Failure of "what-me worry?" arguments for non-elementary charges.

The conventional wisdom is thus: don't be too concerned about loose ends connected with radiation reaction... they will either turn out to ① be negligible, or ② be unobservable. At least for elementary particles.

OK. Even though lack of a precise theory of radiation reaction might seem to be too big a "loose end" to let go in a theory as monolithic and comprehensive as Classical electrodynamics, we can accept the arguments against excessive anxiety about how the theory doesn't work for elementary particles -- but only for elementary particles. Arguments ①&② above are not convincing if the particle (q,m) is macroscopic. Suppose (q,m) is a "super-electron"...

$$\rightarrow (q,m) = (-ne,\ nm_e + M) \quad \Big\lceil\ n = 1,2,3,\ldots,\infty;\ m_e = \text{electron mass};\ M = \text{some mass of a binding matrix}.$$

So/ $\left.\dfrac{\text{distance scale}}{\text{charge radius}} = \right\}\ r_0 = q^2/mc^2 = \left(\dfrac{n}{1+(M/nm_e)}\right)\dfrac{e^2}{m_e c^2},$

$$\left.\frac{\text{time scale in}}{\text{Eq. (12)}}\right\}\ \tau_0 = \frac{2}{3}r_0/c = \left(\frac{n}{1+(M/nm_e)}\right)\cdot\frac{2}{3}e^2/m_e c^2. \qquad (14)$$

Evidently these scales can be made as large as we wish by letting $n \to$ large. In that case, the "negligible size" argument ① fails. Also the "QMly unobservable" argument ② fails similarly. For (q,m) = "super-electron" as above, Eq. (13) is:

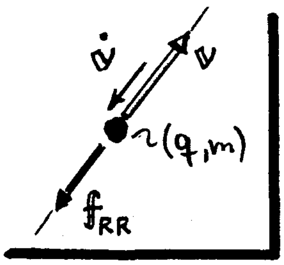
$$\rightarrow \Delta E \sim (205/n^2)\,\mathcal{M}c^2 \quad \Big\lceil\ \mathcal{M} = nm_e + M = \text{total mass of super-electron},\ \text{numerical coeff: } 205 = 1\Big/\tfrac{2}{3}(e^2/\hbar c). \qquad (15)$$

for the energy uncertainty generated by measurement down to $\Delta t \sim \tau_0$ of Eq. (14). We can make $\Delta E < \mathcal{M}c^2$ by choosing $n = 15$, and $\Delta E < 0.01\,\mathcal{M}c^2$ for $n = 144$.

The point of this argument is that while any glitches in radiation reaction theory may well be ignorable for elementary particles, they cannot be ignored (and will be measurable) for classical charged "pithballs" of charge $\sim 100e$.

Derivation of the Abraham-Lorentz Equation.

5) The conventional way of introducing an RR (radiation reaction) force $\mathbf{f}_{RR}$ into the nonrelativistic eqtn-of-motion for (q,m) is recounted in Jk^n Sec. 17.2. As in the derivation leading to Eq. (5) above, $\mathbf{f}_{RR}$ is linked to the Larmor radiation loss rate, but the link is handled differently. As follows.



Larmor loss rate for decelerating q $\Big\}$ $\displaystyle \frac{d\mathcal{E}_{radn}}{dt} = -\left(\frac{2q^2}{3c^3}\right)\dot{\mathbf{v}}^2 = \mathbf{f}_{RR}\cdot\mathbf{v},$

so// $\displaystyle \int_0^{\Delta t} \mathbf{f}_{RR}\cdot\mathbf{v}\,dt = -m\tau_0\int_0^{\Delta t}\dot{\mathbf{v}}\cdot(\dot{\mathbf{v}}\,dt)$ for deceleration during $0\to\Delta t$. $\underline{\tau_0 = 2q^2/3mc^3}$. (16)

Partial-integrate RHS of (16) and rearrange terms in the eqtn to write:

$$\rightarrow \underline{\underline{\int_0^{\Delta t}(\mathbf{f}_{RR} - m\tau_0\ddot{\mathbf{v}})\cdot\mathbf{v}\,dt = -m\tau_0(\mathbf{v}\cdot\dot{\mathbf{v}})\Big|_{t=0}^{t=\Delta t}}} \rightarrow 0. \tag{17}$$

Now the RHS of this eqtn is claimed to be negligible, i.e. zero, on grounds: (A) the motion is periodic (so the time-average of $(\mathbf{v}\cdot\dot{\mathbf{v}})$ is zero on average), or (B) the motion is chaotic (so $\mathbf{v}$ & $\dot{\mathbf{v}}$ are uncorrelated, and $\langle\mathbf{v}\cdot\dot{\mathbf{v}}\rangle = 0$, again on time-average), or (C) it just happens that $(\mathbf{v}\cdot\dot{\mathbf{v}}) = 0$ @ $t=0$ & $t=\Delta t$. This reasoning is ~ flabby, and should work only for SHO motion [case (A)]. But if we put the RHS of (17) $= 0$, then $\underline{\underline{\mathbf{f}_{RR} = m\tau_0\ddot{\mathbf{v}}}}$, and if (q,m) is being acted on by an external force $\mathbf{F}_{ext}$ we write its eqtn-of-motion as [Jk^n Eq. (17.9)]:

$$\boxed{m\dot{\mathbf{v}} = \mathbf{F}_{ext} + m\tau_0\ddot{\mathbf{v}}}, \quad \text{w/ } \tau_0 = 2q^2/3mc^3. \tag{18}$$

This is known as the Abraham-Lorentz Equation (1906); the form of $\mathbf{f}_{RR}$ was first obtained by Larmor (1897). The term in $\ddot{\mathbf{v}}$ is usually called the Schott term, after it was studied extensively by Schott (1912).

$\boxed{\text{REMARKS}}$ <u>on Abraham-Lorentz eqtn-of-motion, Eq. (18). The Schott term.</u>

1. Our new toy, the Schott term in $\ddot{\mathbf{v}}$ in Eq. (18), has the seeming advantage that it does not depend on the <u>structure</u> of (q,m) -- e.g. that (q,m) have any

Remarks on the Schott term (cont'd). Runaway Solutions.

particular size. The Schott term thus will be present for point (q,m)'s. Its structure-independence was demonstrated by Lorentz (1904) in a more elaborate calculation -- see Jk^n. Sec. 17.3. The Schott term is the only possible radiative correction to (q,m)'s motion which does not depend on structure.

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2. The Schott term is small enough to be a minor nuisance, usually.

if (q,m) is accelerated by field $\mathbb{E}$, then: $\dot{\mathbb{v}} \simeq \frac{q\mathbb{E}}{m}$ & $\ddot{\mathbb{v}} \simeq \frac{q\dot{\mathbb{E}}}{m}$;

$$\text{so}\quad \underline{\underline{|\mathbb{f}_{RR}/\mathbb{F}_{ext}| \simeq |\tau_0 q \dot{\mathbb{E}}/q\mathbb{E}| \sim \omega\tau_0}} \qquad \text{(} \omega \text{ is a typical frequency for changes in } \mathbb{E} = \mathbb{E}(t)\text{).} \tag{19}$$

If $(q,m) = (-e, m_e)$ is a single electron, $\tau_0 \simeq 6.26\times10^{-24}$ sec, and $f_{RR} \sim F_{ext}$ only at very high frequencies $\omega \sim 1/\tau_0 = 2\pi\times 2.54\times10^{22}$ Hz.* BUT, by the arguments on p. RR6, we can make the competition between f_{RR} & F_{ext} much more realistic by letting $(q,m)\to$ super-electron.

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3. The Schott term in Eq. (18) is objectionable in that it renders the eqtn-of-motion non-Newtonian. To solve Eq. (18) fully, we now need initial-value information on position $\mathbb{r}$, velocity $\mathbb{v} = \dot{\mathbb{r}}$, and acceleration $\mathbb{a} = \dot{\mathbb{v}}$. A Newtonian eqtn-of-motion of course requires initial values of $\mathbb{r}$ & $\mathbb{v}$ only.¶

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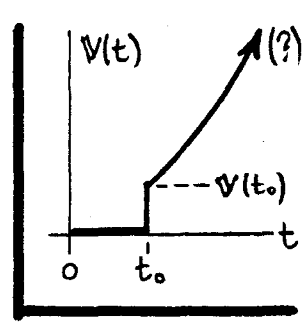
4. The Schott term can generate "runaway solutions", which are totally silly. Let (q,m) be in empty space, where all $\mathbb{F}_{ext} \equiv 0$. Then Eq. (18) reads...

$$\rightarrow m\dot{\mathbb{v}} = \tau_0 \frac{d}{dt}(m\dot{\mathbb{v}}) \Rightarrow \underline{\underline{\mathbb{v}(t) = \mathbb{v}(t_0)\, e^{\frac{1}{\tau_0}(t-t_0)}}}. \tag{20}$$



The history of the motion is sketched at right. If ever (q,m) acquires a velocity -- say $\mathbb{v}(t_0)$ at time t_0 -- it will immediately accelerate off to ∞, in a characteristic time τ_0. Huh?

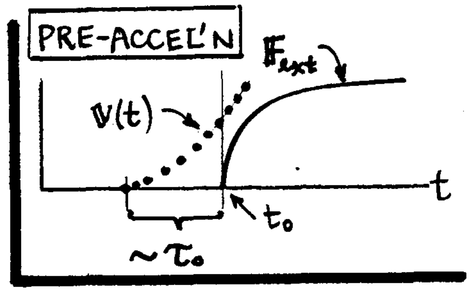
¶ Note that our first RR correction attempt, Eq. (5), gives a Newtonian eqtn.

* These are photons at energy: $\mathcal{E} = \hbar\omega \sim \hbar/\tau_0 = \left[\frac{3}{2}(\hbar c/e^2)\right] m_e c^2 \simeq 105$ MeV.

Remarks on Schott term (cont'd). Pre-Acceleration. Solution for SHO.

5. The runaway solutions are so silly that they must be eliminated. Details of how to do this are given in Jk^n Sec. 17.6. The Abraham-Lorentz Eqtn, viz: $m\dot{\mathbb{V}} = \mathbb{F}_{ext} + m\tau_0\ddot{\mathbb{V}}$, can be converted to an integro-differential eqtn which can be written in Newtonian form as...

$$\rightarrow \underline{\underline{m\dot{\mathbb{V}}(t) = \int_0^\infty e^{-s}\,\mathbb{F}_{ext}(t+\tau_0 s)\,ds}} = \mathbb{F}_{ext}(t) + \sum_{n=1}^{\infty}\Big(\underbrace{\tau_0}_{\tau_0 = 2q^2/3mc^3}\frac{d}{dt}\Big)^n \mathbb{F}_{ext}(t). \qquad (21)$$



This looks OK -- we get back Newton's $m\dot{\mathbb{V}} = \mathbb{F}_{ext}$ in the limit that $\tau_0 \to 0$ (i.e. the charge vanishes). There are no runaway solutions because $\dot{\mathbb{V}} \equiv 0$ in the absence of any $\mathbb{F}_{ext}$. And when $\tau_0 > 0$, the RR corrections appear as a series of terms which are small if $\mathbb{F}_{ext}(t)$ changes slowly. BUT, the integral form makes it clear that (q,m)'s motion $\mathbb{V}(t)$ at time t depends on values of the force $\mathbb{F}_{ext}(t+\tau_0 s)$ at times in the future. Thus Eq.(21) is acausal. In particular, if $\mathbb{F}_{ext}$ is applied at time t_0 (i.e. $\mathbb{F}_{ext} \equiv 0$ @ all $t < t_0$), the particle begins to move at times $\sim(t_0 - \tau_0)$ prior to the action of $\mathbb{F}_{ext}$. This is called pre-acceleration, and is also very silly.

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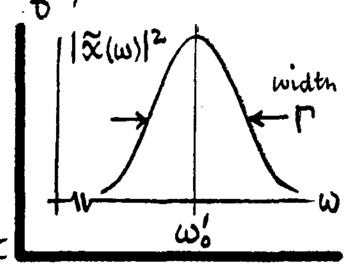
6. One place where the Schott term does work, w/o creating any apparent havoc, is in describing the motion of a SHO. No surprise (?); the SHO is the only system where the neglect of the RHS of Eq.(17) can be justified. For a SHO in 1D, one writes $F_{ext} = -m\omega_0^2 x$ in Eq. (21), w/ ω_0 = natural freq., and solves:

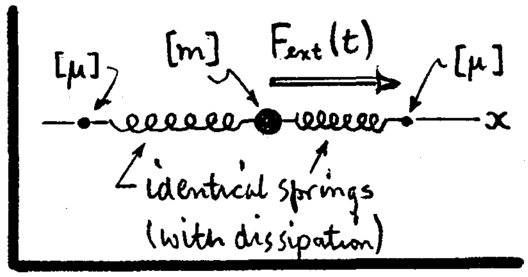
$$\rightarrow \underline{\underline{\ddot{x}(t) + \omega_0^2\int_0^\infty e^{-s}\,x(t+\tau_0 s)\,ds = 0}}. \qquad (22)$$



Details appear in Jk^n Sec. 17.7. One finds solutions $x(t) \sim \exp[-(\frac{\Gamma}{2} \pm i\omega_0')t]$, where -- for $\omega_0\tau_0 \ll 1$ -- the damping cnst $\Gamma = \omega_0^2\tau_0$, and the shifted frequency $\omega_0' = \omega_0[1 - \frac{5}{8}(\omega_0\tau_0)^2]$. This is rational behavior for a radiation-damped SHO. The Fourier spectrum $|\tilde{x}(\omega)|^2$ is Lorentzian.

Last Schott: a mechanical analogy. Do we get saved by relativity?

7. The Schott term in $\ddot{v}$ in Eq. (18) produces more debits than credits in the theory, and it would be "nice" to make it go away. However, there appear to be fundamental reasons for why a $\ddot{v}$ correction must show up in the eqtn-of-motion for (q, m). Consider the 1D motion of the mechanical system shown at right: a central mass m is attached front and back by identical, dissipative springs to masses μ. Mass m is now accelerated



to the right by an external force F_{ext} = fcn of time t. It is easy to show that the complete system obeys Newton II, i.e. $M\ddot{x}_{cm} = F_{ext}$, w/ $M = m + 2\mu$ = total mass, and x_{cm} is the system center-of-mass coordinate. BUT, if we seek an eqtn-of-motion for only a part of the system, say m alone, we find a non-Newtonian result:

$$\rightarrow m\ddot{x} = \left(1 - \frac{2\mu}{M}\right) F_{ext}(t) + \underbrace{(4\beta\mu/M\omega_0^2)\,\dot{F}_{ext}(t)}_{\text{Schott term}} + \ldots \tag{23}$$

(ω_0 is the natural freq. of the springs, and β is the dissipation cnst). The add-on RHS is a Schott term, since it is proportional to $\dot{F}_{ext} \simeq \frac{d}{dt}(m\ddot{x}) = m\dddot{x}$. So we get Schott terms whenever we try to write an eqtn-of-motion just one part of a system that has internal & dissipative degrees of freedom. That is what we have been trying to do for our particle (q, m) -- the implication is that the Schott term appears in the Abraham-Lorentz Eq. (18) as a remnant (and an incomplete remnant) of the dissipative degrees of freedom in (q, m)'s self-fields.

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6) Do we get saved by relativity? I.e. can we modify the covariant Lorentz Law: $m\dot{u}^\alpha = (q/c)F^{\alpha\beta}u_\beta$ [Eq. (3) above] by a sensible RR correction? The answer seems to be "... no, but ..." Dirac took up the question in a 1938 paper, where he derived a covariant eqtn-of-motion for a point (q, m) which included a RR correction. His derivation was consistent with Maxwell's Eqtns, and with conservation of momentum & energy for (q,m) + the EM field. Result:

$$\boxed{\underbrace{m\dot{u}^\alpha}_{\text{mass is "renormalized"}} = \underbrace{\frac{q}{c}F^{\alpha\beta}_{ext}\,u_\beta}_{\text{Standard Lorentz Law for extl fields}} + \frac{2q^2}{3c^3}\Big[\underbrace{\ddot{u}^\alpha}_{\text{Schott term (covariant)}} + \underbrace{\frac{1}{c^2}(\dot{u}^\lambda\dot{u}_\lambda)\,u^\alpha}_{\text{Larmor term } (\alpha=0 \Rightarrow \text{Larmor rate})}\Big] + \underbrace{G^\alpha}_{\text{possible add-on}}} \quad \text{LORENTZ-DIRAC EQTN} \quad (24)$$

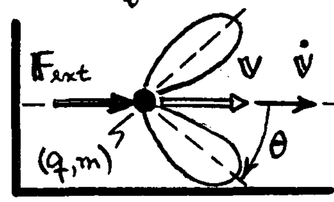
$u^\alpha = \gamma(c, \mathbf{v})$ is m's 4-velocity, and the "$\cdot$" (dots) $\Rightarrow d/d\tau$, w/ τ = m's proper time. $F^{\alpha\beta}_{ext}$ is the external field tensor. The Schott term in $\ddot{u}^\alpha$ is firmly in place, now covariant & irrevocable; in fact the Schott term is there precisely to preserve the Minkowski-force character of this eqtn-of-motion (see ★ below). The Larmor term is a retarding force related physically to the fact that (q,m) radiates total energy at rate $P(t') = -\frac{2q^2}{3c^3}(\dot{u}_\lambda\dot{u}^\lambda)$, and the radiation is preferentially forward (see ★).

UNFORTUNATELY, the L-D Eq. suffers all the debits cited on pp. RR 7-10, and then some. Oops...

★ A way of deriving the Lorentz-Dirac Eq. (24) follows. If (q,m) is arbitrarily accelerated at $\dot{\mathbf{v}}$ by F_{ext}, its radiation (energy) rate per ∞smal solid ∡ $d\Omega$ is [Jk^n Eq. (14.39)]:

$$\frac{dP(t')}{d\Omega} = \frac{q^2\dot{v}^2}{4\pi c^3}\sin^2\theta/(1-\beta\cos\theta)^2$$

Evidently, (q,m) radiates energy preferentially forward.



But radiant energy $\mathcal{E}$ is related to radiant momentum by $p = \mathcal{E}/c$, so the variation of forward momentum rate with $d\Omega$ is: $d[dp/dt'] = \frac{1}{c}\left[\frac{dP(t')}{d\Omega}\right]d\Omega\cos\theta$. The extra $\cos\theta$ factor gives the forward component of dp/dt'. Now integrate over $0 \leq \theta \leq \pi$ to find net forward momentum rate: $d\mathbf{p}/dt' = \left[\frac{1}{c^2}P(t')\right]\mathbf{v}$, w/ $P(t') = \frac{2q^2}{3c^3}(\gamma^3\dot{v})^2$ the total power radiated by (q,m). Identify the RR force as: $\mathbf{f}_{RR} = -\gamma(d\mathbf{p}/dt')$, and write the power $P(t') = -\frac{2q^2}{3c^3}(\dot{u}^\lambda\dot{u}_\lambda)$ in covariant form [Jk^n Eq. (14.24)]. This analysis gives the Larmor term: $f^\alpha_L = (2q^2/3c^3)\frac{1}{c^2}(\dot{u}_\lambda u_\lambda)u^\alpha$, in Eq. (24). But, this Larmor force is not by itself a Minkowski force, since $f^\alpha_L u_\alpha \neq 0$ (the LHS of (24), and the Lorentz term are both Minkowski forces, since $\dot{u}^\alpha u_\alpha = 0$, and $F^{\alpha\beta}u_\beta u_\alpha = 0$). We must add on a term: $f^\alpha_L \to f^\alpha_L + g^\alpha$, such that $(f^\alpha_L + g^\alpha)u_\alpha = 0$. The required add-on, $g^\alpha = \frac{2q^2}{3c^3}\ddot{u}_\alpha$, is just the Schott term indicated on the RHS of (24). Here the Schott term appears as a mathematical artifice -- needed (only?) to preserve the Minkowski force nature of (24).