Kevin-Helmholtz instability in rotating, accreting, magnetic neutron stars

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I. ABSTRACT

There exists a Kevin-Helmholtz instability in the inner edge of the accretion disk of a rotating, magnetic neutron star with a non-compact companion due to the velocity difference between the disk and the neutron star magnetosphere. This instability will cause the disk plasma to diffuse into the magnetosphere. We derive the stability condition from the basic fluid equations for two simplified models (hydrodynamic and magnetohydrodynamic). We extend the derived stability condition for the neutron star and discuss the resulting regions of instability.

II. INTRODUCTION

Kevin-Helmholtz instability is a hydrodynamic instability caused due to the discontinuity in the velocity (shear flow) of two fluids in a relative, irrotational motion. The rarer fluid is superposed on a heavier fluid in order to discount gravity as the generator of instability. Also, viscosity, surface tension of the fluids is also neglected for simplification.
Neutron stars are extremely dense stars formed after the collapse of a $M_{\odot} - 3M_{\odot}$ star. They can be thought as one huge nucleus containing 10^{60} nucleons. Most observable neutron stars are found to be spinning at high speeds with extreme magnetic fields as pulsars or magnetars, or are radiating due to accretion from a companion star.
Rotating, accreting, magnetic neutron stars can be thought as a periodically pulsating X-ray source. Rotation of magnetized neutron star will act as a pulse source, whereas the energy introduced by accretion will generate X-rays. Learning about the transfer of angular momentum from the accretion disk to the neutron star is of vital importance to model the radiation of neutron star.
For accretion, there exists a region where the material of disk co-rotates with the magnetosphere. But away from this region there is a velocity difference between the fluids of the disk and the magnetosphere. This velocity difference will result in Kevin-Helmholtz instability. This instability can be one of the factors that causes turbulent diffusion[6] of disk material into the magnetosphere. We'll look at this K-H instability.
The actual instability is very complex. We'll use simpler models and extend it to give us some idea about the actual neutron star instability. We'll use two models, one is hydrodynamic model where we neglect the magnetic field of the neutron star, and another is Magnetohydrodynamic model where we introduce the magnetic field.

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III. HYDRODYNAMIC MODEL

A. Formulation

We consider a plane interface between two homogeneous inviscid, compressible fluids in Cartesian coordinate as shown in Figure 1. ρ is the fluid density, $\mathbf{u}$ is the

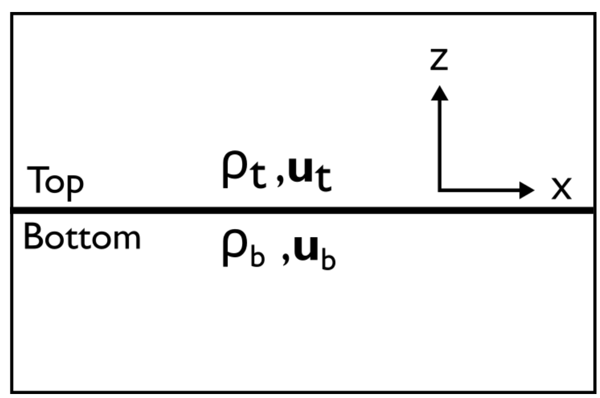

FIG. 1

fluid velocity. Subscript 't' is used for top fluid whereas 'b' for bottom fluid. We use 'p' for pressure , 'c_s' for sound speed, $\mathbf{k}$ for wave vector.

1. Governing Equations

The hydrodynamic continuity equation is:

$$\frac{\partial \rho}{\partial t} + \nabla(\rho\,\mathbf{u}) = 0 \tag{1}$$

The hydrodynamic momentum equation is:

$$\rho\,\frac{\partial \mathbf{u}}{\partial t} + \rho\left(\mathbf{u}\cdot\nabla\right)\mathbf{u} = -\nabla p \tag{2}$$

Linearizing both equations 1 and 2 using

$$\rho = \rho_0 + \rho_1 \qquad \mathbf{u} = \mathbf{u}_0 + \mathbf{u}_1 \qquad p = p_0 + p_1$$

We get

$$\frac{\partial \rho}{\partial t} + \rho_0\,\nabla\mathbf{u}_1 + \mathbf{u}_0\,\nabla\rho_1 = 0 \tag{3}$$

$$\rho_0\,\frac{\partial \mathbf{u}}{\partial t} + \rho_0\,\mathbf{u}_0\,\nabla\mathbf{u}_1 + \nabla p_1 = 0 \tag{4}$$

Since pressure has to be continuous at the boundary, we opt to convert density into pressure using equation of sound speed.

$$\frac{\partial p_1}{\partial t} + \rho_0\,c_s^2\,\nabla\mathbf{u}_1 + u_0\,\frac{\partial p_1}{\partial x} = 0 \tag{5}$$

$$\rho_0\,\frac{\partial \mathbf{u}}{\partial t} + \rho_0\,u_0\,\frac{\partial \mathbf{u}_1}{\partial x} + \nabla p_1 = 0 \tag{6}$$

2. Dispersion Relation

Now, we take perturbations proportional to $\exp i[k_x x + k_y y + k_z z - \omega t]$ where k_x , k_y are real and ω , k_z can be complex.

$$p_1 = A\exp i[k_x x + k_y y + k_z z - \omega t]$$
$$u_{1,i} = A_i\exp i[k_x x + k_y y + k_z z - \omega t]$$

Plugging into equations 5 and 6, we get

$$-i\omega p_1 + \rho_0 c_s^2 (ik_i A_i)\,e^{i[k_x x + k_y y + k_z z - \omega t]} + u_0 i k_x p_1 = 0$$

$$-i\omega\rho_0\mathbf{u}_1 + \rho_0 u_0 (ik_x A_i)\,e^{i[k_x x + k_y y + k_z z - \omega t]} + ik_i p_1 = 0$$

After simplification,

$$-i\omega A + \gamma p_0 (ik_i A_i) + u_0 i k_x A = 0 \tag{7}$$
$$-i\omega\rho_0 A_i + \rho_0 u_0 (ik_x A_i) + ik_i A = 0 \tag{8}$$

Introducing longitudinal wave vector such that $\mathbf{k} = (k_x, k_y, k_z) = (\mathbf{k}_\parallel, k_z)$.

$$-i\omega A + \gamma p_0 (ik_i A_i) + i\mathbf{k}_\parallel u_0 \cos\phi A = 0 \tag{9}$$
$$-i\omega\rho_0 A_i + \rho_0 u_0 \cos\phi (i\mathbf{k}_\parallel A_i) + ik_i A = 0 \tag{10}$$

Multiplying equation 10 by k_i, plugging into 9, and simplifying, we get

$$(\omega - \mathbf{k}_\parallel u_0 \cos\phi)^2 = \frac{\gamma p_0}{\rho_0} k^2 = c_{s,0}^2 k^2$$

For top and bottom fluids, dispersion relation becomes:

$$(\omega - \mathbf{k}_\parallel u_{t,0} \cos\phi)^2 = c_{s,t}^2(\mathbf{k}_\parallel^2 + k_{t,z}^2) \tag{11}$$

$$(\omega - \mathbf{k}_\parallel u_{b,0} \cos\phi)^2 = c_{s,b}^2(\mathbf{k}_\parallel^2 + k_{b,z}^2) \tag{12}$$

3. Stability Condition

For the equation 11 , 12 , we attempt to write k^2 in terms of $\mathbf{k}_\parallel$ using the fact that displacement perpendicular to the interface and the pressure are continuous in the planar boundary.
The z component of displacement ξ will satisfy,

$$\frac{D\xi}{Dt} = u_{1,z} = A_z e^{i[k_x x + k_y y + k_z z - \omega t]} \tag{13}$$

Perturbing $\xi = \xi_0 e^{i[k_x x + k_y y + k_z z - \omega t]}$ and using equation 13, we get

$$\xi_0 = \frac{iA_z}{\omega - \mathbf{k}_\parallel u_0 \cos\phi}$$

Using equation 10,

$$\xi_0 = \frac{ik_z A}{\rho_0(\omega - \mathbf{k}_\parallel u_0 \cos\phi)^2} \tag{14}$$

Pressure continuity together with z-displacement continuity across the interface, we get

$$\frac{k_{t,z}}{\rho_{t,0}(\omega - \mathbf{k}_\parallel u_{t,0}\cos\phi)^2} = \frac{k_{b,z}}{\rho_{b,0}(\omega - \mathbf{k}_\parallel u_{b,0}\cos\phi)^2} \tag{15}$$

Now, we switch to another frame (Galilean) in which velocity of bottom fluid is zero, and the top fluid is moving at velocity Δu.

$$(x, y, z) \longrightarrow (x' = x - u_{b,0}t, y' = y, z' = z)$$

In this new frame,

$$\omega' = \omega - \mathbf{k}_\parallel u_{b,0}\cos\phi \quad \omega - \mathbf{k}_\parallel u_{t,0}\cos\phi = \omega' - \mathbf{k}_\parallel \Delta u \cos\phi \tag{16}$$

where, $\Delta u = (u_{t,0} - u_{b,o})$.

Using equations 11,12,15 and 16, we find the stability condition to be:

$$\boxed{\frac{\dfrac{\omega'^2}{c_{s,b}^2} - \mathbf{k}_\parallel^2}{\gamma_b^2\dfrac{\omega'^4}{c_{s,b}^4}} = \frac{\dfrac{(\omega' - \mathbf{k}_\parallel\Delta u\cos\phi)^2}{c_{s,t}^2} - \mathbf{k}_\parallel^2}{\gamma_t^2\dfrac{(\omega' - \mathbf{k}_\parallel\Delta u\cos\phi)^4}{c_{s,t}^4}}} \tag{17}$$

B. Extension to Neutron Star

In neutron star, let the top fluid be the magnetosphere and bottom fluid be the accretion disk. The adiabatic index γ can be taken as equal in both fluids. Also using phase velocity $\omega = \frac{\omega'}{k_\parallel}$, equation 17 becomes:

$$c_{s,b}^2\frac{(\omega^2 - c_{s,b}^2)}{\omega^4} = c_{s,t}^2\frac{(\omega - \Delta u\cos\phi)^2 - c_{s,t}^2}{(\omega - \Delta u\cos\phi)^4}$$

Using $\dfrac{c_{s,t}}{c_{s,b}} \equiv x$, $\dfrac{\omega}{c_{s,b}} \equiv y$, and $\dfrac{\Delta u \cos\phi}{c_{s,b}} \equiv \mathscr{M}$, above equation on simplification becomes:

$$\Big[\frac{1}{y^2} - \frac{x^2}{(y-\mathscr{M})^2}\Big]\Big[1 - \frac{1}{y} - \frac{x^2}{(y-\mathscr{M})^2}\Big] = 0 \tag{18}$$

All solutions of the first expression of equation 18 are real. But all solutions of second expression are real only if $\mathscr{M}^2 > 8$. i.e.

$$\mathscr{M} = \frac{\Delta u \cos\phi}{c_{s,b}} > \mathscr{M}_{critical} = 2\sqrt{2} \tag{19}$$

In neutron star, this condition is invalid at least for a small ring of accretion disk around the co-rotation radius. The instability can extend further if $\cos\phi << 1$ is satisfied for the waves. In this region of instability, k-modes of the waves can grow exponentially.

IV. MAGNETOHYDRODYNAMIC MODEL

A. Formulation

We now consider a plane interface between two homogeneous inviscid, perfectly conducting compressible fluids in the presence of uniform magnetic fields in Cartesian coordinate as shown in Figure 2. Here $\mathbf{B}$ is the magnetic field. We follow the notation from Hydrodynamic Model.

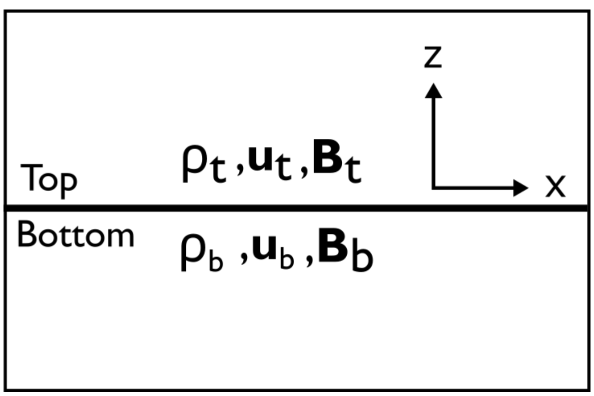

FIG. 2

1. Governing Equations

Linearized Magnetohydrodynamic equations about a homogeneous, static equilibrium are:

$$\frac{\partial \rho_1}{\partial t} + \rho_1 \nabla \mathbf{u}_1 = 0 \tag{20}$$

$$\frac{\partial \mathbf{u}_1}{\partial t} + \Big(\frac{c_s^2}{\rho_0}\Big)\nabla\rho_1 = -\frac{1}{4\pi\rho_0}\mathbf{B}_0 \times (\nabla \times \mathbf{B}_1) \tag{21}$$

$$\frac{\partial \mathbf{B}_1}{\partial t} = \nabla \times (\mathbf{u}_1 \times \mathbf{B}_0) \tag{22}$$

We seek plane wave solutions proportional to $\exp[i(\mathbf{k}\cdot\mathbf{r} - \omega t)]$. The linearized equations become:

$$-\omega\rho_1 + \rho_0\,\mathbf{k}\cdot\mathbf{u}_1 = 0 \tag{23}$$

$$-\omega\mathbf{u}_1 + \Big(\frac{c_s^2}{\rho_0}\Big)\rho_1\mathbf{k} = -\frac{1}{4\pi\rho_0}\mathbf{B}_0 \times (\mathbf{k}\times\mathbf{B}_1) \tag{24}$$

$$-\omega\mathbf{B}_1 = \mathbf{k}\times(\mathbf{u}_1\times\mathbf{B}_0) \tag{25}$$

2. Dispersion relation

Consider l,m,n coordinate system. m-axis be along the $\mathbf{k}$ direction, and mn-plane be the plane of $\mathbf{k}$ and $\mathbf{B}_0$. Then, eliminating ρ_1 from the equation 23 and 24, we get:

$$\omega B_{1,l} = -k\,u_{1,l}B_{0,m} \qquad \omega u_{1,l} = -\frac{\mu_0}{\rho_0}k\,B_{0,m}B_{1,l} \tag{26}$$

$$\omega B_{1,n} = k\,(u_{1,m}B_{0,n} - u_{1,n}B_{0,m}) \quad \omega u_{1,n} = -\frac{\mu_0}{\rho_0}k\,B_{0,m}B_{1,n}$$

$$u_{1,m}\Big(\frac{\omega}{k} - \frac{kc_s^2}{\omega}\Big) = \frac{\mu_0}{\rho_0}B_{0,n}B_{1,n} \tag{27}$$

We find the dispersion relation from equation set 26 and 27 as:

$$\omega^4 - (c_s^2 + V^2)k^2\omega^2 + k^4c_s^2V^2\cos^2\theta = 0 \tag{28}$$

where, θ is the angle between $\mathbf{k}$ and $\mathbf{B}_0$. $\mathbf{V} = \sqrt{\frac{\mu_0}{\rho_0}}\mathbf{B}_0$ is the Alfvén velocity.
In terms of $\mathbf{k} = (k_x, k_y, k_z) = (\mathbf{k}_\parallel, k_z)$,

$$\mathbf{k}_z^2 + \mathbf{k}_\parallel^2 = \frac{\omega^4}{\omega^2(c_s^2+V^2) - c_s^2(\mathbf{k}_\parallel\cdot\mathbf{V})^2} \tag{29}$$

3. Stability Condition

Let $\mathbf{B}_0 = B_0\hat{\mathbf{x}}$ and $p_1 = c_s^2\rho_1$. Now the z-component of equations 25, 24 are:

$$-\omega u_{1,z} + \frac{k_z p_1}{\rho_0} = \Big(\frac{\mu_0}{\rho_0}\Big)B_0(k_xB_{1,z} - k_zB_{1,x}) \tag{30}$$

$$-\omega B_{1,z} = B_0\,k_x\,u_{1,z} \tag{31}$$

Eliminating $u_{1,x}$ from equation 30 and 31, we get:

$$p_1 + \mu_0B_0B_{1,x} = \frac{B_{1,z}\rho_0}{k_zk_xB_0}\Big(\frac{\mu_0B_0^2k_x^2}{\rho_0} - \omega^2\Big) \tag{32}$$

Lets do a Galilean transformation to a coordinate system in which the bottom fluid is at rest and the top fluid is moving at $\mathbf{v}_0$. So, for bottom fluid, $\omega \longrightarrow \omega'$ and for top fluid, $\omega \longrightarrow |\omega' - \mathbf{k}_\parallel\cdot\mathbf{v}_0|$.

We assume no reflection of MHD wave at the interface. Sum of hydrodynamic and magnetic pressure (left expression of equation 32) is continuous at the boundary. So, we can write:

$$\begin{aligned}&\frac{B_{b,z}\rho_{b,0}}{\mathbf{k}_\parallel\cdot\mathbf{B}_{b,0}k_{b,z}}\Big[\frac{\mu_0}{\rho_{b,0}}(\mathbf{B}_{b,0}\cdot\mathbf{k}_\parallel)^2-\omega'^2\Big]\\&=\frac{B_{t,z}\rho_{t,0}}{\mathbf{k}_\parallel\cdot\mathbf{B}_{t,0}k_{t,z}}\Big[\frac{\mu_0}{\rho_{t,0}}(\mathbf{B}_{t,0}\cdot\mathbf{k}_\parallel)^2-(\omega'-\mathbf{k}_\parallel\cdot\mathbf{v}_0)^2\Big]\end{aligned}\tag{33}$$

Let ζ be the displacement in the z-direction. Magnetic field should have no transverse component,

$$B_{b,z}-\mathbf{B}_{b,0}\cdot\nabla\zeta=B_{b,z}+i(\mathbf{B}_{b,0}\cdot\mathbf{k}_\parallel)\zeta=0$$

$$B_{t,z}-\mathbf{B}_{t,0}\cdot\nabla\zeta=B_{t,z}+i(\mathbf{B}_{t,0}\cdot\mathbf{k}_\parallel)\zeta=0$$

Eliminating ζ gives,

$$\frac{B_{b,z}}{\mathbf{B}_{b,0}\cdot\mathbf{k}_\parallel}=\frac{B_{t,z}}{\mathbf{B}_{t,0}\cdot\mathbf{k}_\parallel}\tag{34}$$

Using equation 33 and 34, we get:

$$\begin{aligned}&\frac{\rho_{b,0}}{k_{b,z}}\Big[\frac{\mu_0}{\rho_{b,0}}(\mathbf{B}_{b,0}\cdot\mathbf{k}_\parallel)^2-\omega'^2\Big]\\&=\frac{\rho_{t,0}}{k_{t,z}}\Big[\frac{\mu_0}{\rho_{t,0}}(\mathbf{B}_{t,0}\cdot\mathbf{k}_\parallel)^2-(\omega'-\mathbf{k}_\parallel\cdot\mathbf{v}_0)^2\Big]\end{aligned}\tag{35}$$

Writing above equation in terms of phase velocity $\omega=\frac{\omega'}{k_\parallel}$, Alfvén velocities $\mathbf{V}_b=\sqrt{\frac{\mu_0}{\rho_0}}\mathbf{B}_{0,b}$, $\mathbf{V}_t=\sqrt{\frac{\mu_0}{\rho_0}}\mathbf{B}_{0,t}$ and angles α,β_1,β_2 formed by $\mathbf{k}_\parallel,\mathbf{V}_b,\mathbf{V}_t$ with vector $\mathbf{v}_0$,

$$\begin{aligned}&\frac{\rho_{b,0}}{k_{b,z}}\Big[V_b^2\cos^2(\alpha-\beta_1)-\omega^2\Big]\\&=\frac{\rho_{t,0}}{k_{t,z}}\Big[V_t^2\cos^2(\alpha-\beta_2)-(\omega-v_0\cos\alpha)^2\Big]\end{aligned}\tag{36}$$

The dispersion relations for both fluids using equation 29 is:

$$\mathbf{k}_{b,z}^2+\mathbf{k}_\parallel^2=\frac{\mathbf{k}_\parallel^2\omega^4}{[\omega^2(c_{s,b}^2+V_b^2)-c_{s,b}^2V_b^2\cos^2(\alpha-\beta_1)]}\tag{37}$$

$$\mathbf{k}_{t,z}^2+\mathbf{k}_\parallel^2=\frac{\mathbf{k}_\parallel^2(\omega-v_0\cos\alpha)^4}{[(\omega-v_0\cos\alpha)^2(c_{s,t}^2+V_t^2)-c_{s,t}^2V_t^2\cos^2(\alpha-\beta_2)]}\tag{38}$$

Finally, using equations 36, 37, 38,we get[3]:

$$\boxed{\begin{aligned}&\frac{\rho_{b,0}^2[V_b^2\cos^2(\alpha-\beta_1)-\omega^2]^2}{1-\dfrac{\omega^4}{\omega^2(c_{s,b}^2+V_b^2)-c_{s,b}^2V_b^2\cos^2(\alpha-\beta_1)}}\\&=\frac{\rho_{t,0}^2[V_t^2\cos^2(\alpha-\beta_2)-(\omega-v_0\cos\alpha)^2]^2}{1-\dfrac{(\omega-v_0\cos\alpha)^4}{(\omega-v_0\cos\alpha)^2(c_{s,t}^2+V_t^2)-c_{s,t}^2V_t^2\cos^2(\alpha-\beta_2)}}\end{aligned}}\tag{39}$$

B. Extension to Neutron Star

We take the rotation axis of neutron star normal to the disk whereas its magnetic field axis in the plane of the disk. The magnetic field of the neutron star doesn't penetrate into the disk up to a good approximation because of the high electrical conductivity of the disk. As in Hydrodynamic model, we take the top fluid to be magnetosphere and the bottom fluid to be the accretion disk. For simplicity, we assume the sound velocity in magnetosphere is negligible ($c_{s,t}=0$); Alfvén velocity in the disk is negligible compared to magnetosphere ($V_b=0$). Also using phase velocity $\dfrac{\omega}{c_{s,b}}\equiv y$, and $\dfrac{v_0\cos\alpha}{V_t}\equiv\mathscr{M}$, we get:

$$\frac{y^4}{4}\Big[1-\Big(\frac{y\,c_{s,b}}{V_t-\mathscr{M}}\Big)^2\Big]=\Big[\cos^2(\alpha-\beta_2)-\Big(\frac{y\,c_{s,b}}{V_t-\mathscr{M}}\Big)^2\Big]^2\tag{40}$$

where we have also used the continuity of pressure at the boundary.

$$\rho_{b,0}\,c_{s,b}^2=\Big(\text{Pressure}_{\text{Alfvén,top}}=\frac{1}{2}\rho_{t,0}V_t^2\Big)$$

Stability analysis[7] of equation 40, we find that for $\mathscr{M}^2>2$, all roots are real. i.e.,

$$\mathscr{M}=\frac{v_0\cos\alpha}{V_t}>\mathscr{M}_{critical}=\sqrt{2}\Rightarrow\frac{u_0\cos\alpha}{c_{s,b}}>\sqrt{\frac{4\rho_{b,0}}{\rho_{t,0}}}\tag{41}$$

For hydrodynamic case we found $\dfrac{\Delta u\cos\phi}{c_{s,b}}>2\sqrt{2}$. Generally in a neutron star, $\rho_{b,0}>>\rho_{t,0}$. So equation 41 predicts an extended region of instability around the corotation radius in comparison to previous model.

V. CONCLUSION

We looked at the Kevin-Helmholtz instability near the co-rotation radius of an accreting neutron star using two different models. The magnetohydrodynamic model predicted a wider region of instability than the hydrodynamic model. Also the dependence of stability condition on $\cos\alpha$ meant that some k-modes far away can also be unstable. This instability can generate turbulent diffusion of disk material into the star, which can explain the angular momentum transfer from disk to star. It can also in turn explain the period fluctuations observed in some neutron stars[6].