Failure of classical BB analysis. Planck's hypothesis.

$$\left[\begin{matrix}\text{total energy density}\\ \text{(all frequencies)}\end{matrix}\right\} \quad \underline{\underline{\int_0^\infty U_\nu\, d\nu \propto \int_0^\infty \nu^2 d\nu \to \infty}} \;\Big\|\; \begin{matrix}\text{ULTRAVIOLET}\\ \text{CATASTROPHE}\end{matrix} \tag{8}$$

This is called the "ultraviolet catastrophe", because the high-frequency waves cause the divergence.

4. Note that in the above (disastrous) treatment of BB energy, we have assumed:

a. The mode energy E is a continuous variable [in Eq. (6)].
b. All EM modes give the same average energy contribution $\bar{E} = kT$. (9)
c. The high-freq. modes contribute most of the energy (there are more of them).

In order to bring U_ν(theory) to agree with U_ν(expt), one or more of these assumptions had to be wrong.

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5) Some way had to be found to squelch the high-frequency contribution to the BB energy. It had to be true that the high-frequency modes were not contributing their full kT -- an unavoidable fact if the classical equipartition theorem were true, which rested in turn on the "obvious" assumption that the energy in each EM radiation mode was a continuous variable. So is E discontinuous?

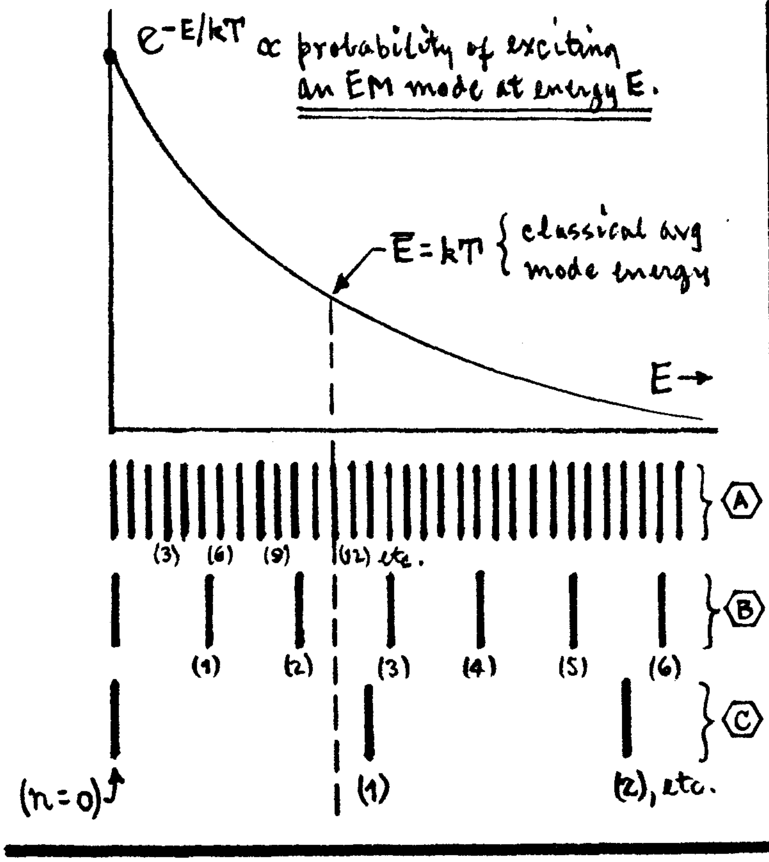
Planck's hypothesis was just that: he assumed that each radiation mode (at freq. ν) was not excited to a continuum of energy states, but only to a set of discrete energy levels, given by

$$\left[\begin{matrix}\text{rad}^{\text{n}}\text{ mode at freq. } \nu \text{ can}\\ \text{occupy only discrete energies } E_n\end{matrix}\right\} \quad \boxed{E_n = nh\nu} \quad \left\{\begin{matrix} n = \text{integer} = 1, 2, 3, \ldots,\\ h = \text{(Planck's) cnst.}\end{matrix}\right. \tag{10}$$

This quantum hypothesis -- so designated because the EM radiation energy, previously considered to be a continuous variable E, has been replaced by a discrete variable E_n -- immediately "explains" the UV catastrophe: the high energy modes do not contribute energy kT

Planck hypothesis ⇒ high-frequency modes are "frozen out."

because there is not enough energy available to excite them, and so the high energy modes are not even present. In pictures...



Ⓐ Low frequency: $h\nu \ll kT$.
The levels $E_n = nh\nu$ are closely spaced, and "many" modes are available for excitation at or near $\bar{E} = kT$.

Ⓑ Intermediate freq.: $h\nu \sim kT$.
Levels $E_n = nh\nu$ less closely spaced. Few modes available for excitation (& occupation) at or near $\bar{E} = kT$.

Ⓒ High frequency: $h\nu \gg kT$.
Energies $E_n = nh\nu$ are widely spaced, and ~ no modes can be excited or occupied at or near $\bar{E} = kT$.

Mainly, the low frequency modes are excited (thermodynamically) by the available energy kT; the energy contribution by the relatively large number of high-frequency modes is ~ negligible because those modes are not occupied.

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4) Now, per Planck, the average energy of the populated modes is quite different than that predicted by the Equipartition Theorem [Eq. (6)]. Assuming the Boltzmann factor is still OK, the average energy of the occupied levels @ ν is:

$$\rightarrow \bar{E} = \sum_{n=0}^{\infty} E_n e^{-E_n/kT} \Big/ \sum_{n=0}^{\infty} e^{-E_n/kT} = h\nu \sum_{n=0}^{\infty} n x^n \Big/ \sum_{n=0}^{\infty} x^n, \quad \text{w/ } x = e^{-h\nu/kT}$$

... but: $\sum_{n=0}^{\infty} x^n = 1/(1-x)$, $\sum_{n=0}^{\infty} n x^n = x/(1-x)^2$...

$$\text{so/ } \bar{E} = h\nu\left[x/(1-x)^2\right]\Big/\left[1/(1-x)\right], \quad \text{or/ } \boxed{\bar{E}(\nu) = h\nu\Big/\left(e^{h\nu/kT} - 1\right)}. \qquad (11)$$

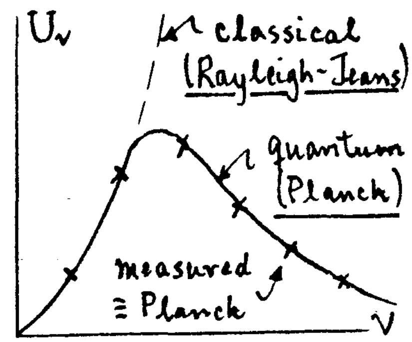
In the low and high-frequency limits, Planck's $\bar{E}(\nu)$ in Eq. (11) agrees with the qualitative picture that the high-frequency modes get frozen out...

$$\left[\begin{array}{ll} \lim_{h\nu \ll kT} : \bar{E} \simeq kT\left(1-\frac{h\nu}{2kT}\right) \to kT & \int \text{same as classical result (equipartition theorem) [low-freq. modes occupied]} \\ \lim_{h\nu \gg kT} : \bar{E} \simeq h\nu\, e^{-h\nu/kT} \longrightarrow 0 & \int \text{a new (quantum) result [high-freq. modes \underline{not} occupied]} \end{array}\right. \quad (12)$$

Planck's quantum hypothesis also implies a quite different result for the energy density of EM radiation in the cavity. Using Eq. (4) for the mode density $d\rho$, but now Eq. (11) for the average mode energy $\bar{E}$, we have...

$$U_\nu d\nu = \bar{E}\, d\rho = \frac{8\pi}{c^3}\nu^2 d\nu \cdot h\nu/\left(e^{h\nu/kT}-1\right),$$

i.e.,
$$\boxed{U_\nu = (8\pi h/c^3)\,\nu^3/\left(e^{h\nu/kT}-1\right)}. \quad (13)$$



This is the quantum result for the BB frequency distribution -- it is called Planck's radiation law. At a given BB temp. T, and with known (classical) cnsts: $c = 3.00\times 10^{10}$ cm/sec = light speed, $k = 1.38\times 10^{-16}$ erg/°C, Planck's law fits measured data if...

$$\rightarrow \underline{\underline{h = 6.63\times 10^{-27}\ \text{erg-sec}}}, \text{ Planck's cnst ("quantum of action").} \quad (14)$$

Also, the total cavity energy density is finite, as...

$$\rightarrow \underline{\underline{\int_0^\infty U_\nu\, d\nu = \frac{4}{c}\sigma T^4}}, \text{ w/ } \sigma = 2\pi^5 k^4/15 h^3 c^2 \text{ (Stefan's Law).} \quad (15)$$

REMARKS On the quantum BB energy distribution.

1. To get to this impressive agreement between theory & experiment, we have given up assumption a in Eq. (9)... now we are treating the radiant energy $E_n = nh\nu$ as a discrete rather than continuous variable.
2. Assumptions b & c in Eq. (9) are also thrown out. The average mode energy $\bar{E}(\nu)$ of Eq. (11) is by no means constant at kT, and the high-

frequency modes contribute little of the cavity energy because they are "frozen out" (i.e. not occupied). So now our truth table looks like...

CLASSICAL	QUANTUM
(a) Energy E is continuous.	$E_n = nh\nu$ is a discrete energy.
(b) All field modes excited equally by kT.	Mode excitation is a fcn of ν.
(c) High-freq. modes carry most energy.	High-freq. modes not occupied.

3. Planck's Law, Eq. (13), contains the classical result, Eq. (7), at low freqs...

$$\rightarrow \lim_{h\nu \ll kT} \Big\} U_\nu(\text{quantum}) \simeq \left[1 - \frac{h\nu}{2kT}\right] \cdot \frac{8\pi kT}{c^3}\nu^2 \rightarrow U_\nu(\text{classical}). \tag{16}$$

At high freqs ($h\nu \gg kT$), $U_\nu(\text{quantum}) \simeq (8\pi h/c^3)\,\nu^3 e^{-h\nu/kT} \rightarrow 0$, as required to agree with experiment. A radical change from U_ν(classical)!

4. The impressive agreement between theory & experiment for BB radiation strongly suggests that Planck's quantum hypothesis is correct, and it also gives us a new fundamental cnst, h in Eq. (14). BUT, this exercise replaces the problem of understanding why the classical formulation fails with the problem of understanding why an EM wave energy appears to be discrete rather than continuous. This was a radical departure from classical dogma, and Planck himself believed it was only an artifice.

As a generalization of Planck's hypothesis, we reach a conclusion entirely antithetical to the classical notion of continuous energy:

> EM waves at frequency ν carry energy in quantized units: $E = h\nu$.
> Thus the waves can act like discrete particles, called photons. (17)

This was the first example of "wave-particle duality", i.e. that the entity (here EM radiation) sometimes acts like a wave -- as in a diffraction expt -- and sometimes like a particle -- as in the BB formula.