M.S. Exam : Dec. 1985 (Robiscoe)

QM Problem : Derivation of Probability Current. 11/16/85

7. A wave function $\psi = \psi(\vec{r}, t)$ obeys the Schrodinger equation

$$i\hbar(\partial\psi/\partial t) = [-(\hbar^2/2m)\nabla^2 + V(\vec{r})]\psi,$$

for a particle of mass m in a real potential $V(\vec{r})$. Show that the probability density $\rho = \psi^*\psi$ then obeys a continuity equation

$$\partial\rho/\partial t + \vec{\nabla}\cdot\vec{J} = 0,$$

where $\vec{J}$ is a "probability current" density. Find $\vec{J}$ explicitly in terms of ψ and its derivatives. Also : comment on what happens to $\vec{J}$ when ψ is a purely real function. What kind of quantum-mechanical problems can be done with ψ = pure real? *

Solution

Take $\psi^* \times$ (original Schr. eqtn) and (original Schr. eqtn)$^* \times \psi$, i.e.

$$i\hbar\,\psi^*\left(\frac{\partial\psi}{\partial t}\right) = -(\hbar^2/2m)\,\psi^*(\nabla^2\psi) + V(\vec{r})\,\psi^*\psi$$

$$-i\hbar\left(\frac{\partial\psi^*}{\partial t}\right)\psi = -(\hbar^2/2m)(\nabla^2\psi^*)\psi + V(\vec{r})\,\psi^*\psi$$

... subtract, and the potential term drops out...

$$i\hbar\underbrace{\left[\psi^*\left(\frac{\partial\psi}{\partial t}\right) + \left(\frac{\partial\psi^*}{\partial t}\right)\psi\right]}_{\frac{\partial}{\partial t}(\psi^*\psi)} = -\frac{\hbar^2}{2m}\underbrace{\left[\psi^*(\nabla^2\psi) - (\nabla^2\psi^*)\psi\right]}_{\vec{\nabla}\cdot[\psi^*\vec{\nabla}\psi - \psi\vec{\nabla}\psi^*]}$$

... rearrange terms ... (over)

$$\underbrace{\frac{\partial}{\partial t}(\psi^* \psi)}_{\substack{\text{prob. density} \\ \rho = \psi^* \psi}} + \vec{\nabla} \cdot \underbrace{\left[\frac{\hbar}{2mi}(\psi^* \vec{\nabla}\psi - \psi\vec{\nabla}\psi^*)\right]}_{\text{prob. current density } \vec{J}} = 0$$

i.e.// $$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0,$$ with// $$\boxed{\vec{J} = \frac{\hbar}{2mi}(\psi^* \vec{\nabla}\psi - \psi\vec{\nabla}\psi^*)}$$

Have shown $\rho = \psi^*\psi$ does obey a continuity eqtn, with probability current density $\vec{J}$ as written.

If ψ = pure real, $\psi^* \equiv \psi$, and $\vec{J} \equiv 0$. There is no probability current density when ψ = pure real.

What kind of QM problems? Define the velocity operator...

$$\vec{v} = \vec{p}/m = -\left(\frac{i\hbar}{m}\right)\vec{\nabla} \leftarrow \text{a pure imaginary operator}$$

So// $$\vec{J} = \frac{1}{2}\left(\frac{-i\hbar}{m}\right)\left[\psi^* \vec{\nabla}\psi - \psi\vec{\nabla}\psi^*\right]$$

$$= \frac{1}{2}\left[\psi^* \vec{v}\psi + \psi\vec{v}^*\psi^*\right] = \mathrm{Re}\left[\psi^* \vec{v}\psi\right].$$

When ψ = pure real, $\vec{J} \equiv 0$, and expectation value $\langle \vec{J} \rangle \equiv 0$. The particle m cannot be moving, and we can only solve problems involving standing waves for ψ -- i.e. statics problems.