ODE

- Separable Equation
  - $\frac{dy}{dx} = f(x)g(y) \rightarrow \int \frac{1}{g(y)}dy = \int f(x)dx$
- Exact Equation
  - $\frac{d}{dx}(f(x)\cdot y) = g(x) \rightarrow f(x)\cdot y = \int g(x)dx$ — y_d 是解, y_i 是 0.
  - 变: $f(x)y' + y f'(x) = g(x)$ $\begin{cases} \frac{d(y f(x))}{dx} = g(x) \\ y f(x) = \int g(x)dx \end{cases}$ 左积=右积.
- Integrating Factor
  - $y' + f(x)y = g(x) \rightarrow I.F. = \exp\left[\int f(x)dx\right]$ 乘了以后可变为 Exact.

Second-order ODEs.

- constant coefficient equation: $ay'' + by' + cy = g(x)$.
  - complementary equation: $ay'' + by' + cy = 0$.
  - General solution of inhomo equation: $y(x) = y_c(x) + y_p(x)$
    - $\begin{cases} y_c(x): \text{general solution of complementary eq.} \\ y_p(x): \text{any particular solution of inhomo eq.} \end{cases}$
- find $y_c(x)$ ⇐ for ... $=0 \Rightarrow g(x) = e^{rx}$ 是△有关
  - $y_1(x), y_2(x)$ 是其2个根. ⇐ 求: 设 $y_{1/2} = e^{kx}$, $ak^2 + bk + c = 0$
  - $\begin{cases} \Delta > 0 & y_c(x) = C_1 y_1(x) + C_2 y_2(x) \quad (y_1, y_2 \text{是两个根}) \\ \Delta = 0 & y_c(x) = (C_1 + xC_2)y(x) \\ \Delta < 0 & y_c(x) = e^{ux}(C_1\cos(vx) + C_2\sin(vx)) \quad r = u \pm iv \end{cases}$
- Find $y_p(x)$:
  - Undetermined Coefficients (UC). $ay'' + by' + cy = g(x)$.

① Find $y_c(x)$

② guess $y_p(x)$ form $\Leftarrow$ $g(x)$. 不

当 $y_p(x)$ 与 y_c 有重复时 "$\times x$"

$Ae^x \to Axe^x$

$Axe^x \to Ax^2e^x$

③ 代入原方程求出 y_p.

④ $y = y_c + y_p$

适用: $g(x)$ 为 finite linear combination of products by exponentials, polynomials and sines or cosines.

$y_c(x)$ 与 $y_p(x)$ 可重复.

Variation of Parameters (VP). $y'' + \underline{p(x)}y' + \underline{q(x)}y = g(x)$

$\hookrightarrow$ 在 $[a,b]$ 连续.

① $y = y_c + y_p$

② find y_c 根据 $g(x)$, 写出 y_1, y_2.

③ 求 $W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1y_2' - y_2y_1'$ $\Leftarrow$ from y_c.

④ $u_1 = -\int \frac{y_2 g(t)}{W} dt$; $u_2 = \int \frac{y_1 g(t)}{W} dt$.

⑤ $y_p = \cancel{y_2}\, u_1 y_1 + u_2 y_2$

Systems of ODEs

- First order differential equations.
  - for any $\dot{x} = Ax \Rightarrow x = PY$ $\left\{ \begin{array}{l} P = \text{the modal matrix of } A \\ Y = \{c_1 e^{\lambda_1 t}, c_2 e^{\lambda_2 t} \dots\}^T \end{array} \right.$
  - Example:
    - $\begin{cases} \dot{x} = 4x + 2y \\ \dot{y} = -x + y \end{cases}$, $x(0) = 1$, $y(0) = 0$.

$$\dot{X} = \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix}, \quad A = \begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \end{bmatrix}.$$

$$\det \begin{bmatrix} 4-\lambda & 2 \\ -1 & 1-\lambda \end{bmatrix} = 0 \Rightarrow (4-\lambda)(1-\lambda) + 2 = 0$$

$$\Rightarrow \begin{cases} \lambda_1 = 3. \\ \lambda_2 = 2. \end{cases}$$

when $\lambda_1 = 3$. $\begin{bmatrix} 1 & 2 \\ -1 & -2 \end{bmatrix}$ $\begin{array}{l} a + 2b = 0. \\ -a - 2b = 0. \end{array}$ $X_1 = \begin{bmatrix} -2b \\ b \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

when $\lambda_2 = 2$ $\begin{bmatrix} 2 & 2 \\ -1 & -1 \end{bmatrix}$ $a + b = 0.$ $X_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$$P = \begin{bmatrix} -2 & -1 \\ 1 & 1 \end{bmatrix} \quad D = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}.$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = PY = \begin{bmatrix} -2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} ke^{3t} \\ ce^{2t} \end{bmatrix} = \begin{bmatrix} -2ke^{3t} - ce^{2t} \\ ke^{3t} + ce^{2t} \end{bmatrix}$$

$$\begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} -2k - c \\ k + c. \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad -k = 1 \quad \begin{array}{l} k = -1 \\ c = 1 \end{array}$$

$$\therefore \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2e^{3t} - e^{2t} \\ -e^{3t} + e^{2t} \end{bmatrix}.$$

- Second order
  - $\ddot{Y} = \begin{bmatrix} \ddot{r} \\ \ddot{s} \end{bmatrix} = DY = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} r \\ s \end{bmatrix}$
    - $\begin{cases} r = k\cos\omega_1 t + L\sin\omega_1 t \\ s = M\cos\omega_2 t + N\sin\omega_2 t. \end{cases}$
  - $X = PY$
  - 代入 X 的各个值.