This exam is open-book, open-notes, and is worth 180 pts. total. For each of the 5 problems, $\boxed{\text{box}}$ the answer on your solution sheet. Number your solution pages in sequence, put your name on p.1, and staple the pages together before handing in.

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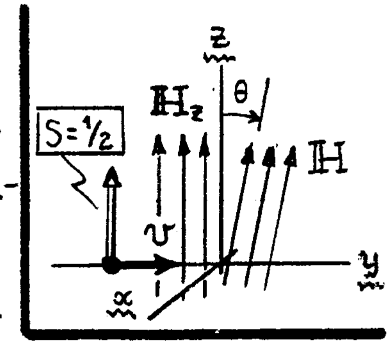
**①** [40 pts.]. A beam of electrons, each with spin $S = ½$ and magnetic moment $\mu = -2\mu_0 \mathbb{S}$, is moving down the $y$-axis of a lab coordinate system at velocity $v$. Along the negative $y$-axis (to the left of the $xz$-plane, in the sketch), there is maintained a magnetic field $\mathbb{H}_z$ that is everywhere parallel to the $z$-axis. Then, in a small interval $\Delta y$ near the origin, this field "tilts" by an $\measuredangle\theta$, so that along the positive $y$-axis (to right of $xz$-plane) there is a field $\mathbb{H}$ oriented at $\measuredangle\theta$ w.r.t. $z$-axis. $\mathbb{H}_z$ & $\mathbb{H}$ have the same magnitude; they differ only in their directions. Assume that the electron beam passes rapidly through the region $\mathbb{H}_z \to \mathbb{H}$, and neglect changes in the electron trajectory due to cyclotron motion.




**(A)** Find the probability of a "spin-flip" induced by the rapid passage $\mathbb{H}_z \to \mathbb{H}$. Specifically, if a spin-up electron enters from the left ($y<0$), what is the probability that it is in a spin-down state as it exits to the right ($y>0$)?

**(B)** The calculation in part (A) should exploit the idea of a rapid passage. Find a criterion (involving $v$, $\Delta\theta = \theta$ in $\Delta y$, etc.) that specifies what "rapid" means here.

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**②** [30 pts.]. For an H-like atom, w/ Coulomb potential $-Ze^2/r$ &/ $r$ = radial coordinate, the expectation value of $1/r^2$ in the eigenstate $|nlm\rangle$ is given by:

$\langle 1/r^2 \rangle = \langle nlm | 1/r^2 | nlm \rangle = (Z/a_0)^2 / n^3 (l + \frac{1}{2})$, w/ $a_0 = \hbar^2/me^2$ = Bohr radius.

Use this result for $\langle 1/r^2 \rangle$ to show that in the same state $\langle 1/r^3 \rangle$ is given by:

$$\boxed{\langle 1/r^3 \rangle = \langle nlm | \tfrac{1}{r^3} | nlm \rangle = (Z/a_0)^3 / n^3 l(l+1)(l+\tfrac{1}{2})}.$$

Do not use explicit wavefcns $|nlm\rangle$. Instead, relate $\langle 1/r^3 \rangle$ & $\langle 1/r^2 \rangle$ by looking at the equations-of-motion for an electron in orbit. (next page)

**③** [30 pts.]. In a certain QM system, it is found the eigenfcns $u_n(x)$ [corresponding to energies $E_n$] are translationally invariant -- i.e. if $u_n(x)$ is a solution to $\mathcal{H} u_n = E_n u_n$, then so is $u_n(x+\Delta x)$, w/ $\Delta x$ = arbitrary displacement of position $x$.

**(A)** As a consequence of this invariance, show that the system's linear momentum operator $p = -i\hbar\,\partial/\partial x$ must commute with the Hamiltonian $\mathcal{H}$, i.e. $[\mathcal{H}, p] = 0$.

**(B)** What sort of "QM system" are you dealing with?

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**④** [40 pts.]. A QM ∡ momentum operator $\mathbb{J} = (J_x, J_y, J_z)$ {obeying the commutation rule $[J_x, J_y] = i J_z$, etc., with $\hbar = 1$} has eigenfcns $|jm\rangle$ such that $\mathbb{J}^2|jm\rangle = j(j+1)|jm\rangle$ and $J_z|jm\rangle = m|jm\rangle$. One can define "ladder operators" $J_\pm = J_x \pm i J_y$, and -- by examining appropriate commutators -- it is easy to show that: $J_+|jm\rangle \propto |j\,m+1\rangle$, $J_-|jm\rangle \propto |j\,m-1\rangle$, so that $J_\pm$ step the $m$-values by $\Delta m = \pm 1$. Here we want to find the constants of proportionality.

**(A)** If: $J_+|jm\rangle = A|j\,m+1\rangle$, show how the constant $A$ is determined.

**(B)** If: $J_-|jm\rangle = B|j\,m-1\rangle$, show how the constant $B$ is determined.

NOTE: This problem does require a derivation. It is not sufficient to just quote the well-known results for A & B.

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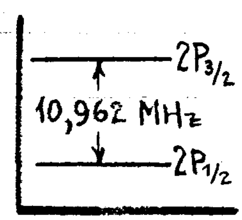
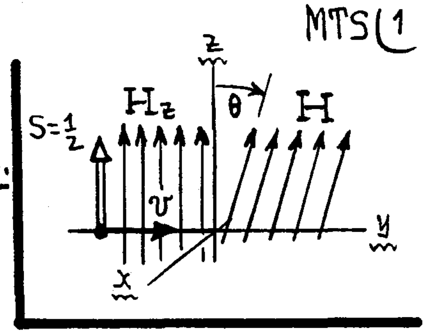
**⑤** [40 pts.]. Per CLASS NOTES p. fs 9, the fine-structure splitting in the $n=2$ level of normal hydrogen is $\Delta\nu = 10{,}962$ MHz (to 0.1%), and the levels that are split are $2P_{3/2}$ & $2P_{1/2}$ [later, we showed that the other $n=2$ level, viz. $2S_{1/2}$, was degenerate with $2P_{1/2}$]. In "normal hydrogen", the electron has spin $s = 1/2$, of course.




**(A)** If the electron spin were turned off, i.e. $s=0$, how would the fine structure splitting $\Delta\nu$ change? HINT: you ought to be able to find a simple proportionality between $\Delta\nu(s=1/2)$ and $\Delta\nu(s=0)$.

**(B)** What levels would be split by the $\Delta\nu(s=0)$ interaction of part (A)?

# $\phi$507 MidTerm Solutions (1994)




**① [40 pts.]. Find the probability for a "spin-flip" for $H_z \to H$.**

(A) 1. In prob<sup></sup>m ⑥⑨, you found the Hamiltonian $\mathcal{H}$, eigenenergies E, and eigenspinors $\psi$ for $S=\frac{1}{2}$ in an arbitrarily oriented magnetic field $\mathbb{H} = H(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$. The results were ($\mu_0 = \frac{e\hbar}{2mc}$)...

$$\rightarrow \mathcal{H} = \mu_0 H \begin{pmatrix} \cos\theta & e^{-i\phi}\sin\theta \\ e^{i\phi}\sin\theta & -\cos\theta \end{pmatrix}, \quad E = \pm\mu_0 H \begin{cases} \nearrow \psi_+ = \begin{pmatrix} \cos(\theta/2) \\ e^{i\phi}\sin(\theta/2) \end{pmatrix}, \text{ spin "up"}; \\ \searrow \psi_- = \begin{pmatrix} -e^{-i\phi}\sin(\theta/2) \\ \cos(\theta/2) \end{pmatrix}, \text{ spin "down"}. \end{cases} \quad (1)$$

In this problem, since $H_z$ is $\parallel$ z-axis ($\theta=0$) @ $y<0$, the system wavefcns there are

$$\rightarrow @\ y<0: \quad \psi_+(y<0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \text{ spin up}; \quad \psi_-(y<0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \text{ spin down}. \quad (2)$$

For $y>0$, with H at $\measuredangle\ \theta$ w.r.t. z-axis, these wavefcns go over to

$$\rightarrow @\ y>0: \quad \psi_+(y>0) = \cos\frac{\theta}{2}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \sin\frac{\theta}{2}\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \leftarrow \text{spin up (energy } +\mu_0 H); \quad (3)$$

$$\psi_-(y>0) = -\sin\frac{\theta}{2}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \cos\frac{\theta}{2}\begin{pmatrix} 0 \\ 1 \end{pmatrix}. \leftarrow \text{spin down (energy } -\mu_0 H).$$

We have put the azimuth $\phi = 0$; $e^{\pm i\phi}$ is an unimportant phase factor.

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2. Now if S passes "rapidly" from $y<0$ to $y>0$, we can use the Sudden Approximation to estimate transition probabilities. From CLASS NOTES, p. tD 20, Eq.(58), the amplitude for a transition $\psi_+(y<0) \to \psi_-(y>0)$, i.e. spin up $\to$ spin down, or a "spin flip", is:

$$b(\downarrow\uparrow) = \langle\psi_-(y>0)|\psi_+(y<0)\rangle = -\sin(\theta/2),$$

so// spin-flip probability: $\boxed{\mathcal{P}(\theta) = |b(\downarrow\uparrow)|^2 = \sin^2(\theta/2)}$. (4)

$\mathcal{P}(\theta=0) = 0$, as should be (for no change in $\mathbb{H}$), while $\mathcal{P}(\theta=\pi) = 1$ is maximum.

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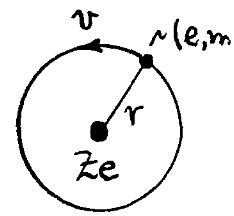
(B) 3. For the sudden approxn to be valid, the system should be ~ stationary during the time $\Delta t$ that the $\mathcal{H}$ change occurs. Here: $\omega\Delta t \ll \Delta\theta$, w/ $\omega = \mu_0 H/\hbar$ is the Larmor frequency, and $\Delta\theta$ is the $\measuredangle$ change which induces the spin flip. So we need:

$\boxed{\omega \ll \Delta\theta/\Delta t = v(\Delta\theta/\Delta y)}$, w/ $v$ = particle velocity, & // $\Delta\theta/\Delta y = \measuredangle$ gradient.

# $\phi$507 MidTerm Solutions (1994, cont'd)

**②** [30 pts.]. For H-like atom, manufacture $\langle 1/r^3\rangle$ from $\langle 1/r^2\rangle$.




1. The equation of the electron orbit at $r$, viz...

$$\rightarrow \; mv^2/r = Ze^2/r^2, \qquad (1)$$

can be written in terms of the orbital ∡ momentum $L = mvr$ as:

$$\rightarrow \; L^2/r^3 = Zme^2/r^2. \qquad (2)$$

Quantum-mechanically, Eq. (2) will hold in an expectation-value sense (by Ehrenfest's Theorem: Sakurai, p.87) and so in the state $|nlm\rangle$

$$\rightarrow \; \langle nlm|\frac{L^2}{r^3}|nlm\rangle = \hbar^2\,\frac{Z}{a_0}\langle nlm|\frac{1}{r^2}|nlm\rangle, \quad a_0 = \hbar^2/me^2. \qquad (3)$$

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2. In Eq. (3), $L^2$ is an operator, which operates on the ∡ cds of $|nlm\rangle$, and which has the eigenvalue $l(l+1)\hbar^2$ in that state. Then (3) reads...

$$\rightarrow \; l(l+1)\langle nlm|\frac{1}{r^3}|nlm\rangle = \frac{Z}{a_0}\langle nlm|\frac{1}{r^2}|nlm\rangle$$

So// $$\langle nlm|\frac{1}{r^3}|nlm\rangle = \frac{Z/a_0}{l(l+1)}\langle nlm|\frac{1}{r^2}|nlm\rangle$$

$$= \underline{\underline{(Z/a_0)^3/n^3\,l(l+1)(l+\tfrac{1}{2})}}, \qquad (4)$$

as required.

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# $\phi$ 507 MidTerm Solutions (1994, cont'd).

## ③ [30 pts.]. Analyse consequences of translational invariance in a QM system.

(A) 1. We are given that:

$$\mathcal{H} u_n(x) = E_n u_n(x), \text{ and } \mathcal{H} u_n(x+\Delta x) = E_n u_n(x+\Delta x). \quad (1)$$

Suppose $\Delta x \to$ infinitesimal, and expand $u_n(x+\Delta x)$ by Taylor series...

$$u_n(x+\Delta x) = u_n(x) + \Delta x \left(\frac{\partial u_n}{\partial x}\right)\bigg|_{\Delta x=0} + \cdots \leftarrow \frac{\partial}{\partial x} = \frac{i}{\hbar} p \text{ (operator)}$$

so, $$u_n(x+\Delta x) = u_n(x) + \frac{i\Delta x}{\hbar} p\, u_n(x) + \cdots \quad (2)$$

The second of Eqs. (1) then gives...

$$\mathcal{H}\left[u_n(x) + \frac{i\Delta x}{\hbar} p\, u_n(x) + \cdots\right] = E_n\left[u_n(x) + \frac{i\Delta x}{\hbar} p\, u_n(x) + \cdots\right] \quad (3)$$

terms cancel

or, $$\cancel{\left(\frac{i\Delta x}{\hbar}\right)} \mathcal{H} p\, u_n(x) = \cancel{\left(\frac{i\Delta x}{\hbar}\right)} p \underbrace{E_n u_n(x)}_{=\mathcal{H} u_n(x)}$$ since $E_n$ commutes with $p$.

i.e., $$[\mathcal{H} p - p \mathcal{H}]\, u_n(x) = 0,$$ so, $\boxed{[\mathcal{H}, p] = 0}$, as required. (4)

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(B) 2. Since $[\mathcal{H}, p] = 0$, then the momentum $p$ is a constant of the motion, as is the total energy $E_n = (p^2/2m) + V$. So the potential is at most a cnst, which can be set to zero. Then $E = p^2/2m$.

<u>The "QM system" under discussion is a free particle.</u>

# $\phi$ 507 MidTerm Solutions (1994, cont'd)

**④** [40 pts.]. For ∡ mom^m ladder operators : $J_\pm|jm\rangle = \left\{\begin{smallmatrix}A\\B\end{smallmatrix}\right\}|j\,m\pm1\rangle$, find consts A & B.

**1.** We must first recall that $\mathbb{J}=(J_x,J_y,J_z)$ is a Hermitian operator; each component $J_k$ is self-adjoint : $J_k^\dagger = J_k$. This follows from the requirement that the ∞smal rotation operator : $R_k(\delta\phi) = 1 - i(\delta\phi)J_k$, for a rotation by $\delta\phi$ about the $k^{th}$ axis, is unitary [i.e. $R_k^\dagger(\delta\phi) = 1 + i(\delta\phi)J_k^\dagger$ is such that $R_k^\dagger R_k = 1$; then $R_k^\dagger(+\delta\phi) = R_k(-\delta\phi)$ is just the inverse rotation, with the same $J_k = J_k^\dagger$]. It follows that although $J_\pm = J_x \pm i J_y$ are not Hermitian, they are in fact the adjoints of each other, i.e.

$$\rightarrow J_+^\dagger = (J_x + iJ_y)^\dagger = J_x^\dagger - iJ_y^\dagger = J_x - iJ_y = J_-, \text{ and } J_-^\dagger = J_+. \tag{1}$$

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(A) **2.** Now, assume $J_+|jm\rangle = A|j\,m+1\rangle$, and that the eigenstates $|jm\rangle$ are orthonormal. Consider a matrix element which isolates A, i.e....

$$\langle jm|J_-J_+|jm\rangle = \langle \overset{J_+}{J_-^\dagger}\, jm|J_+\, jm\rangle = \langle J_+\, jm|J_+\, jm\rangle = |A|^2 \underbrace{\langle j\,m+1|j\,m+1\rangle}_{1}$$

$$\xrightarrow{\text{i.e.}} |A|^2 = \langle jm|J_-J_+|jm\rangle. \tag{2}$$

But : $J_-J_+ = (J_x - iJ_y)(J_x + iJ_y) = \overbrace{J_x^2 + J_y^2}^{\mathbb{J}^2 - J_z^2} + i\overbrace{[J_x, J_y]}^{iJ_z} = \mathbb{J}^2 - J_z^2 - J_z$. So...

$$|A|^2 = \langle jm|\mathbb{J}^2 - J_z^2 - J_z|jm\rangle = j(j+1) - m^2 - m = (j-m)(j+m+1)$$

and
$$\boxed{J_+|jm\rangle = \sqrt{(j-m)(j+m+1)}\,|j\,m+1\rangle}. \tag{3}$$

The desired proportionality cnst A = the $\sqrt{\ }$ here, to within a phase factor.

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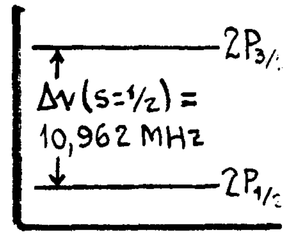
(B) **3.** For $J_-|jm\rangle = B|j\,m-1\rangle$, carry out a similar procedure to get : $|B|^2 = \langle jm|J_+J_-|jm\rangle$, and : $J_+J_- = \mathbb{J}^2 - J_z^2 + J_z$. Then $|B|^2 = j(j+1) - m^2 + m = (j+m)(j-m+1)$, so that :

$$\boxed{J_-|jm\rangle = \sqrt{(j+m)(j-m+1)}\,|j\,m-1\rangle}. \tag{4}$$

B = the $\sqrt{\ }$ here, again to within an arbitrary (uniform) phase factor.

**(5) [40 pts.]. Compare the $n=2$ fs interval $\Delta\nu$ for electron spin $s=1/2$ & spin $s=0$.**

(A) **1.** The work just preceding the quoted value for the $n=2$ fs interval $\Delta\nu(s=1/2)$ in normal hydrogen shows that it is calculated as ($Z=1$):




$$\rightarrow h\,\Delta\nu(s=1/2) = \frac{\alpha^2|E_n|}{n\,l(l+1)}\Bigg|_{\substack{n=2\\ l=1}} = \frac{1}{4}\alpha^2|E_2|, \qquad (1)$$

w/ $h$ = Planck's cnst, $\alpha = e^2/\hbar c \simeq 1/137$ the fs cnst, and $|E_2| = \frac{1}{2}\alpha^2 mc^2/4$ the Bohr energy for $n=2$. We have ignored the correction for the "anomalous" part of the electron $g$-value; it is relatively unimportant here. The fs splitting is $O(\alpha^2)$ relative to the Bohr energy, and it splits $2P_{3/2}$ & $2P_{1/2}$ as shown.

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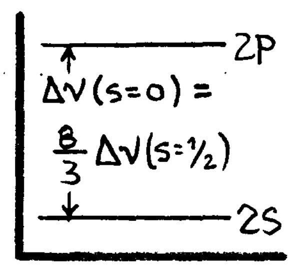
**2.** If the electron spin were $s=0$, then the H-atom obeys the Klein-Gordon Eqtn. We have calculated the exact energy levels of this atom in CLASS NOTES, pp. fs 18-19, and -- in Eq. (22), p. fs 19 -- we found that to $O(\alpha^4)$, or $O(\alpha^2)$ relative to Bohr:

$$\rightarrow \mathcal{E}_{nl} = -|E_n|\left[1 + \frac{\alpha^2}{n}\left(\frac{1}{l+1/2} - \frac{3}{4n}\right)\right]. \qquad (2)$$

The term in $\alpha^2$ inside the [ ] gives the finestructure for this spinless atom, comparable to $\Delta\nu(s=1/2)$ in Eq. (1). The splitting occurs between the levels $2P$ ($n=2$, $l=1$) and $2S$ ($n=2$, $l=0$), and is of size ($Z=1$)...

$$\rightarrow h\,\Delta\nu(s=0) = \mathcal{E}_{21} - \mathcal{E}_{20} = \frac{2}{3}\alpha^2|E_2|. \qquad (3)$$




Comparison of Eqs. (1) & (3) shows that...

$$\boxed{\Delta\nu(s=0) = \frac{8}{3}\Delta\nu(s=1/2) = 29{,}232\ \text{MHz}} \quad \int \text{fs for spinless H-atom.} \qquad (4)$$

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(B) **3.** The levels split by $\Delta\nu(s=0)$ have already been identified above; they are $2P$ ($n=2$, $l=1$) and $2S$ ($n=2$, $l=0$). The splitting is not due to a magnetic interac- as for $\Delta\nu(s=1/2)$ in Eq. (1). Rather, $\Delta\nu(s=0)$ is a relativistic effect, due to slightly different relativistic corrections in the electron motion for $2P$ & $2S$ orbits.