Dirac velocity operator $\mathbf{v} = c\boldsymbol{\alpha}$. QM Eq. of Motion for $\boldsymbol{\alpha}$.

Dirac Equation: Zitter Bewegung.

We now analyse a curious and completely new phenomenon that appears in Dirac theory. It is called Zitter Bewegung (German for "trembling motion"), and it has no classical analogue. We will show that all Dirac particles, even those at rest, exhibit rapid oscillations in space, of amplitude ~ their own Compton wavelength $\hbar/mc$. This oscillation "smears" the particle's position over that dimension, and prevents localizing the particle to better than $\hbar/mc$.

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1) Zitter Bewegung occurs as follows. Provisionally, take the Dirac velocity operator:

$$\rightarrow v_k = c\alpha_k \longleftrightarrow \text{probability current}: J_k = \psi^\dagger v_k \psi. \tag{1}$$

That v_k is a velocity operator is confirmed by the QM Eq. of Motion for position:

$$\frac{d}{dt}x_k = \frac{i}{\hbar}[\mathcal{H}, x_k] \leftarrow \text{let}: \mathcal{H} = \beta mc^2 + c\alpha_j p_j \text{ (free particle)}$$

$$= \frac{i}{\hbar}c\alpha_j[p_j, x_k] \leftarrow \text{but}: [p_j, x_k] = -i\hbar\delta_{jk}$$

$$\xrightarrow{\text{so}} \frac{d}{dt}x_k = c\alpha_j\delta_{jk} = c\alpha_k = v_k \text{, is a velocity operator} \left(\begin{matrix}\text{expectation}\\ \text{value sense}\end{matrix}\right). \tag{2}$$

But **NOTE**: the eigenvalues of α_k are ± 1, so $\langle v_k\rangle = \pm c$. This implies that the (free) particle is always moving randomly at speed c. Needs interpretation!

Look at the QM Eq. of Motion for the α_k. We write (still for a free particle)...

$$\rightarrow \frac{d}{dt}\alpha_k = \frac{i}{\hbar}[\mathcal{H}, \alpha_k] = \frac{i}{\hbar}\left(mc^2[\beta, \alpha_k] + cp_j[\alpha_j, \alpha_k]\right). \tag{3}$$

The [ ]'s here are ordinary commutators. From the anticommutation rules...

$$\left\{\beta, \alpha_k\right\} = 0 \Rightarrow [\beta, \alpha_k] = 2\beta\alpha_k,\ k = 1,2,3;$$

$$\text{and}\quad [\alpha_j, \alpha_k] = \begin{cases} 2\alpha_j\alpha_k, & \text{for } j \neq k, \\ 0, & \text{for } j = k. \end{cases} \tag{4}$$

Use the identities in (4) to process (3)...

Expectation value of $v_k = c\alpha_k$. Appearance of Zitter Bewegung term.

$$\frac{d}{dt}\alpha_k = \frac{2i}{\hbar}(\beta mc^2 + cp_j\alpha_j)\alpha_k \text{, sum over } j \neq k \text{ (w/ } k \text{ fixed)}$$

$$= \frac{2i}{\hbar}(\beta mc^2 + cp_\ell\alpha_\ell - cp_k\alpha_k)\alpha_k \text{, now sum over } \ell = 1,2,3$$

$$\underline{\underline{\frac{d}{dt}\alpha_k = \frac{2i}{\hbar}(\mathcal{H}\alpha_k - cp_k)}}\text{, since } \alpha_k^2 = 1. \qquad (5)$$

Clearly, α_k is not a cnst of the motion for a free particle, so $v_k = c\alpha_k$ can't be the operator for the particle's (group) velocity -- it must have more physics in it.

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2) To see what more physics there is in α_k, consider Eq. (5) as a differential eqtn for α_k [NOTE: the matrices α_k themselves are cnst -- what is changing in time is the expectation value $\langle\alpha_k\rangle$]. For a free particle...

p_k = cnst momentum, $\mathcal{H} = E$ = cnst energy,

so Eq (5) $\Rightarrow$ $\dot{\alpha}_k = \frac{2i}{\hbar}(E\alpha_k - cp_k)$, ‖ for free particle

with solution: $$\boxed{\alpha_k(t) = (cp_k/E) + [\alpha_k(0) - (cp_k/E)]\exp\left[\frac{2i}{\hbar}Et\right].} \qquad (6)$$

But, relativistically (and accurately): $\underline{cp_k/E = \bar{v}_k/c}$, where $\bar{v}_k$ is the actual (cnst) velocity for a free particle of (cnst) momentum p_k & (cnst) energy E. Then, in an expectation value sense, the velocity for a Dirac free particle is...

$$\boxed{v_k(t) = c\alpha_k(t) = \underset{\text{Classical result}}{\bar{v}_k} + \underbrace{[c\alpha_k(0) - \bar{v}_k]\exp\left(\frac{2i}{\hbar}Et\right)}_{\text{Zitter Bewegung term}}}, \quad \bar{v}_k = \frac{c^2 p_k}{E}. \qquad (7)$$

So $v_k = c\alpha_k$ did have a surprise in store: it gives the classical velocity $\bar{v}_k$ plus a new rapidly oscillating add-on term. This Zitter Bewegung term oscillates at very high frequencies: $\omega = 2E/\hbar \sim 2mc^2/\hbar = 2\pi \times 2.5\times 10^{20}$ Hz, e.g. for an electron. The amplitude of this oscillation can be found by integrating the expression for $v_k(t)$ to find the particle's (expected) position...

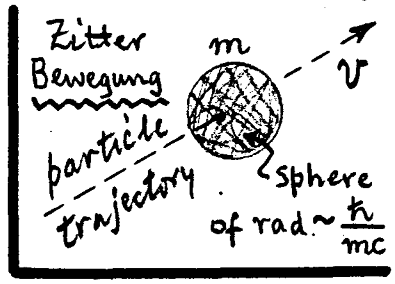
ZitterBewegung Amplitude. Physical manifestation of ZB.

$$x_k(t) = \int_0^t v_k(t')\,dt' = x_k(0) + \overline{v}_k t + [c\alpha_k(0) - \overline{v}_k] \int_0^t e^{(2iE/\hbar)t'}\,dt'$$

or// $\underline{\underline{x_k(t) = \overline{x}_k(t) + \Delta x_k(t)}}$ $\int^{\mathrm{w}/}$ $\overline{x}_k(t) = x_k(0) + \overline{v}_k t$, classical position,

and// $\Delta x_k(t) = A e^{i\omega t} \sin\omega t$, $\omega = E/\hbar \gtrsim mc^2/\hbar$ ← ZITTER BEWEGUNG

ZB amplitude: $\underline{A = (\hbar c/E)[\alpha_k(0) - \frac{\overline{v}_k}{c}]}$. (8)



The maximum ZB amplitude is $\langle |A| \rangle = \hbar c/E \sim \hbar/mc$, i.e. about the size of a Compton wavelength (for an electron, $\hbar/mc = 3.86 \times 10^{-11}$ cm. This is a factor $1/\alpha = 137\times$ larger than the classical electron radius $r_e = e^2/mc^2$). The curious picture emerges that even a "free" particle, as it follows a classical trajectory $\overline{x}_k(t)$, constantly executes random oscillatory excursions about that trajectory. These oscillations are of amplitude $\sim \hbar/mc$, so the particle is smeared out over a sphere whose radius is ~ Compton wavelength.

REMARKS The nature of ZB (ZitterBewegung).

1. What drives ZB? One way to think about it is as a manifestation of the Uncertainty Principle. If the particle were actually localized to within $\Delta x \sim \hbar/mc$, this would generate random momentum components of size $\Delta p \sim \hbar/\Delta x \sim mc$. The particle would then move rapidly and randomly within $\Delta x \sim \hbar/mc$.

2. ZB implies measurable physical effects of the following sort. Suppose the particle is in an external potential $V(\mathbf{r})$. During its ZB, it sees an instantaneous value:

$$\rightarrow \underline{V(\mathbf{r}_0 + \Delta\mathbf{r}) = V(\mathbf{r}_0) + \Delta x_k (\partial V/\partial x_k)_0 + \tfrac{1}{2}\Delta x_j \Delta x_k (\partial^2 V/\partial x_j \partial x_k)_0 + \cdots} \quad (9)$$

where $\mathbf{r}_0$ = particle's mean position, and Δx_k = particle's ZB components. Since ZB is a random motion, the Δx_k are random & uncorrelated, and in a time average $\overline{\Delta x_k} = 0$, for example. The time average of the quadratic term in Eq. (9) does not vanish however, so -- because of ZB -- we get a correction to V:

ZB and the Darwin term. ZB for a free-particle wavepacket.

$$\overline{\Delta V} = [V(\mathbf{r}_0+\Delta\mathbf{r}) - V(\mathbf{r}_0)]\Big|_{\text{time avg.}} = \frac{1}{2}\overline{(\Delta x_k)^2}[\partial^2 V/\partial x_k^2]_0 .$$

... but $\overline{(\Delta x_k)^2} = \frac{1}{3}(\hbar/mc)^2$, for random motion of amplitude $\hbar/mc$...

So/ ZB correction is : $\boxed{\overline{\Delta V} = \frac{1}{6}(\hbar/mc)^2 \nabla^2 V}$ Darwin Interaction (10)

The ZB correction is just the Darwin Interaction term we picked up in our previous $O(v/c)^2$ reduction of the Dirac Eqtn in an external field [See p. DE 23, Eq. (13)] (except the factor $\frac{1}{8} \to \frac{1}{6}$ here). The physics of the Darwin term is clearly due to ZB.

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3) We now do a rather elaborate calculation to show that in Dirac theory, the phenomenon of ZB (Zitter Bewegung) originates in the interference between the required (+)ve & (−)ve energy solutions. We do the calculation explicitly for a free particle wavepacket, to trace how the ZB term arises in $\alpha_k(t)$ of Eq. (6).

① Start by writing the free particle plane waves [p. DE 14, Eqs. (7A) & (7B)] as...

$$\to \psi_{\mathbf{k}}^{(1,2)} = \mathcal{N} U^{(1,2)} e^{i(\mathbf{k}\cdot\mathbf{r} - \omega_p t)} ; \quad \psi_{\mathbf{k}}^{(3,4)} = \mathcal{N} U^{(3,4)} e^{i(\mathbf{k}\cdot\mathbf{r} + \omega_p t)} ;$$

W/ $\mathbf{k} = \mathbf{p}/\hbar$ wavenumber (momentum), $\omega_p = E_p/\hbar = \left[\left(\frac{mc^2}{\hbar}\right)^2 + (c\mathbf{k})^2\right]^{1/2}$ frequency (energy);

&/ $u = \begin{pmatrix}1\\0\end{pmatrix}$, $d = \begin{pmatrix}0\\1\end{pmatrix}$ two component "up" & "down" spinors ; $h = \dfrac{c(\sigma\cdot\mathbf{k})}{\omega_p + (mc^2/\hbar)}$ helicity operator ;

so/ $U_{\mathbf{k}}^{(1)} = \begin{pmatrix}u\\ hu\end{pmatrix}$, $U_{\mathbf{k}}^{(2)} = \begin{pmatrix}d\\ hd\end{pmatrix}$; $U_{\mathbf{k}}^{(3)} = \begin{pmatrix}-hu\\ u\end{pmatrix}$, $U_{\mathbf{k}}^{(4)} = \begin{pmatrix}-hd\\ d\end{pmatrix}$. (11)

With this notation, the normalization in a finite volume V is (μ & ν = 1, 2, 3, 4):

$$\to U_{\mathbf{k}}^{(\mu)\dagger} U_{\mathbf{k}}^{(\nu)} = \delta_{\mu\nu} ; \quad \int_V d^3x\, \psi_{\mathbf{k}}^{(\mu)\dagger} \psi_{\mathbf{k}}^{(\nu)} = \delta_{\mu\nu}, \text{ for } \mathcal{N} = \sqrt{\frac{\omega_p + (mc^2/\hbar)}{2\omega_p V}} . \quad (12)$$

② The most general free particle solution is the superposition of these states, i.e.

$$\Psi(\mathbf{r},t) = \sum_{\mathbf{k}} C_{\mathbf{k}}^{(\mu)} \psi_{\mathbf{k}}^{(\mu)}(\mathbf{r},t)$$ summed over all relevant momenta $\mathbf{k}$ (also summed over μ = 1, 2, 3, 4). (13)

with the $C_{\mathbf{k}}^{(\mu)}$ as Fourier-type expansion coefficients. We can make a localized wavepacket for Ψ by suitable choice of the $C_{\mathbf{k}}^{(\mu)}$. Suppose we begin with...

ZB for a free-particle wavepacket (cont'd).

$$\rightarrow \Psi(\mathbf{r},0) = \sum_{\mathbf{k}} \mathcal{N} C_{\mathbf{k}}^{(\mu)} U_{\mathbf{k}}^{(\mu)} e^{i\mathbf{k}\cdot\mathbf{r}} = \begin{bmatrix} w(\mathbf{r}) \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad (14)$$

this represents some (+)ve energy state @ $t=0$. If $w(\mathbf{r})$ is strongly peaked, then particle is localized.

Pick off coefficients $C_{\mathbf{k}}^{(\nu)}$ by operating through by $\int_V d^3x\, U_{\mathbf{k}'}^{(\nu)\dagger} e^{-i\mathbf{k}'\cdot\mathbf{r}}$ (from left):

$$\rightarrow U_{\mathbf{k}'}^{(\nu)\dagger} \int_V d^3x\, e^{-i\mathbf{k}'\cdot\mathbf{r}} \begin{bmatrix} w(\mathbf{r}) \\ 0 \\ 0 \\ 0 \end{bmatrix} = \sum_{\mathbf{k}} \mathcal{N} C_{\mathbf{k}}^{(\mu)} \underbrace{U_{\mathbf{k}'}^{(\nu)\dagger} U_{\mathbf{k}}^{(\mu)}}_{\delta_{\nu\mu}} \underbrace{\int_V d^3x\, e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}}}_{(2\pi)^3\delta(\mathbf{k}-\mathbf{k}')}$$

$$= (2\pi)^3 \mathcal{N} C_{\mathbf{k}'}^{(\mu)} U_{\mathbf{k}'}^{(\nu)\dagger} U_{\mathbf{k}'}^{(\mu)} = (2\pi)^3 \mathcal{N} C_{\mathbf{k}'}^{(\nu)}$$

So//

$$C_{\mathbf{k}}^{(\mu)} = [1/(2\pi)^3\mathcal{N}]\, U_{\mathbf{k}}^{(\mu)\dagger} \int_V d^3x\, e^{-i\mathbf{k}\cdot\mathbf{r}} \begin{bmatrix} w(\mathbf{r}) \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad (15)$$

Each of the four (scalar) coefficients $C_{\mathbf{k}}^{(\mu)}$ depends on the choice $w(\mathbf{r})$ for the initial packet form. But the ratios of the $C_{\mathbf{k}}^{(\mu)}$ do not. We note in particular that...

$$\boldsymbol{\sigma}\cdot\mathbf{k} = \begin{pmatrix} k_3 & k_1 - ik_2 \\ k_1 + ik_2 & -k_3 \end{pmatrix} \Rightarrow \begin{array}{l} C_{\mathbf{k}}^{(3)}/C_{\mathbf{k}}^{(1)} = -(\hbar)_{11} = -ck_3/[\omega_p + (mc^2/\hbar)], \\ C_{\mathbf{k}}^{(4)}/C_{\mathbf{k}}^{(1)} = -(\hbar)_{21} = -c(k_1+ik_2)/[\omega_p + (mc^2/\hbar)]; \end{array}$$

So//

$$|C_{\mathbf{k}}^{(3)}|^2 + |C_{\mathbf{k}}^{(4)}|^2 = \left(\frac{ck}{\omega_p + (mc^2/\hbar)}\right)^2 |C_{\mathbf{k}}^{(1)}|^2 \sim |C_{\mathbf{k}}^{(1)}|^2, \text{ when } kc \rightarrow \omega_p. \quad (16)$$

This part of the calculation demonstrates the following proposition...

> A (+)ve energy wavepacket $\Psi(\mathbf{r},t)$ must contain (−)ve energy plane wave components [i.e. $C_{\mathbf{k}}^{(3,4)} \neq 0$] in order to be complete. The (−)ve energy components are comparable in size to the (+)ve energy components when Ψ contains Fourier coefficients at momenta $p \sim mc$. Such coefficients are required automatically for initial localizations down to $\Delta x \sim \frac{\hbar}{mc}$. (17)

③ Now we can calculate the expectation value of the Dirac velocity operator α_k w.r.t. a (+)ve energy wavepacket like $\Psi(\mathbf{r},t)$ of Eqs. (13)-(16), keeping in mind that we must retain all four planewave components $\psi_{\mathbf{k}}^{(\mu)}(\mathbf{r},t)$. The integral of interest is...

$$\rightarrow \langle\alpha_k(t)\rangle = \int_V d^3x\, \Psi^\dagger(\mathbf{r},t)\,\alpha_k\,\Psi(\mathbf{r},t) = \sum_{\mathbf{k},\mathbf{k}'} \int_V d^3x \left[C_{\mathbf{k}}^{(\mu)*}\psi_{\mathbf{k}}^{(\mu)\dagger}\right] \alpha_k \left[C_{\mathbf{k}'}^{(\nu)}\psi_{\mathbf{k}'}^{(\nu)}\right]. \quad (18)$$

Split each $\psi_{\mathbf{k}}^{(\mu)}$ & $\psi_{\mathbf{k}'}^{(\nu)}$ into its (+)ve and (−)ve energy parts. Then (18) becomes...

ZB for free-particle wavepacket (cont'd). ZB as an interference term for ±E.

$$\rightarrow \langle \alpha_k(t)\rangle = \sum_{\mathbf{k},\mathbf{k}'} N_{\mathbf{k}} N_{\mathbf{k}'} \int_V d^3x \left[\sum_{\mu=1}^{2} C_{\mathbf{k}}^{(\mu)*} U_{\mathbf{k}}^{(\mu)\dagger} e^{+i\omega_p t} + \sum_{\mu=3}^{4} C_{\mathbf{k}}^{(\mu)*} U_{\mathbf{k}}^{(\mu)\dagger} e^{-i\omega_p t}\right] e^{-i\mathbf{k}\cdot\mathbf{r}} \cdot$$

$$\cdot\, \alpha_k \left[\sum_{\nu=1}^{2} C_{\mathbf{k}'}^{(\nu)} U_{\mathbf{k}'}^{(\nu)} e^{-i\omega_p t} + \sum_{\nu=3}^{4} C_{\mathbf{k}'}^{(\nu)} U_{\mathbf{k}'}^{(\nu)} e^{+i\omega_p t}\right] e^{i\mathbf{k}'\cdot\mathbf{r}}$$

$$= \sum_{\mathbf{k},\mathbf{k}'} N_{\mathbf{k}} N_{\mathbf{k}'} \left[\sum_\mu \sim\right] \alpha_k \left[\sum_\nu \sim\right] \int_V d^3x\, e^{-i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}}$$ the integral gives $(2\pi)^3\delta(\mathbf{k}-\mathbf{k}')$

or/ $$\langle \alpha_k(t)\rangle = (2\pi)^3 \sum_{\mathbf{k}} N_{\mathbf{k}}^2 \left[\sum_\mu \sim\right] \alpha_k \left[\sum_\nu \sim\right]\Big|_{\mathbf{k}'=\mathbf{k}},$$ w/ $$N_{\mathbf{k}}^2 = \frac{\omega_p + (mc^2/\hbar)}{2\omega_p V}. \quad (19)$$

④ The $[\sum_\mu \sim]$ & $[\sum_\nu \sim]$ each have 4 terms, as detailed at top of page. Some of the 16 possible products go like $(e^{+i\omega_p t})(e^{-i\omega_p t}) = 1$ and are time-independent. Other products will go like $e^{\pm 2i\omega_p t}$. A typical time-independent term is...

$$\rightarrow \sum_{\mu,\nu=1}^{2} C_{\mathbf{k}}^{(\mu)*} C_{\mathbf{k}}^{(\nu)} U_{\mathbf{k}}^{(\mu)\dagger} \alpha_k U_{\mathbf{k}}^{(\nu)} = \sum_{\mu,\nu=1}^{2} C_{\mathbf{k}}^{(\mu)*} C_{\mathbf{k}}^{(\nu)} \begin{pmatrix}\varphi_\mu \\ h\varphi_\mu\end{pmatrix}^\dagger \begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0\end{pmatrix} \begin{pmatrix}\varphi_\nu \\ h\varphi_\nu\end{pmatrix}$$ $\varphi_{\mu,\nu} = u$ or d, $h = \dfrac{c\boldsymbol{\sigma}\cdot\mathbf{k}}{\omega_p + (mc^2/\hbar)}$ [from Eq. (11)]

$$= \frac{c\hbar}{\hbar\omega_p + mc^2} \sum_{\mu,\nu=1}^{2} C_{\mathbf{k}}^{(\mu)\dagger} C_{\mathbf{k}}^{(\nu)} \varphi_\mu^\dagger \{\sigma_k(\boldsymbol{\sigma}\cdot\mathbf{k}) + (\boldsymbol{\sigma}\cdot\mathbf{k})\sigma_k\}\varphi_\nu$$

$$= +\left[2cp_k/(E_p + mc^2)\right] \sum_{\mu=1}^{2} |C_{\mathbf{k}}^{(\mu)}|^2.$$

Similarly: $$\sum_{\mu,\nu=3}^{4} C_{\mathbf{k}}^{(\mu)*} C_{\mathbf{k}}^{(\nu)} U_{\mathbf{k}}^{(\mu)\dagger} \alpha_k U_{\mathbf{k}}^{(\nu)} = (-)\left[2cp_k/(E_p + mc^2)\right] \sum_{\mu=3}^{4} |C_{\mathbf{k}}^{(\mu)}|^2. \quad (20)$$

⑤ There are then the cross terms in (19), involving sums like $\sum_{\mu=1}^{2}\sum_{\nu=3}^{4}$ and $\sum_{\mu=3}^{4}\sum_{\nu=1}^{2}$; by inspection, these go like $e^{+2i\omega_p t}$ and $e^{-2i\omega_p t}$ resp. Without reducing the products $U_{\mathbf{k}}^{(\mu)\dagger}\alpha_k U_{\mathbf{k}}^{(\nu)}$ as in Eq. (20), we find for Dirac's velocity operator...

$$\langle \alpha_k(t)\rangle = \sum_{\mathbf{p}} \left[\frac{(2\pi)^3}{V}\sum_{\mu=1}^{2} |C_{\mathbf{p}}^{(\mu)}|^2\right]\left(\frac{cp_k}{+E_p}\right) + \sum_{\mathbf{p}}\left[\frac{(2\pi)^3}{V}\sum_{\mu=3}^{4}|C_{\mathbf{p}}^{(\mu)}|^2\right]\left(\frac{cp_k}{-E_p}\right) + \quad (21)$$

$$+ \sum_{\mathbf{p}}\left[\frac{(2\pi)^3}{V}\left(\frac{E_p + mc^2}{2E_p}\right)\right]\left\{\sum_{\substack{\mu=1,2\\ \nu=3,4}} \left(C_{\mathbf{p}}^{(\mu)*} C_{\mathbf{p}}^{(\nu)} U_{\mathbf{p}}^{(\mu)} \alpha_k U_{\mathbf{p}}^{(\nu)}\right) e^{+2i\omega_p t} + \sum_{\substack{\mu=3,4\\ \nu=1,2}} \left(C_{\mathbf{p}}^{(\mu)*} C_{\mathbf{p}}^{(\nu)} U_{\mathbf{p}}^{(\mu)\dagger}\alpha_k U_{\mathbf{p}}^{(\nu)}\right) e^{-2i\omega_p t}\right\}.$$

$\boxed{\text{ZB terms}}$

The first two terms RHS in (21) are t-indept, and are the group velocities $\pm \frac{cp_k}{E_p}$ of the (±)ve energy components of the packet separately, weighted by their original strengths [per Eq. (15)]. The ZB terms, in $e^{\pm 2i\omega_p t}$, result from the cross terms between (+)ve energy states $\{{\mu=1,2 \atop \nu=1,2}\}$ and (−)ve energy states $\{{\nu=3,4 \atop \mu=3,4}\}$. They are the ± energy interference terms cited on p. DE 27.