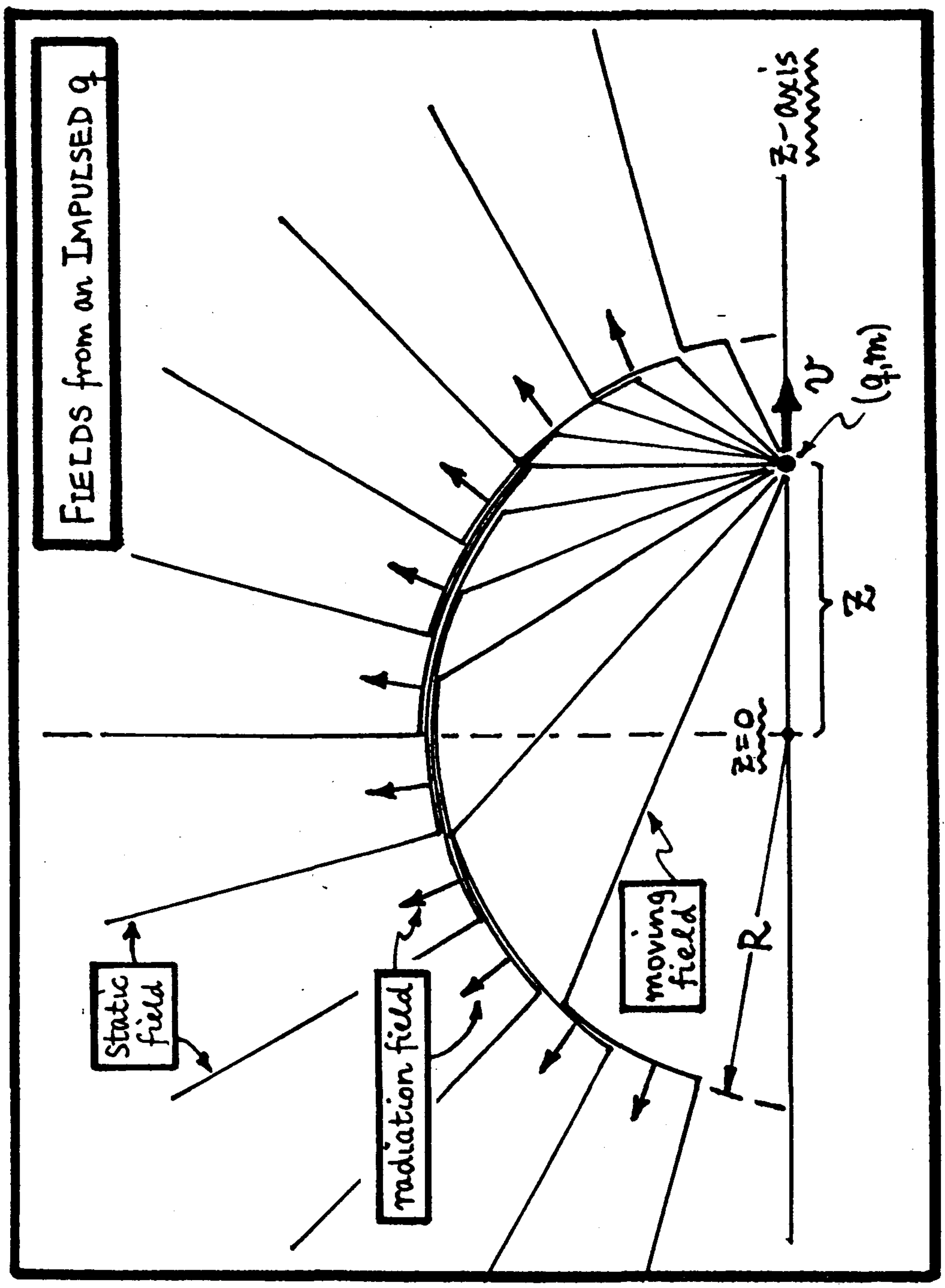
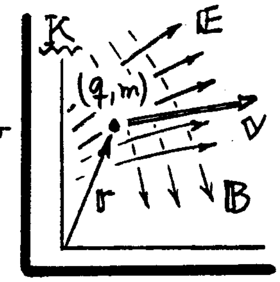
RADIATION REACTION



Incompleteness of the Lorentz Law as an Equation-of-Motion.

The Equation-of-Motion for a Single Charge q. Radiation Reaction.



1) For a classical (spinless) particle of charge q & mass m moving in lab (reference) frame K at velocity $\mathbf{v} = d\mathbf{r}/dt$ (t is K's time), through external electric & magnetic fields $\mathbb{E}$ & $\mathbb{B}$, an eqtn-of-motion is provided by the Lorentz Force Law [Jk$^{\underline{n}}$ Eq. (11.124), or Eq. (1) of ϕ520 notes p. COV1]:

$$\rightarrow \underline{\underline{\mathbb{F} = d\mathbf{p}/dt = q\left(\mathbb{E} + \frac{\mathbf{v}}{c} \times \mathbb{B}\right)}}. \qquad (1)$$

Here $\mathbf{p} = \gamma m \mathbf{v}$ is the particle's relativistic 3-momentum ($\gamma = \frac{1}{\sqrt{1-\beta^2}}$, $\beta = v/c$), and $\mathbb{E}$ & $\mathbb{B}$ are specified at K cds $\mathbf{r}$ & t. In addition to (1), we have...

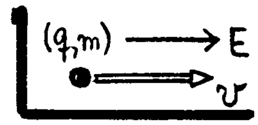
$$\rightarrow \underline{\underline{\frac{dK}{dt} = q\,\mathbb{E}\cdot\mathbf{v}}} \quad \text{w/}\ K = (\gamma-1)mc^2 = \text{particle's relativistic K.E.} \qquad (2)$$

Eq. (2) is the work-energy theorem for (q,m); it tells how fast the particle is gaining the mechanical energy K by action of the $\mathbb{E}$-field. Eqs. (1) & (2) can be combined into a single "manifestly covariant" force law [Jk$^{\underline{n}}$ Eq. (11.144), or Eqs. (7) & (26) of ϕ520 notes pp. COV 2 & 9]. For $F^{\alpha\beta}$ the field tensor...

$$\boxed{m\dot{u}^{\alpha} = (q/c)\,F^{\alpha\beta}\,u_{\beta}}, \qquad u^{\alpha} = \text{4-velocity} = \gamma(c, \mathbf{v}). \qquad (3)$$

The "dot" here means $d/d\tau$, w/ τ = particle's proper time ($d\tau = \frac{1}{\gamma}dt$).

A point to be made at the outset is that all the fields that appear in these eqtns-of-motion are external fields, created in K by sources outside (q,m)... nowhere in Eqs. (1)-(3) is there mention of (q,m)'s own (self) fields. Should there be? (q,m)'s self $\mathbb{E}$-field certainly doesn't accelerate the particle. BUT, these eqtns allow (q,m) to accelerate, and -- in accelerating -- (q,m) must radiate away EM energy, which is carried away and lost to the system [system = (q,m) + the external $\mathbb{E}$ & $\mathbb{B}$] by (q,m)'s own fields. A radiative loss term appears nowhere in Eqs. (1)-(3), but it should because such an energy loss must change (inhibit) the motion of (q,m). So, Eqs (1)-(3) cannot be a complete description of (q,m)'s motion -- there must be missing terms that account for (q,m)'s radiation... terms implicitly dependent on (q,m)'s self-fields.



2) What we are talking about can be seen clearly in the case of a 1D nonrelativistic motion of (q,m) accelerated in an external electric field E. According to Eq. (2), the field does work on the particle at rate qEv. This work will appear as a K.E. increase at rate $\frac{d}{dt}(\frac{1}{2}mv^2)$, and must also supply the radiation energy loss at rate P_{rad}. So Eq. (2) should read...

$$\rightarrow \frac{d}{dt}\left(\frac{1}{2}mv^2\right) + P_{rad} = qEv. \qquad (4)$$

For the radiation loss term we can try the Larmor rate: $P_{rad} = (2q^2/3c^3)\dot{v}^2$ [Jk^n Eq. (14.22)], so that (4) yields...

$$\frac{d}{dt}\left(\frac{1}{2}mv^2\right) + \frac{2q^2}{3c^3}\dot{v}^2 = qEv \quad \surd \text{ do the } \frac{d}{dt}\text{, divide by } v\text{, and define: } \tau_0 = 2q^2/3mc^3 \left\{\text{time scale}\right.$$

$$\text{n/} \quad \underline{\underline{m\dot{v} = qE - f_{RR}}} \quad \text{w/} \quad \underline{\underline{f_{RR} = m\tau_0(\dot{v}^2/v)}}. \qquad (5)$$

Eq. (5) is the force law in this case, now modified by a "radiation reaction" force f_{RR} that opposes (q,m)'s motion. f_{RR} is present because during the motion that produces an acceleration $\dot{v}$, the particle must radiate at rate $f_{RR}v$.

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f_{RR} in the form given in (5) is neither complete nor correct (see ASIDE), even non-relativistically. But the manner in which it enters the energy Eq. (4) and motion Eq. (5) is characteristic: f_{RR} represents a loss term that opposes the motion. It would enter the covariant Lorentz law of Eq. (3) as...

$$\boxed{m\dot{u}^\alpha = (q/c)F^{\alpha\beta}_{ext}u_\beta - f^\alpha_{RR}}. \qquad (6)$$

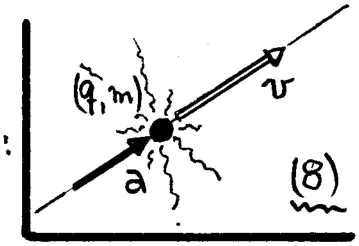
Solution for the Larmor correction in Eq.(5).

ASIDE Solutions to Eq.(5): 1D motion with Larmor loss term.

1. Write (5) as: $\dot{v} = a - (\tau_0/v)\,\dot{v}^2$ ∫ $a = qE/m$ is external acceleration, $\tau_0 = 2q^2/3mc^3$ is a scale time. (7)

2. If $a=0$, have: $\dot{v} = -v/\tau_0 \Rightarrow v(t) = v(0)\exp[-(t/\tau_0)]$. Peculiar... any moving charge spontaneously comes to a stop w/o being decelerated. What?

3. If $a \neq 0$, (7) is a quadratic eqtn for $\dot{v}$, w/ solution:



→ $\dot{v} = \frac{1}{2\tau_0}\left[\sqrt{v^2 + 4a\tau_0 v} - v\right]$ ∫ root sign chosen so that: $\dot{v} \to 0$ when $a \to 0$. (8)

This eqtn can be integrated to find $t = t(v)$. For $v = v_0$ @ $t=0$ & $a = $ cnst...

$\boxed{at = U(v) - U(v_0)}$ ∫ w/ $U(v) = u(v) + a\tau_0 \ln\left[1 + \frac{1}{a\tau_0}u(v)\right]$, and: $u(v) = \frac{1}{2}\left(v + \sqrt{v^2 + 4a\tau_0 v}\right)$. (9)

4. Analyse solution of Eq.(9) for both $\left\{\begin{array}{l}\text{acceleration}: a>0\\ \text{deceleration}: a<0\end{array}\right.$. Two cases of interest...

(A) Acceleration of (q,m) from rest: $v_0 = 0$, and $a = |a| > 0$.

at long times: $t \gg \tau_0,\ v \gg a\tau_0$ } $v(t) \simeq at - a\tau_0 \ln(t/\tau_0)$, $\dot{v}(t) \simeq a[1 - (a\tau_0/v)]$; (10A)

at short times: $t \ll \tau_0,\ v \ll a\tau_0$ } $v(t) \simeq \frac{1}{4}at^2/\tau_0$, $\dot{v}(t) \simeq \frac{1}{2}at/\tau_0$ ∫ NOTE: $\dot{v}^2/v = \frac{a}{\tau_0} =$ cnst. (10B)

The long-time motion is (nearly) Newtonian. Short-time motion is radically new.

NOTE: at short times: $f_{RR} = m\tau_0 \frac{\dot{v}^2}{v} \simeq ma$, which cancels $F_{ext} = ma$. Strange!

(B) Deceleration of a moving (q,m): $v_0 > 0$, and $a = -|a| < 0$.

Per Eq.(9): $|a|t = U(v_0) - U(v)$, but now with: $u(v) = \frac{1}{2}\left(v + \sqrt{v^2 - 4|a|\tau_0 v}\right)$.

At $t > 0$, $v(t)$ declines toward 0. BUT, when $v < 4|a|\tau_0$, the $\sqrt{\ }$ becomes imaginary; solution of Eq.(9) does not apply at the end of the motion, i.e. for velocities in the range: $4|a|\tau_0 > v > 0$. We cannot stop (q,m) in a sensible way.

Incompleteness of an f_{RR} based on radiation zone fields alone.

ASIDE (cont'd) 1D motion with Larmor loss term.

5. The glitch just noted -- our inability to account for the motion (or even whereabouts) of a charged particle which is being decelerated and which has slowed down to a velocity $v < 4|a|\tau_0$ -- certainly suggests that the Larmor form of a radiation reaction force, $f_{RR} = m\tau_0 \dot{v}^2/v$, as written in Eq. (5), is not yet Complete. Some small part of (q,m)'s inertial reaction to an external force is missing. Otherwise the motion analysed in Eqs. (7)-(10) seems plausible.

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3) What is missing in $f_{RR}(\text{Larmor}) = m\tau_0\dot{v}^2/v$ is very likely the effects of (q,m)'s fields "in close", i.e. in the static & induction zones of its radiation field. The Larmor radiation rate $f_{RR}v$ as written only accounts for the radiation zone fields, i.e. those parts of $\mathbb{E}(\text{self})$ & $\mathbb{B}(\text{self})$ which $\propto 1/R$, where R = distance between (q,m) and the observer. But (q,m)'s Lienard Wiechert fields [Jk^{n} Eqs (14.13) & (14.14)] contain parts $\propto 1/R^2$ as well as $1/R$, so a full expansion of (q,m)'s Poynting vector (from whence the radiation loss comes) is:

$$\rightarrow \mathbb{E}(\text{self}) \times \mathbb{B}(\text{self}) \propto \frac{1}{R^2}\begin{pmatrix}\text{radiation}\\ \text{zone}\end{pmatrix} + \frac{1}{R^3}\begin{pmatrix}\text{induction}\\ \text{zone}\end{pmatrix} + \frac{1}{R^4}\begin{pmatrix}\text{static}\\ \text{zone}\end{pmatrix}. \qquad (11)$$

We've used only the first term RHS: this surely constitutes radiation (loss) and surely does hinder (q,m)'s motion via the above f_{RR}. The 2nd & 3rd terms cannot be radiation (since $\oint (1/R^{3,4})\,dS \to 0$ for a large sphere at ∞)... but it would seem that they could contribute to (q,m)'s inertial reaction in an equation of motion. After all, the induction & static zone fields -- with their nontrivial energy densities -- must continually adjust their configurations during an externally applied acceleration.

We shall not pursue these notions (they quickly lead to questions re what happens as $R \to 0$... what is (q,m)'s structure). Instead, we review some orthodoxy.

Time & distance scales for RR. Arguments against "being concerned".

4) It is possible to argue that searching for correct & complete radiation reaction corrections in classical electrodynamics is largely irrelevant -- because these corrections are negligible in most cases of interest, or because quantum-mechanical limitations become dominant long before we reach the distance & time scales at which RR corrections become important. The arguments go as follows.

① Consider nonrelativistic 1D motion of (q,m) at cnst acceleration a. Starting from rest, in time $0 \to T$, the particle (q,m) exhibits:

$$\left[\begin{array}{l}\text{kinetic energy}: K = \frac{1}{2}m(aT)^2, \text{ radiated energy}: \mathcal{E}_{radn} = \left(\frac{2q^2}{3c^3}a^2\right)T;\\ \text{So } \mathcal{E}_{radn} \sim K \text{ only if } T \sim 2\tau_0, \text{ w/ } \underline{\underline{\tau_0 = \frac{2}{3}q^2/mc^3}}.\end{array}\right. \tag{12}$$

τ_0 is the scale time defined in Eq. (5)... it is extremely small for an elementary particle (e.g. $\underline{\tau_0 = \frac{2}{3}e^2/m_ec^3 \simeq 6.26\times10^{-24}\,\text{sec}}$, for an e). The distance traveled during the time when $\mathcal{E}_{radn}$ is significant is also very small; certainly $\ll$ classical charge radius $\underline{\underline{r_0 = q^2/mc^2}} = \frac{3}{2}c\tau_0$ (for an e: $\underline{r_0 = e^2/m_ec^2 = 2.82\times10^{-13}\,\text{cm}}$). Altogether, $\mathcal{E}_{radn}$ has a crucial role in the motion only over negligible times τ_0 & distances r_0.

② The time $\tau_0 = 2q^2/3mc^3$ and distance $r_0 = q^2/mc^2$ scales are QMly inaccessible for measurements on elementary particles. If we try measuring times down to $\Delta t \sim \tau_0$, then we will impart to (q,m) random energies of a size:

$$\rightarrow \Delta E \sim \hbar/\tau_0 = mc^2 \Big/ \left(\frac{2}{3}q^2/\hbar c\right), \tag{13}$$

according to the Uncertainty Principle. If $q = \pm e$, then $\Delta E \sim 205\,mc^2$, so q goes ballistic. Similarly, localization to $\Delta x \sim r_0 \Rightarrow$ uncontrollable momentum up to $\underline{\underline{\Delta p \sim \hbar/r_0 = mc/(q^2/\hbar c)}} \sim 137\,mc$, when $q = \pm e$. Conclusion: measurements at time & distance scales τ_0 & r_0 are impossible.