## $\mathcal{L}_{EM} \Rightarrow$ Source-dependent Maxwell Equations.

i.e.// $\nu=1$ Lagrange Eqtn here $\Rightarrow \left(\frac{1}{c}\mathbb{J} + \frac{1}{4\pi c}\dot{\mathbb{E}}\right)_1 = \frac{1}{4\pi}(\nabla\times\mathbb{B})_1$.

... $\nu=2,3$ eqtns $\Rightarrow$ the 2,3 components of Ampere's Law : $\boxed{\nabla\times\mathbb{B} = \frac{4\pi}{c}\mathbb{J} + \frac{1}{c}\left(\frac{\partial\mathbb{E}}{\partial t}\right)}$ (27)

In Covariant notation, what we have shown here is that...

field-source Lagrange density : $\mathcal{L}_{EM} = -\frac{1}{16\pi}F_{\mu\nu}F^{\mu\nu} - \frac{1}{c}J_\nu A^\nu$,

plus Lagrange Eqtns : $\partial^\mu[\partial\mathcal{L}_{EM}/\partial(\partial^\mu A^\nu)] = \partial\mathcal{L}_{EM}/\partial A^\nu$,

(with components of 4-potential $A^\nu = (\phi, \mathbb{A})$ as generalized cds)

imply the source-dept. Maxwell Eqtns : $\frac{1}{4\pi}\partial^\mu F_{\mu\nu} = \frac{1}{c}J_\nu$. (28)

**REMARK**

We get only the source-dept. Maxwell Eqtns out of the $\mathcal{L}_{EM}$ formalism. What has happened to the other two eqtns, viz $\nabla\cdot\mathbb{B}=0$ & $\nabla\times\mathbb{E} = -\frac{1}{c}\frac{\partial\mathbb{B}}{\partial t}$?

**ANS.** They are "trivially" satisfied by our choice of 4-potential $A^\nu = (\phi, \mathbb{A})$ (and the consequent form of the field tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$) such that Eq. (24) is satisfied, i.e. $\mathbb{E} = -\nabla\phi - \frac{1}{c}(\partial\mathbb{A}/\partial t)$, $\mathbb{B} = \nabla\times\mathbb{A}$. With this way of defining $\phi$ & $\mathbb{A}$, it is automatically true that the Maxwell fields obey $\nabla\cdot\mathbb{B} = \nabla\cdot(\nabla\times\mathbb{A}) = 0$, $\nabla\times\mathbb{E} = -\underbrace{\nabla\times(\nabla\phi)}_{\to 0} - \frac{1}{c}\frac{\partial}{\partial t}(\nabla\times\mathbb{A}) = -\frac{1}{c}\frac{\partial\mathbb{B}}{\partial t}$. From the standpoint of the 4 degrees-of-freedom inherent in the Maxwell field, Eq. (28) gives just as much -- and no more -- information as is needed.

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**19)** The utility of the $\mathcal{L}_{EM}$ formalism does not lie in regurgitating the Maxwell Eqtns -- this is just a check on whether $\mathcal{L}_{EM}$ generates the "right" eqtns-of-motion. The utility of the formalism does lie in being able to quickly decide -- covariantly, of course -- how modifications might be made to EM theory. An example is the Proca Lagrangian [Jk$^{n}$ Eq. (12.91)], including a photon mass:

## Proca Lagrangian for finite photon mass

$$\mathcal{L}_P = -\frac{1}{16\pi} F_{\alpha\beta} F^{\alpha\beta} - \frac{1}{c} J_\alpha A^\alpha + \frac{\mu^2}{8\pi} A_\alpha A^\alpha, \quad \text{w/ } \mu = \frac{mc}{\hbar} \quad \text{w/ } m = \text{finite photon mass.} \tag{29}$$

The photon mass term ③ is added to the usual $\mathcal{L}_{EM}$ (terms ① & ②) in an obviously covariant fashion (since $A_\alpha A^\alpha$ is a Lorentz scalar). The potential $A_\alpha$ venue is taken because the photon -- massive or not -- must mediate a nonlocalized field (represented by $A_\alpha$) rather than a source (like $J_\alpha$).† Eqtn-of-motion is:

$$\rightarrow \partial^\mu[\partial\mathcal{L}_P/\partial(\partial^\mu A^\alpha)] = \frac{\partial\mathcal{L}_P}{\partial A^\alpha} \Rightarrow \frac{1}{4\pi}\partial^\beta F_{\beta\alpha} = \frac{1}{c} J_\alpha - \frac{\mu^2}{4\pi} A_\alpha \tag{30}$$

usual Max. Eqs (from terms ① & ②). mass term (from ③ above).

With $F_{\beta\alpha} = \partial_\beta A_\alpha - \partial_\alpha A_\beta$, have...

$$\rightarrow \partial^\beta F_{\beta\alpha} = \underbrace{(\partial^\beta \partial_\beta)}_{ⓐ} A_\alpha - \partial_\alpha \underbrace{(\partial^\beta A_\beta)}_{ⓑ}$$

ⓐ: $\partial^\beta\partial_\beta = \Box = \frac{1}{c^2}\left(\frac{\partial}{\partial t}\right)^2 - \nabla^2$ {wave operator}

ⓑ: $\partial^\beta A_\beta = \frac{1}{c}\left(\frac{\partial\phi}{\partial t}\right) + \nabla\cdot\mathbb{A} = 0$ ★

(Lorentz Gauge -- req'd by charge consⁿ) (31)

... and so Eq. (30) reads...

$$\boxed{\Box A_\alpha + \mu^2 A_\alpha = (4\pi/c) J_\alpha} \leftarrow \text{Proca's Wave Eqtn} \tag{32}$$

The photon mass term in $\mu$, which we have previously put into the theory "by hand", now appears in the EM wave eqtn -- in virtually the only way possible.

**A solⁿ to (32)** Let: $A_\alpha(x,t) = \tilde{A}_\alpha(x)e^{i\omega t}$, so $\nabla^2\tilde{A}_\alpha - (\mu^2 - \frac{\omega^2}{c^2})\tilde{A}_\alpha = -\frac{4\pi}{c}\tilde{J}_\alpha e^{-i\omega t}$.

Take low freq. limit, $\omega \to 0$, so $\nabla^2\tilde{A}_\alpha - \mu^2\tilde{A}_\alpha = -\frac{4\pi}{c}\tilde{J}_\alpha$.

For a point $q$ at rest, $\tilde{J}_\alpha = 0$, so $\nabla^2\phi - \mu^2\phi = 0 \Rightarrow \boxed{\phi(r) = \frac{q}{r}e^{-\mu r}}$. (33)

Eq. (33) shows how a finite photon mass $\mu$ modifies the EM Coulomb potential. In the general realm of field theory, such a "Yukawa-type" potential will result for any charge-current $\mathcal{L}$ when the field quantum (photon) has finite mass.

† The other obvious add-on, $J_\alpha J^\alpha$, does not give anything new in $\mathcal{L}_P$, because the $\mathcal{L}_P$ eqtns-of-motion involve only derivatives w.r.t. $A^\alpha$. ★ See p 520 Prob. ⑯.