Conversion of Dirac Eqtn for (q,m) in external A_μ to a 2nd order eqtn.

Dirac Equation: Particle in an External Field.

We have previously looked at the Hamiltonian form of the Dirac Eqtn in an external field $A_\mu = (\mathbb{A}, i\phi)$ [pp. DE 20-23], up to identifying the coupling of a Dirac particle with an external magnetic field (w/ intrinsic $g=2$), the spin orbit term (w/ correct Thomas precession factor), and the Darwin Interaction (related to Zitterbewegung, p. DE 27, Eq. (10)). These features were picked up by approxns to $O(v/c)^2$; here we want to do a covariant version of the A_μ problem.

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1) The transition from the free-particle Dirac Eqtn, viz...

$$\underline{(\gamma_\nu p_\nu - imc)\,\psi = 0} \leftarrow \text{free particle } m, \text{ w/ } p_\nu = -i\hbar\,\partial/\partial x_\nu, \tag{1}$$

to the eqtn for (q,m) in an external field $A_\mu = (\mathbb{A}, i\phi)$ is accomplished via...

$$\rightarrow p_\nu \rightarrow \underline{\pi_\nu = p_\nu - (q/c)A_\nu} = -i\hbar\,[(\partial/\partial x_\nu) - i(q/\hbar c)A_\nu];$$

$$\text{So}/\!/ \quad \underline{(\gamma_\nu \pi_\nu - imc)\,\psi = 0} \leftarrow (q,m) \text{ in } (\mathbb{A}, i\phi), \tag{2}$$

as usual. In order to solve for the bispinors φ & χ of $\psi = \begin{pmatrix}\varphi\\ \chi\end{pmatrix}$, we must generate a 2nd-order differential eqtn. We can do this by...

$$\left[\begin{matrix}\text{multiply Eq. (2) on left}\\ \text{by: } (\gamma_\mu \pi_\mu + imc)\end{matrix}\right\} \quad \underline{[\gamma_\mu\gamma_\nu\pi_\mu\pi_\nu + (mc)^2]\,\psi = 0}. \tag{3}$$

The product $\gamma_\mu\gamma_\nu$ in Eq. (3) here is conveniently rewritten in terms of the $\sigma_{\mu\nu}$ matrix defined in Eq. (25), p. DE 35, viz...

$$\left[\begin{matrix}\rightarrow \sigma_{\mu\nu} = -\frac{1}{2}i(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu)\\ \text{and}/\!/ \quad \gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2\delta_{\mu\nu}\end{matrix}\right\} \Rightarrow \underline{\gamma_\mu\gamma_\nu = \delta_{\mu\nu} + i\sigma_{\mu\nu}} \quad \left\{\begin{matrix}\sigma_{\nu\mu} = (-)\sigma_{\mu\nu},\\ \sigma_{\mu\mu} = 0,\\ \sigma_{\mu\nu}^2 = 1.\end{matrix}\right. \tag{4}$$

Use of this form for $\gamma_\mu\gamma_\nu$ in Eq. (3) yields the 2nd order Dirac Eqtn...

$$\underline{\{[\pi_\mu^2 + (mc)^2] + i\sigma_{\mu\nu}\pi_\mu\pi_\nu\}\,\psi = 0.} \tag{5}$$

Derivation of 2nd order Dirac Eqtn (cont'd).

2) NOTE: in Eq (5), the $[\pi_\mu^2 + (mc)^2]$ (× 4 by 4 identity matrix) is the Klein-Gordon operator [ref. Eq (12), p. fs 17 of class notes]. The term in $\sigma_{\mu\nu}\pi_\mu\pi_\nu$ is an add-on, specific to Dirac theory. We process this add-on as follows...

$$\rightarrow \sigma_{\mu\nu}\pi_\mu\pi_\nu = \frac{1}{2}(\sigma_{\mu\nu}\pi_\mu\pi_\nu + \sigma_{\nu\mu}\pi_\nu\pi_\mu) \leftarrow \text{by interchanging indices}$$

$$= \frac{1}{2}(\sigma_{\mu\nu}\pi_\mu\pi_\nu - \overset{\text{antisym}}{\sigma_{\mu\nu}}\pi_\nu\pi_\mu) = \underline{\frac{1}{2}\sigma_{\mu\nu}[\pi_\mu, \pi_\nu]}; \tag{6}$$

... but: $$[\pi_\mu, \pi_\nu] = \left[-i\hbar\frac{\partial}{\partial x_\mu} - \frac{q}{c}A_\mu, -i\hbar\frac{\partial}{\partial x_\nu} - \frac{q}{c}A_\nu\right]$$

$$= i\hbar\frac{q}{c}\left\{\left[\frac{\partial}{\partial x_\mu}A_\nu - \frac{\partial}{\partial x_\nu}A_\mu\right] - \left[A_\nu\frac{\partial}{\partial x_\mu} - A_\mu\frac{\partial}{\partial x_\nu}\right]\right\}\ldots$$

so// $$[\pi_\mu, \pi_\nu]f = i\hbar\frac{q}{c}\left[\left(\frac{\partial A_\nu}{\partial x_\mu}\right) - \left(\frac{\partial A_\mu}{\partial x_\nu}\right)\right]f, \text{ w.r.t. fcns } f; \tag{7}$$

... but: $\underline{F_{\mu\nu} = (\partial A_\nu/\partial x_\mu) - (\partial A_\mu/\partial x_\nu)}$, is the EM field tensor★...

$$\rightarrow \text{so// } [\pi_\mu, \pi_\nu] = (i\hbar q/c)F_{\mu\nu}, \tag{8}$$

and// $$\boxed{i\sigma_{\mu\nu}\pi_\mu\pi_\nu = -(q\hbar/2c)\sigma_{\mu\nu}F_{\mu\nu}}. \tag{9}$$

Using Eq.(9), we can write the 2nd order Dirac Eqtn in Eq.(5) as...

$$\Big\{\underbrace{[\pi_\mu^2 + (mc)^2]}_{\text{KG operator}} - \underbrace{(q\hbar/2c)\sigma_{\mu\nu}F_{\mu\nu}}_{\text{Dirac add-on (due to spin)}}\Big\}\psi = 0. \tag{10}$$

★ With the 4-vector convention in use [$x_\mu = (\mathbf{r}, ict)$, etc.], the field tensor is:

$$F_{\mu\nu} = \begin{bmatrix} 0 & B_3 & -B_2 & -iE_1 \\ -B_3 & 0 & B_1 & -iE_2 \\ B_2 & -B_1 & 0 & -iE_3 \\ iE_1 & iE_2 & iE_3 & 0 \end{bmatrix}$$

See e.g. Jackson "Classical ElectroDynamics" (Wiley, 1st ed., 1962), p. 379 Eq. (11.108). $F_{\mu\nu}$ is still 4×4 and antisymmetric. The fields are defined by $A_\mu = (\mathbf{A}, i\phi)$ in the usual manner, as: $\mathbf{E} = -\nabla\phi - \frac{1}{c}(\partial\mathbf{A}/\partial t)$, $\mathbf{B} = \nabla\times\mathbf{A}$. Maxwell's Eqs. are: $\partial F_{\mu\nu}/\partial x_\nu = \frac{4\pi}{c}J_\mu$.

Derivation of 2nd order Dirac Eqtn (cont'd).

3) The sum $\sigma_{\mu\nu}F_{\mu\nu}$ in Eq. (10) can be further processed. Write...

$$\rightarrow \sigma_{\mu\nu}F_{\mu\nu} = \sigma_{ij}F_{ij} + (\sigma_{k4}F_{k4} + \sigma_{4k}F_{4k}) \tag{11}$$

both $\sigma_{\mu\nu}$ & $F_{\mu\nu}$ are antisym $\Rightarrow$ ( ) $= 2\sigma_{k4}F_{k4}$.

As we have previously seen [Eq. (32), p. DE 37]...

$$\rightarrow \sigma_{ij} = \epsilon_{ijk}\begin{pmatrix}\sigma_k & 0\\ 0 & \sigma_k\end{pmatrix}$$

ϵ_{ijk} = Levi-Civita symbol $= \begin{cases} +1, \text{ for } ijk = \overrightarrow{123}; \\ -1, \text{ for } ijk = \overrightarrow{132}; \\ 0, \text{ otherwise.}\end{cases}$

so//

$$\sigma_{ij}F_{ij} = \begin{pmatrix}\sigma_k & 0\\ 0 & \sigma_k\end{pmatrix}\left(\epsilon_{ijk}\frac{\partial A_j}{\partial x_i} - \epsilon_{ijk}\frac{\partial A_i}{\partial x_j}\right). \tag{12}$$

But: $\epsilon_{ijk}(\partial A_j/\partial A_i) = +\epsilon_{kij}(\partial A_j/\partial x_i) = (\nabla\times\mathbb{A})_k = B_k$;

$\epsilon_{ijk}(\partial A_i/A_j) = -\epsilon_{kji}(\partial A_j/\partial x_i) = -(\nabla\times\mathbb{A})_k = -B_k$;

$$\xrightarrow{\text{So//}} \sigma_{ij}F_{ij} = 2\begin{pmatrix}\sigma_k B_k & 0\\ 0 & \sigma_k B_k\end{pmatrix} = 2\begin{pmatrix}\boldsymbol{\sigma}\cdot\mathbb{B} & 0\\ 0 & \boldsymbol{\sigma}\cdot\mathbb{B}\end{pmatrix}. \tag{13}$$

At this point, Eq. (11) looks like...

$$\rightarrow \sigma_{\mu\nu}F_{\mu\nu} = 2\begin{pmatrix}\boldsymbol{\sigma}\cdot\mathbb{B} & 0\\ 0 & \boldsymbol{\sigma}\cdot\mathbb{B}\end{pmatrix} + 2\sigma_{k4}F_{k4}. \tag{14}$$

The 2nd term RHS is now calculated [see Eq. (39), p. DE 38]:

$$\rightarrow \sigma_{k4} = \begin{pmatrix}0 & \sigma_k\\ \sigma_k & 0\end{pmatrix},\ F_{k4} = -iE_k \Rightarrow \sigma_{k4}F_{k4} = -i\begin{pmatrix}0 & \boldsymbol{\sigma}\cdot\mathbb{E}\\ \boldsymbol{\sigma}\cdot\mathbb{E} & 0\end{pmatrix}.$$

so//

$$\underline{\underline{\frac{1}{2}\sigma_{\mu\nu}F_{\mu\nu} = \begin{pmatrix}\boldsymbol{\sigma}\cdot\mathbb{B} & -i\boldsymbol{\sigma}\cdot\mathbb{E}\\ -i\boldsymbol{\sigma}\cdot\mathbb{E} & \boldsymbol{\sigma}\cdot\mathbb{B}\end{pmatrix}}} \tag{15}$$

$\boldsymbol{\sigma}$ is the Pauli 2×2 matrices: $\underline{\boldsymbol{\sigma} = \left[\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}, \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\right]}$.

Use this result in Eq. (10). After dividing by $2m$, we can write...

$$\boxed{\left\{\underbrace{\frac{1}{2m}\left[\pi_\mu^2 + (mc^2)\right]}_{\text{KG term}} - \underbrace{\mu_0\begin{pmatrix}\boldsymbol{\sigma}\cdot\mathbb{B} & -i\boldsymbol{\sigma}\cdot\mathbb{E}\\ -i\boldsymbol{\sigma}\cdot\mathbb{E} & \boldsymbol{\sigma}\cdot\mathbb{B}\end{pmatrix}}_{\text{Dirac add-on}}\right\}\psi = 0} \tag{16}$$

This is a practical (exact) version of the 2nd order Dirac Eqtn. It shows by how much the Klein-Gordon Eqtn missed describing the electron. The Dirac add-on is entirely connected with the electron spin $\mathbb{S} = \frac{1}{2}\hbar\boldsymbol{\sigma}$, which KG doesn't do.