ϕ 519 Problems { Set #⑤ : Probs. 15-18. Assigned 21 Oct 88; due 28 Oct. 88.

⑤ P6

■ [Jackson Prob. (6.10)]. Discuss conservation of energy and linear momentum, for sources and fields, in a uniform, isotropic, linear medium with dielectric constant ϵ and permeability μ. Show that a "straightforward" (what else?) calculation of energy density u, Poynting vector $\mathbb{S}$, momentum density $\mathbb{g}$, and stress tensor (T_{ik}) yield the Minkowski results (~ Jackson's p. 240):

$$\boxed{\begin{aligned} &u = \frac{1}{8\pi}(\epsilon E^2 + \mu H^2),\quad \mathbb{S} = (c/4\pi)\,\mathbb{E}\times\mathbb{H},\quad \mathbb{g} = (\mu\epsilon/c^2)\,\mathbb{S}, \\ &\text{and}\quad T_{ik} = \frac{1}{4\pi}(\epsilon E_i E_k + \mu H_i H_k) - u\,\delta_{ik}. \end{aligned}}$$

How must the calculation be modified if ϵ and μ depend on position?

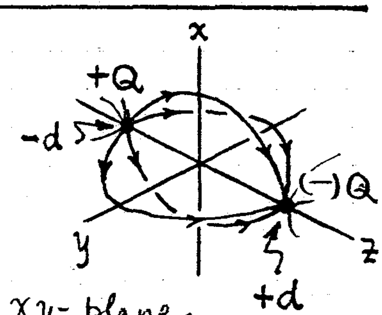
ϕ 519 Problems

[Jackson Prob. (6.11)]. With the same assumptions as in prob. 15, discuss angular momentum conservation. Show that the differential & integral forms of conservation are:

$$\boxed{\frac{\partial}{\partial t}(\mathbb{L}_{mech}+\mathbb{L}_{field})+\operatorname{div}\underset{\sim}{M}=0,\quad \frac{d}{dt}\int_V dv\,(\mathbb{L}_{mech}+\mathbb{L}_{field})+\oint_S d\mathbf{s}\cdot\underset{\sim}{M}=0.}$$

Here $\mathbb{L}_{field}=\mathbf{r}\times\mathbf{g}$ is the ∡ momentum density of the EM field, $\mathbb{L}_{mech}$ is the ∡ momentum density of whatever particles are present, surface S encloses the (~ arbitrary) volume V, and the tensor $\underset{\sim}{M}=\underset{\sim}{T}\times\mathbf{r}$, where $\underset{\sim}{T}$ is the stress tensor. HINT: avoid working with $\underset{\sim}{M}$ until the last moment; recall that $\underset{\sim}{T}$ is symmetric, and $\operatorname{div}\underset{\sim}{T}$ is a vector; work out carefully what $(\operatorname{div}\underset{\sim}{T})\times\mathbf{r}$ means.

Charges $\pm Q$ are fixed at positions $\mp d$ on the z-axis.

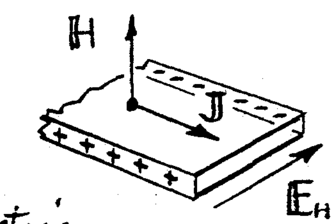


(A) Find the electric field components everywhere in the xy-plane, and write them in terms of plane polar cds (ρ,ϕ).

(B) Find all components T_{ik} of the Maxwell stress tensor in the xy-plane.

(C) Find the force law between the Q's by integrating (T_{ik}) over the surface of a box: one face ≡ xy-plane, other faces at ∞. Compare with Coulomb's law. Reinvent wheel.

[Jackson Prob. (6-16)]. A conductor has current density $\mathbb{J}$ flowing in it, due to an applied electric field $\mathbb{E}$. If, at the same time, a transverse magnetic field $\mathbb{H}$ is applied, then a new (effective) electric field $\mathbb{E}_H$ appears... $\mathbb{E}_H$ is $\perp$ both $\mathbb{J}$ & $\mathbb{H}$, and it generates a voltage between the sides of the conductor. Generation of a voltage this way is called the Hall Effect.



(A) Assume a normal Ohm's Law: $\mathbb{E}=\frac{1}{\sigma}\mathbb{J}$, when $\mathbb{H}=0$. When $\mathbb{H}\neq 0$, use the known symmetries (rotation & reflection) of the fields, plus a Taylor expansion about $\mathbb{H}=0$, to show--for an isotropic medium--that at most: $\mathbb{E}=\frac{1}{\sigma}\mathbb{J}+R(\mathbb{H}\times\mathbb{J})+\beta_1 H^2\mathbb{J}+\beta_2(\mathbb{J}\cdot\mathbb{H})\mathbb{H}$, correct to $O(H^2)$. σ = conductivity of medium, and R (Hall coefficient), β_1 & β_2 are (medium-dept) constants.

B) Discuss the Ohm's Law generalization of part (A) in light of time-reversal invariance.

10/24/84

(15) Derive Minkowski's form of energy & momentum conservation in a linear medium.

1. Energy conservation is done in the book (Sec. 6.8) for a medium with: $\mathbb{D}=\epsilon\mathbb{E}$ and $\mathbb{B}=\mu\mathbb{H}$. Results//

$$\boxed{u=\frac{1}{8\pi}(\epsilon E^2+\mu H^2)\ ,\ \ \mathbb{S}=\frac{c}{4\pi}(\mathbb{E}\times\mathbb{H}).} \tag{1}$$

2. For linear momentum conservation, follow book derivation (from Eq. (6.114) on)...

$$\underline{\frac{d\mathbb{P}_{mech}}{dt}=\int_V\left(\rho\mathbb{E}+\frac{1}{c}\mathbb{J}\times\mathbb{B}\right)d\tau}$$

Lorentz force law -- true microscopically. Use: $\rho=\frac{1}{4\pi}\nabla\cdot\mathbb{D}$, $\frac{1}{c}\mathbb{J}=\frac{1}{4\pi}\left(\nabla\times\mathbb{H}-\frac{1}{c}\frac{\partial\mathbb{D}}{\partial t}\right)$, with $\mathbb{D}=\epsilon\mathbb{E}$ & $\mathbb{B}=\mu\mathbb{H}$ to go macroscopic. Derivation below Eq. (6.115) goes thru, to

$$\rho\mathbb{E}+\frac{1}{c}\mathbb{J}\times\mathbb{B}=\frac{1}{4\pi}\Big[\epsilon(\nabla\cdot\mathbb{E})\mathbb{E}+\mu(\nabla\cdot\mathbb{H})\mathbb{H}-\epsilon\mathbb{E}\times(\nabla\times\mathbb{E})-\mu\mathbb{H}\times(\nabla\times\mathbb{H})-$$

Last term identifiable as field momentum density

$$-\frac{\mu\epsilon}{c}\frac{\partial}{\partial t}(\mathbb{E}\times\mathbb{H})\Big]. \tag{2}$$

i.e. // $\mathbb{g}=\frac{\mu\epsilon}{4\pi c}(\mathbb{E}\times\mathbb{H})=\frac{\mu\epsilon}{c^2}\mathbb{S}$, or// $\boxed{\mathbb{g}=\frac{1}{4\pi c}(\mathbb{D}\times\mathbb{B})}$. (3)

Then, with $\mathbb{P}_{field}=\int_V \mathbb{g}\,d\tau$, the momentum eqtn for particles & fields is...

$$\frac{d}{dt}(\mathbb{P}_{mech}+\mathbb{P}_{field})=\frac{1}{4\pi}\int_V d\tau\Big[\epsilon(\nabla\cdot\mathbb{E})\mathbb{E}+\mu(\nabla\cdot\mathbb{H})\mathbb{H}-\epsilon\mathbb{E}\times(\nabla\times\mathbb{E})-\mu\mathbb{H}\times(\nabla\times\mathbb{H})\Big]. \tag{4}$$

3. Rewriting the [ ] in Eq. (4) as div $\underset{\sim}{T}$ proceeds just as in Jackson's Eq. (6.119); the cnsts ϵ & μ produce nothing new, so long as they are position-independent. Thus:

$$[\text{Eq. (4) above}]_i=\underbrace{\partial T_{ik}/\partial x_k}_{\text{Sum on }k}\ ,\ \ \boxed{T_{ik}=\frac{1}{4\pi}(\epsilon E_iE_k+\mu H_iH_k)-u\,\delta_{ik}}, \tag{5}$$

and/ $\longrightarrow$ $\frac{d}{dt}(\mathbb{P}_{mech}+\mathbb{P}_{field})_i=\oint_S T_{ik}\,dA_k$ $\leftarrow$ momentum conservation: particles + fields. (6)

Eqs (1), (3), (5) are the required "straightforward" Minkowski results for: u, $\mathbb{S}$, $\mathbb{g}$, T_{ik}.

4. If ϵ & μ $\propto$ position depdt, the steps leading to Eq. (5) above are incomplete. We can still define T_{ik} as in Eq. (5), but div $\underset{\sim}{T}$ $\Rightarrow$ another term, viz: $\frac{1}{8\pi}\int_V d\tau\,(E^2\nabla\epsilon+H^2\nabla\mu)_i$, on Eq. (6)-RHS.

10/24/84

Derive angular momentum conservation for particles & fields in a linear medium.

1. Since torque: $\frac{d\mathbb{L}}{dt} = \mathbb{r}\times\mathbb{F}$, on a single particle, it is natural to define...

$$\frac{d}{dt}\int_V \underbrace{\mathbb{L}_{mech}}_{\text{≮ mom. density}} d\tau = \int_V \mathbb{r}\times \underbrace{\mathcal{F}}_{\text{force/volume}} d\tau = \int_V \mathbb{r}\times\left(\rho\mathbb{E}+\frac{1}{c}\mathbb{J}\times\mathbb{B}\right)d\tau. \quad (1)$$

The Lorentz force/vol. can be handled just as in Prob. ⑮, with result

$$\rho\mathbb{E}+\frac{1}{c}\mathbb{J}\times\mathbb{B} = \operatorname{div}\underset{\sim}{T} - \frac{\partial \mathbf{g}}{\partial t}\left\{\begin{array}{l} \mathbf{g} = (\mu\epsilon/c^2)\,\mathbb{S},\quad \underset{\sim}{T} = (T_{ik}),\ \text{with:} \\ T_{ik} = \frac{1}{4\pi}(\epsilon E_i E_k + \mu H_i H_k) - u\,\delta_{ik}. \end{array}\right. \quad (2)$$

Note: $\operatorname{div}\underset{\sim}{T}$ is a vector, with components: $(\operatorname{div}\underset{\sim}{T})_i = \partial T_{ik}/\partial x_k$ (sum on k).

2. $\mathbb{r}\times(\rho\mathbb{E}+\frac{1}{c}\mathbb{J}\times\mathbb{B})$ generates $\mathbb{r}\times(\partial\mathbf{g}/\partial t)$, if we use Eq. (2). Note, however, that in Eq. (1), $\mathbb{r}$ is an integration variable, indpt of time, so we can set $\mathbb{r}\times(\partial\mathbf{g}/\partial t) = \frac{\partial}{\partial t}(\mathbb{r}\times\mathbf{g})$. Then, using Eq. (2) in Eq. (1), and rearranging terms...

$$\longrightarrow \frac{d}{dt}\int_V(\mathbb{L}_{mech}+\mathbb{L}_{field})\,d\tau = \int_V(\mathbb{r}\times\operatorname{div}\underset{\sim}{T})\,d\tau,\quad \underline{\underline{\mathbb{L}_{field} = \mathbb{r}\times\mathbf{g}}} \text{ field ≮ mom. density} \quad (3)$$

3. Calculate $(\operatorname{div}\underset{\sim}{T})\times\mathbb{r}$ explicitly. With: $\operatorname{div}\underset{\sim}{T} = (\partial T_{ik}/\partial x_k)\,\mathbb{e}_i$ & $\mathbb{r} = (x_1, x_2, x_3)$:

$$(\operatorname{div}\underset{\sim}{T})\times\mathbb{r} = \left(x_3\frac{\partial T_{k2}}{\partial x_k} - x_2\frac{\partial T_{k3}}{\partial x_k},\ x_1\frac{\partial T_{k3}}{\partial x_k} - x_3\frac{\partial T_{k1}}{\partial x_k},\ x_2\frac{\partial T_{k1}}{\partial x_k} - x_1\frac{\partial T_{k2}}{\partial x_k}\right) \quad (4)$$

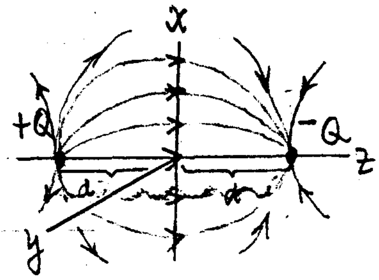
We've used fact that $\underset{\sim}{T}$ is symmetric: $T_{ik} = T_{ki}$. The 1st component can be rewritten:

$$[(\operatorname{div}\underset{\sim}{T})\times\mathbb{r}]_1 = \frac{\partial}{\partial x_1}(T_{12}x_3 - T_{13}x_2) + \frac{\partial}{\partial x_2}(T_{22}x_3 - T_{23}x_2) + \frac{\partial}{\partial x_3}(T_{32}x_3 - T_{33}x_2), \quad (5)$$

using fact that $T_{32} = T_{23}$. This is in the form of the divergence of a tensor $\underset{\sim}{M}$ of comps: $M_{ikl} = T_{ik}x_l - T_{il}x_k$, whose indices are permutations of $\overrightarrow{123}$, like a vector product: $\underline{\underline{\underset{\sim}{M} = \underset{\sim}{T}\times\mathbb{r}}}$, symbolically. In Eq. (3), this gives, as required:

$$\longrightarrow \frac{d}{dt}\int_V(\mathbb{L}_{mech}+\mathbb{L}_{fld})\,d\tau = -\int_V \operatorname{div}\underset{\sim}{M}\,d\tau = -\oint_S \underset{\sim}{M}\,d\mathbb{A},\ \text{or}\ \boxed{\frac{d}{dt}(\mathbb{L}_{mech}+\mathbb{L}_{fld}) + \operatorname{div}\underset{\sim}{M} = 0}. \quad (6)$$

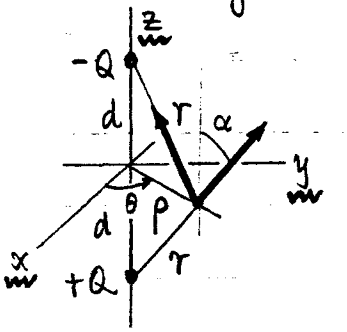
ϕ519 Solutions



(17) (10 pts). Discover Coulomb's Law via Maxwell Stress Tensor.

7 Nov. 84

A. By symmetry, field in xy-plane is everywhere $\perp$ that plane, (i.e. along z-axis), and has magnitude E_z, where E_z is indpt of θ, and...



$$E_z = 2\cdot\frac{Q}{r^2}\cdot\cos\alpha \quad \left\{\begin{array}{l} r^2 = \rho^2 + d^2, \\ \cos\alpha = d/r; \end{array}\right.$$

So// $\boxed{E_z = \frac{2Qd}{(\rho^2+d^2)^{3/2}}}$, on xy plane ($\rho$=radius in xy plane).

B. By def^{n}, elts of $\underset{\sim}{T}$ are: $T_{ik} = \frac{1}{4\pi}E_i E_k - \frac{1}{8\pi}E^2$, with no magnetic field present. On the xy-plane, $\vec{E} = E_z\hat{z}$ only, as in part A, so...

$$\left[T_{zz} = \frac{1}{4\pi}E_z^2 - \frac{1}{8\pi}E_z^2 = \frac{1}{8\pi}\left(\frac{2Qd}{(\rho^2+d^2)^{3/2}}\right)^2 = \frac{Q^2d^2}{2\pi}\cdot\frac{1}{(\rho^2+d^2)^3}\right].$$

Only other non-zero comps are: $T_{xx} = T_{yy} = (-)\,T_{zz}$. These contribute zero to the box integral.

C. On the box surfaces specified, T_{zz} vanishes everywhere but on the xy-plane (since $\rho\to\infty$ everywhere but there), and the integral of interest is..

$$\oint_{box} T_{ik}\, dA_k = \int_{xy\text{-plane}} T_{zz}\, dx\,dy = \int_0^\infty T_{zz}\cdot 2\pi\rho\, d\rho.$$

We've changed to the polar area elt. $dx\,dy \Rightarrow 2\pi\rho\, d\rho$ for obvious reasons.

Then// $$\oint_{box} T_{ik}\, dA_k = Q^2d^2\int_0^\infty \frac{\rho\, d\rho}{(\rho^2+d^2)^3} = \frac{Q^2d^2}{2}\cdot\frac{1}{2(\rho^2+d^2)^2}\Bigg|_{\rho=\infty}^{\rho=0} = \boxed{\frac{Q^2}{(2d)^2}}$$

<u>This is exactly equivalent to the Coulomb law-of-force</u> between $\pm Q$ separated by distance $(2d)$. All wheels appear to be round.

⑱ Construct modified Ohm's Law for Hall Effect, to terms of $O(H^2)$.

10/24/84

1. By def^n: $\mathbb{E}_0 = \frac{1}{\sigma}\mathbb{J} \leftrightarrow$ Ohm's Law when magnetic field $\mathbb{H}=0$. For $\mathbb{H}>0$, the Taylor expansion notion only means we look for corrections $\propto |\mathbb{H}|, |\mathbb{H}|^2, |\mathbb{H}|^3$, etc, i.e. the successively higher powers of $|\mathbb{H}|$ which occur in a Taylor series. Here we need only go to $O(\mathbb{H})^2$.

2. We note that $\mathbb{E}$ and $\mathbb{J}$ are <u>polar</u> vectors, odd under the P (inversion) operation, while $\mathbb{H}$ is an <u>axial</u> vector, which is P-even. Each correction term to $\mathbb{E}_0 = \frac{1}{\sigma}\mathbb{J}$ must then be a polar vector, which involves $\mathbb{J}$ and/or $\mathbb{H}$ (up to $O(\mathbb{H})^2$), and which vanishes when $\mathbb{H}\to 0$. Possible candidates are

(1) $\mathbb{H}\times\mathbb{J}$ ... OK, since (P-even) $\times$ (P-odd) = (P-odd) $\Rightarrow$ $\mathbb{H}\times\mathbb{J}$ is polar;

(2) $\mathbb{H}\times\mathbb{H}$, $\mathbb{J}\times\mathbb{J}$ ... NG, since both are P-even; besides both $\equiv 0$;

(3) $(\mathbb{H}\cdot\mathbb{H})\mathbb{J}$ ... OK, since $\mathbb{J}$ is polar $\Rightarrow$ P-odd;

(4) $(\mathbb{J}\cdot\mathbb{J})\mathbb{H}$ ... NG, since $\mathbb{H}$ is axial $\Rightarrow$ P-even;

(5) $(\mathbb{H}\cdot\mathbb{J})\mathbb{H}$ ... OK, since pseudoscalar $\times$ axialvector = polar $\Rightarrow$ P-odd;

(6) $\mathbb{H}\times(\mathbb{H}\times\mathbb{J}) = (\mathbb{H}\cdot\mathbb{J})\mathbb{H} - (\mathbb{H}\cdot\mathbb{H})\mathbb{J}$ ... OK, but just a comb^n of #(5) & (3).

Other terms of $\leq O(\mathbb{H})^2$ are either equivalent to one of (1)-(5) [as in #(6)] or are $\equiv 0$. Then, from (1), (3) & (5), we can write -- as required ... with R, β_1 & β_2 = cnsts ...

$$\boxed{\mathbb{E} = \frac{1}{\sigma}\mathbb{J} + R(\mathbb{H}\times\mathbb{J}) + \beta_1 H^2\mathbb{J} + \beta_2(\mathbb{H}\cdot\mathbb{J})\mathbb{H}}$$

① ② ③

Term #① is the principal Hall Effect term.

3. $\mathbb{E}$ is T-even, while $\mathbb{J}$ is T-odd; for the other terms: ① = T-even, ② = T-odd, ③ = T-odd. The T-invariance is broken by the <u>dissipative</u> (energy-loss) nature of this current flow.