We seek an approximate solution $y(x)$ to the following differential equation

$$\left(\frac{d^2}{dx^2} + k^2 \right) y(x) = f(x)\, y(x)$$

where the real constant $k > 0$, and f is an arbitrary real function of x.

(a) Determine the most general homogeneous solution $y_\circ(x)$ for $f(x) = 0$.

(b) *Without solving for the Green function* $G(x)$, show explictly that the original differential equation may be written in integral form as

$$y(x) = y_\circ(x) + \int_{-\infty}^{\infty} G(x - x')\, f(x')\, y(x')\, dx' \qquad \text{with} \qquad \left(\frac{d^2}{dx^2} + k^2 \right) G(x) = \delta(x).$$

(c) Analytically solve for $G(x)$ so that $G(x) \propto e^{ikx}$. It will be useful here to recall the following integral expression for a delta function with real α:

$$\delta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\alpha x}\, d\alpha.$$

(d) Now, if $f(x)$ is "small" in some sense, $y(x)$ will be approximately $y_\circ(x)$ plus a small correction:

$$\begin{aligned} y(x) &= y_\circ(x) + \int_{-\infty}^{\infty} G(x - x')\, f(x') \left\{ y_\circ(x') + \int_{-\infty}^{\infty} G(x' - x'')\, f(x'')\, y(x'')\, dx'' \right\} dx' \\ &= y_\circ(x) + \int_{-\infty}^{\infty} G(x - x')\, f(x')\, y_\circ(x')\, dx' + \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G(x - x')\, G(x' - x'')\, f(x')\, f(x'')\, y(x'')\, dx'\, dx'' \\ &\simeq y_\circ(x) + \int_{-\infty}^{\infty} G(x - x') f(x') y_\circ(x')\, dx' \end{aligned}$$

Determine an approximate functional form for $y(x)$ for the case

$$f(x) = \begin{cases} 0 & |x| > a/2 \\ b & |x| < a/2 \end{cases}$$

(e) Develop an analytic constraint on b under which the approximation in part (d) is valid.