Hydrodynamics — Problem set 1. Due. Fri. Sept. 8

This is an exercise to familiarize you with basic concepts used in hydrodynamics: spatial *vs.* temporal dependence, trajectory of a fluid element, mass and energy fluxes, adiabatic *vs.* isentropic, kinetic and thermal energy. It has many parts because there are many concepts to become familiar with. The multi-part format also shows how the inter-related concepts provide an interlocking framework by which results can be checked. Finding an inconsistency is an excellent way to identify error in your work (that's how I did it).

The problem is based on a specified (and somewhat artificial) flow field which is purely radial. Such flows are found in models of astrophysical accretion or (with a change in sign) super nova remnants. These applications are not important for the present exercise. Here you will take the velocity field as given and work out all consequences by formal manipulation of the appropriate fluid equations.

A fluid with $\gamma = 5/3$ has no viscosity and is subject to no heating ($\dot{Q} = 0$). It is, however, subject to an unspecified radial body force, $\mathbf{f} = f_r(r,t)\hat{\mathbf{r}}$, (and no other body forces) which maintains the flow

$$\mathbf{u}(r,t) \;=\; \begin{cases} -\dfrac{\omega r}{(\omega t+1)^2}\hat{\mathbf{r}} & , \quad r < R \\[2ex] -\dfrac{\omega R^3}{r^2(\omega t+1)^2}\hat{\mathbf{r}} & , \quad r > R \end{cases} \tag{1}$$

for $t > 0$ (ω and R are constant parameters of the problem.).

a. The density within the inner region $r < R$ remains always uniform ($\nabla\rho = 0$), and assumes the value $\rho = \rho_0$ at $t = 0$. Use the continuity equation to find the density as a function of time within $r < R$.

b. For the outer region $r > R$ propose a solution of the form

$$\rho(r,t) \;=\; \exp[g(t) + k(r)] \;\;,$$

(a version of separation of variables) and find the solution which matches part a. across $r = R$ (is continuity across $r = R$ physically necessary?). Sketch the density profile at times $\omega t = 0$, 1, 2 and $t \to \infty$.

c. Compute the trajectory of a fluid element, $r(t)$, which crosses the point $r = R$ at time $t = \tau$. Sketch $r(t)$ for cases where $\omega\tau = 0$, 1, 2 and $\tau \to \infty$.

d. Use the results of a–c to find the density $\rho_\tau(t)$ of that fluid element which crosses $r = R$ at $t = \tau$. Write this as an explicit function of t alone — r should not appear.

e. Take the time derivative of $\rho_\tau(t)$ from part d. Where is the density *uniform*? Where is it *constant*? Compare the results to the advective derivative of the density function found in parts a. and b. Verify that the *Lagrangian* form of the continuity equation is satisfied.

f. Compute the mass flux across the surface $r = R$ and integrate this, explicitly, from $t = 0$ to $t = \tau$. Compare this to the mass found by integrating the initial density, $\rho(r, 0)$, over the region $R < r < r_*$. Show that the two expressions agree when $r_* = r(0)$ from part c.: the initial position of that fluid element which will cross $r = R$ at $t = \tau$.

g. Show that within the inner region $r < R$ the energy equation can be satisfied by a parabolic pressure profile

$$p(r,t) \;=\; r^2\, h(t) \quad , \tag{2}$$

for some function of time $h(t)$. Write the pressure explicitly in terms of the initial value $h_0 = h(0)$. (You will not need to find the pressure distribution in the outer region).

h. From the results of parts a. and g. compute the ratio p/ρ^γ within the inner region ($r < R$). Write this as an explicit function of r and t. Is the inner region *isentropic*? How does the ratio vary along the trajectory which crosses $r = R$ at $t = 0$ (i.e. special case of part c.).? Is this fluid element *adiabatic*?

i. Compute the total kinetic energy and thermal energy of the fluid within the inner region $r < R$. Use these to compute the net change in *total fluid energy*, ΔE, from $t = 0$ to $t \to \infty$. Find the value of h_0 (defined in part g.) for which $\Delta E = 0$. For values of h_0 less than that is ΔE positive or negative?

j. Compute the rate of energy change due to *enthalpy* flux across the $r = R$ surface. Integrate this rate of change to find the *net change* due to enthalpy flux alone. How does its sign compare to that from part i? Does its sign depend on the value of h_0 as in part i? If not, how can you explain the difference in signs?