Inter-dependence of Re $\epsilon(\omega)$ and Im $\epsilon(\omega)$. DR

Dispersion Relations for Dielectric Constant $\epsilon(\omega)$ [Jkⁿ Sec. 7.10]

1) The SHO model of a medium's dielectric constant gives $\epsilon(\omega)$ as a complex function of frequency ω [cf. Jkⁿ Eq. (7.51)]. Generally speaking, Re $\epsilon(\omega)$ controls the medium's index of refraction, and thus the reflection/transmission character of an EM wave propagating through the medium. On the other hand, Im $\epsilon(\omega)$ is related to the medium's conductivity, and thus affects the attenuation of the passing EM wave. Both Re $\epsilon(\omega)$ & Im $\epsilon(\omega)$ are required to realistically analyse the passage of an EM wave through a dispersive, lossy material.

Now we shall show that Re $\epsilon(\omega)$ and Im $\epsilon(\omega)$ are closely related: Re $\epsilon(\omega)$ can be derived from a knowledge of Im $\epsilon(\omega)$, and vice versa. The equations relating Re $\epsilon(\omega) \longleftrightarrow$ Im $\epsilon(\omega)$ are known as "dispersion relations". They follow mathematically from the constraints placed on "well-behaved" functions of a complex variable. Physically, the relations go back to the fact that both the Re & Im parts of $\epsilon(\omega)$ originate from the movement of charge by the passing wave... Re $\epsilon(\omega) \leftrightarrow$ polarization of ~bound charge, Im $\epsilon(\omega) \leftrightarrow$ currents of ~free charge.

But the dispersion relations relating Re $\epsilon(\omega)$ & Im $\epsilon(\omega)$ are actually model-independent... i.e. we can derive the relations w/o talking about polarization of or currents in the medium. So we will concentrate on math, not physics. The math trick is this: from the theory of functions $f(z)$ of a complex variable $z = x + iy$, you know that if you write: $f(z) = u(x,y) + i\,v(x,y)$, then in order for $f(z)$ to be "analytic" at z (i.e. have a derivative there), u & v must obey the Cauchy-Riemann equations: $\partial u/\partial x = \partial v/\partial y$, $\partial u/\partial y = -\partial v/\partial x$.* This shows that $u = \mathrm{Re}\,f$ and $v = \mathrm{Im}\,f$ cannot be independent of one another. In fact, given Re f you should be able to find Im f, and vice-versa. It is that relation we will pursue here.

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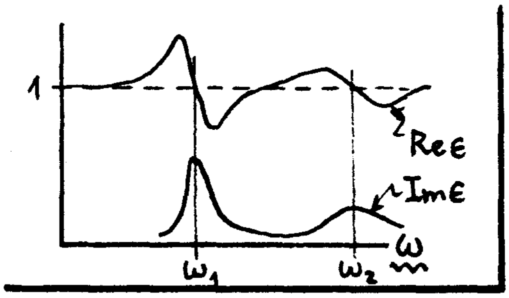
* see, e.g. App. A-2 of Mathews & Walker "Math Methods..." (Benjamin, 2nd ed., 1970).

Standard form for $\epsilon(\omega)$. Conversion: $\mathbb{D}(x,\omega) \rightarrow \mathcal{D}(x,t)$.

2) Consider a medium where the displacement $\mathbb{D}$ is related to the (applied) electric field $\mathbb{E}$ by a frequency-dependent dielectric constant $\epsilon = \epsilon(\omega)$, i.e.

$$\rightarrow \mathbb{D}(x,\omega) = \epsilon(\omega)\,\mathbb{E}(x,\omega) \qquad (1)$$

$\mathbb{D}$ & $\mathbb{E}$ are the fields; x = space coordinate, ω = frequency; $\epsilon(\omega)$: freq.-dept. dielectric const.



Measurements show (and the SHO model in Jk$^{\underline{n}}$ Sec. 7.5 describes) that $\mathrm{Re}\,\epsilon(\omega)$ & $\mathrm{Im}\,\epsilon(\omega)$ typically behave as sketched at left -- they show antiresonant and resonant character at certain frequencies $\omega_1, \omega_2, \ldots$ Clearly $\mathrm{Re}\,\epsilon$ and $\mathrm{Im}\,\epsilon$ are linked. Then the material parameters, viz.

$$\left[\!\left[\begin{matrix}\text{Refractive}\\ \text{Index}\end{matrix}\right\}\ n(\omega) = \mathrm{Re}\sqrt{\mu\epsilon(\omega)}\,, \quad \begin{matrix}\text{Absorption}\\ \text{Coefficient}\end{matrix}\Big\}\ \alpha(\omega) = \frac{2\omega}{c}\,\mathrm{Im}\sqrt{\mu\epsilon(\omega)} \qquad (2)$$

will also be linked†. The "dispersion relations" we are about to derive will show in effect that $n(\omega)$ <u>determines</u> $\alpha(\omega)$, and vice-versa.

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3) Some analytic features of $\epsilon(\omega)$ can be deduced from Eq.(1) w/o appealing to any particular model. Adopt the Fourier Transform convention...

$$F(t) = \int_{-\infty}^{\infty} f(\omega)\, e^{-i\omega t}\, d\omega \longleftrightarrow f(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(t)\, e^{i\omega t}\, dt \qquad (3)$$

Then the Fourier Convolution Theorem reads [Mathews & Walker, Sec. 4-5.7]...

$$\rightarrow \int_{-\infty}^{\infty} a(\omega)\, b(\omega)\, e^{-i\omega t}\, d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} A(\tau)\, B(t-\tau)\, d\tau. \qquad (4)$$

Use the Convolution Theorem to convert $\mathbb{D}(x,\omega)$ of Eq.(1) to real time t, i.e.

$$\mathcal{D}(x,t) = \int_{-\infty}^{\infty} \mathbb{D}(x,\omega)\, e^{-i\omega t}\, d\omega = \int_{-\infty}^{\infty} \epsilon(\omega)\, \mathbb{E}(x,\omega)\, e^{-i\omega t}\, d\omega \qquad (5)$$

... write: $\epsilon(\omega) = 1 + [\epsilon(\omega) - 1]$, in this integral...

(next page)

$\dagger$ If $\epsilon = \epsilon_R + i\epsilon_I$, then: $\sqrt{\epsilon} = \left[\frac{1}{2}\left(\sqrt{\epsilon_R^2+\epsilon_I^2} + \epsilon_R\right)\right]^{1/2} + i\left[\frac{1}{2}\left(\sqrt{\epsilon_R^2+\epsilon_I^2} - \epsilon_R\right)\right]^{1/2}$. Clearly $n \propto \mathrm{Re}\sqrt{\epsilon}$ cannot change (in ϵ_R and/or ϵ_I) without $\alpha \propto \mathrm{Im}\sqrt{\epsilon}$ also changing.

ϵ Dispⁿ Relⁿs — Fields in real time. Kernel for $\epsilon(\omega)$.

So → $\mathcal{D}(x,t) = \mathcal{E}(x,t) + \int_{-\infty}^{\infty} [\epsilon(\omega)-1]\, \mathbb{E}(x,\omega)\, e^{-i\omega t}\, d\omega$

... with: $\mathcal{E}(x,t) = \int_{-\infty}^{\infty} \mathbb{E}(x,\omega)\, e^{-i\omega t}\, d\omega$, real-time electric field...

and

$$\boxed{\begin{aligned} \mathcal{D}(x,t) &= \mathcal{E}(x,t) + \int_{-\infty}^{\infty} K(\tau)\, \mathcal{E}(x,t-\tau)\, d\tau, \\ \text{where: } K(\tau) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} [\epsilon(\omega)-1]\, e^{-i\omega\tau}\, d\omega. \end{aligned}} \tag{6}$$

In the integral for $\mathcal{D}$, $K(\tau)$ is called a "kernel function" (or, in German "Kugel-funktion"). If $\epsilon(\omega)$ has <u>no</u> frequency dependence, i.e. if <u>$\epsilon(\omega) = \epsilon_0 = \text{cnst}$</u>, then $K(\tau)$ reduces to a delta-function, as...

$$\left.\begin{aligned} K(\tau) &= (\epsilon_0 - 1)\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega\tau}\, d\omega = (\epsilon_0 - 1)\,\delta(\tau), \\ \text{So } \mathcal{D}(x,t) &= \mathcal{E}(x,t) + (\epsilon_0 - 1)\int_{-\infty}^{\infty} \delta(\tau)\, \mathcal{E}(x,t-\tau)\, d\tau = \epsilon_0\, \mathcal{E}(x,t). \end{aligned}\right\} \tag{7}$$

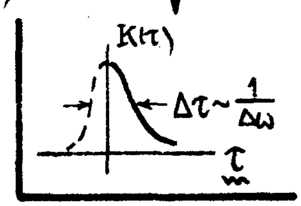
In this case, $\mathcal{D}(x,t)$ and $\mathcal{E}(x,t)$ are related locally in time.

If, on the other hand, $\epsilon(\omega) \neq \text{cnst}$, then $K(\tau)$ acquires a finite width...

IF $\epsilon(\omega)$ has width $\Delta\omega$ near some ω_i, THEN $K(\tau)$ has width $\Delta\tau \sim 1/\Delta\omega$

and → $\mathcal{D}(x,t) = \mathcal{E}(x,t) + \int_{-\infty}^{\infty} K(\tau)\, \mathcal{E}(x,t-\tau)\, d\tau$, is <u>non-local</u> in time;

i.e., $\mathcal{D}(x,t)$ depends on values of $\mathcal{E}$ at times $\neq t$. (8)



This t-nonlocality is a retardation effect: the polarization of the medium that generates $\mathcal{D}$ from $\mathcal{E}$ is <u>delayed</u>, and the delay itself depends on frequency.

Whatever the $\mathcal{E} \to \mathcal{D}$ delay time may be, we <u>must respect causality</u> -- we cannot have $\mathcal{D}(x,t)$ depend on $\mathcal{E}$-values at future times $> t$. So we impose...

CAUSALITY: $K(\tau) \equiv 0$ for $\tau < 0$, so...

$$\left[\mathcal{D}(x,t) = \mathcal{E}(x,t) + \int_0^\infty K(\tau)\,\mathcal{E}(x,t-\tau)\,d\tau\right] \quad \begin{array}{l}\mathcal{D}(x,t) \text{ depends on } \mathcal{E}\text{-values} \\ \text{at times } (t-\tau) \leq t \text{ only.}\end{array} \qquad (9)$$

This property of $K(\tau)$ can be demonstrated explicitly for specific $\epsilon(\omega)$ models.

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4) Take the Fourier inverse of $K(\tau)$ as defined in Eq. (6), and impose the causality condition $K(\tau) \equiv 0$ for $\tau < 0$. Then we get an expression for $\epsilon(\omega)$...

$$\boxed{\epsilon(\omega) = 1 + \int_0^\infty K(\tau)\,e^{i\omega\tau}\,d\tau} \quad (K(\tau) \equiv 0, \text{ for } \tau \leq 0). \qquad (10)$$

Several general properties of $\epsilon(\omega)$ follow from this relation. Noting that $K(\tau)$ is <u>real</u> -- on the assumption that $\mathcal{D}$ & $\mathcal{E}$ in Eq. (9) are <u>real</u> -- we claim:

1. $\epsilon^*(\omega) = \epsilon(-\omega^*)$. Then, for $\omega =$ <u>real</u>...

$$\left[\begin{array}{ll} \operatorname{Re}\epsilon(-\omega) = +\operatorname{Re}\epsilon(\omega), & \operatorname{Re}\epsilon(\omega) \text{ is an even fcn of } \omega; \\ \operatorname{Im}\epsilon(-\omega) = -\operatorname{Im}\epsilon(\omega), & \operatorname{Im}\epsilon(\omega) \text{ is an odd fcn of } \omega. \end{array}\right. \qquad (11)$$

2. If $K(\tau)$ is finite at all $\tau > 0$, then $\epsilon(\omega) = 1 + \int_0^\infty K(\tau)e^{i\omega\tau}\,d\tau$ is regular (i.e. analytic) everywhere in the <u>upper half</u> of the complex ω-plane... it has no poles there. This follows from $\omega = \omega_x + i\omega_y \Rightarrow e^{i\omega\tau} = (e^{i\omega_x\tau})e^{-\omega_y\tau}$, so convergence of $\epsilon(\omega)$ for $|\omega| \to \infty$ is assured for all $\omega_y > 0$ (so long as $K(\tau)$ does not diverge). When $\omega_y = 0$ (on Re ω axis), we must impose $\lim_{\tau\to\infty} K(\tau) = 0$.

3. If we expand the integral in Eq. (10) for high frequencies $|\omega| \to \infty$, we find...¶

$$\rightarrow \epsilon(\omega) - 1 \simeq \frac{i}{\omega}K(0+) - \frac{1}{\omega^2}K'(0+) - \frac{i}{\omega^3}K''(0+) + \mathcal{O}(1/\omega^4). \qquad (12)$$

¶ This follows from a Taylor expansion of $K(\tau)$ about $\tau = 0$, as:

$$\epsilon(\omega) - 1 = \int_0^\infty \left[\sum_{n=0}^\infty \frac{\tau^n}{n!} K^{(n)}(0+)\right] e^{i\omega\tau}\,d\tau = \sum_{n=0}^\infty (i/\omega)^{n+1} K^{(n)}(0+).$$

Asymptotics as $\omega \to \infty$. $\epsilon(\omega)$ as a Cauchy Integral.

For the first term in Eq. (12), we put $K(0+) = 0$.., since $K(0-) \equiv 0$, this avoids a discontinuity in $K(\tau)$ @ $\tau = 0$. Then we have the asymptotic forms

$$\boxed{\mathrm{Re}\,\epsilon(\omega) \simeq 1 - \frac{1}{\omega^2} K'(0+), \quad \mathrm{Im}\,\epsilon(\omega) \simeq -\frac{1}{\omega^3} K''(0+), \text{ as } |\omega| \to \infty.} \tag{13}$$

NOTE: The plasma dispersion relation: $\epsilon = 1 - (\omega_p^2/\omega^2)$, is verified by $\mathrm{Re}\,\epsilon(\omega)$ if we identify $\omega_p^2 = K'(0+)$. We emphasize this result is model-independent.

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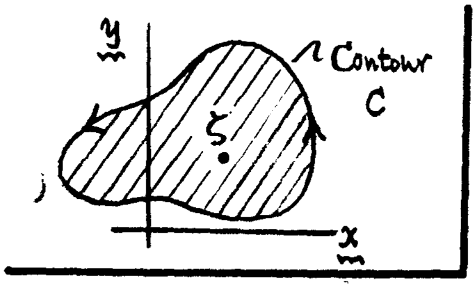
5) With the above properties of $\epsilon(\omega)$ in mind, we now derive the $\epsilon(\omega)$ dispersion relations per se. Recall Cauchy's Integral Theorem...



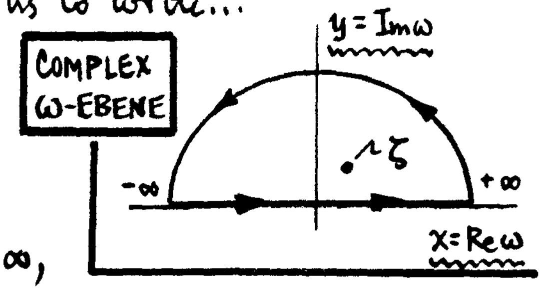
If $f(z)$ is analytic on and within contour C, then...

$$\left[f(\zeta) = \frac{1}{2\pi i} \oint_C \frac{f(z)\,dz}{z - \zeta} \;\; \Big\lmoustache \;\; \begin{matrix} \zeta \text{ within } C, \\ z = x + iy. \end{matrix} \right]. \tag{14}$$

So all values of $f(z)$ at interior points $z = \zeta$ are fixed by the values of $f(z)$ on the boundary curve C, so long as $f(z)$ is regular.

Since we have argued above that $[\epsilon(\omega) - 1]$ is regular in the upper half of the complex ω-plane, then Cauchy's Theorem allows us to write...

$$\rightarrow \epsilon(\zeta) - 1 = \frac{1}{2\pi i} \Big(\underbrace{\int_{-\infty}^{\infty}}_{\text{real axis}} + \underbrace{\oint}_{\text{upper semi-circle}} \Big) \frac{\epsilon(z) - 1}{z - \zeta}\, dz. \tag{15}$$



The contribution from the semi-circle vanishes at ∞, since $\mathrm{Re}\,[\epsilon(z) - 1] \sim \frac{1}{z^2}$ and $\mathrm{Im}\,\epsilon(z) \sim \frac{1}{z^3}$ as $z \to \infty$. Now, to get real frequencies ζ, we let ζ approach the real axis from above, and we have...

$$\left[\zeta = \omega + i\alpha, \text{ w/ } \alpha \to 0, \text{ and } \omega = \text{a real frequency}; \right.$$

$$\text{so/ } \left[\epsilon(\omega) = 1 + \frac{1}{2\pi i} \lim_{\alpha \to 0} \int_{-\infty}^{\infty} \frac{[\epsilon(x) - 1]}{(x - \omega) - i\alpha}\, dx. \right. \tag{16}$$

Plemelj Formula. The Dispersion Relations.

6) The integral denominator in Eq. (16) is a special function in complex analysis, and it is expressed by Plemelj's formula:†

$$\rightarrow \lim_{\alpha\to 0}\left[\frac{1}{(x-\omega)-i\alpha}\right] = \mathcal{P}\left(\frac{1}{x-\omega}\right) + i\pi\,\delta(x-\omega), \tag{17}$$

where $\mathcal{P}$ means "principal value" and δ is the Dirac delta function. Applying formula (17) to the $\epsilon(\omega)$ integral in Eq. (16), we find...¶

$$\boxed{\epsilon(\omega) = 1 + \frac{1}{\pi i}\mathcal{P}\int_{-\infty}^{\infty}\frac{[\epsilon(x)-1]}{x-\omega}dx} \int \omega \text{ is a real frequency, and } x \text{ is on the Re}\,\omega \text{ axis.} \tag{18}$$

The dispersion relations (à la Kramers-Kronig) result from taking the Re and Im parts of Eq. (18). Since x & ω in the integral are both real, we get...

$$\left[\!\left[\mathrm{Re}\,\epsilon(\omega) = 1 + \frac{1}{\pi}\mathcal{P}\int_{-\infty}^{\infty}\frac{\mathrm{Im}\,\epsilon(x)}{x-\omega}dx\ ,\ \ \mathrm{Im}\,\epsilon(\omega) = \frac{1}{\pi}\mathcal{P}\int_{-\infty}^{\infty}\frac{1-\mathrm{Re}\,\epsilon(x)}{x-\omega}dx\ .\right]\!\right] \tag{19}$$

¶ For a general complex fcn $f(z)$, on the real axis ($z\to x$), (18) is: $f(x) = \frac{1}{\pi i}\mathcal{P}\int_{-\infty}^{\infty}\frac{f(\xi)d\xi}{\xi - x}$. The dispersion relations then read, as in Mathews & Walker, Eqs. (5-12):

$$\mathrm{Re}\,f(x) = \frac{1}{\pi}\mathcal{P}\int_{-\infty}^{\infty}\frac{\mathrm{Im}\,f(\xi)}{\xi-x}d\xi,\quad \mathrm{Im}\,f(x) = (-)\frac{1}{\pi}\mathcal{P}\int_{-\infty}^{\infty}\frac{\mathrm{Re}\,f(\xi)}{\xi-x}d\xi\ .$$

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† See Mathews & Walker, App. A2, Eq. (A-18). The function can be factored as...

$$\lim_{\alpha\to 0}\left[\frac{1}{(x-\omega)-i\alpha}\right] = \lim_{\alpha\to 0}\underbrace{\left[\frac{(x-\omega)}{(x-\omega)^2+\alpha^2}\right]}_{①} + i\lim_{\alpha\to 0}\underbrace{\left[\frac{\alpha}{(x-\omega)^2+\alpha^2}\right]}_{②}.$$

As $\alpha\to 0$, [①] behaves everywhere as $1/(x-\omega)$, except at $x=\omega$, where it is zero (for $\alpha = 0+$). This fcn is is denoted $\mathcal{P}[1/(x-\omega)]$, and in an integral it means excluding the singularity at $x=\omega$, viz.: $\mathcal{P}\int_{-\infty}^{\infty}[f(x)/(x-\omega)]dx = \lim_{\alpha\to 0}\left(\int_{-\infty}^{\omega-\alpha} + \int_{\omega+\alpha}^{\infty}\right)\frac{f(x)\,dx}{x-\omega}$ (so-called Cauchy Principal Value of the integral -- it may or may not exist).

As $\alpha\to 0$, [②] is zero everywhere but at $x=\omega$, where it goes as $1/\alpha\to\infty$. The area under this curve is: $\int_{-\infty}^{\infty}[②]\,dx = \pi$, independent of α. Then this curve must be a repⁿ of the Dirac delta fcn, i.e.: $[②] = \pi\,\delta(x-\omega)$. Together with above $\mathcal{P}$, we get (17).

7) The Kramers-Kronig version of the $\epsilon(\omega)$ dispersion relations in Eq. (19) takes advantage of the symmetries in Eq. (11). Straightforwardly get:

$$\mathrm{Re}\,\epsilon(\omega) = 1 + \frac{2}{\pi}\mathcal{P}\int_0^\infty \frac{x\,\mathrm{Im}\,\epsilon(x)}{x^2-\omega^2}\,dx\,, \text{ for } \mathrm{Re}\,\epsilon(\omega) \text{ an } \underline{\text{even}} \text{ fcn of } \omega; \tag{20a}$$

$$\mathrm{Im}\,\epsilon(\omega) = \frac{2\omega}{\pi}\mathcal{P}\int_0^\infty \frac{1-\mathrm{Re}\,\epsilon(x)}{x^2-\omega^2}\,dx\,, \text{ for } \mathrm{Im}\,\epsilon(\omega) \text{ an } \underline{\text{odd}} \text{ fcn of } \omega. \tag{20b}$$

The integrations over negative freqs, $\int_{-\infty}^0 dx$, have been reflected out.

REMARKS

1. As advertised, $\mathrm{Re}\,\epsilon(\omega)$ & $\mathrm{Im}\,\epsilon(\omega)$ are intimately related.

2. To bring Eqs. (20) closer to the optical problem, recall that the (complex) index of refraction is: $n(\omega) = \sqrt{\mu\epsilon(\omega)}$. Then (with $\mu=1$) write...

$\rightarrow n^2(\omega) = \epsilon(\omega) = \mathrm{Re}\,\epsilon(\omega) + i\,\mathrm{Im}\,\epsilon(\omega)$;

and // effective wave #: $\beta(\omega) = \frac{\omega}{c}\mathrm{Re}\,n(\omega)$,

attenuation cnst: $\frac{\alpha(\omega)}{2} = \frac{\omega}{c}\mathrm{Im}\,n(\omega)$.

overall propagation wave # is: $k = \beta + i\frac{\alpha}{2}$ (see forms on p. Waves 16). Refraction index is: $n = \frac{c}{\omega}k$. (21)

Eqs. (20) are then recast as...

$$\boxed{\begin{aligned} &\mathrm{Re}[n^2(\omega)-1] = \frac{2}{\pi}\mathcal{P}\int_0^\infty \frac{\Omega\,\mathrm{Im}[n^2(\Omega)-1]}{\Omega^2-\omega^2}\,d\Omega, \text{ refractive part;} \\ &\mathrm{Im}[n^2(\omega)-1] = (-)\frac{2}{\pi}\mathcal{P}\int_0^\infty \frac{\omega\,\mathrm{Re}[n^2(\Omega)-1]}{\Omega^2-\omega^2}\,d\Omega, \text{ absorptive part.}\end{aligned}}$$

(22a), (22b)

These were the forms originally derived by Kramers & Kronig (1927 & 1926).

3. The KK relations are really just mathematical statements of how the Re & Im parts of an analytic fcn of a complex variable must be related. They just rely on Cauchy's Theorem [Eq. (14)] and the reqt. that $\epsilon(\omega) \to 0$ as $|\omega| \to \infty$ [Eq. (13)]. The only physics we put in was "causality" [$K(\tau) \equiv 0$ for $\tau < 0$, Eq. (9)], and "symmetry" [Eq. (11)]. So Eqs. (22) are basically model-independent.

Some SUM RULES for $\epsilon(\omega)$.

8) We can derive some further model-independent restrictions on the nature of all possible dielectric cnsts $\epsilon(\omega)$. They are called "sum rules". As follows.

(A) Recall the asymptotic behavior from Eq. (13)...

$$\mathrm{Re}\,\epsilon(\omega) \simeq 1 - \frac{1}{\omega^2}K'(0+), \quad \mathrm{Im}\,\epsilon(\omega) \simeq (-)\frac{1}{\omega^3}K''(0+), \text{ as } |\omega| \to \infty.$$

$$\underline{\text{DEFINE}}: \ \omega_p^2 = \lim_{\omega\to\infty}\{\omega^2[1-\epsilon(\omega)]\} = K'(0+). \quad \int \omega_p \text{ is called the "plasma frequency".} \tag{23}$$

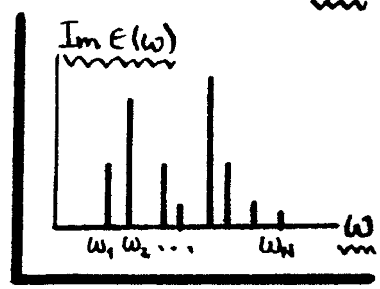
The part of $\epsilon(\omega)$ which fixes ω_p^2 here is $\mathrm{Re}\,\epsilon(\omega)$. We use Eq. (20a) to write:

$$1 - \mathrm{Re}\,\epsilon(\omega) = \frac{2}{\pi}\mathcal{P}\int_0^\infty \frac{x\,\mathrm{Im}\,\epsilon(x)}{\omega^2 - x^2}dx \xrightarrow[(\omega\to\infty)]{} \frac{2}{\pi}\frac{1}{\omega^2}\int_0^\infty x\,\mathrm{Im}\,\epsilon(x)\,dx,$$

So//

$$\boxed{\omega_p^2 = \frac{2}{\pi}\int_0^\infty \omega\,\mathrm{Im}\,\epsilon(\omega)\,d\omega}. \tag{24}$$

This is a "sum rule" in the following sense. Suppose $\mathrm{Im}\,\epsilon(\omega)$ shows a series of sharp absorption resonances @ $\omega_1, \omega_2, \ldots, \omega_N$:



$$\mathrm{Im}\,\epsilon(\omega) = \sum_{j=1}^{N}\frac{f_j}{\omega_j}\delta(\omega - \omega_j) \quad \int f_j = \text{"oscillator strength" for } j^{\text{th}} \text{ absorption resonance.}$$

So//

$$\text{Eq. (24)} \Rightarrow \underline{\underline{\omega_p^2 = \frac{2}{\pi}\sum_{j=1}^{N} f_j = \text{cnst.}}} \leftarrow \text{oscillator strength sum rule.} \tag{25}$$

(B) We can develop an averaging process for $\mathrm{Re}\,\epsilon(\omega)$ from the behavior of $\mathrm{Im}\,\epsilon(\omega)$ as $\omega \to 0$. From Eq. (20b)...

$$\mathrm{Im}\,\epsilon(\omega) = \frac{2\omega}{\pi}\mathcal{P}\int_0^\infty \frac{\mathrm{Re}\,\epsilon(x) - 1}{\omega^2 - x^2}dx = \frac{2}{\pi\omega}\int_0^\infty [\mathrm{Re}\,\epsilon(x) - 1]\overbrace{[1-(x/\omega)^2]^{-1}}^{\text{expand in Taylor Series}}, \tag{26}$$

So//

$$\mathrm{Im}\,\epsilon(\omega) \underset{\omega\to\text{large}}{=} \frac{2}{\pi\omega}\int_0^\infty [\mathrm{Re}\,\epsilon(x) - 1]\,dx + \frac{2}{\pi\omega^3}\int_0^\infty x^2[\mathrm{Re}\,\epsilon(x) - 1]\,dx + \mathcal{O}(1/\omega^5).$$

As $\omega \to \infty$, this expansion must fit: $\mathrm{Im}\,\epsilon(\omega) \simeq (-)\frac{1}{\omega^3}K''(0+)$, from Eq. (13). Evidently, the first term RHS -- of $\mathcal{O}(1/\omega)$ -- must vanish, and the second term

Average value of $\mathrm{Re}\,\epsilon(\omega)$. Dispersion Relations in QM.

gives an expression for $K''(0+)$. Thus, we have...

$$\rightarrow \int_0^\infty [\mathrm{Re}\,\epsilon(x)-1]\,dx = 0, \quad \text{and} \quad K''(0+) = (-)\frac{2}{\pi}\int_0^\infty x^2[\mathrm{Re}\,\epsilon(x)-1]\,dx. \tag{27}$$

(NOTE: by now, we have information on $K(\tau)$ as $\tau \to 0+$, viz. $K(0+)=0$ [Eq.(13)], $K'(0+)=\omega_p^2$ [Eq.(23)], and $K''(0+)$ as given in Eq.(27)).

In the first of Eqs.(27), split the integration into a low and high frequency range, i.e. $0 \leqslant x \leqslant \Omega$ and $\Omega \leqslant x \leqslant \infty$, where Ω is a frequency high enough so that:

$\mathrm{Re}\,\epsilon(x) = 1-\omega_p^2/x^2$, when $x \geqslant \Omega$; [practically $\Omega \geqslant \omega_p$, plasma approxn]

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$$\int_0^\infty [\mathrm{Re}\,\epsilon(x)-1]\,dx = \int_0^\Omega [\mathrm{Re}\,\epsilon(x)-1]\,dx + \int_\Omega^\infty [(-)\omega_p^2/x^2]\,dx = 0$$

or/

$$\int_0^\Omega \mathrm{Re}\,\epsilon(\omega)\,d\omega = \Omega + \frac{\omega_p^2}{\Omega} \Rightarrow \boxed{\langle \mathrm{Re}\,\epsilon(\omega)\rangle_\Omega = \frac{1}{\Omega}\int_0^\Omega \mathrm{Re}\,\epsilon(\omega)\,d\omega = 1 + \frac{\omega_p^2}{\Omega^2}} \tag{28}$$

This result shows that over the frequency range $0 \leqslant \omega \leqslant \Omega$, when $\Omega \gg \omega_p$ (the plasma frequency), <u>the average value of $\mathrm{Re}\,\epsilon(\omega)$ is unity</u>.

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9) Much more can be (and has been) done with dispersion relations... in the 60's & 70's, and in combination with S-matrix theory in QM, they were an industry for high-energy theorists studying <u>scattering and production of exotic particles</u> [see A.S. Davydov "QM" (Pergamon, 2nd ed., 1991 printing), ¶ 123]. Their usefulness is in handling the description of QM systems which are <u>not</u> unitary, e.g. where the energy $E \to \tilde{E} = E + \frac{1}{2}i\hbar\Gamma$ becomes complex, with $\mathrm{Im}\,\tilde{E} = (\hbar/2)\Gamma$ representing an annihilation or production rate. Most of our findings carry over to the scattering amplitudes $f(\tilde{E})$ used extensively in the theory.

Again, the theory of dispersion relations is basically independent of any model. It is ~ remarkable how much can be understood w/o invoking Hooke's famous law: UT TENSIO, SIC VIS. We hardly even drew any pictures.