Shallow water waves — Problem set 3: due Mon. Sept. 25

1. Meteorologists perform weather forecasting using equations for the Earth's atmosphere formally identical to those describing a shallow layer of water.[1] The water layer has a depth $h(x, y, t)$ and velocity $\mathbf{u} = u(x, y, t)\hat{\mathbf{x}} + v(x, y, t)\hat{\mathbf{y}}$, whose evolution is governed by the equations

$$\frac{\partial h}{\partial t} + \nabla \cdot (h\mathbf{u}) = 0 \ , \tag{1}$$

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} + 2\Omega_z \,\hat{\mathbf{z}} \times \mathbf{u} = -g\nabla h \ . \tag{2}$$

The first is a version of continuity where a divergence in the horizontal flow must accompany a decrease in the height of the water. The second is a momentum equation where pressure is given by the "hydraulic head" of the column, gh. The Coriolis effect, important for weather although not for puddles of water, enters through Ω_z, the vertical component of the rotation frequency.

Assume first that the Coriolis effect is uniform — this is the so-called "f-plane approximation" — $\nabla\Omega_z = 0$.

   a. Linearize the governing equations about an equilibrium state of stationary layer, $\mathbf{u}_0 = 0$ of uniform depth h_0. Find the dispersion relations, $\omega(\mathbf{k})$, for the plane-wave normal modes.

   b. Describe the mode of lowest frequency (lowest absolute value). Can you write down a general relation between $\mathbf{u}_1(x, y)$ and $h_1(x, y)$ for this mode? SORT OF HINT: If plane wave modes were combined to form a region of initially enhanced depth – a peak in $h_1(x, y, 0)$ – what would happen to it? What would the flow field be like? (This is a high-pressure system.)

   c. Describe the other wave modes (other than the one in part b.) in the limit of short wave-lengths. Compared to what must $|\mathbf{k}|$ be large for this limit to apply? What is the phase speed of these waves?

The spatial variation of Ω_z becomes important for flows on scales comparable to the planet. This effect is especially important around the equator since all changes in Ω_z are relatively large. The simplest means of including this is the so-called β-plane approximation. Orienting the coordinates so $y = 0$ is the equator consider only linear variation

$$2\Omega_z = \beta\, y \ , \tag{3}$$

where β is a constant. It is a positive constant since $\Omega_z > 0$ in the Northern hemisphere ($y > 0$).

   d. Use the Eikenol approximation to find the ray paths for waves of the high-frequency kind (from part c.). Write as $y(x)$ the ray that crosses the equator ($y = 0$) at a 45° angle. Show that this ray reaches maximum latitudes where $2\Omega_z = \pm\omega/\sqrt{2}$.

[1] Go to `http://www.wpc.ncep.noaa.gov/dailywxmap/` to see contour maps of height $h(x, y)$ of the surface where $p = 0.5$ bar — the so-called 500-millibar map. (Contours are labelled in in dekameters, and have intervals of 6 dm.) Recall that the scale height of Earth's atmosphere is approximately 10 km. From this we would expect the pressure to fall by 50% at a height of roughly 5 km = 500 dm. Labels all fall around that range.

e. Propose a normal mode solution to the full set of linearized equations, and work them into a single ODE for $\hat{u}_{y1}$. Show that this can be expressed in the form

$$\frac{d^2\hat{u}_{y1}}{dy^2} \;-\; \alpha^2\, y^2\, \hat{u}_{y1} \;=\; f(\omega)\, \hat{u}_{y1} \quad , \tag{4}$$

where α is a constant and $f(\omega)$ is a function of the eigenfrequency. This equation is formally identical to Schrödinger's equation for a simple harmonic oscillator; its solutions can be written

$$\hat{u}_{y1} \;=\; H_n(\sqrt{\alpha}\, y)\, e^{-\alpha y^2/2} \quad , \tag{5}$$

where $H_n(x)$ is the Hermite polynomial of order $n = 0,\, 1,\, 2, \ldots,$

f. Use the solution from e. to find an equation for the dispersion relations $\omega_n(k_x)$ for all normal mode branches. How many branches are there for any particular value of k_x and mode number n? How do these relate to the waves found in part a.? Use some mathematical software to solve the equation for $\omega_n(k_x)$ and plot all branches for a few values of n — enough to exhibit the entire variety.

g. Show that at least one branch of solution in part e. recovers the Eikenol solution from part d. in the appropriate limit. Show that the full solution is confined (in some way) to the same range of latitudes as the rays were.

h. Consider the solution branches from f., that correspond to those from part b. By assuming the frequency is sufficiently small, wave numbers sufficiently large, you should be able to drop an ω_n^3 term to arrive at a linear equation for $\omega_n(k_x)$. Verify that this approximate solution matches your graphs from part f. over the appropriate range. Now make a reasonable assignment for the wave number k_y in terms of mode number n (explain your choice), and show that these waves have dispersion relation

$$\omega(k_x, k_y) \;\simeq\; -\,\frac{\beta k_x}{k_x^2 + k_y^2} \quad . \tag{6}$$

This is the dispersion relation for a *Rossby wave*, related to stellar waves known as *r-modes* (the "r" is for Rossby). For which waves is the group velocity directed Westward (i.e. $\hat{\mathbf{x}} \cdot \mathbf{v}_g < 0$) in the Earth's Northern hemisphere? In the Southern hemisphere? (Our analysis has been done in the reference frame of the atmosphere. In the case where there are prevailing winds, as observed from the surface, Rossby waves will be Doppler shifted.)