B.V. Probs II (cont'd) pp. II BV 14–18 optional at this point.

8) We now seek expressions for Green's fcns in spherical symmetry. Recall that a Green's fcn G serves as a bridge between solutions to Laplace' eqtn $\nabla^2\phi = 0$ (in a charge-free region) and Poisson's eqtn $\nabla^2\phi = -4\pi\rho$ in (in charged region), in the following way... (0) Start with solns to $\nabla^2\phi = 0$ in a given D.

REVIEW

(1) $G = G(\mathbf{r},\mathbf{r}')$ is solution for unit pt. source at position $\mathbf{r}'$ in a given domain D $\Big\}$ $\nabla_r^2 G(\mathbf{r},\mathbf{r}') = -4\pi\,\delta(\mathbf{r}-\mathbf{r}')$; (38a)

we desire ptl. $\phi = \phi(\mathbf{r})$ in D, w/ charge-density $= \rho$ $\Big\}$ $\nabla_r^2\phi(\mathbf{r}) = -4\pi\,\rho(\mathbf{r})$. (38b)

(2) Interchange $\mathbf{r}$ & $\mathbf{r}'$ (w/ G symmetric). Mult. (38a) on left by $\phi(\mathbf{r}')$, (38b) on left by $G(\mathbf{r},\mathbf{r}')$ and subtract to get...

$$\underbrace{\phi(\mathbf{r}')\nabla_{r'}^2 G(\mathbf{r},\mathbf{r}') - G(\mathbf{r},\mathbf{r}')\nabla^2\phi(\mathbf{r}')}_{} = -4\pi\left[\phi(\mathbf{r}')\delta(\mathbf{r}-\mathbf{r}') - G(\mathbf{r},\mathbf{r}')\rho(\mathbf{r}')\right]$$

$$\equiv \nabla_{r'}\cdot\left[\phi\,\nabla_{r'}G - G\,\nabla_{r'}\phi\right] \tag{39}$$

(3) Integrate $\int_D d^3x'$ and use Divergence Thm on LHS. If surface S encloses D...

$$\oint_S \left[\phi\frac{\partial G}{\partial n'} - G\frac{\partial\phi}{\partial n'}\right]dS' = -4\pi\left[\phi(\mathbf{r}) - \int_D G(\mathbf{r},\mathbf{r}')\rho(\mathbf{r}')\,d^3x'\right]$$

Soln

$$\boxed{\phi(\mathbf{r}) = \int_D G(\mathbf{r},\mathbf{r}')\rho(\mathbf{r}')\,d^3x' + \frac{1}{4\pi}\oint_S\left[G(\mathbf{r},\mathbf{r}')\frac{\partial\phi}{\partial n'} - \phi(\mathbf{r}')\frac{\partial G}{\partial n'}\right]dS'} \;\star \tag{40}$$

(4) So, if we can solve $\nabla_r^2 G = -4\pi\,\delta(\mathbf{r}-\mathbf{r}')$ in D, we can solve $\nabla_r^2\phi = -4\pi\rho(\mathbf{r})$. The bridge from $\nabla_r^2\phi = 0$ is formed by noting that (sometimes) G is readily compiled from the eigenfcns of the $\nabla_r^2\phi = 0$ problem.

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9) A clear example of how this works is provided by our previous expansion of the sourcept-fieldpt inverse distance $1/R$ in the eigenfcns $Y_{\ell m}$ [Eq. (26), p. II BV9 above]. We claim this gives the spherical Green's fcn on an infinite

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$\star$ Agrees with previous result: Eq. (14), p. ϕ5. Also same as Jackson Eq. (1.42).

B.V. Probs II (cont'd)

G_∞ is expanded in terms of the eigenfcns Y_{lm} of the $\nabla^2\phi = 0$ problem (on an ∞ domain).

(Spherical) domain $D \to \infty$...

$$\to G_\infty(\mathbf{r}, \mathbf{r}') = \frac{1}{|\mathbf{r}-\mathbf{r}'|} = \frac{1}{r}\sum_{l=0}^{\infty}\left(\frac{r'}{r}\right)^l \left[\frac{4\pi}{2l+1}\sum_{m=-l}^{l} Y_{lm}^*(\theta',\varphi') Y_{lm}(\theta,\varphi)\right] \quad \text{for } r'<r. \text{ Interchange when } r<r'. \qquad (41)$$

This is evidently true because we know $\nabla_r^2(1/|\mathbf{r}-\mathbf{r}'|) = -4\pi\delta(\mathbf{r}-\mathbf{r}')$ [per Eq. (44) of Helmholtz Thm notes]. When used in above Green's Thm solution, with boundary surface $S\to\infty$, we get the ∞ domain result...

$$\phi(\mathbf{r}) = \int_{D\to\infty} G_\infty(\mathbf{r},\mathbf{r}')\rho(\mathbf{r}')\,d^3x' = \int_\infty \frac{\rho(\mathbf{r}')\,d^3x'}{|\mathbf{r}-\mathbf{r}'|} \leftarrow \text{obviously true}$$

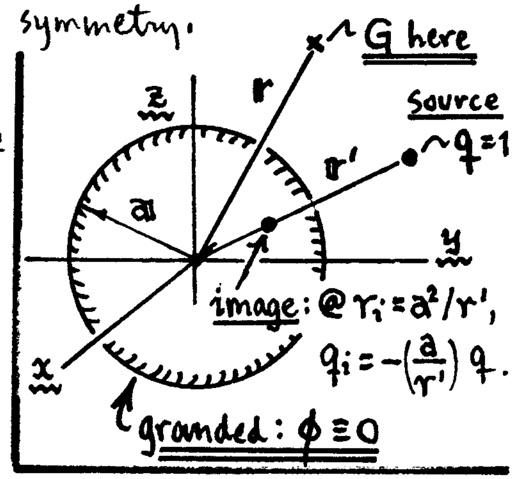
$$\to \phi(\mathbf{r}) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{4\pi}{2l+1} Y_{lm}(\theta,\varphi)\int_\infty d^3x'\,\rho(\mathbf{r}')\underbrace{\left[\frac{1}{r}\left(\frac{r'}{r}\right)^l\right]}_{\text{for } r>r',\text{ etc.}} Y_{lm}^*(\theta',\varphi'). \qquad (42)$$

The symmetry of $\rho(\mathbf{r}')$ may rule out certain l-values... e.g. if the net charge is zero ($\int\rho\,d^3x' = 0$), then the $l=0$ term is missing, and ϕ starts out as a $1/r^2$ term at best.

---

10) $G_\infty(\mathbf{r},\mathbf{r}')$ is "nice" as a paradigm, but not too useful -- principally because it does not obey any interesting B.C. Not with the bounding surface $S\to\infty$. So we look for "more interesting" G's in spherical symmetry.

The point-charge-grounded-conducting-sphere prob^m (Jk^n Sec. (2.2)) gave an "interesting" G-fcn...



$$\left[\begin{aligned} &G_a(\mathbf{r},\mathbf{r}') = \phi(\mathbf{r})\Big|_{q=1} = \frac{1}{|\mathbf{r}-\mathbf{r}'|} - \frac{k}{|\mathbf{r}-k^2\mathbf{r}'|}, \\ &\text{where}: k = a/r' < 1, \ \&\ G_a \equiv 0 \text{ for } \mathbf{r}=\mathbf{a}. \end{aligned}\right. \qquad (43)$$

Both terms in G_a can be expanded by an inverse distance formula like (41)...

$$\to G_a(\mathbf{r},\mathbf{r}') = \sum_{l=0}^{\infty}\frac{4\pi}{2l+1}\left[R_l(r,r';a)\right]\sum_{m=-l}^{+l} Y_{lm}^*(\theta',\varphi')Y_{lm}(\theta,\varphi),$$

(next page)

2nd term is contribution of grounded sphere. When $a \to 0$, $G_a \to G_\infty$ of Eq. (41).

w/

$$R_\ell(r,r';a) = \begin{cases} \frac{1}{r}\left(\frac{r'}{r}\right)^\ell \left[1-\left(\frac{a}{r'}\right)^{2\ell+1}\right], & r > r' \leftarrow \text{vanishes as } r\to\infty; \\ \frac{1}{r'}\left(\frac{r}{r'}\right)^\ell \left[1-\left(\frac{a}{r}\right)^{2\ell+1}\right], & r < r' \leftarrow \text{vanishes at } r = a. \end{cases} \tag{44}$$

$G_a(\mathbf{r},\mathbf{r}')$ is a solution to Laplace Eq. $\nabla^2 G_a = 0$ (with B.C. that $G_a \equiv 0$ @ $\mathbf{r} = \partial\mathcal{D}$) everywhere but at $\mathbf{r} = \mathbf{r}'$, where there is a source $4\pi\,\delta(\mathbf{r}-\mathbf{r}')$.

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11) The construction (or recognition) of a Green's Fcn always depends on the specific B.C. of interest,* but some progress can be made on the general method of construction of G in a system of separable cds. The defining eqtn is always...

$$\boxed{\nabla_r^2\, G(\mathbf{r},\mathbf{r}') = -4\pi\,\delta(\mathbf{r}-\mathbf{r}')} \tag{45}$$

specific B.C. to be incorporated in integration constants of solution for G.

In spherical cds, write...

$$\delta(\mathbf{r}-\mathbf{r}') = \frac{1}{r^2}\,\delta(r-r')\Big[\delta(\cos\theta-\cos\theta')\,\delta(\varphi-\varphi')\Big], \tag{46}$$

$[\ ] = \sum_{\ell,m} Y_{\ell m}^*(\theta',\varphi')\,Y_{\ell m}(\theta,\varphi)$, $\sum$ $\ell = 0 \to \infty$, $m = -\ell \to +\ell$.

So //

$$\int_\infty \delta(\mathbf{r}-\mathbf{r}')\,d^3x_{sph} = \int_0^\infty r^2 dr \int_{-1}^{+1} d(\cos\theta) \int_0^{2\pi} d\varphi\; \delta(\mathbf{r}-\mathbf{r}') = 1.$$

The [ ] in (46) can be represent by the $Y_{\ell m}$ completeness relation as noted, so

$$\longrightarrow \delta(\mathbf{r}-\mathbf{r}') = \frac{1}{r^2}\,\delta(r-r')\sum_{\ell,m} Y_{\ell m}^*(\theta',\varphi')\,Y_{\ell m}(\theta,\varphi). \tag{47}$$

This repn of $\delta(\mathbf{r}-\mathbf{r}')$ is independent of any particular B.C. Now look for a Green's Fcn G in the form...

$$\longrightarrow G(\mathbf{r},\mathbf{r}') = \sum_{\ell,m} A_{\ell m}(r;\mathbf{r}')\,Y_{\ell m}(\theta,\varphi) \tag{48}$$

that $A_{\ell m}$ depends on r (but not θ or φ) anticipates the B.C. will be imposed on surfaces $r = $ cnst, w/ possible θ & φ variation.

Put G of Eq. (48) and δ of Eq. (47) into the diff. eqtn (45) and process...

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* Each $\binom{\text{charge}}{\text{free}}$ probm $\{\nabla^2\phi = 0, + \text{B.C.}\}$ has a $\binom{\text{point}}{\text{source}}$ Green's Fcn $\{\nabla^2 G = -4\pi\,\delta(\mathbf{r}-\mathbf{r}'), + \text{same B.C.}\}$ whose soln $\Rightarrow$ soln to $\{\nabla^2\phi = -4\pi\rho(\mathbf{r}), + \text{same B.C.}\}$ $\binom{\text{charge-present}}{\text{counterpart}}$.

... let : $A_{\ell m}(r;\mathbf{r}') = g_\ell(r,r')\,Y^*_{\ell m}(\theta',\varphi')$ ...

see ★ below

so $$\nabla^2_{sph} G(\mathbf{r},\mathbf{r}') = \sum_{l,m} Y^*_{\ell m}(\theta',\varphi')\left\{\frac{1}{r}\frac{\partial^2}{\partial r^2}(r\ \ ) + \frac{1}{r^2}\nabla^2_\Omega\right\} g_\ell(r,r')\,Y_{\ell m}(\theta,\varphi)$$

use: $\nabla^2_\Omega Y_{\ell m} = -\ell(\ell+1)\,Y_{\ell m}$

or $$\nabla^2_{sph} G = \sum_{l,m}\left\{\frac{1}{r}\frac{\partial^2}{\partial r^2}(r g_\ell) - \frac{\ell(\ell+1)}{r^2} g_\ell\right\} Y^*_{\ell m}(\theta',\varphi')\,Y_{\ell m}(\theta,\varphi), \tag{49}$$

$$= -4\pi\,\delta(\mathbf{r}-\mathbf{r}') = \sum_{l,m}\left\{-\frac{4\pi}{r^2}\delta(r-r')\right\} Y^*_{\ell m}(\theta',\varphi')\,Y_{\ell m}(\theta,\varphi)$$

from Eq. (47)

Identify the { } to get the system ...

$$\left[\!\left[\begin{aligned} &G(\mathbf{r},\mathbf{r}') = \sum_{l,m} g_\ell(r,r')\,Y^*_{\ell m}(\theta',\varphi')\,Y_{\ell m}(\theta,\varphi), \\ &\text{with } \frac{1}{r}\frac{\partial^2}{\partial r^2}(r g_\ell) - \frac{\ell(\ell+1)}{r^2} g_\ell = (-)\frac{4\pi}{r^2}\delta(r-r') \quad \left\{\begin{matrix}\text{for each}\\ \ell=0,1,2,\ldots\end{matrix}\right. \end{aligned}\right]\!\right] \tag{50}$$

The B.C., on surfaces $r=$ cnst, are to be inserted via the integration cnsts of the solution for g_ℓ. Certainly G_∞ & G_a of Eqs (41) & (44) fit this scheme. The system of Eq. (50) is as far as we can go w/o choosing specific B.C.

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12) A particular solution to Eq. (50) goes as follows. Soln to homog^s g_ℓ eqtn is

$$\rightarrow g_\ell(r,r') = \begin{cases} A r^\ell + B r^{-(\ell+1)}, & r<r'; \\ A' r^\ell + B' r^{-(\ell+1)}, & r>r' \end{cases} \quad \Big\| \text{ A, B, A' \& B' are parametric fcns of } r', \text{ free to fit B.C. of choice.} \tag{51}$$

If $G\equiv 0$ @ $r=a$ & $r=b$: $$g_\ell(r,r') = \begin{cases} A r^\ell\left[1-(a/r)^{2\ell+1}\right], & a\leqslant r<r'; \\ B' r^{-(\ell+1)}\left[1-(r/b)^{2\ell+1}\right], & b\geqslant r>r'. \end{cases} \tag{52}$$

Use symmetry: $g_\ell(r',r) \equiv g_\ell(r,r')$, i.e. r & r' can be interchanged. Then ...

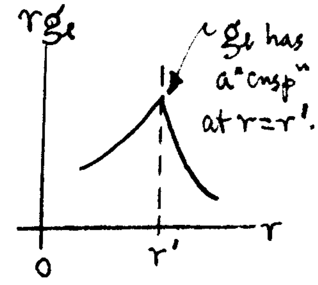
$$\rightarrow g_\ell(r,r') = \text{cnst}\times\frac{1}{r}\left(\frac{r'}{r}\right)^\ell\left[1-\left(\frac{a}{r'}\right)^{2\ell+1}\right]\left[1-\left(\frac{r}{b}\right)^{2\ell+1}\right] \quad \int \begin{matrix}\text{for: } b\geqslant r>r'\geqslant a. \text{ Interchange}\\ r' \,\&\, r \text{ when: } b\geqslant r'>r\geqslant a.\end{matrix} \tag{53}$$

★ $\nabla^2_{sph} = \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\ \ ) + \frac{1}{r^2}\nabla^2_\Omega$, $\nabla^2_\Omega = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}$.

The remaining "cnst" in g_ℓ of Eq. (53) can be determined by integrating through the diff. eqtn for g_ℓ, Eq. (50). Integrate by $\int_{r'-\epsilon}^{r'+\epsilon} dr \cdot r \cdot$, with $\epsilon \to 0+$. Then...

$$\int_{r'-\epsilon}^{r'+\epsilon} \frac{\partial}{\partial r}\left[\frac{\partial}{\partial r}(r g_\ell)\right] dr - \ell(\ell+1)\underbrace{\int_{r'-\epsilon}^{r'+\epsilon} \frac{dr}{r} g_\ell}_{\to 0,\ g_\ell \text{ is cont}^{\underline{s}}} = -4\pi \int_{r'-\epsilon}^{r'+\epsilon} \frac{dr}{r}\,\delta(r-r'),$$

$$\xrightarrow{\text{So}} \lim_{\epsilon\to 0}\left[\frac{\partial}{\partial r}(r g_\ell)\right]\Bigg|_{r=r'-\epsilon}^{r=r'+\epsilon} = -\frac{4\pi}{r'}. \qquad \underline{(54)}$$



This condition fixes the "cnst" in (53). Result is...

$$\text{cnst} = 4\pi/(2\ell+1)\left[1-(a/b)^{2\ell+1}\right], \quad a \leq b \qquad \underline{(55)}$$

Finally, <u>the Green's Fcn for a spherical shell $a \leq r \leq b$ is</u>...

$$G_{a,b}(\mathbf{r},\mathbf{r}') = \sum_{\ell=0}^{\infty} \frac{4\pi}{2\ell+1}\left[R_\ell(r,r';a,b)\right] \sum_{m=-\ell}^{+\ell} Y_{\ell m}^*(\theta',\varphi')\,Y_{\ell m}(\theta,\varphi), \qquad \underline{(56)}$$

W/

$$R_\ell(r,r';a,b) = \frac{1}{\left[1-(a/b)^{2\ell+1}\right]}\left\{\frac{1}{r}\left(\frac{r'}{r}\right)^{\ell}\left[1-\left(\frac{a}{r'}\right)^{2\ell+1}\right]\left[1-\left(\frac{r}{b}\right)^{2\ell+1}\right]\right\}$$

for $a \leq r' \leq r \leq b$. Do $r' \leftrightarrow r$ for $a \leq r \leq r' \leq b$.

This G is of the same <u>form</u> as G(∞ domain) of Eq. (41) & G(sphere a) of Eq. (44). Only the radial fcn R_ℓ is more complicated. In fact, we recover G(∞ domain) when $a \to 0$ & $b \to \infty$, and also recover G(sphere a) when $b \to \infty$.

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13) In <u>Sec. (3.10)</u>, Jackson gives two examples of using $G_{a=0,b}(\mathbf{r},\mathbf{r}')$ of Eq. (52) inside a conducting sphere where there are sources (charged ring & line charge). You can read them with profit. In <u>Sec. (3.11)</u>, Jackson constructs $G_\infty(\mathbf{r},\mathbf{r}')$ in cylindrical cds -- we shall not do this. In <u>Sec. (3.12)</u>, Jackson shows how G is related to eigenfcns of the homog$^{\underline{s}}$ problem (for Schrödinger eqtn) -- we have already done this.