* * * * * * * * * * * *

7. A system of three spin-1/2 particles is pairwise exchange-coupled according to the Ising-model Hamiltonian.

$$H = -JS_1^z S_2^z - j(S_1^z S_3^z + S_2^z S_3^z); \quad |J| >> |j|.$$

a) What must be the signs of J and j if the ground state is ferromagnetic?

b) Determine the eigenstates and their energies for arbitrary J and j.

c) Sketch the energy-level diagram for $J=2j>0$. Also for $J=-2j>0$. What is peculiar about the case $J<0$?

d) Sketch the Zeeman splitting of the ground state in the two cases where $J>0$. If the selection rules for an experiment are $\Delta M = \pm 1$, which case would have detectable transitions within the ground state at very low temperatures?

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Note – the problem was revised. This solution was written for the original version.

a.) $\mathcal{H} = -J\, S_1^z S_2^z - j\,(S_1^z S_3^z + S_2^z S_3^z)$

makes "parallel" arrangement of spins lowest in energy.

b.) States:

1. $|{+}{+}{+}\rangle$ $E_1 = -\frac{J}{4} - \frac{j}{4} - \frac{j}{4} = -\frac{J}{4} - \frac{j}{2}$, $M_s = \frac{3}{2}$
2. $|{+}{+}{-}\rangle$ $E_2 = -\frac{J}{4} + \frac{j}{4} + \frac{j}{4} = -\frac{J}{4} + \frac{j}{2}$, $M_s = \frac{1}{2}$
3. $|{+}{-}{+}\rangle$ $E_3 = \frac{J}{4}$ $M_s = \frac{1}{2}$
4. $|{-}{+}{+}\rangle$ $E_4 = \frac{J}{4}$ $M_s = \frac{1}{2}$
5. $|{-}{-}{+}\rangle$ $E_5 = -\frac{J}{4} + \frac{j}{4} + \frac{j}{4} = -\frac{J}{4} + \frac{j}{2}$ $M_s = -\frac{1}{2}$
6. $|{-}{+}{-}\rangle$ $E_6 = \frac{J}{4}$ $M_s = -\frac{1}{2}$
7. $|{+}{-}{-}\rangle$ $E_7 = \frac{J}{4}$ $M_s = -\frac{1}{2}$
8. $|{-}{-}{-}\rangle$ $E_8 = -\frac{J}{4} - \frac{j}{4} - \frac{j}{4} = -\frac{J}{4} - \frac{j}{2}$ $M_s = -\frac{3}{2}$

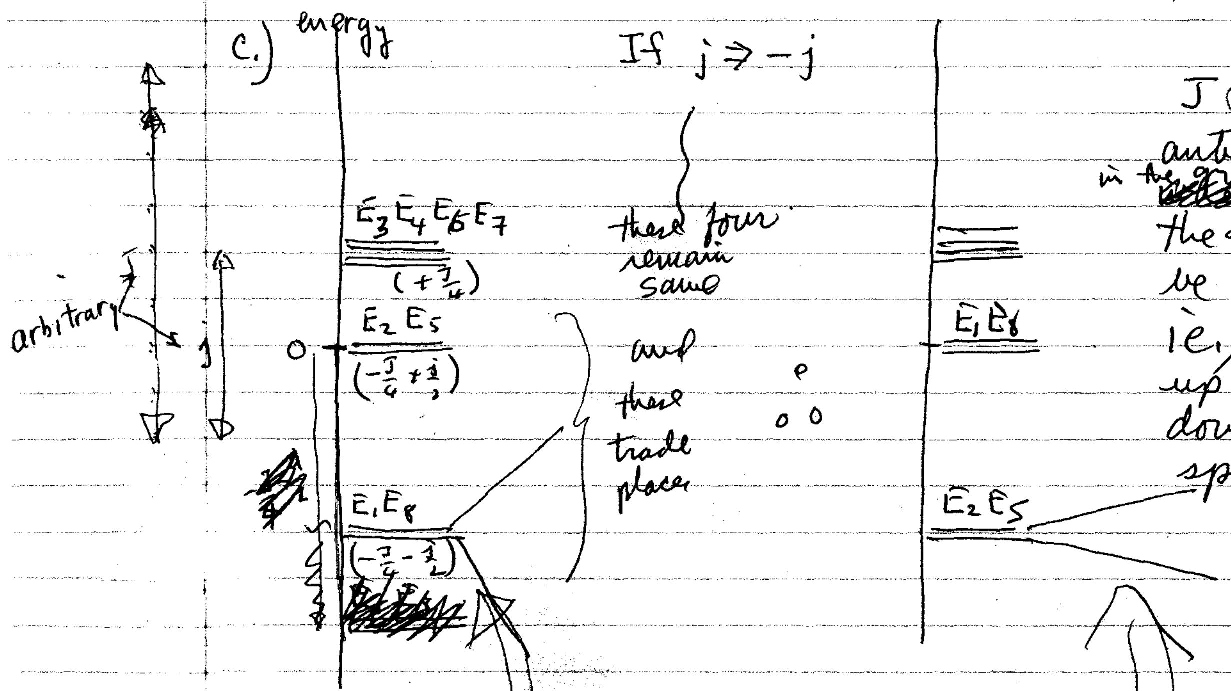
degen.

degen.

degen.

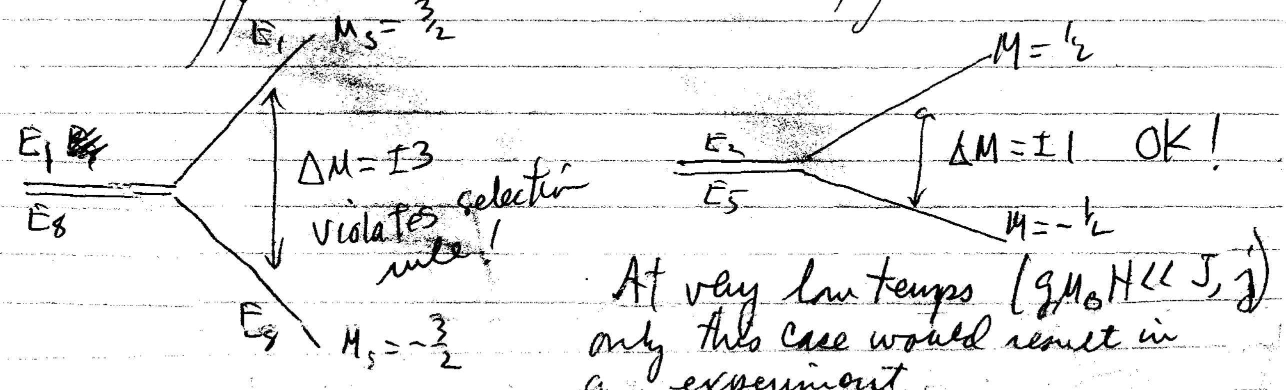
gd. st. has extra degen.

c.)



J cannot be antiferromagnetic in the ground state because the system would be "frustrated", ie, if spin 1 is up and spin 2 is down, what could spin 3 do?

d.) Add $g\mu_B H M_s$



At very low temps ($g\mu_B H \ll J, j$) only this case would result in an experiment.

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8. An electron in the spinor state at t=0,

$$[\chi_o] = \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \end{pmatrix} e^{i\vec{k}\cdot\vec{r}} \qquad [\theta,\ \vec{k} = \text{constant}]$$

is subjected to a uniform magnetic field B in the z-direction, i.e. the quantization axis.

a) Interpret the state $[\chi_o]$.

b) Determine $[\chi]$ for $t>0$ and interpret it.

8. <u>QM Hermanson</u>

"key" "are"

An electron in the Spinor state at $t=0$,

chi

$$[\chi_0] = \begin{pmatrix} [redacted] \\ [redacted] \end{pmatrix} e^{i\underset{\sim}{k}\cdot\underset{\sim}{r}} \qquad [\theta, \underset{\sim}{k} = \text{constant}]$$

is subjected to a uniform magnetic field B in the z direction, i.e. the quantization axis.

a) Interpret the state $[\chi_0]$.

END b) Determine $[\chi]$ for $t>0$ and interpret it.

Sol'n : a) Compute spin projections

$$\langle S_x \rangle = \left(\cos\frac{\theta}{2}, \sin\frac{\theta}{2}\right) \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \end{pmatrix}$$

$$= \frac{\hbar}{2}\left(\cos\frac{\theta}{2}\sin\frac{\theta}{2} + \sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)$$

$$= \frac{\hbar}{2}\sin\theta$$

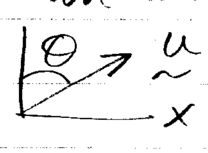
$$\langle S_y \rangle = \frac{\hbar}{2}\left(\cos\frac{\theta}{2}, \sin\frac{\theta}{2}\right) \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \end{pmatrix}$$

$$= 0$$

$$\langle S_z \rangle = \frac{\hbar}{2}\left(\cos\frac{\theta}{2}, \sin\frac{\theta}{2}\right) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \end{pmatrix}$$

$$= \frac{\hbar}{2}\cos\theta$$

The spin is aligned along an axis $\underset{\sim}{u}$ in the xz plane and moves along $\underset{\sim}{k}$.



b) when $B \neq 0$, $H = \frac{p^2}{2m} + H'$

$$H' = -M_z B$$

$$= -g\left(\frac{\mu_B}{\hbar}\right) S_z B \quad ; \quad \mu_B = \frac{q\hbar}{2m_e} < 0 \text{ for } e^-$$

$$= -\left(\frac{g}{2}\right) \frac{qB}{m_e} S_z$$

$$= \omega_0 S_z \quad ; \quad \omega_0 = -\left(\frac{g}{2}\right)\frac{qB}{m_e} > 0$$

Now $[\chi] = e^{-iHt/\hbar} \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \end{pmatrix} e^{i\underset{\sim}{k}\cdot\underset{\sim}{r}}$

$$= e^{i(\underset{\sim}{k}\cdot\underset{\sim}{r} - E_k t/\hbar)} e^{-i\omega_0\left(\frac{S_z}{\hbar}\right)t} \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \end{pmatrix} \begin{matrix} \leftarrow S_z = \frac{\hbar}{2} \\ \leftarrow S_z = -\frac{\hbar}{2} \end{matrix}$$

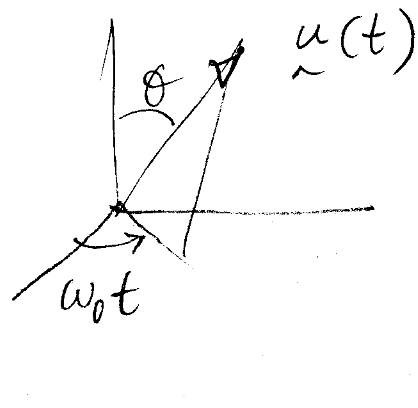
$$= e^{i\phi} \begin{pmatrix} \cos\frac{\theta}{2}\, e^{-i\omega_0 t/2} \\ \sin\frac{\theta}{2}\, e^{i\omega_0 t/2} \end{pmatrix} \quad ; \quad \phi = \underset{\sim}{k}\cdot\underset{\sim}{r} - \frac{\hbar k^2 t}{2m}$$

And $\langle S_x \rangle = \frac{\hbar}{2} \sin\theta \cos\omega_0 t$

$$\langle S_y \rangle = \frac{\hbar}{2} \sin\theta \sin\omega_0 t$$

$$\langle S_z \rangle = \frac{\hbar}{2} \cos\theta$$

The spin precesses about B with frequency ω_0:



DEPARTMENT of PHYSICS

PH.D. COMPREHENSIVE EXAMINATION

TUESDAY, SEPT. 25, 1984, 9-12 AM

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Answer each of the following questions. Each question carries equal weight. Begin your answer to each question on a <u>new</u> sheet of paper; solutions to different questions must <u>not</u> appear on the same sheet of paper. Label <u>each page</u> of your exam as follows:

A. Your name in upper left-hand corner.

B. Problem number, and page number <u>for that problem</u>, in upper right hand corner.

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9. In a crystal lattice of N sites, m Schottky defects are formed, each defect corresponding to one of the N original atoms being removed from the lattice and leaving a vacancy behind. An energy ε ($\varepsilon>0$) is required to form one such defect.

a) Find the average number of Schottky defects when the crystal is at temperature T.

b) Calculate the contribution to the specific heat associated with defect creation.

c) Suppose a volume decrease δ is associated with the formation of each defect. Find the equilibrium volume at temperature T and pressure p, assuming the volume of the perfect crystal is V_o.

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9. In a crystal lattice of N sites, m Schottky defects are formed, each defect corresponding to one of the N original atoms being removed from the lattice and leaving a vacancy behind. An energy ϵ $(\epsilon > 0)$ is required to form one such defect.

uc lc

lc epsilon ε on spinwriter

a) Find the average number of Schottky defects when the crystal is at temperature T. uc

b) Calculate the contribution to the specific heat associated with defect creation.

l.c. delta

c) Suppose a volume decrease δ ~~were~~ is associated with the formation of each defect. Find the equilibrium volume at temperature T and pressure p, assuming the volume of the perfect crystal is V_0.

uc lc

u.c vee

END

Soln: a) Calculate partition fcn., realizing that there are $\binom{N}{n}$ ways of picking the n atoms to be removed:

$$Z = \sum_{n=0}^{N} \binom{N}{n} e^{-n\epsilon/kT} = \left(1 + e^{-\epsilon/kT}\right)^N$$

$$\langle n \rangle = \frac{\partial}{\partial(-\epsilon/kT)} \ln Z = \frac{N \, e^{-\epsilon/kT}}{1 + e^{-\epsilon/kT}} = \frac{N}{1 + e^{\epsilon/kT}}$$

Or calculate $F = U - TS = n\epsilon_0 - T k_B \ln \binom{N}{n}$

$\langle n \rangle$ found from $\left.\frac{\partial F}{\partial n}\right|_{n=\langle n \rangle} = 0$, so $0 = \epsilon_0 - k_B T \ln \frac{N - \langle n \rangle}{\langle n \rangle}$

using Stirling's approx.

b) either write $F = -k_B T \ln Z = -Nk_B T \ln(1+e^{-\epsilon/kT})$

$$\text{\& use } C = T\frac{\partial S}{\partial T} = -T\frac{\partial^2 F}{\partial T^2}$$

or use $C = \frac{\partial U}{\partial T} = \frac{\partial}{\partial T}\langle n\rangle \epsilon$

$$= \frac{\partial}{\partial T}\left(\frac{N\epsilon}{1+e^{\epsilon/kT}}\right) = Nk_B\left(\frac{\epsilon}{kT}\right)^2 \frac{e^{\epsilon/kT}}{\left(1+e^{\epsilon/kT}\right)^2}$$

c) Now

$$Z = e^{-p\frac{V_0}{kT}}\left(1+e^{-\epsilon/kT + p\delta/kT}\right)^N$$

$$\text{\&} \quad G = pV_0 - Nk_B T \ln\left(1+e^{-\epsilon/k_BT + p\delta/k_BT}\right).$$

(where, since the free energy is explicitly a fcn of p, we denote it G for Gibbs)

$$V = \frac{\partial G}{\partial p} = V_0 - \frac{N\delta}{1+e^{\epsilon/kT - p\delta/kT}}$$

10. N atoms of a monatomic gas in a box of volume V have a Maxwell Boltzmann velocity distribution

$$n_o(v) = \frac{N}{V}\left(\frac{m}{2\pi k_B T}\right)^{3/2} 4\pi\ v^2 e^{-mv^2/2k_B T}$$

where $n_o(v)\,dv$ is the number density of atoms with speeds in the interval dv at v, T=absolute temperature, M=mass of each atom, and k_B=Boltzmann's constant. A small hole is made in the box, so that atoms can leak out.

a) Find an expression for the velocity distribution $n'(v)$ of escaping atoms - i.e. the number (per unit time and unit surface area of the hole) escaping with speeds in the interval dv at v. Explain qualitatively why n' differs in functional form from n_o.

b) Find the rms velocity of escaping atoms, and compare it with the rms velocity of atoms inside the container. Based on your result, explain whether the remaining gas will become hotter or colder.

~~Kinetic Theory~~ (G.T.) RC

~~or Stat. Mech~~ uc "vee"

10. uc · N atoms of a monatomic gas in a box of volume V have a Maxwell-Boltzmann velocity distribution

1 space gap

$$n(v) = \left(\frac{N}{V}\right)\left(\frac{m}{2\pi k_B T}\right)^{3/2} 4\pi \; v^2 e^{-mv^2/2k_BT}$$

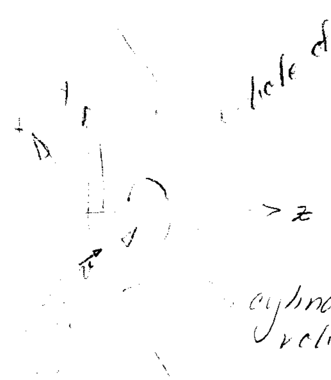
where $n(v)\,dv$ is the number density of atoms with speeds in the interval dv of v, T = absolute temperature, m = mass of each atom and k_B = Boltzmann's constant. A small hole is made in the box, so that atoms can leak out.

a) Find an expression for the velocity distribution $n'(v)$ of escaping atoms – ie, the number (per unit time and unit surface area of the hole) escaping with speeds in the interval dv of v. Explain qualitatively why n' differs in functional form from n. ~~[note: different units, JH]~~

b) Find the rms velocity of escaping ~~atom~~s, and compare it with the rms velocity of atoms inside the container. Based on your result, explain whether ~~will~~ the remaining gas will become hotter or colder.

END

Soln:



In time dt, all atoms with velocity $\vec{v}$ will escape through the hole provided they lie in the slanted cylinder shown at left, with volume $A v_z\,dt$. Thus we need to multiply n_0 by $A v_z\,dt$ to get the number escaping in time dt with vector velocity $\vec{v}$. We then integrate over all angles of $\vec{v}$, subject to the restriction $v_z > 0$. In polar coords, this amounts to $0 \le \varphi < 2\pi$, $0 \le \theta < \pi/2$.

$$n'(v)\, dt\, A = \int_0^{\pi} d\varphi \int_0^{\pi/2} d\theta (\sin\theta)\, v\cos\theta \quad n_0(v)\ A\, dt$$

where the 1st factor of $\cos\theta$ comes from the transcription to polar coordinates, and the 2nd from v_z. Thus.

$$n'(v) = \frac{n_0(v)\, v}{4}$$

This differs from n_0 for the physical reason that faster-moving atoms strike the walls more often, and therefore are more likely to escape.

b) $$\langle v^2\rangle_{esc} = \frac{\int_0^\infty n'(v)\, v^2\, dv}{\int_0^\infty n'(v)\, dv} = \frac{\int_0^\infty v^5 e^{-mv^2/2kT}\, dv}{\int_0^\infty v^3 e^{-mv^2/2kT}\, dv}$$

$$= \left(\frac{2kT}{m}\right) \frac{\int_0^\infty x^5 e^{-x^2}\, dx}{\int_0^\infty x^3 e^{-x^2}\, dx} = \left(\frac{2kT}{m}\right)\cdot 2 = \frac{4kT}{m}$$

so for escaping atoms $\boxed{v_{rms} = \sqrt{\frac{4kT}{m}}}$

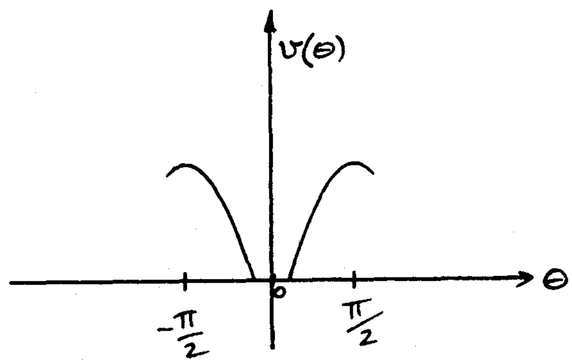
$$\langle v^2\rangle_{cont.} = \frac{\int_0^\infty n_0(v)\, v^2\, dv}{\int_0^\infty n_0(v)\, dv} = \frac{\int_0^\infty v^4 e^{-mv^2/2kT}\, dv}{\int_0^\infty v^2 e^{-mv^2/2kT}} = \left(\frac{2kT}{m}\right)\frac{\int_0^\infty x^4 e^{-x^2}\, dx}{\int_0^\infty x^2 e^{-x^2}\, dx}$$

$$= \left(\frac{2kT}{m}\right)\cdot\frac{3}{2}$$

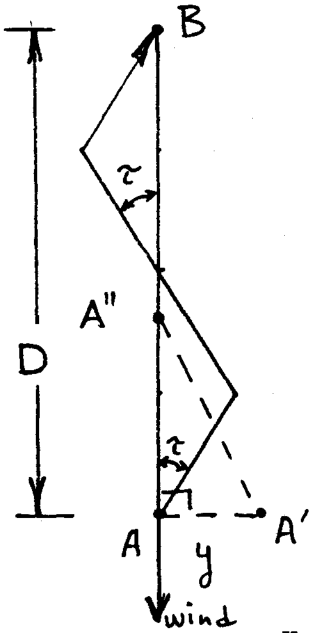
so for atoms in the container $\boxed{v_{rms} = \sqrt{\frac{3kT}{m}}}$

The slower atoms are left behind, and the container cools.

11. A sailboat sails at speed $v(\theta) = -\alpha+\beta \sin|\theta|$ when heading at an angle θ to the wind. $\alpha>0$, $\beta>0$

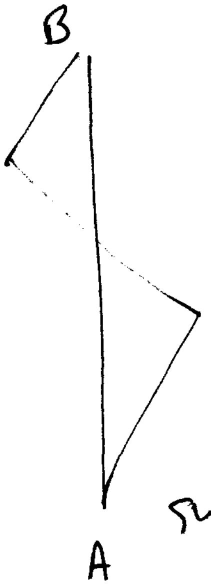


Find the optimum tacking angles τ in order to sail straight upwind at the fastest rate from A to B.



Starting from A' instead of A ($\frac{Y}{D} < \tan\tau$), show that the fastest way to get to B is still to tack at angles τ. (Hint: calculate time to sail from A' to A'' on line AB plus time to sail from A'' to B by optimum tacking, and minimize.)

Solution



$$t = \frac{D}{v(\theta)\cos\theta} \qquad \frac{dt}{d\theta} = 0$$

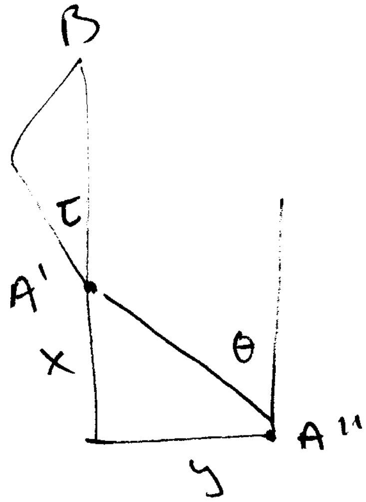
$$\frac{v'(\theta)}{v(\theta)} = \frac{\sin\theta}{\cos\theta} = \frac{\beta\cos\theta}{-\alpha+\beta\sin\theta}$$

$$\sin\theta\left[-\alpha+\beta\sin\theta\right] = \beta\left(1-\sin^2\theta\right)$$

$$2\beta\sin^2\theta - \alpha\sin\theta - \beta = 0$$

$$\sin^2\theta - \frac{\alpha}{2\beta}\sin\theta - \frac{1}{2} = 0$$

$$\boxed{\sin\tau = \frac{\alpha}{4\beta} + \sqrt{\frac{1}{2}+\left(\frac{\alpha}{4\beta}\right)^2}}$$



$$t = \frac{x}{v(\theta)\cos\theta} + \frac{D-x}{v(\tau)\cos\tau} \qquad \frac{y}{x} = \tan\theta$$

$$\frac{dt}{dx} = \left(\frac{1}{v(\theta)\cos\theta} - \frac{1}{v(\tau)\cos\tau}\right) - \frac{x}{v(\theta)\cos\theta}\frac{d}{d\theta}\left(v\cos\theta\right)\frac{d\theta}{dx}$$

$$= 0$$

$\theta = \tau$ fulfills $\frac{dt}{dx} = 0$!

12. ^3H nuclei collide with ^4He nuclei to produce ^6Li nuclei plus neutrons ^1n. Find the kinetic energy threshold for this reaction in the lab frame where the helium nuclei are the targets. Mass defects of the nuclei are:

$\Delta M(^3H) = 15.84$ Mev/c^2

$\Delta M(^4He) = 3.61$ Mev/c^2

$\Delta M(^1n) = 8.37$ Mev/c^2

$\Delta M(^6Li) = 15.86$ Mev/c^2

(The mass defect of a nucleus is:

$$\Delta M = M(A,Z) - A\mu,$$

where $M(A,Z)$ is the actual mass of the nucleus, A is the number of baryons, and μ is the nuclear mass unit.)

Solution: Cons. of energy + momentum in Lab

$$\frac{p^2}{2M_1} + M_1 c^2 + M_2 c^2 = \frac{p^2}{2(M_3+M_4)} + (M_3+M_4)c^2$$

$$\frac{p^2}{2M_1}\left\{1 - \frac{M_1}{M_3+M_4}\right\} = (M_3+M_4-M_1-M_2)\,c^2$$

$$M_1/(M_3+M_4) \cong 3/7$$

$$\frac{p^2}{2M_1} = \frac{7}{4}\left(15.86+8.37-3.61-15.84\right) \text{ MeV}$$

DEPARTMENT of PHYSICS

PH.D. COMPREHENSIVE EXAMINATION

TUESDAY, SEPT. 25, 1984, 2-5 PM

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Answer each of the following questions. Each question carries equal weight. Begin your answer to each question on a new sheet of paper; solutions to different questions must not appear on the same sheet of paper. Label each page of your exam as follows:

A. Your name in upper left-hand corner.

B. Problem number, and page number for that problem, in upper right hand corner.

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13. A rocket has engines which give it a constant acceleration of one g relative to its instantaneous inertial frame as measured by an accelerometer attached to the rocket. The rocket starts from rest near the earth. Ignore all gravitational effects.

Compute the proper time (τ) for the occupants of the rocket ship to travel the 30,000 light years from the earth to the center of the galaxy, assuming that they accelerate at one g for half the trip and decelerate at one g for the remaining half.

Suggestions:

Use the velocity and acceleration four-vectors. Note that $u^\alpha = dx^\alpha/d\tau$, $a^\alpha = du^\alpha/d\tau$, and that $g \approx 1$ year^{-1} in units where $c=1$. Also note that the four-velocity and four acceleration are perpendicular.

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Problem (1) solution

(a) Take the rocket's motion to be along the x-axis. Let t be Earth time and τ proper "ship" time. The initial condition of rest near the Earth is then that

$$u^x = 0 \quad \text{at } t=0 \text{ and/or } \tau = 0$$

We have the following equations for the four-velocity, $\vec{u}$ and four-acceleration $\vec{a}$:

$$\vec{u}\cdot\vec{u} = -1 = -(u^t)^2 + (u^x)^2 \quad \text{(normalization of four velocity)} \quad (1)$$

$$\vec{u}\cdot\vec{a} = 0 = -a^t u^t + a^x u^x \quad (\vec{a} \text{ orthogonal to } \vec{u}) \quad (2)$$

$$\vec{a}\cdot\vec{a} = g^2 = -(a^t)^2 + (a^x)^2 \quad \text{(proper acceleration is } g) \quad (3)$$

(2) $\Rightarrow$ $a^t = a^x \left(\frac{u^x}{u^t}\right)$ substituting this into (3), we get

$$g^2 = (a^x)^2\left[1 - \left(\frac{u^x}{u^t}\right)^2\right] = -\frac{(a^x)^2}{(u^t)^2}\left[-(u^t)^2 + (u^x)^2\right] \quad \text{now use Eq. (1)}$$

$$g^2 = (a^x)^2/(u^t)^2 \quad \text{or} \quad \boxed{a^x = g\,u^t} \quad (4)$$

which further implies that $\boxed{a^t = g\,u^x}$ (5)

Now differentiate Eq. (4) with respect to proper time to get a differential equation for u^x:

$$\underbrace{\frac{da^x}{d\tau}}_{\text{by defn. of } \vec{a}, \vec{u}} = \underbrace{\frac{d^2u^x}{d\tau^2}}_{\text{by Eq. (4)}} = g\,\underbrace{\frac{du^t}{d\tau}}_{\text{by defn. } \vec{a}\ \&\ \vec{u}} = g\,\underbrace{a^t}_{\text{Eqn. (5)}} = g^2 u^x \quad (6)$$

The solutions to this eqn, $\frac{d^2u^x}{d\tau^2} = g^2 u^x$, are, obviously, just

$$\boxed{u^x = A\exp(g\tau) + B\exp(-g\tau)} \quad (7)$$

Since the initial condition is rest at $\tau = 0$, and $a^x = \frac{du^x}{d\tau} = g$ at $\tau = 0$, we must have $A = -B = 1$, so that

$$u^x = \sinh(g\tau) \qquad (8)$$

and, by Eq. (1),

$$u^t = \cosh(g\tau) \qquad (9)$$

To find $x(\tau)$, we integrate Eq. (8), subject to the initial condition that $x=0$ at $\tau=0$:

$$x(\tau) = g^{-1}\left[\cosh(g\tau) - 1\right] \ . \qquad (10)$$

In units with $c = 1 = \frac{3\times10^{10}\text{ cm}}{\text{sec}}$

$$g = 980 \ \frac{\text{cm}}{\text{sec}^2} \cdot \frac{1\text{ sec}}{3\times10^{10}\text{ cm}} = 3.27\times10^{-8}\text{ sec}^{-1} = \frac{1}{3.06\times10^{7}\text{ sec}} \approx \frac{1}{\text{yr.}}$$

To get half-way to the galactic center requires $x = 15{,}000$ light years, so

$$15{,}000 \text{ years} = \frac{1}{1\text{ yr}^{-1}}\left[\cosh(g\tau) - 1\right]$$

or

$$\cosh(g\tau) \approx 15{,}000$$

For such a large value, cosh is well approximated by $\frac{1}{2}\exp(g\tau)$, so

$$g\tau \approx \ln(30{,}000)$$

$$\tau \approx g^{-1}\ln(30{,}000) \approx \underline{10.3 \text{ years}}$$

The deceleration half of the trip is identical, so the total time is

$$\boxed{2\tau = 20.6 \text{ years}}$$

14. Find the quantum-mechanical eigenfunction $\psi_n(k)$ and energy bands $E_n(k)$ of a one-dimensional empty lattice [$V(x)=0$] with lattice constant a; n and k are the band index and wave-vector. Illustrate your results with a sketch of the energy bands. Hint: Use Bloch's theorem to represent $\psi_n(k)$ in terms of its periodic part $u_n(k)$.

$$\psi_n(k) = e^{ikx} u_k^{(n)}(x)$$

$$\nabla_x^2 \psi_n(k) = \frac{2m(V-E)}{\hbar^2} \psi_n(k)$$

$$e^{ikx}\left(\nabla_x^2 + 2i k \nabla_x - k^2\right) u_k^{(n)}$$

$$\left(\nabla_x^2 + 2i k \nabla_x\right) u_k^{(n)} = \left(k^2 - \frac{2mE}{\hbar^2}\right) u_k^{(n)}$$

with BC : $u_k^{(n)}(0) = u_k^{(n)}(a)$

Try plane wave solu: $u_k^{(n)} \propto e^{i\sigma x}$

BC. $e^{i\sigma a} = 1$, $\sigma a = 2\pi n$, $n = 0, \pm 1, \pm 2 \cdots$

put into D. E.

$$\sigma_n^2 - 2\sigma_n k + k^2 = \frac{2m\,E(k,n)}{\hbar^2}$$

$$E(k,n) = \frac{\hbar^2}{2m}(\sigma_n + k)^2 = \frac{\hbar^2}{2m}\left(k + \frac{2\pi n}{a}\right)^2$$

$$u_k^{(n)} = \frac{1}{\sqrt{a}} e^{\frac{2\pi n}{a}x} \qquad n = 0, \pm 1, \pm 2$$

$$-\frac{\pi}{2} < k \leq \pi/a$$

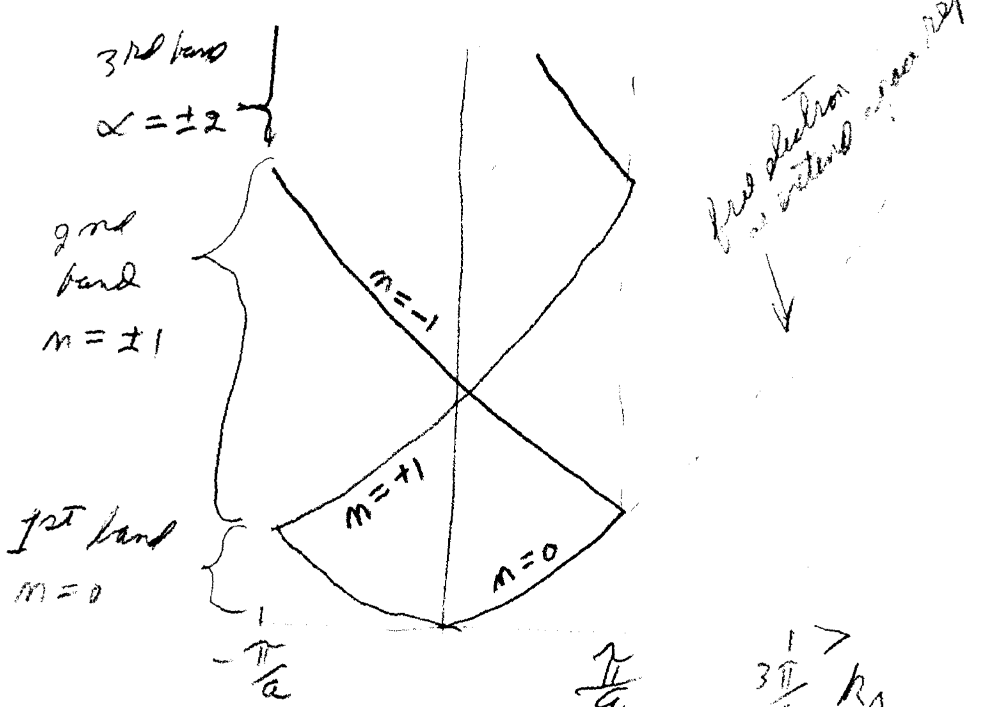
$$\rightarrow \quad \psi_n(k) = \frac{1}{\sqrt{Na}} e^{i\left(k + \frac{2\pi n}{a}\right)x}$$

$n = 1$, $k + \frac{2\pi n}{a} = \frac{\pi}{a}$ for $k = -\frac{\pi}{a}$

$= +\frac{3\pi}{a}$ " $k = +\frac{\pi}{a}$

$n = -1$ $k + \frac{2\pi n}{a} = -\frac{\pi}{a}$ for $k = +\frac{\pi}{a}$

$= -\frac{3\pi}{a}$ " $k = -\frac{\pi}{a}$



quantum mechanical psi l.c. kay

14. Find the quantum mechanical eigenfunction $\psi_{\cancel{m}}(k)$ (n band index) "IN"
and draw the energy bands of a one dimensional
empty lattice with lattice constant a.
Empty lattice means $V(x) = 0$. (Hint use the
wave equation and boundary conditions for $\cancel{u_k}(x)$, "you"
the periodic part of the Block function). "aitch"

Omit

Find the quantum-mechanical eigenfunctions $\psi_n(k)$ and energy bands $E_n(k)$ of a one-dimensional empty lattice [$V(x)=0$] with lattice constant a; n and k are the band index and wave-vector. Hint: Use Bloch's theorem to represent $\psi_n(k)$ in terms of its periodic part $u_n(k)$.

15. The mercury atom has the following energy levels expressed in terms of energy units $1/\lambda$.

$6s^2$	^1S_o	0
$6s6p$	^3P_o	37,645 cm^{-1}
$6s6p$	$^3P_1$	39,412 cm^{-1}
$6s6p$	$^3P_2$	44,043 cm^{-1}
$6s7s$	$^3S_1$	62,350 cm^{-1}

a) Explain the meaning of the spectroscopic notation above.

b) What transitions will occur between these energy levels in a gas discharge? Explain in moderate detail.

c) Briefly outline an experimental method for verifying the total angular momenta J assigned to the levels above.

a) 6s 6p gives nature of indiv. electron wavefn

$\underset{n}{\uparrow}\ \underset{\ell}{\uparrow}$ $\ell = 0, 1, 2, 3, \ldots$ called $s, p, d, f \ldots$

$^3P_0$: $^{2S+1}L_J$ gives nature of multielectron wavefn coupled to give $\vec{L} = \vec{\ell}_1 + \vec{\ell}_2$, $\vec{S} = \vec{s}_1 + \vec{s}_2$ and $\vec{J} = \vec{j}_1 + \vec{j}_2$

b) ① $\left.\begin{array}{l} \Delta S = 0 \\ |\Delta L| \le 1 \\ |\Delta J| \le 1 \\ |\Delta m_J| \le 1 \\ \text{parity change req'd} \end{array}\right\}$ for electric dipole operator $e\vec{r}$

derived on basis of $\longrightarrow$ $\left\{\begin{array}{l} \text{operator } \vec{r} \text{ does not act on } \vec{S} \text{ space} \\ \text{operator } \vec{r} \text{ is a vector (rank 1) operator in } \vec{L} \text{ space} \\ \quad \Rightarrow \text{max change } \pm 1 \text{ or } 0 \end{array}\right.$

② <u>Fluorescence</u> will occur, since the atoms in the discharge are excited by electron collisions

Conclude transitions are allowed between the triplets only (by electric dipole). Furthermore must be $S \to P$ to have parity change

Hence $^3S_1 \to {}^3P_2,\ {}^3P_1,\ {}^3P_0$ only $\boxed{\therefore \text{ 3 lines}}$

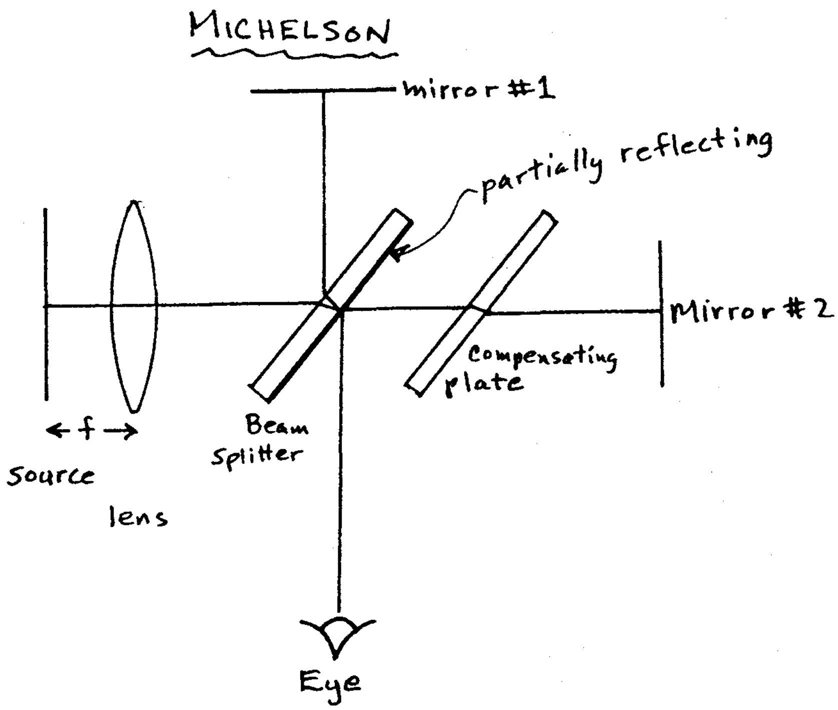
c) Zeeman effect : measure $\left\{\begin{array}{l} \#\text{ sublevels} \\ g_J \text{ factors} \end{array}\right.$

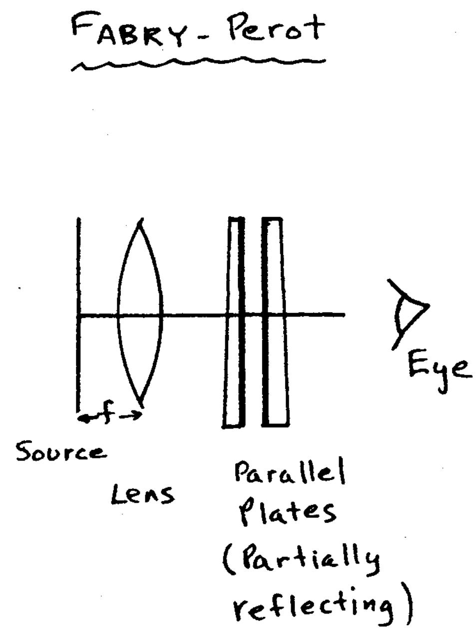
16. Consider the circular pattern of fringes resulting from a Michelson interferometer which is illuminated by an extended monochromatic source and which is viewed by eye. A schematic drawing is given below.

a) If the difference in distance between the beam splitter and the two mirrors is d=2 mm, find the order m of the central fringe for λ=500 nm and discuss whether it is bright or dark. (You may take these numbers to be exact.)

b) Find the angular radius of the 3rd dark fringe seen off-axis.

c) Describe the difference in the fringe pattern for a Fabry-Perot inteferometer (consisting of two parallel partially reflecting mirrors) vs. the Michelson interferometer and comment on their relative usefulness.





#16

a) path difference is $\Lambda = 2d\cos\theta$ (See optics texts) Depending on whether or not there is a phase shift upon reflection, a given Λ value can give <u>either</u> constructive <u>or</u> destructive interference. For example, an uncoated glass beamsplitter would introduce a net phase difference of π. If we <u>ignore</u> this effect, $\Lambda = m\lambda$ is the condition for <u>constructive</u> interference:

$$\Lambda = 2d = m\lambda \qquad m = \frac{4\times10^{-3}\,m}{500\times10^{-9}\,m} = \underline{8000}$$

b) <u>Increasing θ</u> $\Rightarrow$ $\Lambda = 2d\cos\theta$ <u>Decreases</u>

$\therefore$ <u>all</u> off-axis parts of the pattern correspond to <u>smaller</u> m value

1st darkening $m = 7999\frac{1}{2}$

2nd " " $m = 7998\frac{1}{2}$

3rd " " $m = 7997\frac{1}{2}$

$$\text{so} \quad \frac{\Lambda}{\lambda}\cos\theta = 7997.5$$

$$\cos\theta = \frac{7997.5}{8000} \qquad \text{and} \qquad \theta = 1.43^\circ$$

c) Fabry-Perot uses multiple reflections. Multiple beam interference gives much sharper interference than the two-beam case of the Michelson. Hence, Fabry-Perot is more useful for spectroscopy. The Michelson, with two $\perp$ arms, played a key role in confirming special relativity theory. Laser versions have now proven that space is isotropic to better than 2.5 parts in 10^{15}!