Planetary Atmosphere with greenhouse effect — Problem set 2. Due. Fri. Sept. 15

This problem presents a simplified version of a planetary atmosphere. It is too simple to be an accurate model, but illustrates basic principles and provides one of the few tractable examples where the pressure-density relation $\rho(p)$ can be found legitimately, without resorting to *ad hoc* polytropes.

The thickness of any atmosphere is always much smaller than the planetary radius, justifying a planar treatment with uniform downward gravity $\mathbf{g} = -g\hat{\mathbf{z}}$. Solar radiation is mostly visible and we will neglect any absorption by the atmosphere itself. Shorter-wavelengths are absorbed in the upper atmosphere but we will ignore this effect responsible for the richer structure of real atmospheres. To make this a one-dimensional problem (and thus much simpler) we average all quantities over latitude, longitude and time. Averaging the incident radiation gives a flux, $\bar{F}_\odot$, one-quarter[1] the solar bolometric intensity at that orbit. A fraction α (the albedo) of the incident radiation reflects back to space, and the remainder

$$F_0 = (1-\alpha)\bar{F}_\odot \quad , \tag{1}$$

heats the planetary surface. The planet and its atmosphere are in steady state so the same energy flux, F_0, must be somehow radiated back to space. This will be in a black body spectrum of temperature $T_{\rm eff}$, such that

$$F_0 \;=\; \sigma_{\rm SB} T_{\rm eff}^4 \quad , \tag{2}$$

where $\sigma_{\rm SB}$ is the Stefan-Boltzman constant. (The Eath has an effective temperature $T_{{\rm eff},\oplus} \simeq 240$ K, taking $\alpha_\oplus = 0.45$. Venus, because of its higher albedo, is even colder: $T_{\rm eff} \simeq 220$ K. Both cases are quite chilly!). An atmosphere's primary job is to transport this energy from the surface to a layer where it can be radiated away.

The black body spectrum peaks in the far infrared, and a typical infrared photon will propagate through the atmosphere one mean-free path

$$\ell_{\rm mfp} \;=\; \frac{\bar{m}}{\rho\,\sigma_{\rm ir}} \quad , \tag{3}$$

where $\bar{m}$ is the mean molecular mass of the atmosphere and $\sigma_{\rm ir}$ is the infrared scattering cross section per particle[2] — we will take this to be constant here. At low heights, where ρ is large, the mean free path will be small and the atmosphere is *optically thick*. The photons thus random walk, leading to a radiative conductivity, $\kappa^{(\rm rad)}$ (see eq. [1.46] from the notes). The heat flux is thus transported conductively through the lower layer of the atmosphere — called the *troposphere*. We will assume the atmosphere is static so this will be the dominant source of heat transport. We neglect all other heating and cooling within the troposphere, so the upward conductive flux is uniform: F_0.

Higher up the density is low enough that atmosphere becomes *optically thin* and radiates freely into space with a temperature $T_{\rm eff}$. We assume the transition between optically thick and optically thin occurs at a single height $z_{\rm tp}$, defining the *tropopause*, and therefore

$$T(z_{\rm tp}) \;=\; T_{\rm eff} \quad .$$

For added simplicity we take the *entire* atmosphere above this point, called the *stratosphere*, to be *isothermal* with temperature $T_{\rm eff}$.

[1] Take the radiation intercepted by the disk, πR^2, and distribute it uniformly over the entire surface, $4\pi R^2$.

[2] Scattering in the IR is dominated by trace molecules like H_2O and CO_2. The value $\sigma_{\rm ir}$ sums the intrinsic cross sections of these molecules times their per-particle abundances. Increasing an abundance will increase $\sigma_{\rm ir}$.

a. The isothermal layer above the tropopause must be rare enough that infrared photons escape. This condition, known as optical depth one, is formally given by

$$\int_{z_{\rm tp}}^{\infty} \frac{dz}{\ell_{\rm mfp}(z)} \;=\; \int_{z_{\rm tp}}^{\infty} \sigma_{\rm ir}\frac{\rho(z)}{\bar{m}}\,dz \;=\; 1 \quad . \tag{4}$$

Use this to find the pressure at the tropopause, $p_{\rm tp}$, as a function of g, $\bar{m}$, $\sigma_{\rm ir}$, and $T_{\rm eff}$,

b. Show that the photon mean free path, evaluated at the tropopause, exactly matches the pressure scale height there. Without solving (yet) for the structure of the troposphere show that $\ell_{\rm mfp} < H_p$, the pressure scale height, throughout. This is (marginally) consistent with our assumption of optical thickness.

c. If we apply, to the *entire* troposphere, the value of radiative conductivity from the tropopause, then the troposphere becomes a polytropic atmosphere as discussed in §2A.4. Use this (rather poor) approximation to find the value of the polytropic index Γ. You should be able to simplify this to a number. The atmosphere will be mechanically unstable to convection if $\Gamma > \gamma$. Is this condition satisfied?

d. What *lapse rate* does this value predict for the Earth? Taking $T_{0,\oplus} = 300\,{\rm K}$, what is $z_{\rm tp}$?

e. The final boundary condition on the problem comes from the surface pressure, p_0. This is a fixed property of the atmosphere, unaffected by heat transport, so $p(0) = p_0$. Use the poltytropic approximation (i.e. uniform thermal conductivity), described above, to find an expression for the surface temperature of the form

$$T_0 \;=\; T(0) \;=\; T_{\rm eff}\left(\frac{p_0}{C}\right)^b \quad , \tag{5}$$

for some constants b and C. Express those constants in terms of parameters of the problem. Make clear how the surface temperature depends on $\sigma_{\rm ir}$ and p_0 — both of which are considerably larger on Venus than on Earth. If the scattering cross section were doubled, by how much would the surface temperature increase?

f. We next relax our poor assumption of a uniform conductivity, $\kappa^{(\rm rad)}$, and use instead the form given by eq. (1.46) with mean-free path from above — eq. (3). Use this in the heat flux, equal to F_0, and divide by the hydrostatic balance equation. The result will be an equation for

$$\frac{dT/dz}{dp/dz} \;=\; \frac{dT}{dp} \;=\; \mathcal{F}(T,p) \quad , \tag{6}$$

where the right hand side depends on T, p and constants — not ρ or z! Solve the equation with "initial" condition

$$T(p = p_{\rm tr}) \;=\; T_{\rm eff} \quad .$$

What is the surface temperature

$$T_0 \;=\; T(p = p_0) \;\;?$$

g. Use the solution in part f. to write the pressure-density relation $\rho(p)$ for the troposphere. For large pressures this approaches a polytrope. What is the index Γ? Would this be mechanically stable?