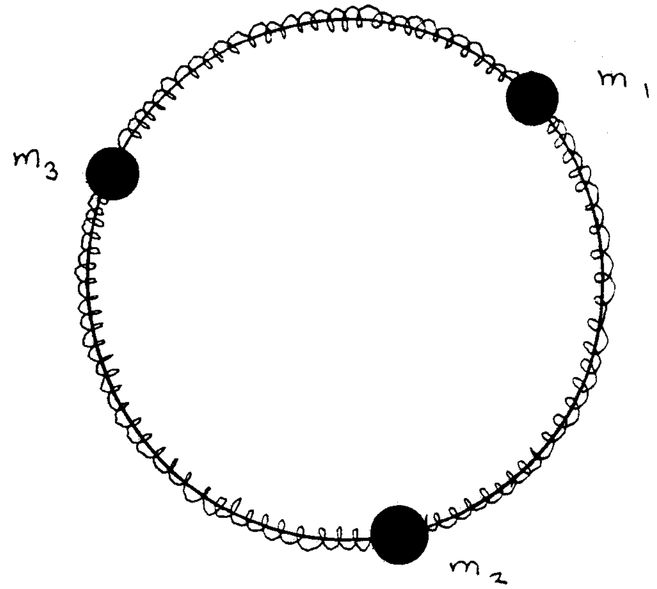
DEPARTMENT OF PHYSICS

PH. D. COMPREHENSIVE EXAMINATION

SEPTEMBER 19-20, 1988

1. Consider a system of three equal masses constrained to move on a ring of radius r. The masses move without friction along the ring and are connected by springs of spring constant k which exert a force between masses which is proportional to the distance along the ring between the masses.



a) Find the equations of motion for the three masses.

b) Find the normal mode solutions of these equations. Choose the modes to be orthogonal if that is possible.

c) Find the most general solution to these equations.

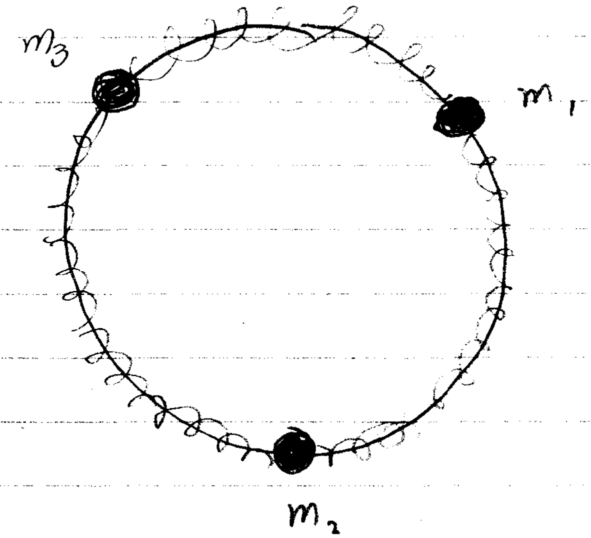
Lee Lindblom

#1 Classical Mechanics

Consider a system of three equal masses constrained to move on a ring of radius r. The masses move without friction along the ring and are connected by springs of spring constant k which exert a force between masses which is proportional to the distance along the ring between the masses.

a) Find the equations of motion for the three masses.

b) Find the normal mode solutions of these equations. Choose the modes to be orthogonal if that is possible.



c) Find the most general solution to these equations.

Solution:

a) Use as coordinates the angular location of each mass: θ_i
The kinetic energy of the system is therefore:

$$T = \frac{1}{2} m r^2 \left(\dot{\theta}_1^2 + \dot{\theta}_2^2 + \dot{\theta}_3^2\right)$$

The potential energy is:

$$V = \frac{1}{2} k r^2 \left\{(\theta_1 - \theta_2)^2 + (\theta_2 - \theta_3)^2 + (\theta_1 - \theta_3)^2\right\}$$

The Lagrangian is $L = T - V$ and Langrange's equations give the equations of motion for the system:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_i}\right) - \frac{\partial L}{\partial \theta_i} = 0$$

$$\begin{cases} m r^2 \ddot{\theta}_1 = -k r^2 (2\theta_1 - \theta_2 - \theta_3) \\ m r^2 \ddot{\theta}_2 = -k r^2 (-\theta_1 + 2\theta_2 - \theta_3) \\ m r^2 \ddot{\theta}_3 = -k r^2 (-\theta_1 - \theta_2 + 2\theta_3) \end{cases}$$

b) The normal mode solutions have time dependence: $e^{i\omega t}$, thus

$$\begin{aligned} -\lambda \theta_1 &= -2\theta_1 + \theta_2 + \theta_3 \\ -\lambda \theta_2 &= \theta_1 - 2\theta_2 + \theta_3 \qquad \text{where } \lambda = \frac{m}{k}\omega^2 \\ -\lambda \theta_3 &= \theta_1 + \theta_2 - 2\theta_3 \end{aligned}$$

or:
$$0 = \begin{pmatrix} \lambda - 2 & 1 & 1 \\ 1 & \lambda - 2 & 1 \\ 1 & 1 & \lambda - 2 \end{pmatrix} \begin{pmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{pmatrix}$$

The determinent of this matrix must vanish if there are to be non trivial solutions, thus:

$$0 = \det \begin{pmatrix} \lambda-2 & 1 & 1 \\ 1 & \lambda-2 & 1 \\ 1 & 1 & \lambda-2 \end{pmatrix} = (\lambda-2)^3 + 2 - 3(\lambda-2)$$

$$= (\lambda^3 - 6\lambda^2 + 12\lambda - 8) + 2 + (-3\lambda + 6)$$

$$= \lambda^3 - 6\lambda^2 + 9\lambda = (\lambda-3)^2\lambda$$

Thus the eigen frequencies of this system are

$$\begin{cases} \omega^2 = 0 \\ \omega^2 = 3\frac{k}{m} \leftarrow \text{doubly degenerate root} \end{cases}$$

The vales of Θ_1, Θ_2 and Θ_3 corresponding to these roots can be found in a straightforeward manner:

$$\omega^2 = 0 \Rightarrow \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}\begin{pmatrix} \Theta_1 \\ \Theta_2 \\ \Theta_3 \end{pmatrix} = 0 \Rightarrow \begin{pmatrix} \Theta_1 \\ \Theta_2 \\ \Theta_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\omega^2 = 3\frac{k}{m} \Rightarrow \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} \Theta_1 \\ \Theta_2 \\ \Theta_3 \end{pmatrix} = 0 \Rightarrow \begin{pmatrix} \Theta_1 \\ \Theta_2 \\ \Theta_3 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} \text{ or } \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

Clearly any linear combination is also a solution, these are chosen to be orthogonal.

c) The equations of motion of this system are linear thus the most general solution of the equations is simply a linear combination of the modes. Or, it would be except for the existence of the zero frequency mode. In this case there is also a solution which is linear in t. Thus the most general solution of the system is the real part of:

$$\begin{pmatrix} \theta_1(t) \\ \theta_2(t) \\ \theta_3(t) \end{pmatrix} = (\alpha + \beta t)\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + (\gamma + i\delta)\, e^{i\left(\frac{3k}{m}\right)^{1/2} t} \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$$

$$+ (\mu + i\nu)\, e^{i\left(\frac{3k}{m}\right)^{1/2} t} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

Where $\alpha, \beta, \gamma, \delta, \mu$ and ν are real constants corresponding to the six initial conditions (positions and velocities) of the three masses.

$\sqrt{3k/m}$ in sol'n ?

2. A thin-walled cylindrical nonmagnetic metal tube has radius R, wall thickness b (b<<R) and length L (L>>R). The tube material has conductivity σ. A coil of N turns per unit length is wound tightly around the outside of the tube. The coil carries a current I.

   a) Find the magnetic induction $\vec{B}$ at the center of the solenoid. (Hint: You may assume the coil and tube to be of infinite length for part a.)

   b) The current, having been maintained at the value I since $t=-\infty$ is switched off at $t=0$, the winding being left open-circuited. Find $\vec{B}(t)$ at the center of the solenoid, neglecting the displacement current.

   c) Under what conditions can the displacement current be neglected safely? Your answer here should be in the form of one or more inequalities involving geometrical and/or material parameters.

E+M Smith

#2 A thin-walled cylindrical nonmagnetic metal tube has radius R, wall thickness b ($b \ll R$) and length L ($L \gg R$). The tube material has conductivity σ. A coil of n turns per unit length is wound tightly around the outside of the tube. The coil carries a current I.

a) Find the magnetic induction $\vec{B}$ at the center of the solenoid. (Hint: You may assume the coil and tube to be of infinite length for part a.)

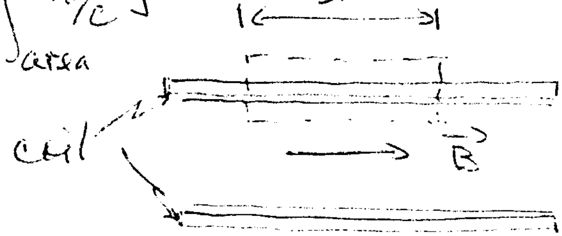
b) The current, having been maintained at the value I since $t = -\infty$ is switched off at $t = 0$, the winding being left open-circuited. Find $\vec{B}(t)$ at the center of the solenoid, neglecting the displacement current.

c) Under what conditions can the displacement current be neglected safely? Your answer here should be in the form of one or more inequalities involving geometrical and/or material parameters.

(a) Get the field, assuming infinitely long tube using

$$\oint \vec{B}\cdot d\vec{l} = \int_{area} \frac{4\pi}{c}\vec{J}$$

for closed circuit



By symmetry, $\vec{B}$ is axial along tube, inside and zero outside

$$Bl = \frac{4\pi}{c} I(nl)$$

$$\boxed{B = \frac{4\pi n I}{c}}$$

(b) Keeping coil open circuit means some charge may build up at ends of wire, but can ignore any current flow in coil.

Neglecting displacement current means here

$$(1) \qquad \vec{\nabla}\times\vec{B} = \frac{4\pi}{c}\vec{J} + \cancel{\frac{1}{c}\frac{\partial E}{\partial t}} \quad \text{neglect}$$

as used in part (a) above. We want to find a diff eqn for $B(t)$. Make use of induced emf, $\mathcal{E}$

$$(2) \qquad \mathcal{E} = -\frac{1}{c}\frac{d\Phi}{dt} = -\frac{1}{c}\pi R^2 \frac{dB}{dt}$$

Using (1) above gives $B(t) = \frac{4\pi}{c} I_{tube}/L$ ← induced

$$\mathcal{E} = -\frac{\pi R^2}{c}\frac{4\pi}{cL}\frac{dI_{tube}}{dt} = I_{tube} Res$$
$$= I_{tube}\frac{1}{\sigma}\frac{2\pi R}{bL}$$

so
$$\frac{dI_{tube}}{I_{tube}} = -\frac{c^2 b}{4\pi^2 R^2}\frac{1}{\sigma}\frac{2\pi R}{bL}dt$$
$$= -\frac{c^2}{2\pi R\sigma b}dt$$
$$I_{tube} = I_{0\,tube}\, e^{-\frac{c^2 t}{2\pi R\sigma b}} + \text{const}$$

At $t = \infty$, $I_{tube} \to 0$ so const $= 0$

Since
$B_{tube} \propto I_{tube}$ we have
$$B(t) = B_0\, e^{-c^2 t/2\pi R\sigma b}$$

at $t=0$ $B_0 = 4\pi n I/c$ from part (a) so
$$\boxed{B(t) = \frac{4\pi n I}{c}\, e^{-c^2 t/2\pi R\sigma b}}$$

(c) To neglect displacement current means consider
$$\frac{1}{c}\partial E/\partial t = \frac{1}{c}\frac{1}{\sigma}\partial J/\partial t = \frac{1}{c\sigma}\frac{1}{bL}\frac{dI}{dt} = -\frac{1}{c\sigma bL}\frac{c^2}{2\pi R\sigma b}I$$
$$= -\frac{c}{2\pi R\sigma^2 b}J \text{ ; we want this} \ll \frac{4\pi}{c}J$$

So require
$$\boxed{\frac{c^2}{8\pi^2 R\sigma^2 b} \ll 1}$$

3. Consider a composite system made up of two spin 1/2 particles. For t<0 the Hamiltonian does not depend on spin, and can be taken to be zero by suitably adjusting the energy scale. For t>0 the Hamiltonian is given by

$$\hat{H} = \frac{4\lambda}{\hbar} \underline{\hat{S}}_1 \cdot \underline{\hat{S}}_2 ,$$

where λ is a real constant with appropriate dimensions, and $\underline{\hat{S}}_1$ and $\underline{\hat{S}}_2$ are the (vector) spin operators for each particle.

a) Suppose the system is in the state $|+-\rangle$ for $t \leq 0$, that is, in a state such that the z-component of the first spin is "up" and the z-component of the second spin is "down". Find, as a function of time, the probability for the system to be found in each of the following states: $|++\rangle$, $|+-\rangle$, $|-+\rangle$, and $|--\rangle$.

b) Can part a) refer equally well to identical or non-identical particles? Explain.

Cone

(Eguiluz)

#3

<u>Quantum</u> <u>Mechanics</u>.

Consider a composite system made up of two spin 1/2 particles. For $t<0$ the Hamiltonian does not depend on spin, and can be taken to be zero by suitably adjusting the energy scale. For $t>0$ the Hamiltonian is given by

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b) Can part a) refer equally well to identical or non-identical particles? Explain.

<u>Solution</u>.

$$\hat{\underset{\sim}{S}} = \hat{\underset{\sim}{S}}_1 + \hat{\underset{\sim}{S}}_2$$

The eigenvectors of $\hat{S}^2$ and $\hat{S}_z$ are the triplet and singlet states:

$$\left.\begin{aligned} |11\rangle &= |++\rangle \\ |10\rangle &= \frac{1}{\sqrt{2}}\left(|+-\rangle + |-+\rangle\right) \\ |1,-1\rangle &= |--\rangle \end{aligned}\right\} \text{Triplet}, \quad S=1$$

$$|0,0\rangle = \frac{1}{\sqrt{2}}\left(|+-\rangle - |-+\rangle\right) \qquad \text{Singlet}, \quad S=0$$

Now:

$$\hat{S}^2 = \hat{S}_1^2 + \hat{S}_2^2 + 2\,\underset{\sim}{\hat{S}_1}\cdot\underset{\sim}{\hat{S}_2}$$

$\therefore$

$$\underset{\sim}{\hat{S}_1}\cdot\underset{\sim}{\hat{S}_2} = \frac{1}{2}\left(\underset{\sim}{\hat{S}}^2 - \underset{\sim}{\hat{S}_1}^2 - \underset{\sim}{\hat{S}_2}^2\right)$$

$\therefore$

$$\hat{H} = \frac{2\lambda}{\hbar}\left(\hat{S}^2 - \hat{S}_1^2 - \hat{S}_2^2\right)$$

Then:

$$\hat{H}|1,m\rangle = $$
$$= \frac{2\lambda}{\hbar}\hbar^2\left[1(1+1) - \frac{1}{2}\left(\frac{1}{2}+1\right) - \frac{1}{2}\left(\frac{1}{2}+1\right)\right]|1,m\rangle$$

$\therefore$

$$\hat{H}|1,m\rangle = \hbar\lambda\,|1,m\rangle$$

$$\hat{H}|0,0\rangle =$$

$$= \frac{2\lambda}{\hbar^2}\hbar^2\left[0 - \frac{1}{2}\left(\frac{1}{2}+1\right) - \frac{1}{2}\left(\frac{1}{2}+1\right)\right]|0,0\rangle$$

$\therefore$

$$\hat{H}|0,0\rangle = -3\hbar\lambda|0,0\rangle$$

Note that $|+-\rangle = \frac{1}{\sqrt{2}}\left(|10\rangle + |0,0\rangle\right)$

The state at time $t>0$ is given by the equation

$$|\psi(t)\rangle = \hat{U}(t,0)|\psi(0)\rangle$$

$$= e^{-\frac{i}{\hbar}\hat{H}t}|+-\rangle = \frac{1}{\sqrt{2}}e^{-\frac{i}{\hbar}\hat{H}t}\left(|10\rangle + |00\rangle\right)$$

$\therefore$

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}}\left[e^{-i\lambda t}|10\rangle + e^{3i\lambda t}|00\rangle\right]$$

Then:

$$\langle ++|\psi(t)\rangle = \frac{1}{\sqrt{2}}\left[e^{-i\lambda t}\underbrace{\langle ++|10\rangle}_{=0} + e^{3i\lambda t}\underbrace{\langle ++|00\rangle}_{=0}\right]$$

$\therefore$

$$\langle ++|\psi(t)\rangle = 0$$

$$\langle +-|\psi(t)\rangle = \frac{1}{\sqrt{2}}\left[e^{-i\lambda t}\underbrace{\langle +-|10\rangle}_{=\frac{1}{\sqrt{2}}} + e^{3i\lambda t}\underbrace{\langle +-|00\rangle}_{=\frac{1}{\sqrt{2}}}\right]$$

$\therefore$

$$\langle +-|\psi(t)\rangle = \frac{1}{2}\left(e^{-i\lambda t} + e^{3i\lambda t}\right)$$

$$= \frac{1}{2}e^{i\lambda t}\left(e^{-2i\lambda t} + e^{2i\lambda t}\right)$$

$$\langle +-|\psi(t)\rangle = e^{i\lambda t}\cos 2\lambda t$$

$$\langle -+|\psi(t)\rangle = \frac{1}{\sqrt{2}}\left[e^{-i\lambda t}\underbrace{\langle -+|10\rangle}_{=\frac{1}{\sqrt{2}}} + e^{3i\lambda t}\underbrace{\langle -+|00\rangle}_{=-\frac{1}{\sqrt{2}}}\right]$$

$\therefore$

$$\langle -+|\psi(t)\rangle = \frac{1}{2}\left(e^{-i\lambda t} - e^{3i\lambda t}\right) = -\frac{e^{i\lambda t}}{2}\left(-e^{-2i\lambda t} + e^{2i\lambda t}\right)$$

$\therefore$

$$\langle -+|\psi(t)\rangle = -i\,e^{i\lambda t}\sin 2\lambda t$$

$$\langle --|\psi(t)\rangle = \frac{1}{\sqrt{2}}\left[e^{-i\lambda t}\underbrace{\langle --|10\rangle}_{=0} + e^{3i\lambda t}\underbrace{\langle --|00\rangle}_{=0}\right]$$

$\therefore$

$$\langle --|\psi(t)\rangle = 0$$

Thus:

Probability to find system in $|++\rangle = 0$

" " " $|--\rangle = 0$

" " " " " $|+-\rangle = \cos^2 2\lambda t$

" " " " " $|-+\rangle = \sin^2 2\lambda t$

b) The initial state, $|+-\rangle$, is unrealizable for a system of identical spin ½ particles, since it is not antisymmetric under the operation of exchanging the two particles.

4. Consider an experimental situation in which the output signal consists of a weak sinusoidal voltage (the response of the system being studied to a sinusoidal driving effect) in addition to a strong noise source having a wide frequency spectrum. Describe in some detail how to instrument this system in order to measure the properties of this weak sinusoidal signal.

   Hint: One approach to this problem would involve the use of a lock-in amplifier. Describe how this device works, and in detail how it would be used to solve this experimental problem.

Hugo Schmidt

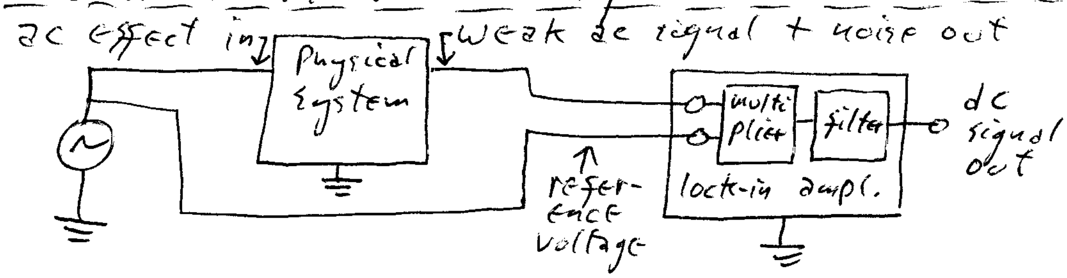
#4 Experimental / Electronics

Consider an experimental situation in which the output signal consists of a weak sinusoidal voltage (the response of the system being studied to a sinusoidal driving effect) in addition to a strong noise source having a wide frequency spectrum. Describe in some detail how to instrument this system in order to measure the properties of this weak sinusoidal signal.

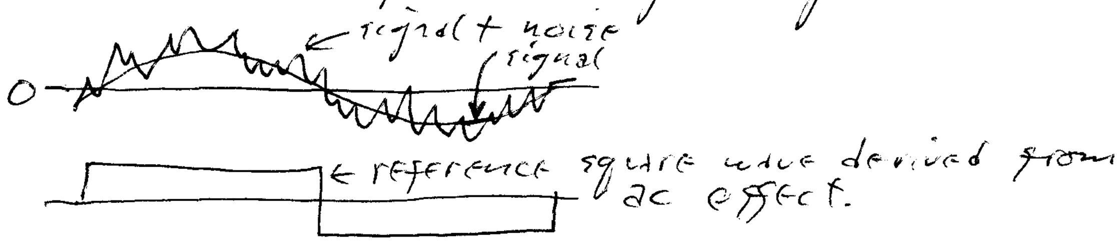
Hint: One approach to this problem would involve the use of a lock-in amplifier. Describe how this device works, and in detail how it would be used to solve this experimental problem.

Experimental / Electronics Solution

The lock-in amplifier is in effect an extremely narrow-band ac amplifier. The basic circuit blocks and operation are illustrated and explained below:

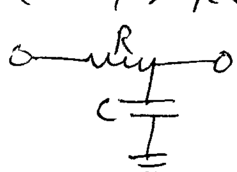


The "ac effect in", whether optical, electrical, or magnetic, etc., can be converted to a reference voltage at the driving frequency. In the lock-in amplifier, the "weak ac signal + noise out" from the physical system is compared with the reference signal by a multiplier:



The product has a dc component: (due to the signal)

The noise at other frequencies gives no dc component. Only noise within the bandpass $(= 1/RC)$ of the low-pass filter will appear at the dc output terminal along with the dc derived from the signal. If the reference phase θ is changed, the output voltage changes as $\cos\theta$, hence the name "phase sensitive detector".



5.

a) Obtain the Green's function, $G(z,z')$ defined by the equation

$$\left(\frac{d^2}{dz^2}-\alpha^2\right) \quad G(zz') \quad = \quad \delta(z-z') \quad ,$$

6 point.

with the boundary conditions that

$$G(z=0,z') \quad = \quad G(z=d,z') \quad = 0 \quad .$$

The parameter α is an arbitrary real number.

b) Using the result of part a) solve the differential equation

$$\frac{d^2}{dz^2}u(z) \quad - \quad \alpha^2 u(z) \quad = \quad z \quad ,$$

4 points.

with the boundary conditions that

$$u(0)=0 \quad ,$$

and

$$u(d)=1 \quad .$$

Mathematical Physics

a) Obtain the Green's function $G(z,z')$ defined by the equation

$$\left(\frac{d^2}{dz^2} - \alpha^2\right) G(z\,z') = \delta(z-z'),$$

with the boundary conditions that

$$G(z=0,z') = G(z=d,z') = 0.$$

The parameter α is an arbitrary real number.

b) Using the result of part a) solve the differential equation

$$\frac{d^2}{dz^2}u(z) - \alpha^2 u(z) = z,$$

with the boundary conditions that

$$u(0) = 0,$$

and

$$u(d) = 1.$$

Solution

a) $G(z\,z') = \begin{cases} A \sinh \alpha z & , \quad \text{for } z < z' \\ B \sinh \alpha (z-d) & , \quad \text{for } z > z' \end{cases}$

"Boundary" conditions at $z = z'$:

i) $G(z,z')$ must be continuous for $z = z'$. Otherwise we would have that $d^2 G(z,z')/dz^2$ would be proportional to $d\,\delta(z-z')/dz$, and the differential equation would not be satisfied for $z = z'$.

ii) Integrating across $z = z'$:

$$\frac{d}{dz} G(z,z') \Big|_{z=z'+0^+} - \frac{d}{dz} G(z,z') \Big|_{z=z'-0^+} = 1$$

Then :

i) : $A \sinh \alpha z' = B \sinh \alpha (z'-d)$

$$\Rightarrow \quad A = B \, \frac{\sinh \alpha (z'-d)}{\sinh \alpha z'}$$

ii) : $B \cosh \alpha (z'-d) - A \cosh \alpha z' = \dfrac{1}{\alpha}$

$$\therefore \quad B \left[\cosh \alpha (z'-d) - \frac{\sinh \alpha (z'-d)}{\sinh \alpha z'} \cosh \alpha z' \right] = \frac{1}{\alpha}$$

$$\sinh\alpha z' \cosh\alpha(z'-d) =$$

$$= \sinh\alpha z' \left(\cosh\alpha z' \cosh\alpha d - \sinh\alpha z' \sinh\alpha d \right)$$

$$\cosh\alpha z' \sinh\alpha(z'-d) =$$

$$= \cosh\alpha z' \left(\sinh\alpha z' \cosh\alpha d - \sinh\alpha d \cosh\alpha z' \right)$$

$$\therefore \; [\;] = \frac{1}{\sinh\alpha z'} \sinh\alpha d \left(\underbrace{-\sinh^2\alpha z' + \cosh^2\alpha z'}_{=1} \right)$$

$$= \frac{\sinh\alpha d}{\sinh\alpha z'}$$

Thus:

$$B = \frac{1}{\alpha} \frac{\sinh\alpha z'}{\sinh\alpha d}$$

and

$$A = \frac{1}{\alpha} \frac{\sinh\alpha(z'-d)}{\sinh\alpha d}$$

Then:

$$G(z,z') = \frac{1}{\alpha \sinh\alpha d} \times \begin{cases} \sinh\alpha z \, \sinh\alpha(z'-d), & \underline{\text{for } z' > z} \\ \sinh\alpha z' \, \sinh\alpha(z-d), & \underline{\text{for } z > z'} \end{cases}$$

b) Homogeneous equation : $\left(\frac{d^2}{dz^2} - \alpha^2\right) u_0(z) = 0$

$\therefore \quad u_0(z) = A e^{\alpha z} + B e^{-\alpha z}$

B.Cs. : $A + B = 0 \quad \rightarrow \quad B = -A$

$$A e^{\alpha d} + B e^{-\alpha d} = 1$$

$$\Rightarrow A\left(e^{\alpha d} - e^{-\alpha d}\right) = 1 \quad \Rightarrow \quad A = \frac{1}{2 \sinh \alpha d}$$

$\therefore$

$$u_0(z) = 2A \sinh \alpha z = \frac{\sinh \alpha z}{\sinh \alpha d}$$

General solution :

$$u(z) = \frac{\sinh \alpha z}{\sinh \alpha d} + \int_0^d dz' \, G(z, z') \, z'$$

Note that $u(z)$ fulfills the required boundary conditions.

Then :

$$u(z) = \frac{\sinh \alpha z}{\sinh \alpha d} +$$

$$+ \frac{1}{\alpha \sinh \alpha d} \sinh \alpha(z-d) \int_0^z dz' \sinh \alpha z' \; z'$$

$$+ \frac{1}{\alpha \sinh \alpha d} \sinh \alpha z \int_z^d dz' \sinh \alpha(z'-d) \; z'$$

$$\int dz' \sinh\alpha z' \; z' = \frac{d}{d\alpha}\int dz' \cosh\alpha z' =$$

$$= \frac{d}{d\alpha}\left(\frac{1}{\alpha}\sinh\alpha z'\right) = -\frac{1}{\alpha^2}\sinh\alpha z' + \frac{z'}{\alpha}\cosh\alpha z'$$

$\therefore$

$$\int_0^z dz' \sinh\alpha z' \; z' = -\frac{\sinh\alpha z}{\alpha^2} + \frac{z}{\alpha}\cosh\alpha z$$

Similarly,

$$\int dz' \sinh\alpha(z'-d) \; z' =$$

$$= -\frac{1}{\alpha^2}\sinh\alpha(z'-d) + \frac{(z'-d)}{\alpha}\cosh\alpha(z'-d)$$

$\therefore$

$$\int_z^d dz' \sinh\alpha(z'-d) \; z' =$$

$$= +\frac{1}{\alpha^2}\sinh\alpha(z-d) - \frac{(z-d)}{\alpha}\cosh\alpha(z-d)$$

Thus:

$$u(z) = \frac{\sinh\alpha z}{\sinh\alpha d} + \frac{1}{\alpha\sinh\alpha d} \times$$

$$\times\left[\sinh\alpha(z-d)\left(\cancel{-\frac{\sinh\alpha z}{\alpha^2}} + \frac{z}{\alpha}\cosh\alpha z\right)\right.$$

$$\left. + \sinh\alpha z\left(\cancel{\frac{\sinh\alpha(z-d)}{\alpha^2}} - \frac{(z-d)}{\alpha}\cosh\alpha(z-d)\right)\right]$$

$\therefore$

$$u(z) = \frac{\sinh \alpha z}{\sinh \alpha d} +$$

$$+ \frac{1}{\alpha^2 \sinh \alpha d} \Big[z \cosh \alpha z \sinh \alpha (z-d)$$

$$- (z-d) \sinh \alpha z \cosh \alpha (z-d) \Big]$$

6. A medium has zero conductivity, unit magnetic permeability, but an anisotropic dielectric tensor $\underline{\varepsilon}$ defined by $\vec{D} = \underline{\varepsilon} \cdot \vec{E}$. In a particular Cartesian (xyz) coordinate system the components of $\underline{\varepsilon}$ are given by:

$$\underline{\varepsilon} = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & \gamma \\ 0 & \gamma & \beta \end{pmatrix}$$

where α, β and γ are constants. Give the polarizations and phase velocities for the two possible normal modes for electromagnetic waves traveling in the x direction.

Lee Lindblom

Electromagnetic Problem

#6

A medium has zero conductivity, unit magnetic permeability, but an anisotropic dielectric tensor $\underset{\sim}{\epsilon}$ defined by $\vec{D} = \underset{\sim}{\epsilon} \cdot \vec{E}$. In a particular Cartesian (xyz) coordinate system the components of ϵ are given by:

$$\underset{\sim}{\epsilon} = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & \gamma \\ 0 & \gamma & \beta \end{pmatrix}$$

where α, β and γ are constants. Give the polarizations and phase velocities for the two possible normal modes for electromagnetic waves traveling in the x direction.

1

Solution:

Maxwell's equations for a dielectric medium are:

$$\vec{\nabla}\cdot\vec{D} = 0 \qquad (1)$$
$$\vec{\nabla}\cdot\vec{B} = 0 \qquad (2)$$
$$\frac{1}{c}\partial_t \vec{D} = \vec{\nabla}\times\vec{H} \qquad (3)$$
$$\frac{1}{c}\partial_t \vec{B} = -\vec{\nabla}\times\vec{E} \qquad (4)$$

We assume that $\vec{D} = \underset{\sim}{\epsilon}\cdot\vec{E}$ and $\vec{B} = \vec{H}$. Normal modes have time dependence $e^{i\omega t}$ while waves traveling in the x direction have spatial dependence e^{-ikx}, thus we assume that $\vec{E}$ and $\vec{B}$ have the form:

$$\vec{E} = \vec{E}_0 e^{i\omega t - ikx}$$
$$\vec{B} = \vec{B}_0 e^{i\omega t - ikx}$$

with $\vec{E}_0$ and $\vec{B}_0$ constants. $\vec{D} = \underset{\sim}{\epsilon}\cdot\vec{E} \equiv \vec{D}_0 e^{i\omega t - ikx}$.

We now use maxwell's equations to find the restrictions on the constants $\vec{E}_0$, $\vec{B}_0$, ω and k in these expressions:

$$(1) \Rightarrow \vec{\nabla}\cdot\vec{D} = -ik D_0^x = 0 \quad\Rightarrow\quad D_0^x = 0 \quad\Rightarrow\quad E_0^x = 0$$

$$(2) \Rightarrow \vec{\nabla}\cdot\vec{B} = -ik B_0^x = 0 \quad\Rightarrow\quad B_0^x = 0$$

$$(3) \Rightarrow \frac{1}{c}\partial_t\vec{D} = \frac{i\omega}{c}\vec{D} = \vec{\nabla}\times\vec{B} = (0, -\partial_x B^z, \partial_x B^y) = ik(0, B^z, -B^y)$$

$$(4) \Rightarrow \frac{1}{c}\partial_t\vec{B} = \frac{i\omega}{c}\vec{B} = -\vec{\nabla}\times\vec{E} = -(0, -\partial_x E^z, \partial_x E^y) = +ik(0, -E^z, E^y)$$

The first two equations ① and ② guarantee that the waves are transverse to the direction of propagation: $E_0^x = B_0^x = 0$.

The last two equations determine the polarization and phase velocity of the waves:

$$③ \Rightarrow \frac{i\omega}{c}\begin{pmatrix}0\\ D_0^y\\ D_0^z\end{pmatrix} = \frac{i\omega}{c}\begin{pmatrix}\alpha & 0 & 0\\ 0 & \beta & \gamma\\ 0 & \gamma & \beta\end{pmatrix}\begin{pmatrix}0\\ E_0^y\\ E_0^z\end{pmatrix} = \frac{i\omega}{c}\begin{pmatrix}0\\ \beta E_0^y + \gamma E_0^z\\ \gamma E_0^y + \beta E_0^z\end{pmatrix} = ik\begin{pmatrix}0\\ B^z\\ -B^y\end{pmatrix}$$

$$④ \Rightarrow \frac{i\omega}{c}\begin{pmatrix}0\\ B_0^y\\ B_0^z\end{pmatrix} = ik\begin{pmatrix}0\\ -E_0^z\\ E_0^y\end{pmatrix}$$

Therefore:
$$\begin{cases}\beta E_0^y + \gamma E_0^z = \frac{ck}{\omega}B^z = \left(\frac{ck}{\omega}\right)^2 E_0^y\\[2mm] \gamma E_0^y + \beta E_0^z = -\frac{ck}{\omega}B^y = \left(\frac{ck}{\omega}\right)^2 E_0^z\end{cases}$$

or:
$$\begin{pmatrix}\beta - \left(\frac{ck}{\omega}\right)^2 & \gamma\\ \gamma & \beta - \left(\frac{ck}{\omega}\right)^2\end{pmatrix}\begin{pmatrix}E_0^y\\ E_0^z\end{pmatrix} = 0$$

The determinent of this matrix must vanish:

$$\left[\beta - \left(\frac{ck}{\omega}\right)^2\right]^2 - \gamma^2 = 0$$

$$\Rightarrow \left(\frac{ck}{\omega}\right)^2 - \beta = \pm\gamma$$

Thus the phase velocities of the two modes are:

$$\frac{\omega}{k} = c\left[\beta \pm \gamma\right]^{-1/2}$$

③

The polerizations are determined by the constants $\vec{E}_0$ and $\vec{B}_0$:

$$0 = \begin{pmatrix} \beta - \left(\frac{ck}{\omega}\right)^2 & \gamma \\ \gamma & \beta - \left(\frac{ck}{\omega}\right)^2 \end{pmatrix} \begin{pmatrix} E_0^y \\ E_0^z \end{pmatrix}$$

$$= \begin{pmatrix} \beta - \beta \mp \gamma & \gamma \\ \gamma & \beta - \beta \mp \gamma \end{pmatrix} \begin{pmatrix} E_0^{y\pm} \\ E_0^{z\pm} \end{pmatrix}$$

$$= \gamma \begin{pmatrix} \mp 1 & 1 \\ 1 & \mp 1 \end{pmatrix} \begin{pmatrix} E_0^{y\pm} \\ E_0^{z\pm} \end{pmatrix}$$

Thus $\vec{E}_0^+ = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ $\quad \vec{E}_0^- = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$

$$\vec{B}_0^+ = (\beta+\gamma)^{1/2} \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \qquad \vec{B}_0^- = (\beta-\gamma)^{1/2} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

are the plane wave modes.

7. A spinless particle moves in one dimension in the presence of a harmonic potential, $m\omega^2\hat{x}^2/2$, where $\hat{x}$ is the position operator for the particle. For t<0 the particle is in the ground state $|0>$ of this potential. For t>0 the particle is subjected to a perturbing potential of the form

$$\hat{V}(t) = V_0 \; e^{-k\hat{x}} \; e^{-\nu t} \quad ,$$

where V_0, k, and ν are real constants with appropriate dimensions ($k,\nu > 0$).

Assuming that V_0 is sufficiently small, calculate in first order perturbation theory the probability for the transition $|0> \to |n>$ to occur for $t \to \infty$. Here $|n>$ is an arbitrary excited state of the oscillator. You must evaluate any matrix elements you encounter explicitly.

You may find the following formulae useful:

$$\hat{x} = \left(\frac{\hbar}{2m\omega}\right)^{1/2} (a^+ + a)$$

$$[a, a^+] = \hat{1}$$

$$a^+ |n> = \sqrt{n+1} \; |n+1>$$

$$a\,|n> = \sqrt{n} \; |n-1>$$

$$\frac{(a^+)^n}{\sqrt{n!}} |0> = |n>$$

For two operators $\hat{A}$ and $\hat{B}$ which commute with their commutator:

$$e^{\hat{A}} e^{\hat{B}} = e^{\hat{A}+\hat{B}} \; e^{1/2[\hat{A},\hat{B}]}$$

Relation between operators in the Schrödinger and interaction pictures:

$$\hat{A}_I(t) = e^{\frac{i}{\hbar}\hat{H}_o t} \; \hat{A}_S \; e^{-\frac{i}{\hbar}\hat{H}_o t}$$

A. Eguiluz

Quantum Mechanics
#7

A spinless particle moves in one dimension in the presence of a harmonic potential, $\frac{1}{2} m\omega^2 \hat{x}^2$, where $\hat{x}$ is the position operator for the particle. For $t<0$ the particle is in the ground state $|0\rangle$ of this potential. For $t>0$ the particle is subjected to a perturbing potential of the form

$$\hat{V}(t) = V_0 \, e^{-k\hat{x}} \, e^{-\nu t} ,$$

where V_0, k, and ν are real constants with appropriate dimensions ($k, \nu > 0$).

Assuming that V_0 is sufficiently small, calculate in first order perturbation theory the probability for the transition $|0\rangle \to |n\rangle$ to occur for $t \to \infty$. Here $|n\rangle$ is an arbitrary excited state of the oscillator. You must evaluate any matrix elements you encounter explicitly. You may find the following formulae useful:

$$\hat{x} = \left(\frac{\hbar}{2m\omega}\right)^{1/2} (a^+ + a) \qquad , \qquad [a, a^+] = \hat{1}$$

$$a^+|n\rangle = \sqrt{n+1}\;|n+1\rangle$$
$$a|n\rangle = \sqrt{n}\;|n-1\rangle$$

$$\frac{(a^+)^n}{\sqrt{n!}}|0\rangle = |n\rangle$$

For two operators $\hat{A}$ and $\hat{B}$ which commute with their commutator:

$$e^{\hat{A}}\, e^{\hat{B}} = e^{\hat{A}+\hat{B}}\; e^{\frac{1}{2}[\hat{A},\hat{B}]}$$

Relation between operators in the Schrödinger and interaction pictures:

$$\hat{A}_I(t) = e^{\frac{i}{\hbar}\hat{H}_0 t}\,\hat{A}_S\,e^{-\frac{i}{\hbar}\hat{H}_0 t}$$

Solution

$$\hat{V}(t) = V_0 \, e^{-k\hat{x}} \, e^{-\nu t}$$

To $O(V)$:

$$|\psi_I(t)\rangle = |0\rangle + \frac{1}{i\hbar}\int_0^t dt' \, \hat{V}_I(t') \, |0\rangle ,$$

where

$$\hat{V}_I(t) = e^{\frac{i}{\hbar}\hat{H}_0 t} \, \hat{V}(t) \, e^{-\frac{i}{\hbar}\hat{H}_0 t}$$

$$\hat{H}_0 |n\rangle = E_n |n\rangle \quad , \quad E_n = \hbar\omega(n+\tfrac{1}{2})$$
$$n = 0, 1, 2, \ldots$$

The amplitude for the transition $|0\rangle \to |n\rangle$ is given by:

$$\langle n|\psi(t)\rangle = \langle n| e^{-\frac{i}{\hbar}\hat{H}_0 t} |\psi_I(t)\rangle =$$

$$= e^{-\frac{i}{\hbar}E_n t} \langle n|\psi_I(t)\rangle$$

$\therefore$

$$|\langle n|\psi(t)\rangle| = |\langle n|\psi_I(t)\rangle|$$

Then, for $n \neq 0$

$$\langle n|\psi_I(t)\rangle = \frac{1}{i\hbar}\int_0^t dt' \, \langle n| \hat{V}_I(t') |0\rangle =$$

$$= \frac{1}{i\hbar}\int_0^t dt' \, e^{\frac{i}{\hbar}(E_n - E_0)t'} \langle n| \hat{V}(t') |0\rangle$$

$\therefore$ $\qquad\qquad\qquad\qquad E_{no} \equiv E_n - E_o$

$$\langle n | \psi_I(t) \rangle =$$

$$= \frac{V_o}{i\hbar} \int_0^t dt' \, e^{\frac{i}{\hbar}(E_n - E_o)t'} \, e^{-\nu t'} \, \langle n | e^{-k\hat{x}} | 0 \rangle$$

$$= \frac{1}{i\hbar} \frac{V_o}{\left(\frac{i}{\hbar} E_{no} - \nu\right)} \, e^{\left(\frac{i}{\hbar} E_{no} - \nu\right)t'} \Big|_0^t \, \langle n | e^{-k\hat{x}} | 0 \rangle$$

$$= - V_o \frac{1}{E_{no} + i\hbar\nu} \left(e^{\left(\frac{i}{\hbar} E_{no} - \nu\right)t} - 1 \right) \langle n | e^{-k\hat{x}} | 0 \rangle$$

For $t \to \infty$:

$$\langle n | \psi_I(t) \rangle \longrightarrow + \frac{V_o}{E_{no} + i\hbar\nu} \langle n | e^{-k\hat{x}} | 0 \rangle$$

Thus, for $t \to \infty$ the probability for the transition $|0\rangle \to |n\rangle$ to occur is given by

$$\mathcal{P}_{0\to n}(t) = |V_o|^2 \frac{|\langle n | e^{-k\hat{x}} | 0 \rangle|^2}{E_{no}^2 + (\hbar\nu)^2}$$

Evaluation of the matrix element $\langle n | e^{-k\hat{x}} | 0 \rangle$

$$\hat{x} = \left(\frac{\hbar}{2m\omega}\right)^{1/2} (a^+ + a) = \beta (a^+ + a)$$

Since $[a, a^+] = \hat{1}$, both a and a^+ commute with their commutator. Then:

$$e^{-k\hat{x}} = e^{-k\beta(a^+ + a)} = \qquad \begin{aligned} A &= -k\beta a^+ \\ B &= -k\beta a \end{aligned}$$

$$= e^{-k\beta a^+} \, e^{-k\beta a} \, e^{-\frac{1}{2}[-k\beta a^+, -k\beta a]}$$

$$= e^{\frac{1}{2}k^2\beta^2} \, e^{-k\beta a^+} \, e^{-k\beta a}$$

$\therefore$

$$\langle n | e^{-k\hat{x}} | 0 \rangle = e^{\frac{1}{2}k^2\beta^2} \langle n | e^{-k\beta a^+} e^{-k\beta a} | 0 \rangle$$

$$= e^{\frac{1}{2}k^2\beta^2} \langle n | e^{-k\beta a^+} | 0 \rangle$$

$$\langle n | e^{-k\beta a^+} | 0 \rangle =$$

$$= \sum_{m=0}^{\infty} \frac{(-k\beta)^m}{m!} \langle n | (a^+)^m | 0 \rangle$$

$$= \sum_{m=0}^{\infty} \frac{(-k\beta)^m}{m!} \sqrt{m!} \, \underbrace{\langle n | m \rangle}_{= \delta_{n,m}}$$

$$= \frac{(-k\beta)^n}{\sqrt{n!}}$$

Thus

$$\langle n | e^{-k\beta a^{+}} | 0 \rangle = \frac{(-k\beta)^n}{\sqrt{n!}}$$

and

$$\langle n | e^{-k\hat{x}} | 0 \rangle = \frac{(-k\beta)^n}{\sqrt{n!}} \, e^{\frac{1}{2}k^2\beta^2}$$

Hence:

$$\mathcal{P}_{0\to n}(t) = \frac{(k\beta)^{2n}}{n!} \, e^{k^2\beta^2} \, \frac{V_0^2}{E_{n0}^2 + \hbar^2\nu^2}$$

8. The imaginary monovalent metal "trillium" has a simple cubic structure with atomic spacing = a_0. Using the free electron model:

a) Find an expression for the density of states in reciprocal space for a sample in the shape of a cube of side L.

b) Derive an expression for the Fermi level at absolute zero temperature.

c) Whether or not you can do parts a) or b), discuss qualitatively the room temperature solution to the apparent paradox that at ordinary temperatures the conduction electrons contribute much less to the heat capacity than the value (3/2)k expected from classical physics.

Hugo Schn...

#8

Solid State Physics Problem

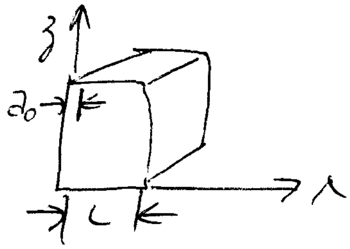
The imaginary monovalent metal "trillium" has a simple cubic structure with atomic spacing $= a_0$. Using the free electron model:

a) Find an expression for the density of states in reciprocal space for a sample in the shape of a cube of side L.

b) ~~Find~~ Derive an expression for the Fermi level in eV and degrees K, at absolute zero.

c) Whether or not you can do parts a) or b), discuss qualitatively the ~~paradox facing physicists at the turn of the century regarding~~ room temperature ~~specific heats of metals.~~ solution to the apparent paradox that at ~~room temperature~~ ordinary temperatures the conduction electrons contribute much less to the heat capacity than the value $\frac{3}{2}k$ expected from classical physics.

Solid State Physics Solution



a) Electron wave function is sinusoidal & vanishes at boundaries, so can have

$$\psi = \psi_0 \sin\left(n_x \tfrac{\pi}{L} x\right) \sin\left(n_y \tfrac{\pi}{L} y\right) \sin\left(n_z \tfrac{\pi}{L} z\right)^{*}, \quad n_i = 1, 2, \ldots$$

the k-space unit cell interval thus is $\frac{\pi}{L}$, and the density of states (with 2 electron spin states) is $\boxed{\frac{2}{(\pi/L)^3} = \frac{16L^3}{\pi^3}}$ (x8 for per. bc.'s over k-space sphere)

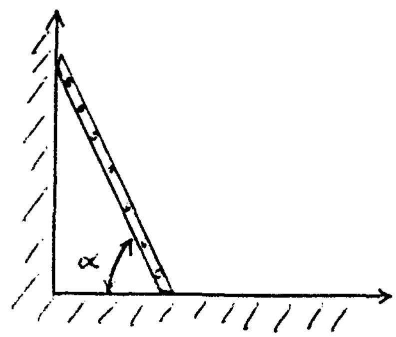
b) There are $(L/a_0)^3 = \left(\frac{10^{-3}\,m}{3\times10^{-10}\,m}\right)^3 = \frac{10^{21}}{27} = N$ electrons in the cube. At $T=0$ only the lowest-$|\vec{k}|$ states are filled (up to the Fermi level).

$$N = \underbrace{\tfrac{4}{3}\pi k_F^3}_{\text{Fermi sphere reciprocal volume}} \frac{16L^3}{\pi^3} = \frac{L^3}{a_0^3} \quad \text{so} \quad k_F^3 = \tfrac{3}{\cancel{16}}\pi^2/a_0^3$$

$$\text{Fermi level} = \frac{\hbar^2 k_F^2}{2m} \left(= \frac{p_F^2}{2m}\right) = \boxed{\frac{\hbar^2}{2m}\left(\tfrac{3}{\cancel{16}}\pi^2/a_0^3\right)^{2/3}}$$

c) According to classical physics (in vogue in 1900), every kinetic energy degree of freedom should have average energy $\frac{1}{2}kT$, so each free electron should contribute $\frac{3}{2}k$ to the heat capacity. But because the Fermi level is much greater than kT at $T=300\,K$, these electrons actually contribute much less than $\frac{3}{2}k$ each. This is in accord with experiments which did not agree with the classical theory's predictions.

9. A ladder rests against a smooth wall and slides without friction on wall and floor. Set up the equation of motion, assuming that the ladder maintains contact with the wall. If initially the ladder is at rest at an angle α with the floor, at what angle, if any, will it leave the wall?

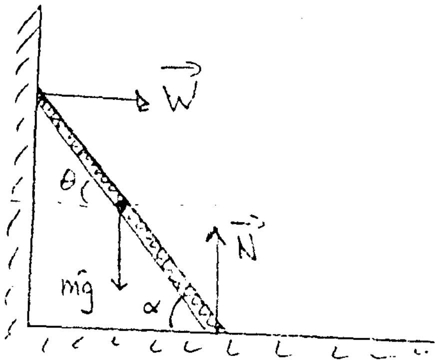


#9

12. A ladder rests against a smooth wall and slides without friction on wall and floor. Set up the equation of motion, assuming that the ladder maintains contact with the wall. If initially the ladder is at rest at an angle α with the floor, at what angle, if any, will it leave the wall?

Smith Mechanics #1

Symon 1-12



Ladder rests against wall, no friction on wall or floor

Length of ladder L
mass m
starting angle, α

a) Set up eqn of motion assuming ladder maintains contact with wall

Consider origin at Center of Mass
Use Θ to orient ladder, and locate CM

$$T = \tfrac{1}{2} m \left(\dot{x}_{cm}^2 + \dot{y}_{cm}^2 \right) + \tfrac{1}{2} I_{cm} \dot{\Theta}^2$$

Hint: Treat ladder as solid bar [illegible]

$$x_{cm} = L/2 \cos\Theta \qquad \dot{x}_{cm} = -L/2 \sin\Theta\, \dot{\Theta}$$
$$y_{cm} = L/2 \sin\Theta \qquad \dot{y}_{cm} = L/2 \cos\Theta\, \dot{\Theta}$$
$$I_{cm} = 1/12\, ML^2$$

So
$$T = \tfrac{1}{2} M \tfrac{L^2}{4} \dot{\Theta}^2 (\sin^2\Theta + \cos^2\Theta) + \tfrac{1}{24} ML^2 \dot{\Theta}^2$$
$$= \tfrac{1}{6} ML^2 \dot{\Theta}^2$$

$$V = Mg\, L/2 \sin\Theta \qquad \text{(cm above floor)}$$

$$L = T - V = \tfrac{1}{6} ML^2 \dot{\Theta}^2 - Mg\, L/2 \sin\Theta$$

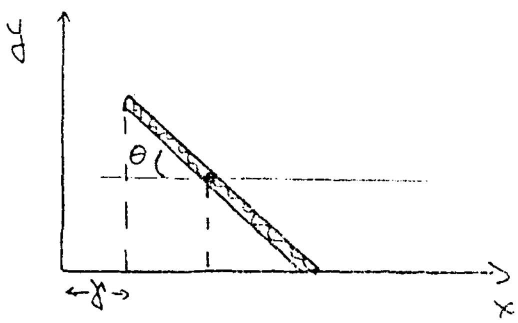
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \Theta} = 0 \text{ gives } \tfrac{1}{3} ML^2 \ddot{\Theta} + Mg\, L/2 \cos\Theta = 0$$

5

or $\boxed{\ddot{\Theta} + 3/2\, g/L \cos\Theta = 0}$ Eqn of Motion

b) At what angle, if any, does ladder leave the wall?

Think of $\vec{W}$ and $\vec{N}$ as constraining forces
Find $\vec{W}$, then if $\bar{W} \to 0$ ladder leaves the wall.



Need another coordinate to change when ladder leaves wall

write

$$x_{cm} = \gamma + L/2 \cos\theta$$

Then $\gamma = 0 \Rightarrow$ Ladder on wall
$\gamma \neq 0 \Rightarrow$ " leaves wall

Now $$T = 1/2 m \left([\dot{\gamma} - L/2 \sin\theta\, \dot{\theta}]^2 + (L/2 \cos\theta\, \dot{\theta})^2 \right) + \frac{1}{24} ML^2 \dot{\theta}$$

$$T = 1/2 m \left(\dot{\gamma}^2 - \dot{\gamma} L \sin\theta\, \dot{\theta} + L^2/4\, \dot{\theta}^2 + \frac{1}{12} L^2 \dot{\theta}^2 \right)$$

$$T = 1/2 m \left(\dot{\gamma}^2 - \dot{\gamma}\dot{\theta} L \sin\theta + 1/3\, L^2 \dot{\theta}^2 \right)$$

Use $$d/dt\left(\frac{\partial T}{\partial \dot{\gamma}}\right) - \frac{\partial T}{\partial \gamma} = Q_\gamma$$

Find Q_γ: $\delta W = Q_\gamma \delta\gamma$ (holding θ const $\Rightarrow \delta\gamma = \delta x$
$= (\bar{W}) \cdot \delta\bar{x} = W \delta x$

$$\therefore \quad Q_\gamma = W$$

So

$$d/dt \left(m\dot{\gamma} - 1/2 M \dot{\theta} L \sin\theta \right) = W$$

$$m\ddot{\gamma} - 1/2 M \ddot{\theta} L \sin\theta - 1/2 M \dot{\theta} L \cos\theta\, \dot{\theta} = W$$

Now put in the constraint $\gamma = 0 = \text{const}$, $\dot{\gamma} = \ddot{\gamma} = 0$

so

$$\boxed{W = \tfrac{1}{2} ML \sin\theta\, \ddot{\theta} - \tfrac{1}{2} ML\, \dot{\theta}^2 \cos\theta}$$

Now set $W = 0$ to get θ at which ladder leaves wall.

Eliminate $\ddot{\theta}$ using part (a)

Eliminate $\dot{\theta}$ using Energy Conservation

$L \neq fn(t)$ so

$$\dot{\theta}\frac{\partial L}{\partial \dot{\theta}} - L = E = T + V = \text{constant} = mgL/2 \sin\alpha \text{ at } t = 0$$

so

$$E = \tfrac{1}{3} ML^2 \dot{\theta}^2 - \tfrac{1}{6} ML^2 \dot{\theta}^2 + MgL/2 \sin\theta$$

$$= \tfrac{1}{6} ML^2 \dot{\theta}^2 + MgL/2 \sin\theta$$

$$\dot{\theta}^2 = [\underbrace{E}_{mgL/2 \sin\alpha} - MgL/2 \sin\theta] \frac{6}{ML^2}$$

$$\dot{\theta}^2 = [\sin\alpha - \sin\theta]\, 3g/L$$

Set $W = 0$ gives $\sin\theta \left(\frac{3}{2} g/L \cos\theta\right) = \cos\theta\, 3g/L [\sin\alpha - \sin\theta]$

or $\sin\theta = 2\sin\alpha - 2\sin\theta$

5 $\boxed{\sin\theta = 2/3 \sin\alpha}$ $\Leftarrow$ gives angle θ at which $W = 0$ or when ladder leaves wall

10.

a) Write down the formal expression for the retarded vector potential due to a source with a current density $\vec{j}(\vec{x};t)$.

b) Derive an approximation for the vector potential given above which is valid in the radiation zone.

c) Obtain the angular distribution of the time-averaged intensity of the electromagnetic radiation emitted by an electric dipole $\vec{p}$ which rotates with angular frequency ω about an axis perpendicular to $\vec{p}$.

Note: For a plane wave the Poynting vector $\vec{S}$ is given by the equation

$$\vec{S} = c\,\frac{B^2}{4\pi}\,\vec{n}\ ,$$

where $\vec{n}$ is the unit vector along the propagation direction.

Eguiluz

Electromagnetism

#10

i) Write down the formal expression for the retarded vector potential due to a source with a current density $\vec{j}(\vec{x};t)$.

ii) ~~Make the dipole~~ approximation for the vector potential ~~in the radiation zone~~.

iii) Obtain the angular distribution of the time-averaged intensity of the electromagnetic radiation emitted by an electric dipole $\vec{p}$ which rotates with angular frequency ω about an axis perpendicular to $\vec{p}$.

Note: For a plane wave the Poynting vector $\vec{S}$ is given by the equation

$$\vec{S} = c\,\frac{E^2}{4\pi}\,\vec{n},$$

where $\vec{n}$ is the unit vector along the propagation direction.

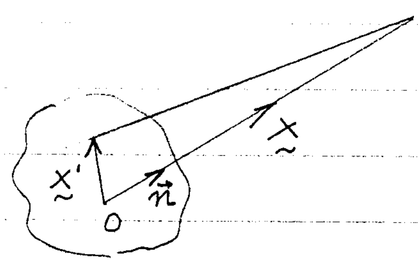
Solution

The retarded vector potential is given, in general, by the equation

$$\vec{A}(\vec{x}t) = \frac{1}{c}\int d^3x' \frac{1}{|\vec{x}-\vec{x}'|} \vec{j}(\vec{x}'t') \ ,$$

where

$$t' = t - \frac{|\vec{x}-\vec{x}'|}{c}$$



Now, for $|\underset{\sim}{x}| \gg |\underset{\sim}{x}'|$:

$$|\underset{\sim}{x}-\underset{\sim}{x}'| = |\underset{\sim}{x}| - \vec{n}.\vec{x}'$$

To first order in $|\vec{x}|^{-1}$:

$$\vec{A}(\vec{x}t) = \frac{1}{cR}\int d^3x' \ \vec{J}(\vec{x}'t') \ ,$$

where now

$$t' = t - \frac{R}{c} + \frac{\vec{n}.\vec{x}'}{c} \ ,$$

with

$$R \equiv |\underset{\sim}{x}| .$$

Dipolar approximation : Set $t' \cong t - \frac{R}{c}$.

This is justified for $\lambda \gg a$, where a is the order of magnitude of the size of the radiating system.

Then:

$$\vec{A}(\vec{x}t) \cong \frac{1}{cR}\int d^3x' \; \vec{J}(\vec{x}'/t - \frac{R}{c})$$

Note that the time argument of the current density is, in the present approximation, independent of the integration variable.

$\therefore$

$$\vec{A}(\vec{x}t) = \frac{1}{cR}\int d^3x' \; \rho(\vec{x}'/t') \; \vec{v}(\vec{x}'/t') =$$

$$= \frac{1}{cR}\sum_i e_i \vec{v}_i = \frac{1}{cR}\frac{d}{dt'}\vec{p}(t')$$

where

$$t' = t - \frac{R}{c} .$$

For the radiation field $\vec{E} = \vec{B}\times\vec{n}$. Thus the vector potential by itself determines the full electromagnetic field.

In the present case:

$$\vec{B}(\vec{x}t) = \frac{1}{c}\frac{\partial}{\partial t}\vec{A}(\vec{x}t) \times \vec{n}$$

$$= \frac{1}{c^2R}\left(\frac{d^2}{dt'^2}\vec{p}(t')\right)\times\vec{n}$$

For a plane wave the Poynting vector $\vec{S}$ is given by the equation

$$\vec{S} = c \frac{B^2}{4\pi} \vec{n}$$

The intensity dI of radiation traversing an element of solid angle $d\Omega$ is given by

$$dI = c \frac{B^2}{4\pi} R^2 d\Omega$$

$\therefore$

$$dI = \frac{1}{4\pi c^3} \left| \frac{d^2}{dt'^2} \vec{P}(t') \times \vec{n} \right|^2 d\Omega$$

Now:

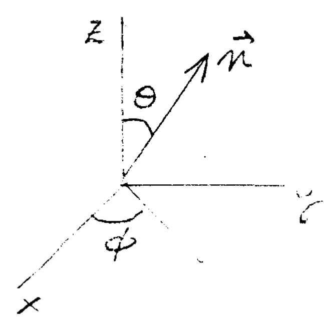
$$\vec{p}(t') = (P \cos\omega t' , P \sin\omega t' , 0)$$

$\therefore$

$$\frac{d^2}{dt'^2} \vec{p}(t') = -\omega^2 (p \cos\omega t' , p \sin\omega t' , 0)$$

$$\vec{n} = (\sin\theta \cos\phi , \sin\theta , \sin\phi , \cos\theta)$$

$$(\vec{p} \times \vec{n})_\alpha = \epsilon_{\alpha\beta\gamma} \, p_\alpha \, n_\gamma$$



$\therefore$

$$(\ddot{\vec{p}} \times \vec{n})_x = \ddot{p}_y n_z = -p\omega^2 \sin\omega t' \cos\theta$$

$$(\ddot{\vec{p}} \times \vec{n})_y = -\ddot{p}_x n_z = -p\omega^2 \cos\omega t' \cos\theta$$

$$\begin{aligned}(\ddot{\vec{p}} \times \vec{n})_z &= \ddot{p}_x n_y - \ddot{p}_y n_x = \\ &= -p\omega^2 \cos\omega t' \sin\theta \sin\phi \\ &\quad - p\omega^2 \sin\omega t' \sin\theta \cos\phi \\ &= -p\omega^2 \sin\theta \sin(\omega t' + \phi)\end{aligned}$$

Then:

$$\left| \frac{d^2\vec{p}(t')}{dt'^2} \times \vec{n} \right|^2 = \begin{aligned}&p^2\omega^4 \sin^2\omega t' \cos^2\theta \\ &+ p^2\omega^4 \cos^2\omega t' \cos^2\theta \\ &+ p^2\omega^4 \sin^2(\omega t' + \phi) \sin^2\theta\end{aligned}$$

We average over one period of the oscillation:

$$\begin{aligned}\overline{|\quad|^2} &= \frac{1}{2} p^2\omega^4 \cos^2\theta + \frac{1}{2} p^2\omega^4 \cos^2\theta \\ &\quad + \frac{1}{2} p^2\omega^4 \sin^2\theta \\ &= \frac{1}{2} p^2\omega^4 (2\cos^2\theta + \sin^2\theta) \\ &= \frac{1}{2} p^2\omega^4 (1 + \cos^2\theta)\end{aligned}$$

Thus:

$$d\bar{I} = \frac{p^2\omega^4}{8\pi c^3} (1 + \cos^2\theta)\, d\Omega$$

11. A quantum-mechanical particle of mass m moves in two dimensions. It is confined by infinitely high walls to the square region

$$|x| \leq \frac{L}{2}, \quad |y| \leq \frac{L}{2} \quad .$$

a) What is the energy and degeneracy of the first excited states? Write down the correctly normalized wavefunctions for these states.

b) The particle is now subjected to a small additional potential

$$V(x,y) \quad = \quad \varepsilon xy$$

Calculate the splitting in the first excited states produced by this perturbation.

Khinersky

#11

QM Problem

A quantum-mechanical particle of mass m moves in two dimensions. It is confined by infinitely high walls to the square region

$|x| \leq L/2$, $|y| \leq L/2$.

a) What is the energy ~~[illegible]~~ and degeneracy of the first excited states? Write down the correctly normalized wavefunctions for these states.

b) The particle is now subjected to a small additional potential

$V(x,y) = \varepsilon xy$

Calculate the splitting in the first excited states produced by this perturbation.

Solution

a) The levels for a 2-D square well are $|n\,m\rangle$, $n,m = 1,2,3,\ldots$

$$E = \frac{\hbar^2\pi^2}{2mL^2}(n^2+m^2).$$

The first excited states are:

$$|12\rangle = \frac{2}{L}\cos\left(\frac{\pi x}{L}\right)\sin\left(\frac{2\pi y}{L}\right)$$

$$|21\rangle = \frac{2}{L}\sin\left(\frac{2\pi x}{L}\right)\cos\left(\frac{\pi y}{L}\right)$$

$$E_1 = \frac{5\hbar^2\pi^2}{2mL^2}$$

b) In first order perturbation theory we look at the matrix elements of V. Since $V(x,y)$ odd in x and odd in y,

$$V = \begin{pmatrix} 0 & X \\ X & 0 \end{pmatrix} \quad \text{where } X = \langle 12|V|21\rangle$$

Choose a basis which diagonalizes V. The eigenvalues of V are $\pm X$ so the perturbed energy levels will be $E_1 \pm X$ and the splitting will be $\boxed{2X}$

$$X = \varepsilon\left(\frac{4}{L^2}\right)\left(\int_{-L/2}^{L/2}\cos\left(\frac{\pi x}{L}\right) x \sin\left(\frac{2\pi x}{L}\right)dx\right)^2$$

identity: $\cos A \sin B = \frac{1}{2}\left(\sin(A+B) - \sin(A-B)\right)$

$$\int_{-L/2}^{L/2} x \sin\left(\frac{3\pi x}{L}\right) dx = \frac{2L^2}{9\pi^2}$$

$$\int_{-L/2}^{L/2} x \sin\left(\frac{\pi x}{L}\right) dx = \frac{2L^2}{\pi^2}$$

$$\Rightarrow \overline{X} = \frac{4}{L^2}\left[\frac{1}{2}\left(\frac{2L^2}{9\pi^2} - \frac{2L^2}{\pi^2}\right)\right]^2 = \frac{256\, L^2}{81\pi^4}$$

12. A $_4Be^7$ nucleus at rest undergoes K-capture.

a) Write the reaction.

b) What velocity (in cm/sec) will the resulting $_3Li^7$ nucleus have?

c) What energy (in eV) will the $_3Li^7$ nucleus have?

<u>Nuclear Data</u>

$_4Be^7$ has $J^P = 3/2-$,
mass excess = 16.380 milli mass units
$_3Li^7$ has $J^P = 3/2-$
mass excess = 16.005 milli mass units

1 mass unit = 931.5 MeV
$m_e = .511$ MeV

You are encouraged to make any simplifying approximations, but be sure they are justified.

— from Bill K.

#12

Particle - Nuclear Problem

A $_4Be^7$ nucleus at rest undergoes K-capture.

a) Write the reaction

How about adding:

b) What velocity (in cm/sec) will the resulting $_3Li^7$ nucleus have? ~~(If you~~ ~~neglect~~ approximate ~~the initial energies of the particles~~ as ~~purely rest mass energy, justify that assumption.)~~

Lu

c) What energy (in eV) will the $_3Li^7$ nucleus have?

Nuclear Data

$_4Be^7$ has $J^P = \frac{3}{2}^-$,
mass excess = 16.380 milli mass units

$_3Li^7$ has $J^P = \frac{3}{2}^-$
mass excess = 16.005 milli mass units

1 mass unit = 931.5 MeV

M_e = .511 MeV

You are encouraged to make any simplifying approximations, but be sure they are justified.

Rationale

1) Know what K-capture is.
2) Know that a neutrino must be emitted
3) Remember to include the m_e
4) Ignore irrelevant J^P data
5) Realize that B.E. of the atomic electron may be neglected.
6) Realize that the neutrino will be relativistic (of course) but the $_3Li^7$ will not be.

Solution

a) $_4Be^7 + e^- \rightarrow {}_3Li^7 + \nu_e$

$$T = m_{Be} + m_e - m_{Li}$$
$$= (16.380 - 16.005)(10^{-3})(931.5) + .511$$
$$= .860 \text{ MeV}$$

E conservation: $T = \frac{1}{2} m_{Li} V^2 + p_\nu c$

p conservation: $m_{Li} V = p_\nu$

$$\Rightarrow \frac{1}{2} m v^2 + m c v - T = 0$$

$$v = \frac{-mc \pm \sqrt{m^2c^2 + 2mT}}{m} \approx \frac{T}{mc}$$

$$\frac{v}{c} = \frac{.860 \text{ MeV}}{(7.016)(931.5) \text{ MeV}} = 1.316 \times 10^{-4}$$

b) $v = (1.316 \times 10^{-4})(3.00 \times 10^{+10}) = \boxed{3.95 \times 10^6 \text{ cm/sec}}$

c) $E = \frac{1}{2} m v^2 = \frac{1}{2}(mc^2)\left(\frac{v}{c}\right)^2$

$$= \frac{1}{2}(7.016)(931.5)(1.316 \times 10^{-4})^2 \text{ MeV}$$
$$= 1.13 \times 10^{-4} \text{ MeV} = \boxed{113 \text{ eV}}$$

13. A smoothly varying function f(x) has been measured at a set of discrete evenly spaced locations $x_n = x_o + nh, \quad n = 0,1,2\ldots$ It is desired to estimate the derivative $f'(x_o)$ from this data.

a) Derive the best finite difference approximation for $f'(x_o)$ in terms of the values of the function at $x_o, \quad x_1$ and x_2.

b) Derive an estimate for the error in your approximation, especially its dependence on the step size h. Under what circumstances will the approximation be larger than the correct value?

Kinnersley

#13

A smoothly varying function $f(x)$ has been measured at a set of discrete evenly spaced locations $x_n = x_0 + nh$, $n = 0, 1, 2, \ldots$ It is desired to estimate $f'(x_0)$ from this data.

a) Construct the best finite difference approximation to $f'(x_0)$ in terms of the values of the function at x_0, x_1 and x_2.

b) Estimate the error in your approximation, especially its dependence on the step size h. Under what circumstances will the approximation be larger than the correct value?

Solution:

$$f_0 = f(x) = f_0$$
$$f_1 = f(x+h) = f_0 + hf_0' + \tfrac{1}{2}h^2 f_0'' + \tfrac{1}{6}h^3 f_0''' + \ldots$$
$$f_2 = f(x+2h) = f_0 + 2h f_0' + 2h^2 f_0'' + \tfrac{4}{3}h^3 f_0''' + \ldots$$

With 3 data points, we can eliminate f_0 and f_0''!

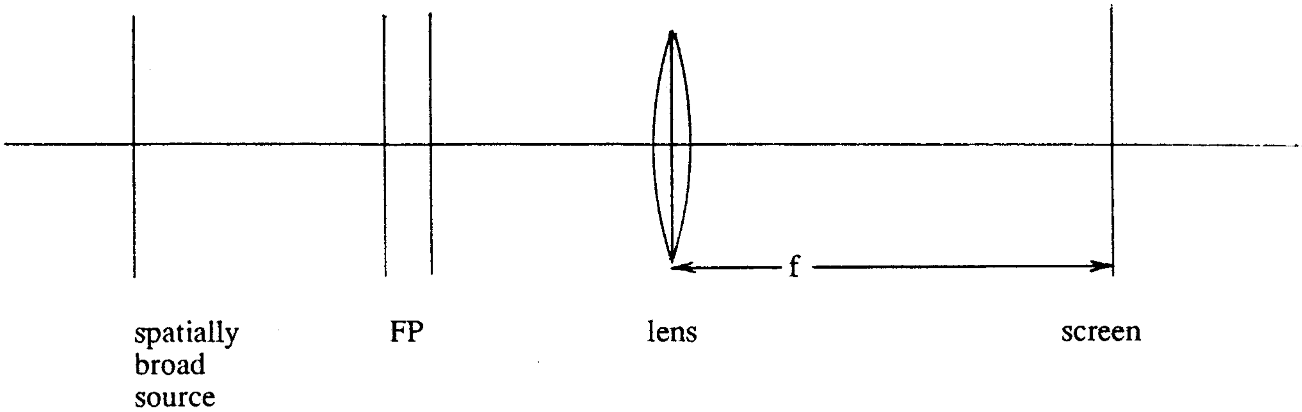
$$f_1 - f_0 = hf_0' + \tfrac{1}{2}h^2 f_0'' + \tfrac{1}{6}h^3 f_0'''$$
$$f_2 - f_0 = 2hf_0' + 2h^2 f_0'' + \tfrac{4}{3}h^3 f_0'''$$
$$(f_2 - f_0) - 4(f_1 - f_0) = -2hf_0' + \tfrac{2}{3}h^3 f_0'''$$

$$f_0' \approx \boxed{\frac{-f_2 + 4f_1 - 3f_0}{2h}} + \tfrac{1}{3}h^2 f_0'''$$

The error in this method is $O(h^2)$
The approximation will be larger than the correct value provided $f_0''' > 0$.

$f_0''' < 0$ I think. L.a.L.

14. A Fabry-Perot interferometer (FP) is illuminated by a spatially broad diffuse source (such as that provided by a gas discharge). The FP is made up of two parallel plates with light amplitude reflectivity r and spacing d. A lens of focal length f is located beyond the FP and is placed a distance f (lens focal length) from the viewing screen.



a) If the source contains only one discrete wavelength, calculate the appropriate condition for constructive interference and then sketch and describe the nature of the interference fringe pattern seen by a human viewer looking at the screen.

b) If the source emits several fixed wavelengths λ_1, λ_2, and λ_3, describe the nature of the interference fringe pattern seen by a human viewer looking at the screen.

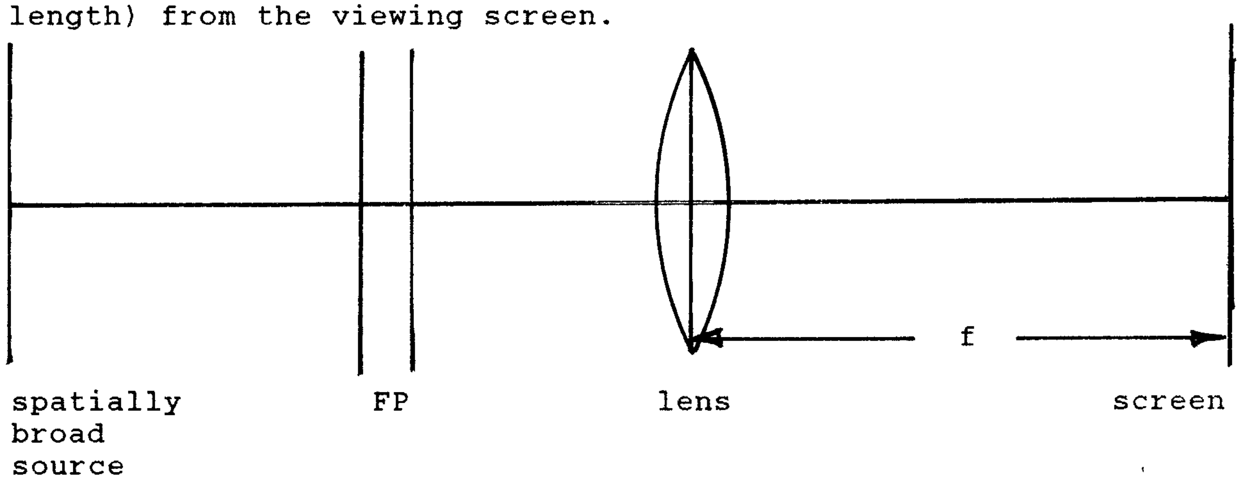
c) If a single discrete but variable wavelength is incident on the FP and the wavelength is slowly but continuously <u>increasing</u> as a function of time, describe the time dependent interference fringe pattern. (Such a variable wavelength could be obtained from a tunable monochromatic laser used in conjunction with a diffusing plate or other optical arrangement.)

d) Assuming that the FP system is illuminated as given in part c, tell what you would observe as a function of time if an electrical light detector were placed behind a small hole of diameter D made at the center of the screen.

#14

OPTICS PROBLEM FOR COMP--CONE

A Fabry-Perot interferometer (FP) is illuminated by a spatially broad diffuse source (such as that provided by a gas discharge). The FP is made up of two parallel plates with light amplitude reflectivity r and spacing d. A lens of focal length f is located beyond the FP and is placed a distance f (lens focal length) from the viewing screen.



a. If the source contains only one discrete wavelength, calculate the appropriate condition for constructive interference and then sketch and describe the nature of the interference fringe pattern seen by a human viewer looking at the screen.

b. If the source simultaneously emits several fixed wavelengths λ_1, λ_2, and λ_3, describe the nature of the interference fringe pattern seen by a human viewer looking at the screen.

c. If a single discrete but variable wavelength is incident on the FP and the wavelength is slowly but continuously <u>increasing</u> as a function of time, describe the time dependent interference fringe pattern. (Such a variable wavelength could be obtained from a tunable monochromatic laser used in conjunction with a diffusing plate or other optical arrangement.)

d. Assuming that the FP system is illuminated as given in part c, tell what you would observe as a function of time if an electrical light detector were placed behind a small hole of diameter a made at the center of the screen.

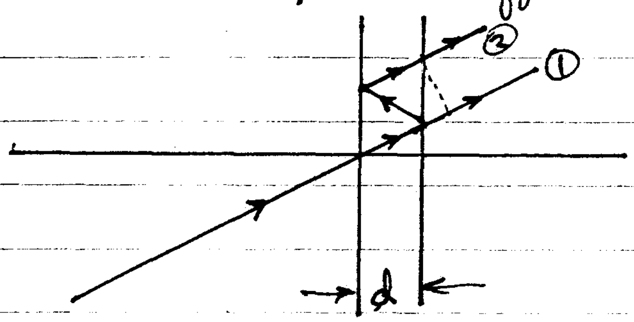
Optics Solution - Cone

(a) For a particular point on the screen, consider the path differences Λ for rays reaching that point.

Note by symmetry that the fringe pattern will consist of concentric circles.

Basic property of lens says that parallel rays will reach the same point in the focal plane (screen).

Consider path difference for reflected ray:



Through geometry (several steps) show

$$\Lambda = 2d\cos\theta$$

(SHOW WORK)

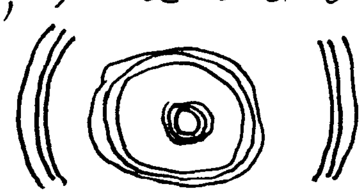
Interference condition for integer m

$$\left\{ \Lambda = m\lambda \right\} \Rightarrow \text{constructive (bright fringe)}$$

$$\boxed{m\lambda = 2d\cos\theta}$$

Bright rings occur at these angles
$\Rightarrow$ concentric bright fringes (not equally spaced)

(b) Since 3 λ_i's are present, there will be 3 sets of concentric rings.



c) Since λ changes, θ must change

$$m\lambda = 2d\cos\theta$$
$$d\lambda = \frac{2d}{m}(-\sin\theta)\,d\theta$$

Here $\sin\theta > 0$ so $\boxed{d\theta \propto -d\lambda}$

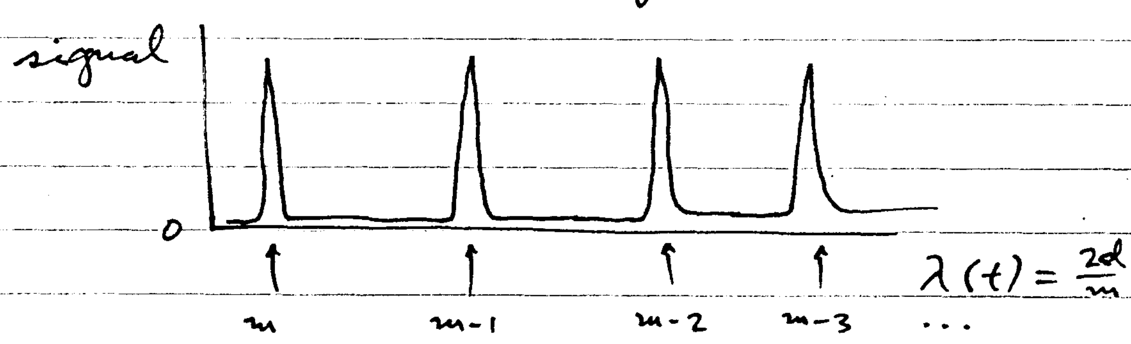
The rings will <u>shrink</u> in diameter as λ increases.

d) When there is a fringe at $\theta \approx 0$, there will be light &; hence, a strong signal.

$$\boxed{m\lambda = 2d}$$

Otherwise, fringes occur away from the center & signal will be zero.

Signal is a periodic function of $\lambda(t)$



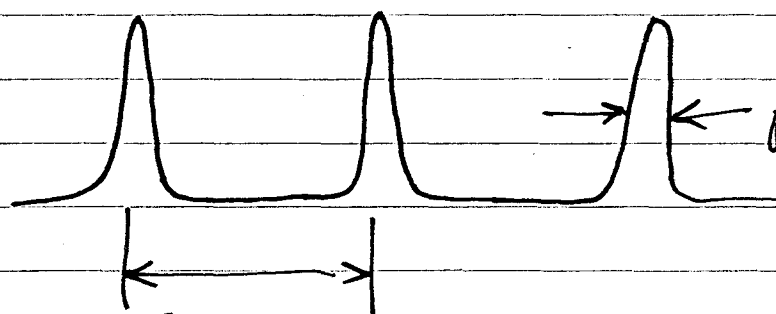
Device has <u>finite</u> resolving power dependent on

r	(want high r)
l	(want large l)
d	(want small d)

In many real cases, you would measure the finite resolution of the FP instrument instead of the $\Delta\lambda$ of the source.

For example, you might discuss the

a) Airy Function response of the FP



b) free spectral range — dependent on d

c) size of fringe relative to diameter of hole D.

15. Consider a system of N protons in a 10 T magnetic field. At this field the transition frequency between the $m_I = \pm\frac{1}{2}$ energy levels is 426 MHz.

a) Derive an expression for the internal energy of this system as a function of temperature.

b) Derive an expression for the heat capacity of this system.

c) Show that the heat capacity plotted as function of temperature has a maximum (the so-called Schottky anomaly).

H. Schmidt

#15

Consider a system of N protons in a 10 T magnetic field. At this field the transition frequency between the $m_I = \pm\frac{1}{2}$ energy levels is 426 MHz.

a) Derive an expression for the internal energy of this system as a function of temperature

b) Derive an expression for the heat capacity of this system.

c) Show that the heat capacity plotted as function of temperature has a maximum (the so-called Schottky anomaly).

Hugo Schmidt

Statistical Mechanics Problem

A system containing protons is in a 10 T magnetic field. At this field, the transition frequency between the $m_I = \pm\frac{1}{2}$ energy levels is 426 MHz. Show that the ~~specific~~ heat capacity plotted as a function of temperature has a maximum (the so-called Schottky anomaly). ~~Find within 10% the temperature at which the maximum occurs for this system.~~

Cone – v. good, but too short

perhaps expand by ① specifying N & asking for values
② asking for T_{max}
③ add entropy calc.
④ ... ?

Statistical Mechanics Solution

The energy levels are 0 and $h\nu$, where $\nu = 426$ MHz.

— $h\nu$

— 0

For N protons the internal energy, controlled by the Boltzmann factor, is

$$U = Nh\nu \frac{e^{-h\nu/kT}}{1+e^{-h\nu/kT}}$$

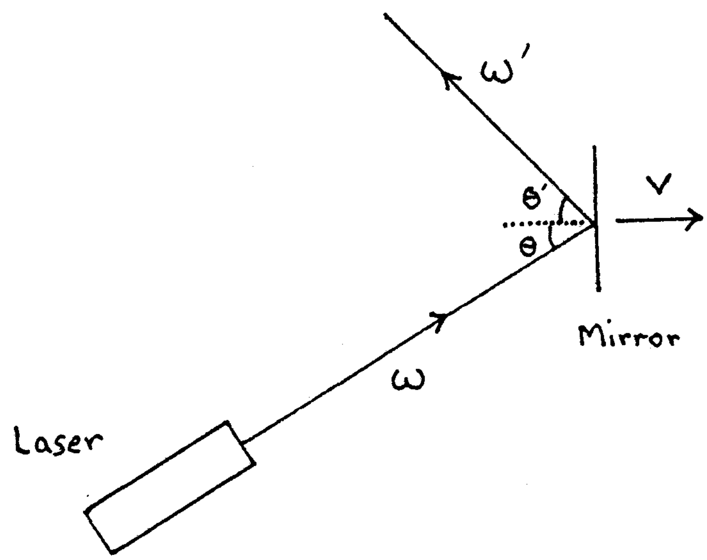
The heat capacity is

$$C = \frac{dU}{dT} = Nh\nu\left[\frac{\frac{h\nu}{kT^2}e^{-h\nu/kT}}{1+e^{-h\nu/kT}} - \frac{e^{-h\nu/kT}\,\frac{h\nu}{kT^2}e^{-h\nu/kT}}{(1+e^{-h\nu/kT})^2}\right]$$

$$= \frac{N(h\nu)^2}{kT^2}\,\frac{e^{h\nu/kT}}{(e^{h\nu/kT}+1)^2}$$

The exponentials dominate over the T^{-2} dependence, so there is a maximum near where $T = h\nu/k$.

16. Consider an ideal plane mirror which moves in the direction of its normal (say the x direction) at a velocity v (with respect to the laboratory) which is not necessarily small compared to the speed of light. Assume that at a particular instant of time a beam from a laser at rest in the laboratory strikes the mirror at an angle θ as measured in the laboratory having a frequency ω as measured in the laboratory. The beam is reflected at an angle θ' with frequency ω' (again measured in the laboratory frame).



a) Find expressions for ω' and θ' as functions of ω and θ.

b) Show that your expressions reduce to the "standard" results

$$\omega' = \frac{c - v}{c + v}\omega \quad , \quad \theta' = 0 \quad \text{in the limit} \quad \theta \to 0 \quad ,$$

and

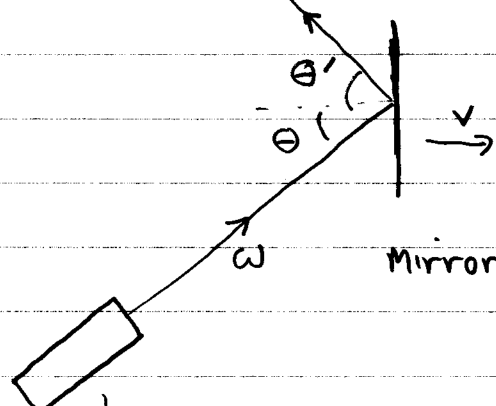
$$\omega' = \omega \quad , \quad \theta' = \theta \quad \text{in the limit} \quad \frac{v}{c} \to 0 \quad .$$

Lee Lindblom

Relativity

#16

Consider an ideal plane mirror which moves in the direction of its normal (say the x direction) at a velocity v (with respect to the laboratory) which is not necessarily small compared to the speed of light. Assume that at a particular instant of time a beam from a laser at rest in the laboratory strikes the mirror at an angle Θ as measured in the laboratory having a frequency ω as measured in the laboratory. The beam is reflected at an angle Θ' with frequency ω' (again measured in the laboratory frame).



a) Find expressions for ω' and Θ' as functions of ω and Θ

b) Show that your expressions reduce to the "standard" results

$$\omega' = \frac{c-v}{c+v}\,\omega \,, \quad \Theta' = 0 \quad \text{in the limit } \Theta \to 0$$

and

$$\omega' = \omega \,, \quad \Theta' = \Theta \quad \text{in the limit } \frac{v}{c} \to 0 .$$

Solution:

The wave vector of the incoming photons will be given by

$$k^a = \omega\left(t^a + \cos\theta\, x^a + \sin\theta\, y^a\right)$$

where t^a, x^a, y^a are unit vectors $(1 = x^a x_a = y^a y_a = -t^a t_a)$ pointing in the t, x, and y directions respectively. This wave vector will be transformed by some linear operator, call it $M^a{}_b$, as it reflects from the mirror.

$$k'^a = M^a{}_b k^b$$

$M^a{}_b$ has the property that it simply reverses the sign of the component of k^a which is parallel to the normal vector to the surface, call it n^a, in the rest frame of the mirror.

Thus $M^a{}_b$ will have the form

$$M^a{}_b = \delta^a{}_b - 2n^a n_b$$

where $\delta^a{}_b$ is the identity matrix and n^a is the appropriate unit normal to the surface

We need now only compute n^a. The four velocity of the mirror is

$$u^a = \gamma\left(t^a + \frac{v}{c} x^a\right) \quad \text{with } \gamma = \left(1 - v^2/c^2\right)^{-\frac{1}{2}}$$

which is the unit vector which points in the time direction to an observer at rest with the mirror. The normal to the mirror n^a must be orthogonal to this, consequently

$$n^a = -\gamma\left(\frac{v}{c}t^a + x^a\right)$$

We can no compute the reflected wave vector straightforewardly:

$$k'^a = M^a{}_b k^b = k^a - 2n^a(n^b k_b)$$

Since $n_b k^b = \gamma\omega\left(\frac{v}{c} - \cos\theta\right)$ we have:

$$k'^a = \omega\left(t^a + \cos\theta\, x^a + \sin\theta\, y^a\right)$$

$$+ 2\omega\gamma^2\left(\frac{v}{c} - \cos\theta\right)\left(\frac{v}{c}t^a + x^a\right)$$

$$= \omega\gamma^2\left\{1 - 2\frac{v}{c}\cos\theta + \frac{v^2}{c^2}\right\} t^a$$

$$- \omega\gamma^2\left\{\left(1 + \frac{v^2}{c^2}\right)\cos\theta - 2\frac{v}{c}\right\} x^a + \omega\sin\theta\, y^a$$

In terms of ω' and θ', k'^a would of course have the form

$$k'^a = \omega'\left(t^a - \cos\theta' x^a + \sin\theta' y^a\right)$$

Thus equating components we have:

$$\omega' = \omega\gamma^2\left\{1 - 2\frac{v}{c}\cos\theta + \frac{v^2}{c^2}\right\} \xrightarrow[\theta \to 0]{} \omega\,\frac{1 - \frac{v}{c}}{1 + v/c}$$

$$\cos\theta' = \frac{\left(1 + \frac{v^2}{c^2}\right)\cos\theta - 2\frac{v}{c}}{1 - 2\frac{v}{c}\cos\theta + \frac{v^2}{c^2}} \xrightarrow[\theta \to 0]{} 1$$