DIFFERENTIAL COHOMOLOGY AND GERBES: AN INTRODUCTION TO HIGHER DIFFERENTIAL GEOMETRY 1/3, 2/3

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Abstract. This is a lecture note for a minicourse given at the IUT Mathematics and Statistics Research Seminar. This version contains the notes for the lecture given on December 11th, 2023.

1. Čech cohomology and characteristic classes

Definition 1.1. Let G be a Lie group. A **principal G-bundle** over a smooth manifold M is a smooth map $\pi\colon P \to M$ and a right G-action on P satisfying

(1) π is G-invariant; i.e., $\pi(p \cdot g) = \pi(p)$ for all $p \in P$ and $g \in G$.
(2) On each fiber G acts freely and transitively from the right.
(3) P is locally trivial via G-equivariant trivialization; i.e., at every $m \in M$ there exists an open subset $U \subset M$ and a diffeomorphism $\varphi\colon \pi^{-1}(m) \to U \times G$ such that $p \mapsto (\pi(p), \phi(p))$ satisfying $p \cdot g \mapsto (\pi(p), \phi(p) \cdot g)$.

The conditions (1) and (2) means that the G-orbits are fibers of π. This is equivalent to saying $P \times G \to P \times_M P$, $(p, g) \mapsto (p, p \cdot g)$ is a diffeomorphism; i.e., P is a *G-torsor*.

Definition 1.2. A **bundle map** of principal G-bundles from $\pi_1\colon P_1 \to M$ to $\pi_2\colon P_2 \to M$ is a diffeomorphism $f\colon P_1 \to P_2$ that preserves the fiber and G-equivariant; i.e., $f(p \cdot g) = f(p) \cdot g$ and $\pi_2 \circ f = \pi_1$.

Principal G-bundles over M with maps form a groupoid (a category whose morphisms are invertible) and it is denoted by $\mathrm{Prin}_G(M)$. We will also use the notation $\mathrm{Bun}_{\mathbb{C}^n}(M)$ to denote the groupoid of rank n complex vector bundles over M.

Example 1.3. Let $G = GL_n(\mathbb{C})$. Consider $\pi\colon P \to M$ and take an associated fiber bundle $E(P) \to M$ with a fiber $\mathbb{C}^n$ defined by $E(P) := (P \times \mathbb{C}^n)/G$ with a diagonal G-action: $(p, v) \mapsto (pg, g^{-1}v)$. The bundle $E(P)$ is a complex vector bundle over M of rank n. On the other hand, let $E \in \mathrm{Bun}_{\mathbb{C}^n}(M)$. At each $x \in M$, consider the set $\mathrm{Fr}(E)_x$ of all bases of the vector space E_x; equivalently the set of all $\mathbb{C}$-linear maps $p\colon \mathbb{C}^n \to E_x$. Then the smooth map $\pi\colon \mathrm{Fr}(E) \to M$ with $\pi^{-1}(x) = P(E)_x$ and a right G-action on $\mathrm{Fr}(E)$ defined by $p \mapsto p \circ g$ is a principal G-bundle over

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M. It leads to the following equivalence of categories.

$$\mathrm{Prin}_{GL_n(\mathbb{C})}(M) \underset{\mathrm{Fr}}{\overset{E}{\rightleftarrows}} \mathrm{Bun}_{\mathbb{C}^n}(M)$$

For this reason, in what follows, we don't distinguish a $\mathbb{C}^\times$-, S^1-, or a $U(1)$-bundle and a complex line bundle.

Notation 1.4. We shall use the notation $U_{i_1\cdots i_n}$ to denote the n-fold intersection $U_{i_1} \cap \cdots \cap U_{i_n}$.

Definition 1.5. Let G be an abelian group, M a topological space, and $\mathcal{U} = \{U_i\}_{i\in\Lambda}$ an open cover of M. The set $\check{C}^p(\mathcal{U}; G) = \{f_{i_0\cdots i_p} \colon U_{i_0\cdots i_p} \to G\}_{i_0,\cdots,i_p\in\Lambda}$ inherited the operation of the group G is degree p **Čech cochain group**. Together with the map $\delta_p : \check{C}^p(\mathcal{U}; G) \to \check{C}^{p+1}(\mathcal{U}; G)$, $(f)_{i_0\cdots i_p} \mapsto (\delta f)_{i_0\cdots i_{p+1}} := f_{\widehat{i_0} i_1\cdots i_{p+1}} - f_{i_0\widehat{i_1}\cdots i_{p+1}} + \cdots + (-1)^{p+1}_{i_0 i_1\cdots i_p \widehat{i_{p+1}}}$, the sequence of groups $(\check{C}^\bullet(\mathcal{U}; G), \delta_\bullet)$ is the **Čech cochain complex**. (It is easy to verify that $\delta^2 = 0$. Here the hat means an omission). The cohomology of this complex $\check{H}^\bullet(\mathcal{U}; G) := \ker(\delta_\bullet)/\mathrm{Im}(\delta_{\bullet-1})$ is the Čech cohomology of M defined on an open cover $\mathcal{U}$.

Now if the group G in the definition above is not abelian, in general, the coboundary maps δ are not group homomorphisms, neither $\ker\delta$ nor $\mathrm{Im}\delta$ form a group, and if we apply δ to a cocycle, we do not get $\delta^2 = 1$. We shall see below what goes on starting from the lowest degree.

- $p = 0$: There is no problem. $\check{H}^0(\mathcal{U}; G) = \{f \in \check{C}^0(\mathcal{U}; G) : \delta(f)_{ij} = 0\} = \mathrm{Map}(M, G)$. This is a group under a pointwise group multiplication.
- $p = 1$: Neither $\ker\delta_1$ nor $\mathrm{Im}\delta_0$ form a group. On the set $\ker\delta_1$, we may impose an equivalence relation defined by the action of 0-cochains
$$g_{ij} \sim g'_{ij} \quad \text{if and only if} \quad g'_{ij} = f_i^{-1} g_{ij} f_j.$$
So we may define $\check{H}^1(\mathcal{U}; G)$ as the pointed set $\ker\delta_1/\sim$ with a distinguished element the constant map $g_{ij} \equiv 1$. Notice that the set $\check{H}^1(\mathcal{U}; G)$ is precisely the set of isomorphism classes of principal G-bundles over M defined on the open cover U (see Remark below). For this reason, principal G-bundles are geometric models of a degree 1 nonabelian cohomology of M with coefficients in a group G.
- $p \geq 2$: There is no reasonable way to make sense of $\check{H}^p(\mathcal{U}; G)$.

Remark 1.6. We shall closely look into how the set $\check{H}^1(M; G)$ classifies principal G-bundle over M up to isomorphism. Recall that every principal G-bundle is locally trivial and diffeomorphic to $U \times G$ for some open $U \subset M$. That means if we are given a family of *transition functions* on every double overlap $U_{ij} \in \mathcal{U} = \{U_{ij}\}_{i,j\in\Lambda}$, i.e., $\{g_{ij} \colon U_{ij} \to G : i, j \in \Lambda\}$, we can rebuild the principal G-bundle. Since the transition functions satisfy

$$g_{ij}(x) \cdot g_{jk}(x) \cdot g_{ki}(x) = 1, \text{ for all } x \in U_{ijk} \tag{1.1}$$

The equation (1.1) is called the *cocycle condition* of a principal G-bundle. So if we have a principal bundle P over M, we have a family of transition functions $\{g_{ij}\}_{i,j\in\Lambda}$ satisfying the condition (1.1) and vice versa (under a mild condition). Likewise, if we have a bundle map $f\colon P \to P'$ covering M, we have a family of functions on open sets in the cover $\{f_i\}_{i\in\Lambda}$ satisfying that $g'_{ij}(x) = f_j^{-1}(x)g_{ij}(x)$ for all $x \in U_{ij}$ and vice versa (under the same mild condition). Here the mild condition is that the open cover $\mathcal{U}$ has to be a good cover. A **good cover** (a.k.a. Leray's covering) is an open cover of M if all open sets and their intersections are contractible. Such a covering always exists (see [7, Prop. A.1] and references therein). An open cover $(\mathcal{V}, \imath)$ is a **refinement** of $\mathcal{U}$ if $\imath\colon \mathcal{V} \to \mathcal{U}$ such that $V \subseteq \imath(V)$ for all $V \in \mathcal{V}$. A refinement induces a map $\mathrm{res}_{\mathcal{V},\mathcal{U}}\colon \check{H}^1(\mathcal{U}; G) \to \check{H}^1(\mathcal{V}; G)$, and it satisfies $\mathrm{res}_{\mathcal{W},\mathcal{U}} = \mathrm{res}_{\mathcal{W},\mathcal{V}} \circ \mathrm{res}_{\mathcal{V},\mathcal{U}}$. So we can define the set $\check{H}^1(M; G)$ a direct limit over refinements of open cover; i.e.,

$$\check{H}^1(M; G) = \varinjlim_{\mathcal{U}} \check{H}^1(\mathcal{U}; G).$$

If the cover $\mathcal{U}$ is good, the restriction map $\check{H}^1(\mathcal{U}; G) \xrightarrow{\cong} \check{H}^1(M; G)$ is an isomorphism. Therefore, we conclude that

$$\begin{aligned} \pi_0\mathrm{Prin}_G(\mathcal{U}) &\to \check{H}^1(\mathcal{U}; G) \\ [P] &\mapsto (g_{ij}). \end{aligned} \tag{1.2}$$

If we remove the abelian assumption of groups, the long exact sequence induced by a short exact sequence of groups cannot go any further than the degree $p = 1$.

Proposition 1.7. Let

$$1 \longrightarrow K \xrightarrow{i} \widetilde{G} \xrightarrow{j} G \longrightarrow 1 \tag{1.3}$$

be a short exact sequence of groups. We have the following long exact sequence of groups and pointed sets

$$\begin{tikzcd} 1 \arrow[r] & \check{H}^0(\mathcal{U}; K) \arrow[r, "i_*"] & \check{H}^0(\mathcal{U}; \widetilde{G}) \arrow[r, "j_*"] & \check{H}^0(\mathcal{U}; G) \arrow[dll] \\ & \check{H}^1(\mathcal{U}; K) \arrow[r, "i_*"] & \check{H}^1(\mathcal{U}; \widetilde{G}) \arrow[r, "j_*"] & \check{H}^1(\mathcal{U}; G) \end{tikzcd}$$

However, in a special case that the second term in the sequence is an abelian group whose image is in the center of the third, we can extend the long exact sequence just one term further. We have the following propositions.

Proposition 1.8. If the group K in the short exact sequence (1.3) is abelian and $i(K)$ belongs to the center of $\widetilde{G}$, then the long exact sequence in the Proposition 1.7 extends to $\check{H}^2(\mathcal{U}; K)$:

$$\begin{tikzcd} 1 \arrow[r] & \check{H}^0(\mathcal{U}; K) \arrow[r, "i_*"] & \check{H}^0(\mathcal{U}; \widetilde{G}) \arrow[r, "j_*"] & \check{H}^0(\mathcal{U}; G) \arrow[dll] \\ & \check{H}^1(\mathcal{U}; K) \arrow[r, "i_*"] & \check{H}^1(\mathcal{U}; \widetilde{G}) \arrow[r, "j_*"] & \check{H}^1(\mathcal{U}; G) \arrow[dll, dashed, "\alpha"] \\ & \check{H}^2(\mathcal{U}; K) & & \end{tikzcd}$$

Proposition 1.9 (Dixmier–Douady)**.** If the sheaf $\underline{\widetilde{G}}_M$ is soft, then

$$\alpha : \check{H}^1(\mathcal{U}; G) \to \check{H}^2(\mathcal{U}; K)$$

is a bijection.

Proof. See Dixmier–Douady [5, Lemme 22, p.278] or Brylinski [1, Prop. 4.1.8, p.162] □

In the above, $\underline{G}_M$ is a sheaf such that $\underline{G}_M(U)$ is a group of smooth functions $f : U \to G$ for each open $U \subseteq M$. A sheaf $\underline{G}_M$ is **soft** if $\underline{G}_M(M) \to \underline{G}_M(C)$ is onto for every closed $C \subset M$. Here, we can think of $\underline{G}_M(C) = \lim_U \underline{G}_M(U)$ (since M is paracompact) where the direct limit is taken over all open neighborhoods of C.

Example 1.10. (1) Consider a short exact sequence

$$1 \longrightarrow SO_n \xrightarrow{i} O_n \xrightarrow{\det} \mathbb{Z}_2 \longrightarrow 1.$$

The induced map $w_1 : \check{H}^1(M; O_n) \to \check{H}^1(M; \mathbb{Z}_2)$ is a correspondence $[P] \in \pi_0 \mathrm{Prin}_{O_n}(M) \mapsto w_1([P])$ which is the *first Stifel–Whitney class.* So $w_1([P]) = 0$ if and only if P comes from an SO_n-bundle; i.e., P is orientable. Equivalently the obstruction for transition maps of a Euclidean vector bundle lift to SO_n is the first Stifel–Whitney class.

(2) Consider a short exact sequence

$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathrm{Spin}_n \longrightarrow SO_n \longrightarrow 1.$$

The induced map $w_2 : \check{H}^1(M; SO_n) \to \check{H}^2(M; \mathbb{Z}_2)$ is a correspondence $[P] \in \pi_0 \mathrm{Prin}_{SO_n}(M) \mapsto w_2([P])$ which is the *second Stifel–Whitney class.* So $w_2([P]) = 0$ if and only if P comes from a Spin_n-bundle. Equivalently the obstruction for transition maps of an oriented Euclidean vector bundle lift to Spin_n is the second Stifel–Whitney class. Here one can think of Spin_n as a double cover of SO_n, which is also a universal cover. For a construction of Spin_n in terms of Clifford algebras, see [11, Section 1.2].

Remark 1.11. The Whitehead tower of O_n is of particular interest. The **Whitehead tower** of a space X is a factorization of the point inclusion $\mathrm{pt} \to X$

$$\mathrm{pt} \simeq \lim_{n\to\infty} X_n \longrightarrow \cdots \longrightarrow X_2 \longrightarrow X_1 \longrightarrow X_0 \simeq X$$

such that each X_n is $(n-1)$-connected (i.e., all homotopy groups π_k vanish for $k \leq n-1$) and each map $X_n \to X_{n-1}$ is a fibration which is an isomorphism on all π_k for $k \geq n$. For the space O_n, we have a Whitehead tower as follows:

$$\mathrm{pt} \longrightarrow \cdots \longrightarrow \mathrm{FiveBrane}_n \longrightarrow \mathrm{String}_n \longrightarrow \mathrm{Spin}_n \longrightarrow SO_n \longrightarrow O_n$$

Here String_n is a 6-connected cover of Spin_n

$$1 \longrightarrow K(\mathbb{Z}, 2) \longrightarrow \mathrm{String}_n \longrightarrow \mathrm{Spin}_n \longrightarrow 1.$$

and $\mathrm{FiveBrane}_n$ is a 7-connected cover of String_n

$$1 \longrightarrow K(\mathbb{Z},6) \longrightarrow \mathrm{FiveBrane}_n \longrightarrow \mathrm{String}_n \longrightarrow 1.$$

It is known that the obstruction to lift a Spin_n-bundle to a String_n-bundle is the *first fractional Pontryagin class* $\frac{1}{2}p_1$ and a String_n-bundle to a $\mathrm{FiveBrane}_n$-bundle the *second fractional Pontryagin class* $\frac{1}{6}p_2$ and so on. See [6] for more details.

Example 1.12. (3) Consider a short exact sequence

$$1 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{R} \longrightarrow S^1 \longrightarrow 1.$$

Note that $\underline{\mathbb{R}}_M$ is a soft sheaf (recall Tietze extension theorem). The induced map $c_1\colon \check{H}^1(M;S^1) \xrightarrow{\cong} \check{H}^2(M;\mathbb{Z})$ is a correspondence $[L] \in \pi_0\mathrm{Prin}_{S^1}(M) \mapsto c_1([L])$ which is the *first Chern class.* Note that if the group G is abelian and $\mathcal{G}$ is a sheaf of locally constant functions in G, $\check{H}^p(M;\mathcal{G})$ and $H^p(M;G)$ the degree p singular cohomology with coefficients in G are the same. Since the group $\mathbb{Z}$ is discrete, we can identify $\check{H}^p(M;\mathbb{Z})$ and $H^p(M;\mathbb{Z})$ for any degree p.

Proposition 1.13. (Dixmier–Douady) Let $\mathcal{H}$ be a complex separable Hilbert space. The sheaf $\underline{U(\mathcal{H})}_M$ is soft.

Proof. See Dixmier–Douady [5, Lemme 4, p.252] or Brylinski [1, Cor. 4.1.6, p.162] □

Example 1.14. (4) Consider a short exact sequence

$$1 \longrightarrow U_1 \longrightarrow U(\mathcal{H}) \longrightarrow PU(\mathcal{H}) \longrightarrow 1.$$

Since $U(\mathcal{H})$ is a soft sheaf, the induced map $DD\colon \check{H}^1(M;PU(\mathcal{H})) \xrightarrow{\cong} \check{H}^2(M;S^1) \xrightarrow{\cong} H^3(M;\mathbb{Z})$ is a correspondence $[P] \in \pi_0\mathrm{Prin}_{PU(\mathcal{H})}(M) \mapsto DD([P])$ which is the *Dixmier–Douady class* of a gerbe.

Definition 1.15. A **characteristic class** of a principal G-bundle P over M is an assignment

$$\begin{aligned} c\colon \pi_0\mathrm{Prin}_G(M) &\to H^\bullet(M;A) \\ [P] &\mapsto c(P) \end{aligned}$$

that is natural; i.e. $f^*c(P) = c(\overline{f}^*P)$ for

$$\begin{tikzcd} P' \arrow[r, "\overline{f}"] \arrow[d, "\pi'"'] \arrow[dr, phantom, "\circlearrowright"] & P \arrow[d, "\pi"] \\ M' \arrow[r, "f"] & M \end{tikzcd}$$

Here A is an abelian group.

Since $\mathrm{Prin}_G(-) : \mathbf{Man}^{\mathrm{op}} \to \mathbf{Sets}$ is representable by BG, by Yoneda Lemma (See MacLane [12]) we have the following proposition

Proposition 1.16. An assignment

$$\{\text{Characteristic class of principal } G\text{-bundles}\} \longrightarrow H^\bullet(BG;A)$$

is one-to-one and onto.

Remark 1.17. There is an alternative way to define characteristic classes using a "geometric datum ," i.e. a connection ∇ on $P \in \mathrm{Prin}_G(M)$. This is the *Chern–Weil theory.* For example, given a line bundle with connection (L,∇), the first Chern class of ∇ is defined by a Chern–Weil form $\frac{i}{2\pi}\mathrm{curv}(\nabla)$. Here $\mathrm{curv}(\nabla)$ is the curvature 2-form of the connection ∇. Chern–Weil theorem shows that the cohomology class of a Chern–Weil form does not depend on the choice of connection. So $\left[\frac{i}{2\pi}\mathrm{curv}(\nabla)\right] \in H^2(M;\mathbb{R})$ is a topological invariant of a line bundle. A priori the class $\left[\frac{i}{2\pi}\mathrm{curv}(\nabla)\right]$ is a class in $H^2(M;\mathbb{C})$, but it can be shown that it is actually a class in $H^2(M;\mathbb{R})$. The realification of the first Chern class Example 1.12 above is equal to the first Chern class $\left[\frac{i}{2\pi}\mathrm{curv}(\nabla)\right]$ from the Chern–Weil theory. See Morita [13, Chapter 5] to learn more about Chern–Weil theory of characteristic classes.

We have seen that, up to isomorphism, complex line bundles are classified by $H^2(M;\mathbb{Z})$ via the first Chern class (Example 1.12) and principal $PU(\mathcal{H})$-bundles are by $H^3(M;\mathbb{Z})$ via the Dixmier–Douady class (Example 1.14). We can ask the following question: What classifies (higher) line bundles with connection? For example, if we consider a groupoid $\mathrm{Bun}_{\mathbb{C}}^{\nabla}(M)$ whose objects are line bundles with connection (L,∇) and morphisms are bundle isomorphism preserving the connection, what classifies the isomorphism classes of $\mathrm{Bun}_{\mathbb{C}}^{\nabla}(M)$? This question leads us to "differential cohomology." Up to isomorphism, line bundles with connection are classified by the degree 2 differential cohomology $\widehat{H}^2(M)$, gerbes with connection by $\widehat{H}^3(M)$, 2-gerbes with connection by $\widehat{H}^4(M)$, and so on.

2. Cheeger–Simons differential characters

In this section we introduce differential extension of singular cohomology theory $H^*(-;\mathbb{Z})$ on the site of smooth manifolds. Among various known models, we shall introduce the model by Cheeger and Simons [4] which is one of historical landmarks. Interested readers are referred to the homotopy theoretic model by Hopkins and Singer [10], a spark complex model by Harvey, Lawson, and Zweck [8], and a novel construction using ∞-sheaves of spectra by Bunke, Nikolaus, and Völkl [2].

Notation 2.1. We shall define some notations which will be used throughout this section. Let M be a smooth manifold and R a commutative ring with unity.

- $C^k(M;R)$: smooth singular k-cochains in M with coefficients in R.
- $Z^k(M;R)$: smooth singular k-cocycles in M with coefficients in R.
- $\Omega^k(M)$: differential k-forms on M.
- $\int\colon \Omega^k(M) \to C^k(M;\mathbb{R})$ is a $\mathbb{R}$-linear map $\omega \mapsto \int\omega$, where $\int\omega\colon C_k(M;\mathbb{R}) \to \mathbb{R}$ is a pairing of a singular k-chain and a differential k-form.

- $\Omega^k_{\text{cl}}(M)_{\mathbb{Z}}$: closed differential k-forms with integral periods; i.e., $\omega \in \Omega^k_{\text{cl}}(M)_{\mathbb{Z}}$ if and only if $d\omega = 0$ and $\int \omega\big|_{Z_k(M)} \in \mathbb{Z}$.
- $\sim$ is the natural map $\mathbb{R} \to \mathbb{R}/\mathbb{Z}$.

A nonvanishing differential form does not take its values in a proper subring $\Lambda \subset \mathbb{R}$. Hence we have the following:

Proposition 2.2. The map

$$\int : \Omega^k(M) \to C^k(M; \mathbb{R}/\mathbb{Z})$$
$$\omega \mapsto \widetilde{\int \omega}$$

is one-to-one.

Definition 2.3 (Cheeger and Simons [4])**.** Let M be a smooth manifold. The group $\widehat{H}^k(M)$ of **differential characters** of degree k consists of pairs (χ, ω) where $\chi \in \operatorname{Hom}_{\mathbb{Z}}(Z_{k-1}(M), \mathbb{R}/\mathbb{Z})$ and $\omega \in \Omega^k(M)$ satisfying that

$$\chi \circ \partial D = \int_D \omega \mod \mathbb{Z}, \text{ for all } D \in C_k(M; \mathbb{Z}),$$

where the group structure is the componentwise addition.

Remark 2.4. The degree of the $\widehat{H}^k(M)$ in the above definition is different from the one appears in Cheeger and Simons [4] which defines the same group as degree $k+1$. A consequence of adopting their convention would be a mismatch of degree in the group of differential characters and real cohomology, so the forgetful map (see below for a definition) would be $I\colon \widehat{H}^k(M) \to H^{k+1}(M; \mathbb{R})$. We stick to our convention for the sake of consistency with literature in recent years.

The main goal of this section is to understand the following diagram known as the *differential cohomology hexagon diagram.*

Proposition 2.5. The group of differential characters $\widehat{H}^k(M)$ satisfies the following diagram; i.e. all square and triangles are commutative and the diagonal, upper and lower sequences of arrows are exact sequences.

$$
\begin{tikzcd}
0 \arrow[dr] & & & & 0 \\
& H^{k-1}(M;\mathbb{R}/\mathbb{Z}) \arrow[rr, "-B"] \arrow[dr, hook, "e"] & & H^k(M;\mathbb{Z}) \arrow[ur] \arrow[dr, "r"] & \\
H^{k-1}(M;\mathbb{R}) \arrow[ur, "\sim"] \arrow[dr, "\mathrm{rep}"'] \arrow[rr, phantom, "\circlearrowleft"] & & \widehat{H}^k(M) \arrow[ur, "I"] \arrow[dr, "R"] \arrow[rr, phantom, "\circlearrowleft"] \arrow[u, phantom, "\circlearrowleft"] \arrow[d, phantom, "\circlearrowleft"] & & H^k(M;\mathbb{R}) \\
& \dfrac{\Omega^{k-1}(M)}{\Omega^{k-1}_{\mathrm{cl}}(M)_{\mathbb{Z}}} \arrow[ur, "a"] \arrow[rr, "d"'] & & \Omega^k_{\mathrm{cl}}(M)_{\mathbb{Z}} \arrow[ur, "\int\circ \mathrm{dR}"'] \arrow[dr] & \\
0 \arrow[ur] & & & & 0
\end{tikzcd}
$$

Proof. We shall divide the proof into several parts and enumerate them.

(1) I and R maps: We begin with some algebra facts.

A1. A subgroup of a free abelian group is free.

A2. An abelian group G is **divisible** if, for any $x \in G$ and any $n \in \mathbb{Z}^+$, there exists $y \in G$ such that $x = ny$.

A3. An abelian group G is divisible if and only if the group G is an injective object in the category of abelian groups; If $f\colon A \to G$ and $A \subset B$, there exists a map $\widetilde{f}\colon B \to G$ that satisfies $\widetilde{f}|_A = f$.

Take $(\chi, \omega) \in \widehat{H}^k(M)$ and consider $\chi\colon Z_{k-1}(M) \to \mathbb{R}/\mathbb{Z}$. Since $Z_{k-1}(M)$ is a subgroup of a free abelian group $C_{k-1}(M;\mathbb{Z})$, it is a free (**A1**), and hence projective. We have the following commutative diagram:

$$
\begin{tikzcd}
& \mathbb{R} \arrow[d] \\
Z_{k-1}(M) \arrow[ur, dotted, "\overline{\chi}"] \arrow[r, "\chi"] & \mathbb{R}/\mathbb{Z} \arrow[d] \arrow[u, phantom, "\circlearrowleft" near start, shift right=15] \\
& 0
\end{tikzcd}
$$

Now since $\mathbb{R}$ is divisible (**A2**), it is injective (**A3**). Hence $\overline{\chi} : Z_{k-1}(M) \to \mathbb{R}$ lifts to the map T satisfying the following commutative diagram:

$$
\begin{tikzcd}
0 \arrow[d] & \\
Z_{k-1}(M) \arrow[r, "\overline{\chi}"] \arrow[d, hook] & \mathbb{R} \\
C_{k-1}(M) \arrow[ur, dotted, "T"'] \arrow[ur, phantom, "\circlearrowleft", near start, shift left=8] &
\end{tikzcd}
$$

So $\widetilde{T|_{Z_{k-1}(M)}} = \chi$. It follows that $\widetilde{\delta T} = \delta\widetilde{T} = \widetilde{T}\circ\partial = \int\omega \mod \mathbb{Z}$ Here the first equality is simply $\sim\circ(T\circ\partial_k) = (\sim\circ T)\circ\partial_k$. Thus, there exists $c\in C^k(M;\mathbb{Z})$ such that

$$\delta T = \int\omega - c. \tag{2.1}$$

Note that $0=\delta^2 = \int d\omega - \delta c$, so $\int d\omega = \delta c$. Since a real differential form cannot take its value in a proper subring of $\mathbb{R}$, this means $d\omega \equiv 0 = \delta c$. It is readily seen that ω has an integral period. We define the maps I and R as follows:

$$\begin{aligned} I : \widehat{H}^k(M) &\to H^k(M;\mathbb{R}) & \qquad R:\widehat{H}^k(M) &\to \Omega^k_{\mathrm{cl}}(M)_{\mathbb{Z}} \\ (\chi,\omega) &\mapsto [c] & (\chi,\omega) &\mapsto \omega \end{aligned}$$

Let's verify that these maps are well-defined. Since the choice of lifts are not unique, we have to verify that the above definition does not depend on the choices we made. Suppose T' is another lift satisfying $\delta T' = \int\omega' - c'$. Then $\widetilde{T' - T|_{Z_{k-1}(M)}} = 0$, so $T' = T+\delta s + d$ for some $d\in C^{k-1}(M;\mathbb{Z})$ and $s\in C^{k-2}(M;\mathbb{R})$. So $\delta T' = \delta T + 0 + \delta d$ if and only if $\int\omega' - c' = \int\omega - c + \delta d$ if and only if $\int(\omega'-\omega) = c' - c + \delta d$. Again, since real differential form cannot take its value in a proper subring of $\mathbb{R}$, this means $\omega\equiv\omega'$ and $[c'] = [c]$.

We show that R are surjective. Let $r\colon H^k(M;\mathbb{Z})\to H^k(M;\mathbb{R})$ be the realification map (which is from the universal coefficient theorem for cohomology; see [9, Section 3.1]). Notice that, given $\omega\in\Omega^k_{\mathrm{cl}}(M)_{\mathbb{Z}}$, there exists $u\in H^k(M;\mathbb{Z})$ such that $r(u) = [\int\omega]$. Since ω has integral periods, $\delta\int\omega = \int\omega\circ\partial\in\mathbb{Z}$ is an integral cochain and since ω is closed, $\delta\int\omega = \int d\omega = 0$ (Stokes' theorem). Now let $u=[c]$ for some $c\in C^k(M;\mathbb{Z})$. Then $\int\omega - c = \delta\lambda$ for some $\lambda\in C^{k+1}(M;\mathbb{R})$. Define $\chi := \widetilde{\lambda|_{Z_{k-1}(M)}}$. So R is surjective.

The map I is also surjective. Given any $[c]\in H^k(M;\mathbb{Z})$, $\delta c = 0$ as real cochains. By the de Rham theorem, there exists a $\omega\in\Omega^k_{\mathrm{cl}}(M)$ such that $\int\omega - c = \delta\mu$ for some $\mu\in C^{k-1}(M;\mathbb{R})$. Define $\chi := \widetilde{\mu|_{Z_{k-1}(M)}}$. So the map I is surjective.

(2) The e map: We define the e map as follows:

$$\begin{aligned} e : H^{k-1}(M;\mathbb{R}/\mathbb{Z}) &\to \widehat{H}^k(M) \\ [x] &\mapsto (x|_{Z_{k-1}(M)}, 0) \end{aligned}$$

The map e is well-defined. If we take a different representative $x+\delta y$, the restriction of δy to $Z_{k-1}(M)$ vanishes. The map e is one-to-one: Let $\Lambda\subset\mathbb{R}$ a proper subring. From the universal coefficient theorem we have $H^k(X;\mathbb{R}/\Lambda)\cong \mathrm{Hom}_{\mathbb{Z}}(H_k(X),\mathbb{R}/\Lambda)$, since $\mathrm{Ext}(H_{n-1}(X),\mathbb{R}/\Lambda) = 0$, from $n(\mathbb{R}/\Lambda) = (n\mathbb{R})/\Lambda = \mathbb{R}/\Lambda$, for any $n\in\mathbb{Z}$. Since $B_k\to Z_k\to H_k\to 0$ is exact if and only if $B_k^*\leftarrow Z_k^*\leftarrow H_k^*\leftarrow 0$ is exact, $\mathrm{Hom}_{\mathbb{Z}}(H_k(X),\mathbb{R}/\Lambda)\hookrightarrow \mathrm{Hom}_{\mathbb{Z}}(Z_k(X),\mathbb{R}/\Lambda)$ is an injection.

(3) The a map: We define the a map as follows:

$$\begin{aligned} a : \frac{\Omega^{k-1}(M)}{\Omega^{k-1}_{\mathrm{cl}}(M)_{\mathbb{Z}}} &\to \widehat{H}^k(M) \\ [\alpha] &\mapsto (\widetilde{\int\alpha|_{Z_{k-1}(M)}}, d\alpha) \end{aligned}$$

It is obvious that the map a is well-defined and the subgroup $\Omega^{k-1}_{\text{cl}}(M)_{\mathbb{Z}}$ is the kernel of the map $\Omega^{k-1}(M) \to \widehat{H}^k(M)$, $\alpha \mapsto (\int \alpha|_{Z_{k-1}(M)}, d\alpha)$.

(4) Diagonals are exact: First $\mathrm{Im} e = \ker R$. The inclusion $\subseteq$ is clear. To see $\supseteq$, take (χ, ω) such that $\omega = 0$. Then $\chi = T|_{Z_{k-1}(M)}$ satisfying that $\delta T = c$, so T is a $\mathbb{R}/\mathbb{Z}$-valued cocycle, representing a class in $H^{k-1}(M; \mathbb{R}/\mathbb{Z})$, and $T|_{Z_{k-1}(M;\mathbb{R}/\mathbb{Z})} = \chi$.

Now $\mathrm{Im} a = \ker I$. Again the inclusion $\subseteq$ is clear. To see $\supseteq$, take (χ, ω) such that $\chi = T|_{Z_{k-1}(M)}$ satisfying $\delta T = \int \omega - c$. By assumption, $c = \delta d$ for some $d \in C^{k-1}(M; \mathbb{Z})$. From $\int \omega = \delta(T + d)$, we have $\omega = d\alpha$ for some $\alpha \in \Omega^{k-1}(M)$, and $\int \alpha = T + d + \delta f$ for some $f \in C^{k-2}(M; \mathbb{R})$. Then δf vanishes when we restrict it to $Z_{k-1}(M)$ and d also vanishes modulo $\mathbb{Z}$. Thus, the preimage of I is $(\int \alpha|_{Z_{k-1}(M)}, d\alpha)$.

(5) Squares commute: The map rep is defined as follows.

$$\begin{aligned} \text{rep} : H^{k-1}(M; \mathbb{R}) &\to \frac{\Omega^{k-1}(M)}{\Omega^{k-1}_{\text{cl}}(M)_{\mathbb{Z}}} \\ [\beta] &\mapsto \beta + \Omega^{k-1}_{\text{cl}}(M)_{\mathbb{Z}} \end{aligned}$$

which does not depend on the choice of representatives since all exact forms are closed forms with integral periods. From this it is clear that the square on the left is commutative. Notice that the Equation (2.1) shows the commutativity of the square on the right.

(6) Triangles commute: Two triangle diagrams below commute.

$$\begin{tikzcd}
H^{k-1}(M;\mathbb{R}/\mathbb{Z}) \arrow[rr, "-B"] \arrow[dr, "e"] & \circlearrowleft & H^k(M;\mathbb{Z}) \\
& \widehat{H}^k(M) \arrow[ur, "I"] \arrow[dr, "R"] & \\
\frac{\Omega^{k-1}(M)}{\Omega^{k-1}_{\text{cl}}(M)_{\mathbb{Z}}} \arrow[ur, "a"] \arrow[rr, "d"] & \circlearrowleft & \Omega^k_{\text{cl}}(M)_{\mathbb{Z}}
\end{tikzcd}$$

The commutativity of the lower triangle is obvious. Take a $\mathbb{R}/\mathbb{Z}$-valued cocycle x and consider $(x|_{Z_{k-1}(M)}, 0) \in \widehat{H}^k(M)$. There exists $T \in C^{k-1}(M; \mathbb{R})$ such that $x|_{Z_{k-1}(M)} = T|_{Z_{k-1}(M)}$ satisfying $\delta T = -c$ for some $c \in C^k(M; \mathbb{Z})$, so $I(x|_{Z_{k-1}(M)}, 0) = c = -\delta T = -B([x])$. $\square$

(7) Upper and lower sequences are exact: It is readily seen that the following are exact sequences.

$$H^{k-1}(M;\mathbb{R}) \xrightarrow{\sim} H^{k-1}(M;\mathbb{R}/\mathbb{Z}) \xrightarrow{-B} H^k(M;\mathbb{Z}) \xrightarrow{r} H^k(M;\mathbb{R})$$

$$H^{k-1}(M;\mathbb{R}) \xrightarrow{\text{rep}} \frac{\Omega^{k-1}(M)}{\Omega^{k-1}_{\text{cl}}(M)_{\mathbb{Z}}} \xrightarrow{d} \Omega^k_{\text{cl}}(M)_{\mathbb{Z}} \xrightarrow{\int \circ \text{dR}} H^k(M;\mathbb{R})$$

Immediately from the definition, $\widehat{H}^0(M) = 0$ and $\widehat{H}^1(M) = C^\infty(M, \mathbb{R}/\mathbb{Z})$. Also note that $\widehat{H}^k(M) = 0$ if $k > \dim(M)$. When $k = 2$, we have the following proposition:

Proposition 2.6. The following assignment is a one-to-one correspondence:

$$\pi_0 \mathrm{Prin}_{S^1,\nabla}(M) \longrightarrow \widehat{H}^2(M)$$
$$[(P,\theta)] \mapsto (\chi, \frac{1}{2\pi} d\theta)$$

where, for any loop γ in M, χ is defined by the holonomy of the loop γ; i.e.,

$$\chi(\gamma) := \mathrm{Hol}(\gamma)$$

and for any $D \in C_2(M;\mathbb{Z})$ bounding γ,

$$\chi(\partial D) = \frac{1}{2\pi}\int_D d\theta \mod \mathbb{Z}$$

which is extended to all $Z_1(M)$ by setting $\chi(x) = \chi(\gamma) + \frac{1}{2\pi}\int d\theta(y)$ for any $x = \gamma + \partial y$.

Given $d\theta \in \Omega^2_{\mathrm{cl}}(M)_{\mathbb{Z}}$, as we have seen in the surjectivity of R, there exists $[c] \in H^2(M;\mathbb{Z})$ such that $[\int d\theta] = r([c])$. The class $[c]$ is the characteristic class that classifies P; i.e., the first Chern class.

The above proposition addresses the question at the end of Section 1 at least for degree 2. What is a higher analogue of Proposition 2.6? How can one define a map? In the following section, we shall see that the isomorphism classes of gerbes with connection is in one-to-one correspondence with $\widehat{H}^3(M)$, and To establish the correspondence one has to construct χ; i.e., a holonomy of gerbe.

Remark 2.7. Although we do not go into details, the differential cohomology group $\widehat{H}^\bullet(M)$ has a ring structure (See Cheeger and Simons [4, p.56, Theorem 1.11]).

In differential cohomology, the hexagon diagram plays an important role. One uses the hexagon diagram in Proposition 2.5 to compute differential cohomology groups. Furthermore, it is known that the hexagon diagram uniquely characterizes the differential cohomology. Phrasing slightly differently, If there are two $\widehat{H}^k(M)$ fitting into the middle of the hexagon diagram, then they are naturally isomorphic. This is a theorem of Simons and Sullivan [14] which is generalized by Bunke and Schick [3] and Stimpson [15] to the uniqueness of the differential extension of all exotic cohomology theories under some mild assumptions.

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