The true value of the above approach lies in the insight it provides into the propagation of information through the linearized system. It shows that the in the limit of very short wave lengths, solutions to linear systems locally resemble plane wave solutions. The spatial variation is described by a wave vector, $\mathbf{k}$, which will vary gradually in space with equilibrium properties. The possible waves at any point are the same as those possible in a homogeneous equilibrium with the same properties.

There are characteristics, or ray paths, in the linear system describing how information moves through it. According to eq. (4.90), the ray is parallel to the *group velocity*

$$\mathbf{v}_g(\mathbf{k},\mathbf{x}) \;=\; \frac{\partial \omega_n}{\partial \mathbf{k}} \quad . \tag{4.93}$$

The group velocity therefore characterizes the propagation of information by linear waves, at least waves of short wavelengths. In many cases this direction differs from that of the wave vector, $\mathbf{k}$.

4F.4 Examples of the eikenol method

Spatially varying sound speed

We illustrate the eikonal method using a fluid with sound speed $c_s(\mathbf{x})$ varying in space. We must assume that variation is smooth, and think of waves whose wave-length is much smaller than the scale of variation in c_s. Forward propagating acoustic modes are described by the dispersion relation

$$\omega_4(\mathbf{k},\mathbf{x}) \;=\; c_s(\mathbf{x})|\mathbf{k}| \quad . \tag{4.94}$$

The group velocity of this mode,

$$\mathbf{v}_g(\mathbf{k},\mathbf{x}) \;=\; \frac{\partial \omega_4}{\partial \mathbf{k}} \;=\; c_s(\mathbf{x})\,\hat{\mathbf{k}} \quad , \tag{4.95}$$

turns out to be in the same direction as the wave vector; the speed of propagation is the sound speed. Dividing eq. (4.90), by $d\ell/d\tau \;=\; |\mathbf{v}_g| \;=\; c_s$ gives an equation for the ray parameterized by its length, ℓ. The component perpendicular to $\hat{\mathbf{k}}$ gives the curvature of that path

$$\frac{d\hat{\mathbf{k}}}{d\ell} \;=\; \left.\frac{1}{|\mathbf{k}|c_s}\,\frac{d\mathbf{k}}{d\tau}\right)_{\perp} \;=\; -\,\nabla_{\perp}\ln(c_s) \quad . \tag{4.96}$$

This shows that the ray path is refracted in the direction of lower sound speed.

An equilibrium with flow

For a slightly more complicated example we consider an equilibrium with uniform pressure, p_0, but non-uniform density, $\rho_0(\mathbf{x})$. We will also assume the system is not static, but has an equilibrium flow $\mathbf{u}_0(\mathbf{x})$. To be in equilibrium we insist that the flow be incompressible, perpendicular to the density gradient, and have no centrifugal force

$$\nabla\cdot\mathbf{u}_0 \;=\; 0 \quad , \qquad \mathbf{u}_0\cdot\nabla\rho_0 \;=\; 0 \quad , \qquad (\mathbf{u}_0\cdot\nabla)\mathbf{u}_0 \;=\; 0 \quad . \tag{4.97}$$

Linearizing about this non-uniform equilibrium leads to more complicated equations

$$\partial_t\rho_1 + \underbrace{\mathbf{u}_0\cdot\nabla\rho_1}_{1.1} = -\underbrace{\mathbf{u}_1\cdot\nabla\rho_0}_{1.2} - \underbrace{\rho_0(\nabla\cdot\mathbf{u}_1)}_{1.3} \quad , \tag{4.98}$$

$$\partial_t\mathbf{u}_1 + \underbrace{(\mathbf{u}_0\cdot\nabla)\mathbf{u}_1}_{2.1} = -\underbrace{(\mathbf{u}_1\cdot\nabla)\mathbf{u}_0}_{2.2} - \underbrace{\rho_0^{-1}\nabla p_1}_{2.3} \quad , \tag{4.99}$$

$$\partial_t p_1 + \underbrace{\mathbf{u}_0\cdot\nabla p_1}_{3.1} = -\underbrace{\gamma p_0(\nabla\cdot\mathbf{u}_1)}_{3.2} \quad . \tag{4.100}$$

Terms have been numbered for reference and terms 1.1, 2.1 and 3.1 have been moved to the left of the equal sign for convenience.

The eikenol approximation means those terms where the derivative acts on a first order quantity, namely 1.1, 1.3, 2.1, 2.3, 3.1 and 3.2, will be far larger than terms where the derivative acts on the equilibrium: 1.2 and 2.2. Those terms may thus be dropped and the gradient in the first set of terms will be replaced by $i\nabla\phi = i\mathbf{k}$. The resulting system

$$-i(\omega - \mathbf{u}_0\cdot\mathbf{k})\hat{\rho}_1 = -\rho_0\, i\mathbf{k}\cdot\hat{\mathbf{u}}_1 \quad , \tag{4.101}$$

$$-i(\omega - \mathbf{u}_0\cdot\mathbf{k})\hat{\mathbf{u}}_1 = -\frac{i\mathbf{k}}{\rho_0}\,\hat{p}_1 \quad , \tag{4.102}$$

$$-i(\omega - \mathbf{u}_0\cdot\mathbf{k})\hat{p}_1 = -\gamma p_0\, i\mathbf{k}\cdot\hat{\mathbf{u}}_1 \quad , \tag{4.103}$$

is formally identical to eq. (4.30) except that the frequency ω, on the left, has been replaced by the *doppler-shifted* frequency $\omega - \mathbf{u}_0\cdot\mathbf{k}$.

The system thus has the same eigenvectors, i.e. the same normal modes, but with doppler-shifted eigenfrequencies. The two shear modes and the entropy mode now have non-zero eigenfrequnecies

$$\omega_1(\mathbf{k},\mathbf{x}) = \omega_2(\mathbf{k},\mathbf{x}) = \omega_3(\mathbf{k},\mathbf{x}) = \mathbf{k}\cdot\mathbf{u}_0(\mathbf{x}) \quad ; \tag{4.104}$$

they are no longer zero-frequency modes. Their group velocity

$$\mathbf{v}_g = \frac{\partial\omega_1}{\partial\mathbf{k}} = \mathbf{u}_0(\mathbf{x}) \quad , \tag{4.105}$$

is the same as the equilibrium flow velocity. This means that any perturbation to the entropy (i.e. for mode 1) will move with the equilibrium flow. The ray path of the entropy mode is therefore the same as the stream line.

The acoustic modes each have doppler-shifted frequencies

$$\omega_{4,5} = \mathbf{k}\cdot\mathbf{u}_0(\mathbf{x}) \pm c_s(\mathbf{x})\,|\mathbf{k}| \quad . \tag{4.106}$$

Their group velocities

$$\mathbf{v}_g = \frac{\partial\omega_{4,5}}{\partial\mathbf{k}} = \mathbf{u}_0(\mathbf{x}) \pm c_s(\mathbf{x})\,\hat{\mathbf{k}} \quad , \tag{4.107}$$

no longer point in the same direction as the wave vector. The group velocity vectors of all possible waves, i.e. all directions $\hat{\mathbf{k}}$, lie on a sphere of radius c_s centered on $\mathbf{u}_0$. If the equilibrium flow is supersonic, $|\mathbf{u}_0| > c_s$, then these vectors all lie within a cone, known as the *Mach cone*, whose axis is along $\mathbf{u}_0$. Those velocity vectors tangent to the sphere define the opening half-angle

$$\alpha_{\rm Mach} \;=\; \sin^{-1}\left(\frac{c_s}{|\mathbf{u}_0|}\right) \;=\; \sin^{-1}(1/M) \quad , \tag{4.108}$$

where $M > 1$ is the Mach number. Thus even a forward wave (ω_4) directed upstream: $\hat{\mathbf{k}} \parallel -\mathbf{u}_0$ sends information *downstream*. No information moves upstream.

For subsonic flows, $|\mathbf{u}_0| < c_s$, on the other hand the sphere of group velocities encloses the origin so $\mathbf{v}_g$ can be made to point in any direction by the appropriate choice of $\hat{\mathbf{k}}$. Upstream-directed waves send information upstream, albeit at speed $c_s - |\mathbf{u}_0|$: slower than the speed of sound.

An equilibrium with symmetry: sound in a stratified atmosphere

Equations (4.90) and (4.91) are formally identical with Hamilton's equations used in classical mechanics, with $\omega(\mathbf{k},\mathbf{x})$ playing the role of the Hamiltonian. As in that more familiar case, we may solve the equations by appealing to constants of the motion arising through any symmetries in the problem. If, for example, the spatial coordinate x does not appear in $\omega(\mathbf{k},\mathbf{x})$ then eq. (4.91) tells us that

$$\frac{dk_x}{d\tau} \;=\; -\frac{\partial\omega}{\partial x} \;=\; 0 \quad , \tag{4.109}$$

so k_x will be constant along the ray path. Combined with ω, which is itself a constant, we may be able to fully solve for the ray paths in closed form.

To see this at work we consider the case of a polytropic atmosphere with a uniform lapse rate, T', given in eq. (2.18). To simplify the math we use a *downward* depth coordinate z for which $z = 0$ is the top of the atmosphere where $T = 0$. The temperature is therefore $T(z) = zT'$, and the sound speed is

$$c_s(z) \;=\; \sqrt{\gamma\frac{p(z)}{\rho(z)}} \;=\; \sqrt{\frac{\gamma k_{\rm B}}{\bar{m}}\,T(z)} \;=\; \sqrt{a\,z} \quad , \tag{4.110}$$

where $a = \gamma k_{\rm B}T'/\bar{m}$ is a constant with units of acceleration; using eq. (2.18) shows it to be proportional to the gravitational acceleration g.

The dispersion relation for a sound wave in this atmosphere can be written explicitly

$$\omega(k_x,k_z,x,z) \;=\; c_s(\mathbf{x})\,|\mathbf{k}| \;=\; \sqrt{az}\,\sqrt{k_x^2+k_z^2} \quad , \tag{4.111}$$

where we are neglecting the y direction for simplicity. It is clear that $\partial\omega/\partial x = 0$ in this case, so k_x is a constant. The two components of eq. (4.90) are

$$\frac{dx}{d\tau} \;=\; \frac{\partial\omega}{\partial k_x} \;=\; \sqrt{az}\,\frac{k_x}{\sqrt{k_x^2+k_z^2}} \tag{4.112}$$

$$\frac{dz}{d\tau} \;=\; \frac{\partial\omega}{\partial k_z} \;=\; \sqrt{az}\,\frac{k_z}{\sqrt{k_x^2+k_z^2}} \quad , \tag{4.113}$$

which are the components of eq. (4.95): the ray propagates at the sound speed along the direction of $\mathbf{k}$. Since the time-like variable τ never appears explicitly, it is always possible to write our ray-equations in terms of any spatial variable

$$\frac{dx}{dz} = \frac{dx/d\tau}{dz/d\tau} = \frac{\partial\omega/\partial k_x}{\partial\omega/\partial k_z} = \frac{k_x}{k_z} \quad . \tag{4.114}$$

We could next use eqs. (4.91) to obtain an equation for dk_z/dz, and then try to solve the two in tandem. But since we have a second constant of the motion, namely $\omega(k_z, z)$, this is not necessary. Instead we use that constant to find an explicit relationship $k_z(z)$ which may be used in eq. (4.114). Solving eq. (4.111) for k_z yields exactly this relationship

$$k_z(z) = \pm\sqrt{\frac{\omega^2}{az} - k_x^2} \quad , \tag{4.115}$$

where both ω and k_x are constants in the expression. This shows that $k_z = 0$ at the depth

$$z_{\rm tp} = \frac{\omega^2}{ak_x^2} = \frac{\omega^2\Gamma}{\gamma(\Gamma-1)g\,k_x^2} \quad , \tag{4.116}$$

where the final expression uses the polytripoc expression eq. (2.18). It is evident that $z \le z_{\rm tp}$ for k_z to be real. For $z > z_{\rm tp}$ the vertical wave-number k_z is imaginary, so the wave packet will decay exponentially. This is known as a region of evanescent waves. The region is not, however, accessible to a ray path, so the ray is *trapped* in the region $0 < z < z_{\rm tp}$, the only region which admits waves of the specified frequency. Using this definition and substituting $k_z(z)$ into eq. (4.114) yields a closed first order ODE

$$\frac{dx}{dz} = \pm\frac{1}{\sqrt{(z_{\rm zp}/z) - 1}} = \pm\frac{\sqrt{z}}{\sqrt{z_{\rm tp} - z}} \quad , \tag{4.117}$$

where the upper (lower) sign applies as the ray path is moving downward (upward) — increasing (decreasing) z in our upside-down coordinate system,

It can be verified, with some effort, that the equation is satisfied by the cycloid-like curve

$$x(z) - x_0 = z_{\rm tp}\sin^{-1}\left(\sqrt{\frac{z}{z_{\rm tp}}}\right) \mp \sqrt{z}\sqrt{z_{\rm tp} - z} \quad , \tag{4.118}$$

where the principal branch of $\sin^{-1}$ ($\sin^{-1} < \pi/2$) is used on the downward trip, and the other branch ($\sin^{-1} > \pi/2$) is used for the upward trip. The single constant of integration, x_0, is the point the ray leaves the top of the atmosphere, $z = 0$. This point is one *cusp* of the cycloid, where k_z passes from $-\infty$ to $+\infty$, representing a reflection form upward to downward. The divergence in k_z is a byproduct of a wave with finite (fixed) frequency encountering a point of zero wave speed.

Solution (4.118) is plotted in fig. 4.1. It is evident that the ray is refracted away from hotter depths as eq. (4.96) had stipulated. The ray is thus refracted towards the surface, from which it reflects, only to be refracted back and reflect again. The distance between reflections

$$\Delta x = \pi\, z_{tp} = \frac{\pi\omega^2}{ak_x^2} \quad , \tag{4.119}$$

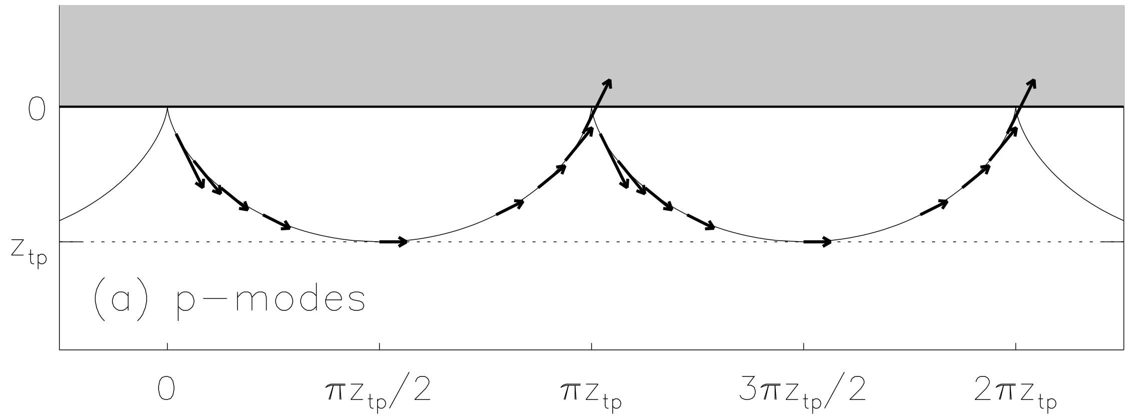

Figure 4.1: The ray of a sound wave, otherwise known as a p-mode, in an atmosphere with uniform lapse rate. The white region is the atmosphere, $z > 0$, and the grey region is space above the atmosphere. The solid curve is the ray path and the arrows show the wave vector $\mathbf{k}$. The dotted curve shows the turning points, z_{tp}.

is found from setting $z = 0$ in eq. (4.118), and noting that the downward and upward encounters use branches of $\sin^{-1}$ differing by π. This distance scales inversely with horizontal wavenumber *squared*. Recall that the assumption underlying the eikenol approach, $k_x \Delta x \sim \omega^2/ak_x \gg 1$, means the space between reflections is *many* wavelengths.