of course you can always show this when you apply the boundary conditions but anticipating the form of the answer simplifies the path to the solution. Thus we start by writing the solution as:

$$\phi(x,y) = \sum\left(a_1e^{\alpha x} + a_2e^{-\alpha x}\right)\left(b_1e^{i\alpha y} + b_2e^{-i\alpha y}\right)$$

where the sum is over values of α that will satisfy the boundary conditions.

Now since $\phi(y_o) = \phi(-y_o) = 0$, then $b_1 = b_2$ so that

$$\left(b_1e^{i\alpha y} + b_2e^{-i\alpha y}\right) = 2b\frac{\left(e^{i\alpha y} + e^{-i\alpha y}\right)}{2} = 2b\cos(\alpha y)$$

Also since $\phi(y_o) = \phi(-y_o) = 0$, then $\cos(\alpha y_o) = 0$ which will require that $\alpha y_o = \frac{n\pi}{2}$ with n odd. Thus our potential becomes:

$$\phi(x,y) = \sum_{n\,odd}\left(a_1e^{\frac{n\pi}{2y_o}x} + a_2e^{-\frac{n\pi}{2y_o}x}\right)2b\cos\left(\frac{n\pi}{2y_o}y\right)$$

Now for x=0 and x=x_o, ϕ=ϕ_o and is symmetric about $x = \frac{x_o}{2}$ so we can let $a_1 = \frac{a}{2}e^{-\frac{n\pi x_o}{4y_o}}$ and $a_2 = \frac{a}{2}e^{+\frac{n\pi x_o}{4y_o}}$ so that

$$\left(a_1e^{\frac{n\pi}{2y_o}x} + a_2e^{-\frac{n\pi}{2y_o}x}\right) = \frac{a}{2}\left(e^{\frac{n\pi}{2y_o}\left(x-\frac{x_o}{2}\right)} + e^{-\frac{n\pi}{2y_o}\left(x-\frac{x_o}{2}\right)}\right) = a\cosh\left(\frac{n\pi}{2y_o}\left(x-\frac{x_o}{2}\right)\right)$$

Thus our potential ϕ becomes:

$$\phi(x,y) = \sum_{n\,odd} A_n\cosh\left(\frac{n\pi}{2y_o}\left(x-\frac{x_o}{2}\right)\right)\cos\left(\frac{n\pi}{2y_o}y\right)$$

where A_n=2ab to combine the two constants into one. Now all that is left is to find how the constant A_n depends on n. To do this we use $\phi(x_o,y)=\phi_o$.

$$\phi(x_o,y) = \phi_o = \sum_{n\,odd} A_n\cosh\left(\frac{n\pi x_o}{4y_o}\right)\cos\left(\frac{n\pi}{2y_o}y\right)$$

Now we multiply both sides of the equation by $\cos\left(\frac{m\pi}{2y_o}y\right)$ and then integrate y from $-y_o$ to $+y_o$.

$$\int_{-y_o}^{+y_o}\phi_o\cos\left(\frac{m\pi}{2y_o}y\right)dy=\sum_{n\,odd}A_n\cosh\left(\frac{n\pi x_o}{4y_o}\right)\int_{-y_o}^{+y_o}\cos\left(\frac{n\pi}{2y_o}y\right)\cos\left(\frac{m\pi}{2y_o}y\right)dy$$

$$\phi_o\left[\sin\left(\frac{m\pi}{2y_o}y\right)\right]\left(\frac{2y_o}{m\pi}\right)\Bigg|_{y=-y_o}^{y=+y_o}=\sum_{n\,odd}A_n\cosh\left(\frac{n\pi x_o}{4y_o}\right)\left[\frac{1}{2}2y_o\right]\delta_{nm}$$

$$\left(\frac{2y_o\phi_o}{m\pi}\right)\left[\sin\left(\frac{m\pi}{2y_o}y_o\right)-\sin\left(-\frac{m\pi}{2y_o}y_o\right)\right]=A_m y_o\cosh\left(\frac{m\pi x_o}{4y_o}\right),\ m\,odd$$

$$\left(\frac{4\phi_o}{m\pi}\right)\left[\sin\left(\frac{m\pi}{2}\right)\right]=A_m\cosh\left(\frac{m\pi x_o}{4y_o}\right),\ m\,odd$$

Solving for A_m and writing it as A_n we get:

$$A_n=\frac{4\phi_o\sin\left(\frac{n\pi}{2}\right)}{n\pi\cosh\left(\frac{n\pi x_o}{4y_o}\right)},\ n\,odd$$

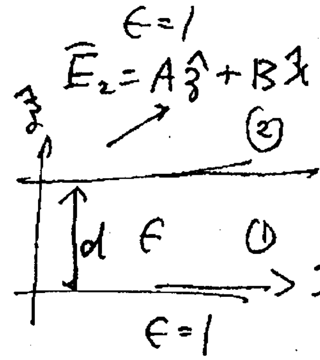
Thus our final solution becomes:

$$\boxed{\phi(x,y)=\sum_{n\,odd}\frac{4\phi_o\sin\left(\frac{n\pi}{2}\right)}{n\pi}\frac{\cosh\left(\frac{n\pi}{2y_o}\left(x-\frac{x_o}{2}\right)\right)}{\cosh\left(\frac{n\pi x_o}{4y_o}\right)}\cos\left(\frac{n\pi}{2y_o}y\right)}$$

#14

Comp/10 Key

E&M S. Tsuruta



Apply boundary condition

(a) $D_{\perp 1} = D_{\perp 2} \to E_{2z} = A = \epsilon E_{1z}$

$\to$ so $E_{1z} = A/\epsilon$

$E_{\parallel 1} = E_{\parallel 2}$ so $E_{x2} = B = E_{x1}$

$E_{y1} = E_{y2} = 0$

so $\bar{E}_1 = E_{1z}\hat{z} + E_{1x}\hat{x} = \boxed{\frac{A}{\epsilon}\hat{z} + B\hat{x}}$ ans

(b). $\bar{P}_1 = \frac{(\epsilon-1)}{4\pi}\bar{E}_1 = \boxed{\frac{(\epsilon-1)}{4\pi}\left(\frac{A}{\epsilon}\hat{z} + B\hat{x}\right)}$ ans

$\cancel{\rho_{P_1} = \frac{(\epsilon-1)}{4\pi}\bar{E}_1}$ $\rho_{p_1} = -\bar{\nabla}\cdot\bar{P}_1 = 0$ // ans

since A, B, ϵ are all constant

(c) $\sigma_p = -(P_{\perp 2} - P_{\perp 1}) = +P_{\perp 1}$

$= \boxed{\left(\frac{(\epsilon-1)}{4\pi\epsilon}\right)A \text{ at top at } z=d}$ ans

$\sigma_p = (P_{\perp 2} - P_{\perp 1}) = -P_{\perp 1} = -P_{z1}$

$= \boxed{-\left(\frac{(\epsilon-1)}{4\pi\epsilon}\right)A \text{ at bottom at } z=0}$

ans.

Answer of problem 15

Introduce x axis normal to the barrier (from left to right), and y axis along the barrier (optional).

(a) The Hamiltonian should include the kinetic energy part,

$$\frac{\hat{p}^2}{2m} = -\frac{\hbar^2}{2m}\nabla^2 = -\frac{\hbar^2}{2m}(\nabla_x^2 + \nabla_y^2)\,;$$

and the magnetic moment energy in the field to the right of $x = 0$ i.e $\sim \theta(x)$, which is lowered if magnetic moment (spin) is along the field and increased if the moment is anti-parallel to the field, i.e. $\mathcal{H}_B = -\mathbf{m}\cdot\mathbf{B}$, where $\mathbf{m} = \mu\boldsymbol{\sigma}$. This means that the Hamiltonian is

$$\mathcal{H} = -\frac{\hbar^2}{2m}\nabla^2 - \mu\mathbf{B}\cdot\boldsymbol{\sigma}\,\theta(x)\,,$$

and the boundary conditions on the wave function at $x = 0$ follow; integration of Schrödinger equation across the barrier gives the continuity of the wave function and its x-derivative at $x = 0$:

$$\Psi_L(0,y) = \Psi_R(0,y)$$
$$\nabla_x\Psi_R(0,y) = \nabla_x\Psi_L(0,y)$$

which should hold for any y along the barrier.

(b) Spin polarization of the reflected beam we calculate as

$$P = |R_\uparrow|^2 - |R_\downarrow|^2\,.$$

To determine the reflection amplitudes we need to use the boundary conditions. Since neither in the Hamiltonian, nor in the boundary conditions the two spins mix, we can consider two projections of spin independently.

The solutions to the wave function on the left is independent of spin projection, and follows from Hamiltonian of a free particle (the normal incidence)

$$\mathcal{H}_L = -\frac{\hbar^2}{2m}\nabla_x^2 \qquad \Rightarrow \qquad \Psi_L(x) = e^{ip_x x/\hbar} + R_\sigma e^{-ip_x x/\hbar}\,, \qquad E = \frac{p_x^2}{2m}$$

and is a sum of incoming wave and reflected wave with reflection coefficient R_σ. Note, that the reflection amplitude will depend on spin through the boundary conditions. On the right, for each spin projection $\sigma = \pm 1$ we have

$$\mathcal{H}_R = -\frac{\hbar^2}{2m}\nabla_x^2 - \mu B\,\sigma \qquad \Rightarrow \qquad \Psi_R(x) = T_\sigma e^{ik_x x/\hbar}\,, \qquad E = \frac{k_x^2}{2m} - \mu B\,\sigma$$

Here T_σ is the transmission amplitude for given spin projection.

Continuity gives,

$$\Psi(0) = 1 + R_\sigma = T_\sigma$$

and from the derivative equation we have after substituting expressions for $\Psi_{L,R}$,

$$k_x T_\sigma = p_x(1 - R_\sigma)$$

Solving these two equations for R_σ we have

$$R_\sigma = \frac{p_x - k_x}{p_x + k_x}$$

where the dependence on spin projection comes through the energy-momentum relation:

$$E = \frac{p_x^2}{2m} = \frac{k_x^2}{2m} - \mu B\,\sigma \qquad \Rightarrow \qquad \frac{k_x}{p_x} = \sqrt{1 + \frac{\mu B}{E}\sigma}\,.$$

Note that for the down-spin the square root will give purely imaginary k_x for $E < \mu B$, which results in $|R_\downarrow|^2 = 1$ - total reflection of down-spins and exponential decay of the wave function into the classicaly forbidden region. The polarization:

$$P = |R_\uparrow|^2 - |R_\downarrow|^2 = \left|\frac{p_x - k_{x\uparrow}}{p_x + k_{x\uparrow}}\right|^2 - \left|\frac{p_x - k_{x\downarrow}}{p_x + k_{x\downarrow}}\right|^2$$

and in terms of energy of incident particles,

$$\boxed{P = \left|\frac{1 - k_{x\uparrow}/p_x}{1 + k_{x\uparrow}/p_x}\right|^2 - \left|\frac{1 - k_{x\downarrow}/p_x}{1 + k_{x\downarrow}/p_x}\right|^2 = \left|\frac{1 - \sqrt{1 + 1/\varepsilon}}{1 + \sqrt{1 + 1/\varepsilon}}\right|^2 - \left|\frac{1 - \sqrt{1 - 1/\varepsilon}}{1 + \sqrt{1 - 1/\varepsilon}}\right|^2 \qquad \text{with } \varepsilon = \frac{E}{\mu B}}\,.$$

Characteristic energy here is $E^* = \mu B$ - energy of spin in magnetic field, and if energy of incident particles is less than that, they cannot enter the field region, which can occur for down-spin particles. Now consider limits of small and large energies.

For $\varepsilon \ll 1$,

$$P(\varepsilon \ll 1) \approx \left|\frac{1 - \sqrt{1/\varepsilon}}{1 + \sqrt{1/\varepsilon}}\right|^2 - 1 \approx \left|\frac{1 - \sqrt{\varepsilon}}{1 + \sqrt{\varepsilon}}\right|^2 - 1 \approx (1 - 2\sqrt{\varepsilon})^2 - 1 \approx -4\sqrt{\varepsilon}$$

- starting from zero it quickly becomes large and negative. Negative, since spin-up particles penetrate into the field region very well (actually pulled into it) while spin-down particles are fully reflected for $\varepsilon < 1$, $|R_\downarrow(\varepsilon < 1)| = 1$.

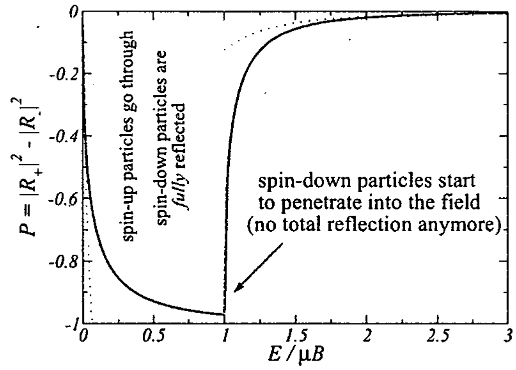
Performing expansion for $\varepsilon \gg 1$,

$$P(\varepsilon \gg 1) \approx \left|\frac{1 - (1 + \frac{1}{2\varepsilon} - \frac{1}{8\varepsilon^2})}{1 + (1 + \frac{1}{2\varepsilon} - \frac{1}{8\varepsilon^2})}\right|^2 - \left|\frac{1 - (1 - \frac{1}{2\varepsilon} - \frac{1}{8\varepsilon^2})}{1 + (1 - \frac{1}{2\varepsilon} - \frac{1}{8\varepsilon^2})}\right|^2 \approx \left|\frac{\frac{1}{2\varepsilon} - \frac{1}{8\varepsilon^2}}{2 + \frac{1}{2\varepsilon} - \frac{1}{8\varepsilon^2}}\right|^2 - \left|\frac{\frac{1}{2\varepsilon} + \frac{1}{8\varepsilon^2}}{2 - \frac{1}{2\varepsilon} - \frac{1}{8\varepsilon^2}}\right|^2$$

Eventually,

$$P(\varepsilon \gg 1) \approx \frac{1}{16\varepsilon^2}\left|\frac{1-\frac{1}{4\varepsilon}}{1+\frac{1}{4\varepsilon}}\right|^2 - \frac{1}{16\varepsilon^2}\left|\frac{1+\frac{1}{4\varepsilon}}{1-\frac{1}{4\varepsilon}}\right|^2 \approx -\frac{1}{8\varepsilon^3}$$

The main point is that at higher energies the polarization of reflected beam very quickly, $\sim O(1/\varepsilon^3)$, drops to zero. This drop in polarization reflects the fact that for $\varepsilon > 1$ spin-down particles no longer totally reflected and start easily penetrate into the region of magnetic field. We can estimate the value of P at $\varepsilon = 1$, and its behavior slightly above $\varepsilon = 1$. For this we only need to know $\sqrt{2} \approx 1.4$; then the full expression for P gives,



$$P(\varepsilon = 1) = \left|\frac{1-\sqrt{2}}{1+\sqrt{2}}\right|^2 - 1 \approx \left(\frac{0.4}{2.4}\right)^2 - 1 = \frac{1}{36} - 1 \approx -0.97$$

The behavior slightly above $\varepsilon = 1$ we get by setting $\varepsilon = 1 + \epsilon$, $\epsilon \ll 1$,

$$P(\varepsilon = 1+\epsilon) \approx \left|\frac{1-\sqrt{2-\epsilon}}{1+\sqrt{2-\epsilon}}\right|^2 - \left|\frac{1-\sqrt{\epsilon}}{1+\sqrt{\epsilon}}\right|^2 \approx P(\varepsilon = 1) + 4\sqrt{\epsilon} + O(\epsilon)$$

The two asymptotic behaviors and this estimate give a good idea of the exact function, which is shown in the figure.