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Exponentially Smoothing the Skewed Laplace Distribution for Value-at-Risk Forecasting

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ABSTRACT
Value-at-risk (VaR) is a standard measure of market risk in financial markets. This paper proposes a novel, adaptive and efficient method to forecast both volatility and VaR. Extending existing exponential smoothing as well as GARCH formulations, the method is motivated from an asymmetric Laplace distribution, where skewness and heavy tails in return distributions, and their potentially time-varying nature, are taken into account. The proposed volatility equation also involves novel time-varying dynamics. Back-testing results illustrate that the proposed method offers a viable, and more accurate, though conservative, improvement in forecasting VaR compared to a range of popular alternatives. Copyright © 2013 John Wiley & Sons, Ltd.

KEY WORDS asymmetric Laplace distribution; exponential smoothing; forecasting; skewness and heavy tails; time-varying parameters; value-at-risk (VaR)

INTRODUCTION

Value-at-risk (VaR) is a widely used measure of market risk for financial assets or portfolios, mainly prompted by the globalization of the world economy, financial innovation and the growth and sophistication of the world's financial centres. Regulators have long urged market participants to make major efforts to understand and control financial risk, with a benchmark of VaR, in order to underpin the solvency and stability of the world banking system. The global financial crisis (GFC) once again calls into question financial risk management methods, models and practice. This paper proposes a novel and flexible parametric model to more accurately estimate and forecast VaR for financial returns.

Risk management has experienced a revolution in recent years, started by VaR, which was developed in response to the financial/derivative disasters of the late 1980s and early 1990s. VaR was pioneered in 1993, part of the 'Weatherstone 4:15 pm' daily risk assessment report, in the RiskMetrics model at JP Morgan (1996). Subsequently, the Group of Thirty (G-30) advised financial institutions to value positions using market prices and to assess financial risks via VaR. The Basel Committee on Banking Supervision in 1996 endorsed the use of VaR, by 'internal models'. For reviews of VaR see, for example, Duffie and Pan (1997), Hull and White (1998a,b), Jorion (2001), and Kaut *et al.* (2007). Despite some criticism of VaR—for example, it does not measure the magnitude of the loss for violations and it is not 'coherent' (Artzner *et al.*, 1999)—it is recommended in Basel II and is widely used in industry.

One reason for the popularity of VaR is the public release of JP Morgan's (1996) RiskMetrics™ Technical Document, where an exponentially weighted moving average (EWMA) estimator was used to forecast return volatility. This is a special non-stationary case of Bollerslev's (1986) general autoregressive conditional heteroskedasticity (GARCH) model, called IGARCH (i.e. integrated GARCH). EWMA methods, around since the 1950s, are still popular forecast tools used in business. We call this method 'standard-EWMA'; it is optimal for series that are conditionally Gaussian. However, it is well known (e.g. see Poon and Granger, 2003) that returns are fat-tailed and mildly skewed, and often empirically have a time-varying distribution. Thus standard-EWMA is inefficient and is highly sensitive to extreme returns. Many authors have suggested solutions to this and similar issues, including Bollerslev (1987), who proposed a GARCH model with Student-t errors, spurring a literature on non-Gaussian conditional return distributions, e.g. the skewed-t of Hansen (1994); while Zakoian (1994) introduced the asymmetric GARCH model, where conditional standard deviations are modelled and driven by absolute, not squared, returns. This paper builds on both these approaches, employing a conditionally asymmetric Laplace (AL) distribution, and a standard deviation IGARCH formulation. The AL density has fat tails compared to a Gaussian, and a sharp peak at its central mode. While logic suggests that a return distribution would not have a sharp peak, when estimating VaR the accuracy in the tails of the distribution is of primary importance. Thus we propose the AL mainly because it can

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accommodate relatively fat tails compared to other distributions, and thus may better account for the tail behaviour in real financial return series.

An approach that combines the nature of these with the spirit of EWMA, that we draw on here, is in Guermat and Harris (2001), who proposed a 'robust-EWMA' procedure, derived from the naturally fat-tailed Laplace (or double exponential) distribution. Their empirical applications showed clear improvements over the standard-EWMA. However, the Laplace is symmetric and has a constant kurtosis, which is not consistent with the time-varying and/or mildly skewed nature of many real financial return series.

The objective of this paper is to propose a new parsimonious model and VaR forecasting method, termed skewed-EWMA. This model extends the RiskMetrics standard-EWMA and the robust-EWMA, as well as the standard deviation GARCH of Zakoian (1994), first by employing an AL error distribution. The AL can capture both skewness and heavy-tailed behaviour in a parsimonious fashion. More importantly, this paper develops a procedure to allow an adaptive adjusting, or time variation, in the shape parameter that controls the skewness and kurtosis in the AL distribution. This involves an EWMA prediction of this shape and can thus potentially adapt to the time-varying nature of skewness and heavy tails in forecasting actual financial returns. Further, the proposed model extends the standard deviation GARCH model by also allowing the volatility dynamics parameters to evolve over time.

The performance of the proposed time-varying skewed-EWMA method is assessed, and compared to a range of existing alternative VaR forecast methods as benchmarks, across various foreign exchange rate and market index return series, over a period that includes the recent GFC. These include exchange rates between the US dollar and British pound, Japanese yen and Australian dollar, as well as market indices in the USA, Hong Kong, UK, China and Australia. The models are assessed and compared in terms of standard back-testing measures and tests widely applied in the literature, with results generally favouring the proposed skewed-EWMA method. Importantly, a long forecast period, consisting of 1000 days, is considered, allowing suitable sample size to properly distinguish between the methods considered.

The structure of the paper follows. The next section outlines some existing approaches to VaR, including RiskMetrics, the semi-parametric CAViaR of Engle and Manganelli (2004), and the robust-EWMA; the third section focuses on developing our proposed methodology in depth, with an introduction to the AL distribution from which the proposed VaR model—skewed-EWMA estimator—is motivated. The fourth section examines and evaluates the VaR forecasts from the empirical applications. The last section concludes.

METHODS FOR VaR FORECASTING

A range of different methods have been developed for VaR estimation and forecasting in the literature, which can be categorized in several ways: first, *indirect* and *direct* methods, and second, parametric, semi-parametric and non-parametric approaches (see Manganelli and Engle, 2004, for definitions. Indirect methods first specify, and estimate if needed, the conditional return distribution and then calculate the implied VaR from the distributional properties assumed. Indirect methods include all parametric approaches, e.g. GARCH with Gaussian or Student-t errors, RiskMetrics, robust-EWMA and Monte Carlo simulation. Parametric GARCH-type models with specified error distributions are often used as benchmarks in VaR studies (e.g. Kuester *et al.*, 2006; Gerlach *et al.*, 2011).

The mild skewness of the unconditional return distribution has been well documented in the financial literature (e.g. see Harvey and Siddique, 1999, 2000; Ait-Sahalia and Brandt, 2001; Chen, 2001). For the conditional return distribution, Hansen (1994) considered the skewed-t, Theodossiou (1998) proposed the generalized t distribution, Zhu and Galbraith (2009) proposed a generalized asymmetric Student-t, with separate parameters in each tail; Griffin and Steel (2006) and Jensen and Maheu (2010) employed Dirichlet process mixtures, while Aas and Haff (2006) used a generalized hyperbolic and (Dark, 2010) allowed time-varying skewness. Naturally, many of these proposals may be too complex or time-consuming for practical real-time use in the markets.

Alternatively, direct methods estimate VaR without assuming a specific distribution, and include methods such as the semi-parametric dynamic quantile-based CAViaR models of Engle and Manganelli (2004), or the non-parametric historical simulation (i.e. taking sample percentiles) method that is used by many financial institutions. Such methods can also capture skewness in the return distribution.

Kuester *et al.* (2006) conducted a review of some competing VaR models, but considered only one (very long) dataset. Gerlach *et al.* (2011) extended the CAViaR models of Engle and Manganelli (2004) and compared these to a range of methods, using data prior to the GFC. Both these studies used a mix of parametric, semi-parametric and non-parametric, as well as direct and indirect, VaR methods. Based on their work we do not consider non-parametric historical simulation here, since it is a highly inaccurate method.

In this paper we develop a time-varying skewed-EWMA forecast model—an indirect, parametric method. We compare this model with a range of common indirect, parametric and direct, semi-parametric estimators, including two CAViaR models. First, a review of the basic ideas in standard-EWMA and robust-EWMA is presented as motivation and background for our proposed skewed-EWMA method.

Standard-EWMA and robust-EWMA

It is common practice in VaR modelling to set the mean of the return series as zero, in estimating low quantiles such as those in VaR. We follow that convention.

JP Morgan's standard-EWMA assumes conditional Gaussianity for returns, r_t, with volatility modelled via a 0-intercept IGARCH(1,1) equation, specifically:

$$\mathrm{VaR}_{t+1} = -\sigma_{t+1}\Phi^{-1}(\alpha) \tag{1}$$

$$\sigma_{t+1}^2 = \lambda\sigma_t^2 + (1-\lambda)r_t^2,\ 0 < \lambda < 1 \tag{2}$$

where α is the desired confidence level; $\Phi^{-1}(\cdot)$ is the inverse cumulative distribution function (cdf) of the standard Gaussian; and (2) can equivalently be expressed as

$$\sigma_{t+1}^2 = (1-\lambda)\sum_{i=0}^{\infty}\lambda^i r_{t-i}^2$$

which is an EWMA estimate of the sample variance $\widehat{\sigma^2} = \frac{1}{n}\sum_{i=1}^{n} r_i^2$ (assuming mean 0), putting geometrically declining weights on past (squared) returns. Nelson and Foster (1994) pointed out that when returns are conditionally Gaussian standard-EWMA is optimal, if volatility is stochastic.

Empirically, the conditional distribution of financial returns is often skewed and heavy-tailed, with departures from a Gaussian (cf. Hull and White, 1998a; Poon and Granger, 2003). To accommodate the heavy-tailed aspect, Guermat and Harris (2001) employed a Laplace distribution, in a robust-EWMA model of the form

$$\mathrm{VaR}_{t+1} = -\frac{\sigma_{t+1}}{\sqrt{2}}\ln\{2(1-\alpha)\} \tag{3}$$

$$\sigma_{t+1} = \lambda\sigma_t + (1-\lambda)\sqrt{2}|r_t|, 0 < \lambda < 1 \tag{4}$$

The robust-EWMA volatility process is an EWMA version of the maximum likelihood estimator (MLE) of the standard deviation, $\widehat{\sigma} = \frac{1}{n}\sum_{t=1}^{n}\sqrt{2}|r_t|$ of the Laplace distribution; i.e. this is a special case of a standard deviation IGARCH model: it can account for heavy tails, but not skewness or time-varying higher moments.

For the above EWMA models, as shown in Theorem 1 of Nelson (1990), the unconditional variance of the return does not exist: the models are non-stationary in volatility. In practice, primary interest is in the accuracy of conditional volatility (and VaR) forecasting; as Nelson (1990, p. 320) stressed: 'only the conditional model is of interest'.

CAViaR and indirect quantile approaches

Quantile-based methods provide a semi-parametric alternative to GARCH-type models. A general dynamic regression model may be written

$$r_t = f_t(\boldsymbol{\beta}, \boldsymbol{x}_{t-1}) + u_t$$

where $\boldsymbol{x}_{t-1}$ are explanatory variables; $\boldsymbol{\beta}$ are unknown parameters and u_t is an error term, i.e. a suitable prediction for u_t is 0. The function $f_t(.)$ then estimates the conditional α level quantile:

$$q_\alpha(r_t|\boldsymbol{\beta}, \boldsymbol{x}_{t-1}) = f_t(\boldsymbol{\beta}_\alpha, \boldsymbol{x}_{t-1})$$

where $\boldsymbol{\beta}_\alpha$ obtains the minimum of the quantile loss function, i.e.

$$\min_{\boldsymbol{\beta}} \sum_t \rho_\alpha\left(r_t - f_t(\boldsymbol{\beta}, \boldsymbol{x}_{t-1})\right) \tag{5}$$

Here $\rho(.)$ is specified as $\rho_\alpha(u) = u(\alpha - I(u < 0))$.

Engle and Manganelli (2004) propose various dynamic quantile functions $f(.)$, called CAViaR models, of which we consider two. The first is the *symmetric absolute value (SAV):*

$$f_t(\boldsymbol{\beta}) = \beta_1 + \beta_2 f_{t-1}(\boldsymbol{\beta}) + \beta_3|r_{t-1}| \tag{6}$$

where the dynamic quantile function responds symmetrically to the lagged return r_{t-1}. This equation is equivalent to the quantile function for a symmetric standard deviation GARCH model (e.g. see Zakoian, 1994). To account for financial market asymmetry, via the leverage effect (see Black, 1976), Engle and Manganelli (2004) proposed the *asymmetric slope (AS):*

$$f_t(\boldsymbol{\beta}) = \beta_1 + \beta_2 f_{t-1}(\boldsymbol{\beta}) + (\beta_3 I_{(r_{t-1}>0)} + \beta_4 I_{(r_{t-1}<0)})|r_{t-1}| \tag{7}$$

where the dynamic quantile function responds asymmetrically to positive and negative returns, called sign asymmetry. This threshold nonlinear model corresponds to a standard deviation GJR-GARCH model.

In practice, the coefficients of the above CAViaR models may not be constant, but varying with some market regimes (cf. Huang *et al.*, 2010).

SKEWED-EWMA

In this section, a skewed-EWMA VaR model, via an AL distribution, is proposed, taking into account skewness and heavy tails in financial returns. Further, an EWMA-based procedure is proposed that adaptively adjusts the shape parameter, allowing time-varying skewness and kurtosis and time-varying GARCH dynamic parameters.

Motivation from an asymmetric Laplace distribution

The AL distribution has been found useful in defining quantiles and performing quantile regression in the literature. For example, the standard quantile regression function minimization, as in Koenker and Bassett (1978), is equivalent to the maximization of a log-likelihood based on the AL distribution (ALD). There are now many papers recognizing and building on this link; see, for example, Koenker and Machedo (1999), Yu *et al.* (2003), Yu and Zhang (2005) and Geraci and Bottai (2007), among others. Differently from these, we employ a newly parametrized AL to model the conditional distribution of financial returns. Our methods are thus indirect and parametric, and are not closely related to semi-parametric direct quantile regression or CAViaR procedures. We define an ALD with an alternative, more interpretable parametrization to that of Kotz *et al.* (2002) and Yu and Zhang (2005).

Definition 1. If a random variable X has the following probability density function (pdf), it is asymmetric Laplace distributed (ALD), denoted $X \sim \mathrm{AL}(\mu, \sigma, p)$:

$$f(x|\mu, \sigma, p) = \frac{k}{\sigma} \exp\left\{ -\left(\tfrac{1}{1-p} I_{[x>\mu]} + \tfrac{1}{p} I_{[x<\mu]} \right) \tfrac{k}{\sigma} |x-\mu| \right\} \tag{8}$$

where μ, σ and p are location, scale and shape parameters, respectively, and $k = k_p = \sqrt{p^2 + (1-p)^2}$.

Here μ, σ, p are different from the parameters in Kotz *et al.* (2002) and Yu and Zhang (2005, p. 1867) and have the convenient meanings: the shape p is the probability that $X < \mu$, where μ is the mode of the ALD; the squared scale parameter σ^2 is the variance of X, i.e. $\mathrm{var}(X) = \sigma^2$, thanks to including the term k_p in (8).

The parameter μ is the mode; thus for all p the modal return is μ in our model. Most financial returns are close to 0 and have a modal point at 0, and 0 appears to be a sensible mode based on the real data we considered. As such, we set $\mu = 0$ for simplicity. We subsequently denote $X \sim \mathrm{AL}(\sigma, p)$. If there is a lot of skewness, the mode still does not change, but instead the mean of the AL would be affected. This issue does not really occur for most financial return data, since skewness is usually very mild, which helps to explain why the AL and Laplace have similar overall performance at VaR forecasting in our empirical study.

Under (8), the shape parameter p controls the skewness and kurtosis, plus all other moments, e.g. via the formulas

$$S_k = \frac{2\left[(1-p)^3 - p^3\right]}{k^3}, \quad K_u = \frac{9(1-p)^4 + 6(1-p)^2p^2 + 9p^4}{k^4} \tag{9}$$

Different values of p thus lead to positive or negative skewness, or even symmetry. With $\mu = 0$, if $p < 0.5$, $|x|$ has a larger weight for $x > 0$, causing the density function to skew to the right, i.e. positive skewness; if $p = 0.5$, we have the symmetric Laplace distribution; while if $p > 0.5$, the ALS is skewed to the left, i.e. negative skewness. The range of skewness and kurtosis for the ALD is $[-2, 2]$ and $[6, 9]$ respectively.

Naturally, the question of whether skewness is required in a VaR model is important, which may be hard to test (cf. Peiro, 1999; Bai and Ng, 2005), but is addressed in our empirical section, where the skewed-EWMA model is compared to the symmetric robust-EWMA model and other competitors.

The variance σ^2 and shape parameter p are to be both stochastic and conditionally changing with time, by EWMA procedures, to be subsequently developed. As in JP Morgan Guaranty Trust Company (1996) and Guermat and Harris (2001), our skewed-EWMA model is motivated from the equations for the MLEs of σ and p in the ALD (8) (with $\mu = 0$), by applying EWMA estimation to these quantities instead. Suppose the return observations are denoted $r_1, \cdots, r_n$. Then the likelihood under the assumption of i.i.d. $\mathrm{AL}(\sigma, p)$ errors is

$$L(\sigma, p) = \frac{k^n}{\sigma^n} \exp\left\{ -\sum_{i=1}^{n} \left(\frac{1}{1-p} I_{[r_i>0]} + \frac{1}{p} I_{[r_i<0]} \right) \frac{k}{\sigma} |r_i| \right\}$$

The log-likelihood is then

$$\ell(\sigma, p) = n \ln k - n \ln \sigma - \sum_{i=1}^{n} \left\{ \left(\frac{1}{1-p} I_{[r_i>0]} + \frac{1}{p} I_{[r_i<0]} \right) \frac{k}{\sigma} |r_i| \right\} \tag{10}$$

Setting the first-order partial derivatives of (10), with respect to σ and p, equal to zero, leads to

$$\hat{\sigma} = \frac{1}{n}\sum_{i=1}^{n}\left(\frac{k}{p}I_{[r_i<0]} + \frac{k}{1-p}I_{[r_i>0]}\right)|r_i|,$$
$$\hat{p} = \frac{1}{1+\sqrt{u/v}}, \tag{11}$$

where

$$u = \sum_{i=1}^{n}|r_i|I_{[r_i>0]}, \quad v = \sum_{i=1}^{n}|r_i|I_{[r_i<0]} \tag{12}$$

and u is the sum of the positive returns, while v is the sum of the absolute values of the negative returns. Naturally, the larger u is compared to v, the better, for investment return performance, but $u > v$ also indicates positive skewness, and thus $p < 0.5$. These formulae are pivotal in developing the skewed-EWMA forecasting method below.

Skewed-EWMA-based VaR modelling

Based on the ALD in (8), if the parameters σ and p are known, then the VaR value at the confidence level α can be derived as

$$\mathrm{VaR}_\alpha = \begin{cases} \sigma\frac{p}{k}\log\left(\frac{\alpha}{p}\right); & 0 \le \alpha < p \\ -\sigma\frac{(1-p)}{k}\log\left(\frac{1-\alpha}{1-p}\right); & p \le \alpha < 1 \end{cases} \tag{13}$$

The appeal of the ALD lies simultaneously in its simplicity and its flexibility. Its simplicity is similar to the Gaussian distribution, but its flexibility allows both skewness and excess kurtosis to be taken into account. Further, return volatility, σ^2, is empirically clustered and time-varying. We also allow the shape parameter p to change with time, and assess the importance of this in practice. This means that the (implied) unconditional distribution of returns is itself not ALD, but is a mixing distribution of the ALD with respect to the stochastic (σ, p) in general.

For σ, an EWMA estimator corresponding to (4), from (11), is

$$\sigma_{t+1} = (1-\lambda)\sum_{i=0}^{\infty}\lambda^i\left(\frac{k}{1-p}I_{[r_{t-i}>0]} + \frac{k}{p}I_{[r_{t-i}<0]}\right)|r_{t-i}|$$

where k is defined as in (8), and λ is a decay factor, the choice of which will be discussed later on. Through iteration, this can be re-expressed as

$$\sigma_{t+1} = \lambda\sigma_t + (1-\lambda)\left(\frac{k}{1-p}I_{[r_t>0]} + \frac{k}{p}I_{[r_t<0]}\right)|r_t| \tag{14}$$

Clearly, if $p = 0.5$, corresponding to the symmetric Laplace distribution, then (14) reduces to (4). However, if $p \neq 0.5$, then the contribution of the positive/negative values of r_t to σ_{t+1} is quite different—allowing the effects of 'good news' ($r_t > 0$) and 'bad news' ($r_t < 0$), well characterized in (14), to be different and thus capture volatility asymmetry. Further, this skewed-EWMA estimate in (14) is a special case of the first-order threshold GARCH (TGARCH) model of Zakoian (1994), which has the form

$$\sigma_{t+1} = \alpha_0 + \alpha_1^+ r_t^+ + \alpha_1^- r_t^- + \beta_1\sigma_t \tag{15}$$

where $r_t^+ = |r_t|I_{[r_t>0]}$, $r_t^- = |r_t|I_{[r_t<0]}$, and $\alpha_0, \alpha_1^+, \alpha_1^-$ and β_1 are parameters, satisfying $\alpha_0 \ge 0$, $\alpha_1^+ \ge 0$, $\alpha_1^- \ge 0$ and $\beta_1 \ge 0$. Obviously, if $\alpha_0 = 0$, $\alpha_1^+ = \frac{k(1-\lambda)}{1-p}$, $\alpha_1^- = \frac{k(1-\lambda)}{p}$ and $\beta_1 = \lambda$, then the TGARCH(1,1) model (15) is equivalent to the skewed-EWMA estimate of volatility in (14).

The parameter p in the skewed-EWMA, (14), is constant. In this case, the conditional return skewness and kurtosis, defined in (9), also remain constant. In the subsequent sections, we call this the skewed-EWMA (fix p) model. An alternative extended time-varying EWMA-based estimate for p is now proposed.

First, we define two EWMA estimates for u and v from (12), as

$$u_{t+1} = (1-\beta_u)\sum_{i=0}^{\infty}\beta_u^i|r_{t-i}|I_{[r_{t-i}>0]},$$
$$v_{t+1} = (1-\beta_v)\sum_{i=0}^{\infty}\beta_v^i|r_{t-i}|I_{[r_{t-i}<0]}$$

where β_u and β_v are two EWMA decay factors, different from the decay factor λ used in (14), to be further specified below. The iterative forms for u_{t+1} and v_{t+1} are thus

$$\begin{aligned} u_{t+1} &= \beta_u u_t + (1-\beta_u)|r_t| I_{[r_t>0]}, \\ v_{t+1} &= \beta_v v_t + (1-\beta_v)|r_t| I_{[r_t<0]} \end{aligned} \tag{16}$$

Then, employing (11) an EWMA-based estimate for p_{t+1} follows:

$$p_{t+1} = \frac{1}{1+\sqrt{u_{t+1}/v_{t+1}}} \tag{17}$$

With p now time-varying, with dynamics implied from (16), the conditional skewness and kurtosis, defined in (9), will be time-varying and potentially able to adapt to the changing nature of financial returns (see also Guermat and Harris, 2002, for time-varying kurtosis, but no skewness).

The intuition regarding (16) follows: u_{t+1} and v_{t+1} are the EWMA estimates of the sum of the positive and negative parts of the historical returns up to time t, respectively. We let the data speak for whether or not u_t and v_t are similar in value and thus p_t is close to 0.5. This is different from Guermat and Harris (2001), who assume $u_t \equiv v_t$ at all times and thus take $p_t \equiv 0.5$. The proposed method should be more flexible and sensible.

The proposed full skewed-EWMA-based estimate of VaR is, from (13),

$$\sigma_{t+1} = \lambda\sigma_t + (1-\lambda)\left(\frac{k_{t+1}}{1-p_{t+1}} I_{[r_t>0]} + \frac{k_{t+1}}{p_{t+1}} I_{[r_t<0]}\right)|r_t| \tag{18}$$

where $k_{t+1} = k(p_{t+1}) = \sqrt{p_{t+1}^2 + (1-p_{t+1})^2}$, and in view of (13),

$$\mathrm{VaR}_\alpha = \begin{cases} \sigma_{t+1}\frac{p_{t+1}}{k_{t+1}} \log\left(\frac{\alpha}{p_{t+1}}\right); \ 0 \le \alpha < p_{t+1} \\ -\sigma_{t+1}\frac{(1-p_{t+1})}{k_{t+1}} \log\left(\frac{1-\alpha}{1-p_{t+1}}\right); \ p_{t+1} \le \alpha < 1 \end{cases} \tag{19}$$

Applying these formulas and the MLEs above, we can calculate the skewed-EWMA forecasts of VaR under a conditional ALD with dynamic moments and shape.

Just as JP Morgan's standard-EWMA (2) is a special case of the GARCH(1,1) model of Bollerslev (1986), this skewed-EWMA estimate, when p is constant, is a special (extended) case of the standard deviation T-GARCH(1,1) model of Zakoian (1994), which in general reduces the number of parameters (four parameters in (15)). In fact, in our empirical application below, we find we can further take $\beta_u = \beta_v = \beta$ for simplicity, to reduce the number of decay factors in the skewed-EWMA forecasting, which thus has only two decay factors: β in (16) and λ in (18). We update the value of the parameter p by an EWMA-based procedure, so our proposed skewed-EWMA in (18) is a varying-coefficient T-IGARCH model that can adjust the skewness and kurtosis conveniently in calculating the VaR by the EWMA-based procedure and also is more efficient and automatically adaptive in modelling the real time-varying financial system.

Maximum likelihood estimation

In the skewed-EWMA estimate of VaR proposed above, there are important decay factors λ and β that need to be specified appropriately in application. We suggest applying the likelihood principle to estimate these decay factors.

Based on a return series of size T, $r_1, \ldots, r_T$, that is conditionally AL, the likelihood, conditional on r_1, is

$$f(r_2,\ldots,r_T|r_1) = \prod_{t=2}^{T} \frac{k_t}{\sigma_t} \exp\left\{-\left(\frac{1}{1-p_t} I_{[r_t>0]} + \frac{1}{p_t} I_{[r_t<0]}\right)\frac{k_t}{\sigma_t}|r_t|\right\} \tag{20}$$

where p_t is a function of $\boldsymbol{\beta} = (\beta_u, \beta_v)$ and σ_t is a function of $\lambda, \boldsymbol{\beta}$; i.e. $p_t = p_t(\boldsymbol{\beta})$, $\sigma_t = \sigma_t(\lambda, \boldsymbol{\beta})$, and $k_t = \sqrt{(1-p_t)^2 + p_t^2}$. For simplicity, we take $r_1 = 0$. Therefore, the likelihood, based on the ALD, depends upon both λ and β. As usual, we consider the log-likelihood

$$\mathrm{LL}(\lambda, \boldsymbol{\beta}) = \sum_{t=2}^{T}\left\{\ln k_t - \ln\sigma_t - \left(\frac{1}{1-p_t} I_{[r_t>0]} + \frac{1}{p_t} I_{[r_t<0]}\right)\frac{k_t}{\sigma_t}|r_t|\right\} \tag{21}$$

We take the maximizers, $\hat{\lambda}$ and $\hat{\boldsymbol{\beta}}$ of (21) as the estimated decay factors, satisfying

$$\mathrm{LL}\left(\hat{\lambda}, \hat{\boldsymbol{\beta}}\right) \ge \mathrm{LL}(\lambda, \boldsymbol{\beta}), \forall(\lambda, \boldsymbol{\beta}) \in \Theta \tag{22}$$

where Θ is the parameter space of $(\lambda, \boldsymbol{\beta})$, which is a unit square. To find the MLEs, the function 'fminunc' was used in Matlab, under default conditions. Since both $(\lambda, \boldsymbol{\beta})$ are restricted (to [0, 1]), a logistic transformation of each was applied during optimization, i.e. the unconstrained MLE for x_1, x_2 is found, where $\lambda = \exp(x_1)(1 + \exp(x_1))$ and $\beta = \exp(x_2)(1 + \exp(x_2))$.

Any model selection can be made by using Akaike's information criterion: $\text{AIC} = -2 \times \text{LL}(\lambda_0, \boldsymbol{\beta}_0) + 2d$, where d is the number of the parameters in $(\lambda, \boldsymbol{\beta})$.

FORECASTING AND EVALUATION

This section considers forecast accuracy for VaR models and then presents and discusses the empirical forecasting study over nine financial return series.

Forecast accuracy for VaR methods

Like volatility, VaR is unobservable for real data, so testing for accuracy of VaR must be done indirectly (i.e. not comparing to the true VaR). In the relevant literature, back-testing techniques are common, including two likelihood ratio (LR) tests: one for unconditional coverage (UC) (Kupiec, 1995) and two joint tests of independence of violations (returns more extreme than the quantile forecast) and conditional coverage, the conditional coverage (CC) test of Christofferson (1998) and the dynamic quantile (DQ) test of Engle and Manganelli (2004). Since the DQ test is well known to be more powerful than the CC test (see, for example, Berkowitz *et al.* (2012), we focus only on the UC and DQ tests here.

1. *Unconditional coverage test.* The most basic requirement for a VaR model is that the proportion of violations, i.e. the number of returns more extreme than the VaR, should be close to , or equal to, the nominal level α; i.e. a VaR model should facilitate a correct unconditional coverage (UC). The proportion of violations is called the violation rate (VRate). Kupiec (1995) proposed an LR test that the true VRate $= \alpha$, based on the fact that the number (N) of violations, in a sample of size T, should be binomially distributed, and hence the LR statistic is
$$LR_u = -2 \ln L_\alpha + 2 \ln L_N \tag{23}$$
where $L_\alpha = (1-\alpha)^{T-N}\alpha^N, L_N = (1 - N/T)^{T-N}(N/T)^N$. As T tends to infinity, LR_u is asymptotically $\chi^2(1)$ distributed, under the null hypothesis that the true VR $= \alpha$ and violations are independent.
2. *Dynamic quantile test.* This is a joint test of independence and correct coverage. It employs a regression-based model of the violation-related variable 'hits', defined as $I(r_t < \text{VaR}_t) - \alpha = I_t - \alpha$, which will on average be α, if unconditional coverage is correct. A regression-type test is then employed to examine whether the 'hits' are correlated to lagged 'hits', lagged VaR forecasts, or some other relevant regressors, over time; a model producing accurate and independent violations and 'hits' will not be related to these lagged variables. The DQ test statistic simultaneously tests that the average 'hit' is α and that 'hits' are not correlated over time and/or related to other lagged regressors, using a statistic that is very much like the usual regression F-test, and which is asymptotically a $\chi^2(c)$, where c is the number of regressors in the model for 'hits'.

We now examine and compare the empirical performance of the proposed skewed-EWMA method, with a range of competitors, using some real financial return series. The methods considered include the standard-EWMA (RiskMetrics), robust-EWMA, GARCH(1,1) and GJR-GARCH models with Gaussian and Student-t errors, as well as the semi-parametric symmetric absolute value (SAV) and asymmetric slope (AS) CAViaR models: Engle and Manganelli (2004). To examine whether higher moments change with time or not, we also consider the skewed-EWMA model with a fixed value of p.

The standard-EWMA, used in RiskMetrics, has been close to an industry standard in financial risk management. It may be fair to say that it is the RiskMetrics system that made VaR a popular measure of market risk in the financial industry. In this section, we demonstrate that our skewed-EWMA viably outperforms it and does at least as well as other competitors considered here.

Data

We consider and examine VaR forecasting for nine financial return series, including three exchange rates: the British pound/US dollar, Japanese yen/US dollar and Australian/US dollar (denoted by BP, JY and AD, respectively) and six stock exchange indices: US S&P500, Hong Kong Hang Seng index, Shanghai Composite Index, Australian All Ordinaries, UK FTSE100 and the Japan Nikkei (denoted by US, HK, SH, AU, UK and JP, respectively). The full datasets run for 12 years: from 1 January 1999 to 6 January 2011. The last four years of each series are used as a forecast sample, being January 2007 to January 2011, while the first eight years are the initial training sample. The stock indices were obtained from the Yahoo Finance website, while the exchange rates were obtained via Thomson Reuters tick history or the OANDA website. We are concerned with the daily percentage return series, defined by $100 \times (\ln P_t - \ln P_{t-1})$, where P_t is the closing value on day t. Days where there was no trading in a market were removed, on an individual market basis. Thus the sample sizes in each market are not the same.

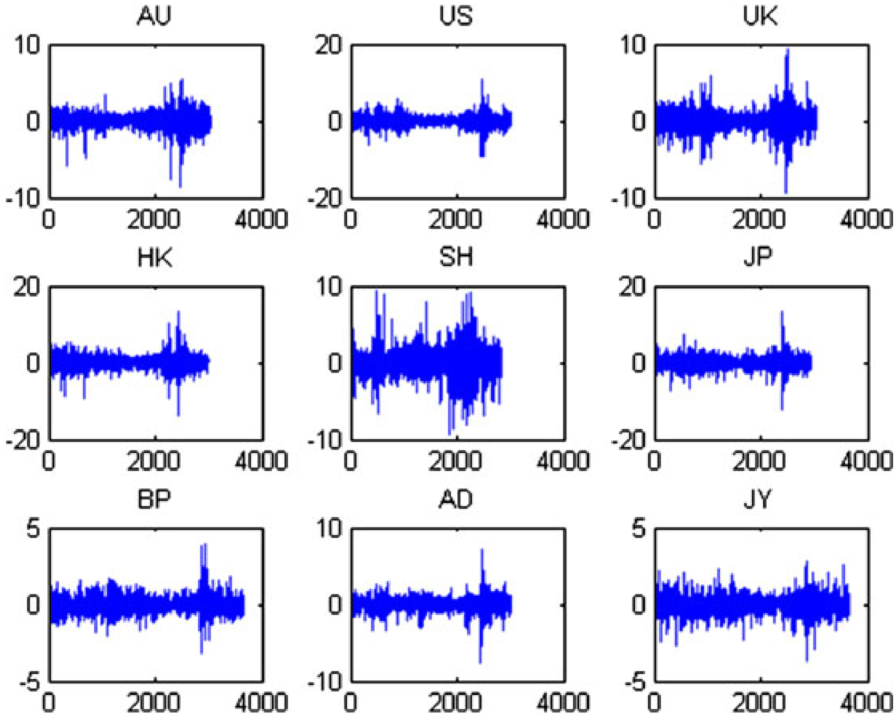

Figure 1. Nine return series from January 1999 to January 2011

Table I. Summary statistics for in-sample and out-of-sample observations over the nine asset return series

Market	In-sample: 1999–2006				Out-of-sample: 2007–2010			
	Mean	SD	Skew	Kurtosis	Mean	SD	Skew	Kurtosis
AU	0.034	0.706	−0.65	7.53	−0.015	1.408	−0.47	6.66
US	0.007	1.131	0.10	5.27	−0.011	1.727	−0.20	9.89
UK	0.003	1.136	−0.15	5.69	−0.003	1.595	−0.06	8.92
HK	0.035	1.362	−0.27	6.28	0.017	2.136	0.10	8.98
JP	0.011	1.394	−0.11	4.60	−0.050	1.921	−0.37	10.07
SH	0.039	1.363	0.58	8.27	0.006	2.188	−0.37	4.83
BP	−0.014	0.437	0.01	4.51	0.017	0.546	0.69	10.11
AD	0.013	0.696	−0.26	4.47	0.023	1.127	−0.36	9.81
JY	0.002	0.483	−0.07	5.64	−0.025	0.532	−0.25	7.52

Figure 1 shows time plots of the nine series, over the entire period from January 1999 to January 2011. All the series seem to have high-volatility periods towards their end, in each case corresponding to the effects of the GFC. Table I shows summary statistics for each of the nine series, including the mean, standard deviation (SD), skewness and kurtosis, separately based on either the in-sample or out-of-sample period data, in each case. Clearly the coefficients of skewness and kurtosis of the six return series are mostly quite large, compared to the Gaussian's 0 and 3, indicating that the distributions of the six return series are non-Gaussian. This is confirmed in all cases by a Jarque–Bera test having p-values close to 0 (not shown to save space). Similarly, Ljung–Box tests applied to squared returns (at 5 lags) all returned p-values close to 0, indicating significant heteroskedasticity via ARCH effects in each series.

Comparing in-sample to out-of-sample periods, in seven of the nine series, the average return was lower in the out-of-sample period, while in all series the level of variation was higher in the out-of-sample period. This is not surprising given that the forecast sample period includes the GFC, and the higher volatility is evident in each series in Figure 1.

Parameter estimation results

For the datasets under study, according to the AIC, we find that we can take $\beta_u = \beta_v$ in each series, so as to reduce the number of parameters. This is reasonable as both $\beta_u = \beta_v$ are in general quite large and close to 1. Thus we denote $\beta_u = \beta_v = \beta$ below. Table II shows the MLEs, and their standard errors (SE), for λ and β using only each initial training sample. Further, we include the skewed-EWMA model when the shape p is fixed and not time-varying, in this study. From this model, the MLEs for p are also shown (the MLEs for λ were the same, to three decimal places, as those for full skewed-EWMA, and so are not shown).

Table II. Skewed-EWMA parameter estimates over the nine asset return series

Market	λ		β		p (fixed)	
	MLE	SE	MLE	SE	MLE	SE
AU	0.944	0.011	0.998	0.0031	0.479	0.008
US	0.956	0.009	1.000	0.00002	0.492	0.008
UK	0.930	0.011	0.998	0.0025	0.493	0.008
HK	0.972	0.008	1.000	0.0002	0.487	0.008
JP	0.957	0.011	0.998	0.0031	0.492	0.008
SH	0.928	0.012	0.995	0.0029	0.491	0.009
BP	0.985	0.005	1.000	0.0001	0.512	0.008
AD	0.963	0.008	1.000	0.0003	0.492	0.008
JY	0.985	0.005	1.000	0.0001	0.499	0.008

The training sample data clearly indicate that the variables u and v may not change over time, and hence that the shape parameter p also may not change over time, since the MLEs for β are all close to 1 (or equal 1–3 decimal places); thus we include the skewed-EWMA model with fixed p in the group of methods under consideration in the forecast study below. From this fixed p model, the MLEs for p are all close to 0.5 and not significantly different from 0.5 in all series (except Australia). Naturally, this aspect may change during the forecast sample, especially during a crisis period.

Forecast accuracy results

For the EWMA-based approaches, robust EWMA and skewed EWMA, we consider two approaches to parameter estimation in forecasting. First, as is done in JP Morgan's RiskMetrics system with a fixed decay factor of 0.94, we also fix the parameters $\beta = \beta_u = \beta_v$ and λ equal to their MLEs in each market, estimated only over the initial training sample (January 1999 to December 2006), for calculation of VaR in the empirical analysis below, over the entire forecast sample. Results from this method are labelled 'no up' in the subsequent tables. Not re-estimating parameters is clearly not standard practice, and likely not optimal either, in forecasting, where parameters are often re-estimated or updated when generating each new day's forecast. We also consider the more usual technique, daily re-estimation, with all the forecast methods considered. However, given that financial institutions often generate VaR for thousands of assets and portfolios daily, if acceptable forecast accuracy could be achieved using parameter values, e.g. as in RiskMetrics, this may save an enormous amount of time for these calculations for institutions.

Since VaR is most often concerned with the measurement of downside risk, we focus on three lower quantile levels: 5%, 1% and 0.5%. Table III shows the violation rates (VRates) for each model in each series at $\alpha = 0.005$. Models with VRate ratios closer to α have achieved closer to nominal coverage. Bolding in this table indicates a VRate where the UC test rejected the nominal coverage rate, while a box indicates the model with VRate closest to $\alpha = 0.005$.

A cursory glance at the results in Table III reveals that, for $\alpha = 0.005$, the RiskMetrics model significantly underestimates risk levels in all markets during the forecast sample, except the USGB exchange rate, as do the GARCH and GJR with Gaussian error models. Underestimation of VaR will result in too many violations and violations rates that are above $\alpha = 0.005$. Both CAViaR models also significantly underestimate risk levels in seven of the nine series. Only models with Student-t errors, plus the rob-EWMA and skewed-EWMA sets of models, seem to be able

Table III. Sample estimates of VRate = $\hat{\alpha}$ at $\alpha = 0.005$

$\alpha = 0.005$	AU	US	UK	HK	JP	SH	BP	AD	JY
RiskMetrics	**0.015**	**0.019**	**0.017**	**0.011**	**0.014**	**0.019**	$\boxed{0.006}$	**0.016**	**0.019**
Rob-EWMA	0.002	$\boxed{0.004}$	0.003	**0.000**	0.003	$\boxed{0.006}$	**0.000**	$\boxed{0.005}$	0.008
Rob-EWMA (no up)	0.002	$\boxed{0.004}$	$\boxed{0.004}$	**0.000**	0.003	0.008	**0.000**	$\boxed{0.005}$	0.008
Sk-EWMA	$\boxed{0.004}$	0.002	0.002	**0.001**	0.004	0.007	**0.002**	$\boxed{0.005}$	**0.010**
Sk-EWMA (no up)	$\boxed{0.004}$	0.003	0.002	**0.000**	0.004	0.009	**0.000**	$\boxed{0.005}$	**0.010**
Sk-EWMA (fix p)	$\boxed{0.004}$	0.003	0.003	0.002	0.004	**0.010**	**0.002**	$\boxed{0.005}$	**0.010**
GARCH-n	**0.016**	**0.016**	**0.018**	**0.011**	**0.012**	**0.023**	$\boxed{0.006}$	**0.017**	**0.017**
GJR-n	**0.018**	**0.014**	**0.020**	**0.012**	**0.010**	**0.020**	0.008	**0.018**	**0.016**
GARCH-t	0.010	0.007	**0.015**	$\boxed{0.006}$	**0.010**	0.007	**0.001**	**0.012**	$\boxed{0.005}$
GJR-t	**0.011**	0.010	**0.016**	$\boxed{0.006}$	0.007	0.007	**0.001**	**0.012**	$\boxed{0.005}$
SAV-CAV	**0.013**	**0.016**	**0.011**	$\boxed{0.006}$	$\boxed{0.005}$	**0.014**	**0.014**	**0.022**	**0.010**
AS-CAV	**0.011**	**0.017**	**0.016**	0.009	0.007	**0.014**	**0.014**	**0.023**	**0.012**

Note: Boxes indicate proportion closest to $\boldsymbol{\alpha = 0.005}$ in that market; bold indicates that the model is rejected by the UC test (at 5% level), for each market; 'no up' indicates that parameters are estimated once, then not updated, during the forecast period.

Table IV. Sample estimates of VRate $= \hat{\alpha}$ at $\alpha = 0.01$

$\alpha = 0.01$	AU	US	UK	HK	JP	SH	BP	AD	JY
RiskMetrics	**0.020**	**0.032**	**0.025**	0.016	**0.021**	**0.023**	0.011	**0.024**	**0.028**
Rob-EWMA	0.005	0.008	0.008	0.005	0.008	0.014	**0.003**	0.005	0.014
Rob-EWMA (no up)	0.005	0.008	0.008	0.005	0.008	0.014	**0.003**	0.005	0.012
Sk-EWMA	$\boxed{0.008}$	0.008	0.008	0.006	$\boxed{0.009}$	$\boxed{0.013}$	**0.003**	$\boxed{0.009}$	0.012
Sk-EWMA (no up)	$\boxed{0.008}$	$\boxed{0.009}$	0.008	0.006	0.008	0.015	**0.002**	0.005	0.012
Sk-EWMA (fix p)	$\boxed{0.008}$	$\boxed{0.009}$	$\boxed{0.010}$	0.006	$\boxed{0.009}$	0.014	**0.002**	$\boxed{0.009}$	0.012
G-n	**0.024**	**0.029**	**0.024**	0.015	**0.017**	**0.032**	0.008	**0.023**	**0.027**
GJR-n	**0.027**	**0.028**	**0.029**	0.016	**0.018**	**0.032**	$\boxed{0.010}$	**0.024**	**0.026**
G-t	0.017	**0.021**	**0.020**	$\boxed{0.013}$	0.014	0.015	**0.002**	**0.019**	$\boxed{0.010}$
GJR-t	**0.024**	**0.020**	**0.026**	$\boxed{0.013}$	0.015	0.015	**0.005**	**0.020**	0.009
CAViaR	0.015	**0.028**	**0.018**	0.014	0.015	**0.024**	**0.033**	**0.028**	**0.016**
AS-CAV	**0.023**	**0.032**	**0.022**	0.014	0.014	**0.023**	**0.033**	**0.027**	**0.016**

Note: Boxes indicate VRate closest to $\alpha = 0.01$ in that market; bold indicates that the model is rejected by the UC test (at 5% level), for each market.

Table V. Sample estimates of VRate $= \hat{\alpha}$ at $\alpha = 0.05$

$\alpha = 0.05$	AU	US	UK	HK	JP	SH	BP	AD	JY
RiskMetrics	0.058	**0.068**	0.061	0.060	**0.073**	0.060	$\boxed{0.045}$	0.053	**0.068**
R-EWMA	0.049	**0.068**	0.051	0.046	0.063	0.054	**0.035**	0.048	**0.071**
R-EWMA(no up)	$\boxed{0.050}$	$\boxed{\mathbf{0.065}}$	0.051	0.043	0.060	$\boxed{0.053}$	**0.033**	0.046	**0.068**
Sk-EWMA	0.049	**0.067**	$\boxed{0.049}$	0.043	$\boxed{0.054}$	0.057	**0.032**	0.048	**0.069**
Sk-EWMA(no up)	0.053	$\boxed{\mathbf{0.065}}$	0.052	0.047	0.059	0.054	**0.031**	0.047	**0.069**
Sk-EWMA(fix p)	0.054	**0.069**	0.055	$\boxed{0.052}$	0.058	$\boxed{0.053}$	**0.031**	$\boxed{0.050}$	**0.069**
G-n	0.064	**0.068**	0.061	0.062	**0.071**	0.065	0.040	0.054	**0.068**
GJR-n	**0.074**	**0.070**	**0.066**	**0.067**	**0.070**	0.062	**0.036**	0.057	**0.069**
G-t	**0.066**	**0.069**	0.062	**0.065**	**0.068**	**0.078**	0.044	0.057	**0.071**
GJR-t	**0.081**	**0.070**	**0.065**	**0.071**	**0.069**	**0.079**	0.044	0.060	**0.071**
CAViaR	**0.071**	**0.071**	0.060	0.056	**0.065**	**0.074**	**0.076**	**0.067**	$\boxed{0.066}$
AS-CAV	**0.075**	**0.076**	0.063	**0.070**	0.062	**0.072**	**0.087**	**0.074**	**0.074**

Note: Boxes indicate VRrate closest to $\alpha = 0.05$ in that market; bold indicates that the model is rejected by the UC test (at 5% level), for each market.

to accurately forecast risk levels for $\alpha = 0.005$ consistently in these nine series. In four of the nine series a robust-EWMA model ranked closest to the nominal rate of 0.5%. Clearly, fat tails are important when considering such an extreme quantile level.

Table IV shows violation rates for $\alpha = 0.01$. These reveal that again the RiskMetrics, GARCH and GJR with Gaussian errors (except for the HK returns and USGB rate data) and both CAViaR models underestimate risk levels in all markets, significantly in most. Again, only the models with Student-t errors, plus the rob-EWMA and skewed-EWMA models, seem able to accurately forecast risk levels for $\alpha = 0.01$ reasonably consistently across these series. In six of the nine series a skewed-EWMA model ranked closest to the nominal rate of 1%.

The results in Table V for $\alpha = 0.05$ reveal that the RiskMetrics, GARCH with Gaussian errors and the rob-EWMA and skewed-EWMA models all seem to be able to accurately forecast risk levels for $\alpha = 0.05$ more consistently in these series. However, the GARCH with Student-t error and CAViaR models mostly significantly underestimated risk levels across the series. Further, all models were rejected for having too high a violation rate in the US asset return and USJP exchange rate series. In six of the nine series a skewed-EWMA model ranked closest to the nominal rate of 5%.

At this point we assess each model's performance via the UC and DQ tests. Table VI shows counts of the number of rejections via each test, at a 5% significance level, over the nine series, for each model. Clearly, the robust-EWMA and skewed-EWMA models are among the least rejected models at each level of risk. These models are typically rejected by either or both tests for three or four of the series. It is difficult to separate or distinguish the performance among the five rob-EWMA and skewed-EWMA formulations for any risk level. However, it is clear that these five methods are rejected less than the other models as a group at each risk level. The RiskMetrics does very well, and equally as well as the rob-EWMA and skewed-EWMA models, for $\alpha = 0.05$ only, but quite poorly for the other risk levels. Clearly the GARCH and GJR formulations cannot accurately forecast either the level of risk (UC test) or the dynamics of changing risk (DQ test) in these nine markets, using either Gaussian or Student-t errors, over the data period considered. Models with Student-t errors do marginally better than Gaussian error models at $\alpha = 0.005, 0.01$ but not at 0.05.

Table VI. Counts of model rejections, at the 5% significance level, at risk levels $\alpha = 0.005,\ 0.01,\ 0.05$

Test	$\alpha = 0.005$		$\alpha = 0.01$		$\alpha = 0.05$	
	UC	DQ	UC	DQ	UC	DQ
RiskMetrics	**8**	7	**7**	**7**	$\boxed{3}$	$\boxed{3}$
R-EWMA	$\boxed{2}$	$\boxed{1}$	$\boxed{1}$	4	$\boxed{3}$	$\boxed{3}$
R-EWMA(no up)	$\boxed{2}$	2	$\boxed{1}$	$\boxed{3}$	$\boxed{3}$	$\boxed{3}$
Sk-EWMA	3	2	$\boxed{1}$	$\boxed{3}$	$\boxed{3}$	4
Sk-EWMA(no up)	3	3	$\boxed{1}$	4	$\boxed{3}$	$\boxed{3}$
Sk-EWMA(fix p)	3	3	$\boxed{1}$	$\boxed{3}$	$\boxed{3}$	$\boxed{3}$
G-n	**8**	7	**7**	**7**	4	6
GJR-n	**8**	**8**	**7**	**7**	**7**	5
G-t	3	3	4	6	6	6
GJR-t	4	5	5	5	**7**	**7**
CAViaR	7	7	6	5	**7**	6
AS-CAV	7	7	**7**	**7**	**7**	**7**

Note: Boxes indicate the favoured model; bold indicates the least favoured model. The DQ column shows the maximum number of rejections over the lag 1 and lag 2 DQ tests.

Table VII. Summary statistics for VRate $= \hat{\alpha}$ at risk levels $\alpha = 0.005,\ 0.01,\ 0.05$

Test	$\alpha = 0.005$		$\alpha = 0.01$		$\alpha = 0.05$	
	Mean	Std.	Mean	Std.	Mean	Std.
RiskMetrics	0.0149	0.0113	0.0220	0.0141	0.0607	0.0142
R-EWMA	0.0034	$\boxed{0.0031}$	0.0078	0.0047	0.0540	0.0123
R-EWMA(no up)	0.0037	0.0032	0.0075	0.0045	0.0523	$\boxed{0.0113}$
Sk-EWMA	0.0040	$\boxed{0.0031}$	0.0085	$\boxed{0.0035}$	$\boxed{0.0522}$	0.0118
Sk-EWMA(no up)	0.0041	0.0036	0.0081	0.0045	0.0532	0.0118
Sk-EWMA(fix p)	$\boxed{0.0047}$	0.0032	$\boxed{0.0089}$	0.0037	0.0547	0.0124
G-n	**0.0150**	**0.0116**	0.0220	0.0147	0.0615	0.0154
GJR-n	0.0149	0.0113	0.0232	**0.0157**	0.0634	0.0182
G-t	0.0080	0.0053	0.0145	0.0075	0.0645	0.0181
GJR-t	0.0082	0.0056	0.0162	0.0095	0.0679	0.0219
CAViaR	0.0122	0.0092	0.0211	0.0137	0.0674	0.0195
AS-CAV	0.0136	0.0103	**0.0225**	0.0150	**0.0724**	**0.0249**

Note: Boxes indicate the favoured model; bold indicates the least favoured model. The DQ column shows the maximum number of rejections over the lag 1 and lag 2 DQ tests; 'Std.' is the square root of the average squared distance from alpha.

Table VII summarizes the results in Tables III–V regarding the VRates at each risk level. This table may allow us to distinguish among the models more finely than the statistical tests did. The table shows average ('Mean') of the VRates across the nine series for each model. Models with these quantities closest to α are preferred and are boxed. Further, the column labelled 'Std.' shows the square root of the average squared distance between the nine VRates and α, the desired value. This is effectively a standard deviation around α. As noted above, results are quite consistent across the three risk levels $\alpha = 0.005,\ 0.01,\ 0.05$. The Riskmetrics, GARCH and GJR with Gaussian and Student-t errors and both CAViaR models consistently under-forecast risk levels and have typical VRates above α and furthest from α in standard deviation. At $\alpha = 0.005,\ 0.01$ models with Student-t errors forecast more accurately than models with Gaussian errors, as expected, but this result is reversed at $\alpha = 0.05$.

Table VII shows that the five rob-EWMA and skewed-EWMA specifications are always closer to α in mean and standard deviation of VRates taken over the nine markets, at all risk levels. None of these five models consistently outperforms the others by these measures. At $\alpha = 0.005$, the skewed-EWMA model with a fixed shape parameter p had an average VRate closest to α, with standard deviation third smallest (behind the dynamic skewed-EWMA with parameter updating and the rob-EWMA with updating). The two rob-EWMA models both ranked first in three out of nine markets with VRates closest to α.

At $\alpha = 0.01$, the skewed-EWMA model with a fixed shape parameter p had average VRate closest to α, with standard deviation second smallest (behind the dynamic skewed-EWMA with parameter updating). The skewed-EWMA model with a fixed shape ranked first in five out of nine markets, while the dynamic skewed-EWMA with updating ranked first in four markets, and had the mean VRate second closest to α of the models. These two models were each rejected in four series, which was the minimum over the models. Clearly, these two models performed best at $\alpha = 0.01$.

At $\alpha = 0.05$, the dynamic skewed-EWMA model with updating had average VRate closest to α, with standard deviation second smallest (behind the rob-EWMA with no updating). This model ranked first in two out of nine markets, while both the skewed-EWMA with fixed p and the rob-EWMA without updating ranked first in three markets. Clearly, these three models performed best at $\alpha = 0.05$.

Overall, two models seem to perform among the very best over the nine series at each of the three risk levels: the dynamic skewed-EWMA with parameter updating and the skewed-EWMA with fixed p. The dynamic skewed-EWMA did outperform the fixed p model at $\alpha = 0.05$, with highly comparable performances at $\alpha = 0.005, 0.01$. All measures considered, these two models marginally outperformed the robust-EWMA with no parameter updating, especially for forecasting VaR at $\alpha = 0.01$, and significantly outperformed all GARCH and CAViaR models, and RiskMetrics, at all risk levels on virtually all measures considered.

Performance of models during crisis and non-crisis periods

The forecast sample period covers the well-known GFC that, by all accounts, started in 2008, or perhaps originated in 2007. The performance of the models may vary between the financial-crisis period and the pre- and post-crisis periods. We thus present a pre- and post-crisis comparison of the models' VaR forecasting performance with their performance during the crisis.

A specific date for the start of the crisis must be chosen, but there is little reason for this date to be exactly the same in each market. From news media accounts and Wikipedia, it is largely agreed that the effects of the crisis are initially apparent during September and/or October:2008; however, there are clear effects of the 'credit crunch' in the data for the Australian market, starting in January 2008. To make a clear metric in each market, we defined the crisis period as the set of consecutive trading days, being at least 200 days in length, and not within 100 days from the start or the end of the forecast sample, that had the largest sample variance of returns. This means that in each series the 'crisis' period could not start in the first half of 2007, and that it must be finished before the second half of 2010. This resulted in the 'crisis' periods being chosen as in Table VIII. The remaining data at the start and end of each series was joined together to form the 'non-crisis' periods.

Table IX shows the violation rates for the non-crisis periods and the crisis period for $\alpha = 0.005$. These rates are summarized in Tables XII and XIII. The results for the non-crisis periods are highly consistent with that of the whole forecast sample, as expected, since there is large overlap. Clearly, the set of five rob-EWMA and skewed-EWMA models all do best in terms of accurately forecasting risk levels, assessed informally and by the UC test, at $\alpha = 0.005$. None of these five models can be rejected during the crisis period, and are rejected in the least number of series during the non-crisis periods. A skewed-EWMA model had closest VRate to $\alpha = 0.005$, both in crisis and non-crisis periods, and in terms of average and standard deviation, and these models performed about as well during non-crisis and crisis periods. On the contrary, all other models underestimated risk, leading to double or more numbers of violations than expected in all periods, and all these other models also did worse during the crisis than the non-crisis periods.

Table X shows VRates for the non-crisis periods and the crisis period for $\alpha = 0.01$. These rates are summarized in Tables XII and XIII. The results for the non-crisis periods are again highly consistent with that of the whole forecast sample. The set of five rob-EWMA and skewed-EWMA models again do best in terms of forecast accuracy, assessed informally and by the UC test, at $\alpha = 0.01$. These five models are rejected the least number of times during the crisis and non-crisis periods. The full skewed-EWMA model and the Rob-EWMA, with no updating of parameter estimates, do the best in terms of average VRate close to $\alpha = 0.01$; again these five models performed equally well during non-crisis and crisis periods. On the contrary, all other models underestimated risk, leading to one and a half times or more actual compared to expected numbers of violations, again performing worse during the crisis period.

Table XI shows VRates for separate periods for $\alpha = 0.05$; again summarized in Tables XII and XIII. Similar comments to those for $\alpha = 0.005, 0.01$ apply. The set of five rob-EWMA and skewed-EWMA models again do best in terms of forecast accuracy and are rejected the least number of times during all periods. The Rob-EWMA, with no

Table VIII. Crisis periods in each series (day/month/year)

Market	Crisis period	
	Start	End
AU	21/01/2008	2/12/2008
US	4/09/2008	22/06/2009
UK	19/08/2008	5/06/2009
HK	21/08/2008	10/06/2009
JP	22/07/2008	20/05/2009
SH	21/01/2008	19/11/2008
BP	9/08/2008	6/03/2009
AD	3/09/2008	24/06/2009
JY	21/05/2008	20/12/2008

Table IX. Sample estimates of VRate $= \hat{\alpha}$ at $\alpha = 0.005$ in non-crisis periods and in the crisis period

Non-crisis	AU	US	UK	HK	JP	SH	BP	AD	JY
RiskMetrics	**0.018**	**0.020**	**0.017**	0.010	**0.014**	**0.018**	$\boxed{0.005}$	**0.015**	**0.016**
Rob-EWMA	0.003	0.004	0.004	**0.000**	0.003	$\boxed{0.005}$	**0.000**	$\boxed{0.005}$	0.007
Rob-EWMA (no up)	0.003	0.004	$\boxed{0.005}$	**0.000**	0.003	0.008	**0.000**	$\boxed{0.005}$	0.007
Sk-EWMA	$\boxed{0.005}$	0.001	0.003	0.001	$\boxed{0.004}$	0.007	**0.000**	$\boxed{0.005}$	0.010
Sk-EWMA (no up)	$\boxed{0.005}$	0.003	0.003	**0.000**	$\boxed{0.004}$	0.009	**0.000**	$\boxed{0.005}$	0.009
Sk-EWMA (fix *p*)	$\boxed{0.005}$	0.003	0.004	0.003	$\boxed{0.004}$	0.010	**0.000**	$\boxed{0.005}$	0.010
GARCH-n	**0.015**	**0.016**	**0.017**	0.010	**0.012**	**0.022**	0.003	**0.016**	**0.015**
GJR-n	**0.016**	**0.014**	**0.018**	**0.011**	0.009	**0.021**	0.004	**0.016**	**0.014**
GARCH-t	0.010	$\boxed{0.006}$	**0.015**	0.006	0.009	0.008	**0.000**	0.010	0.004
GJR-t	**0.011**	0.010	**0.015**	0.007	0.006	0.008	**0.000**	0.010	0.004
SAV-CAV	**0.011**	**0.015**	0.010	$\boxed{0.005}$	$\boxed{0.004}$	**0.016**	0.006	**0.019**	$\boxed{0.006}$
AS-CAV	0.009	**0.015**	**0.015**	0.010	0.003	**0.013**	0.008	**0.019**	0.007
Crisis	**AU**	**US**	**UK**	**HK**	**JP**	**SH**	**BP**	**AD**	**JY**
RiskMetrics	$\boxed{0.005}$	0.015	0.015	0.015	0.015	**0.020**	0.010	**0.020**	**0.033**
Rob-EWMA	0.000	$\boxed{0.005}$	$\boxed{0.000}$	0.000	$\boxed{0.005}$	0.010	0.000	$\boxed{0.005}$	$\boxed{0.009}$
Rob-EWMA (no up)	0.000	$\boxed{0.005}$	$\boxed{0.000}$	0.000	$\boxed{0.005}$	0.010	0.000	$\boxed{0.005}$	$\boxed{0.009}$
Sk-EWMA	0.000	$\boxed{0.005}$	$\boxed{0.000}$	0.000	$\boxed{0.005}$	0.010	0.010	$\boxed{0.005}$	$\boxed{0.009}$
Sk-EWMA (no up)	0.000	$\boxed{0.005}$	$\boxed{0.000}$	0.000	$\boxed{0.005}$	0.010	0.000	$\boxed{0.005}$	0.014
Sk-EWMA (fix *p*)	0.000	$\boxed{0.005}$	$\boxed{0.000}$	0.000	$\boxed{0.005}$	0.010	0.010	$\boxed{0.005}$	$\boxed{0.009}$
GARCH-n	**0.018**	0.015	**0.020**	0.015	0.015	**0.025**	**0.020**	**0.020**	**0.033**
GJR-n	**0.023**	0.015	**0.025**	0.015	0.015	0.015	**0.025**	**0.025**	**0.028**
GARCH-t	0.009	0.010	0.015	$\boxed{0.005}$	0.015	$\boxed{0.005}$	$\boxed{0.005}$	**0.020**	$\boxed{0.009}$
GJR-t	0.009	0.010	**0.020**	0.000	0.010	$\boxed{0.005}$	$\boxed{0.005}$	**0.020**	$\boxed{0.009}$
SAV-CAV	**0.018**	**0.020**	0.015	0.010	0.010	0.010	**0.055**	**0.035**	**0.033**
AS-CAV	**0.018**	**0.025**	**0.020**	$\boxed{0.005}$	**0.025**	**0.020**	**0.050**	**0.040**	**0.038**

Note: Boxes indicate proportion closest to $\alpha = 0.005$ in that market; bold indicates that the model is rejected by the UC test (at a 5% level), for each market; 'no up' indicates that parameters are estimated once, then not updated, during the forecast period.

Table X. Sample estimates of VRate $= \hat{\alpha}$ at $\alpha = 0.01$ in non-crisis periods and in the crisis period

Non-crisis	AU	US	UK	HK	JP	SH	BP	AD	JY
RiskMetrics	**0.021**	**0.032**	**0.025**	0.016	**0.022**	**0.023**	$\boxed{0.010}$	**0.024**	**0.025**
Rob-EWMA	0.006	0.007	0.007	0.005	0.008	0.013	**0.001**	0.005	0.014
Rob-EWMA (no up)	0.006	0.007	0.007	0.005	0.008	0.014	**0.001**	0.005	$\boxed{0.010}$
Sk-EWMA	$\boxed{0.010}$	0.007	0.009	0.006	$\boxed{0.009}$	$\boxed{0.012}$	**0.000**	$\boxed{0.008}$	0.011
Sk-EWMA (no up)	$\boxed{0.010}$	$\boxed{0.009}$	0.007	0.006	0.008	0.014	**0.000**	0.005	0.011
Sk-EWMA (fix *p*)	$\boxed{0.010}$	$\boxed{0.009}$	$\boxed{0.010}$	0.006	$\boxed{0.009}$	0.013	**0.000**	$\boxed{0.008}$	0.011
GARCH-n	**0.025**	**0.026**	**0.023**	0.015	0.017	**0.029**	0.005	**0.023**	**0.023**
GJR-n	**0.027**	**0.025**	**0.028**	0.016	**0.018**	**0.031**	0.006	**0.023**	**0.023**
GARCH-t	0.016	**0.020**	0.020	$\boxed{0.012}$	0.014	0.016	**0.001**	0.018	0.008
GJR-t	**0.027**	**0.021**	**0.025**	$\boxed{0.012}$	0.015	0.017	**0.002**	**0.019**	0.007
SAV-CAV	0.015	**0.022**	0.017	0.014	0.012	**0.021**	0.017	**0.025**	0.009
AS-CAV	**0.019**	**0.030**	**0.018**	0.016	$\boxed{0.009}$	**0.020**	**0.018**	**0.024**	0.011
Crisis	**AU**	**US**	**UK**	**HK**	**JP**	**SH**	**BP**	**AD**	**JY**
RiskMetrics	$\boxed{0.014}$	**0.030**	0.025	$\boxed{0.015}$	0.020	0.020	0.020	0.025	**0.042**
Rob-EWMA	**0.000**	$\boxed{0.010}$	$\boxed{0.010}$	0.005	$\boxed{0.010}$	0.020	0.015	0.005	$\boxed{0.019}$
Rob-EWMA (no up)	**0.000**	$\boxed{0.010}$	$\boxed{0.010}$	0.005	$\boxed{0.010}$	0.015	0.015	0.005	$\boxed{0.019}$
Sk-EWMA	**0.000**	$\boxed{0.010}$	0.005	0.005	$\boxed{0.010}$	0.020	0.020	$\boxed{0.015}$	$\boxed{0.019}$
Sk-EWMA (no up)	**0.000**	$\boxed{0.010}$	$\boxed{0.010}$	0.005	$\boxed{0.010}$	0.020	$\boxed{0.010}$	0.005	$\boxed{0.019}$
Sk-EWMA (fix *p*)	**0.000**	$\boxed{0.010}$	$\boxed{0.010}$	0.005	$\boxed{0.010}$	0.020	0.015	$\boxed{0.015}$	$\boxed{0.019}$
GARCH-n	0.018	**0.040**	0.025	$\boxed{0.015}$	0.020	**0.044**	0.025	0.025	**0.047**
GJR-n	**0.040**	**0.040**	**0.030**	$\boxed{0.015}$	0.020	**0.035**	**0.030**	**0.030**	**0.047**
GARCH-t	0.018	0.025	0.020	$\boxed{0.015}$	0.015	0.015	$\boxed{0.010}$	0.025	$\boxed{0.019}$
GJR-t	**0.027**	0.015	**0.030**	$\boxed{0.015}$	0.015	$\boxed{0.010}$	0.020	0.025	$\boxed{0.019}$
SAV-CAV	$\boxed{0.014}$	**0.050**	0.020	$\boxed{0.015}$	**0.030**	**0.035**	**0.124**	**0.040**	**0.056**
AS-CAV	**0.036**	**0.040**	**0.035**	0.005	**0.035**	**0.035**	**0.119**	**0.040**	**0.042**

Note: Boxes indicate proportion closest to $\alpha = 0.01$ in that market; bold indicates that the model is rejected by the UC test (at a 5% level), for each market; 'no up' indicates that parameters are estimated once, then not updated, during the forecast period.

Table XI. Sample estimates of VRate $= \hat{\alpha}$ at $\alpha = 0.05$ in non-crisis periods and in the crisis period

Non-crisis	AU	US	UK	HK	JP	SH	BP	AD	JY
RiskMetrics	0.064	0.065	0.058	0.063	**0.075**	0.053	$\boxed{0.046}$	0.049	**0.067**
Rob-EWMA	$\boxed{0.053}$	**0.067**	$\boxed{0.049}$	0.049	0.063	$\boxed{0.048}$	0.034	0.044	**0.071**
Rob-EWMA (no up)	0.054	$\boxed{0.064}$	$\boxed{0.049}$	0.046	0.060	$\boxed{0.048}$	0.032	0.044	**0.068**
Sk-EWMA	0.054	0.065	0.047	0.046	$\boxed{0.053}$	0.053	**0.028**	0.046	**0.069**
Sk-EWMA (no up)	0.058	$\boxed{0.064}$	$\boxed{0.049}$	$\boxed{0.050}$	0.059	$\boxed{0.048}$	0.029	0.045	**0.065**
Sk-EWMA (fix p)	0.059	**0.068**	0.054	0.057	0.058	0.045	**0.028**	0.046	**0.069**
GARCH-n	**0.069**	$\boxed{0.064}$	0.058	**0.071**	**0.068**	0.056	0.038	0.049	**0.066**
GJR-n	**0.073**	**0.067**	0.059	**0.070**	0.066	0.053	0.033	$\boxed{0.051}$	**0.067**
GARCH-t	0.071	0.065	0.059	**0.066**	**0.067**	**0.066**	0.041	$\boxed{0.051}$	**0.069**
GJR-t	**0.084**	**0.068**	0.058	**0.071**	0.066	**0.067**	0.039	0.054	**0.069**
SAV-CAV	**0.078**	**0.068**	0.054	0.057	0.062	0.055	**0.056**	0.061	$\boxed{0.056}$
AS-CAV	**0.074**	**0.070**	0.055	**0.070**	0.059	0.061	**0.067**	**0.066**	0.062
Crisis	**AU**	**US**	**UK**	**HK**	**JP**	**SH**	**BP**	**AD**	**JY**
RiskMetrics	0.036	0.080	**0.075**	$\boxed{0.050}$	0.070	**0.084**	0.040	0.069	0.075
Rob-EWMA	0.036	0.075	$\boxed{0.060}$	0.035	0.065	0.079	0.040	$\boxed{0.049}$	$\boxed{0.070}$
Rob-EWMA (no up)	0.036	$\boxed{0.070}$	$\boxed{0.060}$	0.035	$\boxed{0.060}$	$\boxed{0.074}$	0.040	0.055	$\boxed{0.070}$
Sk-EWMA	0.032	0.075	$\boxed{0.060}$	0.035	$\boxed{0.060}$	$\boxed{0.074}$	0.055	0.055	$\boxed{0.070}$
Sk-EWMA (no up)	0.036	$\boxed{0.070}$	0.065	0.035	$\boxed{0.060}$	0.079	0.040	0.055	**0.094**
Sk-EWMA (fix p)	0.036	0.075	$\boxed{0.060}$	0.035	$\boxed{0.060}$	**0.084**	$\boxed{0.050}$	0.064	$\boxed{0.070}$
GARCH-n	0.045	**0.085**	0.075	0.030	**0.085**	**0.098**	$\boxed{0.050}$	0.074	0.080
GJR-n	0.077	**0.085**	**0.094**	0.055	**0.090**	**0.094**	0.055	0.079	0.080
GARCH-t	0.045	**0.085**	0.075	0.060	0.075	**0.123**	0.065	0.079	0.084
GJR-t	0.072	0.080	**0.094**	0.070	**0.085**	**0.123**	0.075	**0.084**	0.084
SAV-CAV	$\boxed{0.049}$	**0.085**	**0.085**	0.055	0.080	**0.148**	**0.189**	**0.089**	**0.127**
AS-CAV	0.077	0.099	**0.094**	0.070	0.075	**0.113**	**0.194**	**0.104**	**0.141**

Note: Boxes indicate proportion closest to $\boldsymbol{\alpha = 0.05}$ in that market; bold indicates that the model is rejected by the UC test (at a 5% level), for each market; 'no up' indicates that parameters are estimated once, then not updated, during the forecast period.

Table XII. Summary statistics for VRate$/\alpha$ at $\alpha = 0.005,\ 0.01,\ 0.05$ for each model across the nine series, during the non-crisis periods

Method	$\alpha = 0.005$		$\alpha = 0.01$		$\alpha = 0.05$	
	Mean	Std.	Mean	Std.	Mean	Std.
RiskMetrics	**0.015**	**0.011**	**0.022**	0.017	0.064	**0.014**
R-EWMA	0.0033	$\boxed{0.0030}$	0.0074	0.0066	0.0565	0.0049
R-EWMA(no up)	0.0037	0.0031	0.0071	$\boxed{0.0059}$	$\boxed{0.0554}$	0.0048
Sk-EWMA	0.0039	0.0033	$\boxed{0.0080}$	0.0074	0.0571	$\boxed{0.0041}$
Sk-EWMA(no up)	0.0041	0.0035	0.0078	0.0063	0.0591	0.0047
Sk-EWMA(fix p)	$\boxed{0.0047}$	0.0034	0.0084	0.0066	0.0592	$\boxed{0.0041}$
G-n	0.014	0.011	0.021	0.023	0.069	0.013
GJR-n	0.014	0.011	0.021	0.025	0.079	0.014
G-t	0.008	0.005	0.014	0.010	0.077	0.007
GJR-t	0.008	0.005	0.016	0.012	0.085	0.010
SAV-CAV	0.010	0.008	0.017	**0.049**	0.101	0.009
AS-CAV	0.011	0.008	0.018	0.047	**0.107**	0.010

Note: Boxes indicate the favoured model; bold indicates the least favoured model, in each column; 'Std.' is the square root of the average squared distance from alpha.

updating of parameter estimates does the best in terms of average VRate close to $\alpha = 0.05$, closely followed by the other four Sk-EWMA and Rob-EWMA models, while these five models performed equally well in all periods. On the contrary, all other models underestimated risk, leading to at one and a half times or more actual compared to expected numbers of violations, again performing worse during the crisis period.

From Table XII, on most criteria the skewed-EWMA models are marginally better than the rob-EWMA models. All other models consistently and often significantly underestimate risk levels. Once again the skewed-EWMA models are marginally favoured over the rob-EWMA models, and clearly favoured over all other models, for forecasting VaR levels during the non-crisis periods.

Results for the crisis period tell much the same story, but are now even more strongly in favour of the rob-EWMA and skewed-EWMA models. Clearly, the set of five rob-EWMA and skewed-EWMA models again all do best in

Table XIII. Summary statistics for VRate/α at $\alpha = 0.005, 0.01, 0.05$ for each model across the nine series, during the crisis periods

Method	$\alpha = 0.005$		$\alpha = 0.01$		$\alpha = 0.05$	
	Mean	Std.	Mean	Std.	Mean	Std.
RiskMetrics	0.0163	0.014	0.023	0.017	0.064	0.023
R-EWMA	0.0038	$\boxed{0.0042}$	0.0104	0.0066	0.0565	0.0183
R-EWMA(no up)	0.0038	$\boxed{0.0042}$	$\boxed{0.0098}$	$\boxed{0.0059}$	$\boxed{0.0554}$	$\boxed{0.0162}$
Sk-EWMA	$\boxed{0.0049}$	$\boxed{0.0042}$	0.0115	0.0074	0.0571	0.0173
Sk-EWMA(no up)	0.0043	0.0051	$\boxed{0.0098}$	0.0063	0.0591	0.022
Sk-EWMA(fix p)	$\boxed{0.0049}$	$\boxed{0.0042}$	0.0115	0.0066	0.0592	0.020
G-n	0.020	0.017	0.029	0.023	0.069	0.030
GJR-n	0.021	0.017	0.032	0.025	0.079	0.034
G-t	0.010	0.008	0.018	0.010	0.077	0.036
GJR-t	0.010	0.008	0.019	0.012	0.085	0.041
SAV-CAV	0.023	0.024	**0.043**	**0.049**	0.101	0.070
AS-CAV	**0.027**	**0.027**	**0.043**	0.047	**0.107**	**0.073**

Note: Boxes indicate the favoured model; bold indicates the least favoured model, in each column; 'Std.' is the square root of the average squared distance from alpha.

terms of accurately forecasting risk levels, assessed informally and by the UC test, at $\alpha = 0.005,\ 0.01$ and 0.05 during the GFC period. These models are rarely rejected, and certainly are rejected in the least number of series (0 or 1) at all risk levels. All other models consistently and often significantly underestimate risk levels, on a scale that is much increased compared to that during the non-crisis periods. In particular, the two CAViaR models have performed by far the worst during the crisis period. From Table XII, on most criteria the skewed-EWMA models are marginally better than the rob-EWMA models, for $\alpha = 0.005$; it is hard to separate them at $\alpha = 0.05,\ 0.01$. During the crisis, in the extreme tail ($\alpha = 0.005$), the skewed-EWMA models are marginally favoured over the rob-EWMA models, while for $\alpha = 0.05,\ 0.01$ all rob-EWMA and skewed-EWMA models perform comparably, for forecasting VaR levels during the crisis periods.

In summary, in most cases the full dynamic shape skewed-EWMA model is one of the most favoured overall and during both crisis and non-crisis periods. During the crisis the rob-EWMA models did just as well, but the skewed-EWMA model is marginally favoured during the non-crisis periods. Further, skewed-EWMA models with fixed shape, or where parameter estimates were not updated during the forecast period, also performed well and were only marginally behind the full skewed-EWMA model. Note that the in-sample period has six series with kurtosis below 6, while all series have kurtosis above 6 in the forecast period. This may explain why the sk-EWMA and rob-EWMA methods give generally conservative, but close to nominal, VaR levels, leading to slightly fewer than expected violations on average. In times of crisis, this could be seen as a good property for a method, being close to nominal but erring on the side of the conservative. Indeed, such conservativeness would help financial institutions to stay viable during crisis periods.

Models with Gaussian errors performed the worst at VaR forecasting, including RiskMetrics, while CAViaR models did the worst during the crisis periods. We suspect this performance across models is mainly due to the GFC period in our forecast sample and the subsequent high volatility and many outlying returns that occurred then. For example, the models with Student-t errors did much better in the non-crisis period, as did the CAViaR model. CAViaR methods, like all quantile regression-based techniques, are less efficient than parametric methods that fit the data well. The accuracy of parameter estimation seems particularly affected during the crisis period, where extreme outlying returns can have big influences on parameter estimation and efficiency. Manganelli and Engle (2004) find that the numerical optimization procedure for quantile models is inefficient and problematic at very low quantiles for CAViaR models. This can only be exacerbated in times of high volatility and many outlying returns, as in the GFC period. The naturally fat-tailed AL and Laplace distributions have coped much better during this period and those surrounding. Apparently, the tails of these distributions were much more accurate than those of the Gaussian and Student-t with estimated degrees of freedom, during the GFC period. However, they also were more accurate during the non-GFC period, highlighting the flexibility of our proposed model and its ability to adapt to very different levels of volatility in the market.

CONCLUSION

VaR, since it was proposed in 1993, has seen various methods developed to estimate it, among which the direct, parametric approach has been most prevalent and popular. In particular, via the RiskMetrics system, JP Morgan Guaranty Trust Company (1996) developed a simple-to-apply parametric method that is still used extensively by financial practitioners.

In this paper, a new VaR forecasting model is proposed, named skewed-EWMA. It is a generalization of RiskMetrics and takes into account skewness and heavy tails in the conditional distribution, in a parsimonious manner. An adaptive adjustment for the shape parameter is also proposed, to capture the time-varying nature of skewness and kurtosis in practice. This adjustment also allows for the leverage effect and for time-varying volatility dynamics parameters. The performance of the proposed skewed-EWMA method is compared with a range of parametric and semi-parametric competitors, including RiskMetrics, standard GARCH and CAViaR models, as well as some simplified skewed-EWMA models, via empirical applications to the forecasting of VaR for nine financial return series. Results illustrate that the proposed method significantly outperformed RiskMetrics, the GARCH and the CAViaR models at three risk levels, while marginally outperforming robust-EWMA models. These results held even when the parameters of the skewed-EWMA (and robust-EWMA) model were not re-estimated during the forecast sample. However, a skewed-EWMA model with a fixed shape parameter, i.e. fixed higher moments, did almost equally as well as the dynamic skewed-EWMA model across the nine series. This result held during both crisis and non-crisis data periods.

Skewed-EWMA dynamic forecasting should be promising in modelling risk in dynamic portfolios and nonlinear derivatives, extensions to which are non-trivial (cf. Lu and Li, 2011). The parameter p acts jointly with λ to determine the degree of volatility asymmetry in our model, while p controls all the moments of the ALD; thus our work could be extended by adding one or two more parameters for flexibility. We leave these topics for future research.

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