DEPARTMENT OF PHYSICS

M.S. COMPREHENSIVE and PH.D. QUALIFYING EXAM

MONDAY, 26 MARCH 1984, 8 AM-12 NOON

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Answer each of the following 8 questions. All questions are of equal weight. Begin your answer to each question on a <u>new</u> sheet of paper, solutions to different questions must <u>not</u> appear on the same sheet of paper. Label each page of your answer sheets as follows:

A. Your name in upper left-hand corner.

B. Problem number, and page number <u>for that problem</u>, in upper right-hand corner.

1. One of Kepler's laws states that the area swept out per unit time by the radius vector from the sun to a planet is constant. Prove this statement.

Short version

Smith

OK KLN
OK RC

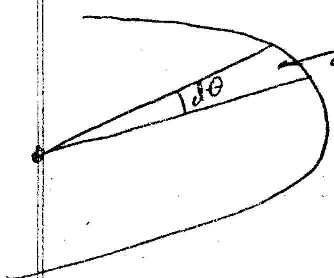
One of Kepler's Laws states that the area swept out per unit time by the radius vector from the sun to a planet is constant. Prove this statement

—— " ——

For planetary motion have gravitational Force

$$F = -G\frac{m_1 m_2}{r^2}\hat{r}, \text{ a central force}$$

So $$\vec{N} = 0 = \frac{d\vec{L}}{dt} \Rightarrow \vec{L} = \text{constant} = m r^2 \dot{\theta}$$



area $dS = \frac{1}{2} r (r\,d\theta)$

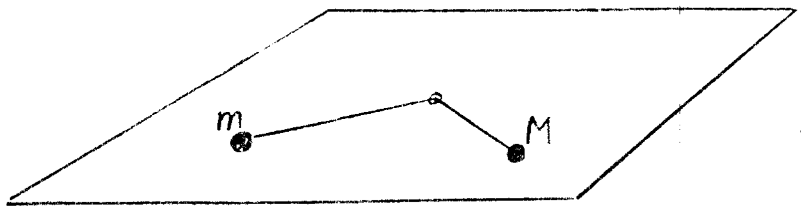
so

$$\frac{dS}{dt} = \frac{1}{2} r^2 \dot{\theta} = \frac{L}{2m} = \text{constant}$$

—— " ——

Problem 1 3/84

2. A string of negligible mass and length l has masses m and M attached to the ends. The string passes through a small frictionless ring fastened at the center of a large horizontal table (the ring pivots freely). Both masses move freely on the frictionless surface of the table. You can neglect gravity.



a) Reduce the problem to an equivalent problem of a single particle moving in a one-dimensional effective potential.

b) Find the equilibrium solutions.

c) Find the period of small oscillations.

d) Find the tension in the string.

e) Find the net force exerted on the ring at the center of the table.

2) $L = T = \frac{1}{2} m_1(\dot{r}_1^2 + r_1^2 \dot{\theta}_1^2) + \frac{1}{2} m_2 (\dot{r}_2^2 + r_2^2 \dot{\theta}_2^2)$

$$L_1 = m_1 r_1^2 \dot{\theta}_1 \qquad\qquad r_1 + r_2 = \ell$$
$$L_2 = m_2 r_2^2 \dot{\theta}_2 \qquad\qquad \dot{r}_1 = -\dot{r}_2$$

a) $$E = \underbrace{\tfrac{1}{2}(m_1+m_2)\dot{r}_1^2}_{\text{"T"}} + \underbrace{\frac{L_1^2}{2m_1 r_1^2} + \frac{L_2^2}{2m_2 r_2^2}}_{\text{"V"}}$$

Equilibrium when $\frac{\partial \text{"V"}}{\partial r_1} = 0$

$$\Rightarrow \quad \frac{L_1^2}{m_1 r_1^3} = \frac{L_2^2}{m_2(\ell - r_1)^3}$$

$$\frac{r_1}{\ell - r_1} = \frac{L_1^2}{L_2^2}\frac{m_2}{m_1} \left(\equiv A \text{ for short}\right)$$

b) $$\boxed{r_1 = \frac{\ell A}{1+A}} \qquad r_2 = \frac{\ell}{1+A}$$

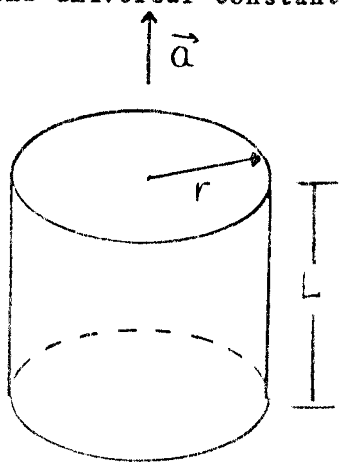
c) $$T = \frac{2\pi}{\omega}, \quad \omega = \sqrt{\frac{(\partial^2 \text{"V"}/\partial r_1^2)}{m_1 + m_2}}.$$

$$\frac{\partial^2 \text{"V"}}{\partial r_1^2} = \frac{3L_1^2}{m_1 r_1^4} + \frac{3L_2^2}{m_2(\ell - r_1)^4}$$
$$= \frac{3L_1^2}{m_1}\left[\left(\frac{1+A}{A\ell}\right)^4 + \left(\frac{1+A}{\ell}\right)^4\right]$$

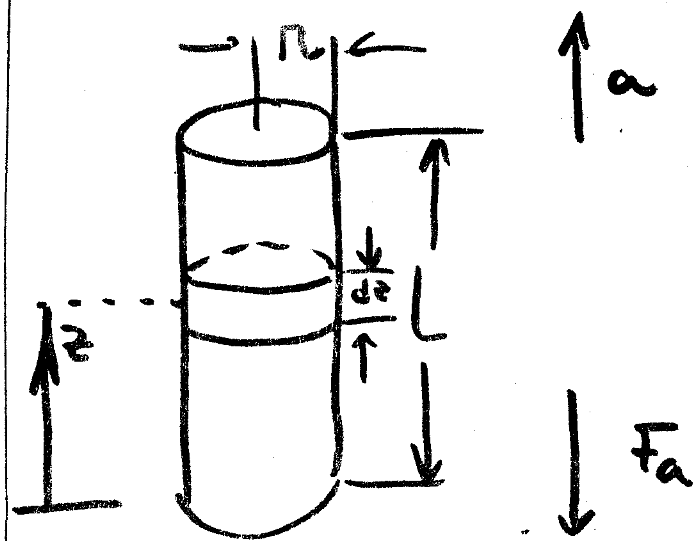
d) $$\tau = \frac{L_1^2}{m_1 r_1^3} = \frac{L_2^2}{m_2(\ell - r_1)^3} = \frac{L_1^2}{m_1 \ell^3}\left(\frac{1+A}{A}\right)^3$$

e) $$\vec{F} = \vec{F}_1 + \vec{F}_2 = 2\tau \sin\left(\frac{\theta_1 - \theta_2}{2}\right)$$

3. A closed cylinder of length, L, and radius, r, is filled with one mole of an ideal gas of molecular weight, m. The cylinder is translated along its axis in empty space with constant acceleration, $\vec{a}$. Assuming a steady state at temperature, T, and no convection, find the pressure profile inside the cylinder. Express the result in terms of the given parameters and universal constants.



ANSWER:



a ⟺ gravity force $F_a = \text{MASS} \cdot a$

MECHANICAL EQUILIBRIUM:

$$\pi R^2 \left[P(z - \tfrac{1}{2}dz) - P(z + \tfrac{1}{2}dz) \right] =$$

$$= \rho(z)\, \pi R^2\, dz \cdot a$$

$$\Rightarrow \boxed{-\frac{dP(z)}{dz} = a\,\rho(z)}$$

THERMODYNAMIC EQUILIBRIUM:

$$P(z)\, \pi R^2 dz = n(z) RT$$

$$n(z) = \rho(z) \pi R^2 dz / N_A \cdot m$$

$$\therefore \boxed{P(z) = \frac{\rho(z)}{N_A \cdot m} RT}$$

ELIMINATING $\rho(z)$

$$\Rightarrow P(z) = P(0)\, e^{-\frac{a N_A m}{RT} z} = P(0)\, e^{-\frac{a m}{k_0 T} z}$$

P1 3

$$\pi R^2 \int_0^L \rho(z)\,dz = m N_A$$

$$\pi R^2 \int_0^L P(0)\,\frac{N_A m}{RT}\, e^{-\frac{a m}{k_B T} z}\, dz = m N_A$$

$$\pi R^2 \frac{P(0)}{RT}\left(+\frac{k_B T}{a m}\right)\left[1 - e^{-\frac{a m}{k_B T} L}\right] = 1$$

3 $$\boxed{P(0) = \cancel{RT} N_A a m / \left[\pi R^2 \cdot \left(1 - e^{-\frac{a m}{k_B T} L}\right)\right]}$$

4. a) An alpha particle of kinetic energy, T_α, makes a head-on collision with a nucleus of atomic number, Z, and mass number, A. Calculate the distance of closest approach, taking into account the recoil of the nucleus.

b) A proton with energy 0.2 MeV makes a head-on collision with an alpha particle at rest. What is the distance of closest approach (in Fermi)?

c) If an alpha particle makes a head-on collision with a proton at rest in the lab, what must the kinetic energy of the alpha particle be so that the distance of closest approach is identical to that in case (b), above?

Solution:

a) KE of $\alpha + (Z,A)$ in CM $= T_{LAB} - T$ of CM $= T_\alpha - \frac{1}{2}(M_\alpha + M_A) v_0^2$

v_0 of CM $= M_\alpha v_\alpha / (M_\alpha + M_A)$

$$T \text{ in CM} = T_\alpha - \frac{1}{2}(M_\alpha + M_A)(M_\alpha v_\alpha)^2/(M_\alpha + M_A)^2$$

$$= T_\alpha - \frac{1}{2} M_\alpha v_\alpha^2 \frac{M_\alpha}{M_\alpha + M_A} = T_\alpha - T_\alpha \frac{M_\alpha}{M_\alpha + M_A} = \frac{T_\alpha}{M_\alpha + M_A}(M_\alpha + M_A - M_\alpha)$$

$$= \frac{M_A}{M_\alpha + M_A} T_\alpha$$

If in LAB, D is closest approach $T_\alpha = \frac{2Ze^2}{D}$, but taking into account recoil of nucleus, only $\frac{M_A}{M_\alpha + M_A} T_\alpha$ is available so

$2Ze^2/D = \frac{A}{A+4} T_\alpha$ (using mass numbers A + 4 as approximate masses in atomic mass units is adequate in most cases).

$$\boxed{\text{Then } D = \frac{2Ze^2}{T_\alpha} \frac{A+4}{A}}$$

b). Proton in lab $T_p = \frac{1 Ze^2}{D}$ but taking into account recoil

$T_{CM} = \frac{M_A}{M_A + M_p} T_p = \frac{4}{4+1} T_p = \frac{4}{5} T_p$ available for $p (M_p \simeq 1)$ on α $(M_A \simeq 4)$

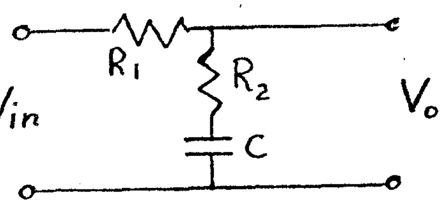
$D = \frac{Ze^2}{T_p} \frac{5}{4} = \frac{2Ze^2}{T_p} 5/4$, $e = 4.8 \times 10^{-10}$ esu, $T_p = 0.2$ MeV, $eV = 1.6 \times 10^{-12}$ erg

$$D = \frac{2(4.8 \times 10^{-10})}{0.2(1.6 \times 10^{-6})} \frac{5}{4} = 1.8 \times 10^{-12} \text{ cm} = 18 \times 10^{-13} \text{ cm} = \boxed{18 F}$$

c) Incident α in LAB, $T_\alpha = 2e^2/D$ but $T_{CM} = \frac{1}{4+1} T_{LAB} = \frac{1}{5} T_\alpha$

$D = \frac{2e^2}{T_\alpha} 5$ must be same as $\frac{2e^2}{T_p}\frac{5}{4} = \frac{2e^2}{T_\alpha} 5 \rightarrow T_\alpha = 4T_p = 4(0.2) = \boxed{0.8 \text{ MeV}}$

5. a. If $H(s)=V_o(s)/V_{in}(s)$ where $V_o(s)$ is the output voltage and $V_{in}(s)$ is the input votlage, $V_{in}(s) = Ae^{st}$ and $s= \sigma + j\omega$, find the H(s) for the circuit shown below ($j = \sqrt{-1}$).



b) Find the poles and zeros of H(s).

c) Make rough sketches of the frequency response of $H(j\omega)$ in the form: (1) log $|H(j\omega)|$ vs. log ω and (2) phase $H(j\omega)$ vs. log ω. Label any constant values and asymptotic slopes.

d) It is possible for a closely related circuit to act as an ideal integrator. What H(s) would be expected in that case?

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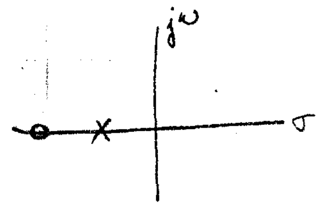
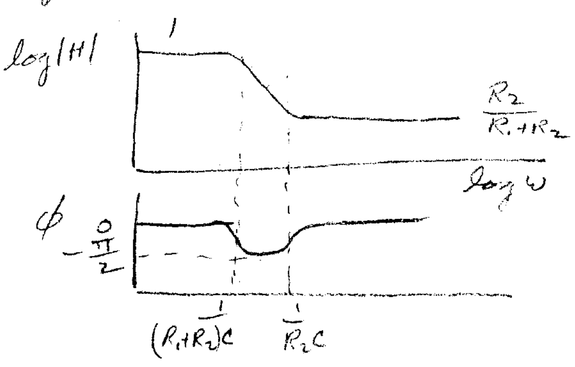
a) $H(s) = \dfrac{(R_2 + \frac{1}{sC})}{(R_1)+(R_2 + \frac{1}{sC})} = \dfrac{1 + R_2Cs}{1 + (R_1+R_2)Cs}$

b) zero: $s = -\dfrac{1}{R_2C}$ pole: $-\dfrac{1}{(R_1+R_2)C}$



c)



d) $\int e^{st} = \frac{1}{s}e^{st}$

so $H(s) = \text{const} \cdot \frac{1}{s}$

6. An electron in the Coulomb field of a proton is initially (t=0) in a state described by the wave function.

Correction given at start of exam.

$$\psi(\vec{r},0) = 1/6\ [4\ \psi_{100} + 3\,\psi_{211} - \psi_{210} + \sqrt{10}\ \psi_{21\bar{1}}]$$

ψ_{nlm} is normalized to 1

with ψ_{nlm}, the energy eigenfunctions, satisfying

$$\left(\frac{\vec{P}^{\,2}}{2m} - \frac{e^2}{r}\right)\psi_{nlm} = E_n\psi_{nlm} \text{ where}$$

$$E_n = -\frac{R_\infty}{n^2} \quad \text{and } \bar{m} = -m$$

1) Is ~~ψ_{nlm}~~ $\psi(\vec{r},0)$ normalized? Give evidence for your answer. $\langle\psi|\psi\rangle = \frac{1}{36}[16+9+1+10] = 1$

2) What is the probability of measuring an eigenvalue of L^2 to be $2\hbar^2$. $l=1$

$\frac{1}{36}[9+1+10] = \frac{5}{9}$

3) What is the probability of measuring an eigenvalue of L_z to be $-\hbar$?

$m=-1$ $\quad \frac{1}{36}[10] = \frac{5}{18}$

4) What is the expectation value of the energy? $\langle H\rangle = \frac{-1}{36}\left[16 + \frac{9+1+10}{4}\right]R_\infty = -\frac{7}{12}R_\infty$

5) What is the expectation value of L_z? $\langle L_z\rangle = \frac{1}{36}[9-10]\hbar = -\frac{1}{36}\hbar$

6) What is the probability of measuring an energy of $-(1/4)R_\infty$? $\frac{9+1+10}{36} = \frac{5}{9}$

7) What is the wave function at time t?

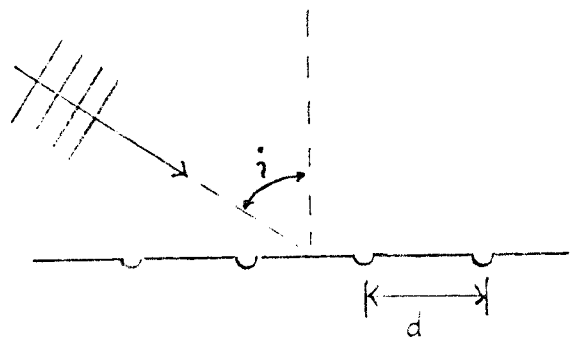
$\psi(\vec{r},t) = \frac{1}{6}\left[4\psi_{100}e^{\frac{-iE_1t}{\hbar}} + (3\psi_{211} - \psi_{210} + \sqrt{10}\,\psi_{21\bar{1}})e^{\frac{-iE_2t}{\hbar}}\right]$

8) Will any of the answers to parts 1 through 6 be different at time t? If so, which ones? Give your reasoning.

L_z, H ... constants of motion

Note corrections

7. a) Derive the condition for constructive interference when a plane wave is incident on a reflective plane grating at an angle of incidence i as shown. The groove spacing is d.



b) As a lecture demonstration, Arthur Schawlow has used an engraved metal ruler, or ''machinist's scale'', at an angle near grazing incidence to measure the wavelength, λ, for a He–Ne laser (λ = 632.8 nm). Explain with a sketch how this would be done and describe in detail the required observations and analysis.

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Measuring the Wavelength of Light with a Ruler

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(Received 29 June 1965)

A simple lecture demonstration is described, in which light from a gas-discharge laser is diffracted at grazing incidence by the rulings of a steel scale. The wavelength of light is obtained by measuring the pattern spacings and the distance from the ruler to the screen.

IT is well known that a reflecting grating with widely spaced grooves gives good diffraction spectra if the light is incident nearly parallel to the surface of the grating. Thus, several orders of reflection spectra may be seen by sighting along a good ruled scale at a small, distant light. It is even possible to estimate the spacing of the diffraction orders by viewing them against a background with scale markings, and so to measure the wavelength.[1]

However, with a small helium–neon gas-discharge laser, a good Fraunhofer diffraction pattern from a ruler can be projected for viewing by a large class. The arrangement is shown in Fig. 1. A good machinists' steel scale is positioned approximately horizontally in front of the laser, so that the light beam passes nearly parallel to, and just above one of the finer sets of rulings. The laser is then tilted downward so that part of the beam grazes the last few inches of the rulings. Several sharp diffraction orders are then seen on the screen. If some of the beam passes over the end or side of the ruler, the direct beam can also be seen. The spots from the direct beam and the zero-order diffracted (specularly reflected) beam define the angle of incidence, and so no direct measurement of angles is needed. All information required to measure the light wavelength comes from the spacings of the spots on the screen, and the distance from screen to ruler. These can be measured with another ruler, or even with the same one if the spot positions are marked.

Figure 2 shows the angles and distances involved. If i and θ are on opposite sides of the normal, and are both taken as positive quantities, the grating equation may be written

$$n\lambda = d(\sin i - \sin\theta),$$

where n is an integer, λ is the wavelength, and d is the spacing between rulings. It is more convenient to use the complements of these angles, i.e., $\alpha = 90° - i$, $\beta = 90° - \theta$ so that the equation becomes

$$n\lambda = d(\cos\alpha - \cos\beta).$$

In the experiment, the distance to the screen x_0, and the distance between spots on the screen are measured. Since $\alpha = \beta_0$ (the zeroth order is specularly reflected), the intersection of the plane of the grating with the screen lies half way between the spots of the direct beam and

[1] This was pointed out to me some years ago by Professor R. R. Richmond, University of Toronto, who remarked to some students on an appropriate occasion "If you don't behave, I'll make you measure the wavelength of light with a ruler."

the zeroth order diffracted beam. Take this point as the origin for measuring distances along the screen. Then the intersection of the direct beam is at $-y_0$, of the zeroth-, first-, second-, $\cdots$ order diffracted beams are at y_0, y_1, $y_2 \cdots$. For any of these, $\tan\beta = y/x_0$; but β is small, so that $\tan\beta \approx \sin\beta$,

$$\cos\beta = [1-\sin^2\beta]^{\frac{1}{2}} \approx [1-(y/x_0)^2]^{\frac{1}{2}}$$

$$= 1 - \tfrac{1}{2}(y^2/x_0^2),$$

$$\cos\alpha = \cos\beta_0 = 1 - \tfrac{1}{2}(y_0^2/x_0^2),$$

$$\cos\beta_n = 1 - \tfrac{1}{2}(y_n^2/x_0^2),$$

$$n\lambda = d(\cos\alpha - \cos\beta_n)$$

$$= d\left(1 - \frac{1}{2}\frac{y_0^2}{x_0^2} - 1 + \frac{1}{2}\frac{y_n^2}{x_0^2}\right) = \tfrac{1}{2}d\left(\frac{y_n^2 - y_0^2}{x_0^2}\right).$$

Thus, $(y_n^2 - y_0^2)$ is linearly proportional to n.

Since the spot spacings increase as d decreases, a widely spaced pattern is obtained by diffraction from a fine scale, such as $\frac{1}{32}''$ or even $\frac{1}{100}''$. It is worth demonstrating that coarser scales give finer patterns. One good way to show this is to slide the ruler a short distance sideways. Then part of the beam is diffracted from the coarser markings used to set-off every second, fourth, or fifth division of the fine scale. Extra spots appear between those seen originally; they are higher orders from the widely spaced long marks on the scale.

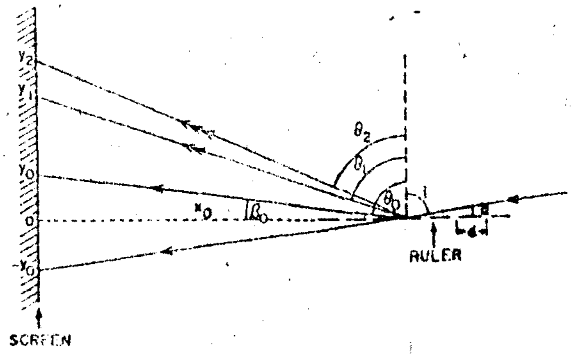


FIG. 2. Angles and distances for calculating wavelength from the diffraction pattern.

It is also possible to demonstrate that the pattern is not clearly developed if the screen is too near the ruler, thus showing that the phenomenon is a case of Fraunhofer diffraction.

In a lecture-demonstration experiment, the distances x_0 and y_n were measured very hastily. Nevertheless, the differences of $y_{n+1}^2 - y_n^2$ all were constant to within two percent of their average value. With a little care, the wavelength of light could be measured to an accuracy about one percent, in a lecture experiment taking only a few minutes, and with all measurements clearly visible to the students. If desired, the theory and calculations can be given as an exercise.

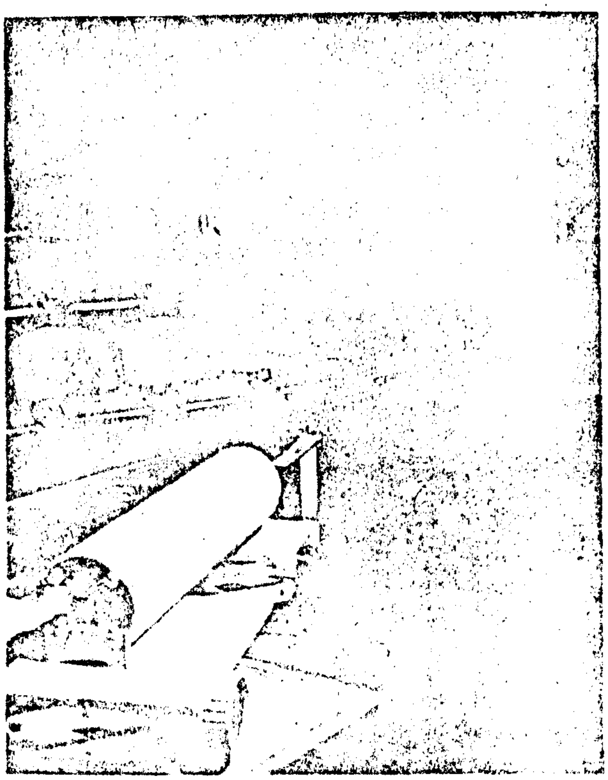
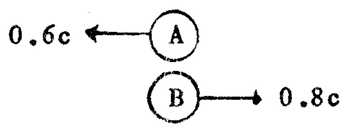


FIG. 1. Arrangement of the apparatus. The laser in the foreground shines on the rulings of a ruler, producing the diffraction pattern on the far wall.

ACKNOWLEDGMENTS

I wish to thank K. Sherwin for constructing the laser, and L. Snyder and R. Rosewood for assistance in setting up the demonstration and in taking the photograph. Optics Technology, Inc. kindly made available a prototype discharge tube of the kind to be used in their model 170 gas laser.

8. Two clocks pass, one going 0.6c to the left, and the other at speed 0.8c to the right. They both read zero when they pass. When clock A reads t = 1 sec, it sends a light pulse toward B. The pulse has frequency f_0 as measured by clock A when the pulse is sent.



a) What time does clock B read when the light pulse reches clock B?

b) What frequency does the light pulse have as measured by clock B?

Now clock B reflects the pulse back to clock A.

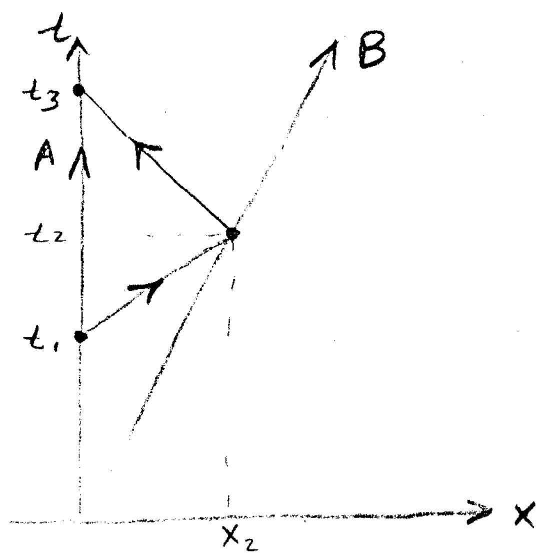
c) What time does clock A read when the reflected pulse reaches clock A?

d) What frequency does clock A measure for the reflected light pulse?

8)

View the system in the rest frame of A. The velocity of B is the relative velocity

$$v = \frac{v_1 + v_2}{1 + v_1 v_2/c^2} = \frac{3/5 + 4/5}{1 + 12/25} = \frac{35}{37}c$$



$t_1 = 1$ sec.

To find the location of (x_2, t_2):

$$\begin{cases} x_2 = v t_2 \\ x_2 = c(t_2 - 1) \end{cases}$$

$$\Rightarrow \quad t_2 = \frac{1}{1 - v/c} \qquad x_2 = \frac{v}{1 - v/c}$$

In the rest frame of B, the proper time is less than t_2 by a factor $1/\gamma$

$$t_2' = t_2\sqrt{1 - \frac{v^2}{c^2}} = \boxed{\sqrt{\frac{1 - v/c}{1 + v/c}} = \frac{1}{6}\text{ sec}} \quad (a)$$

$$t_3 - t_2 = t_2 - t_1 \Rightarrow t_3 = t_1 + 2(t_2 - t_1)$$

$$t_3 = \boxed{\frac{c+v}{c-v} = 36 \text{ sec}} \qquad (c)$$

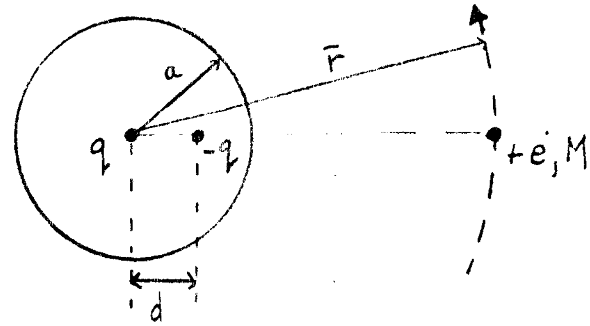
B sees the light redshifted by:

$$f_1 = f_0 \sqrt{\frac{c-v}{c+v}} = \boxed{\frac{f_0}{6}} \qquad (b)$$

The returning pulse is again redshifted by the same factor:

$$f_2 = f_0 \left(\frac{c-v}{c+v}\right) = \boxed{\frac{f_0}{36}} \qquad (d)$$

9. In the vacuum of space, a proton of charge, $+e$, and mass M orbits at radius $\bar{r}$ about an uncharged massive spherical metal shell of radius a, under the influence of the image charges q and $-q$ located respectively at the center of the shell, and at a distance d away from the shell center, as shown.



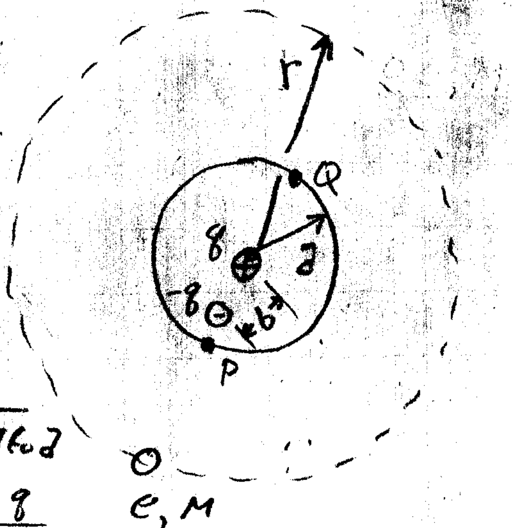
a) Find q and d.

b) Given q and d, find the angular frequency ω of the orbital motion of e around the shell.

c) Will non-circular orbits obey Kepler's laws? Explain briefly.

7H5 Solution B

(a) To make surface of sphere an equipotential, it is enough to have points P and Q at the same potential:



$$V_P = \frac{e}{4\pi\epsilon_0(r-a)} - \frac{q}{4\pi\epsilon_0(a-b)} + \frac{q}{4\pi\epsilon_0 a}$$

$$= V_Q = \frac{e}{4\pi\epsilon_0(r+a)} - \frac{q}{4\pi\epsilon_0(a+b)} + \frac{q}{4\pi\epsilon_0 a}$$

Can set $V_P = V_Q = \frac{q}{4\pi\epsilon_0 a}$, so then $q = \frac{(a-b)}{(r-a)} e = \frac{a+b}{r+a} e$

or $(a-b)(r+a) = (a+b)(r-a)$, or $b(r-a+r+a) = a(r+a-r+a)$,
or $\boxed{b = a^2/r}$

$$q = \left(\frac{a-b}{r-a}\right)e = \left(\frac{a - \frac{a^2}{r}}{r-a}\right)e = \frac{a(r-a)}{r(r-a)}e = \boxed{\frac{a}{r}e = q}$$

(b) $F = \frac{Mv^2}{r} = Mr\omega^2 = \frac{qe}{4\pi\epsilon_0}\left[\frac{1}{(r-b)^2} - \frac{1}{r^2}\right]$

$$= \frac{e^2 a}{4\pi\epsilon_0 r}\left[\frac{r^2-(r^2-2br+b^2)}{r^2(r-b)^2}\right]$$

? RJS

$$= \frac{e^2 a}{4\pi\epsilon_0 r}\left[\frac{2a^2 + a^4/r^2}{r^2\left(\frac{r^2-a^2}{r}\right)^2}\right] = \frac{e^2 a\,(2a^2 + a^4/r^2)}{4\pi\epsilon_0 r\,(r^2-a^2)^2}$$

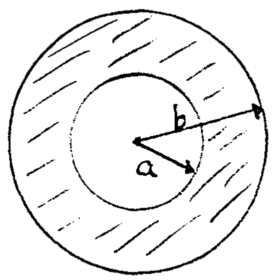
$$\boxed{\omega = \frac{e}{r(r^2-a^2)}\sqrt{\frac{2a^3+a^5/r^2}{4\pi\epsilon_0 M}}}$$

(c) Not inverse-square-law force, so won't obey Kepler's Laws.

10. Calculate the capacitance of a <u>spherical</u> capacitor of inner radius a, and outer radius b, which is filled with a dielectric varying as

$$\varepsilon=\varepsilon_0 + \varepsilon_1 \cos^2\theta$$

where θ is the polar angle.



Sol'n.

$$C = \frac{q}{V}$$

Need $V = -\int_{R_2}^{R_1} \bar{E} \cdot d\bar{l}$

To get E_r:

$$\oint \bar{D} \cdot d\bar{S} = q$$

$$\oint E(\varepsilon_0 + \varepsilon_1 \cos^2\theta) \sin\theta \, d\theta \, d\phi \, r^2 = q \qquad dS = r^2 \sin\theta \, d\theta \, d\phi$$

$$2\pi r^2 E \int_0^{\pi} (\varepsilon_0 + \varepsilon_1 \cos^2\theta) \sin\theta \, d\theta = q$$

$$2\pi r^2 E \left(2\varepsilon_0 \mp \varepsilon_1 \int_1^{-1} x^2 dx \right) = q \qquad \begin{array}{l} x = \cos\theta \\ dx = -\sin\theta \, d\theta \end{array}$$

$$2\pi r^2 E \left(2\varepsilon_0 + \varepsilon_1 \frac{2}{3} \right) = q$$

$$E = \frac{q}{4\pi(\varepsilon_0 + \varepsilon_1/3) r^2}$$

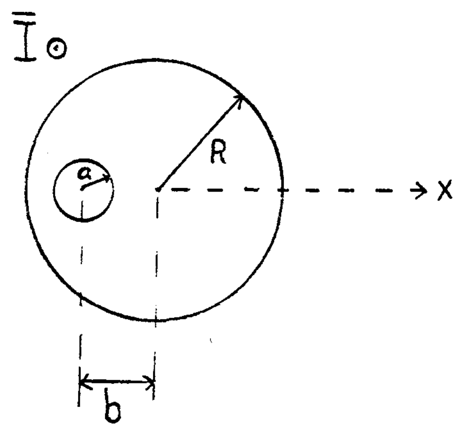
Then

$$V = -\int_{R_2}^{R_1} \frac{q}{4\pi(\varepsilon_0 + \varepsilon_1/3)} \frac{dr}{r^2} = \frac{q}{4\pi(\varepsilon_0 + \varepsilon_1/3)} \left(\frac{1}{R_1} - \frac{1}{R_2} \right)$$

and

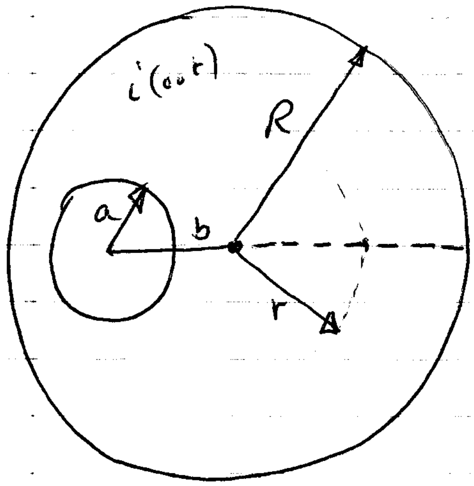
$$C = 4\pi\left(\varepsilon_0 + \frac{\varepsilon_1}{3}\right) \frac{R_1 R_2}{R_2 - R_1}$$

11. A long fat wire of radius R has a cylindrical hole of radius a at a distance b from the center, as shown. The wire carries a total current, $\vec{I}$, directed out of the paper. Calculate the $\vec{B}$-field at all points along the positive x-axis (dashed line). Make a rough plot of $\vec{B}$ vs. x.



Sol'n.

Total current out is i, $\therefore$ current per unit area is $\frac{i}{\pi(R^2-a^2)}$.



Consider the B-field to be the superposition of that from a single fat wire (R) carrying total current $\frac{i}{\pi(R^2-a^2)}\pi R^2$ out

and a smaller wire carrying current $\frac{i}{\pi(R^2-a^2)}\pi a^2$ in.

B (r<R): fat wire

$$\oint \vec{B}\cdot d\vec{l} = \mu_0 i''$$

$$B\,2\pi r = \mu_0 i \frac{R^2}{R^2-a^2}\frac{\pi r^2}{\pi R^2} = \mu_0 i \frac{r^2}{R^2-a^2}$$

$$B = \frac{\mu_0 i}{2\pi}\frac{r}{R^2-a^2} \quad (\text{~~downward~~ upward ie, } +B_y)$$

B (r+b): small wire

$$\oint \vec{B}\cdot d\vec{l} = \mu_0 i''$$

$$B\,2\pi(r+b) = \mu_0 i \frac{a^2}{R^2-a^2}$$

$$B = \frac{\mu_0 i}{2\pi}\frac{1}{R^2-a^2}\frac{a^2}{r+b} \quad (\text{~~upward~~ downward ie, } -B_y)$$

valid for $b>a$

B (total): (r<R)

$$B_x = 0$$

$$B_y = \frac{\mu_0 i}{2\pi}\frac{1}{R^2-a^2}\left(-\frac{a^2}{r+b} + r\right)$$

Valid ~~for~~ $r \leq R$

B tot (r>R): total wire $\quad B\,2\pi r = \mu_0 i \frac{R^2}{R^2-a^2}$

$$B = \frac{\mu_0 i}{2\pi}\frac{1}{R^2-a^2}\frac{R^2}{r}$$

B (total) $(r > R)$: $B_x = 0$

$$B_y = \frac{\mu_0 I}{2\pi} \frac{1}{R^2 - a^2}\left(-\frac{a^2}{r+b} + \frac{R^2}{r}\right)$$

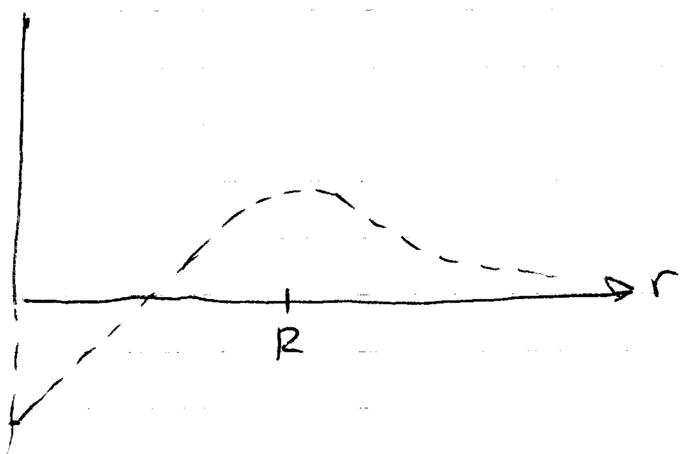
$\underline{r > R}$

At $r = 0$ $\quad B_y = \text{const}\left(-\frac{a^2}{b}\right)$

$r = R$ $\quad B_y = \text{const}\left(R - \frac{a^2}{R+b}\right)$

which is > 0 if $R^2 + Rb > a^2$.

B.



12. Determine the transmission and reflection coefficients for the one-dimensional scattering of a particle of mass m from the potential

$$V(x) = g\ \delta(x), \qquad g = \text{real constant}$$

[Hint: Note that there is a discontinuity in the slope of the wave function at $x = 0$.]

Ph. D. Qual. Spr. 1984 12 3/84

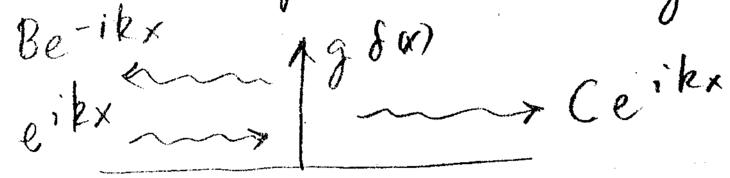
1. Q. Mech. J. Hermanson

Determine the transmission and reflection coefficients for 1D scattering of a particle of mass m from the potential

$$V(x) = g\,\delta(x) \quad ; \quad g = \text{real constant.}$$

[Hint: note the slope discontinuity of the WF at x = 0]

Sol'n:



At $x=0$,

a) match value: $1+B=C$ (3) (1)

b) match slope but include discontinuity $\frac{2mg}{\hbar^2}\psi(0)$ —

$$ik(1-B) = ikC - \frac{2mg}{\hbar^2}C \qquad (2) \quad (3)$$

$$= (ik-\alpha)C \qquad , \ \alpha \equiv 2mg/\hbar^2$$

$$= (ik-\alpha)(1+B) \qquad \text{From (1)}$$

$$(2ik-\alpha)B = ik - ik + \alpha$$

OK KW

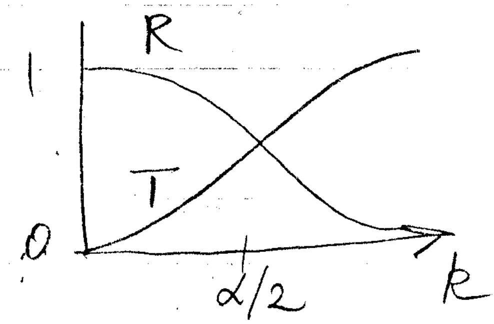
OK RC, perhaps needed hint

$$\begin{cases} B = \dfrac{\alpha}{2ik-\alpha} \\ C = 1+B = \dfrac{2ik}{2ik-\alpha} \end{cases}$$

coefficients

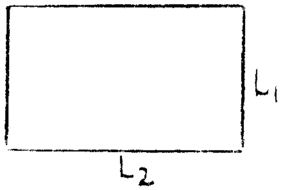
$$R = |B|^2 = \frac{\alpha^2}{4k^2+\alpha^2}$$

$$T = |C|^2 = \frac{4k^2}{4k^2+\alpha^2}$$

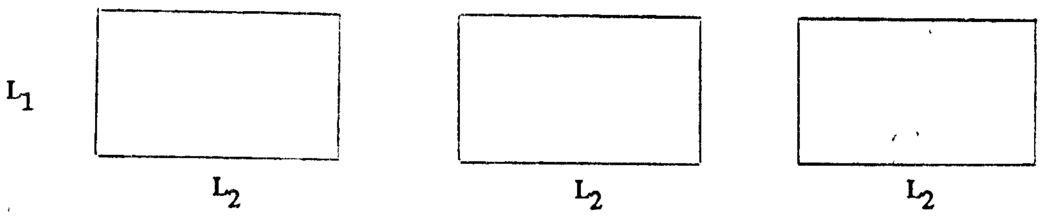


(4)

13. a. A box of dimensions $L_1 \times L_2$ has a particle of mass M in it. Find the energy for the ground state, the 1st excited state, and the 2nd excited state, if $L_2 = \sqrt{5/3}\, L_1$.



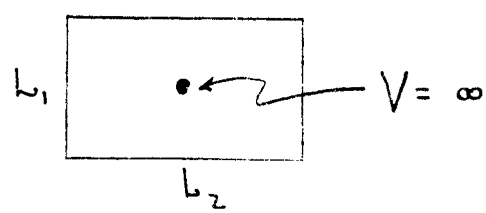
b. Describe the spatial variation of the particle's wave functions over the rectangle by drawing in the nodes in the rectangle for each case.



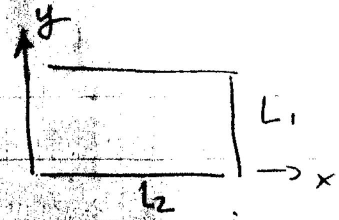
Ground state | 1st excited state | 2nd excited state

c. The 3rd excited state is degenerate for the case $L_2 = \sqrt{5/3}\, L_1$. Find the energy of the 3rd excited state and describe the wave functions over the rectangle for each of the degenerate wavefunctions.

d. If a small repulsive circle were added in the exact center of the rectangle, indicate which degenerate state would be perturbed the most.



13 Box of dimensions $L_1 \times L_2$



Periodic Boundary condition with zero at wall means

$$\psi(x,y) = A \sin\left(\frac{n\pi x}{L_2}\right) \sin\left(\frac{m\pi y}{L_1}\right)$$

Free particle energy is $-\frac{\hbar^2}{2M}\nabla^2\psi = E_{n,m}\psi$

$$E_{n,m} = \hbar^2/2m \left(\left(\frac{n\pi}{L_2}\right)^2 + \left(\frac{m\pi}{L_1}\right)^2 \right)$$

$$= \frac{\hbar^2\pi^2}{2m} \left(\left(n/L_2\right)^2 + \left(m/L_1\right)^2 \right)$$

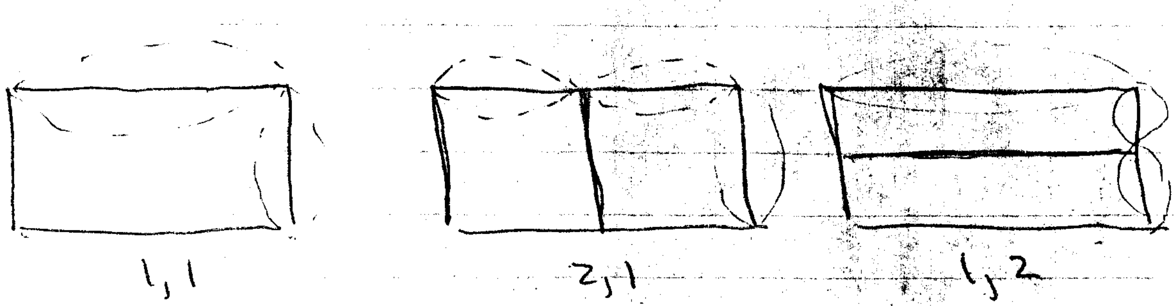
Let $L_2 = \sqrt{5/3}\, L_1 \Rightarrow E_{n,m} = (\text{const})\left(\frac{3}{5}n^2 + m^2\right)$

(a)
$E_{1,1} \sim 3/5 + 1 = 1.6$ ⇐ Ground State
$E_{1,2} \sim 3/5 + 4 = 4.6$ ⇐ 2nd Excited State
$E_{2,1} = 3/5 \cdot 4 + 1 = 3.4$ ⇐ 1st Excited State

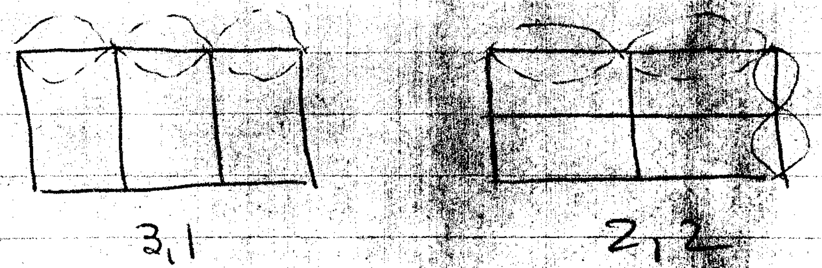
$E_{1,3} = 3/5 + 9 = 9.6$

(c)
$E_{3,1} = 3/5 \cdot 9 + 1 = 6.4$ ⇐ Degenerate 3rd Excited State
$E_{2,2} = 3/5 \cdot 4 + 4 = 6.4$ ⇐

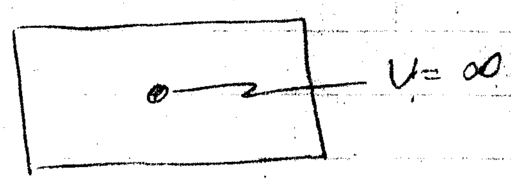
(b)



(c) cont'd

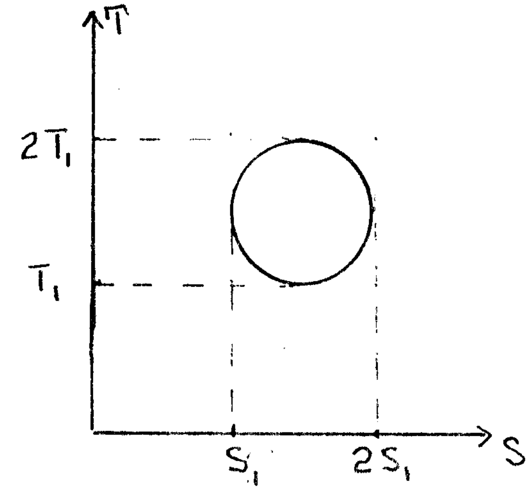


(d) repulsive barrier at center



at a node for 2,2 but at an anti node for 3,1 so 3,1 is disturbed most.

14. Consider a cyclic thermodynamic process in which the working substance traces out the circular path in the T-S (temperature-entropy) plane, shown at right.



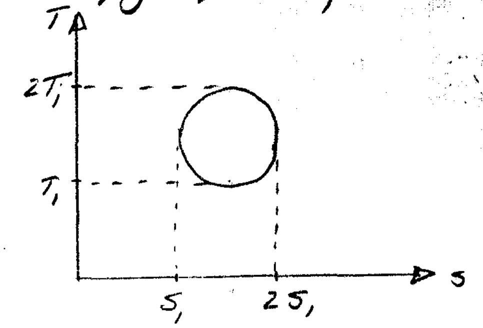
a) What direction does the system follow around the path if it operates as an <u>engine</u>? State your reasoning.

b) What is the net work done in one cycle of the engine in terms of T_1 and S_1?

c) Calculate the thermodynamic efficiency of this engine.

d) Compare your result for part(c) with the thermodynamic efficiency of a Carnot engine operating between heat reservoirs at $2T_1$ and T_1.

Consider a cyclic thermodynamic process in which the working substance traces out the circular path in the T-S (temperature - entropy) plane, shown below



OK KW
OK RC

a) What direction does the system follow around the path if it operates as an engine? (State your reasoning.)

b) What is the net work done in one cycle of the engine, in terms of T_1 and S_1

c) Calculate the thermodynamic efficiency of this engine.

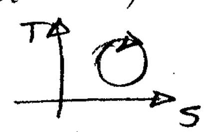
d) Compare your result for c) with the thermodynamic efficiency of a Carnot engine operating between heat reservoirs at $2T_1$ and T_1.

Answers:

a) An engine does positive net work W, where

$W = \oint p\,dV = \oint (T\,dS - dU) = \oint T\,dS$ since U (int'l energy) is a state function

For $\oint T\,dS$ to be positive, the direction of travel must be clockwise



b) $W = \oint T\,dS$ = area of circle. Bearing in mind the fact that the horiz & vertical axes have different dimensions,

we find
$$W = \pi \, \frac{S_1}{2} \cdot \frac{T_1}{2} = \frac{\pi S_1 T_1}{4}$$

c) efficiency $\eta = \dfrac{W}{Q_{in}}$

Now Q_{in} is just the integral $\int T\,dS$ taken over the part of the path for which S is increasing, so

$$Q_{in} = \frac{3T_1}{2} S_1 + \frac{1}{2}\left(\frac{\pi S_1 T_1}{4}\right) = T_1 S_1 \left(\frac{3}{2} + \frac{\pi}{8}\right)$$

$$\eta = \frac{W}{Q_{in}} = \frac{\pi/4 \; S_1 T_1}{S_1 T_1 \left(\frac{3}{2} + \frac{\pi}{8}\right)} = \frac{\pi}{6 + \frac{\pi}{2}} = 0.415$$

d) Carnot effic. $\eta_c = \dfrac{T_{hot} - T_{cold}}{T_{hot}} = \dfrac{2T_1 - T_1}{2T_1} = \dfrac{1}{2}$

$$\frac{\eta}{\eta_{car}} = \underline{0.83}$$

15. Sum (evaluate) the infinite series below:

$$S = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \frac{1}{4\cdot 5} + \cdots$$

MSU M.S. Comp. -- Nov. 83 #15 3/84 OK Ken ✓ OK - RC 15 Nov. 83

(R. Robiscoe) "

Arithmetic Problem

" Sum the infinite series

$$S = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \frac{1}{4\cdot 5} + \cdots "$$

Solution

Can write: $S = \sum_{n=1}^{\infty} 1/n(n+1)$.

Consider: $S(x) = \sum_{n=1}^{\infty} x^n/n(n+1)$. Want $S(1) = S$.

Multiply thru by x and differentiate...

$$\frac{d}{dx}[x\,S(x)] = \frac{d}{dx}\sum_{n=1}^{\infty}\frac{x^{n+1}}{n(n+1)} = \sum_{n=1}^{\infty}\frac{x^n}{n} = -\ln(1-x)$$

so// $S(x) = -\frac{1}{x}\int \ln(1-x)\,dx$ + cnst

The cnst = 0, since $x\,S(x)$ vanishes as $x \to 0$. Then...

$$S(x) = -\frac{1}{x}\int_0^x \ln(1-\xi)\,d\xi \Rightarrow S(1) = -\int_0^1 \ln(1-\xi)\,d\xi$$

or// $S = S(1) = -\int_0^1 \ln u\,du = (u - u\ln u)\Big|_{u=0}^{u=1} = 1$

↑ tabulated

so// $\boxed{\sum_{n=1}^{\infty}\frac{1}{n(n+1)} = 1}$

16. Consider a gas of N non-interacting, spin-1/2 particles (Fermi gas) in a cube with sides of length L.

a) Derive the energy distribution of states (i.e. the number of particles with energy between E and E + dE) for this gas.

b) Determine the energy of the most energetic particle in the gas at temperature T=0 (Fermi Energy) as a function of particle density (N/L^3).

Dick Smith

Exam #1 ✓

KN good

RC ✓ [omit hint?]

12

Consider a gas of N non-interacting spin-1/2 particles (Fermi gas) in a cube of side L, volume $V = L^3$. Derive the energy distribution of states (density of states) for this gas and determine the energy, E_F, of the most energetic particle at temperature $T = 0$ (Fermi energy) as a function of (N/V).

? (Hint: Use periodic boundary conditions on a plane wave representation)

— " —

Free particle : $\psi = C e^{i \bar{k} \cdot \bar{x}}$

PBC $\Rightarrow$ $k_x = \frac{2\pi}{L} n_x$ $n_x =$ integer

$k_y = \frac{2\pi}{L} n_y$ $n_y =$ "

$k_z = 2\pi/L \, n_z$ $n_z =$ "

So volume associated with $\underline{\text{one}}$ state is

K-space volume per state $\frac{1}{(2\pi/L)^3}$ $\Rightarrow$ $g(k) = (L/2\pi)^3 = \frac{V}{(2\pi)^3} \times 2$ ↑ spin

Free particle $\Rightarrow$ $E = \frac{\hbar^2 k^2}{2m}$

Energy distribution of states: $g(\epsilon)\, d\epsilon = g(k) \underbrace{d^3k}_{\text{volume at appropriate } k} = g(k) \underbrace{4\pi k^2 dk}_{\text{volume of shell, radius } k}$

$$= g(k)\, 4\pi k^2 \frac{dk}{d\epsilon} d\epsilon = \frac{V}{(2\pi)^3} 2 \, 4\pi k^2 \left(\frac{m}{\hbar^2 k}\right) d\epsilon$$

$$g(\epsilon) = \frac{V}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)\frac{1}{k} = \boxed{\frac{V}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} \epsilon^{1/2}}$$

#1 cont'd

Fill up the states to E_F at $T=0$

$$N = \text{\# particles} = \int_0^{E_F} g(\epsilon)\,d\epsilon = \frac{V}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} E_F^{3/2} \cdot 2/3$$

$$\boxed{E_F = \left(3\pi^2 N/V\right)^{2/3} \frac{\hbar^2}{2m}}$$