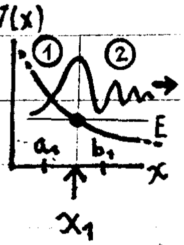
WKB (cont'd) First WKB Connection Formula. A Quantization Condition.

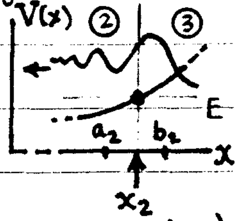
18) Now we know how a ψ (WKB exp^l) connects to a ψ (WKB (osc^y)) through a turning point, namely...



$$\boxed{\begin{aligned} &\psi_1(x \leq a_1) = \left(A/\sqrt{\kappa(x)}\right) e^{-\int_x^{x_1} \kappa(x')dx'}, \text{ in region ①;} \\ &\text{thru } x_1 \\ &\psi_2(x \geq b_1) = \left(2A/\sqrt{k(x)}\right) \sin\left(\int_{x_1}^{x} k(x')dx' + \frac{\pi}{4}\right), \text{ in region ②.} \end{aligned}} \quad (47)$$

where: $\hbar\kappa(x) = \sqrt{2m[V(x)-E]}$ & $\hbar k(x) = \sqrt{2m[E-V(x)]}$, for QM problem (as above). So ψ evolves from an exponential $\rightarrow$ oscillation, the amplitude $A \rightarrow 2A$, and the oscillation picks up a phase factor of $\pi/4$.

We can repeat the procedure at the other turning point, i.e. at $x = x_2$ in diagram on p. WKB 12. This just amounts to changing notation in Eq. (47). Have...



$$\boxed{\begin{aligned} &\psi_2(x \leq a_2) = \left(2C/\sqrt{k(x)}\right) \sin\left(\int_x^{x_2} k(x')dx' + \frac{\pi}{4}\right), \text{ in region ②,} \\ &\text{thru } x_2 \\ &\psi_3(x \geq b_2) = \left(C/\sqrt{\kappa(x)}\right) e^{-\int_{x_2}^{x} \kappa(x')dx'}, \text{ in region ③.} \end{aligned}} \quad (48)$$

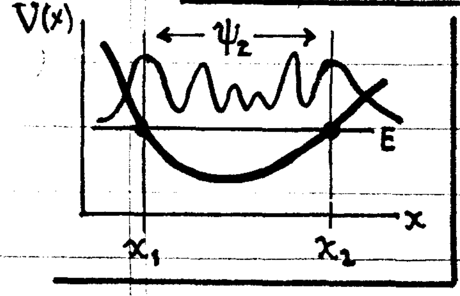
EXAMPLE

19) This "bookkeeping" actually has some physical content. We have two equivalent expressions for ψ_2 in the interior region: $b_1 < x < a_2$. By continuity of ψ, claim:

$$\underbrace{(2A/\sqrt{k}) \sin\left(\int_{x_1}^{x} k(x')dx' + \frac{\pi}{4}\right)}_{\text{from left: ①→②, Eq. (47)}} = \psi_2(x) = \underbrace{(2C/\sqrt{k}) \sin\left(\int_x^{x_2} k(x')dx' + \frac{\pi}{4}\right)}_{\text{from right: ②←③, Eq. (48)}}$$

$\Rightarrow A \sin\left(\int_{x_1}^{x} \sim + \frac{\pi}{4}\right) = C \sin\left(\int_x^{x_2} \sim + \frac{\pi}{4}\right)$ use: $\int_x^{x_2} = \int_{x_1}^{x_2} - \int_{x_1}^{x}$, define: $\phi = \int_{x_1}^{x} + \frac{\pi}{4}$

$$\Rightarrow \boxed{A \sin\phi = C \sin(\phi_0 - \phi)}, \text{ where: } \phi_0 = \int_{x_1}^{x_2} k(x')dx' + \frac{\pi}{2}. \quad (49)$$



This identity ensures ψ_2 is continuous in the interior. It can only be true (for all interior x & ϕ) if we have:

$$\rightarrow \phi_0 = (n+1)\pi, \text{ and: } C = (-)^n A, \; n = 0, 1, 2, \ldots \quad (50)$$

So the WKB phase integral ϕ_0 is quantized as a result of continuity in ψ:

$$\phi_0 = (n+1)\pi \Rightarrow \boxed{\int_{x_1}^{x_2} k(x)\,dx = \left(n+\tfrac{1}{2}\right)\pi}, \quad n=0,1,2,\ldots \tag{51}$$

This is a classical result... involving use of $\psi \sim \psi(\text{WKB})$ & ψ continuous, only.

In QM, we write momentum $p = \hbar k$, so that this condition is

$$\boxed{\int_{x_1}^{x_2} p(x)\,dx = \int_{x_1}^{x_2} \sqrt{2m[E-V(x)]}\,dx = \left(n+\tfrac{1}{2}\right)\pi\hbar}, \quad n=0,1,2,3\ldots \tag{52}$$

This condition can be satisfied only for quantized values of the total energy, i.e. $E = E_n$. So every QM particle m in a "well" $V(x)$ has quantized E_n's. This important result is known as the Bohr-Sommerfeld Quantization.

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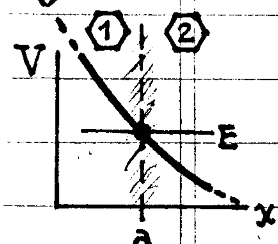
20) The Connection Formulas in Eq. (47) & (48) connect an exponentially decreasing WKB solution to an oscillatory one across a turning point. For completeness, we also need the connection for exponentially increasing WKB → oscillatory WKB. Calculations similar to the above produce† the following results (in a form suitable for QM problems):

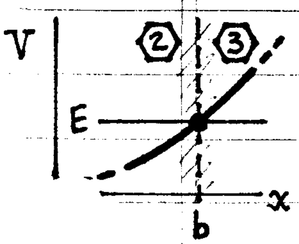
CONNECTION FORMULAS

Let: $\hbar k(x) = \sqrt{2m[E-V(x)]}$, $\hbar\kappa(x) = \sqrt{2m[V(x)-E]} = i\,k(x)$.

Then WKB solutions to $\begin{cases} \psi'' + k^2\psi = 0 \leftarrow \text{bound-state regions} \\ \psi'' - \kappa^2\psi = 0 \leftarrow \text{"forbidden" regions} \end{cases}$ are...



$$\begin{cases} \psi_1(x<a) = \dfrac{A}{\sqrt{\kappa}}\, e^{-\int_x^a \kappa(\xi)\,d\xi} \longrightarrow \psi_2(x>a) = \dfrac{2A}{\sqrt{k}} \sin\left(\int_a^x k(\xi)\,d\xi + \dfrac{\pi}{4}\right), \\ \psi_1(x<a) = \dfrac{\tilde{A}}{\sqrt{\kappa}}\, e^{+\int_x^a \kappa(\xi)\,d\xi} \longrightarrow \psi_2(x>a) = \dfrac{\tilde{A}}{\sqrt{k}} \cos\left(\int_a^x k(\xi)\,d\xi + \dfrac{\pi}{4}\right); \end{cases} \tag{53}$$



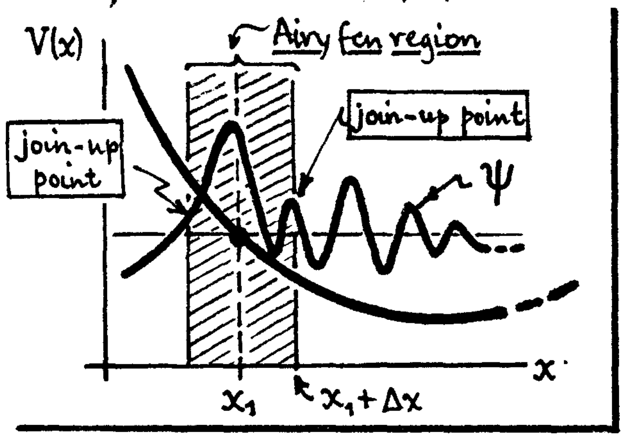
$$\begin{cases} \psi_2(x<b) = \dfrac{2C}{\sqrt{k}} \sin\left(\int_x^b k(\xi)\,d\xi + \dfrac{\pi}{4}\right) \longleftarrow \psi_3(x>b) = \dfrac{C}{\sqrt{\kappa}}\, e^{-\int_b^x \kappa(\xi)\,d\xi}, \\ \psi_2(x<b) = \dfrac{\tilde{C}}{\sqrt{k}} \cos\left(\int_x^b k(\xi)\,d\xi + \dfrac{\pi}{4}\right) \leftarrow \psi_3(x>b) = \dfrac{\tilde{C}}{\sqrt{\kappa}}\, e^{+\int_b^x \kappa(\xi)\,d\xi}. \end{cases} \tag{54}$$

$A, \tilde{A}$ & $C, \tilde{C}$ are all adjustable cnsts. 1st & 3rd connections are Eqs. (47) & (48); 2nd & 4th connections are "similar calculations".

† See e.g. "J. Powell & B. Crasemann "QM" (Addison-Wesley 1961), p. 148 et seq.

WKB: A Physical Accuracy Criterion

21) We can state a "physical" criterion for accuracy of the WKB approxn in terms of the <u>de Broglie wavelength</u> $\lambda = 2\pi/k$ of the particle (mass m) described by ψ. Recall that on p. 13 we found that ψ could be continued thru a turning point by means of the Airy-fcn analysis if we joined up the WKB solutions to an appropriate Airy fcn in the asymptotic region $|\xi|^{\frac{3}{2}} \gg \frac{1}{2}$ (to left & right of turning pt x_1 shown).



In fact, in that notation, $|\xi|^{\frac{3}{2}} \gg \frac{1}{2}$, was equivalent to the WKB "goodness" criterion $|k'/k^2| \ll 1$. This asymptotic condition can be converted to a statement about the size of the well in units of λ.

Consider a "join-up point" (Airy$\rightarrow$WKB) @ $x_1 + \Delta x$ as shown. Compare the size of Δx with $\lambda = 2\pi/k$, where $k = \sqrt{(2mF_1/\hbar^2)\Delta x}$ at that point. Then...

$$\left[\!\left[\frac{\Delta x}{\lambda} = \frac{1}{2\pi}\left(\sqrt{(2mF_1/\hbar^2)\Delta x}\right)\Delta x = \frac{1}{2\pi}\left[\left(\frac{2mF_1}{\hbar^2}\right)^{\frac{1}{3}}\Delta x\right]^{\frac{3}{2}} = \frac{1}{2\pi}|\xi|^{3/2} \gg 1. \right.\right. \qquad (55)$$

We have recognized ξ by its definition in Eq. (33), p. 13 [note $\hbar^2$ there]. This condition says that a successful Airy$\leftrightarrow$WKB join-up can only occur when well is big enough so that there are allowed regions <u>$\Delta x \gg \lambda$</u> on either side of a turning point. To the extent this condition is weakened, the WKB approxn to ψ will become less accurate.

In these terms, we can see immediately that for the bound state problem we have done, <u>WKB will be accurate only if</u> the energy E is high enough so that the distance between the turning points $(x_2 - x_1) \gg \lambda$. This condition is successively weakened as the particle sinks down to the bottom of the well, since $(x_2 - x_1)$ decreases while λ increases. So WKB results here are expected to be ~ poor for the lowest lying states, but they improve as E increases.