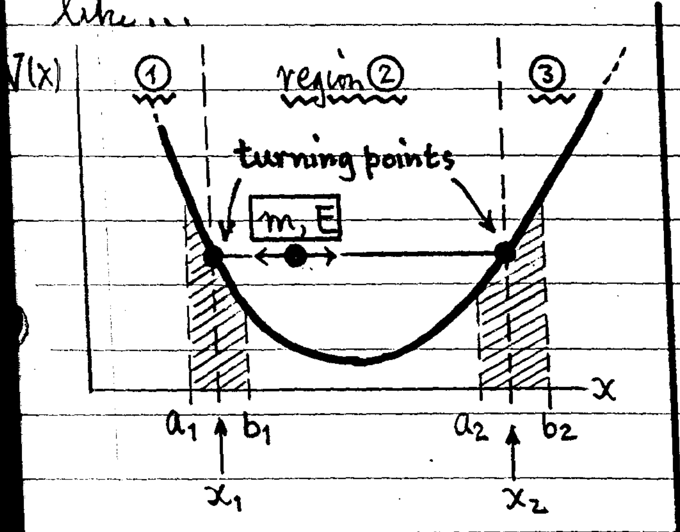
# WKB (cont'd) Back to the turning point problem.

11) [We have remarked before (e.g. on p. WKB 5) that the WKB (approximate) solution to $\psi'' + k^2\psi = 0$ does not work when $k^2 \to 0$... the solns $\propto 1/\sqrt{k}$ diverge, the "slowly-varying" condition $|k'/k^2| \ll 1$ can't be met, etc... everything seems to be a mess. Here we will see how something can be rescued from this mess.

It is easiest to begin sorting out the mess by talking about a physical example. We turn to QM... where a particle of mass $m$ & energy $E$ is trapped in a "potential well", i.e. a potential energy fcn $V(x)$ [1D motion] which looks like...




$x_1$ & $x_2$ are "turning points" of the motion (classical $m$ turns around there), where...

$$V(x_1) = E = V(x_2),$$

$$\Rightarrow \hbar k(x) = \sqrt{2m[E - V(x)]} = 0, \text{ @ } x_1 \text{ \& } x_2,$$

and WKB fails (for $\psi'' + k^2(x)\psi = 0$),

in regions: $a_1 < x < b_1$, $a_2 < x < b_2$. (29)

12) A classical $m$ would never be found in regions ①&③, where $V(x) > E$; it would have to have (−)ve kinetic energy & imaginary velocity. This is reflected in QM by claiming the wavefcn $\psi(x)$ [with $|\psi|^2 \propto$ "presence" of $m$] must be "small" in ①&③ [$m$ may be there, but not very often]. So we choose WKB forms

$$\left.\begin{array}{ll} \underline{\text{in region ①}}: \ \psi_1(x) = \dfrac{A}{\sqrt{K(x)}}\, e^{-\int_x^{x_1} K(\xi)\,d\xi}, & \underline{x < a_1} \\[2ex] \underline{\text{in region ③}}: \ \psi_3(x) = \dfrac{C}{\sqrt{K(x)}}\, e^{-\int_{x_2}^{x} K(\xi)\,d\xi}, & \underline{x > b_2} \end{array}\right\} \hbar K(x) = \sqrt{2m[V(x) - E]}. \tag{30}$$

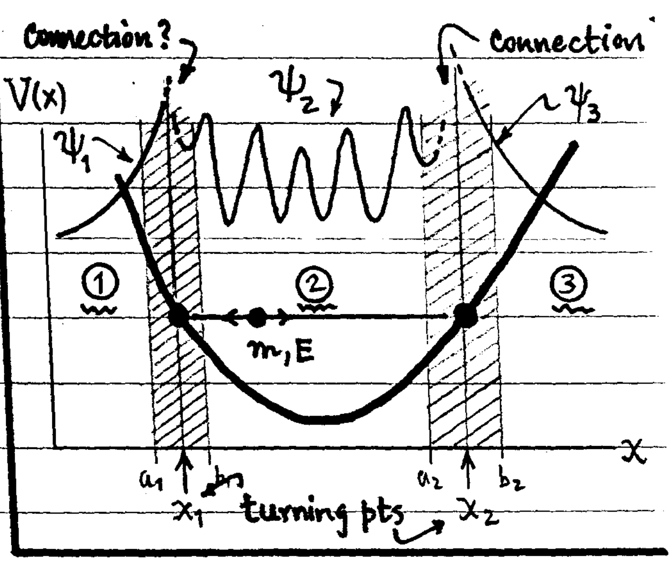
Both of these get suitably small as $|x| \to$ large. Anyway, we are adopting the point of view that WKB is $\sim$ OK as long as we exclude the shaded regions: $a_1 < x < b_1$ & $a_2 < x < b_2$ (size to be fixed later). In the

# WKB (cont'd) Need for Connection Formulas across turning points.

Same spirit, we claim that in region ②, where m is most likely to be found, the suitable WKB soln -- with two more indpt cnsts B & $\beta$ -- is given by

$$\text{in region ②}: \quad \psi_2(x) = \frac{B}{\sqrt{k(x)}} \sin\left( \int_{x_1}^{x} k(\xi)\,d\xi + \beta \right), \quad \text{for } b_1 < x < a_2, \quad \text{w/ } \hbar k(x) = \sqrt{2m[E - V(x)]}. \tag{31}$$

Pictorially, we have the problem at right. We have valid WKB $\psi$'s everywhere but in shaded regions, near where k & $k \to 0$. But in those regions, we know the "real" $\psi$ must be continuous (and $\psi'$ continuous). So what we want is a way of connecting $\psi_1$ to $\psi_2$, and $\psi_2$ to $\psi_3$ across the turning point barriers.




STRATEGY: $\psi_1, \psi_2, \psi_3$ of Eqs. (30) & (31) contain $\underline{\underline{4}}$ arbitrary cnsts A, B & $\beta$, C. Only $\underline{\underline{2}}$ are necessary in soln to $\psi'' + k^2(x)\psi = 0$. We will use the freedom of the two extra cnsts to connect $\psi_1$ to $\psi_2$ at $x_1$, and $\psi_2$ to $\psi_3$ at $x_2$. This will result in what are called the WKB Connection Formulas, and it will solve the turning point problem.

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**13)** Look at the Schrödinger problem in neighborhood of a turning point. Have...

... in LH shaded region, $a_1 < x < b_1$ ...

Exact eqtn is: $\psi'' + \frac{2m}{\hbar^2}[E - V(x)]\psi = 0$ — the [ ] is (+)ve or (−)ve.

Near $x_1$: $V(x) = \underbrace{\cancel{V(x_1)}}_{\to E} + \underbrace{V'(x_1)(x - x_1)}_{\text{this is (−) ve}} + \underbrace{\cancel{\tfrac{1}{2}V''(x_1)(x-x_1)^2 + \cdots}}_{\to 0,\ \text{ignore (small)}}$

so/ $$\boxed{\psi'' + \frac{2mF_1}{\hbar^2}(x - x_1)\psi \simeq 0}, \quad \text{near } x = x_1, \quad \text{w/ } F_1 = |V'(x_1)|. \tag{32}$$

WKB (cont'd) Airy Eqtn near turning point.

It is convenient to write this eqtn in dimensionless form, as...

Airy's eq. $$\boxed{\frac{d^2\psi}{d\xi^2} - \xi\psi = 0, \quad \text{w/} \quad \xi = \left(\frac{2m}{\hbar^2}F_1\right)^{\frac{1}{3}}(x_1 - x) \quad [\text{as } x \to x_1]}. \tag{33}$$

TACTICS: Solve this for $\psi = \psi(\xi)$; Connect $\begin{cases} \psi \text{ to } \psi_1 \text{ @ } x = a_1 \\ \psi \text{ to } \psi_2 \text{ @ } x = b_1 \end{cases}$.

Solutions to Eq. (33) thus provide the needed bridge $\psi_1 \curvearrowright \underset{(x_1)}{\overset{\psi}{\uparrow}} \curvearrowright \psi_2$. It is clear that $a_1$ & $b_1$ should be chosen so that...

$$\left.\begin{matrix} \psi_1(\text{WKB}) \text{ "good" up to } x = a_1, \\ \psi_2(\text{WKB}) \text{ "good" down to } x = b_1; \end{matrix}\right\} \psi(\text{Eq.}(33)) \text{ "good" in } a_1 \leq x \leq b_1. \tag{34}$$

$\to$ requires: $\left|\frac{1}{k^2}(dk/dx)\right| \ll 1$ @ $x = a_1$ & $x = b_1$,

... with: $\hbar k(x) = \sqrt{2m[E - V(x)]} = \sqrt{2mF_1(x - x_1)}$ here...

WKB "goodness" requires: $$\left|\left(\frac{2mF_1}{\hbar^2}\right)^{1/2}(x - x_1)^{3/2}\right| = \boxed{|\xi|^{3/2} \gg \frac{1}{2}}. \tag{35}$$
(at $x = a_1, b_1$)

This is a big relief ... it means we need only asymptotic solutions to Eq. (33): $\psi'' - \xi\psi = 0$, for $|\xi| \to$ large, at the endpoints $a_1$ & $b_1$.

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**14)** The eqtn $\psi'' - \xi\psi = 0$ is solved most efficiently by Fourier transforms. We look for a solution in terms of a Fourier integral...

$$\left[\!\left[\psi(\xi) = \int_{-\infty}^{\infty} \varphi(k)\, e^{ik\xi}\, dk \leftarrow \varphi(k) \text{ to be found, to satisfy: } \psi'' - \xi\psi = 0.\right.\right. \tag{36}$$

Show spectrum fcn is: $\varphi(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \psi(\xi)\, e^{-ik\xi}\, d\xi$ (Fourier inverse).

If we can find an eqtn for $\varphi(k)$, and solve it, we can at least write $\psi(\xi)$ as an integral.

# WKB (cont'd) Airy Eqtn solved by Fourier Integral.

To convert the Airy Eqtn [Eq. (33)] to a Fourier problem, note identities...

① $i\left(\frac{d\varphi}{dk}\right) = \frac{1}{2\pi}\int_{-\infty}^{\infty}[\xi]\psi(\xi)\,e^{-ik\xi}\,d\xi$ ← just differentiate under the $\int_{-\infty}^{\infty}$;

② $\frac{1}{2\pi}\int_{-\infty}^{\infty}\psi'(\xi)\,e^{-ik\xi}\,d\xi = ik\,\varphi(k)$ ← partial integration★ (assume $\psi \to 0$ as $|\xi|\to\infty$).

③ $\frac{1}{2\pi}\int_{-\infty}^{\infty}[\psi''(\xi)]\,e^{-ik\xi}\,d\xi = -k^2\varphi(k)$ ← repeat ② ($\xi\ \psi' \to 0$ as $|\xi|\to\infty$).

Then can convert the 2nd order $\psi$ eqtn to a 1st order $\varphi$ eqtn...

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}(\underbrace{\psi''}_{\text{use ③}} - \underbrace{\xi\psi}_{\text{use ①}})\,e^{-ik\xi}\,d\xi \Rightarrow \boxed{\frac{d\varphi}{dk} = +ik^2\varphi} \tag{37}$$

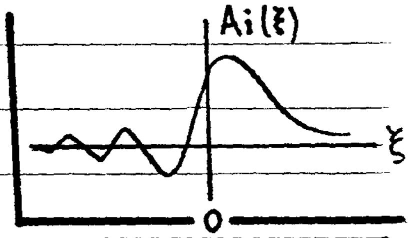
The $\varphi$ eqtn is trivial, and has solution: $\underline{\varphi(k) = \text{cnst}\cdot e^{\frac{1}{3}ik^3}}$. Then the solution to Eq. (33): $\psi'' - \xi^2\psi = 0$ takes the Fourier form [Eq. (36)]:

$$\psi(\xi) = \int_{-\infty}^{\infty}\varphi(k)\,e^{ik\xi}\,d\xi = \text{cnst}\cdot\int_{-\infty}^{\infty}e^{i(\xi k + \frac{1}{3}k^3)}\,dk$$

← exp. is odd in k

$$\text{w/}\ \boxed{\psi(\xi) = \text{cnst}\cdot Ai(\xi),\quad Ai(\xi) = \frac{1}{\pi}\int_0^{\infty}\cos(\xi k + \tfrac{1}{3}k^3)\,dk}. \tag{38}$$

15) $Ai(\xi)$ is called an "Airy Function"; it is closely related to Bessel fcns of order $\nu = \pm 1/3$.¶ Asymptotic forms for $|\xi|\to$ large are...




$$\to Ai(\xi) \sim \begin{cases} (1/2\sqrt{\pi})\,\xi^{-1/4}\,e^{-\zeta}, & \text{for } \xi \gg +1;\ [\text{exponential}] \\ (1/\sqrt{\pi})\,|\xi|^{-1/4}\sin(|\zeta| + \frac{\pi}{4}), & \xi \ll -1;\ [\text{Oscillatory}] \end{cases}$$

$$\text{w/}\ \underline{\zeta = (2/3)\,\xi^{3/2}}. \tag{39}$$

¶ NBS Math Handbook, Ch. 10, Sec. 4. E.g. $Ai(z) = (1/\pi\sqrt{3})\,z^{1/2}\,K_{\frac{1}{3}}(\frac{2}{3}z^{3/2})$.

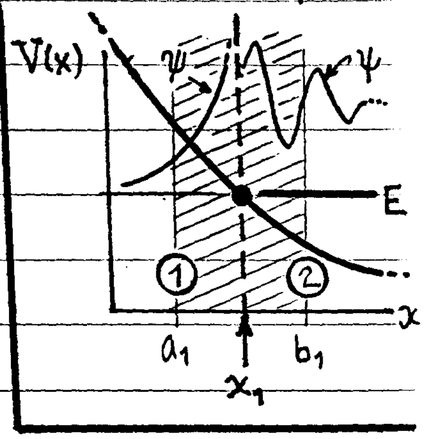
★ $\int_{-\infty}^{\infty}\psi'(\xi)e^{-ik\xi}\,d\xi = \int e^{-ik\xi}\,d\psi = \cancel{\psi e^{-ik\xi}}\Big|_{\xi=-\infty}^{\xi=+\infty} - \int_{-\infty}^{\infty}\psi(\xi)\frac{d}{d\xi}e^{-ik\xi}\,d\xi \Rightarrow$ ②.

## WKB (Cont'd) Asymptotic forms for $\psi(\xi)$ near turning point.

The asymptotic forms for $Ai(\xi)$ in Eq. (39) can be verified by direct substitution.*

SUMMARY: we have now solved the Schrödinger problem near the LH turning pt:

$\rightarrow d^2\psi/d\xi^2 - \xi\psi = 0$, w/ $\xi = (2mF_1/\hbar^2)^{\frac{1}{3}}(x_1 - x)$, and $F_1 = \left|\frac{dV}{dx}\right|_{x=x_1}$.




$\Rightarrow$ Region ①: $x < x_1$, and $\xi \gg +1$ for $x \to a_1$:

$$\rightarrow \psi(\xi) \propto \frac{1}{2}\xi^{-1/4} e^{-\frac{2}{3}\xi^{3/2}} \quad \text{exponential decline.} \tag{40}$$

Region ②: $x > x_1$, and $\xi \ll (-)1$ for $x \to b_1$:

$$\longrightarrow \psi(\xi) \propto |\xi|^{-1/4} \sin\left(\tfrac{2}{3}|\xi|^{3/2} + \underbrace{\tfrac{\pi}{4}}_{\text{phase is important!}}\right) \quad \text{oscillatory character.} \tag{41}$$

Now we need to join up all the pieces of $\psi$ [$\psi$(Airy) from left & right, and $\psi$(WKB) from left & right], smoothly, in the neighborhood $x \sim x_1$.

6) Since the $\psi$(WKB)'s are quoted in terms of $K = \sqrt{\frac{2m}{\hbar^2}(V-E)}$ & $k = \sqrt{\frac{2m}{\hbar^2}(E-V)}$, it is convenient to express the $\psi$(Airy)'s in the same terms.

In ①: $a_1 < x < x_1$, and: $K(x) = [(2mF_1/\hbar^2)(x_1 - x)]^{1/2}$.

So// $\frac{2}{3}\xi^{3/2} = \frac{2}{3}\sqrt{\frac{2mF_1}{\hbar^2}}(x_1 - x)^{3/2} = \int_x^{x_1} K(x')\,dx'$, ← nice trick!

and// $\xi^{-1/4} \propto 1/\sqrt{K(x)}$.

$$\psi(\xi) = \frac{D}{2\sqrt{K(x)}} e^{-\int_{x\to a_1}^{x_1} K(x')\,dx'} \tag{42}$$

* For $\xi \to +\infty$, put $\psi(\xi) = \xi^{-1/4} \cdot e^{-\frac{2}{3}\xi^{3/2}}$ into Airy's Eqtn [Eq. (33)]...

$\psi'' = \xi\left(1 + \frac{5}{16}\xi^{-3}\right)\psi$, so: $\psi'' - \xi\psi \simeq 0$, neglecting $\mathcal{O}(\xi^{-3})$. OK

Do same with asymptotic form for $\xi \to (-)\infty$. Note that...

$\xi < 0 \Rightarrow |\xi| = -\xi$, and: $\xi^{3/2} = -i|\xi|^{3/2}$, $\xi^{-1/4} = e^{-i\frac{\pi}{4}}|\xi|^{-1/4}$;

So// $\xi^{-1/4} e^{-\frac{2}{3}\xi^{3/2}} = |\xi|^{-\frac{1}{4}} e^{i\left(\frac{2}{3}|\xi|^{3/2} - \frac{\pi}{4}\right)} \xrightarrow[\text{part}]{\text{Re}} |\xi|^{-\frac{1}{4}} \sin\left(|\xi| + \frac{\pi}{4}\right)$. OK

This result is ~ pleasing, because it resembles the $\psi_1$(WKB) form we wrote down in Eq. (30)... $\psi_1$ exponentially declining @ $x < a_1$. As for $x > x_1$...

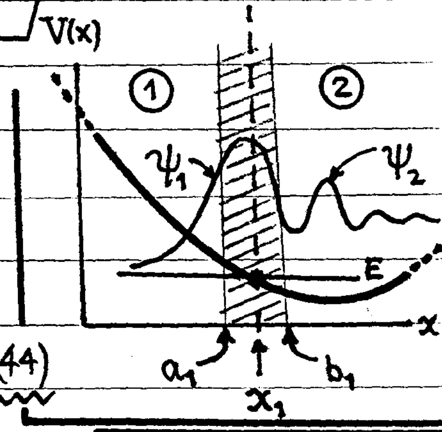
In ②: $x_1 < x < b_1$, and: $k(x) = [(2mF_1/\hbar^2)(x-x_1)]^{1/2}$.

So// $\frac{2}{3}|\xi|^{3/2} = \frac{2}{3}\sqrt{\frac{2mF_1}{\hbar^2}}(x-x_1)^{3/2} = \int_{x_1}^{x} k(x')\,dx'$, ← same trick works

and// $|\xi|^{-1/4} \propto 1/\sqrt{k(x)}$.

$$\psi(\xi) = \frac{D}{\sqrt{k(x)}} \sin\left(\int_{x_1}^{x\to b_1} k(x')\,dx' + \frac{\pi}{4}\right). \tag{43}$$

Again ~ pleasing... because $\psi$(Airy) resembles the oscillatory $\psi_2$(WKB) form in Eq. (31). NOTE: the same amplitude cnst $D$ is used in both $\psi(\xi)$'s here [Eqs. (42) & (43)], because both $\psi$'s refer to the same solution. Also... we still don't have a valid $\psi$ at $x = x_1$ (this would entail the $|\xi|\to 0$ version of $Ai(\xi)$ in Eq. (38), rather than the $|\xi|\to\infty$ version we have used). But we don't need $\psi$(Airy) at $x = x_1$; it is sufficient for matching purposes to know how $\psi$(Airy) behaves at the WKB boundaries $x \to a_1$ & $x \to b_1$. Just such information is provided by Eqs. (42) & (43).




7) Now, finally, we can connect solutions. We have...

→ REGION ①: declining exponential.

$$[30)]\ \text{WKB}\ (x \leq a_1):\ \psi_1(x) = \frac{A}{\sqrt{k(x)}} e^{-\int_x^{x_1} k(x')\,dx'},$$

$$[42)]\ \text{Airy}\ (x \geq a_1):\ \psi(x) = \frac{1}{2}\frac{D}{\sqrt{k(x)}} e^{-\int_x^{x_1} k(x')\,dx'}. \tag{44}$$

→ REGION ②: distorted oscillation.

$$[43)]\ \text{Airy}\ (x \leq b_1):\ \psi(x) = \frac{D}{\sqrt{k(x)}} \sin\left(\int_{x_1}^{x} k(x')\,dx' + \frac{\pi}{4}\right),$$

$$[31)]\ \text{WKB}\ (x \geq b_1):\ \psi(x) = \frac{B}{\sqrt{k(x)}} \sin\left(\int_{x_1}^{x} k(x')\,dx' + \beta\right). \tag{45}$$

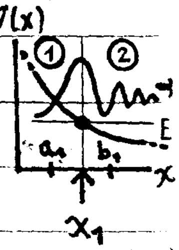
$\psi$ is continuous across boundaries at $a_1$ & $b_1$ (and even in $a_1 \leq x \leq b_1$)

So// $\psi$ continuous at $x = a_1$ & $x = b_1$ ⇒ $\boxed{2A = D = B, \text{ and } \beta = \frac{\pi}{4}}$. (46)

# WKB (cont'd) First WKB Connection Formula. A Quantization Condition.



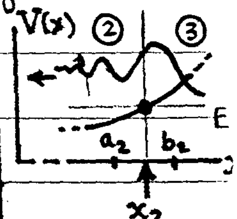
18) Now we know how a $\psi$(WKB exp$^l$) connects to a $\psi$(WKB osc$^y$) through a turning point, namely...




$$\boxed{\begin{aligned} &\psi_1(x \leq a_1) = \left(A/\sqrt{\kappa(x)}\right) e^{-\int_x^{x_1} \kappa(x')\,dx'}, \text{ in region ①;} \\ &\text{thru } x_1 \\ &\psi_2(x \geq b_1) = \left(2A/\sqrt{k(x)}\right) \sin\left(\int_{x_1}^{x} k(x')\,dx' + \frac{\pi}{4}\right), \text{ in region ②.} \end{aligned}} \tag{47}$$

Where: $\hbar \kappa(x) = \sqrt{2m[V(x)-E]}$ & $\hbar k(x) = \sqrt{2m[E-V(x)]}$, for QM problem (as above). So $\psi$ evolves from an exponential $\rightarrow$ oscillation, the amplitude $A \rightarrow 2A$, and the oscillation picks up a phase factor of $\pi/4$.

We can repeat the procedure at the other turning point, i.e. at $x = x_2$ in diagram on p. WKB 12. This just amounts to changing notation in Eq. (47). Have...



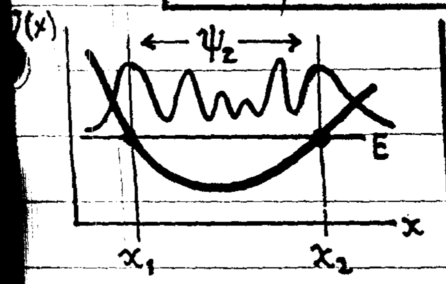

$$\boxed{\begin{aligned} &\psi_2(x \leq a_2) = \left(2C/\sqrt{k(x)}\right) \sin\left(\int_x^{x_2} k(x')\,dx' + \frac{\pi}{4}\right), \text{ in region ②,} \\ &\text{thru } x_2 \\ &\psi_3(x \geq b_2) = \left(C/\sqrt{\kappa(x)}\right) e^{-\int_{x_2}^{x} \kappa(x')\,dx'}, \text{ in region ③.} \end{aligned}} \tag{48}$$

19) This "bookkeeping" actually has some physical content. We have two equivalent expressions for $\psi_2$ in the interior region: $b_1 < x < a_2$. By continuity of $\psi$, claim:

$$\underbrace{\left(2A/\sqrt{k}\right) \sin\left(\int_{x_1}^{x} k(x')\,dx' + \frac{\pi}{4}\right)}_{\text{from left: ①→②, Eq.(47)}} = \psi_2(x) = \underbrace{\left(2C/\sqrt{k}\right) \sin\left(\int_{x}^{x_2} k(x')\,dx' + \frac{\pi}{4}\right)}_{\text{from right: ②←③, Eq.(48)}}$$

$$\Rightarrow A \sin\left(\int_{x_1}^{x} \sim + \frac{\pi}{4}\right) = C \sin\left(\int_{x}^{x_2} \sim + \frac{\pi}{4}\right) \leftarrow \text{use: } \int_x^{x_2} = \int_{x_1}^{x_2} - \int_{x_1}^{x}, \text{ define: } \phi = \int_{x_1}^{x} + \frac{\pi}{4}$$

$$\Rightarrow \boxed{A \sin\phi = C \sin(\phi_0 - \phi)}, \text{ where: } \phi_0 = \int_{x_1}^{x_2} k(x')\,dx' + \frac{\pi}{2}. \tag{49}$$




This identity ensures $\psi_2$ is continuous in the interior. It can only be true (for all interior $x$ & $\phi$) if we have:

$$\rightarrow \phi_0 = (n+1)\pi, \text{ and: } C = (-)^n A,\ n = 0, 1, 2, \ldots \tag{50}$$

So the WKB phase integral $\phi_0$ is <u>quantized</u> as a result of continuity in $\psi$:

$$\phi_0 = (n+1)\pi \Rightarrow \boxed{\int_{x_1}^{x_2} k(x)\,dx = \left(n+\tfrac{1}{2}\right)\pi}, \quad n = 0,1,2,\ldots \tag{51}$$

This is a <u>classical</u> result... involving use of $\psi \sim \psi(\text{WKB})$ & $\psi$ continuous, only. In QM, we write momentum $p = \hbar k$, so that this condition is

$$\boxed{\int_{x_1}^{x_2} p(x)\,dx = \int_{x_1}^{x_2}\sqrt{2m[E-V(x)]}\,dx = \left(n+\tfrac{1}{2}\right)\pi\hbar}, \quad n = 0,1,2,3\ldots \tag{52}$$

This condition can be satisfied only for <u>quantized</u> values of the total energy, i.e. $E = E_n$. So <u>every</u> QM particle $m$ in a "well" $V(x)$ has quantized $E_n$'s. This important result is known as the Bohr-Sommerfeld Quantization.

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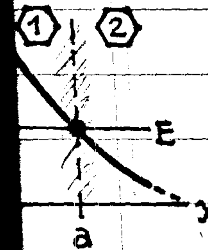
**20)** The Connection Formulas in Eq. (47) & (48) connect an exponentially <u>decreasing</u> WKB solution to an oscillatory one across a turning point. For completeness, we also need the connection for exponentially <u>increasing</u> WKB → oscillatory WKB. Calculations similar to the above produce† the following results (in a form suitable for QM problems)

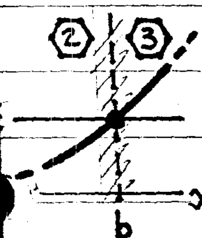
CTION MULAS

Let: $\hbar k(x) = \sqrt{2m[E-V(x)]}$, $\hbar\kappa(x) = \sqrt{2m[V(x)-E]} = i\,k(x)$.

Then WKB solutions to $\begin{cases}\psi'' + k^2\psi = 0 \leftarrow \text{bound-state regions} \\ \psi'' - \kappa^2\psi = 0 \leftarrow \text{"forbidden" regions}\end{cases}$ are...



$$\begin{cases}\psi_1(x<a) = \dfrac{A}{\sqrt{\kappa}}\, e^{-\int_x^a \kappa(\xi)\,d\xi} \longrightarrow \psi_2(x>a) = \dfrac{2A}{\sqrt{k}}\sin\left(\int_a^x k(\xi)\,d\xi + \dfrac{\pi}{4}\right), \\ \psi_1(x<a) = \dfrac{\tilde{A}}{\sqrt{\kappa}}\, e^{+\int_x^a \kappa(\xi)\,d\xi} \longrightarrow \psi_2(x>a) = \dfrac{\tilde{A}}{\sqrt{k}}\cos\left(\int_a^x k(\xi)\,d\xi + \dfrac{\pi}{4}\right);\end{cases} \tag{53}$$



$$\begin{cases}\psi_2(x<b) = \dfrac{2C}{\sqrt{k}}\sin\left(\int_x^b k(\xi)\,d\xi + \dfrac{\pi}{4}\right) \longleftarrow \psi_3(x>b) = \dfrac{C}{\sqrt{\kappa}}\, e^{-\int_b^x \kappa(\xi)\,d\xi}, \\ \psi_2(x<b) = \dfrac{\tilde{C}}{\sqrt{k}}\cos\left(\int_x^b k(\xi)\,d\xi + \dfrac{\pi}{4}\right) \longleftarrow \psi_3(x>b) = \dfrac{\tilde{C}}{\sqrt{\kappa}}\, e^{+\int_b^x \kappa(\xi)\,d\xi}\end{cases} \tag{54}$$

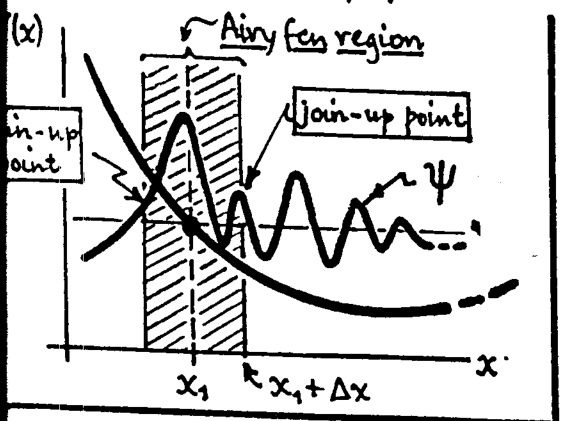
$A, \tilde{A}$ & $C, \tilde{C}$ are all adjustable cnsts. 1st & 3rd connections are Eqs. (47) & (48); 2nd & 4th connections are "similar calculations".

† See e.g. "J. Powell & B. Crasemann" "QM" (Addison-Wesley 1961), p. 148 et seq.

# WKB: A Physical Accuracy Criterion



21) We can state a "physical" criterion for accuracy of the WKB approxn in terms of



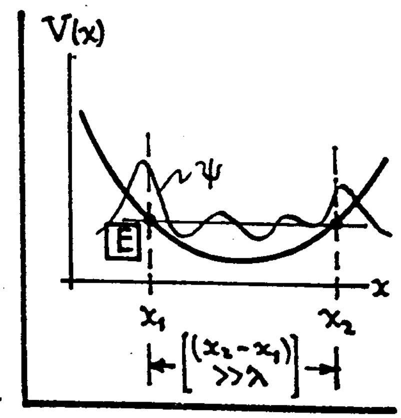

the de Broglie wavelength $\lambda = 2\pi/k$ of the particle (mass $m$) described by $\psi$. Recall that on p. 13 we found that $\psi$ could be continued thru a turning point by means of the Airy-fcn analysis if we joined up the WKB solutions to an appropriate Airy fcn in the asymptotic region $|\xi|^{\frac{3}{2}} >> \frac{1}{2}$ (to left & right of turning pt $x_1$ shown). In fact, in that notation, $|\xi|^{\frac{3}{2}} >> \frac{1}{2}$, was equivalent to the WKB "goodness" criterion $|k'/k^2| << 1$. This asymptotic condition can be converted to a statement about the size of the well in units of $\lambda$.

Consider a "join-up point" (Airy → WKB) @ $x_1 + \Delta x$ as shown. Compare the size of $\Delta x$ with $\lambda = 2\pi/k$, where $k = \sqrt{(2mF_1/\hbar^2)\Delta x}$ at that point. Then...

$$\left[\frac{\Delta x}{\lambda} = \frac{1}{2\pi}\left(\sqrt{(2mF_1/\hbar^2)\Delta x}\right)\Delta x = \frac{1}{2\pi}\left[\left(\frac{2mF_1}{\hbar^2}\right)^{\frac{1}{3}}\Delta x\right]^{\frac{3}{2}} = \frac{1}{2\pi}|\xi|^{3/2} >> 1. \qquad (55)\right.$$

We have recognized $\xi$ by its definition in Eq. (33), p. 13 [note $\hbar^2$ there]. This condition says that a successful Airy ↔ WKB join-up can only occur when well is big enough so that there are allowed regions $\underline{\Delta x >> \lambda}$ on either side of a turning point. To the extent this condition is weakened, the WKB approxn to $\psi$ will become less accurate.

In these terms, we can see immediately that for the bound state problem we have done, <u>WKB will be accurate only if</u> the energy E is high enough so that the distance between the turning points $(x_2 - x_1) >> \lambda$. This condition is successively weakened as the particle sinks down to the bottom




of the well, since $(x_2 - x_1)$ decreases while $\lambda$ increases. So WKB results here are expected to be ~ poor for the lowest lying states, but they improve as E increases.