Effect of Finite σ on Wave Propagation.

6) We backtrack a bit to see what effect finite conductivity σ has on the wave propagation. From Eqs. (3) above, for $\rho=0$ and in 1D, $\mathbb{E}$ & $\mathbb{B}$ components obey:

$\rightarrow u_{xx} - \alpha u_t - (1/v^2)\, u_{tt} = 0$ ... define: $\beta = \alpha v^2 = \frac{4\pi\mu\sigma}{c^2}\cdot\frac{c^2}{\mu\epsilon} = \frac{4\pi\sigma}{\epsilon}$;

so// $\left[\!\left[u_{tt} + \beta u_t - v^2 u_{xx} = 0 \right]\!\right]$ ... substitute: $u(x,t) = e^{-\gamma t}\,\psi(x,t)$; (13)

... with substitution: $u = \psi e^{-\gamma t}$, and a bit of arithmetic, get...

$$\psi_{tt} + (\beta - 2\gamma)\psi_t + \gamma(\gamma-\beta)\psi - v^2\psi_{xx} = 0 \leftarrow \text{choose } \gamma = \beta/2;$$

so// $\left[\!\left[\psi_{tt} - v^2\psi_{xx} - \frac{1}{4}\beta^2\psi = 0 \right]\!\right] \leftarrow$ for: $u = \psi e^{-\frac{1}{2}\beta t}$ (14)

Try solutions to Eq. (14) in the form: $\psi(x,t) \sim e^{i(kx - \Omega t)}$. Then (14) $\Rightarrow$

$\left[\Omega^2 - (k^2v^2 - \frac{1}{4}\beta^2)\right]\psi = 0$ ... OK, if: $\Omega = \pm\omega\sqrt{1-(\beta/2\omega)^2}$. (15)

$\uparrow \omega = kv$, as before.

So, for finite $\beta = (4\pi/\epsilon)\sigma$, the solutions are...

$$\boxed{u(x,t) = A(k)\,e^{-\frac{1}{2}\beta t}\cdot e^{i(kx \mp \omega t\sqrt{1-(\beta/2\omega)^2})}} \quad (16)$$

$\uparrow$ attenuation $\qquad \uparrow$ frequency cutoff ... (freqs. $0 \leq \omega \leq \beta/2$ do not propagate; they get wasted)

Evidently, conductivity $\sigma > 0$ changes the wave propagation radically. We shall return to these effects later. For now, we go back to assuming $\sigma \to 0$.

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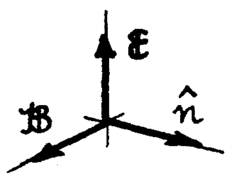
7) Return to unattenuated planewaves. They are classified w.r.t. polarization...

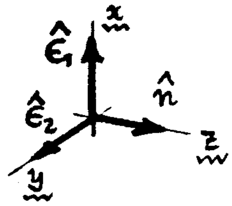
$$\left[\!\left[\begin{array}{l} (\mathbb{E}, \mathbb{B}) = (\mathcal{E}, \mathcal{B})\, e^{i(k\hat{n}\cdot\mathbf{r} - \omega t)}, \ \text{w/}\ \omega = kv, \\ \text{with: } \hat{n}\cdot\mathcal{E} = \hat{n}\cdot\mathcal{B} = 0,\ \mathcal{B} = \sqrt{\mu\epsilon}\,(\hat{n}\times\mathcal{E}). \end{array} \right.$$ (17)



If $\hat{n}$ (propagation direction) is real, the last relation [see Eq. (10c)] shows $\mathbb{E}$ & $\mathbb{B}$ are in phase. Customarily, one defines two "polarization directions" $\hat{\epsilon}_1$ & $\hat{\epsilon}_2$ to form an

Polarization for plane waves. (TAXONOMY)

orthonormal triad with $\hat{n}$: $\hat{\epsilon}_1 \times \hat{\epsilon}_2 = \hat{n}$, etc.¶ There are then two "polarizations" for $\mathbb{E}$:



$$\left[\begin{array}{l}\{\text{polarization \#1}\}\ \mathcal{E}_1 = E_1\hat{\epsilon}_1,\ \mathbb{B}_1 = (\sqrt{\mu\epsilon}\,E_1)\hat{\epsilon}_2\,; \\ \{\text{polarization \#2}\}\ \mathcal{E}_2 = E_2\hat{\epsilon}_2,\ \mathbb{B}_2 = -(\sqrt{\mu\epsilon}\,E_2)\hat{\epsilon}_1\,.\end{array}\right. \tag{18}$$

These waves are independent (E_1 & E_2 are indpt amplitudes). NOTE: a "polarization" always refers to the direction of the lightwave's E-field (not B-field). The polarizations in Eq.(18) are called "linear polarizations" because $\mathbb{E}$ is fixed along $\hat{\epsilon}_1$ or $\hat{\epsilon}_2$.

Waves can also be circularly or elliptically polarized. That story goes as follows.

$$\left[\begin{array}{l}\text{let}/\ \mathbb{E}_m = \hat{\epsilon}_m E_m\, e^{i(\mathbb{k}\cdot\mathbb{r}-\omega t)},\ m = 1\,\&\,2,\ \text{be two (linear) indpt waves;} \\ \text{w}/\!/\ \text{associated } \mathbb{B}_m = \sqrt{\mu\epsilon}\ \hat{n}\times\mathbb{E}_m\,;\ \text{s}/\ E_m = \text{arbitrary complex amplitudes.}\end{array}\right\} \tag{19}$$

Combine the $\mathbb{E}_m$'s to form the most general planewave going in the direction $\mathbb{k}$:

$$\rightarrow \mathbb{E} = \mathbb{E}_1 + \mathbb{E}_2 = (\hat{\epsilon}_1 E_1 + \hat{\epsilon}_2 E_2)\, e^{i(\mathbb{k}\cdot\mathbb{r}-\omega t)}\,. \tag{20}$$

To keep track of the amplitude of $\mathbb{E}$, for arbitrary relative phase of $\mathbb{E}_1$ & $\mathbb{E}_2$,

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¶ $\hat{n}$ does not have to be real (usu. the k in $k\hat{n}$ is not real, due to attenuation). Write

$$\hat{n} = \hat{n}_R + i\hat{n}_I \Rightarrow e^{i(k\hat{n}\cdot\mathbb{r}-\omega t)} = \underbrace{(e^{-k\hat{n}_I\cdot\mathbb{r}})}_{\text{attenuation}}\ \underbrace{e^{i(k\hat{n}_R\cdot\mathbb{r}-\omega t)}}_{\text{ordinary planewave}}$$

This complex $\hat{n}$ has the following algebra...

$$\hat{n}\cdot\hat{n} = (n_R^2 - n_I^2) + 2i\,\hat{n}_R\cdot\hat{n}_I = 1 \Rightarrow \begin{cases} n_R^2 - n_I^2 = 1, \\ \hat{n}_R\cdot\hat{n}_I = 0. \end{cases}$$

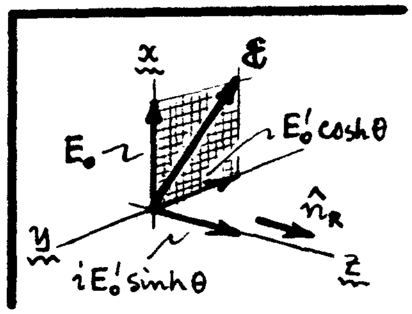
NOTE: $|\hat{n}|^2 = n_R^2 + n_I^2 \neq 1$. Instead, we impose $\hat{n}\cdot\hat{n} = 1$.

So/ $\boxed{\hat{n} = \hat{e}_z\cosh\theta + i\,\hat{e}_y\sinh\theta}$, satisfies these conditions.

An $\mathcal{E}$ field obeying transversality ($\hat{n}\cdot\mathcal{E} = 0$) is then...

$$\rightarrow \mathcal{E} = \hat{e}_x E_0 + (i\,\hat{e}_z\sinh\theta - \hat{e}_y\cosh\theta)\,E_0'$$

Here E_0 & E_0' = cnsts. An $\mathbb{E}$-wave propagating along a complex $\hat{n}$ generally has a longitudinal component.



Circularly Polarized Planewaves. (TAXONOMY)

We define new unit vectors $\hat{\epsilon}_\pm$ as combinations of $\hat{\epsilon}_1$ & $\hat{\epsilon}_2$...

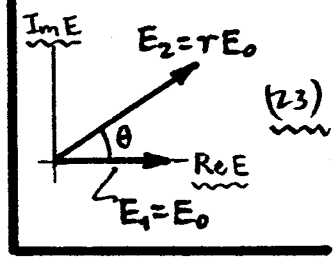
$$\left[\hat{\epsilon}_\pm = \frac{1}{\sqrt{2}}(\hat{\epsilon}_1 \pm i\hat{\epsilon}_2) \Rightarrow \begin{matrix} \hat{\epsilon}_\pm^* \cdot \hat{\epsilon}_\pm = 1 \\ \hat{\epsilon}_\pm^* \cdot \hat{\epsilon}_\mp = 0 \end{matrix}, \text{ and } \begin{cases} \hat{\epsilon}_1 = \frac{1}{\sqrt{2}}(\hat{\epsilon}_- + \hat{\epsilon}_+), \\ \hat{\epsilon}_2 = \frac{i}{\sqrt{2}}(\hat{\epsilon}_- - \hat{\epsilon}_+). \end{cases}\right. \tag{21}$$

Put these last expressions for $\hat{\epsilon}_{1,2}$ into the combined wave of Eq. (20) to get...

$$\rightarrow \boxed{\mathbb{E} = (\hat{\epsilon}_- E_- + \hat{\epsilon}_+ E_+)\, e^{i(\mathbb{k}\cdot\mathbb{r} - \omega t)}}, \text{ w/ } E_\mp = \frac{1}{\sqrt{2}}(E_1 \pm i E_2). \tag{22}$$

This form for $\mathbb{E}$ is particularly convenient for keeping track of phases.

$$\text{Let } \begin{cases} E_1 = E_0, \\ E_2 = r E_0 e^{i\theta}. \end{cases} \Bigg\| \begin{matrix} E_0 \text{ \& } r \text{ (arbitrary) are } \underline{\text{real}}; \\ \theta = \underline{\text{relative phase}} \text{ between } E_1 \text{ \& } E_2; \end{matrix} \tag{23}$$

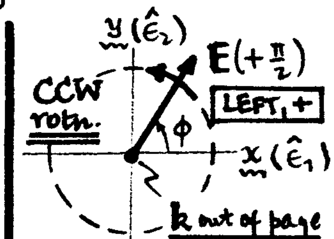


$$\xrightarrow{\text{so}} E_\mp = \frac{E_0}{\sqrt{2}}\left[1 + r e^{i(\theta \pm \frac{\pi}{2})}\right], \text{ in Eq. (22).} \tag{24}$$

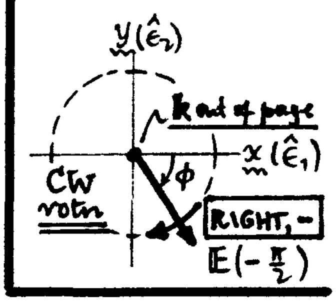
Now, we can look at the waves which result from some specific choices of θ & r. As a shorthand, let $\phi = \omega t - \mathbb{k}\cdot\mathbb{r}$. The principal choices & polarization classes are:

① $r = 1$, $\{\theta = \pm\frac{\pi}{2}\}$. Wave is $\left\{\begin{matrix}\text{left}\\\text{right}\end{matrix}\right\}$ circularly polarized, with $\left\{\begin{matrix}\text{positive}\\\text{negative}\end{matrix}\right\}$ "helicity".

$$\left[\begin{aligned} &\mathbb{E}\{\theta = \pm\tfrac{\pi}{2}\} = [(1 \mp r)\hat{\epsilon}_- + (1 \pm r)\hat{\epsilon}_+]\frac{E_0}{\sqrt{2}} e^{-i\phi} = \hat{\epsilon}_\pm(\sqrt{2}E_0)e^{-i\phi}, \\ &\xrightarrow{\text{and}} \operatorname{Re}\mathbb{E}\{\theta = \pm\tfrac{\pi}{2}\} = E_0(\hat{\epsilon}_1\cos\phi \pm \hat{\epsilon}_2 \sin\phi). \end{aligned}\right. \tag{25}$$

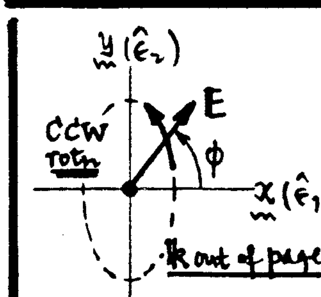


When viewed head-on, the waves for $\theta = \pm\frac{\pi}{2}$ have an $\mathbb{E}$-vector which appears to rotate CCW & CW resp., as sketched at right. The "helicity" is the relative sign between the propagation direction $\mathbb{k}$ and the ∡ momentum $\mathbb{L}$ carried by the (plane) wave.



② $r \neq 1$, $\theta = \frac{\pi}{2}$. Wave is elliptically polarized.

$$\left[\begin{aligned} &\mathbb{E} = [(1-r)\hat{\epsilon}_- + (1+r)\hat{\epsilon}_+]\frac{E_0}{\sqrt{2}}e^{-i\phi} = (\hat{\epsilon}_1 + ir\hat{\epsilon}_2)E_0 e^{-i\phi}, \\ &\xrightarrow{\text{and}} \operatorname{Re}\mathbb{E} = (E_0\cos\phi)\hat{\epsilon}_1 + (rE_0\sin\phi)\hat{\epsilon}_2. \end{aligned}\right. \tag{26}$$



We sketch the case $r > 1$ at right ($E_y = rE_x > E_x$). The choices in Eqs. (18), (24) & (25) represent all prototype polarizations.