1. SOLUTION:

a. Far away ($d \gg R$) the ring appears as a magnetic dipole with moment $\mathbf{m} = \pi R^2 I_0 \hat{\mathbf{z}}$. The field at the sphere ($\mathbf{x} = -d\hat{\mathbf{z}}$) is therefore

$$B_z(0,0,-d) \;=\; \frac{\mu_0}{2\pi}\frac{m}{d^3} \;=\; \frac{\mu_0 I_0 R^2}{2d^3} \;=\; B_0 \quad . \tag{1}$$

A paramagnetic sphere in a uniform external field, $\mathbf{B} = B_0\mathbf{z}$, will have a field given by $\mathbf{H} = -\nabla\Phi_m$ where

$$\Phi_m(r,\theta) \;=\; \begin{cases} \mu_0^{-1}\, C\, r\, \cos\theta & , \quad r < a \\ \mu_0^{-1}\left[Ar^{-2} - B_0 r\right]\cos\theta & , \quad r > a \end{cases}$$

in coordinates centered at the sphere. Continuity of H_θ across $r = a$ is equivalent to continuity of Φ_m which requires

$$A/a^2 - B_0 a \;=\; Ca \quad .$$

Continuity of $B_r = -\mu\partial\Phi_m/\partial r$ demands

$$2A/a^3 + B_0 \;=\; -\,(\mu/\mu_0)C \quad .$$

These two can be combined to eliminate C and yield

$$(2\mu_0 + \mu)A/a^3 \;+\; (\mu_0 - \mu)B_0 \;=\; 0 \quad ,$$

from which

$$A \;=\; \frac{\mu - \mu_0}{\mu + 2\mu_0}\, a^3\, B_0 \quad . \tag{2}$$

The dipole magnetic field, on axis, from the sphere is thus

$$B_{\rm dip} \;=\; -\frac{\partial}{\partial r}\left(\frac{\mu - \mu_0}{\mu + 2\mu_0}\,\frac{a^3\,B_0}{r^2}\right) \;=\; \frac{2(\mu - \mu_0)}{\mu + 2\mu_0}\,\frac{a^3\,B_0}{r^3} \quad . \tag{3}$$

Comparison to the dipole field in eq. (1) reveals the dipole moment on the sphere

$$\boxed{\mathbf{m}_{\rm sph} \;=\; \frac{4\pi(\mu - \mu_0)}{\mu + 2\mu_0}\,\frac{a^3\,B_0}{\mu_0}\hat{\mathbf{z}} \;=\; \frac{2\pi(\mu - \mu_0)}{\mu + 2\mu_0}\,\frac{a^3 R^2\, I_0}{d^3}\hat{\mathbf{z}}} \tag{4}$$

which is *upward*, parallel to that of the loop, since $\mu > \mu_0$ in a paramagnetic material.

b. Equation (3) can be used to evaluate the additional magnetic field at the loop, due to the sphere,

$$\mathbf{B}_{\rm dip}(0,0,0) \;=\; \frac{2(\mu - \mu_0)}{\mu + 2\mu_0}\,\frac{a^3\,B_0}{d^3}\hat{\mathbf{z}} \;=\; \frac{\mu - \mu_0}{\mu + 2\mu_0}\,\frac{a^3 R^2 \mu_0\, I_0}{d^6}\hat{\mathbf{z}} \tag{5}$$

Integrating this over the area of the loop gives an additional (upward) flux

$$\boxed{\Delta\psi \;=\; \pi R^2\,\hat{\mathbf{z}}\cdot\mathbf{B}_{\rm dip}(0,0,0) \;=\; \frac{\pi(\mu - \mu_0)}{\mu + 2\mu_0}\,\frac{a^3 R^4 \mu_0\, I_0}{d^6} \quad .} \tag{6}$$

c. Beginning with $\psi = \mathcal{L} I_0$ we can relate the change in flux to a change in inductance

$$\Delta\mathcal{L} = \frac{\Delta\psi}{I_0} = \frac{\pi(\mu-\mu_0)}{\mu+2\mu_0}\frac{a^3 R^4 \mu_0}{d^6} \quad . \tag{7}$$

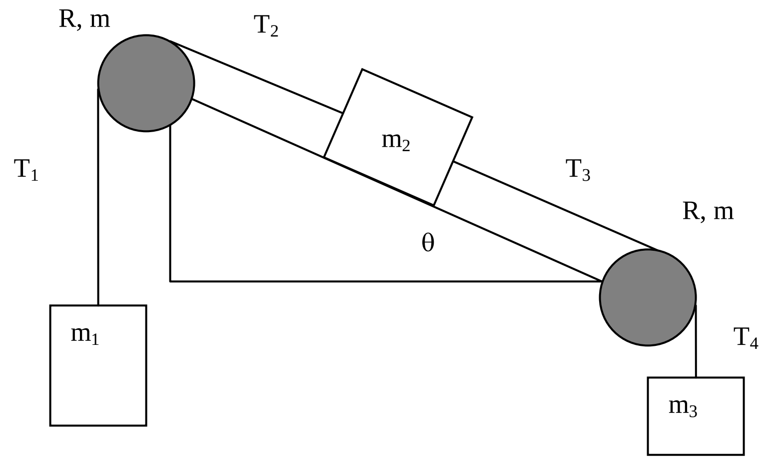
This then gives

$$\boxed{\frac{\Delta\mathcal{L}}{\mathcal{L}} = \frac{\Delta\mathcal{L}}{\alpha\mu_0 R} = \frac{\pi(\mu-\mu_0)}{\alpha(\mu+2\mu_0)}\frac{a^3 R^3}{d^6} \quad .} \tag{8}$$

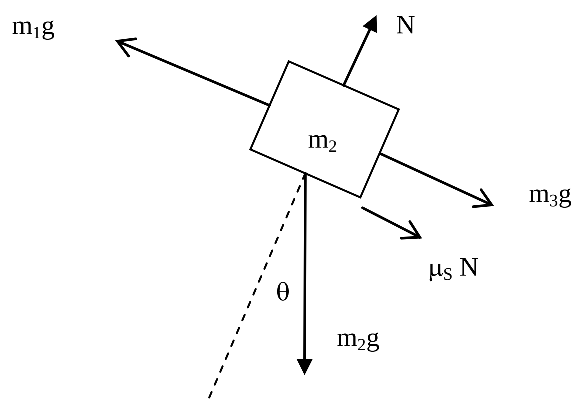
The inductance is *increased* by a factor scaling with the inverse *sixth* power of distance. The relative change is proportional to the *volume* of the sphere and the third power of the loop's circumference.

2. <u>Solution</u>

a) First we label all the tensions in the ropes as shown. Then we note that without any motion of the top pully, $T_1 = T_2$. And for the lower pully, $T_3 = T_4$.



Thus we can draw the following free body diagram for the mass m_2 when it is not moving (when the acceleration is zero). In this case all the forces balance and we get two equations for the two orthogonal directions.



$$m_1 g = m_3 g + \mu_S N + m_2 g \sin\theta$$
$$N = m_2 g \cos\theta$$

Combining these two equations, we can solve for the mass m_1 that will just balance the other forces.

$$\boxed{m_1 = m_3 + \mu_S m_2 \cos\theta + m_2 \sin\theta}$$

When m_1 increases above this value the mass m_2 will start to move up the slope.

b) Now to find the acceleration up the slope, we will have a free body diagram for each mass that will lead to these five equations, where we take positive to be the direction of a falling mass m_1.

$$m_1g - T_1 = m_1a$$
$$\tau_1 - \tau_2 = I\alpha$$
$$T_2 - T_3 - \mu_K m_2 g\cos\theta - m_2 g\sin\theta = m_2a$$
$$\tau_3 - \tau_4 = I\alpha$$
$$T_4 - m_3g = m_3a$$

We used Newton's second law for the first, third and fifth equations and Newton's second law in angular form for the second and fourth equations. Putting in the torques and the moment of inertia for a solid disc pulley and relating the angular acceleration to the linear acceleration, we get:

$$m_1g - T_1 = m_1a$$
$$R\left(T_1 - T_2\right) = \left(\tfrac{1}{2}mR^2\right)\left(\frac{a}{R}\right)$$
$$T_2 - T_3 - \mu_K m_2 g\cos\theta - m_2 g\sin\theta = m_2a$$
$$R\left(T_3 - T_4\right) = \left(\tfrac{1}{2}mR^2\right)\left(\frac{a}{R}\right)$$
$$T_4 - m_3g = m_3a$$

Note that all the terms in R now drop out and we get for the five equations:

$$m_1g - T_1 = m_1a$$
$$T_1 - T_2 = \tfrac{1}{2}ma$$
$$T_2 - T_3 - \mu_K m_2 g\cos\theta - m_2 g\sin\theta = m_2a$$
$$T_3 - T_4 = \tfrac{1}{2}ma$$
$$T_4 - m_3g = m_3a$$

At this point it is really easy to add all the equations together (since all the tensions will cancel!) to get one equation.

$$m_1g - \mu_K m_2 g\cos\theta - m_2 g\sin\theta - m_3 g = m_1a + \tfrac{1}{2}ma + m_2a + \tfrac{1}{2}ma + m_3a$$

We can now solve for the acceleration a:

$$\boxed{a = \frac{g\left(m_1 - \mu_K m_2\cos\theta - m_2\sin\theta - m_3\right)}{m_1 + m_2 + m_3 + m}}$$

where m_1 is given by the result in part a) since that was the value of m_1 that was just big enough to start the motion.

$$m_1 = m_3 + \mu_S m_2\cos\theta + m_2\sin\theta$$

c) Now to find the power lost to friction, we start with the work W done by the constant friction force $F_{friction}$ over a distance x:

$$W = F_{friction}x = \left(\mu_K m_2 g\cos\theta\right)x$$

This gives the energy lost to heat in the friction process. To get the power P we take the time derivative to get:

$$P = \frac{d}{dt}W = \left(\mu_K m_2 g\cos\theta\right)\frac{d}{dt}x = \mu_K m_2 g\cos\left(\theta\right)v\left(t\right)$$

Since the acceleration 'a' is constant, we can use our kinematic equation for v

$$v\left(t\right) = v(0) + at = 0 + \frac{g\left(m_1 - \mu_K m_2\cos\theta - m_2\sin\theta - m_3\right)}{m_1 + m_2 + m_3 + m}t$$

Thus for the power lost, we get

$$\boxed{P = \mu_K m_2 g \cos(\theta) \left[\frac{g\left(m_1 - \mu_K m_2 \cos\theta - m_2 \sin\theta - m_3\right)}{m_1 + m_2 + m_3 + m} \right] t}$$

where m_1 is again given by the result in part a) since that was the value of m_1 that was just big enough to start the motion.

$$m_1 = m_3 + \mu_S m_2 \cos\theta + m_2 \sin\theta$$

3. Solution

a. When ^3H decays through $^3H \rightarrow {}^3He^+ + e^- + \bar{\nu}$ transition the electron orbiting the nucleus experience a change in the nuclear charge from $Z=1$ to $Z=2$. As a result of this the hydrogen wave functions are modified slightly in that while orbiting the ^3H nucleus the Bohr radius of the electron is a_0 , after the transition it reduces to $a_0/2$ around $^3He^+$ nucleus. This change in the wave function modifies the radial wave functions slightly: Initially, electron was in the ground state of the ^3H isotope ($Z=1$) described by the wave function $\phi_{100} = R_{10}(r)Y_{00}(\theta,\phi) = \frac{1}{\sqrt{\pi}}\left(\frac{1}{a_0}\right)^{3/2} e^{-r/a_o}$, where $Y_{00}(\theta,\phi) = \frac{1}{\sqrt{4\pi}}$ is a constant, independent of angular variables. This means that this part of the ^3H wave function also represents the $Y_{00}(\theta,\phi)$ spherical harmonics associated with the wave functions of $^3He^+$ ion with nuclear charge $Z=2$. The probability amplitude sought by the problem can now be found by the overlap integral:

$$A = \int \phi_{100}\ \phi_{210} r^2 dr\ d\Omega = \int_0^\infty R^{^3H}{}_{10}(r) R^{^3He^+}{}_{20}(r) r^2 dr \int Y_{00}(\theta,\phi) Y_{10}{}^*(\theta,\phi) d\Omega$$

, and the probability is simply given by $P = |A|^2$. However, spherical harmonics are orthogonal functions represented by $\int Y_{l^{'}m^{'}}(\theta,\phi) Y_{lm}{}^*(\theta,\phi) d\Omega = \delta_{ll^{'}}\delta_{mm^{'}}$, which means $A \equiv 0$ regardless of the value of the radial integration. Therefore, probability of finding the $^3He^+$ ion in the state $\phi_{nlm} = R_{nl}(r)Y_{lm}(\theta,\phi) = \phi_{210}$ where $l \neq 0$ is zero.

b. This time $l=0$ and hence $m=0$ the probability amplitude A as given below

$$A = \int \phi_{100}\ \phi_{200} r^2 dr\ d\Omega = \int_0^\infty R^{^3H}{}_{10}(r) R^{^3He^+}{}_{20}(r) r^2 dr \int Y_{00}(\theta,\phi) Y_{00}{}^*(\theta,\phi) d\Omega$$

will not be zero.

This is because it involves $\int Y_{00}(\theta,\phi) Y_{00}{}^*(\theta,\phi) d\Omega \equiv 1$ and $\int_0^\infty R^{^3H}{}_{10}(r) R^{^3He^+}{}_{20}(r) r^2 dr \neq 0$ non-zero integrals, even though the radial functions are orthogonal functions and $\int_0^\infty R_{nl}(r) R^*_{n^{'}l^{'}}(r) r^2 dr = \delta_{nn^{'}}\delta_{ll^{'}}$. In our case, however, orthogonality does not apply because each radial function with different nl represents electron orbiting around a different hydrogen-like nucleus, but different nuclear charges. Using the functions provided in the *hint* we now can evaluate the probability amplitude:

$$A = \int \phi_{100}\ \phi_{200} r^2 dr\ d\Omega = 4\pi \int_0^\infty \frac{1}{\sqrt{\pi}}\left(\frac{1}{a_0}\right)^{3/2} e^{-r/a_0} \frac{1}{4\sqrt{2\pi}}\left(\frac{2}{a_0}\right)^{3/2}\left(2 - \frac{2r}{a_0}\right) e^{-r/a_0} r^2 dr$$

which results in $A = -1/2$. This yields a probability of $P = |A|^2 = 0.25$.

Similarly, we would have found that the probabilities that the $^3He^+$ ion is in the states of $\phi_{nlm} = \phi_{100}$ and $\phi_{nlm} = \phi_{300}$, respectively, are 0.702 and 0.013. Because of the restriction $\int Y_{00}(\theta,\phi) Y_{lm}{}^*(\theta,\phi) d\Omega = \delta_{0,l}\delta_{0m}$,in general, the only states the $^3He^+$ ion can couple to are those states having a spherical symmetry described by $\phi_{nlm} = R_{n0}(r)Y_{00}(\theta,\phi) = \phi_{n00}$ functions as exemplified above.

4. <u>Solution</u>

With the B-field in the z-direction, it is appropriate to quantize the spins in the $+z$ (up) and $-z$ (down) directions. These can be denoted by $\vec{u} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\vec{d} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Acting on these states with the Pauli interaction, $H = \mu B_0 \sigma_z$, gives

$$H\vec{u} = \hbar\omega\vec{u} \qquad H\vec{d} = -\hbar\omega\vec{d} \qquad \hbar\omega = \mu B_0$$

The eigenstates of σ_x are $\psi_x = \frac{1}{\sqrt{2}}(\vec{u} + \vec{d})$ for the (+x)-direction and $\psi_{\bar{x}} = -\frac{1}{\sqrt{2}}(\vec{u} - \vec{d})$ for the (-x)-direction since $\sigma_x \psi_x = \psi_x$ and $\sigma_x \psi_{\bar{x}} = -\psi_{\bar{x}}$. Similarly the eigenstates of σ_y are $\psi_y = \frac{1}{\sqrt{2}}(\vec{u} + i\vec{d})$ and $\psi_{\bar{y}} = \frac{1}{\sqrt{2}}(\vec{u} - i\vec{d})$ since $\sigma_y \psi_y = \psi_y$ and $\sigma_y \psi_{\bar{y}} = -\psi_{\bar{y}}$.

At time t_0, we start in state ψ_x and allow it to time evolve so that the state becomes

$$\Psi_x(t) = \frac{1}{\sqrt{2}}\left(\vec{u}e^{-i\omega t} + \vec{d}e^{i\omega t}\right).$$

The amplitude of the projection of this state into the *negative y*-direction is found by calculating the matrix element

$$\left\langle \psi_{\bar{y}} \middle| \Psi_x(t) \right\rangle = \frac{1}{2}\left(e^{-i\omega t} + ie^{i\omega t}\right).$$

The probability is the square of the absolute magnitude of the matrix element

$$P_{x\bar{y}}(t) = \frac{1 - \sin(2\omega t)}{2} = \cos^2\left(\omega t + \frac{\pi}{4}\right) = \cos^2\left(\frac{B_0 \mu t}{\hbar} + \frac{\pi}{4}\right).$$

5. Solution

For r > a, no free source, so $\nabla^2 \Phi = 0$. So, $\Phi = R(r)\,\Theta(\theta)$ (1)

Spherical with azimuthal symmetry. So spherical symmetry with m = 0. Then for r > a the general solution is:

$$\Phi = \sum_{\ell=0}^{\infty}[A_\ell\ r^{\ell} + B_\ell\ r^{-(\ell+1)}]\,P_\ell\ (\cos\theta)$$
$$= \sum_{\ell=0}^{\infty} B_\ell\ r^{-(\ell+1)}\,P_\ell\ (\cos\theta),\quad \text{since } \Phi = 0 \text{ at infinity.}$$

Apply the boundary condition at r = a: $0 \le \theta \le \pi/2$, , $\Phi = V_N$
$\pi/2 \le \theta \le \pi$, , $\Phi = V_S$,

get $V_N = \sum_{\ell=0}^{\infty} B_\ell\, a^{-(\ell+1)}\ P_\ell\ (\cos\theta)$, $0 \le \theta \le \pi/2$ $(1 \ge \cos\theta \ge 0)$
$V_S = \sum_{\ell=0}^{\infty} B_\ell\, a^{-(\ell+1)}\ P_\ell\ (\cos\theta)$, $\pi/2 \le \theta \le \pi$.

Muliply the last two equations by P_ℓ (cos θ) d(cos θ) and with cos θ = x,

Get $\int_{-1}^{0} V_S\ P_\ell\ (x)dx + \int_{0}^{1} V_N\ P_\ell\ (x)dx = \frac{2}{2\ell+1} B_\ell a^{-(\ell+1)}$

By using $P^\ell(x)$ orthonormality equation,

$$B_\ell = a^{(\ell+1)}\,\frac{2\ell+1}{2}\,[V_N + (-1)^\ell V_S]\int_0^1 P^\ell(x)dx.$$

Using [Hint] relation for $\int_0^1 P^\ell(x)dx$

get $B_\ell = a^{(\ell+1)}\,\frac{1}{2}(V_N - V_S)\,\frac{(2\ell+1)(\ell-1)!}{\left\{\frac{(\ell+1)}{2}\right\}!\left\{\frac{(\ell-1)}{2}\right\}!2^\ell}\,(-1)^{(\ell-1)/2}$ for ℓ odd,

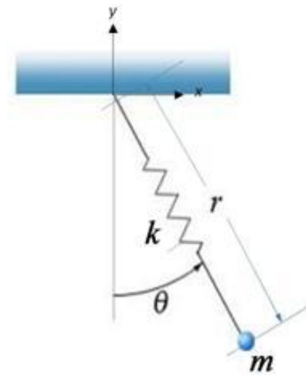
$B_0 = a\,\frac{1}{2}(V_N + V_S)$ for $\ell = 0$, and $B_\ell = 0$ for ℓ even.

So,

$$\Phi(r,\theta) = a\,\frac{1}{2}(V_N + V_S)\,\frac{1}{r} + \sum_{\ell=\text{odd}}^{\infty}\left(\frac{a}{r}\right)^{(\ell+1)}\frac{1}{2}(V_N - V_S)\,\frac{(2\ell+1)(\ell-1)!}{\left\{\frac{(\ell+1)}{2}\right\}!\left\{\frac{(\ell-1)}{2}\right\}!2^\ell}$$
$$\times\ (-1)^{(\ell-1)/2}\ P_\ell\ (\cos\theta).$$

6. Solution

a. Lagrangian $L = K - V$ can easily be found (refer to the figure on the right for details) as: $L = \frac{1}{2}m(\dot{r}^2 + (r\dot{\theta})^2) - \frac{1}{2}k(r - r_o)^2 + mgrCos\theta$, where θ and r are independent variables.



The equations of motion can be found from $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \left(\frac{\partial L}{\partial q}\right) = 0$:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \left(\frac{\partial L}{\partial \theta}\right) = m\frac{d}{dt}\left(r^2\dot{\theta}\right) + mgrSin\theta = 0 \;\Rightarrow$$

$$m\frac{d}{dt}\left(r^2\dot{\theta}\right) + mgrSin\theta = 0 \quad \cdots \quad (1)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \left(\frac{\partial L}{\partial r}\right) = m\frac{d}{dt}\left(\dot{r}\right) - mr\dot{\theta}^2 + k(r - r_o) - mgCos\theta = 0 \quad \Rightarrow$$

$$m\ddot{r} - mr\dot{\theta}^2 + k(r - r_o) - mgCos\theta = 0 \quad \cdots \quad (2)$$

b. Equation (1) is nothing but the net torque acting on particle m being equal to the rate of change in the angular momentum:

Equation (1): $m\frac{d}{dt}\left(r^2\dot{\theta}\right) + mgrSin\theta = \frac{d}{dt}\left(mr\dot{\theta}r\right) + mgrSin\theta = 0$

$$\Rightarrow \quad \frac{d}{dt}\left(rmv\right) = -rmgSin\theta \quad \Rightarrow \quad \frac{d}{dt}\left(\vec{r} \times \vec{p}\right) = \vec{r} \times m\vec{g} = \frac{d}{dt}\vec{L} = \vec{\tau}$$

Equation (2) is nothing but the net force acting on the particle along the $\hat{r}$ direction being equal to the mass times the net acceleration along the $\hat{r}$ direction, as described below:

$$m\ddot{r} - mr\dot{\theta}^2 + k(r - r_o) - mgCos\theta = 0 \quad \Rightarrow \; m\ddot{r} - mr\dot{\theta}^2 = -k(r - r_o) + mgCos\theta$$

c. The small amplitude approximation for Equation (1) yields:

$$m\frac{d}{dt}\left(r^2\dot{\theta}\right) + mgrSin\theta = 0 \cdots \Rightarrow \frac{d}{dt}\left((r_e + u)^2\dot{\theta}\right) + g(r_e + u)\theta = 0$$

$$\frac{d}{dt}\left((1 + u/r_e)^2\dot{\theta}\right) + (g/r_e)(1 + u/r_e)\theta = 0 \cdots \Rightarrow \frac{d}{dt}\left((1 + 2u/r_e)\dot{\theta}\right) + (g/r_e)(1 + u/r_e)\theta = 0$$

where $r_e = r_o + \frac{mg}{k}$ is the equilibrium position of the spring under the influence of gravity. By rearranging terms in the last equation above we obtain:

$$\ddot{\theta} + (g/r_e)\theta + \frac{d}{dt}\left(2u\dot{\theta}/r_e\right) + gu\theta/r_e^{\,2} = 0 \quad ..... \quad (3).$$

The small amplitude approximation for Equation (2) yields:

$$m\ddot{r} - mr\dot{\theta}^2 + k(r - r_o) - mgCos\theta = 0 \quad \rightarrow \quad \ddot{u} + \frac{k}{m}u + g - r_e\dot{\theta}^2 - g(1 - \frac{\theta^2}{2}) = 0$$

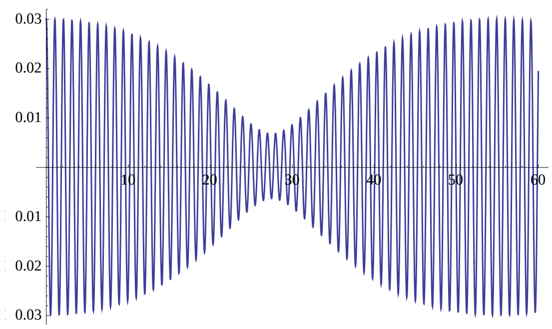
Rearranging the last equation yields:

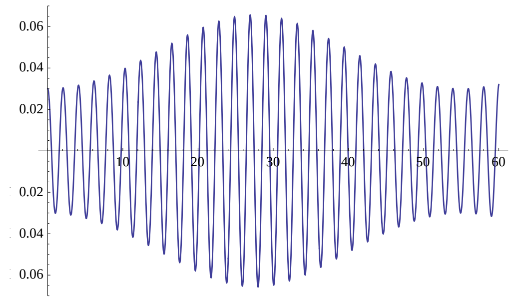
$$\ddot{u} + \frac{k}{m}u + g\frac{\theta^2}{2} - r_e\dot{\theta}^2 = 0 \quad ......(4),$$ where we use $u = r - r_e$. $Cos\theta \simeq 1 - \frac{\theta^2}{2}$ and $r\dot{\theta}^2 \simeq r_e\dot{\theta}^2$

d. If we ignore the second-order terms in the amplitudes of the motions in u and θ in the equations described by (3) and (4) this decouples the u and θ motions, and the equations of motion become:

$\ddot{\theta}+\frac{g}{r_e}\theta=0$ and $\ddot{u}+\frac{k}{m}u=0$. The solutions of these equations are two decoupled simple harmonic oscillators whose angular frequencies are given by $\omega_r=\sqrt{\frac{k}{m}}$ and $\omega_\theta=\sqrt{\frac{g}{r_e}}$. The two motions are totally independent. However, in reality these motions are highly coupled as suggested by the small amplitude equations of motion described equations (3) and (4). Below I illustrated this by an example where all units are in SI.

The exact solutions for $u(t)$ and $\theta(t)$ for small amplitude approximations using Mathematica are shown below. The graph on the left shows $u(t)$ vs. t, and the one on the right gives $r_e\theta(t)$ vs. t. I used $m=1$ kg, $k=36$ N/m, $\mathrm{r}_o=0.818$ m, and $\mathrm{r}_e=1.09$ m.





7. Solution

a) To find the capacitance per unit length C, we first put a charge per unit length +Q on the inner conductor and a –Q on the inner surface of the outer conductor. Then use Gauss' Law and symmetry to get the electric field E between the conductors:

$$\oint_S \vec{E}\cdot\hat{n}dA = \frac{Q\ell}{\varepsilon_o} \rightarrow E(2\pi r\ell) = \frac{Q\ell}{\varepsilon_o} \rightarrow \vec{E} = \frac{Q}{2\pi\varepsilon_o}\frac{1}{r}\hat{r}$$

Then integrate over r to find the electric potential V between the conductors:

$$V = V(a) - V(b) = \int_a^b \vec{E}\cdot d\vec{r} = \int_a^b \frac{Q}{2\pi\varepsilon_o}\frac{1}{r}\hat{r}\cdot\hat{r}dr = \frac{Q}{2\pi\varepsilon_o}\ln\left(\frac{b}{a}\right)$$

Now we can find the capacitance per unit length C using:

$$C = \frac{Q}{V} = \frac{Q}{\frac{Q}{2\pi\varepsilon_o}\ln\left(\frac{b}{a}\right)}$$

$$\boxed{C = \frac{2\pi\varepsilon_o}{\ln\left(\frac{b}{a}\right)}}$$

Note that both the capacitance and the charge are given per unit length so the units come out fine.

b) To find the inductance per unit length L, we put a current I out of the page for the center conductor and into the page for the outer conductor. This generates a magnetic field B that we can find using Ampere's Law and symmetry to get B between the conductors:

$$\oint_C \vec{B}\cdot d\vec{s} = \mu_o I \rightarrow B(2\pi r) = \mu_o I \rightarrow \vec{B} = \frac{\mu_o I}{2\pi}\frac{1}{r}\hat{\phi}$$

Then we integrate over a surface perpendicular to the magnetic field to get the magnetic flux Φ_m between the conductors:

$$\Phi_m = \int_A \vec{B}\cdot\hat{n}dA = \int_0^\ell\int_a^b \frac{\mu_o I}{2\pi}\frac{1}{r}\hat{\phi}\cdot\hat{\phi}drdz = \frac{\mu_o I}{2\pi}\int_0^\ell dz\int_a^b \frac{1}{r}dr = \frac{\mu_o I\ell}{2\pi}\ln\left(\frac{b}{a}\right)$$

Now we can find the inductance per unit length L using:

$$L = \frac{1}{\ell}\frac{\Phi_m}{I} = \frac{1}{\ell}\frac{\frac{\mu_o I\ell}{2\pi}\ln\left(\frac{b}{a}\right)}{I}$$

$$\boxed{L = \frac{\mu_o}{2\pi}\ln\left(\frac{b}{a}\right)}$$

c) Now to find the characteristic impedance of the transmission line, we use the hint. Combining the impedance of the capacitor with Z_o in parallel, we get:

$$\frac{1}{Z_1} = \frac{1}{Z_C} + \frac{1}{Z_o} = i\omega Cdx + \frac{1}{Z_o} = \frac{Z_o i\omega Cdx + 1}{Z_o} \rightarrow Z_1 = \frac{Z_o}{Z_o i\omega Cdx + 1}$$

Then combining this impedance in series with the impedance of the inductor, we should get Z_o back:

$$Z_o = Z_1 + Z_L = Z_1 + i\omega L dx = \frac{Z_o}{Z_o i\omega C dx + 1} + i\omega L dx = \frac{Z_o + i\omega L dx \left(Z_o i\omega C dx + 1\right)}{Z_o i\omega C dx + 1}$$
$$= \frac{Z_o + i\omega L dx Z_o i\omega C dx + i\omega L dx}{Z_o i\omega C dx + 1} = \frac{Z_o + i\omega L dx}{Z_o i\omega C dx + 1}$$

Note that we dropped the nonlinear term in dx^2 since it is negligible for small dx. Continuing to simplify our equation, we get:

$$Z_o \left(Z_o i\omega C dx + 1\right) = Z_o + i\omega L dx \rightarrow Z_o^2 i\omega C dx + Z_o = Z_o + i\omega L dx \rightarrow Z_o^2 = \frac{i\omega L dx}{i\omega C dx}$$

Thus our solution for Z_o is given by:

$$\boxed{Z_o = \sqrt{\frac{L}{C}}}$$

Putting in our values of L and C from parts a) and b) we get:

8. Solution

a) Pressure: From the ideal gas law, $PV = nRT \ (or) \ Nk_B T$, for an isothermal expansion, the temperature is a constant and therefore

$$P_f = P_0 \frac{V_0}{V_f} = P_0 \frac{V_0}{2V_0} = \frac{P_0}{2}.$$

b) Temperature: Isothermal process requires $T_f = T_0$.

c) Heat Absorbed: 2nd law of thermodynamics states $-\Delta E = W - Q$, but the internal energy is only a function of the temperature for an ideal gas, so $W = Q$, and therefore
$Q_{absorbed} = nRT_0 \ln 2 \ (or) \ Nk_B T_0 \ln 2$

d) Work: Work is the integral of the PV-curve

$$W = \int_{V_0}^{V_f} PdV = \int_{V_0}^{2V_0} \frac{nRT_0}{V} dV = nRT_0 \ln 2 \ (or) \ Nk_B T_0 \ln 2$$

e) Change in Internal Energy: The internal energy is only a function of the temperature for an ideal gas so $\Delta E = 0$

f) Change in entropy: $\Delta S = \dfrac{\Delta Q}{T_0} = nR\ln 2 \ (or) \ Nk_B \ln 2$

For an adiabatic free expansion, no heat is absorbed.

a) Pressure: From the ideal gas law $PV = nRT \ (or) \ Nk_B T$, for an adiabatic free expansion, the temperature is a constant (see below) and therefore

$$P_f = P_0 \frac{V_0}{V_f} = P_0 \frac{V_0}{2V_0} = \frac{P_0}{2}.$$

b) Temperature: for an adiabatic free expansion, no work is done and no heat is absorbed. The 2nd law requires $\Delta E = 0$ (see below), and since for an ideal gas, *E* is only a function of *T*, requires $T_f = T_0$.

c) Heat Absorbed: For an adiabatic expansion, no heat is absorbed $Q_{absorbed} = 0$.

d) Work: for a free expansion, no work is done $W=0$.

e) Change in Internal Energy: 2nd law of thermodynamics states $-\Delta E = W - Q$, so $\Delta E = 0$.

f) Change in entropy: The entropy does increase. In this case we must determine the entropy change from a reversible process, such as the isothermal expansion of the gas.
$\Delta S = \dfrac{\Delta Q}{T_0} = nR\ln 2 \ (or) \ Nk_B \ln 2.$

9. SOLUTION:

a. The two-dimensional SHO has energy-eigenfunctions, $|\,m,n\,\rangle = \phi_m(x)\phi_n(y)$, such that

$$\hat{H}|\,m,n\,\rangle \;=\; E_{m,n}|\,m,n\,\rangle \;=\; \hbar\Omega_0(m+n+1)|\,m,n\,\rangle \quad . \tag{2}$$

The wave function in eq. (1) can be simply expanded to form

$$\begin{aligned} |\,\psi\,\rangle &= \left[\frac{\sqrt{3}}{2}\phi_1(x) \;-\; \frac{1}{2}\phi_2(x)\right]\left[\frac{1}{\sqrt{2}}\phi_0(y) \;-\; \frac{i}{\sqrt{2}}\phi_1(y)\right] \\ &= \frac{\sqrt{3}}{2\sqrt{2}}\,|\,1,0\,\rangle \;-\; \frac{i\sqrt{3}}{2\sqrt{2}}\,|\,1,1\,\rangle \;-\; \frac{1}{2\sqrt{2}}\,|\,2,0\,\rangle \;+\; \frac{i}{2\sqrt{2}}\,|\,2,1\,\rangle \quad . \end{aligned} \tag{3}$$

It is clear that the state is a superposition of *four* energy eigenstates for which $m+n =$ 1, 2, 2, and 3, in that order (the second and third states are degenerate). An energy measurement might yield one of the three energy eigenvalues, namely

$$E_{m,n} \;=\; 2\Omega_0\hbar \quad , \quad 3\Omega_0\hbar \quad , \quad 4\Omega_0\hbar \quad . \tag{4}$$

The probability of measuring the eigenvalue of a given eigenstate is equal to the square amplitude of its coefficient in the wave function. The results may be read off to make up the table of probability

E	probability
$2\Omega_0\hbar$	$3/8$
$3\Omega_0\hbar$	$3/8+1/8=1/2$
$4\Omega_0\hbar$	$1/8$

b. The expected energy is

$$\boxed{\langle\,E\,\rangle \;=\; \frac{3}{8}(2\Omega_0\hbar) \;+\; \frac{4}{8}(3\Omega_0\hbar) \;+\; \frac{1}{8}(4\Omega_0\hbar) \;=\; \frac{22}{8}\Omega_0\hbar \;=\; \frac{11}{4}\Omega_0\hbar} \tag{5}$$

c. The angular momentum operator acts on energy eigenstates

$$\hat{L}_z|\,m,n\,\rangle \;=\; i\hbar\sqrt{m(n+1)}\;|\,m-1,n+1\,\rangle \;-\; i\hbar\sqrt{(m+1)n}\;|\,m+1,n-1\,\rangle \quad .$$

It does not change the energy of the state (i.e. $m+n$ is unchanged), and only *different* degenerate energy eigenstates will be coupled. Considering only the two degenerate terms in eq. (3) with $m+n=2$ we find

$$\begin{aligned} \hat{L}_z|\,\psi\,\rangle &= -\frac{i\sqrt{3}}{2\sqrt{2}}\,\hat{L}_z|\,1,1\,\rangle \;-\; \frac{1}{2\sqrt{2}}\,\hat{L}_z|\,2,0\,\rangle \;+\; \cdots \\ &= -\frac{\hbar\sqrt{3}}{2\sqrt{2}}\sqrt{2}\,|\,2,0\,\rangle \;-\; \frac{i\hbar}{2\sqrt{2}}\sqrt{2}\,|\,1,1\,\rangle \;+\; \cdots \\ &= -\hbar\left(\frac{\sqrt{3}}{2}\,|\,2,0\,\rangle \;+\; \frac{i}{2}\,|\,1,1\,\rangle\right) \;+\; \cdots \quad , \end{aligned}$$

where the dots denote kets not present in $|\,\psi\,\rangle$. The transpose of eq. (1) is

$$\langle\,\psi\,| \;=\; \frac{\sqrt{3}}{2\sqrt{2}}\,\langle\,1,0\,| \;+\; \frac{i\sqrt{3}}{2\sqrt{2}}\,\langle\,1,1\,| \;-\; \frac{1}{2\sqrt{2}}\,\langle\,2,0\,| \;-\; \frac{i}{2\sqrt{2}}\,\langle\,2,1\,| \quad .$$

The expectation is therefore

$$\boxed{\begin{aligned} \langle L_z \rangle &= \langle \psi | \hat{L}_z | \psi \rangle = -\hbar \left(\frac{i\sqrt{3}}{2\sqrt{2}} \langle 1,1 | - \frac{1}{2\sqrt{2}} \langle 2,0 | \right) \left(\frac{\sqrt{3}}{2} | 2,0 \rangle + \frac{i}{2} | 1,1 \rangle \right) \\ &= -\hbar \left(-\frac{\sqrt{3}}{4\sqrt{2}} - \frac{\sqrt{3}}{4\sqrt{2}} \right) = \frac{\sqrt{3}}{2\sqrt{2}} \hbar = \frac{\sqrt{6}}{4} \hbar \qquad (6) \end{aligned}}$$

10. <u>Solution</u>

(a) Since the gas is nonrelativistic, $v = p/m$. Insert this v into Equation (1), and together with Equation (2) integrate (1) from 0 to infinity, and we obtain Equation (3).

(b) Since the object is spherical, spherical symmetry applies, and then parameters vary along radial distance r only. Then, for the star to be stationary (i.e., no contraction nor expansion), at a point distance r away from the center, gravitational force F_G
$= -G\, M(r)\, dm / r^2$ (downward) on a mass element dm, must be balanced by pressure force $F_P = P\, dA - (P + dP)\, dA = -dP\, dA$ (upward). Here M(r), the mass of the sphere within the radial distance r, $= \int_0^r \rho(r)\, 4\pi\, r^2\, dr$, while the mass element $dm = \rho(r)\, dA\, dr$ at a distance r from the center. For mechanical balance $F_G = F_P$. Dividing this equation by dA dr, we obtain 1^{st} of Equation (4). 2^{nd} of Equation (4) is just definition of density, from $M(r) = \int_0^r \rho(r)\, 4\pi\, r^2\, dr$, or $\rho(r) = dM(r)/dV$, and $dV = 4\pi\, r^2\, dr$, for spherical shell at r.

11. SOLUTION:

a. Introducing the form $\psi(x) = A\,x^\nu$ into the homogeneous form of eq. (1), $\mathcal{L}\psi = 0$, leads to the equation

$$A\nu(\nu+1)\,x^\nu \;-\; 2Ax^\nu \;=\; A(\nu^2+\nu-2) \;=\; A(\nu-1)(\nu+2) \;=\; 0 \quad , \tag{2}$$

which requires $\nu = +1$ or $\nu = -2$. Thus the general homogeneous solution is

$$\boxed{\psi_h(x) \;=\; A\,x \;+\; B\,x^{-2} \quad . \tag{3}}$$

b. The Green's function $G(x,y)$ should satisfy the equation

$$\mathcal{L}G \;=\; \delta(x-y) \quad , \tag{4}$$

with homogenous BCs

$$G(1,y) \;=\; 0 \quad , \quad G(x,y) \to 0 \;\text{ as }\; x \to \infty \quad . \tag{5}$$

Since eq. (4) corresponds to the homogeneous equation, $\mathcal{L}G = 0$, for $x < y$ and $x > y$ we can propose the general solution.

$$G(x,y) \;=\; \begin{cases} A_-\,x \;+\; B_-\,x^{-2} & , \quad 1 \le x < y \\ A_+\,x \;+\; B_+\,x^{-2} & , \quad x > y \end{cases} \tag{6}$$

Satisfying the BC, eq. (5), requires

$$A_- \;+\; B_- \;=\; 0 \quad \text{and} \quad A_+ \;=\; 0 \quad ,$$

which can be used to eliminate A_+ and B_- from the Green's function

$$G(x,y) \;=\; \begin{cases} A_-\,(x - x^{-2}) & , \quad 1 \le x < y \\ B_+\,x^{-2} & , \quad x > y \end{cases} \tag{7}$$

Since $\mathcal{L}$ is a second order operator $G(x,y)$ must be continuous at $x = y$, requiring

$$A_-(y - y^{-2}) \;=\; B_+ y^{-2} \quad \Longrightarrow \quad B_+ \;=\; (y^3 - 1)\,A_- \quad .$$

Integrating eq. (4) across an infinitesimal interval spanning $x = y$ leads to

$$\begin{aligned}
\lim_{\epsilon\to 0}\left[\int_{y-\epsilon}^{y+\epsilon} \frac{d}{dx}(x^2 G_x)\,dx \;-\; \int_{y-\epsilon}^{y+\epsilon} 2G\,dx\right] \;&=\; \lim_{\epsilon\to 0}\int_{y-\epsilon}^{y+\epsilon} \delta(x-y)\,dx \\
\lim_{\epsilon\to 0}\Big[x^2 G_x(x,y)\Big]\Big|_{x=y-\epsilon}^{y+\epsilon} \;-\; \underbrace{\lim_{\epsilon\to 0}\,[4\epsilon G(y,y)]}_{\to 0} \;&=\; 1 \\
\lim_{\epsilon\to 0}\Big[\,y^2 G_x(y+\epsilon,y) - y^2 G_x(y-\epsilon,y)\Big] \;&=\; 1 \\
-y^2\,2B_+ y^{-3} \;-\; y^2 A_-(1 + 2y^{-3}) \;&=\; 1 \quad ,
\end{aligned} \tag{8}$$

where subscripts denote partial differentiation. The two conditions may be combined to yield

$$1 \;=\; -\,A_-\Big[2(y^3-1)/y \;+\; y^2(1+2y^{-3})\Big] \;=\; -\,3y^2A_-$$

The complete solutions are thus

$$A_- \;=\; -\,\frac{1}{3y^2} \quad , \quad B_+ \;=\; \frac{y^{-2}-y}{3} \quad .$$

and the full Green's function is

$$\boxed{G(x,y) \;=\; \begin{cases} \dfrac{x^{-2}-x}{3y^2} & , \quad 1\le x<y \\[2ex] \dfrac{y^{-2}-y}{3x^2} & , \quad x>y \end{cases}} \tag{9}$$

c. The particular solution is found from convolution with the Green's function

$$\begin{aligned}
\psi_p(x) \;&=\; \int_1^\infty G(x,y)f(y)\,dy \;=\; \int_1^x \frac{y^{-2}-y}{3x^2}\,\frac{1}{y^2}dy \;+\; \int_x^\infty \frac{x^{-2}-x}{3y^2}\,\frac{1}{y^2}dy \\
&=\; \frac{1}{3x^2}\int_1^x (y^{-4}-y^{-1})\,dy \;+\; \frac{x^{-2}-x}{3}\int_x^\infty y^{-4}\,dy \\
&=\; \frac{1}{3x^2}\left(-\frac{1}{3y^3}-\ln y\right)\Bigg|_{y=1}^{x} \;+\; \frac{x^{-2}-x}{3}\left(-\frac{1}{3y^3}\right)\Bigg|_{y=x}^{\infty} \\
&=\; \frac{1}{3x^2}\left(\frac{1}{3}-\frac{1}{3x^3}-\ln x\right) \;+\; \frac{x^{-2}-x}{9\,x^3} \;=\; -\,\frac{\ln x}{3x^2}
\end{aligned}$$

This has the property, $\psi_p(1)=0$, which was imposed on the Green's function. To satifiy the BC of the problem we must add a homogeneous solution, eq. (3) to yield

$$\boxed{\psi(x) \;=\; \frac{1}{x^2} \;-\; \frac{\ln x}{3x^2} \quad .} \tag{10}$$

12. <u>Solution</u>

This problem comes from an example in section 2.3 of <u>Classical</u> <u>Mechanics</u> by Goldstein, Poole, and Safko.

Using the Lagrangian approach with generalized co-ordinates r, θ where r is the distance from the center of the hemisphere to the object and θ is the angle measured from the vertical position to the object (the initial conditions and anticipated motion allow us to ignore the rotational coordinate). To treat the constraint, we will use the method of Lagrange multipliers. For unconstrained motion, the Lagrangian of the object is

$$L = T - V = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - mgr\cos\theta$$

There are two forces acting on the cylinder, the gravitational force, $F_g = mg$ and the normal force, F_N. The last force is a force of constraint and acts in the radial direction only. The constraint for sliding along the hemisphere is $r = a$.

The Lagrange equations become

for *r*, $$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} - \lambda_r \frac{\partial F_N}{\partial r} = m\ddot{r}^2 - mr\dot{\theta}^2 + mg\cos\theta - \lambda = 0$$

for θ, $$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} - \lambda_\theta \frac{\partial F_N}{\partial \theta} = mr^2\ddot{\theta} + mgr\sin\theta = 0$$

(the generalized coordinate for the force of constraint is in the r-direction only).

Using the equation of constraint $\left(r = a \text{ and } \ddot{r} = 0\right)$ yields 2 equations

$$ma\dot{\theta}^2 - mg\cos\theta + \lambda = 0 \quad \text{and} \quad ma^2\ddot{\theta} + mga\sin\theta = 0$$

The second equation, $\ddot{\theta} + \frac{g}{a}\sin\theta = 0$, can be solved ($\dot{\theta}$, not θ) by different methods. The easiest is to assume that the solution has the form $\dot{\theta}^2 = c_1\cos\theta + c_2\sin\theta + c_3$, then differentiating to obtain $2\dot{\theta}\ddot{\theta} = -c_1\sin\theta\dot{\theta} + c_2\cos\theta\dot{\theta}$ or $\ddot{\theta} = -\tfrac{1}{2}c_1\sin\theta + \tfrac{1}{2}c_2\cos\theta$. Comparing to the original equation establishes the unknown coefficients.

$$\dot{\theta}^2 = \frac{2g}{a}(1 - \cos\theta) \quad \text{and} \quad \lambda = mg(3\cos\theta - 2)$$

The object leaves the surface when the force of constraint is zero or when

$$\lambda = 0 \quad or \quad \theta = \cos^{-1}\left(\tfrac{2}{3}\right).$$

13. <u>Solution</u>

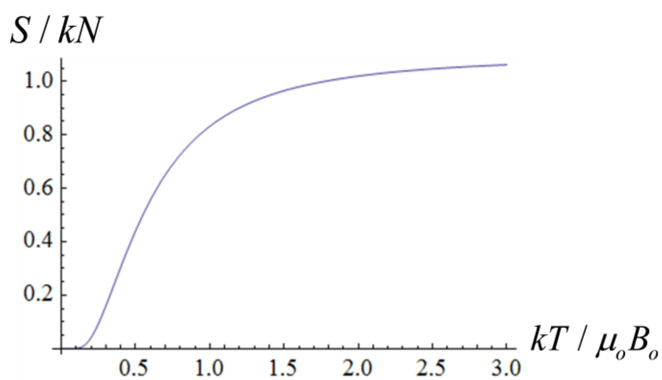
a. Recall that $S = -\left(\frac{\partial F}{\partial T}\right)_{N,V}$ where $F = -kT\ln Z$. The partition function Z can be found easily by

$Z = Z_1^{\ N}$ (particles are distinguishable) where $Z_1 = \sum_j e^{-\beta\mu_o B_o j} = e^{\beta\mu_o B_o} + 1 + e^{-\beta\mu_o B_o}$ and $\beta = \frac{1}{kT}$.

The Helmholtz free energy can be found from $F = -kT\, lnZ = -kTN\ln(e^{\beta\mu_o B_o} + 1 + e^{-\beta\mu_o B_o})$. The entropy can be calculated from

$$S = -\left(\frac{\partial F}{\partial T}\right)_{N,V} = kN\left(\ln(e^{\beta\mu_o B_o} + 1 + e^{-\beta\mu_o B_o}) + \left(\beta\mu_o B_o\right)\left(\frac{e^{-\beta\mu_o B_o} - e^{\beta\mu_o B_o}}{e^{\beta\mu_o B_o} + 1 + e^{-\beta\mu_o B_o}}\right)\right).$$

The plot of $\left(S/kN\right)$ vs. $\left(kT/\mu_o B_o\right)$ is given below. As expected at *T=0* the entropy goes to zero. This is because all the dipoles align with the magnetic field and the system has only a single accessible state. On the other hand, when $kT \gg \mu_o B_o$ the entropy approaches to a constant value given by the limit $\lim_{x\to\infty}\left(S/kN\right) \to \ln 3 \simeq 1.1$ where $x = kT/\mu_o B_o$. This is also anticipated because high temperature randomizes the dipoles that are trying to align with a weak magnetic field. Since each magnetic dipole can take up one of three values with nearly equal probability. Hence, each particle will have three equally accessible states (Ω_1 =3) hence at high temperatures the entropy approaches to its maximum value of $S = kN\ln\Omega_1 = kN\ln 3$.



b. As the magnetic field is reduces from B_o to $(B_o/100)$ adiabatically this process maintains the entropy of the system constant at its initial value which is given by the expression derived in part (a) above:

$S = kN\left(\ln(e^{\beta\mu_o B_o} + 1 + e^{-\beta\mu_o B_o}) + \left(\beta\mu_o B_o\right)\left(\frac{e^{-\beta\mu_o B_o} - e^{\beta\mu_o B_o}}{e^{\beta\mu_o B_o} + 1 + e^{-\beta\mu_o B_o}}\right)\right)$. As one notices *S* is a function of *N* and $\beta\mu_o B_o = \mu_o B_o / kT$. This means when the magnetic field is reduced by a factor 100 the temperature must also reduce by a factor of 100 in order to keep the entropy constant. Therefore, the final temperature of the system will be $T_f = T/100$. This is the foundation of the modern cryogenics where they reduce the temperatures closer to absolute zero by adiabatic demagnetization of paramagnetic salts.

14. Solution

To solve this problem, we first need to know the energy of the ground state in the well. Since $L \gg \frac{\hbar}{\sqrt{mU_o}}$, the ground state E_1 can be estimated from the ground state of the infinite square well. Also, since $W \gg \frac{\hbar}{\sqrt{mU_o}}$ the decay will be small and the wave function will not be strongly perturbed. Thus for E_1 we get

$$E_1 = \frac{\pi^2\hbar^2}{2mL^2}$$

Using this value of E_1, we now know the energy difference to the top of the barrier which is

$$\Delta E = U_o - E_1 = U_o - \frac{\pi^2\hbar^2}{2mL^2}$$

Using this energy difference ΔE, we can estimate the transmission through the barrier using:

$$T = \exp\left[-2\kappa W\right]$$

where the factor of 2 comes from squaring the amplitude to get the transmission probability. The decay coefficient is

$$\kappa = \frac{\sqrt{2m\Delta E}}{\hbar}$$

Putting all this together we get:

$$T = \exp\left[-2\sqrt{\left(\frac{2mU_o}{\hbar^2} - \frac{\pi^2}{L^2}\right)}W\right]$$

Now to get the rate of decay, we need an estimate of how often the electron interacts with the barrier. We can estimate this by using the de Broglie wavelength λ, which is, for the ground state of the infinite square well, λ=2L. Using this de Broglie wavelength λ, we can estimate the velocity v of the electron as it bounces back and forth in the well.

$$v = \frac{p}{m} = \frac{h}{m\lambda} = \frac{h}{m(2L)}$$

Using this velocity, we can estimate the time difference Δt between these bounces at the barrier.

$$\Delta t = \frac{2L}{v} = \frac{2L}{\frac{h}{m(2L)}} = \frac{4mL^2}{h}$$

Combining this with the transmission T will give us the desired rate of escape R.

$$R = \frac{T}{\Delta t} = \frac{\exp\left[-2\sqrt{\left(\frac{2mU_o}{\hbar^2} - \frac{\pi^2}{L^2}\right)}W\right]}{\frac{4mL^2}{h}}$$

which reduces to:

$$\boxed{R = \frac{h}{4mL^2}\exp\left[-2\sqrt{\left(\frac{2mU_o}{\hbar^2} - \frac{\pi^2}{L^2}\right)}W\right]}$$

Alternative solution using the probability current.

The above solution is the usual solution presented at the Serway level. However, we can also use the probability current J that we first learn about at the senior level and is given by:

$$J = \frac{i\hbar}{2m}\left(\psi\frac{\partial\psi^*}{\partial x} - \psi^*\frac{\partial\psi}{\partial x}\right)$$

Here the wave function ψ at the right side of the barrier will be given by:

$$\psi(x,t) = A\sqrt{T}\exp(ikx - iEt/\hbar)$$

where A is the amplitude of the wave function inside the well and E is the energy of the electron. Note that this energy E is the same inside the well and on the right side of the barrier. Note that the amplitude A inside the well is attenuated by $\sqrt{T}$ to form the traveling wave leaking out of the barrier. Since we are using the infinite square well solution for the ground state wave function inside the well, we might expect to have

$$A_{SW} = \sqrt{\frac{2}{L}}$$

However, keep in mind that this amplitude is for the standing wave solution inside the well, which is a sum of a left traveling wave with a right traveling wave. With the leakage through the barrier to the right, we are interested in only the wave traveling to the right inside the well, so we need to use an A with half the amplitude, which gives for A in this case:

$$A = \frac{1}{2}\sqrt{\frac{2}{L}}$$

Now we can proceed to find the probability current J as:

$$J = \frac{i\hbar}{2m}\left(\psi\frac{\partial\psi^*}{\partial x} - \psi^*\frac{\partial\psi}{\partial x}\right) = \frac{i\hbar}{2m}\left(-ik\psi\psi^* - ik\psi^*\psi\right) = \frac{\hbar k}{m}|\psi|^2 = \frac{\hbar k}{m}|A|^2 T = \frac{\hbar k}{m}\frac{1}{2L}T$$

Recalling that for the ground state in the infinite square well, $k = \frac{2\pi}{\lambda} = \frac{2\pi}{2L} = \frac{\pi}{L}$, we get:

$$J = \frac{\hbar k}{m}\frac{1}{2L}T = \frac{\hbar}{m}\frac{\pi}{L}\frac{1}{2L}T = \frac{\hbar 2\pi}{4mL^2}T = \frac{h}{4mL^2}T$$

Then, putting in our earlier value for T, we get:

$$\boxed{J = \frac{h}{4mL^2}\exp\left[-2\sqrt{\left(\frac{2mU_o}{\hbar^2} - \frac{\pi^2}{L^2}\right)}W\right]}$$

which is the same answer we got for R using the Serway multiple bounce approach.

15. <u>Solution</u>

(a)

$$q_{\ell m} = \int Y^*_{\ell m}(\theta, \varphi)\, r^{\ell}\, \rho(\mathbf{x})\, d^3x,$$

and for the setting of Figure 1,

$$\rho(\mathbf{x}) = \frac{1}{r^2 \sin\theta}(-2q\delta(r)\delta(\theta) + q\delta(r-a)\delta(\theta) + q\delta(r-a)\delta(\theta-\pi))\,\delta(\varphi)$$

$q_{00} = 0$, because $-2q$ at $r = 0$, and q_{00} at $+a = -q_{00}$ at $-a$.

Then, $q_{\ell m} = q\, a^{\ell} \int\{Y^*_{\ell m}(0,\varphi) + Y^*_{\ell m}(\pi,\varphi)\}\, d\varphi\, \delta(\varphi)$.

But azimuthal symmetry, and so $q_{\ell m} = 0$ for $m \neq 0$. So,

$q_{\ell 0} = q\, a^{\ell} \int\{Y^*_{\ell 0}(0,\varphi) + Y^*_{\ell 0}(\pi,\varphi)\}\, d\varphi\, \delta(\varphi)$.

and $Y_{\ell 0}(\theta,\varphi) = \sqrt{\frac{2\ell+1}{4\pi}}\, P^0_{\ell}(\cos\theta)$ for $m = 0$.

So, $q_{\ell 0} = q\, a^{\ell} \sqrt{\frac{2\ell+1}{4\pi}} \int (P^0_{\ell}(\cos 0) + P^0_{\ell}(\cos\pi))\, \delta(\varphi)\, d\varphi$

$= = q\, a^{\ell} \sqrt{\frac{2\ell+1}{4\pi}}\, (P^0_{\ell}(1) + P^0_{\ell}(-1))$,

and $q_{\ell 0} = 0$ if ℓ is odd.

So, $q_{\ell 0} = 2\, q\, a^{\ell} \sqrt{\frac{2\ell+1}{4\pi}}$, with ℓ even. Answer

(b)

Potential $\Phi = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{+\ell} \frac{4\pi}{2\ell+1}\, r^{-\ell-1}\, Y_{\ell m}(\theta,\varphi)\, q_{\ell m}$.

Since $r > a$, and $\Phi \to \infty$ as $r \to \infty$.

From last equation in (a) and m=0, we get

$$\Phi = \sum_{\ell=2}^{\infty} \frac{4\pi}{2\ell+1}\, 2q\, \frac{a^{\ell}}{r^{\ell+1}}\, Y_{\ell 0}(\theta,\varphi) \quad \text{for even } \ell \qquad \text{Answer.}$$

By evaluating $Y_{\ell 0}(\theta,\varphi) = \sqrt{\frac{2\ell+1}{4\pi}}\, P^0_{\ell}(\cos\theta)$ in (a), and keeping $\ell = 2$ only (the lowest term in the expansion of $Y_{\ell 0}$ in the above equation), we obtain

$$\Phi = q\, \frac{a^2}{r^3}\, (3\cos^2\theta - 1), \qquad \text{valid for } r \gg a. \qquad \text{Answer}$$

(c) On the x-y plane $\theta = 0$, so

$$\Phi = -\, q\, \frac{a^2}{r^3}$$

(d) Potential from three changes in Figure 1 from Coulomb's law on the x-y plane is

$$\Phi = \frac{q}{\sqrt{r^2+a^2}} + \frac{q}{\sqrt{r^2+a^2}} - \frac{2q}{r},$$

for +q at +a(1st term), +q at –a(2nd term) and –2q at origin(3rd term) $\rightarrow$

$$. \Phi = 2q \left\{ \frac{1}{\sqrt{r^2+a^2}} - \frac{1}{r} \right\}, \qquad \text{Answer}$$

which will reduce to the form in (c) as r→∞, as we expand { } with a/r.