2025. 8. 12.

Lecture 3. Congruent conics and normal forms.

Recap Why is $\left(\mathrm{Cong}(\mathbb{R}^2), \circ\right)$ a group?

(1) $\circ : \mathrm{Cong}(\mathbb{R}^2) \times \mathrm{Cong}(\mathbb{R}^2) \longrightarrow \underset{\psi}{\mathrm{Cong}}(\mathbb{R}^2)$ ✓

$(\phi_1, \phi_2) \longmapsto \phi_1 \circ \phi_2$

$\because$ Let $\phi_1(-) = R_1(-) + T_1$, $\phi_2(-) = R_2(-) + T_2$

$$\phi_1 \circ \phi_2(-) = R_1\left(R_2(-)+T_2\right)+T_1$$

$$= \underbrace{R_1R_2}_{\text{rotation}}(-) + \underbrace{R_1T_2+T_1}_{\text{translation}}$$

(2) ϕ_i $(i=1,2,3) \in \mathrm{Cong}(\mathbb{R}^2)$

$$\phi_1 \circ (\phi_2 \circ \phi_3) = (\phi_1 \circ \phi_2) \circ \phi_3$$

(Nothing to prove! Equality holds as planar maps)

(3) The identity $I = R(0)(-) + \vec{0}$

$\forall\, \phi \in \mathrm{Cong}(\mathbb{R}^2)$ $\quad \phi \circ I = \phi \circ I = \phi$. ✓

(4) For given $\underset{R(\theta)(-)+T}{\overset{\shortparallel}{\phi}} \in \mathrm{Cong}(\mathbb{R}^2)$, there exists $\phi^{-1} := R(-\theta)(-)+$

So that

$$\phi^{-1} \circ \phi = I \iff \phi^{-1}\left(\underline{R(\theta)(-)+T}\right) = I$$

$$\phi \circ \phi^{-1} = I \qquad \phi^{-1} = R(-\theta)(-) + \hat{T}$$

$$= R(-\theta)R(\theta)(-) + \underbrace{R(-\theta)T + \hat{T}}_{=0}$$

$$-R(-\theta)T = \hat{T}.$$

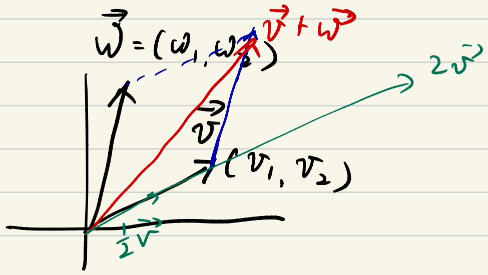
$$R(\theta)\left(R(-\theta)(-) - R(-\theta)T\right) + T$$

$$= \underbrace{R(\theta)R(-\theta)(-)}_{I} - \cancel{R(\theta)R(-\theta)T} + \cancel{T} = I \quad ✓$$

"Goro Shimura, The map of my life"

1. Some properties of Congruence

<u>Aside</u> Let $\vec{v}, \vec{w} \in \mathbb{R}^2$ be vectors.

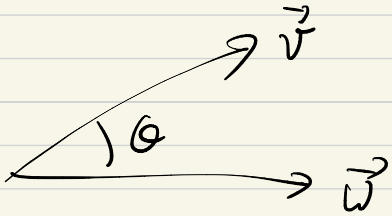


<u>Def</u> $\vec{v} \cdot \vec{w} = v_1 w_1 + v_2 w_2$

inner product, dot product.

<u>Exercise</u> (1) $\vec{v} \cdot \vec{v} = v_1^2 + v_2^2 = (\underbrace{\text{length of } \vec{v}}_{\|\vec{v}\|})^2$

(2)



<u>Thm</u>: $\vec{v} \cdot \vec{w} = \|\vec{v}\| \, \|\vec{w}\| \cos\theta$.

<u>Proof</u> : You!

<u>Proposition</u>: Let R be a rotation. Then for any $\vec{z}, \vec{w} \in \mathbb{R}^2$

$$R(\vec{z}) \cdot R(\vec{w}) = \vec{z} \cdot \vec{w}$$

<u>Proof</u> : LHS $= \begin{pmatrix} c & -s \\ s & c \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \cdot \begin{pmatrix} c & -s \\ s & c \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = (cz_1 - sz_2, sz_1 + cz_2)$

$R = \begin{pmatrix} c & -s \\ s & c \end{pmatrix}$

$\cdot \, (cw_1 - sw_2, sw_1 + cw_2)$

$= \vec{z} \cdot \vec{w}$.

You!

Corollary: Rotations preserve distance and angle.

E.g. $\|R(z)\| = \|z\|$

Pf: $\|R(z)\|^2 = R(z) \cdot R(z) = z \cdot z = \|z\|^2$ ✓

Def: An invertible planar map that preserves distances is an isometry. (등거리 함수, 등장사상)

Example: Any congruence ϕ is an isometry.

Let $\phi = \underset{\substack{\uparrow \\ \text{rotation}}}{R(-)} + \underset{\substack{\uparrow \\ \text{translation}}}{T}$.

$$\|\phi(\vec{z}) - \phi(\vec{z}')\|$$
$$= \|R\vec{z} + T - (R\vec{z}' + T)\|$$
$$= \|R\vec{z} - R\vec{z}'\|$$
$$= \|R(\vec{z} - \vec{z}')\|$$
$$= \|\vec{z} - \vec{z}'\| \quad \text{for any } \vec{z}, \vec{z}' \in \mathbb{R}^2$$

2. Congruent lines

A linear function is a function $L: \mathbb{R}^2 \to \mathbb{R}$

$(x, y) \mapsto ax + by + c$

A straight line in the plane is zero set of $\mathcal{L}$.

Def: Two linear functions $\mathcal{L}, \mathcal{M}$ are <u>congruent</u> if there exists $\phi \in \mathrm{Cong}(\mathbb{R}^2)$ and a nonzero constant $\mu \in \mathbb{R}$ satisfying

$$\mathcal{M} = \mu(\mathcal{L} \circ \phi)$$

i.e. $\phi : (X, Y) \longmapsto (x, y)$

$\mathcal{L}$을 회전+평행이동 한 것.

$$\mathcal{M}(X, Y) = \mu \mathcal{L}(x, y) = \mu \boxed{\mathcal{L}(\phi(X, Y))}$$

E.g. $\mathcal{L} = ax + by + c$

$$\begin{cases} x = (\cos\theta X - \sin\theta Y + u) \\ y = (\sin\theta X + \cos\theta Y + v) \end{cases} \qquad T = (u, v)$$

Proposition Congruence of lines is an equivalence relation.

What is equivalent relation $\sim$?

반사적

(1) $x \sim x$ (reflexivity)

대칭적

(2) $x \sim y \Rightarrow y \sim x$ (Symmetry)

추이적

(3) $x \sim y,\ y \sim z \Rightarrow x \sim z$. (transitivity)

E.g. "Scalar multiple" is an equivalence relation on linear functions.

$$\mathcal{L}_1 \sim \mathcal{L}_2 \iff \exists \lambda \neq 0 \quad \mathcal{L}_1 = \lambda \mathcal{L}_2$$

(1) $\mathcal{L}_1 \sim \mathcal{L}_1$. ($\because \lambda = 1$.)

(2) $\mathcal{L}_1 \underset{\lambda}{\sim} \mathcal{L}_2 \Rightarrow \mathcal{L}_2 \underset{1/\lambda}{\sim} \mathcal{L}_1$

(3) $\mathcal{L}_1 \underset{\lambda_1}{\sim} \mathcal{L}_2,\ \mathcal{L}_2 \underset{\lambda_2}{\sim} \mathcal{L}_3 \Rightarrow \mathcal{L}_1 \underset{\lambda_1\lambda_2}{\sim} \mathcal{L}_3$.

Congruence? (1) $\mathcal{L} \sim \mathcal{L}$.

$$\mathcal{L} = \underset{1}{\mu}\, \mathcal{L} \circ \underset{I}{\phi} \quad ✓$$

phi 의

(2) $\mathcal{M} = \mu \mathcal{L} \circ \phi \Rightarrow \mathcal{L} = \frac{1}{\mu} \mathcal{M} \circ \phi^{-1}$

$\mathcal{L} \sim \mathcal{M}$, $\mathcal{M} \sim \mathcal{N} \Rightarrow \mathcal{L} \sim \mathcal{N}$

(3) $\mathcal{M} = \mu \mathcal{L} \circ \phi$, $\mathcal{N} = \lambda \mathcal{M} \circ \psi$

psi

$$\mathcal{N} = \lambda \mathcal{M} \circ \psi = \lambda(\mu \mathcal{L} \circ \phi) \circ \psi$$
$$= \underbrace{\lambda \mu}_{\neq 0} \mathcal{L} \circ \underbrace{(\phi \circ \psi)}_{\in \mathrm{Cong}(\mathbb{R}^2)}$$

$\mathrm{Lin}(\mathbb{R}^2, \mathbb{R}) / \sim$ ← scalar multiple 각 $\boxed{\text{원소}}$: 직선

set of all linear functions

$\mathrm{Lin}(\mathbb{R}^2, \mathbb{R}) / \sim$ ← Congruence "증명" 단 하나의 원소뿐!

Prop Any two lines $\mathcal{L}, \mathcal{M}$ are congruent.

Proof: It is enough to prove that any given line $\mathcal{M}$ is congruent to $x = 0$. ($\because$ transitivity).

Write down $\mathcal{M} = \mu(\mathcal{L} \circ \phi)$

$\mathcal{L} = ax + by + c$

$\mathcal{M} = Ax + By + C$

$\phi = R(\theta)(-) + \binom{u}{v}$

You $\Rightarrow$ $\begin{cases} A = \mu(a\cos\theta + b\sin\theta) \\ B = \mu(-a\sin\theta + b\cos\theta) \\ C = (au + bv + c) \end{cases}$

Here $\mathcal{L}: x = 0$. i.e. $a = 1, b = 0, c = 0$.

$$\mathcal{M} = Ax + By + C \qquad \begin{array}{l} A = \mu\cos\theta \\ B = -\mu\sin\theta \\ C = \mu u. \end{array}$$

Want to prove: There exists θ, (u, v), μ such that

$$\mathcal{M} = \mu(\mathcal{L} \circ \phi)$$

We may take $\mu > 0$ such that $\underline{A^2 + B^2 = \mu^2}$ (one of A, B : nonzero)

$\to \mu = \sqrt{A^2+B^2} > 0$ ✓

So Take $u = \dfrac{C}{\mu}$ $\qquad \left\{ \begin{array}{l} \cos\theta = \dfrac{A}{\mu} \\ \sin\theta = -\dfrac{B}{\mu}. \end{array} \right\} \to$ Find θ ✓

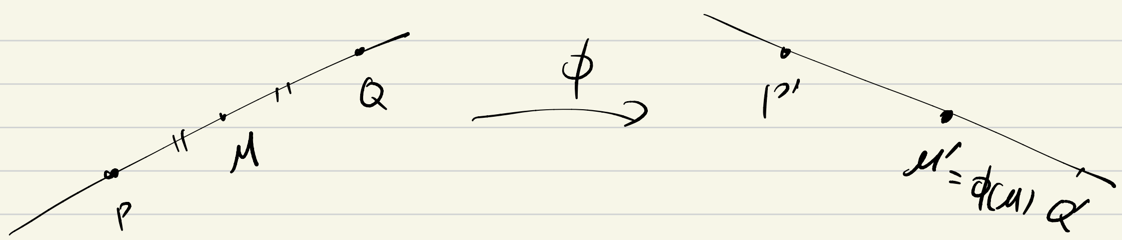
□

Prop (1) If ϕ: congruence maps lines $\mathcal{L}, \mathcal{M}$ to $\mathcal{L}', \mathcal{M}'$

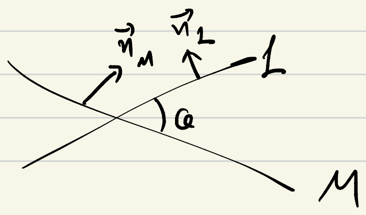
then $P = \mathcal{L} \cap \mathcal{M} \iff P' = \mathcal{L}' \cap \mathcal{M}'$

$\mathcal{L} \parallel \mathcal{M} \iff \mathcal{L}' \parallel \mathcal{M}'$

(2) If ϕ: congruence, then it preserves the midpoints of segments.



(3) If ϕ : Congruence, then it preserves the angles between lines.



$$\cos\theta = \frac{\vec{n}_M \cdot \vec{n}_L}{\|\vec{n}_M\| \, \|\vec{n}_L\|} \overset{\checkmark}{=} \frac{R\vec{n}_M \cdot R\vec{n}_L}{\|R\vec{n}_M\| \, \|R\vec{n}_L\|}$$

3. Congruent Conics.

<u>Def</u>: Two quadratic functions Q, R are <u>Congruent</u> if there exists a $\phi \in \mathrm{Cong}(\mathbb{R}^2)$ and a nonzero constant $\mu \in \mathbb{R}$ such that

$$R = \mu (Q \circ \phi)$$

i.e. if $\phi : (\bar{X}, \bar{Y}) \mapsto (x, y)$

$$R(\bar{X}, \bar{Y}) = \mu Q(x, y) = \mu Q(\phi(\bar{X}, \bar{Y}))$$

Two conics are <u>translationally congruent</u> if ϕ is a translation and <u>rotationally congruent</u> if ϕ is a rotation.

<u>Example</u>: Rotation with an angle $\tan\theta = -\frac{4}{3}$.

$\sin\theta = \frac{-4}{5}$, $\cos\theta = \frac{3}{5}$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \underbrace{\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}}_{R} \begin{pmatrix} \bar{X} \\ \bar{Y} \end{pmatrix} = \begin{pmatrix} \frac{3}{5} & \frac{4}{5} \\ -\frac{4}{5} & \frac{3}{5} \end{pmatrix} \begin{pmatrix} \bar{X} \\ \bar{Y} \end{pmatrix} = \begin{pmatrix} \frac{3X+4Y}{5} \\ \frac{-4X+3Y}{5} \end{pmatrix}.$$

$$Q(x, y) = 11x^2 + 24xy + 4y^2 - 5.$$

$$\downarrow$$

$$R = \mu Q \circ \phi(\bar{X}, \bar{Y}) = 11\left(\frac{3X+4Y}{5}\right)^2 + 24\left(\frac{3X+4Y}{5}\right)\left(\frac{-4X+3Y}{5}\right) + 4\left(\frac{-4X+3Y}{5}\right)^2 - 5.$$

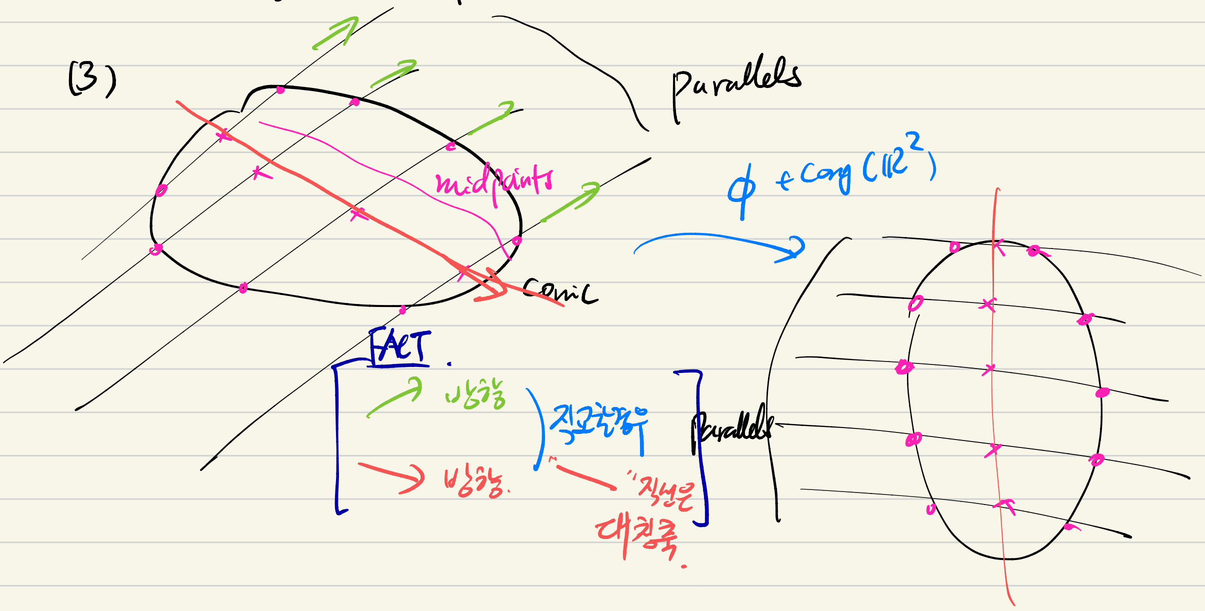
$$= -5(X^2 - 4Y^2 + 1)$$

Proposition:

(1) Congruence is an equivalence relation.

(2) Congruence preserves the zero set of conics.

(3)



4. Classifying Conics and normal forms.

Theorem: Let $Q(x,y) = ax^2 + 2hxy + by^2 + 2gx + 2fy + c$.

Any such conic Q is rotationally congruent to one which the coefficient of the cross term xy is zero.

Proof: The effect of rotation:

$$\begin{cases} x = X\cos\theta - Y\sin\theta \\ y = X\sin\theta + Y\cos\theta \end{cases}$$

$$Q' = Q(R(X,Y))$$

$$Q' = Q(x,y)$$

The cross term $-2a\cos\theta\sin\theta + 2b\sin\theta\cos\theta + 2h(\cos^2\theta - \sin^2\theta)$

//

$$2h' = (b-a)\sin 2\theta + 2h\cos 2\theta.$$

$$h' = \frac{1}{2}(b-a)\sin 2\theta + h\cos 2\theta.$$

Assume $h \neq 0$ (obviously!) Seek θ such that $h' = 0$.

(i) $a = b \Rightarrow \cos 2\theta = 0. \Rightarrow \theta = \frac{\pi}{4}.$

(ii) $\tan 2\theta = \frac{2h}{a-b}.$ $\Leftrightarrow \frac{2T}{1-T^2} = \frac{2h}{(a-b)}$

$\quad = \frac{2\tan\theta}{1-\tan^2\theta}$ $\Leftrightarrow hT^2 + (a-b)T - h = 0.$

$\tan\theta = T.$

$D > 0.$ Two real root.

$\quad \parallel$

$(a-b)^2 - 4h(-h) > 0.$

Example $Q(x,y) = 41x^2 - 24xy + 34y^2 - 90x + 5y + 25.$

$$\tan 2\theta = \frac{2h}{a-b} = \frac{2(\frac{1}{2}\cdot(-24))}{41 - 34} = -\frac{24}{7}.$$

$\Leftrightarrow 12T^2 - 7T - 12 = 0.$

$\Leftrightarrow \underbrace{T = \frac{4}{3}}, \; -\frac{3}{4}.$

$\hookrightarrow \sin\theta = \frac{4}{5}$, $\cos\theta = \frac{3}{5}.$

$x = \frac{3X - 4Y}{5}, \; y = \frac{-4X + 3Y}{5}.$

$Q'(X,Y) = X^2 + 2Y^2 - 2X + 3Y + 1!$