Generalization of $\mathbb{A}$ for mode $(\mathbf{k};\hat{\epsilon})$.

7) Generalization of $\mathbb{A}$. Note that in addition to the "cos" solution we've used for $\mathbb{A}$ [Eq. (16)] we could also use a linearly independent "sin" solution...

$$\left[\begin{array}{l} \mathbb{A}_1 = \sqrt{2}\,C\,\hat{\epsilon}\,(a_1^\dagger + a_1)\cos(\mathbf{k}\cdot\mathbf{r}) \\ \mathbb{A}_2 = \sqrt{2}\,C\,\hat{\epsilon}\,(a_2^\dagger + a_2)\sin(\mathbf{k}\cdot\mathbf{r}) \end{array}\right\| \begin{array}{l} \text{w/ } \underline{C} = (2\pi\hbar c/Vk)^{1/2}.\ \mathbb{A}_1\ \&\ \mathbb{A}_2 \text{ are indpt solns} \\ \text{to: } (\nabla^2 + k^2)\mathbb{A} = 0, \text{ for same wave } (\mathbf{k};\hat{\epsilon}). \end{array} \tag{32}$$

The operators a_1 & a_2 are independent (like independent amplitudes), each with the "normalization" $[a_j, a_j^\dagger] = 1$, for $j = 1$ & 2. Define a new pair of operators:

$$\left[\begin{array}{l} a_+ = \frac{1}{\sqrt{2}}(a_1 - i a_2), \quad a_- = \frac{1}{\sqrt{2}}(a_1 + i a_2). \\ \underline{\text{NOTE}}: \text{ if/ } [a_j, a_j^\dagger] = 1, \text{ and lin. indpt, } \text{ then/ } [a_\pm, a_\pm^\dagger], \text{ and lin. indpt.} \end{array}\right\} \tag{33}$$

Then we can add $\mathbb{A}_1$ & $\mathbb{A}_2$ of Eq. (32) to form...

$$\rightarrow \underline{\mathbb{A} = \mathbb{A}_1 + \mathbb{A}_2 = C\hat{\epsilon}\left\{(a_+ + a_-^\dagger)e^{+i\mathbf{k}\cdot\mathbf{r}} + (a_- + a_+^\dagger)e^{-i\mathbf{k}\cdot\mathbf{r}}\right\}}. \tag{34}$$

By adjusting the mixture of the independent "amplitudes" a_+ & a_-, we can make this sum into a rightward or leftward traveling wave. Note that the new Hamiltonian is:

$$\rightarrow \mathcal{H}_\omega = \sum_{j=1}^{2}(a_j^\dagger a_j + \tfrac{1}{2})\hbar\omega = (a_+^\dagger a_+ + a_-^\dagger a_- + 1)\hbar\omega. \tag{35}$$

These are the contributions from two independent (Bose-Einstein) fields, one corresponding to each of $a_\pm$. The eigenfcns of $\mathcal{H}_\omega$ are now expanded to product states, viz. $|N_+, N_-\rangle = |N_+\rangle \otimes |N_-\rangle$, w/ separate # operators: $a_+^\dagger a_+ = N_+$, and $a_-^\dagger a_- = N_-$.

To get an explicit time-dependence, use the QM eqtn-of-motion for the $a_\pm$'s, giving:

$$\rightarrow i\hbar\dot{a}_\pm = [a_\pm, \mathcal{H}_\omega] = [a_\pm, a_\pm^\dagger a_\pm]\hbar\omega = [a_\pm, a_\pm^\dagger]a_\pm\hbar\omega,$$

$$\text{i.e./ } \dot{a}_\pm = -i\omega a_\pm, \text{ w/ } \boxed{a_\pm(t) = a_\pm(0)e^{-i\omega t}}. \tag{36}$$

Putting this into $\mathbb{A}$ of Eq. (34), we get the most general $\mathbb{A}$ for a plane wave at $(\mathbf{k};\hat{\epsilon})$, as a sum of rightward & leftward traveling waves... (next page)

Traveling waves for $\mathbb{A}$ at $(\mathbb{k};\hat{e})$. Poynting vector for $(\mathbb{k};\hat{e})$.

rightward wave

$$\left[\begin{array}{l}\mathbb{A} = c\hat{e}\left\{[a_+(0)e^{+i(\mathbb{k}\cdot\mathbb{r}-\omega t)} + a_+^\dagger(0)e^{-i(\mathbb{k}\cdot\mathbb{r}-\omega t)}] + \right. \\ \qquad\qquad \text{leftward wave} \rightarrow \left.[a_-^\dagger(0)e^{+i(\mathbb{k}\cdot\mathbb{r}+\omega t)} + a_-(0)e^{-i(\mathbb{k}\cdot\mathbb{r}+\omega t)}]\right\}.\end{array}\right. \tag{37}$$

Choice of "amplitude" $a_-(0)=0 \Rightarrow$ pure rightward wave, while $a_+(0)=0 \Rightarrow$ pure left-ward wave (either choice can annihilate or create photons via $a_\pm$ & $a_\pm^\dagger$). In what follows, we shall -- for convenience & choice -- work mainly w/ rightward waves. We can also think of doing the calculation for leftward waves, or both together.

Also, in what follows, we shall rarely write down the time dependence of the $a_\pm$ explicitly, but will understand the symbol $a_\pm$ to mean $a_\pm(0)e^{-i\omega t}$.

---

8) It is instructive to calculate the Poynting vector (energy flux) for $\mathbb{A}$ of (37). Fields are:

$$\left[\begin{array}{l}\mathbb{E} = -\frac{1}{c}\partial\mathbb{A}/\partial t = ikc\hat{e}\,[(a_+ - a_-^\dagger)e^{i\mathbb{k}\cdot\mathbb{r}} - (a_+^\dagger - a_-)e^{-i\mathbb{k}\cdot\mathbb{r}}], \\ \mathbb{B} = \nabla\times\mathbb{A} = i(\mathbb{k}\times\hat{e})c\,[(a_+ + a_-^\dagger)e^{i\mathbb{k}\cdot\mathbb{r}} - (a_+^\dagger + a_-)e^{-i\mathbb{k}\cdot\mathbb{r}}];\end{array}\right\} \tag{38}$$

So $\rightarrow$ $$\$ = \frac{c}{4\pi}(\mathbb{E}\times\mathbb{B}) = (\hbar c^2/V)\,\mathbb{k}\,\Big[(a_+^\dagger a_+ - a_-^\dagger a_-) - \tfrac{1}{2}(a_+a_+ - a_-^\dagger a_-^\dagger)e^{2i\mathbb{k}\cdot\mathbb{r}} - \tfrac{1}{2}(a_+^\dagger a_+^\dagger - a_-a_-)e^{-2i\mathbb{k}\cdot\mathbb{r}}\Big]. \tag{39}$$

Now do a time-average. Terms like a_+a_+ have a time factor $e^{-2i\omega t}$, and average to zero. We get:

$$\left[\langle\$\rangle = \frac{c^2}{V}(a_+^\dagger a_+ - a_-^\dagger a_-)\hbar\mathbb{k} = \frac{c^2}{V}(N_+ - N_-)\underbrace{\hbar\mathbb{k}}_{\text{photon momentum}}\right]. \tag{40}$$

number operators

This $\langle\$\rangle$ represents a net transport of photons @ momentum $\hbar\mathbb{k}$, N_+ traveling to right, N_- to left. It is obviously consistent with the idea of photons.

One more technical matter, before we generalize $\mathbb{A}$ to a $\mathbb{k}$-spectrum. To count photon modes possible in the volume V, we impose periodic boundary conditions (i.e. "box normalisation") on the fields, i.e.

(next page)

Counting photon modes via "box normalization". Further generalization of $\mathbb{A}$.

Let volume V of Eq. (19) be a box of side L. Impose periodic B.C.:

$$e^{ik_j x_j} = e^{ik_j(x_j+L)} \Rightarrow k_j L = 2\pi n_j \begin{cases} n_j = \text{integer} = 0,1,2,3,\ldots \\ j = 1,2,3 = \text{component of } \mathbb{r}; \end{cases}$$

$$\Rightarrow \text{quantized wave vectors}: \mathbb{k}_s = 2\pi \mathbb{n}_s / L, \quad \mathbb{n}_s = (n_1, n_2, n_3). \tag{41}$$

Can have $L \to \infty$ for the box, so $\mathbb{k}_s$ is "barely quantized". But the quantization of $\mathbb{k}_s$ helps with counting photon modes that can exist in V. The $\mathbb{k}_s$ vectors of (41) define a cubic lattice in $\mathbb{k}$-space, w/ lattice spacing $\Delta k_j = 2\pi/L$. The # allowable $\mathbb{k}_s$ vectors -- each one corresponding to a different independent oscillator -- in the "volume element" d^3k is given by:

$$\left.\begin{matrix}\text{\# field modes}\\ \text{in "volume" } d^3k\end{matrix}\right\} \left(\frac{dk_1}{2\pi/L}\right)\left(\frac{dk_2}{2\pi/L}\right)\left(\frac{dk_3}{2\pi/L}\right) = \frac{V}{(2\pi)^3}\, d^3k, \text{ dimensionless.} \tag{42}$$

Actually, the # allowed radiation field modes is just 2x this, because there are two independent (mutually ⊥) polarizations $\hat{\varepsilon}$ for each $\mathbb{k}_s$. However, we will count the polarizations separately. The importance of this "box normalization" is that when we sum over a $\mathbb{k}$-spectrum, in the limit that the box volume $V \to \infty$ (we should let $V \to \infty$ at the end of the calculation, to make the results independent of the box choice of B.C. in Eq. (41)) we can convert sums to integrals via//

$$\text{when } V \to \infty: \sum_{\mathbb{k}_s} \text{ goes over to } [V/(2\pi)^3] \int d^3k. \tag{43}$$

---

9) Further generalization of $\mathbb{A}$. The most general rightward-traveling wave at $\mathbb{k}_s$ is:

$$\mathbb{A}_s(\mathbb{r},t) = \sum_{\sigma=1}^{2} (2\pi\hbar c/V k_s)^{1/2}\, \hat{\varepsilon}_{s\sigma} \left[a_{s\sigma}(t)e^{i\mathbb{k}_s\cdot\mathbb{r}} + a_{s\sigma}^\dagger(t) e^{-i\mathbb{k}_s\cdot\mathbb{r}}\right], \tag{44}$$

w/ $a_{s\sigma}(t) = a_{s\sigma}(0)e^{-ik_s ct}$, and: $[a_{s\sigma}, a_{s\sigma}^\dagger] = 1$;

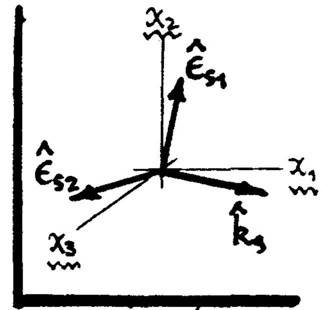
$\sigma = 1,2 \leftrightarrow$ two mutually ⊥ polarization directions $\hat{\varepsilon}_{s1}$ & $\hat{\varepsilon}_{s2}$ for the wave;

&// eigenfcn $|N_{s1}, N_{s2}\rangle$, $N_{s\sigma}$ = # plane-wave photons @ $\begin{cases}\text{wave vector } \mathbb{k}_s, \\ \text{polarization } \hat{\varepsilon}_{s\sigma}.\end{cases}$

Note: we've inserted: $C = (2\pi\hbar c/V k_s)^{1/2}$, from Eq (32).

A relation for triad $(\hat{\epsilon}_{s1}, \hat{\epsilon}_{s2}, \mathbb{k}_s)$. Generalize $\mathbb{A}_s$ to a $\mathbb{k}_s$-spectrum.

ASIDE The orthonormal triad $(\hat{\epsilon}_{s1}, \hat{\epsilon}_{s2}, \hat{k}_s)$.



The unit wavevector $\hat{k}_s = \mathbb{k}_s/k_s$ and two polarization vectors $\hat{\epsilon}_{s\sigma}$ form an orthonormal triad for the (plane) wave propagating at $\mathbb{k}_s$. From this, and the notion of direction cosines, we can write down a useful relation between the components of $(\hat{\epsilon}_{s1}, \hat{\epsilon}_{s2}, \mathbb{k}_s)$. A unit vector along the x_i-axis has components in the $\hat{\epsilon}$-$\hat{k}$ system given by:

$$\rightarrow \hat{x}_i = \left((\hat{\epsilon}_{s1})_i, (\hat{\epsilon}_{s2})_i, (\hat{k}_s)_i\right) \;;\; (\hat{\epsilon}_{s1})_i = \hat{x}_i \cdot \hat{\epsilon}_{s1} \Rightarrow \cos\measuredangle(\hat{x}_i, \hat{\epsilon}_{s1}), \text{ etc} \quad (45a)$$

Similarly for $\hat{x}_j$. Then, since: $\hat{x}_i \cdot \hat{x}_j = \begin{cases} 0, \text{ if } j \neq i \\ 1, \text{ if } j = i \end{cases}$, we can write

$$(\hat{\epsilon}_{s1})_i(\hat{\epsilon}_{s1})_j + (\hat{\epsilon}_{s2})_i(\hat{\epsilon}_{s2})_j + (\hat{k}_s)_i(\hat{k}_s)_j = \delta_{ij}$$

$$\text{or } \boxed{\sum_{\sigma=1}^{2} (\hat{\epsilon}_{s\sigma})_i (\hat{\epsilon}_{s\sigma})_j = \delta_{ij} - (k_{si}\, k_{sj})/k_s^2} \quad (45b)$$

We will use this relation in later calculations.

END of ASIDE

Now, add the $\mathbb{A}_s$'s in Eq. to form the most general rightward wave, i.e.

$$\mathbb{A}(\mathbb{r}, t) = \sum_s \mathbb{A}_s(\mathbb{r}, t) = \sum_{s,\sigma} \left(\frac{2\pi\hbar c}{V k_s}\right)^{\frac{1}{2}} \hat{\epsilon}_{s\sigma}\left[a_{s\sigma}(t)\, e^{i\mathbb{k}_s\cdot\mathbb{r}} + a^\dagger_{s\sigma}(t)\, e^{-i\mathbb{k}_s\cdot\mathbb{r}}\right]. \quad (46)$$

Sum over all possible wave-vectors $\mathbb{k}_s$. Later, an integral, per Eq. (43).

This is basically the generalization [of single-mode $\mathbb{A}$ of Eq. (30)] we were after... we will use this $\mathbb{A}$ in the atom $\leftrightarrow$ field coupling $\mathcal{H}_{int} \propto \mathbb{A}\cdot\mathbb{p}$. In (46), the $\{a_{s\sigma}\}$ are (operator) amplitudes, some of which may be chosen $\equiv 0$, which obey

$$\rightarrow [a_{s\sigma}, a^\dagger_{s'\sigma'}] = \delta_{ss'}\delta_{\sigma\sigma'} \;;\; \text{all other commutators} \equiv 0 \quad (47)$$

this is for independent field modes $(\mathbb{k}_s; \hat{\epsilon}_{s\sigma})$.

The $a_{s\sigma}$ have implicit time dependence: $a_{s\sigma} = a_{s\sigma}(t) = a_{s\sigma}(0)\, e^{-ik_s ct}$.

NOTE If, in $\mathbb{A}$ of (46), you consider the $a_{s\sigma}$ as just amplitudes, then $\mathbb{A}$ conforms to Fourier's Theorem (for a RW wave)-- the most general solution to the wave eqtn for $\mathbb{A}$ is a superposition of (RW) plane waves, discrete if $\mathbb{k}_s$ is.

QM details for the quantized radiation field.

10) Now the QM of the quantized radiation field is "simple". The actual quantization resides in the photon annihilation & creation operators $a_{s\sigma}$ & $a_{s\sigma}^\dagger$ in Eq. (46); they obey the SHO commutator $[a_{s\sigma}, a_{s\sigma}^\dagger] = 1$. The state of the field is specified by a set of occupation numbers $\{N_{s\sigma}\}$, with $N_{s\sigma}$ = # photons in any one of the ∞ number of possible modes ($\mathbb{k}_s$ (wavevector); $\hat{\varepsilon}_{s\sigma}$ (polarization)). The field eigenfcns are direct products of the independent basis states $|N_{s\sigma}\rangle$ for each mode, denoted by//

$$\rightarrow |\ldots; \underbrace{N_{s1}, N_{s2}}_{\text{photons at } \mathbb{k}_s}; \cdots; \underbrace{N_{s'1}, N_{s'2}}_{\text{photons at } \mathbb{k}_s'}; \ldots\rangle = |\underbrace{(N)}_{\text{denotes } \infty \text{ set of occupation \#'s } N_{s\sigma}}\rangle \qquad (48)$$

With each mode assumed orthonormal: $\langle M|N\rangle = \delta_{MN}$, we also have for the overall state: $\langle (M)|(N)\rangle = \delta_{(M)(N)}$. Matrix elements build on single mode results, e.g.

operates on mode s1 — op. on s1

$$\langle \ldots; M_{s1}, M_{s2}; \ldots; M_{s'1}, M_{s'2}; \ldots | a_{s1} | \ldots; N_{s1}, N_{s2}; \ldots; N_{s'1}, N_{s'2}; \ldots\rangle$$

$$= \sqrt{N_{s1}} \langle \ldots; *, M_{s2}; \cdots; M_{s'1}, M_{s'2}; \ldots | \ldots; *, N_{s2}; \ldots; \ldots; N_{s'1}, N_{s'2}; \ldots\rangle$$

mode s1 is integrated out

if $M_{s1} = N_{s1} - 1$, and zero otherwise [see Eq. (29)]. (49)

In this quantized radiation field, we have a total Hamiltonian (field energy):

$$\rightarrow \mathcal{H}_{rad} = \sum_{s,\sigma} (N_{s\sigma} + \tfrac{1}{2}) \hbar c k_s, \quad \text{w/ } \underline{\underline{N_{s\sigma} = a_{s\sigma}^\dagger a_{s\sigma}}} = \#\text{ operator for mode } s\sigma, \qquad (50)$$

[see Eqs (10) & (31)]. And we can find quantities like the total field momentum:

$$\rightarrow \mathbb{P}_{rad} = \frac{V}{c^2}\langle \mathbb{S}\rangle = \sum_{s,\sigma} N_{s\sigma} \hbar \mathbb{k}_s, \text{ for RW waves [see Eq. (40)]}. \qquad (51)$$

The radiation electric & magnetic fields which follow from $\mathbb{A}$ of Eq. (46) are:

$$\mathbb{E} = -\frac{1}{c}\partial \mathbb{A}/\partial t = i \sum_{s,\sigma} C_s k_s \hat{\varepsilon}_{s\sigma} [a_{s\sigma} e^{i\mathbb{k}_s\cdot\mathbb{r}} - a_{s\sigma}^\dagger e^{-i\mathbb{k}_s\cdot\mathbb{r}}],$$

$$\mathbb{B} = \nabla \times \mathbb{A} = i \sum_{s,\sigma} C_s (\mathbb{k}_s \times \hat{\varepsilon}_{s\sigma}) [a_{s\sigma} e^{i\mathbb{k}_s\cdot\mathbb{r}} - a_{s\sigma}^\dagger e^{-i\mathbb{k}_s\cdot\mathbb{r}}]; \qquad (52)$$

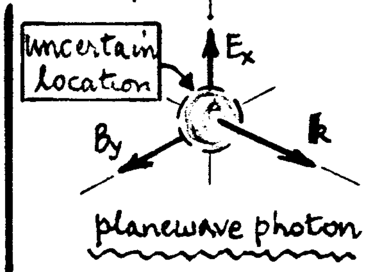
w/ $C_s = (2\pi\hbar c/Vk_s)^{1/2}$. The a's still obey: $[a_{s\sigma}, a_{s\sigma}^\dagger] = 1$. Etc.

QM Consequences of Fields as Operators.

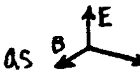
$\boxed{\text{ASIDE}}$ Fields as operators. Fluctuations & commutators¶.

1. Because of the noncommuting operators $a_{s\sigma}$ & $a_{s\sigma}^{\dagger}$ present in the radiation field $\mathbb{A}$ of Eq. (46), $\mathbb{E}$ & $\mathbb{B}$ of Eq. (52), and the photon # operator $N = \sum_{s,\sigma} a_{s\sigma}^{\dagger} a_{s\sigma}$, none of these quantities will inter-commute in general. For example, can show

$$\rightarrow [E_x(\mathbf{r},t),\ B_y(\mathbf{r}',t)] = i\hbar c \frac{\partial}{\partial z}\delta(\mathbf{r}-\mathbf{r}'). \tag{53}$$



planewave photon

This means that at a given time, we cannot determine $\mathbb{E}$ & $\mathbb{B}$ at the same point in space to arbitrary accuracy. Our old picture of a planewave photon, in sketch, acquires a fuzziness right at its origin -- we should not draw it as  any more than we should should draw electron orbits  in an atom.

2. Eq. (53) also ⇒ that if try to specify the photon momentum $\hbar\mathbf{k} \propto \mathbb{E}\times\mathbb{B}$ to arbitrary accuracy, we will lose all information on its position. Conversely, locating the photon's position accurately ($\mathbf{r}' \to \mathbf{r}$) introduces huge momentum uncertainties. The photon thus obeys a position-momentum uncertainty relation $\Delta x \Delta p \gtrsim \hbar$ much like a "classical" QM particle.

3. In QM, noncommuting operators always obey an uncertainty relation (e.g. for x and $p = -i\hbar\,\partial/\partial x$, have: $[x,p] = i\hbar \leftrightarrow \Delta x \Delta p \gtrsim \hbar$), and if one of the pair is accurately fixed, the other becomes ∞ly uncertain ($\Delta x \to 0 \Rightarrow \Delta p \sim \hbar/\Delta x \to \infty$). Thus, since $\mathbb{E}$ and the # operator N do not commute, we expect that★

$$\rightarrow (\Delta\mathbb{E})_0^2 = \langle 0|\mathbb{E}\cdot\mathbb{E}|0\rangle \rightarrow \infty. \tag{54}$$

★ For operator Q, the uncertainty ΔQ is defined as: $(\Delta Q)^2 = \langle Q^2\rangle - \langle Q\rangle^2$ (mean sq. deviation in expectation values). For $Q \to \mathbb{E}$, and the field vacuum state $|0\rangle$, can show $\langle 0|\mathbb{E}|0\rangle$. Then $(\Delta\mathbb{E})^2 = \langle 0|\mathbb{E}\cdot\mathbb{E}|0\rangle$, as used in Eq. (54).

¶ For a more detailed discussion, see J.J. Sakurai "Advanced QM" (Addison-Wesley, 1967), Sec. 2.3. Sakurai also discusses photon phase uncertainties, for example.

Vacuum Field Fluctuations. The Classical Limit.

This says that when we specify N precisely, as in the field vacuum state $|0\rangle$, we generate large and uncontrollable fluctuations $(\Delta\mathbb{E})^2 \to \infty$ in the vacuum fields. This is a bit alarming... we have just filled up the vacuum with a huge amount of random flap! This can be ameliorated a bit by allowing $\mathbb{E}$ in Eq. (54) to be averaged over a small volume

$$\bar{\mathbb{E}} = (1/\Delta V)\int_{\Delta V} \mathbb{E}\, d^3x\,,\quad \Delta V = \text{volume of linear dimension } \Delta l, \tag{55}$$

rather than being considered at a point. Then the vacuum fluctuations are

$$\rightarrow (\Delta\bar{\mathbb{E}})_0^2 = \langle 0|\bar{\mathbb{E}}\cdot\bar{\mathbb{E}}|0\rangle \sim \hbar c/(\Delta l)^4. \tag{56}$$

---

4. The above QM uncertainties & fluctuations in the EM radiation field are very alarming if they obviate the orderly workings of macroscopic fields. E.g. how can a radio work with all this going on? The answer is that the QM effects dominate when only a very few photons/unit volume are present in the fields, while classical (smooth) behavior is associated with a large number of photons/unit volume in (Maxwell-type) fields. We can see this by comparing the vacuum fluctuations in (56) with the fields present in a typical radio wave. Let the volume for comparison be a sphere of radius $\lambda = 2\pi\lambda\!\!\!^-$, the photon wavelength, and let the radio wave fields have $\bar{n}$ photons/unit volume in the sphere. Then

$$(\Delta\bar{\mathbb{E}})_0^2 \sim \hbar c/\lambda\!\!\!^{-4},\quad \langle \mathbb{E}_{rf}^2\rangle = \bar{n}\hbar c/\lambda\!\!\!^- \quad\Big\|\quad \text{Classical fields dominate QM fluctuations if: } \langle \mathbb{E}_{rf}^2\rangle \gg (\Delta\bar{\mathbb{E}})_0^2 \Rightarrow \boxed{\bar{n}\lambda\!\!\!^{-3} \gg 1}. \tag{57}$$

For a typical FM station (@ 100 MHz $\Rightarrow \lambda\!\!\!^- \simeq 48$ cm) broadcasting at 100 kW, the # photons per $\lambda\!\!\!^{-3}$ at 5 mi. distance from the antenna is $\bar{n} \simeq 10^{17}$. So $\bar{n}\lambda\!\!\!^{-3} \gg 1$ is very well satisfied, and you can listen in peace.

This formulation completes Topic II as listed on p. QF 1 above.