Accretion disks — Problem set 4

An accretion disk is composed of ionized plasma ($\bar{m} = m_p/2$) in an axisymmetric ($\partial/\partial\phi = 0$), equilibrium ($\partial/\partial t = 0$). Its fluid velocity is mostly azimuthal ($\mathbf{u} \propto \hat{\boldsymbol{\phi}}$), with a much smaller inward radial component (the accretion). Many accretion disks are treated in the *thin disk* limit, where axial distance z is assumed much smaller than the cylindrical radius ϖ, and expansions are truncated at their lowest order in $z/\varpi \ll 1$. The dominant, azimuthal flow velocity is written

$$\mathbf{u} \;=\; \varpi\,\Omega(\varpi)\,\hat{\boldsymbol{\phi}} \quad , \tag{1}$$

where $\Omega(\varpi)$ is the angular rotation rate whose possible z dependance is neglected in the spirit of the thin disk. The gravitational force is only from the point-like central object, of mass M_*, and is expanded in the thin disk limit

$$\Psi(\varpi, z) \;=\; -\frac{GM_*}{\sqrt{\varpi^2 + z^2}} \;\simeq\; -\frac{GM_*}{\varpi} \;+\; \frac{GM_*}{2\varpi^3}\,z^2 \;=\; -\frac{GM_*}{\varpi} \;+\; \tfrac{1}{2}\,\Omega_{\rm kep}^2(\varpi)\,z^2 \quad , \tag{2}$$

where $\Omega_{\rm kep}$ is classical Keplerian rotation. The plasma temperature is also taken to depend on radius alone, $T(\varpi)$.

a. Write down the axial ($\hat{\mathbf{z}}$) component of the **inviscid** momentum equation using expressions (1) and (2). Use the fact that T is independent of z to find $p(\varpi, z)$ within the disk. Show that the vertical scale height is $h = c_{s,i}/\Omega_{\rm kep}$, where $c_{s,i} = \sqrt{k_{\rm B}T/\bar{m}}$ is the isothermal sound speed. For an accreting object of $M_* = 100 M_\odot$, what is the scale height at a radius of 1 AU ($= 1.5 \times 10^{13}$ cm) if $T = 10^4$ K there? Can this be considered thin?

b. Use the results of part a. write the disk's surface density

$$\Sigma(\varpi) \;=\; \int_{-\infty}^{\infty} \rho(\varpi, z)\,dz \quad , \tag{3}$$

in terms of $h(\varpi)$ and the equatorial mass density $\rho(\varpi, 0)$.

c. Write down the radial ($\hat{\boldsymbol{\varpi}}$) component of the momentum equation evaluated in the equatorial plane, $z = 0$. You must take care evaluating the advective term, $(\mathbf{u}\cdot\nabla)\mathbf{u}$, using either curls and gradients or recalling that $\partial\hat{\boldsymbol{\phi}}/\partial\phi = -\hat{\boldsymbol{\varpi}}$. Use scaling arguments to show that for a thin disk, i.e. $h \ll \varpi$, then the pressure term is negligible and therefore the azimuthal velocity must have a *Keplerian* profile: $\Omega(\varpi) \simeq \Omega_{\rm kep}(\varpi)$. Does a small *inward* pressure gradient create *faster* or *slower* rotation?

d. The presence of a small radial flow component, $u_\varpi(\varpi)$, will not affect the previous parts. Use it in the continuity equation to show that $\varpi u_\varpi \Sigma$ is a constant — independent of ϖ. Compute the mass flux, $\dot{M}$, across a given radius — also known as the *accretion rate*.

e. Now introduce the radial flow component into the azimuthal ($\hat{\boldsymbol{\phi}}$) momentum equation evaluated in the equatorial plane, $z = 0$. Keep only terms to leading order in $u_\varpi/u_\phi \ll 1$. Here you must introduce viscosity. Use the simplified for form $\mu\nabla^2\mathbf{u}$, keep only leading order term (i.e. only u_ϕ) and be (once again) careful to evaluate this correctly in the curvilinear coordinates. Find the accretion rate $\dot{M}$ in terms of μ and other properties of the disk. Use the viscosity for an unmagnetized, fully-ionized plasma, given in Table 1.1 of the notes, to evaluate $\dot{M}$ for the point on the disk used in part a. You can convert this to an accretion luminosity, $\dot{M}c^2$, to obtain something which can be easily compared to other objects visible in our universe — such as the Sun.

f. Assuming you found a ridiculously small accretion rate in part e., *actual* accretion cannot be driven by genuine viscosity. It must instead occur through *turbulent viscosity.*[1] This is exactly analogous to turbulent conductivity, and its diffusion rate scales exactly the same way

$$\nu_{\rm turb} \sim \tilde{\kappa}_{\rm turb} \sim v_{\rm turb}^2 \tau_{\rm cor} \quad . \tag{4}$$

where $v_{\rm turb}$ is the root-mean-squared velocity of the turbulence and $\tau_{\rm cor}$ is its correlation time. In all likelihood, the turbulent velocity will scale with the isothermal sound speed, $c_{s,i}$. Arguing along the same lines as for mixing-length theory, eddies will be comparable to the thickness of the disk, $\ell_{\rm eddy} \sim h$, and the correlation time will be the eddy-turnover time (the hydrodynamic time scale) $\tau_{\rm cor} \sim \ell_{\rm eddy}/v_{\rm turb} \sim h/c_{s,i}$. The upshot is a kinematic viscosity

$$\nu_{\rm turb} = \alpha \, c_{s,i} \, h \quad , \tag{5}$$

where α is a dimensionless parameter ($\alpha \sim 1$) accounting for all the $\sim$s used in the foregoing argument. Use this in the expression you found in part e. to obtain an expression for the inward radial velocity u_ϖ. Show that for a thin disk $|u_\varpi| \ll u_\phi$ and $|u_\varpi| \ll c_{s,i}$.

g. To provide a concrete example we choose a particular temperature distribution. In particular we take the isothermal speed to be

$$c_{s,i}(\varpi) = \sqrt{\frac{k_{\rm B}T(\varpi)}{\bar{m}}} = \xi \sqrt{GM_*} \, \varpi^{-1/2} \quad . \tag{6}$$

What is required of the dimensionless constant ξ in order that this describe a *thin disk*? Use this to compute the time it takes for a fluid element starting at $\varpi = R$ to fall onto the central mass. Express your result in terms of $T_{\rm kep}(R)$, the initial orbital period of the fluid element. Compute the *trajectory* of this in falling mass, $\varpi(\phi)$.

h. The disk can have stationary, spiral density perturbations from *standing acoustic waves*. We solve for these in the eikonal limit, by proposing a phase function

$$\varphi(\varpi, \phi) = m \Big[f(\varpi) - \phi \Big] \quad , \tag{7}$$

where m is the azimuthal mode number and $f(\varpi)$ is a function we must find. An acoustic wave of frequency ω must have a phase function satisfying the doppler-shifted dispersion relation

$$\omega = \mathbf{u}(\varpi) \cdot \nabla\varphi \pm c_{s,a}(\varpi) \, |\nabla\varphi| \quad , \tag{8}$$

where $c_{s,a} = \sqrt{\gamma} c_{s,i}$ is the adiabatic sound speed, and we continue to use the isothermal sound speed given in eq. (6). A standing wave will satisfy this relation with $\omega = 0$. Neglect doppler shifts from the radial flow and solve for $f(\varpi)$. Use this to show that density ridges ($\varphi = 0$) follow a spiral. Is this spiral tighter or looser than the accretion trajectory?

[1] Or magnetic forces. . . or forces from a turbulent magnetic field, which take the form of turbulent viscosity.