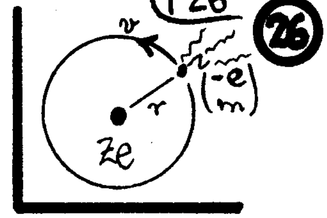
1992 ϕ520 Problems **Assigned: 4/27. Due: 5/4/92.**

file (P26) 26



(81) An electron [mass m, charge (-)e] in a hydrogenlike atom [w/ stationary nucleus of charge Ze] moves in a circular orbit of radius r. Treat this system classically, and assume the electron velocity is $v \ll c$.

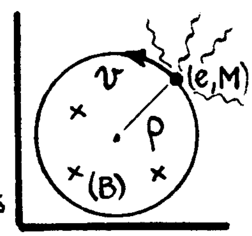
(A) Find an expression for the electron's total orbit energy E in terms of r alone.

(B) Assume the electron radiates energy $\Delta E \ll |E|$, per orbit. Find the radiated power P in terms of r alone. By equating P to the rate of loss of orbit energy, obtain a differential equation for the decrease in r, as a fcn of time, due to radiation.

(C) Calculate the elapsed time for the electron to spiral into the nucleus if it starts from $r = a_0$. Put $Z=1$, $a_0 = 0.53$ Å (Bohr radius), and find a number for the collapse time.

---

(82) [Jackson Eq. (14.26)]. In Eqs. (14.23)-(14.26), p. 660, Jackson indicates how the nonrelativistic Larmor radiation rate: $P(\text{Larmor}) = \frac{2}{3}(q^2/c)|\dot{\boldsymbol{\beta}}|^2$, for a charge q with acceleration $\dot{\boldsymbol{\beta}} = \dot{\mathbf{v}}/c$, can be generalized to a relativistic version, viz $P(\text{Liénard}) = \frac{2}{3}(q^2/c)\gamma^6[\dot{\boldsymbol{\beta}}^2 - (\boldsymbol{\beta}\times\dot{\boldsymbol{\beta}})^2]$, w/ $\gamma = 1/\sqrt{1-\beta^2}$. Fill in the missing steps for this derivation. In particular, decide what the "dots" mean in the Liénard result of (14.26)... they are time derivatives, but w.r.t. what time?



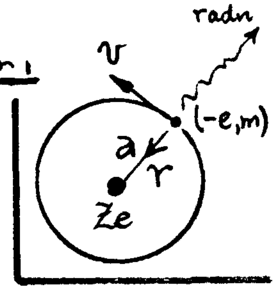
(83) Consider a large synchrotron that maintains a beam of highly relativistic protons [charge e, rest energy $\mathcal{E}_0 = Mc^2 = 938$ MeV] in a circular orbit of radius ρ at total energy $\mathcal{E} \gg \mathcal{E}_0$. The machine supplies energy to the beam at constant rate (in lab) $\frac{dU}{dz}\left(\frac{\text{MeV}}{\text{meter}}\right)$, per proton, and has magnets with high enough B-fields to contain the proton orbits for any "reasonable" $\mathcal{E}$ (see Jackson, Sec. 12.3). Assume at first that the limit on $\mathcal{E}$ results from radiation losses.

(A) Find the limiting value for $\gamma = \mathcal{E}/Mc^2$ under these circumstances.

(B) If $dU/dz = 10$ MeV/m, and $\rho = 15$ km {~SSC specs}, calculate a number for γ_{max}.

(C) What magnetic field B is required for the orbit? What γ's can be maintained for available magnets?

(81) Calculate radiative time-of-collapse of the classical planetary atom.



(A) Centripetal force = Coulomb force $\Rightarrow$ $\frac{mv^2}{r} = \frac{Ze^2}{r^2}$, which implies...

$\rightarrow$ kinetic energy : $K = \frac{1}{2}mv^2 = Ze^2/2r$. (1)

The potential energy for the orbiting electron is : $V = -\frac{Ze^2}{r}$, so total orbit energy is :

$\rightarrow$ $\underline{\underline{E = K + V = -Ze^2/2r}}$, $\leftarrow$ total orbit energy. (2)

(B) With the centripetal acceleration : $a = \frac{v^2}{r}$, the electron must radiate at a rate...

$$\rightarrow P = \frac{2e^2}{3c^3}|a|^2 = \frac{2e^2}{3c^3}|Ze^2/mr^2|^2 = \frac{2}{3}\left(\frac{e^2(Ze^2)^2}{m^2c^3}\right)\frac{1}{r^4}. \quad (3)$$

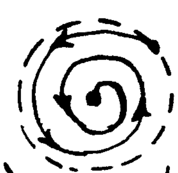
We've used : $v^2/r = \frac{1}{m}(Ze^2/r^2)$, from part (A). Now equate the radiative loss rate P of Eq. (3) to the rate of loss of E(orbit) from Eq. (2), i.e. ...

$$\rightarrow \frac{d}{dt}\left(-\frac{Ze^2}{2r}\right) = (-)\frac{2}{3}\left(\frac{e^2(Ze^2)^2}{m^2c^3}\right)\frac{1}{r^4} \Rightarrow \boxed{\frac{dr}{dt} = (-)\frac{4}{3}\left(\frac{Ze^4}{m^2c^3}\right)\frac{1}{r^2}} \quad (4)$$

So $r = r(t)$ is a decreasing fcn of time; the electron spirals in towards the nucleus. Note that (4) can be written : $\underline{\frac{1}{c}(dr/dt) = (-)\frac{4Z}{3}(r_0/r)^2}$, w/ $r_0 = \frac{e^2}{mc^2} = 2.8\times10^{-13}$ cm. r_0 is the classical electron radius, and is $\sim$ size of nuclei (e.g. proton size). This means the radial velocity of the electron is $\ll c$ until it essentially hits the nucleus; the radial shrinkage is $\sim$ "small" to the same point, and this justifies the nonrelativistic treatment herein. The Larmor approxn is "good enough" here.

(C) The total time for the electron to spiral down from orbit radius $r=R$ to $r=0$ is :

$$\rightarrow T(\text{collapse}) = \int_{r=R}^{r=0} dt = \frac{3}{4}\left(\frac{m^2c^3}{Ze^4}\right)\int_0^R r^2\,dr = \frac{1}{4Z}(m^2c^3/e^4)R^3,$$



or// $\boxed{T(\text{collapse}) = \frac{1}{4Z}(r_0/c)(R/r_0)^3}$, w// $r_0 = \frac{e^2}{mc^2} = 2.82\times10^{-13}$ cm (electrons) (5)

For $Z=1$ (hydrogen), and $R = a_0 = 0.53\times10^{-8}$ cm (1st Bohr orbit), have $\frac{R}{r_0} = 1.88\times10^4$.

Then : $T(\text{collapse}) = \frac{1}{4}\cdot\frac{2.82}{3}\times10^{-23}\times(1.88\times10^4)^3 = \underline{\underline{1.6\times10^{-11}\text{ sec.}}}$ Atoms are evanescent.

● Fill in missing steps in derivation of P(Liénard) of Jackson's Eq. (14.26).

1. The footnote on p. 660 makes it ~ clear what Jackson is doing in Eqs. (14.23) → (14.25). First missing step is in (14.25), where he replaces $\left[\frac{1}{c}\left(\frac{dE}{d\tau}\right)\right]^2$ by $\left[\beta\left(\frac{dp}{d\tau}\right)\right]^2$. Follows from...

$$\left.\begin{matrix} \mathbf{p}=\gamma mc\boldsymbol{\beta} \\ E=\gamma mc^2 \end{matrix}\right\} \text{w/ } \gamma = 1/\sqrt{1-\beta^2}, \text{ and: } \frac{d\gamma}{d\tau} = \gamma^3\beta\frac{d\beta}{d\tau}, \tag{1}$$

$$\text{since: } \frac{dp}{d\tau} = mc\frac{d}{d\tau}(\gamma\beta) = mc\left[\gamma^3\beta^2\frac{d\beta}{d\tau} + \gamma\frac{d\beta}{d\tau}\right] = mc\gamma^3\left(\frac{d\beta}{d\tau}\right),$$

$$\frac{dE}{d\tau} = mc^2\frac{d\gamma}{d\tau} = c\beta\left[mc\gamma^3\frac{d\beta}{d\tau}\right] = c\beta\frac{dp}{d\tau}, \text{ as required.} \tag{2}$$

With this identity, Jackson's Eq. (14.24) can be written

$$\rightarrow P = \frac{2}{3}(e^2/c)\left\{\left[\frac{d}{d\tau}(\gamma\boldsymbol{\beta})\right]^2 - \beta^2\left[\frac{d}{d\tau}(\gamma\beta)\right]^2\right\}. \tag{3}$$

---

2. Now reduce the { } in Eq. (3). Let "dot" signify $\frac{d}{d\tau}$ at this point. Then

$$\left[\frac{d}{d\tau}(\gamma\boldsymbol{\beta})\right]^2 = \gamma^2(\dot{\boldsymbol{\beta}} + \gamma^2\beta\dot{\beta}\boldsymbol{\beta})^2, \quad \beta^2\left[\frac{d}{d\tau}(\gamma\beta)\right]^2 = \beta^2(\gamma^3\dot{\beta})^2; \tag{4}$$

So $\{\text{Eq.}(3)\}/\gamma^2 = \dot{\boldsymbol{\beta}}^2 + 2\gamma^2\beta\dot{\beta}(\boldsymbol{\beta}\cdot\dot{\boldsymbol{\beta}}) + \underbrace{\left[\gamma^4\beta^4\dot{\beta}^2 - \gamma^4\beta^2\dot{\beta}^2\right]}_{\text{Combine to form: } -\gamma^2\beta^2\dot{\beta}^2}$

$$= (1-\gamma^2\beta^2)\dot{\beta}^2 + 2\gamma^2\beta\dot{\beta}(\boldsymbol{\beta}\cdot\dot{\boldsymbol{\beta}}) \ldots \text{ in first term: } (1-\gamma^2\beta^2) = (1-2\beta^2)\gamma^2$$

$$\xrightarrow{\text{or}} \frac{1}{\gamma^4}\{\text{Eq.}(3)\} = (1-2\beta^2)\dot{\beta}^2 + 2\beta\dot{\beta}(\boldsymbol{\beta}\cdot\dot{\boldsymbol{\beta}}) = \dot{\beta}^2 - 2\beta\dot{\beta}[\beta\dot{\beta} - \boldsymbol{\beta}\cdot\dot{\boldsymbol{\beta}}]. \tag{5}$$

But $\boldsymbol{\beta}\cdot\dot{\boldsymbol{\beta}} = \frac{1}{2}\frac{d}{d\tau}(\boldsymbol{\beta}\cdot\boldsymbol{\beta}) = \beta\dot{\beta}$, so the [ ] in (5) vanishes, and (3) reads...

$$\boxed{P = \frac{2}{3}(e^2/c)\gamma^4[d\beta/d\tau]^2}. \tag{6}$$

---

3. When P of Eq. (6) is converted to q's lab time: $dt' = \gamma d\tau$, we will evidently get the γ^6 factor, i.e. $P(t') = \frac{2}{3}(e^2/c)\gamma^6[d\beta/dt']^2$. It is also clear that in fact $P(t') \rightarrow P(\text{Larmor})$ when $v \ll c$. What is not clear is how to convert $P(t')$ to $P(t)$, where t = observer time; this involves an integral over $\measuredangle$'s [recall: $dt/dt' = (1-\hat{n}\cdot\boldsymbol{\beta})_{t'}$, etc.].

So we have to do something different.

ϕ520 Solutions (82 cont'd)

4. Where $a^\alpha = du^\alpha/d\tau$ is q's 4-acceleration, write Jackson's Eq. (14.24) as

$$\rightarrow P = -(2e^2/3c^3)\, a_\alpha a^\alpha. \tag{7}$$

The 4-acceleration is given by [class notes, p. SRT 14, Eq. (9)]:

$$\rightarrow a^\alpha = c\gamma^2\left(\gamma^2(\boldsymbol\beta\cdot\dot{\boldsymbol\beta}),\ \dot{\boldsymbol\beta} + \gamma^2(\boldsymbol\beta\cdot\dot{\boldsymbol\beta})\boldsymbol\beta\right) \int \begin{matrix}\gamma = 1/\sqrt{1-\beta^2},\\ \underline{\dot{\boldsymbol\beta} = d\boldsymbol\beta/dt.}\end{matrix} \tag{8}$$

NOTE: $\dot{\boldsymbol\beta}$ & $\boldsymbol\beta$ here are referred to observer time t, already. Contraction is:

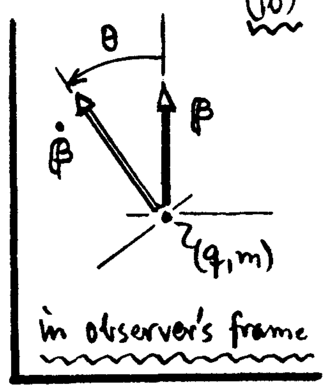
$$\rightarrow (-)a_\alpha a^\alpha = c^2\gamma^4\left\{[\dot{\boldsymbol\beta} + \gamma^2(\boldsymbol\beta\cdot\dot{\boldsymbol\beta})\boldsymbol\beta]^2 - \gamma^4(\boldsymbol\beta\cdot\dot{\boldsymbol\beta})^2\right\} \tag{9}$$

$$= c^2\gamma^4\left\{\dot\beta^2 + 2\gamma^2(\boldsymbol\beta\cdot\dot{\boldsymbol\beta})^2 + \underbrace{[\gamma^4\beta^2(\boldsymbol\beta\cdot\dot{\boldsymbol\beta})^2 - \gamma^4(\boldsymbol\beta\cdot\dot{\boldsymbol\beta})^2]}_{=-\gamma^2(\boldsymbol\beta\cdot\dot{\boldsymbol\beta})^2}\right\}$$

i.e.//

$$\left[\!\left[(-)a_\alpha a^\alpha = c^2\gamma^4\left\{\dot\beta^2 + \gamma^2(\boldsymbol\beta\cdot\dot{\boldsymbol\beta})^2\right\}.\right]\!\right] \tag{10}$$

Let $\theta = \measuredangle(\boldsymbol\beta, \dot{\boldsymbol\beta})$, in observer's frame. Then (10) is:



$$\rightarrow (-)a_\alpha a^\alpha = c^2\gamma^4\dot\beta^2\left\{1 + \gamma^2\beta^2\underbrace{\cos^2\theta}_{=1-\sin^2\theta}\right\}$$

$$= c^2\gamma^4\dot\beta^2\left\{\underbrace{(1+\gamma^2\beta^2)}_{=\gamma^2} - \gamma^2\beta^2\sin^2\theta\right\}$$

$$= c^2\gamma^6\dot\beta^2\left\{1 - \beta^2\sin^2\theta\right\},$$

or// $\underline{\underline{(-)a_\alpha a^\alpha = c^2\gamma^6\left\{\dot{\boldsymbol\beta}^2 - (\boldsymbol\beta\times\dot{\boldsymbol\beta})^2\right\}}}$, all in observer time t. (11)

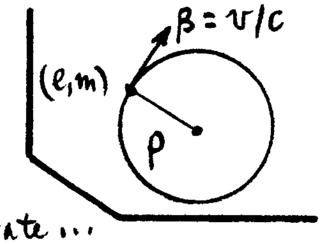
---

5. With the result of Eq. (11), the radiated power of Eq. (7) becomes...

$$\boxed{P(\text{Liénard}) = (2e^2/3c)\gamma^6\left[\dot{\boldsymbol\beta}^2 - (\boldsymbol\beta\cdot\dot{\boldsymbol\beta})^2\right]}, \quad \text{w//}\ \dot{\boldsymbol\beta} = \left.\frac{d\boldsymbol\beta}{dt}\right|_{\text{LAB TIME}} \tag{12}$$

This is Jackson's Eq. (14.26). His "derivation" is misleading in that his Eq. (14.25) appears to be totally irrelevant.

ϕ 520 Problem Solutions



(83) Radiation limit to synchrotron energy.

(A) From class notes, or Jkⁿ Eq. (14.46), each proton radiates at rate...

$$\rightarrow P_{rad} = \frac{2}{3}(e^2 c/\rho^2)\beta^4\gamma^4, \quad \gamma = \mathcal{E}/mc^2 \quad (\mathcal{E} = \gamma mc^2 = \text{total energy}). \qquad (1)$$

This is relativistically correct, and it's what is seen in lab. The radiation energy loss during one orbit period $\Delta t = \frac{2\pi\rho}{\beta c}$ is $P_{rad}\Delta t$, and it must be less than the energy supplied during that circuit, viz $(dU/dz)\times 2\pi\rho$. So

$$\left[\!\left[P_{rad}\Delta t < \left(\frac{dU}{dz}\right)\cdot 2\pi\rho \Rightarrow \underline{\gamma^4 < \frac{3}{2}\left(\frac{dU}{dz}\right)\frac{\rho^2}{e^2}\cdot\frac{1}{\cancelto{1}{\beta^3}}}\right. \text{ (highly relativistic)}. \qquad (2)$$

(B) For numbers for the γ limit in Eq. (2), let the units of $\left(\frac{dU}{dz}\right)$ be $\frac{\text{MeV}}{\text{m}}$ and measure ρ in units of km. Then...

$$\left[\!\left[\gamma^4 < 1.043\times 10^{21}\,\rho^2(dU/dz), \text{ or } \underline{\gamma < 1.797\times 10^5\,[\overset{\text{in km}}{\rho^2}(\overset{\text{in MeV/m}}{dU/dz})]^{\frac{1}{4}}}.\right. \qquad (3)$$

If $dU/dz = 10\,\frac{\text{MeV}}{\text{m}}$ and $\rho = 15$ km, then: $\boxed{\gamma < 1.24\times 10^6}$. The corresponding proton energy is $\mathcal{E} = \gamma\,\mathcal{E}_0 = 1160$ TeV, which is <u>very</u> robust. But this "limiting" energy is $\sim 10^3\times$ max. design energy for the SSC. So <u>something else fails</u> before the radⁿ limit is reached on this machine.

(C) The B-field needed to maintain the orbit is found from Jkⁿ (12.39):

$$\rightarrow \omega_B = \frac{v}{\rho} = \frac{eB}{\gamma mc} \Rightarrow B = \gamma\cancelto{1}{\beta}\,\frac{mc^2}{e\rho} = 31.3\,\gamma/\rho \quad \left\{\begin{array}{l}\text{B is in Gauss,}\\ \text{for } \rho \text{ in km.}\end{array}\right. \qquad (4)$$

If $\rho = 15$ km, then $B = 2.09\,\gamma$, Gauss. The beam magnets are capable of supplying (perhaps) $B \simeq 20{,}000$ G [this a big field for earthlings], and so the proton orbit can be held in place only up to $\gamma \sim 10^4$ (i.e. 10 TeV). <u>We cannot yet build a radiation-limited synchrotron</u>, for lack of adequate magnets. Available B-flds are too small by $\sim 100\times$.