DEPARTMENT OF PHYSICS

2003 COMPREHENSIVE EXAM

25 August-27 August 2003

Answer each of the following questions, all of which carry equal weight. Begin your answer to each question on a new sheet of paper; solutions to different questions must not appear on the same sheet. Each sheet of paper must be labeled with your name and the problem number in the upper right hand corner. When more than one sheet is submitted for a problem, be sure the pages are ordered properly.

PHYSICAL CONSTANTS

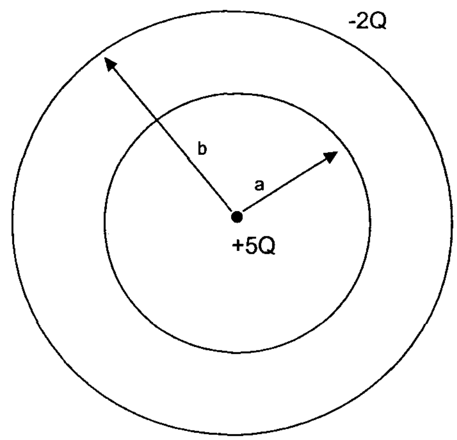
Quantity	Symbol	Value
acceleration due to gravity	g	9.8 m s^{-2}
gravitational constant	G	$6.67 \times 10^{-11} \text{ N m}^2 \text{ kg}^{-2}$
permittivity of vacuum	ε_o	$8.85 \times 10^{-12} \text{ C}^2 \text{ N}^{-1} \text{ m}^{-2}$
permeability of vacuum	μ_o	$4\pi \times 10^{-7} \text{ N A}^{-2}$
speed of light in vacuum	c	$3.00 \times 10^8 \text{ m s}^{-1}$
elementary charge	e	$1.602 \times 10^{-19} \text{ C}$
mass of electron	m_e	$9.11 \times 10^{-31} \text{ kg}$
mass of proton	m_p	$1.673 \times 10^{-27} \text{ kg}$
Planck constant	h	$6.63 \times 10^{-34} \text{ J s}$
Avogadro constant	N_A	$6.02 \times 10^{23} \text{ mol}^{-1}$
Boltzmann constant	k	$1.38 \times 10^{-23} \text{ J K}^{-1}$
molar gas constant	R	$8.31 \text{ J mol}^{-1} \text{ K}^{-1}$
standard atmospheric pressure		$1.013 \times 10^5 \text{ Pa}$

1. Pictured below is a hollow spherical conducting shell with thickness b-a and inner radius a. The conductor has a net charge of –2Q. A 5Q charge is placed at the center of the sphere.
   a. After equilibrium is reached, how would the charge on the conductor be distributed?
   b. Use Gauss's law to find the electric field in the three regions:

      i) $r < a$

      ii) $a < r < b$

      iii) $r > b$
   c. Indicate (or state) the direction **E** in each region, and provide a rough plot of **E** as a function of r.
   d. Find the electric potential V outside of the spherical shell and inside of the shell taking the potential an infinite distance from the shell as zero.



1.

Pictured below is a hollow spherical conducting shell with thickness b-a and inner radius a. It has a net charge of $-2Q$. A $5Q$ charge is placed at the center of the sphere.

a. After equilibrium is reached, how would the charge on the conductor be distributed?

b. Use Gauss's law to find the E-field in the three regions:

i) $r < a$

ii) $a < r < b$

iii) $r > b$

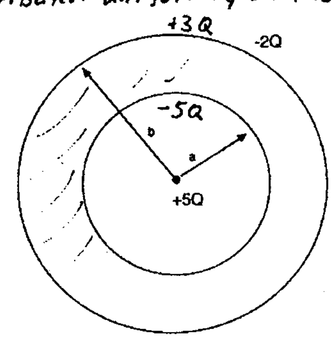
c. Indicate (or state) the direction **E** in each region, and provide a rough plot of E as a function of r.

d. Find the electric potential V outside of the spherical shell and inside of the shell taking the potential an infinite distance from the shell as zero.

a) A charge of $-5Q$ would be distributed uniformly on the inside

$+3Q$ on the outside

This leads to $E = 0$ in the conducting shell



b)

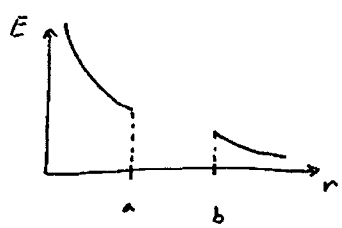
i) $\oint \vec{E}\cdot d\vec{A} = \frac{5Q}{\epsilon_0} \Rightarrow E = \frac{5Q}{4\pi\epsilon_0 r^2}$

pointing radially outward

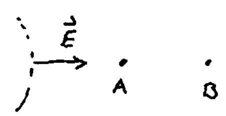
ii) $\oint \vec{E}\cdot d\vec{A} = 0 \Rightarrow E = 0$ since no charge is enclosed

iii) $\oint \vec{E}\cdot d\vec{A} = \frac{3Q}{\epsilon_0} \Rightarrow E = \frac{3Q}{4\pi\epsilon_0 r^2}$ directed radially outward.

c) Direction noted in (b)



d) Outside we consider the integral $V_B - V_A = -\int_A^B \vec{E}\cdot d\vec{r}$



$V_B - V_A = \frac{3Q}{4\pi\epsilon_0}\left[\frac{1}{r_B} - \frac{1}{r_A}\right]$ let $r_B \to \infty$

$V = \frac{3Q}{4\pi\epsilon_0 r_A}$ or $V = \frac{3Q}{4\pi\epsilon_0 r}$

Inside the shell $V =$ constant

$= \frac{3Q}{4\pi\epsilon_0 r_b}$

2. A particle of mass m is contained in a one-dimensional impenetrable box extending from x = -L/2 to x = +L/2. The particle is initially in its quantum-mechanical ground state.

    a. Find the energy eigenfunctions of the ground state **and** the first excited state. Make a sketch for each of these two eigenfunctions.

    b. The walls of the box are moved outward **instantaneously** to form a box extending from x = -L to x = +L. Sketch the new ground state and new first excited state eigenfunctions.

    c. Calculate the probability that the particle will stay in the ground state during this sudden expansion (NB this is not a perturbation problem).

    d. Calculate the probability that the particle jumps from the initial ground state (before the expansion) to the first excited final state (after the expansion).

    e. Would your answer change if the expansion occurred very slowly? Determine the probability that the particle will stay in the ground state during this slow expansion.

2. SOLUTION:

a) Starting from the one-dimensional Schrodinger equation with infinite boundary conditions, a particle confined in a box extending from x = -L/2 to x = +L/2 has eigenfunctions of the form

$$\psi_n(x) = A_n e^{ik_n x} + B_n e^{-ik_n x} \quad for \ |x| \le \frac{L}{2}$$

$$\psi_n(x) = 0 \qquad\qquad for \ |x| \ge \frac{L}{2}$$

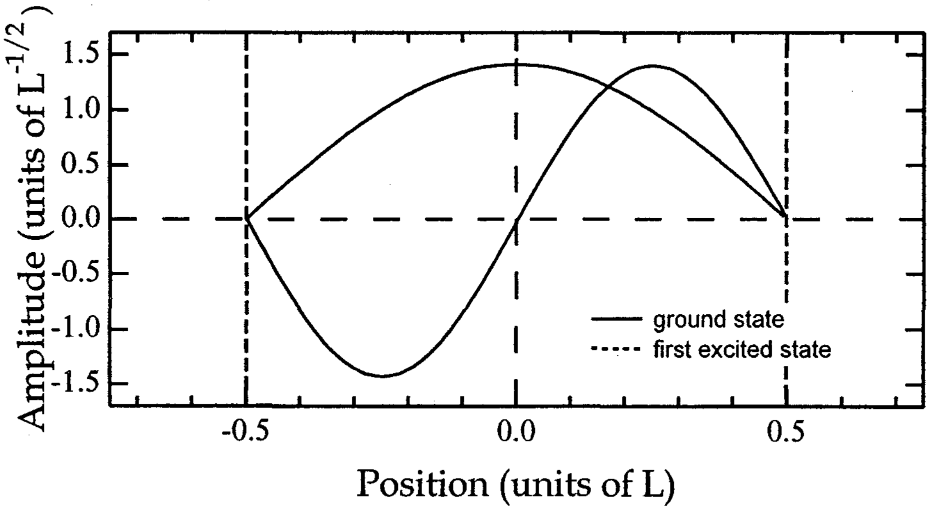
The boundary conditions at $x = \pm\frac{L}{2}$ determine that $k_n = \frac{n\pi}{L}$ and they also determine that that the two coefficients satisfy $A_n^{\ 2} = B_n^{\ 2}$. Furthermore, from the wavefunction normalization condition $\int dx \ |\psi_n(x)|^2 \ =$, the coefficients are related by $A_n^{\ 2} = B_n^{\ 2} = \frac{2}{L}$. The ground state and first excited state are therefore determined to be

$$\psi_0 = \sqrt{\frac{2}{L}} \ \cos\frac{\pi x}{L}$$

$$\psi_1 = \sqrt{\frac{2}{L}} \ \sin\frac{2\pi x}{L}$$

All this (except for the normalization constants) can be intuitively determined from what physics graduate students know about the ground state and 1st excited states and their boundary conditions.

SKETCH



b) The sketch is the same as above except the limits in the x-direction go from –1 to +1 and the limits in the y-direction are +1 and –1 using units from above.

c) In the sudden approximation, the probability that a particle starts in one initial state and goes to a final state is simply the square of the overlap integral between these states, $P_{if} = |I_{if}|^2$. The overlap integral is calculated as

$$I_{if} = \int_{-L/2}^{+L/2} \psi'_f(x)\ \psi_i(x)\ dx$$

The amplitude of the particle to remain in the ground state is given by

$$\begin{aligned}
I_{00} &= \int_{-L/2}^{+L/2} \psi'_0(x)\ \psi_0(x)\ dx \\
&= \int_{-L/2}^{+L/2} dx \left(\frac{1}{\sqrt{L}}\cos\frac{\pi x}{2L}\right)\left(\frac{\sqrt{2}}{\sqrt{L}}\cos\frac{\pi x}{L}\right) \\
&= \frac{1}{\sqrt{2}L}\int_{-L/2}^{+L/2} dx \left(\cos\frac{\pi x}{2L} + \cos\frac{3\pi x}{2L}\right) \\
&= \frac{1}{\sqrt{2}L}\ \frac{4L}{\pi}\left(\sin\frac{\pi x}{2L} + \frac{1}{3}\sin\frac{3\pi x}{2L}\right)\Bigg|_{x=L/2} \\
&= \frac{8}{3\pi}
\end{aligned}$$

and $P_{00} = \left(\frac{8}{3\pi}\right)^2$.

d) Similarly, the probability that the particle starts in the initial ground state and goes to the final 1st excited state can be determined from that overlap integral

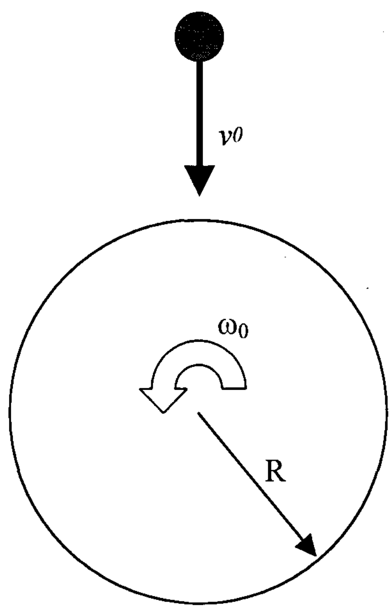
$$\begin{aligned}
I_{01} &= \int_{-L/2}^{+L/2} \psi'_1(x)\ \psi_0(x)\ dx \\
&= \int_{-L/2}^{+L/2} dx \left(\frac{1}{\sqrt{L}}\sin\frac{\pi x}{L}\right)\left(\frac{\sqrt{2}}{\sqrt{L}}\cos\frac{\pi x}{L}\right) \\
&= \frac{\sqrt{2}}{L}\int_{-L/2}^{+L/2} dx\ \sin\frac{\pi x}{L}\ \cos\frac{\pi x}{L} \quad = 0
\end{aligned}$$

because the integrand is odd parity over an even integral. Hence $P_{01} = 0$.

e) If the expansion were to occur very slowly (adiabatically), the wavefunction would stay in the ground state throughout the adiabatic expansion. The probability to stay in the ground state changes from the sudden approximation and is now equal to 1, whereas the probability to move to the first excited final state remains zero.

3. A spherical planet of radius R, mass M has a moment of inertia $I_0 = \frac{1}{10}MR^2$ (its mass distribution is spherically symmetric but not uniform). The planet is spinning at angular frequency ω_0, when it is struck at the equator by a massive asteroid, mass $m = \frac{1}{9}M$, traveling at velocity v_0, normal to the planet's surface. The collision is inelastic, and afterwards the asteroid is a point mass fixed to the planet's equator at radius R. (In spite of its great mass, the asteroid may be considered a point.)

   a. What is the velocity of the system's center of mass following the collision? Take a reference frame where the planet's center was at rest prior to the collision.
   b. How far from the planet's center is the center-of-mass of the combined system and what is the angular velocity about this center-of-mass?
   c. How much energy was dissipated (irreversibly lost) as a result of the collision?



3.

a) The momentum of the system is $p = mv_0 = \frac{1}{9}Mv_0$. The mass of the composite system is $M + m = \frac{10}{9}M$, so its velocity following collision is

$$\boxed{v = \frac{p}{M+m} = \frac{1}{10}v_0}$$

b) The center of mass of the composite system is located at

$$\mathbf{x}_{\text{com}} = \frac{M\mathbf{x}_p + m\mathbf{x}_a}{M+m} = \frac{1}{10}R\hat{\mathbf{r}}$$

where $\mathbf{x}_p = 0$ is the planet's center and $\mathbf{x}_a = R\hat{\mathbf{r}}$ is the location of the asteroid. The center-of-mass is a distance R/10 from the planet's center. Using the parallel axis theorem gives the moment of inertia of the planet, about the center-of-mass, to be

$$I_p = I_0 + M\,|\,\mathbf{x}_{\text{com}}\,|^2 = \frac{11}{100}MR^2$$

The moment of inertia of the point mass is

$$I_a = m\,|\,\mathbf{x}_{\text{com}} - R\hat{\mathbf{r}}\,|^2 = \frac{9}{100}MR^2$$

Therefore the total angular momentum of the composite system, about its center-of-mass, is $I = I_p + I_a = \frac{2}{10}MR^2$. Conservation of angular momentum requires $I_0\omega_0 = I\omega$ so the final angular velocity is

$$\boxed{\omega = \frac{I_0}{I}\omega_0 = \frac{1}{2}\omega_0}$$

c) The initial energy of the system is a combination of translational and rotational kinetic energy

$$E_0 = \tfrac{1}{2}mv_0^2 + \tfrac{1}{2}I_0\omega_0^2 = \frac{1}{18}Mv_0^2 + \frac{1}{20}MR^2\omega_0^2$$

The final energy of the system is

$$E = \tfrac{1}{2}(m+M)v^2 + \tfrac{1}{2}I\,\omega^2 = \frac{1}{180}Mv_0^2 + \frac{1}{40}MR^2\omega_0^2$$

The difference has been lost in the inelastic collision

$$\boxed{\Delta E = E_0 - E = \frac{1}{20}Mv_0^2 + \frac{1}{40}MR^2\omega_0^2}$$

4. An isolated charge q with mass m is initially at rest at coordinates $(x,y,z)=(0,0,0)$ in deep outer space. The charge is acted upon by an electromagnetic plane wave with a step wave-front, whose electric field is:

$$\mathbf{E}=\begin{cases} E_0\hat{\mathbf{y}}\cos(kz-\omega t) & , \quad z<ct \\ 0 & , \quad z>ct \end{cases}$$

   a. Ignoring back-reaction, write down the equations describing the position of the charge as a function of time.
   b. Solve the equations of motion to zeroth order in v/c.
   c. Solve the equations of motion to second order in v/c.
   d. Describe in words what would happen if back reaction were included.

④ Solution

$$\nabla \times \underset{\sim}{E} + \frac{1}{c}\frac{\partial \underset{\sim}{B}}{\partial t} = 0 \quad , \quad \omega^2 = c^2 k^2$$

$$\nabla \times \underset{\sim}{E} = E_0 \hat{x}\, k \sin(kz - \omega t) = -\frac{1}{c}\frac{\partial \underset{\sim}{B}}{\partial t}$$

$$\Rightarrow \quad \underset{\sim}{B} = \begin{cases} -E_0 \hat{x} \cos(kz - \omega t) & z < ct \\ 0 & z > ct \end{cases}$$

a)

$$\frac{d\underset{\sim}{p}}{dt} = \underset{\sim}{F} = q\left(\underset{\sim}{E} + \underset{\sim}{v}/c \times \underset{\sim}{B}\right)$$

where $\underset{\sim}{p} = \gamma m \underset{\sim}{v}$, $\gamma = \frac{1}{\sqrt{1 - v^2/c^2}} \simeq 1 + \frac{1}{2}\frac{v^2}{c^2}$

$$\therefore \quad \frac{d(\gamma \underset{\sim}{v})}{dt} = \frac{qE_0}{m}\cos(kz - \omega t)\left(\hat{y} - \underset{\sim}{v}/c \times \hat{x}\right)$$

and $\underset{\sim}{v} = \frac{d\underset{\sim}{x}}{dt}$, $\underset{\sim}{x}(0) = \underset{\sim}{0}$, $\underset{\sim}{v}(0) = \underset{\sim}{0}$

b) zeroth order in v/c

$$\Rightarrow \quad \frac{d\underset{\sim}{v}}{dt} = \frac{qE_0}{m}\hat{y}\cos(kz - \omega t)$$

$$\Rightarrow \quad \underset{\sim}{v} = \frac{qE_0}{m\omega}\sin(\omega t)\,\hat{y}$$

$$\boxed{\text{define } v_0 = \frac{qE_0}{m\omega}}$$

$$\Rightarrow \quad \underset{\sim}{x}(t) = -\frac{qE_0}{m\omega^2}\left[\cos(\omega t) - 1\right]\hat{y}$$

c) Explict EOM

$$\begin{cases} \dfrac{d(\gamma v_x)}{dt} = 0 \\[2ex] \dfrac{d(\gamma v_y)}{dt} = \dfrac{qE_0}{m}\cos(kz - \omega t)\left(1 - \dfrac{v_z(t)}{c}\right) \\[2ex] \dfrac{d(\gamma v_z)}{dt} = \dfrac{qE_0}{m}\cos(kz - \omega t)\dfrac{v_y(t)}{c} \end{cases}$$

(4)

c) cont. $\quad U_x(t) = 0 = x(t)$ to all orders in v/c

At zeroth order we had

$$U_y = U_0 \sin(\omega t), \quad y = \frac{U_0}{\omega}(1-\cos(\omega t))$$
$$U_z = 0, \quad z = 0, \quad \gamma = 1$$

Moving to first order we have

$$\frac{dU_z}{dt} = \frac{\omega U_0^2}{c}\cos(\omega t)\sin(\omega t) = \frac{\omega U_0^2}{2c}\sin(2\omega t)$$

$$\Rightarrow \quad U_z(t) = -\frac{U_0^2}{4c}\cos(2\omega t), \quad z(t) = -\frac{U_0^2}{8\omega c}\sin(2\omega t)$$

Thus $(kz-\omega t) \simeq -\left(\omega t + \frac{U_0^2}{8c^2}\sin(2\omega t)\right)$ and $\gamma \simeq 1 + \frac{U_0^2}{2c^2}\sin^2\omega t$

Next correction to $z(t)$ is order $(v/c)^3$ $\Rightarrow$ only have to consider $y(t)$

$$\frac{d(\gamma U_y)}{dt} = U_0\omega\cos\left(\omega t + \frac{U_0^2}{8c^2}\sin 2\omega t\right)\left(1 + \frac{U_0^2}{4c^2}\cos(2\omega t)\right)$$
$$= U_0\omega\left(\cos\omega t + \frac{U_0^2}{8c^2}\left(2\cos\omega t\cos 2\omega t - \sin\omega t\sin 2\omega t\right)\right)$$

$$\Rightarrow \quad \gamma U_y = U_0\left(\sin\omega t + \frac{U_0^2}{16c^2}\left(\sin(\omega t) + \sin(3\omega t)\right)\right)$$

$$\Rightarrow \quad U_y = U_0\left(\sin\omega t + \frac{U_0^2}{4c^2}\left(\sin\omega t - 3\sin^3\omega t\right)\right)$$

d) Because the electron is accelerated by the incident wave it will radiate. In other words, the electron first absorbs energy and momentum from the incident wave, then re-radiates some of this energy and momentum. The equations of motion will be those of a damped and driven oscillator. The solution will no longer be purely periodic, and secular terms will develop. In particular, the electron will aqrire a net momentum in the +z direction. You can think of this being due to radiation reaction.

5. Two processes affect the carbon-14 abundance in the upper atmosphere. These are: 1) beta-decay of ^{14}C to ^{14}N, with a half-life of 5730 years, and 2) the creation of new ^{14}C from cosmic rays via the reaction, $^{14}N + n \rightarrow {}^{14}C + p$. The rate of creation changes with time due to variations in the solar luminosity and to the strength of magnetic fields near the Earth. If the rate of creation is given by $f(t)$, then the atmospheric abundance $N(t)$ of ^{14}C is determined by the requirement that it satisfy the differential equation

$$\frac{dN}{dt} = -kN + f(t) \quad ,$$

where k is the decay constant due to beta-decay.

a. Find the general solution to the homogenous equation: $\dfrac{dN}{dt} + kN = 0$.

b. Find the Green's function for the decay operator, $d/dt + k$, by solving Green's equation for an impulsive source at time $t' > 0$,

$$\frac{dG(t,t')}{dt} + kG(t,t') = \delta(t - t') \quad ,$$

with homogenous boundary condition ($G = 0$ at $t = 0$). [Hint: $G(t,t')$ will be of the form

$$G(t,t') = \begin{cases} 0 & , \quad t < t' \\ ? & , \quad t > t' \end{cases}$$

c. Write $N(t)$ as an integral over the Green's function,

$$N(t) = \int_0^\infty G(t,t') f(t') dt' \quad ,$$

and show the solution to the particular case $f(t) = F =$ constant.

5a] $\frac{dN}{dt} + kN = 0 \Rightarrow \frac{dN}{N} = -k\,dt \Rightarrow \ln N = -kt + c$

$$\Rightarrow \underline{N = N_0 e^{-kt}}$$

b] The Green's ~~equation~~ function satisfies the homogeneous equation at all points except $t' = t$, so it is of the form

$$G(t,t') = \begin{cases} N_- e^{-kt} & t < t' \\ N_+ e^{-kt} & t > t' \end{cases}$$

If $G = 0$ at $t = 0$, then $N_- = 0$.

To find N, integrate Green's equation from $t' - \epsilon$ to $t' + \epsilon$

$$\int_{t'-\epsilon}^{t'+\epsilon} \frac{dG}{dt}\,dt + k\int_{t'-\epsilon}^{t'+\epsilon} G\,dt = \int_{t'-\epsilon}^{t'+\epsilon} \delta(t-t')\,dt = 1$$

$$\int_{t'-\epsilon}^{t'+\epsilon} G\,dt = \int_{t'-\epsilon}^{t'} G_<\,dt + \int_{t'}^{t'+\epsilon} G_>\,dt \approx \epsilon G_< + \epsilon G_> \to 0 \text{ as } \epsilon \to 0$$

$$\int_{t'-\epsilon}^{t'+\epsilon} \frac{dG}{dt}\,dt = G\Big|_{t'+\epsilon} - G\Big|_{t'-\epsilon} = 1 \Rightarrow N_+ e^{-k(t'+\epsilon)} - 0 = 1 \quad \text{as } \epsilon \to 0$$

$$\Rightarrow N_+ = e^{kt'} \Rightarrow \underline{G(t,t') = \begin{cases} 0 & t < t' \\ e^{k(t'-t)} & t > t' \end{cases}}$$

c] $N(t) = \int_0^\infty G(t,t')\,f(t')\,dt' = \underline{\int_0^t e^{k(t'-t)} f(t')\,dt'}$

$f(t') = F = \text{const} \Rightarrow N(t) = F\int_0^t e^{k(t'-t)}\,dt' = \frac{F}{k} e^{k(t'-t)}\Big|_0^t$

$$\underline{N(t) = \frac{F}{k}\left(1 - e^{-kt}\right)}$$

6.

a. Prove that the average number of photons of mode frequency v in thermal and diffusive equilibrium with the walls of a cavity at temperature T is given by the Plank distribution function:

$$<n>=\frac{1}{e^{\alpha}-1}, \text{ where } \alpha=\frac{hv}{kT}$$

b. Prove that the root-mean-square fluctuation in the number of photons described above is always greater than the average number $<n>$, and is given by:

$$\sqrt{<\Delta n^2>}=e^{\alpha/2}<n>, \text{ where } <\Delta n^2>=<(n-<n>)^2>$$

(Ignore the zero point energy of photon.)

6. Solution:

a. $\varepsilon_n = nh\nu,\ where\ n = 0,1,2,..\ \ Partition\ function: z = \sum_{n=0}^{\infty} e^{-n\alpha} = \frac{1}{1-e^{-\alpha}}$. The probability that the photon is in state n is given by the Boltzmann probability factor: $P_n = \frac{e^{-n\alpha}}{z}$. Therefore the average number of photons in this mode can easily be obtained by $<n> = \sum_{n=0}^{\infty} nP_n = -\frac{\partial \ln z}{\partial \alpha} = -\frac{\partial}{\partial \alpha} ln\left(\frac{1}{1-e^{-\alpha}}\right) = \frac{1}{e^{\alpha}-1}$. This is known as the Plank distribution function, which can be obtained (as expected) from the Bose-Einstein distribution by setting the photon chemical potential $\mu = 0$.

b. The mean-square average is defined as: $<\Delta n^2> = <(n-<n>)^2> = <n^2> - <n>^2$, which can be written as:

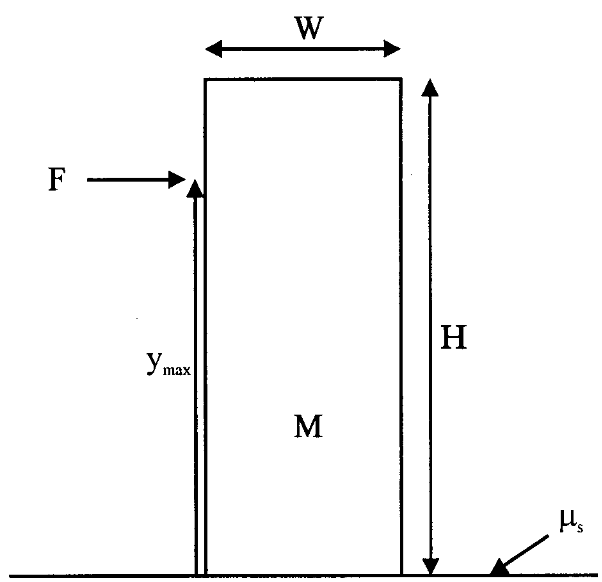
$$<\Delta n^2> = \sum n^2 P_n - (\sum nP_n)^2 = \frac{\frac{\partial^2 z}{\partial \alpha^2}}{z} - \frac{\left(\frac{\partial z}{\partial \alpha}\right)^2}{z^2} = \frac{z\frac{\partial^2 z}{\partial \alpha^2} - \left(\frac{\partial z}{\partial \alpha}\right)^2}{z^2} = \frac{\partial}{\partial \alpha}\left(\frac{1}{z}\frac{\partial z}{\partial \alpha}\right) = -\frac{\partial <n>}{\partial \alpha}$$

This immediately yields the desired result:

$$\sqrt{<\Delta n^2>} = \sqrt{-\frac{\partial <n>}{\partial \alpha}} = \sqrt{\frac{e^{\alpha}}{\left(e^{\alpha}-1\right)^2}} = e^{\alpha/2} <n>$$

Notice that this result is radically different from the fluctuations in the number of, say, gas particles in a volume in thermal and diffusive equilibrium with a reservoir (assume an imaginary volume of *1cm³* of a room is in equilibrium with the rest of the room). In this case we can easily show that $\sqrt{<\Delta n^2>} = <n>^{1/2}\ (= <n>)$, which is an extremely small fraction of the average number of particles in that volume.

7. You want to push a refrigerator across the floor. The refrigerator has a height H, a width W and a mass M. Let the coefficient of static friction between the refrigerator and the floor be μ_s.



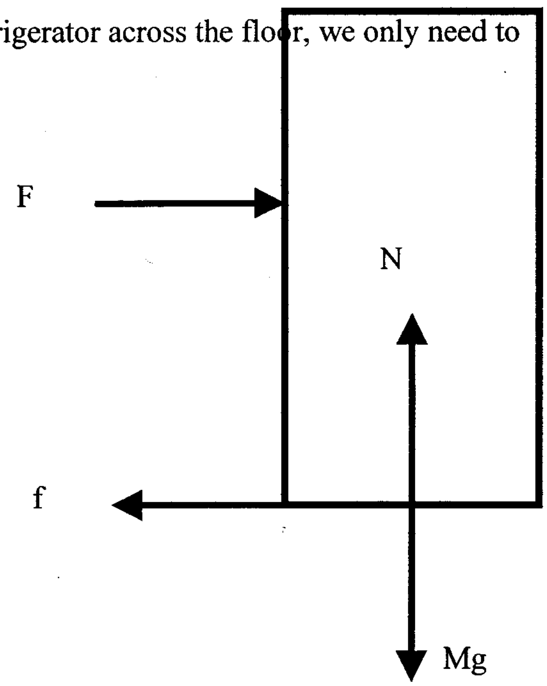
a. How much force F is needed to start to slide the refrigerator across the floor?

b. What is the maximum height y_{max} that you can push on the refrigerator without the refrigerator tipping over?

7. Solution

A. To find the force F needed to slide the refrigerator across the floor, we only need to overcome the force of static friction f.

Thus we only need $F > f$



$$f = \mu N$$
$$= \mu Mg$$

Thus for the refrigerator to slide, we at least need to have:

$$F > \mu Mg$$

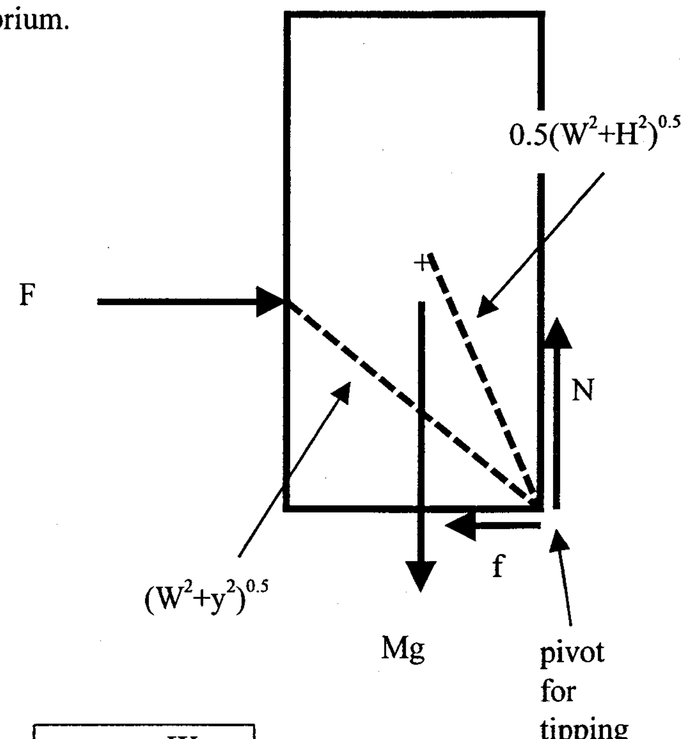
B. Now to find the maximum height that we can push on the refrigerator without tipping, we use Newton's second law for rotational equilibrium.

$\sum \tau = 0$ just before tipping.

$$\tau = rF\sin\theta$$

$$\tau_1 = +\tfrac{1}{2}\left(W^2 + H^2\right)^{0.5} Mg \frac{W}{\left(W^2 + H^2\right)^{0.5}}$$
$$= +\frac{MgW}{2}$$

$$\tau_2 = -\left(W^2 + y^2\right)^{0.5} F \frac{y}{\left(W^2 + y^2\right)^{0.5}}$$
$$= -Fy$$



Thus

$$Fy = \frac{MgW}{2}$$
$$y_{MAX} = \frac{MgW}{2F} = \frac{MgW}{2\mu Mg} = \frac{W}{2\mu}$$

$$\boxed{y_{MAX} = \frac{W}{2\mu}}$$

8. An electron is in a uniform magnetic field $\mathbf{B} = B\hat{\mathbf{z}}$, which interacts with the electron's spin through the Hamiltonian

$$\hat{H} = -\mu B\hat{S}_z$$

where $\hat{S}_z$ is the spin operator and μ is the Bohr magneton. The eigenstates of $\hat{S}_z$, denoted $|+\rangle$ and $|-\rangle$, have eigenvalues $+\frac{1}{2}$ and $-\frac{1}{2}$ respectively. Experimental equipment is in place to measure the x-component of the electron's spin, an observable described by the operator

$$\hat{S}_x = \frac{1}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

in the $\{|+\rangle, |-\rangle\}$ basis. At time t=0 the electron is found to be in the $s_x = +\frac{1}{2}$ state by the experiment.

a. Express the state of the electron at time t=0 in the $\{|+\rangle, |-\rangle\}$ basis.

b. The experiment is repeated at later time t. What is the probability that the electron is found to be in the $s_x = +\frac{1}{2}$ state this time?

a) The eigenvalues and eigenvectors of $\hat{S}_x$ can be found directly by proposing

$$\hat{S}_x|\psi\rangle = \frac{1}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} \psi_+ \\ \psi_- \end{bmatrix} = s_x|\psi\rangle$$

The solutions are

$$s_x = +\tfrac{1}{2} \quad \text{for} \quad |\psi\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}}\left(|+\rangle + |-\rangle\right)$$

$$s_x = -\tfrac{1}{2} \quad \text{for} \quad |\psi\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \frac{1}{\sqrt{2}}\left(|+\rangle - |-\rangle\right)$$

Since the initial measurement found the electron to be in the $s_x = +\frac{1}{2}$ state, it's state is

$$|\psi(0)\rangle = \frac{1}{\sqrt{2}}\left(|+\rangle + |-\rangle\right)$$

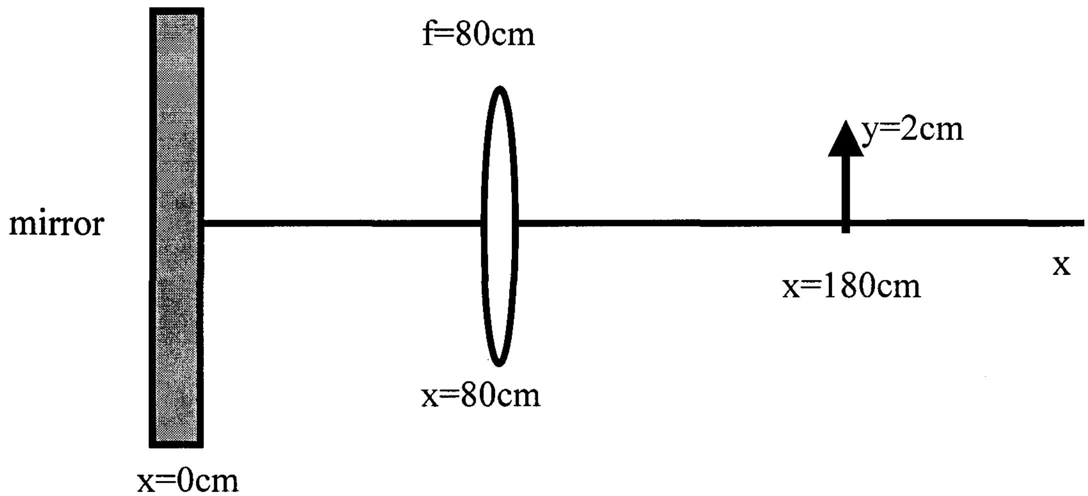
b) The wave function evolves according to the Hamiltonian, whose eigenvalues can be written $E_\pm = \mp\frac{1}{2}\mu B = \mp\hbar\omega$. The time-dependant wave function is therefore

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}}\left(e^{-i\omega t}|+\rangle + e^{+i\omega t}|-\rangle\right)$$

The probability of observing this wave function in state $|s_x = +\frac{1}{2}\rangle$ is given by

$$\left|\langle s_x = +\tfrac{1}{2}|\psi(t)\rangle\right|^2 = \left|\tfrac{1}{2}(e^{-i\omega t} + e^{i\omega t})\right|^2 = \cos^2\left(\frac{\mu B t}{2\hbar}\right)$$

9. A plane mirror is located at the origin as shown. A converging lens of focal length 80cm is located 80cm from the mirror. An object of height 2cm is located at x=180cm (i.e. 100cm from the lens).



a. Find the location of the image formed by the lens-mirror combination.

b. Determine the height of the image formed and also state whether the image is vertical or inverted.

9. Solution

A. This is equivalent to a two-lens problem. To solve for the location of an image we use the thin lens equation:

$$\frac{1}{d_o}+\frac{1}{d_i}=\frac{1}{f}$$

$$\frac{1}{d_i}=\frac{1}{f}-\frac{1}{d_o}=\frac{1}{80}-\frac{1}{100}=\frac{1}{400}$$

Thus the first image is located 400cm past the first lens. Since there is 160cm between the "two lenses" when the reflection off the plane mirror is taken into account, the object distance for the "second lens" will be 240cm past the location of the second lens. Thus $d_o=-240$. Now we can apply the thin lens equation for the second lens:

$$\frac{1}{d_o}+\frac{1}{d_i}=\frac{1}{f}$$

$$\frac{1}{d_i}=\frac{1}{f}-\frac{1}{d_o}=\frac{1}{80}-\frac{1}{-240}=\frac{1}{60}$$

Thus the image is located 60 cm past the second lens or at x=140cm in the coordinate shown in the original figure.

B. To find the image size and orientation, we use the magnification, which is given by:

$$M=-\frac{d_i}{d_o}$$

Thus for the first image:

$$M1=-\frac{400}{100}=-4$$

Thus the first image is inverted and magnified by 4.

For the second image:

$$M2=-\frac{60}{-240}=+\frac{1}{4}$$

10. The operation of a gasoline engine is roughly similar to the Otto cycle as shown in the figure below where S represents the entropy and V represents the piston's volume. Assume that gas in the piston behaves as an ideal gas with an adiabatic exponent γ (as in $PV^{\gamma} = const.$) and has a temperature-independent heat capacity $C_V = \left(\frac{\partial E}{\partial T}\right)_{V,N}$.

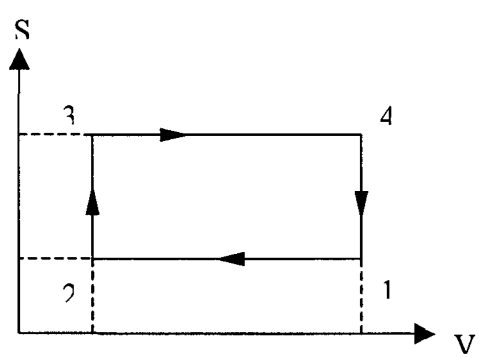
The engine operates between a hot reservoir at temperature T_3 and a cold reservoir at temperature T_1 corresponding to points 1 and 3 in the figure below.

a. Determine the entropy **entering** into the engine between points 2 and 3 and the entropy **leaving** the engine between points 4 and 1. Is the net change in entropy **of the engine** zero after a complete cycle? Please explain.

b. Determine the change in the entropy **of the universe** per cycle and determine whether this change is positive, negative, or zero.

c. Qualitatively describe what causes a reduction in the efficiency of a general heat engine, and comment on why the efficiency of the Otto cycle is expected to be less than that of the Carnot cycle, which is known to be the most efficient heat engine possible.

(Hints: Efficiency is defined as work (W) done per unit of heat (Q_{in}) received by the engine: $e = \frac{W}{Q_{in}}$; $x + \frac{1}{x} \geq 2$ for any real positive x)



10. Solution:

a. The first law of thermodynamics: $dE = \delta Q + \delta W = TdS - pdV$ (reversible heat engine). Along path 2-3 $dV=0$ hence $dE = TdS$. On the other hand from the definition $C_V = \left(\frac{\partial E}{\partial T}\right)_{V,N}$ we obtain $dE = C_V dT$. This immediately yields:

$dS = C_V \frac{dT}{T} \;\rightarrow\; \Delta S_{in} = C_V \ln\frac{T_3}{T_2}$ (entropy gain). Similarly entropy loss during path 4-1 is given by: $dS = C_V \frac{dT}{T} \;\rightarrow\; \Delta S_{out} = C_V \ln\frac{T_1}{T_4}$. The processes along path 3-4 and 1-2 are adiabatic, i.e. entropy is constant. This means along path 3-4 we have: $T_3V_3^{\gamma-1} = T_4V_4^{\gamma-1}$ and along 1-2 we have: $T_2V_3^{\gamma-1} = T_1V_4^{\gamma-1}$. Comparing (dividing) these two relations we get $\frac{T_3}{T_2} = \frac{T_4}{T_1}$. Using this equality in the net change in entropy of engine per cycle yields: $\Delta S_{net} = \Delta S_{in} + \Delta S_{out} = 0$. This is not surprising because entropy is a state function and it is only a function of end points. Therefore, a cycle returns the system to its original state hence engine neither gains nor loses entropy. If engine starts gaining entropy this means your engine is in need of immediate attention; otherwise it will burn out in a short time. Make sure you have enough coolant and oil in your car to prevent this from happening.

b. Universe exchanges entropy with the engine via hot- and cold-reservoirs. The entropy loss by the universe is: $\Delta S_{loss} = \frac{Q_{in}}{T_3}$, while entropy gain by the universe is $\Delta S_{gain} = \frac{Q_{out}}{T_1}$. The heat-in and heat-out can easily be calculated using the 1st law along paths 2-3 and 4-1, respectively: $dE = dQ = C_V dT \;\rightarrow\; Q = C_V \Delta T$. This immediately yields: $\Delta S_{loss} = \frac{Q_{in}}{T_3} = \frac{C_V(T_3 - T_2)}{T_3}$, while $\Delta S_{gain} = \frac{Q_{out}}{T_1} = \frac{C_V(T_4 - T_1)}{T_1}$. Here all quantities are defined as positive. The net change in the entropy of the universe is given by $\Delta S_{uni.} = \Delta S_{gain} - \Delta S_{loss} = C_V(\frac{T_4}{T_1} + \frac{T_2}{T_3} - 2) \geq 0$. The reason $\Delta S_{uni.}$ is greater than zero because $\frac{T_3}{T_2} = \frac{T_4}{T_1}$ and $x + \frac{1}{x} \geq 2$, where $x = \frac{T_4}{T_1}$. Net result is that the universe gains entropy at each cycle while engine's entropy remains constant. Therefore, an engine working in Otto cycle plus the universe is *not* a reversible system. Carnot cycle plus the universe is the only heat engine produces no change in the entropy of the universe during its operation and hence is the most efficient as explained below.

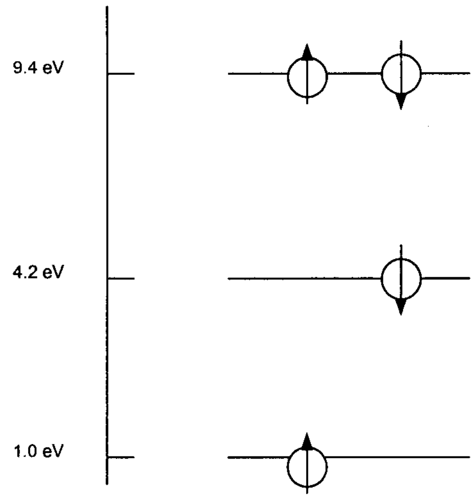
c. The most important factor that effects the efficiency of a heat engine is the amount of heat used to throw away the entropy from the engine during the exhaust cycle. The Carnot cycle puts out as much entropy into the cold reservoir as it receives from the hot reservoir. Hence the entropy of the universe remains unchanged, i.e. it is a reversible engine. However, as proved in part (b) Otto cycle is not a reversible engine and exhausts into the cold reservoir more entropy than it receives from the hot reservoir. This means engine spends extra heat to clear the entropy from the engine during the exhaust cycle. Hence system uses extra energy to remove entropy instead of doing work. This makes Otto engine slightly less efficient than the Carnot engine. Though not required the mathematical proof of this fact is given below:

Mathematical Proof (not required but could have been): As explained in part (b) Otto cycle plus the universe is *not* a reversible system. This means the efficiency $e = \frac{W}{Q_{in}} = \frac{Q_{in} - Q_{out}}{Q_{in}} = 1 - \frac{Q_{out}}{Q_{in}}$ will be smaller than the Carnot efficiency $(= 1 - \frac{T_1}{T_3})$ because the Otto cycle uses more Q_{out} relative to the Q_{in} than does the Carnot cycle for throwing out the entropy generated during the heat intake cycle (path 2-3). This is easily proven as follows: $e = 1 - \frac{Q_{out}}{Q_{in}} \leq 1 - \frac{T_1}{T_3}$. Using the values for Q_{in} and Q_{out} from part (b) above we get $e = 1 - \frac{T_4 - T_1}{T_3 - T_2} \leq 1 - \frac{T_1}{T_3}$ or

$\frac{T_4 - T_1}{T_3 - T_2} \geq \frac{T_1}{T_3} \rightarrow \frac{T_4 - T_1}{T_3 - T_2} - \frac{T_1}{T_3} \geq 0$. This immediately can be converted to

$\frac{\frac{T_4}{T_1} - 1}{1 - \frac{T_2}{T_3}} - 1 \geq 0$, and this is equivalent to $x + \frac{1}{x} - 2 \geq 0$, where $x = \frac{T_4}{T_1} = \frac{T_3}{T_2}$.

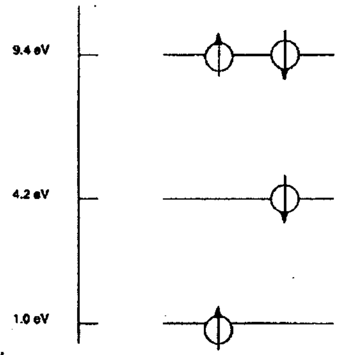
11. Answer the following
    a. What are the energies of the photons that would be emitted when the four-electron system in the figure below returns to its ground state (sketch the various transitions)?



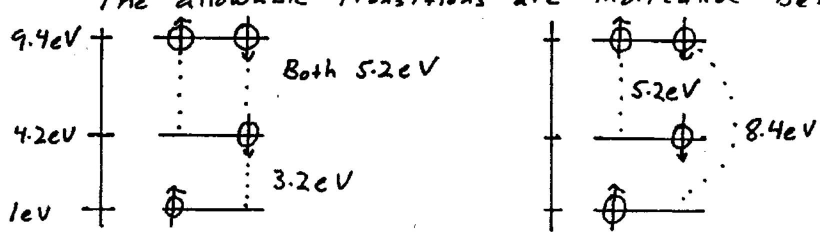
    b. Once the system is in its ground state, a magnetic field of magnitude 2 Tesla is applied. Explain how this affects the energy levels by drawing a new energy diagram and providing any necessary calculations.

11.

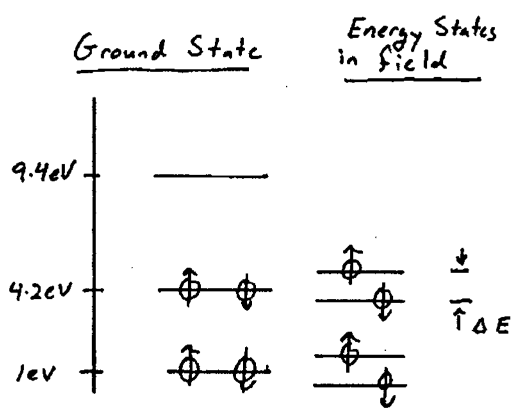
(a) What are the energies of the photons that would be emitted when the four-electron system in the figure below returns to its ground state (sketch the various transitions)?



The allowable transitions are indicated below...



(b) Once the system is in its ground state, a magnetic field of magnitude 2 tesla is applied. Explain how this affects the energy levels by drawing a new energy diagram and providing any necessary calculations.



The magnetic field leads to a splitting of the energy levels as indicated. The new energy levels are given by...

$$E(B) = E_i \pm \frac{e\hbar}{2m} B,$$

where e is the electron charge, m its mass, and $\hbar$ is the Planck constant over 2π.

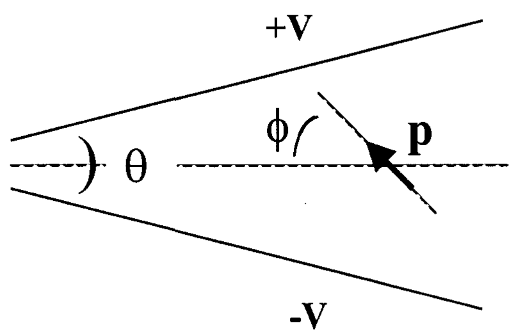
$$\frac{e\hbar}{2m} B = 5.8 \times 10^{-5}\, eV$$

so $\Delta E = 1.2 \times 10^{-4}\, eV$

12. An electric dipole of moment **p** (which can be described as two point charges +q and –q separated by some small distance d) is placed at an arbitrary angle, ϕ, and equidistant between two semi-infinite plates that make an angle θ with each other (see figure).

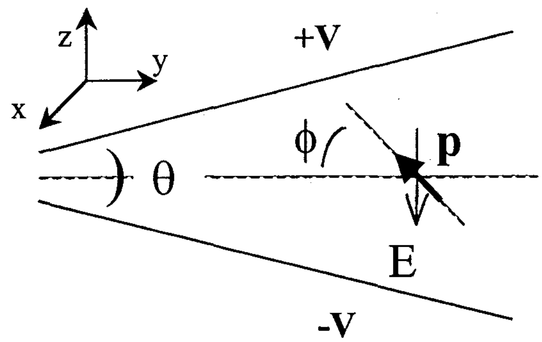
   a. Describe the initial behavior of the dipole for the situation where the two plates have a potential of +V and –V, respectively (as shown in the figure). You do not need to solve for its behavior.

   b. Describe the steady state or long time behavior of the dipole.



12. SOLUTION:

a) From the symmetry of the problem, we can see that the lines of equipotential are straight lines that intercept at what would be the vertex of the two plates. Therefore the electric field, $\vec{E} = -\vec{\nabla} V$, must be radial arcs centered at the same vertex. At the dipole position (see figure), the electric field looks, to first order, as a field pointed in the negative z-direction (downward).



The dipole has no monopole term, so there is no force acting on the dipole due to the constant electric field (there is a force due to the gradient of the electric field), but there is a torque. The torque is simply $\vec{N} = \vec{p} \times \vec{E} = |p||E|\cos\phi$, directed in the plus x-direction (out of the paper). (We have the cos ϕ term because the angle between **p** and **E** is the complementary angle to ϕ.) In addition, as mentioned above, the gradient of the electric field produces a net force given by $\vec{F} = (\vec{p} \bullet \vec{\nabla})\vec{E} = |p||\vec{\nabla} E_y| \sin\phi$, directed in the +y-direction (to the right). Therefore, the initial motion is a rotation of the dipole to align with the applied field and motion of the dipole to the right.

b) The initial motion of the dipole is to align with the applied field. When the dipole is aligned, both the torque and the electric field gradient force are zero. In the absence of loss mechanisms, the rotational energy of the dipole will move the dipole beyond alignment with the field direction, generating a torque opposite the original torque AND an electric field gradient force opposite the original force (a restoring force). The dipole will therefore oscillate back and forth both in its position along the y-axis and its orientation with respect to the E-field direction. If loss mechanisms are present (damping), the final configuration will be the dipole pointed along the electric field direction (minus z-direction) with no net force acting on the dipole. In addition, eddy current damping within the plates due to the moving charge will inevitably stop the dipole, creating a static dipole with no net velocity.

13. A pendulum is constructed from a point mass m that is attached to a massless string of length L, both of which rest on a frictionless inclined plane. The other end of the string is attached to the inclined plane. The plane makes an angle α with the horizontal, and there is a uniform gravitational field g that acts in the downward vertical direction.

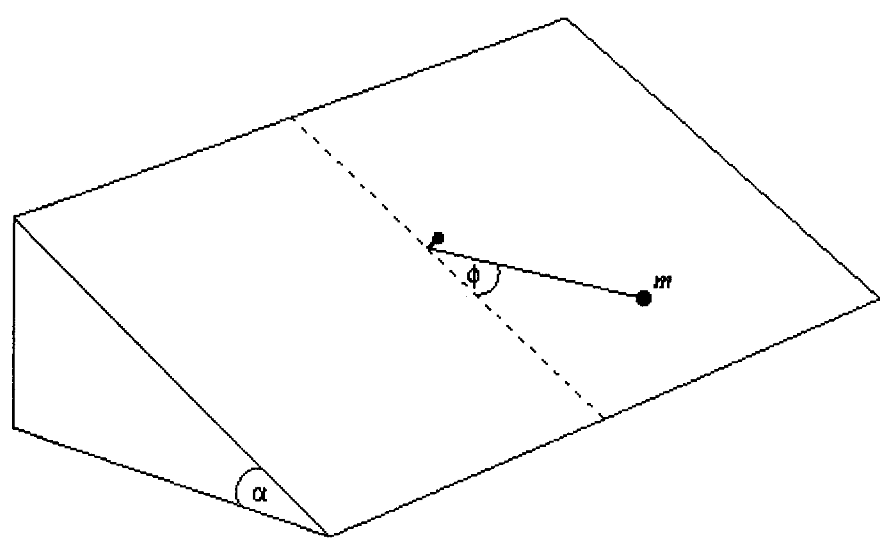
a. Set up a Lagrangian for the system using an appropriate set of generalized coordinates and show that the tension in the string is equal to

$$\tau = mL\dot{\phi}^2 + mg\cos\phi\sin\alpha$$

b. Find the equations of motion and any constants of the motion. Solve the equations of motion for small oscillations about the stable equilibrium point and find the angular frequency, ω, of the oscillations.

c. Calculate the quantity $\Delta E = E - E_0$ for small oscillations, where E is the energy and E_0 is the energy of the system at equilibrium. Express your answer in terms of ω and the amplitude of the small oscillations, A.

d. If the inclination angle α is changed on a time-scale long compared to $1/\omega$, the system has an adiabatic invariant $J = \oint p_\phi \, d\phi$, where ϕ is the generalized coordinate that best describes the oscillations p_ϕ is its conjugate momentum, and the integral is taken over one complete oscillation. Calculate J and use it to show that $\Delta E/\omega$ remains constant when the inclination of the plane is changed slowly. How does the amplitude of the oscillation vary with ω?



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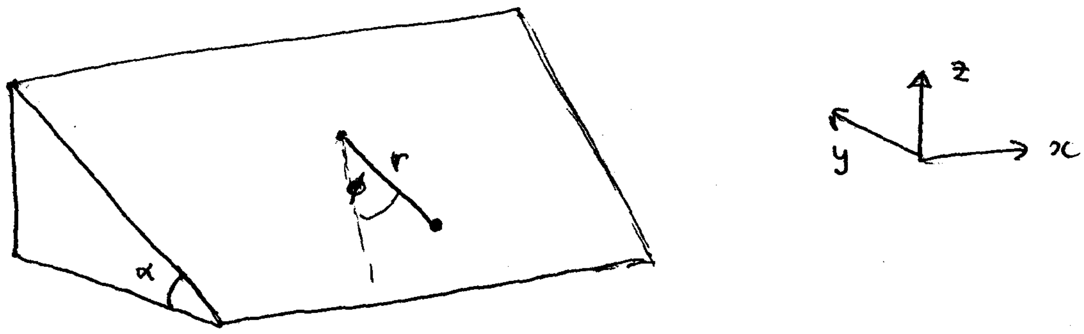
Solution

a) + b)



$x = r\sin\phi$

$y = r\cos\phi\cos\alpha$

$z = r\cos\phi\sin\alpha$

constraint $r = L$ enforced by $f = r - L = 0$

$V = -mgz$

$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2)$

$L = T - V = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) + mgr\cos\phi\sin\alpha$

Euler - Lagrange

$$\begin{cases} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = \lambda\frac{\partial f}{\partial r} = \lambda \\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) - \frac{\partial L}{\partial \phi} = 0 \end{cases}$$

EOM
$$\begin{cases} r = L \\ m\ddot{r} - mr\dot{\phi}^2 - mg\cos\phi\sin\alpha = \lambda \\ m\frac{d}{dt}(r^2\dot{\phi}) + mgr\sin\phi\sin\alpha = 0 \end{cases}$$

Tension, $\tau = -\lambda = mL\dot{\phi}^2 + mg\cos\phi\sin\alpha$

$\frac{\partial L}{\partial t} = 0 \Rightarrow H = \frac{1}{2}mL^2\dot{\phi}^2 - mgL\cos\phi\sin\alpha = E$

$= \text{constant}.$

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b) cont. Equations of motion boil down to

$$\ddot{\phi} = -\frac{g}{L}\sin\phi\sin\alpha$$

Equilibrium at $\phi = 0$ and $\phi = \pi$. Former is stable

Small oscillations $\ddot{\phi} = -\left(\frac{g}{L}\sin\alpha\right)\phi$

$$\Rightarrow \quad \phi = A\cos\omega t \quad , \quad \omega^2 = \frac{g}{L}\sin\alpha$$

c) $E_0 = -mgL\sin\alpha$

$$\Delta E = E - E_0 = \tfrac{1}{2}mL^2\dot{\phi}^2 + mgL\sin\alpha(1-\cos\phi)$$

Small oscillations, $\phi, \dot{\phi}$ small

$$\Rightarrow \quad \Delta E \simeq \tfrac{1}{2}mL^2\dot{\phi}^2 + \tfrac{1}{2}mgL\sin\alpha\,\phi^2$$

$$= \tfrac{1}{2}mL^2A^2\omega^2\sin^2\omega t + \tfrac{1}{2}mgLA^2\cos^2\omega t\sin\alpha$$

$$= \tfrac{1}{2}mgLA^2\sin\alpha = \text{const.} \checkmark$$

$$= \tfrac{1}{2}mL^2\omega^2A^2$$

d) $J = \oint p_\phi\, d\phi \qquad p_\phi = \frac{\partial L}{\partial\dot{\phi}} = mL^2\dot{\phi}$

$$\Rightarrow \quad J = 4\int_0^A mL^2\dot{\phi}\, d\phi$$

$$= -4\omega mL^2\int_0^A \sqrt{A^2-\phi^2}\, d\phi$$

$$= -4mL^2\omega A^2 \quad \Rightarrow \quad A \propto \frac{1}{\sqrt{\omega}}$$

$$\frac{\Delta E}{\omega} = \tfrac{1}{2}mL^2\omega A^2 = -\frac{J}{8} = \text{const. for adiabatic changes.}$$

14. The electric and magnetic fields, **E** and **B**, may be written in terms of a scalar potential Φ and a vector potential **A** as

$$\mathbf{E} = -\nabla\Phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}$$
$$\mathbf{B} = \nabla \times \mathbf{A}$$

a) Show what guage freedom exists in the definitions of **A** and Φ by showing how they may be modified in some manner determined by a scalar guage function Λ without changing the fields.

b) Find the equation that a guage function Λ must satisfy if one wishes to choose the guage so that the Lorentz condition

$$\nabla \cdot \mathbf{A} + \frac{1}{c}\frac{\partial \Phi}{\partial t} = 0$$

is satisfied.

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a) $\vec{A} \to \vec{A}' = \underline{\vec{A} + \vec{\nabla}\Lambda}$ gives a $\vec{B}$ field that is unchanged, since

$$\vec{B}' = \vec{\nabla}\times(\vec{A}+\vec{\nabla}\Lambda) = \vec{\nabla}\times\vec{A} + \cancel{\vec{\nabla}\times(\vec{\nabla}\Lambda)}^{\,0} = \vec{\nabla}\times\vec{A} = \vec{B}$$

The $\vec{E}$ field would then be

$$\vec{E}' = -\vec{\nabla}\phi' - \frac{1}{c}\frac{\partial \vec{A}'}{\partial t} = -\vec{\nabla}\phi' - \frac{1}{c}\frac{\partial \vec{A}}{\partial t} - \frac{1}{c}\frac{\partial}{\partial t}(\vec{\nabla}\Lambda)$$

The $\vec{E}$-field is therefore unchanged if $\underline{\phi' = \phi - \frac{1}{c}\frac{\partial \Lambda}{\partial t}}$

for then $$\vec{E}' = -\vec{\nabla}\phi + \cancel{\frac{1}{c}\nabla\frac{\partial\Lambda}{\partial t}} - \frac{1}{c}\frac{\partial\vec{A}}{\partial t} - \cancel{\frac{1}{c}\frac{\partial}{\partial t}\vec{\nabla}\Lambda} = \vec{E}$$

b) Suppose $\vec{\nabla}\cdot\vec{A} + \frac{1}{c}\frac{\partial\phi}{\partial t} \neq 0$. Then define $\begin{cases} \vec{A}' = \vec{A} + \vec{\nabla}\Lambda \\ \phi' = \phi - \frac{1}{c}\frac{\partial\Lambda}{\partial t} \end{cases}$

and require $\vec{\nabla}\cdot\vec{A}' + \frac{1}{c}\frac{\partial\phi'}{\partial t} = 0$

$$\Rightarrow \quad \vec{\nabla}\cdot\vec{A} + \nabla^2\Lambda + \frac{1}{c}\frac{\partial\phi}{\partial t} - \frac{1}{c^2}\frac{\partial^2\Lambda}{\partial t^2} = 0$$

which is satisfied if Λ satisfies

$$\underline{\nabla^2\Lambda - \frac{1}{c^2}\frac{\partial^2\Lambda}{\partial t^2} = -\vec{\nabla}\cdot\vec{A} - \frac{1}{c}\frac{\partial\phi}{\partial t}}$$

15. A spherical conducting spacecraft of radius R and mass M is bathed in monochromatic light of wavelength λ, sufficiently short to cause electron emission by the photoelectric effect. Photoelectric emission occurs when the energy of a photon exceeds the work-function E_w of the conductor. As it leaves the surface, the emitted electron carries away kinetic energy equal to the difference between the photon energy and the work function. The spacecraft acquires a net charge as a result of this emission. When the spacecraft is fully charged no emitted electron can reach infinity. **Express the net charge of the spacecraft in terms of those factors given above and other physical constants.**

15.

Photons of wavelength λ have energy $E_\nu = \dfrac{hc}{\lambda}$, where h is Planck's constant. The kinetic energy of the emitted electron is therefore $E_e = E_\nu - E_w$. When fully charged the spacecraft will be at a positive potential $V_0 = E_e / e$ relative to infinity, in order that no emitted electron escape (e is the magnitude of the electron charge). The electrostatic potential outside a spherical conductor of radius R is

$$V(r) = \frac{V_0 R}{r}$$

making the electric field

$$\mathbf{E}(r) = -\nabla V = \frac{V_0 R}{r^2}\hat{\mathbf{r}}$$

The net charge on the spacecraft is therefore

$$Q_{\text{s/c}} = \varepsilon_0 \oint \mathbf{E} \cdot d\mathbf{a} = 4\pi\varepsilon_0 V_0 R = \frac{4\pi\varepsilon_0 R}{e}\left(\frac{hc}{\lambda} - E_w\right)$$