(39) Bessel's ODE is: $y'' + \frac{1}{x}y' + (1 - \frac{\nu^2}{x^2})y = 0$, w/ ν = real cnst. Find an <u>approximate</u> solution for the Bessel fcn $y \simeq J_\nu(x)$ by the WKB method. Then find an asymptotic form for $J_\nu(x)$ as $x \to$ "large" (specifically: $x \gg |\nu|$). You may assume $|\nu| \gg \frac{1}{2}$.

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(40) In CLASS NOTES pp. WKB 7-10, we solved the WKB problem $\ddot{v} + \Omega^2 v = 0$ by transforming variables: $t \to s = \int \Omega(t)\,dt$, $v \to u = v\sqrt{\Omega}$; then: $u'' + [1 + b(s)]u = 0$, w/ $b(s)$ defind in Eq. (20) of NOTES. $b(s) = 0$ gives the zeroth-order (WKB) solution: $u(s) \simeq u_0(s) = Ae^{+is} + Be^{-is}$. One iteration gave: $u_1 \simeq u_0 + \int_0^s u_0 K\,d\sigma$, w/ K defined in Eq. (27). After n+1 iterations: $u_{n+1} = u_n + \int_0^s u_n K\,d\sigma$, etc. The problem: write u_{n+1} <u>explicitly</u> as a series of (n+2) terms, in successively higher powers of b(s). Show that:

$$u_{n+1}(s) = u_0(s) + \sum_{k=1}^{n+1} \binom{n+1}{k} \int_0^s d\sigma_1 \int_0^{\sigma_1} d\sigma_2 \cdots \int_0^{\sigma_{k-1}} d\sigma_k\, u_0(\sigma_k)\, \mathcal{H}(\sigma_k, \ldots, \sigma_1; s).$$

Find $\mathcal{H}$.

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(41) A QM particle of mass m and energy E moves in a 1D SHO potential $V(x) = \frac{1}{2}m\omega^2 x^2$; w/ ω = SHO natural frequency. Use Bohr-Sommerfeld quantization (NOTES, p. WKB 18) to find the eigenenergies for the motion. How do E_n(WKB) and E_n(actual) compare?

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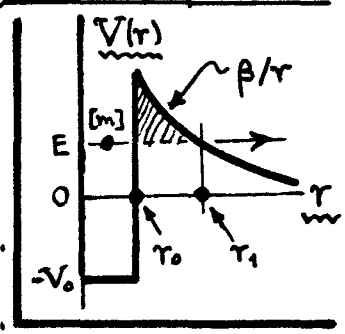
(42) [20 pts] A particle (mass m & energy E > 0) is initially bound in a nuclear potential well of depth V_0 and width r_0. m tunnels thru the Coulomb barrier β/r, and emerges at r_1 w/ zero ∢ momentum.



(A) Per WKB, calculate the probability T(E) that tunneling occurs. For high barriers ($E \ll \beta/r_0$), show: $T(E) \simeq \exp\{-\frac{\pi\beta}{\hbar}\sqrt{2m/E}\}$, independent of r_0.

(B) Consider deuterium fusion: ${}_1H^2 + {}_1H^2 \to {}_2He^3 + n$ (3.2 MeV), by collisions of ${}_1H^2$ nuclei. Find the tunneling factor for ${}_1H^2 \to {}_1H^2$ penetration at room temperature (300°K).

(C) Consider ${}_1H^2$ gas at STP, w/ density n & thermal speed $\bar{v}$. The probability/time of ordinary collisions is $\Gamma_0 = n\sigma_A\bar{v}$, w/ σ_A = <u>atomic</u> collision cross-section. The fusion rate is: $\Gamma_f = n\sigma_D\bar{v}T(\bar{v})$, w/ σ_D = ${}_1H^2$ <u>nuclear</u> cross-section. Approximate σ_A & σ_D by simple geometrical cross-sections; then estimate Γ_f/Γ_0. Is "cold fusion" plausible?

(39) Find an asymptotic form for the Bessel fcn $J_\nu(x)$, $x \to$ "large", via WKB.

1) Bessel's Eqtn: $y'' + (1/x)y' + [1 - (\nu^2/x^2)]y = 0$, converts to WKB form, via:

$\longrightarrow y(x) = \psi(x)\exp\left(-\frac{1}{2}\int\frac{dx}{x}\right) = \psi(x)/\sqrt{x}$,

$$\Rightarrow \boxed{\psi'' + k^2(x)\psi = 0, \ \ ^{w/} \ k(x) = \left[1 - \frac{1}{x^2}\left(\nu^2 - \frac{1}{4}\right)\right]^{1/2}.} \tag{1}$$

This eqtn is exact. A WKB approxn to $\psi(x)$ [and thus to $y = \psi/\sqrt{x}$] will work at values of x where k is "slowly-varying", i.e.

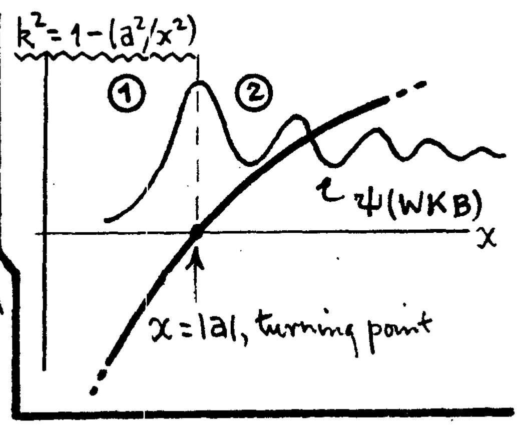
$\longrightarrow \left|\frac{1}{k^2}(dk/dx)\right| = \left|\frac{1}{k^3x^3}\left(\nu^2 - \frac{1}{4}\right)\right| \ll 1$, or $|(kx)^3| \gg |\nu^2 - \frac{1}{4}|$

This works OK when $|x| \to$ "large", so long as i.e. $\underline{\underline{\left|x^2 - \left(\nu^2 - \frac{1}{4}\right)\right|^{\frac{3}{2}} \gg \left|\nu^2 - \frac{1}{4}\right|}}$. (2)

ν = some cnst. Then a WKB form for ψ should be good for $\frac{1}{2} \ll |\nu| \ll x \to \infty$.

2) Let $\underline{\underline{a}} = (\nu^2 - \frac{1}{4})^{1/2}$, so $k(x) = [1 - (a^2/x^2)]^{\frac{1}{2}}$.

$x = |a|$ is a "turning point" for the prob$^{\underline{m}}$ [$k(a) = 0$], and we want ψ(WKB) for $x > |a|$. To be an acceptable solution, ψ should decrease exponentially in region ①, and oscillate in region ②. So we write:



$\longrightarrow \psi(x > |a|) \simeq (A/\sqrt{k})\sin\left(\int_a^x k(\xi)d\xi + \beta\right)$; k as above, ampl. A & phase β = cnsts;

but $\int_a^x k(\xi)d\xi = \int_a^x \frac{d\xi}{\xi}(\xi^2 - a^2)^{1/2} = (x^2 - a^2)^{1/2} - a\cos^{-1}\left(\frac{a}{x}\right) \simeq x - \frac{\nu\pi}{2}$ $\int$ for $x \gg |a|$ & $a \simeq \nu$.

So since $k \simeq 1$ as $x \to$ "large", then: $\underline{\underline{\psi(x > |a|) \simeq A\sin\left(x - \frac{\nu\pi}{2} + \beta\right)}}$. (3)

3) Since $y = \psi/\sqrt{x}$, the WKB solution to Bessel's Eqtn, for $\frac{1}{2} \ll |\nu| \ll x \to \infty$, is

$$\boxed{y(x) = J_\nu(x) \simeq \frac{\text{cnst}}{\sqrt{x}}\sin\left(x - \frac{\nu\pi}{2} + \beta\right)} \tag{4}$$

When the phase $\beta = \pi/4$, this is a standard result; see NBS Math. Handbook # (9.2.1). The phase β can be fixed by the WKB Connection Formulas.

(40) Iterate the Neumann series for $u_{n+1}(s)$ [from p. 10 of "Notes on WKB Method"].

1) Start from the $\underline{m=1^{st}}$ iteration [Eq. (27) of "Notes on the WKB Method]: $u_{n+1}(s) = u_n(s) + \int_0^s d\sigma_1\, u_n(\sigma_1) K(\sigma_1, s)$, and insert: $u_n(x) = u_{n-1}(x) + \int_0^x d\sigma_2 u_{n-1}(\sigma_2) K(\sigma_2, x)$. So:

$$\rightarrow u_{n+1}(s) = u_{n-1}(s) + 2\int_0^s d\sigma_1\, u_{n-1}(\sigma_1) K(\sigma_1, s) + \int_0^s d\sigma_1 \int_0^{\sigma_1} d\sigma_2\, u_{n-1}(\sigma_2) K(\sigma_2, \sigma_1) K(\sigma_1, s). \quad (1)$$

This is the $\underline{m=2^{nd}}$ iteration. Put $u_{n-1}(x) = u_{n-2}(x) + \int_0^x d\sigma_3\, u_{n-2}(\sigma_3) K(\sigma_3, x)$ into Eq. (1) and again collect like terms to find for the $\underline{m=3^{rd}}$ iteration...

$$\rightarrow u_{n+1}(s) = u_{n-2}(s) + 3\int_0^s d\sigma_1\, u_{n-2}(\sigma_1) K(\sigma_1, s) + 3\int_0^s d\sigma_1 \int_0^{\sigma_1} d\sigma_2\, u_{n-2}(\sigma_2) K(\sigma_2, \sigma_1) K(\sigma_1, s) +$$

etc.

$$+ \int_0^s d\sigma_1 \int_0^{\sigma_1} d\sigma_2 \int_0^{\sigma_2} d\sigma_3\, u_{n-2}(\sigma_3) K(\sigma_3, \sigma_2) K(\sigma_2, \sigma_1) K(\sigma_1, s). \quad (2)$$

---

2) In the $m=1$ iteration above, there are 2 terms, with numerical coefficients [1, 1]. For $m=2$ in Eq. (1), we got 3 terms, with coefficients [1, 2, 1], and for $m=3$ in Eq. (2), we got 4 terms, with coefficients [1, 3, 3, 1]. These sets are the binomial coefficients $\binom{m}{k} = m!/k!(m-k)!$, with m = iteration order #, and $k = 0, 1, \ldots, m$. After the m^{th} such operation as in Eq. (2) above, we will have the series...

$$u_{n+1}(s) = u_{n+1-m}(s) + \sum_{k=1}^{m} \binom{m}{k} \underbrace{\int_0^s d\sigma_1 \int_0^{\sigma_1} d\sigma_2 \cdots \int_0^{\sigma_{k-1}} d\sigma_k}_{k \text{ integrations}}\, u_{n+1-m}(\sigma_k)\, \underbrace{K^{(k)}}_{k \text{ factors of } K}(\sigma_k, \ldots, \sigma_1, s),$$

$$\text{where: } \underline{K^{(k)}(\sigma_k, \ldots, \sigma_1, s) = K(\sigma_k, \sigma_{k-1}) K(\sigma_{k-1}, \sigma_{k-2}) \cdots K(\sigma_2, \sigma_1) K(\sigma_1, s)}. \quad (3)$$

Since $K(x,y) = b(x)\sin(x-y)$, then $K^{(k)}$ is of order $(b)^k$ in the small factor b.

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3) The iteration in Eq. (3) can be done a maximum of $m = n+1$ times. Then...

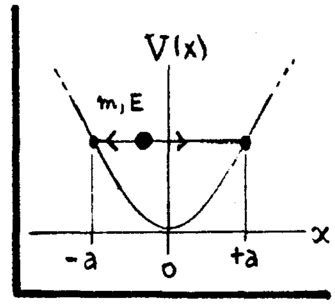
$$\boxed{u_{n+1}(s) = u_0(s) + \sum_{k=1}^{n+1} \binom{n+1}{k} \int_0^s d\sigma_1 \int_0^{\sigma_1} d\sigma_2 \cdots \int_0^{\sigma_{k-1}} d\sigma_k\, u_0(s)\, K^{(k)}(\sigma_k, \ldots, \sigma_1, s).} \quad (4)$$

This allows expressing $u_{n+1}(s)$ in terms of the WKB approximate for $u_0(s)$, with correction terms of order $K, (K)^2, \ldots, (K)^{n+1}$. Note that in Eq. (4), $n = 0, 1, 2, \ldots, \infty$.

(41) Quantization of the SHO via Bohr-Sommerfeld rule [by way of WKB approxn].

1) With $V(x) = \frac{1}{2}m\omega^2 x^2$, the QM version of the WKB interior phase integral is

$$\int_{x_1}^{x_2} k(x)\,dx = \int_{x_1}^{x_2} \left[\frac{2m}{\hbar^2}\left(E - \tfrac{1}{2}m\omega^2 x^2\right)\right]^{1/2} dx = \left(n+\tfrac{1}{2}\right)\pi, \quad (1)$$



with $n = 0, 1, 2, \ldots$, and $x_{1,2}$ "turning points"... i.e. points at which $E = V(x) = \frac{1}{2}m\omega^2 x^2$. Define these to be at $x = \pm a$...

$$\rightarrow E = \tfrac{1}{2}m\omega^2 a^2 \longleftrightarrow \text{turning points at } x_1 = (-)a,\ x_2 = +a. \quad (2)$$

---

2) Eq. (1), the Bohr-Sommerfeld quantization, now amounts to...

$$\frac{m\omega}{\hbar}\int_{-a}^{+a} (a^2 - x^2)^{1/2}\,dx = \left(n+\tfrac{1}{2}\right)\pi \quad (3)$$

... but: $\displaystyle\int_{-a}^{+a} (a^2-x^2)^{1/2}\,dx = \frac{1}{2}\left[x(a^2-x^2)^{1/2} + a^2 \sin^{-1}\left(\frac{x}{a}\right)\right]\Bigg|_{x=-a}^{x=+a}$

$$= \tfrac{1}{2}a^2\left[\sin^{-1}(+1) - \sin^{-1}(-1)\right] = \tfrac{1}{2}a^2\pi,$$

So, $\displaystyle\frac{m\omega}{\hbar}\cdot\frac{1}{2}a^2\pi = \left(n+\tfrac{1}{2}\right)\pi$, ^{n}// $\underline{\underline{\tfrac{1}{2}m\omega^2 a^2 = \left(n+\tfrac{1}{2}\right)\hbar}}$. (4)

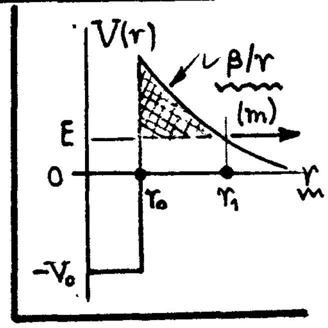
---

3) By defn of a, in Eq. (2), we see that $\frac{1}{2}m\omega a^2 = E$ in Eq. (4). So the quantized energies of the SHO, via Bohr-Sommerfeld (á la WKB) are...

$$\boxed{E_n = \left(n+\tfrac{1}{2}\right)\hbar\omega,\quad n = 0, 1, 2, \ldots}. \quad (5)$$

These are the <u>exact</u> energies of a QM SHO (consult any telephone book, or QM directory, etc.). WKB (Bohr-Sommerfeld) quantization is usually a ~ good approxn, but not always this good. This is the only instance -- that I know of -- where the WKB energies agree <u>exactly</u> with the QM result.

⑫ [20 pts]. Penetration of a Coulomb barrier (via WKB). Will "cold fusion" work?



(A) 1) The centrally symmetric problem reduces to a 1D motion along the radial direction r, and if the tunneling particle (m, E) has zero ∡ momentum, there is no centrifugal barrier term -- the potential in the tunneling region is just β/r. We can therefore use the transmission coefficient T of Eq.(11), p WKB 23 of classnotes directly:

$$\rightarrow T = \exp\left\{-\frac{2}{\hbar} J(E)\right\}, \text{ w/ } J(E) = \int_{r_0}^{r_1} \sqrt{2m[(\beta/r) - E]}\, dr. \tag{1}$$

2) The initial barrier contact point is r_0 = nuclear radius, and the exit point r_1 is such that $\beta/r_1 = E$, i.e. $r_1 = \beta/E$. By a simple change of variables...

$$\rightarrow u = \frac{\beta}{Er} \Rightarrow J(E) = \beta\sqrt{\frac{2m}{E}} \int_1^{u_0} \frac{du}{u^2}\sqrt{u-1}, \text{ w/ } u_0 = \beta/Er_0. \tag{2}$$

Integrals of this form are tabulated, and the result for J(E) is...

$$\rightarrow J(E) = \beta\sqrt{\frac{2m}{E}}\left[\tan^{-1}\sqrt{u_0 - 1} - \frac{1}{u_0}\sqrt{u_0 - 1}\right], \quad \underline{\underline{u_0 = \frac{\beta}{r_0}/E.}} \tag{3}$$

Note that u_0 = ratio of initial barrier height to particle energy. In the limits...

$$\begin{cases} E \to 0+,\ u_0 \to \infty :\ J(E) \simeq \frac{\pi}{2}\beta\sqrt{\frac{2m}{E}}\left[1 - \frac{4}{\pi}(1/\sqrt{u_0})\right]; \\ E \to \frac{\beta}{r_0}-,\ u \to 1+ :\ J(E) \simeq \beta\sqrt{2m/E}\,(u_0 - 1)^{3/2}/u_0. \end{cases} \tag{4}$$

For high barriers, $\beta/r_0 \gg E$, $u_0 \to$ large, and the tunneling probability is

$$\boxed{T(E) \simeq \exp\left(-\frac{\pi\beta}{\hbar}\sqrt{2m/E}\right)}. \tag{5}$$

(B) 3) If the emergent particle is not relativistic (~ always true), then in Eq.(5): $E = \frac{1}{2}mv_{out}^2$, and: $T(v_{out}) \simeq \exp(-2\pi\beta/\hbar v_{out})$, where v_{out} is the velocity of m outside the barrier. Furthermore, $\beta = e^2 \times$(some factor f), so...

 fs cnst: $\alpha = e^2/\hbar c \simeq 1/137$.

$$\rightarrow T(v_{out}) \simeq \exp\left(-2\pi f \frac{e^2}{\hbar c}\frac{c}{v_{out}}\right) = \exp[-2\pi f\alpha(c/v_{out})]. \quad (6)$$

For ${}_1H^2$ at room temperature (300°K), the K.E. is (1/38.7) eV, so

$$\frac{v_{out}}{c} = \sqrt{\frac{2E_{out}}{mc^2}} = \sqrt{\frac{2\times(1/38.7)}{2\times 932\times 10^6}} = 1/1.9\times 10^5. \quad (7)$$

(we've take $m = 2$ a.m.u. for ${}_1H^2$). With $f=1$ in Eq. (6) for a barrier penetration of ${}_1H^2$ by ${}_1H^2$ (both charged at $+e$), we find the tunneling factor in Eq. (6): $\underline{\underline{T(v_{out}) = e^{-871.4} = 3.6\times 10^{-379}}}$. Which is kinda small.

(C) 4) For ${}_1H^2$ gas at STP, $n = 2.7\times 10^{19}/cm^3$ (Loschmidt #), and $\bar{v} = c/190 = 1.58\times 10^8$ cm/sec. But these numbers drop out when we take the ratio of the collision rates...

$$\left.\begin{matrix}\Gamma(\text{fusion}) = n\sigma_D \bar{v}\, T(\bar{v}) \\ \Gamma(\text{atomic}) = n\sigma_A \bar{v}\end{matrix}\right\} \quad \frac{\Gamma(\text{fusion})}{\Gamma(\text{atomic})} = \left(\frac{\sigma_D}{\sigma_A}\right) T(\bar{v}). \quad (8)$$

So it doesn't much matter whether we work with liquid or gaseous ${}_1H^2$. The geometrical cross-sections are: $\sigma_D \sim \pi\times(2\times 10^{-13}\,cm)^2$ *, $\sigma_A = \pi\times(0.53\times 10^{-8}\,cm)^2$; so: $\sigma_D/\sigma_A \sim 1.42\times 10^{-9}$, and the relative fusion reaction rate is

$$\rightarrow \Gamma(\text{fusion})/\Gamma(\text{atomic}) \sim 1.42\times 10^{-9}\; T(\bar{v}). \quad (9)$$

For room temp, $T(\bar{v}) = 3.6\times 10^{-379}$, as calculated in part (B), so then this ratio is: $\underline{\underline{\Gamma(\text{fusion})/\Gamma(\text{atomic}) \sim 5\times 10^{-388}}}$. At room temp, fusions occur "spontaneously" ~ one time per 2×10^{387} collisions. Does not appear too promising.

To make the fusion work, you have to heat the ${}_1H^2$ gas, to increase $T(E)$. At a temp ~ 300×10^6 °K, $E = 26$ keV, and $T(E) \simeq 1.64\times 10^{-4}$. Then $\Gamma(\text{fusion})/\Gamma(\text{atomic}) \sim 2\times 10^{-13}$, which begins to approach the realm of the possible.

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* Ref. A. Arya "Fund^{ls} of Nuclear ϕ" (Allyn-Bacon 1966), p. 123: $r \simeq (1.35\times 10^{-13}\,cm)\times A^{1/3}$.