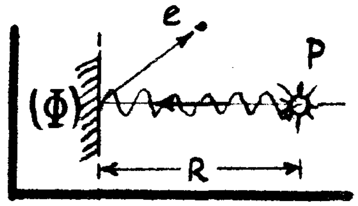
ϕ506 Problems



④ A lightbulb, radiating total power $P = 10W$, is placed at distance $R = 1m$ from a large metal plate. The metal has a work function $\Phi = 3eV$. An electron in the metal can absorb the incident radiation over an area adjacent to it -- suppose that area is a circle of radius $r \sim$ several atomic radii (say 5). Treat this problem classically.

(A) Calculate the time Δt required for the electron to absorb enough energy to be ejected from the surface. It is observed that $\Delta t < 1ns$. Comment.

(B) With all else the same, what lightbulb power is needed to eject the electron in 1ns? Compare this with the output of a typical commercial power plant.

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⑤ (A) Show that a photon cannot transfer all of its energy to a free electron.

(B) Suppose a photon, incident on a free electron, has energy $\mathcal{E}_i = \mathcal{N}mc^2$, where m is the electron rest mass and $\mathcal{N}$ is a numerical factor. What value of $\mathcal{N}$ is required for the photon to transfer 99% of its energy? What is λ for this photon?

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⑥ Apply the Correspondence Principle to radiation from a transition $n \to n-1$ in a hydrogenlike Bohr atom. The average power radiated is : $\underline{P_n = \frac{1}{\tau_n} \Delta E(n \to n-1)}$, w/ $\Delta E =$ energy emitted, and $\tau_n =$ "lifetime" of state n. As $n \to$ large, P_n must become the classical Larmor radiation rate : $\underline{P_n = (2e^2/3c^3)|a_n|^2}$, for a charge e undergoing (centripetal) acceleration a_n. Assume circular orbits, and define the dimensionless constant : $\alpha = e^2/\hbar c \simeq 1/137\ldots$ α is the "fine-structure" cnst.

(A) Let $\Gamma_n = 1/\tau_n$ be the "transition probability / unit time" for $n \to n-1$. Show that : $\Gamma_n = \Gamma_1/n^5$, where : $\Gamma_1 = \frac{2}{3} Z^4 \alpha^5 \cdot (mc^2/\hbar)$.

(B) For hydrogen ($Z = 1$), compare the result in part (A) for the n^{th} state lifetime $\tau_n = 1/\Gamma_n$ with known values : $\tau(n = 2 \to 1) = 1.6ns$, $\tau(n = 4 \to 3) = 73ns$, and $\tau(n = 6 \to 5) = 610ns$. Comment on the trend of the comparisons.

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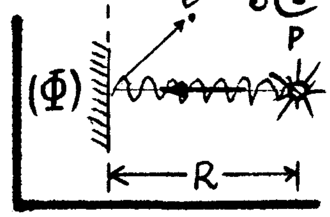

④ Time & power scales for the classical photo-electric effect.

(A) The radiation incident on the plate delivers a flux of size:

$$\rightarrow \frac{\text{energy/time}}{\text{area}} = P/4\pi R^2. \qquad (1)$$

If a given electron gathers energy from an area = circle of radius na, w/ a = atomic radius and n = numerical factor, it absorbs energy at a rate...

$$\rightarrow \frac{\text{energy}}{\text{time}} = (P/4\pi R^2)\cdot\pi(na)^2 = P\cdot(na/2R)^2. \qquad (2)$$

In order to be ejected in time Δt, the electron must gather an energy exceeding the work function barrier, i.e.

$$\rightarrow \left(\frac{\text{energy}}{\text{time}}\right)\Delta t = P\Delta t\cdot(na/2R)^2 \geqslant \Phi \Rightarrow \Delta t \geqslant \frac{\Phi}{P}\cdot(2R/na)^2. \qquad (3)$$

For the given numbers...

$\Phi = 3\,eV = 4.81\times10^{-19}\,J$, $P = 10\,W = 10\,J/sec$

$R = 1\,m = 100\,cm$; $a = 0.53\times10^{-8}\,cm$ $\left(\begin{smallmatrix}\text{Bohr}\\ \text{radius}\end{smallmatrix}\right)$ & $n = 5$...

$$\underline{\underline{\Delta t}} = \frac{4.81\times10^{-19}}{10}(2\times100/5\times0.53\times10^{-8})^2 = \underline{\underline{2.74\ \text{sec}}} \qquad (4)$$

The observed $\Delta t < 1\,ns$. Clearly the classical picture of the electron <u>continuously</u> absorbing the radiant energy is monumentally WRONG. Enter Einstein, center stage: the "absorption" is actually a collision.

(B) If $\Delta t = 2.74$ sec is to be decreased to 1 ns, then -- per Eq. (3) -- the lightbulb power P must be increased by a factor $2.74/10^{-9}$. This gives a required power...

$$\rightarrow \text{for } \Delta t = 1\,ns:\ \underline{\underline{P}} = 2.74\times10^9\times10\,W = \underline{\underline{27{,}400\ MW}}. \qquad (5)$$

A typical commercial power plant puts out ~ 100–300 MW. So the power required in Eq. (5) is equivalent to the output of ~100 such plants.

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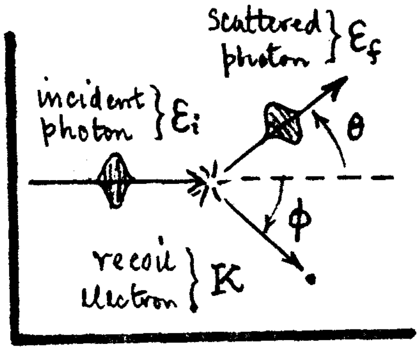
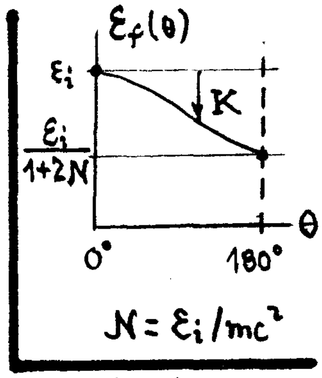
⑤ Most energetic photon → electron collision.



(A) Ref. CLASS NOTES, p. Intro. 12, Eqs. (26)-(27). There, for a photon → free electron collision, we have written energy & momentum conservation eqtns, and have found the final photon frequency. Since energy $\mathcal{E} = h\nu$ for the photon, Eq. (27) is:

$$\underline{\underline{\mathcal{E}_f = \mathcal{E}_i / [1 + (\mathcal{E}_i/mc^2)(1-\cos\theta)]}}, \quad \theta = \text{scattering } \measuredangle \text{ for photon}. \tag{1}$$



The photon transfers a recoil (kinetic) energy K to the e, of an amount...

$$\rightarrow K = \mathcal{E}_i - \mathcal{E}_f = \mathcal{E}_i \left[\frac{N(1-\cos\theta)}{1+N(1-\cos\theta)}\right] < \mathcal{E}_i, \text{ w/ } N = \frac{\mathcal{E}_i}{mc^2}. \tag{2}$$

K increases monotonically with photon scattering $\measuredangle\, \theta$, reaching its max. value @ $\theta = 180°$ (photon backscattering), whence

$$\rightarrow K_{MAX} = K(\theta = 180°) = \mathcal{E}_i \left[\frac{2N}{1+2N}\right] < \mathcal{E}_i. \tag{3}$$

For any finite photon energy N, $K_{max} < \mathcal{E}_i$, and -- with this max. energy transfer -- the photon still has energy $\mathcal{E}_f = \mathcal{E}_i/(1+2N)$. So, indeed, the photon <u>cannot</u> transfer all its energy to the free e... that transaction would violate conservation of momentum.

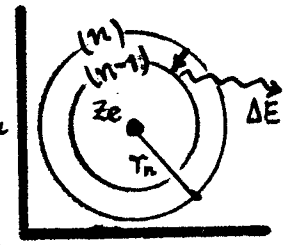
(B) If the transfer $\mathcal{E}_i \rightarrow K$ is to be 99% complete, then the least value of N necessary is -- from Eq. (3)...

$$\rightarrow K_{MAX} = \mathcal{E}_i \left[\frac{2N}{1+2N}\right] = 0.99\,\mathcal{E}_i \Rightarrow \underline{\underline{N}} = \frac{1}{2}\left(\frac{1}{\epsilon} - 1\right)\Big|_{\epsilon=0.01} = \underline{\underline{49.5}}. \tag{4}$$

This is a very robust photon, with incident energy $\mathcal{E}_i = Nmc^2 = 25.3$ MeV. Its wavelength is...

$$\underline{\underline{\lambda}} = hc/\mathcal{E}_i = \frac{1}{N}(h/mc) = \frac{1}{49.5} \times 2.43 \times 10^{-10}\,\text{cm} = \underline{\underline{4.91 \times 10^{-4}\,\text{Å}}} \tag{5}$$

⑥ <u>H-atom transition rates per the Correspondence Principle.</u>



(A) $P_n = \Gamma_n \Delta E(n \to n-1) \to \frac{2}{3}(e^2/c^3)|a_n|^2$, so we want to calculate

$$\left[\Gamma_n = \frac{2}{3}(e^2/c^3)|a_n|^2 / \Delta E(n \to n-1)\right] \int \begin{array}{l} a_n = \text{accel}^{\text{n}} \text{ in } n^{\text{th}} \text{ orbit}, \\ \Delta E = n \to n-1 \text{ transition energy}. \end{array} \quad (1)$$

Assume circular orbits of radii r_n. Bohr model gives [Eq. (16), p. Duality 7]:

$$\left[\begin{array}{l} \left.\begin{array}{l}\text{orbit}\\ \text{energy}\end{array}\right\} E_n = -\frac{1}{2}(Z\alpha)^2 mc^2/n^2 \Rightarrow \underline{\underline{\Delta E(n \to n-1)}} = E_n - E_{n-1} \underset{n \gg 1}{\simeq} \underline{\underline{\frac{(Z\alpha)^2 mc^2}{n^3}}}; \quad (2) \\ \left.\begin{array}{l}\text{orbit}\\ \text{radius}\end{array}\right\} r_n = \frac{n^2 a_0}{Z}, \text{ w/ } a_0 = \frac{\hbar^2}{me^2} = \text{Bohr radius. NOTE: } \alpha = e^2/\hbar c. \quad (3) \end{array}\right.$$

The electron's centripetal accel$^{\text{n}}$ a_n is found from: $ma_n = -\frac{Ze^2}{r_n^2}$, i.e.

$$\to a_n = -\frac{1}{m}Ze^2/r_n^2 = -Z^3 e^6 m/n^4\hbar^4, \text{ w/ } \underline{\underline{a_n = -Z^3 e^2 \alpha^2 mc^2/n^4 \hbar^2}}. \quad (4)$$

so/ Larmor power: $P_n = \frac{2}{3}(e^2/c^3)|a_n|^2 = \frac{2}{3}Z^6\alpha^7(mc^2)^2/n^8\hbar$. (5)

Then Eq. (1) yields...

$$\to \Gamma_n = P_n[\text{Eq.}(5)]/\Delta E[\text{Eq.}(2)] = \frac{2}{3}Z^4\alpha^5 mc^2/n^5\hbar,$$

or/ $\boxed{\Gamma_n = \Gamma_1/n^5, \text{ w/ } \Gamma_1 = \frac{2}{3}Z^4\alpha^5 mc^2/\hbar}$. <u>QED</u> (6)

(B) The lifetime for $n \to n-1$ is: $\underline{\tau_n = 1/\Gamma_n = n^5\tau_1}$, where: (7)

$$\tau_1 = 1/\Gamma_1 = \frac{1}{Z^4}\cdot\left(\frac{3}{2}\hbar/\alpha^5 mc^2\right) \int \begin{array}{l} \alpha = e^2/\hbar c \simeq 1/137, \\ \hbar/mc^2 = 1.29 \times 10^{-21} \text{ sec.} \end{array}$$

so/ numerically: $\underline{\tau_1 = \frac{1}{Z^4}\cdot 0.0932 \text{ ns}}$. (8)

For $Z=1$, we calculate the #s at right by use of $\tau_n = n^5\tau_1$ in Eq. (7). The agreement with known values improves as n increases; this is in accord with the Correspondence Principle feature that the classical result becomes more accurate as the quantum system becomes larger.

TRANSITION	THIS CALCN τ_n, ns	KNOWN τ_n, ns	ratio
$n = 2 \to 1$	3.0	1.6	1.88
$n = 4 \to 3$	95	73	1.30
$n = 6 \to 5$	725	610	1.19