Problem 1 Solution

Consider a two-state quantum system with a time-independent Hamiltonian:

$$\mathcal{H}\,|\varphi_1\rangle = \varepsilon_1\,|\varphi_1\rangle \qquad\qquad \mathcal{H}\,|\varphi_2\rangle = \varepsilon_2\,|\varphi_2\rangle\,,$$

where the real constants obey $\varepsilon_1 < \varepsilon_2$, and both states $|\varphi_1\rangle$ and $|\varphi_2\rangle$ are normalized. Define two other states of the system

$$|\Phi_1\rangle = \frac{1}{\sqrt{2}}\Big(|\varphi_1\rangle + i\,|\varphi_2\rangle\Big) \qquad \text{and} \qquad |\Phi_2\rangle = \frac{1}{\sqrt{2}}\Big(|\varphi_1\rangle - i\,|\varphi_2\rangle\Big).$$

(a) Prove that $|\varphi_1\rangle$ and $|\varphi_2\rangle$ are orthogonal using definitions of Hermitian adjoints and operators.

We calculate the matrix element of the Hamiltonian between the two orthogonal states, acting to the right with the Hermitian Hamiltonian on the state $|\varphi_1\rangle$*:*

$$\langle\varphi_2\,|\mathcal{H}|\,\varphi_1\rangle = \varepsilon_1\,\langle\varphi_2\,|\,\varphi_1\rangle$$

This must be equal to the result obtained by acting with the Hamiltonian to the left ...

$$\langle\varphi_2\,|\mathcal{H}|\,\varphi_1\rangle = \varepsilon_2\,\langle\varphi_2\,|\,\varphi_1\rangle$$

Because these two expressions must be equal, and because the two eigenvalues are not equal, the only possible conclusion is that the inner product, which appears in both expressions, is zero.

(b) Explicitly demonstrate that $|\Phi_1\rangle$ and $|\Phi_2\rangle$ are also orthogonal and normalized.

$$\begin{aligned}
\langle\Phi_1\,|\,\Phi_1\rangle &= \left\{\frac{1}{\sqrt{2}}\Big(\langle\varphi_1| - i\,\langle\varphi_2|\Big)\right\}\left\{\frac{1}{\sqrt{2}}\Big(|\varphi_1\rangle + i\,|\varphi_2\rangle\Big)\right\}\\
&= \frac{1}{2}\left\{\underbrace{\langle\varphi_1\,|\,\varphi_1\rangle}_{=1} + i\underbrace{\langle\varphi_1\,|\,\varphi_2\rangle}_{=0} - i\underbrace{\langle\varphi_2\,|\,\varphi_1\rangle}_{=0} - i^2\underbrace{\langle\varphi_2\,|\,\varphi_2\rangle}_{=1}\right\} = 1 \quad \text{normalized}\\
\langle\Phi_2\,|\,\Phi_2\rangle &= \left\{\frac{1}{\sqrt{2}}\Big(\langle\varphi_1| + i\,\langle\varphi_2|\Big)\right\}\left\{\frac{1}{\sqrt{2}}\Big(|\varphi_1\rangle - i\,|\varphi_2\rangle\Big)\right\}\\
&= \frac{1}{2}\left\{\underbrace{\langle\varphi_1\,|\,\varphi_1\rangle}_{=1} - i\underbrace{\langle\varphi_1\,|\,\varphi_2\rangle}_{=0} + i\underbrace{\langle\varphi_2\,|\,\varphi_1\rangle}_{=0} - i^2\underbrace{\langle\varphi_2\,|\,\varphi_2\rangle}_{=1}\right\} = 1 \quad \text{normalized}\\
\langle\Phi_1\,|\,\Phi_2\rangle &= \left\{\frac{1}{\sqrt{2}}\Big(\langle\varphi_1| - i\,\langle\varphi_2|\Big)\right\}\left\{\frac{1}{\sqrt{2}}\Big(|\varphi_1\rangle - i\,|\varphi_2\rangle\Big)\right\}\\
&= \frac{1}{2}\left\{\underbrace{\langle\varphi_1\,|\,\varphi_1\rangle}_{=1} - i\underbrace{\langle\varphi_1\,|\,\varphi_2\rangle}_{=0} - i\underbrace{\langle\varphi_2\,|\,\varphi_1\rangle}_{=0} + i^2\underbrace{\langle\varphi_2\,|\,\varphi_2\rangle}_{=1}\right\} = 0 = \langle\Phi_2\,|\,\Phi_1\rangle^*
\end{aligned}$$

...so $|\Phi_1\rangle$ *and* $|\Phi_2\rangle$ *are orthogonal.*

(c) At time $t = 0$, the state of the system is known to be $|\Phi_1\rangle$. For $t \geq 0$, calculate the product of the uncertainty in energy, and the time interval between a minimum and the subsequent maximum that the system will be found in the state $|\Phi_2\rangle$. Does this product satisfy a Heisenberg uncertainty relation? If not, explain why not. If so, discuss its physical significance considering that there is no "time operator" in quantum mechanics.

Recall that, for an observable represented as an operator $\mathcal{A}$, the uncertainty in the expectation value $\langle \mathcal{A} \rangle$ is defined as $\sqrt{\langle \mathcal{A}^2 \rangle - \langle \mathcal{A} \rangle^2}$.

Because we have a time-independent Hamiltonian, which commutes with itself, all energy measures are time-independent quantities. This means we can simplify the analysis, and calculate the values at $t = 0$, when the state is $|\Phi_1\rangle$. For further simplicity, $|\Phi_1\rangle$ is expanded in terms of eigenstates of $\mathcal{H}$ (and of $\mathcal{H}^2$), so we can use the squares of the moduli of the expansion coefficients.

$$\begin{aligned}
\langle E \rangle &= \langle \Phi_1 | \mathcal{H} | \Phi_1 \rangle = \left| \frac{1}{\sqrt{2}} \right|^2 \varepsilon_1 + \left| \frac{i}{\sqrt{2}} \right|^2 \varepsilon_2 = \frac{1}{2} (\varepsilon_1 + \varepsilon_2) \\
\left\langle E^2 \right\rangle &= \left\langle \Phi_1 \left| \mathcal{H}^2 \right| \Phi_1 \right\rangle = \left| \frac{1}{\sqrt{2}} \right|^2 \varepsilon_1^2 + \left| \frac{i}{\sqrt{2}} \right|^2 \varepsilon_2^2 = \frac{1}{2} \left(\varepsilon_1^2 + \varepsilon_2^2 \right) \\
\Delta E &= \sqrt{\langle E^2 \rangle - \langle E \rangle^2} = \sqrt{\frac{1}{2}\varepsilon_1^2 + \frac{1}{2}\varepsilon_2^2 - \frac{1}{4} \left(\varepsilon_1^2 + 2\varepsilon_1\varepsilon_2 + \varepsilon_2^2 \right)} = \frac{1}{2}(\varepsilon_2 - \varepsilon_1)
\end{aligned}$$

The probability to find the state in $|\Phi_2\rangle$ is the modulus squared of the inner product of it and the time-evolved state. Using the time-evolution operator, ...

$$\begin{aligned}
|\psi(t)\rangle &= e^{-i\mathcal{H}t/\hbar} |\Phi_1\rangle = \frac{1}{\sqrt{2}} \left(e^{-i\varepsilon_1 t/\hbar} |\varphi_1\rangle + i e^{-i\varepsilon_2 t/\hbar} |\varphi_2\rangle \right) \\
\langle \Phi_2 | \psi(t) \rangle &= \frac{1}{\sqrt{2}} \Big(\langle \varphi_1 | + i \langle \varphi_2 | \Big) \frac{1}{\sqrt{2}} \left(e^{-i\varepsilon_1 t/\hbar} |\varphi_1\rangle + i e^{-i\varepsilon_2 t/\hbar} |\varphi_2\rangle \right) \\
&= \frac{1}{2} \left(e^{-i\varepsilon_1 t/\hbar} - e^{-i\varepsilon_2 t/\hbar} \right) \\
|\langle \Phi_2 | \psi(t) \rangle|^2 &= \frac{1}{4} \left(e^{-i\varepsilon_1 t/\hbar} - e^{-i\varepsilon_2 t/\hbar} \right) \left(e^{+i\varepsilon_1 t/\hbar} - e^{+i\varepsilon_2 t/\hbar} \right) \\
&= \frac{1}{4} \left[1 - e^{-i(\varepsilon_2 - \varepsilon_1)t/\hbar} - e^{i(\varepsilon_2 - \varepsilon_1)t/\hbar} + 1 \right] = \frac{1}{2} \left\{ 1 - \cos \left[\frac{(\varepsilon_2 - \varepsilon_1)t}{\hbar} \right] \right\}
\end{aligned}$$

The probability is minimized when the cosine is +1, and maximized when the cosine is −1. Let n be any non-negative integer ...

$$\boxed{t_{\text{min},n} = \frac{2n\pi\hbar}{\varepsilon_2 - \varepsilon_1}} \qquad \boxed{t_{\text{max},n} = \frac{(2n+1)\pi\hbar}{\varepsilon_2 - \varepsilon_1}}$$

The significance of the time-energy uncertainty principle is that the minimum uncertainty in energy is proporational to the inverse of the smallest time over which the system changes appreciably. The system clearly changes appreciably if the probability for finding $|\Phi_2\rangle$ changes from 0 to 1. Define Δt as the corresponding time interval, and choose $n = 0$.

$$\begin{aligned}
\Delta t &= t_{\text{max},0} - t_{\text{min},0} = \frac{\pi\hbar}{\varepsilon_2 - \varepsilon_1} \\
(\Delta E)\,(\Delta t) &= \left[\frac{1}{2}(\varepsilon_2 - \varepsilon_1) \right] \left[\frac{\pi\hbar}{\varepsilon_2 - \varepsilon_1} \right] = \boxed{\frac{\pi\hbar}{2}} > \frac{\hbar}{2}
\end{aligned}$$

#2. Full Solution

a) To find the Lagrangian, first find the **kinetic energy T**.

$$T = \frac{1}{2}m\left(\dot{x}_{cm}^2 + \dot{y}_{cm}^2\right) + \frac{1}{2}I\omega^2$$

Relate the center of mass (cm) coordinates of the ladder to θ and S.

$$x_{cm} = S + \tfrac{1}{2}\ell\cos\theta \qquad y_{cm} = \tfrac{1}{2}\ell\sin\theta$$

Take the time derivative.

$$\dot{x}_{cm} = \dot{S} - \tfrac{1}{2}\ell(\sin\theta)\dot{\theta} \qquad \dot{y}_{cm} = \tfrac{1}{2}\ell(\cos\theta)\dot{\theta}$$

Using: $I = \tfrac{1}{12}m\ell^2$ and $\omega = \dot{\theta}$

we get for T:

$$T = \frac{1}{2}m\left[\left(\dot{S} - \tfrac{1}{2}\ell(\sin\theta)\dot{\theta}\right)^2 + \left(\tfrac{1}{2}\ell(\cos\theta)\dot{\theta}\right)^2\right] + \frac{1}{2}\left(\tfrac{1}{12}m\ell^2\right)\left(\dot{\theta}\right)^2$$

$$= \tfrac{1}{2}m\left[\dot{S}^2 + \tfrac{1}{4}\ell^2\dot{\theta}^2\left(\sin^2\theta + \cos^2\theta\right) - \dot{S}\dot{\theta}\ell\sin\theta\right] + \tfrac{1}{24}m\ell^2\dot{\theta}^2$$

$$= \tfrac{1}{2}m\dot{S}^2 + \tfrac{1}{6}m\ell^2\dot{\theta}^2 - \tfrac{1}{2}m\dot{S}\dot{\theta}\ell\sin\theta$$

For **the potential V** we have: $V = mg\dfrac{\ell}{2}\sin\theta$

Thus for the **Lagrangian L** we have:

$$\boxed{L = T - V = \tfrac{1}{2}m\dot{S}^2 + \tfrac{1}{6}m\ell^2\dot{\theta}^2 - \tfrac{1}{2}m\dot{S}\dot{\theta}\ell\sin\theta - \tfrac{1}{2}mg\ell\sin\theta}$$

b) Using the method of Lagrange multipliers, we use **Lagranges modified equation for S:**

$$\frac{\partial L}{\partial S} - \frac{d}{dt}\frac{\partial L}{\partial \dot{S}} + \lambda\frac{\partial f}{\partial S} = 0$$

where the **equation of constraint** in this case is: $f(S,\theta) = 0 \rightarrow S = 0$

Thus $\dfrac{\partial f}{\partial S} = 1$

Also:

$$\frac{\partial L}{\partial S} = 0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{S}}\right) = \frac{d}{dt}\left(m\dot{S} - \tfrac{1}{2}m\dot{\theta}\ell\sin\theta\right) = m\ddot{S} - \tfrac{1}{2}m\ddot{\theta}\ell\sin\theta - \tfrac{1}{2}m\dot{\theta}^2\ell\cos\theta$$

Thus Lagrange's modified equation for S becomes:

$$\lambda = m\ddot{S} - \tfrac{1}{2}m\ddot{\theta}\ell\sin\theta - \tfrac{1}{2}m\dot{\theta}^2\ell\cos\theta$$

Lagranges modified equation for θ:

$$\frac{\partial L}{\partial \theta} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} + \lambda\frac{\partial f}{\partial \theta} = 0$$

In this case $\dfrac{\partial f}{\partial \theta} = 0$

Also:

$$\frac{\partial L}{\partial \theta} = -\tfrac{1}{2}m\dot{S}\dot{\theta}\ell\cos\theta - \tfrac{1}{2}mg\ell\cos\theta$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta}\right)=\frac{d}{dt}\left(\tfrac{1}{3}m\ell^2\dot\theta-\tfrac{1}{2}m\dot S\ell\sin\theta\right)=\tfrac{1}{3}m\ell^2\ddot\theta-\tfrac{1}{2}m\ddot S\ell\sin\theta-\tfrac{1}{2}m\dot S\dot\theta\ell\cos\theta$$

Thus Lagrange's modified equation for θ becomes:

$$-\tfrac{1}{2}m\dot S\dot\theta\ell\cos\theta-\tfrac{1}{2}mg\ell\cos\theta-\tfrac{1}{3}m\ell^2\ddot\theta+\tfrac{1}{2}m\ddot S\ell\sin\theta+\tfrac{1}{2}m\dot S\dot\theta\ell\cos\theta=0$$

Cancelling the first and last term we get Lagrange's equation for θ:

$$-\tfrac{1}{2}mg\ell\cos\theta-\tfrac{1}{3}m\ell^2\ddot\theta+\tfrac{1}{2}m\ddot S\ell\sin\theta=0$$

Now that we have taken the derivatives to get our equations, we can **apply the constraint** when the ladder loses contact with the wall.

$$S=0\rightarrow\ddot S=0$$

Thus our **coupled equations** become:

$$\lambda=-\tfrac{1}{2}m\ddot\theta\ell\sin\theta-\tfrac{1}{2}m\dot\theta^2\ell\cos\theta$$

$$-\tfrac{1}{2}mg\ell\cos\theta-\tfrac{1}{3}m\ell^2\ddot\theta=0\rightarrow\ddot\theta=-\frac{3g}{2\ell}\cos\theta$$

Now we need to **combine these two equations.** Note that we can use the second equation to eliminate $\ddot\theta$ in the first equation, but we still need to eliminate $\dot\theta$. To find an expression for $\dot\theta$ from $\ddot\theta$, we use:

$$\ddot\theta=\frac{d}{dt}\left(\dot\theta\right)=\frac{d\dot\theta}{d\theta}\frac{d\theta}{dt}=\dot\theta\frac{d\dot\theta}{d\theta}$$

Thus we can get $\dot\theta$ from the following integration:

$$\dot\theta\frac{d\dot\theta}{d\theta}=-\frac{3g}{2\ell}\cos\theta\rightarrow\qquad\dot\theta d\dot\theta=-\frac{3g}{2\ell}\cos\theta d\theta$$

$$\int_0^{\dot\theta}\dot\theta d\dot\theta=-\frac{3g}{2\ell}\int_{\theta_o}^{\theta}\cos\theta d\theta$$

$$\frac{1}{2}\dot\theta^2=-\frac{3g}{2\ell}\left(\sin\theta-\sin\theta_o\right)\rightarrow\ \dot\theta^2=\frac{3g}{\ell}\left(\sin\theta_o-\sin\theta\right)$$

Thus we can finally eliminate all the derivatives and get

$$\lambda=-\tfrac{1}{2}m\left(-\frac{3g}{2\ell}\cos\theta\right)\ell\sin\theta-\tfrac{1}{2}m\left[\frac{3g}{\ell}\left(\sin\theta_o-\sin\theta\right)\right]\ell\cos\theta$$

Now to get the angle when the ladder loses constact, **set λ=0.**

$$\tfrac{1}{2}\sin\theta=\sin\theta_o-\sin\theta\rightarrow\qquad\tfrac{3}{2}\sin\theta=\sin\theta_o$$

Solving for θ, the angle where the ladder loses contact, we get:

$$\boxed{\theta=\sin^{-1}\left(\tfrac{2}{3}\sin\theta_o\right)}$$

c) For the specific angle of θ_o=45°, we get(using the sine tables in the CRC):

$$\theta=\sin^{-1}\left(\tfrac{2}{3}\sin(45)\right)=\sin^{-1}\left(\tfrac{2}{3}0.707\right)=\sin^{-1}\left(0.471\right)\approx28^\circ$$

$$\boxed{\theta\approx28^\circ}$$

Problem 3

(a) Setting $dg/dt = 0$ and $dh/dt = 0$ results in the simultaneous equations

$$h - 2g^2 = 0 \ , \tag{1}$$
$$4g^2 - 2g - gh = 0 \ . \tag{2}$$

Eq. (1) yields $h = 2g^2$. Substituting this in eq. (2) gives the cubic

$$4g^2 - 2g - 2g^3 = -2g(g^2 - 2g + 1) = -g(g-1)^2 = 0 \ . \tag{3}$$

The only non-trvial solution to this is $g = 1$ from which we have the steady state

$$g_0 = 1 \ , \quad h_0 = 2 \ . \tag{4}$$

(b) Substituting $g(t) = g_0 + \delta g(t)$ and $h(t) = h_0 + \delta h(t)$ and keeping only terms to linear order yields the equations

$$\begin{aligned} \frac{d(\delta g)}{dt} &= \delta h - 4\delta g \\ \frac{d(\delta h)}{dt} &= 8\delta g - 2\delta g - \delta h - 2\delta g = 4\delta g - \delta h \end{aligned}$$

(c) The coupled equations above can be written in matrix form

$$\frac{d}{dt}\begin{bmatrix} \delta g \\ \delta h \end{bmatrix} = \begin{bmatrix} -4 & 1 \\ 4 & -1 \end{bmatrix} \cdot \begin{bmatrix} \delta g \\ \delta h \end{bmatrix} \tag{5}$$

Proposing a solution

$$\begin{bmatrix} \delta g(t) \\ \delta h(t) \end{bmatrix} = \begin{bmatrix} \delta g_0 \\ \delta h_0 \end{bmatrix} e^{\gamma t}$$

and introducing it to the equations yields the eigenvaue equation

$$\gamma \begin{bmatrix} \delta g_0 \\ \delta h_0 \end{bmatrix} = \begin{bmatrix} -4 & 1 \\ 4 & -1 \end{bmatrix} \cdot \begin{bmatrix} \delta g_0 \\ \delta h_0 \end{bmatrix} \tag{6}$$

for which γ is the eigenvalue of the matrix. The eigenvalue must satisfy the equation

$$(\gamma + 4)(\gamma + 1) - 4 = \gamma^2 + 5\gamma = 0$$

The two roots of this quadratic equation give the two eigenvalues

$$\gamma_1 = 0 \ , \quad \gamma_2 = -5 \ . \tag{7}$$

Since the system has a no positive eigenvalue ($\gamma_1 = 0$) the equilibrium is *stable*. The system will **not** diverge from the equilibrium.

(d) Virtually any perturbation will consist of a mixture of the two eignvectors. The presence of a $\gamma_1 = 0$ means the solution never approaches the equilibrium, so the characteritic time is ∞.

Problem 4 Solution

A *Buckminsterfullerene*, or "buckyball", is a spherically shaped molecule composed entirely of carbon atoms. The most common, naturally occurring buckyball is C_{60}, which contains 60 carbon atoms and has a diameter of roughly 1 nm. We consider here the interference of these massive C_{60} molecules as they pass through a double-slit apparatus (*e.g.* O. Nairz, M. Arndt, and A. Zeilinger, Am. J. Phys. **71** (4), 2003). For this experiment, assume a beam of molecules with an average speed of 200 m/s, a slit separation of 100 nm, and a slit-to-detector distance of 1 m. Ignore the single-slit interference pattern.

(a) To one significant figure, estimate the spatial separation between interference maxima near the undeviated beam in the observation plane. Which is larger: the spatial separation between interference maxima, or the size of the molecules? Describe the observational consequences if the reverse were true.

For double-slit interference, the maxima are located at $m\lambda = d\sin\theta$, where m is the order of the maximum, λ is the (deBroglie) wavelength, d is the slit separation, and θ is the angle of maxima with respect to the undeviated beam.

We use the small-angle approximation, and let L be the distance between the observation plane and the slits. The mass of the C_{60} is approximately 60 times the mass of one carbon atom, which is approximately 12 times the mass of a proton.

$$\begin{aligned}\Delta y &\simeq L\Delta\theta \simeq \frac{L\lambda}{d} = \frac{Lh}{mvd} = \frac{(1m)(7\times10^{-34}\ \mathrm{J\cdot s})}{(60\cdot12\cdot2\times10^{-27}\ \mathrm{kg})(2\times10^{2}\ \mathrm{m/s})(1\times10^{-7}\ \mathrm{m})} \\ &\simeq \boxed{3\times10^{-5}\ \mathrm{m}}\end{aligned}$$

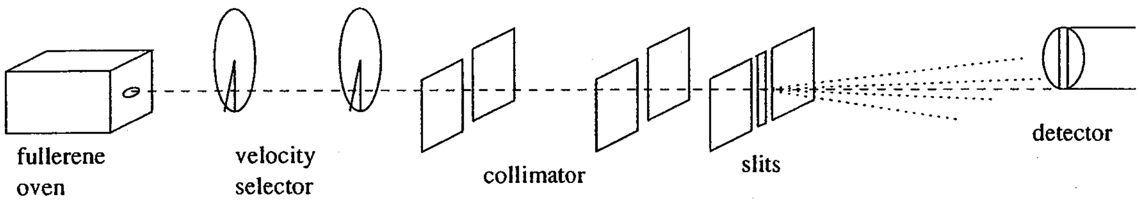
The spatial separation of the fringes Δy is four orders of magnitude larger than the size of the molecules. The detector aperture needs to be large enough to accommodate a molecule, so if the reverse were true, the aperture would be larger than the distance between fringes, so the fringes would not be resolved.

(b) The velocity distribution of the molecules incident on the slits has a width in both the longitudinal and the transverse directions. Quantitatively characterize how each of these widths affects the interference pattern.

To avoid destroying the interference pattern, the transverse velocity must correspond to a deBroglie wavelength that is larger than the slit separation, so the transverse speed must be smaller than 0.006 m/s.

To observe an arbitrary maximum, the spread in the longitudinal velocity must correspond to an uncertainty in wavelength that is smaller than the path length difference, the smallest of which is equal to the wavelength. Therefore, the spread in longitudinal speeds must be small compared to the average speed of 200 m/s.

(c) Beginning with a supply of a powder of the C_{60} molecules, draw a block diagram of an experimental apparatus that you might use to create and observe a double slit diffraction pattern of these molecules. Include experimental components that minimize the effects discussed in part (b). Label each component in your diagram and describe its basic function.



The fullerene oven heats up a source of C_{60}, some of which emerge through the aperture. The oven temperature is chosen to maximize the population at the desired speed.

The velocity selector then rejects those C_{60}s at the wrong speed, and might take the form of rotating blades separated by some distance with a tuned rotation speed. Velocity selection can be improved by reducing the size of the slots in the rotating disks of the velocity selector. Next, the collimators to ensure the beam direction. The transverse speeds can be reduced by placing the collimator far from the slits, so that the incident beam is a plane wave. Both of the methods suggested for minimizing the spread in beam velocity also result in fewer C_{60} passing through the slits, so a large supply of C_{60} is necessary.

Then, the slits or a grating.

The narrow-aperture detector's function is to count the number of C_{60}s arriving at a particular location in the observation plane. A possible detector type would involve an ionizing laser orthogonal to the diffraction plane followed by a charged particle detector.

Problem 5

(a) Defining $\hat{\mathbf{x}}$ as the orientation of the wire loop, the dipole moment of the spinning bar magnet is

$$\mathbf{m}_{\rm mag}(t) = m\cos(\omega t)\hat{\mathbf{x}} + m\sin(\omega t)\hat{\mathbf{y}} \ .$$

The magnetic field at the wire loop is

$$\mathbf{B} = -\frac{\mu_0}{4\pi d^3}\mathbf{m}_{\rm mag} = -\frac{\mu_0 m}{4\pi d^3}[\,\hat{\mathbf{x}}\cos(\omega t) + \hat{\mathbf{y}}\sin(\omega t)\,] \ .$$

This means the flux enclosed by the wire loop is

$$\Phi = A\hat{\mathbf{x}}\cdot\mathbf{B} = -\frac{\mu_0 m A}{4\pi d^3}\cos(\omega t) \ ,$$

from which we find the EMF

$$\mathcal{E} = -\frac{d\Phi}{dt} = -\frac{\mu_0 m A\omega}{4\pi d^3}\sin(\omega t) \ .$$

The current in the wire is therefore

$$I = \frac{\mathcal{E}}{R} = -\frac{\mu_0 m A\omega}{4\pi R d^3}\sin(\omega t) \ . \qquad (1)$$

(b) The dipole moment of the loop carrying current I is

$$\mathbf{m}_{\rm loop} = AI\hat{\mathbf{x}} = -\frac{\mu_0 m A^2\omega}{4\pi R d^3}\sin(\omega t)\hat{\mathbf{x}}$$

The magnetic field at the magnet is then

$$\mathbf{B}_{\rm mag} = -\frac{\mu_0}{4\pi d^3}\mathbf{m}_{\rm loop} = \frac{\mu_0^2 m A^2\omega}{(4\pi)^2 R d^6}\sin(\omega t)\hat{\mathbf{x}}$$

The torque on the magnet is

$$\mathbf{N} = \mathbf{m}_{\rm mag}\times\mathbf{B}_{\rm mag} = -\frac{\mu_0^2 m^2 A^2\omega}{(4\pi)^2 R d^6}\sin^2(\omega t)\hat{\mathbf{z}} \qquad (2)$$

(c) The instantaneous power delivered to the magnet is

$$P = -\mathbf{N}\cdot\boldsymbol{\omega} = \frac{\mu_0^2 m^2 A^2\omega^2}{(4\pi)^2 R d^6}\sin^2(\omega t) \ .$$

The time average of this, over a single rotation of the magnetic is,

$$\bar{P} = \frac{\mu_0^2 m^2 A^2\omega^2}{2(4\pi)^2 R d^6} = \frac{\mu_0^2 m^2 A^2\omega^2}{32\pi^2 R d^6} \qquad (3)$$

The instantaneous and average power delivered to the magnet matches the Ohmic power lost in the wire.

Problem 6.

a. The X ray's $\vec{E}$ and $\vec{B}$ fields will exert a Lorentz force on the conduction electron of gold and this is given in Gaussian units by $\vec{F} = -e(\vec{E} + \frac{v}{c}\vec{B})$. Remember also that in a Gaussian system the units of the $\vec{E}$ and $\vec{B}$ fields are the same and that, furthermore, for a plane wave $\frac{|\vec{E}|}{|\vec{B}|} = 1$; hence the Lorentz force exerted by the $\vec{B}$ field is negligibly small as compared to that exerted by the $\vec{E}$ field because *v/c~0.01* (typical conduction electron velocities are on the order of ~10^6 *m/s*). Therefore, we can ignore the contribution of the X ray's $\vec{B}$ field to the dynamics of electron-X-ray interactions. Using Newton's second law of motion, $\vec{F} \simeq -e\vec{E} = m\frac{d\vec{v}}{dt}$, leads to a relation between the oscillating $\vec{E} = \vec{E}_o e^{i(\vec{k}.\vec{r}-\omega t)}$ field and the velocity, $\vec{v}$, of the electron: $\vec{v} = \frac{e}{i\omega m}\vec{E}$. Remember that current density by definition is given by $\vec{J} = -e\rho\vec{v}$ where ρ is the density of conduction electrons per unit volume and is a *constant.*

Combining these two results a relation between $\vec{J}$ and $\vec{E}$ can be found: $\vec{J} = \frac{-e^2\rho}{i\omega m}\vec{E} = \sigma\vec{E}$ where

$\sigma = \frac{-e^2\rho}{i\omega m}$ is the conductivity of the metal (in this case gold). Our objective is to find the index of refraction of metal using relation $n= ck/\omega$. This means we need to find the dispersion relation between k and ω. This is typically done by using Maxwell's equations. This is what we will do next: The equation $\vec{\nabla}\times\vec{E} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t}$ immediately yields $\vec{B} = \frac{c}{i\omega}\vec{\nabla}\times\vec{E}$, where $\vec{E} = \vec{E}_o e^{i(\vec{k}.\vec{r}-\omega t)}$ is used. We now plug this into $\vec{\nabla}\times\vec{B} = \frac{1}{c}\frac{\partial\vec{E}}{\partial t} + \frac{4\pi}{c}\vec{J}$ together with $\vec{J} = \sigma\vec{E}$: $\frac{c}{i\omega}\vec{\nabla}\times\vec{\nabla}\times\vec{E} = (\frac{-i\omega}{c} + \frac{4\pi\sigma}{c})\vec{E}$. Using $\vec{\nabla}\times\vec{\nabla}\times\vec{E} = \vec{\nabla}(\vec{\nabla}.\vec{E}) - \nabla^2\vec{E}$, $\vec{\nabla}(\vec{\nabla}.\vec{E}) = -4\pi e\vec{\nabla}\rho = 0$ and $-\nabla^2\vec{E} = k^2\vec{E}$ we obtain $\frac{k^2c}{i\omega}\vec{E} = (\frac{-i\omega}{c} + \frac{4\pi\sigma}{c})\vec{E}$ where $\sigma = \frac{-e^2\rho}{i\omega m}$. Combining all these results and rearranging the equations we obtained the desired dispersion relation:

$k^2 = \frac{\omega^2}{c^2}(1 - \frac{\omega_p^{\ 2}}{\omega^2})$, where $\omega_p = \frac{4\pi\rho e^2}{m}$ is the plasmon frequency of the metal (in our case gold).

This gives the sought result. By comparing $n= ck/\omega$ we obtain the index of refraction n of the metal:

$$\boxed{n = (1 - \frac{\omega_p^{\ 2}}{\omega^2})^{1/2}}$$

Notice that as long as $\omega > \omega_p$ the index of refraction is always a real number and is always $n<1$. In fact, this is the secret for the total external reflection of X rays from metal surfaces below the critical angle of grazing incidence. As far as X rays are concerned air is denser than the metal.

b. Snell's relation asserts that $\sin(\pi/2-\theta) = n\sin(\pi/2-\phi)$ where θ and ϕ are the grazing angle of incidence and the angle of refraction measured from the surface of the metal. The critical angle then can be found by setting $\phi = 0$. This immediately gives the relation between critical angle of incidence θ_c and n:

$$\boxed{\cos\theta_c = n = (1-\frac{\omega_p^2}{\omega^2})^{1/2}. \text{ Plugging in the numbers suggested gives: } \theta_c \simeq 2.2^o.}$$

As you can see one can increase the critical angle by choosing materials with high plasmon energies. For example, the bulk plasmon of Si is about 17 eV, which gives a critical angle of $\theta_c(Si) \simeq 9.8^o$.

Comp 09

Problem #7 - Key

(a) Einstein's photon energy eqn: $E = \hbar\omega$ ①

" relativistic mass-energy eqn:

$$E^2 = p^2c^2 + m_\gamma^2 c^4 \quad ②$$

$P = \hbar k$ ③ momentum of photon.

So ①②③ →

$$\hbar^2\omega^2 = \hbar^2 k^2 c^2 + m_\gamma^2 c^4 \quad ④$$

④ → $\begin{cases} \omega^2 = k^2c^2 + \omega_0^2 & ⑤ \\ \text{where } \omega_0 = \dfrac{m_\gamma c^2}{\hbar} = \text{const.} & ⑥ \end{cases}$ Ans

(b) given $m_\gamma = 4\times10^{-48}\,gm$ ⑦

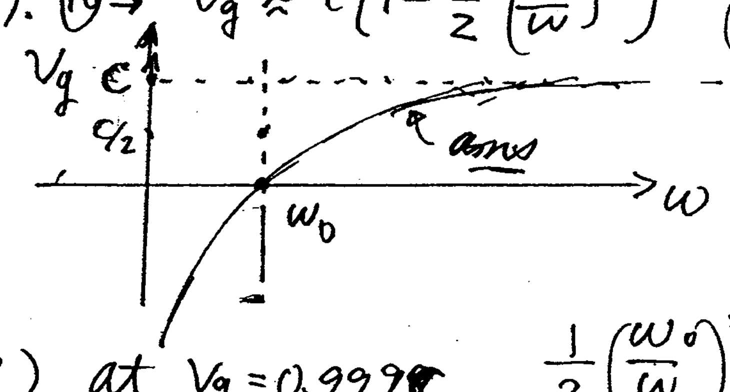
So ⑥ & ⑦ → $\omega_0 = \dfrac{4\times10^{-51}\,Kg\,(3\times10^8\,m/s)^2}{1.05\times10^{-34}\,Js} = 3.4\,rad/s$ ⑧ // Ans.

(c) Group velocity V_g.

⑤ → $\omega = \sqrt{k^2c^2 + \omega_0^2}$ ⑨

⑨⑤ → $V_g = \dfrac{\partial\omega}{\partial k} = \dfrac{kc^2}{\sqrt{k^2c^2+\omega_0^2}} = \dfrac{c\sqrt{\omega^2-\omega_0^2}}{\omega} = c\left[1-\left(\dfrac{\omega_0}{\omega}\right)^2\right]^{1/2}$ ⑩ // Ans.

(d). ⑩ → $V_g \approx c\left[1-\frac{1}{2}\left(\frac{\omega_0}{\omega}\right)^2\right]$ $\left(\frac{\omega_0}{\omega} << 1\right)$



$\begin{cases} \omega = 0 & V_g = -\infty \\ \omega = \omega_0 & V_g = 0 \leftarrow ⑩ \\ \omega = \infty & V_g = c \end{cases}$

(e) at $V_g = 0.999c$ $\quad \frac{1}{2}\left(\frac{\omega_0}{\omega}\right)^2 = 0.0001$

→ with ⑧ → $\omega \cong 240\,rad/s.$

$\lambda = \dfrac{2\pi}{\omega}\cdot c = \dfrac{2\pi}{240}\times3\times10^8 = 7.8\times10^6\,m$ // Ans

#8. Full Solution

a) To find thecapacitance for a length L of the cable we will use $Q \rightarrow E_o \rightarrow E \rightarrow V \rightarrow C$. For a length L of cable, we put a charge Q on the cable with +Q on the inner wire and –Q on the outer mesh.

Now we find the electric field E_o in the region between the center wire and the mess in the absence of the dielectric. We will add the dielectric later. To find the electric field, we use Gauss' Law. For a gaussian cylindrical surface of length L and of radius ρ, with $a<\rho<b$, the enclosed charge is +Q. Thus Gauss's law gives

$$\oint \vec{E} \cdot d\vec{A} = \frac{1}{\varepsilon_o} Q_{enc}$$

$$E_o (2\pi\rho L) = \frac{Q}{\varepsilon_o}$$

$$\vec{E}_o = \frac{Q}{(2\pi\rho L)\varepsilon_o} \hat{\rho}$$

Next we find the electric field in the presence of the dielectric κ.

$$\vec{E} = \frac{\vec{E}_o}{\kappa} = \frac{Q}{2\pi\varepsilon_o \kappa L} \frac{1}{\rho} \hat{\rho}$$

Now we find the potential difference V between the inner wire and the outer mesh.

$$V = -\int_b^a \vec{E} \cdot d\vec{\rho} = \int_a^b \frac{Q}{2\pi\varepsilon_o \kappa L} \frac{1}{\rho} \hat{\rho} \cdot \hat{\rho} d\rho = \frac{Q}{2\pi\varepsilon_o \kappa L} \ln\left(\frac{b}{a}\right)$$

Next we use Q=CV to get the capacitance.

$$C = \frac{Q}{V} = \frac{Q}{\frac{Q}{2\pi\varepsilon_o \kappa L} \ln\left(\frac{b}{a}\right)} = \frac{2\pi\varepsilon_o \kappa L}{\ln\left(\frac{b}{a}\right)}$$

Dividing by the length we get the capacitance per unit length C'

$$C' = \frac{C}{L} = \frac{2\pi\varepsilon_o \kappa}{\ln\left(\frac{b}{a}\right)}$$

b) Now we put in some numbers.

$$\varepsilon_o = 8.85\text{x}10^{-12}\,\frac{C^2}{N\cdot m^2}$$

$$\kappa = 2.09$$

$$\frac{b}{a} = \frac{0.114\text{in}/2}{0.035\text{in}/2} \approx 3$$

$$C' = \frac{2\pi\varepsilon_o\kappa}{\ln\left(\frac{b}{a}\right)} \approx \frac{2\pi\left(8.85\text{x}10^{-12}\,\frac{C^2}{N\cdot m^2}\right)(2.09)}{\ln(3)} \approx \frac{(6)\left(9\text{x}10^{-12}\,\frac{C^2}{N\cdot m^2}\right)(2)}{1.09} \approx 10^{-10}\,\frac{F}{m}$$

c) Finally put C' in puffs per foot

$$C' \approx 10^{-10}\,\frac{F}{m} = 10^{-10}\,\frac{F}{m}\left(\frac{10^{12}\,\text{pF}}{F}\right)\left(\frac{1\text{m}}{100\text{cm}}\,\frac{2.54\text{cm}}{1\text{in}}\,\frac{12\text{in}}{\text{ft}}\right) \approx 30\,\frac{\text{pF}}{\text{ft}}$$

or about 30 puffs per foot.

Prob. 9

Thermodynamics Comprehensive 2009 – Solution

Energy is transferred by heat conduction through the ice at a rate

$$P = \frac{dQ}{dt} = \kappa_{ice} A \frac{dT}{dx}$$
$$= \kappa_{ice} A \frac{T_{hot} - T_{cold}}{l}$$

where κ_{ice} is the thermal conductivity of ice, A is the area of the pond, and l is the thickness of the ice.

At the water/ice interface, water is transformed into ice according to

$$dQ = L_{l-to-s} dm = L_{l-to-s} \rho_{ice} dV = L_{l-to-s} \rho_{ice} A dl$$

where $L_{l\text{-}to\text{-}s}$ is the latent heat of solidification and ρ_{ice} is the density of ice (we see that the units for the heat of solidification are, as expected, J/kg).

Substituting this energy into the heat diffusion equation, we can determine the rate of ice formation

$$\frac{L_{l-to-s} \rho_{ice} A dl}{dt} = \kappa_{ice} A \frac{T_{hot} - T_{cold}}{l}$$
$$or \quad l \; dl = \frac{\kappa_{ice}(T_{hot} - T_{cold})}{L_{l-to-s} \rho_{ice}} \; dt$$

Integrating both sides and using mks units gives

$$\int_{0.04m}^{0.08m} l \; dl = \frac{\kappa_{ice}(10 \; K)}{L_{l-to-s} \rho_{ice}} \int_{0 \sec}^{36000 \sec} dt$$
$$and \quad L_{l-to-s} = \frac{(2 \; W/m-K)(10 \; K)}{(0.0024 \; m^2)(1000 \; kg/m^3)}(36000 \; \sec)$$
$$or \quad L_{l-to-s} = 3 \times 10^5 J/kg$$

Problem 10

(a) The inner product with arbitrary states $f(\mathbf{x})$ and $g(\mathbf{x})$

$$\begin{aligned}\langle f|\hat{A}|g\rangle &= \int f^*(\mathbf{x})\,(\hat{A}g)\,r^2dr\,d\Omega \;=\; \int f^*(\mathbf{x})\,i\frac{\partial g}{\partial r}\,r^2dr\,d\Omega \\ &= i\int \left(f^*gr^2\right)\Big|_0^\infty d\Omega \;-\; i\int g(\mathbf{x})\left(r^2\frac{\partial f^*}{\partial r}+2rf^*\right)dr\,d\Omega \\ &= -i\int g(\mathbf{x})\left(\frac{\partial f^*}{\partial r}+\frac{2}{r}f^*\right)r^2dr\,d\Omega\ .\end{aligned}$$

Setting this equal to $\langle g|\hat{A}^\dagger|f\rangle^*$ we find

$$\hat{A}^\dagger \;=\; i\left(\frac{\partial}{\partial r}+\frac{2}{r}\right) \tag{1}$$

(b) The operator $\hat{B} = \frac{1}{2}\hat{A}+\frac{1}{2}\hat{A}^\dagger = i(\partial/\partial r + 1/r)$, is Hermetian. Taking its Hermitian conjugate

$$\hat{B}^\dagger \;=\; \tfrac{1}{2}\hat{A}^\dagger+\tfrac{1}{2}(\hat{A}^\dagger)^\dagger \;=\; \tfrac{1}{2}\hat{A}^\dagger+\tfrac{1}{2}\hat{A} \;=\; \hat{B}\ .$$

An operator that is its own Hermitian conjugate is a "Hermitian operator".

(c) The Hamiltonian of a free particle is

$$\begin{aligned}\hat{H} &= -\frac{\hbar^2}{2m}\nabla^2 \\ &= -\frac{\hbar^2}{2m}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)-\frac{\hbar^2}{2m\,r^2}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)\;+\;\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]\end{aligned}$$

First we find the commutator

$$\left[\hat{B},\frac{1}{r^2}\right]f \;=\; \left[i\frac{\partial}{\partial r},\frac{1}{r^2}\right]f \;=\; i\frac{\partial}{\partial r}\left(\frac{f}{r^2}\right)-\frac{i}{r^2}\frac{\partial f}{\partial r} \;=\; -\frac{2i}{r^3}f\ .$$

This alone shows that

$$\left[\hat{B},\hat{H}\right] \;=\; \text{something}+\frac{i\hbar^2}{m\,r^3}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)\;+\;\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]$$

where "something" involves only r and $\partial/\partial r$. In other words nothing will cancel out the θ and ϕ derivatives so $\left[\hat{B},\hat{H}\right]\neq 0$. The operators **do not** commute.

(d) The eigenvalue equation is

$$\hat{B}\psi \;=\; i\frac{\partial\psi}{\partial r}+i\frac{\psi}{r} \;=\; \lambda\psi \quad\text{or}\quad \frac{\partial\psi}{\partial r} \;=\; -\left(\frac{1}{r}+i\lambda\right)\psi\ .$$

We solve this by multiplying both sides by the integrating factor $e^{a(r)}$, where

$$a(r) \;=\; \int^r\left(\frac{1}{r}+i\lambda\right)dr \;=\; \ln(r)+i\lambda r\ ,$$

to give

$$\frac{\partial}{\partial r}\left(e^{a(r)}\psi\right) \;=\; \frac{\partial}{\partial r}\left(re^{i\lambda r}\psi\right) \;=\; 0$$

From which we find

$$\psi(r) \;=\; \frac{e^{-i\lambda r}}{r}\ . \tag{2}$$

#11 Full Solution

a) We know $\Delta E_{int} = Q - W$. For an isotherm, $\Delta E_{int} = 0$ so we have $W = Q$.

For Q_H we have $Q_H = W_{a\to b} = nRT_H \ln\left(\frac{V_b}{V_a}\right)$

For Q_L we have $Q_L = W_{c\to d} = nRT_L \ln\left(\frac{V_a}{V_b}\right) = -nRT_L \ln\left(\frac{V_b}{V_a}\right)$

For the work done in one cycle, we note that there is no work done for b→c or d→a, so

$$W = W_{a\to b} + W_{c\to d} = |Q_H| - |Q_L| = nR(T_H - T_L)\ln\left(\frac{V_b}{V_a}\right)$$

To find the Q for the isovolumetric paths b→c or d→a, we use

$$Q = nC_v\Delta T = n\left(\frac{3}{2}R\right)(T_H - T_L)$$

To find the efficiency e we use

$$e = \frac{W}{Q_{in}} = \frac{nR(T_H - T_L)\ln\left(\frac{V_b}{V_a}\right)}{nRT_H \ln\left(\frac{V_b}{V_a}\right) + n\left(\frac{3}{2}R\right)(T_H - T_L)} = \frac{(T_H - T_L)\ln\left(\frac{V_b}{V_a}\right)}{T_H \ln\left(\frac{V_b}{V_a}\right) + \left(\frac{3}{2}\right)(T_H - T_L)}$$

$$= \frac{(T_H - T_L)}{T_H + \left(\frac{3}{2}\right)\frac{(T_H - T_L)}{\ln\left(\frac{V_b}{V_a}\right)}} = \frac{\left(1 - \frac{T_L}{T_H}\right)}{1 + \left(\frac{3}{2}\right)\frac{\left(1 - \frac{T_L}{T_H}\right)}{\ln\left(\frac{V_b}{V_a}\right)}}$$

b) Now we put in the values and get a rough estimate:

$$e = \frac{\left(1 - \frac{300K}{850K}\right)}{1 + \left(\frac{3}{2}\right)\frac{\left(1 - \frac{300K}{850K}\right)}{\ln\left(\frac{4L}{2L}\right)}} \approx \frac{\left(1 - \frac{1}{3}\right)}{1 + \left(\frac{3}{2}\right)\frac{\left(1 - \frac{1}{3}\right)}{\ln(2)}} \approx \frac{\left(\frac{2}{3}\right)}{1 + \left(\frac{3}{2}\right)\frac{\left(\frac{2}{3}\right)}{0.693}} \approx \frac{0.7}{1 + \frac{1}{0.7}} \approx \frac{0.7}{1 + 1.4} \approx \frac{0.7}{2.4} \approx 30\%$$

c) Now we find the efficiency e_{Carnot} for the Carnot engine to compare.

$$e = 1 - \frac{T_L}{T_H} = 1 - \frac{300K}{850K} \approx 1 - \frac{1}{3} \approx 0.66 \approx 65\%$$

Problem 12

(a) Using a coordinate z which increases downward from the hook,

$$z_{\rm cm} = \frac{M(L/2) + ML}{2M} = \tfrac{3}{4}L \ . \tag{1}$$

The moments of inertia of the components about the hook are

$$I_{\rm shaft} = \tfrac{1}{12}ML^2 + M\left(\frac{L}{2}\right)^2 = \tfrac{1}{3}ML^2$$
$$I_{\rm head} = ML^2$$

The moment of interia of the sledge hammer is therefore

$$I_{\rm sledge} = I_{\rm shaft} + I_{\rm head} = \tfrac{4}{3}ML^2 \tag{2}$$

(b) The change in the projectile's angular momentum (about the hook) is

$$\Delta\mathbf{L}_{\rm proj.} = \mathbf{r} \times \Delta\mathbf{p}_{\rm proj.} = (L/2)\Delta p\,\hat{\mathbf{y}} \ ,$$

since $\mathbf{r} = (L/2)\hat{\mathbf{z}}$ (pointing downward) during the impact. Since the hook exerts no torque about itself, angular momentum of the system must be conserved during the collision. The angular momentum of the sledge hammer is therefore $\mathbf{L}_{\rm sledge} = -(L/2)\Delta p\hat{\mathbf{y}}$ immediately after the impact. Its angular velocity at this time is

$$\boldsymbol{\omega} = \frac{\mathbf{L}_{\rm sledge}}{I_{\rm sledge}} = -\frac{3\Delta p}{8ML}\hat{\mathbf{y}} \ . \tag{3}$$

The momentum of the sledge hammer is therefore

$$\mathbf{p}_{\rm sledge} = 2M\mathbf{v}_{\rm cm} = -2M\omega\left(\tfrac{3}{4}L\right)\hat{\mathbf{x}} = -\tfrac{9}{16}\Delta p\,\hat{\mathbf{x}} \ . \tag{4}$$

This is in the same direction projectile was initially moving: $-\hat{\mathbf{x}}$.

By Newton's third law the impulse of the projectile on the sledge hammer is

$$\Delta\mathbf{p}_{ps} = \int \mathbf{F}_{ps}(t)\,dt = -\Delta p\,\hat{\mathbf{x}} \ .$$

To this the hook must have added an impulse

$$\Delta\mathbf{p}_{hs} = \tfrac{7}{16}\Delta p\,\hat{\mathbf{x}} \ , \tag{5}$$

so that the net momentum change was that from eq. (4).

(c) Striking at an unknown position z (distance from the hook), the projectile gives the sledge hammer angular momentum

$$\mathbf{L}_{\rm sledge} = -z\Delta p\,\hat{\mathbf{y}} \ ,$$

and therefore angular velocity

$$\boldsymbol{\omega} = \frac{\mathbf{L}_{\rm sledge}}{I_{\rm sledge}} = -\frac{3z\Delta p}{4ML^2}\hat{\mathbf{y}} \ . \tag{6}$$

The final momentum of the sledge hammer is therefore

$$\mathbf{p}_{\rm sledge} = 2M\mathbf{v}_{\rm cm} = -2M\omega\left(\tfrac{3}{4}L\right)\hat{\mathbf{x}} = -\tfrac{9}{8}\frac{z}{L}\Delta p\,\hat{\mathbf{x}} \ . \tag{7}$$

If the hook delivered no impulse, then this, the final momentum, must equal the impulse delivered by the projectile, $-\Delta p\,\hat{\mathbf{x}}$. Imposing this shows that the projectile must strike at a distance $z = \frac{8}{9}L$ from the hook.

Prob. 13

There are a number of ways to solve this problem including using simple basic freshman level physics (see below). Let us start with the more challenging way first and apply the Maxwell-Boltzmann distribution to a single particle by assuming that each particle in the cylindrical volume will be governed by the single particle distribution function in the configuration space, which is given by configuration probability density $\phi(r)$. In order to determine $\phi(r)$, we need to determine the classical Hamiltonian $H = \frac{1}{2}mv^2 + U(r)$ for a single particle at position r, where U is the potential energy at position *r*. *U* can be determined by considering the rotating frame of the particles where each particle is stationary (aside from its kinetic behavior) but experiences a central force (centripetal force) given by $F(r) = -\frac{dU}{dr} = -m\omega^2 r$. This immediately suggests that in the laboratory frame each particle has a potential energy given by:

$U = -m\omega^2 r^2/2$.

a. Then $\phi(r) = \frac{e^{-\beta U}}{\int d^3\vec{r}\, e^{-\beta U}}$ gives the probability density (probability of finding a particle per unit volume) at configuration space point $\vec{r}$. In cylindrical coordinates this takes the form:

$$\phi(r) = \frac{e^{\beta m\omega^2 r^2/2}}{\int_0^R 2\pi h\, e^{\beta m\omega^2 r^2/2} r\,dr} = \frac{\beta m\omega^2 e^{\beta m\omega^2 r^2/2}}{2\pi h(e^{\beta m\omega^2 r^2/2} - 1)}$$

where $\phi(r)\, 2\pi h r dr$ represents the probability of finding one of the *N* particles in the volume element specified by cylindrical shell $2\pi h r dr$ at location *r* from the center. Here, *h* represents the height of the spinning cylinder, *R* its radius, $\beta=1/k_BT$, and N is the total number of particles in the cylinder. Therefore, it is clear that $n(r) = N\phi(r)$ is the particle density at position *r*. The local pressure at position *r* can then be found using the ideal gas law $p(r)=n(r)\, k_BT= N\phi(r)/\beta$. This immediately yields

$$\boxed{p(r,\omega) = \frac{Nm\omega^2 e^{\beta m\omega^2 r^2/2}}{2\pi h(e^{\beta m\omega^2 R^2/2} - 1)} = p(0,\omega)e^{\beta m\omega^2 r^2/2}} \quad \text{.............(1)}$$

The pressure $p(0,\omega)$ at *r=0* relative to the atmospheric pressure $p(0,0) = N/\beta V$ can readily be determined by inserting *r=0* in the above equation, and taking the ratio to the atmospheric pressure at $=0$ we obtain: $p(0,\omega)/p(0,0) = p(0,\omega)/(N/\beta V) = \frac{Nm\omega^2}{2\pi h(e^{\beta m\omega^2 R^2/2} - 1)}\frac{\beta V}{N}$. Using $V = \pi R^2 h$, we immediately obtain

$$\boxed{p(0,\omega)/p(0,0) = \frac{(\beta m\omega^2 R^2/2)}{(e^{\beta m\omega^2 R^2/2} - 1)} = \frac{u}{e^u - 1}}\text{, where } u = \beta m\omega^2 R^2/2.$$

b. Plugging in the numbers suggested gives

$p(0,\omega)/p(0,0)$=0.58. The pressure at the center is $p(0,\omega)$ =*0.58 atm.*

a. Now let us solve the same problem *the easy way* using freshman level physics. Consider a spinning volume element dV at position r and apply Newton's second law of motion to the mass dm contained within volume element dV. Mass dm then is given by $dm = m\, n(r)\, dV$ where $dV = 2\pi h r dr$ is the volume element associated with a cylindrical shell at position r and $n(r)$ is the local density of air molecules at position r. Remember that the local density is related to the local pressure $p(r)$ by the ideal gas law: $n(r) = \beta p(r)$. Now using Newton's second law of motion applied to mass dm at position r spinning with angular velocity ω yields immediately: $dF = dm\,\omega^2 r$ where dF is the local net central force (centripetal force) acting on mass element dm and is given by $dF = dp\, S(r) = dp 2\pi rh$, where dp is the pressure drop across dr at position r and $S(r)$ is the surface area of the cylindrical shell at position r. Rearranging Newton's equation at position r gives: $\frac{dp}{p} = \beta m\omega^2 r\, dr$, which immediately yields the same relation as in Equation (1) above: $p(r,\omega) = p(0,\omega)\, e^{\beta m\omega^2 r^2/2}$ where $p(0,\omega)$ can be found from the normalization condition in that $n(r,\omega) = \beta p(r,\omega)$, and the total number of particles contained in the spinning cylinder is given by

$$N = \int_0^R n(r,\omega)\, 2\pi h r dr = \int_0^R \beta\; p(0,\omega) e^{\beta m\omega^2 r^2/2} 2\pi h r dr.$$

This immediately yields $p(0,\omega)$

$$\boxed{p(0,\omega) = \frac{N}{2\pi h\beta \int_0^R e^{\beta m\omega^2 r^2/2}\, rdr} = \frac{N m\omega^2}{2\pi h(e^{\beta m\omega^2 R^2/2} - 1)} \rightarrow p(r,\omega) = p(0,\omega)\, e^{\beta m\omega^2 r^2/2}}$$

as in equation (1) above.

Prob. 14

Quantum Mechanics – Solution

The complete solution to the barrier problem has solutions to three separate regions with boundary condition matching between each. Dividing the x-axis into three region, region I for $x \le 0$ is prior to the barrier, region II for $0 \le x \le a$ is within the barrier of height V, and region III for $x \ge a$ is after the barrier (here we have assumed particles are initially traveling in the $+x$-direction and are therefore incident on the barrier from negative x).

The general solution to the Schroedinger equation for particles with positive energy ($E > 0$) are traveling waves in both directions

$$\psi_I(x) = Ae^{ik_I x} + Be^{-ik_I x} \qquad \textit{for region } I$$
$$\psi_{II}(x) = Ce^{ik_{II} x} + De^{-ik_{II} x} \qquad \textit{for region } II$$
$$\psi_{III}(x) = Ee^{ik_{III} x} + Fe^{-ik_{III} x} \qquad \textit{for region } III$$

Substituting these solutions into the Schroedinger equation with $V(x) = V$ in region II only and zero elsewhere determines that

$$k_I = k_{III} = \sqrt{\frac{2mE}{\hbar^2}} \qquad \textit{and} \qquad k_{II} = \sqrt{\frac{2m(E-V)}{\hbar^2}}$$

Note that for $E < V$, k_{II} is imaginary and the solutions in region II become decaying exponentials. Matching boundary conditions for $\psi(x)$ and $d\psi(x)/dx$ at the points $x = 0$ and $x = a$, and recognizing that $F = 0$ (no particles incident from positive x), we can determine the parameters B, C, D, and E in terms of A (which represents the incident flux intensity of the particles on the barrier).

The transmission co-efficient can then be determined, after much algebra, as

$$T = \frac{E^*E}{A^*A} = \left[1 + \frac{\sin^2(ak_{II})}{4\frac{E}{V}\left(\frac{E}{V}-1\right)}\right]^{-1} \textit{ for } E > V \quad \textit{and} \quad T = \frac{E^*E}{A^*A} = \left[1 + \frac{\sinh^2(ak_{II})}{4\frac{E}{V}\left(1-\frac{E}{V}\right)}\right]^{-1} \textit{ for } E < V$$

For $E = V$, $T = ½$. The second solution is always less than 1, but the first solution has $T = 1$ for $k_{II} = n\pi/a$ where n is an integer greater than zero. The lowest energy that results in $T = 1$ occurs for $n = 1$ and has an energy $E = \left(\frac{\hbar^2\pi^2}{2ma^2} + V\right)$.

This is simply the condition that the barrier width is equal to a value that is one-half of the de Broglie wavelength of the particle when it is inside the barrier.

Problem 15 Solution

Consider the electrostatic potential $\Phi(\vec{r})$ at an arbitrary field point $\vec{r}$ due to a positive point charge q located above the xy plane at $x = y = 0, z = a$, and a negative point charge $-q$ located below the xy plane at $x = y = 0, z = -a$. Treat the vector $\vec{r}$ in terms of spherical coordinates: r is the distance of the field point from the origin, θ is the polar angle (down from the $+z$ axis), and ϕ is the azimuthal angle (counterclockwise about the $+z$ axis).

(a) Determine the exact expression for $\Phi(r, \theta, \phi)$ assuming that the potential is zero as $r \to \infty$. Explain why $\Phi(\vec{r})$ is independent of ϕ.

For $\theta < \pi/2$ and in MKS units:

$$\Phi(r, \theta) = \frac{q}{4\pi\epsilon_o}\left\{\frac{1}{\sqrt{r^2 + a^2 - 2ra\cos\theta}} - \frac{1}{\sqrt{r^2 + a^2 + 2ra\cos\theta}}\right\}$$

If $\pi/2 < \theta < \pi$, then the signs of the cosine terms in the radicands switch. The potential is independent of ϕ because the two-charge system is azimuthally symmetric.

(b) For $r \gg a$, determine the first two non-zero terms in the series expansion of $\Phi(\vec{r})$ in powers of a/r for arbitrary θ. Explain the physical significance of these first two terms.

This is the classic multipole expansion. We expect that the monopole term will vanish because there is zero net charge and that the dipole term will dominate. We do a binomial expansion of the square roots, treating the arguments of the square-roots as $1 + x$, where $x \ll 1$.

$$\begin{aligned}
\Phi(\vec{r}) &= \frac{q}{4\pi\epsilon_o r}\left\{\frac{1}{\sqrt{1 + \left(\frac{a}{r}\right)^2 - 2\left(\frac{a}{r}\right)\cos\theta}} - \frac{1}{\sqrt{1 + \left(\frac{a}{r}\right)^2 + 2\left(\frac{a}{r}\right)\cos\theta}}\right\} \\
&= \frac{q}{4\pi\epsilon_o r}\left\{1 + \left(-\frac{1}{2}\right)\left[\left(\frac{a}{r}\right)^2 - 2\left(\frac{a}{r}\right)\cos\theta\right] + \left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left[\left(\frac{a}{r}\right)^2 - 2\left(\frac{a}{r}\right)\cos\theta\right]^2\right. \\
&\quad + \left(\frac{1}{3!}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\left[\left(\frac{a}{r}\right)^2 - 2\left(\frac{a}{r}\right)\cos\theta\right]^3 + \ldots \\
&\quad -1 - \left(-\frac{1}{2}\right)\left[\left(\frac{a}{r}\right)^2 + 2\left(\frac{a}{r}\right)\cos\theta\right] - \left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left[\left(\frac{a}{r}\right)^2 + 2\left(\frac{a}{r}\right)\cos\theta\right]^2 \\
&\quad \left. - \left(\frac{1}{3!}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\left[\left(\frac{a}{r}\right)^2 + 2\left(\frac{a}{r}\right)\cos\theta\right]^3 + \ldots\right\} \\
&= \frac{q}{4\pi\epsilon_o r}\left\{(1-1)\left(\frac{a}{r}\right)^0 + (1+1)\cos\theta\left(\frac{a}{r}\right)^1 + \left(-\frac{1}{2} + \frac{1}{2} + \frac{3}{2}\cos^2\theta - \frac{3}{2}\cos^2\theta\right)\left(\frac{a}{r}\right)^2\right. \\
&\quad \left. + \left(-\frac{3}{2}\cos\theta - \frac{3}{2}\cos\theta + \frac{5}{2}\cos^3\theta + \frac{5}{2}\cos^3\theta\right)\left(\frac{a}{r}\right)^3 + \ldots\right\} \\
&= \frac{2q}{4\pi\epsilon_o r}\left\{P_1(\cos\theta)\left(\frac{a}{r}\right)^1 + P_3(\cos\theta)\left(\frac{a}{r}\right)^3 + \ldots\right\}
\end{aligned}$$

The first-two surviving terms are the dipole term – as expected – and the octupole term. The quadrupole term is not present because of the odd symmetry in the charge configuration. Similarly, all even Legendre polynomial terms will not appear in the expansion.