Problem Set 4

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b) The energy density is given in the problem statement as

$$\mathcal{E}_{nt} = \frac{1}{2} m_e \int_0^\infty v^2 f(v)\, dv$$

Taking the time derivative gives

$$\frac{\partial \mathcal{E}_{nt}}{\partial t} = \frac{1}{2} m_e \int_0^\infty v^2 \frac{\partial f}{\partial t}\, dv$$

Plug in definition of $\partial f/\partial t$ given in the problem statement

$$\Rightarrow \frac{\partial \mathcal{E}_{nt}}{\partial t} = \frac{1}{2} m_e \int_0^\infty v^2 \frac{\partial}{\partial v}\left[\frac{K(v^2 - 2v_{th}^2)}{v^4} f + \left(\frac{K v_{th}^2}{v^3} + D^{(turb)} \right) \frac{\partial f}{\partial v} \right] dv$$

Integrate by parts

$$\mu = v^2 \qquad \nu = \frac{K(v^2 - 2v_{th}^2)}{v^4} f + \left(\frac{K v_{th}^2}{v^3} + D^{(turb)} \right) \frac{\partial f}{\partial v}$$

$$d\mu = 2v\, dv \qquad d\nu = \frac{\partial}{\partial v}\left[\frac{K(v^2 - 2v_{th}^2)}{v^4} f + \left(\frac{K v_{th}^2}{v^3} + D^{(turb)} \right) \frac{\partial f}{\partial v} \right]$$

$$\Rightarrow \frac{\partial \mathcal{E}_{nt}}{\partial t} = \frac{1}{2} m_e \left\{ \mu\nu \Big|_0^\infty - \int_0 \nu\, d\mu \right\}$$

$$= \frac{m_e}{2} \left\{ A - \int_0^\infty 2v \left[\frac{K(v^2 - 2v_{th}^2)}{v^4} f + \left(\frac{K v_{th}^2}{v^3} + D^{(turb)} \right) \frac{\partial f}{\partial v} \right] dv \right.$$

Where the surface term, denoted by A, is

$$A = \left[\frac{K(v^2 - 2v_{th}^2)}{v^2} f + \left(\frac{K v_{th}^2}{v} + v^2 D \right) \frac{\partial f}{\partial v} \right] \Bigg|_0^\infty$$

$$= \left[K\left(1 - \cancel{\frac{2v_{th}^2}{v^2}}\right) f + \left(\cancel{\frac{K v_{th}^2}{v}} + v^2 D \right) \frac{\partial f}{\partial v} \right|^\infty - \left[K\left(1 - \frac{2v_{th}^2}{v^2}\right) f + \left(\frac{K v_{th}^2}{v} + v^2 D \right) \frac{\partial f}{\partial v} \right|^0$$

Using the limits

$$\lim_{v\to 0} f(v) = Cv^2$$

$$\lim_{v\to\infty} f(v) = \alpha e^{-\beta v}$$

and

$$\lim_{v\to 0} v^2 D = 0$$

given in the problem statement, this surface term becomes

$$\Rightarrow A = \left[\cancel{K\alpha e^{-\beta v}} + \cancel{v^2 D\frac{\partial}{\partial v}(\alpha e^{-\beta v})}\right]\Big|_\infty - \left[\cancel{KCv^2} - 2KCv_{th}^2 + \left(\frac{Kv_{th}^2}{v} + \cancel{v^2 D}\right)(2Cv)\right]\Big|_0$$

$$= 0 - \left[-2KCv_{th}^2 + 2KCv_{th}^2\right]$$

$$= 0$$

So the change in energy density becomes

$$\frac{\partial \mathcal{E}_{nt}}{\partial t} = -m_e\int_0^\infty \frac{K(v^2 - 2v_{th}^2)}{v^3} f\,dv - B$$

Where

$$B = m_e\int_0^\infty \left(\frac{Kv_{th}^2}{v^2} + vD\right)\frac{\partial f}{\partial v}\,dv$$

Integrate the second term, B, by parts once more

$$\mu = \left(\frac{Kv_{th}^2}{v^2} + vD\right) \qquad \nu = f$$

$$d\mu = \left(-\frac{2Kv_{th}^2}{v^3} + D + v\frac{\partial D}{\partial v}\right)dv \qquad d\nu = \frac{\partial f}{\partial v}$$

$$\Rightarrow B = m_e C - m_e\int_0^\infty \left(-\frac{2Kv_{th}^2}{v^3} + D + v\frac{\partial D}{\partial v}\right) f\,dv$$

Where the surface term, C, is

$$C = \left[\left(\frac{K v_{th}^2}{v^2} + vD\right) f\right|_0^\infty$$

$$= \cancel{\left[\left(\frac{K v_{th}^2}{v^2} + vD\right) \alpha e^{-\beta v}\right|^\infty} - \left[C K v_{th}^2 + \cancel{C v^3 D}\right|^0$$

$$= -C K v_{th}^2$$

Then the change in energy density becomes

$$\frac{\partial \mathcal{E}_{nt}}{\partial t} = -m_e \int_0^\infty \frac{K(v^2 - \cancel{2 v_{th}^2})}{v^3} f \, dv + m_e C K v_{th}^2 + m_e \int_0^\infty \left(-\cancel{\frac{2K v_{th}^3}{v^3}} + D + v \frac{\partial D}{\partial v}\right) f \, dv$$

$$\boxed{\frac{\partial \mathcal{E}_{nt}}{\partial t} = m_e C K v_{th}^2 - m_e \int_0^\infty \left[\frac{K}{v} - \frac{\partial}{\partial v}\left(v D\right)\right] f(v) \, dv}$$