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PHSX 591 Solar Flares & CMEs
Problem Set 4

Clear variables from memory

```
In[697]:= Clear["Global`*"]
```

Define assumptions

```
In[698]:= $Assumptions = vnt > 0 && K0 > 0 && v > 0 && vth > 0 && n > 0 &&
    k > 0 && T > 0 && m > 0 && C0 > 0 && β > 0 && δ > 0 && G > 0 && nnt > 0;
```

Part a.

The Fokker-Planck equation is given in the problem statement as

In[699]:= $\text{FP} = \partial_t f[v, t] == \partial_v \left(\frac{\text{K0}\,(v^2 - 2\,\text{vth}^2)}{v^4} f[v, t] + \left(\frac{\text{K0}\,\text{vth}^2}{v^3} + \text{Dturb} \right) \partial_v f[v, t] \right)$ // Hold;

On page 29 of Lecture 17 the Maxwellian distribution is given as

In[700]:= $\text{fm}[v, t] = \frac{4\,\pi\,n}{(2\,\pi\,k\,T\,/\,m)^{3/2}} v^2\,\text{Exp}\left[-\frac{m\,v^2}{2\,k\,T}\right];$

Check that this distribution satisfies the Fokker-Planck equation by taking Dturb to zero, evaluating v at v_{th} and using the definition of v_{th}.

In[701]:= (FP /. f[v, t] → fm[v, t] /. Dturb → 0 // ReleaseHold) /. v → vth /. vth → $\sqrt{k\,T\,/\,m}$ //
   Simplify // Framed

Out[701]= $\boxed{\text{True}}$

Since this expression evaluates to True, the Fokker-Planck equation is indeed satisfied.

b) The energy density is given in the problem statement as

$$\mathcal{E}_{nt} = \frac{1}{2} m_e \int_0^\infty v^2 f(v)\, dv$$

Taking the time derivative gives

$$\frac{\partial \mathcal{E}_{nt}}{\partial t} = \frac{1}{2} m_e \int_0^\infty v^2 \frac{\partial f}{\partial t}\, dv$$

Plug in definition of $\partial f/\partial t$ given in the problem statement

$$\Rightarrow \frac{\partial \mathcal{E}_{nt}}{\partial t} = \frac{1}{2} m_e \int_0^\infty v^2 \frac{\partial}{\partial v}\left[\frac{K(v^2 - 2v_{th}^2)}{v^4} f + \left(\frac{K v_{th}^2}{v^3} + D^{(turb)}\right)\frac{\partial f}{\partial v}\right] dv$$

Integrate by parts

$$\mu = v^2 \qquad \nu = \frac{K(v^2 - 2v_{th}^2)}{v^4} f + \left(\frac{K v_{th}^2}{v^3} + D^{(turb)}\right)\frac{\partial f}{\partial v}$$

$$d\mu = 2v\, dv \qquad d\nu = \frac{\partial}{\partial v}\left[\frac{K(v^2 - 2v_{th}^2)}{v^4} f + \left(\frac{K v_{th}^2}{v^3} + D^{(turb)}\right)\frac{\partial f}{\partial v}\right]$$

$$\Rightarrow \frac{\partial \mathcal{E}_{nt}}{\partial t} = \frac{1}{2} m_e \left\{ \mu\nu\Big|_0^\infty - \int_0 \nu\, d\mu \right\}$$

$$= \frac{m_e}{2}\left\{ A - \int_0^\infty 2v \left[\frac{K(v^2 - 2v_{th}^2)}{v^4} f + \left(\frac{K v_{th}^2}{v^3} + D^{(turb)}\right)\frac{\partial f}{\partial v}\right] dv\right.$$

Where the surface term, denoted by A, is

$$A = \left[\frac{K(v^2 - 2v_{th}^2)}{v^2} f + \left(\frac{K v_{th}^2}{v} + v^2 D\right)\frac{\partial f}{\partial v}\right]\Bigg|_0^\infty$$

$$= \Big[K\left(1 - \cancel{\frac{2v_{th}^2}{v^2}}\right) f + \left(\cancel{\frac{K v_{th}^2}{v}} + v^2 D\right)\frac{\partial f}{\partial v}\Big|^\infty - \Big[K\left(1 - \frac{2v_{th}^2}{v^2}\right) f + \left(\frac{K v_{th}^2}{v} + v^2 D\right)\frac{\partial f}{\partial v}\Big|^0$$

Using the limits

$$\lim_{v\to 0} f(v) = Cv^2$$

$$\lim_{v\to \infty} f(v) = \alpha e^{-\beta v}$$

and

$$\lim_{v\to 0} v^2 D = 0$$

given in the problem statement, this surface term becomes

$$\Rightarrow A = \left[\cancel{K\alpha e^{-\beta v}} + \cancel{v^2 D \frac{\partial}{\partial v}\left(\alpha e^{-\beta v}\right)}\right]\Big|_{\infty} - \left[\cancel{KCv^2} - 2KC v_{th}^2 + \left(\frac{K v_{th}^2}{v} + \cancel{v^2 D}\right)(2Cv)\right]\Big|_{0}$$

$$= 0 - \left[-2KC v_{th}^2 + 2KC v_{th}^2\right]$$

$$= 0$$

So the change in energy density becomes

$$\frac{\partial E_{nt}}{\partial t} = -m_e \int_0^\infty \frac{K\left(v^2 - 2v_{th}^2\right)}{v^3} f \, dv - B$$

Where

$$B = m_e \int_0^\infty \left(\frac{K v_{th}^2}{v^2} + vD\right)\frac{\partial f}{\partial v}\, dv$$

Integrate the second term, B, by parts once more

$$\mu = \left(\frac{K v_{th}^2}{v^2} + vD\right) \qquad \nu = f$$

$$d\mu = \left(-\frac{2K v_{th}^2}{v^3} + D + v\frac{\partial D}{\partial v}\right) dv \qquad d\nu = \frac{\partial f}{\partial v}$$

$$\Rightarrow B = m_e C - m_e \int_0^\infty \left(-\frac{2K v_{th}^2}{v^3} + D + v\frac{\partial D}{\partial v}\right) f \, dv$$

Where the surface term, C, is

$$C = \left[\left(\frac{K v_{th}^2}{v^2} + vD\right) f\right]_0^\infty$$

$$= \left[\cancel{\left(\frac{K v_{th}^2}{v^2} + vD\right) x e^{-\beta v}}\right]^\infty - \left[C K v_{th}^2 + \cancel{C v^3 D}\right]^0$$

$$= - C K v_{th}^2$$

Then the change in energy density becomes

$$\frac{\partial \mathcal{E}_{nt}}{\partial t} = - m_e \int_0^\infty \frac{K(v^2 - \cancel{2 v_{th}^2})}{v^3} f \, dv + m_e C K v_{th}^2 + m_e \int_0^\infty \left(-\cancel{\frac{2K v_{th}^2}{v^3}} + D + v \frac{\partial D}{\partial v}\right) f \, dv$$

$$\boxed{\frac{\partial \mathcal{E}_{nt}}{\partial t} = m_e C K v_{th}^2 - m_e \int_0^\infty \left[\frac{K}{v} - \frac{\partial}{\partial v}(v D)\right] f(v) \, dv}$$

Part c.

Define $f(v)$ as a Maxwellian with thermal speed v_{nt}

In[702]:= $fc = \frac{4\pi n}{(2\pi k T / m)^{3/2}} v^2 \text{Exp}\left[-\frac{m v^2}{2 k T}\right];$

where v_{nt} is

In[703]:= $fc = fc \;/.\; (m / (k T)) \to (1/vnt^2) \;/.\; \left(\frac{k T}{m}\right) \to vnt^2 \;/.\; n \to C0 \frac{vnt^3}{\sqrt{2/\pi}}$ // Simplify

Out[703]= $C0\, e^{-\frac{v^2}{2\, vnt^2}} v^2$

The collisional contribution found in Part b. is

In[704]:= $d\varepsilon dt = m\, C0\, K0\, vth^2 - m \int_0^\infty \left(\frac{K0}{v} - \partial_v (v\, Dturb)\right) f \, dv$ // Hold;

Plug in the Maxwellian expression for $f(v)$ and take Dturb to zero

In[705]:= dεdt /. Dturb → 0 /. f → fc // ReleaseHold // Simplify // Framed

Out[705]= $\boxed{C0\, K0\, m\, (-vnt^2 + vth^2)}$

If $v_{nt} > v_{th}$ then the change in non-thermal energy is negative

Part d.

The form of turbulent diffusion given in the problem statement is

In[706]:= $Dturbd = G / v;$
(*$G = (2\,\pi\, e\, /\, m)^2 \varepsilon turb\, /\, \bar{k}$;*)

and the steady state distribution given in the problem statement is

In[707]:= $fd = C0\, v^2\, (1 + \beta\, v^2)^{-(\delta+1)};$

Show that this distribution exactly solves the Fokker-Planck equation in the steady state by plugging the above into the Fokker-Planck equation and the change in energy density

In[708]:=
```
eq1 = FP /. f[v, t] → fd /. Dturb → Dturbd // ReleaseHold // FullSimplify
eq2 = 0 == dεdt /. f → fd /. Dturb → Dturbd // ReleaseHold // FullSimplify
```

Out[708]= $2\,G\left(-2+v^2\,\beta\,\delta\right)+K0\left(-1-v^2\,\beta+2\,vth^2\,\beta\,(2+\delta)\right)==0$

Out[709]= $2\,vth^2\,\beta\,\delta==1$

Solve this system for

In[710]:=
```
(dsol = Part[Solve[{eq1, eq2}, {β, δ}], 1] // FullSimplify) // Framed
```

Out[710]= $\boxed{\left\{\beta\to\frac{G}{K0\,vth^2},\ \delta\to\frac{K0}{2\,G}\right\}}$

Check that these values of β and δ` solve the Fokker-Planck equation

In[711]:=
```
eq1 /. dsol // FullSimplify // Framed
```

Out[711]= $\boxed{\text{True}}$

Part e.

Demonstrate that the distribution found in part d. assumes a Maxwellian in the limit $G\to 0$.

In[712]:=
```
fe = fd /. dsol // FullSimplify
```

Out[712]= $C0\,v^2\left(1+\frac{G\,v^2}{K0\,vth^2}\right)^{-1-\frac{K0}{2\,G}}$

In[713]:=
```
Limit[fe, G → 0] // Framed
```

Out[713]= $\boxed{C0\,e^{-\frac{v^2}{2\,vth^2}}\,v^2}$

This is a Maxwellian with width v_{th}

Part f.

Perform the first integral in Equation 1.

In[714]:= $nntf = nnt == \int_0^\infty fd \, dv$

Out[714]= $\text{ConditionalExpression}\left[nnt == \frac{C0\sqrt{\pi}\,\text{Gamma}\left[-\frac{1}{2}+\delta\right]}{4\,\beta^{3/2}\,\text{Gamma}[1+\delta]},\ \delta > \frac{1}{2}\right]$

Perform the second integral in Equation 1

In[715]:= $\varepsilon ntf = \frac{3}{2} m\, nnt\, vnt^2 == \frac{1}{2} m \int_0^\infty v^2\, fd \, dv$

Out[715]= $\text{ConditionalExpression}\left[\frac{3}{2} m\, nnt\, vnt^2 == \frac{3\, C0\, m\sqrt{\pi}\,\text{Gamma}\left[-\frac{3}{2}+\delta\right]}{16\,\beta^{5/2}\,\text{Gamma}[1+\delta]},\ \delta > \frac{3}{2}\right]$

Apparently $\delta > \frac{3}{2}$ to make sure the energy is positive

In[716]:= $\$Assumptions = \$Assumptions \;\&\&\; \delta > 3/2 \;\&\&\; \frac{K0}{2\,G} > \frac{3}{2};$

Solve for the constant C0

In[717]:= $\beta f = dsol[[1]] \;/.\; (G/K0) \to 2\,\delta;$

$(C0f = \text{Part}[\text{Solve}[nntf \;/.\; \beta f \;//\; \text{Simplify},\ C0],\ 1,\ 1]) \;//\; \text{Framed}$

Out[718]= $\boxed{C0 \to \frac{8\, nnt\sqrt{\frac{2}{\pi}}\;\delta^{3/2}\,\text{Gamma}[1+\delta]}{vth^3\,\text{Gamma}\left[-\frac{1}{2}+\delta\right]}}$

Solve for the non-thermal velocity v_{nt}, and plug in the constant C0 found on the line above.

In[719]:= $(vntf = \text{Part}[\text{Solve}[\varepsilon ntf \;/.\; \beta f \;//\; \text{Simplify},\ vnt],\ 2,\ 1] \;/.\; C0f \;//\; \text{FullSimplify}) \;//\; \text{Framed}$

Out[719]= $\boxed{vnt \to \frac{vth}{\sqrt{2}\,\sqrt{\delta\,(-3+2\,\delta)}}}$

Check this answer in the limit $\delta \to 0$

In[720]:=
```
Limit[vntf[[2]], δ → ∞] // Framed
```

Out[720]= $\boxed{0}$

As $\delta \to \infty$ the non-thermal velocity goes to zero. This is because as $\delta \to \infty$, $D^{turb} \to 0$, so there is not enough turbulence to maintain the population of non-thermal electrons.

Part g.

The change in turbulence is given as

In[721]:=
```
dfdt = ∂v (G/v ∂v f) // Hold;
```

Plug in the distribution found in part d.

In[722]:=
```
dfdt = dfdt /. f → fe /. C0f /. dsol // ReleaseHold // FullSimplify
```

Out[722]=
$$\left(8\sqrt{G}\,K0^{7/2}\,\text{nnt}\,v\left(1+\frac{G\,v^2}{K0\,\text{vth}^2}\right)^{-\frac{K0}{2G}}\left(v^2-4\,\text{vth}^2\right)\text{Gamma}\left[2+\frac{K0}{2G}\right]\right)\Big/\left(\sqrt{\pi}\,\text{vth}\left(G\,v^2+K0\,\text{vth}^2\right)^3\text{Gamma}\left[\frac{1}{2}\left(-1+\frac{K0}{G}\right)\right]\right)$$

The turbulence adds particles when the above expression is greater than zero, solve this condition for v

In[723]:=
```
Reduce[dfdt > 0 // Simplify, v] // Simplify // Framed
```

Out[723]= $\boxed{v > 2\,\text{vth}}$