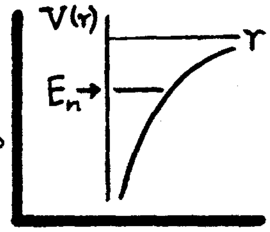
Φ507 Final Exam (in class, 3hrs.) — Tues. 11 May 1993

This exam is open-book, open-notes, and is worth 300 points total. There are 6 problems on 3 pages, with point-values as marked. For each problem, put a box around your answer. Number your solution pages consecutively, write your name on p. 1, and staple the pages together before handing them in.

---

① [50pts.]. Use the Bohr-Sommerfeld energy quantization rule to find bound-state energies E_n for an electron $(-e, m)$ in a Coulomb well: $V(r) = -Ze^2/r$, w/ $r \geq 0$. (The total particle energy is (-)ve). Assume the electron is in an S-state, w/ orbital ∡ momentum $l = 0$. Compare your result for E_n (Bohr Sommerfeld) with the known E_n for the Bohr atom.



---

② [50pts]. The mu meson, μ^+, is an elementary particle with charge $+e$, mass $= 207\, m_e$ (m_e = electron mass), spin $\frac{1}{2}\hbar$, a normal Dirac g-value: $g_\mu = 2$, and a lifetime (against $\mu^+ \to e^+ + \nu_e + \bar{\nu}_\mu$) of 2.2×10^{-6} sec in its rest frame. Despite its short life, it is possible for the μ^+ to capture an electron, e^-, to form a bound system $\mu^+ e^-$ called "muonium"; this is an exotic H-atom, with μ^+ replacing the proton.

(A) For an $n = 3 \to 2$ transition in a normal H-atom, the light emitted is the Balmer α line at wavelength: $\lambda_\alpha = 656.3$ nm. What is λ_α for muonium?

(B) For a normal H-atom, the hyperfine splitting in the ground state is $\Delta\nu_{hfs} = 1420$ MHz. What is $\Delta\nu_{hfs}$ for the ground state of muonium?

---

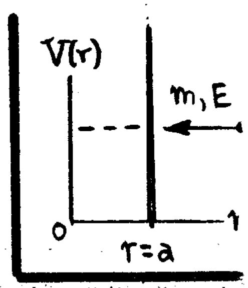
③ [50pts]. Let $p_k = -i\hbar\, \partial/\partial x_k$ be the k^{th} component of the momentum operator (the x_k, $k = 1$ to 3, are spatial coordinates). Calculate the expectation value $\langle p_k \rangle$ for a Dirac free-particle wavepacket [the wavepacket $\Psi(\mathbf{r}, t)$ is described in class notes, pp. DE 27-28, Eqs. (11)-(17)]. Does $\langle p_k \rangle$ show oscillatory terms corresponding to Zitter Bewegung? Comment on your result... is it permissible for $\langle v_k \rangle = \langle c\alpha_k \rangle$ to show ZB, while $\langle p_k \rangle$ does not? or does?

(next page)

④ [50pts]. The so-called Darwin term appears in an $\mathcal{O}(1/c^2)$ reduction of the Dirac Eqtn [ref. Eq.(10), p. DE 23 of class notes]. This term adds an interaction energy $W_D = -\frac{1}{8} q (\hbar/mc)^2 \nabla \cdot \mathbf{E}$ to a Schrödinger-like Hamiltonian for a particle of charge q & mass m in an external electric field $\mathbb{E}$. Let $(q, m) = (-e, m_e)$ be an electron in the Coulomb field of a massive, point-like nucleus of charge $+e$... i.e. consider the nonrelativistic H-atom.

(A) Treat W_D as a perturbation on the Bohr energies $E_n = -\frac{1}{2}\alpha^2 mc^2/n^2$. Calculate the first-order energy shift $\Delta E_n^{(D)} = \langle n | W_D | n \rangle$ caused by the Darwin term in the n^{th} hydrogenic state. **HINT**: information on the H-atom wavefunctions ψ_n appears in Davydov Sec. 38.

(B) Compare $\Delta E_n^{(D)}$ to the Bohr energy $|E_n|$ in state n. Roughly speaking, how does $\Delta E_n^{(D)}$ compare with the spin-orbit energies? Comment on the claim: $\Delta E_n^{(D)}$ actually is a (special) kind of spin-orbit energy.



⑤ [50pts]. A particle of mass m and energy E is incident on a hard spherical shell of radius "a", which is fixed at the origin. The shell's scattering potential is taken to be: $V(r) = V_0 a \delta(r-a)$, where V_0 & a are constants, r is the radial coordinate, and δ is the Dirac delta fcn. Treat the scattering by first Born Approximation.

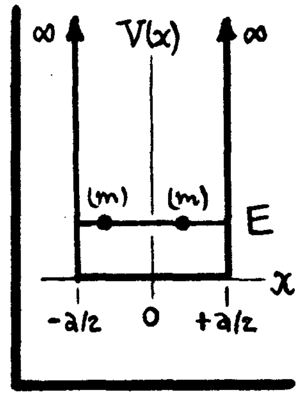
(A) What condition on V_0 ensures that the Born Approximation is valid at all energies E? Assume this condition is satisfied in what follows.

(B) Find the differential scattering cross-section $\frac{d\sigma}{d\Omega}$ as a fcn of momentum transfer q. Sketch $\frac{d\sigma}{d\Omega}$ vs. q over the allowed range of q. **NOTE**: $\frac{d\sigma}{d\Omega}$ vanishes at certain values of q. Is there any physics in this?

(C) Express the total scattering cross-section σ as an integral over q. Find the leading terms in σ (incl. E-dependence) in the low-energy limit.

(next page)

ϕ507 Final (cont'd) 5/11/93



⑥ [50 pts.]. Two identical spin $\frac{1}{2}$ fermions (each of mass m) move in one dimension in a QM "box" of length a as shown. The box is represented by infinite potential walls, $V(x) \to \infty$, at $x = \pm a/2$. For parts (A) & (B), assume the particles do not interact.

(A) Find the ground-state energy (i.e. lowest permitted energy) when the particles are in a spin triplet configuration. Call this energy E_T.

(B) Find the ground-state energy (lowest energy possible) when the particles are in a spin singlet configuration. Call this energy E_S.

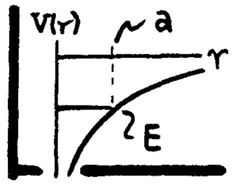
(C) Suppose now that the particles interact by a strong, attractive, short-range potential: $V(x_1, x_2) = -\lambda\,\delta(x_1 - x_2)$, w/ $\lambda = (+)$ve const, and x_1 & x_2 the positions of the particles in the box. Use first order perturbation theory (sic) to discuss what happens to the energies E_T and E_S obtained in parts (A) & (B).

HINT: beware of Pauli's Exclusion Principle.

ϕ507 Final Exam Solutions (1993)

1 [50pts]. S-state energies of the H-atom via Bohr-Sommerfeld quantization.

1) With E the total electron ($\substack{\text{charge } -e \\ \text{mass } m}$) energy, the BS rule is:★



$$\rightarrow \int_0^a \sqrt{2m[E-V(r)]}\,dr = (n+\tfrac{1}{2})\pi\hbar, \quad n=0,1,2,\ldots \qquad (1)$$

for a 1D motion (in this case along the radial cd. r). Let $E=(-)E_n$ be a bound state, so the RH turning point is at $E_n = Ze^2/a$, i.e. $a = Ze^2/E_n$. The LH turning point is at $r=0$, because $r \geqslant 0$ by defⁿ.† Put in $V(r) = -Ze^2/r$, so...

$$\rightarrow (n+\tfrac{1}{2})\pi\hbar = \int_0^a \sqrt{2m[-E_n + Ze^2/r]}\,dr = \sqrt{2mE_n}\int_0^a \left[\frac{a}{r}-1\right]^{1/2} dr, \qquad (2)$$

is the BS prescription of Eq. (1). The integral looks potentially divergent.

---

2) In (2), change variables to $x = a/r$ in the integral. Then $dr = -a\,\frac{dx}{x^2}$, and $0 \leqslant r \leqslant a \leftrightarrow \infty \geqslant x \geqslant 1$. Eq. (2) yields...

$$\rightarrow (n+\tfrac{1}{2})\pi\hbar / a\sqrt{2mE_n} = \int_1^\infty \frac{dx}{x^2}\sqrt{x-1}. \qquad (3)$$

The integral is tabulated [e.g. Dwight # (194.21) & (192.11)]...

$$\rightarrow \int_1^\infty \frac{dx}{x^2}\sqrt{x-1} = \cancelto{0}{-\frac{1}{x}\sqrt{x-1}\Big|_{x=1}^{x=\infty}} + \frac{1}{2}\int_1^\infty \frac{dx}{x\sqrt{x-1}} = \tan^{-1}(\sqrt{x-1})\Big|_{x=1}^{x=\infty} = \frac{\pi}{2}. \qquad (4)$$

Put this result into (3) to get the quantization via the BS rule...

$$(n+\tfrac{1}{2})\cancel{\pi}\hbar = a\sqrt{2mE_n}\cdot\frac{\cancel{\pi}}{2} \leftarrow \text{put in } a = Ze^2/E_n \text{ as defined...}$$

So // $(n+\tfrac{1}{2})\hbar = Ze^2\sqrt{m/2E_n}$, $\;$ w// $\boxed{E_n = \tfrac{1}{2}(Z\alpha)^2 mc^2/(n+\tfrac{1}{2})^2}$. $\qquad (5)$

$\alpha = e^2/\hbar c \simeq 1/137$ is the finestructure cnst. These E_n are to be compared with the actual (Bohr) energies: $E_n(\text{Bohr}) = \frac{1}{2}(Z\alpha)^2 mc^2/n^2$, ● w// $n=1,2,3,\ldots$ The BS rule replaces Bohr's n by $(n+\frac{1}{2})$. This is not bad as $n \to$ large, but for the ground state: $E_n(\text{BS})|_{n=0} = 4\times E_n(\text{Bohr})|_{n=1}$, which is not too stunning.

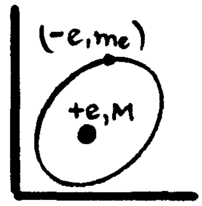
---

† For states with $l \neq 0$, there would be a centrifugal barrier as $r \to 0$ w// $r>0$ on LH side.

★ class notes, p. WKB 18, Eq. (52). ● class notes, p. H6, Eq. (20).

ϕ 507 Final Exam Solutions (1993)

② [50 pts]. Some spectroscopic features of μ^+e^- (muonium).



(A) 1) For purposes of this problem the μ^+ acts (so long as it lives) just like a replacement proton, except it is lighter ($M_\mu = 207\, m_e$ vs. $M_p = 1836\, m_e$), and it has a normal Dirac g-value ($g_\mu = 2$ vs $g_p = 2\times 2.79$). The Bohr energy levels for any such bound $(-e, m_e) \leftrightarrow (+e, M)$ system are:

$$\underline{E_n = -\tfrac{1}{2}\alpha^2 mc^2/n^2} \quad \int \alpha = e^2/\hbar c \simeq 1/137;\ n = 1,2,3,\ldots;\ \text{and:} \quad \underline{m = m_e/[1+(m_e/M)]} \leftarrow \text{electron reduced mass.} \tag{1}$$

The only thing that changes here, upon replacing p^+ by μ^+, is the mass M. For the Balmer α transition $n = 3 \to 2$, the emitted energy & photon wavelength are

$$\Delta E_\alpha = E_3 - E_2 = \frac{5}{72}\alpha^2 mc^2, \quad \lambda_\alpha = \frac{hc}{\Delta E_\alpha} = (72/5\alpha^2)\frac{h}{mc}$$

So// $$\underline{\lambda_\alpha = \left[1 + \frac{m_e}{M}\right]\cdot(72/5\alpha^2)(h/m_e c)} \quad \int \text{upon putting in reduced mass } m \text{ of Eq. (1).} \tag{2}$$

Then λ_α for muonium and λ_α for normal hydrogen are related by...

$$\frac{\lambda_\alpha(\text{muonium})}{\lambda_\alpha(\text{hydrogen})} = \frac{1+(m_e/M_\mu)}{1+(m_e/M_p)} = \frac{1+(1/207)}{1+(1/1836)} = 1.004\,284$$

NOTE: the difference $\Delta\lambda_\alpha = 2.8$ nm is readily detected.

$$\ldots \text{if } \underline{\lambda_\alpha(H) = 656.3\ \text{nm}}, \text{ then: } \boxed{\lambda_\alpha(\mu) = 659.1\ \text{nm}}. \tag{3}$$

(B) 2) Recall [from ϕ 507 prob^m #35] that the ground state hyperfine splitting for a hydrogenic atom, with a spin-$\frac{1}{2}$ nucleus characterized by g-value g_p, was...

$$\to \Delta\nu_{hfs} = \frac{8}{3}|g_n|\alpha^2 c R_\infty, \quad R_\infty = \text{Rydberg cnst for } \underline{\text{infinite}} \text{ mass nucleus.} \tag{4}$$

In replacing p^+ by μ^+, the only parameter that changes is $|g_n|$. <u>Important</u>: the way g_n is defined, it includes the <u>mass ratio</u>: $g_n = g(\text{nucleus})\cdot(m_e/M)$. So: $|g_n|_{\text{proton}} = 2\times 2.79\cdot(m_e/M_p)$, $|g_n|_{\text{muon}} = 2\times 1\cdot(m_e/M_\mu)$, and the ratio is: $|g_n|_{\text{muon}}/|g_n|_{\text{proton}} = (1/2.79)(M_p/M_\mu) = 3.179$. Then, for the hfs interval...

$$\to \frac{\Delta\nu_{hfs}(\mu)}{\Delta\nu_{hfs}(H)} = \frac{|g_n|_{\text{muon}}}{|g_n|_{\text{proton}}} = 3.179, \quad \text{and}// \quad \boxed{\Delta\nu_{hfs}(\mu) = 3.179\,\Delta\nu_{hfs}(H) = 4514\ \text{MHz}} \tag{5}$$

ϕ507 Final Exam Solutions (1993)

③ [50pts]. Expectation value of $p_k = -i\hbar\,\partial/\partial x_k$ for a Dirac wavepacket.

1) As described in Eqs. (11)-(17), pp. DE 27-28 of class notes, the wavepacket is:

$$\rightarrow \Psi(\mathbf{r},t) = \sum_{\mathbf{k},\mu} C_{\mathbf{k}}^{(\mu)} N_{\mathbf{k}} U_{\mathbf{k}}^{(\mu)} e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} \Big/ \text{w/ } \mathbf{k} = \mathbf{p}/\hbar = \text{cnst},\ \omega = \begin{cases} +\omega_p, & \text{for } \mu = 1,2; \\ -\omega_p, & \text{for } \mu = 3,4. \end{cases} \quad (1)$$

The 4-spinors $U_{\mathbf{k}}^{(\mu)}$ are orthonormal: $U_{\mathbf{k}}^{(\mu)\dagger} U_{\mathbf{k}}^{(\nu)} = \delta_{\mu\nu}$, and the norm cnst is $N_{\mathbf{k}} = \left([|\omega| + (mc^2/\hbar)]/2|\omega|V\right)^{1/2}$, w/ $|\omega| = \omega_p = [(\mathbf{k}c)^2 + (mc^2/\hbar)^2]^{1/2}$.

---

2) The required expectation value of $p_j = -i\hbar\,\partial/\partial x_j$ is...

$$\langle p_j \rangle = \int d^3x\, \Psi^\dagger(\mathbf{r},t) \left\{-i\hbar \frac{\partial}{\partial x_j}\right\} \Psi(\mathbf{r},t)$$

or/

$$\langle p_j \rangle = \int d^3x \sum_{\mathbf{k}',\nu} C_{\mathbf{k}'}^{(\nu)*} N_{\mathbf{k}'} U_{\mathbf{k}'}^{(\nu)\dagger} e^{-i(\mathbf{k}'\cdot\mathbf{r}-\omega t)} \left\{-i\hbar\frac{\partial}{\partial x_j}\right\} \cdot$$

$$\cdot \sum_{\mathbf{k},\mu} C_{\mathbf{k}}^{(\mu)} N_{\mathbf{k}} U_{\mathbf{k}}^{(\mu)} e^{+i(\mathbf{k}\cdot\mathbf{r}-\omega t)}. \quad (2)$$

cancel

The operation $\{\partial/\partial x_j\}\, e^{+i(\mathbf{k}\cdot\mathbf{r}-\omega t)} = i\left(\frac{p_j}{\hbar}\right) e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$, w/o affecting the time variation. In fact the $e^{+i\omega t}$ from the LH $\Psi^\dagger$ <u>cancels</u> the $e^{-i\omega t}$ from the RH Ψ, and Eq. (2) yields...

$$\langle p_j \rangle = \sum_{\mathbf{k}',\nu} \sum_{\mathbf{k},\mu} C_{\mathbf{k}'}^{(\nu)*} N_{\mathbf{k}'} C_{\mathbf{k}}^{(\mu)} N_{\mathbf{k}}\, \underbrace{U_{\mathbf{k}'}^{(\nu)\dagger} U_{\mathbf{k}}^{(\mu)}}_{=\,\delta_{\mu\nu},\ \text{when } \mathbf{k}'=\mathbf{k}} \{p_j\} \underbrace{\int d^3x\, e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}}}_{=\,(2\pi)^3\,\delta(\mathbf{k}-\mathbf{k}')}$$

So/

$$\boxed{\langle p_j \rangle = (2\pi)^3 \sum_{\mathbf{k},\mu} N_{\mathbf{k}}^2 \left|C_{\mathbf{k}}^{(\mu)}\right|^2 \{p_j\}} \Big/ \text{since } \mathbf{k}' = \mathbf{k}, \text{ and by norm: } U^{(\mu)\dagger} U^{(\mu)} = 1. \quad (3)$$

---

3) The expectation value $\langle p_j \rangle$ of Eq. (3) is a time-independent cnst for a Dirac wavepacket; <u>$\langle p_j \rangle$ shows <u>no</u> oscillatory terms corresponding to Zitter-Bewegung</u>. But the velocity operator $v_k = c\alpha_k$, occurring in the Dirac probability current $J_k = \Psi^\dagger c\alpha_k \Psi$, <u>did</u> show ZB. Not to worry... p_k <u>must</u> be the correct momentum because $\langle p_k \rangle$ <u>is</u> cnst (w/o ZB) for free particles, and $\langle p_k \rangle$ does $\rightarrow (-)\langle p_k \rangle$ under charge conjugation. That $\langle c\alpha_k \rangle$ shows ZB emphasizes the fact that $J_k \neq$ cnst for free particles.

Φ507 Final Exam Solutions (1993)

④ [50pts]. The Darwin term as a perturbation on H-atom spectrum.

(A) 1) $\nabla \cdot \mathbb{E} = 4\pi \rho(\mathbf{r})$, w/ $\rho(\mathbf{r})$ = charge density of the source of $\mathbb{E}$. In our case, the source is a pointlike nucleus of charge e, so $\rho(\mathbf{r}) = e\,\delta(\mathbf{r})$, and...

$$\underline{W_D = +\frac{1}{8} e(\hbar/m_e c)^2 \cdot 4\pi e\,\delta(\mathbf{r})} \quad (1)$$

← Darwin term for $(-e, m_e)$ in presence of pointlike nucleus.

The 1st order energy perturbation due to W_D in state n is then...

$$\Rightarrow \Delta E_n^{(D)} = \int d^3x\, \Psi_n^*(\mathbf{r}) \{W_D\} \Psi_n(\mathbf{r}) = \frac{\pi}{2} e^2 (\hbar/m_e c)^2 |\Psi_n(0)|^2. \quad (2)$$

---

2) Consult Davydov, Sec. 38. Normalized H-atom radial wavefcns $f_{nl}(\rho)$ appear on p. 156 (middle of page, unnumbered eqtn). Since the confluent fcn $F = 1$ when $\rho = 0$ [ref. Eq. (21b), p. H7 class notes], then $f_{nl}(\rho) \propto \rho^l$ as $\rho \to 0$, and all $f_{nl}(0) \equiv 0$, except when $l = 0$. The only nonvanishing $|\Psi_n(0)|^2$ in Eq. (2) are for S-states, $l = 0$, and for them the radial wavefcn $f_{n0}(0) = N_{n0}$, where -- per Davydov -- the norm cnst $N_{n0} = \frac{1}{\sqrt{2}} (2Z/na_0)^{3/2}$ [we've restored the unit of length: $a_0 = \hbar^2/m_e e^2$ = Bohr radius]. Divide $|f_{n0}(0)|^2$ by 4π for the ∡ norm, so:

$$\rightarrow |\Psi_n(0)|^2 = \frac{1}{4\pi} |f_{n0}(0)|^2 = \frac{1}{\pi} (Z/na_0)^3 \quad (3)$$

← S-states ($l = 0$) only.

Set $Z = 1$, and put this in Eq. (2). Noting $\frac{\hbar}{m_e c} \cdot \frac{1}{a_0} = \frac{e^2}{\hbar c} = \alpha$, we find...

$$\boxed{\Delta E_n^{(D)} = \frac{1}{2}\alpha^2 e^2/n^3 a_0 = \frac{1}{2}\alpha^4 mc^2/n^3} \quad (4)$$

← S states ($l = 0$) only.

(B) 3) Relative to the Bohr energies E_n, have: $\underline{\Delta E_n^{(D)} = (\alpha^2/n)|E_n|}$. But order $\frac{\alpha^2}{n}$ relative to the $|E_n|$ is precisely the size of a spin-orbit energy [ref. class notes, p. fs 11 Eq. (23)], so the Darwin energy $\Delta E_n^{(D)}$ enters the energy spectrum at the same level as the spin-orbit energies. In fact, it *is* a special kind of spin-orbit term for just the S-states... it does for them what the $\zeta(\mathbb{S} \cdot \mathbb{L})$ term does for the $l \neq 0$ states. The physics is different though: the S-state electron has to be in *contact* with the nucleus to get it's $\Delta E_n^{(D)}$ boost.

[50 pts.]. Analyse scattering from potential $V(r) = V_0 a\, \delta(r-a)$, via Born Approxn.

1. Let $k = \sqrt{2mE/\hbar^2}$ be m's incident wave #. Born Approxn validity requires:

(A) $\rightarrow \left| \int_0^\infty [e^{2ikr} - 1]\, V(r)\, dr \right| \ll \hbar v = \hbar^2 k/m$ ← Davydov Eq. (106.16), or Class notes p. ScT 10, Eq (22). (1)

... for $V(r) = V_0 a\, \delta(r-a)$, Eq (1) $\Rightarrow$ $\left(\frac{\sin ka}{ka}\right) V_0 \ll \frac{1}{2m}(\hbar/a)^2$. (2)

Born Approxn is good at all energies (even $E \to 0$) if $\boxed{V_0 \ll \frac{1}{2m}(\hbar/a)^2}$. (3)

---

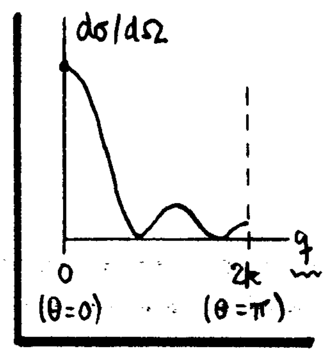
2. By class notes p. ScT (13), Eq. (31), the differential scattering cross-section is:

(B) $\rightarrow \frac{d\sigma}{d\Omega} = \left(\frac{m}{2\pi\hbar^2}\right)^2 |\tilde{V}(q)|^2$, w/ $q = 2k \sin(\theta/2)$ ∫ q = momentum transfer, θ = scattering angle. (4)

and// $\tilde{V}(q) = \frac{4\pi}{q} \int_0^\infty r\, V(r) \sin qr\, dr = [4\pi V_0 a^3] \left(\frac{\sin qa}{qa}\right)$ ∫ for the given: $V(r) = V_0 a\, \delta(r-a)$. (5)

In (5), $[4\pi V_0 a^3] = \int_0^\infty V(r) \cdot 4\pi r^2 dr = \Lambda$, the "volume" of $V(r)$. So we get...

$$\boxed{\frac{d\sigma}{d\Omega} = \left(\frac{m\Lambda}{2\pi\hbar^2}\right)^2 \left(\frac{\sin qa}{qa}\right)^2} \quad \int^{\text{w/}} \Lambda = 4\pi V_0 a^3,\ q = 2k \sin(\theta/2). \quad (6)$$



By the inequality in (3), the coefficient $(m\Lambda/2\pi\hbar^2)^2 \ll a^2$. $(d\sigma/d\Omega)$ vs. q is sketched at right -- the scattering vanishes when $qa = n\pi$, $n = 1, 2, \ldots$ (and $q \leq 2k$). At these points, there is a sort of resonance condition, where an integral # of half-wavelengths of q fit inside the scattering potential, i.e. $n \cdot \frac{1}{2}(2\pi/q) = a$, and $V(r)$ appears to be transparent.

---

3. The solid ∡ $d\Omega = 2\pi \sin\theta\, d\theta = (2\pi/k^2)\, q\, dq$ [prob. 21], so the total cross-section is:

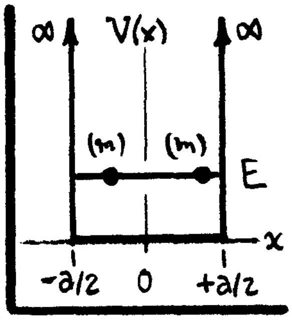
(C) $\rightarrow \sigma = \int_{4\pi} (d\sigma/d\Omega)\, d\Omega = \left(\frac{m\Lambda}{2\pi\hbar^2}\right)^2 \frac{2\pi}{k^2} \int_0^{2k} \left(\frac{\sin qa}{qa}\right)^2 q\, dq = \frac{2\pi}{k^2 a^2} \left(\frac{m\Lambda}{2\pi\hbar^2}\right)^2 \int_0^{2ka} \frac{dx}{x} \sin^2 x$. (7)

The integral is not an elementary fcn. When $a \to 0$ ($ka \ll 1$), put $\sin^2 x \simeq [x(1 - \frac{x^2}{6})]^2$ so that $\int_0^{2ka} (\sin^2 x) \frac{dx}{x} \simeq \int_0^{2ka} x(1 - \frac{x^2}{3})\, dx = 2(ka)^2 [1 - \frac{2}{3}(ka)^2]$. Then leading terms in σ.

$$\boxed{\sigma \simeq 4\pi (m\Lambda/2\pi\hbar^2)^2 \left[1 - \tfrac{2}{3} k^2 a^2\right]} \quad (8)$$

σ falls off slowly with energy (at low energy).

ϕ507 Final Exam Solutions (1993)



[50 pts]. Analyse case of two identical fermions in a box.

The particle-in-a-box problem is solved everywhere, e.g. in Davydov, Sec. 25. For a single particle of mass m, the eigenenergies are $E_n = n^2 E_1$, w/ $E_1 = \pi^2\hbar^2/2ma^2$, and $n = 1, 2, 3, \ldots$ The normalized eigenfcns are: $\phi_n(x) = \sqrt{\frac{2}{a}}\cos(n\pi x/a)$, n = odd; $\phi_n(x) = \sqrt{\frac{2}{a}}\sin(n\pi x/a)$, n = even.

(A) & (B) 1) For a spin triplet ($\uparrow\uparrow$), the Exclusion Principle forbids both fermions being in the same space state, and symmetrization requires the overall system wavefcn $u(x_1, x_2)$ be odd under exchange, $x_1 \leftrightarrow x_2$. So: $u_T(x_1, x_2) = \frac{1}{\sqrt{2}}[\phi_m(x_1)\phi_n(x_2) - \phi_n(x_1)\phi_m(x_2)]$ with $m \neq n$. For a spin singlet ($\uparrow\downarrow$): $u_S(x_1, x_2) = \frac{1}{\sqrt{2}}[\phi_m(x_1)\phi_n(x_2) + \phi_n(x_1)\phi_m(x_2)]$, and $m = n$ is allowed. The energies in these states are -- where H_1 & H_2 are the Hamiltonians for particles 1 & 2 that give the above eigenenergies E_n --

$$\rightarrow E_{T,S} = \langle u_{T,S}|H_1 + H_2|u_{T,S}\rangle = (E_m + E_n) \mp (E_m + E_n)\langle\phi_m(x_1)\phi_n(x_2)|\phi_n(x_1)\phi_m(x_2)\rangle. \tag{1}$$

The upper ($-$) sign on the RHS is for triplets; the lower (+) is for singlets. In doing this calculation, we have assumed the $\phi_n(x)$ are orthonormal.

---

2) For T states, $m \neq n$ in Eq. (1), and the overlap term vanishes. Lowest energy is:

$$\rightarrow \text{triplet ground state energy: } E_T = E_1 + E_2 = 5E_1, \text{ w/ } E_1 = \frac{\pi^2\hbar^2}{2ma^2}. \tag{2}$$

For S states, can have $m = n = 1$ in Eq. (1). The overlap integral is $\equiv 1$, and we get:

$$\rightarrow \text{singlet ground state energy: } E_S = (E_1 + E_1) + (E_1 + E_1) = 4E_1, \text{ w/ } E_1 = \frac{\pi^2\hbar^2}{2ma^2}. \tag{3}$$

The singlet is lower in energy, as it is for the He ground state.

(C) 3) Using the symmetry of the δ-fcn, and by the sort of calculation as in Eq. (1)...

$$\rightarrow V_{T,S} = -\lambda\langle u_{T,S}|V|u_{T,S}\rangle = -2\lambda\left\{\int dx\,|\phi_m(x)|^2|\phi_n(x)|^2 \mp \int dx\,|\phi_m(x)|^2|\phi_n(x)|^2\right\} \tag{4}$$

The perturbation V_T on the triplet state E_T vanishes. The singlet state E_S is reduced by an amount: $V_S = -4\lambda\int_{-a/2}^{+a/2} dx\,|\phi_1(x)|^4 = -6\lambda/a$.