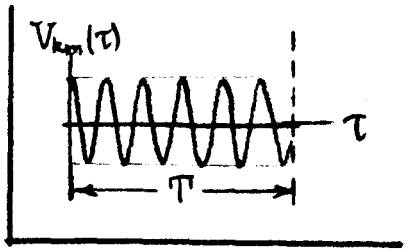
6) With Eq. (16) in hand, we do an <u>example</u> of how a time-dependent coupling $V(x,t)$ actually drives transitions out of state m into states $k \neq m$.

EX. PULSED HARMONIC PERTURBATION

$$V(x,t) = \hbar\Omega(x)\left[e^{+i\omega t} + e^{-i\omega t}\right] = 2\hbar\Omega(x)\cos\omega t \begin{cases} \text{for } 0 \le t \le T, \text{ and} \\ 0, \text{ otherwise} \end{cases}$$

Say $\longrightarrow V_{km}(\tau) = \hbar\Omega_{km}\left[e^{i\omega\tau} + e^{-i\omega\tau}\right]$, over $0 \le \tau \le T$

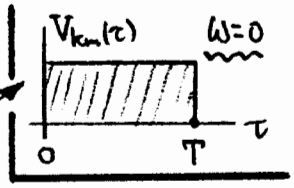


where: $\hbar\Omega_{km} = \langle \phi_k(x) | V(x) | \phi_m(x) \rangle$. (17)

Example of Pulsed Harmonic Perturbation

Such a coupling $V(x,t)$ could represent the effect of a laser pulse (monochromatic, at freq. ω) shining on an atom [initially in state m] for some finite time T.

Notice that when $\omega \to 0$, then $V_{km}(\tau) = 2\hbar\Omega_{km} = \text{const}$, $0 \le \tau \le T$, and zero, otherwise... so the "atom" is exposed to a const field pulse.



Put $V_{km}(\tau)$ of Eq. (17) into the first-order amplitude of Eq. (13)...

$$a_k^{(1)}(t \gtrsim T) = -\frac{i}{\hbar}\hbar\Omega_{km}\int_0^T \left(e^{i\omega\tau} + e^{-i\omega\tau}\right) e^{i\omega_{km}\tau}\, d\tau,$$

$$\xrightarrow{\text{Sol'n}} a_k^{(1)}(t \gtrsim T) = \Omega_{km}\left[\underbrace{\frac{1-e^{i(\omega_{km}+\omega)T}}{\omega_{km}+\omega}}_{①} + \underbrace{\frac{1-e^{i(\omega_{km}-\omega)T}}{\omega_{km}-\omega}}_{②}\right] \quad (18)$$

$t \gtrsim T$ means pulse V is finished.

REMARKS on Eq. (18).

1. V "small" here $\Rightarrow |\Omega_{km}| << |\omega_{km}|$. Then $a_k^{(1)}(t \gtrsim T)$ is appreciable only at the "resonances": $\omega_{km} \simeq \mp\omega$, when: $a_k^{(1)}(t \gtrsim T) \simeq -i\Omega_{km}T$. This can become large as $T \to$ large (and requires higher order pert'n theory to handle completely).

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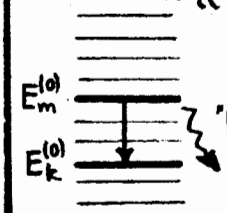
2. The resonant condition $\omega_{km} \simeq \mp\omega$ can be written in terms of energies as...

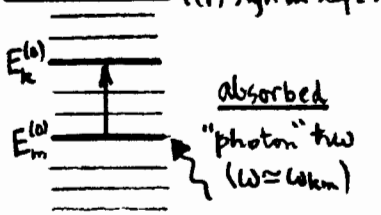
(19) $$\boxed{E_k^{(0)}[\text{final state}] = E_m^{(0)}[\text{initial state}] \mp \hbar\omega}$$

Interpretation: the resonant behavior of $a_k^{(1)}(t \gtrsim T)$ in Eq. (18) is associated with either the emission (term ①) or absorption (term ②) of a "photon" of energy $\hbar\omega$, $\omega \simeq |\omega_{km}|$, from V-field.

EMISSION {term ① in (18), (−) sign at left.



ABSORPTION {term ② in (18), (+) sign at left.



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3. If we are interested in absorptive processes only (which pump energy into the "atom"), then in Eq. (18) only term ② is important, and we can write...

for $E_m^{(0)} \to E_k^{(0)} > E_m^{(0)}$, absorption, the $m \to k$ transition probability is:

$$|a_k^{(1)}|^2 = 4|\Omega_{km}|^2 \sin^2\tfrac{1}{2}(\omega_{km}-\omega)T \,/\, (\omega_{km}-\omega)^2 \quad (20)$$

for irradiation by pulse of length T @ frequency ω.

This is the lowest order result. The emission probability is gotten via $\omega \to (-)\omega$ here.

Further general remarks on 1st order amplitude

7) Two other general remarks can be made about the first-order amplitude in Eq.(13):

4. Even though emittive or absorptive transitions $m \to k$ are induced by V, the population of the initial state m is **barely depleted** (in this order of pert^{b}n theory). The argument goes as follows. By time t after turn-on, V has induced the state

$$\to \psi(x,t) = \underbrace{\phi_m(x)\, e^{-i\omega_m t}}_{\text{initial state}} + \underbrace{\sum_n a_n^{(1)}(t)\, \phi_n(x)\, e^{-i\omega_n t}}_{\text{admixture due to } V}, \quad \text{to } O(V). \tag{21}$$

The amplitude of ϕ_m still present in ψ is ...

$$\to \langle \phi_m | \psi \rangle = e^{-i\omega_m t}\left[1 + a_m^{(1)}(t)\right] = e^{-i\omega_m t}\left[1 - \frac{i}{\hbar}\int_{t_0}^{t} V_{mm}(\tau)\, d\tau\right]$$

$$\text{Say } \langle \phi_m | \psi \rangle = e^{-i\omega_m t}\, e^{-\frac{i}{\hbar}\int_{t_0}^{t} V_{mm}(\tau) d\tau}, \qquad = e^{-\frac{i}{\hbar}\int V_{mm}\, d\tau}, \text{ to } O(V);$$

$$\left.\begin{array}{l}\text{only population of initial}\\ \text{state } m \text{ at time } t\end{array}\right\} \quad |\langle \phi_m | \psi \rangle|^2 = \left| e^{-\frac{i}{\hbar}\int V_{mm} d\tau} \right|^2 \to 1. \tag{22}$$

All that is needed here is to assume V is real (necessary to ensure $\mathcal{H}_{\text{at}} + V$ is Hermitian). The depletion of state m by transitions $m \to k$ is at most an $O(V^2)$ effect.

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5. Consider $m \to k$ couplings via $V_{km}(\tau)$ which last a "long" time [e.g. could be $V_{mk}(\tau) \propto e^{-\alpha\tau^2}$, a Gaussian; all we really need is duration $\Delta\tau \gg 1/|\omega_{km}|$].

Formally, the $m \to k$ transitions are not finished till $t \to \infty$, so we need:

$$\to a_k^{(1)}(\infty) = -\frac{i}{\hbar}\int_{-\infty}^{\infty} V_{km}(\tau)\, e^{i\omega_{km}\tau}\, d\tau, \tag{23}$$

to find out the final $m \to k$ transfer. Except for the factor $(i\hbar)^{-1}$ out in front, the required amplitude is just the **Fourier Transform** of $V_{km}(\tau)$. So the way a quantum system reacts to an impressed time-dependent perturbation $V(\tau)$ is to **Fourier analyse** $V(\tau)$ w.r.t. its own natural freqs. ω_{km}.

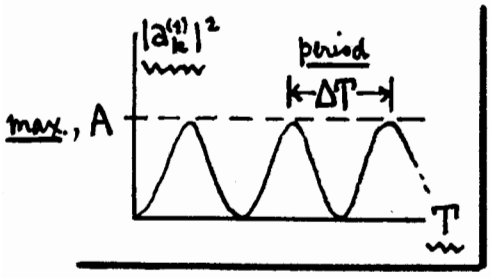
$m \to k$ transition dynamics (leading order)

8) We backtrack to the Pulsed Harmonic Perturbation of Eqs. (17)–(20) above, in order to learn more about the $m \to k$ transition dynamics.

Consider absorptive probability for $m \to k$ per Eq. (20), viz.

$$\boxed{|a_k^{(1)}|^2 = \frac{4|\Omega_{km}|^2}{(\omega_{km}-\omega)^2} \sin^2 \tfrac{1}{2}(\omega_{km}-\omega)T} \quad \int \text{for } m \to k \text{ via pulse of duration } T \text{ @ freq. } \omega. \qquad (24)$$

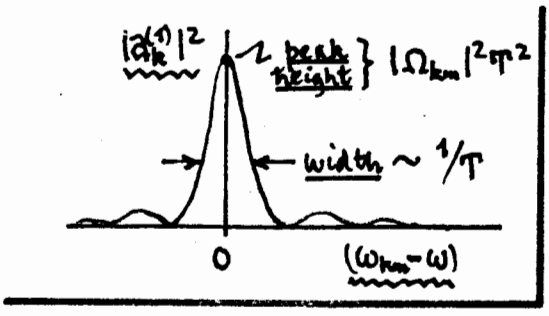
A. Plot $|a_k^{(1)}|^2$ vs. pulse duration T; all other parameters fixed.



max.: $A = 4|\Omega_{km}|^2/(\omega_{km}-\omega)^2$; period: $\Delta T = 2\pi/|\omega_{km}-\omega|$.

System exhibits a "quantum oscillation" $m \rightleftarrows k$ out of and back into initial state m @ freq. $|\omega_{km}-\omega|$.

B. Plot $|a_k^{(1)}|^2$ vs. off-resonance freq. $\omega_{km}-\omega$; all other parameters fixed



Area under curve ~ peak height × width ~ T. As $T \to$ large, transition $m \to k$ with $E_k^{(0)} = E_m^{(0)} + \hbar\omega$ becomes more certain. Curve width ~ $1/T$ is consistent with energy uncertainty $\Delta E \sim \hbar/T$ for $m \to k$.

NOTE: $(|a_k^{(1)}|^2/T)$ is a nascent delta-function.

C. Suppose the absorption is $m \to \{k\}$, a set of final states k. Then consider:

$$\left[\begin{array}{l}\text{total transition}\\ \text{probability: } m \to \{k\}\end{array}\right\} \quad P_m = \sum_k |a_k^{(1)}|^2, \quad \text{w/} \; E_k^{(0)} \simeq E_m^{(0)} + \hbar\omega. \qquad (25)$$

[Keep in mind we are working to lowest order (leading order) in V].

Now suppose the final states $\{k\}$ lie close together in energy, and define

$\rightarrow \rho(k)\, dE_k$ = number of final states k with energy in range $E_k^{(0)}$ to $E_k^{(0)} + dE_k$. (26)

$\rho(k)$ is called the "density of states" function. Now in these terms, write P_m