DEPARTMENT OF PHYSICS

2007 COMPREHENSIVE EXAM

Answer each of the following questions. Begin your answer to each question on a new sheet of paper; solutions to different questions must not appear on the same sheet. Each sheet of paper must be labeled with your name and the problem number in the upper right hand corner. When more than one sheet is submitted for a problem, be sure the pages are ordered properly.

PHYSICAL CONSTANTS

Quantity	Symbol	Value
acceleration due to gravity	g	$9.8\ \mathrm{m\,s^{-2}}$
gravitational constant	G	$6.67 \times 10^{-11}\ \mathrm{N\,m^2\,kg^{-2}}$
permittivity of vacuum	ϵ_0	$8.85 \times 10^{-12}\ \mathrm{C^2\,N^{-1}\,m^{-2}}$
permeability of vacuum	μ_0	$4\pi \times 10^{-7}\ \mathrm{N\,A^{-2}}$
speed of light in vacuum	c	$3.00 \times 10^{8}\ \mathrm{m\,s^{-1}}$
elementary charge	e	$1.602 \times 10^{-19}\ \mathrm{C}$
mass of electron	m_e	$9.11 \times 10^{-31}\ \mathrm{kg}$
mass of proton	m_p	$1.673 \times 10^{-27}\ \mathrm{kg}$
Planck constant	h	$6.63 \times 10^{-34}\ \mathrm{J\,s}$
Avogadro constant	N_A	$6.02 \times 10^{23}\ \mathrm{mol^{-1}}$
Boltzmann constant	k	$1.38 \times 10^{-23}\ \mathrm{J\,K^{-1}}$
molar gas constant	R	$8.31\ \mathrm{J\,mol^{-1}\,K^{-1}}$
standard atmospheric pressure		$1.013 \times 10^{5}\ \mathrm{Pa}$

Problem #1

$N+1$ balls, with masses $m_0, m_1, m_2, ...m_N$, move together along a line toward a (perpendicular) wall, at a constant velocity v, with m_0 at the head of the line. Assume that all collisions are perfectly elastic.

A. Find a recursion relation to determine $m_1, m_2, ...m_N$ in terms of m_0 so that m_N will have all of the kinetic energy of the system once all the collisions are complete.

B. Find the final speed of m_N in terms of v.

#1

Solution to Problem #1.

A. The ith ball (except $i = 0$) collides with ball $i - 1$, leaving the latter at rest. Let $-v_i$ denote the velocity of m_i after collision with m_{i-1} (the minus sign indicates that the ball is traveling away from the wall). Since the collisions are elastic, we apply conservation of momentum and energy. In the following equations, the left and right hand sides denote quantities before and after the collision, respectively:

$$m_i v - m_{i-1} v_{i-1} = -m_i v_i,$$
$$m_i v^2 + m_{i-1} v_{i-1}^2 = m_i v_i^2.$$

Combining these equations to eliminate v_i and solving for m_i,

$$m_i = \frac{v_{i-1}}{2v + v_{i-1}} m_{i-1}$$

The recursion relation is not yet useful, because the v_{i-1} are unknown. However, if we substitute this result back into the momentum equation, it turns out that

$$v_i = v + v_{i-1}.$$

The collision of ball 0 with the wall is a special case, giving it velocity $-v_0 = -v$. Thus, we have

$$v_i = (i + 1)v.$$

Putting this into the equation for m_i, we find the recursion relation

$$m_i = \frac{i}{i+2} m_{i-1}.$$

Given m_0, the rest of the masses are specified.

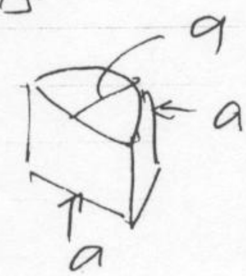
B. The final speed of m_N, based on the above results, is simply $(N + 1)v$.

Problem #2

A half cylinder of metal of length a and radius a is initially at temperature T_0. At time $t = 0$ the metal is immersed in a bath which maintains the surface at temperature $T = 0$. Find an approximate expression for $T(\rho, \theta, z, t)$ for large t.

2007 Comprehensive Exams

$\boxed{\#2}$ Key.



$T = T_0 \quad t < 0$

$T = 0$ at surface at $t \geq 0$

The correct eqn to use is Diffusion Eqn:

$$\nabla^2 T = C \frac{\partial T}{\partial t} \quad ① \qquad C = \text{const.}$$

Use separation of variables

$$T = R(\rho)\Theta(\theta)\, Z(z)\, T(t) \quad ②$$

The normal soln is: in the form of

$$T \propto J_m(K\rho)\sin\left(\frac{2\pi z}{a}\right)e^{-\lambda t} \quad ③$$

To satisfy the boundary condition at a, we must choose Ka to be a root of J_m. Also, the relation between λ, n, & K is

$$K^2 + \left(\frac{n\pi}{a}\right)^2 = \frac{\lambda}{n} \quad ④$$

The general soln will be a linear combination of these with coefficients $C_{m,n,k}$.

To satisfy the initial condn at $T = T_0$, we can integrate over a full cylinder if we use

$$f(\rho, z, \theta) = \begin{cases} +T_0 & \text{for } 0 < \theta < \pi \\ -T_0 & \text{for } \pi < \theta < 2\pi \end{cases} \quad ⑤$$

at $t = 0$,

$$f = \Sigma\, C_{m,n,k} \sin\left(\frac{n\pi z}{a}\right)\sin m\theta\, J_m(K\rho) \quad ⑥$$

$$\& \quad T = f e^{-\lambda t} \quad ⑦$$

multiply ⑥ by $\sin\left(\frac{n'\pi z}{a}\right)\sin m'\theta\, J_m(K'\rho)$
where $K'a$ is also a root of J_m, and integrate over the volume of the cylinder. Use the relation:

$$\int_0^a J_m(K_n x)J_m(K_p x)\,x\,dx = \delta_{mp}\frac{a^2}{2}\left[J_{m+1}(K_p a)\right]^2 \quad ⑧$$

From R.H.S. we get

$$\sum C_{nmk}\delta_{nn'}\left(\frac{a}{2}\right)\delta_{mm'}(\pi)\delta_{kk'}\left(\frac{a^2}{2}\right)\left[J_{m+1}(K'a)\right]^2$$

$$= C_{m'n'k'}\frac{a^3\pi}{4}\left[J_{m+1}(K'a)\right]^2 \quad ⑧$$

From L.H.S. we get

$$\int_0^a \rho J_m(K'\rho)\,d\rho\,(T_0)\left(\frac{a}{n'\pi}\right)\left\{\int_0^\pi \sin m'\theta\,d\theta - \int_\pi^{2\pi}\sin m'\theta\,d\theta\right\} = \frac{aT_0}{n'\pi}\int_0^a \rho J_m(K'\rho)\,d\rho\left\{\frac{2}{m'}(1-\cos m'\pi)\right\}$$

$$\left.\begin{cases} = 0 \text{ for } m' \text{ even} \\ = \dfrac{4aT_0}{m'n'\pi}\displaystyle\int_0^a \rho J_m(K'\rho)\,d\rho \end{cases}\right\} ⑨$$

$$⑧ = ⑨ \Rightarrow \frac{4T_0 a}{m'n'\pi}\int_0^a \rho J_m(K'\rho)\,d\rho = C_{m'n'k'}\frac{a^3\pi}{4}\left[J_{m+1}(K'a)\right]^2 \quad ⑩$$

For large t, the lowest mode will survive. So,

$$⑩ \rightarrow C_{11K} = \frac{16T_0}{\pi^2 a^2}\int_0^a \rho J_1(K\rho)\,d\rho \Big/ \left[J_2(Ka)\right]^2 \quad ⑪$$

So, the leading term of T at large t is: ⑦ & ⑪ ⇒

$$\boxed{T = f e^{-\lambda t} = \frac{16T_0}{\pi^2 a^2}\frac{\int_0^a \rho J_1(K\rho)\,d\rho}{\left[J_2(Ka)\right]^2}\exp\left[-\left(K^2+\frac{\pi^2}{a^2}\right)t\right]} \quad ⑫$$

~~correct answer.~~

note: (i) K is obtained ~~as~~ from Ka = a root of $J_1(Ka)$

(ii) λ is eliminated from ④

$\rightarrow \lambda = K^2 + \left(\frac{\pi}{a}\right)^2$ with $n=1$.

Problem #3

The glow element of a *24 V* incandescent lamp is made of *0.1 mm* diameter tungsten wire. At the nominal temperature, $T=3200\ ^0C$ the lamp consumes *100 W* of electrical power. The electrical resistivity of tungsten is, $\rho = \rho_0 + \alpha(T - T_0)$, where $T_0 = 20\ ^0C$, $\rho_0 = 5.5\ 10^{-8}\ \Omega m$, $\alpha = 2.7\ 10^{-10}\ \Omega m\ K^{-1}$. The emitted black body radiation constitutes 40% of the consumed electrical power, independent of the temperature. Density of tungsten is $\rho_m = 19.3\ g\ cm^{-3}$, and the specific heat of tungsten is $c_p = 1.67\ J\ K^{-1}\ g^{-1}$.

How long does it take for the wire to heat up to $T=3200\ ^0C$, if at the moment the lamp is switched on the temperature is $T=20\ ^0C$?

Solution #3

Electrical power $P = \frac{U^2}{R}$ $\quad R_0 = \frac{(24)^2}{100} = 5.76$

$$R = \rho \cdot \frac{\text{length}}{\text{area}} = \left[\rho_0 + \alpha(T - T_0)\right] \frac{\ell}{S}$$

$$\underbrace{\Delta T \cdot m \cdot c_p}_{\text{heat energy}} = 0.6 \cdot \frac{U^2}{R} \cdot \Delta t$$

$$\Delta T\, \rho_m \cdot \cancel{S} \cdot \ell \cdot c_p \left[\rho_0 + \alpha(T - T_0)\right] \frac{\ell}{\cancel{S}} \frac{1}{0.6 \cdot U^2} = \Delta t$$

$$\frac{\ell}{S} = \frac{5.76\,\Omega}{\rho_0 + \alpha(T - T_0)} = \frac{5.76\,\Omega}{5.5 \cdot 10^{-8}\,\Omega \cdot m + 2.7 \cdot 10^{-10} \cdot 3180\,\Omega \cdot m}$$

$$= \frac{5.76}{9.14} \cdot 10^7 = 0.63 \cdot 10^7\, m^{-1}$$

$$S = \pi r_0^2 = \pi (0.5 \cdot 10^{-4}\, m)^2 = 0.785 \cdot 10^{-8}\, m^2$$

$$\ell = 0.63 \cdot 10^7\, m^{-1} \cdot 0.785 \cdot 10^{-8}\, m^2 = 0.05\, m$$

$$t = \frac{\rho_m c_p \cdot \ell^2}{0.6 \cdot U^2} \int_{T_0}^{T} (\rho_0 - \alpha T_0 + \alpha T)\, dT$$

$$= \frac{\rho_m c_p \ell^2}{0.6 \cdot U^2} \left[(\rho_0 - \alpha T_0)(T - T_0) + \frac{\alpha}{2}(T - T_0)^2\right]$$

$$\frac{19.3\, g \cdot cm^{-3} \cdot 1.67 \cdot J \cdot K^{-1} g^{-1} \cdot (5\, cm)^2}{0.6 \cdot (24\, V)^2} = 2.33 \frac{J}{V^2 \cdot K \cdot cm}$$

$$[\ \ldots\ ] = 0.0013\, \Omega \cdot m \cdot K$$

$$t = 0.003 \frac{J \cdot \Omega}{V^2} \frac{m}{cm} = \underline{\underline{0.3\, s}}$$

Problem #4

A particle of charge $\boldsymbol{q}$ and mass $\boldsymbol{m}$, is moving in a one-dimensional harmonic potential of frequency $\boldsymbol{\omega}$. The particle is subject to a *weak* electric field $\boldsymbol{E}$.

Find the energy eigenvalues of the system
(a) By exact solution of the Schroedinger equation;
(b) By calculate the first nonzero correction using perturbation theory.

Solution #4

Charged harmonic oscillator in electric field

$$\hat{H} = -\frac{\hat{p}_x^2}{2m} + \frac{kx^2}{2} + \underbrace{qEx}_{\text{el. static potential}} \qquad \omega^2 = \frac{k}{m}$$

$$= -\frac{\hat{p}_x^2}{2m} + \frac{\omega^2 m}{2}\underbrace{\left(x^2 + \frac{2qE}{\omega^2 m}x + \left(\frac{qE}{\omega^2 m}\right)^2\right)}_{\left(x + \frac{qE}{\omega^2 m}\right)^2} - \frac{q^2E^2}{2\omega^2 m}$$

$$= -\frac{\hat{p}_x^2}{2m} + \frac{\omega^2 m}{2}\underbrace{\left(x + \frac{qE}{\omega^2 m}\right)}_{=y}^2 - \underbrace{\frac{q^2E^2}{2\omega^2 m}}_{U_E = \text{const}} \qquad y = x + \frac{qE}{\omega^2 m}$$

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial y}\cdot\frac{\partial y}{\partial x} = \frac{\partial}{\partial y}$$

$$= -\frac{\hat{p}_y^2}{2m} + \frac{\omega^2 m}{2}y^2 - U_E$$

Exact solution:

Energy levels: $E_n = \left(n + \frac{1}{2}\right)\hbar\omega - U_E$

Eigenfunctions: $|n\rangle = \Psi_n\left(x - \frac{qE}{\omega^2 m}\right)$

Perturbation solution: $\hat{H}' = qEx$

1^{st} order correction $E_n^{(1)} = \langle n|\hat{H}'|n\rangle = 0$

due to parity

2^{nd} order correction $E_n^{(2)} = \sum_{m\neq n} \frac{|\langle m|\hat{H}'|n\rangle|^2}{E_n^{(0)} - E_m^{(0)}}$

Non-zero matrix elements:

$\langle n\pm1|x|n\rangle = \sqrt{n\pm1}\sqrt{\frac{\hbar}{2m\omega}}$; $\langle n-1|x|n\rangle = \sqrt{n}\sqrt{\frac{\hbar}{2m\omega}}$

#4 cont.

$$E_n^{(2)} = q^2 E^2 \left(\frac{(n+1)\frac{\hbar}{2m\omega}}{-\hbar\omega} + \frac{n\frac{\hbar}{2m\omega}}{+\hbar\omega} \right)$$

$$= -\frac{q^2E^2}{2m\omega^2}$$

Problem #5

A. A uniform external electric field $E_0\hat{z}$ is applied to an uncharged conducting sphere of radius R. What is the resulting electric dipole moment of the sphere?

B. A second, identical sphere is now introduced at a distance $d \gg R$. Show that the field due to the first sphere evaluated at the second sphere is much less than E_0. Thus, the polarization of the two spheres is practically independent.

C. A dielectric solid is approximated by a lattice of conducting spheres, radius R, with number density n (m^{-3}). Assuming $nR^3 \ll 1$, what is the electric susceptibility χ_e of the material?

#5

Solution to Problem #5.

A. The boundary conditions on the potential $\Phi(r, \theta)$ are:

$$\Phi(R, \theta) = 0 \qquad \text{(the sphere is uncharged)}, \tag{1}$$

$$\Phi(r \to \infty, \theta) = -E_0 r \cos\theta \qquad \text{(uniform field at large } r\text{)}. \tag{2}$$

Total potential is a sum of the uniform background and the sphere's dipole:

$$\Phi = -E_0 R \cos\theta + \frac{\hat{r} \cdot \mathbf{p}}{4\pi\epsilon_0 r^2} = \left(-E_0 R + \frac{p}{4\pi\epsilon_0 r^2}\right) \cos\theta.$$

The second boundary condition is automatically satisfied. We now pick the dipole moment $\mathbf{p}$ to ensure that the first boundary condition is satisfied:

$$\mathbf{p} = 4\pi\epsilon_0 R^3 E_0 \hat{z}.$$

B. The field due to the sphere's polarization is

$$\mathbf{E}_{dip} = -\nabla\Phi = -\nabla\left(\frac{E_0 R^3 \cos\theta}{r^2}\right) = \frac{E_0 R^3}{r^3}\left(2\hat{r}\cos\theta + \hat{\theta}\sin\theta\right).$$

At a distance $r = d \gg R$, this will all go as $E_0(R/d)^3 \ll E_0$. The sphere at $r = d$ feels only very slight influence from the dipole at the origin.

C. Since we are given $R^3 n \ll 1$, we know that the spheres are not in contact and that they each experience an approximately uniform field that varies slowly from sphere to sphere. The dipole moment per unit volume (the polarization) is

$$\mathbf{P} = n\mathbf{p} = 4\pi\epsilon_0 R^3 n E_0 \hat{z}.$$

Now, $E_0\hat{z}$ may be viewed as the field inside the "dielectric". Compare the above with the definition of the electric susceptibility χ:

$$\mathbf{P} = \epsilon_0 \chi \mathbf{E} \tag{3}$$

$$\therefore \quad \chi = 4\pi R^3 n. \tag{4}$$

Problem #6

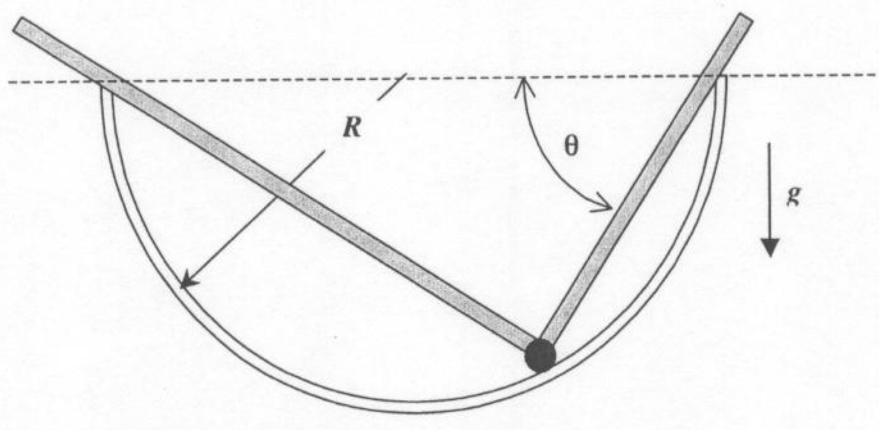
Two uniform bars are hinged together at one end with a massless hinge and are free to slide in contact with a smooth, semicircular bowl of radius R (as shown below). One rod has mass M_1 and length $2l_1$ and the second rod has mass M_2 and length $2l_2$. The bowl is attached to a table and does not move. The system is acted on by gravity. Each rod has two points of contact with the bowl, at the hinge point and at the lip of the bowl (neither rod falls inside the bowl). For the following consider motion only in the plane of the two rods (2 dimensional problem).

a) How many independent degrees of freedom are there in this system?

b) Determine the potential energy of the two bar system.

c) Determine the Lagrangian for the two bar system.

d) Determine the angle θ made by the first bar with respect to the horizontal after the two bar system reaches equilibrium (motion is damped out).



#6

Problem #6 (2007) Solution

For this two dimensional problem, it first seems that there are 2 independent variables (listed as θ_1 and θ_2 in the drawing below) but it turns out that these two are simply related by geometry, giving only one independent variable. Formally, each bar has 3 degrees of freedom, 2 to identify the center-of-mass position and 1 to determine the orientation of each bar, giving a total of 6 degrees of freedom. In addition, there are 5 constraints on this problem, 2 for the contact points for each bar (total 4) and another 1 that two contact points must be the same (for the restriction forced by the hinge). That leaves one independent degree of freedom, θ.

The potential energy is simply $V = -M_1gh_1 - M_2gh_2$, where we have taken the zero of potential energy to be at the top of the bowl. Noting that $h_1 = (s_1 - l_1)\cos\theta_1 = (s_1 - l_1)\cos\theta$ and that $h_2 = (s_2 - l_2)\cos\theta_2 = (s_2 - l_2)\sin\theta$ because $\theta_1 = \theta$ and $\theta_2 = (\pi - \theta_1)$, and that

$$s_1 = 2R\cos\theta \qquad\qquad s_2 = 2R\sin\theta$$

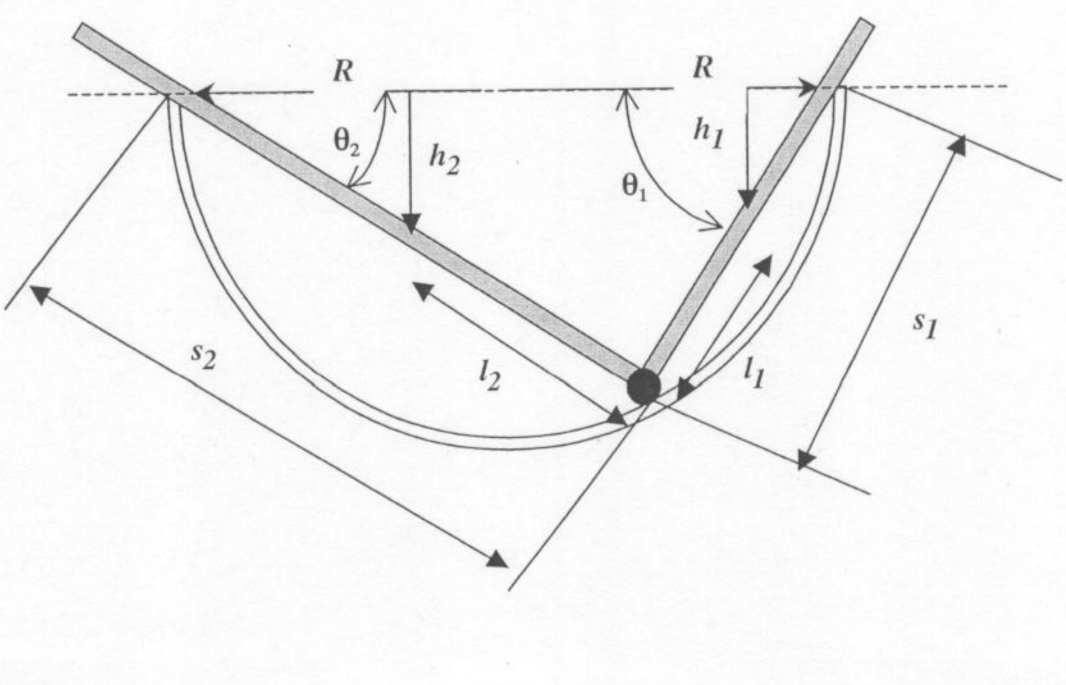
we find $\qquad V = M_1gl_1\sin\theta + M_2gl_2\cos\theta - 2(M_1 + M_2)gR\cos\theta\sin\theta$

The lagrangian is simply $L = T\text{-}V$, where T is the kinetic energy (translation and rotation).

Once the potential is known, the equilibrium position can be determined by setting the derivative of the potential with respect to θ to zero and solving indirectly for θ.

$$\tan\theta = \frac{2(M_1 + M_2)R\cos\theta - M_1l_1}{2(M_1 + M_2)R\sin\theta - M_2l_2}$$

once all the parameters are known, θ can be determined graphically.



Problem #7

An electron neutrino ν_e created in subatomic interactions can later be observed as a muon neutrino ν_μ, and vice versa, in a process known as *neutrino oscillation*. Such oscillations occur **not** as decays, but as a result of a mismatch between the neutrino *flavor eigenstates* (ν_e and ν_μ, with definite lepton numbers) and the *mass eigenstates* ν_1 and ν_2 (with definite masses m_1 and m_2, respectively). The eigenstates in these complete, orthonormal bases[1] are coupled via an arbitrary real unitary matrix:

$$U = \begin{pmatrix} \langle \nu_e | \nu_1 \rangle & \langle \nu_e | \nu_2 \rangle \\ \langle \nu_\mu | \nu_1 \rangle & \langle \nu_\mu | \nu_2 \rangle \end{pmatrix} \equiv \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

where the unknown *mixing angle* θ is real.

[1]We ignore here the tau neutrino ν_τ and any possible *sterile* neutrinos.

(a) Briefly discuss any physical constraints that require U to be unitary.

(b) Calculate the probability as a function of time t that a ν_e at time $t = 0$ with momentum p will transform into a ν_μ, expressing your answer in terms of θ and the energies $E_1 = \sqrt{p^2c^2 + m_1^2c^4}$ and $E_2 = \sqrt{p^2c^2 + m_2^2c^4}$ of the two mass eigenstates.

(c) Consider now the *ultrarelativistic limit*, i.e., $pc \gg mc^2$ for both mass eigenstates. Re-express your answer from part (a) in terms of p, θ, $\Delta m^2 \equiv m_2^2 - m_1^2$, and distance traveled $L \simeq ct$.

(d) Make an accurate plot of probability as a function of p for given values of L, θ, and Δm^2, and describe an experiment or set of experiments that could be used to determine both θ and Δm^2.

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[1] We ignore here the tau neutrino ν_τ and any possible *sterile* neutrinos.

(a) Briefly discuss any physical constraints that require U to be unitary.

Unitary transformations in quantum mechanics are necessary to conserve probabilities – directly related to the norm of the state vectors – when changing from one basis to another. For example, let $|\Psi\rangle = U\,|\psi\rangle$ and calculate the norm of the state vector $|\Psi\rangle$:

$$\langle\Psi|\,\Psi\rangle = \left(\langle\psi|\,U^\dagger\right)\left(U\,|\psi\rangle\right) = \left\langle\psi\left|U^\dagger U\right|\psi\right\rangle = \langle\psi|\,\psi\rangle$$

where we have used the definition of a unitary operator, namely that $U^\dagger U \equiv \mathcal{I}$, the identity operator.

(b) Calculate the probability as a function of time t that a ν_e at time $t = 0$ with momentum p will transform into a ν_μ, expressing your answer in terms of θ and the energies $E_1 = \sqrt{p^2c^2 + m_1^2c^4}$ and $E_2 = \sqrt{p^2c^2 + m_2^2c^4}$ of the two mass eigenstates.

Time evolution is easily performed for energy eigenstates. Since we have a momentum eigenstate (definite p), we can only form energy eigenstates from the mass eigenstates ν_1 and ν_2.

$$\begin{aligned}
|\nu_e\rangle &= \cos\theta\,|\nu_1\rangle + \sin\theta\,|\nu_2\rangle \qquad\qquad |\nu_\mu\rangle = -\sin\theta\,|\nu_1\rangle + \cos\theta\,|\nu_2\rangle \\
|\nu_e\rangle_t &= \cos\theta e^{-iE_1t/\hbar}\,|\nu_1\rangle + \sin\theta e^{-iE_2t/\hbar}\,|\nu_2\rangle \\
\langle\nu_\mu|\,\nu_e\rangle_t &= \left(-\sin\theta\,\langle\nu_1| + \cos\theta\,\langle\nu_2|\right)\left(\cos\theta e^{-iE_1t/\hbar}\,|\nu_1\rangle + \sin\theta e^{-iE_2t/\hbar}\,|\nu_2\rangle\right) \\
&= -\sin\theta\cos\theta e^{-iE_1t/\hbar} + \cos\theta\sin\theta e^{-iE_2t/\hbar} = \frac{1}{2}\sin 2\theta\left(e^{-iE_2t/\hbar} - e^{-iE_1t/\hbar}\right) \\
P_{e\to\mu}(t) &= |\langle\nu_\mu|\,\nu_e\rangle_t|^2 = \frac{\sin^2 2\theta}{4}\left[e^{-iE_2t/\hbar} - e^{-iE_1t/\hbar}\right]\left[e^{+iE_2t/\hbar} - e^{+iE_1t/\hbar}\right] \\
&= \frac{\sin^2 2\theta}{4}\left[2 - e^{-i(E_2-E_1)t/\hbar} - e^{+i(E_2-E_1)t/\hbar}\right] = \frac{\sin^2 2\theta}{2}\left\{1 - \cos\left[(E_2 - E_1)t/\hbar\right]\right\} \\
&= \boxed{\sin^2 2\theta\,\sin^2\left[\frac{(E_2-E_1)t}{2\hbar}\right]}
\end{aligned}$$

(c) Consider now the *ultrarelativistic limit*, i.e., $pc \gg mc^2$ for both mass eigenstates. Re-express your answer from part (b) in terms of p, θ, $\Delta m^2 \equiv m_2^2 - m_1^2$, and distance traveled $L \simeq ct$.

We'll need to do a binomial expansion in $m_ic^2/pc \ll 1$.

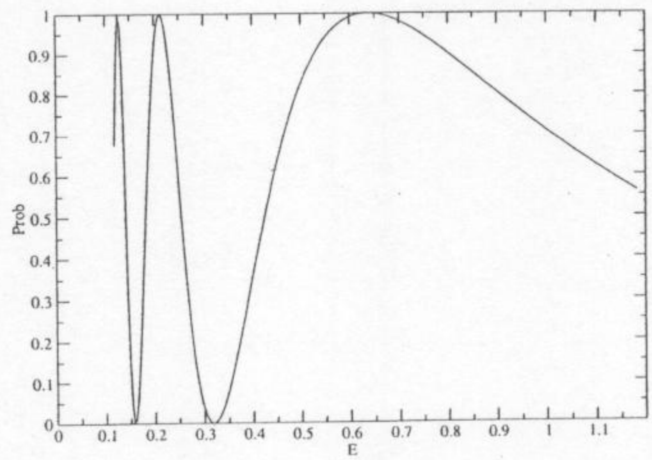
$$E_i = pc\sqrt{1 + \frac{m_i^2c^4}{p^2c^2}} \simeq pc\left(1 + \frac{m_i^2c^4}{2p^2c^2}\right) = pc + \frac{m_i^2c^4}{2pc}$$

#7 cont.

$$E_2 - E_1 = \frac{m_2^2c^4}{2pc} - \frac{m_1^2c^4}{2pc} = \frac{\Delta m^2c^4}{2pc}$$

$$P_{e\to\mu} = \boxed{\sin^2 2\theta\ \sin^2\left[\frac{\Delta m^2 c^4 L}{4\hbar c\, pc}\right]}$$

(d) Make an accurate plot of probability as a function of p for given values of L, θ, and Δm^2, and describe an experiment or set of experiments that could be used to determine both θ and Δm^2.



Here, the units of $E \simeq pc$ are $4\hbar c/\Delta m^2 L$, and the units of probability are $\sin^2 2\theta$.

An obvious experiment based on this plot is to place a ν_μ detector a fixed distance L from the ν_e production source, and measure the number of ν_μs detected as a function of prodution momentum. The absolute height of the oscillation tells us the mixing angle, and the location of maxima and minima tells us Δm^2.

The MiniBOONE experiment at Fermilab, for example, recently published results from precisely this experiment at $L \simeq 0.5$ km (PRL **98***, 231801 (2007)). Their best estimates for the parameters are $\sin^2 2\theta = 10^{-3}$ and $\Delta m^2 = 4$ eV^2. This indicates a tiny oscillation probability, and that the minimum indicated on the plot above near $1/\pi$ on the horizontal axis occurs at 810 MeV.*

Problem #8

In 1960, physicist Freeman Dyson theorized that an advanced civilization might build a rigid sphere enclosing a star, to capture all the energy from the star.

A. Suppose we built a Dyson sphere of radius $R = 200$ million km around our own Sun, which has luminosity $L_\odot = 3.8 \times 10^{26}$ W. Assuming that $L_\odot$ remains constant and that the outside surface of the sphere is a perfect absorber, what is the equilibrium temperature of the outside of the Dyson sphere?[1]

B. In 1970, Larry Niven's novel Ringworld described a rigid ring built around a star. MIT students at the 1971 World Science Fiction Convention chanted, "The Ringworld is unstable!"

   i. Sketch the potential energy of the Dyson sphere of radius R when its center is displaced a distance r from the Sun.

   ii. The potential $U(r)$ for in-plane displacements r of the Ringworld cannot be calculated analytically. To gain some insight, consider the case $r \sim R$, so that the ring can be treated as an infinitely long rod. Based on what you learn from this analogy, sketch $U(r)$ for the Ringworld.

   iii. If the Dyson sphere is displaced slightly ($r/R = \epsilon \ll 1$), and is left at rest, does the perturbation tend to grow? What about for the Ringworld?

   iv. Does the rotation of the Ringworld affect its stability?

[1] Stefan's constant is $\sigma = 2\pi^5 k^4/15h^3c^2$.

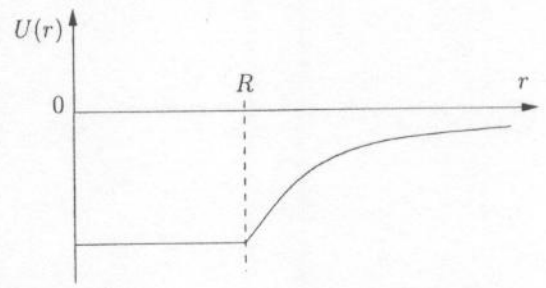
#8

Solution to Problem #8.

A. In equilibrium, the Dyson sphere must radiate the same power as the Sun. Since it is a perfect absorber, it is also a perfect black body so

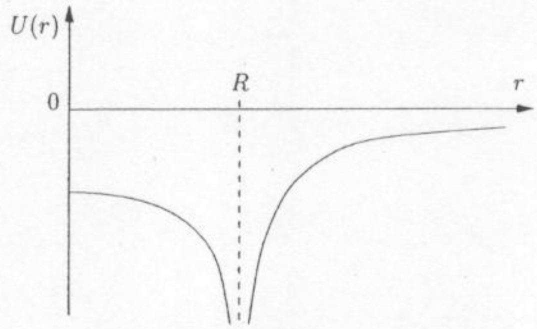
$$\begin{aligned} L_\odot &= (4\pi R^2)\sigma T^4 \\ \implies T &= \left(\frac{L_\odot}{(4\pi R^2)\sigma}\right)^{\frac{1}{4}} \\ &= \left(\frac{3.8\times 10^{26}}{4\pi(2\times 10^{11})^2\times 5.67\times 10^{-8}}\right)^{\frac{1}{4}} = 340\,\mathrm{K} \end{aligned}$$

B. i. The gravitational force on an object inside a uniform spherical shell is zero. Hence, the force of the Sun on the Dyson sphere is also zero, and the potential is flat for $r < R$. (For $r > R$, the potential goes like $-1/r$.)



ii. if the ring comes close to the Sun ($|R - r| = \rho \ll R$), then the potential approximates that of a rod near a point mass. The point mass is attracted to the rod, with a force that goes like $1/\rho$, so the potential goes like $\ln\rho = \ln|r - R|$. By symmetry, the potential should also be an even function of r. This and the existence of the potential well at $r = R$ implies that the potential is concave-down near $r = 0$.

#8 cont.



iii. The Dyson sphere experiences no force from the Sun. The perturbation does not grow. However, the Ringworld has a concave-down potential. Small perturbations grow, so it is unstable.

iv. The rotational and translational motions are not coupled in any way, so they can be treated independently. Therefore the rotation of the Ringworld does not affect its stability.

Problem #9

Our universe is filled with black body radiation at a temperature of T= 3 K. This radiation is thought to be a relic from the "big bang" now filling the continuously expanding and cooling universe. Answer the following questions:

a. Express the photon number density analytically in terms of T, universal constants and numerical cofactors.
b. Now determine n numerically in terms of photons/cm^3.

(Hint: The Bose-Einstein distribution for photons is given by $\frac{1}{e^{\beta\hbar\omega}-1}$, the integral $\int_0^\infty \frac{x^2 dx}{e^x - 1} \simeq 2.4$, and $d^3\boldsymbol{n} = \frac{V}{(2\pi)^3} d^3\boldsymbol{k}$)

#9

#9

Solution:

a. The Bose-Einstein distribution for photons is given by $<n>=\dfrac{1}{e^{\beta\hbar\omega}-1}$, where $\beta = 1/kT$. This is better known as the *Planck distribution function* and <n> simply gives the average number of photons per *mode* $\hbar\omega$ in volume V. The total number of photons in the volume can be found by integrating over all modes. $N = 2\int <n> d^3\boldsymbol{n}$, where $d^3\boldsymbol{n}$ is the number of modes within a small volume $d^3\boldsymbol{k}$ in $\boldsymbol{k}$-space for a given polarization and is given by $d^3\boldsymbol{n}=\dfrac{V}{(2\pi)^3}\, d^3\boldsymbol{k}$, where V is the volume of the universe. The factor 2 in front of the integral is due to there being two polarizations per $\hbar\omega$. Using the relation $\omega = ck$ and converting to spherical coordinates in $\boldsymbol{k}$-space we immediately find $d^3\boldsymbol{k}=4\pi k^2 dk=4\pi\dfrac{\omega^2}{c^2}\dfrac{d\omega}{c}=\dfrac{4\pi\omega^2}{c^3}d\omega$. Setting $x=\beta\hbar\omega$ and arranging the terms in the integral we obtain $N = 2\dfrac{V}{\pi^2}\left(\dfrac{k_B}{\hbar c}\right)^2 I\,T^3$, where $I = \int_0^\infty \dfrac{x^2dx}{e^x-1} \simeq 2.4$, and $x = \beta\hbar\omega$. The number density, n, can be obtained from the last relations:

$$n = \frac{N}{V} \simeq \frac{1}{\pi^2}\left(\frac{k_B}{\hbar c}\right)^3 I\,T^3 = 0.24\left(\frac{k_B}{\hbar c}\right)^3 T^3$$

b. Setting $k_B = 1.38\times10^{-23}$ J/K, $\hbar = 1.05\times10^{-34}$ J.s, $c = 3.0\times10^8$ m/s, and $T = 3$ K in the last equation for n, we obtain $n \simeq 1.84\times10^8$ photons/m^3 $\simeq 200$ photons/cm^3.

Problem #10

A point charge q is located in free space at distance d from the center of a dielectric sphere of radius a ($a<d$) and dielectric constant ε.

(a) Use *appropriate boundary conditions* to verify which one of the following functions is a suitable Green's function describing the electrostatic potential in this problem (Only one function is correct – all others are false).

$$G^{(1)}(\vec{x},\vec{x}') = 4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{Y^*_{lm}(\theta',\phi')Y_{lm}(\theta,\phi)}{(2l+1)}\times\begin{cases}\text{outside} & \dfrac{r_<^l}{r_>^{l+1}}-\dfrac{l(\varepsilon-1)}{[l(\varepsilon+1)-2]}\dfrac{a^{2l+1}}{(r_>r_<)^{l+1}}\\ \text{inside } (r<a, r'>a) & \dfrac{2(l-1)}{[l(\varepsilon+1)-2]}\dfrac{r'^l}{r'^{l+1}}\end{cases}$$

$$G^{(2)}(\vec{x},\vec{x}') = 4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{Y^*_{lm}(\theta',\phi')Y_{lm}(\theta,\phi)}{(2l+1)}\times\begin{cases}\text{outside} & \dfrac{r_<^l}{r_>^{l+1}}-\dfrac{l(\varepsilon-1)}{[l(\varepsilon+1)-1]}\dfrac{a^{2l+1}}{(r_>r_<)^{l+1}}\\ \text{inside } (r<a, r'>a) & \dfrac{2l-1}{[l(\varepsilon+1)-1]}\dfrac{r^l}{r'^{l+1}}\end{cases}$$

$$G^{(3)}(\vec{x},\vec{x}') = 4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{Y^*_{lm}(\theta',\phi')Y_{lm}(\theta,\phi)}{(2l+1)}\times\begin{cases}\text{outside} & \dfrac{r_<^l}{r_>^{l+1}}-\dfrac{l(\varepsilon-1)}{[l(\varepsilon+1)+1]}\dfrac{a^{2l+1}}{(r_>r_<)^{l+1}}\\ \text{inside } (r<a, r'>a) & \dfrac{2l+1}{[l(\varepsilon+1)+1]}\dfrac{r^l}{r'^{l+1}}\end{cases}$$

$$G^{(4)}(\vec{x},\vec{x}') = 4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{Y^*_{lm}(\theta',\phi')Y_{lm}(\theta,\phi)}{(2l+1)}\times\begin{cases}\text{outside} & \dfrac{r_<^l}{r_>^{l+1}}-\dfrac{l(\varepsilon+1)}{[l(\varepsilon-1)-1]}\dfrac{a^{2l+1}}{(r_>r_<)^{l+1}}\\ \text{inside } (r<a, r'>a) & \dfrac{2l+1}{[1-l(\varepsilon-1)]}\dfrac{r^l}{r'^{l+1}}\end{cases}$$

(b) Calculate the electrostatic field near the center of the sphere up to and including the $l=1$ terms. Verify your answer in the limits $\varepsilon\to 1$ and $\varepsilon\to\infty$. (Hint: $\theta'=0$)

(c) Calculate the l=*2* correction to the el. field in Cartesian coordinates.

2007 Comprehensive Exams

#10 Key

(a) Boundary conditions to check are:

① $G(r, r')$ is continuous at $r=a$,

② $\left(\frac{\partial G}{\partial r}\right)^{out}_{r=a+0} = \epsilon \left(\frac{\partial G}{\partial r}\right)^{in}_{r=a-0}$

Apply these condns to $G^{(i)}$; $i=1, 2, 3, 4$.

The result:

	①	②	
$G^{(1)}$	No	No	False ×
$G^{(2)}$	Yes	No	False ×
$G^{(3)}$	Yes	Yes	TRUE! ✓
$G^{(4)}$	No	Yes	False ×

(b) So, Using $G^{(3)}$,

Potential is:

$$\Phi = q\int_{-\infty}^{+\infty} G^{(3)}(x, x')\rho(x')\,dx'$$

$$\Phi^{in}(r,\theta,\phi) = \frac{1}{4\pi\epsilon_0}\cdot 4\pi \sum_{l=0}^{\infty}\sum_{m=-l}^{+l} Y^*_{lm} Y_{lm} \frac{1}{(l(\epsilon+1)+1)}\frac{r^l}{d^{l+1}} \quad ③$$

Close to the center, $r \ll d$, $l=0,1$.

$$\Phi^{in}(r,\theta,\phi) \approx \frac{q}{\epsilon_0}\Big\{\frac{1}{4\pi}\cdot\frac{1}{d} + \frac{1}{\epsilon+2}\Big[\frac{3}{4\pi}\cos\theta\cos\theta'$$

$$+\frac{3}{8\pi}\sin\theta\sin\theta' e^{i(\phi-\phi')} + \frac{3}{8\pi}\sin\theta\sin\theta' e^{-i(\phi-\phi')}\Big]$$

$$\times\frac{r}{d^2}\Big\}$$

$$= \frac{q}{4\pi\epsilon_0}\Big\{\frac{1}{d} + \frac{3}{(\epsilon+2)}\left[\cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi-\phi')\right]\frac{r}{d^2}\Big\} \quad ④$$

Then, for $\theta'=0$, $\Phi^{in} \approx \frac{q}{4\pi\epsilon_0}\left(\frac{1}{d} + \frac{3}{\epsilon+2}\frac{z}{d^2}\right)$ ⑤

2007 Comp $\boxed{\#10}$ contd.

$$\left.\begin{aligned} &\text{Then, } E_r^{in} = E_z^{in} = -\frac{\partial \Phi^{in}}{\partial z} \approx -\frac{q}{4\pi\epsilon_0}\frac{1}{d^2}\frac{3}{\epsilon+2} \\ &E_\theta = E_\phi = 0 \end{aligned}\right\} \text{Ans. } (6)$$

$$\left.\begin{aligned} &\text{If} \quad \epsilon = 1 \quad (6) \rightarrow E_z^{in} = -\frac{q}{4\pi\epsilon_0 d^2} = E_r^{in} \\ &\qquad \epsilon = \infty \quad (6) \rightarrow E_z^{in} = 0 = E_r^{in} \end{aligned}\right\} (7) \text{ Ans}$$

(c). $l = 2$ correction:

$$\left.\begin{aligned} \Phi_c^{in} = &\frac{q}{\epsilon_0}\frac{1}{2(\epsilon+1)+1}\frac{r^2}{d^3}\left[\frac{1}{4}\left(\frac{5}{4\pi}\right)(3\cos^2\theta - 1)\cdot 2\right] \\ &+ \frac{q}{\epsilon_0}\frac{1}{3(\epsilon+1)+1}\frac{r^3}{d^4}\left[\frac{1}{4}\left(\frac{7}{4\pi}\right)(5\cos^3\theta - 3\cos\theta)\, 2\right] \end{aligned}\right\} (8)$$

$$= \frac{q}{4\pi\epsilon_0 d}\left[\frac{5}{2\epsilon+3}\left(\frac{r}{d}\right)^2\frac{1}{2}(3\cos^2\theta - 1) + \frac{7}{3\epsilon+4}\left(\frac{r}{d}\right)^3\frac{1}{2}\left[5\cos^3\theta - 3\cos\theta\right]\right] \quad l = 3$$

$$\left.= \frac{q}{4\pi\epsilon_0 d}\left[\frac{5}{2\epsilon+3}\frac{1}{d^2}\left(\frac{3}{2}z^2 - \frac{1}{2}r^2\right) + \frac{7}{3\epsilon+4}\frac{1}{d^3}\left(\frac{5}{2}z^3 - 3zr^2\right)\right] \quad l = 3 \right\} (9)$$

Then, electric fld is:

$$\left.\begin{aligned} E_z^c &= -\frac{\partial \Phi_c^{in}}{\partial z} = \frac{-q}{4\pi\epsilon_0 d}\left[\frac{5}{2\epsilon+3}\frac{2z}{d^2}\right] \\ &= -\frac{q}{4\pi\epsilon_0 d^2}\frac{5}{2\epsilon+3}\left(\frac{2z}{d}\right) \\ E_x^c &= -\frac{\partial \Phi_c^{in}}{\partial x} = +\frac{q}{4\pi\epsilon_0 d^2}\left(\frac{x}{d}\right) \\ E_y^c &= -\frac{\partial \Phi_c^{in}}{\partial y} = +\frac{q}{4\pi\epsilon_0 d^2}\left(\frac{y}{d}\right) \end{aligned}\right\} \text{Ans. } (10)$$

Problem #11

(a) Figure 1 below shows a non-polarized optical beam passing through an optical system consisting of two crossed polarizers. A third freely rotating polarizer is placed between of the two. Considering all polarizers to be ideal, i.e. without any spurious absorption or scatter. At what angle α_1 is the transmitted intensity the largest? What is the largest transmitted intensity?
(b) Figure 2 shows a similar arrangement but with two freely rotationg polarizers. At what angles α_1, α_2, is the transmitted intensity the largest and what is the largest transmitted intensity?
(c) The same as in (b) but with N polarizers at angles, α_1, α_2, α_3,... α_N .
(d) What is the transmitted intensity in the limit $N \rightarrow \infty$

Figure 1.

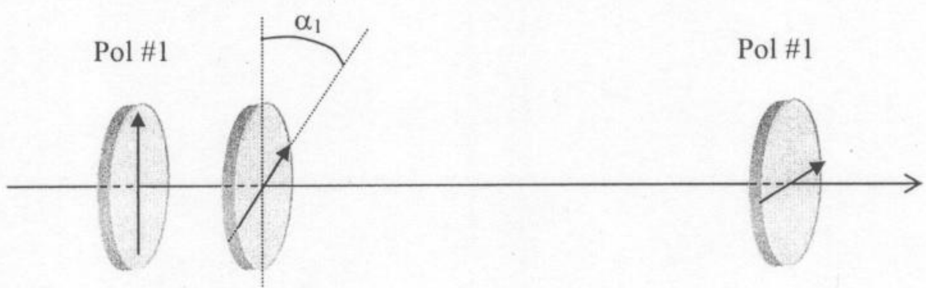


Figure 2.

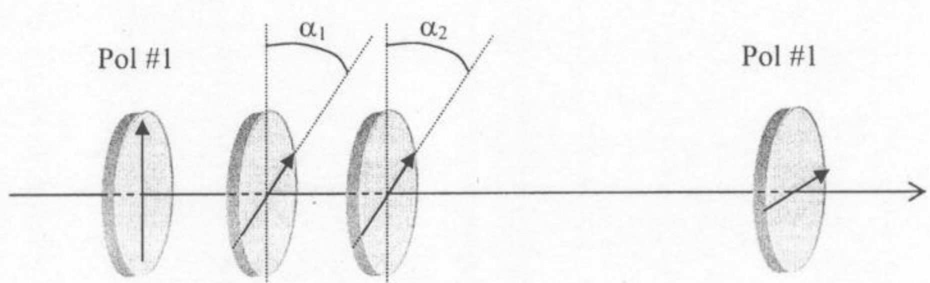
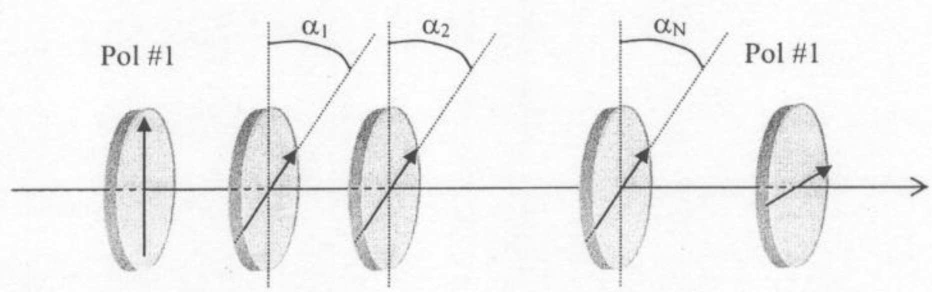


Figure 3.



Solution #11

(a) $I(\alpha_1) = \left[\cos\alpha_1 \cos\left(\frac{\pi}{2}-\alpha_1\right)\right]^2$

$= \cos^2\alpha_1 \sin^2\alpha_1 = \frac{1}{4}\sin^2 2\alpha_1$

(b) $I(\alpha_1,\alpha_2) = \left[\cos\alpha_1 \cos(\alpha_2-\alpha_1)\cos\left(\frac{\pi}{2}-\alpha_2\right)\right]^2$

By symmetry, extremum (maximum) occurs when $\alpha_1 = \frac{1}{2}\alpha_2 = \frac{\pi}{6}$

$I_{max} = \left[\cos\frac{\pi}{6}\cos\frac{\pi}{6}\cos\frac{\pi}{6}\right]^2 = \left(\cos\frac{\pi}{6}\right)^6 = \left(\frac{\sqrt{3}}{2}\right)^6 = \frac{27}{16^2}$

(c) $I(\alpha_1,\alpha_2,\ldots,\alpha_N) = \left(\cos\frac{\pi}{2N}\right)^{2N}$

(d) $\lim\limits_{N\to\infty}\left(\cos\frac{\pi}{2N}\right)^{2N} = 1$

This can be shown in two steps.

i) $\cos x = 1 - \frac{x^2}{2!} + \underbrace{\frac{x^4}{4!} - \frac{x^6}{6!} + \ldots}_{>0} \geq 1 - \frac{x^2}{2!}$

$\Rightarrow \cos\frac{\pi}{2N} \geq 1 - \frac{1}{2}\left(\frac{\pi}{2N}\right)^2 = 1 - \left(\frac{\pi}{\sqrt{2}}\frac{1}{2N}\right)^2$

$= \left(1 - \frac{\pi}{\sqrt{2}}\frac{1}{2N}\right)\left(1 + \frac{\pi}{\sqrt{2}}\frac{1}{2N}\right)$

$1 \geq \left(\cos\frac{\pi}{2N}\right)^{2N} \geq \left(1 - \frac{\pi}{\sqrt{2}}\frac{1}{2N}\right)^{2N}\left(1 + \frac{\pi}{\sqrt{2}}\frac{1}{2N}\right)^{2N}$

ii) $\lim\limits_{N\to\infty}\left(1 \pm \frac{x}{N}\right)^N = e^{\pm N}$

$1 \geq \lim\limits_{N\to\infty}\left(\cos\frac{\pi}{2N}\right)^{2N} \geq e^{-\frac{\pi}{\sqrt{2}}}\cdot e^{+\frac{\pi}{\sqrt{2}}} = 1$

Problem #12

Fermat's principle states that a ray of light will follow the path that requires the shortest traveling time. For a two dimensional case, that path is obtained by minimizing the integral

$$\int_{x_1}^{x_2} n(x,y)\sqrt{1+y'^2}\,dx$$

where $n(x,y)$ is the index of refraction and $y' = dy/dx$.

(a) For the special case that the integrand F does not depend explicitly on x, use the *Euler-Lagrange Equation* to prove that

$$F - y'\frac{\partial F}{\partial y'} = \text{constant.}$$

(b) Find all possible $y(x)$ for the particular case $n = 1 + a|y|$ with $a > 0$.

(c) Take the limit $a \to 0$ of your $y(x)$ from part (b), and discuss the results.

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#12

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(a) For the special case that the integrand F does not depend explicitly on x, use the *Euler-Lagrange Equation* to prove that

$$F - y'\frac{\partial F}{\partial y'} = \text{constant}.$$

$$\begin{aligned}
0 &= \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial y} && \text{Euler-Lagrange}\\
\frac{dF}{dx} &= \underbrace{\frac{\partial F}{\partial x}}_{=0} + y'\frac{\partial F}{\partial y} + y''\frac{\partial F}{\partial y'} && \text{chain rule}\\
&= y'\left[\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right] + y''\frac{\partial F}{\partial y'} && \text{using Euler-Lagrange}\\
&= \frac{d}{dx}\left[y'\left(\frac{\partial F}{\partial y'}\right)\right] - y''\frac{\partial F}{\partial y'} + y''\frac{\partial F}{\partial y'} && \text{chain rule again}\\
0 &= \frac{d}{dx}\left[F - y'\left(\frac{\partial F}{\partial y'}\right)\right] && \Rightarrow \text{const} = F - y'\left(\frac{\partial F}{\partial y'}\right)
\end{aligned}$$

(b) Find all possible $y(x)$ for the particular case $n = 1 + a|y|$ with $a > 0$.

First, we'll develop the solution for $y > 0$, defining α as the constant.

$$\begin{aligned}
\frac{\partial F}{\partial y'} &= (1+ay)\frac{y'}{\sqrt{1+y'^2}}\\
\alpha &= (1+ay)\sqrt{1+y'^2} - (1+ay)\frac{y'^2}{\sqrt{1+y'^2}} = \frac{1+ay}{\sqrt{1+y'^2}}\\
\alpha^2(1+y'^2) &= (1+ay)^2 \qquad \Rightarrow 1 = \left(\frac{1}{\alpha} + \frac{a}{\alpha}y\right)^2 - y'^2\\
\text{Let } q &\equiv \frac{1}{\alpha} + \frac{a}{\alpha}y \qquad \Rightarrow q' = \frac{a}{\alpha}y'\\
1 &= q^2 - \left(\frac{\alpha}{a}\right)^2 q'^2 \qquad \Rightarrow q(x) = \cosh\left(\frac{ax}{\alpha} + \beta\right)\\
y(x) &= \frac{\alpha}{a}\cosh\left(\frac{ax}{\alpha} + \beta\right) - \frac{1}{a}
\end{aligned}$$

with α and β as undetermined constants.

To treat the $y < 0$ case, we simply change the sign of a everywhere, recalling that hyperbolic cosine is even and β is still undetermined (so we may change its sign, too).

$$\boxed{y(x) = \pm\frac{\alpha}{a}\cosh\left(\frac{ax}{\alpha} + \beta\right) \mp \frac{1}{a}}$$

#12 cont.

(c) Take the limit $a \to 0$ of your $y(x)$ from part (b), and discuss the results.

For the $y > 0$ case:

$$
\begin{aligned}
1 + ay &= \alpha \cosh\left(\frac{ax}{\alpha} + \beta\right) = \frac{\alpha}{2}\left[e^{\left(\frac{ax}{\alpha}+\beta\right)} + e^{-\left(\frac{ax}{\alpha}+\beta\right)}\right] \\
&= \frac{\alpha}{2}\left[1 + \left(\frac{ax}{\alpha} + \beta\right) + \frac{1}{2}\left(\frac{ax}{\alpha} + \beta\right)^2 + \ldots + 1 - \left(\frac{ax}{\alpha} + \beta\right) + \frac{1}{2}\left(\frac{ax}{\alpha} + \beta\right)^2 + \ldots\right] \\
&\simeq \frac{\alpha}{2}\left[2 + \left(\frac{ax}{\alpha} + \beta\right)^2\right] = \frac{\alpha}{2}\left[2 + \beta^2 + 2\frac{\beta a x}{\alpha} + \frac{a^2x^2}{\alpha^2}\right] \simeq \frac{\alpha}{2}(2+\beta^2) + \beta a x \\
y(x) &= \frac{\frac{\alpha}{2}(2+\beta^2) - 1}{a} + \beta x
\end{aligned}
$$

Linear, as it must be since the index of refraction is unity. An easier method is to take the equation for y' and set $a = 0$ there. This shows that y' must be a constant, so y must be linear.

Problem #13

An infinite one-dimensional square-well potential defined as

$$V(x) = 0, \quad 0 \le x \le a,$$
$$V(x) = \infty, \quad 0 > x > a$$

has well-defined normalized energy eigenfunctions given by:

$\varphi_n(x) = \sqrt{\frac{2}{a}} Sin(nk_1 x)$, where $k_1 = \frac{\pi}{a}$. Answer the following questions:

a. For the 3[rd] excited state (n=3) what is the probability $P_n(x)$ that a particle is located between the interval at $x = \frac{a}{3}$ and $x + dx$, where $dx = \frac{a}{1000}$?

b. Now consider the momentum space representation of the wave functions for the one-dimensional square-well potential described above and let us call these functions $\phi_n(p)$, where p is the linear momentum of the particle in the n[th] excited state. Determine $\phi_n(p)$, and explain briefly the physical meaning of $\phi_n(p)$.

c. Determine the probability of finding a particle in the 3[rd] excited state with a momentum between $p = 2\hbar k_1$ and $p+dp$, where $dp = \frac{\hbar k_1}{1000}$.

#13

Quantum

Solution:

a. In quantum mechanics $P_n(x) = |\varphi_n(x)|^2$ represents the probability density which, when multiplied by small increment dx, gives the probability of finding a particle, described by $\varphi_n(x)$, in the narrow interval between x and $x+dx$. Therefore,

$P_3(\frac{a}{3}) = \left|\sqrt{\frac{2}{a}} Sin(\cancel{3}\frac{\pi}{\cancel{a}}\frac{\cancel{a}}{\cancel{3}})\right|^2 = 0$, which means that the probability of finding a particle in the 3rd excited state in the vicinity of $x = \frac{a}{3}$ is zero.

b. To determine the normalized momentum space representation, $\phi_n(p)$, of the eigenfunctions we have to find the Fourier transform of the real space wave functions, $\varphi_n(x)$. This is simply done by $\phi_n(p) = (\frac{1}{2\pi\hbar})^{1/2}\int_{-\infty}^{+\infty}\varphi_n(x)e^{(\frac{ipx}{\hbar})}dx$.

Because $\varphi_n(x)$ is zero everywhere except $0 \le x \le a$, the above is reduced to

$\phi_n(p) = (\frac{1}{2\pi\hbar})^{1/2}\int_0^a \varphi_n(x)e^{(\frac{ipx}{\hbar})}dx = (\frac{1}{a\pi\hbar})^{1/2}\int_0^a Sin(nk_1x)\, e^{\frac{ipx}{\hbar}}\, dx$. This integration is readily performed by noting that $Sin(nk_1x) = \frac{e^{ink_1x} - e^{-ink_1x}}{2i}$, thus

$$\phi_n(p) = (\frac{1}{a\pi\hbar})^{1/2}\frac{1}{2i}\int_0^a e^{i(\frac{p}{\hbar}+nk_1)x} - e^{i(\frac{p}{\hbar}-nk_1)x}dx = (\frac{1}{a\pi\hbar})^{1/2}\frac{1}{2i}\left[\left(\frac{e^{i(\frac{p}{\hbar}+nk_1)a}-1}{i(\frac{p}{\hbar}+nk_1)}\right) - \left(\frac{e^{i(\frac{p}{\hbar}-nk_1)a}-1}{i(\frac{p}{\hbar}-nk_1)}\right)\right]$$

Noting that $e^{\pm ink_1a} = e^{\pm in\pi} = (-1)^n$, $\phi_n(p)$ can easily be reduced to

$\phi_n(p) = (\frac{\pi a}{\hbar})^{1/2}\frac{n(1-(-1)^n e^{\frac{ipa}{\hbar}})}{(n\pi)^2 - (\frac{pa}{\hbar})^2}$. The physical meaning of $\phi_n(p)$ can be related to probability density $Q_n(p) = |\phi_n(p)|^2$ in momentum space, which can be related to the probability of finding a particle in the nth energy eigenstate with a momentum between p and p+dp, which is given by $Q_n(p)\, dp$.

c. For $n=3$ and $p = 2\hbar k_1$ we can determine the probability density easily:

$Q_n(p) = |\phi_n(p)|^2 = \left|(\frac{\pi a}{\hbar})^{1/2}\frac{n(1-(-1)^n e^{\frac{ipa}{\hbar}})}{(n\pi)^2 - (\frac{pa}{\hbar})^2}\right|^2 = (\frac{6}{5})^2\frac{1}{\pi^2}(\frac{a}{\pi\hbar})$; therefore, the probability of finding a particle in the 3rd excited state with a momentum

#13 cont.

Quantum

between $p = 2\hbar k_1$ and $p+dp$, where $dp = \dfrac{\hbar k_1}{1000}$, is given by

$$Q_n(p)\,dp = (\frac{6}{5})^2 \frac{1}{\pi^2}(\frac{\cancel{a}}{\cancel{\pi}\cancel{\hbar}})\frac{\cancel{\hbar}\cancel{\pi}}{1000\cancel{a}} = 1.46\times10^{-4}.$$

Problem #14

An off-centered hole of radius a is bored parallel to the axis of a long metallic cylinder of radius b $(b>a)$. With the exception of the bored hole the cylinder is assumed to be full. The two axes are at distance d apart as shown in **Fig. 1** below. Uniform current I with current density $\boldsymbol{J}$ flows in the cylinder out of the plane of the paper and perpendicular to the paper as shown in **Fig. 1**. Answer the following questions:

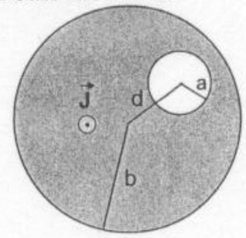


Fig. 1

a. What are the magnitude and direction of magnetic field $\boldsymbol{B}$ at the center of the hole?
b. Now assume that the current through the long metallic cylinder is generated by the circuit shown in **Fig. 2**, where $\boldsymbol{S}$ represents a switch, $\boldsymbol{C}$ a large capacitor, $\boldsymbol{V}$ a battery, and $\boldsymbol{R}$ the total resistance of the circuit. A square conducting loop of side L is placed at distance r from the center of the metallic cylinder and in the plane formed by the axes of the two cylinders, of radius a and b. Assume also that the conducting loop has mass $\boldsymbol{m}$ and resistance $\boldsymbol{R_L}$ and is located on a frictionless horizontal plane. Now let us assume that we suddenly open the circuit by disconnecting switch $\boldsymbol{S}$ from the circuit. For this problem, ignore the displacement current, and assume that all the wires and the circuit elements are far from the metallic cylinder and the loop. Using only the first principles, show on the figure the direction of the current in the loop and the acceleration $\boldsymbol{a}$ exerted on the loop immediately after the switch is opened. Briefly explain your reasoning.
c. Determine the acceleration of the loop immediately after the switch is opened.

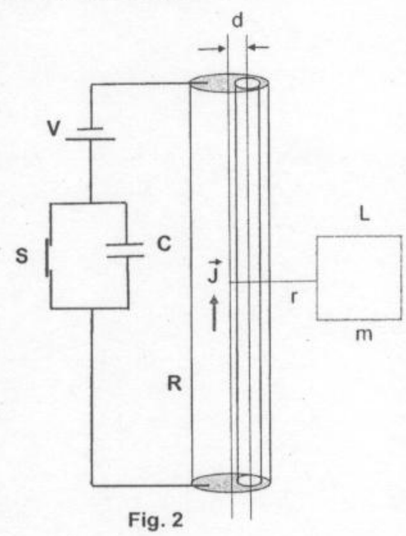
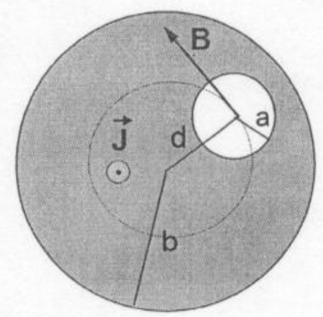


Fig. 2

Solution:

a. The direction of magnetic field $\boldsymbol{B}$ is easily determined using the symmetry and the right-handed screw rule; the direction is shown in the figure below. The magnitude of $\boldsymbol{B}$ can easily be determined using the principle of superposition, which is implied by the linearity of Maxwell's equations, in this case Ampere's Law, $\oint \vec{B}.d\vec{s} = \mu_o I = \mu_o \iint \vec{J}.d\vec{S}$. Let us assume that current I has two components, I_1 and I_2, where I_1 flows through a solid cylinder of radius b while I_2 flows in the opposite direction through a solid cylinder of radius a, located inside the bore hole. The superposition of the two currents must be equal to the current flowing through the cylinder with the bore hole, $I = I_1 + I_2$, where $I_1 = \iint \vec{J}_1.d\vec{S} = \pi b^2 J_1$ and $I_2 = \iint -\vec{J}_2.d\vec{S} = -\pi a^2 J_2$, where $I = \pi(b^2 J_1 - a^2 J_2)$.



Furthermore, in order to produce zero current in the bore-hole region J_1 must be equal to J_2, which must be equal to $J = J_1 = J_2 = \dfrac{I}{\pi(b^2 - a^2)}$. Now applying the principle of superposition we assert that the magnetic field at the center of the bore hole has contributions from $\boldsymbol{J}_1$ and $\boldsymbol{J}_2 = -\boldsymbol{J}_1$. Let us call these fields $\boldsymbol{B}_1$ and $\boldsymbol{B}_2$, respectively; the net magnetic field at the center is then given by $\boldsymbol{B} = \boldsymbol{B}_1 + \boldsymbol{B}_2$.

It is clear that $\boldsymbol{B}_2 = \boldsymbol{0}$ because the current through an arbitrarily small area $\left(I' = J\pi r^2 = \dfrac{I\, r^2}{(b^2 - a^2)} \to 0 \text{ as } r \to 0\right)$ can be made zero, which produces zero $\boldsymbol{B}_2$.

$\boldsymbol{B}_1$ can easily be calculated from

$\oint \vec{B}_2.d\vec{s} = \mu_o \iint \vec{J}.d\vec{S}$, which yields $2\pi d B_2 = \dfrac{\mu_o \pi d^2 I}{\pi(b^2 - a^2)} \to B_2 = \dfrac{\mu_o I\, d}{2\pi(b^2 - a^2)}$.

Therefore, $B = B_1 + B_2 = B_2 = \dfrac{\mu_o I\, d}{2\pi(b^2 - a^2)}$.

b. The directions of current i and acceleration $\boldsymbol{a}$ are as shown in **Fig. 3**. The reason for these choices is Lenz' Law, which asserts that both the direction of the current and that of the acceleration should be such as to resist the changes in magnetic flux through the conducting loop. Once the switch is open, current density J, and hence the magnetic flux through the loop, will decrease exponentially. In order to oppose this reduction, the induced current in the loop must be flowing in the clockwise direction so that the induced magnetic field will line up parallel to the decreasing field in the loop. At the same time, the loop itself should move towards the conducting cylinder, where magnetic fields are stronger.

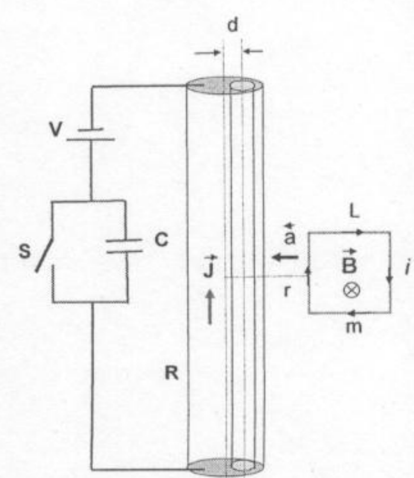


Fig. 3

#14 cont. 2

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c. We only consider the contribution of $\boldsymbol{J}$ in the metallic cylinder as suggested by the problem. The magnitude of the acceleration can easily be found using Newton's 2nd Law: $F=ma$, where F is the magnitude of the total force on the loop and m is the mass of the loop. Since there is no friction on the loop the net force is entirely due to magnetic force. It is clear that the net force on the loop is $F=F_1-F_2$. This is because $F_3=F_4$ and F_3 and F_4 oppose each other (**Fig. 4**). The magnetic force on a current segment of length dl directed along current i is given by: $d\vec{F} = i\, d\vec{l} \times \vec{B}$. Thus, the net force is given by $F = i\,L\,(B(r) - B(r+L))$, directed to the left. Using Ampere's law ignoring the displacement current, $\oint \vec{B}.d\vec{l} = \mu_o I$, and the principle of superposition, the total magnetic field $B(r)$ at location r in the plane of the loop can easily be

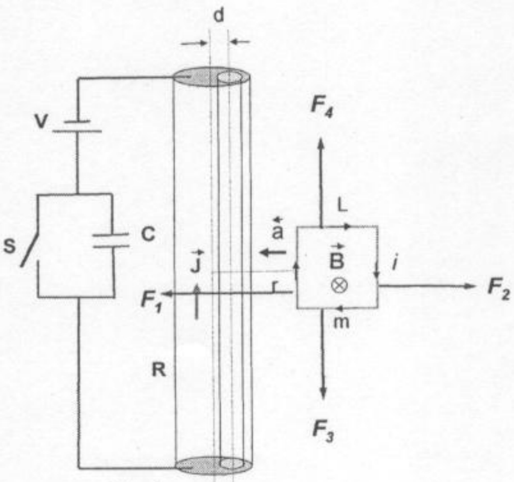


Fig. 4

determined as $B(r) = \dfrac{\mu_o}{2\pi}\left(\dfrac{I_1(t)}{r} - \dfrac{I_2(t)}{r-d}\right) = \dfrac{\mu_o J(t)}{2}\left(\dfrac{b^2}{r} - \dfrac{a^2}{r-d}\right)$ where

$I_1 = \pi b^2 J,\ I_2 = \pi a^2 J$ and $J = \dfrac{I(t)}{\pi(b^2-a^2)}$. Therefore, the acceleration is given by

$$a = \frac{F}{m} = \frac{i\,L}{m}(B(r) - B(r+L)) = \frac{\mu_o i\, L^2 I}{2\pi m(b^2-a^2)}\left(\frac{b^2}{r(r+L)} - \frac{a^2}{(r-d)(r-d+L)}\right).$$ Now,

all we have to do is to find the induced current, $i(t)$, at $t=0^+$, immediately after the switch $\boldsymbol{S}$ is opened. The induced current can be found using Ohm's law and Faraday's law of induction: $i = \dfrac{\varepsilon}{R_L}$, where $\varepsilon = -\dfrac{d\phi_m}{dt}$. In the last equation, ϕ_m is the total magnetic flux passing through the conducting loop and is given by

$$\phi_m = \int \vec{B}.d\vec{A} = \int_r^{r+L} L\, B(r)\, dr = \frac{L\mu_o J(t)}{2}\left(b^2 \ln\left(\frac{r+L}{r}\right) - a^2 \ln\left(\frac{r-d+L}{r-d}\right)\right).$$ In the last

equation, $J(t)$ corresponds to the uniform current density across the conducting cylinder associated with a charging capacitor and is given

by $J(t) = \dfrac{Ie^{-\frac{t}{RC}}}{\pi(b^2-a^2)}$, where $I = \dfrac{V}{R}$. From this last result we can find the induced current in the conducting loop at $t=0$; it is given by

$$i(t=0) = \frac{\mu_o IL}{2\pi R_L RC(b^2-a^2)}\left(b^2 \ln\left(\frac{r+L}{r}\right) - a^2 \ln\left(\frac{r+L-d}{r-d}\right)\right).$$ From this the

acceleration, a, is determined as:

$$a = \frac{\mu_o^{\,2} I^2 L^3}{4\pi^2 m(b^2-a^2)^2 R_L RC}\left(\frac{b^2}{r(r+L)} - \frac{a^2}{(r-d)(r-d+L)}\right)\left(b^2 \ln\left(\frac{r+L}{r}\right) - a^2 \ln\left(\frac{r+L-d}{r-d}\right)\right)$$

Problem #15

A turbine is used to reduce the gas pressure in a chamber (the "low pressure" region). A separate vacuum pump keeps the "high pressure" region at 7.6×10^{-3} torr. The turbine consists of a series of thin, diagonal, 5 mm blades moving at speed v_f (see sketch). Let $T = 300\,\mathrm{K}$ everywhere.[1]

A. Show that the mean free path of the gas in the high pressure region is greater than the dimensions of the turbine blades (thus collisions between molecules can be ignored).

B. Molecules in the high pressure region have a Maxwellian distribution,

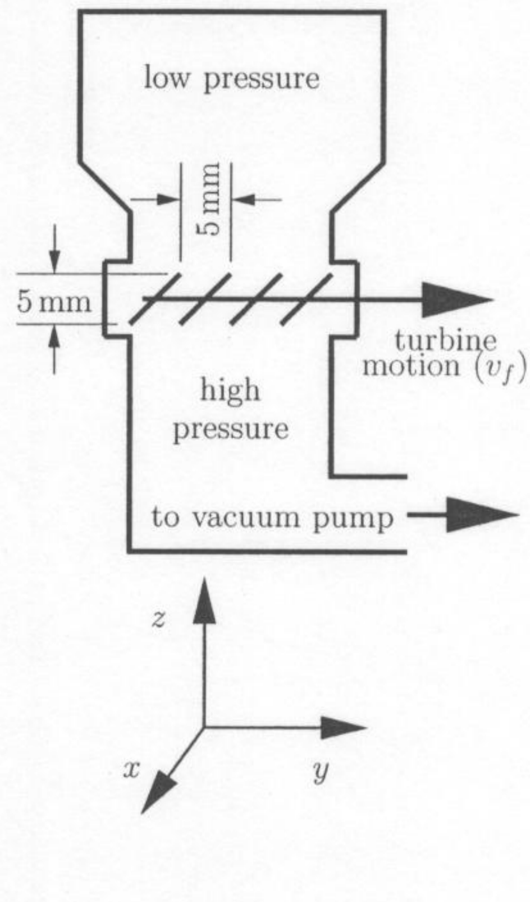
$$\frac{dn_H}{dv_z} = \frac{n_H}{c\sqrt{2\pi}}\, e^{-v_z^2/2c^2},$$

where n_H is the molar density (mol m^{-3}) of molecules and c is the thermal speed. Express the rate of molecules incident on a unit area of the turbine (mol m^{-2} s^{-1}) from the high pressure region in the velocity range v_z to $v_z + dv_z$, in terms of the above variables.

C. Let $v_f = 10^3\,\mathrm{m/s}$. Assume that all molecules with $v_z > v_f$ will "backstream" (cross over from the high to low pressure region), and calculate the total rate (mol m^{-2} s^{-1}).

D. Assume that *all* molecules incident on the turbine from the low pressure region (which has molar density n_L) pass "forward" through the turbine. Express this forward streaming rate in terms of n_L.

E. In a steady state, the forward and backstreaming rates balance. Calculate the pressure (in torr) of the low pressure region.



[1] One mole of air at 300 K, 760 torr has a volume of ~22.4 liters and masses 0.029 kg.

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#15

Solution to Problem #15.

A. Let σ be the cross-section for molecular collisions. If we have $\mathcal{N}$ particles per m^3, the mean free path (based on geometrical considerations) must be approximately $1/\mathcal{N}\sigma$. The exact formula happens to be $\lambda = 1/\sqrt{2}\mathcal{N}\sigma$. The number density at atmospheric pressure is

$$\mathcal{N}_{atm} = \frac{N_A}{22.4\,\mathrm{l/mol}} = \frac{6.02 \times 10^{23}}{22.4 \times 10^{-3}\,\mathrm{m}^3} = 2.69 \times 10^{25}\,\mathrm{m}^{-3}.$$

In the high pressure region, then,

$$\mathcal{N}_H = \frac{7.6\,\mathrm{millitorr}}{760\,\mathrm{torr}} \mathcal{N}_{atm} = 2.69 \times 10^{20}\,\mathrm{m}^{-3}.$$

For the cross-section, we might estimate that a molecule of N_2 or O_2 will collide with another when they pass within 2 Å. The cross-section is then

$$\sigma = \pi(2 \times 10^{-10}\,\mathrm{m})^2 = 1.3 \times 10^{-19}\,\mathrm{m}^2.$$

The resulting mean free path is $\lambda = 0.02\,\mathrm{m}$, which is significantly larger than the 0.005 m size of the turbine blades.

B. The n_H on the right-hand side of the equation for the distribution is, as the problem states, the molar density. This is a constant. The dn_H on the left-hand side is an infinitesimal piece of the total density contributed by molecules in the velocity range v_z to $v_z + dv_z$. The normalization is thus

$$\int_{-\infty}^{\infty} \frac{dn_H}{dv_z}\, dv_z = \int_{-\infty}^{\infty} dn_H = n_H,$$

which is easily verified from the given distribution.

The units of the distribution dn_H/dv_z are molar density per velocity. For particles in the range v_z to $v_z + dv_z$, they will have molar density

$$dn_H = \frac{n_H}{c\sqrt{2\pi}}\, e^{-v_z^2/2c^2}\, dv_z,$$

so the rate in moles per square meter per second is

$$d\varphi = v_z\, dn_H = \frac{n_H\, v_z}{c\sqrt{2\pi}}\, e^{-v_z^2/2c^2}\, dv_z.$$

#15 cont.

C. To find the backstreaming rate, we integrate the differential rate over the range of velocities that backstream:

$$\begin{aligned}\varphi_B &= \int_{v_z=v_f}^{\infty} d\varphi = \int_{v_f}^{\infty} \frac{n_H\, v_z}{c\sqrt{2\pi}}\, e^{-v_z^2/2c^2}\, dv_z \\ &= -\frac{n_H\, c}{\sqrt{2\pi}}\, e^{-v_z^2/2c^2}\bigg|_{v_f}^{\infty} = \frac{n_H\, c}{\sqrt{2\pi}}\, e^{-v_f^2/2c^2}.\end{aligned}$$

D. The forward streaming rate from the low pressure region is the same sort of integral, using n_L and integrating over all negative v_z:

$$\varphi_F = \int_{-\infty}^{0} \frac{n_L\, v_z}{c\sqrt{2\pi}}\, e^{-v_z^2/2c^2}\, dv_z = -\frac{n_L\, c}{\sqrt{2\pi}},$$

where the minus sign indicates that the movement is downward.

E. Steady state occurs when there is no net flow across the turbine: $\varphi_B + \varphi_F = 0$. Using the results of the previous two parts, we find

$$n_L = n_H\, e^{-v_f^2/2c^2}.$$

We have all we need to solve for this numerically. The thermal speed (considering only motions in the z-direction) is given by equipartition,

$$\frac{1}{2}mc^2 = \frac{1}{2}RT,$$

where m is the molar mass. For air, the footnote tells us that $m = 0.029\,\mathrm{kg\,mol^{-1}}$. This leads to $c^2 = 8.6 \times 10^4\,\mathrm{m^2\,s^{-2}}$. Plugging in the numbers,

$$\frac{n_L}{n_H} = e^{-v_f^2/2c^2} = 0.003.$$

But since the temperatures are the same, in the two regions, the ratio of pressures must be the ratio of densities:

$$P_L = \frac{n_L}{n_H}\, P_H = 0.003 \times 7.6\,\text{millitorr} = 2.3 \times 10^{-5}\,\text{torr}.$$