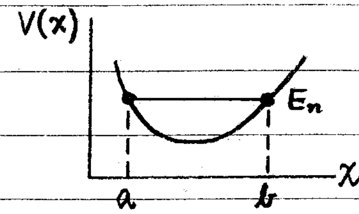
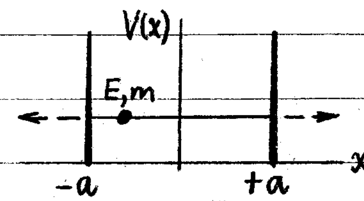
Physics 506 Hour Exam

9 Feb. 1972



Prob. # ③
φ507 MIDTERM
(Spr.'92)

① Consider a particle bound in an arbitrary attractive 1D potential V(x) as indicated. Show that the spacing between adjacent energy levels E_n is given, in WKB approximation, by $\Delta E_n = \hbar\omega_n$, where ω_n is the natural vibration frequency of the n^{th} level.



Prob. # ⑥
φ506 Final
(Dec. 1993)

② A particle of mass m and energy E is trapped in a 1D box of length 2a. The walls of the box (at ±a) may be represented by δ-fcns of strength C, i.e. the potential is $V(x) = C\,[\delta(x+a) + \delta(x-a)]$. Estimate the lifetime of the particle in the box, i.e. how long before it penetrates one of the barriers and gets out?

③ A QM system in state $\psi(x)$ at time $t=0$ is subjected for $t>0$ to an interaction H which generates two discrete eigenstates ϕ_n with eigen-energies E_n, such that $E_2 - E_1 = \hbar\Omega \neq 0$. The energy spectrum of H is therefore discrete, with values

$$W_n = \left| \int \phi_n^*(x)\,\psi(x)\,dx \right|^2, \quad n = 1 \,\&\, 2.$$

Assume $\sum_n W_n = 1$ for convenience. Calculate the probability P(t) for finding the original state $\psi(x)$ at times $t>0$. What is the oscillation period between points of maximum probability?

Note: Problem ④ is on the next page.

④ Start from the definition of the S-matrix in the form

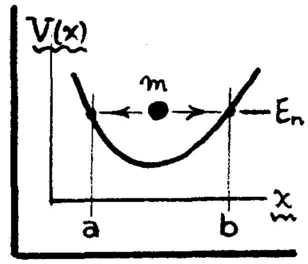
$$\psi_\alpha(x', t') = \sum_\beta S_{\beta\alpha} \phi_\beta(x', t'),$$

which describes the evolution of a free particle state $\phi_\alpha(x,t)$ in the distant past to the state $\psi_\alpha(x', t')$ in the distant future. Suppose the ϕ_β are orthonormal, and that the total interaction is at all times Hermitian. Then the normalization and orthogonality of the ψ_α must be time-independent. Use this fact to show that the S-matrix is unitary, i.e.

$$S^\dagger S = 1, \quad \text{or} \quad (S^\dagger S)_{ij} = \sum_\beta S^\dagger_{i\beta} S_{\beta j} = \sum_\beta S^*_{\beta i} S_{\beta j} = \delta_{ij}$$

③ [30 pts.]. Find the spacing ΔE_n of WKB bound-state energy levels ($n \to$ large).

1. The bound state energies E_n are found from the Bohr-Sommerfeld rule:



$$\rightarrow \int_a^b \sqrt{2m[E_n - V(x)]}\, dx = (n+\tfrac{1}{2})\pi\hbar. \qquad (1)$$

When $n \to$ large, E_n and n become quasi-continuous functions (e.g. $\Delta n/n \to 0$, for unit steps), so we differentiate (1) by $\frac{\partial}{\partial n}$ to get...

$$\int_a^b \frac{1}{2}\left(2m[E_n - V(x)]\right)^{-\frac{1}{2}} \cdot 2m\left(\frac{\partial E_n}{\partial n}\right) dx \simeq \pi\hbar, \text{ for } n \to \text{large};$$

$$\xrightarrow{\text{or}} \underline{\underline{m\left(\frac{\partial E_n}{\partial n}\right)\int_a^b \frac{dx}{p_n(x)} \simeq \pi\hbar}}, \text{ w/ } p_n(x) = \sqrt{2m[E_n - V(x)]}. \qquad (2)$$

$p_n(x)$ is the momentum of m in level E_n.

2. The natural period of the (quasi-oscillatory) motion of m in level E_n is $\tau_n = 2\int_a^b dx/v_n(x)$, with $v_n(x) = m$'s velocity. Set $v_n(x) = p_n(x)/m$, and put $\tau_n = 2\pi/\omega_n$, where ω_n is the ($\measuredangle$ lar) oscillation frequency. Then...

$$\rightarrow \frac{2\pi}{\omega_n} = 2\int_a^b \frac{dx}{p_n(x)/m}, \text{ so, } \underline{\underline{m\int_a^b \frac{dx}{p_n(x)} = \frac{\pi}{\omega_n}}}. \qquad (3)$$

3. Using Eq. (3) in Eq. (2), we obtain...

$$\rightarrow \left(\frac{\partial E_n}{\partial n}\right) \cdot \frac{\pi}{\omega_n} \simeq \pi\hbar, \text{ or } \underline{\underline{\frac{\partial E_n}{\partial n} \simeq \hbar\omega_n}}. \qquad (4)$$

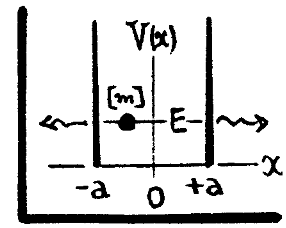
Then, to a first approximation (and for $n \to$ large), the spacing between adjacent levels, $\Delta n = 1$ around energy E_n, is given by

$$\boxed{\Delta E_n \simeq (\partial E_n/\partial n)\, \cancelto{1}{\Delta n} \simeq \hbar\omega_n}, \qquad (5)$$

where the frequency ω_n is defined in Eq. (3). n must be large enough here (i.e. terms of $O(1/n) \to$ negligible) to justify the derivatives take in Eq. (2). The result of Eq. (5) certainly does not work for the low-n states.

⑥ [45 pts]. Lifetime for a particle trapped in a semi-permeable box.

1. The decay rate for trapping is: $\underline{\Gamma = \left(\frac{1}{\tau/2}\right) T}$, where τ is the natural period of m's motion inside the box, and T is the transmission coefficient at one of the walls. The required lifetime is: $\underline{\Delta t = 1/\Gamma}$. Since m is free inside the box, we can write: $\tau = 2\cdot(2a)/v$, where m's velocity $v = \sqrt{2E/m}$. So: $\tau/2 = \sqrt{2ma^2/E}$, and the trapping lifetime is:

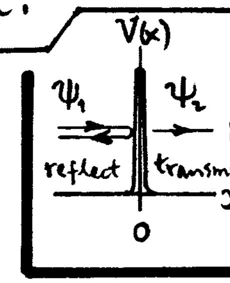


$$\rightarrow \Delta t = \frac{\tau}{2}\Big/T = \sqrt{\frac{2ma^2}{E}}\Big/T. \qquad (1)$$

If the wall transmission coefficient $T \rightarrow 0$, $\Delta t \rightarrow \infty$ and m remains forever trapped in the box. BUT, as we show below, T is finite for a δ-fcn wall.

---

2. Find T for a δ-fcn wall, with potential $V = C\,\delta(x)$. If m is incident at energy E, with momentum $\hbar k = \sqrt{2mE}$, wavefcns are:



$$\rightarrow x<0:\ \psi_1(x) = e^{ikx} + A e^{-ikx};\quad x>0:\ \psi_2(x) = B e^{ikx}. \qquad (2)$$

We want $T = |B|^2$. Impose the continuity conditions (see prob^m ㉘ for II)...

I. ψ continuous @ $x=0$: $1 + A = B$. (3)

II. ψ' discontinuous @ $x=0$: $\psi_2'(0+) - \psi_1'(0-) = \frac{2mC}{\hbar^2}\psi(0)$,

i.e.// $ik[B - (1-A)] = \left(\frac{2mC}{\hbar^2}\right)B$, ^w// $1 - A = \left(1 - \frac{2mC}{ik\hbar^2}\right)B$. (4)

Add (3) & (4) to eliminate A. Get: $B = 1/[1 + i(mC/\hbar^2 k)]$. Then, using $(\hbar k)^2 = 2mE$, we find the transmission coefficient...

$$\rightarrow T = |B|^2 = 1/[1 + (mC^2/2\hbar^2 E)]. \qquad (5)$$

Put T of Eq. (5) to find the required trapping lifetime...

$$\boxed{\Delta t = \sqrt{\frac{2ma^2}{E}}\left[1 + (mC^2/2\hbar^2 E)\right]} \qquad (6)$$

As $E \rightarrow$ large, $\Delta t \rightarrow 0$... m escapes rapidly. $E \rightarrow 0 \Rightarrow$ trapping forever.