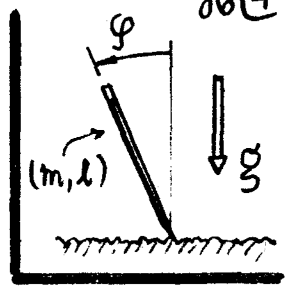
ϕ506 Problems



⑩ [15 pts]. A pencil of mass $m = 10\,gm$ and length $l = 15\,cm$ is balanced vertically on its point, on a horizontal (completely rough) surface. Neglect all complicating effects. Find the <u>maximum</u> <u>time</u> that elapses before the pencil must make an $\measuredangle\,\varphi = 5°$ with the vertical. <u>HINT</u>: verify and use the uncertainty relation in the form: $\Delta L \Delta\varphi \sim \hbar$, w/ L = $\measuredangle$ momentum of the pencil about its point.

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⑪ (A) An electron wave packet is originally confined to an atomic dimension of size $\sim 10^{-8}$ cm. How long will it take for the packet to spread out to <u>twice</u> its original dimension? How long before the electron packet exceeds the size of the solar system? (B) A 50 kgm person is seated at a desk, and is thereby localized to ~ 10 cm. Assume the person can be described by a wave packet (sic!), and sits at the desk for time T = known age of the universe. By how much does the person-packet increase in size during this interval? (C) In this time T, why don't the electron packets inside the person-packet just expand and make the person disappear? <u>NOTE</u>: this would save administrative costs.

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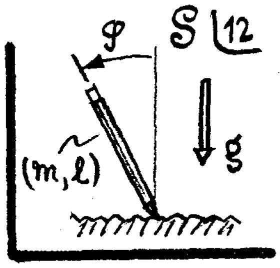
⑫ [15 pts]. Consider the Klein-Gordon (KG) Eq.: $\left[\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \left(\frac{mc}{\hbar}\right)^2\right]\phi = 0$.

(A) In analogy with the Schrödinger (Sch.) Eq., find a probability density ρ and current density $\mathbf{J}$ that obey the continuity eqtn: $\underline{\underline{\partial\rho/\partial t + \nabla\cdot\mathbf{J} = 0}}$.

(B) Show that with a proper choice of multiplicative cnsts, $\mathbf{J}$(KG Eq.) is the <u>same</u> as $\mathbf{J}$(Sch. Eq.). In this case, what is ρ(KG Eq.)? Is this ρ necessarily positive definite? Is "probability" conserved for the KG Eq.?

(C) Transform the KG wavefcn ϕ by: $\phi = \psi \exp[-i(mc^2/\hbar)t]$. Show that in the nonrelativistic limit ($c \to \infty$), ρ(KG Eq, in ψ) reduces to ρ(Sch. Eq).

Φ506 Solutions



⑩ [15 pts]. **Balancing a QM pencil on its point.**

1. The useful uncertainty relation is: $\underline{\underline{\Delta L\, \Delta\varphi \sim \hbar}}$, w/ $\varphi = \measuredangle$ of rotation about its point (the point remains fixed on a perfectly rough surface), and $L = \measuredangle$ momentum of pencil about its point. This relation is just $\underline{\Delta p\, \Delta x \sim \hbar}$ applied to the free end of the pencil, since -- with ℓ = pencil length -- the free end's momentum is $p = L/\ell$, after moving through an arc length $x = \ell\varphi$. Then $\Delta p\, \Delta x = \left(\frac{\Delta L}{\ell}\right)(\ell\Delta\varphi) = \Delta L\Delta\varphi \sim \hbar$.

Now $L = I\dot{\varphi}$, where I = pencil's moment of inertia about its point. If the pencil's mass m is distributed $\sim$ uniformly along its length ℓ, then we can put $I = \frac{1}{3}m\ell^2$. The above uncertainty relation $\Delta L\, \Delta\varphi \sim \hbar$ then yields...

$$\underline{\underline{\Delta\dot{\varphi}\, \Delta\varphi \sim \hbar/I}}, \quad I = \tfrac{1}{3}m\ell^2. \tag{1}$$

$\Delta\varphi$ & $\Delta\dot{\varphi}$ are the initial uncertainties in the pencil's $\measuredangle$ position & $\measuredangle$ velocity.

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2. As the pencil begins to fall, pivoted at its (fixed) point, its equation-of-motion is: $\underline{\underline{I\ddot{\varphi} = \frac{1}{2}mg\ell\sin\varphi}} \simeq \underline{\underline{\frac{1}{2}mg\ell\,\varphi}}$, for "small" $\measuredangle$s φ. The solution is

$$\rightarrow \varphi(t) = \varphi(0)\cosh\omega t + \frac{1}{\omega}\dot{\varphi}(0)\sinh\omega t, \quad \text{w/ } \underline{\underline{\omega^2}} = \tfrac{1}{2}mg\ell/I = 3g/2\ell. \tag{2}$$

$\varphi \sim$ small, even as $\omega t \rightarrow$ large, so that: $\varphi(t) \simeq \left[\varphi(0) + \frac{1}{\omega}\dot{\varphi}(0)\right]e^{\omega t}$...

$$\Rightarrow \underline{\underline{t(\varphi) \simeq \frac{1}{\omega}\ln\left\{\omega\varphi/[\omega\varphi(0) + \dot{\varphi}(0)]\right\}}}. \tag{3}$$

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3. $t(\varphi)$ is the time required to rotate through a (small) $\measuredangle\,\varphi$. Classically, $\varphi(0)$ & $\dot{\varphi}(0)$ are not correlated, but QMly they are, through Eq. (1). We take $\underline{\varphi(0) = \Delta\varphi}$, $\underline{\dot{\varphi}(0) = \Delta\dot{\varphi} = (\hbar/I)/\Delta\varphi}$, form: $t(\Delta\varphi) \simeq \frac{1}{\omega}\ln\left\{\omega\varphi/\left[\omega\Delta\varphi + \frac{\hbar/I}{\Delta\varphi}\right]\right\}$, and maximize t by imposing: $\partial t/\partial\Delta\varphi = 0 \Rightarrow \Delta\varphi = \sqrt{\hbar/I\omega}$. Then...

$$\boxed{t_{max}(\varphi) = \frac{1}{\omega}\ln\left\{\frac{\varphi}{2}\sqrt{I\omega/\hbar}\right\}}. \tag{4}$$

$t_{max}(\varphi)$ is the max. time permitted before rotating through $\measuredangle\,\varphi$. For $m = 10$ gm, $\ell = 15$ cm & $\varphi = 5°$, we find: $\boxed{t_{max}(5°) = 3.4\text{ sec.}}$

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(11) Spreading of QM wavepackets.

For a free-particle wavepacket with a Gaussian spectral fcn [CLASS NOTES, pp. Pack 5-6, Eqs. (16)-(18)], the packet width spreads in time according to...

$$\rightarrow \delta x \simeq \delta x_0 \left[1+(t/t_0)^2\right]^{1/2}, \quad \underline{\underline{t_0 = \frac{m}{\hbar}(\delta x_0)^2}} \;\;\&\;\; \delta x_0 = \text{width at } t=0. \qquad (1)$$

To get this form, we have used the GVD factor: $\alpha = \frac{\partial^2}{\partial k^2}(\hbar k^2/2m) = \hbar/m$, for a free particle. The functional form of δx depends on choice of spectral fcn... for the generic form in problem (9): $\delta x \simeq \delta x_0\left[1+\frac{1}{2}(t/t_0)\right]$. In all (reasonable) cases, however, $t_0 = \frac{m}{\hbar}(\delta x_0)^2$ is the scale time for spreading.

(A) For an electron: $m = 9.1\times10^{-28}$ gm, and: $\hbar = 1.05\times10^{-27}$ cgs, so -- with δx_0 in cm -- the scale time is: $t_0 = 0.87\,(\delta x_0/1\,\text{cm})^2$, sec. Then, using Eq. (1):

$$\rightarrow \delta x_0 \sim 10^{-8}\,\text{cm, and } \delta x = 2\,\delta x_0 \text{ when: } \underline{\underline{t_D \simeq \sqrt{3}\,t_0 = 1.5\times10^{-16}\,\text{sec}}}. \qquad (2)$$

The size of the solar system is the diameter of Pluto's orbit: $D_{pluto} \simeq 1.2\times10^{10}$ km $= 1.2\times10^{15}$ cm. With $D_{pluto} \gg \delta x_0$ the condition is met when...

$$\rightarrow \delta x \simeq D_{pluto} \simeq \delta x_0\left(\frac{t}{t_0}\right), \text{ when: } \underline{\underline{t}} \simeq t_0\left(\frac{D_{pluto}}{\delta x_0}\right) = \underline{1.0\times10^{7}\,\text{sec}}\ (121\,\text{days}) \qquad (3)$$

(B) For a person of mass $m = 50$ kgm, the scale time is: $t_0 = 4.8\times10^{31}\left(\frac{\delta x_0}{1\,\text{cm}}\right)^2$ sec, and for localization to $\delta x_0 \sim 10$ cm: $\underline{t_0 = 4.8\times10^{33}\,\text{sec}}$. The known (?) age of the universe is: $\underline{T \simeq 20\times10^9\,\text{yrs} = 6.3\times10^{17}\,\text{sec}}$. With $t = T \ll t_0$, Eq. (1) yields for the fractional personpacket increase in size...

$$\rightarrow \underline{\underline{(\delta x - \delta x_0)/\delta x_0 \simeq \frac{1}{2}(T/t_0)^2 = 0.86\times10^{-32}}}. \qquad (4)$$

(C) To do the above numbers, we've assumed the electron (and the person) is free. The electrons inside the personpacket are bound, however, and don't obey Eq. (1). In fact the binding maintains the electron packet width at a constant level.

(12) [15 pts] Continuity Eqtn for the Klein-Gordon Wave Eqtn.

(A) KG Eq. is: $\left[\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \left(\frac{mc}{\hbar}\right)^2\right]\phi = 0$, so $\left[\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \left(\frac{mc}{\hbar}\right)^2\right]\phi^* = 0$. (1)

Multiply the first of Eqs. (1) on the left by ϕ^*, the second by ϕ, and subtract...

$$\rightarrow \underbrace{(\phi^*\nabla^2\phi - \phi\nabla^2\phi^*)}_{=\nabla\cdot(\phi^*\nabla\phi - \phi\nabla\phi^*)} - \frac{1}{c^2}\underbrace{\left(\phi^*\frac{\partial^2}{\partial t^2}\phi - \phi\frac{\partial^2}{\partial t^2}\phi^*\right)}_{=\frac{\partial}{\partial t}\left(\phi^*\frac{\partial}{\partial t}\phi - \phi\frac{\partial}{\partial t}\phi^*\right)} = 0. \quad (2)$$

To make the first term look like $\mathbf{J}$(Schrödinger), multiply by $\frac{\hbar}{2im}$. Then...

$$\underline{\underline{\nabla\cdot\mathbf{J} + \frac{\partial\rho}{\partial t} = 0}} \int^{\text{w/}} \mathbf{J} = \frac{\hbar}{2im}(\phi^*\nabla\phi - \phi\nabla\phi^*), \quad \rho = \frac{i\hbar}{2mc^2}\left(\phi^*\frac{\partial}{\partial t}\phi - \phi\frac{\partial}{\partial t}\phi^*\right). \quad (3)$$

We have Schrödinger's $\mathbf{J}$ exactly, but our ρ is much different than $\phi^*\phi$.

(B) The KG probability density in Eq. (3) can be written as...

$$\rightarrow \rho = (\hbar/mc^2)\,\mathrm{Im}\,[\phi\,(\partial\phi/\partial t)^*] = \rho(\mathrm{KG}). \quad (4)$$

Now ϕ & $\partial\phi/\partial t$ must be independent (like position & velocity), because the KG Wave Eq. is a 2nd order PDE. In Eq (4), this means that $\rho(\mathrm{KG})$ can be +ve, −ve, or zero, depending on its starting value. In short, $\rho(\mathrm{KG})$ is not positive definite, quite unlike $\rho(\mathrm{Schr.}) = \phi^*\phi \geqslant 0$.

Nevertheless $\int_\infty \rho\, d^3r$ is a cnst for the KG Eqtn, since -- by (3) -- we have:

$$\rightarrow \frac{\partial}{\partial t}\int_\infty \rho\, d^3r = -\int_\infty \nabla\cdot\mathbf{J}\, d^3r = -\oint_\infty \mathbf{J}\cdot d\mathbf{S} \rightarrow 0 \int \text{for all } \phi\text{'s that vanish at } \infty. \quad (5)$$

Then: $\int_\infty \rho(\mathrm{KG})\, d^3r = $ cnst, and in this sense "probability" is conserved.

(C) A transform $\phi = \psi\, e^{-i(mc^2/\hbar)t}$ makes $\rho(\mathrm{KG})$ of Eq. (4) look like...

$\underline{\rho(\mathrm{KG}) = \psi^*\psi + (\hbar/mc^2)\,\mathrm{Im}\,[\psi(\partial\psi/\partial t)^*]}$. (6) In the NR limit ($c\rightarrow\infty$), the 2nd term RHS here becomes negligible, and indeed $\rho(\mathrm{KG}) \rightarrow \psi^*\psi = \rho(\mathrm{Schr.})$. We shall consider $\rho(\mathrm{KG})$ in more detail later in this course.