1)

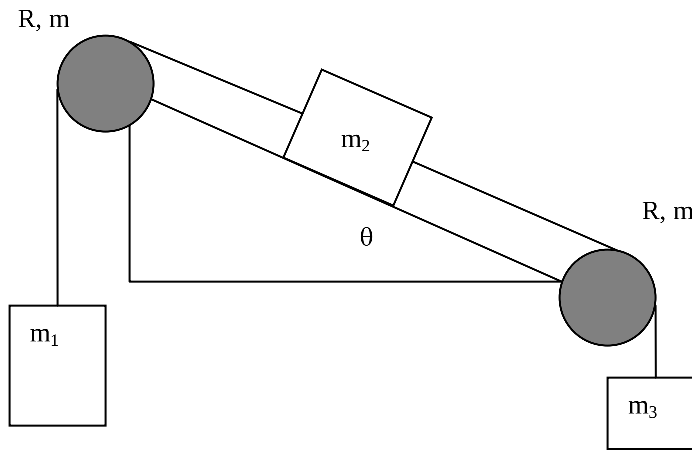
A metal detector is made from a single horizontal circular loop of radius R in which a constant current I_0 is maintained using an ideal current generator.

A metal sphere of radius $a<<R$ is buried a depth $d>>R$ beneath the soil (soil has $\mu = \mu_o$). Consider the sphere to be made of linear paramagnetic material of permeability μ. In the following retain only the leading order in small parameters a/R and R/d.

a. What dipole moment is induced in the sphere when the coil is at ground level directly above it?
b. By how much does the flux inside the current loop change as a result of moving it to the point above the sphere from somewhere far away?
c. Denote the self-inductance of the loop, with no magnetic material around, $L_o = \alpha\mu_o R$ where α is a constant. What is the *relative change* in its self-inductance, $\Delta L / L_0$, when it is moved to the point over the metal sphere? Note whether the change is an increase or decrease.

2)

Three masses m_1, m_2 and m_3 are connected by massless ropes over two pulleys as shown. The solid disc pulleys have a mass m and a radius R. The slope has a coefficient of kinetic friction μ_K and a coefficient of static friction μ_S. The incline is at an angle θ to the horizontal.



a) Find the minimum mass m_1 for which the mass m_2 will begin to move up the incline.
b) Once the mass m_2 starts to slide, find the steady acceleration 'a' of m_2.
c) Find the power lost to heat by the friction of the mass m_2 on the incline as a function of time t.

3)

The tritium ^3H atom is a metastable isotope of the ^{1}H atom with a half-life of 12.33 years. It undergoes a nuclear transition and transforms into a $^3He^+$ ion through a *beta* (e^-) and an *antineutrino* ($\bar{\nu}$) decay, $^3H \rightarrow {}^3He^+ + e^- + \bar{\nu}$. One of the neutrons in the nucleus of ^3H is converted into a proton. The energetic electron generated by the beta decay escapes the atom, leaving the *He* in a positively charged state. The electron that was in the ground state orbiting the ^3H atom suddenly finds itself orbiting a $^3He^+$ ion. Assume that the *beta* decay is too fast for the orbiting electron to adjust to the new configuration until after the decay is complete. Answer the following questions:

a. Assuming that a ^3H atom starts out in the ϕ_{100} ground state, prove that the probability of the electron orbiting the $^3He^+$ ion in the $\phi_{nlm} = R_{nl}(r)Y_{lm}(\theta,\phi) = \phi_{210}$ state of the $^3He^+$ ion is zero. Is this expected? Explain.

b. The result found in (a) would suggest that finding the $^3He^+$ ion in any excited state as a result of nuclear transition is zero. Rather than rushing to this conclusion, now determine the probability that the $^3He^+$ ion is in the $\phi_{nlm} = \phi_{200}$ state. From this result draw a general conclusion on the symmetry of the excited state wave functions.

(Hint. Useful functions: $\phi_{100} = \frac{1}{\sqrt{\pi}}\left(\frac{Z}{a_0}\right)^{3/2} e^{-Zr/a_o}$, $\phi_{210} = \frac{1}{4\sqrt{2\pi}}\left(\frac{Z}{a_0}\right)^{3/2} \frac{Zr}{a_0} e^{-Zr/2a_o}\cos\theta$,

$\phi_{200} = \frac{1}{4\sqrt{2\pi}}\left(\frac{Z}{a_0}\right)^{3/2}\left(2 - \frac{Zr}{a_0}\right)e^{-Zr/2a_o}$, $Y_{00}(\theta,\phi) = \frac{1}{\sqrt{4\pi}}$)

4)

A spin-1/2 particle interacts with a magnetic field $\vec{B} = B_o\hat{z}$ through the Pauli interaction $H = \mu\vec{\sigma}\cdot\vec{B}$, where μ is the magnetic moment and $\vec{\sigma} = \left(\sigma_x, \sigma_y, \sigma_z\right)$ are the Pauli spin matrices. At *t* = 0, a measurement determines that the spin is pointing along the positive *x*-axis. Calculate the probability that it will be pointing along the negative *y*-axis at a later time *t*.

NOTE: The Pauli spin matrices are given by

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

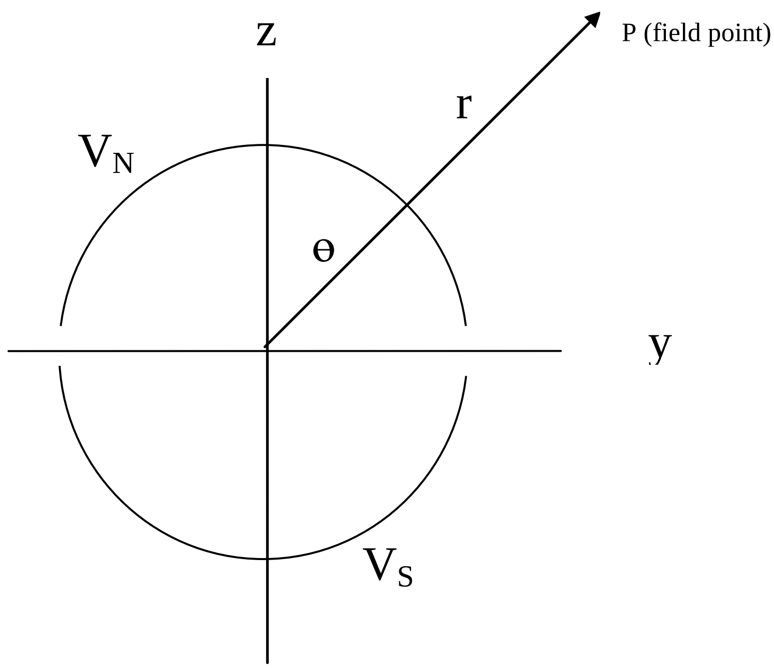
5)
Let a conducting sphere of radius **a** be split into hemispheres, held at potentials V_N and V_S, respectively (see Figure 1). Find the potential Φ (r, θ) <u>outside</u> the sphere. Let $\Phi(r=\infty,\theta)=0$.

Hint: $\int_0^1 P^{\ell}(x)dx = \frac{(-1)^{(\ell-1)/2}\,(\ell-1)!}{2^{\ell}\,\{(\ell+1)/2\}!\,\{(\ell-1)/2\}!}$, if ℓ = odd

$= 0$ if ℓ = even

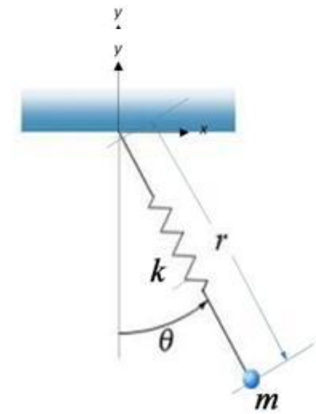
$= 1$ if $\ell = 0$

Figure 1



6)

The figure below shows a spring pendulum consisting of a massless spring of spring constant k and unstretched length r_o . One end of the spring is pivoted from the ceiling while the other is attached to a point mass *m*. The pendulum is restricted to movement in an x-y plane as shown in the figure. The pendulum is pulled off its equilibrium and set into motion. Answer the following questions:



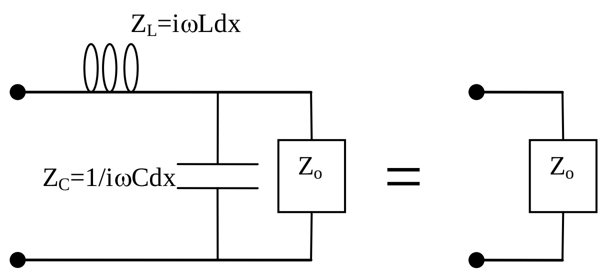
a. Write down the Lagrangian and determine the equations of motion for *m* using polar coordinates r and θ , shown in the figure.
b. Identify each component in the equations found in part (a) in terms of Newton's laws of motion.
c. Now make small amplitude approximations in the Lagrangian equations associated with the $r\ (= r_e + u)$ and θ motions but keep the terms in the **second order** in the amplitudes of motion (i.e. in θ and u / r_e and in their derivatives) and write down the equations of motion for $\theta(t)$ and $u(t)$; here r_e is the equilibrium position around which *m* oscillates.
d. Now discuss the coupling between the u and θ motions implied in (c) and compare this motion with the motions in u and θ, but this time keeping only the **first-order** terms in the amplitudes of the motion.

7)
A long coaxial cylinder transmission line is made of an inner conductor of radius a surrounded by an outer conductor of inner radius b. Assume that the region between a and b is filled with vacuum and that the conductors are perfectly conducting.

a) Find an expression for the capacitance per unit length C of the coaxial cylinder in terms of a and b.

b) Find an expression for the inductance per unit length L of the coaxial cylinder in terms of a and b.

c) Find the characteristic impedance Z_o for the coaxial cylinder transmission line. Hint: The two circuits to the right are equivalent. Use this fact to find the impedance Z_o for the transmission line. For the circuit on the left side, the impedances for the inductance and capacitance of a short length dx of the line are also shown. You may assume dx is small and keep only appropriate terms.



8)
Initially n moles of an ideal, monatomic gas are at pressure P_o, volume V_o, and temperature T_o. The gas undergoes an isothermal expansion that doubles its volume. What are the final values for each of the following quantities in terms of the initial values of these quantities?

a. pressure
b. temperature
c. heat absorbed
d. work done
e. change in internal energy
f. change in entropy

Find these same quantities if the gas undergoes an adiabatic **free** expansion from volume V_o to $2\ V_o$.

9)

A two-dimensional simple harmonic oscillator has oscillation frequency Ω_0. It is prepared in the state

$$\psi(x,y) = \left[\tfrac{\sqrt{3}}{2}\varphi_1(x) - \tfrac{1}{2}\varphi_2(x)\right]\left[\tfrac{1}{\sqrt{2}}\varphi_0(y) - \tfrac{i}{\sqrt{2}}\varphi_1(y)\right] \qquad (1)$$

where φ_n is the n^{th} energy eigenfunction of the *one-dimensional* simple harmonic oscillator - φ_0 is the ground state.

a. If the energy of this state were measured, what possible values could be found? (Express them in terms of Ω_0.) With what probability will each energy be observed?
b. What is the expected energy $\langle E \rangle$?
c. The angular momentum of the particle is represented by the operator

$$\hat{L}_z = i\hbar\left(\hat{a}_x\hat{a}_y^\dagger - \hat{a}_x^\dagger\hat{a}_y\right)$$

written in terms of raising and lower operators for the 1d SHO satisfying the following relationships

$$\hat{a}_x^\dagger\,\varphi_n(x) = \sqrt{n+1}\,\varphi_{n+1}(x) \quad , \quad \hat{a}_x\,\varphi_n(x) = \sqrt{n}\,\varphi_{n-1}(x)$$

and similarly for $\hat{a}_y^\dagger$ and $\hat{a}_y$ acting on $\varphi_n(y)$ (You do not need to show this -- it is true.) What is the expectation $\langle L_z \rangle$ for a particle in state given by eq. (1).

10)

(a) Starting with the equation for mechanical pressure P:

$$P = \frac{1}{3}\int_0^\infty p\, v\; n(\mathbf{p})\, \mathbf{dp}, \qquad (1)$$

where p is the momentum, v is the velocity of the constituent particles and n is the momentum distribution of the particles.

Utilizing the momentum distribution in thermal equilibrium:

$$n\,(p)\,dp = \frac{4\,\pi\,N\,p^2\,dp}{V(2\,\pi\,m\,k\,T)^{3/2}}\;\exp\,\{-\,p^2/\,(2\,m\,k\,T)\}, \qquad (2)$$

show that the equation of state of a perfect nonrelativistic gas is expressed as:

$$PV = N\,k\,T, \qquad (3)$$

where T is temperature, N is the number of particles in a volume V and m is the mass of a particle.

(b) From the expression for force balance, show that the structure of a stationary spherical object, e.g., a star, in hydrostatic equilibrium is found from:

$$\frac{dP(r)}{dr} = -\frac{\rho(r)\;G\;M(r)}{\boldsymbol{r^2}},$$

$$\frac{dM(r)}{dr} = \rho(r)\;4\pi\;r^2 \qquad (4)$$

where P(r) and ρ(r) are pressure and density at distance r from the center, M(r) is the total mass within a sphere of radius r, and G is the gravitational constant.

11)

Consider the inhomogeneous equation with the differential operator $\mathcal{L}$

$$\mathcal{L}\,\psi = \frac{d}{dx}\left(x^2\,\frac{d\psi}{dx}\right) - 2\psi = f(x)$$

a. Find a complete set of **homogeneous** solutions by proposing $\psi = Ax^{\lambda}$

b. For the boundary value problem

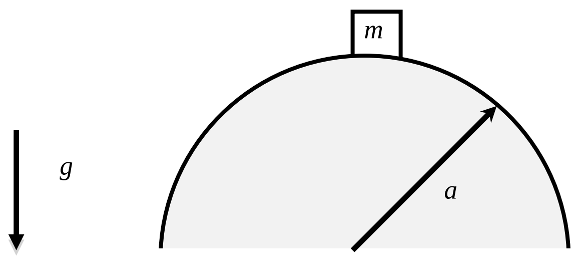
$$\mathcal{L}\,\psi = f(x) \quad , \quad \psi(1) = 1 \quad , \quad \psi \to 0 \quad \text{as} \quad x \to \infty \tag{1}$$

find the Green's function for the operator $\mathcal{L}$ within the region $1 < x$.

c. Use the Green's function from b. to solve eq. (1) for the choice of RHS $f(x)=x^{-2}$

12)

A smooth hemispherical surface of radius a is placed with its flat side down and fastened to the Earth whose gravitational acceleration is g. A small object of mass m is placed on top of the hemisphere and given a very small push ($v_{in} \approx 0$). If the surface is frictionless, use the method of Lagrange multipliers to calculate at what point the small object leaves the surface of the hemisphere.



13)

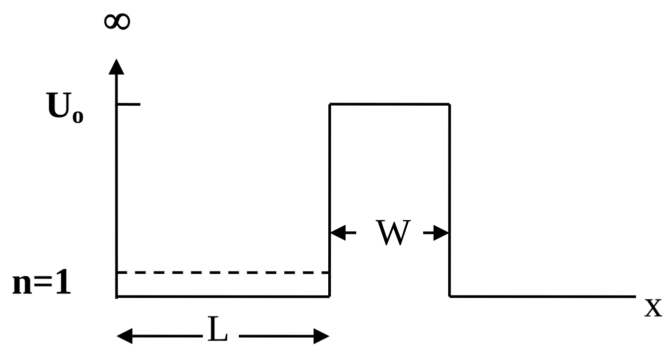
A paramagnetic material is subjected to a magnetic field, $\vec{B} = B_o\hat{z}$, and is in thermal equilibrium with a reservoir at temperature *T*. Assume that the paramagnetic material is made up of *N* distinguishable magnetic particles, with each particle having a net angular momentum of *j=1* and a magnetic dipole moment of $\vec{\mu} = \mu_o\vec{j}$. Answer the following questions.

a. Determine the entropy of the paramagnetic material under the conditions described above.
b. Now assume that the paramagnetic material is isolated from the reservoir and the magnetic field is reduced adiabatically to $\vec{B}_f = (B_o/100)\hat{z}$. Determine the final temperature of the system in terms of *T* as a result of this adiabatic demagnetization.

(Hint: $F = -kT\,lnZ$ and $dF = -SdT - pdV + \mu dN$, where *F* is the Helmholtz free energy and *Z* is the canonical partition function.)

14)

An electron is in the lowest state (n=1) of a 1-D quantum well of width L as shown in the figure. The potential wall at x=0 is infinitely high, while the potential wall on the right at x=L is finite with a height of U_o and width W forming a barrier. Estimate the initial rate of escape of the electron through the barrier.



Assume that $L \gg \dfrac{\hbar}{\sqrt{mU_o}}$ so that the initial state can be approximated by the ground state of an infinite 1-D square well.

Also assume that $W \gg \dfrac{\hbar}{\sqrt{mU_o}}$ when finding the initial rate of escape through the barrier.

15)

(a) Calculate the multipole moment $q_{\ell m}$ of the charge distributions shown in Figure 1 (positive charge q on the z axis distance **a** above and below the origin, and negative 2q at the origin). Obtain results for nonvanishing moments valid for all ℓ.

(b) For the charge distribution shown in Figure 1 write down the multipole expansion for the potential ϕ, for r > a.

(c) What is the potential if you keep only the lowest order term in the expansion (valid for r >> a)?

(d) How does the potential in (c) behave as function of distance r on the x-y plane?

(e) Calculate directly from Coulomb's law the exact potential for **(a)** in the x-y plane, and compare the behavior with the solution in (d) as $r \to \infty$..

Hints: $q_{\ell m} = \int Y^*_{\ell m}(\theta, \varphi)\, r^\ell \rho(\mathbf{x})\, d^3x.$

$$\phi = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{+\ell} \frac{4\pi}{2\ell+1} r^{-\ell-1} Y_{\ell m}(\theta, \varphi) q_{\ell m}$$

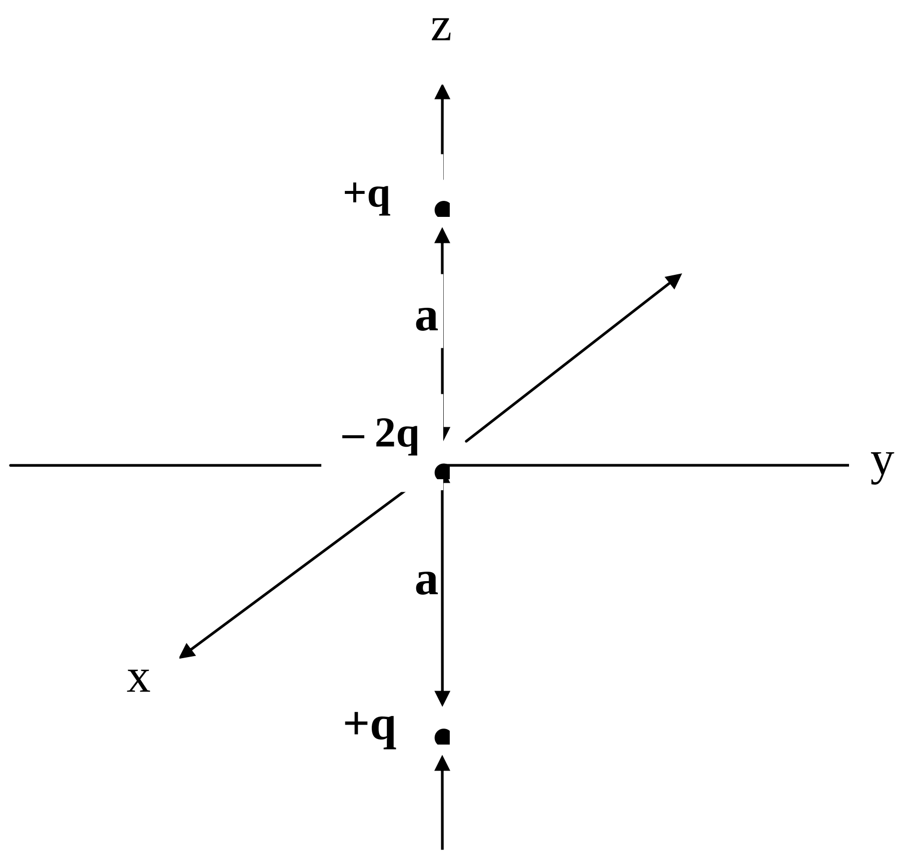


Figure 1