1T4: Some basic mathematics

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Sinusoidal functions (sinewaves)

$$x[n] = A\cos(\omega nT + \phi) = A\cos(2\pi f\, nT + \phi)$$

A : amplitude

ω : angular frequency in radians/seconds

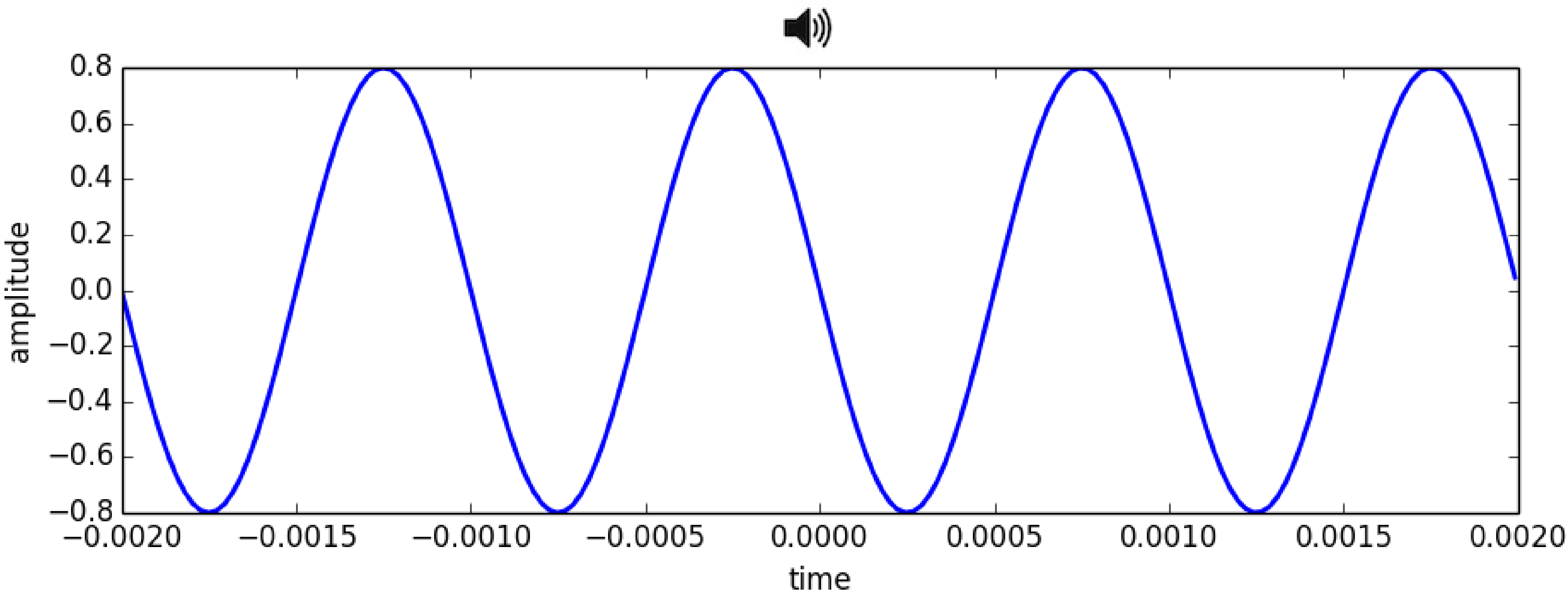
$f = \omega / 2\pi$: frequency in Hertz (cycles/seconds)

ϕ : initial phase in radians

n : time index

$T = 1/f_s$: sampling period in seconds $(t = nT = n/f_s)$

Sinewave plot




```
A = .8
f0 = 1000
phi = np.pi/2
fs = 44100
t = np.arange(-.002, .002, 1.0/fs)
x = A * np.cos(2*np.pi*f0*t+phi)
```

Complex numbers

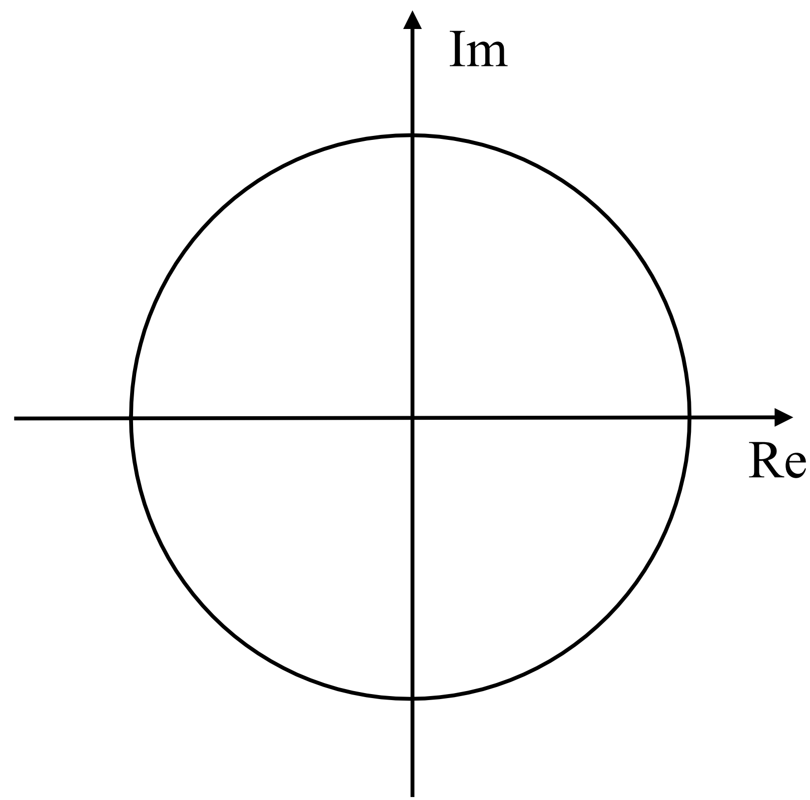
(a + jb) *a*, *b*: real numbers

$j = \sqrt{-1}$: imaginary unit

Complex plane:

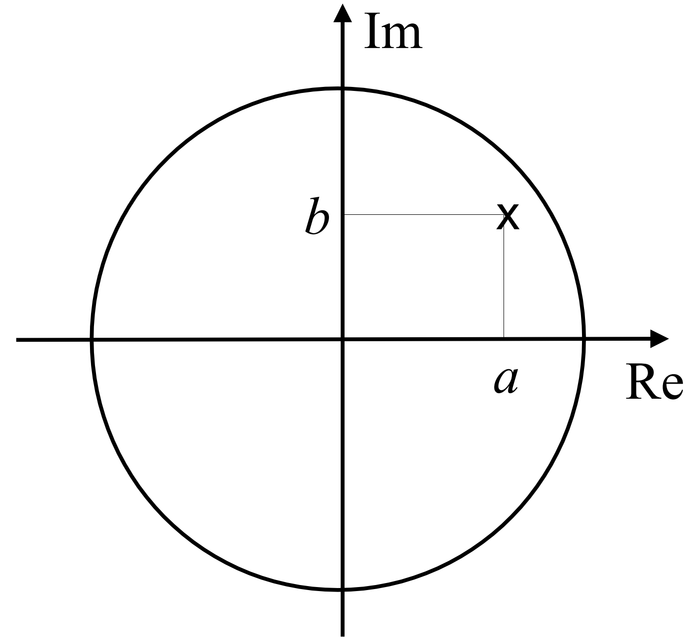
Re (real axis)

Im (imaginary axis)



Rectangular form:

$$(a + jb)$$



Polar form:

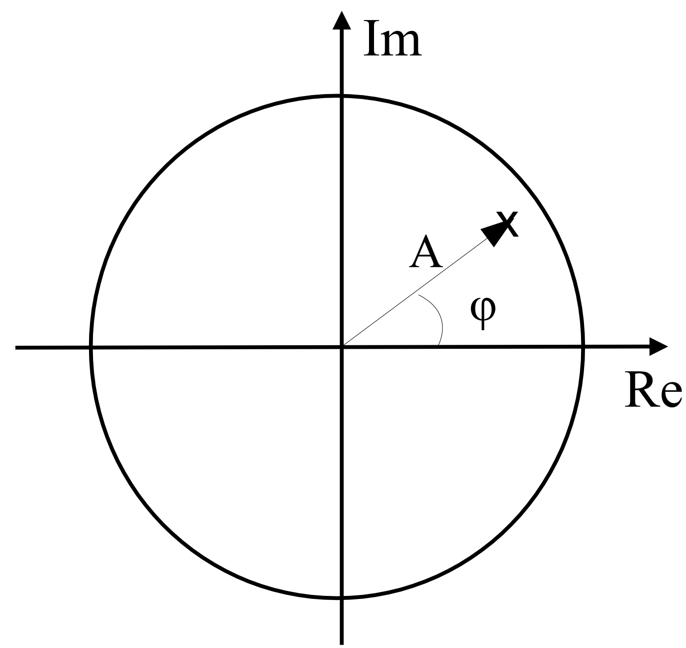
$$A = \sqrt{a^2 + b^2}$$

$$\phi = atan\,2\left(\frac{b}{a}\right)$$

where:

if $(a > 0)$ $\quad atan\,2\left(\frac{b}{a}\right) = \tan^{-1}\left(\frac{b}{a}\right)$

else if $(a < 0)$ $\quad atan\,2\left(\frac{b}{a}\right) = \tan^{-1}\left(\frac{b}{a}\right) - pi$

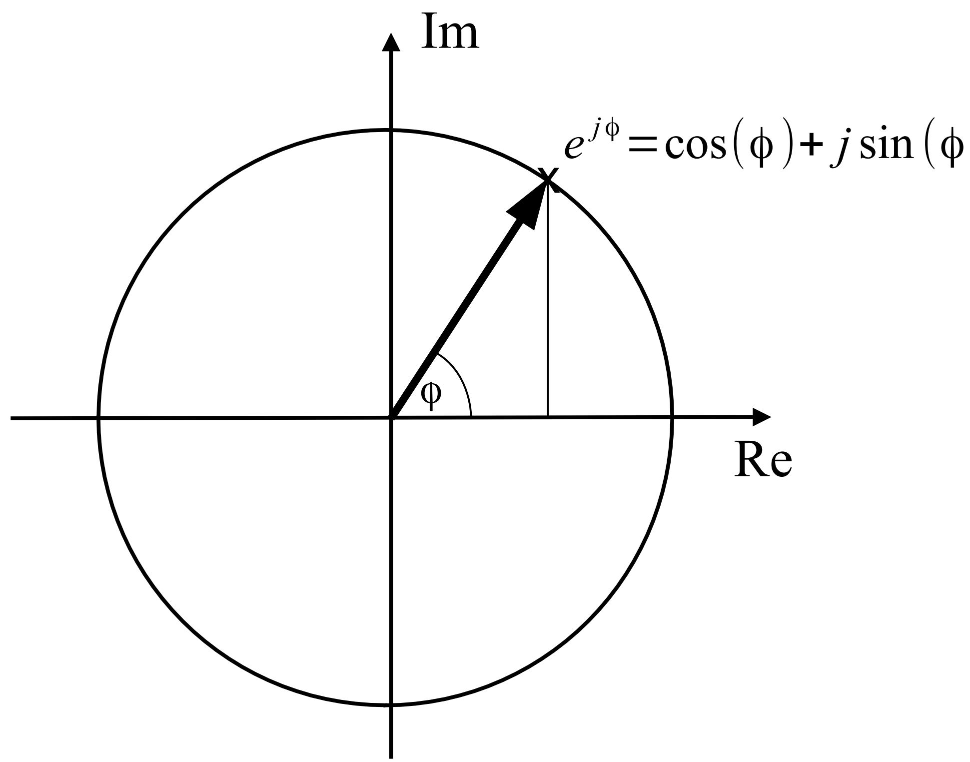


Euler's formula

$$e^{j\phi} = \cos\phi + j\sin\phi$$

$$\cos\phi = \frac{e^{j\phi} + e^{-j\phi}}{2}$$

$$\sin\phi = \frac{e^{j\phi} - e^{-j\phi}}{2j}$$

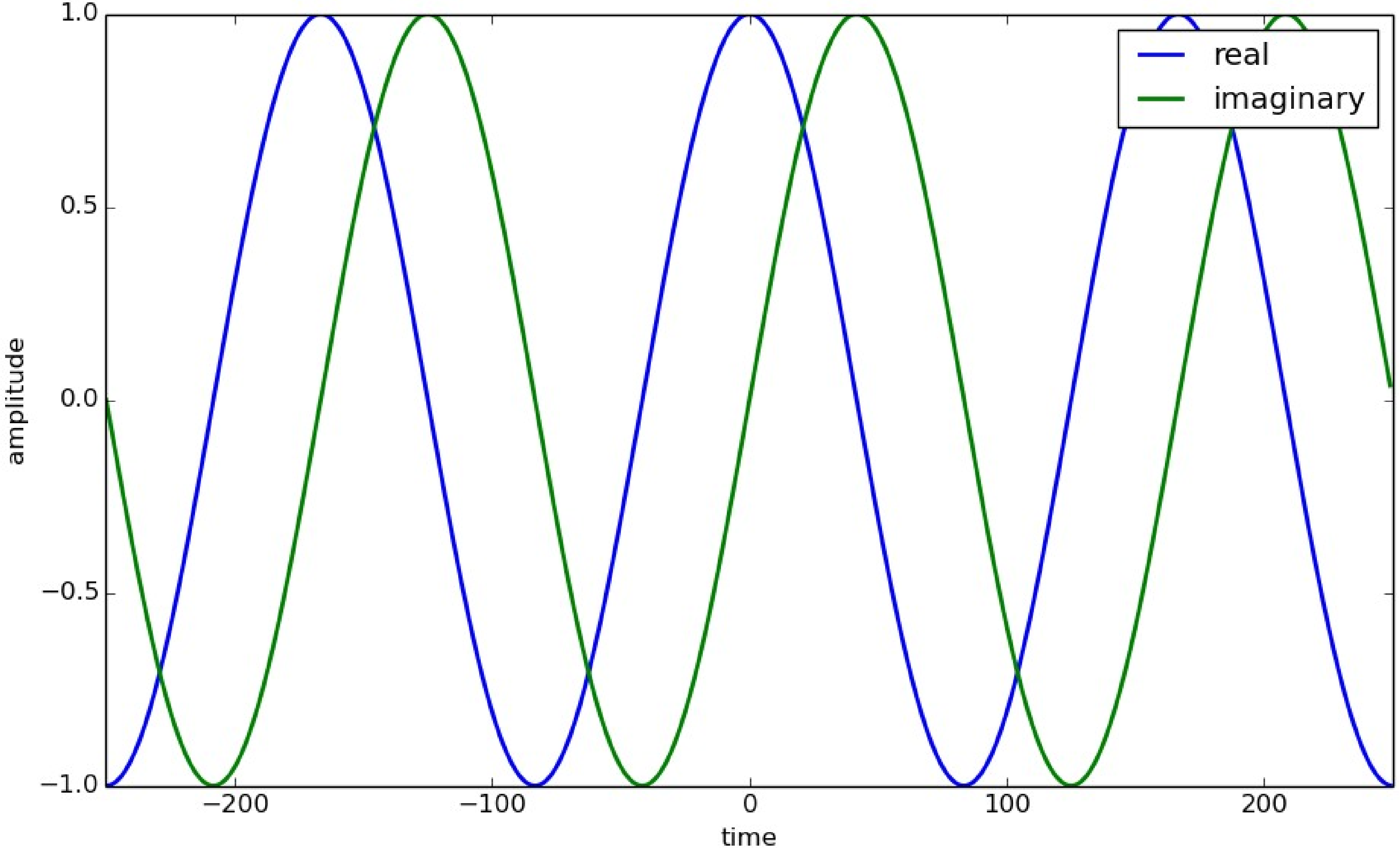


Complex sinewave

$$\bar{x}[n] = A e^{j(\omega nT + \phi)} = A e^{j\phi} e^{(j\omega nT)} = X e^{j(\omega nT)}$$
$$= A\cos(\omega nT + \phi) + jA\sin(\omega nT + \phi)$$

Real sinewave:

$$x[n] = A\cos(\omega nT + \phi) = A\left(\frac{e^{j(\omega nT + \phi)} + e^{-j(\omega nT + \phi)}}{2}\right)$$
$$= \frac{1}{2} X e^{j(\omega nT)} + \frac{1}{2} X^* e^{-j(\omega nT)} = \frac{1}{2}\bar{x}[n] + \frac{1}{2}\bar{x}^*[n]$$
$$= \Re\{\bar{x}[n]\}$$



1.0
0.5
0.0
−0.5
−1.0
amplitude
−200
−100
0
100
200
time
real
imaginary

Scalar (dot) product of sequences

$$\langle x, y\rangle = \sum_{n=0}^{N-1} x[n]\, y^{*}[n]$$

Example:

$$x[n] = [0, j, 1];\; y[n] = [1, j, j]$$
$$\langle x, y\rangle = 0\times 1 + j\times(-j) + 1\times(-j) = 0 + 1 + (-j) = 1 - j$$

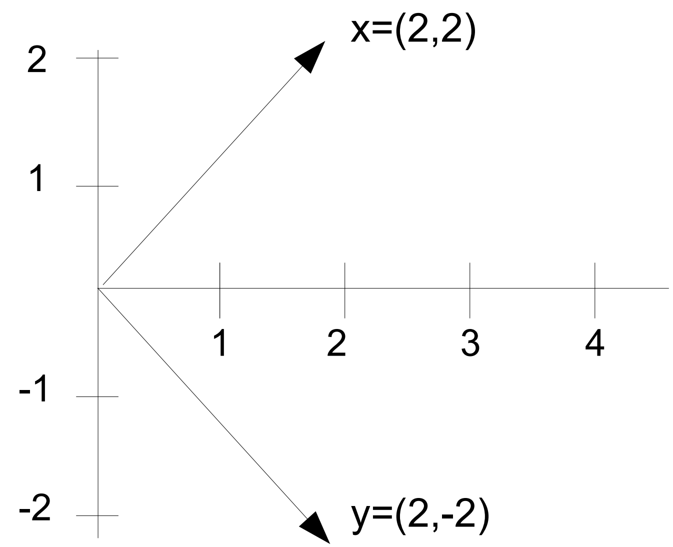
Orthogonality of sequences

$$x \perp y \Leftrightarrow \langle x , y \rangle = 0$$

Example:

$$x[n] = [2,2]; y[n] = [2, -2]$$

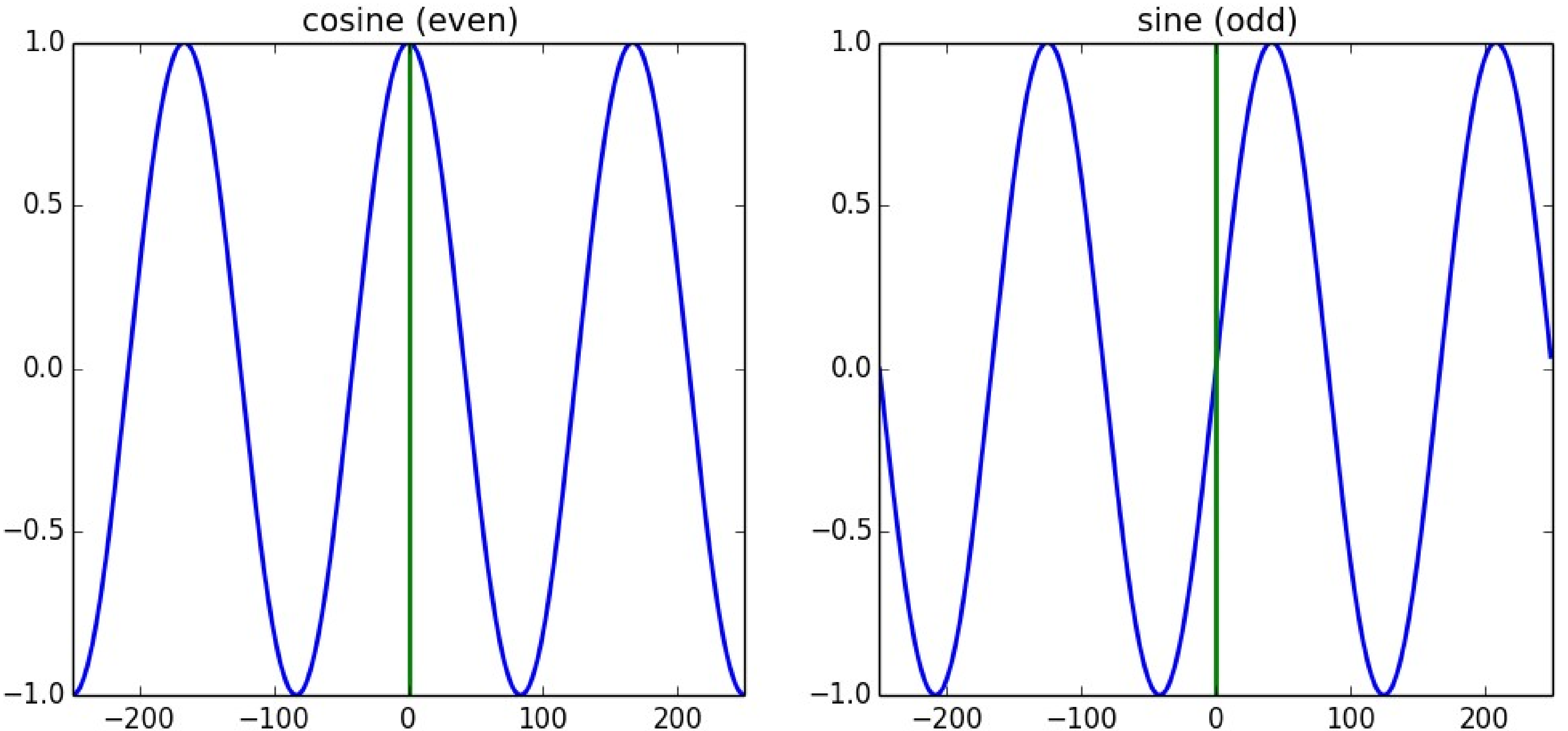
$$\langle x , y \rangle = 2 \times 2^* + 2 \times (-2)^* = 4 - 4 = 0$$



Even and odd functions

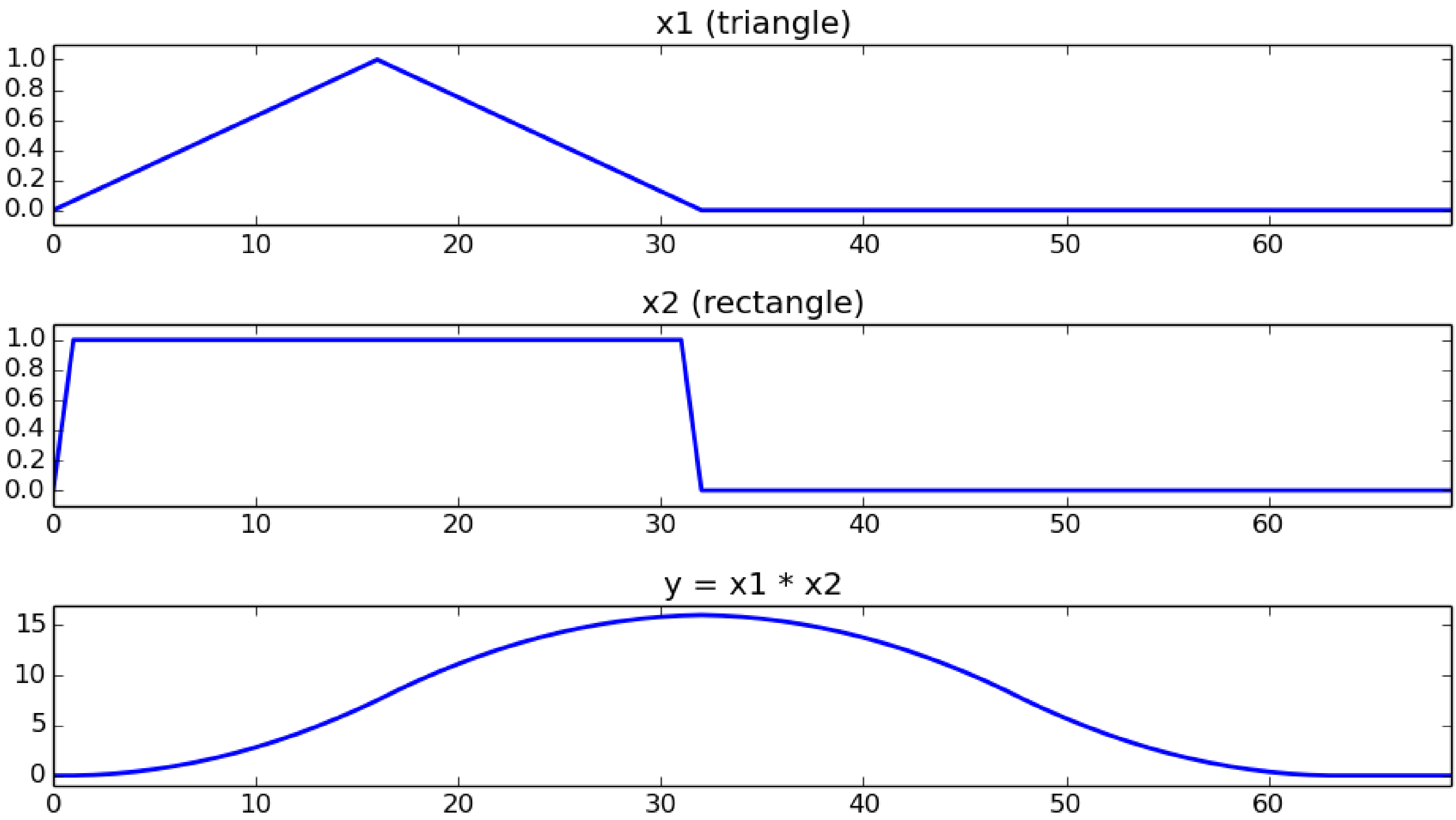
f [n] is *even* if f [-n] = f [n] [symmetric]

f [n] is *odd* if f [-n] = -f [n] [antisymmetric]



Convolution

$$y[n]=\left(x_1[n]*x_2[n]\right)_n=\sum_{m=0}^{N-1} x_1[m]x_2[n-m]$$



References and credits

- More information in:
  - https://en.wikipedia.org/wiki/Sinusoid
  - https://en.wikipedia.org/wiki/Complex_numbers
  - https://en.wikipedia.org/wiki/Euler_formula
  - http://en.wikipedia.org/wiki/Dot_product
  - https://en.wikipedia.org/wiki/Convolution
- Reference for the mathematics of the DFT by Julius O. Smith: https://ccrma.stanford.edu/~jos/mdft/
- Slides released under CC Attribution-Noncommercial-Share Alike license and code under Affero GPL license; available from https://github.com/MTG/sms-tools

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