Derivation of Maxwell Eqtns from $\mathcal{L}_{EM}$

just four degrees-of-freedom, as represented by the potential $A^\alpha = (\phi, \mathbb{A})$

and// $\longrightarrow \mathbb{E} = -\nabla\phi - \frac{1}{c}(\partial\mathbb{A}/\partial t)$, $\mathbb{B} = \nabla \times \mathbb{A}$. (24)

So we match the # degrees-of-freedom of the Maxwell field by assigning the 4-vector components $(\phi, \mathbb{A})$ to the continuum coordinates $\xi^{(\nu)}$ in the Lagrange eqtn-of-motion, viz.: $\partial^\mu[\partial\mathcal{L}/\partial(\partial^\mu\xi^{(\nu)})] = \partial\mathcal{L}/\partial\xi^{(\nu)}$; this is Eq. (16), p. I & II 13 for independent coordinates $\xi^{(\nu)}$, $\nu = 0,1,2,3$. Then, also, the fact that $\xi^{(\nu)} = A^\nu$ is a 4-vector means that the $\mathcal{L}_{EM}$ eqtns-of-motion will be Lorentz covariant (footnote, p. I & II 13). If, in fact, the $\mathcal{L}_{EM}$ eqtns-of-motion are the Maxwell Equations, we know (and require) already that they are Lorentz covariant. A double check!

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18) Now, with $\xi^{(\nu)} = (\phi, \mathbb{A})$, we want to show $\mathcal{L}_{EM}$ of Eq. (23) gives the "right" eqtns-of-motion, namely the Maxwell Equations for E & B. Our system is:

$$\left[\!\left[\mathcal{L}_{EM} = \frac{1}{8\pi}(E^2 - B^2) - \rho\phi + \frac{1}{c}\mathbb{J}\cdot\mathbb{A}, \text{ w/ } \frac{\partial}{\partial t}\left(\frac{\partial\mathcal{L}_{EM}}{\partial\xi_t^{(\nu)}}\right) + \underbrace{\frac{\partial}{\partial x_k}}_{\text{sum on } k}\left(\frac{\partial\mathcal{L}_{EM}}{\partial\xi_{x_k}^{(\nu)}}\right) = \frac{\partial\mathcal{L}_{EM}}{\partial\xi^{(\nu)}}. \right]\!\right] \quad (25)$$

for $\nu = 0$ cd., i.e. $\xi^{(0)} = \phi$. Have: $\partial\mathcal{L}_{EM}/\partial\phi = -\rho$, $\partial\mathcal{L}_{EM}/\partial\phi_t = 0$, so...

$$-\rho = \frac{\partial}{\partial x_k}\left(\frac{\partial\mathcal{L}_{EM}}{\partial\phi_{x_k}}\right) = -\frac{\partial}{\partial x_k}\left(\frac{\partial\mathcal{L}_{EM}}{\partial E_k}\right) = -\frac{1}{4\pi}\underbrace{\frac{\partial}{\partial x_k}E_k}_{\text{div }\mathbb{E}}, \text{ w/ } \underbrace{\boxed{\nabla\cdot\mathbb{E} = 4\pi\rho}}_{\text{Gauss' Law}} \quad (26)$$

use $\phi_{x_k} = \partial\phi/\partial x_k = -E_k$

for $\nu = 1$ cd., i.e. $\xi^{(1)} = A_1$. Have: $\frac{\partial\mathcal{L}_{EM}}{\partial A_1} = \frac{1}{c}J_1$, $\frac{\partial\mathcal{L}_{EM}}{\partial A_{1t}} = -\frac{\partial\mathcal{L}_{EM}}{c\,\partial E_1} = -\frac{E_1}{4\pi c}$, so:

$$\frac{1}{c}J_1 = -\frac{\partial}{\partial t}\left(\frac{E_1}{4\pi c}\right) + \frac{\partial}{\partial x_k}\left(\frac{\partial\mathcal{L}_{EM}}{\partial(\partial A_1/\partial x_k)}\right) \longleftarrow \text{use: } \mathbb{B} = \left(\frac{\partial A_3}{\partial x_2} - \frac{\partial A_2}{\partial x_3}, \frac{\partial A_1}{\partial x_3} - \frac{\partial A_3}{\partial x_1}, \frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2}\right)$$

So// $$\frac{1}{c}J_1 + \frac{1}{4\pi c}\left(\frac{\partial E_1}{\partial t}\right) = \frac{\partial}{\partial x_2}\left(\frac{\partial\mathcal{L}_{EM}}{\partial(\partial A_1/\partial x_2)}\right) + \frac{\partial}{\partial x_3}\left(\frac{\partial\mathcal{L}_{EM}}{\partial(\partial A_1/\partial x_3)}\right) = \frac{1}{4\pi}\left(\frac{\partial B_3}{\partial x_2} - \frac{\partial B_2}{\partial x_3}\right)$$

$$= -\frac{\partial\mathcal{L}_{EM}}{\partial B_3} = +\frac{B_3}{4\pi} \qquad = +\frac{\partial\mathcal{L}_{EM}}{\partial B_2} = -\frac{B_2}{4\pi}$$

[next page]

$\mathcal{L}_{EM} \Rightarrow$ Source-dependent Maxwell Equations.

i.e.// $\nu=1$ Lagrange Eqtn here $\Rightarrow \left(\frac{1}{c}\mathbb{J} + \frac{1}{4\pi c}\dot{\mathbb{E}}\right)_1 = \frac{1}{4\pi}(\nabla\times\mathbb{B})_1$.

... $\nu=2,3$ eqtns $\Rightarrow$ the 2,3 components of Ampere's Law : $\boxed{\nabla\times\mathbb{B} = \frac{4\pi}{c}\mathbb{J} + \frac{1}{c}\left(\frac{\partial\mathbb{E}}{\partial t}\right)}$ (27)

In covariant notation, what we have shown here is that...

field-source Lagrange density : $\mathcal{L}_{EM} = -\frac{1}{16\pi}F_{\mu\nu}F^{\mu\nu} - \frac{1}{c}J_\nu A^\nu$,

plus Lagrange Eqtns : $\partial^\mu[\partial\mathcal{L}_{EM}/\partial(\partial^\mu A^\nu)] = \partial\mathcal{L}_{EM}/\partial A^\nu$,

(with components of 4-potential $A^\nu = (\phi, \mathbb{A})$ as generalized cds)

imply the source-dept. Maxwell Eqtns : $\frac{1}{4\pi}\partial^\mu F_{\mu\nu} = \frac{1}{c}J_\nu$. (28)

REMARK

We get only the source-dept. Maxwell Eqtns out of the $\mathcal{L}_{EM}$ formalism. What has happened to the other two eqtns, viz $\nabla\cdot\mathbb{B}=0$ & $\nabla\times\mathbb{E} = -\frac{1}{c}\frac{\partial\mathbb{B}}{\partial t}$?

ANS. They are "trivially" satisfied by our choice of 4-potential $A^\nu = (\phi, \mathbb{A})$ (and the consequent form of the field tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$) such that Eq. (24) is satisfied, i.e. $\mathbb{E} = -\nabla\phi - \frac{1}{c}(\partial\mathbb{A}/\partial t)$, $\mathbb{B} = \nabla\times\mathbb{A}$. With this way of defining ϕ & $\mathbb{A}$, it is automatically true that the Maxwell fields obey $\nabla\cdot\mathbb{B} = \nabla\cdot(\nabla\times\mathbb{A}) = 0$, $\nabla\times\mathbb{E} = -\underbrace{\nabla\times(\nabla\phi)}_{\to 0} - \frac{1}{c}\frac{\partial}{\partial t}(\nabla\times\mathbb{A}) = -\frac{1}{c}\frac{\partial\mathbb{B}}{\partial t}$. From the standpoint of the 4 degrees-of-freedom inherent in the Maxwell field, Eq. (28) gives just as much -- and no more -- information as is needed.

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19) The utility of the $\mathcal{L}_{EM}$ formalism does not lie in regurgitating the Maxwell Eqtns -- this is just a check on whether $\mathcal{L}_{EM}$ generates the "right" eqtns-of-motion. The utility of the formalism does lie in being able to quickly decide -- covariantly, of course -- how modifications might be made to EM theory. An example is the Proca Lagrangian [Jkⁿ Eq. (12.91)], including a photon mass: