Jackson 7.19

Part b

Our function for this problem is

In[31]:=
```
f = N0 E^(-α^2 x^2/4);
```

The energy density is

In[32]:=
```
u0 = f E^(I k0 x);
```

The amplitude of the wavenumber spectrum is given by

In[33]:=
```
A[u_] := 1/Sqrt[2 π] Integrate[u E^(-I k x), {x, -Infinity, Infinity}];
```

Plug in our energy density

In[34]:=
```
Ab = FullSimplify[A[u0], Assumptions → α ∈ Reals && α > 0]
```

Out[34]=
```
$Aborted
```

Evaluate the wavenumber spectrum

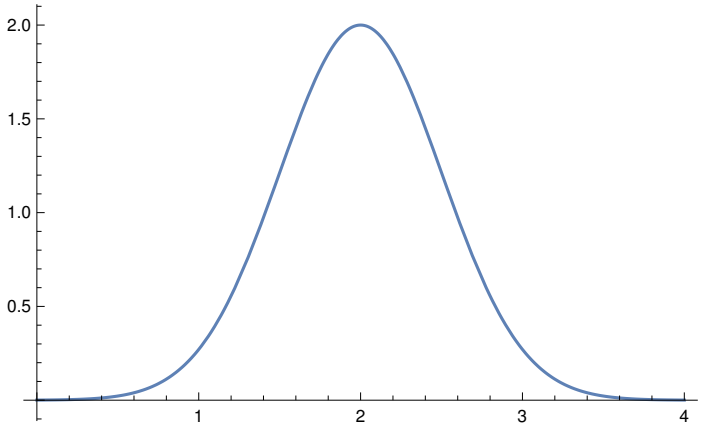
In[16]:=
```
kSpec = Ab^2
```

Out[16]=
$$\frac{2\, e^{-\frac{2\,(k-k0)^2}{\alpha^2}}\, N0^2}{\alpha^2}$$

Plot the wavenumber spectrum

In[19]:=
```
Plot[kSpec /. k0 → 2 α /. α → 1 /. N0 → 1, {k, 0, 4}]
```

Out[19]=



Evaluate the modulus squared of the energy density

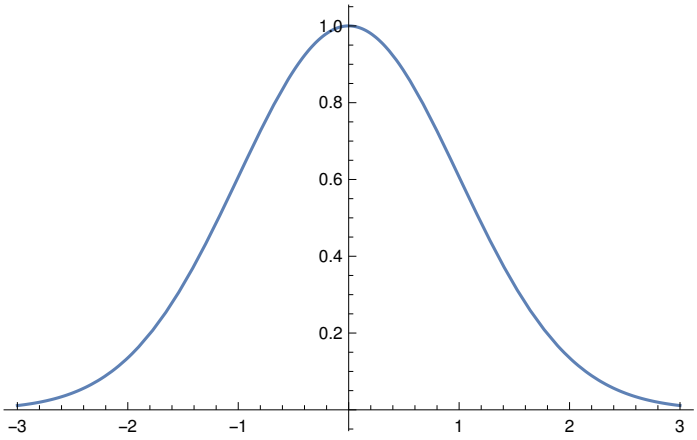
In[21]:=
```
uSpec = FullSimplify[Abs[u0]^2, Assumptions → N0 > 0 && α > 0 && x ∈ Reals && k0 ∈ Reals]
```

Out[21]=
$$e^{-\frac{1}{2} x^2 \alpha^2}\, N0^2$$

Plot the modulus squared of the energy density

In[25]:=
```
Plot[uSpec /. α → 1 /. N0 → 1, {x, -3, 3}]
```

Out[25]=



Find the spread in the wavenumber

In[27]:=
```
Δk = FullSimplify[Sqrt[Integrate[k^2 kSpec, {k, -Infinity, Infinity}] / Integrate[kSpec, {k, -Infinity, Infinity}]], Assumptions → α > 0]
```

Out[27]=
$$\frac{1}{2}\sqrt{4\,k0^2+\alpha^2}$$

In[28]:=
```
Δx = FullSimplify[Sqrt[Integrate[x^2 uSpec, {x, -Infinity, Infinity}] / Integrate[uSpec, {x, -Infinity, Infinity}]], Assumptions → α > 0]
```

Out[28]=
$$\frac{1}{\alpha}$$

Part c

Our function for this problem is

In[63]:=
```
f = N0 (-HeavisideTheta[x - a] + HeavisideTheta[x + a]);
Plot[f /. N0 → 1 /. a → 1, {x, -2, 2}];
```

The energy density is

In[65]:=
```
u0 = f E^(I k0 x);
```

Plug in our energy density

In[67]:=
```
Ac = FullSimplify[A[u0], Assumptions → Im[k0] > Im[k]]
```

Out[67]=
$$\frac{N0\sqrt{\frac{2}{\pi}}\,\text{Sin}\left[a\,(k-k0)\right]}{k-k0}$$

Evaluate the wavenumber spectrum

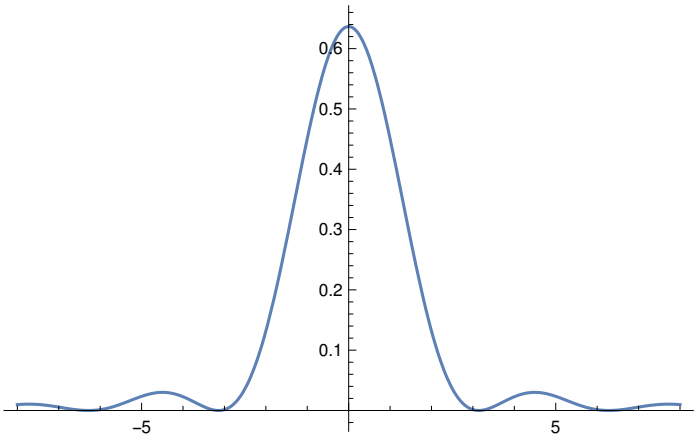
In[68]:=
```
kSpec = Ac^2
```

Out[68]=
$$\frac{2\,N0^2\,\text{Sin}\left[a\,(k-k0)\right]^2}{(k-k0)^2\,\pi}$$

Plot the wavenumber spectrum

In[76]:=
```
Plot[kSpec /. k0 → 0 /. a → 1 /. N0 → 1, {k, -8, 8}]
```

Out[76]=



Evaluate the modulus squared of the energy density

In[71]:=
```
(uSpec = FullSimplify[Abs[u0]^2,
    Assumptions → N0 > 0 && a > 0 && x ∈ Reals && k0 ∈ Reals]) // TraditionalForm
```

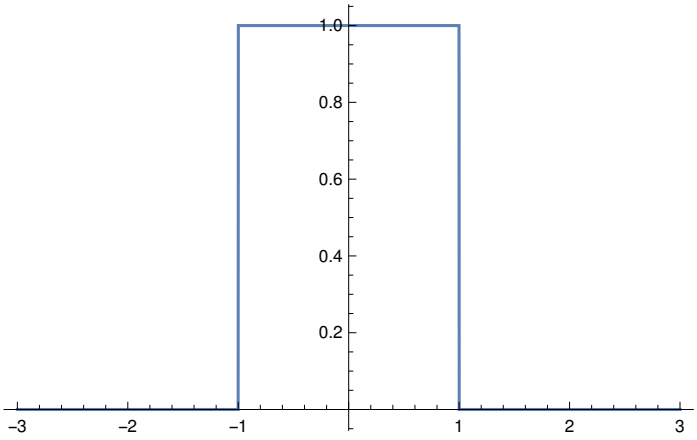
Out[71]//TraditionalForm=

$$N0^2 \left|\theta(x-a)-\theta(a+x)\right|^2$$

Plot the modulus squared of the energy density

In[72]:=
```
Plot[uSpec /. a → 1 /. N0 → 1, {x, -3, 3}]
```

Out[72]=



Find the spread in the wavenumber

In[78]:=
```
Δk = FullSimplify[ √( Integrate[ k^2 kSpec /. k0 → 0, {k, -Infinity, Infinity}] / Integrate[kSpec /. k0 → 0, {k, -Infinity, Infinity}] ) ]
```

... Integrate: Integral of $\frac{2\,N0^2\,\mathrm{Sin}[a\,k]^2}{\pi}$ does not converge on {−∞, ∞}.

Out[78]=
```
ConditionalExpression[ √( ∫_-∞^∞ (2 N0^2 Sin[a k]^2)/π dk / N0^2 ) / (√2 √Abs[a]), a ∈ Reals]
```

In[79]:=
$$\Delta x = \text{FullSimplify}\left[\sqrt{\frac{\text{Integrate}\left[x^2\, \text{uSpec}, \{x, -\text{Infinity}, \text{Infinity}\}\right]}{\text{Integrate}\left[\text{uSpec}, \{x, -\text{Infinity}, \text{Infinity}\}\right]}}\right]$$

Out[79]=
$$\frac{\sqrt{a^2}}{\sqrt{3}}$$