PHSX 565 Astrophysical Plasma Physics
Problem Set 5 - Force-free Magnetic Fields
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Prepare Mathematica environment

Clear variables

In[663]:=
```
Clear["Global`*"]
vals = {};
```

Define shortcut to convert rule to equation

In[665]:=
```
r2e = Rule → Equal;
```

Use preprint variable to print derivatives in traditional form

In[666]:=
```
$PrePrint = # /. Derivative[id__][f_][args__] :→
      TraditionalForm[HoldForm@D[f[args], #] &[Sequence @@
         (DeleteCases[Transpose[{{args}, {id}}], {_, 0}] /. {x_, 1} :→ x)]] &;
(*$PrePrint = TraditionalForm*)
```

Define the del operator

In[667]:=
```
$x = {x, y, z};
∇ /: ∇ f_ := Grad[f, $x]
∇ · f_ := Div[f, $x]
∇ /: ∇ × f_ := Curl[f, $x]
∇ · ∇ := Δ
∇ /: ∇^2 := Δ
Δ /: Δ f_ := Laplacian[f, $x]
CenterDot /: (v_ · ∇) f_ := Grad[f, $x].v
```

In[675]:=

Part a.

Find the constant-α field

We are given the following boundary conditions in the Problem Statement.

```
In[676]:= bc1 = Bz[y, 0] → B0 Cos[π/L y];
          bc2 = Bz[y, 2 L] → 0;
          bc3 = By[-L, z] → 0;
          bc4 = By[L, z] → 0;

In[680]:= $Assumptions = B0 > 0;
```

Constant-α force-free fields with cartesian symmetry and boundary conditions are solutions to Longcope 10.41, given as

```
In[681]:= f$a[y_, z_] := f[y, z] → B1 Sin[k y] Cos[Sqrt[α^2 - k^2] z] ;
          $Assumptions = $Assumptions && α ≥ π/L &&
             y ≥ -L && y ≤ L && z ≥ 0 && z ≤ 2 L && α > k && k > 0;
```

Where C0 is a unknown constant and the wavenumber, k, is

```
In[683]:= k$a = k → π/L;

In[684]:= $Assumptions = $Assumptions && L > 0;
```

The magnetic field can be calculated through Longcope 10.27

```
In[685]:= B$a[y_, z_] = B[y, z] → (∇ (f[y, z] /. f$a[y, z])) × (∇ x)
```

Out[685]= $B[y, z] \to \left\{0, -B1\sqrt{-k^2+\alpha^2}\,\mathrm{Sin}[k\,y]\,\mathrm{Sin}\left[z\sqrt{-k^2+\alpha^2}\right], -B1\,k\,\mathrm{Cos}[k\,y]\,\mathrm{Cos}\left[z\sqrt{-k^2+\alpha^2}\right]\right\}$

Which is a solution to the Helmholtz equation, Longcope 10.41

```
In[686]:= $1041 = Δ f[y, z] == - α^2 f[y, z]
```

Out[686]= $\frac{\partial^2 f(y,z)}{\partial y^2} + \frac{\partial^2 f(y,z)}{\partial z^2} == -\alpha^2 f[y, z]$

Use the boundary conditions to find the constants C0 and α

In[687]:=
```
e1$a = Bz[y, 0] == B$a[y, 0][[2, 3]];
e2$a = Bz[y, 2 L] == B$a[y, 2 L][[2, 3]];
e3$a = By[-L, z] == B$a[-L, z][[2, 2]];
e4$a = By[L, z] == B$a[L, z][[2, 2]];
```

In[691]:=
```
$Assumptions = $Assumptions && n ∈ Integers;
{{B1$a, α$a1}, {B1$a, α$a2}} =
  (Solve[Reduce[{e1$a /. bc1 , e2$a /. bc2, e3$a /. bc3, e4$a /. bc4} /. k$a] /.
      C[1] → n // FullSimplify, {B1, α}] // FullSimplify // Quiet);
```

In[693]:=
```
B1$a
α$a1
α$a2
```

Out[693]= $B1 \to -\frac{B0\ L}{\pi}$

Out[694]= $\alpha \to \frac{\sqrt{17 + 8\ n\ (-1 + 2\ n)}\ \pi}{4\ L}$

Out[695]= $\alpha \to \frac{\sqrt{17 + 8\ n\ (1 + 2\ n)}\ \pi}{4\ L}$

Plug constants into solution

In[696]:=
```
(f$a1 = f$a[y, z] /. B1$a /. α$a1 /. k$a // FullSimplify) // Framed
(f$a2 = f$a[y, z] /. B1$a /. α$a2 /. k$a // FullSimplify) // Framed
```

Out[696]= $\boxed{f[y, z] \to -\frac{B0\ L\ \mathrm{Cos}\left[\frac{(1-4\ n)\ \pi\ z}{4\ L}\right]\ \mathrm{Sin}\left[\frac{\pi\ y}{L}\right]}{\pi}}$

Out[697]= $\boxed{f[y, z] \to -\frac{B0\ L\ \mathrm{Cos}\left[\frac{(1+4\ n)\ \pi\ z}{4\ L}\right]\ \mathrm{Sin}\left[\frac{\pi\ y}{L}\right]}{\pi}}$

Evaluate magnetic field

In[698]:=
```
B$a1[y_, z_] = B$a[y, z] /. B1$a /. α$a1 /. k$a // FullSimplify
B$a2[y_, z_] = B$a[y, z] /. B1$a /. α$a2 /. k$a // FullSimplify
```

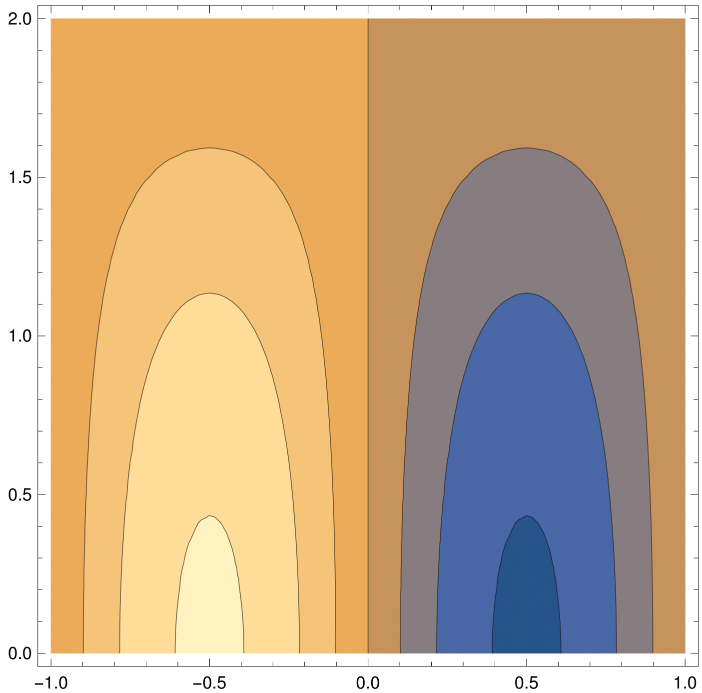
Out[698]= $B[y, z] \to \left\{0, \frac{1}{4}\ B0\ (1 - 4\ n)\ \mathrm{Sin}\left[\frac{\pi\ y}{L}\right]\ \mathrm{Sin}\left[\frac{(1 - 4\ n)\ \pi\ z}{4\ L}\right], B0\ \mathrm{Cos}\left[\frac{\pi\ y}{L}\right]\ \mathrm{Cos}\left[\frac{(1 - 4\ n)\ \pi\ z}{4\ L}\right]\right\}$

Out[699]= $B[y, z] \to \left\{0, \frac{1}{4}\ B0\ (1 + 4\ n)\ \mathrm{Sin}\left[\frac{\pi\ y}{L}\right]\ \mathrm{Sin}\left[\frac{(1 + 4\ n)\ \pi\ z}{4\ L}\right], B0\ \mathrm{Cos}\left[\frac{\pi\ y}{L}\right]\ \mathrm{Cos}\left[\frac{(1 + 4\ n)\ \pi\ z}{4\ L}\right]\right\}$

Plot solution

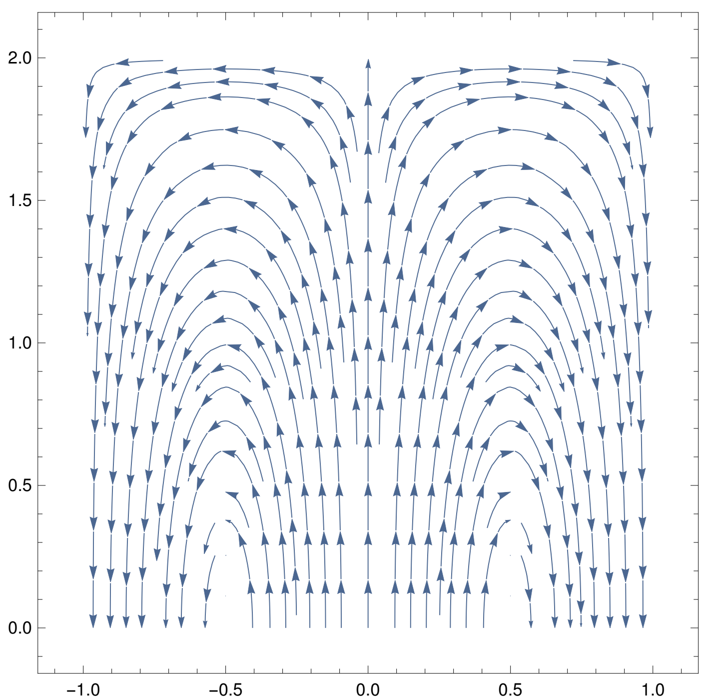
In[700]:=
```
ContourPlot[f$a1[[2]] /. B0 → 1 /. L → 1 /. n → 0, {y, -1, 1}, {z, 0, 2}]
```

Out[700]=



In[701]:=
```
StreamPlot[(B$a1[y, z][[2]] /. B0 → 1 /. L → 1 /. n → 0)[[2 ;; 3]],
 {y, -1, 1}, {z, 0, 2}]
```

Out[701]=



Part b.

Compute the magnetic energy per unit length

The magnetic energy is calculated using Longcope 8.21

In[702]:=
```
n$b = Solve[α$a2 /. r2e, n][[1]];
(EM$b =
   EM → (1/(8 π) Integrate[B[y, z].B[y, z] /. B$a[y, z] /. B1$a /. k$a , {y, -L, L}, {z, 0,
            2 L}] /. α$a2 // FullSimplify) /. n$b // FullSimplify) // Framed
```

Out[703]=
$$\boxed{EM \to \frac{B0^2 L^4 \alpha^2}{8 \pi^3}}$$

Part c.

Find the field line

The magnetic field line has the boundary conditions

In[704]:=
```
bc$c = z[L/2 + ε] == 0;
$Assumptions = $Assumptions && ε > 0 && ε < L;
```

A magnetic field line satisfies Longcope 9.1

In[706]:=
```
$91y = ∂s y[s] == B[y[s], z[s]].(∇ y)
$91z = ∂s z[s] == B[y[s], z[s]].(∇ z)
```

Out[706]=
$$\frac{\partial y(s)}{\partial s} == B[y[s], z[s]].\{0, 1, 0\}$$

Out[707]=
$$\frac{\partial z(s)}{\partial s} == B[y[s], z[s]].\{0, 0, 1\}$$

Form a ratio of the coupled differential equations

In[708]:=
```
eq$c = D[z[y], y] == B$a1[y, z[y]][[2, 3]] / B$a1[y, z[y]][[2, 2]]
```

Out[708]= $\frac{\partial z(y)}{\partial y} == \frac{4\,\mathrm{Cot}\left[\frac{\pi\, y}{L}\right]\,\mathrm{Cot}\left[\frac{(1-4\,n)\,\pi\, z[y]}{4\,L}\right]}{1-4\,n}$

Solve the ODE for z as a function of y

In[709]:=
```
{z$c1, z$c2} = (DSolve[{eq$c, bc$c}, z[y], y] // Quiet)[[ ;; , 1]];
z$c1
z$c2
```

Out[710]= $z[y] \to -\frac{4\,L\,\mathrm{ArcCos}\left[\mathrm{Csc}\left[\frac{\pi\, y}{L}\right]\,\mathrm{Sin}\left[\frac{\pi\left(\frac{L}{2}+\varepsilon\right)}{L}\right]\right]}{(-1+4\,n)\,\pi}$

Out[711]= $z[y] \to \frac{4\,L\,\mathrm{ArcCos}\left[\mathrm{Csc}\left[\frac{\pi\, y}{L}\right]\,\mathrm{Sin}\left[\frac{\pi\left(\frac{L}{2}+\varepsilon\right)}{L}\right]\right]}{(-1+4\,n)\,\pi}$

How does the location of the other footpoint vary with α?

Set expression above and solve for y.

In[712]:=
```
{z0$c1, z0$c2 } =
 FullSimplify[Solve[z$c1[[2]] == 0, y], Assumptions → C[1] == 0][[ ;; , 1]]
```

Out[712]= $\left\{y \to L - \frac{L\,\mathrm{ArcSin}\left[\mathrm{Cos}\left[\frac{\pi\,\varepsilon}{L}\right]\right]}{\pi},\; y \to \frac{L\,\mathrm{ArcSin}\left[\mathrm{Cos}\left[\frac{\pi\,\varepsilon}{L}\right]\right]}{\pi}\right\}$

We find that the results are independent of α

In[713]:=
```
(z0$c1 = y → Series[z0$c1[[2]], {ε, 0, 1}] // Normal) // Framed
(z0$c2 = y → Series[z0$c2[[2]], {ε, 0, 1}] // Normal) // Framed
```

Out[713]= $\boxed{y \to \frac{L}{2} + \varepsilon}$

Out[714]= $\boxed{y \to \frac{L}{2} - \varepsilon}$

How does the apex vary with α?

Set the derivative equal to zero and solve for the y-value

In[715]:=
```
dz0$c = Solve[D[z$c1[[2]], y] == 0, y][[2, 1]] // Quiet
```

Out[715]= $y \to \frac{L}{2}$

Plug in y-value to find height

In[716]:=
```
zL$c = z$c1 /. dz0$c /. n$b // FullSimplify // Framed
```

Out[716]= $\boxed{z\left[\frac{L}{2}\right] \to \frac{2\,\pi\,\varepsilon}{\pi + 2\sqrt{-\pi^2 + L^2\,\alpha^2}}}$

Part d.

Find solutions with homogenous boundary conditions

Homogenous boundary conditions

In[717]:=
```
bc1$d = Bz[y, 0] → 0;
bc2$d = Bz[y, 2 L] → 0;
bc3$d = By[-L, z] → 0;
bc4$d = By[L, z] → 0;
```

Propose even and odd flux function solution

In[721]:=
$$\texttt{f\$d1[y\_, z\_] := f[y, z]} \to \texttt{B1 Sin[k y] Sin}\left[\sqrt{\alpha^2 - k^2}\; z\right];$$
$$\texttt{f\$d2[y\_, z\_] := f[y, z]} \to \texttt{B1 Cos[k y] Sin}\left[\sqrt{\alpha^2 - k^2}\; z\right];$$

With wavenumber

In[723]:=
$$\texttt{k\$d = k} \to \frac{l\,\pi}{2\,L};$$
```
$Assumptions = $Assumptions && l ∈ Integers && m ∈ Integers;
```

Use the boundary conditions to find α

In[725]:=
```
{α$d1, α$d2} = Solve[2 L Sqrt[-k^2 + α^2] == n π, α][[ ;; , 1]]
```

Out[725]= $\left\{\alpha \to -\frac{\sqrt{4\,k^2\,L^2 + n^2\,\pi^2}}{2\,L},\ \alpha \to \frac{\sqrt{4\,k^2\,L^2 + n^2\,\pi^2}}{2\,L}\right\}$

Plug into flux function

In[726]:=
```
(f$d3[y_, z_] = f$d1[y, z] /. α$d2 /. k$d /. l → 2 m // Simplify) // Framed
(f$d4[y_, z_] = f$d2[y, z] /. α$d2 /. k$d /. l → (2 m + 1) // Simplify) // Framed
```

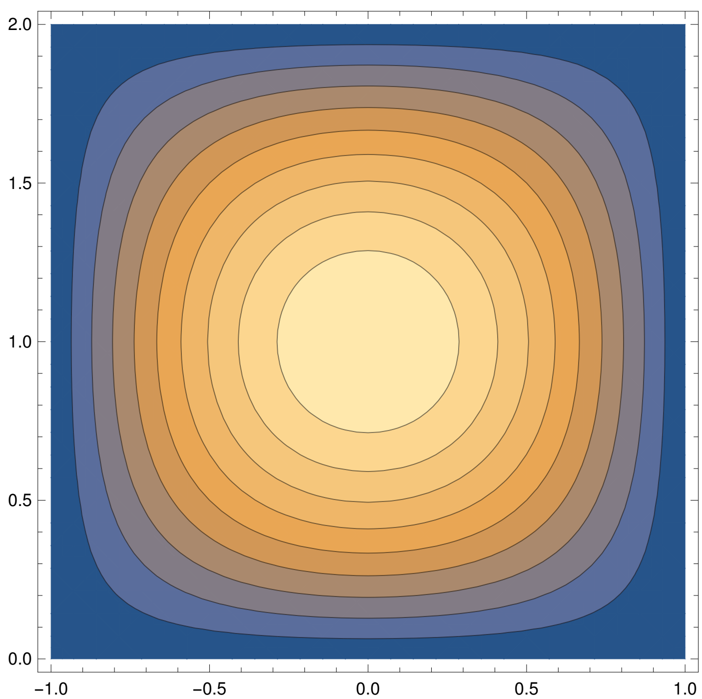
Out[726]= $\boxed{f[y, z] \to B1\,\mathrm{Sin}\left[\frac{m\,\pi\,y}{L}\right]\,\mathrm{Sin}\left[\frac{\pi\,z\,\mathrm{Abs}[n]}{2\,L}\right]}$

Out[727]= $\boxed{f[y, z] \to B1\,\mathrm{Cos}\left[\frac{(1 + 2\,m)\,\pi\,y}{2\,L}\right]\,\mathrm{Sin}\left[\frac{\pi\,z\,\mathrm{Abs}[n]}{2\,L}\right]}$

Plot flux function

In[728]:=
```
ContourPlot[
 f$d4[y, z][[2]] /. L → 1 /. B1 → 1 /. m → 0 /. n → 1, {y, -1, 1}, {z, 0, 2}]
```

Out[728]=



Evaluate magnetic field

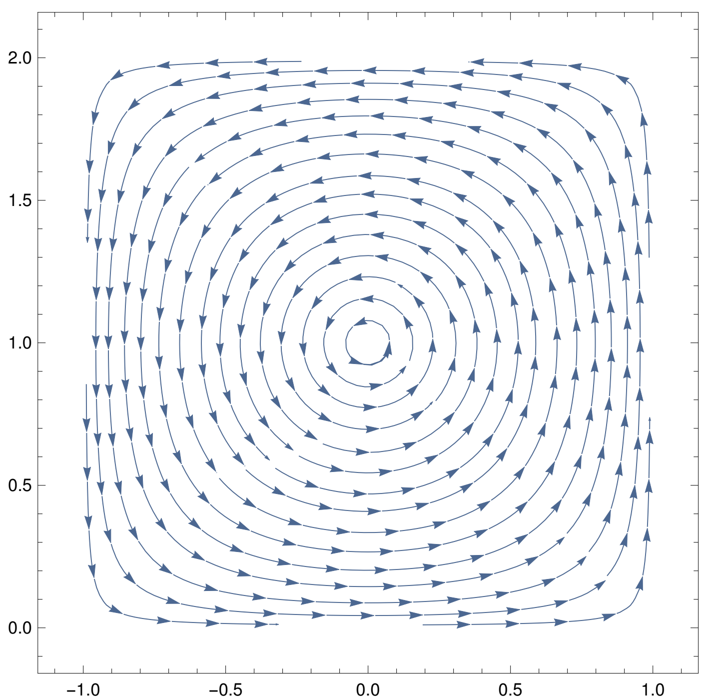
In[729]:=
```
B$d3[y_, z_] = B[y, z] → (∇ (f[y, z] /. f$d3[y, z])) × (∇ x) // Simplify
B$d4[y_, z_] = B[y, z] → (∇ (f[y, z] /. f$d4[y, z])) × (∇ x) // Simplify
```

Out[729]= $B[y,z] \to \left\{0, \frac{B1\,\pi\,\text{Abs}[n]\,\text{Cos}\left[\frac{\pi\,z\,\text{Abs}[n]}{2\,L}\right]\,\text{Sin}\left[\frac{m\,\pi\,y}{L}\right]}{2\,L}, -\frac{B1\,m\,\pi\,\text{Cos}\left[\frac{m\,\pi\,y}{L}\right]\,\text{Sin}\left[\frac{\pi\,z\,\text{Abs}[n]}{2\,L}\right]}{L}\right\}$

Out[730]= $B[y,z] \to \left\{0, \frac{B1\,\pi\,\text{Abs}[n]\,\text{Cos}\left[\frac{(1+2\,m)\,\pi\,y}{2\,L}\right]\,\text{Cos}\left[\frac{\pi\,z\,\text{Abs}[n]}{2\,L}\right]}{2\,L}, \frac{B1\,(1+2\,m)\,\pi\,\text{Sin}\left[\frac{(1+2\,m)\,\pi\,y}{2\,L}\right]\,\text{Sin}\left[\frac{\pi\,z\,\text{Abs}[n]}{2\,L}\right]}{2\,L}\right\}$

In[731]:=
```
StreamPlot[B$d4[y, z][[2, 2 ;; 3]] /. L → 1 /. B1 → 1 /. m → 0 /. n → 1,
 {y, -1, 1}, {z, 0, 2}]
```

Out[731]=



Part e.

Find magnetic field line parallel to x-axis

Solve for the y and z values where the magnetic field is zero (not including the edges of the domain)

In[732]:=
```
{y0$d, z0$d} =
  (Solve[{B$d4[y, z][[2]] == 0, B$d4[y, z][[2]] == 0} /. m → 0 /. n → 1, {y, z}] /.
      C[1] → 0 /. C[2] → 0 // FullSimplify)[[2]]
```

Out[732]= $\{y \to 0, z \to L\}$

Solve Longcope 9.1 for the field line

In[733]:=
```
eq$d = D[z[y], y] == B$d4[y, z[y]][[2, 3]] / B$d4[y, z[y]][[2, 2]] /. n → 1 /. m → 0
bc$d = z[y0$d[[2]]] == z0$d[[2]] - ε
```

Out[733]= $\frac{\partial z(y)}{\partial y} == \mathrm{Tan}\left[\frac{\pi y}{2 L}\right] \mathrm{Tan}\left[\frac{\pi z[y]}{2 L}\right]$

Out[734]= $z[0] == L - \varepsilon$

In[735]:=
```
(z$d = DSolve[{eq$d, bc$d}, z[y], y][[1, 1]] // Quiet) // Framed
```

Out[735]= $\boxed{z[y] \to \frac{2 L \,\mathrm{ArcSin}\left[\mathrm{Sec}\left[\frac{\pi y}{2 L}\right] \mathrm{Sin}\left[\frac{\pi (L-\varepsilon)}{2 L}\right]\right]}{\pi}}$

Part f.

Construct constant-α solution as a combination of homogenous and inhomogenous solutions

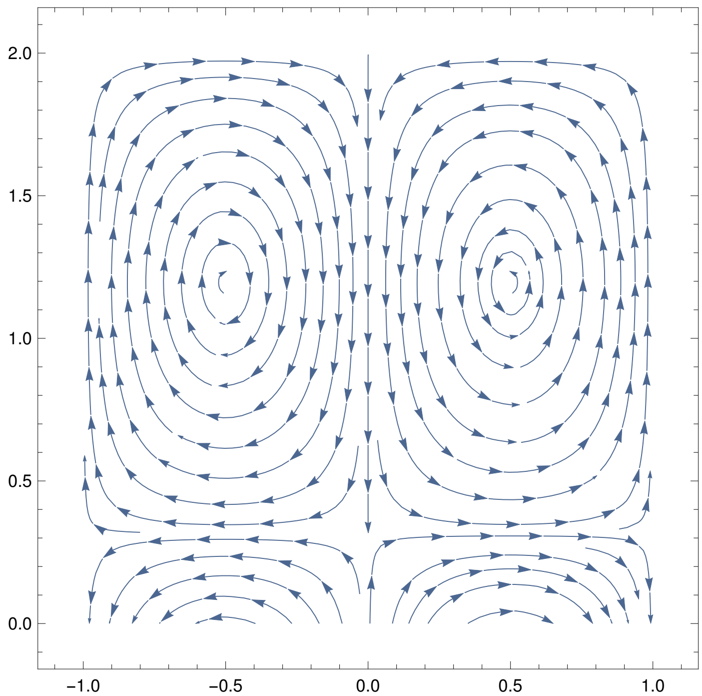
Sum magnetic fields found in part a. and d.

In[736]:=
```
B$e[y_, z_] = B[y, z] → B$a1[y, z][[2]] + B$d3[y, z][[2]]
```

Out[736]= $B[y, z] \to \left\{0, \frac{B1\, \pi\, \mathrm{Abs}[n]\, \mathrm{Cos}\left[\frac{\pi z\, \mathrm{Abs}[n]}{2 L}\right] \mathrm{Sin}\left[\frac{m \pi y}{L}\right]}{2 L} + \frac{1}{4} B0\, (1 - 4 n)\, \mathrm{Sin}\left[\frac{\pi y}{L}\right] \mathrm{Sin}\left[\frac{(1 - 4 n)\, \pi z}{4 L}\right], B0\, \mathrm{Cos}\left[\frac{\pi y}{L}\right] \mathrm{Cos}\left[\frac{(1 - 4 n)\, \pi z}{4 L}\right] - \frac{B1\, m\, \pi\, \mathrm{Cos}\left[\frac{m \pi y}{L}\right] \mathrm{Sin}\left[\frac{\pi z\, \mathrm{Abs}[n]}{2 L}\right]}{L}\right\}$

In[737]:=
```
StreamPlot[B$e[y, z][[2, 2 ;; 3]] /. L → 1 /. B0 → 1 /. B1 → 1/2 /. m → 1 /. n → 1,
 {y, -1, 1}, {z, 0, 2}]
```

Out[737]=



In[738]:=
$$\left(\text{EM\$d} = \text{EM} \to \frac{1}{8\,\pi}\,\text{Integrate}[\text{B}[y, z].\text{B}[y, z] \;/.\; \text{B\$e}[y, z]\;, \{y, -\text{L}, \text{L}\}, \{z, 0, 2\,\text{L}\}] \;/.\; \left(17 - 8\,n + 16\,n^2\right) \to 4\,\alpha^2\,\frac{\text{L}}{\pi} \;//\; \text{Simplify}\right) \;//\; \text{Framed}$$

Out[738]=
$$\boxed{\text{EM} \to \frac{\text{B1}^2\,\left(4\,m^2 + n^2\right)\,\pi^3 + \text{B0}^2\,\text{L}^3\,\alpha^2}{32\,\pi^2}}$$