DEPARTMENT OF PHYSICS

M.S. COMPREHENSIVE / PH. D. QUALIFYING EXAMINATION

MARCH 27, 1989

1. Find the general solution of the differential equation:

$$a^2 y''^2 = (1 + y'^2)^3$$

where a is a constant.

Q. Find the general solution:

$$a^2 y''^2 = (1+y'^2)^3$$

where a is a constant.

A. Note that y is absent. Let $u = y'$

$$a^2 u'^2 = (1+u^2)^3$$

$$\frac{u'}{(1+u^2)^{3/2}} = \frac{1}{a}$$

Integrate both sides

$$\frac{u}{(1+u^2)^{1/2}} = \frac{x-c}{a}$$

where C is an integration constant.
Solving for u,

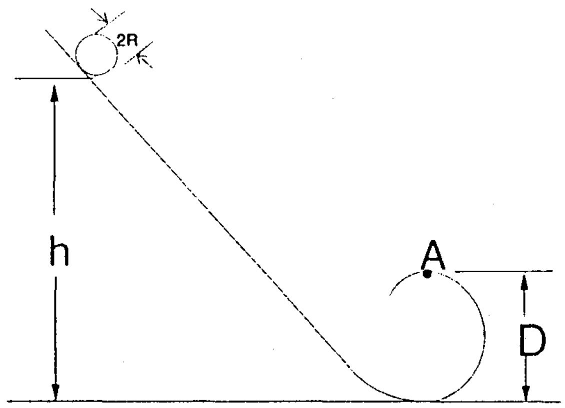
$$u = \frac{x-c}{\sqrt{a^2-(x-c)^2}} = \frac{dy}{dx}$$

Integrate again:

$$y = -\sqrt{a^2-(x-c)^2} + D$$

$$\boxed{(y-D)^2 + (x-c)^2 = a^2}$$

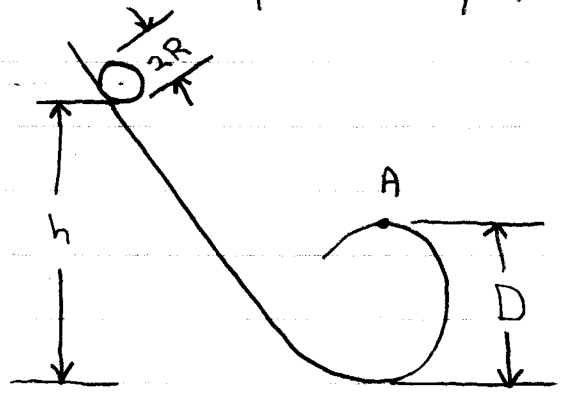
2. Consider a uniform density cylinder of mass M, radius R and length L which rolls without slipping down an incline from rest at an initial height h. If the lower part of the incline is cylindrical in shape with diameter D (with R << D), what is the minimum initial height that will guarantee that the cylinder does not leave the track at the top of the loop at point A?



Lee Lindblom

Mechanics

Consider a uniform density cylindar of mass M, radius R and length L which rolls without slipping down an incline from rest at an initial height h. If the lower part of the incline is cylindrical in shape with diameter D (with $R << D$), what is the minimum initial height that will guarantee that the cylinder does not leave the track at the top of the loop at point A?



Solution:

The kinetic energy of the cylinder has a translational part

$$T_T = \tfrac{1}{2} M v^2$$

where v is the velocity of the center of the cylinder, and a rotational part

$$T_R = \tfrac{1}{2}(2\pi L)\int_0^R \left(\frac{M}{\pi R^2 L}\right) r^2 \omega^2 r\,dr = \tfrac{1}{4} M R^2 \omega^2$$

where $\omega = v/R$ is the angular velocity of the cylindar. The total kinetic energy of the cylindar, therefore, is given by:

$$T = T_T + T_R = \tfrac{3}{4} M v^2$$

The gravitational potential energy of the cylinder is

$$V = -Mg(h-z)$$

where z is the height of the cylinder at a given point.

Since energy is conserved in this system, the velocity of the cylinder at the top of the loop will be:

$$E = 0 = \tfrac{3}{4}Mv^2 - Mg(h-D) \quad \text{for} \quad R \ll D$$

$$v^2 = \tfrac{4}{3}g(h-D)$$

If the cylinder does not leave the track, it will be accelerated around the lower loop at a rate

$$a = v^2/\tfrac{1}{2}D \qquad \text{for} \quad R \ll D$$

This acceleration must be a result of the gravitational force on the cylinder, Mg, and the contact force of the track F_{track}:

$$Ma = 2Mv^2/D = F = Mg + F_{track} \geq Mg$$

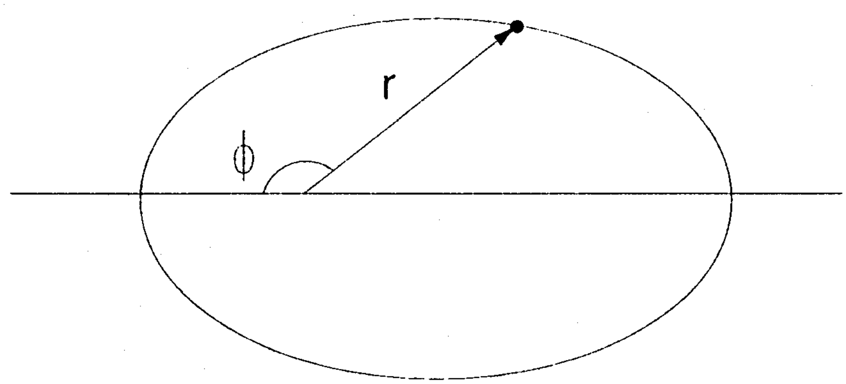
Therefore $$\frac{2v^2}{D} = \frac{8}{3}g\left(\frac{h}{D}-1\right) \geq g$$

$$\Rightarrow \boxed{h \geq \frac{11}{8}D}$$

3. A particle moves in a spherical potential V(r) along an elliptical path with one focus at the center of the potential: *i.e.*

$$r = \frac{a(1-\varepsilon^2)}{1+\varepsilon\cos\phi}$$

where a and $\varepsilon \neq 0$ are constants. Use the conservation of energy and angular momentum to find the forms of the potential energy that are consistent with this type of orbit.



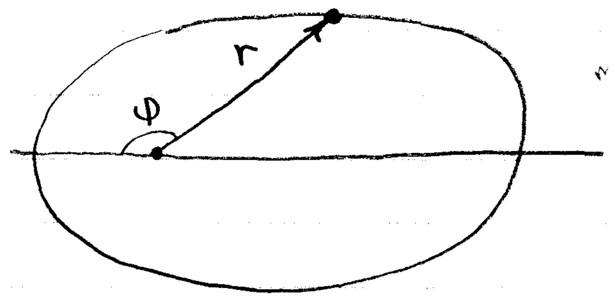
Lee Lindblom

Mechanics

A particle moves in a spherical potential $V(r)$ along an elliptical path with one focus at the center of the potential: i.e.

$$r = \frac{a(1-\epsilon^2)}{1+\epsilon\cos\varphi}$$

where a and $\epsilon \neq 0$ are constants. Use the conservation of energy and angular momentum to find the forms of the potential energy that are consistent with this type of orbit.



Solution:

The kinetic energy of the particle is

$$T = \tfrac{1}{2} m \left(\dot r^2 + r^2 \dot\varphi^2\right)$$

and the angular momentum is

$$J = m r^2 \dot\varphi$$

Use the equation for the orbit to express $\dot r$ in terms of $\dot\varphi$ and replace the $\dot\varphi$'s by using the angular momentum:

(2)

$$\dot{r} = \frac{a\epsilon(1-\epsilon^2)\sin\varphi\,\dot{\varphi}}{(1+\epsilon\cos\varphi)^2}$$

Therefore:

$$T = \tfrac{1}{2}m(\dot{r}^2 + r^2\dot{\varphi}^2)$$

$$= \tfrac{1}{2}m\left\{\frac{a^2\epsilon^2(1-\epsilon^2)^2\sin^2\varphi}{(1+\epsilon\cos\varphi)^4} + \frac{a^2(1-\epsilon^2)^2}{(1+\epsilon\cos\varphi)^2}\right\}\dot{\varphi}^2$$

$$= \tfrac{1}{2}m\,\frac{a^2(1-\epsilon^2)^2}{(1+\epsilon\cos\varphi)^4}\left\{\epsilon^2\sin^2\varphi + (1+\epsilon\cos\varphi)^2\right\}\frac{J^2}{m^2r^4}$$

$$= \tfrac{1}{2}m\,\frac{r^4}{a^2(1-\epsilon^2)^2}\left\{1+2\epsilon\cos\varphi+\epsilon^2\right\}\frac{J^2}{m^2r^4}$$

$$= \frac{1}{2m}\,\frac{J^2}{a(1-\epsilon^2)}\left\{2\,\frac{1+\epsilon\cos\varphi}{a(1-\epsilon^2)} - \frac{1}{a}\right\}$$

$$= \frac{1}{2m}\,\frac{J^2}{a(1-\epsilon^2)}\left\{\frac{2}{r} - \frac{1}{a}\right\}$$

The total energy of the particle is the sum of the kinetic and potential energies:

$$E = T + V(r)$$

Thus $$V(r) = E - \frac{1}{2m}\,\frac{J^2}{a(1-\epsilon^2)}\left\{\frac{2}{r} - \frac{1}{a}\right\}$$

Thus the general form of the potential energy consistent with elliptical orbits with one focus at the center of the potential is

$$V(r) = V_0 + \frac{K}{r}$$

where V_0 and K are constants.

4. The Berkeley Bevatron was designed to produce proton-antiproton pairs by bombarding stationary protons with high-energy protons. The nuclear physicist would write this as

$$p + p \rightarrow p + p + (p + \bar{p})$$

Each particle has rest mass mc^2, so $2(mc^2)$ units of rest mass must be created. Calculate the minimum kinetic energy required in the laboratory system in order that the reaction may go (this is called the threshold energy).

PROBLEM:

The Berkeley Bevatron was designed to produce proton-antiproton pairs by bombarding stationary protons with high-energy protons. The nuclear physicist would write this as

$$p + p \rightarrow p + p + (p + \bar{p})$$

Each particle has rest mass mc^2, so $2(mc^2)$ units of rest mass must be created. Calculate the minimum kinetic energy required in the laboratory system in order that the reaction may go (This is called the threshold energy).

~~(Hint: Solve the problem in the center of mass system and then transform back to the laboratory.)~~

2 relativistic energy
2 mom conserved
3 energy conserved
4 solve for p or T in lab

 Smith

<u>Solution</u> $p + p \longrightarrow p + p + (p + \bar{p})$

(1) Easy way: First solve in cm system then transform back.

In CM System: Before collision

$P_{TOTAL} = 0$

$E_{TOTAL} = 2\gamma_{cm} mc^2$

AFTER Collision: $P_{TOT} = 0$ (all 4 particles at rest)

$E_{TOT} = 4\,mc^2$

$\therefore\ 2\gamma_{cm} mc^2 = 4\,mc^2 \Rightarrow \boxed{\gamma_{cm} = 2}$

In LAB:

Before collision $(T_p + m_pc^2) + m_pc^2 = E_{TOT}$

After collision $E_{TOT} = 4\gamma_{cm} m_pc^2$

(4 particles moving at v_{cm})

so $T_p + 2m_pc^2 = 8m_pc^2$

$\boxed{T_p = 6m_pc^2}$ Threshold K.E for $p\bar{p}$ creation

(2) Harder way (more direct?) Solve in lab frame.

$\vec{P_i}$ at rest

Before

P_{final}

After

Write $P_i = 4P_{final}$

Energy conserv. $\left[p_i^2c^2+m^2c^4\right]^{1/2} + mc^2 = 4\left[p_f^2c^2+m^2c^4\right]^{1/2}$

Solve for p_f: using $p_i = 4p_f$

$$16p_f^2c^2+m^2c^4+m^2c^4+2mc^2\left[16p_f^2c^2+m^2c^4\right]^{1/2}$$
$$= 16p_f^2c^2+m^2c^4\,16$$

$$\left[16p_f^2c^2+m^2c^4\right]^{1/2} = 7mc^2$$

Solve:

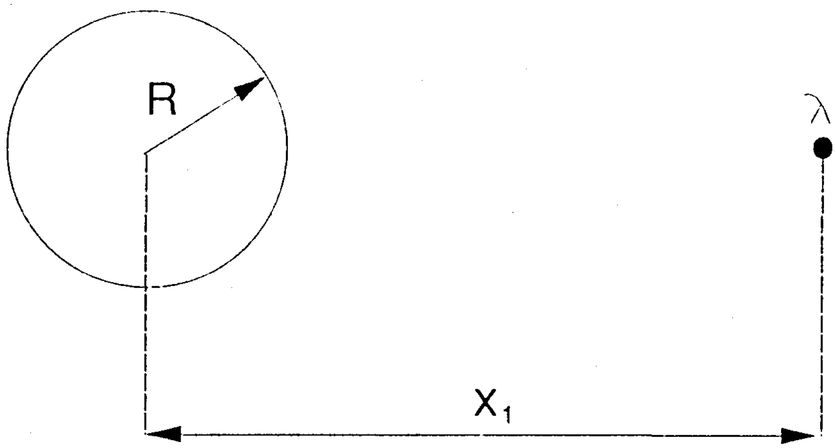
$$\Rightarrow \quad p_f^2c^2 = 3m^2c^4$$

Then $T_i + mc^2 = \left[p_i^2c^2+m_i^2c^2\right]^{1/2}$

$$= \left[16p_f^2c^2+m^2c^4\right]^{1/2}$$
$$= \left[48m^2c^4+m^2c^4\right]^{1/2}$$
$$= 7mc^2$$

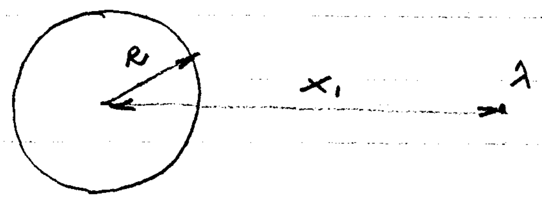
$\boxed{T_i = 6mc^2}$ as in case 1

5. Consider a thin wire with a charge per unit length λ placed in front of a perfect conductor in the shape of a very long cylinder of radius R. The cylinder is kept at a constant potential ϕ_0. Using the method of images, obtain the electrostatic potential for all points of space outside the cylinder.
Hint: Find the location of the image wire inside the cylinder.

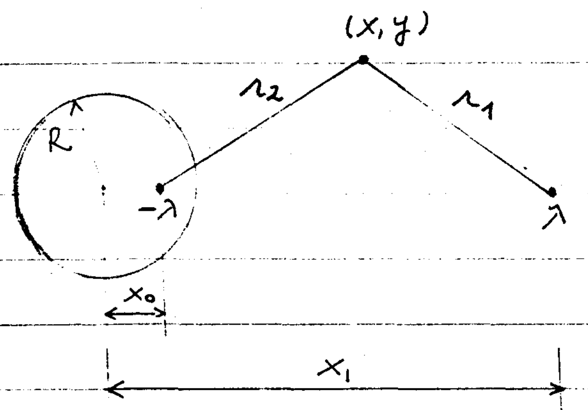


Consider a thin wire with a charge per unit length λ placed in front of a perfect conductor in the shape of a very long cylinder of radius R. The cylinder is kept at a constant potential ϕ_0. Using the method of images, obtain the electrostatic potential for all points of space outside the cylinder.

Hint: Find the location of the image wire inside the cylinder, ~~such that the surface of the cylinder is an equipotential surface.~~



i) The image charge set up in the conductor will be a line of charge with (linear) density $-\lambda$. By symmetry, it must be located as shown in the figure. The distance x_0 is found as follows.



Superposing the potentials due to each line of charge we have that

$$\Phi(x,y) = -2\lambda \ln r_1 + 2\lambda \ln r_2$$

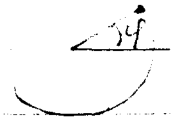
$$= -\lambda \ln[(x-x_1)^2+y^2] + \lambda \ln[(x-x_0)^2+y^2]$$

We impose the condition that the surface of the cylinder must be an equipotential surface:

$$\left.\frac{\partial}{\partial\varphi}\Phi(x,y)\right|_{x^2+y^2=R^2} =$$

$$= -\lambda \frac{2x_1 R \sin\varphi}{x_1^2+R^2-2x_1R\cos\varphi} + \lambda \frac{2x_0 R\sin\varphi}{x_0^2+R^2-2x_0R\cos\varphi}$$

$$= 0$$



where φ is the polar angle.

Then:

$$\frac{x_0}{x_0^2 + R^2 - 2x_0 R\cos\varphi} = \frac{x_1}{x_1^2 + R^2 - 2x_1 R\cos\varphi}$$

$\therefore$

$$1 + \left(\frac{R}{x_0}\right)^2 - 2\left(\frac{R}{x_0}\right)\cos\varphi =$$

$$= \frac{x_1}{x_0}\left[1 + \left(\frac{R}{x_1}\right)^2 - 2\left(\frac{R}{x_1}\right)\cos\varphi\right]$$

Note that the coefficients of $\cos\varphi$ are the same on both sides of the equation. This guarantees that the required condition is fulfilled for all points of the surface (i.e., for all φ). Mathematically, this is a consequence of our having guessed that the image charge has linear density $-\lambda$. If we had left this linear density unspecified, i.e., if we had called it λ', the requirement that the coefficients of $\cos\varphi$ be equal would have given us $\lambda' = -\lambda$.

Then:

$$1 + \left(\frac{R}{x_0}\right)^2 = \frac{x_1}{x_0}\left[1 + \left(\frac{R}{x_1}\right)^2\right]$$

$$\therefore \quad (x_0 - x_1) = \frac{R^2}{x_1 x_0}(x_0 - x_1)$$

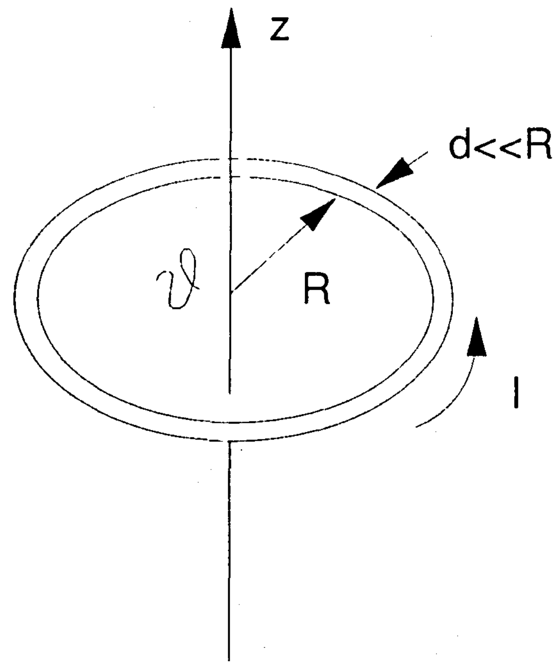
$$\Rightarrow \quad \frac{R^2}{x_1 x_0} = 1$$

Thus the image charge density $-\lambda$ must be placed at

$$x_0 = \frac{R^2}{x_1}$$

ii) Since the solutions of Laplace's equation can be superposed, and by virtue of the uniqueness theorem for the solution of this equation satisfying a Dirichlet boundary condition (Φ = constant on the boundary), when the cylinder is kept at a finite potential Φ_0, all we have to do is add to the above solution the difference $(\Phi_0 - \Phi_{cyl.})$, where Φ_{cyl} is the value of the potential at the cylinder due to the two linear charges.

6. a) State Ampere's Law and the Biot-Savart Law. Which Law would you use to calculate $\vec{B}$ at arbitrary z along the axis of the current loop shown, with z=0 in the plane of the loop?



b) Find $\vec{B}(z)$.

E&M Problem

(a) State Ampere's Law and the Biot-Savart Law.

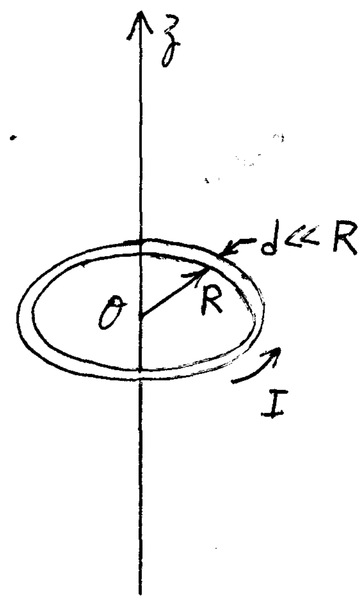
Ampere's Law in MKS units is

$$\oint_c \vec{B}\cdot d\vec{l} = \mu_0 \int_s \vec{J}_f \cdot d\vec{a} = \mu_0 I_{encl}.$$

to "Solution"

The Biot-Savart Law in MKS units is

$$\vec{B} = \frac{\mu_0}{4\pi} I \oint \frac{d\vec{l}\times\hat{r}}{r^2}.$$



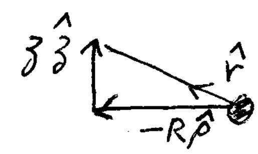
(a) Which Law would you use to calculate $\vec{B}$ at arbitrary z along the axis of the current loop shown, with $z=0$ in the plane of the loop?

(b) Find $\vec{B}(z)$.

E+M Solution

(a) Biot-Savart Law; symmetry required for use of Ampere's Law is lacking!

(b) In $\vec{B} = \frac{\mu_0}{4\pi} I \oint \frac{d\vec{l} \times \hat{r}}{r^2}$,



$$d\vec{l} = R\hat{\phi}\, d\phi, \quad \hat{r} = \frac{-R\hat{\rho} + z\hat{z}}{\sqrt{R^2+z^2}}$$

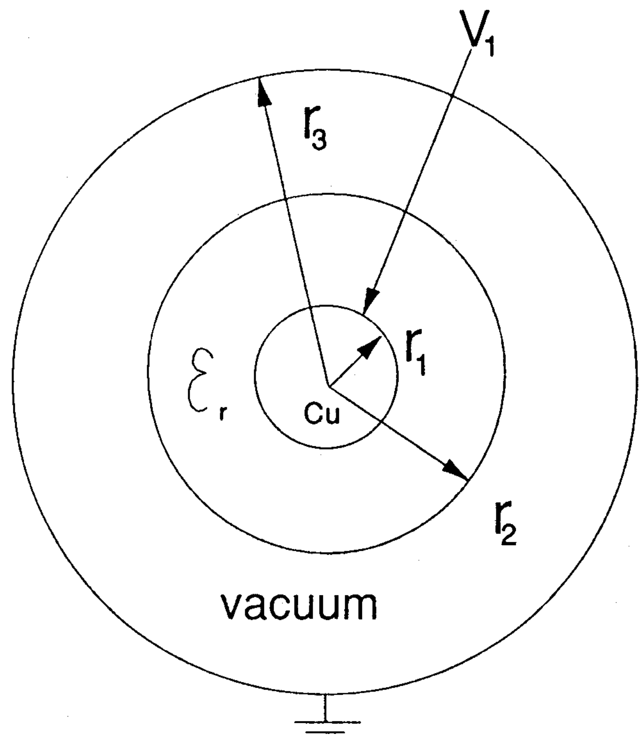
$$r = \sqrt{R^2+z^2}$$

$$d\vec{l} \times \hat{r} = \frac{R\, d\phi}{\sqrt{R^2+z^2}} \left(\hat{z}R + \hat{\rho} z\right)$$

$\hat{\rho}$ terms cancel upon integrating around loop, so we get

$$\vec{B} = \frac{\mu_0 I}{4\pi} \frac{2\pi R^2 \hat{z}}{\left(\sqrt{R^2+z^2}\right)^3} = \frac{\mu_0 I R^2 \hat{z}}{2\left(R^2+z^2\right)^{3/2}}$$

7. A Cu wire of radius r_1 is surrounded by an insulator of relative permittivity ε_r and outside radius r_2. This insulated wire runs along the center of a grounded metal tube of inside radius r_3. The Cu wire is at potential V_1 relative to ground.

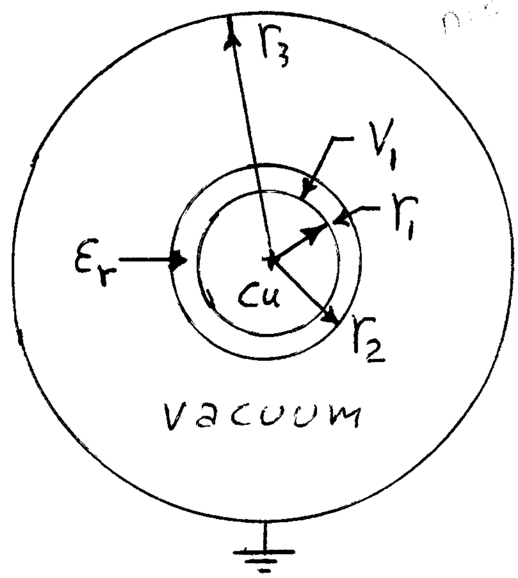


Find the electric field as a function of r,

(a) in the insulator where $r_1 < r < r_2$,

(b) in the vacuum where $r_2 < r < r_3$.

E & M Problem



A Cu wire of radius r_1 is surrounded by an insulator of relative permittivity ε_r and outside radius r_2. This insulated wire runs along the center of a grounded metal tube of inside radius r_3. The Cu wire is at potential V_1 relative to ground.

Find the electric field as a function of r,

(a) in the insulator where $r_1 < r < r_2$,

(b) in the vacuum where $r_2 < r < r_3$.

E&M Solution

Because of the cylindrical symmetry and the fact that $\vec{D}$ lines originate only on free charges, we note that

$$D = \sigma r_1 / r, \quad \text{for all} \quad r_1 < r < r_3,$$

where σ is the free surface charge density on the Cu wire.

Knowing D, we can find $E = D/\epsilon$, where $\epsilon = \epsilon_0$ for $r_2 < r < r_3$ but $\epsilon = \epsilon_0 \epsilon_r$ for $r_1 < r < r_2$.

We find σ from the boundary condition $V(r_1) = V_1$, upon integrating $E(r)$ to find V_1.

$$V_1 = -\int_{r_3}^{r_1} E\,dr = +\int_{r_1}^{r_3} E\,dr = \int_{r_1}^{r_2} \frac{\sigma r_1\,dr}{\varepsilon_0 \varepsilon_r r} + \int_{r_2}^{r_3} \frac{\sigma r_1\,dr}{\varepsilon_0 r}$$

$$= \frac{\sigma r_1}{\varepsilon_0}\left(\frac{1}{\varepsilon_r} \ln\frac{r_2}{r_1} + \ln\frac{r_3}{r_2}\right)$$

so $\sigma = \varepsilon_0 V_1 / r_1 \left(\frac{1}{\varepsilon_r} \ln\frac{r_2}{r_1} + \ln\frac{r_3}{r_2}\right)$, and

$\vec{E} = \hat{r}\,\dfrac{\sigma r_1}{\varepsilon_0 \varepsilon_r r}$ for $r_1 < r < r_2$, while

$\vec{E} = \hat{r}\,\dfrac{\sigma r_1}{\varepsilon_0 r}$ for $r_2 < r < r_3$.

8. (a) Suppose that you have linearly polarized light which propagates along the +z direction with its polarization along x. You desire linearly polarized light with polarization at 30° to x, *i.e.* along

$$e = x \cos 30^\circ + y \sin 30^\circ.$$

(1) How can you obtain it with some loss of intensity?
(2) How can you obtain it without any loss of intensity?

(b) A linearly polarized light beam propagates along the +z direction with E initially polarized along the +x direction. Separately consider each of the three cases below where the light beam encounters a single wave plate. Describe carefully the final polarization state for each case. All orientations are in the first quadrant of the x-y plane. **Explain each case briefly.**

(1) fast axis of $\lambda/4$ plate at 45°.
(2) fast axis of $\lambda/4$ plate at 30°.
(3) fast axis of $\lambda/2$ plate at 45°.

Optional hint: A wave plate is birefringent device which introduces phase retardation.

Optics Problem 2 -- Cone

(a) Suppose that you have linearly polarized light which propagates along the $+ \mathbf{z}$ direction with its polarization along $\mathbf{x}$. You desire linearly polarized light with polarization at 30° to x, i.e. along

$$\mathbf{e} = \mathbf{x} \cos 30^\circ + \mathbf{y} \sin 30^\circ .$$

(1) How can you obtain it with some loss of intensity?
(2) How can you obtain it without any loss of intensity?

(b) A linearly polarized light beam propagates along the + z direction with $\mathbf{E}$ initially polarized along the $+ \mathbf{x}$ direction. Separately consider each of the three cases below where the light beam encounters a single wave plate. Describe carefully the final polarization state for each case. All orientations are in the first quadrant of the x-y plane. **Explain each case briefly.**

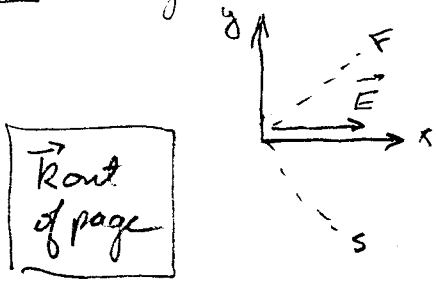
(1) fast axis of $\lambda/4$ plate at 45°
(2) fast axis of $\lambda/4$ plate at 30°
(3) fast axis of $\lambda/2$ plate at 45°.

Optional hint: A wave plate is a birefringent device which introduces phase retardation.

(a) ① polarizer at 30° angle

② $\frac{\lambda}{2}$ plate at 15° angle (like case 3 below)

(b) Case 1 gives circular polarization



① $\vec{E}$ has components along F & S

② Component along F will lead by $\pi/2$

$\Rightarrow$ ③ $\vec{E}$ rotates cw

$\therefore$ RT CIRCULAR

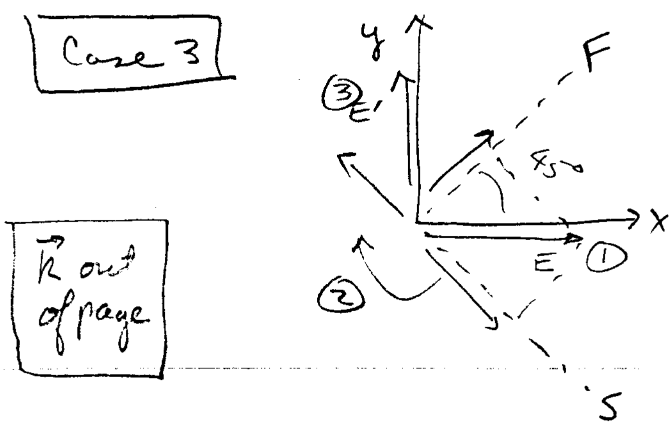
Case 2 More subtle case, but similar to $\boxed{1}$.

① $\frac{\pi}{2}$ phase shift $\Rightarrow$ circular

② 30° $\Rightarrow$ F & S components are not equal

$\therefore$ { Linear (excess F component) plus RT CIRCULAR }

Case 3



① $\vec{E}$ resolved into F & S components

② S component changes sign by π lag

③ Result $\vec{E}'$ is rotated $2(45°) = 90°$ giving $\vec{E}'$ vertical linear polarization

9. Consider a spinless particle of mass m subjected to a one-dimensional potential of the form

$$V(x) = \begin{cases} 0 & \text{for} \quad 0 < x < a \\ +\infty & \text{for} \quad x \geq a \quad \text{and} \quad x \leq 0 \end{cases} .$$

a) Find the normalized energy eigenfunctions and eigenvalues.

b) Suppose that at t=0 the state of the particle is given by the wave function

$$\psi(x) = \frac{1}{\sqrt{4}}\psi_1(x) + \frac{i}{\sqrt{4}}\psi_2(x) + \frac{1}{\sqrt{2}}\psi_3(x) \quad ,$$

where $\psi_1(x)$ refers to the ground state, and $\psi_2(x)$ and $\psi_3(x)$ to the first and second excited states, respectively.

At a later time t we measure the energy of the particle. What are the possible results of this measurement, and their respective probabilities? If the measurement had been carried out at t=0, would the results, and their probabilities, have been different? Why?

c) Evaluate the mean value of the above measurement of the energy carried out at time t.

Aftermon

Q.M.

Consider a spinless particle of mass m subjected to a one-dimensional potential of the form

$$V(x) = \begin{cases} 0 & \text{for } 0 < x < a \\ +\infty & \text{for } x \geq a \text{ and } x \leq 0 . \end{cases}$$

a. Find the normalized energy eigenfunctions and eigenvalues. $2^{1/2}$ points.

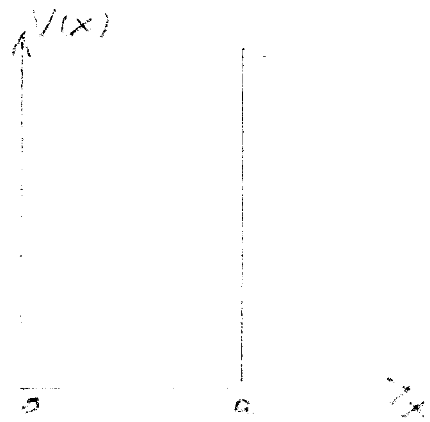
b. Suppose that at $t=0$ the state of the particle is given by the wave function

$$\psi(x) = \frac{1}{\sqrt{4}} \varphi_1(x) + \frac{i}{\sqrt{4}} \varphi_2(x) + \frac{1}{\sqrt{2}} \varphi_3(x)$$

where $\varphi_1(x)$ refers to the ground state, and $\varphi_2(x)$ and $\varphi_3(x)$ the first and second excited states respectively.

At a later time t we measure the energy of the particle. What are the possible results (of the measurements being taken), and their respective probabilities? If the measurement had been carried out at $t=0$, would the results, and their probabilities has been different? Why? 5 points.

c. Evaluate the mean value of the above measurement of the energy carried out at time t. $2^{1/2}$ points a.



For $0 < x < a$

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x) = E\,\psi(x)$$

$$\therefore \left(\frac{d^2}{dx^2} + k^2\right)\psi(x) = 0$$

$$E = \frac{\hbar^2 k^2}{2m}$$

$$\Rightarrow \quad \psi(x) = A\sin kx$$

$$\psi(0) = 0$$

$$\psi(a) = A\sin ka = 0 \quad \Rightarrow \quad ka = n\pi$$

$$\Rightarrow \quad E_n = \frac{\hbar^2}{2m}\left(\frac{n\pi}{a}\right)^2 \qquad n = 1, 2, 3, \dots$$

$$\langle\psi|\psi\rangle = 1 = |A|^2\int_0^a dx\,|\psi(x)|^2 =$$

$$= |A|^2\int_0^a dx\,\sin^2 kx =$$

$$= |A|^2\,\frac{1}{2}\int_0^a dx\left(1 - \cos 2\frac{n\pi}{a}x\right)$$

$$= |A|^2\,\frac{a}{2}$$

$$\therefore \quad A = \left(\frac{2}{a}\right)^{1/2} \qquad \text{(picked phase} = 0)$$

$$\Rightarrow \quad \psi_n(x) = \left(\frac{2}{a}\right)^{1/2}\sin\frac{n\pi}{a}x \qquad n = 1, 2, 3, \dots$$

b) $|\psi(0)\rangle = \frac{1}{\sqrt{4}}|1\rangle + \frac{i}{\sqrt{4}}|2\rangle + \frac{1}{\sqrt{2}}|3\rangle$

$$|\psi(t)\rangle = e^{-\frac{i}{\hbar}\hat{H}t}|\psi(0)\rangle$$

$$= \frac{e^{-\frac{i}{\hbar}E_1 t}}{\sqrt{4}}|1\rangle + i\frac{e^{-\frac{i}{\hbar}E_2 t}}{\sqrt{4}}|2\rangle + \frac{e^{-\frac{i}{\hbar}E_3 t}}{\sqrt{2}}|3\rangle$$

Possible results are

$$E_1 = \frac{\hbar^2}{2m}\left(\frac{\pi}{a}\right)^2 \; ; \; E_2 = \frac{\hbar^2}{2m}\left(\frac{2\pi}{a}\right)^2 \; ; \; E_3 = \frac{\hbar^2}{2m}\left(\frac{3\pi}{a}\right)^2$$

Respective probabilities are

$$\mathcal{P}(E_1) = \frac{1}{4} \; ; \; \mathcal{P}(E_2) = \frac{1}{4} \; ; \; \mathcal{P}(E_3) = \frac{1}{2}$$

If we had measured at $t=0$ we would have obtained the same results and probabilities because the Hamiltonian is time-independent.

c) $\langle\psi(t)|\hat{H}|\psi(t)\rangle =$

$$= \frac{\hbar^2}{2m}\left(\frac{\pi}{a}\right)^2\left[1\times\frac{1}{4} + 4\times\frac{1}{4} + 9\times\frac{1}{2}\right]$$

$$= \frac{\hbar^2}{2m}\left(\frac{\pi}{a}\right)^2\left(\frac{5}{4} + \frac{9}{2}\right)$$

$$= \frac{23}{4}\frac{\hbar^2}{2m}\left(\frac{\pi}{a}\right)^2$$

10. A quantum-mechanical particle of mass m moves in two dimensions. It is confined by infinitely high walls to the square region

$$|x| \leq L/2, \quad |y| \leq L/2.$$

a) What is the energy and degeneracy of the first excited level? Write down the correctly normalized wavefunctions for this level.

b) The particle is now subjected to a small additional potential

$$V(x,y) = \varepsilon x y$$

Calculate the splitting in the first excited state produced by this perturbation.

QM Problem

A quantum-mechanical particle of mass m moves in two dimensions. It is confined by infinitely high walls to the square region

$$|x| \le L/2, \quad |y| \le L/2.$$

a) What is the energy level and degeneracy of the first excited states? Write down the correctly normalized wavefunctions for these states.

b) The particle is now subjected to a small additional potential

$$V(x,y) = \varepsilon xy$$

Calculate the splitting in the first excited states produced by this perturbation.

Solution

a) The levels for a 2-D square well are $|n\,m\rangle$, $n,m = 1,2,3,\ldots$

$$E = \frac{\hbar^2\pi^2}{2mL^2}(n^2+m^2).$$

The first excited states are:

$$|12\rangle = \frac{\sqrt{2}}{L}\cos\left(\frac{\pi x}{L}\right)\sin\left(\frac{2\pi y}{L}\right)$$

$$|21\rangle = \frac{\sqrt{2}}{L}\sin\left(\frac{2\pi x}{L}\right)\cos\left(\frac{\pi y}{L}\right)$$

$$E_1 = \frac{5\hbar^2\pi^2}{2mL^2}$$

b) In first order perturbation theory we look at the matrix elements of V. Since $V(x,y)$ is odd in x and odd in y,

$$V = \begin{pmatrix} 0 & X \\ X & 0 \end{pmatrix} \quad \text{where } X = \langle 12|V|21\rangle$$

Choose a basis which diagonalizes V. The eigenvalues of V are $\pm X$, so the perturbed energy levels will be $E_1 \pm X$ and the splitting will be $\boxed{2X}$

$$X = \varepsilon\left(\frac{2}{L^2}\right)\left(\int_{-L/2}^{L/2}\cos\left(\frac{\pi x}{L}\right) x \sin\left(\frac{2\pi x}{L}\right)dx\right)^2$$

identity: $\cos A \sin B = \frac{1}{2}\left(\sin(A+B) - \sin(A-B)\right)$

$$\int_{-L/2}^{L/2} x \sin\left(\frac{3\pi x}{L}\right) dx = -\frac{18L^2}{\pi^2}$$

$$\int_{-L/2}^{L/2} x \sin\left(\frac{\pi x}{L}\right) dx = \frac{2L^2}{\pi^2}$$

$$\Rightarrow \overline{X} = \frac{2\varepsilon}{L^2}\left[\frac{1}{2}\left(-\frac{18L^2}{\pi^2} - \frac{2L^2}{\pi^2}\right)\right]^2 = \frac{200\varepsilon L^2}{\pi^4}$$

11. In a Stern-Gerlach experiment, a well-collimated beam of silver atoms in their ground state ($^2S_{1/2}$) emerges from an oven inside which the atoms are in thermal equilibrium at temperature T. The beam enters a region of length l, in which there is a strong magnetic field B and a constant gradient of field $\partial B/\partial z$ perpendicular to the axis of the beam. After leaving this region, the beam travels a further distance l' in a field-free region to a detector. Show that in the plane of the detector the deflection S_α of those atoms which had the most probable speed $\alpha = \sqrt{\frac{2kT}{m}}$ in the oven is

$$S_\alpha = \pm \frac{\mu_B}{4kT} (\partial B/\partial z)\,(l^2 + 2ll')$$

where μ_B is the Bohr magneton.

(Optional hint: Think of the force as arising from the gradient of the potential energy.)

Quantum Mechanics/Atomic Physics Problem 1--Cone

In a Stern-Gerlach experiment, a well-collimated beam of silver atoms in their ground state ($^2S_{1/2}$) emerges from an oven inside which the atoms are in thermal equilibrium at temperature T. The beam enters a region of length l, in which there is a strong magnetic field B and a constant gradient of field $\partial B/\partial z$ perpendicular to the axis of the beam. After leaving this region, the beam travels a further distance l' in a field-free region to a detector. Show that in the plane of the detector that the deflection s_α of those atoms which had the most probable speed $\alpha = \sqrt{\frac{2kT}{m}}$ in the oven is

$$s_\alpha = \pm \frac{\mu_B}{4kT}\left(\frac{\partial B}{\partial z}\right)\left(l^2 + 2ll'\right)$$

where μ_B is the Bohr magneton.

(Optional hint: Think of the force as arising from the gradient of the potential energy.)

Magnet region

$$\begin{cases} \text{Energy } E = -\vec{\mu}\cdot\vec{B} & \mu_z = \pm\mu_B \\ \quad \text{since: } \vec{\mu} = g_s \mu_B \vec{S}_z = 2\mu_B \vec{S}_z \end{cases}$$

$$(\text{Force along } z) = -\frac{\partial E}{\partial z} = 2\mu_B S_z \frac{\partial B}{\partial z} = \pm\mu_B \frac{\partial B}{\partial z}$$

Apply mechanics to trajectory with constant accel.

magnet region l

$$z_1 = \tfrac{1}{2} a t^2 \qquad \text{where} \begin{cases} a = \frac{1}{m}\left(\pm\mu_B \frac{\partial B}{\partial z}\right) \\ t = \frac{l}{\alpha} \end{cases}$$

$$z_1 = \frac{1}{2}\left[\frac{1}{m}\left(\pm\mu_B \frac{\partial B}{\partial z}\right)\right]\frac{l^2}{\alpha^2}$$

field-free region l'

$$z_2 = (v_z)(t') \qquad \text{where} \quad v_z = at \quad \text{from above}$$

$$t' = \frac{l'}{\alpha}$$

$$z_2 = \left[\frac{1}{m}\left(\pm\mu_B \frac{\partial B}{\partial z}\right)\right]\frac{l\,l'}{\alpha^2}$$

$$S_\alpha = z_1 + z_2 = \left(\frac{1}{2}\right)\left(\frac{m}{2kT}\right)\frac{1}{m}\left(\pm\mu_B \frac{\partial B}{\partial z}\right)\left[l^2 + 2 l l'\right]$$

$$\boxed{S_\alpha = \pm\frac{\mu_B}{4kT}\left(\frac{\partial B}{\partial z}\right)\left(l^2 + 2 l l'\right)}$$

12. a) Consider two equivalent p electrons in an atom. (Here, equivalent means that each has the same n and l quantum numbers.) This *configuration* is often denoted np^2. Find all of the states of total angular momentum L and total spin angular momentum S which are allowed by the exclusion principle. You may find it helpful, but not sufficient, to consider the total number of allowed states. Summarize your results in a block at the end and explain your reasoning.

b) According to Hund's rules, which state do you expect to be lowest in energy (the *ground* state)?

c) If the two electrons are inequivalent, that is if they have different n quantum numbers, what additional SL states are allowed?

Quantum Mechanics/Atomic Physics Problem 2--Cone

a) Consider two equivalent p electrons in an atom. (Here, equivalent means that each has the same n and l quantum numbers.) This *configuration* is often denoted np^2. Find all of the states of total angular momentum L and total spin angular momentum S which are allowed by the exclusion principle. You may find it helpful, but not sufficient, to consider the total number of allowed states. Summarize your results in a block at the end and explain your reasoning.

b) According to Hund's rules, which state do you expect to be lowest in energy (the *ground* state)?

c) If the two electrons are inequivalent, that is if they have different n quantum numbers, what additional SL states are allowed?

QM/AP #2

a) Clever way $(L+S)$ must be even

$\Rightarrow$ ${}^1S$, ${}^3P$, ${}^1D$ only (max $L=2$)

Basic Way list all possible pairs of quantum numbers & pick out allowed states by elimination

$\boxed{\text{See attachment for details}}$

b) (1) Max S then (2) max L

$\therefore$ ${}^3P$ lowest

c) ${}^3S$, ${}^1P$, & ${}^3D$ also allowed

13. In a harmonic oscillator the energy eigenvalues are $(n + 1/2)\hbar\omega$ and each n-state has equal *a priori* probability of being occupied. Using the Boltzmann factor for actual probability of occupation, find the mean energy of a harmonic oscillator of angular frequency ω at temperature T.

Show that $\langle E \rangle \approx kT$ at high temperature, as required by Bohr's Correspondence Principle.

Statistical Mechanics Problem

In a harmonic oscillator the energy eigenvalues are $(n+\frac{1}{2})\hbar\omega$ and each n-state has equal *a priori* probability of being occupied. Using the Boltzmann factor for actual probability of occupation, find the mean energy of a harmonic oscillator of angular frequency ω at temperature T.

Show that $\langle E \rangle \simeq kT$ at high temperature as required by Bohr's Correspondence Principle.

Statistical Mechanics Solution

Normalization: $\sum P_n = 1 = P_0\left(1 + e^{-\hbar\omega/kT} + e^{-\frac{2\hbar\omega}{kT}} + \cdots\right)$

$= \dfrac{P_0}{1 - e^{-\hbar\omega/kT}}$ so $P_0 = 1 - e^{-\hbar\omega/kT}$ is probability that $n=0$ state is occupied.

$$\langle E \rangle = \sum E_n P_n = \sum_{n=0}^{\infty} (n+\tfrac{1}{2})\hbar\omega\left(1 - e^{-\hbar\omega/kT}\right)e^{-n\frac{\hbar\omega}{kT}}$$

$$= \tfrac{1}{2}\hbar\omega + \hbar\omega\left(1 - e^{-\hbar\omega/kT}\right)\sum_{n=0}^{\infty} n e^{-n\hbar\omega/kT}$$

But $\dfrac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \cdots = \sum_{n=1}^{\infty} n x^{n-1}$

and $\langle E \rangle = \tfrac{1}{2}\hbar\omega + \hbar\omega\left(1 - e^{-\hbar\omega/kT}\right)e^{-\hbar\omega/kT}\sum_{n=1}^{\infty} n e^{-(n-1)\frac{\hbar\omega}{kT}}$

$$= \tfrac{1}{2}\hbar\omega + \hbar\omega\left(1 - e^{-\hbar\omega/kT}\right)e^{-\hbar\omega/kT}\sum_{n=1}^{\infty} n\left(e^{-\frac{\hbar\omega}{kT}}\right)^{n-1}$$

so $x \equiv e^{-\frac{\hbar\omega}{kT}}$ and

$$\langle E \rangle = \tfrac{1}{2}\hbar\omega + \frac{\hbar\omega e^{-\hbar\omega/kT}}{1 - e^{-\hbar\omega/kT}}$$

$$= \boxed{\tfrac{1}{2}\hbar\omega + \frac{\hbar\omega}{e^{\hbar\omega/kT} - 1}}$$

At high T, $\langle E \rangle \simeq \tfrac{1}{2}\hbar\omega + \dfrac{\hbar\omega}{1 + \frac{\hbar\omega}{kT} - 1}$

$= \tfrac{1}{2}\hbar\omega + kT \simeq kT$

14. Do part a) <u>or</u> b) - not both.

a) Sketch a typical vacuum system consisting of a mechanical forepump, a diffusion pump, necessary valves, and an evacuated chamber. Assume that you have just finished an experiment at high vacuum, that you have to bring the chamber up to atmospheric pressure to change the sample, keeping the diffusion pump running (hot), and that you then must evacuate the chamber again. List the steps necessary to accomplish this -- what valves do you open and close, and in what order?

b) You want to evaporate Au onto a crystal surface and need a mean free path of 5 cm to perform this task. Estimate the pressure in torr (1 atmosphere = 760 torr) in the vacuum chamber needed to achieve this mean free path, at room temperature. Show all your reasoning in making this estimate -- simply writing down a pressure you may remember will not suffice.

Experimental Problem

Do part A or B; not both. ~~if you do both, the better one will be graded.~~

A. Sketch a typical vacuum system consisting of a mechanical forepump, a diffusion pump, necessary valves, and an evacuated ~~container~~ chamber. ~~covered by a bell jar. List all the steps necessary to evacuate the system and then bring it back to atmospheric pressure.~~ Emphasize precautions to take in order not to damage the system.

B. You want to evaporate Au onto a crystal surface and need a mean free path of 5 cm to perform this task. Estimate the pressure in torr (1 atmosphere = 760 torr) in the vacuum chamber needed to achieve this mean free path, at room temperature. Show all your reasoning in making this estimate – simply writing down a pressure you may remember will not suffice.

Assume that you have just finished an experiment at high vacuum, that you have to bring the ~~[illegible]~~ chamber up to atmospheric pressure to change the sample, (see next page) ~~without~~ keeping the diffusion pump running (hot),