4) To make a <u>direct</u> connection between what we've done with image charges and the Official Solution to $\nabla^2\phi = -4\pi\rho$ via Green's Fcns, note Jk$^{\text{n}}$ Sec. (2.6).

In Eq. (7) above, we have found the sphere–pt. charge potential for points <u>outside</u> the sphere as...

$$\rightarrow \phi_p(\mathbf{r}) = q\left\{\underbrace{\frac{1}{|\mathbf{r}-\mathbf{r}'|}}_{\substack{\text{outside sphere}\\ \text{(i.e. } q \text{ itself)}}} + \underbrace{F(\mathbf{r},\mathbf{r}')}_{\substack{\text{inside sphere}\\ \text{(image charge)}}}\right\} \quad \text{at all } r \geqslant a, \quad \text{w/ } F(\mathbf{r},\mathbf{r}') = \frac{(-)\,k}{|\mathbf{r}-k^2\mathbf{r}'|}, \; k = \frac{a}{r'} \leqslant 1. \tag{8}$$

Set the $\{\ \} = G(\mathbf{r},\mathbf{r}')$, and note for <u>outside</u> points

$$\nabla^2\{\ \} = \nabla^2 G = -4\pi\,\delta(\mathbf{r}-\mathbf{r}') + \cancel{\nabla^2 F}^{\,\to\, 0,\ \text{image charge is } \underline{\text{not}} \text{ outside sphere;}}$$

$$\rightarrow \text{so}/\!/ \quad \underline{G = \{\ \}\ \text{qualifies as a Green's fcn for } r \geqslant a.} \tag{9}$$

The general solution for ϕ <u>outside</u> the sphere is, from Eq. (1) above...

$$\phi_p(\mathbf{r}) = \int_{V(\text{outside})} G(\mathbf{r},\mathbf{r}')\,\rho(\mathbf{r}')\,d^3x' + \frac{1}{4\pi}\oint_{\text{Sphere}:a}\left[\cancel{G(\mathbf{r},\mathbf{r}')\frac{\partial\phi}{\partial n'}}^{\,\to\, 0,\ \text{on sphere surface}} - \phi(\mathbf{r}')\frac{\partial G}{\partial n'}\right]dS' \tag{10}$$

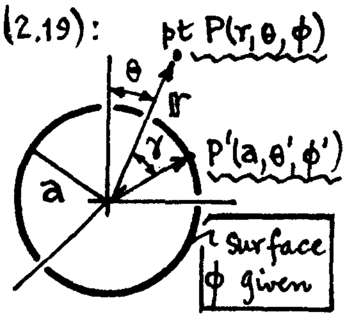
or/\!/

$$\left[\begin{aligned} &\phi_p(\mathbf{r}) = \int_V G\rho\,d^3x' + \frac{1}{4\pi}\oint_{\text{sphere}}\phi\,\left[(-)\,\partial G/\partial n'\right]dS' \\ &\text{w/ } G(\mathbf{r},\mathbf{r}') = \frac{1}{|\mathbf{r}-\mathbf{r}'|} - \frac{a}{r'}\left(\frac{1}{|\mathbf{r}-(a/r')^2\mathbf{r}'|}\right), \text{ all } r \,\&\, r' \geqslant a. \end{aligned}\right\} \tag{11}$$

This solution to $\nabla^2\phi = -4\pi\rho$ will now work for all points in V <u>outside</u> the sphere, where ρ is the density in V, and in RHS integral ϕ is potential on the sphere surface. In particular, note Jk$^{\text{n}}$ Eq. (2.19):

$$\left.\begin{aligned} &\underline{\rho \equiv 0 \text{ in } V;} \\ &\underline{\phi \text{ given on sphere;}} \\ &\underline{z = \tfrac{r}{a},\ dS' = a^2 d\Omega'} \end{aligned}\right\} \phi(\mathbf{r}) = \frac{(z^2-1)}{4\pi}\oint_{\text{Sphere}}\frac{\phi(a,\theta',\phi')\,d\Omega'}{(1-2z\cos\gamma+z^2)^{3/2}}, \tag{12}$$

$$\text{w/ } \cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi-\phi').$$



5) There are other methods of solving our electrostatics prob$^{\underline{m}}$: $\underline{\nabla^2\phi = -4\pi\rho}$. This PDE (after separation of variables) can be attacked with a considerable arsenal of techniques from the theory of special functions.

In Secs. 2.8–2.10, and 3.1–3.12 (except 3.4, 3.8, 3.10) Jackson hauls out the arsenal. You have (or will) see most of it in ϕ 566, so we will not go over all the details -- just the highlights.

A useful overview is provided by Sturm-Liouville theory...*

Review of Sturm-Liouville Theory

1. Most general linear, 2nd-order differential equation, on $a \leqslant x \leqslant b$:

$$\rightarrow p_2(x)u'' + p_1(x)u' + p_0(x)u = 0 \quad \left\{ \begin{array}{l} \text{coeff}^{\underline{t}}\text{s } p_i(x) \text{ given,} \\ \text{solution } u = u(x) \text{ desired.} \end{array} \right. \tag{13}$$

The $p_i(x)$ have to be "well-behaved" (in some sense) on $[a,b]$ for useful solutions $u(x)$ to exist. For the particular choice:

$$p_2' = p_1 = p(x), \quad p_0(x) = q(x) + \lambda w(x) \quad \left\{ \begin{array}{l} \text{prime } (\,'\,) \text{ means} \\ d/dx \end{array} \right., \tag{14}$$

Eq. (13) can be written as the "STURM-LIOUVILLE EQUATION":

$$\left[\begin{array}{l} \boxed{\mathcal{L}(u) + \lambda w(x) u = 0}, \quad x \in [a,b], \\ \text{w/ } \mathcal{L}(u) = \frac{d}{dx}[p(x)u'] + q(x)u \quad \left\{ \begin{array}{l} \text{S-L operator } \mathcal{L} \\ \text{is "self-adjoint";} \end{array} \right. \\ w(x) \text{ is called a "weighting fcn" (will be clear momentarily);} \\ \lambda = \text{constant, and may be restricted to "eigenvalues" by boundary conditions on } u(x). \end{array}\right\} \tag{15}$$

* Mathews & Walker "Math. Methods of Physics" (Benjamin: 2nd ed, 1970) p. 264, 334, 338. G. Arfken "Math. Methods for Physicists" (Academic: 3rd ed., 1985) Chap. 9.

- Many (most!) of the ODE's of canonical math physics can be put in this S-L form... e.g. Legendre's & Bessel's Eqtns, LaGuerre & Hermite, etc.★ And -- fortunately -- it is possible to learn a great deal about the nature of the solutions to all of these ODE's by studying the nature of solutions to this generic S-L problem.

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3. Acceptable solutions $u(x)$ & $v(x)$ to the S-L problem usually (must!) obey certain "boundary conditions" at the endpts of $[a, b]$... e.g. if $a = 0$ & $b \to \infty$, we might require: $u(0) =$ finite & $u(\infty) = 0$. Generally, when such "boundary conditions" are imposed, the solutions $u(x)$ are limited to certain "eigenfunctions" $u_n(x)$, $n = 0, 1, 3 \to \infty$. Corresponding to this quantization of the u's, the permissible values of λ are limited to a discrete set of "eigenvalues" λ_n. The B.C. may be one of the following types (for the 1D problem):

A. Dirichlet : $u(a)$ & $u(b)$ given.

B. Neumann : $u'(a)$ & $u'(b)$ given.

C. Cauchy : u & u' given at a and/or b.

One of these B.C. (usually) cause the following products to vanish:

$$u p u'\big|_{x=a} = 0, \quad u p u'\big|_{x=b} = 0. \tag{16}$$

We will adopt a less restrictive assumption for the B.C., viz:

$$\longrightarrow \boxed{v p u'\big|_{x=a} = v p u'\big|_{x=b}}, \quad u \,\&\, v = \text{any solns to S-L Eq. (15)}. \tag{17}$$

With this choice of B.C., it is easy to show (by partial-integration) that:

$$\longrightarrow \boxed{\int_a^b v \mathcal{L}(u)\, dx = \int_a^b u \mathcal{L}(v)\, dx.} \tag{18}$$

This gives meaning to the "self-adjoint" character of $\mathcal{L}$. The relation is reminiscent of a Hermitian operator $\mathcal{H}$, w/ $\int \psi_\alpha^* (\mathcal{H} \psi_\beta)\, dx = \int \psi_\beta (\mathcal{H} \psi_\alpha)^* dx$.

★ Both the HyperGeometric & Confluent Hypergeometric Eqs. are of S-L type.

$$\frac{d}{dx}[pu'] + (q+\lambda w)u = 0 \left\{ \begin{array}{l} \text{S-L} \\ \text{Eqtn.} \end{array} \right.$$

1. With this statement of the Sturm-Liouville problem, the following generals results can be proved:

A. For $\mathcal{L}$ a real operator (p & q real), and w a real weighting fcn, but eigenfcns u_n & eigenvalues λ_n possibly complex, it happens that the eigenvalues λ_n are in fact real.

B. There is a denumerable infinity of eigenvalues λ_n, which can be ordered: $\lambda_0 < \lambda_1 < \lambda_2 < \cdots < \lambda_\infty$, and which correspond to a denumerably infinite set of eigenfcns: $u_0(x), u_1(x), u_2(x), \ldots, u_\infty(x)$. The "large" eigenvalues behave as: $\lambda_n \propto n^2$ ($[a,b]$ finite), or $\lambda_n \propto n$ $[a,b]$ infinite) as $n \to \infty$.

C. The eigenfcns $u_n(x)$ are orthogonal: $\int_a^b u_m(x)\,u_n(x)\,w(x)\,dx = 0$, when $m \neq n$.† By appropriate normalization of u_n (i.e. $u_n \to C_n u_n$ such that $\int_a^b [C_n u_n(x)]^2 w(x)\,dx = 1$) can have the u_n orthonormal:

$$\longrightarrow \boxed{\int_a^b [u_m(x)\,u_n(x)]\,w(x)\,dx = \delta_{mn}} \text{ (Kronecker delta).} \qquad (19)$$

Notice how $w(x)$ "weights" the integration interval.

D. Completeness of the eigenfcn set $\{u_n(x)\}$? There are two ways to discuss this problem, which is concerned with the possibility of expanding:

$$\left[\begin{array}{l} f(x) = \sum_{n=0}^{\infty} c_n u_n(x), \quad f(x) \text{ on } [a,b]; \\ \text{w/} \quad c_n = \int_a^b f(\xi)\,u_n(\xi)\,d\xi, \text{ expansion coefficients.} \end{array}\right\} \qquad (20)$$

This is $\sim$ expanding a vector $\mathbf{f}$ in terms of a set of unit vectors $\{\hat{u}_n\}$, i.e.

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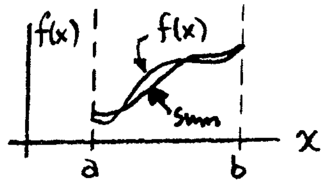
† Case of degeneracy excluded. In degenerate case, use Schmidt orthogonalization.

$f = \sum_n c_n \hat{u}_n$, and asking whether the $\{\hat{u}_n\}$ "span" the space of f. Do the $u_n(x)$ in Eq.(20) "span" the space (domain of definition) of $f(x)$? Can show...

① A partial sum expansion: $f(x) \simeq \sum_{n=0}^{N} c_n u_n(x)$ "converges in-the-mean", i.e.

$$\longrightarrow \lim_{N\to\infty} \int_a^b \Big[f(x) - \sum_{n=0}^{N} c_n u_n(x)\Big]^2 w(x)\,dx = 0. \qquad (21)$$



The mean-square difference vanishes as $N \to \infty$.

② Do a recursion: put the c_n's in Eq. (20) back into the sum for $f(x)$:

$$\longrightarrow f(x) = \sum_{n=0}^{\infty} \Big[\overbrace{\int_a^b f(\xi) u_n(\xi)\,d\xi}^{c_n}\Big] u_n(x), \text{ expansion possible on } [a,b];$$

or// $$f(x) = \int_a^b f(\xi) \underbrace{\Big[\sum_{n=0}^{\infty} u_n(x) w(\xi) u_n(\xi)\Big]}_{\text{this acts precisely as } \delta(x-\xi) \text{ on } [a,b]} d\xi$$

i.e.// if $f(x)$ expansion is possible on $[a,b]$ $\Big\}$ $$\boxed{\sum_{n=0}^{\infty} u_n(x) w(\xi) u_n(\xi) = \delta(x-\xi).} \qquad (22)$$

Last result is known as CLOSURE RELATION for the $\{u_n\}$. Either Eq.(21) or Eq.(22) is enough to establish the $\{u_n\}$ as a "complete set" on $[a,b]$.

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5. Specific example of above account is the Associated Legendre Equation:

$$(1-x^2)u'' - 2xu' + \Big[l(l+1) - \frac{m^2}{1-x^2}\Big]u = 0 \quad \int x = \cos\theta,\ [a,b] = [-1,+1]; \qquad (23)$$

S-L Eqtn, with: $p(x) = (1-x^2)$, $q(x) = \frac{-m^2}{1-x^2}$, $w(x) = 1$, $\lambda = l(l+1)$.

Eigenfcn solutions are well-known Associated Legendre polynomials, i.e. $u_{lm}(x) = P_{lm}(\cos\theta)$ $\Big\{ {l = 0,1,\ldots,\infty; \atop |m| \leq l.}$ (the P_{lm}'s are finite on $[-1,+1]$; there are also Q_{lm}'s which are singular at endpts). The P_{lm}'s are ∞ in number, and obey orthogonality [like Eq.(19)]. Also, $\lambda = l^2$ as $l \to \infty$, per claim 4.B above.

END Sturm Liouville Review

Jackson Math Topics 9/13/91

Sec	Topic
2.8	Orthogonal Fcns & Expansions.
2.9	Separation of Variables: $\nabla^2\phi = 0$.
2.10	Fourier Series
3.1	$\nabla^2\phi = 0$ in Spherical Cds
3.2	Legendre Eq. & Legendre Polynomials
3.3	Boundary-Value Probs & Symmetries
3.5	Spherical Harmonics $Y_\ell^m(\theta,\phi)$.
3.6	Addition Theorem for the $Y_\ell^m(\theta,\phi)$.
3.7	$\nabla^2\phi = 0$ in Cyl. Cds (Bessel Eq.).
3.9	Green's Fcns in Spherical Cds.
3.11	" " " Cylindrical Cds
3.12	Eigenfcn expansions for Green's Fcn.

What is overlap with ϕ566 topics?
... ϕ506 ...