

# Homework 5

## Problem 1

```
In[304]:= x =  $\frac{1}{2} (u^2 - v^2)$ 
```

```
y = u v
```

```
Out[304]=  $\frac{1}{2} (u^2 - v^2)$ 
```

```
Out[305]= u v
```

```
In[306]:= A = {x y, y^2 - x^2} // FullSimplify
```

```
Out[306]=  $\left\{ \frac{1}{2} u (u - v) v (u + v), \frac{1}{4} (-u^4 + 6 u^2 v^2 - v^4) \right\}$ 
```

```
In[307]:= first = X == x
```

```
second = Y == y
```

```
Out[307]=  $X == \frac{1}{2} (u^2 - v^2)$ 
```

```
Out[308]=  $Y == u v$ 
```

```
In[309]:= rev = Solve[first && second, {u, v}] // Expand // FullSimplify
```

```
Out[309]=  $\left\{ \left\{ u \rightarrow -\sqrt{X - \sqrt{X^2 + Y^2}}, v \rightarrow -\frac{Y}{\sqrt{X - \sqrt{X^2 + Y^2}}} \right\}, \left\{ u \rightarrow \sqrt{X - \sqrt{X^2 + Y^2}}, v \rightarrow \frac{Y}{\sqrt{X - \sqrt{X^2 + Y^2}}} \right\}, \right.$   
 $\left. \left\{ u \rightarrow -\sqrt{X + \sqrt{X^2 + Y^2}}, v \rightarrow -\frac{Y}{\sqrt{X + \sqrt{X^2 + Y^2}}} \right\}, \left\{ u \rightarrow \sqrt{X + \sqrt{X^2 + Y^2}}, v \rightarrow \frac{Y}{\sqrt{X + \sqrt{X^2 + Y^2}}} \right\} \right\}$ 
```

```
In[310]:= uBasis = {D[rev[[4, 1, 2]], X] // FullSimplify, D[rev[[4, 1, 2]], Y] // FullSimplify}
```

```
vBasis = {D[rev[[4, 2, 2]], X] // FullSimplify, D[rev[[4, 2, 2]], Y] // FullSimplify}
```

```
Out[310]=  $\left\{ \frac{\sqrt{X + \sqrt{X^2 + Y^2}}}{2 \sqrt{X^2 + Y^2}}, \frac{Y}{2 \sqrt{X^2 + Y^2} \sqrt{X + \sqrt{X^2 + Y^2}}} \right\}$ 
```

```
Out[311]=  $\left\{ -\frac{Y}{2 \sqrt{X^2 + Y^2} \sqrt{X + \sqrt{X^2 + Y^2}}}, \frac{\sqrt{X + \sqrt{X^2 + Y^2}}}{2 \sqrt{X^2 + Y^2}} \right\}$ 
```

```
In[312]:= StreamPlot[uBasis, {X, -3, 3}, {Y, -3, 3}];
```

```
StreamPlot[vBasis, {X, -3, 3}, {Y, -3, 3}];
```

```
In[314]:= transU = du == D[rev[[4, 1, 2]], X] dx + D[rev[[4, 1, 2]], Y] dy // FullSimplify
transV = dv == D[rev[[4, 2, 2]], X] dx + D[rev[[4, 2, 2]], Y] dy // FullSimplify
```

$$\text{Out[314]}= du == \frac{dy Y + dx \left( X + \sqrt{X^2 + Y^2} \right)}{2 \sqrt{X^2 + Y^2} \sqrt{X + \sqrt{X^2 + Y^2}}}$$

$$\text{Out[315]}= dv == \frac{\sqrt{X + \sqrt{X^2 + Y^2}} \left( dy Y + dx \left( X - \sqrt{X^2 + Y^2} \right) \right)}{2 Y \sqrt{X^2 + Y^2}}$$

```
In[316]:= transXY = Solve[transU && transV, {dx, dy}] // FullSimplify
```

$$\text{Out[316]}= \left\{ \left\{ dx \rightarrow \frac{-dv Y + du \left( X + \sqrt{X^2 + Y^2} \right)}{\sqrt{X + \sqrt{X^2 + Y^2}}}, dy \rightarrow \frac{\sqrt{X + \sqrt{X^2 + Y^2}} \left( dv Y + du \left( -X + \sqrt{X^2 + Y^2} \right) \right)}{Y} \right\} \right\}$$

```
In[317]:= transXY[[1, 1, 2]]
```

$$\text{Out[317]}= \frac{-dv Y + du \left( X + \sqrt{X^2 + Y^2} \right)}{\sqrt{X + \sqrt{X^2 + Y^2}}}$$

```
In[318]:= col = {transXY[[1, 1, 2]], transXY[[1, 2, 2]]}
```

$$\text{Out[318]}= \left\{ \frac{-dv Y + du \left( X + \sqrt{X^2 + Y^2} \right)}{\sqrt{X + \sqrt{X^2 + Y^2}}}, \frac{\sqrt{X + \sqrt{X^2 + Y^2}} \left( dv Y + du \left( -X + \sqrt{X^2 + Y^2} \right) \right)}{Y} \right\}$$

```
In[319]:= Aij = {{Y, -2 X}, {X, 2 Y}}
```

$$\text{Out[319]}= \left\{ \{Y, -2 X\}, \{X, 2 Y\} \right\}$$

```
In[320]:= Aj = Aij.col // FullSimplify
```

$$\text{Out[320]}= \left\{ \frac{du Y \left( -X + \sqrt{X^2 + Y^2} \right) - dv \left( Y^2 + 2 X \left( X + \sqrt{X^2 + Y^2} \right) \right)}{\sqrt{X + \sqrt{X^2 + Y^2}}}, \frac{dv Y \left( X + 2 \sqrt{X^2 + Y^2} \right) + du \left( 2 Y^2 + X \left( X + \sqrt{X^2 + Y^2} \right) \right)}{\sqrt{X + \sqrt{X^2 + Y^2}}} \right\}$$

```
In[321]:= FullSimplify[Ajtrans = Aj /. X -> x /. Y -> y, Assumptions -> u \in Reals && v \in Reals]
```

$$\text{Out[321]}= \left\{ \left( -dv u^3 + du v^3 \right) \text{Sign}[u], \frac{1}{2} \left( dv v \left( 3 u^2 + v^2 \right) + du \left( u^3 + 3 u v^2 \right) \right) \text{Sign}[u] \right\}$$

## Part e.

```
In[322]:= eu = {D[x, u], D[y, u]}
```

```
ev = {D[x, v], D[y, v]}
```

$$\text{Out[322]}= \{u, v\}$$

$$\text{Out[323]}= \{-v, u\}$$

```
In[324]:= g = {{eu.eu, eu.ev}, {ev.eu, ev.ev}}
```

$$\text{Out[324]}= \left\{ \{u^2 + v^2, 0\}, \{0, u^2 + v^2\} \right\}$$

```

In[325]:= gradA = {{D[A[[1]], u], D[A[[1]], v]], {D[A[[2]], u], D[A[[2]], v]}} -

$$\frac{1}{u^2 - v^2} (\{u A[[1]], v A[[1]]\}, \{u A[[2]], v A[[2]]\}) +$$


$$\{u A[[1]], u A[[2]]\}, \{v A[[1]], v A[[2]]\}) -$$


$$\{u A[[1]] + v A[[2]], 0\}, \{0, u A[[1]] + v A[[2]]\}) // \text{Expand} // \text{FullSimplify}$$

Out[325]= 
$$\left\{ \left\{ \frac{3 u^4 v + v^5}{4 u^2 - 4 v^2}, \frac{3 u^5 - 16 u^3 v^2 + 9 u v^4}{4 u^2 - 4 v^2} \right\}, \left\{ -\frac{3 u^5 - 8 u^3 v^2 + 9 u v^4}{4 u^2 - 4 v^2}, \frac{15 u^4 v - 24 u^2 v^3 + 5 v^5}{4 u^2 - 4 v^2} \right\} \right\}$$


```