Hale COLLAGE problem set 1 (Due Wed. Feb 15)

1. The first lectures used simple problems, formulated in two dimensions, to illustrate the role of magnetic reconnection in releasing stored energy. Here we explore such model for a compact flare — no flux rope or eruption. The model involves two identical magnetic bipoles, whose field is shown in fig. 1a. Any magnetic field in the (y, z) plane, can be written using a flux function $A(y, z)$

$$\mathbf{B}(y,z) \;=\; \nabla A \times \hat{\mathbf{x}} \;=\; \frac{\partial A}{\partial z}\,\hat{\mathbf{y}} \;-\; \frac{\partial A}{\partial y}\,\hat{\mathbf{z}} \quad . \tag{1}$$

Contours of the function show field lines, and its value at any point ($z \geq 0$) gives the flux (per ignorable length) ψ passing between that point and the origin $(y, z) = (0, 0)$. (If you have no experience working with flux functions, you might try convincing yourself of these important properties using vector calculus.)

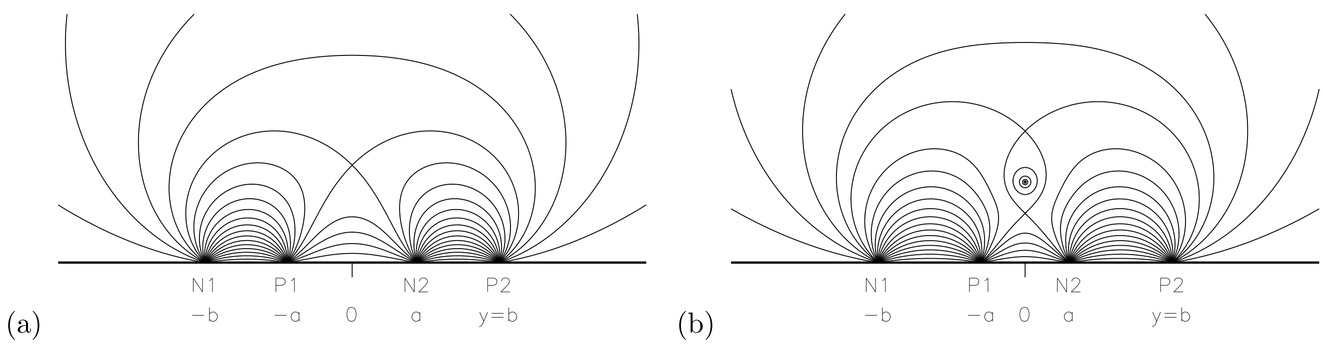

Figure 1: The field above a pair of line-bipoles $P1$–$N1$ (left) and $P2$–$N2$ (right). (a) Shows the potential field defined by eq. (2). (b) Shows the field with a single island, given by eq. (3), used as a simple model of a current sheet. The flux linking $P1$ to $N2$, denoted ψ_{12} is the same in both examples.

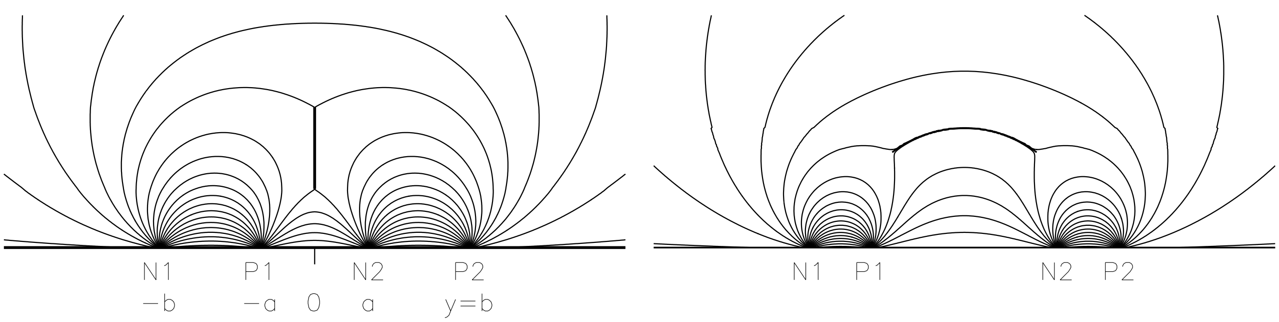

Figure 2: The field when the inner separation, $2a$, is decreased (left) and increased (right). The null point has been deformed into a current sheet, shown with a dark curve. The flux ψ_{12} connected $P1$ to $N2$ is the same in each case as in the potential field, fig. 1a.

The pair of bipoles consists of four sources, two positive and two negative, located on the photosphere ($z = 0$) and arranged symmetrically about the z axis ($y = 0$) at points

$y = \pm a$ and $y = \pm b$ (see fig. 1a). The potential field, $\nabla \times \mathbf{B} = 0$, above this distribution is generated by the flux function

$$A^{(p)}(y,z) = \lambda \tan^{-1}\left(\frac{y+b}{z}\right) - \lambda \tan^{-1}\left(\frac{y+a}{z}\right) + \lambda \tan^{-1}\left(\frac{y-a}{z}\right) - \lambda \tan^{-1}\left(\frac{y-b}{z}\right) \quad , \tag{2}$$

for $z > 0$, where the magnetic flux (per ignorable length) of each sources in $\pi\lambda$. (You can see from the definition that $A^{(p)} = \pi\lambda$ at points on the photosphere $-b < y < -a$ and $a < y < b$.)

a. The potential field has a single X-point at $(y,z) = (0, z_x)$. Find the value of z_x explicitly in terms of a, b, and λ. Evaluate the flux function at that point to obtain the amount of flux, $\psi_{12}^{(p)}$, connecting $P1$ to $N2$ in a potential field. Verify that this expression has the correct values in limits $a/b \to 1$ and $a/b \to 0$.

b. Say we begin with $a = b/2$, and then change a by a small amount $\Delta a \ll a$. Compute the change $\Delta\psi_{12}^{(p)}$ to lowest order in $\Delta a/a \ll 1$.

c. In an ideal plasma, where $E' = 0$, the flux connecting $P1$ to $N2$, ψ_{12}, will not change. If we change a, as in part b, the field will develop a current sheet in place of the null point, as shown in fig. 2. It is possible to approximate the sheet with a set of wires whose islands link together to form a chain. The simplest such model has a *single* wire, located at $z = h$, and carrying current $I_{\rm cs}$. This creates a single island, as shown in fig. 1b. The field is given by a potential

$$A(y,z) = A^{(p)}(y,z) + \frac{I_{\rm cs}}{c} \ln\left[\frac{y^2 + (z+h)^2}{y^2 + (z-h)^2}\right] \quad , \tag{3}$$

where $A^{(0)}(y,z)$ is given by eq. (2). (Note that the added term vanishes along $z = 0$, so the wire does not affect the vertical photospheric field. This is because the added terms consists of a wire, at $z = h$, and an oppsing *image current* at $z = -h$.)

The field generated by eq. (3), contains two null points in place of the single null in $A^{(0)}(y,z)$. To model a current sheet these *must* occur at the same value of A, as in fig. 1b. This value gives the flux ψ_{12} in the presence of the current sheet. The separation between the two nulls approximates the extent of the actual current sheet, $2L$, as in fig. 2.

For which sign of $I_{\rm cs}$ will these null points both fall on the z axis as in fig. 1b? Explain your answer in words. Use this sign and find the null positions in the limit of small current. In this limit you may take $h = z_x$, from part a., and expand expressions in powers of $z - z_x$, to obtain a location to leading order in $I_{\rm cs}/c\lambda \ll 1$. Find both nulls and verify that they each have the same value of A (justifying our assignment of $h = z_x$).

d. Use the results of c. to compute the flux difference from a potential field in the case $b = a/2$ to leading order in $I_{\rm cs}$. This should take a form

$$\Delta\psi_{12} = \psi_{12} - \psi_{12}^{(p)} \propto I_{\rm cs} \ln\Big(\alpha |I_{\rm cs}|\Big) \quad , \tag{4}$$

where you need to find the constant of proportionality *and* the value of α. Next find the distance between X-points, as an approximation to the full length of the current sheet, $2L$. Find this to leading order in current, explicitly in terms of $I_{\rm cs}$, λ and a.

e. The energy (per ignorable length) released by complete reconnection of a current sheet ($\Delta\psi_{12} \to 0$) can be found from the electromagnetic work integral

$$\Delta\mathcal{E}_M \;=\; \frac{1}{c}\int\limits_0^{\Delta\psi_{12}} I_{\rm cs}(\Delta\psi_{12})\, d(\Delta\psi_{12}) \quad . \tag{5}$$

Use the approximate expression from part d. to perform the integral explicitly and obtain an explicit expression in terms of $I_{\rm cs}$. (This might require an integration by parts.)

f. We may use this simple model to obtain the magnetic field strength B_i just outside the actual current sheet. Ampère's law,

$$\frac{4\pi |I_{\rm cs}|}{c} \;=\; \oint \mathbf{B}\cdot d\mathbf{l} \;\simeq\; 4L\, B_i \quad , \tag{6}$$

can be used, in conjunction with the length and current from e., to obtain an explicit expression for B_i in terms of $I_{\rm cs}$, λ and a, for the case $b = 2a$. Assuming some uniform mass density ρ_0 write down the value of the Alfvén-transit time

$$\tau_{\rm A} \;=\; \frac{2L}{v_A} \quad , \tag{7}$$

for the current sheet.

g. We now use the results above to obtain numerical values for a simplified model of a compact flare. Consider a case where $a = 3 \times 10^9$ cm, and $b = 2a = 6 \times 10^9$ cm. We will assign a finite extent, $L_x = 10^{10}$ cm in the previously ignorable direction, but continue to use the two-dimensional expressions obtained above. Assign the parameter λ so that every source has a total flux of 10^{22} Mx. The current sheet builds up as the inner sources each move by $\Delta a = 10^9$ cm under *ideal* conditions ($E' = 0$). Assume the motion is in the direction which produces a *vertical* current sheet. Use the expressions derived for $\Delta a \ll a$ to find the length $2L$ of the resulting current sheet, the height of its center, h, the current it carries $|I_{\rm cs}|$ (express this in Amps), and the energy available for release by magnetic reconnection. Finding the current will require you to solve a transcendental equation. You may do this approximately in whichever manner you prefer.

h. Following the storage phase (part g.) there is sudden and complete reconnection of the current sheet, restoring a potential field ($\Delta\psi_{12} \to 0$). This produces flare ribbons in a region whose photospheric, vertical field strength is $B_{z,0} = 300$ G (use this instead of the pathological values you would obtain using eq. (3) at $z = 0$). Each flare ribbon moves horizontally with a mean velocity $v_{\rm rib} = 3$ km/s. What is the time, $\tau_{\rm rx}$, required for complete reconnection? Use this to compute the average power released in the flare, the mean reconnection electric field (expressed in V/m), and the energy flux incident on each ribbon (erg/cm^2/s). Finally assume a mass density $\rho_0 = 10^{-15}$ g/cm^3 at the current sheet and compute the Alfvén Mach number of the reconnection.