Comp 2013
Solution 1

a) To find the coefficients c_n, we apply the destruction operator $\hat{a}$ to the superposition of the number states :

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$$
$$\hat{a}\sum_{n=0} c_n |n\rangle = \alpha\sum_{n=0} c_n |n\rangle$$
$$\sum_{n=0} c_n \hat{a}|n\rangle = \alpha\sum_{n=0} c_n |n\rangle$$
$$\sum_{n=1} c_n \sqrt{n}\,|n-1\rangle = \alpha\sum_{n=0} c_n |n\rangle$$

Note that in this last step, we run the first sum from n=1 since the number state $|n=-1\rangle$ does not exist.

Now we can use the last equation to compare terms between the two sums to get the recursion relation:

$$c_n\sqrt{n} = \alpha c_{n-1}$$

Now we can use this last equation to find all the c_n terms all relative to the coefficient c_o by repeated application of the above equation. Thus we have:

$$\boxed{c_n = \frac{\alpha^n}{\sqrt{n!}} c_o}$$

b) To find the value of c_o we only need to use the normalization condition $\langle\alpha|\alpha\rangle = 1$:

$$\langle\alpha|\alpha\rangle = 1$$
$$\left(\sum_{m=0} c_m^* \langle m|\right)\left(\sum_{n=0} c_n |n\rangle\right) = 1$$
$$\left(\sum_{m=0} \frac{\left(\alpha^*\right)^m}{\sqrt{m!}} c_o^* \langle m|\right)\left(\sum_{n=0} \frac{\alpha^n}{\sqrt{n!}} c_o |n\rangle\right) = 1$$
$$\left(\sum_{m=0}\sum_{n=0} \frac{\left(\alpha^*\right)^m}{\sqrt{m!}} \frac{\alpha^n}{\sqrt{n!}} |c_o|^2 \langle m|n\rangle\right) = 1$$
$$\left(\sum_{m=0}\sum_{n=0} \frac{\left(\alpha^*\right)^m}{\sqrt{m!}} \frac{\alpha^n}{\sqrt{n!}} |c_o|^2 \delta_{mn}\right) = 1$$
$$|c_o|^2\left(\sum_{n=0} \frac{|\alpha|^{2n}}{n!}\right) = 1$$
$$|c_o|^2 \exp\left(|\alpha|^2\right) = 1$$
$$\boxed{c_o = \exp\left(-\frac{|\alpha|^2}{2}\right)}$$

to within an overall phase factor.

c) Now we can write the coherent state as a superposition of number states with the constants expressed only in terms of α.

$$\boxed{\left|\alpha\right\rangle = \exp\left(-\frac{\left|\alpha\right|^2}{2}\right)\sum_{n=0}\frac{\alpha^n}{\sqrt{n!}}\left|n\right\rangle}$$

d) Now to find the time dependence into the coherence state, we put in the time dependence of each number state of the superposition state in the usual way with the specific energy of each number state in the exponential:

$$\begin{aligned}
\left|\alpha(t)\right\rangle &= \exp\left(-\frac{\left|\alpha\right|^2}{2}\right)\sum_{n=0}\frac{\alpha^n}{\sqrt{n!}}\left|n\right\rangle \exp\left(-\frac{iE_n t}{\hbar}\right) \\
&= \exp\left(-\frac{\left|\alpha\right|^2}{2}\right)\sum_{n=0}\frac{\alpha^n}{\sqrt{n!}}\left|n\right\rangle \exp\left(-i\left(n+\tfrac{1}{2}\right)\omega t\right) \\
&= \exp\left(-\frac{\left|\alpha\right|^2}{2}\right)\exp\left(-\frac{i\omega t}{2}\right)\sum_{n=0}\frac{\left[\alpha\exp(-i\omega t)\right]^n}{\sqrt{n!}}\left|n\right\rangle
\end{aligned}$$

Note in the final line that the time dependence has been distributed in a way that shows that the form is very similar to the form at t=0 but with an extra overall phase term. This shows that the coherent state remains a coherent state for all times but that the eigenvalue $\alpha(t)$ depends on time.

end

Comp 2013
Solution 2

The electric field in between the plates, neglecting edge effects, is $E = \sigma/\varepsilon_o$. In Gaussian units $E = 4\pi\sigma$.

The Maxwell equation to be utilized is: $\oint B \bullet ds = \mu_0 I + \varepsilon_0 \mu_0 \frac{d\phi_E}{dt}$

This becomes, since the $\mu_0 I$ term is zero:

$B2\pi r = \varepsilon_o \mu_o \pi r^2 (I/(\varepsilon_o \pi b^2))$, which can be solved to yield: $B = (\mu_o I r)/(2\pi b^2)$
In Gaussian units: $B = 2Ir/(cb^2)$.

3. Solution

a. The moth's velocity

$$\mathbf{v} \;=\; \dot{r}\,\hat{\mathbf{r}} \;+\; r\dot{\phi}\,\hat{\phi} \;=\; -\,v_0\cos\alpha\,\hat{\mathbf{r}} \;+\; v_0\sin\alpha\,\hat{\phi} \quad ,$$

gives $\mathbf{v}\cdot\hat{\mathbf{r}}/v_0 \;=\; -\cos\alpha$ as required. The radial equation, $\dot{r} \;=\; -v_0\cos\alpha$, is easily solved

$$r(t) \;=\; r_0 \;-\; v_0\cos\alpha\,t \quad . \tag{1}$$

The angular equation

$$\dot{\phi} \;=\; \frac{v_0\sin\alpha}{r} \;=\; \frac{v_0\sin\alpha}{r_0 \;-\; v_0\cos\alpha\,t} \quad ,$$

is solved by

$$\phi(t) \;=\; \phi_0 \;-\; \tan\alpha\,\ln\left[1 \;-\; v_0\cos\alpha\,t/r_0\right] \;=\; \phi_0 \;-\; \tan\alpha\,\ln(r/r_0) \quad . \tag{2}$$

The two can be combined into an expression for the flight path

$$\boxed{r(\phi) \;=\; r_0\exp\left[-\cot\alpha\,(\phi-\phi_0)\right] \quad .} \tag{3}$$

Special cases: $\alpha = \pi/2$ gives a circle, $r(\phi) = r_0$. $\alpha = 0$ gives a straight line along $\phi = \phi_0$.

b. The acceleration vector is

$$\boxed{\begin{aligned} \mathbf{a} \;&=\; \frac{d\mathbf{v}}{dt} \;=\; -\,v_0\cos\alpha\,\dot{\phi}\,\hat{\phi} \;-\; v_0\sin\alpha\,\dot{\phi}\,\hat{\mathbf{r}} \\ &=\; -\frac{v_0^2}{r}\Big[\sin^2\alpha\,\hat{\mathbf{r}} \;+\; \sin\alpha\cos\alpha\,\hat{\phi}\Big] \end{aligned}} \tag{4}$$

c. Since the moth is flying at constant speed its acceleration is purely centripetal: $|\mathbf{a}| = v_0^2/r_c$, where r_c is the radius of curvature. Solving for r_c gives

$$\boxed{r_c \;=\; \frac{v_0^2}{|\mathbf{a}|} \;=\; \frac{r}{\sin\alpha} \quad .} \tag{5}$$

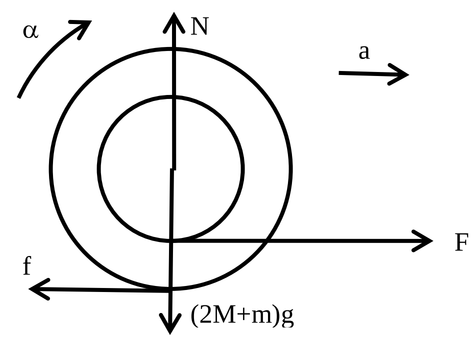
EXTRA:

Expression (3) is known as a *logarithmic spiral*.[1] The moth spirals into the flame, hitting it at $t = r_0/v_0\cos\alpha$. It has been hypothesized that this suicidal moth behavior arrises from an instinct to fly at a fixed angle to a light source. Had the light been the Sun, rather than a flame, this instinct would assure that the moth flies in a straight line. Perhaps this is what moths really wish to do, to get wherever it is moths need to go.

[1] Written in this form the term "exponential spiral" would seem more appropriate. The logarithmic spiral differs from an *arithmetic spiral*, also called an Archimedean spiral, which is described by the equation $r(\phi) = r_0 - \beta(\phi - \phi_0)$, where β is a constant. For this curve, the tangent relative to the radial $\tan\alpha = -v_\phi/v_r = -r/r' = r/\beta$, decreases as the spiral approaches the origin.

Comp 2013
Solution 4

a) To solve for the maximum acceleration of the spool, we start by drawing a free body diagram for the spool. Included are the pulling force F, the friction force f, the gravitational force (2M+m)g, the normal force N, the acceleration a and the angular acceleration α. We assume that the acceleration is to the right. This assumption will be correct if our acceleration comes out to be positive in the end.



We start with Newton's second law in the horizontal direction:

$$\sum F_x = ma_x \rightarrow F - f = (2M+m)a$$

Next we look at Newton's second law for the angular rotation:

$$\sum \tau = I\alpha$$

$$Rf - rF = \left(\frac{1}{2}MR^2 + \frac{1}{2}MR^2 + \frac{1}{2}mr^2\right)\frac{a}{R}$$

$$\frac{R}{r}f - F = \left(M\frac{R}{r} + \frac{m}{2}\frac{r}{R}\right)a$$

Note that for the moment of inertia I we just added moments of inertia for three solid discs together.

Now we can combine these two equations to eliminate F and solve for a:

$$F - f = (2M+m)a$$

$$\frac{R}{r}f - F = \left(M\frac{R}{r} + \frac{m}{2}\frac{r}{R}\right)a$$

$$\rightarrow f\left(\frac{R}{r} - 1\right) = \left[M\left(2 + \frac{R}{r}\right) + m\left(1 + \frac{r}{2R}\right)\right]a$$

$$\rightarrow a = \frac{f\left(\frac{R}{r} - 1\right)}{\left[M\left(2 + \frac{R}{r}\right) + m\left(1 + \frac{r}{2R}\right)\right]}$$

But we still need to find the frictional force f. To do this we start with Newton's second law in vertical direction to get the normal force N:

$$\sum F_y = ma_y \rightarrow N - (2M+m)g = 0 \rightarrow N = (2M+m)g$$

Using the normal force N, we can calculate the maximum frictional force using:

$$f = \mu N = \mu(2M+m)g$$

Now we use this to find the final expression for the acceleration a.

$$\boxed{a = \frac{\left[\mu(2M+m)g\right]\left(\frac{R}{r}-1\right)}{\left[M\left(2+\frac{R}{r}\right)+m\left(1+\frac{r}{2R}\right)\right]}}$$

b) Note that for R>r, which is the case here, the acceleration will be positive and hence to the right so that the spool is winding up as the spool accelerates to the right.

c) Now we take the limit when μ=1 and R>>r and M>>m to get:

$$a = \frac{\left[\mu(2M+m)g\right]\left(\frac{R}{r}-1\right)}{\left[M\left(2+\frac{R}{r}\right)+m\left(1+\frac{r}{2R}\right)\right]} \rightarrow a = \frac{\left[(2M)g\left(\frac{R}{r}\right)\right]}{\left[M\left(\frac{R}{r}\right)\right]}$$

$$\boxed{\rightarrow a = 2g}$$

In this case, surprisingly, we should be able to accelerate at twice the acceleration of gravity as the threads winds up on the spool!

end

Comp 2013

Solution 5

a. As suggested by the hint we can start with the probability that there are n particles occupying state of energy ε, which is given by $P(n)=\frac{e^{-\beta(\varepsilon(n)-n\mu)}}{Z}=\frac{e^{-\beta(n\varepsilon-n\mu)}}{Z}$. The average n is then given by $<n>=N=\sum_n nP(n)$ where $\varepsilon(n)=n\varepsilon$, $\beta=\frac{1}{kT}$ and $Z=\sum_n e^{-\beta(n\varepsilon-n\mu)}$. One can easily see that $N=\sum_n nP(n)=\frac{1}{\beta}\left(\frac{\partial \ln Z}{\partial\mu}\right)_\beta$. Alternatively, we can use the hint that $d\Phi=E\,dT-p\,dV-N\,d\mu$, which implies that the average number of particles occupying the Fermion state can be found from $N=-\left(\frac{\partial\Phi}{\partial\mu}\right)_{T,V}$ where $\Phi=-kT\ln Z$. This yields the same formula for N as the first approach. We know that in a Fermion gas a given state cannot be occupied by more than a single particle. Therefore, there are only two possible terms associated with Z, as n can only be 0 or 1. Thus, for a Fermi gas $Z=\sum_{n=0}^{1}e^{-\beta(n\varepsilon-n\mu)}=1+e^{-\beta(\varepsilon-\mu)}$; this immediately yields: $\Phi=-kT\ln(1+e^{-\beta(\varepsilon-\mu)})$ and $N=-\left(\frac{\partial\Phi}{\partial\mu}\right)_{T,V}=\frac{e^{-\beta(\varepsilon-\mu)}}{1+e^{-\beta(\varepsilon-\mu)}}=\frac{1}{e^{\beta(\varepsilon-\mu)}+1}$. This is known as the **Fermi-Dirac** distribution function and, as you can see, the average occupation number is $N\le 1$ as expected.

b. We use a very similar procedure to determine the average number of particles occupying a given state in a Boson gas. However, in a Boson system a single state can in principle be occupied by all the particles available in the system (this is observed routinely for the ground state of Boson gases such as ^{4}He at low temperatures and is known as Boson condensation). Therefore, the number of particles n that can occupy state of energy ε can be any number from 0 to ∞; hence $Z=\sum_{n=0}^{\infty}e^{-n\beta(\varepsilon-\mu)}$, since $\varepsilon>\mu$. The partition function becomes $Z=\frac{1}{1-e^{-\beta(\varepsilon-\mu)}}$ and the grand potential is $\Phi=kT\ln(1-e^{-\beta(\varepsilon-\mu)})$. This immediately yields the average number of particles occupying a Boson state: $N=-\left(\frac{\partial\Phi}{\partial\mu}\right)_{T,V}=\frac{e^{-\beta(\varepsilon-\mu)}}{1-e^{-\beta(\varepsilon-\mu)}}=\frac{1}{e^{\beta(\varepsilon-\mu)}-1}$. This last expression for N, known as the **Bose-Einstein** distribution function, can in principle be a very large number $(N>>1)$ in such systems as superconductors, superfluids and lasers.

Comp 2013
Solution 6

Answer:

(a) We got B(z) = $B_z(0)$ (1 – k $(z/a)^4$ + higher order terms). (1)

dB = $\frac{1}{c}$ I **dl** x $\boldsymbol{x}$ / $|x|^3$.

Here field near axis (i.e., $\rho \to 0$). So, $|x|$ = | 0 – a| = a.
So, dB_z = $\frac{1}{c}$ I dl a (f(z) + f(−z)),
where f(z) = 1/ $g(z)^{3/2}$; f(−z) = 1/ $g(-z)^{3/2}$,
g(z) = $a^2 + (\frac{b}{2} + z)^2$; g(−z) = $a^2 + (\frac{b}{2} - z)^2$.

Then, $B_z = \int dB_z = (1/c)\ [(\int I\, dl\, a\, (f(z) + f(-z))]$ →

B_z = (2πNa^2I/c)(f(z) + f(−z)). (2)

Expand f(z) near z=0 as
f(z) = f(0) + $f(0)'$z + $f(0)''z^2/2!$ + $f(0)'''\ z^3/3!$ + $f(0)''''\ z^4\ /4!$ + higher order terms), where (') means the first derivative, ('') 2nd derivative, etc.
Then , B_z = (2πNa^2I/c)(f(z) + f(−z))

→ B_z = (2πNa^2I/c)(2f(0) + 2$f(0)''z^2/2!$+ 2$f(0)''''\ z^4\ /4!$ + higher oder terms). (3)

By comparing (1) and (3), $f(0)''$ must be 0, which gives

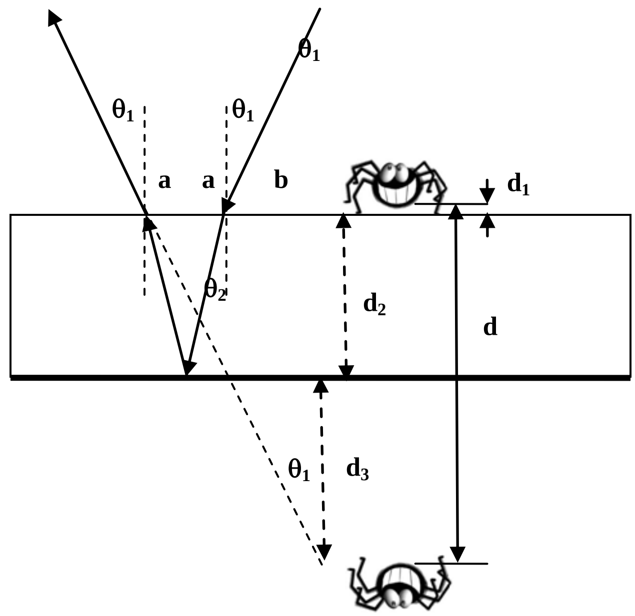
b = a. Answer.

(b) From **∇·B** = 0 → $\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho\, B_\rho) = -\frac{\partial B_z}{\partial z} = B_z(0)\ k\ 4\ z^3/a^4$,
we get $B_\rho = (2\, B_z(0)\ k\ \ z^3/a^4)\ \rho$

From (2) B_z = (2πNa^2I/c) (2/$g(0)^{3/2}$), with g(0) = $a^2 + (\frac{b}{2})^2$.

Comp 2013
Solution 7

We want the location of the virtual image of the spider below the mirror. To solve this let's follow a ray from the spider, through the glass, off the mirror and out of the glass. Knowing this exit angle and the distance from the spider's vertical ray will allow us to find the vertical distance from the mirror to the virtual image of the spider. Call the angle of the ray that leaves the spider θ_1. Since the spider is close to the glass, I expanded the rays a bit so I could label the angles. This first ray from the spider will hit the glass at a horizontal distance b from the vertical given by



$$b = d_1 \tan\theta_1 \approx d_1\theta_1$$

Then the ray enters the glass and the angle of refraction is given by Snell's law (using the suggested small angle approximation and also setting n_{air}=1.0) :

$$n_{air} \sin\theta_1 = n_{glass} \sin\theta_2$$
$$\rightarrow n_{air}\theta_1 = n_{glass}\theta_2$$
$$\rightarrow \theta_2 = \frac{\theta_1}{n_{glass}}$$

Now using θ_2 we can find the horizontal distance a:

$$a = d_2 \tan\theta_2 \approx d_2\theta_2$$

Now we can form the triangle with the vertical side equal to d_2 +d_3 and a horizontal side equal to 2a+b to get:

$$\tan\theta_1 = \frac{2a+b}{d_2+d_3} \rightarrow \theta_1 \approx \frac{2a+b}{d_2+d_3}$$

$$\rightarrow d_3 \approx \frac{2a+b}{\theta_1} - d_2 = \frac{2(d_2\theta_2)+(d_1\theta_1)}{\theta_1} - d_2 = \frac{\left(2d_2\frac{\theta_1}{n_{glass}}\right)+(d_1\theta_1)}{\theta_1} - d_2 = \left(\frac{2d_2}{n_{glass}} + d_1\right) - d_2$$

$$d_3 \approx d_2\left(\frac{2}{n_{glass}} - 1\right) + d_1 = (10\text{mm})\left[\frac{2}{(1.5)} - 1\right] + 1\text{mm} = (10\text{mm})(0.333) + 1\text{mm}$$
$$= 3.33\text{mm} + 1\text{mm} = 4.33\text{mm}$$

$$d = d_1 + d_2 + d_3 = 1\text{mm} + 10\text{mm} + 4.33\text{mm} = 15.33\text{mm}$$

$$\boxed{d = 15.33\text{ mm}}$$

end

Comp 2013
Solution 8

Assume that the heat bath is large, so that its temperature does not change.

Heat flow into water $Q_W = mC(100\ K) = 418{,}000\ J$

Entropy change of heat bath $\Delta S = -Q_W/(373\ K) = -1121\ J/K$
(A decrease in entropy for the reservoir since heat flows out.)

For the water:

$$S_w = \int_{273\ K}^{373\ K} \frac{mCdT}{T} = \left(4180\frac{J}{K}\right)\ln\left(\frac{373}{273}\right) = 1305\ J/K$$

Add these two answers to find the total entropy change for the water and heat bath:

$S_{total} = 1305\ J/K - 1121\ J/K = 184\ J/K$

Comp 2013

Solution 9

The kinetic energy of the photoelectrons, $K = p^2 / 2m$, is given by the conservation of energy, $K = p^2 / 2m = h\nu - E_o$, where p is the momentum of the photoelectrons ejected from the gold surface. The de Broglie wavelength of these photoelectrons is given by $\lambda = h / p$ where h is Planck's constant. The reflection condition of these electron waves from the crystal planes is determined by the constructive interference condition of $n\lambda = 2d\,Sin\theta$ where *n= 1,2, 3..*is an integer. For a minimum θ the integer *n* has to be given by *n=1*. Therefore, we can determine E_o using $E_o = h\nu - p^2 / 2m = h\nu - (h/\lambda)^2 / 2m$ where *m* is the mass of the electron and $\lambda = 2d\,Sin\theta$. Inserting the numbers given we obtain

$$E_o = 190\ (eV) - \frac{(6.63\times 10^{-34})^2}{(2\times 2.355\times 10^{-10}\ Sin15^o)^2 \times 2\times 9.11\times 10^{-31}} \times \frac{10^{19}}{1.6}\,(eV) = 88.5\,eV\ .$$

10. Solution

a. Denoting the positions of the particles, x_1 and x_2, the Lagrangian (in cgs units) is

$$L(x_1, \dot{x}_1, x_2, \dot{x}_2) \;=\; \tfrac{1}{2}m\dot{x}_1^2 + \tfrac{1}{2}m\dot{x}_2^2 + \frac{q^2}{\sqrt{(x_1 - x_2)^2 + d^2}} \tag{1}$$

The Euler-Lagrange equations are

$$\begin{aligned} m\ddot{x}_1 &= \frac{\partial L}{\partial x_1} = -\frac{q^2(x_1 - x_2)}{[(x_1 - x_2)^2 + d^2]^{3/2}} \\ m\ddot{x}_2 &= \frac{\partial L}{\partial x_2} = \frac{q^2(x_1 - x_2)}{[(x_1 - x_2)^2 + d^2]^{3/2}} \end{aligned}$$

Any case with $x_1 = x_2$ will therefore be an equilibrium. Expanding about this gives the coupled equations

$$\frac{d^2}{dt^2}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{q^2}{md^3}\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

The eigenfrequencies of this are

$$\omega = 0 \;, \quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad , \quad \omega = \sqrt{\frac{2q^2}{md^3}} \;, \quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad ,$$

The first of these correspond to translation of the center of mass, the second is small oscillation at

$$\boxed{\omega \;=\; \sqrt{\frac{2q^2}{md^3}} \;.} \tag{2}$$

b. A linear combination of the two normal modes above give

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = V_1 t \begin{bmatrix} 1 \\ 1 \end{bmatrix} \;+\; X_2 \sin(\omega t) \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

where V_1 and X_2 are unknown constants. The form satisfies the initial conditions $x_1(0) = x_2(0) = 0$. Designative the upper (lower) row as the positive (negative) ball makes the initial velocities

$$\begin{bmatrix} \dot{x}_1(0) \\ \dot{x}_2(0) \end{bmatrix} = V_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + X_2\omega \begin{bmatrix} 1 \\ -1 \end{bmatrix} \;=\; \begin{bmatrix} v_0 \\ 0 \end{bmatrix}$$

Solving these gives $V_1 = X_2\omega$ and $2V_1 = v_0$. The second row of the complete solution gives the position of the negative ball

$$\boxed{x_2(t) \;=\; \tfrac{1}{2}v_0 \left[t \;-\; \frac{1}{\omega}\sin(\omega t)\right] \;.} \tag{3}$$

Comp 2013
Solution 11

a) To find the expectation value of the position operator <x> we use

$$\langle x\rangle = \int \psi^*(x)\,x\,\psi(x)\,dx = \int_{-\infty}^{\infty}\left[\frac{1}{\sqrt{2a}}e^{-|x|/(2a)}\right]x\left[\frac{1}{\sqrt{2a}}e^{-|x|/(2a)}\right]dx = \frac{1}{2a}\int_{-\infty}^{\infty} xe^{-|x|/(a)}dx$$

Note that this last form of the integral is odd around the origin so that we immediately know that

$$\boxed{\langle x\rangle = 0}$$

b) Now to find the expectation value of the square of the position operator <x^2>, we will have to do the integral

$$\langle x^2\rangle = \int \psi^*(x)\,x^2\psi(x)\,dx = \int_{-\infty}^{\infty}\left[\frac{1}{\sqrt{2a}}e^{-|x|/(2a)}\right]x^2\left[\frac{1}{\sqrt{2a}}e^{-|x|/(2a)}\right]dx = \frac{1}{2a}\int_{-\infty}^{\infty} x^2e^{-|x|/a}dx = \frac{2}{2a}\int_{0}^{\infty} x^2e^{-x/a}dx$$

$$= \frac{1}{a}\left[2!a^{2+1}\right]$$

$$\boxed{\langle x^2\rangle = 2a^2}$$

c) Finding the expectation of the momentum operator <p> is easy using Ehrenfest

$$\boxed{\langle p\rangle = m\frac{d}{dt}\langle x\rangle = 0}$$

d) In this part we find the expectation value of the square of the momentum operator <p^2>. This is more challenging since there is a kink (change in slope) in wave function at the origin. We start by looking at the integral

$$\langle p^2\rangle = \int \psi^*(x)\,p^2\psi(x)\,dx = \int_{-\infty}^{\infty}\left[\frac{1}{\sqrt{2a}}e^{-|x|/(2a)}\right]\left(-i\hbar\frac{d}{dx}\right)^2\left[\frac{1}{\sqrt{2a}}e^{-|x|/(2a)}\right]dx$$

$$= -\frac{\hbar^2}{2a}\int_{-\infty}^{\infty}\left[e^{-|x|/(2a)}\right]\frac{d^2}{dx^2}\left[e^{-|x|/(2a)}\right]dx$$

Before doing the integral, let's find the first derivative

$$\frac{d}{dx}e^{-|x|/(2a)} = \frac{d}{dx}\begin{Bmatrix} e^{-x/(2a)}, & x>0 \\ e^{+x/(2a)}, & x<0\end{Bmatrix} = \begin{cases} -\dfrac{1}{2a}e^{-x/(2a)}, & x>0 \\ +\dfrac{1}{2a}e^{+x/(2a)}, & x<0\end{cases}$$

Note that this function has a negative step of -1/a as we go from x<0 to x>0. Thus when we take the second derivative, we will need to add a $-(1/a)\delta(x)$ to account for this change in the slope.

$$\frac{d^2}{dx^2}e^{-|x|/(2a)} = \frac{d}{dx}\begin{Bmatrix} -\dfrac{1}{2a}e^{-x/(2a)}, & x>0 \\ +\dfrac{1}{2a}e^{+x/(2a)}, & x<0\end{Bmatrix} = \begin{Bmatrix} \dfrac{1}{(2a)^2}e^{-x/(2a)}, & x>0 \\ -\dfrac{1}{a}\delta(x), & x=0 \\ \dfrac{1}{(2a)^2}e^{+x/(2a)}, & x<0\end{Bmatrix}$$

You can check this is correct by integrating this second derivative to get back to the first derivative. The delta function will lead to the negative step in the first derivative.
Now we are ready to do the integral

$$\left\langle p^2\right\rangle = -\frac{\hbar^2}{2a}\int_{-\infty}^{\infty}\left[e^{-|x|/(2a)}\right]\frac{d^2}{dx^2}\left[e^{-|x|/(2a)}\right]dx = -\frac{\hbar^2}{2a}\int_{-\infty}^{\infty} e^{-|x|/(2a)}\left\{\begin{array}{ll}\frac{1}{(2a)^2}e^{-x/(2a)}, & x>0\\ -\frac{1}{a}\delta(x), & x=0\\ \frac{1}{(2a)^2}e^{+x/(2a)}, & x<0\end{array}\right\}dx$$

$$= -\frac{\hbar^2}{2a}\left\{\int_{-\infty}^{-\varepsilon} e^{+x/(2a)}\frac{1}{(2a)^2}e^{+x/(2a)}dx + \int_{-\varepsilon}^{\varepsilon}\left[e^{-|x|/(2a)}\right]\left[-\frac{1}{a}\delta(x)\right]dx + \int_{\varepsilon}^{\infty} e^{-x/(2a)}\frac{1}{(2a)^2}e^{-x/(2a)}dx\right\}$$

$$= -\frac{\hbar^2}{2a}\left\{\frac{1}{(2a)^2}\int_{-\infty}^{-\varepsilon} e^{+x/a}dx - \frac{1}{a}e^0\int_{-\varepsilon}^{\varepsilon}\delta(x)dx + \frac{1}{(2a)^2}\int_{\varepsilon}^{\infty}e^{-x/a}dx\right\}$$

$$= -\frac{\hbar^2}{2a}\left\{\frac{2}{(2a)^2}\int_0^{\infty}e^{-x/a}dx - \frac{1}{a}\right\}$$

In the last step above, we took the limit of $\varepsilon\to 0$ after integrating the delta function and noting that the first and last integrals are the same by symmetry. The last integral is easy and we get

$$\left\langle p^2\right\rangle = -\frac{\hbar^2}{2a}\left\{\frac{2}{(2a)^2}\int_0^{\infty}e^{-x/a}dx - \frac{1}{a}\right\} = -\frac{\hbar^2}{2a}\left\{\frac{2}{(2a)^2}(-a)\left[e^{-x/a}\right]_0^{\infty} - \frac{1}{a}\right\} = -\frac{\hbar^2}{2a}\left\{\frac{2}{(2a)^2}(-a)[0-1] - \frac{1}{a}\right\}$$

$$= -\frac{\hbar^2}{2a}\left\{\frac{2a}{4a^2} - \frac{1}{a}\right\} = -\frac{\hbar^2}{2a}\left\{\frac{1}{2a} - \frac{1}{a}\right\} = -\frac{\hbar^2}{2a}\left\{-\frac{1}{2a}\right\}$$

$$\boxed{\left\langle p^2\right\rangle = \frac{\hbar^2}{4a^2}}$$

e) Now to check the Heisenberg uncertainty principle, we calculate σ_x and σ_p using

$$\sigma_x = \sqrt{\left\langle x^2\right\rangle - \left\langle x\right\rangle^2} = \sqrt{2a^2 - 0} = \sqrt{2}a$$

$$\sigma_p = \sqrt{\left\langle p^2\right\rangle - \left\langle p\right\rangle^2} = \sqrt{\frac{\hbar^2}{4a^2} - 0} = \frac{\hbar}{2a}$$

Looking at the product we get

$$\boxed{\sigma_x\sigma_p = \left(\sqrt{2}a\right)\left(\frac{\hbar}{2a}\right) = \sqrt{2}\left(\frac{\hbar}{2}\right) \geq \frac{\hbar}{2}}$$

telling us that the uncertainty is 1.414 larger than the minimum allowed. Only the Gaussian shape is at the minimum so we expect this function to be larger than the minimum as it is.

end

Comp 2013

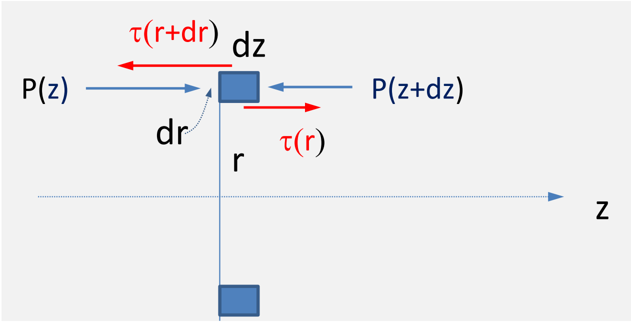
Solution 12

1. The figure shows the free-body diagram of a ring-shaped differential fluid element of radius $r \le R$, thickness dr and length dz oriented coaxially with the horizontal pipe. At the steady state the net force acting on the fluid element must be zero. This means:

$$(P(z)-P(z+dz))dA_1+\tau(r)dA_2-\tau(r+dr)dA_3=0$$

where

$$dA_1=2\pi r dr, \quad dA_2=2\pi r dz, \quad dA_3=2\pi(r+dr)dz$$



Reorganizing the equation yields: $\frac{dP}{dz}+\frac{1}{r}\frac{d(\tau r)}{dr}=0$

where $dP=P(z+dz)-P(z)$ and

$d(\tau r)=\tau(r+dr)(r+dr)-\tau(r)r$. Inserting $\tau(r)=-\mu\frac{du}{dr}$ and rearranging the equation, we obtain

$\frac{dP}{dz}-\frac{\mu}{r}\frac{d}{dr}\left(r\frac{du}{dr}\right)=0$. Since $\frac{dP}{dz}$ is constant we can easily solve this equation as:

$u(r)=\frac{1}{4\mu}\left(\frac{dP}{dz}\right)r^2+c_1\ln r+c_2$, where c_1 and c_2 are integration constants and can be found from the boundary conditions where $u(0)=u_m$ is the maximum velocity and is finite. This immediately yields $c_1=0$. This yields $u_m=c_2$. The second boundary condition is $u(R)=0$, which yields

$c_2=u_m=-\frac{R^2}{4\mu}\left(\frac{dP}{dz}\right)$, which results in $u(r)=u_m(1-r^2/R^2)$.

2. Since the flow rate is constant and is given by $Q=u_oA$ we can calculate it by integrating the flow rates of the ring-shaped differential fluid elements of radius $r \le R$ over the area $A=\pi R^2$. This means $Q=u_oA=u_o\pi R^2=\int_0^R u(r)dA=\int_0^R u(r)2\pi r dr$. This immediately yields

$Q=\int_0^R u(r)2\pi r dr=-\frac{\pi R^2}{2\mu}\left(\frac{dP}{dz}\right)\int_0^R r(1-r^2/R^2)dr=-\frac{\pi R^4}{8\mu}\left(\frac{dP}{dz}\right)$. This immediately suggests

that the average speed of the flow is given by $u_o=-\frac{R^2}{8\mu}\left(\frac{dP}{dz}\right)$

Comment: In freshman physics you learned that for an incompressible fluid such as water the Bernoulli's equation is given by $u^2/2+g\,z+P/\rho=const.$ where u is the average speed at a point on a streamline, g is the acceleration due to gravity, z is the elevation relative to a reference, P is the pressure at the chosen point and ρ is the density of the fluid. This suggests that for a constant flow speed pressure P along the horizontal pipe must also be constant? However, in practice this equation is incomplete because unless we are dealing with a superfluid most fluids are viscous. We just proved above that for a viscous fluid the flow speed u is proportional to the pressure gradient $\frac{dP}{dz}$ along the horizontal pipe whereas Bernoulli's equation predicts that pressure is constant and $\frac{dP}{dz} = 0$ along the horizontal uniform pipe!

13. Solution

a. Initially both masses have identical velocities, $\dot{x}_1 = \dot{x}_2 = v_0$. Following the elastic collision, $\dot{x}_1 = -v_0$ while $\dot{x}_2 = +v_0$, so the total momentum is

$$p = m_1\dot{x}_1 + m_2\dot{x}_2 = -(m_1 - m_2)v_0 \quad .$$

This makes the COM velocity

$$\boxed{v_{\rm cm} = \frac{p}{m_1 + m_2} = -\frac{m_1 - m_2}{m_1 + m_2}\,v_0 \quad .} \tag{1}$$

b. The instant the spring is fully extended both masses have the same velocity, $\dot{x}_1 = \dot{x}_2 = v_{\rm cm}$, giving the system kinetic energy

$$K_f = \tfrac{1}{2}m_1\dot{x}_1^2 + \tfrac{1}{2}m_2\dot{x}_2^2 = \tfrac{1}{2}(m_1 + m_2)v_{\rm cm}^2 = \frac{(m_1 - m_2)^2}{2(m_1 + m_2)}\,v_0^2$$

The total energy of the system is conserved under the elastic collision so the potential energy must be the difference

$$V_f = K_i - K_f = \tfrac{1}{2}(m_1 + m_2)\,v_0^2 - \frac{(m_1 - m_2)^2}{2(m_1 + m_2)}\,v_0^2 = \frac{2m_1m_2}{m_1 + m_2}\,v_0^2 \quad .$$

The potential energy of a spring is $V = k\Delta x^2/2$, so the spring's extension, Δx, at is maximum is

$$\boxed{\Delta x = \sqrt{\frac{2V_f}{k}} = 2\sqrt{\frac{m_1m_2}{k(m_1 + m_2)}}\,v_0 = \frac{2v_0}{\omega} \quad ,} \tag{2}$$

where ω is the frequency of oscillation.

c. The instant after collision the velocity of m_1 *relative to the system's center of mass* is

$$\dot{x}_1' = \dot{x}_1 - v_{\rm cm} = -v_0 - v_{\rm cm} = -\frac{2m_2}{m_1 + m_2}\,v_0 \quad .$$

Since the spring is still unstretched at that instant this is the maximum inward motion of mass m_1. The maximum *outward* motion occurs exactly one half oscillation period later when the relative velocity is equal and opposite

$$\dot{x}_1' = +\frac{2m_2}{m_1 + m_2}\,v_0 \quad .$$

Its lab-frame velocity at this instant is

$$\dot{x}_1 = \dot{x}_1' + v_{\rm cm} = \frac{3m_2 - m_1}{m_1 + m_2}\,v_0 \quad .$$

This is the maximum velocity achieved by m_1 and it will be positive under the conditions

$$\boxed{m_1 < 3m_2 \quad .} \tag{3}$$

Comp 2013
Solution 14

Answer:

Since f(z) is an analytic function, it satisfies the Cauchy-Riemann equation : $\frac{\delta u}{\delta x} = \frac{\delta v}{\delta y}$ and $\frac{\delta u}{\delta y} = -\frac{\delta v}{\delta x}$.

(a) Since u(x,y) = e^x cos y, $\frac{\delta u}{\delta y} = -\frac{\delta v}{\delta x} = -e^x$ sin y.
Integrate this equation (partially) with respect to x gives
v(x,y)= e^x sin y + w(y), where w(y) is to be found.
Also $\frac{\delta u}{\delta x} = \frac{\delta v}{\delta y} =$ e^x cos y = e^x cos y + $\frac{\delta w}{\delta y}$. So $\frac{\delta w}{\delta y}$=0.
So w must be a constant. Then if C is any complex constant so that w is iC,

The required solutions are:

v(x,y)= e^x sin y+ C, and

f(z) =u + iv = e^{x+iy} + iC. = e^z + iC.

(b) Since v(x,y) = y $(3x^2 - y^2 - 1)$,
$\frac{\delta u}{\delta x} = \frac{\delta v}{\delta y} = (3x^2 - 3y^2 - 1)$.
Integrate this equation (partially) with respect to x gives
u(x,y)= x^3 – 3x$y^2 -$ x + w(y), with w(y) to be found.
Also $\frac{\delta u}{\delta y} = -\frac{\delta v}{\delta x}$ $\rightarrow$ $-$ 6xy +$\frac{\delta w}{\delta y} = -$ 6xy $\rightarrow$ $\frac{\delta w}{\delta y}$ = 0.
Then w must be a constant. Then if C is any complex constant, the required solutions are:

u(x,y)= x^3 – 3x$y^2 -$ x + C, and

f(z) =u + iv = $(z^3 -$ z) + C after some algebra.

Comp 2013
Solution 15

(Students should know the charge magnitude associated with singly-ionized and Avogadro's #.)
(a) To counter the B-field force, the E-field must point up and in the page's plane. Balancing the forces yields E = vB = 40,000 V/m.
(b) It will hit the bottom detector, as determined using q**v** X **B.** $qvB = mv^2/r$ so $m = qBr/v = 2.67 \times 10^{-26}$ kg
(c) Multiply the answer in (b) by Avogadro's number to find 16 g/mole, so it is oxygen.