Solving Every Sudoku Puzzle

by Peter Norvig

In this essay I tackle the problem of solving every Sudoku puzzle. It turns out to be quite easy (about one page of code for the main idea and two pages for embellishments) using two ideas: **constraint propagation** and **search**.

Sudoku Notation and Preliminary Notions

First we have to agree on some notation. A Sudoku puzzle is a *grid* of 81 squares; the majority of enthusiasts label the columns 1-9, the rows A-I, and call a collection of nine squares (column, row, or box) a *unit* and the squares that share a unit the *peers*. A puzzle leaves some squares blank and fills others with digits, and the whole idea is:

> *A puzzle is solved if the squares in each unit are filled with a permutation of the digits 1 to 9.*

That is, no digit can appear twice in a unit, and every digit must appear once. This implies that each square must have a different value from any of its peers. Here are the names of the squares, a typical puzzle, and the solution to the puzzle:

```
 A1 A2 A3| A4 A5 A6| A7 A8 A9    4 . . |. . . |8 . 5     4 1 7 |3 6 9 |8 2 5
 B1 B2 B3| B4 B5 B6| B7 B8 B9    . 3 . |. . . |. . .     6 3 2 |1 5 8 |9 4 7
 C1 C2 C3| C4 C5 C6| C7 C8 C9    . . . |7 . . |. . .     9 5 8 |7 2 4 |3 1 6
---------+---------+---------    ------+------+------    ------+------+------
 D1 D2 D3| D4 D5 D6| D7 D8 D9    . 2 . |. . . |. 6 .     8 2 5 |4 3 7 |1 6 9
 E1 E2 E3| E4 E5 E6| E7 E8 E9    . . . |. 8 . |4 . .     7 9 1 |5 8 6 |4 3 2
 F1 F2 F3| F4 F5 F6| F7 F8 F9    . . . |. 1 . |. . .     3 4 6 |9 1 2 |7 5 8
---------+---------+---------    ------+------+------    ------+------+------
 G1 G2 G3| G4 G5 G6| G7 G8 G9    . . . |6 . 3 |. 7 .     2 8 9 |6 4 3 |5 7 1
 H1 H2 H3| H4 H5 H6| H7 H8 H9    5 . . |2 . . |. . .     5 7 3 |2 9 1 |6 8 4
 I1 I2 I3| I4 I5 I6| I7 I8 I9    1 . 4 |. . . |. . .     1 6 4 |8 7 5 |2 9 3
```

Every square has exactly 3 units and 20 peers. For example, here are the units and peers for the square `C2`:

```
    A2   |         |                    |         |            A1 A2 A3|         |
    B2   |         |                    |         |            B1 B2 B3|         |
    C2   |         |            C1 C2 C3| C4 C5 C6| C7 C8 C9   C1 C2 C3|         |
---------+---------+---------  ---------+---------+---------  ---------+---------+---------
    D2   |         |                    |         |                    |         |
    E2   |         |                    |         |                    |         |
    F2   |         |                    |         |                    |         |
---------+---------+---------  ---------+---------+---------  ---------+---------+---------
    G2   |         |                    |         |                    |         |
    H2   |         |                    |         |                    |         |
    I2   |         |                    |         |                    |         |
```

We can implement the notions of units, peers, and squares in the programming language Python (2.5 or later) as follows:

```python
def cross(A, B):
    "Cross product of elements in A and elements in B."
    return [a+b for a in A for b in B]

digits   = '123456789'
rows     = 'ABCDEFGHI'
cols     = digits
squares  = cross(rows, cols)
unitlist = ([cross(rows, c) for c in cols] +
            [cross(r, cols) for r in rows] +
            [cross(rs, cs) for rs in ('ABC','DEF','GHI') for cs in ('123','456','789')])
units = dict((s, [u for u in unitlist if s in u])
             for s in squares)
peers = dict((s, set(sum(units[s],[]))-set([s]))
             for s in squares)
```

If you are not familiar with some of the features of Python, note that a `dict` or dictionary is Python's name for a hash table that maps each key to a value; that these are specified as a sequence of (key, value) tuples; that `dict((s, [...]) for s in squares)` creates a dictionary which maps each square `s` to a value that is the list `[...]`; and that the expression `[u for u in unitlist if s in u]` means that this value is the list of units `u` such that the square `s` is a member of `u`. So read this assignment statement as "`units` is a dictionary where each square maps to the list of units that contain the square". Similarly, read the next assignment statement as "`peers` is a dictionary where each square `s` maps to the set of squares formed by the union of the squares in the units of `s`, but not `s` itself".

It can't hurt to throw in some tests (they all pass):

```
def test():
    "A set of unit tests."
    assert len(squares) == 81
    assert len(unitlist) == 27
    assert all(len(units[s]) == 3 for s in squares)
    assert all(len(peers[s]) == 20 for s in squares)
    assert units['C2'] == [['A2', 'B2', 'C2', 'D2', 'E2', 'F2', 'G2', 'H2', 'I2'],
                           ['C1', 'C2', 'C3', 'C4', 'C5', 'C6', 'C7', 'C8', 'C9'],
                           ['A1', 'A2', 'A3', 'B1', 'B2', 'B3', 'C1', 'C2', 'C3']]
    assert peers['C2'] == set(['A2', 'B2', 'D2', 'E2', 'F2', 'G2', 'H2', 'I2',
                               'C1', 'C3', 'C4', 'C5', 'C6', 'C7', 'C8', 'C9',
                               'A1', 'A3', 'B1', 'B3'])
    print 'All tests pass.'
```

Now that we have squares, units, and peers, the next step is to define the Sudoku playing grid. Actually we need two representations: First, a textual format used to specify the initial state of a puzzle; we will reserve the name *grid* for this. Second, an internal representation of any state of a puzzle, partially solved or complete; this we will call a *values* collection because it will give all the remaining possible values for each square. For the textual format (*grid*) we'll allow a string of characters with 1-9 indicating a digit, and a 0 or period specifying an empty square. All other characters are ignored (including spaces, newlines, dashes, and bars). So each of the following three grid strings represent the same puzzle:

```
"4.....8.5.3..........7......2.....6.....8.4......1.......6.3.7.5..2.....1.4......"

"""
400000805
030000000
000700000
020000060
000080400
000010000
000603070
500200000
104000000"""

"""
4 . . |. . . |8 . 5
. 3 . |. . . |. . .
. . . |7 . . |. . .
------+------+------
. 2 . |. . . |. 6 .
. . . |. 8 . |4 . .
. . . |. 1 . |. . .
------+------+------
. . . |6 . 3 |. 7 .
5 . . |2 . . |. . .
1 . 4 |. . . |. . .
"""
```

Now for *values*. One might think that a 9 x 9 array would be the obvious data structure. But squares have names like `'A1'`, not `(0,0)`. Therefore, *values* will be a dict with squares as keys. The value of each key will be the possible digits for that square: a single digit if it was given as part of the puzzle definition or if we have figured out what it must be, and a collection of several digits if we are still uncertain. This collection of digits could be represented by a Python `set` or `list`, but I chose instead to use a string of digits (we'll see why later). So a grid where `A1` is `7` and `C7` is empty would be represented as `{'A1': '7', 'C7': '123456789', ...}`.

Here is the code to parse a grid into a *values* dict:

```
def parse_grid(grid):
    """Convert grid to a dict of possible values, {square: digits}, or
    return False if a contradiction is detected."""
    ## To start, every square can be any digit; then assign values from the grid.
    values = dict((s, digits) for s in squares)
    for s,d in grid_values(grid).items():
        if d in digits and not assign(values, s, d):
            return False ## (Fail if we can't assign d to square s.)
    return values

def grid_values(grid):
    "Convert grid into a dict of {square: char} with '0' or '.' for empties."
    chars = [c for c in grid if c in digits or c in '0.']
    assert len(chars) == 81
    return dict(zip(squares, chars))
```

Constraint Propagation

The function `parse_grid` calls `assign(values, s, d)`. We could implement this as `values[s] = d`, but we can do more than just that. Those with experience solving Sudoku puzzles know that there are two important strategies that we can use to make progress towards filling in all the squares:

> *(1) If a square has only one possible value, then eliminate that value from the square's peers.*
> *(2) If a unit has only one possible place for a value, then put the value there.*

As an example of strategy (1) if we assign 7 to `A1`, yielding `{'A1': '7', 'A2':'123456789', ...}`, we see that `A1` has only one value, and thus the 7 can be removed from its peer `A2` (and all other peers), giving us `{'A1': '7', 'A2': '12345689', ...}`. As an example of strategy (2), if it turns out that none of `A3` through `A9` has a 3 as a possible value, then the 3 must belong in `A2`, and we can update to `{'A1': '7', 'A2':'3', ...}`. These updates to `A2` may in turn cause further updates to its peers, and the peers of those peers, and so on. This process is called *constraint propagation*.

The function `assign(values, s, d)` will return the updated *values* (including the updates from constraint propagation), but if there is a contradiction--if the assignment cannot be made consistently--then `assign` returns `False`. For example, if a grid starts with the digits `'77...'` then when we try to assign the 7 to `A2`, `assign` would notice that 7 is not a possibility for `A2`, because it was eliminated by the peer, `A1`.

It turns out that the fundamental operation is not assigning a value, but rather eliminating one of the possible values for a square, which we implement with `eliminate(values, s, d)`. Once we have `eliminate`, then `assign(values, s, d)` can be defined as "eliminate all the values from `s` except `d`".

```
def assign(values, s, d):
    """Eliminate all the other values (except d) from values[s] and propagate.
    Return values, except return False if a contradiction is detected."""
    other_values = values[s].replace(d, '')
    if all(eliminate(values, s, d2) for d2 in other_values):
        return values
    else:
        return False

def eliminate(values, s, d):
    """Eliminate d from values[s]; propagate when values or places <= 2.
    Return values, except return False if a contradiction is detected."""
    if d not in values[s]:
        return values ## Already eliminated
    values[s] = values[s].replace(d,'')
    ## (1) If a square s is reduced to one value d2, then eliminate d2 from the peers.
    if len(values[s]) == 0:
        return False ## Contradiction: removed last value
    elif len(values[s]) == 1:
        d2 = values[s]
        if not all(eliminate(values, s2, d2) for s2 in peers[s]):
```

```
            return False
    ## (2) If a unit u is reduced to only one place for a value d, then put it there.
    for u in units[s]:
        dplaces = [s for s in u if d in values[s]]
        if len(dplaces) == 0:
            return False ## Contradiction: no place for this value
        elif len(dplaces) == 1:
            # d can only be in one place in unit; assign it there
            if not assign(values, dplaces[0], d):
                return False
    return values
```

Now before we can go much further, we will need to display a puzzle:

```
def display(values):
    "Display these values as a 2-D grid."
    width = 1+max(len(values[s]) for s in squares)
    line = '+'.join(['-'*(width*3)]*3)
    for r in rows:
        print ''.join(values[r+c].center(width)+('|' if c in '36' else '')
                      for c in cols)
        if r in 'CF': print line
    print
```

Now we're ready to go. I picked the first example from a list of easy puzzles from the fine Project Euler Sudoku problem and tried it:

```
>>> grid1 = '003020600900305001001806400008102900700000008006708200002609500800203009005010300'

>>> display(parse_grid(grid1))
4 8 3 |9 2 1 |6 5 7
9 6 7 |3 4 5 |8 2 1
2 5 1 |8 7 6 |4 9 3
------+------+------
5 4 8 |1 3 2 |9 7 6
7 2 9 |5 6 4 |1 3 8
1 3 6 |7 9 8 |2 4 5
------+------+------
3 7 2 |6 8 9 |5 1 4
8 1 4 |2 5 3 |7 6 9
6 9 5 |4 1 7 |3 8 2
```

In this case, the puzzle was completely solved by rote application of strategies (1) and (2)! Unfortunately, that will not always be the case. Here is the first example from a list of hard puzzles:

```
>>> grid2 = '4.....8.5.3..........7......2.....6.....8.4......1.......6.3.7.5..2.....1.4......'

>>> display(parse_grid(grid2))
   4      1679   12679  |  139     2369    269   |   8      1239     5
 26789     3    1256789 | 14589   24569   245689 | 12679    1249   124679
  2689   15689   125689 |   7     234569  245689 | 12369   12349   123469
------------------------+------------------------+------------------------
  3789     2     15789  |  3459   34579    4579  | 13579     6     13789
  3679   15679   15679  |  359      8     25679  |   4     12359   12379
 36789     4     56789  |  359      1     25679  | 23579   23589   23789
------------------------+------------------------+------------------------
  289      89     289   |   6      459      3    |  1259     7     12489
   5      6789     3    |   2      479      1    |   69     489     4689
   1      6789     4    |  589     579     5789  | 23569   23589   23689
```

In this case, we are still a long way from solving the puzzle--61 squares remain uncertain. What next? We could try to code more sophisticated strategies. For example, the *naked twins* strategy looks for two squares in the same unit that both have the same two possible digits. Given `{'A5': '26', 'A6':'26', ...}`, we can conclude that 2 and 6 must be in `A5` and `A6` (although we don't know which is where), and we can therefore eliminate 2 and 6 from every other square in the `A` row unit. We could code that strategy in a few lines by adding an `elif len(values[s]) == 2` test to `eliminate`.

Coding up strategies like this is a possible route, but would require hundreds of lines of code (there are dozens of these strategies), and we'd never be sure if we could solve *every* puzzle.

Search

The other route is to *search* for a solution: to systematically try all possibilities until we hit one that works. The code for this is less than a dozen lines, but we run another risk: that it might take forever to run. Consider that in the `grid2` above, `A2` has 4 possibilities (`1679`) and `A3` has 5 possibilities (`12679`); together that's 20, and if we keep multiplying, we get $4.62838344192 \times 10^{38}$ possibilities for the whole puzzle. How can we cope with that? There are two choices.

First, we could try a brute force approach. Suppose we have a very efficient program that takes only one instruction to evaluate a position, and that we have access to the next-generation computing technology, let's say a 10GHz processor with 1024 cores, and let's say we could afford a million of them, and while we're shopping, let's say we also pick up a time machine and go back 13 billion years to the origin of the universe and start our program running. We can then compute that we'd be almost 1% done with this one puzzle by now.

The second choice is to somehow process much more than one possibility per machine instruction. That seems impossible, but fortunately it is exactly what constraint propagation does for us. We don't have to try all 4×10^{38} possibilities because as soon as we try one we immediately eliminate many other possibilities. For example, square H7 of this puzzle has two possibilities, 6 and 9. We can try 9 and quickly see that there is a contradiction. That means we've eliminated not just one possibility, but fully *half* of the 4×10^{38} choices.

In fact, it turns out that to solve this particular puzzle we need to look at only 25 possibilities and we only have to explicitly search through 9 of the 61 unfilled squares; constraint propagation does the rest. For the list of 95 hard puzzles, on average we need to consider 64 possibilities per puzzle, and in no case do we have to search more than 16 squares.

What is the search algorithm? Simple: first make sure we haven't already found a solution or a contradiction, and if not, choose one unfilled square and consider all its possible values. One at a time, try assigning the square each value, and searching from the resulting position. In other words, we search for a value `d` such that we can successfully search for a solution from the result of assigning square `s` to `d`. If the search leads to an failed position, go back and consider another value of `d`. This is a *recursive* search, and we call it a *depth-first* search because we (recursively) consider all possibilities under `values[s] = d` before we consider a different value for `s`.

To avoid bookkeeping complications, we create a new copy of `values` for each recursive call to `search`. This way each branch of the search tree is independent, and doesn't confuse another branch. (This is why I chose to implement the set of possible values for a square as a string: I can copy `values` with `values.copy()` which is simple and efficient. If I implemented a possibility as a Python `set` or `list` I would need to use `copy.deepcopy(values)`, which is less efficient.) The alternative is to keep track of each change to `values` and undo the change when we hit a dead end. This is known as *backtracking search*. It makes sense when each step in the search is a single change to a large data structure, but is complicated when each assignment can lead to many other changes via constraint propagation.

There are two choices we have to make in implementing the search: *variable ordering* (which square do we try first?) and *value ordering* (which digit do we try first for the square?). For variable ordering, we will use a common heuristic called *minimum remaining values*, which means that we choose the (or one of the) square with the minimum number of possible values. Why? Consider `grid2` above. Suppose we chose `B3` first. It has 7 possibilities (`1256789`), so we'd expect to guess wrong with probability 6/7. If instead we chose `G2`, which only has 2 possibilities (`89`), we'd expect to be wrong with probability only 1/2. Thus we choose the square with the fewest possibilities and the best chance of guessing right. For value ordering we won't do anything special; we'll consider the digits in numeric order.

Now we're ready to define the `solve` function in terms of the `search` function:

```
def solve(grid): return search(parse_grid(grid))

def search(values):
    "Using depth-first search and propagation, try all possible values."
    if values is False:
        return False ## Failed earlier
```

```
    if all(len(values[s]) == 1 for s in squares):
        return values ## Solved!
    ## Chose the unfilled square s with the fewest possibilities
    n,s = min((len(values[s]), s) for s in squares if len(values[s]) > 1)
    return some(search(assign(values.copy(), s, d))
                for d in values[s])

def some(seq):
    "Return some element of seq that is true."
    for e in seq:
        if e: return e
    return False
```

That's it! We're done; it only took one page of code, and we can now solve any Sudoku puzzle.

Results

You can view the complete program. Below is the output from running the program at the command line; it solves the two files of 50 easy and 95 hard puzzles (see also the 95 solutions), eleven puzzles I found under a search for [hardest sudoku], and a selection of random puzzles:

```
% python sudo.py
All tests pass.
Solved 50 of 50 easy puzzles (avg 0.01 secs (86 Hz), max 0.03 secs).
Solved 95 of 95 hard puzzles (avg 0.04 secs (24 Hz), max 0.18 secs).
Solved 11 of 11 hardest puzzles (avg 0.01 secs (71 Hz), max 0.02 secs).
Solved 99 of 99 random puzzles (avg 0.01 secs (85 Hz), max 0.02 secs).
```

Analysis

Each of the puzzles above was solved in less than a fifth of a second. What about really hard puzzles? Finnish mathematician Arto Inkala described his 2006 puzzle as "the most difficult sudoku-puzzle known so far" and his 2010 puzzle as "the most difficult puzzle I've ever created." My program solves them in 0.01 seconds each (`solve_all` will be defined below):

```
>>> solve_all(from_file("hardest.txt")[0:2], 'Inkala')
8 5 . |. . 2 |4 . .
7 2 . |. . . |. . 9
. . 4 |. . . |. . .
------+------+------
. . . |1 . 7 |. . 2
3 . 5 |. . . |9 . .
. 4 . |. . . |. . .
------+------+------
. . . |. 8 . |. 7 .
. 1 7 |. . . |. . .
. . . |. 3 6 |. 4 .

8 5 9 |6 1 2 |4 3 7
7 2 3 |8 5 4 |1 6 9
1 6 4 |3 7 9 |5 2 8
------+------+------
9 8 6 |1 4 7 |3 5 2
3 7 5 |2 6 8 |9 1 4
2 4 1 |5 9 3 |7 8 6
------+------+------
4 3 2 |9 8 1 |6 7 5
6 1 7 |4 2 5 |8 9 3
5 9 8 |7 3 6 |2 4 1

(0.01 seconds)

. . 5 |3 . . |. . .
8 . . |. . . |. 2 .
. 7 . |. 1 . |5 . .
------+------+------
```

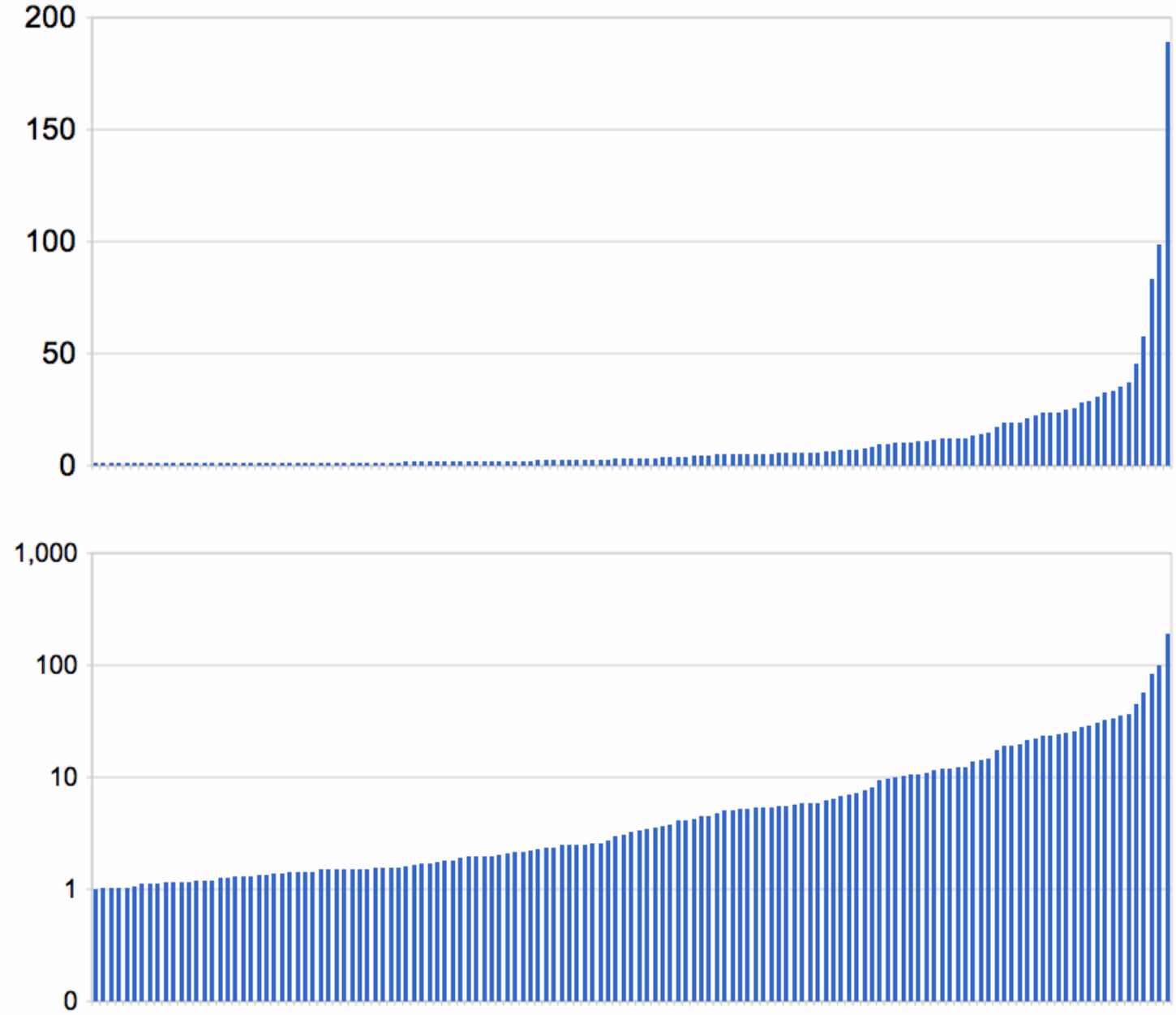
```
4 . . |. . 5 |3 . .
. 1 . |. 7 . |. . 6
. . 3 |2 . . |. 8 .
------+------+------
. 6 . |5 . . |. . 9
. . 4 |. . . |. 3 .
. . . |. . 9 |7 . .

1 4 5 |3 2 7 |6 9 8
8 3 9 |6 5 4 |1 2 7
6 7 2 |9 1 8 |5 4 3
------+------+------
4 9 6 |1 8 5 |3 7 2
2 1 8 |4 7 3 |9 5 6
7 5 3 |2 9 6 |4 8 1
------+------+------
3 6 7 |5 4 2 |8 1 9
9 8 4 |7 6 1 |2 3 5
5 2 1 |8 3 9 |7 6 4

(0.01 seconds)

Solved 2 of 2 Inkala puzzles (avg 0.01 secs (99 Hz), max 0.01 secs).
```

I guess if I want a really hard puzzle I'll have to make it myself. I don't know how to make hard puzzles, so I generated a million random puzzles. My algorithm for making a random puzzle is simple: first, randomly shuffle the order of the squares. One by one, fill in each square with a random digit, respecting the possible digit choices. If a contradiction is reached, start over. If we fill at least 17 squares with at least 8 different digits then we are done. (Note: with less than 17 squares filled in or less than 8 different digits it is known that there will be duplicate solutions. Thanks to Olivier Grégoire for the fine suggestion about 8 different digits.) Even with these checks, my random puzzles are not guaranteed to have one unique solution. Many have multiple solutions, and a few (about 0.2%) have no solution. Puzzles that appear in books and newspapers always have one unique solution.

The average time to solve a random puzzle is 0.01 seconds, and more than 99.95% took less than 0.1 seconds, but a few took much longer:

0.032%	(1 in 3,000) took more than 0.1 seconds
0.014%	(1 in 7,000) took more than 1 second
0.003%	(1 in 30,000) took more than 10 seconds
0.0001%	(1 in 1,000,000) took more than 100 seconds

Here are the times in seconds for the 139 out of a million puzzles that took more than a second, sorted, on linear and log scales:



It is hard to draw conclusions from this. Is the uptick in the last few values significant? If I generated 10 million puzzles, would one take 1000 seconds? Here's the hardest (for my program) of the million random puzzles:

```
>>> hard1  = '.....6....59.....82....8....45........3........6..3.54...325..6..................'
>>> solve_all([hard1])
. . . |. . 6 |. . .
. 5 9 |. . . |. . 8
2 . . |. . 8 |. . .
------+------+------
. 4 5 |. . . |. . .
. . 3 |. . . |. . .
. . 6 |. . 3 |. 5 4
------+------+------
. . . |3 2 5 |. . 6
. . . |. . . |. . .
. . . |. . . |. . .

4 3 8 |7 9 6 |2 1 5
6 5 9 |1 3 2 |4 7 8
2 7 1 |4 5 8 |6 9 3
```

```
------+------+------
8 4 5 |2 1 9 |3 6 7
7 1 3 |5 6 4 |8 2 9
9 2 6 |8 7 3 |1 5 4
------+------+------
1 9 4 |3 2 5 |7 8 6
3 6 2 |9 8 7 |5 4 1
5 8 7 |6 4 1 |9 3 2

(188.79 seconds)
```

Unfortunately, this is not a true Sudoku puzzle because it has multiple solutions. (It was generated before I incorporated Olivier Grégoire's suggestion about checking for 8 digits, so note that any solution to this puzzle leads to another solution where the 1s and 7s are swapped.) But is this an intrinsicly hard puzzle? Or is the difficulty an artifact of the particular variable- and value-ordering scheme used by my `search` routine? To test I randomized the value ordering (I changed `for d in values[s]` in the last line of `search` to be `for d in shuffled(values[s])` and implemented `shuffled` using `random.shuffle`). The results were starkly bimodal: in 27 out of 30 trials the puzzle took less than 0.02 seconds, while each of the other 3 trials took just about 190 seconds (about 10,000 times longer). There are multiple solutions to this puzzle, and the randomized `search` found 13 different solutions. My guess is that somewhere early in the search there is a sequence of squares (probably two) such that if we choose the exact wrong combination of values to fill the squares, it takes about 190 seconds to discover that there is a contradiction. But if we make any other choice, we very quickly either find a solution or find a contradiction and move on to another choice. So the speed of the algorithm is determined by whether it can avoid the deadly combination of value choices.

Randomization works most of the time (27 out of 30), but perhaps we could do even better by considering a better value ordering (one popular heuristic is *least-constraining value*, which chooses first the value that imposes the fewest constraints on peers), or by trying a smarter variable ordering.

More experimentation would be needed before I could give a good characterization of the hard puzzles. I decided to experiment on another million random puzzles, this time keeping statistics on the mean, 50th (median), 90th and 99th percentiles, maximum and standard deviation of run times. The results were similar, except this time I got two puzzles that took over 100 seconds, and one took quite a bit longer: 1439 seconds. It turns out this puzzle is one of the 0.2% that has no solution, so maybe it doesn't count. But the main message is that the mean and median stay about the same even as we sample more, but the maximum keeps going up--dramatically. The standard deviation edges up too, but mostly because of the very few very long times that are way out beyond the 99th percentile. This is a *heavy-tailed* distribution, not a normal one.

For comparison, the tables below give the statistics for puzzle-solving run times on the left, and for samples from a normal (Gaussian) distribution with mean 0.014 and standard deviation 1.4794 on the right. Note that with a million samples, the max of the Gaussian is 5 standard deviations above the mean (roughly what you'd expect from a Gaussian), while the maximum puzzle run time is 1000 standard deviations above the mean.

Samples of Puzzle Run Time

N	mean	50%	90%	99%	max	std. dev
10	0.012	0.01	0.01	0.01	0.02	0.0034
100	0.011	0.01	0.01	0.02	0.02	0.0029
1,000	0.011	0.01	0.01	0.02	0.02	0.0025
10,000	0.011	0.01	0.01	0.02	0.68	0.0093
100,000	0.012	0.01	0.01	0.02	29.07	0.1336
1,000,000	0.014	0.01	0.01	0.02	1439.81	1.4794

Samples of *N*(0.014, 1.4794)

N	mean	50%	90%	99%	max	std. dev
10	0.312	1.24	1.62	1.62	1.62	1.4061
100	0.278	0.18	2.33	4.15	4.15	1.4985
1,000	0.072	0.10	1.94	3.38	6.18	1.4973
10,000	0.025	0.05	1.94	3.45	6.18	1.4983
100,000	0.017	0.02	1.91	3.47	7.07	1.4820
1,000,000	0.014	0.01	1.91	3.46	7.80	1.4802

Here is the impossible puzzle that took 1439 seconds:

```
. . . |. . 5 |. 8 .
. . . |6 . 1 |. 4 3
. . . |. . . |. . .
------+------+------
```

```
. 1 . |5 . . |. . .
. . . |1 . 6 |. . .
3 . . |. . . |. . 5
------+------+------
5 3 . |. . . |. 6 1
. . . |. . . |. . 4
. . . |. . . |. . .
```

Here is the code that defines `solve_all` and uses it to verify puzzles from a file as well as random puzzles:

```
import time, random

def solve_all(grids, name='', showif=0.0):
    """Attempt to solve a sequence of grids. Report results.
    When showif is a number of seconds, display puzzles that take longer.
    When showif is None, don't display any puzzles."""
    def time_solve(grid):
        start = time.clock()
        values = solve(grid)
        t = time.clock()-start
        ## Display puzzles that take long enough
        if showif is not None and t > showif:
            display(grid_values(grid))
            if values: display(values)
            print '(%.2f seconds)\n' % t
        return (t, solved(values))
    times, results = zip(*[time_solve(grid) for grid in grids])
    N = len(grids)
    if N > 1:
        print "Solved %d of %d %s puzzles (avg %.2f secs (%d Hz), max %.2f secs)." % (
            sum(results), N, name, sum(times)/N, N/sum(times), max(times))

def solved(values):
    "A puzzle is solved if each unit is a permutation of the digits 1 to 9."
    def unitsolved(unit): return set(values[s] for s in unit) == set(digits)
    return values is not False and all(unitsolved(unit) for unit in unitlist)

def from_file(filename, sep='\n'):
    "Parse a file into a list of strings, separated by sep."
    return file(filename).read().strip().split(sep)

def random_puzzle(N=17):
    """Make a random puzzle with N or more assignments. Restart on contradictions.
    Note the resulting puzzle is not guaranteed to be solvable, but empirically
    about 99.8% of them are solvable. Some have multiple solutions."""
    values = dict((s, digits) for s in squares)
    for s in shuffled(squares):
        if not assign(values, s, random.choice(values[s])):
            break
        ds = [values[s] for s in squares if len(values[s]) == 1]
        if len(ds) >= N and len(set(ds)) >= 8:
            return ''.join(values[s] if len(values[s])==1 else '.' for s in squares)
    return random_puzzle(N) ## Give up and make a new puzzle

def shuffled(seq):
    "Return a randomly shuffled copy of the input sequence."
    seq = list(seq)
    random.shuffle(seq)
    return seq

grid1  = '003020600900305001001806400008102900700000008006708200002609500800203009005010300'
grid2  = '4.....8.5.3..........7......2.....6.....8.4......1.......6.3.7.5..2.....1.4......'
hard1  = '.....6....59.....82....8....45........3........6..3.54...325..6..................'

if __name__ == '__main__':
    test()
    solve_all(from_file("easy50.txt", '========'), "easy", None)
    solve_all(from_file("top95.txt"), "hard", None)
    solve_all(from_file("hardest.txt"), "hardest", None)
    solve_all([random_puzzle() for _ in range(99)], "random", 100.0)
```

Why?

Why did I do this? As computer security expert Ben Laurie has stated, Sudoku is "a denial of service attack on human intellect". Several people I know (including my wife) were infected by the virus, and I thought maybe this would demonstrate that they didn't need to spend any more time on Sudoku. It didn't work for my friends (although my wife has since independently kicked the habit without my help), but at least one stranger wrote and said this page worked for him, so I've made the world more productive. And perhaps along the way I've taught something about Python, constraint propagation, and search.

Translations

This code has been reimplemented by several people in several languages:

- C# with LINQ version by Richard Birkby
- Clojure version by Justin Kramer
- Erlang version by Andreas Pauley
- Haskell version by Emmanuel Delaborde
- Java version by Johannes Brodwall
- Javascript version by Brendan Eich
- Javascript version by Pankaj Kumar
- Ruby version by Luddite Geek
- Ruby version by Martin-Louis Bright

You can see a Korean translation of this article by JongMan Koo, or use the translation widget below:

Select Language ▼ Powered by Google™ Translate

Peter Norvig

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Anonymous

Ahh, but there is an advantage to playing sudoku (as opposed to just completing the puzzle with a computer program). That advantage is making your brain think in a logical fashion, not just work from rote. And it is this logical thought process that gets the blood flowing to sometimes under-utilized portions of the brain, and can help stave off both boredom and possibly even dementia.

1 month ago 1 Like Like Reply

naeg

This article should be called "how to win sudoku".

7 months ago 2 Likes Like Reply

Ravi Annaswamy

Genius is fluency. Genius is going to a distance where no one has gone and then going further and further away.

Peter Norvig teases me. As though if you give him an hour, he can take any issue and make a giant step. Wow! I memorized the first version of your sudoku.py to learn Python idioms. I spent a couple of months with that code and it led me to functional programming, lisp and so many beautiful things. Proof that if you go deep, you encompass wide. Thank you.
Ravi

1 year ago 2 Likes Like Reply

Bill Zwicky

Your link for "easy 50" has the wrong file name and wrong format, and the program no longer works with them. To fix, use this from_file which supports regular expressions:

```
def from_file(filename, sep='\n'):
"Parse a file into a list of strings, separated by sep."
s = file(filename).read().strip()
l = re.split(sep, s)
l = filter(lambda z: len(z)>0, l)
return l
```

and this test line:

```
solve_all(from_file("easy50.txt", "Grid [0-9]+"), "easy", None)
```

11 months ago 1 Like Like Reply

Peter Norvig

Bill, you are exactly right: I tried to give more credit to project Euler by linking directly to their file, when I should have linked to my local version of the file, in which "Grid ..." is replaced by a constant '======' marker. Your solution using re.split is a good one and more general than my solution, but I wanted to keep it simple.

11 months ago in reply to Bill Zwicky 3 Likes Like Reply

Michel Lespinasse

I like your writeup & your concise implementation.

When doing my own sudoku solver, I used a similar approach but with a twist in the constraint propagation. I had constraints not just on cell contents like in yours, but also in values that MAY or MUST be seen within blocks of 3 consecutive cells (horizontally or vertically). I found that this allowed my algorithm to make quick progress in cases where the search method would not - for example, I can find out that your 1439 seconds puzzle has no solution without having to invoke search even once.

As an example of how this works:
If the possible values for 3 consecutive cells don't include a given value, one can note that the corresponding cell block may not include that value.
if two aligned cell blocks may not include a given value, then the third one must include that value (even if we don't know which of the 3 cells will have the value in).
Since we know the value will be present in that third cell block, it may not be present in any of the other 6 cells in the corresponding 3x3 block.
This is a good heuristic to have because it's very complementary with the search - it works well in cases where the search does not.

1 year ago 1 Like Like Reply

Gareth

I was keen to try the "impossible puzzle"... only to find that it truly is. This is because the centre box bottom row needs to contain three numbers in the middle and centre box... being 1,5 and 6.

6 days ago Like Reply

Bojan

I could almost agree with you because I have the same attitude towards chess. I use the word almost because there is a huge difference in the way human brain and algorithm tackle sudoku or chess. I cannot help but admire the elegant approach peter's code takes and would argue that human doesn't contribute any value added. In chess, however, chess engines still rely heavily on brute force technique and cannot emulate thinking processes of leading grandmasters. Sure, they are getting better and better in this respect also, but human and chess engine would still beat chess engine.
Also, you are being unfair in your comparison. Why not compare playing sudoku as opposed to solving it once and for all? I guess that gets much more blood flowing to the brain.

1 week ago Like Reply

Folders Printing, Online Printing Company

great article

2 weeks ago Like Reply

Mehpic

This was a very interesting article, but the code seems unnecessarily verbose. Here's my suggestion of how to solve sudoku with 10 lines of Python.

1 month ago Like Reply

TonyVeijalainen

Same as Peter, I rearranged your code more regular function first order and added from top95 of norvig the first puzzle, I had not patience to wait a result, seemed to hang my computer (simple list comprehension to put line to your format: input_board = [0 if n=='.' else int(n) for n in '4.....8.5.3.....7.....2.....6.....8.4.....1.....6.3.7.5..2.....1.4.....'])

If you want to see another simplest half-brute force possible solution, which does arrive mostly to solution (not always though) can be found in my post from DaniWeb (as earlier posted here) http://www.daniweb.com/code/sn... (uses http://www.daniweb.com/softwar... for pretty printing, but it does not output the solutions one by one as is). The point was to check how simple the sudoku algorithm can get and still mostly arrive to solution in reasonable time. It takes however 11.5 s to found solution to above puzzle and even longer 4th and fifth in top95. Funny is that hardest set is actually easier to my program as is the multisolution one Peter posted.

1 month ago in reply to Mehpic Like Reply

Peter Norvig

Good point! The naive search algorithm you describe is indeed simpler, both in terms of lines of code and in conceptual complexity. And for many puzzles, it arrives at the solution in less than a second, so it has a lot to recommend it. But for other puzzles, it takes many minutes. Consider the puzzle I show above, which you would encode as [4, 0, 0, 0, 0, 0, 8, 0, 5,
0, 3, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 7, 0, 0, 0, 0, 0,
0, 2, 0, 0, 0, 0, 0, 6, 0,
0, 0, 0, 0, 8, 0, 4, 0, 0,
0, 0, 0, 0, 1, 0, 0, 0, 0,
0, 0, 0, 6, 0, 3, 0, 7, 0,
5, 0, 0, 2, 0, 0, 0, 0, 0,
1, 0, 4, 0, 0, 0, 0, 0, 0] I started to run this with your code, but after 20 minutes it still hadn't completed and I interrupted the process. The algorithm I describe is over 10,000 times faster on hard puzzles like this. That's why I dismissed the naive search approach, although I should have covered it in my article.

1 month ago in reply to Mehpic Like Reply

Christian Jauvin

The code is the exact opposite of verbose: it solves the problem in a very efficient and elegant way. I suggest you simply try to run your own code on the "grid2" or "hard1" problems cited above, you'll see why efficiency matters!

1 month ago in reply to Mehpic Like Reply

Caro Eardley

How rare and wonderful! An exceptional demonstration of The Simple & The Easy at work…if only this standard were ubiquitous in design/programming!

I'd normally fear to tread in such hallowed halls of scholarship but as Dr Inkala has graciously stated I've developed "a new approach to playing Sudoku at any level" in my ebook, 'Win At Sudoku', I'd be so interested to hear any comment on my techniques for Constraint Propagation. There's no rocket science – not my goal! – but the method definitely works…

- Count the occurrences of Given Numbers (GNs). Whip through puzzle starting with the GN with the most occurrences and work your way down
to the GN with the least. This blindingly-obvious technique cut approximately 30% of instructions off my Worked Examples and also made very hard puzzles much less complex to solve.
- On the Grid: Write Options for any Missing Number (MN) when there are only 2 possible positions for an MN in a Box, Row, or Column.
- Off the Grid: Switch focus from GNs to MNs. Note these down off the grid for each Box, Row, and Column. I developed a new puzzle layout to accommodate this additional info.
- Study this info carefully. Positional clues appear from this info eg 2 Boxes contain Left & Middle Options for a MN? 3rd Box position must be Right.
- Continue building Options: the book explains Fixed, Deduced, Long, Holding and Half Options in addition to my Power Tools, Joyful Options, which currently include Twins, Trips and Quads which always follow specific patterns and, more importantly, exclude groups of cells from further consideration, until more info is acquired as game progresses. I've dithered over Quins because I know they do work but have currently excluded them as surplus to requirements – except maybe for Dr Inkala's top quartile hardest puzzles, which are truly fiendishly cunning.

I used to do my own programming but that was then and this is now – so is there anyone interested in testing/applying these concepts? It would be so great to see some hard proof of how my method works 'off paper' and I'd love to connect with any interested puzzle-makers and/or game-designers…

Anyone who'd like to take a closer look at my method (especially you, Peter, to say thanks for the wonderful essay) is welcome to a complimentary copy of my (short – 50xA4 theory, 50xA4 Worked Examples) book.

2 months ago Like Reply

Christian Jauvin

What a wonderful essay, with complex ideas so clearly explained and coded!

First, even if I knew about the article, I refrained from reading it, because I wanted to figure Sudoku solving by myself:

http://cjauvin.blogspot.com/20...

Then I read it, and got curious to know if there was a way to adapt my previous findings:

http://cjauvin.blogspot.com/20...

I hope this will have been a good warm-up for the AI class starting next week..

2 months ago Like Reply

Gaurav Singh

I appreciate your essay.
I have developed a Sudoku algorithm that will solve every Sudoku puzzle possible. If the puzzle has multiple solutions, it will provide all of them.Also if the puzzle has no solution, it will tell.The main property of this algorithm is that it solves even the hardest of puzzles having unique solutions within .01 seconds. It provides solutions to puzzles with multiple solutions also very quickly.Even if the algorithm is applied on a blank Sudoku, it will provide all possible solutions.I applied my algorithm in C++,and later in C# and it works perfectly.May i ask how and where can I publish my (finding)?

2 months ago Like Reply

Peter Norvig

I'm glad you enjoyed the essay, and even more glad that it helped inspire you to create a program of your own. You could publish it on a blog of your own, and/or on a code-hosting site like github or code.google.com/projecthosting . A maximum of .01 seconds is excellent; congratulations. I did a version in Java (which I will write about later) that has a worst-case time of .03 seconds, after testing on 100,000 examples. But be careful: if I only test on 1000 examples, the worst time is .0006 seconds -- the maximum time is highly skewed, depending on the number of boards you test. (My average rate was 40,000 boards per second.)

2 months ago in reply to Gaurav Singh Like Reply

Gaurav Singh

may you please tell me how many solutions your solver affords in the case of blank grid. My program running on my algorithm affords 2 lac solutions per second on my i3 laptop. Also the aforementioned hardest impossible puzzle that took you 1439 seconds takes .082 seconds on my program. Also the aforementioned Sudoku
000006000059000008200008000045000000003000000006003054000325006000000000000000000
that took your program 188 seconds to find one solution, took mine 1077.373 seconds to identify all 148357268 solutions this puzzle has. I think I have found an algorithm that can change how people look at Sudoku and other NP complete problems in general.

1 month ago in reply to Peter Norvig Like Reply

Celio Guimaraes

It seems that "real sudoku puzzles" have only one solution as Inkala's 2010 and 2006 "hardest puzzles": they use about 23 squares and all 9 digits; Norvig's hard1 has above 2 million solutions; using his random puzzle generator I found only 2

puzzles with unique solutions out of a million (they fill 26 squares and use all 9 digits; I found none using only 23 squares):

```
. . . |. 5 . |. 6 .
7 5 . |8 . . |9 . 1
9 6 . |. . 4 |. . .
------+------+------
. 1 . |. . . |3 8 .
. 9 5 |. 1 . |. . .
. 7 . |. . . |. . .
------+------+------
. . . |6 . . |. . 7
. 4 . |. 2 3 |. 5 .
. 2 . |. 9 . |. . 6
```

and

```
. . 2 |. 6 . |4 . .
. . . |. . 3 |. . 5
. 1 . |. 7 . |3 . .
------+------+------
. . . |. 3 . |. 8 .
7 8 3 |. 5 . |. 9 2
. . 9 |2 . 8 |. . .
------+------+------
. . . |. . 6 |. 5 .
6 . . |. 8 . |. . .
. 5 . |1 . . |2 . .
```

3 months ago Like Reply

Vit Mițul

Hi thanks for this post. I'm new to Python and curently i'm learning it.
could you tell me how to find the number of solutions that a sudoku puzzle have?
For example, how can i find how many solutions "hard1" have?
thanks

4 months ago Like Reply

Patrick

I very much appreciate your essay, Dr. Norvig. It encouraged me to finish my Sudoku program (with similarly speedy results) and now I too am no longer "infected".

If I may ask one question, do you see Chess in a similar light as Sudoku, a "DoS attack on human intellect"?

5 months ago Like Reply

Peter Norvig

That's a great question. Certainly devoting your life to chess is more socially acceptable than devoting your life to Sudoku. Why is that? In some sense chess is more challenging, so if we just think about within-the-game, I'd say chess is more worthy of study. But outside-the-game, I'm not convinced the efforts devoted to chess are worth it (although I admit I don't know enough about chess history. I can't think of a lot of chess champions who go on to other things and can point to their chess expertise as enabling them to do other things (the exception being Kasparov in politics). On the other hand, a lot of non-champions say that that studying chess taught them the value of studying, concentrating, and hard work. So I would say that being obsessed with chess within limits can be useful.

5 months ago in reply to Patrick 3 Likes Like Reply

Preetam D'Souza

Wrote up a Ruby solver with the help of the CP techniques in this article. Decided to use a list of numbers instead of strings to represent possibilities and use a different strategy for generating the units. It is a bit shorter than the other Ruby versions out there. Check it out here (included runtimes with both Ruby and JRuby): https://gist.github.com/101379...

6 months ago Like Reply

Jonas

Here's another Ruby version. Not that exciting except if you like shorter code and it's maybe a tad bit more idiomatic Ruby than some earlier examples.
https://gist.github.com/914326

8 months ago Like Reply

Jonas Elfström

And in CoffeeScript https://gist.github.com/927782

7 months ago in reply to Jonas Like Reply

Jean-François Comeau

Very inspirational!

I know it's not always about numbers but here's another pretty fats client-side solver: http://solvesudoku4.me/

The javascript implementation is rock solid.

8 months ago Like Reply

Mike

There's a JavaScript solver here: http://sudoku.mytestbench.com/

8 months ago Like Reply

Jesse

A problem with your randomly generated puzzles is that they may have multiple solutions. A true sudoku puzzle has exactly one solution. The hardest you mention in this article has over 500 possible solutions, according to http://www.sudokuwiki.org/sudo...

Perhaps a better way is to randomly fill a completed puzzle, then erase one random cell at a time as long at it still has exactly one solution.

Another way is to take any legal puzzle, and apply random legal transformations.

A legal transformation is one of:
* swap any two digits everywhere (ex: 4 <=> 7)
* swap two columns in a set of three (ex: columns 1 and 3)
* swap two rows in a set of three (ex: rows E and F)
* swap two box-columns (ex: columns 123 with 789)
* swap two box-rows (ex: rows ABC with DEF)
* flip entire puzzle horizontally, vertically, or diagonally

With these legal transformations, one legal puzzle becomes many similar (though apparently dissimilar) puzzles.

-Jesse

9 months ago Like Reply

Tony Veijalainen

I had to strugle how to be original after your beautiful code. So I did one as different solver as I could think of for Project Euler to have something self made, not just using your solution. Interestingly it is doing quite well in the cases most difficult for your solver. Had to also include empirical maximum failure count to find when solution did not exist. Thanks for sample of impossible one.
Quest for most stupid Sudoku solver:http://www.daniweb.com/code/sn...

10 months ago Like Reply

Lukas Vermeer

Cool! Interesting your algorithm performs so well for this particular puzzle.

My hypothesis is that these long execution times are not the result of difficulty, but rather unlucky guesses/assumptions made during the search for a solution. Your algorithm makes different assumptions, so it would make sense that the performance for this particular puzzle would be different (and since this is an outlier, probably the performance would be better).

If you were to generate many random puzzles, like Peter and I did, I would expect you would encounter some puzzles your program finds 'difficult' to solve. This is the nature of a search strategy; sometimes you end up searching in the wrong alley.

You wrote that "the problem solves the sudoku '.....6....59.....82....8....45.....3.....6..3.54...325..6.....' in 0.13 s which was very hard for the recurvive search and propagation version."

I ran this puzzle through both Peter's algorithm and my modified version and found 109 seconds and 0.015 seconds respectively.

10 months ago in reply to Tony Veijalainen Like Reply

Tony Veijalainen

Actually if you run the sudoku's top95.txt, you get very bad performance from my primitive loop version. With givng up disabled the program did finnish them. That is not unexpected as I only wanted to see how to get 'good enough' for simpler sudoku's with primitive algorithm. Interesting was also, that I tried to reduce pushing to stack and optimize putting to stack and taking out immidiately. But as side effect the code started push much solutions to stack. So I must have hitted by luck way to 'fail fast enough' for significant number of sudokus. But generally my primitive way does very bad on average and especially in worst case. Interesting is though that the ones my algorithm finds difficult are not only those rated 'hard'. Interesting collection of sudokus can be found from: http://mapleta.maths.uwa.edu.a...
The sudoku mentioned is the hard1 in Peter's code. His randomizing also solves this case. My code with command line interface original Peters cod egives time of 270810.65 ms, your modification code takes 18.5 ms and Peter's randomized version around 17 ms.

10 months ago in reply to Lukas Vermeer Like Reply

Lukas Vermeer

Great post! Delightful read. I've managed to improve the performance of your sovler by quite a bit by generalizing your second rule.

"If a unit has only N possible places for values d1 to dN, then those are the only options in those places."

For the case N=1 this is equal to your rule two, but this also works for cases where N>1.

http://lukasvermeer.wordpress....

10 months ago Like Reply

Tony Veijalainen

For some strange reason your log of hard1 run has 6 at wrong position in starting position print. As my own excersise I coded another boggle word finder, but I had not courage to use your elimination method. You are welcome to comment on my solution in DaniWeb: http://www.daniweb.com/code/sn...

1 year ago Like Reply

Peter Norvig

You're right, there was an error in the hard1 starting position -- I must have hit a ^T in the wrong place while editing the file. I've corrected it; thanks.

1 year ago in reply to Tony Veijalainen Like Reply

Ufkapano

The program is great but the sudoku hard1 have multiple solutions.

1 year ago Like Reply

mory

this man is fierce

1 year ago Like Reply

Peter Norvig

I've updated the page to show the solution for both the 2006 and 2010 Inkala puzzles. However, I only show the final solution; I don't show the path used to arrive at it.

1 year ago Like Reply

Mark

Truly brilliant stuff Peter. Kudos.

1 year ago in reply to Peter Norvig Like Reply

Transient

Truly a work of art!!
Tried it on the new Arto Inkala puzzle of 2010 claimed by the author (as is his wont) to be the hardest sudoku ever and it solved it in 0.02 seconds flat. Thought you would be glad to know that Peter.

1 year ago Like Reply

Mark

I have been manually trying to solve Inkala's puzzle and have sort of hit a roadblock. You got an answer to it that you can share with me? Would be much appreciated.

1 year ago in reply to Transient Like Reply

Tony Veijalainen

I posted slightly modified code to DaniWeb Python forum. Really fine code, only needed little modernizing (all def not needed, if else cleaner than and or).
http://www.daniweb.com/code/sn...

1 year ago Like Reply

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