Dirac Equation: Non-relativistic Reduction.

The appearance of an intrinsic spin in the Dirac Eqtn is confirmed by looking at its nonrelativistic limit, as $c\to\infty$. We shall now show that in this limit the Dirac Eqtn reduces to the Schrödinger Eqtn plus a "familiar" spin interaction term.

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1) Start from the Dirac Eqtn for particle (q,m) in an external field $(\mathbb{A}, i\phi)$:

$$\rightarrow i\hbar\,\partial\psi/\partial t = \left[\beta mc^2 + q\phi + c\alpha\cdot\left(\mathbb{p}-\tfrac{q}{c}\mathbb{A}\right)\right]\psi. \tag{1}$$

Set $\psi = \begin{pmatrix}\varphi\\ \chi\end{pmatrix}$. Assume ψ is a 4-spinor of energy E: $i\hbar\frac{\partial}{\partial t}\begin{pmatrix}\varphi\\ \chi\end{pmatrix} = E\begin{pmatrix}\varphi\\ \chi\end{pmatrix}$.

then// Eq. (1) $\Rightarrow$ $E\begin{pmatrix}\varphi\\ \chi\end{pmatrix} = \begin{pmatrix} q\phi+mc^2 & c\sigma\cdot(\mathbb{p}-\frac{q}{c}\mathbb{A})\\ c\sigma\cdot(\mathbb{p}-\frac{q}{c}\mathbb{A}) & q\phi-mc^2\end{pmatrix}\begin{pmatrix}\varphi\\ \chi\end{pmatrix}$,

or//

$$\boxed{\begin{aligned}(E-mc^2-q\phi)\varphi &= c\sigma\cdot(\mathbb{p}-\tfrac{q}{c}\mathbb{A})\chi\\ (E+mc^2-q\phi)\chi &= c\sigma\cdot(\mathbb{p}-\tfrac{q}{c}\mathbb{A})\varphi\end{aligned}} \tag{2}$$

This is still exact. Set $E=mc^2+\epsilon$, w/ ϵ = usual nonrelativistic eigen-energy. Solve for χ from 2nd eqtn and substitute into 1st eqtn to get a quasi-relativistic system...

$$\chi = \frac{1}{2mc}\left[\frac{\sigma\cdot(\mathbb{p}-\frac{q}{c}\mathbb{A})}{1+\left(\frac{\epsilon-q\phi}{2mc^2}\right)}\right]\varphi \sim \left[\mathcal{O}\left(\frac{v}{c}\right)\right]\cdot\varphi, \text{ for } \begin{matrix}v\ll c\\ (\epsilon\ll mc)\end{matrix} \tag{2A}$$

χ is the "small" comp^{t} of the Dirac ψ; φ is the "large" component;

$$(\epsilon-q\phi)\varphi = \frac{1}{2m}\left[\sigma\cdot(\mathbb{p}-\tfrac{q}{c}\mathbb{A})\right]\left\{\frac{1}{1+(\epsilon-q\phi)/2mc^2}\right\}\left[\sigma\cdot(\mathbb{p}-\tfrac{q}{c}\mathbb{A})\right]\varphi$$

$$\simeq \frac{1}{2m}\left[\sigma\cdot(\mathbb{p}-\tfrac{q}{c}\mathbb{A})\right]\left\{1-\left(\frac{\epsilon-q\phi}{2mc^2}\right)\right\}\left[\sigma\cdot(\mathbb{p}-\tfrac{q}{c}\mathbb{A})\right]\varphi. \tag{2B}$$

For $c\to\infty$ (i.e. $\epsilon\ll mc^2$), the correction term in $\{\ \}$ of (2B) is negligible, and:

$$(\epsilon-q\phi)\varphi = \frac{1}{2m}\left[\sigma\cdot(\mathbb{p}-\tfrac{q}{c}\mathbb{A})\right]^2\varphi, \text{ neglecting } \mathcal{O}\left(\frac{v}{c}\right)^2 \text{ corrections.} \tag{3}$$

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2) Now work on (3). By use of the Dirac Identity:

$$\rightarrow\left[\sigma\cdot(\mathbb{p}-\tfrac{q}{c}\mathbb{A})\right]^2 = (\mathbb{p}-\tfrac{q}{c}\mathbb{A})^2 + i\sigma\cdot\left[(\mathbb{p}-\tfrac{q}{c}\mathbb{A})\times(\mathbb{p}-\tfrac{q}{c}\mathbb{A})\right]. \tag{4A}$$

(next page)

But: $(\mathbb{p}-\frac{q}{c}\mathbb{A})\times(\mathbb{p}-\frac{q}{c}\mathbb{A}) = \underbrace{\mathbb{p}\times\mathbb{p}}_{0} - \frac{q}{c}(\mathbb{p}\times\mathbb{A}+\mathbb{A}\times\mathbb{p}) + \frac{q^2}{c^2}\underbrace{\mathbb{A}\times\mathbb{A}}_{0,\text{ obviously}}$ (4B)

because $(\mathbb{p}\times\mathbb{p})f \propto \nabla\times(\nabla f)\equiv 0$, for all scalar fcns f.

And: $(\mathbb{p}\times\mathbb{A})f = -i\hbar\nabla\times(\mathbb{A}f) = -i\hbar[(\nabla\times\mathbb{A})f - \mathbb{A}\times(\nabla f)]$

$= -i\hbar\,\mathbb{B}f - (\mathbb{A}\times\mathbb{p})f$, w/ $\mathbb{B}=\nabla\times\mathbb{A}$ the extl. mag. fld. (4C)

So: $(\mathbb{p}\times\mathbb{A}+\mathbb{A}\times\mathbb{p})f = -i\hbar\,\mathbb{B}f$. ← Use this in (4B), in operator sense. (4D)

Thus: $\frac{1}{2m}[\sigma\cdot(\mathbb{p}-\frac{q}{c}\mathbb{A})]^2 = \frac{1}{2m}(\mathbb{p}-\frac{q}{c}\mathbb{A})^2 - \frac{q\hbar}{2mc}\sigma\cdot\mathbb{B}$. (4E)

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3) Use of (4E) in (3) shows that as $c\to\infty$ (and neglecting corrections of $\mathcal{O}(v/c)^2$) the Dirac Eqtn for the "large" bispinor φ, viz [Eq.(3)]...

$$\to\left\{\frac{1}{2m}[\sigma\cdot(\mathbb{p}-\frac{q}{c}\mathbb{A})]^2 + q\phi\right\}\varphi = \epsilon\varphi, \quad \text{for } \epsilon \ll mc^2, \tag{5}$$

... reduces to a Schrödinger-like system...

$\{\mathcal{H}_s + \mathcal{E}_{mag}\}\varphi = \epsilon\varphi$,

w/ $\mathcal{H}_s = \frac{1}{2m}(\mathbb{p}-\frac{q}{c}\mathbb{A})^2 + q\phi$ ← usual Schrödinger EM Hamiltonian;

&/ $\mathcal{E}_{mag} = -(q\hbar/2mc)\,\sigma\cdot\mathbb{B}$ ← particle interaction w/ extl. mag. fld $\mathbb{B}$;

w/ $\varphi = a\binom{1}{0} + b\binom{0}{1} = \binom{a}{b}$ ← a bispinor for a spin ½ particle: a = spin "up", b = spin "down" amplitudes. (6)

The remarkable feature of this reduction is the automatic appearance of the magnetic interaction $\mathcal{E}_{mag}$, in lowest order $\mathcal{O}(1/c)$. For an electron, $q=-e$, so...

w/ $e\hbar/2mc = \mu_0$, Bohr magneton, &/ $\frac{1}{2}\sigma = \mathbb{S}$, electron spin,

$\to \mathcal{E}_{mag.} = +\mu_0\,\sigma\cdot\mathbb{B} = -\mu\cdot\mathbb{B}$, w/ $\boxed{\mu = -g\mu_0\mathbb{S}}$, and $g=2$. (7)

This reduction of the Dirac Eqtn thus includes the usual Schrödinger theory for an electron in an external field $(\mathbb{A}, i\phi)$, but also includes two new features:

Dirac Eq. ↔ electrons. Next approximation: to $O(v/c)^2$.

(a) electrons *must* be described by *two*-component spinors $\varphi = \begin{pmatrix} a \\ b \end{pmatrix}$.

(b) electrons interact with a magnetic field $\mathbb{B}$ via: $\mathcal{E}_{mag} = -\boldsymbol{\mu}\cdot\mathbb{B}$, w/ $\boldsymbol{\mu} = -\underline{\underline{2}}\mu_0 \mathbb{S}$.

NOTE: $g=2$

CONCLUSION: Dirac's wave equation describes particles with spin $S=\frac{1}{2}$, and with magnetic moment $\boldsymbol{\mu} = -2\mu_0\mathbb{S}$. Note that the g-value has evolved from the classical (nonrelativistic) value of 1 to Dirac's $g=2$.

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4) In the next order of approxn [i.e. Dirac Eq. to $O(v/c)^2$], we get a new term that does not appear in Schrödinger theory (not even in the add-on version we did). To see how this works, start from $\mathbb{A}=0$ (for simplicity)¶, and Eq. (2B)...

$$\left[\begin{aligned} &(\epsilon - q\phi)\varphi \simeq \frac{1}{2m}(\sigma\cdot\mathbb{p})\left\{1 - \left(\frac{\epsilon - q\phi}{2mc^2}\right)\right\}(\sigma\cdot\mathbb{p})\varphi \quad \text{for } \mathbb{A}=0, \text{ and to } O(v/c)^2, \\ &\qquad\qquad \text{now keep this term} \\ &\text{w/}\ \chi = \frac{1}{2mc}\left\{\sigma\cdot\mathbb{p}\Big/\left[1+\left(\frac{\epsilon-q\phi}{2mc^2}\right)\right]\right\}\varphi \simeq (\sigma\cdot\mathbb{p}/2mc)\varphi, \text{ neglecting } O\left(\tfrac{v}{c}\right)^3. \end{aligned}\right\} \quad (8)$$

$\epsilon = E - mc^2$ (for φ the +ve E solution) is the conventional eigenenergy. Now normalization for the Dirac wavefcn requires

$$\left[\begin{aligned} &\int \psi^\dagger \psi\, d^3x = \int (\varphi^\dagger\varphi + \chi^\dagger\chi)\, d^3x = 1 \leftarrow \text{use } \chi \text{ from Eq. (8)...} \\ &\text{or// } \int \varphi^\dagger \left[1 + (\mathbb{p}^2/4m^2c^2)\right]\varphi\, d^3x = 1, \text{ to } O(v/c)^2. \end{aligned}\right. \quad (9)$$

$$\left[\begin{aligned} &\text{Define: } \Phi = \Omega\varphi, \text{ w/ } \Omega = 1 + (\mathbb{p}^2/8m^2c^2), \text{ to } O(v/c)^2 \text{ terms} \\ &\text{Then: } \int \Phi^\dagger \Phi\, d^3x = 1, \text{ to } O(v/c)^2. \end{aligned}\right.$$

By using Φ, rather than φ, we preserve the norm w/o adjustment. Now put $\varphi = [1 - (\mathbb{p}^2/8m^2c^2)]\Phi$ into Eq. (8), and discard terms of order higher than $(v/c)^2$. The result is a Schrödinger-like equation which is correct to $O(\frac{v}{c})^2$ [compare with Eq. (3.83) in Sakurai's "Advanced QM" (1967)]:

¶ $\mathbb{A}=0 \Rightarrow$ mag. fld. $\mathbb{B}\equiv 0$. Thus we will *not* pick up the $\mathcal{E}_{mag} = -\boldsymbol{\mu}\cdot\mathbb{B}$ term in this case.

Dirac Eqtn to $\Theta(v/c)^2$ terms. The Darwin Term.

note spinor

$$\left[\left(\frac{\mathbf{p}^2}{2m} + q\phi\right) - \frac{\mathbf{p}^4}{8m^3c^2} - \frac{q\hbar\boldsymbol{\sigma}\cdot(\mathbf{E}\times\mathbf{p})}{4m^2c^2} - \frac{q\hbar^2}{8m^2c^2}(\nabla\cdot\mathbf{E})\right]\Phi = \epsilon\Phi, \tag{10}$$

① ② ③ ④

w/ $\mathbf{E} = -\nabla\phi$, the external electric field (and $\nabla\cdot\mathbf{E} = 4\pi\rho$ its source density).

① is the usual (nonrelativistic) Schrödinger Hamiltonian,

② is Pauli's $\Theta(v/c)^2$ correction to the energy ϵ,[¶] as we can see by expanding:

$$\rightarrow \epsilon = [(mc^2)^2 + (c\mathbf{p})^2]^{\frac{1}{2}} - mc^2 = (\mathbf{p}^2/2m) - \frac{1}{8}(\mathbf{p}^4/m^3c^2) + \ldots \tag{11}$$

this is term ②

③ is the spin-orbit interaction, previously manufactured.[§] It can be written as...

$$-\frac{q\hbar\boldsymbol{\sigma}\cdot(\mathbf{E}\times\mathbf{p})}{4m^2c^2} = -\frac{1}{2}\boldsymbol{\mu}\cdot\mathbf{B}_{mot} \int \begin{array}{l}\boldsymbol{\mu} = (q\hbar/2mc)\boldsymbol{\sigma}, \text{ is } q\text{'s mag. moment} \\ \mathbf{B}_{mot} = \mathbf{E}\times\frac{\mathbf{v}}{c}, \text{ motional mag. field}\end{array} \tag{12}$$

The factor ½ is the Thomas precession factor.

All these terms appear in the patchwork version of a corrected Schrödinger theory that we have looked at before. The new term is #4 in Eq. (10)... if the interaction potential for q in the external $\mathbf{E}$ is V, i.e. $q\mathbf{E} = -\nabla V$, then

$$④ = +\frac{1}{8}(\hbar/mc)^2\nabla^2 V = \frac{1}{8}(\partial/\partial\xi_k)^2 V, \text{ w/ } \xi_k = \frac{x_k}{\lambda}, \lambda = \frac{\hbar}{mc}. \tag{13}$$

This is called the "Darwin Term"; it is important only when the potential V changes rapidly over lengths ~ Compton wavelength λ. More on this later.

The nonrelativistic reduction of the Dirac Eqtn, pp. DE 20-23, has thus shown...

- Dirac theory includes Schrödinger (nonrelativistic) theory for a spinor electron;
- The theory identifies the electron as a spin ½ particle w/ magnetic momentum $\boldsymbol{\mu} = -\mu_0\boldsymbol{\sigma}$;
- The theory produces the previous (expected) $\Theta(v/c)^2$ corrections;
- It generates the spin-orbit interaction w/ a correct Thomas precession factor;
- It introduces a new "Darwin Interaction" proportional to $\nabla\cdot\mathbf{E}$.

¶ class notes, p. fs 12, Eq. (27). § class notes, pp. fs 6, Eq. (5), Thomas factor in Eq. (22).