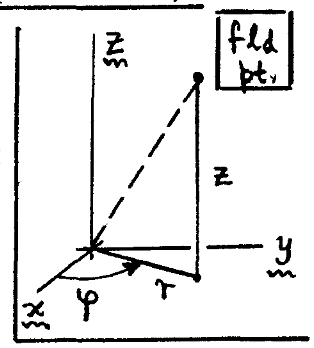
7) $\nabla^2\phi = 0$ in cylindrical polar cds (r,φ,z): Jackson Secs. (3.7)–(3.8).



1. Cd. $z \equiv$ the z of rect^r cds, $\measuredangle\varphi \equiv$ azimuth of sph^l cds, and we use r (instead of Jkn's ρ) as the radius in the xy-plane. Then, in these (r,φ,z) cds, Laplace' problem is...¶

$$\rightarrow \nabla^2\phi = \left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2} + \frac{\partial^2}{\partial z^2}\right]\phi = 0; \tag{27}$$

... put $\phi(r,\varphi,z) = R(r)\,Q(\varphi)\,Z(z)$, so $\frac{1}{\phi}\nabla^2\phi = 0$ yields...

$$\frac{1}{r^2}\left[\frac{r}{R}\frac{d}{dr}\left(r\frac{dR}{dr}\right) + \underbrace{\frac{1}{Q}\left(\frac{d^2Q}{d\varphi^2}\right)}_{\substack{=-\nu^2,\ \text{cnst} \\ (\text{fcn } \varphi \text{ only})}}\right] + \underbrace{\frac{1}{Z}\left(\frac{d^2Z}{dz^2}\right)}_{\substack{=k^2,\ \text{cnst} \\ (\text{fcn } z \text{ only})}} = 0, \tag{28}$$

i.e.,
$$\left.\begin{aligned} Z'' - k^2 Z = 0 &\Rightarrow Z(z) = e^{\pm kz}\ (\text{or } \cosh kz\ \&\ \sinh kz), \\ Q'' + \nu^2 Q = 0 &\Rightarrow Q(\varphi) = e^{\pm i\nu\varphi}\ (\text{or } \cos\nu\varphi\ \&\ \sin\nu\varphi). \end{aligned}\right\} \tag{29}$$

As for spherical cds, two sep^n cnsts and two simple eqtns, followed by

$$\boxed{R'' + \frac{1}{r}R' + \left(k^2 - \frac{\nu^2}{r^2}\right)R = 0} \leftarrow \text{Bessel's ODE}. \tag{30}$$

Which is non-simple. As before, we get one "hard" ODE as the price of separation.

¶ See Arfken "Math. Methods for Physicists" (3rd ed., 1985), Ch. 2. For general curvilinear (orthogonal) cds: $q_k = f_k(x,y,z)$, $^{w/}$ $k=1,2,3$, the line element in k^{th} direction is: $ds_k = h_k\,dq_k$, $^{w/}$ $h_k^2 = (\partial x/\partial q_k)^2 + (\partial y/\partial q_k)^2 + (\partial z/\partial q_k)^2$. If $\hat{e}_k$ is the unit vector along q_k, then in q-cds the gradient operator is: $\nabla_q = \sum_{k=1}^{3}(\hat{e}_k/h_k)\frac{\partial}{\partial q_k}$. The $\hat{e}_k$ & h_k are generally fcns of (x,y,z). Calculation shows:

$$\left[\!\left[\nabla_q^2 = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial q_1}\left(\frac{h_2h_3}{h_1}\frac{\partial}{\partial q_1}\right) + \frac{\partial}{\partial q_2}\left(\frac{h_3h_1}{h_2}\frac{\partial}{\partial q_2}\right) + \frac{\partial}{\partial q_3}\left(\frac{h_1h_2}{h_3}\frac{\partial}{\partial q_3}\right)\right],\right]\!\right]$$

This general result gives the ∇^2 forms quoted on inside back cover of Jackson.

REMARKS

A. For the φ-variation: $Q(\varphi) = e^{\pm i\nu\varphi}$, and -- for $Q(\varphi)$ to be single-valued when $\varphi \to \varphi + 2\pi$ -- we impose: $\underline{\nu = n = 0, 1, 2, 3, \ldots}$. No such restriction applies to the k-value in $Z(z) = \{\cosh, \sinh\}(kz)$, although usually the cosh or sinh is selected via B.C. POINT: in Bessels ODE, Eq. (30), ν is not free to be an eigenvalue, but k is.

B. The radial variable r in Bessel's ODE is generally defined over $0 \leq r \leq a$, (where, sometimes, $a \to \infty$). Change variables in Eq. (30), as...

$$\left[\begin{array}{l} r \to \xi = (\alpha/a) r, \quad a = r_{max} \ \& \ \alpha = \text{cnst}; \\ \text{so} \quad \frac{d}{d\xi}\left(\xi \frac{dR}{d\xi}\right) + \left(\lambda\xi - \frac{\nu^2}{\xi}\right) R = 0, \ \text{w/} \ \lambda = \left(\frac{ka}{\alpha}\right)^2 \ \& \ \nu = n \ (\text{usu.}). \end{array}\right. \quad (31)$$

Bessel's ODE is clearly a Sturm-Liouville type, with $p(\xi) = \xi$, $q(\xi) = -\nu^2/\xi$, weighting fcn $w(\xi) = \xi$, and eigenvalues λ related to the scale a of the domain of definition. ν is just a parameter (as was m in the associated Legendre eqtn); ν serves as an index for the solutions $R(\xi)$.

C. In order to fully qualify as a S-L diff. eqtn., the solutions $R_\nu(\xi)$ to Eq. (31) must obey S-L B.C.: at the endpts $r = 0$ & a of the domain...

$$\longrightarrow R_\nu(\xi)\, p(\xi)\, R_\mu'(\xi)\Big|_{\xi=0} = R_\nu(\xi)\, p(\xi)\, R_\mu'(\xi)\Big|_{\xi=\alpha},$$

$$\Longrightarrow \left\{\begin{array}{l} p(\xi) = \xi, \text{ and} \\ R(0) \text{ non-singular} \end{array}\right\} \quad R_\nu(\alpha)\, R_\mu'(\alpha) = 0 \int \begin{array}{l} \text{gives solns } \alpha = \alpha_{\nu n} \\ \text{with } R_\nu(\alpha_{\nu n}) = 0. \end{array} \quad (32)$$

Role of α is now clear: $\underline{\alpha \to \alpha_{\nu n} \text{ is the } n^{\text{th}} \text{ zero of } R_\nu(\alpha)}$. The quantization of α this way is similar to: $\sin(\alpha x/a)\big|_{x=a} = 0 \Rightarrow \alpha = n\pi = \alpha_n$.

D. The lore of Sturm-Liouville theory now prescribes that R_ν's belonging to different eigenvalues $\alpha_{\nu m}$ & $\alpha_{\nu n}$ will be orthogonal, and that the R_ν's form a complete set on $[0, a]$. Details are worked out in Jackson Eqs. (3.93)-(3.97) [or see Mathews & Walker (2nd ed., 1970), pp. 181-3] with the following results:

B. V. Probs II (cont'd)

$$\left[\!\left[\int_0^a R_\nu\left(\alpha_{\nu m}\frac{r}{a}\right) R_\nu\left(\alpha_{\nu n}\frac{r}{a}\right) r\,dr = \frac{a^2}{2}\left[R_\nu'(\alpha_{\nu n})\right]^2 \delta_{mn} \right]\!\right]. \tag{33}$$

(this is similar to : $\int_0^a \sin\left(m\pi\frac{x}{a}\right)\sin\left(n\pi\frac{x}{a}\right)dx = \frac{a}{2}\left[\cos n\pi\right]^2\delta_{mn}$). Also...

$$\rightarrow f(r) = \sum_{n=1}^{\infty} A_{\nu n} R_\nu\left(\alpha_{\nu n}\frac{r}{a}\right) \longleftrightarrow A_{\nu n} = \frac{2}{\left[a R_\nu'(\alpha_{\nu n})\right]^2}\int_0^a f(r) R_\nu\left(\alpha_{\nu n}\frac{r}{a}\right) r\,dr. \tag{34}$$

The R_ν's (ξ) to be used here are those which are regular (non-singular) at $\xi=0$. They are usually denoted by $J_\nu(\xi)$.

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2. Bessels ODE, Eq.(31), generally has two indept solns $R_\nu(\xi)$: usu. one is regular at $\xi=0$, and the other blows up. A series solution can be developed. Results are:

fcn $R_\nu(\xi)$	name	definition	asymptote : $\xi \ll 1$	asymptote: $\xi \gg 1$
$J_\nu(\xi)$	Bessel fcn (1st kind)	$\left(\frac{\xi}{2}\right)^\nu \sum_{n=0}^{\infty} \frac{(-\xi^2/4)^n}{n!\,\Gamma(n+\nu+1)}$	$\frac{1}{\Gamma(\nu+1)}(\xi/2)^\nu$	$\sqrt{\frac{2}{\pi\xi}}\cos\left(\xi-\frac{\pi}{2}(\nu+\frac{1}{2})\right)$
$N_\nu(\xi)$ [or $Y_\nu(\xi)$]	Neumann fcn	$\frac{J_\nu(\xi)\cos\nu\pi - J_{-\nu}(\xi)}{\sin\nu\pi}$	$(2/\pi)\ln(\xi/2)$, $\nu=0$; $-\frac{1}{\pi}\Gamma(\nu)\left(\frac{2}{\xi}\right)^\nu$, $\nu\neq0$.	$\sqrt{\frac{2}{\pi\xi}}\sin\left(\xi-\frac{\pi}{2}(\nu+\frac{1}{2})\right)$
$H_\nu^{(1)}(\xi)$	Hankel fcn (1st kind)	$J_\nu(\xi)+i\,N_\nu(\xi)$	$(2i/\pi)\ln(\xi/2)$, $\nu=0$; $-\frac{i}{\pi}\Gamma(\nu)(2/\xi)^\nu$, $\nu\neq0$.	$\sqrt{2/\pi\xi}\,e^{+i(\xi-\frac{\pi}{2}(\nu+\frac{1}{2}))}$ ←[outgoing wave]→
$H_\nu^{(2)}(\xi)$	Hankel fcn (2nd kind)	$J_\nu(\xi)-i\,N_\nu(\xi)$	$-(2i/\pi)\ln(\xi/2)$, $\nu=0$; $+\frac{i}{\pi}\Gamma(\nu)(2/\xi)^\nu$, $\nu\neq0$.	$\sqrt{2/\pi\xi}\,e^{-i(\xi-\frac{\pi}{2}(\nu+\frac{1}{2}))}$ →[incoming wave]←
$I_\nu(\xi)$	modified Bessel (1st kind)	$i^{-\nu}J_\nu(i\xi)$	$\frac{1}{\Gamma(\nu+1)}(\xi/2)^\nu$	$\sqrt{\frac{\pi}{2\xi}}\,e^{+\xi}\left[1+O(1/\xi)\right]$
$K_\nu(\xi)$	modified Bessel (2nd kind)	$\frac{\pi}{2}i^{\nu+1}H_\nu^{(1)}(i\xi)$	$-\ln(\xi/2)+\cdots$, $\nu=0$; $\frac{1}{2}\Gamma(\nu)(2/\xi)^\nu$, $\nu\neq0$.	$\sqrt{\frac{\pi}{2\xi}}\,e^{-\xi}\left[1+O(1/\xi)\right]$

(35)

Any of the pairs (J_ν, N_ν), $(H_\nu^{(1)}, H_\nu^{(2)})$, (I_ν, K_ν) are linearly independent for all ν, and--in linear combination [e.g. $AJ_\nu(\xi)+BN_\nu(\xi)$]--serve as a complete soln to Bessel's Eqtn. Much more information can be found in Ch. 9 of the NBS Handbook (ed. Abramowitz & Stegun).

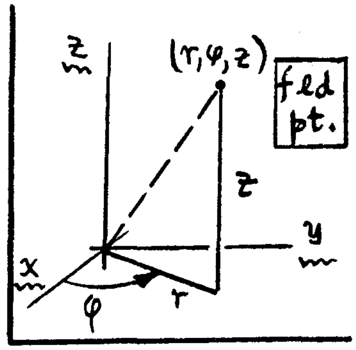
3. All this arithmetic gives us a <u>fully separated solution in cylindrical cds</u>:

$$\phi(r,\varphi,z) = \sum_{\nu,k} R_\nu(kr)\,Q_\nu(\varphi)\,Z_k(z); \qquad (36)$$

w/ $Q_\nu(\varphi) = \left\{\begin{smallmatrix}\sin\\ \cos\end{smallmatrix}\right\}(\nu\varphi)$, $\nu = m = 0, 1, 2, \ldots$ (Q_ν single valued);

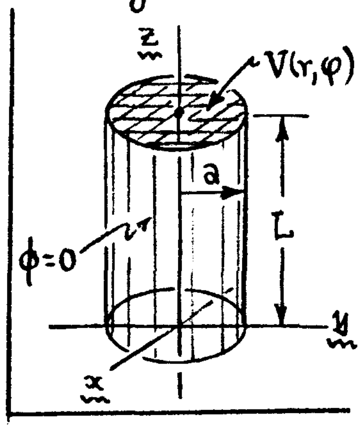
$Z_k(z) = \left\{\begin{smallmatrix}\sinh\\ \cosh\end{smallmatrix}\right\}(kz)$, k is free;

&// $R_\nu(kr) = \left\{\begin{smallmatrix}J_\nu\\ N_\nu\end{smallmatrix}\right\}(kr)$ — solutions to Bessel's Eqtn, in form: $R_\nu'' + \frac{1}{x}R_\nu' + \left(1 - \frac{\nu^2}{x^2}\right)R_\nu = 0$, $x = kr$.



Now we can do problems like the example cited in Jackson Eqs. (3.105) – (3.109)... <u>cylindrical counterpart of the rect^r box problem</u> in Fig. 2.9.

A conducting cylinder of radius a and length L is held at potential $\phi = 0$ everywhere but on its top cap, where: $\phi(r,\varphi,z=L) = V(r,\varphi)$. The problem is to find ϕ everywhere <u>inside</u> the (charge-free) cylinder, i.e. $\phi(r,\varphi,z)$ for $0 \leq r \leq a$, $0 \leq \varphi \leq 2\pi$, $0 \leq z \leq L$. To "sculpt" a solution out of Eq. (36), note...



(1) $Q_\nu(\varphi)$ single-valued $\Rightarrow \nu = m = 0, 1, 2, \ldots$ & $Q_\nu(\varphi) = \left\{\begin{smallmatrix}\sin\\ \cos\end{smallmatrix}\right\} m\varphi$.

(2) $\phi \equiv 0$ at $z = 0 \Rightarrow Z_k(z) = \sinh kz$ only.

(3) ϕ regular @ $r = 0$ & $\phi \equiv 0$ @ $r = a \Rightarrow R_\nu(kr) = J_m(k_{mn}r)$ only, with $k \to k_{mn} = \alpha_{mn}/a$, quantized in terms of zeros α_{mn} of J_m [i.e. $J_m(\alpha_{mn}) \equiv 0$].

(4) The series of Eq. (36) assumes the form, for this problem...

$$\rightarrow \phi(r,\varphi,z) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} J_m(k_{mn}r)\,\sinh k_{mn}z\,[A_{mn}\sin m\varphi + B_{mn}\cos m\varphi]. \qquad (37)$$

[NOTE: must keep both sin & cos here to accommodate $V(r,\varphi)$]. The A_{mn} & B_{mn} are fixed by the B.C.: $V(r,\varphi) = \sum_{m,n}[\sinh k_{mn}L]\,J_m(k_{mn}r)\,[A_{mn}\sin m\varphi + B_{mn}\cos m\varphi]$, using orthogonality in the usual fashion. Results are given in Jk^n Eq. (3.109). They are not exceedingly lovely.