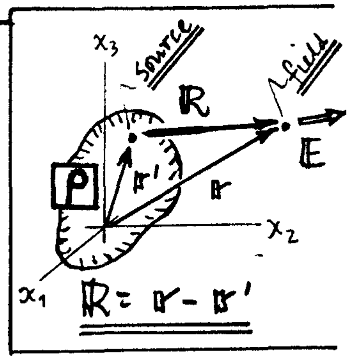
The Electrostatic Potential ϕ

Following is a paraphrase of Jackson's Ch. 1 on "Introd$^{\underline{n}}$ to Electrostatics"



1) An electro<u>static</u> field is defined via Maxwell's Eqs. by:

$$\left[\begin{array}{ll} \nabla\cdot\mathbb{E} = 4\pi\rho, & \rho = \text{time-indpt (static) charge density;} \\ \nabla\times\mathbb{E} = 0, & \text{because } \dfrac{\partial\mathbb{B}}{\partial t} = 0 \text{ (all fields are static).} \end{array}\right. \qquad (1)$$

So, by Helmholtz' Thm, $\mathbb{E}$ must be derivable from a scalar potential ϕ: $\mathbb{E} = -\nabla\phi + \cancel{\nabla\times\mathbb{A}}^{\,0}$ ($\mathbb{A}$ goes out because $\nabla\times\mathbb{E}=0$), i.e.

$$\rightarrow \mathbb{E}(\mathbb{r}) = -\nabla_{\text{fld}}\,\phi(\mathbb{r}), \quad \text{w/} \quad \phi(\mathbb{r}) = \int_{\text{all sources}} \frac{\rho(\mathbb{r}')}{R}\, d^3x_i', \quad R = |\mathbb{r}-\mathbb{r}'|. \qquad (2)$$

Suppose $\rho(\mathbb{r}')$ due to <u>point</u> charges q_i at positions $\mathbb{r}_i$ *

$$\rho(\mathbb{r}') = \sum_i q_i\,\delta(\mathbb{r}_i - \mathbb{r}') \Rightarrow \phi(\mathbb{r}) = \sum_i \frac{q_i}{R_i}, \quad \mathbb{R}_i = \mathbb{r} - \mathbb{r}_i;$$

so// $\mathbb{E}(\mathbb{r}) = -\nabla\phi(\mathbb{r}) = \sum_i q_i\left\{-\nabla\left(\frac{1}{R_i}\right)\right\} = \sum_i q_i\left\{\frac{\mathbb{R}_i}{R_i^3}\right\}$

i.e.// $$\boxed{\mathbb{E}(\mathbb{r}) = \sum_i \mathbb{E}_i, \quad \mathbb{E}_i = \left(\frac{q_i}{R_i^2}\right)\hat{\mathbb{R}}_i} \quad \text{Coulomb's Law for point charges } q_i \text{ at } \mathbb{r}_i. \qquad (3)$$

So $\nabla\cdot\mathbb{E} = 4\pi\rho \Rightarrow$ Coulomb's Law. This also works in reverse: starting from $\mathbb{E} = (q/R^2)\hat{\mathbb{R}}$, you can easily show that $\nabla\cdot\mathbb{E} = 4\pi\rho$, which is the way Jackson does it in his Secs. (1.2) – (1.4).

<u>NOTE</u> If Coulomb's Law were <u>not</u> inverse square, then Gauss' Law could <u>not</u> be written as $\nabla\cdot\mathbb{E} = 4\pi\rho$, $\rho = q\,\delta(\mathbb{r})$ for a point charge. Easy to see this for $\mathbb{E} = [q f(r)](\mathbb{r}/r^3)$, with $f(0)=1$, but $f(r)$ with some variation at $r>0$.

* <u>Point</u> charge q at position $\mathbb{a}$ has singular density $\rho(\mathbb{r}') = q\,\delta(\mathbb{a}-\mathbb{r}')$, in that $\int_{\text{all space}} \rho(\mathbb{r}')\,d^3x' = q\int_\infty \delta(\mathbb{a}-\mathbb{r}')\,d^3x' = q$ <u>identifies</u> q, but also ρ has <u>zero</u> spatial extent.

Electrostatic ϕ (cont'd)

2) REMARKS

1. The (Scalar) electrostatic potential ϕ which bridges the first Maxwell Eq. [viz. $\nabla\cdot\mathbb{E} = 4\pi\rho$] and Coulomb's Law [viz $\mathbb{E} = (q/R^2)\hat{R}$] must in itself obey an interesting eqtn...

$$\left[\begin{array}{l}\mathbb{E} = -\nabla\phi \\ \nabla\cdot\mathbb{E} = 4\pi\rho\end{array}\right\} \Rightarrow \nabla\cdot(\nabla\phi) \quad \begin{array}{l}\nabla^2\phi = -4\pi\rho, \ \text{POISSON EQ: } \phi \text{ in presence of } \rho \neq 0; \\ \nabla^2\phi = 0, \ \text{LAPLACE EQ: for } \rho \equiv 0.\end{array} \tag{4}$$

All of electrostatics is just an exercise in solving Poisson's Eq. for some given distribution of charge ρ. "Boundary conditions" play a big role here: they are conditions specifying where ϕ = cnst (e.g. conductor surface). More, later.

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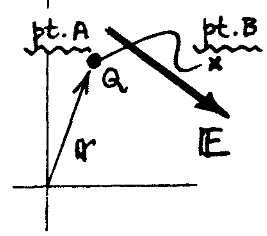
2. Writing $\mathbb{E} = -\nabla\phi$ makes an important statement about $\mathbb{E}$ as a "conservative" force field. Suppose we move a (small) test charge Q from pt. A to pt. B in a predetermined $\mathbb{E}$...



$$\left.\begin{array}{l}\text{work done on Q} \\ \text{during pt. A} \to \text{B}\end{array}\right\} W(A\to B) = -\int_A^B Q\,\mathbb{E}\cdot d\mathbb{r} = +Q\int_A^B (\nabla\phi)\cdot d\mathbb{r}. \tag{5}$$

But $(\nabla\phi)\cdot d\mathbb{r} = \left(\frac{\partial\phi}{\partial x_i}\right)\cdot(dx_i) = \sum_i \frac{\partial\phi}{\partial x_i}dx_i = d\phi$, perfect differential;

So, $\boxed{W(A\to B) = Q\int_A^B d\phi = Q(\phi_B - \phi_A)}$

and $W(B\to A) = Q(\phi_A - \phi_B) = (-)W(A\to B)$

W is indpt of path between A & B, and is exactly reversible: $\mathbb{E}$ is conservative.

That $\mathbb{E}$ is conservative follows more generally from $\nabla\times\mathbb{E} = 0$. (6)

ADVANTAGES of ϕ TO DATE

(1) Bridge: Coulomb Law $\rightleftarrows$ $\nabla\cdot\mathbb{E} = 4\pi\rho$.

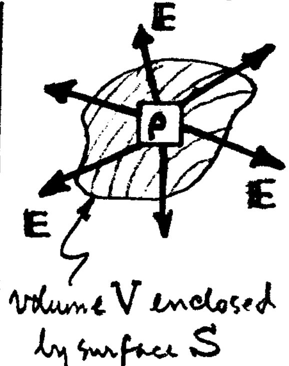
(2) Reduces electrostatics to soln of scalar Poisson eqtn: $\nabla^2\phi = -4\pi\rho$.

(3) $\mathbb{E} = (-)\nabla\phi$ (& $\nabla\times\mathbb{E} = 0$) instantly identifies $\mathbb{E}$ as a conservative field.

(4) Work done by $\mathbb{E}$ on Q is appealingly simple: $W(A\to B) = Q(\phi_B - \phi_A)$.

Electrostatic ϕ (cont'd)

3. In passing, we note that there is an integral form of the Maxwell Eq. we are discussing viz...



$$\nabla\cdot\mathbb{E} = 4\pi\rho \Rightarrow \int_V (\nabla\cdot\mathbb{E})\,dV = 4\pi\int_V \rho\,dV$$

... by Divergence Thm: $\int_V (\nabla\cdot\mathbb{E})\,dV = \oint_S \mathbb{E}\cdot d\mathbb{S}$ ← electric flux exiting $\mathbb{S}$

... by def^n of ρ: $\int \rho\,dV = Q_{in}$ ← Q_{in} = charge inside surface $\mathbb{S}$;

So/ $$\boxed{\nabla\cdot\mathbb{E} = 4\pi\rho \longleftrightarrow \oint \mathbb{E}\cdot d\mathbb{S} = 4\pi Q_{in}} \quad (7)$$

The integral form is sometimes called Gauss' Law; it is equivalent to our differential form. POINT: ϕ plays no direct role in the integral form.