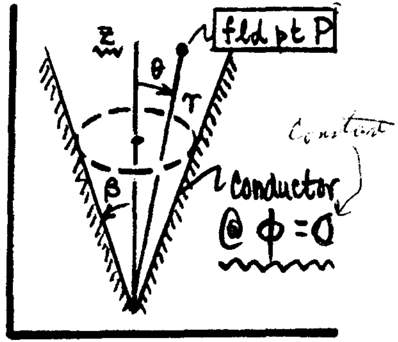
5) Next, in Sec.(3.4), Jackson does a 3D version of the 2D wedge problem -- which was Jk$^{\text{s}}$ Sec(2.11), and which we "did" on pp. I BV 13-14 above. This is problem of a <u>conical hole in a conductor</u>. Qualitatively, the results are not different from the 2D wedge [fields & charge densities are weak when $r \to 0$ and $\beta \to$ small (deep hole); E & $\sigma \to$ strong when $r \to 0$ and $\beta \to \pi$ (sharp point)]. Quantitatively, though, the math has some new twists -- we get to use solutions $P_\nu(x)$ with $\nu \neq$ integer.



The reason why we chose $\nu = l = 0, 1, 2, \ldots$ in the standard solution (p. II BV 3) was to ensure $P_\nu(x)$ converged @ $x^2 = \cos^2\theta = 1$, i.e. $\theta = 0$ & π. In the above conical hole problem, $0 \leq \theta \leq \beta$, so θ generally does not run to π. We need $P_\nu(x)$ convergent at $x = 1$ $(\theta = 0)$, but not necessarily at $x = -1$. Then <u>we do not have to fix ν = integer</u>; we can generalize $P_l(x) \to P_\nu(x)$.

from J.D. Jackson
"Classical ElectroDynamics"
(Wiley, 2nd ed., 1975)

An important expansion is that of the potential at $\mathbf{x}$ due to a unit point charge at $\mathbf{x}'$:

$$\boxed{\frac{1}{|\mathbf{x}-\mathbf{x}'|}=\sum_{l=0}^{\infty}\frac{r_<^l}{r_>^{l+1}}P_l(\cos\gamma)} \tag{3.38}$$

where $r_<$ ($r_>$) is the smaller (larger) of $|\mathbf{x}|$ and $|\mathbf{x}'|$, and γ is the angle between $\mathbf{x}$ and $\mathbf{x}'$, as shown in Fig. 3.3. This can be proved by rotating axes so that $\mathbf{x}'$ lies along the z axis. Then the potential satisfies the Laplace equation, possesses azimuthal symmetry, and can be expanded according to (3.33), except at the point $\mathbf{x}=\mathbf{x}'$:

$$\frac{1}{|\mathbf{x}-\mathbf{x}'|}=\sum_{l=0}^{\infty}(A_l r^l+B_l r^{-(l+1)})P_l(\cos\gamma)$$

If the point $\mathbf{x}$ is on the z axis, the right-hand side reduces to (3.37), while the left-hand side becomes:

$$\frac{1}{|\mathbf{x}-\mathbf{x}'|}=\frac{1}{(r^2+r'^2-2rr'\cos\gamma)^{1/2}}\rightarrow\frac{1}{|r-r'|}$$

EXPANSION of INVERSE DISTANCE

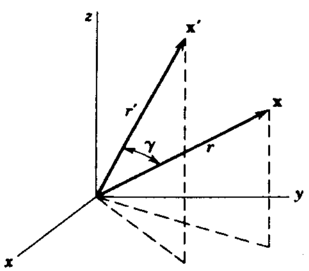
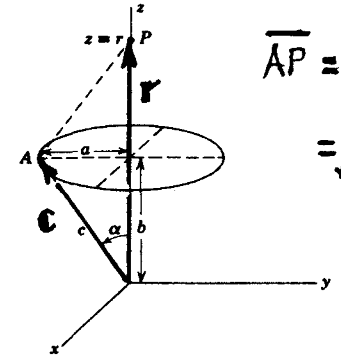


Fig. 3.3

NOTE...

$$\frac{1}{z-z'}=\begin{cases}\frac{1}{z}\left(1-\frac{z'}{z}\right)^{-1}=\frac{1}{z}\sum_{n=0}^{\infty}(z'/z)^n, & \text{for } |z'|<|z|;\\ \frac{1}{z'}\left(1-\frac{z}{z'}\right)^{-1}=\frac{1}{z'}\sum_{n=0}^{\infty}(z/z')^n, & \text{for } |z|<|z'|.\end{cases}$$

CHARGED CIRCULAR RING



$\overline{AP} = |\mathbf{r}-\mathbf{c}|$

$= \sqrt{r^2+c^2-2rc\cos\alpha}$

Fig. 3.4 Ring of charge of radius a and total charge q located on the z axis with center at $z=b$.

Expanding, we find, for $\mathbf{x}$ on axis,

$$\boxed{\frac{1}{|\mathbf{x}-\mathbf{x}'|}=\frac{1}{r_>}\sum_{l=0}^{\infty}\left(\frac{r_<}{r_>}\right)^l}$$

For points off the axis it is only necessary, according to (3.33) and (3.37), to multiply each term by $P_l(\cos\gamma)$. This proves the general result (3.38).

Another example is the potential due to a total charge q uniformly distributed around a circular ring of radius a, located as shown in Fig. 3.4, with its axis the z axis and its center at $z=b$. The potential at a point P on the axis of symmetry with $z=r$ is just q divided by the distance AP:

$$\Phi(z=r)=\frac{q}{(r^2+c^2-2cr\cos\alpha)^{1/2}}$$

denom. $= |\mathbf{r}-\mathbf{c}|$

where $c^2=a^2+b^2$ and $\alpha=\tan^{-1}(a/b)$. The inverse distance AP can be expanded using (3.38). Thus, for $r>c$,

$$\Phi(z=r)=q\sum_{l=0}^{\infty}\frac{c^l}{r^{l+1}}P_l(\cos\alpha)$$

For $r<c$, the corresponding form is:

$$\Phi(z=r)=q\sum_{l=0}^{\infty}\frac{r^l}{c^{l+1}}P_l(\cos\alpha)$$

The potential at *any point* in space is now obtained by multiplying each member of these series by $P_l(\cos\theta)$:

$$\boxed{\Phi(r,\theta)=q\sum_{l=0}^{\infty}\frac{r_<^l}{r_>^{l+1}}P_l(\cos\alpha)P_l(\cos\theta)}$$

where $r_<$ ($r_>$) is the smaller (larger) of r and c.

"Was man nicht im Kopf haben,
Er müsst im Beinen tragen." (Anon.)

II BV5A

Jackson shows how this is done in his Eq.s (3.39)–(3.42)...

new variable: $\xi = \frac{1}{2}(1-x) \Rightarrow$ Legendre Eq : $\frac{d}{d\xi}\left[\xi(1-\xi)\frac{dP}{d\xi}\right] + \nu(\nu+1)P = 0$; (11)

$\begin{pmatrix} \theta = 0\ (x=+1) \Rightarrow \xi = 0 \\ \theta = \pi\ (x=-1) \Rightarrow \xi = 1 \end{pmatrix}$ soln $P_\nu(\xi) = 1 + \frac{(-\nu)(\nu+1)}{1!}\frac{\xi}{1!} + \frac{(-\nu)(-\nu+1)(\nu+1)(\nu+2)}{2!}\frac{\xi^2}{2!} + \cdots$

This series converges @ $\xi = 0$ for all ν, and also converges for all ν when $\xi < 1$.† Since we never get to $\xi = 1$ in the problem at hand, we can let ν be free. The solution for the potential which is finite as $r \to 0$, and $\nu > 0$, is of form...

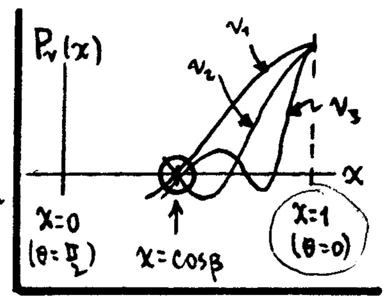
$$\rightarrow \phi(r,\theta) = \sum_\nu A_\nu r^\nu P_\nu(\cos\theta) \quad (12)$$

The local B.C. (that $\phi = 0$ @ $\theta = \beta$) will quantize ν. Distant B.C. will determine the A_ν (as they did for the 2D wedge); we won't do A_ν explicitly.

The local B.C. : $\phi = 0$ @ $\theta = \beta$ requires quantized ν's...

$$\rightarrow P_\nu(\cos\beta) = 0 \Rightarrow \text{solns: } \nu = \nu_1,\ \nu_2 > \nu_1,\ \nu_3 > \nu_2, \text{ etc.} \quad (13)$$

pole



As $r \to 0$, the dominant term in ϕ of Eq.(12) is the lowest power of r, i.e. $\phi \simeq A r^{\nu_1} P_{\nu_1}(\cos\theta)$, where ν_1 is the first ν-value for which $P_\nu(\cos\beta) = 0$. An approximate analysis [Jk$^{\text{n}}$ Eqs.(3.48)] shows...

$$\rightarrow \nu_1 \simeq \frac{2.405}{\beta} - \frac{1}{2}, \text{ as } \beta \to 0+ \ ; \quad \nu_1 \simeq 1/2\ln\left(\frac{2}{\pi-\beta}\right), \text{ as } \beta \to \pi- \ ; \quad (14)$$

then

$$(E_r, E_\theta) = -\left(\frac{\partial\phi}{\partial r}, \frac{1}{r}\frac{\partial\phi}{\partial\theta}\right) = A r^{\nu_1 - 1}\begin{cases} (-)\,\nu_1 P_{\nu_1}(\cos\theta), \\ +(\sin\theta)\,P'_{\nu_1}(\cos\theta); \end{cases}$$

$$\sigma(r) = -\frac{1}{4\pi}E_\theta\Big|_{\theta=\beta} = -\frac{1}{4\pi}A r^{\nu_1-1}(\sin\beta)\,P'_{\nu_1}(\cos\beta). \quad (15)$$

exponents resemble those of 2D wedge (sec 2.11)

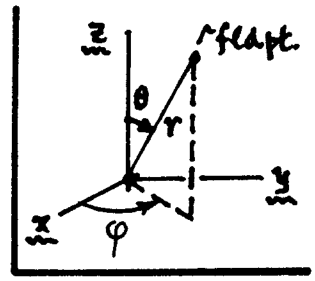
$\left.\begin{matrix}\beta \to 0+ \text{ (deep} \\ \text{conical hole)}\end{matrix}\right\} \Rightarrow (\nu_1 - 1) \gg 1$ ∫ E_r, E_θ & σ all vanish as $r \to 0$ ‖ $\left.\begin{matrix}\beta \to \pi- \text{ (sharp} \\ \text{conical point)}\end{matrix}\right\} (\nu_1 - 1) < 0$ ∫ E_r, E_θ & σ all $\sim 1/r \to \infty$ as $r \to 0$.

---

†Actually $P_\nu(\xi) = {}_2F_1(-\nu, \nu+1; 1; \xi)$, a "hypergeometric function" (w/ $a = -\nu$, $b = \nu+1$, $c = 1$). The hypergeometric eqtn is : $\xi(1-\xi)y'' + [c - (a+b+1)\xi]\,y' - ab\,y = 0$, w/ a, b, c = cnsts. The solution which is regular @ $\xi = 0$ is denoted : $y = {}_2F_1(a,b;c;\xi) = 1 + \left(\frac{ab}{c}\right)\xi + \left(\frac{a(a+1)\,b(b+1)}{c(c+1)}\right)\frac{\xi^2}{2!} + \cdots = \sum_{n=0}^{\infty}[(a)_n(b)_n/(c)_n]\,\xi^n/n!$ See Abramowitz & Stegun, Ch.15.

6) φ (azimuthal) variation: the $P_\ell^m(\cos\theta)$ and spherical harmonics $Y_{\ell m}(\theta,\varphi)$.*

1. In ¶ **2)**, pp. II BV2-4, we outlined the solution to $\nabla^2\phi=0$ in spherical cds (r,θ,φ), finding it convenient along the way to suppress the φ (azimuthal ∡)-dependence. We did this by setting $m=0$ in Legendre's Equation, viz...



$$\rightarrow \frac{d}{dx}\left[(1-x^2)\frac{dP}{dx}\right]+\left[l(l+1)-\frac{m^2}{1-x^2}\right]P=0; \qquad (16)$$

recall $x=\cos\theta$, and $m=0,\pm1,\pm2,\ldots$. Now we look at solutions to Eq. (16) for $m\neq0$, so that we are not confined to problems with azimuthal symmetry.

---

2. We use the method of solution-by-proclamation. Solutions to (16) are:

$$\left[\begin{array}{l} P_\ell^m(x)=(-)^m(1-x^2)^{\frac{m}{2}}\left(\frac{d}{dx}\right)^m P_\ell(x)=\frac{(-)^m}{2^\ell \ell!}(1-x^2)\left(\frac{d}{dx}\right)^{\ell+m}(x^2-1)^\ell; \\ \text{where: } \ell=0,1,2,\ldots; \text{ and } m=0,\pm1,\ldots,\pm\ell \ (2\ell+1 \text{ values max.}) \end{array}\right\} \qquad (17)$$

The $P_\ell^m(x)$ are called "associated Legendre polynomials". ℓ is quantized as noted in order to ensure the P_ℓ^m are finite over the entire range $|x|\leq1$. m is bounded by $|m|\leq\ell$ simply because $P_\ell^m\equiv0$ for $|m|>\ell$ $[(d/dx)^m P_\ell(x)\equiv0$ when $m>\ell$, since P_ℓ is a polynomial of order $\ell]$. Although the Rodrigues formula in (17) defines P_ℓ^{-m}, it is handy to know:

$$\rightarrow P_\ell^{-|m|}(x)=(-)^{|m|}\left[(\ell-|m|)!/(\ell+|m|)!\right]P_\ell^{|m|}(x). \qquad (18)$$

We also proclaim the $P_\ell^m(x)$ to be orthogonal on $|x|\leq1$; we know this to be true from Sturm-Liouville theory. The orthogonality integral is

$$\rightarrow \int_{-1}^{+1} P_\ell^m(x)\,P_\lambda^m(x)\,dx=\frac{2}{2\ell+1}\,\frac{(\ell+m)!}{(\ell-m)!}\,\delta_{\ell\lambda}. \qquad (19)$$

---

* Paraphrase of Jackson's Secs. (3.5)-(3.6).

3. The $\measuredangle$lar variation for the $\nabla^2\phi = 0$ problem in spherical cds will now be represented by product functions $P_\ell^m(\cos\theta)e^{im\varphi}$, $\ell = 0,1,2,\ldots$ & $|m| \leq \ell$. Since these fcns show up in all problems involving spherical symmetry†, they are given a special name, i.e. "Spherical Harmonics"★

$$\left[\!\!\left[\begin{array}{l} Y_{\ell m}(\theta,\varphi) = N_{\ell m}\,P_\ell^m(\cos\theta)\,e^{im\varphi}, \\ \text{w/}\;\; N_{\ell m} = \sqrt{\frac{2\ell+1}{4\pi}\left[(\ell-m)!/(\ell+m)!\right]}\,; \end{array}\right.\right. \left\{\begin{array}{l}\text{and: } \ell = 0,1,2,\ldots \\ \text{with: } |m| \leq \ell. \end{array}\right\} \tag{20}$$

The norm constant $N_{\ell m}$ is chosen to make orthogonality "look nice", i.e.

$$\rightarrow \int_{4\pi} d\Omega\, Y_{\ell m}^*(\theta,\varphi)\,Y_{\lambda\mu}(\theta,\varphi) = \delta_{\ell\lambda}\,\delta_{m\mu}\,; \tag{21}$$

$$\text{w/}\;\int_{4\pi} d\Omega = \int_0^{2\pi} d\varphi \int_0^{\pi} \sin\theta\, d\theta, \text{ integration over all solid } \measuredangle^s\ d\Omega.$$

The $Y_{\ell m}$ have the symmetry: $Y_{\ell,-|m|}(\theta,\varphi) = (-)^{|m|}\,Y_{\ell|m|}^*(\theta,\varphi)$, and they completely span the (θ,φ) space $[0 \leq \theta \leq \pi,\ 0 \leq \varphi \leq 2\pi]$ in that

$$\rightarrow \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{+\ell} Y_{\ell m}^*(\theta',\varphi')\,Y_{\ell m}(\theta,\varphi) = \delta(\cos\theta - \cos\theta')\,\delta(\varphi-\varphi'). \tag{22}$$

With this closure relation, an arbitrary $\measuredangle$lar fcn $g(\theta,\varphi)$ can be expanded in a series of spherical harmonics in the "usual" fashion, viz.

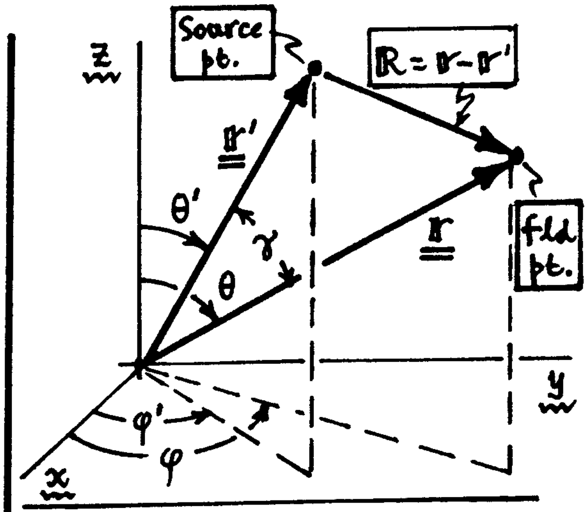
$$\left[\!\!\left[g(\theta,\varphi) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} A_{\ell m}\,Y_{\ell m}(\theta,\varphi) \longleftrightarrow A_{\ell m} = \int_{4\pi} d\Omega\, g(\theta,\varphi)\,Y_{\ell m}^*(\theta,\varphi)\right]\!\!\right]. \tag{23}$$

Finally, our $\nabla^2\phi = 0$ problem in spherical cds has the full solution:

$$\boxed{\phi(r,\theta,\varphi) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{+\ell}\left[A_{\ell m}\,r^\ell + B_{\ell m}\,r^{-(\ell+1)}\right]Y_{\ell m}(\theta,\varphi)}. \tag{24}$$

★ Jackson lists explicit forms for the $Y_{\ell m}$ through $\ell = 3$ on his pp. 99–100.

† For the generalized Helmholtz Eqtn: $[\nabla^2 + k^2 f(\mathbf{r})]\phi = 0$ (this includes Schrodinger), whenever f is a fcn of $r = |\mathbf{r}|$ only, the $\measuredangle$ variation goes as the $Y_{\ell m}(\theta,\varphi)$.



Source pt. @ $\mathbf{r}' = (r', \theta', \varphi')$;
field pt. @ $\mathbf{r} = (r, \theta, \varphi)$.
$R = |\mathbf{r} - \mathbf{r}'| = [r^2 + r'^2 - 2rr'\cos\gamma]^{1/2}$.

4. Back as far as Helmholtz' Theorem, we encountered the field point - source point distance $R = |\mathbf{r} - \mathbf{r}'|$ as an integral part of the solution for the potential (recall the solution to $\nabla^2\phi = (-)4\pi\rho$ was: $\phi(\mathbf{r}) = \int d\tau' \rho(\mathbf{r}')/R$). And recently we have expanded $1/R$ in a Legendre series...

$$\rightarrow \frac{1}{R} = \frac{1}{r}\sum_{l=0}^{\infty}\left(\frac{r'}{r}\right)^l P_l(\cos\gamma) \quad \int \text{for } r' < r. \text{ When } r' > r, \text{ interchange } r' \text{ with } r. \qquad (25)$$

[this is Jackson Eq. (3.38)]. Such a series is useful when we want $\phi(\mathbf{r}) = \int d\tau' \rho(\mathbf{r}')/R$ as a sum of successively smaller terms. But what's still clumsy here is that the $\measuredangle(\mathbf{r}, \mathbf{r}') = \gamma$ appears; in terms of the $\measuredangle$s θ' & φ' natural to $\mathbf{r}'$, and θ & φ for $\mathbf{r}$, γ has the forbidding form: $\cos\gamma = \cos\theta'\cos\theta + \sin\theta'\sin\theta\cos(\varphi' - \varphi)$.

It would be "nice" to express the $\measuredangle$ variation of $\frac{1}{R}$ in terms of the "natural" $\measuredangle$s θ' & φ' and θ & φ instead of γ; in (25), this amounts to finding an expression for $P_l(\cos\gamma)$ in terms of the fcns $Y_{lm}(\theta', \varphi')$ and $Y_{lm}(\theta, \varphi)$ which span the space of the $\mathbf{r}'$ and $\mathbf{r}$ directions. In fact this can be done by the Addition Theorem for Spherical Harmonics*, with the result...

$$\boxed{\frac{1}{R} = \frac{1}{r}\sum_{l=0}^{\infty}\left(\frac{r'}{r}\right)^l \underbrace{\left[\frac{4\pi}{2l+1}\sum_{m=-l}^{+l} Y^*_{lm}(\theta', \varphi') Y_{lm}(\theta, \varphi)\right]}_{\text{the [ ] } \equiv P_l(\cos\gamma), \text{ in natural } \measuredangle\text{'s}}} \quad \int \text{for } r' < r. \text{ When } r' > r, \text{ interchange } r' \text{ with } r. \qquad (26)$$

$1/R$ is now in a completely factored form for the positions $\left\{\begin{array}{l}\mathbf{r}' = (r', \theta', \varphi'),\\ \mathbf{r} = (r, \theta, \varphi).\end{array}\right.$ We can use this result later when we study so-called "multipole expansions".

* Proofs are given in many places, e.g. Jackson (2nd ed.) pp. 100-102, Arfken (3rd ed.) pp. 693-696, Mathews & Walker (2nd ed.) pp. 176-178. Arfken has some ~ interesting applications of the Addition Theorem in his problems for Sec. (12.8).