1993 ϕ507 Problems **Assigned: 2/19. Due: 2/26/93.** ~~HB 6~~ file

(6)

(17) A QM system consists of two particles, w/ masses m_1 & m_2. Express the operators for total momentum $\hat{\mathbb{P}} = \hat{\mathbb{p}}_1 + \hat{\mathbb{p}}_2$, and total ∡ momentum $\hat{\mathbb{L}} = \hat{\mathbb{l}}_1 + \hat{\mathbb{l}}_2$, in terms of the relative cd. $\mathbb{r} = \mathbb{r}_1 - \mathbb{r}_2$ and center-of-mass cd. $\mathbb{R} = (m_1\mathbb{r}_1 + m_2\mathbb{r}_2)/(m_1+m_2)$. Show that the K.E. part of the Hamiltonian, viz. $\hat{K} = \frac{1}{2m_1}\hat{\mathbb{p}}_1^2 + \frac{1}{2m_2}\hat{\mathbb{p}}_2^2$, can be put in the form: $\underline{\underline{\hat{K} = -(\hbar^2/2M)\nabla_R^2 - (\hbar^2/2\mu)\nabla_r^2}}$, w/ $M = (m_1+m_2)$ & $\mu = m_1 m_2/(m_1+m_2)$.

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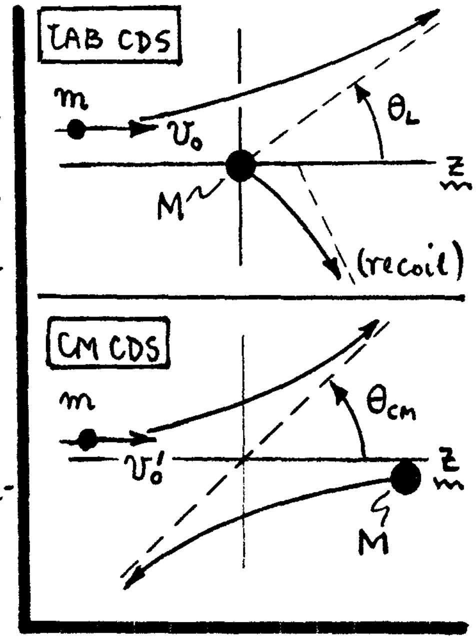
(18) [15 pts]. Most 2-body scattering events [w/ m(projectile) incident on M(target)] are described in terms of the scattering ∡ θ_{cm} in the <u>center-of-mass</u> (CM) <u>system</u>. When m/M is finite, θ_{cm} is generally $\neq \theta_L$, the actual scattering ∡ of m in the <u>lab</u> (L) <u>system</u>, because of M's recoil. Here we wish to relate θ_L to θ_{cm} for a classical <u>elastic</u> scattering event. Assume that M is initially at rest on the z-axis in lab, and m is incident at velocity $v_0 \parallel$ z-axis. Assume axial symmetry.



(A) After finding the CM velocity w.r.t. lab, and m's final velocity in CM (for an elastic event), show that: $\boxed{\tan\theta_L = \sin\theta_{cm}/[\cos\theta_{cm} + (m/M)]}$, is the required relation. Evidently $\theta_L \simeq \theta_{cm}$ when $m \ll M$. What is the relation when $m = M$?

(B) If $\frac{d\sigma}{d\Omega}$ is the differential scattering cross-section (# particles m scattered into solid ∡ $d\Omega = 2\pi \sin\theta\, d\theta$, per $d\Omega$), show: $\underline{\underline{(d\sigma/d\Omega)_L = (d\sigma/d\Omega)_{cm} \cdot \frac{d\cos\theta_{cm}}{d\cos\theta_L}}}$. What does this relation reduce to when $m = M$? What is the maxm. θ_L in this case?

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(19) [15 pts]. Use the Born approx$^{\underline{n}}$ to find the total cross section for an elastic scattering by a spherical well: $\underline{V(r) = (-)V_0,\ r < a;\ V(r) \equiv 0,\ r > a}$. <u>NOTE</u>: it is handy to verify and use Eq. (31), p. ScT 13, of class notes -- following from Eq. (14), p. ScT 7, for elastic & spherically symmetric events. Find limiting forms for $\sigma(k)$ for low energies [$ka \ll 1$] and high energies [$ka \gg 1$].

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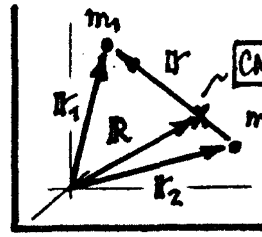
(17) QM system of m_1 & m_2: express total $\hat{\mathbb{P}}$, $\hat{\mathbb{L}}$ & $\hat{\mathbb{K}}$ in cds $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$ & $\mathbb{R}_{CM}$.

1) The CM transformation and its inverse are (w/ $M = m_1 + m_2$):



$$\left[\begin{array}{l} \mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2, \\ \mathbb{R} = \frac{1}{M}(m_1\mathbf{r}_1 + m_2\mathbf{r}_2); \end{array}\right\} \leftrightarrow \left\{ \begin{array}{l} \mathbf{r}_1 = \mathbb{R} + (m_2/M)\mathbf{r}, \\ \mathbf{r}_2 = \mathbb{R} - (m_1/M)\mathbf{r}. \end{array}\right. \qquad (1)$$

Symbolically: $\frac{\partial}{\partial \mathbf{r}_1} = \left(\frac{\partial \mathbf{r}}{\partial \mathbf{r}_1}\right)\frac{\partial}{\partial \mathbf{r}} + \left(\frac{\partial \mathbb{R}}{\partial \mathbf{r}_1}\right)\frac{\partial}{\partial \mathbb{R}} = \frac{\partial}{\partial \mathbf{r}} + \left(\frac{m_1}{M}\right)\frac{\partial}{\partial \mathbb{R}}$, i.e. $\nabla_1 = \nabla_r + \left(\frac{m_1}{M}\right)\nabla_R$; this works component-by-component. Treating $\partial/\partial\mathbf{r}_2$ similarly, we can write...

$$\rightarrow \nabla_1 = +\nabla_r + (m_1/M)\nabla_R, \quad \nabla_2 = -\nabla_r + (m_2/M)\nabla_R. \qquad (2)$$

2) The total system momentum is just that of the CM, since...

$$\hat{\mathbb{P}} = \hat{\mathbf{p}}_1 + \hat{\mathbf{p}}_2 = -i\hbar(\nabla_1 + \nabla_2) = -i\hbar\left(\frac{m_1+m_2}{M}\right)\nabla_R = \underline{\underline{-i\hbar\nabla_R}}. \qquad (3)$$

The total system $\measuredangle$ momentum is that of the CM (about origin) plus that of the particles about the CM, since...

$$\hat{\mathbb{L}} = \hat{\mathbb{L}}_1 + \hat{\mathbb{L}}_2 = \mathbf{r}_1 \times \hat{\mathbf{p}}_1 + \mathbf{r}_2 \times \hat{\mathbf{p}}_2$$

$$= -i\hbar\left\{\left(\mathbb{R} + \frac{m_2}{M}\mathbf{r}\right)\times\left(\nabla_r + \frac{m_1}{M}\nabla_R\right) + \left(\mathbb{R} - \frac{m_1}{M}\mathbf{r}\right)\times\left(-\nabla_r + \frac{m_2}{M}\nabla_R\right)\right\}$$

$$\hat{\mathbb{L}} \doteq \underline{\underline{-i\hbar\{\mathbb{R}\times\nabla_R + \mathbf{r}\times\nabla_r\}}} = \mathbb{R}\times\hat{\mathbb{P}} + \mathbf{r}\times\hat{\mathbf{p}} \;\Big\backslash\; \begin{array}{l} \mathbb{P} = -i\hbar\nabla_R, \text{ CM}; \\ \mathbf{p} = -i\hbar\nabla_r, \text{ relative.} \end{array} \qquad (4)$$

This is just what happens in the CM transform of classical mechanics.

3) The kinetic energy operator transforms as...

cross-terms cancel

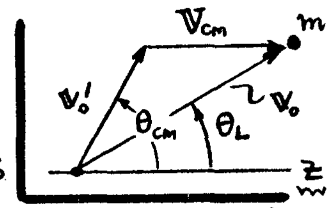
$$\hat{\mathbb{K}} = \frac{1}{2m_1}\hat{\mathbf{p}}_1^2 + \frac{1}{2m_2}\hat{\mathbf{p}}_2^2 = -\frac{\hbar^2}{2}\left\{\frac{1}{m_1}\left(\nabla_r + \frac{m_1}{M}\nabla_R\right)^2 + \frac{1}{m_2}\left(\nabla_r - \frac{m_2}{M}\nabla_R\right)^2\right\}$$

So $\boxed{\hat{\mathbb{K}} = -(\hbar^2/2M)\nabla_R^2 - (\hbar^2/2\mu)\nabla_r^2} \;\Big\backslash\; \begin{array}{l} M = (m_1+m_2), \text{ total mass;} \\ \mu = m_1m_2/(m_1+m_2), \text{ reduced mass.} \end{array} \qquad (5)$

As required. If the system Hamiltonian is: $\hat{\mathcal{H}} = \hat{\mathbb{K}} + V(r)$, where the interaction potential $V(r)$ depends on the relative cd $r = |\mathbf{r}_1 - \mathbf{r}_2|$, then we have shown: $\hat{\mathcal{H}} = \hat{\mathcal{H}}_{cm} + \hat{\mathcal{H}}_{rel}$, w/ $\hat{\mathcal{H}}_{cm} = -(\hbar^2/2M)\nabla_R^2$ is the free motion of the CM, and $\hat{\mathcal{H}}_{rel} = -(\hbar^2/2\mu)\nabla_r^2 + V(r)$ is the interaction in relative cds.

ϕ507 Solutions

(18) [15 pts]. Transform scattering $\measuredangle$s θ from CM to lab system.



(A) 1. For an elastic scattering event, the outgoing projectile velocity $\mathbb{v}_0'$ in the CM system has the same magnitude, namely v_0', as the incoming velocity (in CM), and: $\mathbb{v}_0 = \mathbb{v}_0' + \mathbb{V}_{cm}$, is the outgoing projectile velocity in the lab system, where $\mathbb{V}_{cm}$ is the CM velocity. From the diagram...

$$\left.\begin{matrix} v_0 \sin\theta_L = v_0' \sin\theta_{cm} \\ v_0 \cos\theta_L = v_0' \cos\theta_{cm} + V_{cm} \end{matrix}\right\} \Rightarrow \tan\theta_L = \frac{\sin\theta_{cm}}{\cos\theta_{cm} + (V_{cm}/v_0')}. \qquad (1)$$

2. So we need the ratio V_{cm}/v_0'. From prob$^{\underline{m}}$ (17), and with the target M initially at rest, the CM velocity is $\mathbb{V}_{cm} = \dot{\mathbb{R}} = [m/(m+M)]\,\mathbb{v}_0(in)$, w/ $\mathbb{v}_0(in)$ m's incident velocity. Then: $\underline{V_{cm} = (\mu/M)v_0}$, along the z-axis, w/ $\mu = mM/(m+M)$ the reduced mass. The momentum of m in CM is: $mv_0' = \mu v_0$, so m's approach velocity in CM is: $\underline{v_0' = (\mu/m)v_0}$. Thus we have...

$$\left.\begin{matrix} V_{cm} = (\mu/M)v_0 \\ v_0' = (\mu/m)v_0 \end{matrix}\right\} \Rightarrow \frac{V_{cm}}{v_0'} = \frac{m}{M}, \text{ and: } \boxed{\tan\theta_L = \frac{\sin\theta_{cm}}{\cos\theta_{cm} + (m/M)}}, \qquad (2)$$

as required. Generally $\theta_L < \theta_{cm}$. The discrepancy is largest for $m=M$, when $\tan\theta_L = \frac{\sin\theta_{cm}}{\cos\theta_{cm}+1} = \tan(\theta_{cm}/2)$, i.e. $\underline{\underline{\theta_L = \frac{1}{2}\theta_{cm}}}$. Clearly, with $\theta_{cm}(\text{max.}) = 180°$, have $\theta_L(\text{max.}) = 90°$, for the $m=M$ case.

(B) 3. By def$^{\underline{n}}$ of $(d\sigma/d\Omega)$, the total # of particles scattered into range $d\theta$ at θ is $(d\sigma/d\Omega)\cdot 2\pi\sin\theta\, d\theta$, for axial symmetry. This # must be invariant for a given direction (which appears different in CM & L systems), so...

$$\left(\frac{d\sigma}{d\Omega}\right)_L \cdot 2\pi\sin\theta_L\, d\theta_L = \left(\frac{d\sigma}{d\Omega}\right)_{cm} \cdot 2\pi\sin\theta_{cm}\, d\theta_{cm} \Rightarrow \boxed{\left(\frac{d\sigma}{d\Omega}\right)_L = \left(\frac{d\sigma}{d\Omega}\right)_{cm} \cdot \frac{d\cos\theta_{cm}}{d\cos\theta_L}}. \qquad (3)$$

We've used $\sin\theta\, d\theta = -d\cos\theta$. When $m=M$, $\theta_L = \frac{1}{2}\theta_{cm}$, and $\cos\theta_{cm} = 2\cos^2\theta_L - 1$, so...

$$\underline{\underline{\left(\frac{d\sigma}{d\Omega}\right)_L = [4\cos\theta_L]\left(\frac{d\sigma}{d\Omega}\right)_{cm}}}, \text{ for } m=M. \qquad (4)$$

θ_L ranges over $0 \to 90°$ (max.), and $(d\sigma/d\Omega)_L$ should vanish at $90°$.

ϕ507 Solutions

(19) [15 pts]. Born Approxn for scattering from a spherical well: $V(r) = \begin{cases} (-)V_0, & r<a; \\ 0, & r>a. \end{cases}$

1. Eq. (31), p. ScT 13, of class notes is (of course) correct, and it specifies the differential scattering cross-section for elastic scattering from a sph. symmetric potl. V:

$$\rightarrow \frac{d\sigma}{d\Omega} = \left|(m/2\pi\hbar^2)\tilde{V}(q)\right|^2, \text{ w/ } \tilde{V}(q) = \frac{4\pi}{q}\int_0^\infty r\,V(r)\sin qr\,dr, \tag{1}$$

w/ $\underline{q = 2k\sin(\theta/2)}$, the momentum transfer, θ = scattering $\measuredangle$, and $\hbar k = \sqrt{2mE}$ the incident momentum of particle m. If V(r) is the given spherical well, then:

$$\tilde{V}(q) = -\frac{4\pi V_0}{q}\int_0^a r\sin qr\,dr = -\frac{4\pi V_0}{q^3}[\sin qa - qa\cos qa],$$

$$\text{So}/\!/\quad d\sigma/d\Omega = 4a^2(mV_0a^2/\hbar^2)^2\,[\sin qa - qa\cos qa]^2/(qa)^6. \tag{2}$$

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2. The total cross-section is : $\sigma(k) = \int_{4\pi}(d\sigma/d\Omega)\,d\Omega$, w/ $d\Omega = 2\pi\sin\theta\,d\theta$. Since:

$$\rightarrow d\Omega = 2\pi\times\left(2\sin\frac{\theta}{2}\right)d\left(\sin\frac{\theta}{2}\right) = (2\pi/k^2)\,q\,dq, \int \text{and: } 0\leq\theta\leq\pi \Rightarrow 0\leq q\leq 2k; \tag{3}$$

Then/

$$\rightarrow \sigma(k) = \int_0^{2k}\left(\frac{d\sigma}{d\Omega}\right)\frac{2\pi}{k^2}q\,dq = \frac{2\pi}{k^2a^2}\cdot 4a^2\left(\frac{mV_0a^2}{\hbar^2}\right)^2 \underbrace{\int_0^{2ka}\frac{1}{x^6}[\sin x - x\cos x]^2 x\,dx}_{I(2ka)}. \tag{4}$$

Let $\underline{\alpha = 2ka}$. Do a partial integration on $I(\alpha)$...

$$\rightarrow I(\alpha) = \int_0^\alpha \frac{dx}{x^5}[\sin x - x\cos x]^2 = -\frac{1}{4}\left\{\left(\frac{\sin x}{x^2} - \frac{\cos x}{x}\right)^2\Big|_{x=0}^{x=\alpha} - 2\int_0^\alpha dx\left(\frac{\sin x}{x^2} - \frac{\cos x}{x}\right)\frac{\sin x}{x}\right\}. \tag{5}$$

Can continue with trig integrals, or recognize spherical Bessel fcns $j_0(x) = \frac{\sin x}{x}$ & $j_1(x) = \frac{\sin x}{x^2} - \frac{\cos x}{x}$, and use tables [e.g. G & R # (5.55), p. 634]. Result is:

$$I(\alpha) = \frac{1}{4}\left\{1 - \frac{1}{\alpha^2} + \frac{1}{\alpha^3}\sin 2\alpha - \frac{1}{\alpha^4}\sin^2\alpha\right\},\quad \alpha = 2ka; \tag{6}$$

$$\text{So}/\!/\quad \boxed{\sigma(k) = \frac{2\pi}{k^2}(mV_0a^2/\hbar^2)^2\left\{1 - \frac{1}{\alpha^2} + \frac{1}{\alpha^3}\sin 2\alpha - \frac{1}{\alpha^4}\sin^2\alpha\right\}}. \tag{7}$$

LOW ENERGY $(\alpha \ll 1) \Rightarrow \{\ \} \rightarrow \frac{2}{9}\alpha^2$, and: $\underline{\sigma(k) \rightarrow \frac{16\pi a^2}{9}(mV_0a^2/\hbar^2)^2}$; (8A)

HIGH ENERGY $(\alpha \gg 1) \Rightarrow \{\ \} \rightarrow 1$, and: $\underline{\sigma(k) \rightarrow \frac{2\pi}{k^2}(mV_0a^2/\hbar^2)^2}$. (8B)