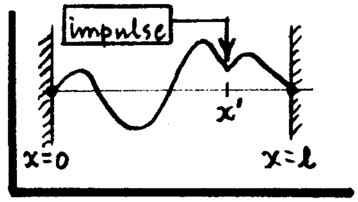
(11) [15 pts]. The Green's function $G(x,x')$ for a 1D simple harmonic oscillator (SHO) is defined by a point-source equation. With k = constant the system wavenumber...

$$\boxed{\frac{d^2G}{dx^2} + k^2 G = \alpha\,\delta(x-x'), \text{ on } [0,\ell].}$$



α is a constant available for normalization. The x-interval is $[0,\ell]$, and the endpoints are fixed, so $G \equiv 0$ @ $x=0$ & $x=\ell$. G is continuous on $[0,\ell]$.

(A) Show that : $G(x,x') = \begin{cases} A(x') \sin kx, & \text{for } 0 \leq x < x', \\ B(x') \sin k(\ell - x), & x' < x \leq \ell. \end{cases}$ ‖ $A(x')$ & $B(x')$ are to be determined.

(B) $G' = dG/dx$ is not continuous on $[0,\ell]$. Show that the discontinuity in G' at $x = x'$ is measured by : $\lim_{\lambda\to 0} [G'(x'+\lambda, x') - G'(x'-\lambda, x')] = \alpha$.

(C) Now find the coefficients $A(x')$ & $B(x')$ explicitly.

(D) What is G good for? Show that for the inhomogeneous SHO problem (on $[0,\ell]$, w/ $y(0) = y(\ell) = 0$), viz. $y'' + k^2 y = F(x)$, a particular integral is readily provided by : $y(x) = \frac{1}{\alpha}\int_0^\ell G(x,x')\,F(x')\,dx'$. What choice of α should you make?

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(12) [Jackson Prob. (2.1)], [15 pts]. Point charge q at distance d from ∞ conducting plane. By images, find : (A) σ (surface-charge density), and plot ; (B) force ($q \to$ plane) by Coulomb's Law ; (C) force ($q \to$ plane) by σ integration ; (D) work to remove q : $W(d \to \infty)$; (E) potential energy between q and its image ; (F) find work of part (D) in eV when $d = 1$ Å.

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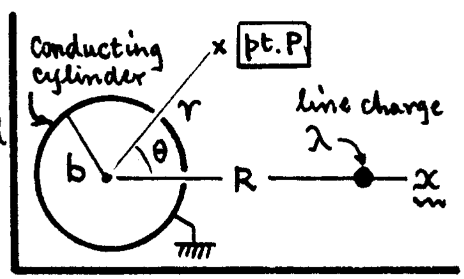
(13) [~Jackson Prob. (2.7)], [15 pts]. A long line with uniform charge density λ lies ‖ to the axis of a long, grounded conducting cylinder of radius b; the separation is $R > b$.



(A) Find the magnitude and position of the image charge λ'.

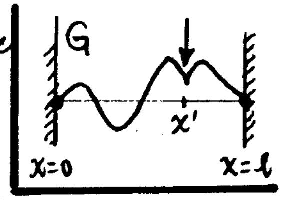
(B) Find the potential ϕ[pt. P] in the polar coordinates shown ($r > b$). As $r \to \infty$, $\phi(r) \sim$ what?

(C) Find the surface charge density $\sigma(b,\theta)$ induced on the cylinder. Plot σ vs. θ for $\frac{R}{b} = 2$.

(D) Find the force acting between the line and the cylinder.

(11) [15 pts]. Derive Green's Fcn $G(x,x')$ for SHO problem: $y''+k^2y=F(x)$ $\begin{cases} x\in[0,\ell], \\ \text{pegged.}\end{cases}$

(A) G defined via: $G''+k^2G=\alpha\,\delta(x-x')$, k & α = consts. Everywhere but at $x=x'$, G satisfies homogeneous eqtn: $G''+k^2G=0$, with obvious solutions: $G\propto \sin kx$, $\cos kx$. To satisfy boundary conditions, viz $G\equiv 0$ @ $x=0$ & $x=\ell$, evidently: $G\propto \sin kx$, $x<x'$ & $G\propto \sin k(\ell-x)$, $x>x'$.



Add multiplicative coefficients A & B (which can depend on x'): $\boxed{G(x,x')=\begin{cases} A\sin kx, & \text{for } 0\leq x<x'; \\ B\sin k(\ell-x), & x'<x\leq \ell.\end{cases}}$ (1)

(B) Integrate the G'' eqtn in neighborhood of x':

$$\underbrace{\int_{x'-\lambda}^{x'+\lambda}\frac{d}{dx}(G')\,dx}_{①} + \underbrace{k^2\int_{x'-\lambda}^{x'+\lambda} G\,dx}_{②} = \alpha\int_{x'-\lambda}^{x'+\lambda}\delta(x-x')\,dx = \alpha$$

With G continuous on $[0,\ell]$, term ② vanishes as $\lambda\to 0$. Term ① integrates directly for desired result.

So $\to$ $\underline{G'(x'+\lambda,x')-G'(x'-\lambda,x')}=\alpha$, as $\lambda\to 0$. (2)

(C) Now have two conditions to find the two coefficients A & B in Eq. (1), viz.

G is continuous at $x=x'$: $B\sin k(\ell-x')=A\sin kx'$,

G' discontinuous at $x=x'$: $-kB\cos k(\ell-x')-kA\cos kx'=\alpha$. (3)

Simple arithmetic (plus a trig identity) gives the solutions...

$$\boxed{A=-\frac{\alpha}{k}\frac{\sin(\ell-x')}{\sin k\ell},\quad B=-\frac{\alpha}{k}\frac{\sin kx'}{\sin k\ell};} \quad (4)$$

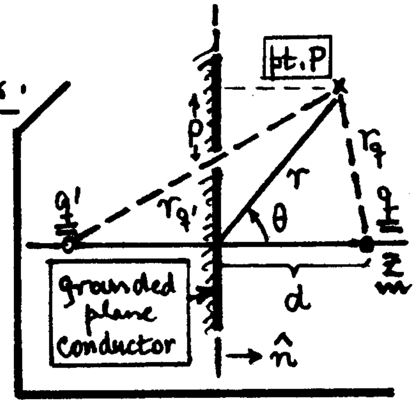
So $$G(x,x')=(-)\frac{\alpha}{k\sin k\ell}\cdot\begin{cases}\sin kx\,\sin k(\ell-x'), & 0\leq x\leq x'; \\ \sin kx'\sin k(\ell-x), & x'\leq x\leq \ell.\end{cases}$$

[redacted] G is symmetric: $G(x',x)=G(x,x')$. (5)

(D) $G''+k^2G=\alpha\,\delta(x-x')$, $y''+k^2y=F(x)$. ‖ Multiply 1st eqtn on left by y, 2nd eqtn on left by G, and subtract to get: $yG''-Gy''=\alpha y\,\delta(x-x')-GF$.

Recognize LHS term as a derivative, viz. $\frac{d}{dx}(yG'-Gy')$. Now interchange x & x'; both G and the δ-fcn are unaffected. Integrate $\int_0^\ell dx'$, noting that the term $(yG'-Gy')\big|_{x'=0}^{x'=\ell}=0$, by the boundary conditions. Then: $\boxed{y(x)=\frac{1}{\alpha}\int G(x,x')F(x')\,dx'}$, as advertised. Since G itself contains α (Eq. (5)), α drops out; it is unimportant.

(12) [15 pts]. Analyse pointcharge – grounded plane problem by images.



The image $q' = (-)q$ is located at distance d to the left.

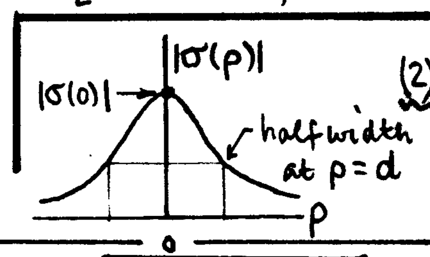
(A) In terms of spherical polar cds (r, θ) as shown, potential is:

$$\rightarrow \phi = \frac{q}{r_q} - \frac{q}{r_{q'}}, \quad \text{w/} \quad r_q = (r^2 + d^2 - 2rd\cos\theta)^{1/2}, \quad r_{q'} = (r^2 + d^2 + 2rd\cos\theta)^{1/2}. \qquad (1)$$

The $\mathbb{E}$-field is everywhere normal to the plane, and (by Gauss' Law) the surface charge density in the plane obeys: $4\pi\sigma = \mathbb{E}\cdot\hat{n} = (-)\hat{n}\cdot\nabla\phi\big|_{\text{plane}}$. Since by symmetry ϕ does not depend on the azimuthal ∢ about the z-axis, then...

$$\rightarrow \sigma = -\frac{1}{4\pi}\hat{n}\cdot\left(\hat{e}_r\frac{\partial\phi}{\partial r} + \hat{e}_\theta\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right)\Big|_{\theta=\frac{\pi}{2}} = +\frac{1}{4\pi r}\left(\frac{\partial\phi}{\partial\theta}\right)\Big|_{\theta=\frac{\pi}{2}} \quad \text{in the plane: } \hat{n}\cdot\hat{e}_r = 0,\ \hat{n}\cdot\hat{e}_\theta = -1;$$

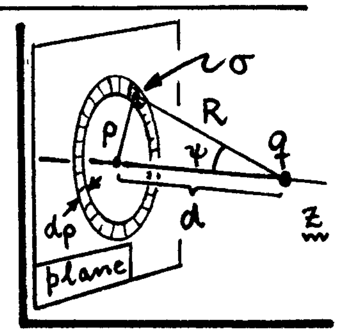
$$\text{w/} \quad \boxed{\sigma(\rho) = (-)\frac{qd}{2\pi}\Big/(\rho^2+d^2)^{3/2}} \quad \rho = \text{radius in xy-plane} \Rightarrow \qquad (2)$$



$\sigma(\rho) = \sigma(0)/[1+(\rho/d)^2]^{3/2}$, $\sigma(0) = -q/2\pi d^2$, is sketched...

(B) q & image $q' = (-)q$ are separated by distance $2d$, so force: $\boxed{F_{qq'} = (-)\frac{q^2}{4d^2}}$ (3) by Coulomb's Law. $F_{qq'}$ is of course attractive.

(C) Find $F_{qq'}$ by integrating over σ. The (attractive) force between q and the annular ring $\rho \to \rho + d\rho$ as shown is clearly:



$dF_{q\sigma} = -\frac{q}{R^2}(|\sigma| 2\pi\rho\, d\rho)\cos\psi$; only the component along z-axis counts. With $R = (\rho^2 + d^2)^{1/2}$ and $\cos\psi = \frac{d}{R}$, total force is:

$$\rightarrow F_{q\sigma} = \int_{\rho=0}^{\rho=\infty} dF_{q\sigma} = -\cancel{2\pi}q\int_0^\infty \frac{1}{R^2}\left(\frac{qd}{\cancel{2\pi}}\frac{1}{(\rho^2+d^2)^{3/2}}\right)\left[\frac{d}{R}\right]\rho\, d\rho = -q^2d^2\int_0^\infty \frac{\rho\, d\rho}{(\rho^2+d^2)^3},$$

$$\text{or} \quad \boxed{F_{q\sigma} = (-)q^2/4d^2 \equiv F_{qq'}}. \qquad (4)$$

The force attracting q to the plane is "supplied" equivalently by <u>either</u> the image q' <u>or</u> the induced surface charge σ.

⑫ (cont'd)

(D) The attractive force $q \to$ plane at distance z is $F = -q^2/4z^2$. Work done to move q from $z=d$ to $z=\infty$ is then...

$$\rightarrow W(d\to\infty) = -\int_{z=0}^{z=\infty} F\,dz = \frac{q^2}{4}\int_d^\infty \frac{dz}{z^2}, \text{ i.e. } \boxed{W(d\to\infty) = q^2/4d}. \quad (5)$$

(E) The ptl. of q (at position d) in the presence of its image $q' = -q$ is...

$$\varphi = -\frac{q}{r_{q'}}\bigg|_{\substack{r=d\\ \theta=0}}, \text{ or } \boxed{\varphi = -q/2d}. \quad (6)$$

So the qq' P.E. is: $q\varphi = -q^2/2d$. It is not true for this φ that the energy-of-assembly is $q\varphi = -W(d\to\infty)$, since the image charge moves when q & q' (or σ) are brought together. The average P.E. during assembly, viz. $\frac{1}{2}q\varphi = -q^2/4d$, does agree with $(-)W(d\to\infty)$ of Eq. (5).

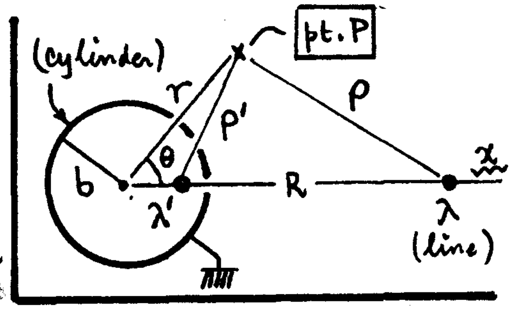
(F) For $q=-e$, $e = 4.803\times10^{-10}$ esu the electronic charge, and d measured in Å ($1\,\text{Å} = 10^{-8}$ cm), W of Eq. (5) is [in cgs units]:

$$W = (4.803\times10^{-10})^2/4d\times10^{-8} = (5.77\times10^{-12})/d, \text{ ergs}. \quad (7)$$

But: $1\,\text{erg} = 10^{12}/1.602$ eV, so the work is

$$\boxed{W = 3.60/d,\ \text{eV}}, \ d \text{ is in Å}. \quad (8)$$

For $d = 1\,\text{Å}$, $W = 3.60$ eV. This is (at least) the order-of-magnitude of the "work function" for typical metals.



⑬ [15 pts] Line charge $\lambda \parallel$ axis of grounded cylinder.

This is the 2D analog of the pointcharge – sphere problem.

(A) The potential at radial distance ρ from an isolated line charge λ is: $\phi(\rho) = -2\lambda \ln\rho + \phi_0$, $\phi_0 = $ cnst.

With an image line λ' positioned on x-axis @ $R' < b$, potential at pt $P(r \geqslant b)$ is:

$$\rightarrow \phi_P = \phi_0 - 2[\lambda \ln\rho + \lambda' \ln\rho'] \quad \text{w/} \quad \rho = \sqrt{R^2 + r^2 - 2Rr\cos\theta}, \quad \rho' = \sqrt{R'^2 + r^2 - 2R'r\cos\theta}. \qquad (1)$$

At the cylinder surface $(r = b)$, impose $\phi_P \equiv 0 \Rightarrow \phi_0 = 2[\lambda \ln\rho_s + \lambda' \ln\rho_s'] = $ cnst, w/ ρ_s & ρ_s' the values at $r = b$. In order that ϕ_0 does not depend on $\cos\theta$, evidently $\rho_s' = (\text{cnst}) \times \rho_s$, and $\lambda' = (-)\lambda$. Correct placement of λ' is at $R' = b^2/R$, whence

$$\boxed{\lambda' = (-)\lambda, @\ R' = b^2/R} \Rightarrow \rho_s' = (b/R)\rho_s, \text{ and: } \phi_0 = 2\lambda \ln(R/b). \qquad (2)$$

(B) Plug into Eq. (1) to find the complete potential at pt. $P(r \geqslant b)$...

$$\rightarrow \phi_P = 2\lambda \ln\left(\frac{R}{b}\right) - 2\lambda[\ln\rho - \ln\rho'], \text{ or: } \boxed{\phi(r,\theta) = 2\lambda \ln\left[\left(\frac{R}{b}\right)\frac{\rho'}{\rho}\right]}. \qquad (3)$$

ρ is defined in Eq. (1); ρ' there is now evaluated with $R' = b^2/R < R$. As $r \to \infty$, it is easy to show: $\rho'/\rho \simeq 1 + \frac{1}{r}(R - R')\cos\theta$, to 1ST order in $\frac{R}{r} \ll 1$. Then:

$$\rightarrow \phi(r,\theta) \simeq 2\lambda\left[\frac{R}{b}\left\{1 + \frac{R}{r}\left(1 - \frac{b^2}{R^2}\right)\cos\theta\right\}\right] \simeq \underbrace{\phi_0}_{\text{cnst}} + \underbrace{\frac{2\lambda R}{r}\left(1 - \frac{b^2}{R^2}\right)\cos\theta}_{\text{dipole term}}, \quad r \to \infty \qquad (4)$$

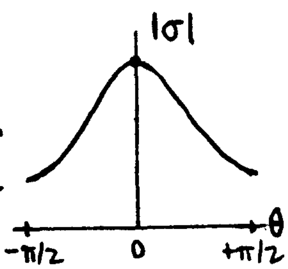
(C) As usual, the surface charge density on a conductor obeys: $4\pi\sigma = -\hat{n} \cdot \nabla\phi|_{\text{surface}}$, or

$$\sigma(b,\theta) = -\frac{\lambda}{2\pi}\frac{\partial}{\partial r}\ln\left[\left(\frac{R}{b}\right)\frac{\rho'}{\rho}\right]\Big|_{r=b} = \ldots$$

or//

$$\boxed{\sigma(b,\theta) = (-)\frac{\lambda}{2\pi b}\,\frac{N^2 - 1}{N^2 + 1 - 2N\cos\theta}, \quad N = \frac{R}{b}}. \qquad (5)$$

for $N = 2$, have... $|\sigma| = \frac{\lambda}{2\pi b}\left(\frac{3}{5 - 4\cos\theta}\right)$. Sketched



D) The force between the lines λ & λ' is attractive, and of size $\mathcal{F} = \lambda|E_{\lambda'}|$, per unit length. Since the distance between λ & λ' is $R - R'$, have: $\boxed{\mathcal{F} = -[2\lambda^2 R/(R^2 - b^2)]\,\hat{x}}$.

♠ See any elementary E&M text. The field is: $\mathbf{E} = -\nabla\phi = (2\lambda/\rho)\hat{\rho}$.