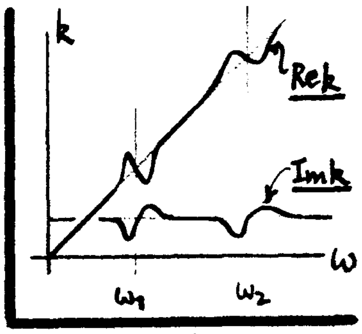
Notion of a Dispersion Relation

Dispersion Properties of Wave Packets [Jkⁿ Secs 7.8 & 7.9]

1) For EM wave propagation in a material medium, it is clear that most wave properties depend significantly on frequency ω. E.g. from Jkⁿ Eq. (7.70)...

$$\left[\begin{array}{l} \underline{\text{POOR CONDUCTOR}}:\ 4\pi\sigma/\omega\epsilon \ll 1 \quad (\text{w/}\ \mu=1)\ldots \\ \text{wave vector}:\ k \simeq \frac{\omega}{c}\sqrt{\epsilon(\omega)} + i\left(2\pi\sigma/c\sqrt{\epsilon(\omega)}\right), \\ \text{w/}\ \sigma \simeq ne^2/m\gamma_0 \sim \text{cnst},\ \text{but//}\ \epsilon(\omega) = 1 + \omega_p^2 \sum_j \frac{g_j(\omega_j^2-\omega^2)}{(\omega_j^2-\omega^2)^2+(\gamma_j\omega)^2}. \end{array}\right] \quad (1)$$

This SHO model of $\epsilon(\omega)$ is not the only one possible, but it is rich enough in detail to suit our purposes. What we will look at now is the propagation of a "wave packet" (i.e. an EM disturbance with a finite extension in space/time) in a medium with a given "<u>dispersion</u> <u>relation</u>", i.e.



$$\left.\begin{array}{l}\text{DISPERSION}\\ \text{RELATION}\end{array}\right\}\ k = k(\omega),\ \text{w/}\ \omega=\omega(k) \quad \int \begin{array}{l}\text{most trivial example is: } \omega = kc, \\ \text{for wave propagation in free space.}\end{array} \quad (2)$$

The "<u>wave packet</u>" will be a superposition of Fourier-type waves at different frequencies. The term "dispersion" refers to the fact that for nontrivial dependence of ω on k, the individual frequency components of the packet will travel at different speeds; hence the packet comes apart, or disperses, as time goes on.

<u>Dispersion means</u>...

1. Phase velocity $v_{ph} = \omega/k$ = fcn of ω (or k); different ω components fall out of phase.
2. Energy transport velocity: $v_g = \partial\omega/\partial k$ [group velocity] $\neq v_{ph}$.
3. For dissipation [$\text{Im}\,k$ or $\text{Im}\,\omega \neq 0$] as well as dispersion, a wavepacket--or pulse of EM radiation--will show <u>attenuation</u> as well as <u>distortion</u> as it propagates.

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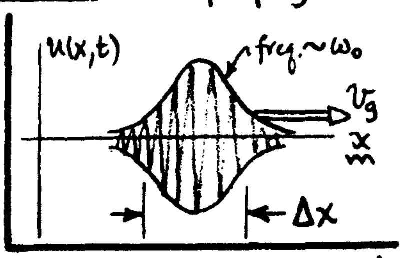
2) Represent the EM pulse, in a 1D dispersive medium, by:

$$\boxed{u(x,t) = \int_{-\infty}^{\infty} A(k)\, e^{i[kx-\omega(k)t]}\, dk}, \quad \omega(k)\ \text{given}; \quad (3)$$



$$\text{w/}\ u(x,0) = \int_{-\infty}^{\infty} A(k)\, e^{ikx}\, dk \leftrightarrow A(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} u(x,0)\, e^{-ikx}\, dx,\ \text{specified (initial condition)}.$$

Consequences of Packet Localization

2) **REMARKS** on: $u(x,t)=\int_{-\infty}^{\infty} A(k)\, e^{i[kx-\omega(k)t]}\,dk$.

1. This Fourier pulse satisfies the wave eqtn: $u_{tt}+\beta u_t - v^2 u_{xx} + \omega_p^2 u = 0$, with β, v & ω_p characteristic of the medium traversed, provided that...

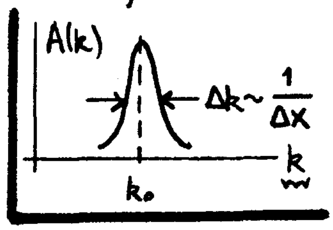
$$\int_{-\infty}^{\infty}[-\omega^2 - i\beta\omega + v^2k^2+\omega_p^2]\,A(k)\,e^{i[kx-\omega(k)t]}\,dk=0$$

So// $\omega^2+i\beta\omega-\omega_p^2 = v^2k^2$, w// $\omega(k)=\pm\sqrt{(\omega_p^2+k^2v^2)-\frac{1}{4}\beta^2}-\frac{i}{2}\beta$. (4)

So the dispersion relation $\omega=\omega(k)$ can be discovered from whatever wave eqtn $u(x,t)$ is supposed to obey, just by plugging the Fourier integral into the eqtn.

2. $u(x,0)$ must (initially) be localized in some region Δx of space in order that the Fourier integral holds. Consequently, the spectrum fcn $A(k)$ is also localized...

[$A(k)$ is localized near $k=k_0$, with width $\boxed{\Delta k \sim 1/\Delta x}$ (5)
⇒ integrand of $u(x,t)$ is "appreciable" only in $k=k_0\pm\Delta k$.



3. This localization (of u to Δx and A to Δk, w// $\Delta k\,\Delta x\sim 1$) ⇒ that in the integrand of $u(x,t)$ we can expand $\omega(k)$ in a Taylor series...

→ $\omega(k)=\omega_0+v_g(k-k_0)+\frac{1}{2}\alpha(k-k_0)^2+\ldots$ w// $\omega_0=\omega(k_0)$, main carrier frequency; $v_g=\left(\frac{d\omega}{dk}\right)_0$, group velocity @ k_0; $\alpha=\left(\frac{d^2\omega}{dk^2}\right)_0$, GVD (group velocity dispersion @ k_0). (6)

The nomenclature will become apparent. Put this expansion into the Fourier integral for $u(x,t)$ to obtain...

→ $u(x,t)\simeq e^{i\phi}\int_{-\infty}^{\infty}dk\,A(k)\,e^{ik\xi}\left[e^{-\frac{1}{2}i\alpha(k-k_0)^2t}\right]$, w// $\phi=(k_0v_g-\omega_0)t$, $\xi=x-v_gt$. (7)

Write the [ ] = 1 − {1 − [ ]}, and split the integral in two parts...

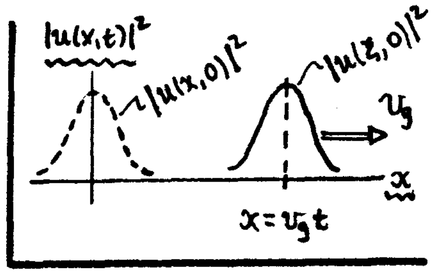
$$u(x,t)\simeq e^{i\phi}\left\{\int_{-\infty}^{\infty}dk\,A(k)\,e^{ik\xi}-\int_{-\infty}^{\infty}dk\,A(k)\,e^{ik\xi}\left[1-e^{-\frac{1}{2}i\alpha t(k-k_0)^2}\right]\right\}$$

this ≡ $u(\xi,0)$, initial pulse; call this part $\Delta u(\xi)$

$$\boxed{u(x,t)\simeq e^{i\phi}\{u(\xi,0)-\Delta u(\xi)\}}. \quad (8)$$

If Δk is not too large, Eq. (8) describes propagation of all pulses u, in any $\omega=\omega(k)$.

Pulse Distortion at Early Times



3) If, in Eq. (8), we set the GVD factor $\alpha = (d^2\omega/dk^2)_0 = 0$...

$\rightarrow u(x,t) \simeq e^{i\phi}\, u(\xi,0)$, so $|u(x,t)|^2 = |u(\xi,0)|^2$, (9)

with: $\xi = x - v_g t$. To this lowest order-of-approxn, the pulse (intensity) just propagates undistorted at the "group velocity" $v_g = \left(\frac{d\omega}{dk}\right)_0$.

Details of the ~~the~~ next order-of-approxn, $\alpha \neq 0$ but small, are carried out in a problem (ϕ 520 Prob. # 54). The results are the following...

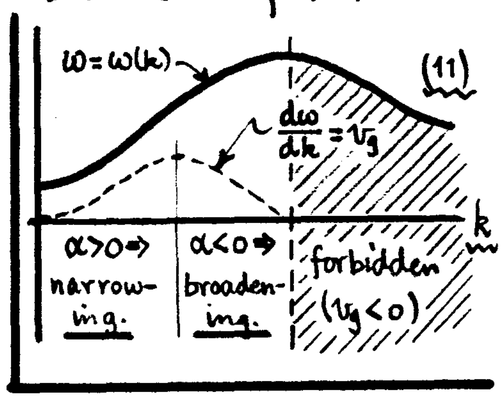
1. Assume $\alpha \neq 0$, and $u(\xi,0)$ is real. Expand: $|u(x,t)|^2 = |u(\xi,0) - \Delta u(\xi)|^2$...

$$\boxed{|u(x,t)|^2 \simeq \underbrace{[u(\xi,0)]^2}_{\text{original pulse}} \left\{1 + \underbrace{2k_0\alpha t\,[u_x(\xi,0)/u(\xi,0)]}_{\text{distortion via GVD term}} + O(\alpha^2)\right\}} \qquad (10)$$

This is to $O(\alpha t)$ and no higher, so it holds only for early times and/or $\alpha \to 0$; we've also assumed $|\alpha t| \ll 1/(\Delta k)^2$. The term in α distorts the pulse shape $u(\xi,0)$; the distortion depends on both α (size & sign) and the pulseshape $u(x,0)$.

2. With the same approxns as in Eq. (10) calculate the HWHM (Half Width at Half Maximum) of $|u|^2$. Get...

$$\boxed{\underbrace{\Delta x(t)}_{\substack{\text{HWHM}\\ \text{of } |u(x,t)|^2}} = \underbrace{\Delta x}_{\substack{\text{HWHM}\\ \text{of } |u(x,0)|^2}} - \underbrace{k_0 t\,(d^2\omega/dk^2)_0}_{\text{GVD correction}}}\text{, to } O(\alpha t) \ll \left(\frac{1}{\Delta k}\right)^2. \qquad (11)$$



At early times $[t \ll 1/\alpha(\Delta k)^2]$, the pulse is distorted, and will either broaden ($\alpha < 0$) or narrow ($\alpha > 0$) depending on the sign of the GVD factor $\alpha = (d^2\omega/dk^2)_0$.

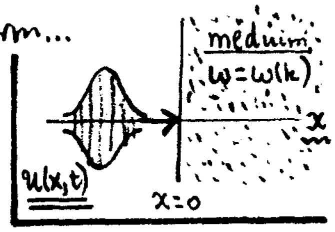
A spectrum $\omega = \omega(k)$ is sketched for which both pulse broadening & narrowing is possible, depending on whether the main carrier frequency ω_0 is high or low. Pulse-narrowing media are of great interest in the construction of optical fibers.

3. To the $O(\alpha t)$ approxn in Eq. (10), the overall pulse energy is unaffected, since:

$$\rightarrow \int_{-\infty}^{\infty} |u(x,t)|^2 dx = \int_{-\infty}^{\infty} |u(x,0)|^2 dx \Rightarrow \text{pulse energy } \int |u|^2 dx = \text{cnst}. \qquad (12)$$

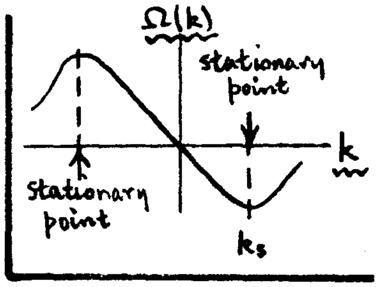
Pulse Behavior as $t \to \infty$.

4) In addition to characterizing the dispersive pulse behavior at very short times, we can analyse how a pulse behaves in a general dispersive medium as $t \to \infty$. Start from the Fourier integral for a pulse moving along the x-axis written in form...



$$\rightarrow u(x,t) = \int_{-\infty}^{\infty} A(k)\, e^{-i\Omega(k)t}\, dk, \quad \text{w/} \quad \Omega(k) = \omega(k) - k\frac{x}{t}. \qquad (13)$$

We are interested in how $u(x,t)$ behaves as t (and hence x) $\to \infty$. To fix ideas, we hold the ratio x/t = cnst... this ratio is essentially a phase velocity for a fixed point on the pulse, so we can follow any point which moves at that velocity.



We use the "method of stationary phase" (related to "method of steepest descent") on the $u(x,t)$ integral -- this method is explained in detail in Jk$^{\text{n}}$ Sec. 7.11(d) [see also Mathews and Walker, Sec. 3.6; Arfken, Sec. 7.4]. At a fixed t, and during the k-integration, the factor $e^{-i\Omega(k)t}$ in Eq.(13) normally oscillates rapidly because $\Omega(k)$ is large and changes quickly; this rapid oscillation $\Rightarrow$ the integral ~ 0.

BUT, whenever $\Omega(k) \simeq$ cnst... i.e. is (nearly) stationary... the integral can accumulate a value. The places where this happens is wherever $\partial\Omega/\partial k = 0$, i.e. when

$$\frac{\partial \Omega}{\partial k} = \omega'(k) - \frac{x}{t} = 0 \Rightarrow \text{"stationary phase point" } k_s \left\{ \begin{array}{l}\text{such}\\ \text{that}\end{array} : \underbrace{\omega'(k_s)}_{\text{from given dispersion}} = \overbrace{\frac{x}{t}}^{\text{choice of phase point}} \right. \qquad (14)$$

In the neighborhood of k_s, $\Omega(k)$ behaves as...

$$\rightarrow \Omega(k) \simeq \Omega(k_s) + \tfrac{1}{2}(k-k_s)^2\, \Omega''(k_s)$$

So

$$u(x,t) \simeq e^{-i\Omega(k_s)t} \int_{-\infty}^{\infty} [A(k_s) + \underbrace{\cancel{(k-k_s)A'(k_s)}}_{\text{ignore as "slowly varying"}}]\, e^{-\frac{1}{2}i\Omega''(k_s)(k-k_s)^2 t}\, dk,$$

$$\simeq A(k_s)\, e^{i[k_s x - \omega(k_s)t]} \underbrace{\int_{-\infty}^{\infty} e^{-ipz^2}\, dz}_{=(\pi/|p|)^{1/2} \exp(-i\frac{\pi}{4}\,\mathrm{sgn}\,p)\ [\text{G\&R \#}(3.323.2)]}, \quad \text{w/} \quad p = \tfrac{1}{2}\Omega''(k_s)\, t$$

only

$$\boxed{u(x,t) \simeq \sqrt{\frac{2\pi}{|\omega_s''|t}}\, A(k_s)\, e^{i[(k_s x - \omega_s t) - \frac{\pi}{4}\,\mathrm{sgn}\,\omega_s'']}} \qquad \begin{array}{l}\omega_s = \omega(k_s),\\ \omega_s'' = \omega''(k_s).\end{array} \qquad (15)$$

Pulse Behavior as $t \to \infty$

REMARKS on $u(x,t)$ of Eq. (15)

1. Besides the explicit x & t dependence shown in Eq. (15), there is an x/t dependence in k_s, since the defining eqtn (14) : $\omega'(k_s) = x/t$ makes k_s a fcn of x/t. ¶

2. Eq. (15) gives the contribution to the pulse $u(x,t)$ from just one stationary point k_s of $\Omega(k)$. If there are several such points (suitably separated by $\Delta k \gg 1/\sqrt{|\omega_s''|t}$, so they don't overlap and interfere), then they each contribute similar terms, so...

$$\left[\!\!\left[\; u(x,t) \simeq \sum_s \sqrt{2\pi/t\omega_s''}\, A(k_s)\, e^{i\left[(k_s x - \omega_s t) - \frac{\pi}{4}\,\mathrm{sgn}\,\omega_s''\right]},\ \text{as } t \to \text{"large"}. \;\right]\!\!\right] \tag{16}$$

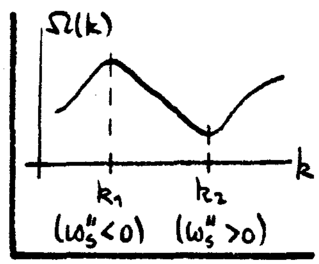
In this expression, $t \to$ "large" means : $\omega_s t \gg 1$ (the pulse has propagated for many cycles). Alternatively, the distance traveled : $x = v_g t \gg \Delta x$ (initial width).

3. NOTE : Just as the early time pulse dispersion was largely governed by the GVD factor $\alpha = \omega''(k)$ [see Eqs. (10) & (11), p. Waves 20], so is the late time dispersion also strongly dependent on $\alpha_s = \omega_s''$. Of course $\omega_s'' \neq 0$ for use of Eq. (16).

4. The $1/\sqrt{t}$ factor in Eq. (16) can be taken outside the $\sum_s$. Then, with x/t fixed in the k_s variation, we have the prediction that in all media (w/ $\omega_s'' \neq 0$) all dispersing pulses will ultimately fall off in intensity as $|u|^2 \propto 1/t$.

EXAMPLE : Two-point problem w/ $\Omega(k)$ has stationary points @ k_1 & k_2 :

$$\rightarrow u(x,t) \simeq \sqrt{2\pi/t}\left[a_1 e^{i(\phi_1 + \frac{\pi}{4})} + a_2 e^{i(\phi_2 - \frac{\pi}{4})}\right] \int \begin{array}{l} a_s = A(k_s)/\sqrt{|\omega_s''|}, \\ \phi_s = (k_s \frac{x}{t} - \omega_s)t;\ s=1,2 \end{array}$$



Assume the a_s are real. Pulse intensity is...

$$\rightarrow |u(x,t)|^2 \simeq \frac{2\pi}{t}\left\{a_1^2 + a_2^2 + 2a_1 a_2 \sin\left[(k_2 - k_1)\frac{x}{t} - (\omega_2 - \omega_1)\right]t\right\}. \tag{17}$$

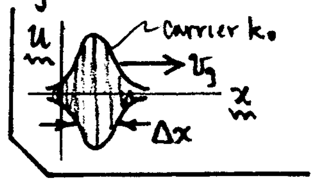
This wave no longer shows any pronounced localization. Although the a_s may vary a bit with choice of position x/t, at a fixed x the intensity just oscillates between the limits $(2\pi/t)(a_1 \pm a_2)^2$. Meanwhile, the wave dies out as $|u|^2 \propto 1/t$.

¶ E.g. for a plasma-type dispersion : $\omega^2 = k^2c^2 + \omega_p^2$, the condition : $\omega'(k) = x/t$ gives a solution : $k = \frac{\omega_p}{c}\,\gamma/\sqrt{1-\gamma^2}$, w/ $\gamma = x/ct < 1$. So indeed k_s is a fcn of x/t.

Dispersive Pulse Example

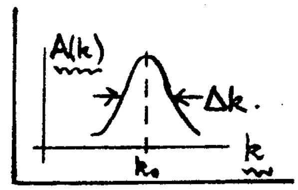
5) As an instance of a dispersing pulse which we can analyse w/o approximation, we do an example similar to that in Jk$^{\underline{n}}$ Sec. 7.9. In the Fourier integral...

$$u(x,t) = \int_{-\infty}^{\infty} dk\, A(k)\, e^{i[kx-\omega(k)t]}, \quad (18)$$



Choose// $u(x,0) = N e^{ik_0 x} e^{-(x/\Delta x)^2}$ ∫ a Gaussian pulse shape, at carrier k_0, initial width Δx;

So// $A(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} u(x,0)\, e^{-ikx}\, dx = \left(\frac{N\Delta x}{2\sqrt{\pi}}\right) e^{-\frac{1}{4}(k-k_0)^2(\Delta x)^2}$.

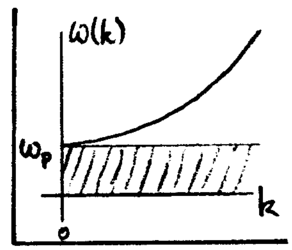


Note -- if u is initially localized to Δx, then $A(k)$ is localized to $\Delta k \sim 1/\Delta x$.

Choose further a dispersion relation of the type (plasma long wavelength limit*):

$\omega(k) = \omega_p\left(1+\frac{1}{2}\lambda_p^2 k^2\right)$ ∫ ω_p = cnst (plasma freq), $\lambda_p = c/\omega_p$ = cnst $\Big\} \lambda_p k \ll 1$;

So// group velocity: $v_g = (d\omega/dk)|_{k=k_0} = (k_0\lambda_p)c \ll c$. (19)



Then our chosen pulse is, at time $t>0$...

$$u(x,t) = \left(\frac{N\Delta x}{2\sqrt{\pi}}\right)\int_{-\infty}^{\infty} dk\, e^{-\frac{1}{4}[(k-k_0)\Delta x]^2 + i[kx-\omega_p(1+\frac{1}{2}\lambda_p^2 k^2)t]}$$

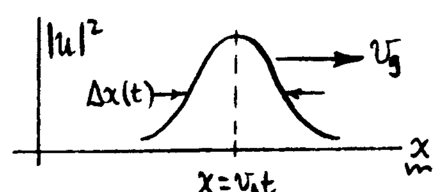
... use: $\int_{-\infty}^{\infty} e^{-p^2y^2 \pm qy}\, dy = (\sqrt{\pi}/p)\, e^{q^2/4p^2}$, for $p>0$ [G&R #(3.323.2)]...

So// $$\left[\!\left[\, u(x,t) = \frac{N e^{i[k_0 x - \omega(k_0)t]}}{\sqrt{1+i\tau}}\, e^{-\left(\frac{x-v_g t}{\Delta x(t)}\right)^2(1-i\tau)} \,\right]\!\right], \text{ evolving pulse.} \quad (20)$$

where: $\underline{\underline{\tau}} = \frac{2}{n^2}(k_0 v_g t)$, w/ $\underline{\underline{n}} = k_0\Delta x$ = # carrier wavelength in initial width Δx, $\underline{\underline{\Delta x(t)}} = \Delta x\sqrt{1+\tau^2}$.

We will not be interested in phases, so look at the absolute value...

$$\boxed{|u(x,t)| = \left[|N|/(1+\tau^2)^{\frac{1}{4}}\right] e^{-\left(\frac{x-v_g t}{\Delta x(t)}\right)^2}} \quad (21)$$



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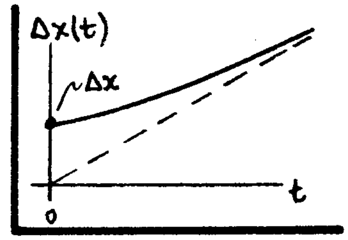
* For a real plasma: $\omega = \sqrt{\omega_p^2 + k^2c^2}$ [Jk$^{\underline{n}}$ Eq. (7.61)]. In the long wavelength limit, $k = 2\pi/\lambda \to$ small, so: $\omega \simeq \omega_p\left[1+\frac{1}{2}(k\lambda_p)^2\right]$, w/ $\lambda_p = c/\omega_p$, and $k\lambda_p \ll 1$.

Dispersive Pulse Example

REMARKS on evolving pulse: $u(x,t) = (N/\sqrt{1+i\tau})\, e^{i(k_0 x-\omega_0 t)}\, e^{-\left(\frac{x-v_g t}{\Delta x(t)}\right)^2 (1-i\tau)}$.

1. This calculation of $u(x,t)$ is exact, but specialized to initial Gaussian shape, and dispersion relation $\omega(k) = a + bk^2$. NOTE that when $t \to 0$, we recover the input: $u(x,t) \to u(x,0) = (N e^{ik_0 x})\, e^{-(x/\Delta x)^2}$. So the arithmetic checks out.

2. The pulsewidth increases with time... ★



$$\rightarrow \Delta x(t) = \Delta x\sqrt{1+\tau^2} \overset{\tau\gg 1}{\simeq} \tau\Delta x = \frac{2}{n} v_g t, \quad \text{w/ } n = 2\pi\frac{\Delta x}{\lambda_0}. \quad (22)$$

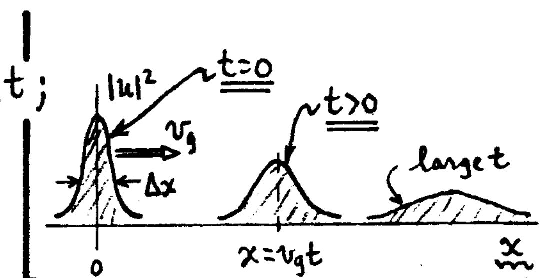
The fractional rate of increase is: $\frac{d}{dt}[\Delta x(t)/\Delta x] = \frac{2}{n}(v_g/\Delta x)$. This pulse broadening is ~ small for initially broad pulses (n & $\Delta x \to$ large), but large for initially narrow pulses... i.e. narrow pulses disperse rapidly.

(Reason... group velocity spread: $\Delta v_g \sim \omega''\Delta k \sim \omega''/\Delta x \to$ large;
... packet dispersion: $\Delta x(t) \sim t\,\Delta v_g \sim (t\,\omega'')/\Delta x \to$ large.)

3. The pulse amplitude decreases with time...

$$|u(x,t)| \overset{\tau\gg 1}{\simeq} (|N|/\sqrt{\tau})\, e^{-\left(\frac{x-v_g t}{\Delta x(t)}\right)^2}, \quad \text{w/ } \Delta x(t) \simeq \frac{2}{n} v_g t;$$

$$\left[\, |u(x,t)| \simeq \left(\frac{|N|\sqrt{\Delta x}}{\sqrt{(2/n)\, v_g t}}\right) e^{-\frac{n^2}{4}\left(\frac{x-v_g t}{v_g t}\right)^2}. \quad (23) \Rightarrow\right.$$



The $1/\sqrt{t}$ behavior is consistent with the stationary phase prediction [Eq.(16) above].
NOTE: intensity height × width, i.e. $|u|^2_{peak}\,\Delta x(t) \sim$ cnst as $t \to \infty$ (pulse remnant).

4. The pulse energy is conserved [if $\omega = \omega(k)$ is real $\Rightarrow$ no dissipation]. In general...

$$\left.\begin{array}{r}\text{energy}\\ \text{@ time } t\end{array}\right\}\ W(t) = \int_{-\infty}^{\infty} dx\, |u(x,t)|^2 = \int_{-\infty}^{\infty} dk\, A^*(k)\, e^{i\omega^* t}\int_{-\infty}^{\infty} dk'\, A(k')\, e^{-i\omega t}\underbrace{\int_{-\infty}^{\infty} dx\, e^{i(k'-k)x}}_{2\pi\delta(k'-k)}$$

So

$$\boxed{W(t) = 2\pi\int_{-\infty}^{\infty} dk\, |A(k)|^2\, e^{+2[\mathrm{Im}\,\omega(k)]t}} \quad (24) \qquad \underline{W = \text{cnst, if } \mathrm{Im}\,\omega(k) = 0.}$$

★ Despite the fact that $\alpha = \omega'' = c^2/\omega_p > 0$ for our medium, there is no pulse narrowing at early times, per Eq.(11) on Waves, p. 20. Reason is: present pulse ≠ real, per Eq.(10).