Plane wave solutions to free particle Dirac Eqtn.

The Dirac Equation: Free Particle Solutions

For a free particle, solutions to the Dirac Eqtn are plane waves, just as for the nonrelativistic case. We now analyse such plane wave solutions.

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1) It is convenient to work with the original Dirac form [pp. DE4-5, Eqs.(6)&(11)]:

$$\underline{i\hbar\,\partial\psi/\partial t = (M\beta + c\boldsymbol{\alpha}\cdot\mathbf{p})\psi}\ \int^{w/}\ M = mc^2,\ \beta = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},\ \boldsymbol{\alpha} = \begin{pmatrix}0 & \boldsymbol{\sigma}\\ \boldsymbol{\sigma} & 0\end{pmatrix};$$

for $\psi = \begin{pmatrix}\varphi\\ \chi\end{pmatrix}$, get

$$\begin{aligned} &i\hbar\,\partial\varphi/\partial t = c(\boldsymbol{\sigma}\cdot\mathbf{p})\chi + M\varphi,\\ &i\hbar\,\partial\chi/\partial t = c(\boldsymbol{\sigma}\cdot\mathbf{p})\varphi - M\chi. \end{aligned} \tag{1}$$

φ & χ are each two-component spinors, and Eqs (1) here are the same as Eqs. (12), p. DE6. Now assume plane wave solutions of the form...

$$\rightarrow \begin{pmatrix}\varphi\\ \chi\end{pmatrix} = \begin{pmatrix}\varphi_0\\ \chi_0\end{pmatrix} e^{\frac{i}{\hbar}(\mathbf{p}\cdot\mathbf{r} - Et)}\ \left\{\begin{array}{l} i\hbar\,\partial/\partial t \leftrightarrow E, \text{ cnst energy (and momentum)},\\ \varphi_0\ \&\ \chi_0 = \text{cnst spinors, free for norm}^{\underline{z}}\text{n}. \end{array}\right. \tag{2}$$

With this Ansatz, Eqs. (1) inter-relate φ_0 & χ_0, as

$$\left[\!\!\left[\begin{array}{lll} E\varphi_0 = M\varphi_0 + c(\boldsymbol{\sigma}\cdot\mathbf{p})\chi_0 & \Rightarrow & \varphi_0 = \left[\dfrac{c(\boldsymbol{\sigma}\cdot\mathbf{p})}{E-M}\right]\chi_0 \leftarrow ①\\ E\chi_0 = -M\chi_0 + c(\boldsymbol{\sigma}\cdot\mathbf{p})\varphi_0 & \Rightarrow & \chi_0 = \left[\dfrac{c(\boldsymbol{\sigma}\cdot\mathbf{p})}{E+M}\right]\varphi_0 \leftarrow ② \end{array}\right.\right. \tag{3}$$

For ① & ② to be self-consistent, the RHS coefficients must obey...

$$\rightarrow \left[\frac{c(\boldsymbol{\sigma}\cdot\mathbf{p})}{E-M}\right]\left[\frac{c(\boldsymbol{\sigma}\cdot\mathbf{p})}{E+M}\right] = 1 \Rightarrow E^2 = M^2 + c^2\underbrace{(\boldsymbol{\sigma}\cdot\mathbf{p})^2}_{(=\mathbf{p}^2,\ \text{by Dirac identity})},\ \text{or}\ \boxed{E = \pm E_p\ \ w/\ E_p = \sqrt{M^2 + (c\mathbf{p})^2}} \tag{4}$$

We recover the energy-momentum relation in this way, and get both (+)ve & (−)ve energy eigenvalues $\pm E_p$. Further, using ②, then ①, we can choose...

$$\left[\!\!\left[\begin{array}{ll} \underline{\text{for } E = +E_p}:\ \varphi_0 \text{ is free,} & \underline{\text{for } E = -E_p}:\ \chi_0 \text{ is free,}\\ \qquad \chi_0 = \left[\dfrac{c(\boldsymbol{\sigma}\cdot\mathbf{p})}{E_p + M}\right]\varphi_0; & \qquad \varphi_0 = (-)\left[\dfrac{c(\boldsymbol{\sigma}\cdot\mathbf{p})}{E_p + M}\right]\chi_0. \end{array}\right]\right\} \tag{5}$$

Details of Dirac Plane-Wave Solutions.

2) Since φ_0 & χ_0 are both 2-component spinors, we can use the elementary forms:

$$\rightarrow \varphi_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \underline{\underline{u}} \int^{\text{"up"}}_{\text{spinor}}, \text{ or } \varphi_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \underline{\underline{d}} \int^{\text{"down"}}_{\text{spinor}}; \text{ also: } \chi_0 = u, \text{ or } d. \quad \text{(6)}$$

Then we construct four independent plane wave solutions, as follows...†

(+)ve ENERGY SOLUTIONS : $E = +E_p$

$$\varphi_0^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = u, \quad \chi_0^{(1)} = \left(\frac{c\sigma\cdot\mathbf{p}}{E_p+M}\right) u = \frac{1}{E_p+M}\begin{pmatrix} cp_3 \\ c(p_1+ip_2) \end{pmatrix}$$

or//

$$\varphi_0^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = d, \quad \chi_0^{(2)} = \left(\frac{c\sigma\cdot\mathbf{p}}{E_p+M}\right) d = \frac{1}{E_p+M}\begin{pmatrix} c(p_1-ip_2) \\ -cp_3 \end{pmatrix}$$

have used first of Eqs. (5) for χ_0 in terms of φ_0.

SO

$$\underline{\underline{\psi^{(1)}}} = \mathcal{N}\begin{bmatrix} u \\ \left(\frac{c(\sigma\cdot\mathbf{p})}{E_p+M}\right) u \end{bmatrix} e^{\frac{i}{\hbar}(\mathbf{p}\cdot\mathbf{r}-E_pt)}, \quad \underline{\underline{\psi^{(2)}}} = \mathcal{N}\begin{bmatrix} d \\ \left(\frac{c(\sigma\cdot\mathbf{p})}{E_p+M}\right) d \end{bmatrix} e^{\frac{i}{\hbar}(\mathbf{p}\cdot\mathbf{r}-E_pt)}. \quad \text{(7A)}$$

(−)ve ENERGY SOLUTIONS : $E = -E_p$

$$\chi_0^{(3)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = u, \quad \varphi_0^{(3)} = -\left(\frac{c\sigma\cdot\mathbf{p}}{E_p+M}\right) u = \frac{-1}{E_p+M}\begin{pmatrix} cp_3 \\ c(p_1+ip_2) \end{pmatrix}$$

or//

$$\chi_0^{(4)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = d, \quad \varphi_0^{(4)} = -\left(\frac{c\sigma\cdot\mathbf{p}}{E_p+M}\right) d = \frac{-1}{E_p+M}\begin{pmatrix} c(p_1-ip_2) \\ -cp_3 \end{pmatrix}$$

have used second of Eqs. (5) for φ_0 in terms of χ_0

SO

$$\underline{\underline{\psi^{(3)}}} = \mathcal{N}\begin{bmatrix} -\left(\frac{c(\sigma\cdot\mathbf{p})}{E_p+M}\right) u \\ u \end{bmatrix} e^{\frac{i}{\hbar}(\mathbf{p}\cdot\mathbf{r}+E_pt)}, \quad \underline{\underline{\psi^{(4)}}} = \mathcal{N}\begin{bmatrix} -\left(\frac{c(\sigma\cdot\mathbf{p})}{E_p+M}\right) d \\ d \end{bmatrix} e^{\frac{i}{\hbar}(\mathbf{p}\cdot\mathbf{r}+E_pt)}. \quad \text{(7B)}$$

In Eqs. (7A) & (7B), $\mathcal{N}$ is a common normalization cnst we are still free to choose. Each of the four planewaves $\psi^{(\nu)}$ above satisfies the field-free Dirac Eq. viz: $\underline{(\gamma_\mu p_\mu - imc)\psi^{(\nu)} = 0}$, $\nu = 1$ to 4. The reason for writing down these $\psi^{(\nu)}$ in such detail is that we can discover new & general dynamical features of Dirac's ψ from them -- e.g. that $\mathbf{p} \to (-)\,\mathbf{p}$ under charge conjugation, that the particle's spin is a cnst of the motion, etc.

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† $\sigma\cdot\mathbf{p} = \sigma_k p_k = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} p_1 + \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} p_2 + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} p_3 = \begin{pmatrix} p_3 & p_1-ip_2 \\ p_1+ip_2 & -p_3 \end{pmatrix}$.

Characteristics of Dirac PlaneWave Solutions.

REMARKS on Dirac planewaves, Eqs.(7A) & (7B).

1. The norm cnst $\mathcal{N}$ can be chosen so that over a finite volume V¶

$$\langle \psi^{(\mu)} | \psi^{(\nu)} \rangle = \int_V \psi^{(\mu)\dagger} \psi^{(\nu)} d^3x = \delta_{\mu\nu}. \tag{8}$$

This relation expresses the orthonormality of the four indpt $\psi^{(\nu)}$. Since $u^\dagger u = d^\dagger d = 1$, and $u^\dagger d = d^\dagger u = 0$, we find by inspection that when $\nu \neq \mu$, $\psi^{(\mu)\dagger} \psi^{(\nu)} = 0$. To get $\mathcal{N}$, we need only look at one integral, e.g.

$$\int_V \psi^{(1)\dagger} \psi^{(1)} d^3x = |\mathcal{N}|^2 \left[1 + \left(\frac{c\boldsymbol{\sigma}\cdot\mathbf{p}}{E_p+M}\right)^2\right] V = 1,$$

$$\Rightarrow \underline{\underline{\mathcal{N}}} = 1 \Big/ \left\{ V\left[1 + \frac{(c\mathbf{p})^2}{(E_p+M)^2}\right]\right\}^{1/2} = \underline{\underline{\sqrt{(E_p+M)/2E_pV}}}. \tag{9}$$

2. In the particle's rest frame: $\mathbf{p} = 0$ & $E_p = M = mc^2$. The planewave solutions of Eqs. (7A) & (7B) reduce (w/ V_0 the norm^z^n volume in rest frame) to:

$$\rightarrow \underline{\underline{\psi^{(1\&2)}_{rest}}} = \frac{1}{\sqrt{V_0}} e^{-\frac{i}{\hbar}Mt} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \& \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \underline{\underline{\psi^{(3\&4)}_{rest}}} = \frac{1}{\sqrt{V_0}} e^{+\frac{i}{\hbar}Mt} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \& \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}. \tag{10}$$

Here the linear independence and orthogonality of the $\psi^{(\nu)}$ is manifest.

3. Per the remark on p. DE6, Eq.(13), the parities of the bispinors φ & χ in $\psi = \begin{pmatrix}\varphi \\ \chi\end{pmatrix}$ turn out to be OK for the Dirac planewaves $\psi^{(\nu)}$ in Eqs.(7A) & (7B)...

$$\left[\begin{array}{ll} \text{(+)ve energy solutions} & \left\{\begin{array}{l} \varphi_0 = u = \binom{1}{0}, \text{ or } d = \binom{0}{1} \text{ has (+)ve parity;} \\ \chi_0 = [c\boldsymbol{\sigma}\cdot\mathbf{p}/(E_p+M)]\,\varphi_0 \text{ has (−)ve parity, since } \boldsymbol{\sigma}\cdot\mathbf{p} \rightarrow (-)\,\boldsymbol{\sigma}\cdot\mathbf{p}. \end{array}\right. \\ \text{(−)ve energy solutions} & \left\{\begin{array}{l} \chi_0 = u \text{ or } d \text{ has (+)ve parity;} \\ \varphi_0 = -[c\boldsymbol{\sigma}\cdot\mathbf{p}/(E_p+M)]\,\chi_0 \text{ has (−)ve parity.} \end{array}\right. \end{array}\right. \tag{11}$$

The relative parity of the upper & lower bispinors φ & χ is always (−)ve.

¶ If $\theta = \begin{bmatrix}\theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4\end{bmatrix}$ is a column matrix, then the Hermitian conjugate $\theta^\dagger$ is a row matrix, complex-conjugated: $\theta^\dagger = [\theta_1^*, \theta_2^*, \theta_3^*, \theta_4^*]$. The operation $\theta^\dagger\theta$ is the usual row-column multiplication: $\theta^\dagger\theta = \theta_\mu^*\theta_\mu$.

Parity of the Dirac $\psi^{(\nu)}$. Large & Small Comps. Charge Conjugation.

$\boxed{\text{REMARKS}}$ on Dirac planewaves (cont'd).

4. By requiring that the Dirac Eqtn is overall parity-invariant (i.e. does not alter its physical content upon changing from right-handed to left-handed cds) we can show that under the parity operation $\mathbb{P}$:

$$\underline{\underline{\mathbb{P}\psi^{(\nu)} = \gamma_4\psi^{(\nu)}}} \Rightarrow \begin{cases} \psi^{(1,2)} \longrightarrow (+)\,\psi^{(1,2)} \ldots \; E = +E_p \text{ solns have } (+) \text{ parity;} \\ \psi^{(3,4)} \longrightarrow (-)\,\psi^{(3,4)} \ldots \; E = -E_p \text{ solns have } (-) \text{ parity.} \end{cases} \quad (12)$$

Details of this proof are left as an exercise for the student.

5. For the solns of Eqs. (7A) & (7B), there is both a "large" and "small" part of the wavefcns. For $v \ll c$ & $E_p \sim M$, we have...

(+)ve ENERGY SOLNS : $|\varphi_0|^2 \sim |u \text{ or } d|^2 = 1$, $|\chi_0|^2 \sim \left|\frac{c\boldsymbol{\sigma}\cdot\mathbf{p}}{E_p+M}\right|^2 \sim \left(\frac{v}{2c}\right)^2 \ll 1$;

(−)ve ENERGY SOLNS : $|\chi_0|^2 \sim |u \text{ or } d|^2 = 1$, $|\varphi_0|^2 \sim \left|\frac{c\boldsymbol{\sigma}\cdot\mathbf{p}}{E_p+M}\right|^2 \sim \left(\frac{v}{2c}\right)^2 \ll 1$;

So

$$\left[\!\!\left[\begin{array}{l} \text{for } E = +E_p \text{ solns : lower bispinor } |\chi_0| \ll \text{ upper bispinor } |\varphi_0|; \\ \text{for } E = -E_p \text{ solns : upper bispinor } |\varphi_0| \ll \text{ lower bispinor } |\chi_0|. \end{array}\right]\right\} \quad (13)$$

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3) We have promised previously [p. DE 12, below Eq. (42)] to show that upon charge conjugation, i.e. $\psi \rightarrow \psi_c = \gamma_2\psi^*$, the particle's momentum reverses: $\mathbf{p} \rightarrow (-)\mathbf{p}$. We now show $\mathbf{p} \rightarrow (-)\mathbf{p}$ for charge conjugated Dirac planewaves. For $\psi^{(1)}$ [Eq. (7A)]:

$$\rightarrow \psi_c^{(1)}(+\mathbf{p}) = \gamma_2\left[\psi^{(1)}(+\mathbf{p})\right]^* = \begin{pmatrix} 0 & -i\sigma_2 \\ i\sigma_2 & 0 \end{pmatrix} \cdot \mathcal{N}\begin{bmatrix} u \\ \left(\frac{c\boldsymbol{\sigma}\cdot\mathbf{p}}{E_p+M}\right)u \end{bmatrix}^* e^{-\frac{i}{\hbar}(\mathbf{p}\cdot\mathbf{r} - E_p t)}$$

$$= \mathcal{N}\begin{bmatrix} -i\sigma_2\left(\frac{c(\boldsymbol{\sigma}\cdot\mathbf{p})^*}{E_p+M}\right)u \\ +i\sigma_2 u \end{bmatrix} e^{\frac{i}{\hbar}[(-\mathbf{p})\cdot\mathbf{r} + E_p t]}, \quad \text{w/ } u = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \quad (14)$$

But : $+i\sigma_2 u = i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} = -d$.

And : $(\boldsymbol{\sigma}\cdot\mathbf{p})^* = \begin{pmatrix} p_3 & p_1 - ip_2 \\ p_1 + ip_2 & -p_3 \end{pmatrix}^* = \begin{pmatrix} p_3 & p_1 + ip_2 \\ p_1 - ip_2 & -p_3 \end{pmatrix}$. the p_k are real.

(next page)

ϕ 507 End Game

25 Apr. 94

DATE	LECTURE	REMARKS
Mon. 4/25	HOLIDAY (Dirac's birthday)	Set #27 assigned.
Wed. 4/27	finish Dirac planewaves, pp DE 17-19. start Dirac Eqtn: nonrel$^{\text{c}}$ reduction, pp. 20-22.	Set #26 due.
Fri. 4/29	finish Dirac Eqtn: nonrel$^{\text{c}}$ reduction, pp. 22-23. start ZitterBewegung, pp. 24-27. Skim pp. 27-29.	–
Mon. 5/2	Lorentz covariance of Dirac Eqtn: skim pp. DE 30-38.	Set #27 due.
Wed. 5/4	Dirac particle in $(\mathbb{A}, i\phi)$: central force prob$^{\text{m}}$, pp. 39-48.	–
Fri. 5/6	Dirac's version of the H-atom: pp. 49-53.	(final preview?)
Mon. 5/9	EXAM WEEK.	–
Wed. 5/11		–
Fri. 5/13		–

The ϕ 507 Final Exam is scheduled for 4-6 PM on Thursday, 12 May, in room AJM 230.

I will try to extend the exam time by one hour -- to either 3-6 PM, or 4-7 PM -- and will inform you of the change ASAP.

Dick Robiscoe

$(+q,+E,+\mathbb{p})_c \to (-q,-E,-\mathbb{p})$. Spin as a constant of the motion.

So: $-i\sigma_2(\sigma\cdot\mathbb{p})^* u = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} p_3 & p_1+ip_2 \\ p_1-ip_2 & -p_3 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = -\begin{pmatrix} p_1-ip_2 \\ -p_3 \end{pmatrix} = -(\sigma\cdot\mathbb{p})d$

Then Eq. (14) can be rewritten as...

note sign change

$$\rightarrow \psi_c^{(1)}(+\mathbb{p}) = (-)\mathcal{N}\begin{bmatrix} \left(\frac{c\sigma\cdot\mathbb{p}}{E_p+M}\right)d \\ d \end{bmatrix} e^{\frac{i}{\hbar}[(-\mathbb{p})\cdot\mathbb{r}+E_pt]} = (-)\psi^{(4)}(-\mathbb{p}). \qquad (15)$$

By similar calculations, we find

$$\rightarrow \psi_c^{(2)}(+\mathbb{p}) = +\psi^{(3)}(-\mathbb{p}),\quad \psi_c^{(3)}(+\mathbb{p}) = +\psi^{(2)}(-\mathbb{p}),\quad \psi_c^{(4)}(+\mathbb{p}) = -\psi^{(1)}(-\mathbb{p}). \qquad (16)$$

In each case, the charge conjugate spinor component belongs to the <u>opposite</u> energy state and has the direction of $\mathbb{p}$ <u>reversed</u>. So the assertion is demonstrated, at least for Dirac planewaves $\psi^{(\nu)}$: $[\psi(+q,+E,+\mathbb{p})]_c = \Psi(-q,-E,-\mathbb{p})$, is a wavefcn describing a particle with signs of q, E and $\mathbb{p}$ all reversed.

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4) Dirac's wave eqtn ultimately describes spin $\frac{1}{2}$ particles (e.g. electrons). We now demonstrate that this is plausible by showing that for Dirac planewaves $\psi^{(\nu)}$ there is enough freedom to accommodate the <u>spin</u> as a cnst of the motion.

PROPOSITION: $\Sigma\cdot\hat{p}$ is a cnst of the motion,

w/ $\Sigma = \begin{pmatrix} \sigma & 0 \\ 0 & \sigma \end{pmatrix}$ ← "super spin" matrix (4×4), &/ $\hat{p} = \mathbb{p}/p$ ← unit vector along direction of motion. (17)

To show this, we use the QM equation-of-motion for the operator $\Omega = \Sigma\cdot\hat{p}$. Since Ω is t-independent, it will be a cnst of the motion iff $[\mathcal{H},\Omega]=0$,¶ i.e. if it commutes with Dirac's Hamiltonian operator $\mathcal{H}$. So, look at...

¶ For wave eqtn $i\hbar\partial\psi/\partial t = \mathcal{H}\psi$, with a Hermitian $\mathcal{H}$ (as per Dirac), the expectation value of an operator Ω is: $\langle\Omega\rangle = \int \psi^\dagger\Omega\psi\, d^3x$. Then, by direct differentiation under the integral: $\frac{d}{dt}\langle\Omega\rangle = \langle\partial\Omega/\partial t\rangle + \frac{i}{\hbar}\langle[\mathcal{H},\Omega]\rangle$. When Ω does <u>not</u> depend explicitly on time, $\partial\Omega/\partial t = 0$, and Ω will be a constant of the motion, in the sense that $\frac{d}{dt}\langle\Omega\rangle = 0$, if and only if $[\mathcal{H},\Omega]=0$.

Eigenvalues of the helicity operator $\Sigma\cdot\hat{p}$.

$[\mathcal{H}, \Omega]$ $\int^{w/}$ $\mathcal{H} = \beta mc^2 + c\alpha\cdot\mathbf{p}$, &/ $\Omega = \Sigma\cdot\hat{p}$,

and: $\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\alpha = \begin{pmatrix} 0 & \sigma \\ \sigma & 0 \end{pmatrix}$, $\Sigma = \begin{pmatrix} \sigma & 0 \\ 0 & \sigma \end{pmatrix}$;

So// $$[\mathcal{H}, \Omega] = mc^2 \underbrace{[\beta, \Sigma\cdot\hat{p}]}_{①} + cp\underbrace{[\alpha\cdot\hat{p}, \Sigma\cdot\hat{p}]}_{②}. \tag{18}$$

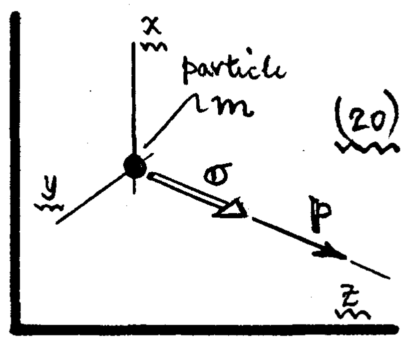
Let $z = \sigma\cdot p$. Then...

$$① = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix} - \begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} z & 0 \\ 0 & -z \end{pmatrix} - \begin{pmatrix} z & 0 \\ 0 & -z \end{pmatrix} \equiv 0,$$

$$② = \begin{pmatrix} 0 & z \\ z & 0 \end{pmatrix}\begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix} - \begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix}\begin{pmatrix} 0 & z \\ z & 0 \end{pmatrix} = \begin{pmatrix} 0 & z^2 \\ z^2 & 0 \end{pmatrix} - \begin{pmatrix} 0 & z^2 \\ z^2 & 0 \end{pmatrix} \equiv 0;$$

and// $[\mathcal{H}, \Sigma\cdot\hat{p}] \equiv 0$, so: $\underline{\langle\Sigma\cdot\hat{p}\rangle = \text{const-of-motion}}$ (for free particles). (19)

Next, what are the eigenvalues of $\Sigma\cdot\hat{p}$? To find out, apply $\Sigma\cdot\hat{p}$ to the Dirac planewaves $\psi^{(\nu)}$. Choose $\mathbf{p}$ along the z-axis, so that



$$\left[\begin{aligned} &\mathbf{p} = (0,0,p),\ \sigma\cdot\hat{p} = \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \text{ and: } \Sigma\cdot\hat{p} = \begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}. \\ &\text{Consider: } \psi^{(1)} = N\begin{bmatrix} u \\ \left(\frac{c\sigma_3 p}{E_p+M}\right)u \end{bmatrix} e^{\frac{i}{\hbar}(pz - E_p t)},\ u = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \end{aligned}\right. \tag{20}$$

So// $$(\Sigma\cdot\hat{p})\psi^{(1)} = N\begin{bmatrix} \sigma_3 u \\ \left(\frac{c\sigma_3 p}{E_p+M}\right)\sigma_3 u \end{bmatrix} e^{\frac{i}{\hbar}(\text{"})} = (+1)\,\psi^{(1)},\ \text{since: } \sigma_3 u = (+1)u. \tag{21}$$

Similarly: $(\Sigma\cdot\hat{p})\psi^{(2)} = (-1)\,\psi^{(2)}$, $(\Sigma\cdot\hat{p})\psi^{(3)} = (+1)\,\psi^{(3)}$, $(\Sigma\cdot\hat{p})\psi^{(4)} = (-1)\,\psi^{(4)}$.

i.e.// $$\underline{\underline{\Sigma\cdot\hat{p} = \begin{pmatrix} +1 & & & \\ & -1 & & \\ & & +1 & \\ & & & -1 \end{pmatrix}}},\ \text{w.r.t. Dirac's free particle wavefcns } \psi^{(\nu)}. \tag{22}$$

$\Sigma\cdot\hat{p}$ is called the "<u>helicity operator</u>". It evidently measures the projection on the particle's momentum $\mathbf{p}$ of an intrinsic quantity σ associated with the particle. The projection of σ on the given axis $\hat{p}$ can have eigenvalues ± 1 only. What else can σ be but the Pauli operators representing spin $\frac{1}{2}$?.

Spin operator in the rest frame. Classification of Dirac $\psi^{(\nu)}$'s.

What is going on here is clearly seen in the particle's rest frame. There [per Eq.(10)]:

$$\underset{(+)\text{ve } E_p \text{ solns}}{\underline{\underline{\psi_{rest}^{(1\,\&\,2)}}}} = N_0\, e^{-\frac{i}{\hbar}Mt}\begin{bmatrix}1\\0\\0\\0\end{bmatrix} \,\&\, \begin{bmatrix}0\\1\\0\\0\end{bmatrix}, \quad \underset{(-)\text{ve } E_p \text{ solns}}{\underline{\underline{\psi_{rest}^{(3\,\&\,4)}}}} = N_0\, e^{+\frac{i}{\hbar}Mt}\begin{bmatrix}0\\0\\1\\0\end{bmatrix} \,\&\, \begin{bmatrix}0\\0\\0\\1\end{bmatrix}. \tag{23}$$

Now define the spin operator (for motion along the z-axis, i.e. 3-axis):

$$\left.\begin{array}{l} \rightarrow S_3 = \frac{\hbar}{2}\boldsymbol{\Sigma}\cdot\hat{\mathbf{p}} = \frac{\hbar}{2}\begin{pmatrix}\sigma_3 & 0\\ 0 & \sigma_3\end{pmatrix}, \quad \text{w/ } \sigma_3 = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}; \\ \text{then// } S_3\psi_{rest}^{(1)} = \left(+\frac{\hbar}{2}\right)\psi_{rest}^{(1)}, \quad S_3\psi_{rest}^{(3)} = \left(+\frac{\hbar}{2}\right)\psi_{rest}^{(3)} \;\surd\; \underline{\psi_{rest}^{(1\,\&\,3)} \text{ are spin } \hbar/2 \text{ "up" states;}} \\ \quad S_3\psi_{rest}^{(2)} = \left(-\frac{\hbar}{2}\right)\psi_{rest}^{(2)}, \quad S_3\psi_{rest}^{(4)} = \left(-\frac{\hbar}{2}\right)\psi_{rest}^{(4)} \;\surd\; \underline{\psi_{rest}^{(2\,\&\,4)} \text{ are spin } \hbar/2 \text{ "down" states.}} \end{array}\right\} \tag{24}$$

There are just two degrees of freedom for S_3 (i.e. eigenvalues $\pm\hbar/2$, or "up" and "down"), so at most S_3 can be a spin ½ operator. It is <u>remarkable</u> that the Dirac Eqtn accommodates an internal spin ½ variable from the outset.

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5) By the assignments we have made, we can now identify the Dirac free-particle (i.e. plane-wave) solutions fully, as follows [consult Eqs(7) for explicit $\psi^{(\nu)}$'s]:

$$\left.\begin{array}{llll} \psi^{(1)} & \text{describes } (+q, +E, +\mathbf{p}) & \text{with spin } +\hbar/2 & (\text{spin "up", (+)ve helicity}); \\ \psi^{(2)} & \quad\text{"} \quad (+, +, +) & \quad\text{"} \quad\text{"} \quad -\hbar/2 & (\text{spin "down", (−)ve helicity}); \end{array}\right\} \tag{25A}$$

$$\left.\begin{array}{llll} \psi^{(3)} & \quad\text{"} \quad (-q, -E, -\mathbf{p}) & \quad\text{"} \quad\text{"} \quad +\hbar/2 & (\text{spin "up", (+)ve helicity}); \\ \psi^{(4)} & \quad\text{"} \quad (-, -, -) & \quad\text{"} \quad\text{"} \quad -\hbar/2 & (\text{spin "down", (−) helicity}). \end{array}\right\} \tag{25B}$$

This classification exhausts the 4 degrees of freedom in these 4-component spinors. Accounting shows that 2 degrees of freedom are needed for the ($\pm$) energies, while the other 2 are required for the ($\pm$) helicities.

<u>NOTE</u> Under charge conjugation: $\psi \rightarrow \psi_c = \gamma_2\psi^*$, have: $\psi^{(1)} \rightarrow -\psi^{(4)}$, $\psi^{(2)} \rightarrow +\psi^{(3)}$, $\psi^{(3)} \rightarrow +\psi^{(2)}$, and $\psi^{(4)} \rightarrow -\psi^{(1)}$, with sign reversal of $(q, E, \mathbf{p})$ and <u>also</u> helicity.