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PHSX 591 Solar Flares & CMEs
Problem Set 4

Clear variables from memory

In[697]:=
```
Clear["Global`*"]
```

Define assumptions

In[698]:=
```
$Assumptions = vnt > 0 && K0 > 0 && v > 0 && vth > 0 && n > 0 &&
   k > 0 && T > 0 && m > 0 && C0 > 0 && β > 0 && δ > 0 && G > 0 && nnt > 0;
```

Part a.

The Fokker-Planck equation is given in the problem statement as

In[699]:= $FP = \partial_t f[v, t] == \partial_v \left(\frac{K0\,(v^2 - 2\,vth^2)}{v^4} f[v, t] + \left(\frac{K0\,vth^2}{v^3} + Dturb \right) \partial_v f[v, t] \right)$ // Hold;

On page 29 of Lecture 17 the Maxwellian distribution is given as

In[700]:= $fm[v, t] = \frac{4\,\pi\,n}{(2\,\pi\,k\,T\,/\,m)^{3/2}} v^2\,Exp\left[-\frac{m\,v^2}{2\,k\,T}\right];$

Check that this distribution satisfies the Fokker-Planck equation by taking Dturb to zero, evaluating v at v_{th} and using the definition of v_{th}.

In[701]:= $(FP\ /.\ f[v, t] \to fm[v, t]\ /.\ Dturb \to 0$ // ReleaseHold$)\ /.\ v \to vth\ /.\ vth \to \sqrt{k\,T\,/\,m}$ //
  Simplify // Framed

Out[701]= $\boxed{\text{True}}$

Since this expression evaluates to True, the Fokker-Planck equation is indeed satisfied.

Part c.

Define *f*(*v*) as a Maxwellian with thermal speed v_{nt}

In[702]:= $fc = \frac{4\pi n}{(2\pi k T / m)^{3/2}} v^2 \text{Exp}\left[-\frac{m v^2}{2 k T}\right];$

where v_{nt} is

In[703]:= $fc = fc \;/.\; (m / (k T)) \to (1/vnt^2) \;/.\; \left(\frac{k T}{m}\right) \to vnt^2 \;/.\; n \to C0 \frac{vnt^3}{\sqrt{2/\pi}} \;//\; \text{Simplify}$

Out[703]= $C0\, e^{-\frac{v^2}{2\, vnt^2}}\, v^2$

The collisional contribution found in Part b. is

In[704]:= $d\varepsilon dt = m\, C0\, K0\, vth^2 - m \int_0^\infty \left(\frac{K0}{v} - \partial_v (v\, Dturb)\right) f\, dv \;//\; \text{Hold};$

Plug in the Maxwellian expression for *f*(*v*) and take Dturb to zero

In[705]:= `dεdt /. Dturb → 0 /. f → fc // ReleaseHold // Simplify // Framed`

Out[705]= $\boxed{C0\, K0\, m\, (-vnt^2 + vth^2)}$

If $v_{nt} > v_{th}$ then the change in non-thermal energy is negative

Part d.

The form of turbulent diffusion given in the problem statement is

In[706]:= `Dturbd = G/v;`
`(*G = (2 π e / m)^2 εturb / k̄;*)`

and the steady state distribution given in the problem statement is

In[707]:= $fd = C0\, v^2 (1 + \beta v^2)^{-(\delta+1)};$

Show that this distribution exactly solves the Fokker-Planck equation in the steady state by plugging the above into the Fokker-Planck equation and the change in energy density

In[708]:=
```
eq1 = FP /. f[v, t] → fd /. Dturb → Dturbd // ReleaseHold // FullSimplify
eq2 = 0 == dεdt /. f → fd /. Dturb → Dturbd // ReleaseHold // FullSimplify
```

Out[708]= $2\,G\left(-2+v^2\,\beta\,\delta\right)+K0\left(-1-v^2\,\beta+2\,vth^2\,\beta\,(2+\delta)\right)=0$

Out[709]= $2\,vth^2\,\beta\,\delta=1$

Solve this system for

In[710]:=
```
(dsol = Part[Solve[{eq1, eq2}, {β, δ}], 1] // FullSimplify) // Framed
```

Out[710]= $\boxed{\left\{\beta\to\frac{G}{K0\,vth^2},\ \delta\to\frac{K0}{2\,G}\right\}}$

Check that these values of β and δ` solve the Fokker-Planck equation

In[711]:=
```
eq1 /. dsol // FullSimplify // Framed
```

Out[711]= $\boxed{\text{True}}$

Part e.

Demonstrate that the distribution found in part d. assumes a Maxwellian in the limit $G\to\ 0$.

In[712]:=
```
fe = fd /. dsol // FullSimplify
```

Out[712]= $C0\,v^2\left(1+\frac{G\,v^2}{K0\,vth^2}\right)^{-1-\frac{K0}{2\,G}}$

In[713]:=
```
Limit[fe, G → 0] // Framed
```

Out[713]= $\boxed{C0\,e^{-\frac{v^2}{2\,vth^2}}\,v^2}$

This is a Maxwellian with width v_{th}

Part f.

Perform the first integral in Equation 1.

In[714]:=
```
nntf = nnt == ∫_0^∞ fd ⅆv
```

Out[714]= $\text{ConditionalExpression}\left[\text{nnt} == \frac{\text{C0}\sqrt{\pi}\ \text{Gamma}\left[-\frac{1}{2}+\delta\right]}{4\beta^{3/2}\ \text{Gamma}[1+\delta]},\ \delta > \frac{1}{2}\right]$

Perform the second integral in Equation 1

In[715]:=
```
εntf = 3/2 m nnt vnt^2 == 1/2 m ∫_0^∞ v^2 fd ⅆv
```

Out[715]= $\text{ConditionalExpression}\left[\frac{3}{2}\text{m nnt vnt}^2 == \frac{3\ \text{C0 m}\sqrt{\pi}\ \text{Gamma}\left[-\frac{3}{2}+\delta\right]}{16\beta^{5/2}\ \text{Gamma}[1+\delta]},\ \delta > \frac{3}{2}\right]$

Apparently $\delta > \frac{3}{2}$ to make sure the energy is positive

In[716]:=
```
$Assumptions = $Assumptions && δ > 3 / 2 && K0/(2 G) > 3/2;
```

Solve for the constant C0

In[717]:=
```
βf = dsol[[1]] /. (G / K0) → 2 δ;
(C0f = Part[Solve[nntf /. βf // Simplify, C0] , 1, 1]) // Framed
```

Out[718]= $\boxed{\text{C0} \to \frac{8\ \text{nnt}\sqrt{\frac{2}{\pi}}\ \delta^{3/2}\ \text{Gamma}[1+\delta]}{\text{vth}^3\ \text{Gamma}\left[-\frac{1}{2}+\delta\right]}}$

Solve for the non-thermal velocity v_{nt}, and plug in the constant C0 found on the line above.

In[719]:=
```
(vntf = Part[Solve[εntf /. βf // Simplify, vnt], 2, 1] /. C0f // FullSimplify) // Framed
```

Out[719]= $\boxed{\text{vnt} \to \frac{\text{vth}}{\sqrt{2}\sqrt{\delta\,(-3+2\,\delta)}}}$

Check this answer in the limit $\delta \to 0$

In[720]:=
```
Limit[vntf[[2]], δ → ∞] // Framed
```

Out[720]= $\boxed{0}$

As $\delta \to \infty$ the non-thermal velocity goes to zero. This is because as $\delta \to \infty$, $D^{turb} \to 0$, so there is not enough turbulence to maintain the population of non-thermal electrons.

Part g.

The change in turbulence is given as

In[721]:= $\text{dfdt} = \partial_v \left(\frac{G}{v} \partial_v f\right)$ // Hold;

Plug in the distribution found in part d.

In[722]:=
```
dfdt = dfdt /. f → fe /. C0f /. dsol // ReleaseHold // FullSimplify
```

Out[722]=
$$\left(8 \sqrt{G}\, K0^{7/2}\, \text{nnt}\, v \left(1 + \frac{G v^2}{K0\, \text{vth}^2}\right)^{-\frac{K0}{2 G}} \left(v^2 - 4\, \text{vth}^2\right) \text{Gamma}\left[2 + \frac{K0}{2 G}\right]\right) \Big/ \left(\sqrt{\pi}\, \text{vth} \left(G v^2 + K0\, \text{vth}^2\right)^3 \text{Gamma}\left[\frac{1}{2}\left(-1 + \frac{K0}{G}\right)\right]\right)$$

The turbulence adds particles when the above expression is greater than zero, solve this condition for v

In[723]:=
```
Reduce[dfdt > 0 // Simplify, v] // Simplify // Framed
```

Out[723]= $\boxed{v > 2\, \text{vth}}$