THEOREM THE MEAN-VALUE THEOREM (SEVERAL VARIABLES)

If f is differentiable at each point of the line segment ab, then there exists on that line segment a point c between a and b such that

$$f(b) - f(a) = \nabla f(c) \cdot (b - a).$$

Proof. As t ranges from 0 to 1, $a + t(b - a)$ traces out the line segment ab. The idea of the proof is to apply the one-variable mean-value theorem to the function

$$g(t) = f(a + t[b - a]), \qquad t \in [0, 1].$$

To show that g is differentiable on the open interval $(0, 1)$, we take $t \in (0, 1)$ and form

$$\begin{aligned} g(t + h) - g(t) &= f(a + (t + h)[b - a]) - f(a + t[b - a]) \\ &= f(a + t[b - a] + h[b - a]) - f(a + t[b - a]) \\ &= \nabla f(a + t[b - a]) \cdot h[b - a] + o(h[b - a]). \end{aligned}$$

Since

$$\nabla f(a + t[b - a]) \cdot h(b - a) = [\nabla f(a + t[b - a]) \cdot (b - a)]h$$

and the $o(h(b - a))$ term is obviously $o(h)$, we can write

$$g(t + h) - g(t) = [\nabla f(a + t[b - a]) \cdot (b - a)]h + o(h).$$

Dividing both sides by h, we see that g is differentiable and

$$g'(t) = \nabla f(a + t[b - a]) \cdot (b - a).$$

The function g is clearly continuous at 0 and at 1. Applying the one-variable mean-value theorem to g, we can conclude that there exists a number t_0 between 0 and 1 such that

$$g(1) - g(0) = g'(t_0)(1 - 0).$$

Since $g(1) = f(b), g(0) = f(a)$, and $g'(t_0) = \nabla f(a + t_0[b - a]) \cdot (b - a)$, the above gives

$$f(b) - f(a) = \nabla f(a + t_0[b - a]) \cdot (b - a).$$

Setting $c = a + t_0[b - a]$, we have

$$f(b) - f(a) = \nabla f(c) \cdot (b - a).$$