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IUT Math & Stat Research Seminar

Differential Cohomology and gerbes:

An introduction to higher differential geometry

1. Čech cohomology and characteristic classes

<u>Def</u>: Let G be a Lie group, M: a smooth mfld.
A <u>principal G-bundle</u> P is a smooth map
$\pi : P \to M$ and a right G-action on P s.t.
(1) π is G-invariant.
(2) On every fiber, G acts on P freely and transitively from the right.
(3) P is locally trivial via G-equivariant trivializatio[n]

<u>Examples</u> (or a <u>prop</u>) when $G = GL_n(\mathbb{C})$

$$\begin{tikzcd} \text{A } G\text{-bundle } P \xrightarrow{\pi} M \arrow[r, bend left, "\text{``associated fiber bundle''}"] & \text{A complex vector bundle } E \to M, \text{ rank } n \arrow[l, bend left, "\text{frame bundle (or bundle of bases)}"] \end{tikzcd}$$

So we don't really distinguish a $U(1)$-bundle / $\mathbb{C}^\times$- / S^1- and complex line bundles.

<u>Def</u>: Let G be an ~~abelian~~ group, M: a top. space.

$\mathcal{U} = \{\mathcal{U}_i\}_{i \in \Lambda}$: an open cover of M.

$$\check{C}^p(\mathcal{U}; G) := \{ f_{i_0 \cdots i_p} : U_{i_0 \cdots i_p} \to G \}_{i_0, \cdots, i_p \in \Lambda}$$

<u>Notation</u>: $\mathcal{U}_{i_1 i_2 \cdots i_n} := U_{i_1} \cap \cdots \cap U_{i_n}$

$$\delta : \check{C}^p(\mathcal{U}; G) \longrightarrow \check{C}^{p+1}(\mathcal{U}; G)$$

$$f \longmapsto (\delta f)_{i_0 \cdots i_p i_{p+1}} := f_{\hat{i}_0 i_1 \cdots i_p i_{p+1}} - f_{i_0 \hat{i}_1 \cdots i_{p+1}}$$

So $(\check{C}^\bullet(\mathcal{U}; G), \delta_\bullet)$ is a complex

<u>Def</u>: $\check{H}^p(\mathcal{U}; G) := \ker \delta / \operatorname{Im} \delta$ <u>Čech cohomology</u> of M defined on $\mathcal{U}$ of degree p

<u>Question</u> what goes wrong if G is not abelian.

- $p = 0$; No prob! $\left(\because \delta f = 0 \Leftrightarrow f_i = f_j, \text{ so } \check{H}^0(\mathcal{U}; G) = \operatorname{Map}(M, G) \right)$

- $p = 1$; $\check{H}^1(\mathcal{U}; G)$ is not a group

  It is a pointed set, $*$ = constant map to 1_G

  Neither $\check{Z}^1(\mathcal{U}; G)$ nor $\check{B}^1(\mathcal{U}; G)$ form a subgroup.

But we can use an equivalence rel.

$$g'_{ij} \sim g_{ij} : U_{ij} \to G \iff g'_{ij} = f_j^{-1} g_{ij} f_i$$

and define $\check{H}^1(\mathcal{U}; G) = \ker\delta / \sim$.

- $p \geq 2$ $\quad \check{H}^p(\mathcal{U}; G)$: no hope.

<u>Prop</u> Let $1 \to K \xrightarrow{i} \tilde{G} \xrightarrow{j} G \to 1 \quad \cdots (*)$ be a S.E.S.

We have a L.E.S.

$$(**)$$

$$\begin{array}{l} 1 \to \check{H}^0(\mathcal{U}; K) \to \check{H}^0(\mathcal{U}; \tilde{G}) \to \check{H}^0(\mathcal{U}; G) \\ \to \check{H}^1(\mathcal{U}; K) \to \check{H}^1(\mathcal{U}; \tilde{G}) \to \check{H}^1(\mathcal{U}; G). \end{array}$$

<u>Prop</u>: If the S.E.S. $(*)$ has K: abelian, and $i(K) \subseteq \mathrm{Center}(\tilde{G})$, then the LES $(**)$ extends to $\check{H}^2(\mathcal{U}; K)$.

$$\begin{array}{l} 1 \to \check{H}^0(\mathcal{U}; K) \to \check{H}^0(\mathcal{U}; \tilde{G}) \to \check{H}^0(\mathcal{U}; G) \\ \to \check{H}^1(\mathcal{U}; K) \to \check{H}^1(\mathcal{U}; \tilde{G}) \to \check{H}^1(\mathcal{U}; G) \\ \xrightarrow{d} \check{H}^2(\mathcal{U}; K) \end{array}$$

Furthermore,

<u>Thm</u> (Dixmier–Douady) If the sheaf $\underline{\tilde{G}}_M$ is soft

(i.e. $\Gamma(M, A) \xrightarrow{\text{res}} \Gamma(C, A)$: onto)

C: any closed set

then $\alpha : \check{H}^1(\mathcal{U}; G) \longrightarrow \check{H}^2(\mathcal{U}; K)$ is a bijection.

Note: $\check{H}^1(M; G)$ classifies principal G-bundles over M up to isomorphism

Having $G \to P \to M$ $\iff$ Having $g_{ij} : U_{ij} \to G$ $U_{ij} \in \mathcal{U}$ s.t. $g_{ki}(x) \cdot g_{jk}(x) \cdot g_{ij}(x) = \mathbb{1}_G \ \forall x \in U_{ijk}$ (cocycle condition)

Having
$$\begin{tikzcd} P \arrow[rr] \arrow[dr] & & P' \arrow[dl] \\ & M & \end{tikzcd}$$
$\iff$ Having $\{ f_i : U_i \to G \}$ s.t. $g_{ij}'(x) = f_j^{-1}(x) g_{ij}(x) f_i(x)$.

In general $\check{H}^1(\mathcal{U}; G) \xrightarrow{\cong ?} \check{H}^1(M; G) := \operatorname{colim}_{\mathcal{U}} \check{H}^1(\mathcal{U}; G)$

If $\mathcal{U}$ is good.

Def: A cover $\mathcal{U}$ of M is good if all open sets and their intersections are contractible.

Examples: (1) $1 \to SO(n) \to O(n) \to \mathbb{Z}_2 \to 1$

Given $[P] \in \mathrm{Bun}_{O(n)}(M)/\cong \ \longmapsto w_1([P])$: the first Stifel-Whitney class.

$w_1 : \check{H}^1(M; O(n)) \longrightarrow \check{H}^1(M; \mathbb{Z}_2)$

<u>Note</u> $w_1(P) = 0 \iff P$ comes from an $SO(n)$-bundle
P: orientable

(2) $1 \to \mathbb{Z}_2 \to Spin(n) \to SO(n) \to 1$.

$$\check{H}^1(M; SO(n)) \longrightarrow \check{H}^2(M; \mathbb{Z}_2)$$
$$[P] \longmapsto w_2(P)$$: the second Stiefel-Whitney class

<u>Note</u>: $w_2(P) = 0 \iff P$ comes from a $Spin(n)$-bundle.

<u>Remark</u> The Whitehead tower of $O(n)$

$$\cdots \text{FiveBrane}(n) \to \text{String}(n) \to Spin(n) \to SO(n) \to O(n)$$

(3) $1 \to \mathbb{Z} \to \mathbb{R} \to S^1 \to 1$

$\mathbb{R}$: soft ($\because$ Tietze extension)

$$c_1 : \check{H}^1(M; S^1) \xrightarrow{\cong} \check{H}^2(M; \mathbb{Z}) \cong H^2_{sing}(M; \mathbb{Z})$$
$$[L] \longmapsto c_1(L)$$: the first Chern class

(4) <u>Thm</u> (Dixmier–Douady) $\underline{U(\mathcal{H})}_M$: soft.

Consider $1 \to U(1) \to U(\mathcal{H}) \to PU(\mathcal{H}) \to 1$

$$DD : \check{H}^1(M; PU(\mathcal{H})) \xrightarrow{\cong} \check{H}^2(M; S^1) \xrightarrow{\cong} H^3_{sing}(M; \mathbb{Z})$$
$$[P] \longmapsto DD(P)$$

Dixmier – Douady class

<u>Def</u> A <u>characteristic class</u> c of a G-bundle $P \to M$ is an assignment

$$c : \mathrm{Bun}_G(M)/\cong \longrightarrow H^*(M; \overset{\text{Ab. gp.}}{A})$$

$$[P] \longmapsto c(P)$$

that is natural (i.e. for
$$\begin{array}{ccc} \bar{P} & \xrightarrow{\bar{f}} & P \\ \downarrow & & \downarrow \\ M' & \xrightarrow{f} & M \end{array} \qquad f^* c(P) = c(\bar{f}^* P) \text{)}$$

<u>Remark</u> (Yoneda Lemma) $\quad \mathrm{Bun}_G(-) \cong \mathrm{Maps}(-, BG)$

An assignment

$$\left\{ \begin{array}{l} \text{Characteristic} \\ \text{class of } P \in \mathrm{Bun}_G M \end{array} \right\} \longrightarrow H^*(BG; A)$$

is 1-1 and onto.

<u>Remark</u> There's an alternative way to define characteristic classes using geom. data. "Chern – Weil"

E.g. $L, \nabla \rightsquigarrow$ Chern-Weil form $c(\nabla) = \frac{i}{2\pi} \mathrm{Curv}(\nabla)$

<u>Thm</u> (Chern-Weil) $[c_1(\nabla)]$ is indep. of choice of ∇.

$H^2(M;\mathbb{Z})$ classifies Line bundles

$H^3(M;\mathbb{Z})$ classifies "Gerbes"

What classifies line bundles w/ conn? (L, ∇) $\hat{H}^2(M)$

gerbes w/ conn? $\hat{H}^3(M)$

higher gerbes w/ conn? $\hat{H}^\bullet(M)$