DEPARTMENT OF PHYSICS

M.S. COMPREHENSIVE / PH. D. QUALIFYING EXAMINATION

MARCH 28, 1988

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MONDAY, MARCH 28, 1988, 8 A.M.-12 NOON

Answer each of the following questions, all of which carry equal weight. Begin your answer to each question on a new sheet of paper; solutions to different questions must not appear on the same sheet. Each sheet of paper must be labeled with your name and the problem number in the upper right hand corner. When more than one sheet is submitted for a problem, be sure the pages are ordered properly. [Problems 1-8].

1. A ball of mass $m=0.20$ kg rests on a vertical post of height $h=5.0$ m. A bullet of mass $m=0.010$ kg travelling at $v_o=500$ m/s passes horizontally through the center of the ball. The ball hits the ground at a distance of $R=20$ m.

a) Where does the bullet hit the ground?

b) What part of the kinetic energy of the bullet was converted to heat?

OK. A.E

Mechanics

LDK

1. A ball of mass $M = 0.20$ kg rests on a vertical post of height $h = 5.0$ m. A bullet of mass $m = 0.010$ kg traveling at $v_0 = 500$ m/s passes horizontally through the center of the ball. The ball hits the ground at a distance of $R = 20$ m.

a) Where does the bullet hit the ground?

b) What part of the kinetic energy of the bullet was converted to heat.

Conservation of momentum

$$m v_0 = m v + M V$$

where v & V are the final velocities of the bullet & ball, respectfully. Time of flight t is

2 $$t = \sqrt{2h/g} = \sqrt{10.0\,m/9.8\,m/s^2} = 1.01\ s$$

1½ $$V = R/t = 20\,m/1.01\,s = 19.8\ m/s$$

2 $$v = v_0 - MV/m = 500\,m/s - \left(\frac{0.20\ kg}{0.010\ kg}\right) 19.8\,m/s = 104\ m/s$$

1½ $$r = vt = 104\ m/s \times 1.01\,s = 105\ m$$

$$KE_0 = \tfrac{1}{2} m v_0^2 = \tfrac{1}{2}(0.010\,kg)(500\,m/s)^2 = 1250\ J$$

3 $$KE_f = \tfrac{1}{2} m v^2 + \tfrac{1}{2} M V^2 = \tfrac{1}{2}\left(0.010\,kg \times (104\,m/s)^2 + \tfrac{1}{2} 0.20\,kg\,(19.8\,m/s)^2\right)$$

$$= 93\ J \Rightarrow \text{loss} = 1157\ J \text{ or } 93\%$$

2. Obtain the energy of the bound state of a very shallow one-dimensional square well.

3. Consider the collision of a photon of wavelength λ with an electron of mass m which is at rest. Determine whether the collision must be treated relativistically if the photon is in the

a) visible range
b) x-ray range
c) microwave range
d) gamma ray range

3. Consider the collision of a photon of wavelength λ with an electron of mass m which is at rest. Determine whether the collision must be treated relativistically if the photon is in

a) visible range

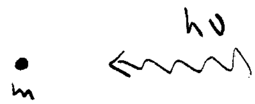
b) the x-ray range

c) microwave range

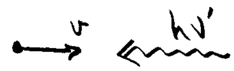
d) gamma ray range

<u>solution</u>

in <u>Lab frame</u>



in <u>cm frame</u>



$\therefore$

$$\gamma m v = \frac{h\nu'}{c} \quad \text{where relativistic Doppler shift is } \nu' = \gamma(1 - v/c)\nu$$

$$= \frac{h\nu}{c}(1 - v/c)\gamma \implies m v c = h\nu(1 - v/c)$$

$$\rightarrow \quad m v c + h\nu \frac{v}{c} = h\nu$$

$$\frac{v}{c}(mc^2 + h\nu) = h\nu \quad \rightarrow \quad \frac{v}{c} = \frac{h\nu}{h\nu + mc^2} \rightarrow 0 \qquad mc^2 \gg h\nu$$

Thus the problem needs to be treated relativistically when $mc^2 \approx h\nu = \frac{hc}{\lambda}$

or for $\lambda \lesssim \frac{hc}{mc^2} = \frac{12400 \text{ eV}\cdot\text{Å}}{.5 \times 10^6 \text{ eV}} = 2.5 \times 10^{-2}$ Å

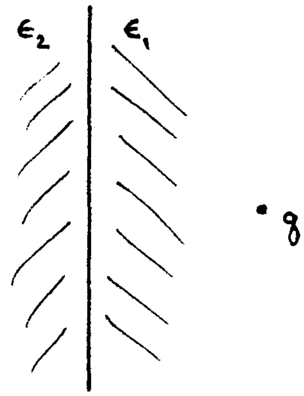
For $\underline{\lambda \lesssim 25 \text{ mÅ}}$ need relativistic treatment

or $h\nu \gtrsim 0.5$ MeV

The approximate ranges for μ-waves, visible, X-rays and γ-rays are listed below

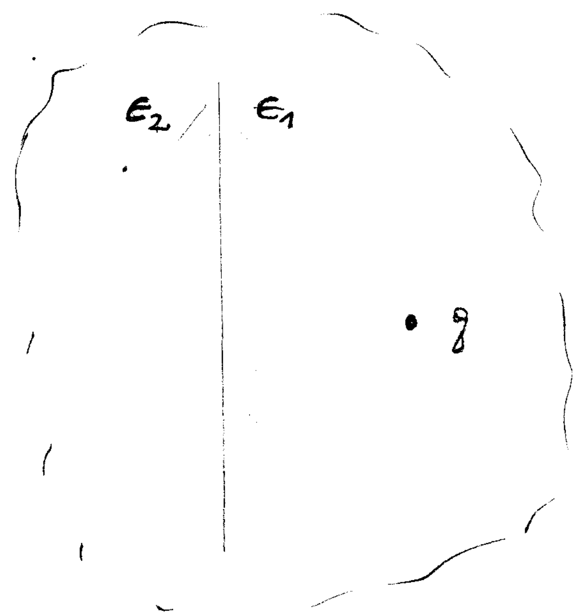
	λ range	hν range	<u>Relativity needed?</u>
μ-waves	1mm – 10 cm		No
visible	1μ – 1000Å	1eV – 10eV	No
X-rays		100eV – 1MeV	probably depending on λ
γ-rays		≳ 1 MeV	Yes

4. Find the electrostatic potential everywhere for the following problem



E and M

4. Find the electrostatic potential everywhere for the following problem



Nice . he

$$\Rightarrow \qquad q' = -\left(\frac{\epsilon_2 - \epsilon_1}{\epsilon_1 + \epsilon_2}\right) q$$

$$q'' = \frac{2\epsilon_2}{\epsilon_1 + \epsilon_2} q$$

For $\epsilon_2 \gg \epsilon_1 \qquad q' = -q$

Let $R = \sqrt{d^2 + x^2}$

$$\frac{\partial}{\partial z}\frac{1}{R_1} = \frac{\partial}{\partial z}\left[(z-d)^2 + x^2\right]^{-1/2} \qquad \frac{\partial}{\partial z}\frac{1}{R_2} =$$

$$= -\tfrac{1}{2}\left[\quad\right]^{-3/2} 2(z-d)$$

$$= -\frac{z-d}{R_1^3} \qquad\qquad = -\frac{z+d}{R_2^3}$$

$$\Big|_{z=0} = \frac{d}{R^3} \qquad\qquad \Big|_{z=0} = \frac{-d}{R^3}$$

$$\Rightarrow \quad \frac{1}{\varepsilon_1}\left[\frac{q}{R} + \frac{q'}{R}\right) = \frac{1}{\varepsilon_2}\frac{q''}{R} \quad \text{or} \quad \left\{ \varepsilon_2 q + \varepsilon_2 q' = \varepsilon_1 q'' \right.$$

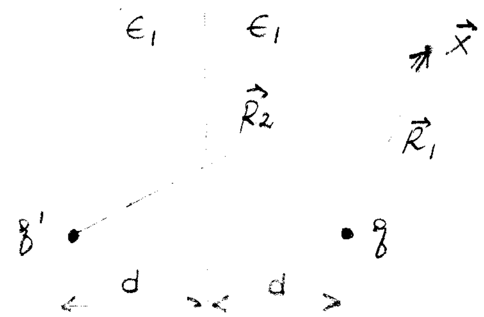
$$\Rightarrow \quad \frac{qd}{R^3} - \frac{q'd}{R^3} = \frac{q''d}{R^3} \quad \text{or} \quad \left. q - q' = q'' \right\}$$

yields above results.

This problem can be solved directly by matching solutions of Laplace's equation (for $z<0$) and Poisson's equation (for $z>0$) at $z=0$.

Another procedure is to use the method of images, based on the uniqueness theorem for Poisson's equation.

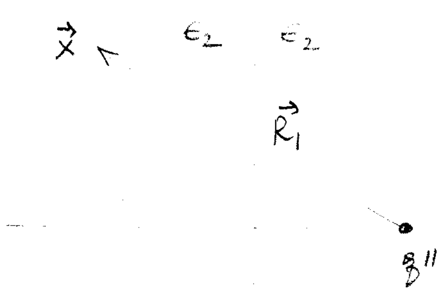
For $\underline{z>0}$ try the following solution



$$\Phi^{>}(\vec{x}) = \frac{1}{\epsilon_1}\left(\frac{q}{R_1} + \frac{q'}{R_2}\right)$$

This function solves Poisson's equation for $z>0$. q' is an unknown image charge.

For $\underline{z<0}$ try the solution



$$\Phi^{<}(\vec{x}) = \frac{1}{\epsilon_2}\,\frac{q''}{R_1}$$

This function solves Laplace's equation for $z<0$. q'' is an unknown image charge.

B. Cs.

$$\Phi^{>}(\vec{x}) = \Phi^{<}(\vec{x}) \Big|_{z=0}$$

$\epsilon E = D$

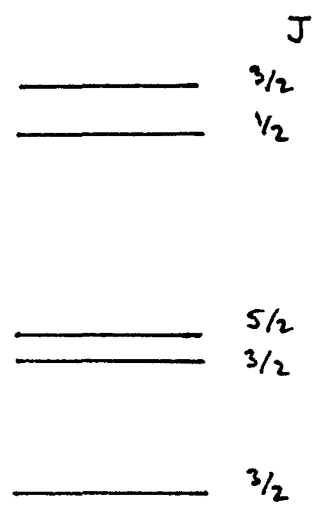
$$\epsilon_2 \frac{\partial \Phi^{<}}{\partial z}\Big|_{z=0^-} = \epsilon_1 \frac{\partial}{\partial z}\Phi^{>}\Big|_{z=0^+}$$

D continuous

5. a) Consider the configuration $n\ell^x$ of equivalent electrons, each of angular momentum ℓ. Show that in LS coupling the largest value of the orbital angular momentum L for terms of highest multiplicity (largest S) is

$$L_{max} = \frac{1}{2}\, x(2\ell+1-x) \qquad \text{for} \qquad x \leq 2\ell+1$$

b) The diagram shows the five levels of the $3p^3$ ground configuration of Fe^{11+} and gives their J values. Suggest further quantum numbers to identify the levels, giving reasons for your choices. (You need not derive the terms of the $3p^3$ configuration from first principles.)



Cone Atomic Physics / QM (Excl. princ.)

5. (a) Consider the configuration nl^x of equivalent electrons, each of angular momentum l. Show that in LS coupling the largest value of the orbital angular momentum L for terms of highest multiplicity (largest S) is

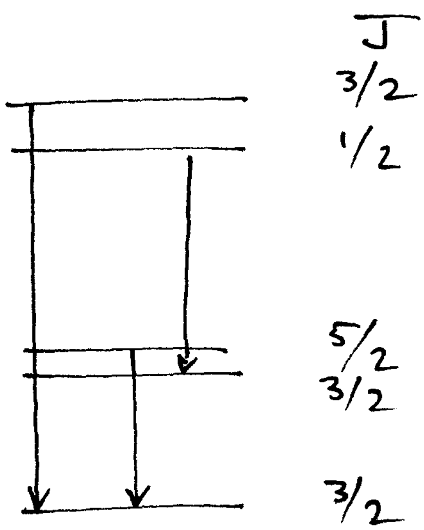
$$L_{max} = \frac{1}{2} x (2l+1-x) \qquad \text{for} \quad x \leq 2l+1.$$

(b) The diagram shows the five levels of the $3p^3$ ground configuration of Fe^{11+} and gives their J values. Suggest further quantum numbers to identify the levels, giving reasons for your choices. (You need not derive the terms of the $3p^3$ configuration from first principles.)

optional

Not used

~~(c) The three emission lines indicated are observed in the spectrum of the solar corona. State what kind of radiation occurs in each case with justification from selection rules~~



Answer (b)

$S = \frac{1}{2}$ then need $L = 1$

1st doublets, 2nd then ($J = L \pm \frac{1}{2}$) gives

$S = \frac{1}{2}$ then need $L = 2$

1st Singlet $\Rightarrow$ one must be zero.
2nd L cannot be half-integral.

$S = \frac{3}{2}$ $L = 0$ (Hunds rule ground state)

a) max spin state $|S, M_S = S\rangle$ has all $m_s = +\frac{1}{2}$

Then exclusion princip.

$$\left.\begin{aligned} m_{l_1} &= l \\ m_{l_2} &= l-1 \\ &\cdots \\ m_{l_X} &= l-X+1 \end{aligned}\right\} \text{ sum gives desired result}$$

b) see p 1

~~c) optional - not used~~

6. Sodium vapor in a gas discharge tube emits a strong yellow line at 5890 Å. If the vapor is at room temperature, estimate roughly how many angstroms broad this line will appear due to Doppler shifts caused by thermal motion.

Useful information: For Na, $Mc^2 \approx 23 \times 10^9$ eV

~~Comp~~ QUAL

Unknown Category - Stat. Mech.? Hiscock

6. Sodium vapor in a gas discharge tube emits a strong yellow line at 5890 Å. If the vapor is at room temperature, estimate roughly how many angstroms broad this line will appear, due to Doppler shifts caused by thermal motion.

Useful information: for Na, $mc^2 = 23 \times 10^9$ eV

NICE JEC

OK JED

easy AE..

easy - JH

easy L.A.L.

solution:

at room temperature, $kT \simeq \frac{1}{40}$ eV & $\frac{1}{2}mv^2 \simeq \frac{3}{2}kT$, so

$mv^2 \simeq \frac{3}{40}$ eV $\qquad mc^2 \simeq 23\times10^9$ eV, so $\frac{v^2}{c^2} \simeq \frac{\frac{3}{40}}{23\times10^9} \simeq 3.26\times10^{-12}$

& $\frac{v}{c} \simeq 1.81\times10^{-6}$

Doppler broadening: wavelengths emitted will be roughly bounded by $\lambda_0\left(1 \pm \frac{v}{c}\right)$; the broadening is then $2\lambda_0\frac{v}{c}$

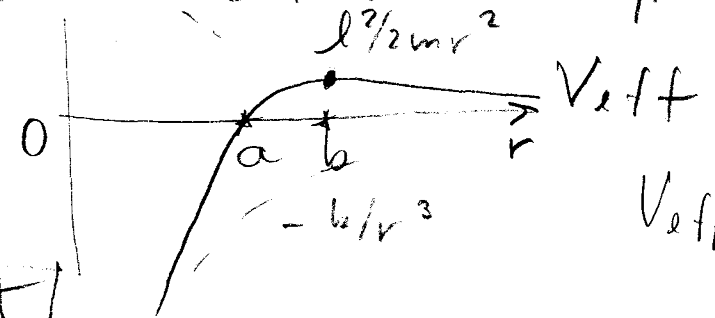
$$\Delta\lambda \simeq 2\lambda_0\frac{v}{c} \simeq 2\cdot 5890\,\text{Å}\cdot\left(1.81\times10^{-6}\right) \simeq \boxed{.0213\,\text{Å}}$$

7. A particle of mass m and zero energy moves in the potential $V(r)=-k/r^3$ where $k>0$. Without solving the equation of the classical motion, describe the motion as completely as you can for non-vanishing angular momentum.

Mechanics . Hermanson

7. A particle of mass m and zero energy moves in the potential $V(r) = -k/r^3$ where $k > 0$. Without solving the equation of the classical motion, describe the motion as completely as you can for non-vanishing angular momentum.

Sol'n From the effective 1D problem



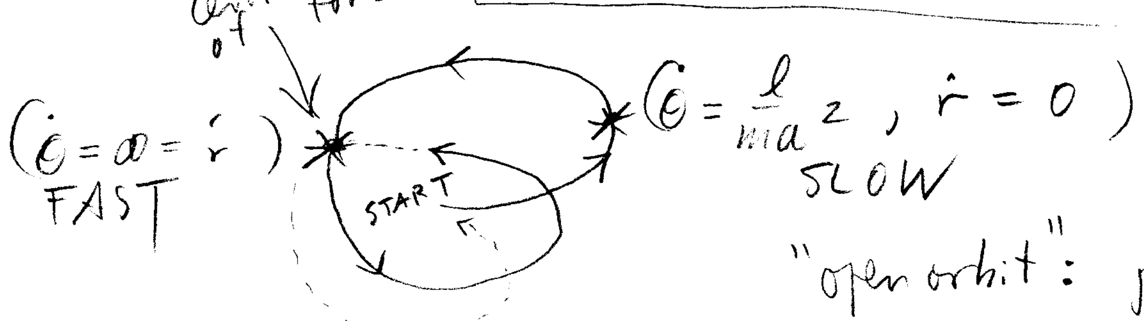
$$V_{eff} = \frac{\ell^2}{2mr^2} - \frac{k}{r^3}$$

2 a) $\boxed{\ell = \text{const}}$

4 b) turning points: $\boxed{r = 0,\ a}$

$$E(a) = -\frac{k}{a^3} + \frac{\ell^2}{2ma^2} = 0 \Rightarrow \boxed{a = \frac{2mk}{\ell^2}}$$

4 c) Since $\dot\theta = \frac{\ell}{mr^2}$ the 3D motion is an open orbit passing thru the center of force



"open orbit": precession

8. Consider a travelling plane wave propagating in a region of totally ionized hydrogen (i.e. protons and electrons). Neglecting the interaction between the electrons and protons,

a) Calculate the position of an electron in the plasma if the local electric field at the point is given by $E=E_o\sin(\omega t)$.

b) Ignoring the motion of the heavier protons, calculate the induced polarization at that point.

c) Calculate the frequency ω_α which will produce a polarization that will exactly cancel the local electric field at that point.

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Problem seems simple. L.A.C.

easy - JH

[illegible] A.E.

Solution

a) The equation of motion for the electron

$$\left.\begin{aligned} F &= m \frac{d^2 x}{dt^2} \\ &= -|e|\, E(t) \end{aligned}\right\} \rightarrow \quad \underline{x(t) = \frac{|e|\, E_0}{m\omega^2} \sin(\omega t)}$$

Note that the electron displacement is about its equilibrium point.

b) To calculate the polarization, we assume the proton does not move since it is more massive.

Thus the polarization is just a sum over induced dipoles,

$$P(t) = -|e|\, n\, x(t) = -\frac{n e^2 E_0}{m\omega^2} \sin(\omega t)$$

c) The Polarization will cancel the electric field when

$E = -4\pi P$.

Thus

$$-4\pi \left(-\frac{n e^2 E_0}{m\omega_\alpha^2} \sin(\omega_\alpha t) \right) = E_0 \sin(\omega_\alpha t)$$

or

$$\frac{4\pi\, n e^2}{m\omega_\alpha^2} = 1 \quad \longrightarrow \quad \underline{\omega_\alpha^2 = \frac{4\pi\, n e^2}{m}}$$

ω_α is of course the plasma frequency below which the induced polarization prevents propagation in the plasma.

DEPARTMENT OF PHYSICS

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9. All elements heavier than Iron are formed in supernova explosions. It is also currently believed that the interstellar shock wave from a supernova explosion initiates the collapse of gas clouds which leads to the formation of stars and planets.

Assume that the initial abundances of U^{235} and U^{238} were equal when the Earth was formed. Today there are about 140 times as many U^{238} atoms as U^{235} on Earth today. Use this information to estimate the age of the Earth.

1/2 life of $U^{235} \simeq 7.07 \times 10^{8}$ yrs

1/2 life of $U^{238} \simeq 4.51 \times 10^{9}$ yrs

Nuclear Physics | Hiscock

9. All elements heavier than Iron are formed in supernova explosions. It is also currently believed that the interstellar shock wave from a supernova explosion initiates the collapse of gas clouds which leads to the formation of stars and planets.

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$$\tfrac{1}{2} \text{ life of } U^{235} \simeq 7.07 \times 10^{8} \text{ yrs}$$

$$\tfrac{1}{2} \text{ life of } U^{238} \simeq 4.51 \times 10^{9} \text{ yrs}$$

Solution: $n(t) = n_0 \exp(-\lambda t) \qquad \lambda = \dfrac{\ln 2}{\tau_{1/2}}$

$$\lambda_{235} = \frac{\ln 2}{7.07\times10^{8}\text{ yrs}} = 9.80\times10^{-10}\text{ yr}^{-1}$$

$$\lambda_{238} = \frac{\ln 2}{4.51\times10^{9}\text{ yrs}} = 1.54\times10^{-10}\text{ yr}^{-1}$$

$$\frac{n_{238}(t)}{n_{235}(t)} = \frac{n^0_{238}\exp(-\lambda_{238}t)}{n^0_{235}\exp(-\lambda_{235}t)} = 1\cdot\exp[(\lambda_{235}-\lambda_{238})t]$$

Today, $t = t_0$, and

$$\frac{n_{238}(t_0)}{n_{235}(t_0)} = 140 = \exp[(\lambda_{235}-\lambda_{238}/t_0]$$

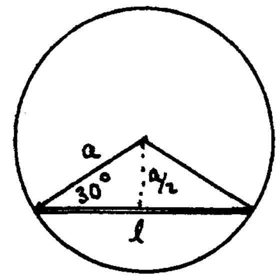
so

$$t_0 = \frac{\ln(140)}{(\lambda_{235}-\lambda_{238})} \simeq \frac{4.94}{8.26\times10^{-10}\text{ yr}^{-1}} = \boxed{5.98\times10^{9}\text{ yrs}}$$

I think this is fun. SLC.
I give it in 230

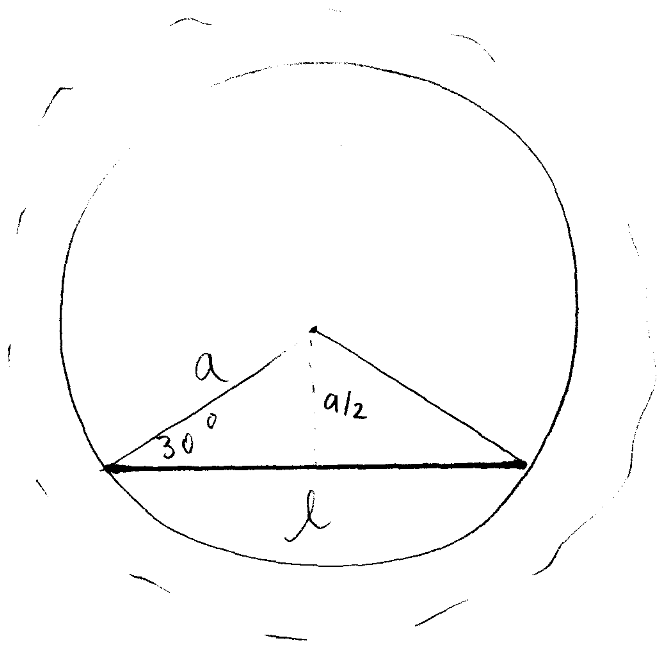
Statement of problem may "scare" students away from it. What is [illegible]

10. A uniform rod of mass m and length ℓ slides with its ends on a frictionless vertical circle. Using a Lagrangian, find the frequency of small oscillations if the rod subtends an angle of 120° at the circle's center. [The rod's moment of inertia about its midpoint is $m\ell^2/12$.]

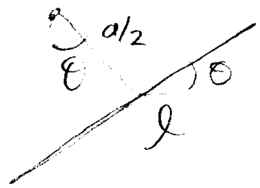


Mechanics (Lagrangian) [illegible]

10. A uniform rod of mass m and length l slides with its ends on a frictionless vertical circle. Using a Lagrangian, find the frequency of small oscillations if the rod subtends an angle of 120° at the circle's center. [The rod's moment of inertia about its midpoint is $ml^2/12$.]



[illegible]



Sol'n

$$T = \frac{1}{2} m v^2 + \frac{1}{2} I \dot\theta^2 \quad \begin{cases} v = \frac{a}{2}\dot\theta \\ I = \frac{ml^2}{12} = \frac{m}{12}(\sqrt{3}a)^2 = \frac{ma^2}{4} \end{cases}$$

$$= \frac{ma^2}{8}\dot\theta^2 + \frac{ma^2}{8}\dot\theta^2 = \frac{ma^2}{4}\dot\theta^2 \left(= \frac{ml^2}{12}\dot\theta^2 \right)$$

$$V = -mg\frac{a}{2}\cos\theta \quad \left(= -mg\frac{l}{2\sqrt{3}}\cos\theta \right)$$

$$L = \frac{ma^2}{2}\left(\frac{1}{2}\dot\theta^2 + \frac{g}{a}\cos\theta\right)$$

5

$$\left(\frac{d}{dt}\frac{\partial}{\partial\dot\theta} - \frac{\partial}{\partial\theta}\right)\left(\frac{1}{2}\dot\theta^2 + \frac{g}{a}\cos\theta\right)$$

$$= \ddot\theta + \frac{g}{a}\sin\theta = 0$$

$$\boxed{\omega = \sqrt{\frac{g}{a}} = \sqrt{\frac{\sqrt{3}g}{l}} = 1.316\sqrt{\frac{g}{l}}}$$

5

11. The wavefunction of an electron in the ground state of atomic hydrogen is given by:

$$\psi_{100}(r,\theta,\phi)=\left[\frac{1}{\pi a_o^3}\right]^{1/2}\exp\left(-\frac{r}{a_o}\right)$$

For this state, calculate

a) the most probable value of r
b) the expectation value of r
c) the expectation value of the potential energy
d) the expectation value of the kinetic energy.

$$[\int_o^\infty x^n \exp(-ax)dx=\frac{n!}{a^{n+1}}$$

where n is a positive integer and $a > 0$.]

Core: Wave Mechanics

1. The wavefunction of an electron in the ground state of atomic hydrogen is given by:

$$\psi_{100}(r,\theta,\phi) = \left[\frac{1}{\pi a_0^3}\right]^{\frac{1}{2}} \exp\left(-\frac{r}{a_0}\right)$$

For this state, calculate

(a) the most probable value of r

(b) the expectation value of r

(d) the expectation value of the kinetic energy

(c) ~~▆~~ the expectation value of the potential energy.

$\left[\int_0^\infty x^n \exp(-ax)\,dx = \frac{n!}{a^{n+1}}\right.$ where n is a positive integer and $a>0$.$\left.\right]$

Solution (a) probability density $[4\pi r^2]\,\psi^2(r)\,dr$

find max by usual derivative method, gives $\underline{r = a_0}$

(b & c) set up & evaluate integrals $\langle r\rangle = \frac{3a_0}{2}$

$\langle V\rangle = -\frac{e^2}{a_0}$

(d) use expectation value of H to avoid derivatives

(otherwise, you need Laplacian in spherical coordinates).

$$\left\langle \frac{p^2}{2m}\right\rangle = +\frac{e^2}{2a_0}$$

Good JLC

Details ⟶ over

(a) Spherical symmetry $\Rightarrow \quad d^3r = (4\pi r^2)\,dr$

$\frac{d}{dr}\left(r^2|\psi|^2\right)$ gives

$$2r e^{-\frac{2r}{a_0}} + r^2\left(-\frac{2}{a_0}\right)e^{-\frac{2r}{a_0}} = 0$$

$$\left(1-\frac{r}{a_0}\right) = 0 \qquad \text{so} \qquad \boxed{r = a_0}$$

(b)

$$\frac{4\pi}{\pi a_0^3}\int_0^\infty r^3 e^{-\frac{2r}{a_0}}\,dr = \left(\frac{4}{a_0^3}\right)\frac{6}{\left(\frac{2}{a_0}\right)^4} = \frac{24}{16}a_0 = \boxed{\frac{3}{2}a_0 = \langle r\rangle}$$

(c)

$$-\left(\frac{4\pi e^2}{\pi a_0^3}\right)\int_0^\infty r e^{-\frac{2r}{a_0}}\,dr = \left(\frac{4e^2}{a_0^3}\right)\frac{1}{\left(\frac{2}{a_0}\right)^2} = \boxed{-\frac{e^2}{a_0} = \langle V\rangle}$$

attraction

(d) 3 ways

1. Brute force
2. Virial Thm.
3. Eigenvalue

Recall $E(n=1) = -\frac{e^2}{2a_0}$

$$\langle V\rangle + \langle T\rangle = E$$

$$\boxed{\langle T\rangle = +\frac{e^2}{2a_0}}$$

12. Evaluate the following integral

$$I = \int_0^{2\pi} d\theta \quad \frac{e^{i\alpha\theta}}{a + b\cos\theta}$$

for $\alpha > 0, \quad a > b > 0.$

Indicate what you would do for $b = a$.

Mathematical Physics

12. Evaluate the following integral

$$I = \int_0^{2\pi} d\theta \; \frac{e^{i\alpha\theta}}{a + b\cos\theta} \;,$$

for $\alpha > 0$, $a > b > 0$.

Indicate what you would do for $b = a$.

$\therefore$

$$I = 4\pi \frac{z_1^{\alpha}}{z_1 - z_2}$$

$$z_1 - z_2 = -\frac{a}{b} + [\quad]^{1/2} + \frac{a}{b} + [\quad]^{1/2}$$

$$= 2\left[\left(\frac{a}{b}\right)^2 - 1\right]^{1/2}$$

$\therefore$

$$I = \frac{2\pi}{b} \frac{z_1^{\alpha}}{\left[\left(\frac{a}{b}\right)^2 - 1\right]^{1/2}}$$

If $a = b$ the poles lie on the contour of integration, and we cannot use the above procedure without some modification. Usually the physics of the problem leads to the definition

$$I_+ = \lim_{\eta \to 0^+} \int_0^{2\pi} d\theta \; \frac{e^{i\alpha\theta}}{a + b\cos\theta + i\eta}$$

For finite η the poles are off the contour (still one inside and one outside). <u>After</u> the integral is performed, we take the limit $\eta \to 0^+$.

$$I = \int_0^{2\pi} d\theta \, \frac{e^{i\alpha\theta}}{a + b\cos\theta} \qquad \alpha > 0 \quad , \quad a > b > 0$$

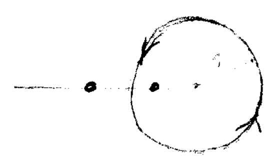
Set $\quad z = e^{i\theta} \quad , \quad dz = i e^{i\theta} d\theta \;\rightarrow\; d\theta = \frac{dz}{iz}$

$$\cos\theta = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right) = \frac{1}{2}\left(z + \frac{1}{z}\right) = \frac{1}{2z}\left(z^2+1\right)$$

$\therefore$

$$I = \oint_{\substack{\text{unit}\\ \text{Circle}}} \frac{dz}{iz} \, \frac{z^\alpha}{a + \frac{b}{2z}(z^2+1)}$$

$$= \oint \frac{2}{i} \, dz \, \frac{z^\alpha}{2az + b(z^2+1)}$$



$\therefore$

$$I = \frac{2}{i} \oint_{\substack{\text{unit}\\ \text{Circle}}} dz \, \frac{z^\alpha}{bz^2 + 2az + b} = \frac{2}{ib} \oint dz \, \frac{z^\alpha}{z^2 + 2\frac{a}{b}z + 1}$$

The poles of the integrand occur for

$$z_{1,2} = -\frac{a}{b} \pm \left[\left(\frac{a}{b}\right)^2 - 1\right]^{1/2}$$

$\Rightarrow$ z_1 is inside the contour of integration (Note that $z_1 z_2 = 1$)

$\therefore$

$$I = \left(\frac{2}{ib}\right) 2\pi i \text{ Residue of integrand at } z = z_1$$

13. Consider a Carnot cycle in which radiation is the working substance. If the initial expansion of the system increases the volume from V_A to V_B and the external heat reservoirs have temperatures T_h and T_c, determine how much work is extracted in each cycle. The radiation entropy satisfies

$$S = \frac{4}{3} b V T^3 \quad , \quad b = \text{const} > 0 \quad .$$

Thermo. Hermanson

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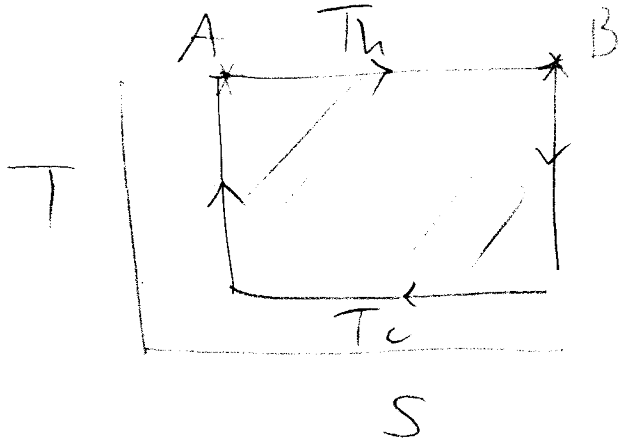


$$Q_h = T_h \Delta S_{AB}$$
$$= T_h \cdot \frac{4}{3} b T_h^3 (V_B - V_A)$$
$$= \text{net heat absorbed at } T_h$$

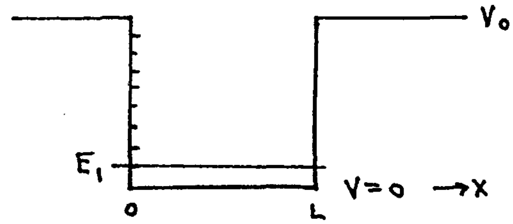
5

$$\text{Now } W = \varepsilon Q_h = \frac{T_h - T_c}{T_h} \cdot \frac{4}{3} b T_h^4 (V_B - V_A)$$

5

$$\boxed{W = \frac{4}{3} b T_h^3 (T_h - T_c)(V_B - V_A)}$$

14. a) Consider the finite square well shown. E_1 marks the ground state energy. For this system sketch in the approximate energies of the expected bound states.

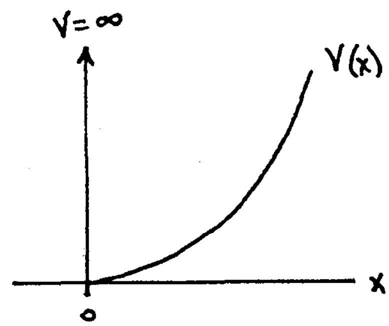


b) Sketch the approximate form of the wave functions associated with these eigenstates.

c) Now consider the potential shown, where

$$V = \frac{1}{2}kx^2 \quad , x > 0$$

$$V = \infty \quad , x < 0$$



Sketch the wave function associated with the first three eigenstates of this system. Label the classical turning points on your sketches.

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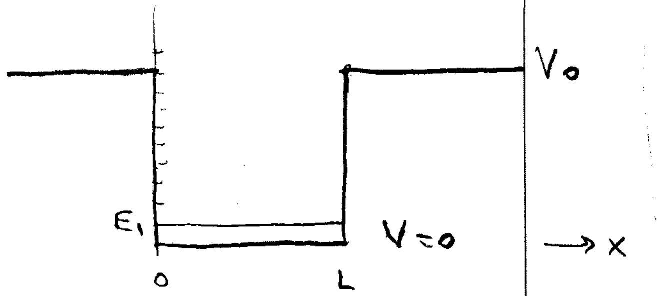
For this system sketch in the approximate energies of the expected bound states.



b) Sketch the approximate form of the wave functions associated with these eigenstates.

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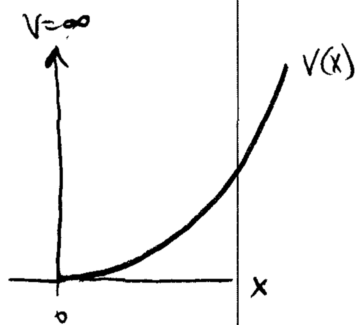
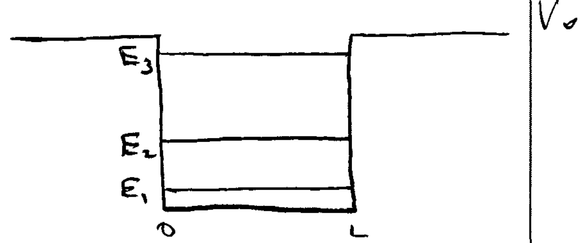


Sketch the wave function associated with the first three eigenstates of this system. Label the classical turning points on your sketches.

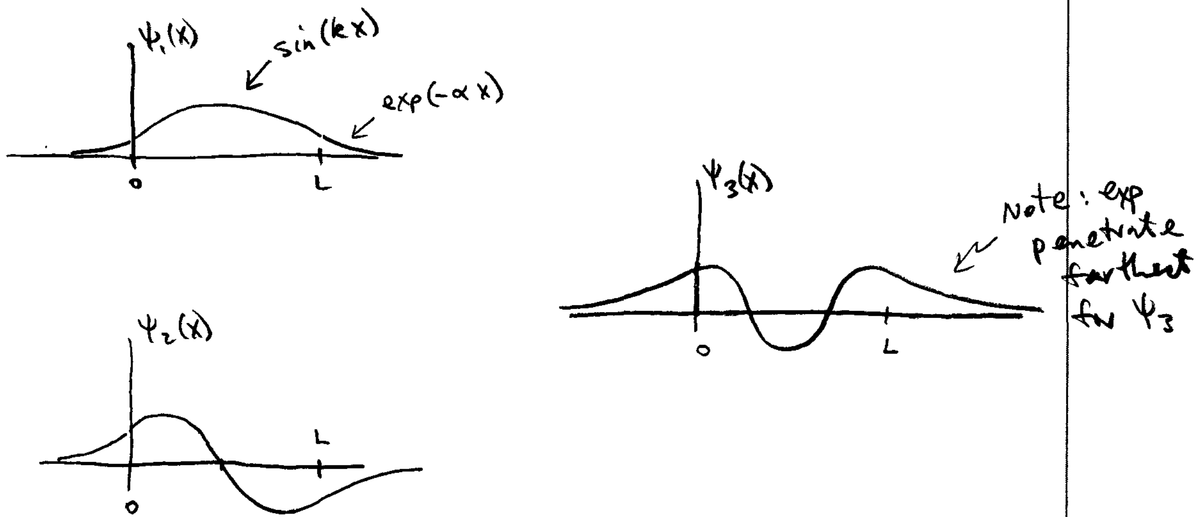
Solution

a) The energy levels for the infinite square well go as n^{-2}. The energy levels for the finite square well will lie somewhat below these since the deBloglie wavelength is larger to match boundary conditions with the exponential decay. Thus we expect the following approximate eigenenergies.

Note that E_3 is just barely bound.

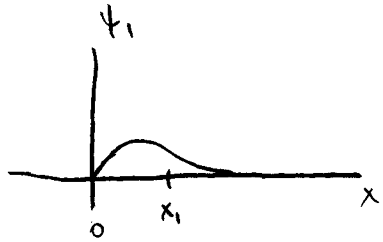


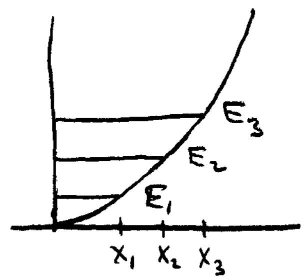
b)

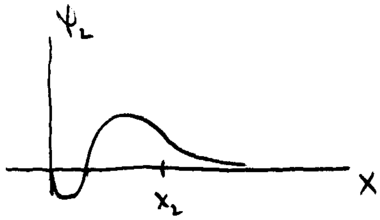


Solution (cont)

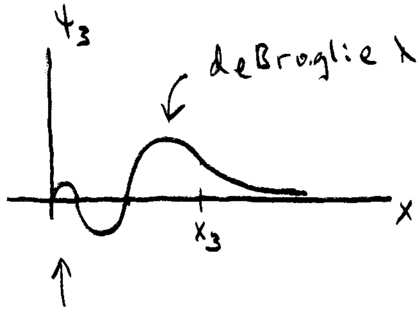
c) To plot the first three wave functions for the half harmonic oscillator, note that $\psi(0)=0$ since $V(0)=\infty$. Also note that the deBroglie λ is shortest near $x=0$ since in this region there is more kinetic energy. Finally note that the amplitude of the wave function is largest near the classical turning points where the kinetic energy is least.







deBroglie λ is longer, K.E. is smaller, amplitude is larger.



note: deBroglie λ is shorter, K.E. is larger, amplitude is lower

15. Power lines run due west from Hoover Dam at Lake Mead to Los Angeles, carrying a D.C. current of 100 amps. Assume the Earth's magnetic field points due north and has a strength of 1 gauss $(10^{-4}T)$.

$$\left(\mu_o = 4\pi \times 10^{-7} J A^{-2} m^{-1}\right)$$

a) What is the force/meter on the power line (magnitude and direction)?

b) If there are two parallel power lines, each carrying 100 amps west and separated by 5 meters, what is the magnetic force/meter between the power lines? Is it attractive or repulsive?

c) In order to transmit a given amount of power (say, 1 megawatt) from Nevada to Los Angeles, is it better to use high voltage/low current or low voltage/high current? Why?

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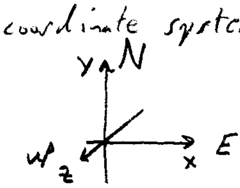
<u>Solution:</u> (a) $\xleftarrow{I = 100\text{ amps}}$ to L.A. W, N $\uparrow |\vec{B}| = 1 \text{ gauss} = 10^{-4}\,T$

$\vec{F}/L = \vec{I} \times \vec{B}$ use right hand rule

$= \boxed{10^{-2}\ N/m\ \underline{\text{downward}}}$

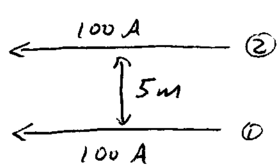
or, more formally: set up a Cartesian coordinate system positive x-axis points East
" y-axis " North
" z-axis " up



Then: $\vec{I} = -100\ A\ \hat{i}$
$\vec{B} = 10^{-4}\ T\ \hat{j}$

$\vec{F}/L = -10^{-2}\,(A \cdot T)\ \hat{i} \times \hat{j} = \boxed{-10^{-2}\left(\frac{N}{m}\right)\hat{k}}$ since $\hat{i} \times \hat{j} = \hat{k}$

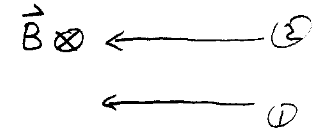
(b) Two parallel power lines, each has $I = 100\ A$.



First use Ampere's law to find the magnetic field caused by power line ① at ② : $|\vec{B}| = \frac{\mu_0 I}{2\pi r}$

$$|\vec{B}| = \frac{\cancel{4}^{2}\cancel{\pi}\times 10^{-7}\ \mathrm{J A^{-2} m^{-1}}\ (100\ \mathrm{A})}{\cancel{2}\cancel{\pi}\ \ 5\ \mathrm{m}} = \boxed{4\times 10^{-6}\ \mathrm{T}}$$

and, by the right hand rule, it points <u>downward</u>



so, in the coordinate system of (a), $\vec{B} = -4\times 10^{-6}\,\mathrm{T}\ \hat{k}$

Now, $\dfrac{\vec{F}}{L} = \vec{I}\times\vec{B} = (-100\,\mathrm{A}\ \hat{i})\times(-4\times10^{-6}\,\mathrm{T}\ \hat{k}) = 4\times10^{-4}\left(\frac{N}{m}\right)(\hat{i}\times\hat{k})$

$\hat{i}\times\hat{k} = -\hat{j}$, so $\boxed{\vec{F}/L = -4\times10^{-4}\left(\frac{N}{m}\right)\hat{j}}$

or: the force on 2 from 1 is of magnitude 4×10^{-4} N/m, and is directed <u>towards</u> 1. There is an equal and opposite force on 1 from 2. The force between the wires is <u>attractive</u>.

(c) It is better to design power lines with $\boxed{\text{high voltage / low current}}$

Reasoning: Non-superconducting power lines have some non-zero resistance R: the amount of power lost to dissipation (heat) in the power line is <u>I^2R</u>. The smaller I is, the smaller the power loss before the consumer is reached. Power transmission is then more efficient for smaller currents I. For a fixed amount of power VI, it is then best to make V large and I small.

16. A box containing a particle is divided into a right and a left compartment by a thin partition. If the particle is known to be on the right or left sides with certainty, the state is represented by the position eigenkets $|R\rangle$ and $|L\rangle$, respectively, where we have neglected spatial variations within each half of the box.

The particle can tunnel through the partition. This tunneling effect is characterized by the Hamiltonian

$$\hat{H} = V(|L\rangle\langle R| + |R\rangle\langle L|),$$

where V is a real number with the dimension of energy.

Suppose at t=0 the particle is on the right with certainty. What is the probability for observing the particle on the left as a function of time?

Quantum Mechanics

A box containing a particle is divided into a right and a left compartment by a thin partition. If the particle is known to be on the right or left side with certainty, the state is represented by the position eigenkets $|R\rangle$ and $|L\rangle$, respectively, where we have neglected spatial variations within each half of the box.
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$$\hat{H} = V\left(|L\rangle\langle R| + |R\rangle\langle L| \right),$$

where V is a real number with the dimension of energy.

Suppose at $t=0$ the particle is on the right with certainty. What is the probability for observing the particle on the left as a function of time?

Let's change the notation : $\hat{H} = V\,(|1\rangle\langle 2| + |2\rangle\langle 1|)$

$|1\rangle \to |L\rangle$

$|2\rangle \to |R\rangle$

$H_{11} = \langle 1|\hat{H}|1\rangle = 0$ $\qquad H_{22} = \langle 2|\hat{H}|2\rangle = 0$

$H_{12} = \langle 1|\hat{H}|2\rangle = V$ $\qquad H_{21} = \langle 2|\hat{H}|1\rangle = V$

∴ in the $\{|1\rangle, |2\rangle\}$ basis, the matrix representing $\hat{H}$ is

$$H = \begin{pmatrix} 0 & V \\ V & 0 \end{pmatrix}$$

We need the eigenvectors and eigenvalues of $\hat{H}$:

$$\hat{H}\,|\chi\rangle = \lambda\,|\chi\rangle\,.$$

We easily find that $\lambda = \pm V$, and

for $\lambda = V$: $|\chi\rangle = |V\rangle = \frac{1}{\sqrt{2}}\,(|1\rangle + |2\rangle)$ Symmetric

for $\lambda = -V$: $|\chi\rangle = |-V\rangle = \frac{1}{\sqrt{2}}\,(|1\rangle - |2\rangle)$ Anti-symmetric

Note that the hopping process splits the energies of the two stationary states. Note also that V is usually negative (In molecules, such as H_2^+, the symmetric state is the ground state).

$t=0$: $|\psi(0)\rangle = |R\rangle = |2\rangle$

$\langle V|2\rangle = \frac{1}{\sqrt{2}}$ $\qquad \langle -V|2\rangle = -\frac{1}{\sqrt{2}}$

$\therefore \quad |2\rangle = \frac{1}{\sqrt{2}}(|V\rangle - |-V\rangle)$

$$|\psi(t)\rangle = e^{-\frac{i}{\hbar}\hat{H}t}|2\rangle = \frac{1}{\sqrt{2}}e^{-\frac{i}{\hbar}\hat{H}t}(|V\rangle - |-V\rangle)$$

$$= \frac{1}{\sqrt{2}}\left(e^{-\frac{i}{\hbar}Vt}|V\rangle - e^{\frac{i}{\hbar}Vt}|-V\rangle\right)$$

$$= \frac{1}{2}\left[e^{-\frac{i}{\hbar}Vt}(|1\rangle + |2\rangle) - e^{\frac{i}{\hbar}Vt}(|1\rangle - |2\rangle)\right]$$

$$= -\frac{1}{2}\left(e^{\frac{i}{\hbar}Vt} - e^{-\frac{i}{\hbar}Vt}\right)|1\rangle + \frac{1}{2}\left(e^{\frac{i}{\hbar}Vt} + e^{-\frac{i}{\hbar}Vt}\right)|2\rangle$$

$\therefore$

$$|\psi(t)\rangle = -i\sin\frac{V}{\hbar}t\,|1\rangle + \cos Vt\,|2\rangle$$

$$\therefore \quad \mathcal{P}_{R\to L}(t) = \sin^2\left(\frac{V}{\hbar}t\right)$$