FLOWFoil Theory Document

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1 Geometry

1.1 Panels

For any geometric input, we assume that the airfoil geometry begins at the trailing egde and proceeds clockwise around the leading edge and back to the trailing edge. It is likely convenient to input geometry data that positions the airfoil leading edge at the origin and has a chord length of one.

Linear panels are defined using the input coordinates as the panel end points. Figure 1 shows the geometry convention used in FLOWFoil as well as the linear paneling of an airfoil.

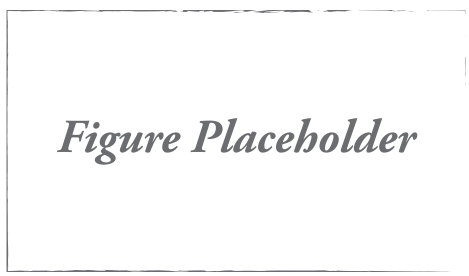

Figure 1: Linear Paneling (see fig1 in mfoil paper)

2 Inviscid Solution Details

2.1 Vortex Influence Coefficients

Following the formulation in [1], the inviscid system is assembled as follows.

$$\begin{aligned}
&\sum_{j=1}^{N}(a_{ij}[\gamma^0,\gamma^{90}]_j) - \Psi_0 = [-z_i, x_i] \\
&[\gamma^0,\gamma^{90}]_1 + [\gamma^0,\gamma^{90}]_N = [0,0] \qquad \text{(Kutta Condition)}
\end{aligned} \tag{1}$$

or

$$\mathbf{A}\gamma = \mathbf{\Psi}^{\infty} \tag{2}$$

where γ^0 and γ^{90} are the perpendicular componenets of vorticity (such that $\mathbf{V}/\mathbf{V}_\infty = \gamma^0 \cos\alpha + \gamma^{90}\sin\alpha$), Ψ_0 is an unknown constant streamfunction at each node, and $\mathbf{A}$ is comprized of the coefficients of influence between the panels and evaluation points (a_{ij}), the Kutta condition, and the influence of Ψ_0.

For a given evaluation point, $i \in \mathbb{N}^N$, and panel, $k \in \mathbb{N}^{N-1}$, (comprised of nodes $j \in \mathbb{N}^{N-1}$ and $j+1$) the streamfunction at the evaluation point due to a linear vortex distribution across the panel (with vortex strengths at the nodes of γ_j and γ_{j+1}, respectively) is

$$\Psi^{\gamma}_{ik} = \mathbf{\Psi}^{\gamma}_{ik}[\gamma_j, \gamma_{j+1}]^{\top}, \tag{3}$$

where

$$\mathbf{\Psi}^{\gamma}_{ik} = \left[\overline{\psi}^{\gamma}_{ik} - \widetilde{\psi}^{\gamma}_{ik}, \widetilde{\psi}^{\gamma}_{ik}\right] \tag{4}$$

Therefore, the influence coefficient of the jth node on the ith node (the ijth element of $\mathbf{A}$) is comprized of portions of the influence seen from each panel of which it is part:

$$a_{ij} = \begin{cases} \overline{\psi}^{\gamma}_{ij} - \widetilde{\psi}^{\gamma}_{ij} & j = 1 \\ \widetilde{\psi}^{\gamma}_{i,j-1} & j = N \\ \widetilde{\psi}^{\gamma}_{i,j-1} + \left(\overline{\psi}^{\gamma}_{ij} - \widetilde{\psi}^{\gamma}_{ij}\right) & \text{otherwise.} \end{cases} \tag{5}$$

Note that panel 1 and panel N-1 do not share nodes 1 and N. That is, node 1 is associated only with panel 1, and node N only with panel N-1.

2.1.1 Influence Geometry

The components of $\mathbf{\Psi}^{\gamma}_{ik}$ are defined as

$$\overline{\psi}^{\gamma}_{ik} = \frac{1}{2\pi}\left(h_{ik}(\theta_{i,j+1} - \theta_{ij}) - d_k + a_{ik}\ln(r_{ij}) - (a_{ik} - d_k)\ln(r_{i,j+1})\right) \tag{6}$$

$$\widetilde{\psi}^{\gamma}_{ik} = \frac{a_{ik}}{d_k}\overline{\psi}^{\gamma}_{ik} + \frac{1}{4\pi d_k}\left(r^2_{i,j+1}\ln(r_{i,j+1}) - r^2_{ij}\ln(r_{ij}) - \frac{1}{2}r^2_{i,j+1} + \frac{1}{2}r^2_{ij}\right). \tag{7}$$

In order to evaluate the the componenets of $\mathbf{\Psi}^{\gamma}_{ik}$ we will need to understand the geometry of the problem.

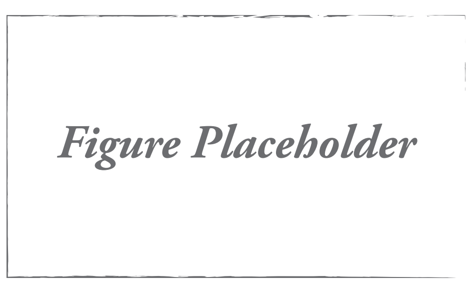

Figure 2: Influence Geometry (see fig10 in mfoil paper)

Figure 2 shows the relative geometry for the problem, and we calculate each of the geometric values as follows.

- The kth panel vector, length, unit tangent, and unit normal, respectively, from the jth to $j+1$th node:

$$\mathbf{d}_k = \mathbf{q}_{j+1} - \mathbf{q}_j \tag{8}$$
$$d_k = ||\mathbf{q}_{j+1} - \mathbf{q}_j|| \tag{9}$$
$$\hat{\mathbf{t}}_k = \frac{\mathbf{d}_k}{d_k} \tag{10}$$
$$\hat{\mathbf{n}}_k = [-\hat{t}_{k_2}, \hat{t}_{k_1}], \tag{11}$$

  where $\mathbf{q}$ is the node position.

- The vector and distance from the jth node to the evaluation point:

$$\mathbf{r}_{ij} = \mathbf{p}_i - \mathbf{q}_j \tag{12}$$
$$r_{ij} = ||\mathbf{p}_i - \mathbf{q}_j||, \tag{13}$$

  where $\mathbf{p}$ is the evaluation point position.

- The natural log of the distance between node and evaluation point:

$$\ln(r_{ij}) = \begin{cases} 0 & \text{if} \quad r_{ij} = 0 \\ \ln(r_{ij}) & \text{otherwise.} \end{cases} \tag{14}$$

- The distance, normal to the panel, from the panel to the evaluation point:

$$h_{ik} = \mathbf{r}_{ij} \cdot \hat{\mathbf{n}}_k, \tag{15}$$

- The distance, tangent to the panel, from the jth node to the evaluation point:

$$a_{ik} = \mathbf{r}_{ij} \cdot \hat{\mathbf{t}}_k, \tag{16}$$

- The angles between the kth panel and the evaluation point centered at the jth and $j+1$th nodes, respectively:

$$\theta_{ij} = \tan^{-1}\left(\frac{h_{ik}}{a_{ik}}\right) \tag{17}$$
$$\theta_{i,j+1} = \tan^{-1}\left(\frac{h_{ik}}{a_{ik} - d_k}\right) \tag{18}$$

2.1.2 Completing the Coefficient Matrix

To complete the coefficient matrix, $\mathbf{A}$ after putting together the vortex influence coefficients, we need to first add in the contribution of the unknown constant,

Ψ_0. We do this by adding a column of -1's to the current $N \times N$ matrix of coefficients

$$\begin{pmatrix} & & & & -1 \\ & & & & -1 \\ & a_{ij} & & & -1 \\ & & & & -1 \\ & & & & -1 \end{pmatrix}. \tag{19}$$

We then add in the kutta condition, $\gamma_1 + \gamma_N = 0$, as an additional row to our matrix

$$\begin{pmatrix} & & & & & -1 \\ & & & & & -1 \\ & & a_{ij} & & & -1 \\ & & & & & -1 \\ & & & & & -1 \\ 1 & 0 & \cdots & 0 & 1 & 0 \end{pmatrix}. \tag{20}$$

We now have an $N+1 \times N+1$ system of equations for γ and Ψ_0

$$\begin{pmatrix} & & & & & -1 \\ & & & & & -1 \\ & & a_{ij} & & & -1 \\ & & & & & -1 \\ & & & & & -1 \\ 1 & 0 & \cdots & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \gamma_1^0 & \gamma_1^{90} \\ \gamma_2^0 & \gamma_2^{90} \\ \vdots & \\ \gamma_N^0 & \gamma_N^{90} \\ \Psi_0^0 & \Psi_0^{90} \end{pmatrix} = \begin{pmatrix} -z_1 & x_1 \\ -z_2 & x_2 \\ \vdots & \\ -z_N & x_N \\ 0 & 0 \end{pmatrix} \tag{21}$$

2.1.3 Trailing Edge Treatment

We have a few modifications to make to our system depending on whether the airfoil has a blunt or sharp trailing edge.

Sharp Trailing Edge In the case of the sharp trailing edge, the first and last nodes are coincident, which leads them to have identical equations, causing the matrix to be singular. In this case, we discard the Nth row and substitute it for an extrapolation of the mean (between upper and lower sides) to the trailing edge as

$$\gamma_1 - 2\gamma_2 + \gamma_3 - \gamma_{N-2} + 2\gamma_{N-1} - \gamma_N = 0. \tag{22}$$

This gives us the Nth row of the coefficient matrix to be

$$\begin{bmatrix} 1 & -2 & 1 & 0 & \cdots & 0 & -1 & 2 & -1 & 0 \end{bmatrix}. \tag{23}$$

This yields the following system of equations for the inviscid solution of an airfoil with a sharp trailing edge:

$$
\begin{pmatrix}
a_{11} & a_{12} & & & \cdots & & & a_{1,N-1} & a_{1N} & -1 \\
a_{21} & a_{22} & & & \cdots & & & a_{2,N-1} & a_{2N} & -1 \\
\vdots & \vdots & & & \ddots & & & \vdots & \vdots & -1 \\
a_{N-1,1} & a_{N-1,2} & & & \cdots & & & a_{N-1,N-1} & a_{N-1,N} & -1 \\
1 & -2 & 1 & 0 & \cdots & 0 & -1 & 2 & -1 & 0 \\
1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 & 0
\end{pmatrix}
\begin{pmatrix}
\gamma_1^0 & \gamma_1^{90} \\
\gamma_2^0 & \gamma_2^{90} \\
\vdots & \\
\gamma_{N-1}^0 & \gamma_{N-1}^{90} \\
\gamma_N^0 & \gamma_N^{90} \\
\Psi_0^0 & \Psi_0^{90}
\end{pmatrix}
=
\begin{pmatrix}
-z_1 & x_1 \\
-z_2 & x_2 \\
\vdots & \\
-z_{N-1} & x_{N-1} \\
0 & 0 \\
0 & 0
\end{pmatrix}
\tag{24}
$$

Blunt Trailing Edge In the case of the blunt trailing edge, the first and Nth nodes are not coincident, which leads to an open body. To remedy this issue, we place an $N+1$th panel from node N to node 1, closing the gap. For this trailing edge panel, we impose a constant vortex and source distribution across the panel.

This augments the system of equations to be

$$
\begin{aligned}
\sum_{j=1}^{N}(a_{ij}\gamma_j) - \Psi_0 + \sum_{j=1,N} a_{i,TE}(\gamma_N - \gamma_1) = V_\infty(z_i \cos\alpha - x_i \sin\alpha) \\
\gamma_1 + \gamma_N = 0 \qquad \text{(Kutta Condition)}
\end{aligned}
\tag{25}
$$

where

$$
a_{i,TE} = \frac{1}{2}\left(\overline{\Psi}^{\sigma}_{i,TE}|\hat{\mathbf{s}}_{TE} \times \hat{\mathbf{d}}_{TE}| - \overline{\Psi}^{\gamma}_{i,TE}|\hat{\mathbf{s}}_{TE} \cdot \hat{\mathbf{d}}_{TE}|\right), \tag{26}
$$

with $\hat{\mathbf{d}}_{TE}$ being the unit vector along the trailing edge panel from node N to node 1, and $\hat{\mathbf{s}}_{TE}$ being the unit trailing-edge bisector vector defined as

$$
\hat{\mathbf{s}}_{TE} = -\mathbf{d}_1||\mathbf{d}_N|| + \mathbf{d}_N|| - \mathbf{d}_1|| \tag{27}
$$

This gives us the following system for the inviscid solution of an airfoil with a blunt trailing edge:

$$
\mathbf{A}\gamma = \mathbf{\Psi}^\infty \tag{28}
$$

where

$$
\mathbf{A} = \begin{pmatrix}
a_{11} + a_{1,TE_1} & a_{12} & \cdots & a_{1,N-1} & a_{1N} + a_{1,TE_N} & -1 \\
a_{21} + a_{2,TE_1} & a_{22} & \cdots & a_{2,N-1} & a_{2N} + a_{12,TE_N} & -1 \\
\vdots & \vdots & \ddots & \vdots & \vdots & -1 \\
a_{N-1,1} + a_{N-1,TE_1} & a_{N-1,2} & \cdots & a_{N-1,N-1} & a_{N-1,N} + a_{N-1,TE_N} & -1 \\
a_{N,1} + a_{N,TE_1} & a_{N,2} & \cdots & a_{N,N-1} & a_{N,N} + a_{N,TE_N} & -1 \\
1 & 0 & \cdots & 0 & 1 & 0
\end{pmatrix} \tag{29}
$$

$$
\gamma = \begin{pmatrix}
\gamma_1^0 & \gamma_1^{90} \\
\gamma_2^0 & \gamma_2^{90} \\
\vdots & \\
\gamma_N^0 & \gamma_N^{90} \\
\Psi_0^0 & \Psi_0^{90}
\end{pmatrix} \tag{30}
$$

$$
\mathbf{\Psi}^{\infty} = \begin{pmatrix}
-z_1 & x_1 \\
-z_2 & x_2 \\
\vdots & \\
-z_N & x_N \\
0 & 0
\end{pmatrix} \tag{31}
$$

2.2 Multi-body System

The multi-body system is treated much the same as a single body system, just with the size of the system increasing. The portions of the system related to the individual airfoils are identical to those above, thus each individual airfoil will have its respective system along the diagonal of the total system. The airfoils' influences on eachother are included in the influence coefficients in the off-diagonal positions of the full system. See equation (32) for a general outline of what a full system might look like.

$$
\mathbf{A} = \begin{pmatrix}
 & & & & & -1 & & & & & & 0 \\
 & & a_{ij}^{11} & & & -1 & & & a_{ij}^{1N} & & & 0 \\
 & & & & & -1 & & & & & & 0 \\
 & & & & & -1 & & & & & & 0 \\
1 & 0 & \cdots & 0 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
 & & & & & 0 & & & & & & -1 \\
 & & a_{ij}^{N1} & & & 0 & & & a_{ij}^{NN} & & & -1 \\
 & & & & & 0 & & & & & & -1 \\
 & & & & & 0 & & & & & & -1 \\
 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 & \cdots & 0 & 1 & 0
\end{pmatrix} \tag{32}
$$

$$
\gamma = \begin{pmatrix}
\gamma_{11}^{0} & \gamma_{11}^{90} \\
\gamma_{21}^{0} & \gamma_{21}^{90} \\
\vdots & \\
\gamma_{N1}^{0} & \gamma_{N1}^{90} \\
\Psi_{01}^{0} & \Psi_{01}^{90} \\
\gamma_{1N}^{0} & \gamma_{1N}^{90} \\
\gamma_{2N}^{0} & \gamma_{2N}^{90} \\
\vdots & \\
\gamma_{NN}^{0} & \gamma_{NN}^{90} \\
\Psi_{0N}^{0} & \Psi_{0N}^{90}
\end{pmatrix} \tag{33}
$$

$$
\mathbf{\Psi}^{\infty} = \begin{pmatrix}
-z_{11} & x_{11} \\
-z_{21} & x_{21} \\
\vdots & \\
-z_{N1} & x_{N1} \\
0 & 0 \\
-z_{1N} & x_{1N} \\
-z_{2N} & x_{2N} \\
\vdots & \\
-z_{NN} & x_{NN} \\
0 & 0
\end{pmatrix} \tag{34}
$$

where a_{ij}^{XY} indicates the influence coefficient of the jth panel of the Yth airfoil on the ith node of the Xth airfoil, γ_{jY} indicates the stregth of the vortex at the jth node on the Yth airfoil, and z_{jY} indicates the boundary condition at the jth node on the Yth airfoil.

Treatment of the trailing edges is very similar to the single airfoil case as well. For sharp trailing edges the Nth row (the row associated with the last node of the airfoil) of the block diagonals are replaced as before in section 2.1.3. The remainder of the row (in the off diagonals) and the relevant row of the right hand side array are set to zeros. For blunt trailing edges, the block diagonals and off diagonals are treated the same. For any node influenced by the trailing edge gap panel the associated row of the matrix recieves the additions explained in section 2.1.3.

References

[1] Fidkowski, K. J., "A Coupled Inviscid–Viscous Airfoil Analysis Solver, Revisited," *AIAA Journal*, Vol. 60, No. 5, May 2022, pp. 2961–2971.

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