DEPARTMENT OF PHYSICS

1998 COMPREHENSIVE EXAM

AUGUST 24 THRU AUGUST 26, 1998

Answer each of the following questions, all of which carry equal weight. Begin your answer to each question on a new sheet of paper; solutions to different questions must not appear on the same sheet. Each sheet of paper must be labeled with your name and the problem number in the upper right hand corner. When more than one sheet is submitted for a problem, be sure the pages are ordered properly.

PHYSICAL CONSTANTS

Quantity	Symbol	Value
acceleration due to gravity	g	9.8 m s^{-2}
gravitational constant	G	$6.67 \times 10^{-11} \text{ N m}^2 \text{ kg}^{-2}$
permittivity of vacuum	ε_o	$8.85 \times 10^{-12} \text{ C}^2 \text{ N}^{-1} \text{ m}^{-2}$
permeability of vacuum	μ_o	$4\pi \times 10^{-7} \text{ N A}^{-2}$
speed of light in vacuum	c	$3.00 \times 10^{8} \text{m s}^{-1}$
elementary charge	e	$1.602 \times 10^{-19} \text{ C}$
mass of electron	m_e	$9.11 \times 10^{-31} \text{ kg}$
mass of proton	m_p	$1.673 \times 10^{-27} \text{ kg}$
Planck constant	h	$6.63 \times 10^{-34} \text{ J s}$
Avogadro constant	N_A	$6.02 \times 10^{23} \text{ mol}^{-1}$
Boltzmann constant	k	$1.38 \times 10^{-23} \text{ J K}^{-1}$
molar gas constant	R	$8.31 \text{ J mol}^{-1} \text{ K}^{-1}$
standard atmospheric pressure		$1.013 \times 10^{5} \text{ Pa}$

QUESTION #1

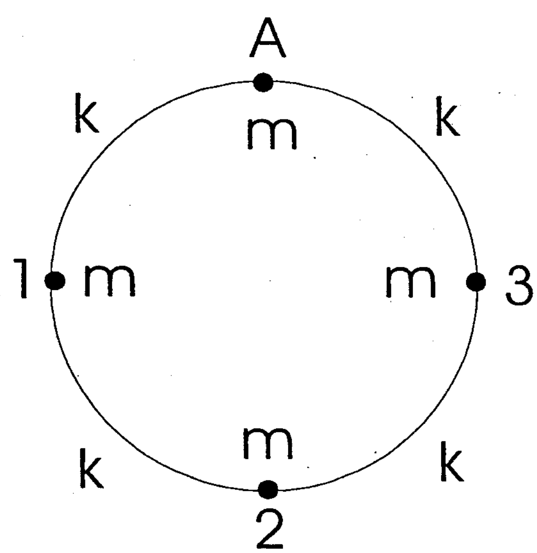
Four beads of mass m slide without friction on a circular hoop of radius a. Each bead interacts with its neighbors via a harmonic potential. For example, the potential between beads 2 and 3 is given by

$$V_{23} = \frac{1}{2}k(x_2 - x_3)^2,$$

where x measures the displacement along the hoop and all of the $k's$ are the same. One bead is constrained to remain at point A. (Ignore gravity)

a. Determine the frequencies for all normal modes.

b. Which frequency is associated with the symmetrical mode in which bead 2 is at rest?



CM Hermanson

Solution:

$$T = \frac{m}{2}(\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2)$$

$$V = \frac{1}{2}k\left(x_1^2 + (x_1-x_2)^2 + (x_2-x_3)^2 + x_3^2\right)$$

$$= k\left(x_1^2 + x_2^2 + x_3^2 - x_1x_2 - x_2x_3\right)$$

mass matrix $\quad \underline{m} = m\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{array}{l} 1 \\ 2 \\ 3 \end{array}$

spring matrix $\quad \underline{V} = k\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$

Euler-Lagrange Equations:

Solve $\underline{m}\,\ddot{\underline{x}} + \underline{V}\,\underline{x} = 0 \quad$ where $\underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

Let $\underline{x} = \rho\, e^{-i\omega t}$, trial solution

$$\Rightarrow \left(-\omega^2\underline{m} + \underline{V}\right)\rho = 0$$

For nontrivial solution, require

$$0 = \left|-\omega^2\underline{m} + \underline{V}\right| = \begin{vmatrix} 2k-\omega^2 m & -k & 0 \\ -k & 2k-\omega^2 m & -k \\ 0 & -k & 2k-\omega^2 m \end{vmatrix}$$

or $\begin{vmatrix} 2-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 2-\lambda \end{vmatrix} = 0 \quad$ where $\lambda = \dfrac{\omega^2 m}{k}$

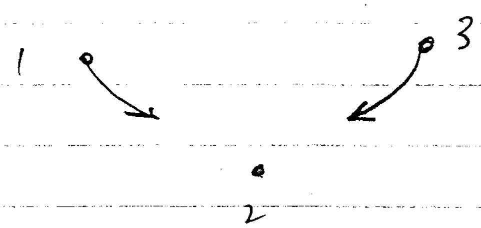
Solve for λ:

$$(2-\lambda)^3 - 2(2-\lambda) = (2-\lambda)\left[(2-\lambda)^2 - 2\right] = 0$$

or $\underbrace{(2-\lambda)}_{\lambda = 2}\underbrace{(\lambda^2 - 4\lambda + 2)}_{\lambda = 2 \pm \sqrt{2^2 - 2} = 2 \pm \sqrt{2}} = 0$

$$\boxed{\omega = \sqrt{\frac{2k}{m}},\ \sqrt{1 \pm \frac{1}{\sqrt{2}}} \cdot \sqrt{\frac{2k}{m}}}$$

The symmetrical mode:



Frequency:
2 springs active for each mass, so $\omega = \sqrt{\frac{2k}{m}}$ expected

Check: $(-\omega^2 \underline{m} + \underline{V})\,\underline{\rho} = 0$, with $\omega = \sqrt{\frac{2k}{m}}$

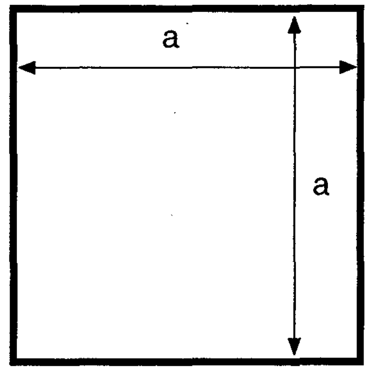
Then $2k - \omega^2 m = 0$ and we must have

$$\begin{pmatrix} 0 & -k & 0 \\ -k & 0 & -k \\ 0 & -k & 0 \end{pmatrix}\begin{pmatrix} \rho_1 \\ \rho_2 \\ \rho_3 \end{pmatrix} = 0$$

or $\rho_2 = 0$, $\rho_1 = -\rho_3$ ✓

QUESTION #2

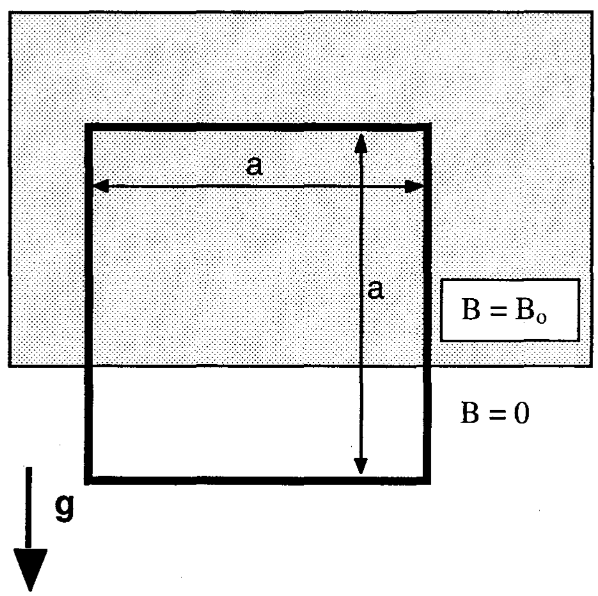
11. A square loop is formed out of superconducting wire (zero resistance). The dimensions of the inside of the loop are a x a, as shown. The radius of the wire is r.



a. Show that for $a >> r$, the self inductance of this loop is

$$L = \frac{2\mu_o a}{\pi} \ln\left(\frac{a}{r}\right)$$

b. The loop is now placed such that part of the loop experiences a region of uniform magnetic field B_o directed into the page, as shown. The loop has mass M, and is released from rest at $t = 0$. (Note: The gravitational force acting on the loop points toward the bottom of the page, and should certainly not be ignored.)



i. Find the current in the loop as a function of time.

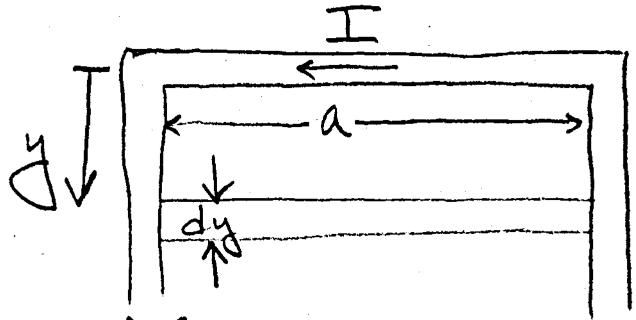
ii. Show that the loop experiences simple harmonic motion, and find the frequency and amplitude of this oscillation as functions of the given parameters.

2 Faraday Oscilator - Solution

A. $L = \frac{\Phi_{tot}}{I}$ (flux through loop per unit of current)

$\Phi_{tot} = 4\,\Phi_{\text{due to one wire}}$

Find $\Phi_{\text{due to one wire}}$:



$d\Phi = B(y)\,dA$

$= \frac{\mu_0 I}{2\pi y}\, a\, dy$

$$\Phi = \int_r^{r+a} d\Phi = \frac{\mu_0 I a}{2\pi}\int_r^{r+a}\frac{dy}{y} = \frac{\mu_0 I a}{2\pi}\ln\left(\frac{r+a}{r}\right)$$

$$\approx \frac{\mu_0 I a}{2\pi}\ln\left(\frac{a}{r}\right) \quad \text{for } a \gg r$$

$$\therefore\ L = \frac{4\Phi}{I} = \frac{2\mu_0 a}{\pi}\ln\left(\frac{a}{r}\right)$$

B. i) At $t = \phi$:
$y = \phi$
$I = \phi$
$v = \phi$

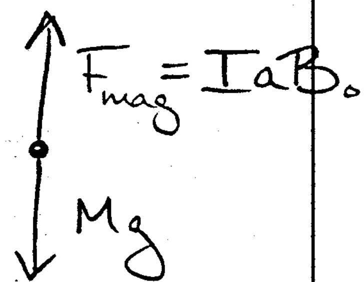
let the positive y direction be down.

(1) $\mathcal{E}_{tot} = I\cancel{R}^{\phi} = \phi$ Given.

$$\therefore \quad \mathcal{E}_{\text{motional}} + \mathcal{E}_{\substack{\text{self}\\\text{inductance}}} = \phi$$

(2) $$B_0 a V - L\frac{dI}{dt} = \phi$$ Now, diff w/ respect to ti

(3) $$B_0 a \frac{F_{net}}{M} - L\frac{d^2 I}{dt^2} = \phi$$



(4) $$F_{net} = Mg - IaB_0$$

(5) $$\therefore \quad \frac{d^2 I}{dt^2} + \frac{B_0^2 a^2}{ML} I = \frac{B_0 a g}{L}$$

The solution is

$$I = I_h + I_p$$

where $$I_h = C_1 \cos \omega_0 t + C_2 \sin \omega_0 t$$

$$\omega_0 = \frac{B_0 a}{\sqrt{ML}}$$

$$I_p = C_3 = \frac{Mg}{B_0 a}$$

Using the condition that $I(t=\phi) = \phi$, we find that $C_1 = -C_3 = -\frac{Mg}{B_0 a}$

Using equation (2) we find that $\frac{dI}{dt} = \phi$ at $t = \phi$, so $C_2 = \phi$.

Finally,

$$\boxed{I(t) = \frac{Mg}{B_0 a}\left(1 - \cos \omega_0 t\right) \qquad \omega_0 = \frac{B_0 a}{\sqrt{ML}}}$$

ii) $\frac{F_{net}}{M} = g - \frac{I a B_0}{M}$ from (4)

$$= g \cos \omega_0 t$$

$$V = +\frac{g}{\omega_0} \sin \omega_0 t + C_4$$

$$V(t=\phi) = \phi \implies C_4 = \phi$$

$$y = -\frac{g}{\omega_0^2} \cos \omega_0 t + C_5$$

$$y(t=\phi) = \phi \implies C_5 = \frac{g}{\omega_0^2}$$

$$y = \frac{g}{\omega_0^2}\left[1 - \cos \omega_0 t\right]$$

Simple Harmonic Motion

Frequency $\omega_0 = \frac{B_0 a}{\sqrt{ML}}$

Amplitude $A = \frac{g}{\omega_0^2} = \frac{MgL}{B_0^2 a^2}$

QUESTION #3

Consider a system of two non-interacting spin-1/2 particles.

a. Describe how you would go about finding the matrix forms of the operators $\hat{S}^2, \hat{S}_z, \hat{S}_x$, and $\hat{S}_y$ describing the absolute value and the three cartesian projections of the total spin of the system, respectively.

b. Assuming that $\hat{S}^2, \hat{S}_x, \hat{S}_y$, and $\hat{S}_z$ are known, describe how you would determine the state of the system when it is known that the expectation values of the z- and x-projections of the total spin are zero and that the expectation value of the absolute value of the total spin and its y-projection have their maximum possible values.

c. Find the matrix forms of $\hat{S}^2, \hat{S}_x, \hat{S}_y$, and $\hat{S}_z$.

Hint: Use the representation (coupled representation) in which $\hat{S}^2$ and $\hat{S}_z$ both have definite values.

d. Find the state vector of the system described in part b.

Quantum mechanics (1)

Solution

1. For the system of two spin-½ particles, appropriate basis is that of coupled representation:

$$
\left.
\begin{array}{lll}
|1\rangle = \alpha(1)\,\alpha(2) & & m_z = 1 \\
|2\rangle = \frac{1}{\sqrt{2}}\left[\alpha(1)\beta(2) + \beta(1)\alpha(2)\right] & & m_z = 0 \\
|3\rangle = \beta(1)\,\beta(2) & & m_z = -1
\end{array}
\right\} S = 1
$$

$$
|4\rangle = \frac{1}{\sqrt{2}}\left[\alpha(1)\beta(2) - \beta(1)\alpha(2)\right] \qquad m_z = 0 \; ; \; S = 0
$$

2. Find matrix form of the operators $\hat{S}^2, \hat{S}_z, \hat{S}_x, \hat{S}_y$:

(a) Absolute value of total spin:

$$
(\langle \hat{S}^2 \rangle)_{kl} = \hbar^2 s(s+1)\,\delta_{kl} = \hbar^2 \cdot 2 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}
$$

(b) z-projection of total spin:

$$
(\langle \hat{S}_z \rangle)_{kl} = \hbar \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}
$$

(c) x-projection of total spin:

$$
\hat{S}_x = \hat{S}_{x1} + \hat{S}_{x2} = \frac{1}{2}\left(\hat{S}_{+1} + \hat{S}_{-1} + \hat{S}_{+2} + \hat{S}_{-2}\right)
$$

(Here we need to evaluate individual matrix elements)

$$\hat{S}_x|1\rangle = \frac{1}{2}\left(\hat{S}_{+1} + \hat{S}_{-1} + \hat{S}_{+2} + \hat{S}_{-2}\right)\alpha(1)\alpha(2)$$

$$= \frac{\hbar}{2}\left[0 + \beta(1)\alpha(2) + 0 + \alpha(1)\beta(2)\right]$$

$$= \frac{\hbar}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}}\left[\alpha(1)\beta(2) + \beta(1)\alpha(2)\right] = \frac{\hbar}{\sqrt{2}}|2\rangle$$

$$\hat{S}_x|2\rangle = \frac{1}{2}\left(\hat{S}_{+1} + \hat{S}_{-1} + \hat{S}_{+2} + \hat{S}_{-2}\right)\frac{1}{\sqrt{2}}\left[\alpha(1)\beta(2) + \beta(1)\alpha(2)\right]$$

$$= \frac{\hbar}{2\sqrt{2}}\left[0 + \alpha(1)\alpha(2) + \beta(1)\beta(2) + \alpha(1)\alpha(2) + 0 + 0 + \beta(1)\beta(\right.$$

$$= \frac{\hbar}{\sqrt{2}}\left[\alpha(1)\alpha(2) + \beta(1)\beta(2)\right] = \frac{\hbar}{\sqrt{2}}\left[|1\rangle + |3\rangle\right]$$

$$\hat{S}_x|3\rangle = \frac{1}{2}\left(\hat{S}_{+1} + \hat{S}_{-1} + \hat{S}_{+2} + \hat{S}_{-2}\right)\beta(1)\beta(2)$$

$$= \frac{\hbar}{2}\left(\alpha(1)\beta(2) + 0 + \beta(1)\alpha(2) + 0\right) = \frac{\hbar}{\sqrt{2}}|2\rangle$$

$$\hat{S}_x|4\rangle = \frac{1}{2}\left(\hat{S}_{+1} + \hat{S}_{-1} + \hat{S}_{+2} + \hat{S}_{-2}\right)\frac{1}{\sqrt{2}}\left[\alpha(1)\beta(2) - \beta(1)\alpha(2)\right]$$

$$= \frac{\hbar}{2\sqrt{2}}\left[0 - \cancel{\alpha(1)\alpha(2)} + \cancel{\beta(1)\beta(2)} - 0 + \cancel{\alpha(1)\alpha(2)} - 0 + 0 - \cancel{\beta(1)\beta(2)}\right.$$

$$= 0$$

$$\left(\langle\hat{S}_x\rangle\right)_{kl} = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

(d) y- projection of total spin:

$$\hat{S}_y = \hat{S}_{y1} + \hat{S}_{y2} = \frac{1}{2i}\left(\hat{S}_{+1} - \hat{S}_{-1} + \hat{S}_{+2} - \hat{S}_{-2}\right)$$

(evaluate individual matrix elements)

$$\hat{S}_y|1\rangle = \frac{1}{2i}\left(\hat{S}_{+1} - \hat{S}_{-1} + \hat{S}_{+2} - \hat{S}_{-2}\right)\alpha(1)\alpha(2)$$

$$= \frac{\hbar}{2i}\left[0 - \beta(1)\alpha(2) + 0 - \alpha(1)\beta(2)\right] = -\frac{\hbar}{\sqrt{2}\,i}|2\rangle$$

$$\hat{S}_y|2\rangle = \frac{1}{2i}\left(\hat{S}_{+1} - \hat{S}_{-1} + \hat{S}_{+2} - \hat{S}_{-2}\right)\frac{1}{\sqrt{2}}\left[\alpha(1)\beta(2) + \beta(1)\alpha(2)\right]$$

$$= \frac{\hbar}{2\sqrt{2}\,i}\left[0 + \alpha(1)\alpha(2) - \beta(1)\beta(2) - 0 + \alpha(1)\alpha(2) + 0 - 0 - \beta(1)\beta(2)\right]$$

$$= \frac{\hbar}{\sqrt{2}\,i}\left[\alpha(1)\alpha(2) - \beta(1)\beta(2)\right] = \frac{\hbar}{\sqrt{2}\,i}\left[|1\rangle - |3\rangle\right]$$

$$\hat{S}_y|3\rangle = \frac{1}{2i}\left(\hat{S}_{+1} - \hat{S}_{-1} + \hat{S}_{+2} - \hat{S}_{-2}\right)\beta(1)\beta(2)$$

$$= \frac{\hbar}{2i}\left[\alpha(1)\beta(2) - 0 + \beta(1)\alpha(2) - 0\right] = \frac{\hbar}{\sqrt{2}\,i}|2\rangle$$

$$\hat{S}_y|4\rangle = \frac{1}{2i}\left(\hat{S}_{+1} - \hat{S}_{-1} + \hat{S}_{+2} - \hat{S}_{-2}\right)\frac{1}{\sqrt{2}}\left[\alpha(1)\beta(2) - \beta(1)\alpha(2)\right]$$

$$= \frac{\hbar}{2\sqrt{2}\,i}\left[0 - \alpha(1)\alpha(2) - \beta(1)\beta(2) + 0 + \alpha(1)\alpha(2) - 0 - 0 + \beta(1)\beta(2)\right]$$

$$= 0$$

$$\left(\langle\hat{S}_y\rangle\right)_{kl} = \frac{\hbar}{\sqrt{2}\,i}\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

3. Normalized spin state function is given in most general form:

$$|\chi\rangle = C_1 |1\rangle + C_2 |2\rangle + C_3 |3\rangle + C_4 |4\rangle,$$ where

Overall phase has been excluded

$$C_1 = |C_1| \quad ; \quad C_2 = |C_2| e^{i\varphi_2}$$
$$C_3 = |C_3| e^{i\varphi_3} \quad ; \quad C_4 = |C_4| e^{i\varphi_4}$$

and $|C_1|^2 + |C_2|^2 + |C_3|^2 + |C_4|^2 = 1$

4. $\langle \chi | \hat{S}^2 | \chi \rangle = 2\hbar^2$ ← maximum possible value of total spin

$$\Rightarrow (C_1^*, C_2^*, C_3^*, C_4^*)\, \hbar^2 \cdot 2 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{pmatrix} = 2\hbar^2$$

$$\Rightarrow |C_1|^2 + |C_2|^2 + |C_3|^2 = 1$$

First equation gives: $C_4 = 0$

5. $\langle \chi | \hat{S}_z | \chi \rangle = 0$

$$\Rightarrow (C_1^*, C_2^*, C_3^*, 0)\, \hbar \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \\ C_3 \\ 0 \end{pmatrix} = 0$$

$$\Rightarrow \quad |C_1|^2 - |C_3|^2 = 0$$

Second equation gives: $|C_1| = |C_3|$

At this point, the state function can be written as:

$$|\chi\rangle = |C_1| |1\rangle + |C_2| e^{i\varphi_2} |2\rangle + |C_1| e^{i\varphi_3} |3\rangle,$$

where $|C_2| = \sqrt{1 - 2|C_1|^2}$

6. $\langle \chi | \hat{S}_x | \chi \rangle = 0$

$$\Rightarrow \left(|C_1|, |C_2| e^{-i\varphi_2}, |C_1| e^{-i\varphi_3}, 0\right) \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} |C_1| \\ |C_2| e^{i\varphi_2} \\ |C_1| e^{i\varphi_3} \\ 0 \end{pmatrix} = 0$$

$$\Rightarrow \frac{\hbar}{\sqrt{2}} \left(|C_1|, |C_2| e^{-i\varphi_2}, |C_1| e^{-i\varphi_3}, 0\right) \begin{pmatrix} |C_2| e^{i\varphi_2} \\ |C_1|(1+e^{i\varphi_3}) \\ |C_2| e^{i\varphi_2} \\ 0 \end{pmatrix} = 0$$

$$\frac{\hbar}{\sqrt{2}} \left[|C_1||C_2| e^{i\varphi_2} + |C_1||C_2| e^{-i\varphi_2} (1 + e^{i\varphi_3}) + |C_1||C_2| e^{-i\varphi_3} e^{i\varphi_2} \right] = 0$$

$$\Rightarrow |C_1||C_2| \Big[\underbrace{e^{i\varphi_2} + e^{-i\varphi_2}}_{2\cos\varphi_2} + \underbrace{e^{i(\varphi_3-\varphi_2)} + e^{-i(\varphi_3-\varphi_2)}}_{2\cos(\varphi_3-\varphi_2)} \Big] = 0$$

$$\boxed{\cos\varphi_2 + \cos(\varphi_3 - \varphi_2) = 2 \cos\frac{\varphi_3}{2} \cos\frac{2\varphi_2-\varphi_3}{2}}$$

$$\Rightarrow |C_1||C_2| \cos\frac{\varphi_3}{2} \cos\frac{2\varphi_2-\varphi_3}{2} = 0$$

3^{rd} equation gives following possibilities;

(a) $C_1 = 0$

(b) $C_2 = 0$

(c) $\cos\frac{\varphi_3}{2} = 0 \Rightarrow \varphi_3 = \pm\pi$

(d) $\cos\frac{2\varphi_2-\varphi_3}{2} = 0 \Rightarrow 2\varphi_2 - \varphi_3 = \pm\pi$

7. $\langle \chi | \hat{S}_y | \chi \rangle = \hbar$ ← maximum value for projection of total spin

$$\left(|C_1|, |C_2| e^{-i\varphi_2}, |C_1| e^{-i\varphi_3}, 0\right) \frac{\hbar}{\sqrt{2}i} \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} |C_1| \\ |C_2| e^{i\varphi_2} \\ |C_1| e^{i\varphi_3} \\ 0 \end{pmatrix} = \hbar$$

$$\Rightarrow \frac{1}{\sqrt{2}i} \left(|C_1|, |C_2| e^{-i\varphi_2}, |C_1| e^{-i\varphi_3}, 0\right) \begin{pmatrix} |C_2| e^{i\varphi_2} \\ |C_1|(-1 + e^{i\varphi_3}) \\ -|C_2| e^{i\varphi_2} \\ 0 \end{pmatrix} = 1$$

$$\Rightarrow \frac{1}{\sqrt{2}i} \left[|C_1||C_2| e^{i\varphi_2} + |C_1||C_2| e^{-i\varphi_2}(e^{i\varphi_3} - 1) - |C_1||C_2| e^{i\varphi_2} e^{-i\varphi_3} \right] = 1$$

$$\Rightarrow \frac{|C_1||C_2|}{\sqrt{2}i} \left[\underbrace{e^{i\varphi_2} - e^{-i\varphi_2}}_{2i \sin\varphi_2} + \underbrace{e^{i(\varphi_3 - \varphi_2)} - e^{-i(\varphi_3 - \varphi_2)}}_{2i \sin(\varphi_3 - \varphi_2)} \right] = 1$$

$$\boxed{\sin\varphi_2 + \sin(\varphi_3 - \varphi_2) = 2 \sin\frac{\varphi_3}{2} \cos\frac{2\varphi_2 - \varphi_3}{2}}$$

$$\Rightarrow \frac{4\cancel{i}|C_1||C_2|}{\sqrt{2}\cancel{i}} \sin\frac{\varphi_3}{2} \cos\frac{2\varphi_2 - \varphi_3}{2} = 1$$

Together with the 3^{rd} equation, we obtain:

$|C_1| \neq 0$; $|C_2| \neq 0$; $\cos\frac{2\varphi_2 - \varphi_3}{2} \neq 0$

$$\boxed{\varphi_3 = \pm\pi} \Rightarrow \cos\frac{2\varphi_2 - \varphi_3}{2} = \cos\left(\varphi_2 \mp \frac{\pi}{2}\right) = \pm \sin\varphi_2$$

$$\boxed{\sqrt{2} \cdot 2 |C_1||C_2| \sin\varphi_2 = 1}$$

8. $$\sqrt{2}\cdot 2\,|C_1|\sqrt{1-2|C_1|^2}\,\sin\varphi_2 = 1$$

$$|C_1|\sqrt{1-2|C_1|^2} = \frac{1}{\sqrt{2}\cdot 2\sin\varphi_2} \equiv A$$

$$|C_1|^2(1-2|C_1|^2) = A^2$$

$$-2|C_1|^4+|C_1|^2-A^2=0$$

$$|C_1|^4-\frac{1}{2}|C_1|^2+\frac{1}{2}A^2=0$$

$$|C_1|^2=\frac{1}{4}\pm\sqrt{\frac{1}{16}-\frac{1}{2}A^2} \Rightarrow \frac{1}{16}\geqslant\frac{1}{2}A^2=\frac{1}{2\cdot 2\cdot 4\sin^2\varphi_2}$$

$$\Rightarrow \boxed{\varphi_2=\frac{\pi}{2};\ |C_1|=\frac{1}{2}}$$

Answer:

$$\boxed{|\chi\rangle = \frac{1}{2}|1\rangle+\frac{i}{\sqrt{2}}|2\rangle-\frac{1}{2}|3\rangle}$$

Quick check of the result:

$$\left(\tfrac{1}{2}, \tfrac{-i}{\sqrt{2}}, -\tfrac{1}{2}, 0\right)\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} \frac{1}{2} \\ \frac{i}{\sqrt{2}} \\ -\frac{1}{2} \\ 0 \end{pmatrix}$$

$$= \left(\tfrac{1}{2}, \tfrac{-i}{\sqrt{2}}, -\tfrac{1}{2}, 0\right)\begin{pmatrix} \frac{i}{\sqrt{2}} \\ 0 \\ \frac{i}{\sqrt{2}} \\ 0 \end{pmatrix} = \left[\frac{i}{2\sqrt{2}} - 0 - \frac{i}{2\sqrt{2}}\right] = \underline{\underline{0}}$$

$$\frac{\hbar}{\sqrt{2}\, i}\left(\tfrac{1}{2}, \tfrac{-i}{\sqrt{2}}, -\tfrac{1}{2}, 0\right)\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} \frac{1}{2} \\ \frac{i}{\sqrt{2}} \\ -\frac{1}{2} \\ 0 \end{pmatrix}$$

$$= \frac{\hbar}{\sqrt{2}\, i}\left(\tfrac{1}{2}, \tfrac{-i}{\sqrt{2}}, -\tfrac{1}{2}, 0\right)\begin{pmatrix} \frac{i}{\sqrt{2}} \\ -1 \\ -\frac{i}{\sqrt{2}} \\ 0 \end{pmatrix} = \frac{\hbar}{\sqrt{2}\, i}\left(\frac{i}{2\sqrt{2}} + \frac{i}{\sqrt{2}} + \frac{i}{2\sqrt{2}}\right)$$

$$= \frac{\cancel{i}\hbar\, \cancel{2}}{\cancel{\sqrt{2}}\cancel{i}\cancel{\sqrt{2}}} = \underline{\underline{\hbar}}$$

Alternative solution to the matrix form of the operators:

Use uncoupled representation

$$|1\rangle = \alpha(1)\,\alpha(2) = |++\rangle$$

$$|2\rangle = \alpha(1)\,\beta(2) = |+-\rangle$$

$$|3\rangle = \beta(1)\,\alpha(2) = |-+\rangle$$

$$|4\rangle = \beta(1)\,\beta(2) = |--\rangle$$

$$\hat{S}_x|++\rangle = \frac{\hbar}{2}\left(|-+\rangle + |+-\rangle\right); \quad \hat{S}_y|++\rangle = \frac{\hbar}{2i}\left(-|+-\rangle - |-+\rangle\right)$$

$$\hat{S}_x|+-\rangle = \frac{\hbar}{2}\left(|++\rangle + |--\rangle\right)$$

$$\hat{S}_x|-+\rangle = \frac{\hbar}{2}\left(|++\rangle + |--\rangle\right)$$

⋮

etc.

⋮

etc.

Matrix representation of operators:

$$\langle\hat{S}_z\rangle_{kl} = \hbar\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \qquad \langle\hat{S}_x\rangle_{kl} = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

$$\langle\hat{S}_y\rangle_{kl} = \frac{\hbar}{2i}\begin{pmatrix} 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & -1 & 0 \end{pmatrix}$$

$$\langle\hat{S}^2\rangle_{kl} = \langle(\hat{S}_z)^2\rangle + \langle\hat{S}_x\rangle^2 + \langle\hat{S}_y\rangle^2 = \hbar^2\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

Alternative solution to finding the state function

Since it is known that $\langle \hat{S}^2 \rangle$ and $\langle \hat{S}_y \rangle$ have both maximum value, it is clear that the state has to be that $| S = 1, m_y = 1 \rangle$. In this state both spins are oriented in y-direction and the state function is a product.

$$| S_{TOT} = 1, m_{y_{TOT}} = 1 \rangle = | s_1 = \tfrac{1}{2} ; m_{1y} = \tfrac{1}{2} \rangle \, | s_2 = \tfrac{1}{2} ; m_{2y} = \tfrac{1}{2} \rangle$$

where $| s_1 = \frac{1}{2} ; m_{1y} = \frac{1}{2} \rangle$ is eigenfunction of $\hat{S}_1^2$ and $\hat{S}_{1y}$

and $| s_2 = \frac{1}{2} ; m_{2y} = \frac{1}{2} \rangle$ —— " —— of $\hat{S}_2^2$ and $\hat{S}_{2y}$.

In uncoupled z-representation we get:

$$\left. \begin{aligned} | s_1 = \tfrac{1}{2} ; m_{1y} = \tfrac{1}{2} \rangle &= \tfrac{1}{\sqrt{2}} \left[\alpha_z(1) + i \beta_z(1) \right] \\ | s_2 = \tfrac{1}{2} ; m_{2y} = \tfrac{1}{2} \rangle &= \tfrac{1}{\sqrt{2}} \left[\alpha_z(2) + i \beta_z(2) \right] \end{aligned} \right\}$$

The state function is:

$$| S_{TOT} = 1, m_{y_{TOT}} = 1 \rangle = \frac{1}{2} \left[\alpha(1) + i\beta(1) \right] \left[\alpha(2) + i\beta(2) \right]$$

$$= \underline{\underline{\frac{1}{2} \left[\alpha(1)\alpha(2) + i\left[\alpha(1)\beta(2) + \alpha(2)\beta(1)\right] - \beta(1)\beta(2) \right]}}$$

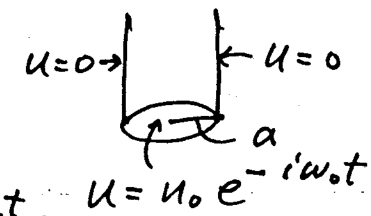
QUESTION #4

A scalar quantity $u(\mathbf{r}, t)$ satisfies the wave equation

$$\nabla^2 u - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = 0,$$

inside a hollow cylindrical pipe of radius a with $u = 0$ on the walls of the pipe. If at the end $z = 0$, $u = u_o e^{-i\omega_o t}$, waves will be sent down the pipe with various spatial distributions (modes). Find the phase velocity of the fundamental mode as a function of the frequency ω_o and interpret the result for small ω_o.

Comp98



MP1: S. Tsuruta   Key.

Boundary condn: $\begin{cases} z=0 \to u=u_0e^{-i\omega_0 t} & ① \\ u=0 \text{ when } \rho=a & ② \end{cases}$

Use cylindrical coordinates:

$$\nabla^2 u - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = 0 \quad ③ \quad \text{given.}$$

① → ③ & get $\nabla^2 u_0 + k_0^2 u_0 = 0$ ④, where $k_0 = \omega_0/c$ ⑤.
④ is Helmholtz eqn in cylindrical coordinates.
So the soln is:

$$u = J_m[\sqrt{k_0^2-\alpha^2}\,\rho]\,e^{i\alpha z - i\omega_0 t} \quad ⑥$$

If the boundary condn at $\rho=a$ is to be obeyed, we must have:
$J_m(x)=0$ at $\rho=a$, where $x \equiv \sqrt{k_0^2-\alpha^2}\,\rho$.
$\therefore\ x=\sqrt{k_0^2-\alpha^2}\,a \to J_m(x)=0$
For the lowest mode, $\sqrt{k_0^2-\alpha^2}\ a = 2.4$.
So $\alpha^2 = k_0^2-(2.4/a)^2 = (\omega_0^2/c^2)-(2.4/a)^2$. ⑦

So the solution is ⑥ with $\alpha = ((\omega_0/c)^2-(2.4/a)^2)^{1/2}$ ⑧.

The phase velocity down the pipe is
$v_{ph} = \frac{dz}{dt} = \frac{\omega_0}{\alpha}$ ⑨ from ⑥.
⑧ → ⑨ & get

$$v_{ph} = \frac{\omega_0 c}{\sqrt{\omega_0^2 - 2.4^2c^2/a^2}} = \frac{c}{\sqrt{1-\left(\frac{2.4^2c^2}{a^2\omega_0^2}\right)}} \quad // \text{ans.}$$

For small ω_0, the expression for α becomes imaginary. This means the wave is damped, and will not propagate for ω_0 less than $\omega_c = 2.4c/a$, the cut-off frequency. Near $\omega \simeq \omega_c$ the phase velocity becomes infinite.

QUESTION #5

Spaceship A travels from the Sun to α-Centauri, a distance of four light-years, at a constant speed $v_0 = c/2$. Spaceship B, starting at rest, departs with Spaceship A, and undergoes constant acceleration in the frame at rest with respect to the Sun. Both spaceships reach α-Centauri at the same time as seen in a frame at rest with respect to the Sun and α-Centauri.

a. How long does the journey take as observed by an astronaut on Spaceship A?

b. How long does the journey take as observed by an astronaut on Spaceship B?

5)

a) Elapsed proper time for space ship A:

$$\tau = (1 - v_0^2/c^2)^{1/2}\, t,$$

where t is the time elapsed in the frame of the Sun (8 yr).

$$\boxed{\tau = 4\sqrt{3}\ \text{yr} = 6.9\ \text{yr}}$$

b) Interval in proper time relative to interval in Sun's frame is given by

$$d\tau = (1 - v^2/c^2)^{1/2}\, dt.$$

Now $v = at$, with a = constant acceleration.

Total elapsed proper time is

$$\tau = \int_0^t (1 - a^2 t'^2/c^2)^{1/2}\, dt'$$

Define $u \equiv at/c$

$$\Rightarrow \tau = \frac{1}{2}\frac{c}{a}\left[u\sqrt{1-u^2} + \sin^{-1} u\right]$$

Since ships A and B reach α-Centauri at same time (in Sun's frame):

$$\frac{at^2}{2} = v_0 t \Rightarrow at = 2v_0,\ u = 1 \Rightarrow \boxed{\tau = \frac{1}{2}\frac{c}{a}\cdot\frac{\pi}{2} = \frac{\pi}{4}t = 2\pi\ \text{yr} \quad 6.3\ \text{yr}}$$

QUESTION #6

Consider a long, uniform hexagonal prism (like a #2 yellow wooden pencil) with mass m and side-length a. The moment of inertia of the prism about its center is $5/12\, ma^2$. The prism is initially at rest with its axis horizontal on an inclined plane that makes an angle θ with the horizontal. Assume that the surfaces of the prism are slightly concave so that the prism only touches the plane at its edges. The prism is given a push so that it begins rolling down the plane. Assume that friction prevents any sliding and that the prism does not lose contact with the plane. The latter assumption means that the collisions of the edges of the prism with the plane are inelastic. (If you cannot complete any part, you may use the previously defined constants in your answer to a subsequent part.)

a. If we consider an infinitesimally short period of time from just before a given edge hits the plane until immediately after it hits, the angular momentum of the prism about this edge is conserved. (Angular momentum about the center of mass is not conserved.) Argue that this is true.

b. Let ω_i denote the angular speed just before a given edge hits the plane and ω_f denote the angular speed immediately after the impact. Use conservation of angular momentum about this edge to show that we may write

$$\omega_f = \alpha\omega_i,$$

where α is a numerical constant. Find α.

c. The kinetic energies just before and after the impact are similary denoted by K_i and K_f. Show that we may write

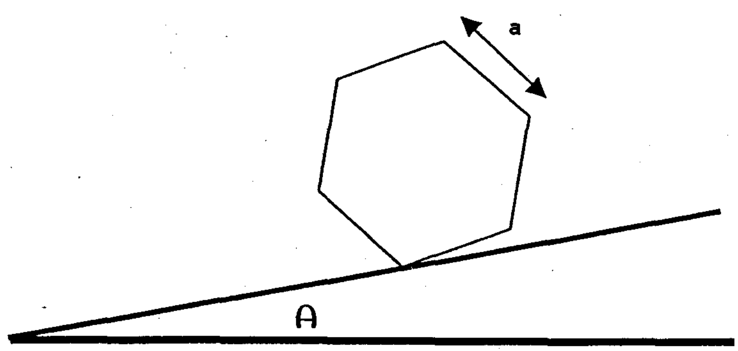
$$K_f = \beta K_i$$

and find the numerical constant β.

d. For the next impact to occur K_i must have a minimum value K_{min} which may be written in the form

$$K_{min} = \gamma mga,$$

where g is the acceleration due to gravity. Find the coefficient γ in terms of the slope angle θ.



6. a. There are only two forces to consider: gravity and the force of the plane acting on the edge. Obviously, the force of the plane cannot produce a torque or an angular impulse about this edge. However, the force of gravity will, in general, produce a torque about the edge. But when we multiple this torque by the time interval, the angular impulse is negligibly small. Therefore, angular momentum is conserved about the edge. Note that this argument does not apply to the center of mass, because the angular impulse of the force of the plane does not vanish. If you wish, you can consider this as being a delta function force.

b. The angular momentum about the edge is given by the angular momentum about the center of mass plus the angular momentum of the center of mass about the edge

$$\vec{L} = \vec{L}_c + m\vec{r}_c \times \vec{v}_c,$$

where the subscript c refers to the center of mass quantity. Note that $\vec{v}_c$ is directed 30° downward relative to the plane before the impact and 30° upward after the impact. Also note that $v_c = a\omega$.

Using conservation of angular momentum about the edge, we obtain

$$L_i = I_c\omega_i + \tfrac{1}{2}ma^2\omega_i = L_f = I_c\omega_f + ma^2\omega_f.$$

Therefore,

$$\tfrac{11}{12}ma^2\omega_i = \tfrac{17}{12}ma^2\omega_f$$

or

$$\alpha = \tfrac{11}{17}.$$

c. The total kinetic energy is given by the kinetic energy of translation plus the kinetic energy of rotation

$$K = \tfrac{1}{2}I_c\omega^2 + \tfrac{1}{2}mv_c^2 = \tfrac{1}{2}\left(\tfrac{5}{12}\right)ma^2\omega^2 + \tfrac{1}{2}ma^2\omega^2 \propto \omega^2.$$

Therefore,

$$K_f = \left(\frac{\omega_f}{\omega_i}\right)^2 K_i = \left(\tfrac{11}{17}\right)^2 K_i$$

and

$$\beta = \alpha^2 = \left(\tfrac{11}{17}\right)^2.$$

d. In order for the next impact to occur, the kinetic energy remaining after the impact must be at least large enough to lift the center of mass vertically over the edge.

$$K_f = \left(\tfrac{11}{17}\right)^2 K_i \geq mga\left[1 - \cos\left(30^\circ - \theta\right)\right].$$

Therefore,

$$\gamma = \left(\tfrac{17}{11}\right)^2\left[1 - \cos\left(30^\circ - \theta\right)\right].$$

QUESTION #7

At $t = 0$, an electron is placed in a constant, uniform magnetic field **B** directed along the x-axis.

a. Give the 2 x 2 matrix representing the spin Hamiltonian $H = -\mu\cdot\mathbf{B}$ in the S_z representation.

b. Find the eigenvalues and eigenkets of H.

c. If the electron has $S_z = -\frac{\hbar}{2}$ at $t = 0$, determine its spin state for $t > 0$.

d. Evaluate $\langle S_z \rangle$ for $t > 0$.

QM Hermanson

Solution

(a) $H = -\vec{\mu}\cdot\vec{B}$ with $\vec{\mu} = \frac{e}{mc}\vec{S}$, $e<0$

$$= -\frac{eB}{mc} S_x = \underline{\underline{\frac{\hbar\omega_0}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}}} \quad ; \quad \omega_0 = -\frac{eB}{mc} > 0$$

(b) $\boxed{E_\pm = \pm\frac{\hbar\omega_0}{2}}$ eigenvalues of H

eigenvector for $E = +\frac{\hbar\omega_0}{2}$:

$$\begin{pmatrix}-1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}a\\ b\end{pmatrix} = 0 \Rightarrow a = b = \frac{1}{\sqrt{2}}$$

$$\boxed{\chi_+ = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ 1\end{pmatrix}}$$

For $E = -\frac{\hbar\omega_0}{2}$:

$$\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}\begin{pmatrix}a\\ b\end{pmatrix} = 0 \Rightarrow a = -b = \frac{1}{\sqrt{2}}$$

$$\boxed{\chi_- = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -1\end{pmatrix}}$$

(c) For $t>0$,

$$\chi = e^{-i\frac{Ht}{\hbar}}\begin{pmatrix}0\\ 1\end{pmatrix}$$

$$= e^{-i\frac{Ht}{\hbar}}\cdot\frac{1}{\sqrt{2}}(\chi_+ - \chi_-)$$

$$= \frac{1}{\sqrt{2}}\left(e^{-i\omega_+ t}\chi_+ - e^{-i\omega_- t}\chi_-\right),$$

where $\omega_\pm = E_\pm/\hbar = \pm\frac{\omega_0}{2}$

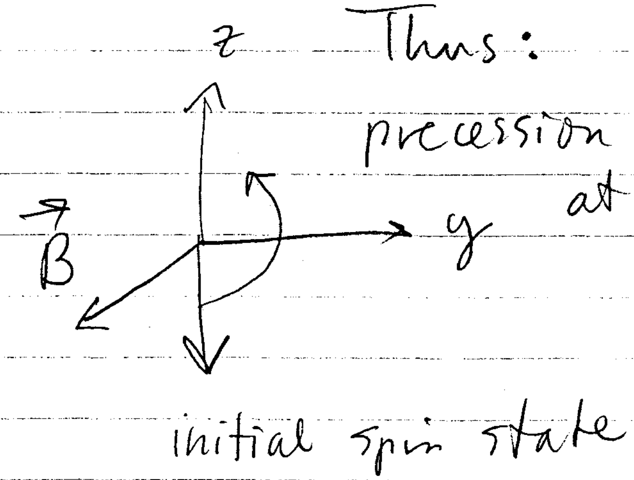
$$\chi = \frac{1}{2}\begin{pmatrix} e^{-i\frac{w_0 t}{2}} - e^{i\frac{w_0 t}{2}} \\ e^{-i\frac{w_0 t}{2}} + e^{i\frac{w_0 t}{2}} \end{pmatrix}$$

$$\boxed{\chi = \begin{pmatrix} -i \sin \frac{\omega_0 t}{2} \\ \cos \frac{\omega_0 t}{2} \end{pmatrix} \begin{matrix} + \\ - \end{matrix}}$$

(d) $\langle S_z \rangle = \frac{\hbar}{2} \sin^2 \frac{\omega_0 t}{2} - \frac{\hbar}{2} \cos^2 \frac{\omega_0 t}{2}$

$$= -\frac{\hbar}{2}\left(\cos^2 \frac{\omega_0 t}{2} - \sin^2 \frac{\omega_0 t}{2}\right)$$

$$\boxed{\langle S_z \rangle = -\frac{\hbar}{2} \cos \omega_0 t}$$



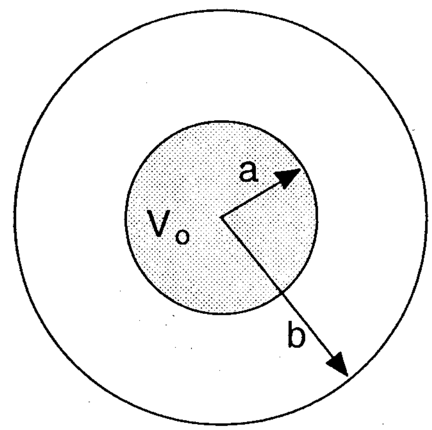
Thus:
precession of $\vec{S}$ about $\vec{B}$ at frequency ω_0 ✓

QUESTION #8

A conducting sphere of radius a, held at potential V_o, is surrounded by a thin concentric spherical shell of radius b, over which someone has glued a surface charge

$$\sigma(\theta) = \sigma_o \cos\theta$$

where σ_o is constant and θ is the usual spherical coordinate between the position vector and the z-axis.



a. Find the electrostatic potential in each region:

   i. $r > b$

   ii. $a < r < b$

b. Find the induced surface charge $\sigma_i(\theta)$ on the conductor.

c. Find the total charge of this system.

d. Show that your answers are consistent with the behavior of V at large r.

Problem #8

a) Solution to Laplace's Eq. with azimuthal symmetry is:

$$V_{out}(r,\theta) = \sum_{\ell=0}^{\infty}\left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}}\right)P_\ell(\cos\theta)$$

for $r>b$, $A_\ell = \phi$ for all ℓ, because $V(r\to\infty)\to$

for $a<r<b$:

$$V_{in}(r,\theta) = \sum_{\ell=0}^{\infty}\left(C_\ell r^\ell + \frac{D_\ell}{r^{\ell+1}}\right)P_\ell(\cos\theta)$$

Since the boundary conditions are either zeroth order or first order in $\cos\theta$, let us try to construct solutions with only the $\ell=0$ and $\ell=1$ terms. This gives us six unknowns: $B_0, B_1, C_0, C_1, D_0, D_1$

The boundary conditions:

1) $V_{in}(a,\theta) = V_0$

2) $V_{in}(b,\theta) = V_{out}(b,\theta)$

3) $\left[\frac{\partial V_{out}}{\partial r} - \frac{\partial V_{in}}{\partial r}\right]_{r=b} = -\frac{\sigma_0\cos\theta}{\epsilon_0}$

will yield the six equations we need.

1) P_0 terms: $C_0 + \frac{D_0}{a} = V_0$

P_1 terms: $C_1 a + \frac{D_1}{a^2} = 0$

2) P_0 terms: $\frac{B_0}{b} = C_0 + \frac{D_0}{b}$

P_1 terms: $\frac{B_1}{b^2} = C_1 b + \frac{D_1}{b^2}$

3) P_0 terms: $-\frac{B_0}{b^2} + \frac{D_0}{b^2} = 0$

P_1 terms: $-\frac{2B_1}{b^3} + \frac{2D_1}{b^3} - C_1 = -\frac{\sigma_0}{\epsilon_0}$

from 3), $B_0 = D_0$ and from 2), $C_0 = 0$. We may then solve 1) for $D_0 = aV_0$.

The $l=1$ coefficients may similarly be found.

$$B_0 = aV_0$$
$$C_0 = 0$$
$$D_0 = aV_0$$
$$B_1 = \frac{-\sigma_0}{3\epsilon_0}(a^3 - b^3)$$
$$C_1 = +\frac{\sigma_0}{3\epsilon_0}$$
$$D_1 = -\frac{\sigma_0 a^3}{3\epsilon_0}$$

$$V_{out}(r,\theta) = \frac{aV_0}{r} - \frac{\sigma_0(a^3-b^3)\cos\theta}{3\epsilon_0 r^2}$$

$$V_{in}(r,\theta) = \frac{aV_0}{r} + \frac{\sigma_0}{3\epsilon_0} r\cos\theta - \frac{\sigma_0 a^3 \cos\theta}{3\epsilon_0 r^2}$$

$$= \frac{aV_0}{r} + \frac{\sigma_0(r^3-a^3)\cos\theta}{3\epsilon_0 r^2}$$

b) $\sigma(a,\theta) = -\epsilon_0 \left.\frac{\partial V_{in}}{\partial r}\right|_{r=a}$

$$= -\epsilon_0\left[-\frac{aV_0}{r^2} + \frac{2\sigma_0(a^3-b^3)\cos\theta}{3\epsilon_0 r^3}\right]_{r=a}$$

$$= -\epsilon_0\left[-\frac{V_0}{a} + \frac{2\sigma_0}{3\epsilon_0}\left(1-\frac{b^3}{a^3}\right)\cos\theta\right]$$

c) $Q_{in} = \int \sigma(a,\theta)\, a^2 \sin\theta \, d\theta\, d\varphi$

$$= 2\pi a^2 \int_0^\pi \sigma(a,\theta)\sin\theta\, d\theta$$

$$= 2\pi a^2 \int_{-1}^{1} \sigma(a,u)\, du \qquad u = \cos\theta$$

$= 4\pi a \epsilon_0 V_0$ as expected.

$Q_{out} = \int \sigma_0 \cos\theta\, a^2 \sin\theta\, d\theta\, d\varphi = 0$

$Q_{tot} = 4\pi a \epsilon_0 V_0$

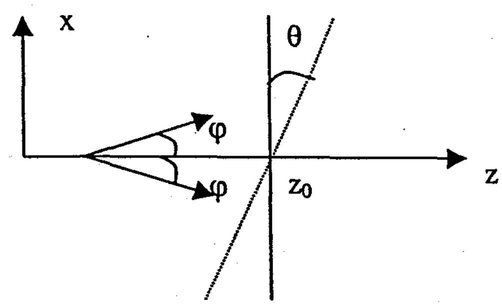
d) for large r, $V(r,\theta) \to \frac{Q_{tot}}{4\pi\epsilon r} = \frac{aV_0}{r}$

QUESTION #9

Consider two monochromatic plane waves of wavelength λ propagating at angles ϕ and $-\phi$ with respect to the positive z-axis. Both waves have unit amplitude and are linearly polarized in the y-direction.

a. Suppose that the waves interfere on a flat screen which is located at $z = z_0$ and is perpendicular to the z-axis. What is the average intensity on the screen as a function of x and y? What is the spacing of the interference fringes on the screen?

b. What is the spacing of the interference fringes if the screen is now inclined at angle θ with respect to the x-axis as shown in the figure?



Optics (3).

(a) The amplitude of the optical field is given by superposition of two propagating waves:

$$E(\vec{r},t) = E_1(\vec{r},t) + E_2(\vec{r},t) \text{ , where}$$

$$E_1(\vec{r},t) = e^{i(\omega t - \vec{k}_1\vec{r})}$$

$$E_2(\vec{r},t) = e^{i(\omega t - \vec{k}_2\vec{r})}$$

and $\omega = \frac{2\pi c}{\lambda}$; $\vec{k}_1 = \frac{2\pi}{\lambda}(\sin\varphi, 0, \cos\varphi)$; $\vec{k}_2 = \frac{2\pi}{\lambda}(-\sin\varphi, 0, c$

The intensity in the plane $z = z_0$ is

$$I(x,y,z=z_0) = \left\{ \left| E(\vec{r},t) \right|^2 \right\}_{z=z_0} = \left\{ \left| e^{i(\omega t - \vec{k}_1\vec{r})} + e^{i(\omega t - \vec{k}_2\vec{r})} \right|^2 \right.$$

$$= \left\{ \left| e^{i(\omega t - \vec{k}_1\vec{r})} \right|^2 + \left| e^{i(\omega t - \vec{k}_2\vec{r})} \right|^2 + 2Re\, e^{i(\vec{k}_1 - \vec{k}_2)\vec{r}} \right\}_{z=z}$$

$$= \left\{ 2 + 2\cos(\vec{k}_1 - \vec{k}_2)\vec{r} \right\}_{z=z_0} = 2\left\{ 1 + \cos\frac{2\pi}{\lambda} \right.$$

$$= 2\left\{ 1 + \cos\frac{2\pi}{\lambda}\left[(\sin\varphi + \sin\varphi)x + \underbrace{(\cos\varphi - \cos\varphi)}_{=0}Z \right]_{z=z_0} \right.$$

$$= 2\left[1 + \cos\frac{2\pi}{\lambda} 2\sin\varphi\, x \right]$$

The period of the fringes is:

$$\frac{2\pi}{\lambda} 2\sin\varphi\, \Lambda = 2\pi \implies \underline{\underline{\Lambda = \frac{\lambda}{2\sin\varphi}}}$$

(b) The inclined plane is described by relation

$$z = z_0 + x \sin\theta$$

The intensity in this plane is:

$$I(x, y, z = z_0 + x\sin\theta) = \left\{2 + 2\cos(\vec{k}_1 - \vec{k}_2)\vec{r}\right\}_{z = z_0 + x\sin\theta}$$

$$= 2\left[1 + \cos\frac{2\pi}{\lambda} 2\sin\varphi x\right] \text{ (i.e. the same as above)}$$

Define the coordinate along the inclined plane:

$$x = x'\cos\theta \rightarrow x' = \frac{x}{\cos\theta}$$

The intensity along the inclined plane becomes

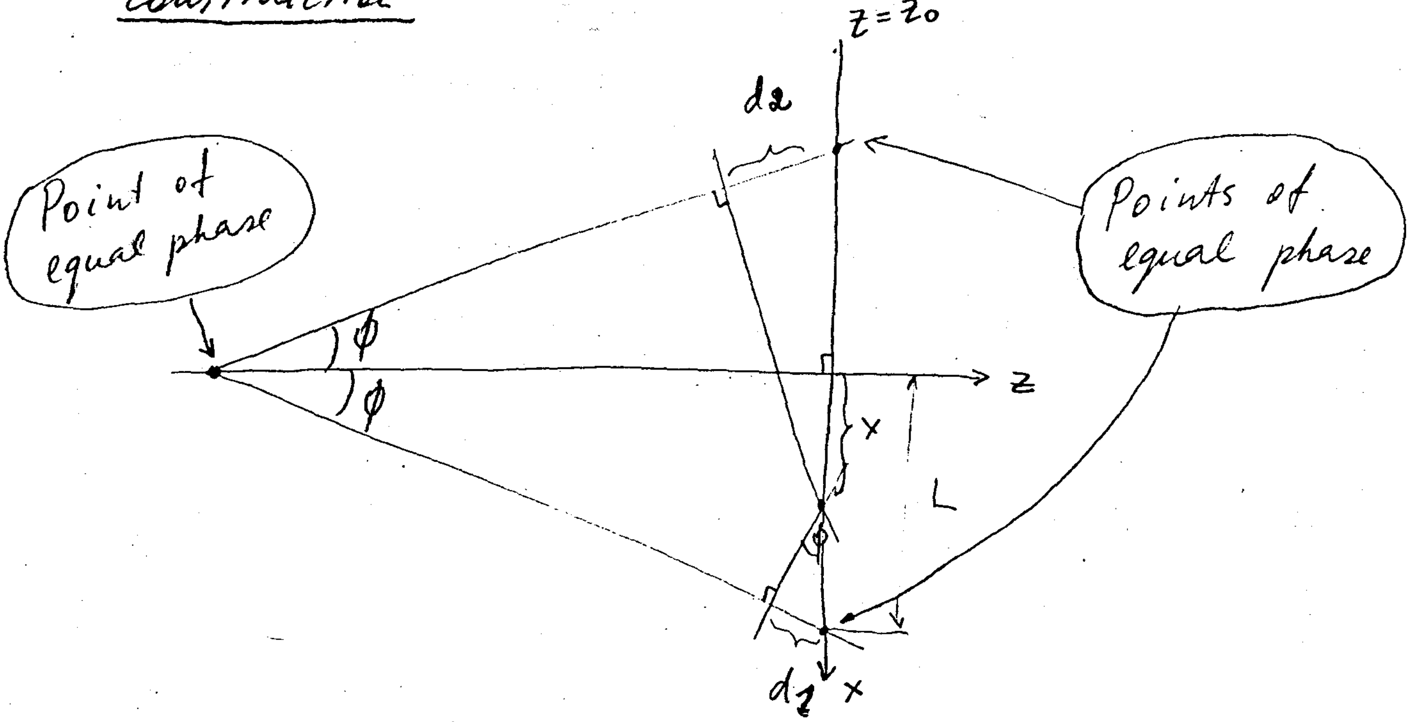
$$I(x') = 2\left[1 + \cos\frac{2\pi}{\lambda} 2\sin\varphi \cdot \cos\theta \cdot x'\right]$$

The period of the fringes is:

$$\frac{2\pi}{\lambda} 2\sin\varphi \cdot \cos\theta \cdot \Lambda = 2\pi$$

$$\Lambda = \frac{\lambda}{2\sin\varphi\cos\theta}$$

Alternative solution to part (a) using geometric construction



$$\Delta\varphi = 2\pi \frac{d_1 - d_2}{\lambda} = 2\pi \left(\frac{-2x \sin\phi}{\lambda}\right)$$

$$d_1 = (L - x)\sin\phi$$

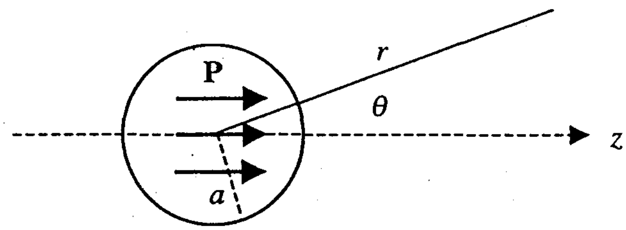
$$d_2 = (L + x)\sin\phi$$

$$\frac{2\Lambda \sin\phi}{\lambda} = 1 \Rightarrow \Lambda = \frac{\lambda}{2\sin\phi}$$

QUESTION #10

A dielectric sphere of radius a has a uniform permanent polarization $\mathbf{P} = P_o\hat{z}$ inside the sphere as shown in the figure below. Find the electric potential ϕ both inside and outside the sphere.

(Hint: The electric potential and the normal component of $\mathbf{D}$ are continuous across the surface of the sphere, and $\mathbf{D} = \mathbf{E} + 4\pi\mathbf{P}$.)



Because of the azimuthal symmetry ϕ can be expanded in Legendre polynomials. Recall that

$$\vec{\nabla}\cdot\vec{E} = \vec{\nabla}\cdot(\vec{D}-4\pi\vec{P}) = \underbrace{\vec{\nabla}\cdot\vec{D}}_{\substack{0 \\ \text{(no external charge)}}} - 4\pi \underbrace{\vec{\nabla}\cdot\vec{P}}_{\substack{0 \\ (\vec{P}=\vec{const.})}} = 0$$

and, $\vec{\nabla}\times\vec{E} = 0$

$$\Longrightarrow \vec{E} = -\vec{\nabla}\phi \Rightarrow \vec{\nabla}^2\phi = 0 \quad \text{Laplace equation.}$$

General solution for the azimuthally symmetric potential is:

$$\phi = \sum_{\ell=0}^{\infty} \left(A_\ell r^\ell + B_\ell r^{-(\ell+1)}\right) P_\ell(x), \quad \text{where} \quad x = \cos\theta$$

ϕ is finite, this means:

$$\phi^{in} = \sum_{\ell=0}^{\infty} A_\ell r^\ell P_\ell(x), \quad r \leq a$$

$$\phi^{out} = \sum_{\ell=0}^{\infty} B_\ell r^{-(\ell+1)} P_\ell(x), \quad r \geq a$$

ϕ is continues at $r=a$:

$$\phi^{in}(r=a) = \phi^{out}(r=a) \Longrightarrow A_\ell a^\ell = B_\ell a^{-(\ell+1)}$$

$$\Longrightarrow \boxed{B_\ell = a^{2\ell+1} A_\ell}, \quad \text{where } \ell = 0, 1, 2, \cdots$$

Normal component of $\vec{D}$ is continues across the surface:

$$D_n^{(out)} = D_n^{(in)} \Rightarrow -\frac{\partial \phi^{(out)}}{\partial r}\Big|_a = -\frac{\partial \phi^{in}}{\partial r}\Big|_a + 4\pi \vec{P}\cdot\hat{r}$$

$$\Rightarrow \sum_\ell B_\ell (\ell+1)\, a^{-(\ell+2)} P_\ell(x) = -\sum_\ell \ell A_\ell a^{\ell-1} P_\ell(x) + 4\pi P \underset{(\cos\theta)}{x}$$

This relation holds for every $x = \cos\theta$. This can only happen if and only if:

$B_\ell = A_\ell = 0$ for all $\ell \neq 1$ (note $P_1 = x = \cos\theta$), and

$2B_1 a^{-3} = -A_1 + 4\pi P$, for $(\ell = 1)$

From p.1: $B_1 = a^3 A_1$

these equations yield:

$$\boxed{A_1 = \frac{4\pi}{3} P, \quad \text{and} \quad B_1 = \frac{4\pi}{3} a^3 P}$$

This gives:

$$\phi^{in} = A_1 r \cos\theta = \frac{4\pi}{3} P r \cos\theta$$

$$\phi^{out} = \frac{B_1 \cos\theta}{r^2} = \frac{\left(\frac{4\pi}{3} a^3 P\right)\cos\theta}{r^2} = \frac{\vec{p}\cdot\vec{r}}{|\vec{r}|^3} \quad \text{(Dipole field)},$$

where $\vec{p} = VP\hat{z}$, $\vec{r} = r\hat{r}$, $V = \frac{4\pi}{3}a^3$

↑ total polarization

— ∘ —

QUESTION #11

The laws of scaling are very important in physics. Work any four of the five scaling problems given below.

a. A typical elephant has seven times the mass of a typical horse. How much larger is the cross-sectional area of an elephant's leg than a horse's leg?

b. A mass hangs from a massless spring and oscillates with a frequency of 1 Hz. If the spring is cut in half, what is the new oscillation frequency?

c. The Bohr radius for the hydrogen atom a_o has the numerical value 0.0529 nm
What would be the new radius if the electron were replaced by a muon with a mass that is 207 times as large? (We assume that the mass of the proton is large compared to the mass of the muon.)

d. The mean temperature on the Earth is $T = 287$ K. What would the new mean equilibrium temperature T' be if the mean distance between the Earth and the Sun were reduced by 1%?

e. On a given day, the air is dry and has a density $\rho = 1.2500$ kg/m^3. The next day the humidity has increased and the air contains 2% water vapor by mass. The pressure and temperature are the same as the day before. What is the new air density ρ'? Assume ideal gas behavior. The mean molecular weight of dry air is 28.8 g/mol and the molecular weight of water is 18 g/mol.

II. Solutions

A. The compression strength of a beam varies as the cross-sectional area of the beam. Because each elephant leg must support 7 times the mass, it must have 7 times the cross-sectional area or $\sqrt{7} = 2.65$ times the diameter.

B. Cutting the spring in half doubles the spring constant. The oscillation frequency of the shorter spring is therefore.

$$f = \frac{1}{2\pi}\sqrt{\frac{2k}{m}} = \sqrt{2} f_o.$$

C. The formula for the Bohr radius is

$$a_o = \frac{\hbar^2}{mke^2},$$

telling us that the Bohr radius is inversely proportional to the mass. Therefore, we have

$$a_\mu = a_o\left(\frac{m_e}{m_\mu}\right) = \frac{a_o}{207} = 0.256 \text{ pm}.$$

D. We match the input radiation to the output radiation because the Earth is in thermal equilibrium. If the power output of the Sun is P the radiation reaching the Earth per unit area is $P/4\pi R^2$. If we denote the Earth's radius by R_E and its reflectance by r, the input power P_{in} to the Earth is

$$P_{in} = (1-r)\frac{P}{4\pi R^2}\pi R_E^2$$

Stefan's Law gives the output power

$$P_{out} = 4\pi R_E^2 \varepsilon\sigma T^4,$$

where ε is the Earth's emissivity and σ is Stefan's constant. Although the emissivity is a function of temperature, the change in temperature is expected to be small and we can neglect this dependence. Therefore,

$$T \propto \sqrt{\frac{1}{R}}$$

and a reduction of 1% in R gives a 0.5% rise in T. For a mean temperature of 287 K, we get a rise of 1.4 K.

E ~~D.~~ Let's use the subscripts d and m for dry and moist air respectively. Then the number of molecules N_d in the dry air is

$$N_d \propto \frac{M_d}{28.8},$$

where M_d is the mass of dry air in a unit volume and the mean molecular mass of dry air is 28.8 g/mol. For moist air, we must account for the proportions of dry air and water vapor. For 2% humidity, we have

$$N_m \propto 0.02\frac{M_m}{18} + 0.98\frac{M_m}{28.8},$$

where the mean molecular mass of water is 18 g/mol.

We know that identical volumes of ideal gases with the same temperature and pressure have the same number of molecules. Therefore,

$$M_d = 1.012\, M_m.$$

Because the densities of equal volumes are proportional to the respective masses,

$$\frac{\rho_m}{\rho_d} = \frac{M_m}{M_d} = 0.988$$

and using $\rho_d = 1.25$ kg/m^3, we get our answer

$$\rho_m = 1.235 \text{ kg/m}^3.$$

F. The mechanical power P of the helicopter is equal to the thrust T times the downward velocity component v of the air below the blades. The thrust is given by the change in momentum of the air per unit time

$$T = v\frac{\mathrm{d}m}{\mathrm{d}t}$$

with

$$\frac{\mathrm{d}m}{\mathrm{d}t} = \rho A v,$$

where ρ is the density of the air and A is the cross-sectional area covered by the blades. Thus,

$$T = \rho A v^2.$$

When the helicopter is hovering, the thrust must be equal to the helicopter's weight. Therefore,

$$v^2 = \frac{T}{\rho A} = \frac{W}{\rho A}.$$

If the size of the helicopter is characterized by a linear dimension L, then $W \propto L^3$, $A \propto L^2$, and $v \propto L^{0.5}$. Thus,

$$P = Tv = Wv \propto L^{3.5}.$$

For a half-scale helicopter, the required power is $0.5^{3.5}P = 0.0884P$.

QUESTION #12

Consider an electron beam with cross-sectional area of radius a, charge density ρ, and velocity v. An electron in this beam will experience a repulsive force by other electrons in the beam, which will tend to expel the electron.

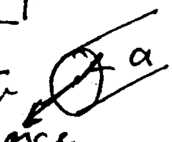
Find this repulsive force as a function of the radial distance r, velocity v, and given and known constants such as the electron charge e, the charge density ρ, and the speed of light c.

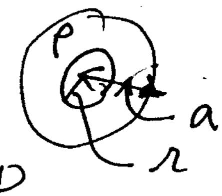
Comp 98

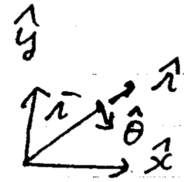
Key

$\boxed{\text{E \& M. / ST}}$ Tsuruta

$\boxed{\#12}$







◎ Lorentz Force

$$\bar{F} = e\left(\bar{E} + \frac{\bar{v}}{c} \times \bar{B}\right) \quad (1)$$

◎ Gauss's Law →

$$\oint \bar{E}\cdot d\bar{S} = 4\pi Q \quad ; \quad \bar{E} = E_r \hat{r} \quad (2)$$

$$\therefore \ 2\pi r E_r = 4\pi \cdot \pi r^2 \rho$$

$$\rightarrow \quad E_r = 2\pi\rho r \quad (3)$$

◎ Ampere's Law →

$$\oint \bar{B}\cdot d\bar{l} = \frac{4\pi}{c} I; \qquad I = i \times \text{area}, \quad i = \rho v. \quad (4)$$

$$\therefore \ \bar{B} = B\hat{\theta}, \ \&$$

$$2\pi r B = \frac{4\pi}{c}\rho v \cdot \pi r^2$$

$$\therefore \quad B = \frac{2\pi\rho v}{c} r = B_\theta \quad (5)$$

$$\therefore \ (1) \rightarrow \quad \bar{F} = F_r \hat{r}$$

$$\& \ F_r = e\left(E_r - \frac{v}{c}B_\theta\right) = e\left(2\pi\rho r - 2\pi\frac{\rho v^2}{c^2} r\right)$$

$$\rightarrow \ \therefore \ \bar{F} = F_r\hat{r} = 2\pi\rho e\left[1 - \frac{v^2}{c^2}\right]\bar{r}. \ // \text{Ans.}$$

QUESTION #13

Recall that the (Helmholtz) free energy of an ideal, classical gas of volume V containing N particles at temperature T is given by

$$F_0 = -kNT\left[1 + \ln \alpha \frac{VT^{3/2}}{N}\right],$$

where α is a constant with appropriate units. The definition of the free energy is $F = E - ST$, and $dF = SdT - PdV + \mu dN$, where E is the energy of the gas and S is its entropy.

Interactions among the particles in the gas produce deviations from ideal-gas behavior and affect the thermodynamic properties. Suppose an interacting gas has a free energy

$$F = F_0 + \varepsilon N^2 \frac{V_0}{V},$$

where ε and V_0 are positive constants with units of energy and volume, respectively.

a. The gas undergoes a reversible, isothermal, *small* change in volume ΔV. Evaluate the work done by the gas.

b. Determine the pressure and energy of the gas. Is the particle interaction attractive or repulsive?

c. The gas undergoes an adiabatic, free expansion from volume V_0 to a volume $V_f >> V_0$. Assume that the system is isolated during this expansion. Following the expansion, interactions of the particles with each other and with the walls of the confining vessel restore the gas to thermal equilibrium. Calculate the change, if any, in the gas temperature.

13)

a) For an isothermal process, with N constant

$$\Delta F = -S\cancelto{0}{\Delta T} - P\Delta V + \mu\cancelto{0}{\Delta N} = -\Delta W$$

$$\boxed{\Delta W = -\Delta F = kTN\frac{\Delta V}{V} + \epsilon N^2 V_0 \frac{\Delta V}{V^2}}$$

b) $$\boxed{P = -\left.\frac{\partial F}{\partial V}\right|_T = \frac{kTN}{V} + \epsilon N^2 \frac{V_0}{V^2}}$$

First term is ideal gas contribution. Second term is positive $\Rightarrow$ repulsive.

$$E = F + ST = F - T\left.\frac{\partial F}{\partial T}\right|_V = -T^2\left(\frac{\partial}{\partial T}\frac{F}{T}\right)_V$$

$$\Rightarrow \quad \boxed{E = \frac{3}{2}NkT + \epsilon N^2 \frac{V_0}{V}}$$

c) $dE = dQ - dW$

$dQ = 0$ (adiabatic)

$dW = 0$ for a free expansion

$\Rightarrow \quad E_i = E_f$

$$\frac{3}{2}NkT_i + \epsilon N^2 = \frac{3}{2}NkT_f$$

$$\boxed{T_f = T_i + \frac{2}{3}\frac{\epsilon N}{k}}$$

Gas heats up.

QUESTION #14

In a so-called quantum wire, semiconductor properties are engineered so that electrons move freely within a very long rectangular channel whose sides a and b are extremely narrow. Assume that the channel walls are infinitely hard, that the channel is infinitely long and oriented along the z-axis, and that only the lowest-energy modes associated with the x- and y-directions are occupied.

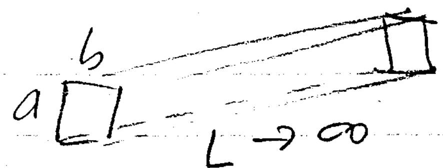
a. Determine the allowed wavefunctions and energy levels.

b. Adopting periodic boundary conditions for the wavefunction along the z-axis, determine the density of allowed states $\frac{\mathrm{d}n}{\mathrm{d}E}$ for the quantum wire.

Solid State Hermanson

Solution:

Quantum wire :



$$\Psi = A \sin \frac{\pi}{a} x \cdot \sin \frac{\pi}{b} y \cdot e^{ikz}$$

$$E = \frac{\hbar^2}{2m}\left[\left(\frac{\pi}{a}\right)^2 + \left(\frac{\pi}{b}\right)^2 + k^2\right] = E_0 + \frac{\hbar^2 k^2}{2m}$$

$$\frac{dn}{dE} = \frac{dn}{dk} \Big/ \frac{dE}{dk}$$

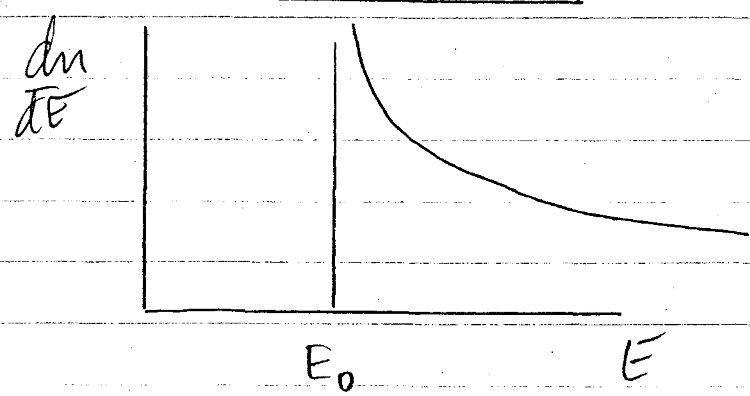
With periodic BC, $k = \frac{2\pi}{L} n$, n = integer

$$\frac{dn}{dk} = \frac{L}{2\pi}$$

$$\frac{dE}{dk} = \frac{\hbar^2 k}{m} = \frac{\hbar^2}{m}\sqrt{\frac{2m}{\hbar^2}(E-E_0)}$$

$$= \hbar \sqrt{\frac{2}{m}(E-E_0)}$$

$$\boxed{\frac{dn}{dE} = \frac{L}{2\pi} \cdot \frac{1}{\hbar}\sqrt{\frac{m}{2(E-E_0)}}}$$



QUESTION #15

This problem focuses on the interaction of an electron with a vector field **A** in a region of space where the total magnetic field **B** is zero. Such a situation is accomplished by two perpendicular sheets of current running along the $-\hat{x}$ and $-\hat{y}$ directions. First consider an infinitely thin sheet of uniform current density confined in the x-z plane and directed in the $-\hat{x}$ direction as shown in the figure below. The current density can be represented as $\mathbf{j} = -\kappa\delta(y)\hat{x}$ where κ is the current per unit length along the z-axis.

a. Taking advantage of the symmetry in **j** prove that **B** and **A** are given by

$$\mathbf{B} = \begin{cases} -\dfrac{\mu_o\kappa}{2}\hat{z}, & y>0 \\ \dfrac{\mu_o\kappa}{2}\hat{z}, & y<0 \end{cases}, \text{ and } \mathbf{A} = \frac{\mu_o\kappa}{2}|y|\hat{x}$$

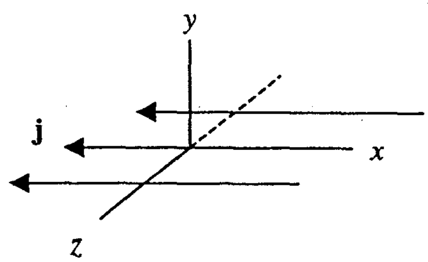
(Hint: $\oint \mathbf{A}\cdot d\ell = \int \mathbf{B}\cdot d\mathbf{a}$, and $\oint \mathbf{B}\cdot d\ell = \mu_o \int \mathbf{J}\cdot d\mathbf{a}$)

b. Now consider two perpendicular sheets of identical current density as described above, one in the $-\hat{x}$ direction and confined in the x-z plane and the other in the $-\hat{y}$ direction and confined in the y-z plane. Show (draw figure to illustrate) that at a point in the first quadrant ($x,y > 0$) **B** and **A** are given by

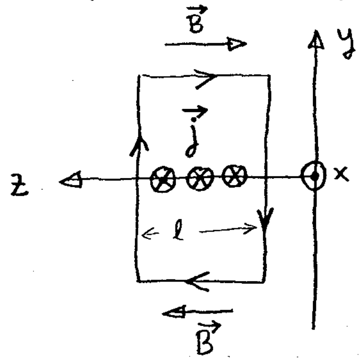
$$\mathbf{B} = 0, \text{ and } \mathbf{A} = \frac{\mu_o\kappa}{2}(y\hat{x} + x\hat{y}).$$

c. Using $H = \left(p - \frac{e}{c}A\right)^2 \Big/ 2m$ write down the Schrödinger equation for an electron in this field and far away from the current sheets in the first quadrant (that is, x, y >> de Broglie wavelength).

d. Find the solution to the Schrödinger equation assuming that the wave function is of the form $\psi_k(x,y,z) = e^{i\alpha xy}\phi_k(z)$, where $\phi_k(z)$, describes the motion of an electron along the z-axis and α is a constant to be determined.



(a) From symmetry it is clear that $\vec{B}$ will be along z-axis, and $\vec{A}$ along x-axis. Let us use Ampere's law for the loop:

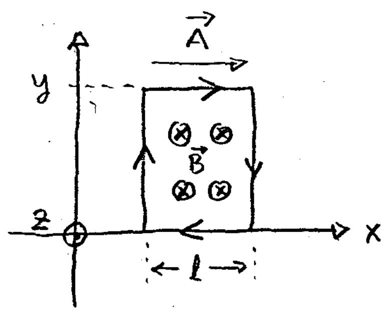


$$\oint \vec{B}\cdot d\vec{\ell} = \mu_0 \int \vec{j}\cdot d\vec{a}$$

$$2B\ell = \mu_0 \ell K \underbrace{\int_{-y}^{+y} \delta(y)\,dy}_{1}$$

$$\boxed{\vec{B} = \begin{cases} -\dfrac{\mu_0 K}{2}\hat{z}, & y>0 \\ +\dfrac{\mu_0 K}{2}\hat{z}, & y<0 \end{cases}}$$

Similarly, $\vec{A}$ can be obtained from $\oint \vec{A}\cdot d\vec{\ell} = \int \vec{B}\cdot d\vec{a}$ for a loop on the left:



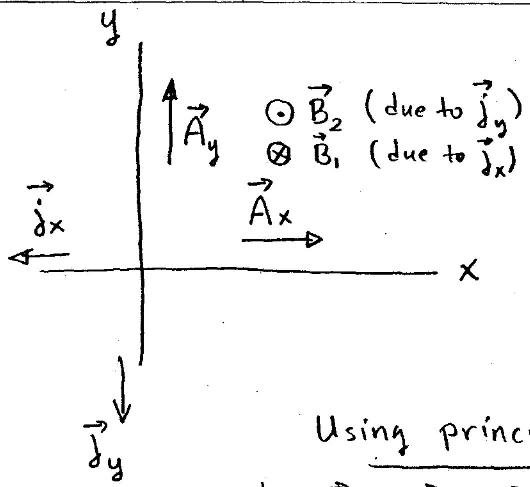
$$A(y)\,\ell - A(0)\,\ell = B\ell y$$

$$A(y) = A(0) + By = By$$

set $A(0)=0$

$$\boxed{\vec{A} = \frac{\mu_0 K}{2}\, y\, \hat{x}}$$

(b)



$\vec{B}_1 = -\vec{B}_2$

Using principle of superposition;

$$\boxed{\begin{aligned}\vec{B} &= \vec{B}_1 + \vec{B}_2 = 0 \\ \vec{A} &= \vec{A}_x + \vec{A}_y = \frac{\mu_0 K}{2}(y\hat{x} + x\hat{y})\end{aligned}}$$

(c) $$H = \frac{\left(\vec{p} - \frac{e}{c}\vec{A}\right)^2}{2m} \leftarrow \text{canonical momentum}$$

$$H = \frac{1}{2m}\left(\frac{\hbar}{i}\vec{\nabla} - \frac{e}{c}\vec{A}\right)^2 = -\frac{\hbar^2}{2m}\left(\vec{\nabla} - \frac{ie}{\hbar c}\vec{A}\right)^2$$

$$\Rightarrow \quad H = -\frac{\hbar^2}{2m}\left\{\left(\frac{\partial}{\partial x} - i\beta y\right)^2 + \left(\frac{\partial}{\partial y} - i\beta x\right)^2 + \frac{\partial^2}{\partial z^2}\right\}, \quad \text{where}$$

$$\beta = \frac{e\mu_0 K}{2\hbar c}$$

$\Rightarrow$ Schrödinger equation: $\boxed{H\Psi = E\Psi}$, where H is given above.

(d) Assume $\Psi = e^{i\alpha xy}\phi_k(z)$

$$H\psi = -\frac{\hbar^2}{2m}\left\{\left(\frac{\partial}{\partial x} - i\beta y\right)(i\alpha y - i\beta y)e^{i\alpha xy}\phi_k(z)\right.$$

$$+\left(\frac{\partial}{\partial y} - i\beta x\right)(i\alpha x - i\beta x)e^{i\alpha xy}\phi_k(z)$$

$$\left. + e^{i\alpha xy}\frac{\partial^2\phi(z)}{\partial z^2}\right\} = E\Psi$$

$$\Rightarrow \frac{\hbar^2}{2m}\left((\alpha-\beta)^2 y^2 + (\alpha-\beta)^2 x^2\right)\psi - \frac{\hbar^2}{2m}\frac{\partial^2\phi}{\partial z^2}e^{i\alpha xy} = E\psi$$

set $\boxed{\alpha = \beta = \frac{e\mu_0 K}{2\hbar c}}$, then $H\Psi = E\Psi$ is solved:

$$\Rightarrow -\frac{\hbar^2}{2m}\frac{\partial^2\phi}{\partial z^2} = E\phi \Rightarrow \phi = c\, e^{\pm ikz}$$

and $\boxed{E = \frac{\hbar^2 k^2}{2m}}$

Assume electron is travelling in $+\hat{z}$ directions

$\int_{-\infty}^{+\infty} dz\, \phi_k \phi_{k'} = \delta(k-k')$ & use $\int e^{iz(k-k')}dz = 2\pi\delta(k-k')$

$$\Rightarrow C = \frac{1}{\sqrt{2\pi}}$$

$$\Rightarrow \boxed{\Psi = \frac{1}{\sqrt{2\pi}} e^{\frac{ie\mu_0 K}{2\hbar c}xy} e^{ikz} \quad ; \quad E = \frac{\hbar^2 k^2}{2m}}$$

Even though $\vec{B}=0$ the electron still interacts with the magnetic field via $\vec{A}\neq 0$. This shows as a phase shift in the plane wave travelling along z-axis. Sach effect is known as Aharonov–Bohm effect.