교사를 위한 기하학 II. Final exam.

#1. (1) F Circles passing through the center O of the circle of inversion get inverted to straight line not passing through the center O.

(2) T

(3) F projective geometry is based on projective properties and not metric properties

(4) F In addition to Axiom 9, there are other axioms of projective geometry (such as incidence axioms) as a part of sufficient conditions.

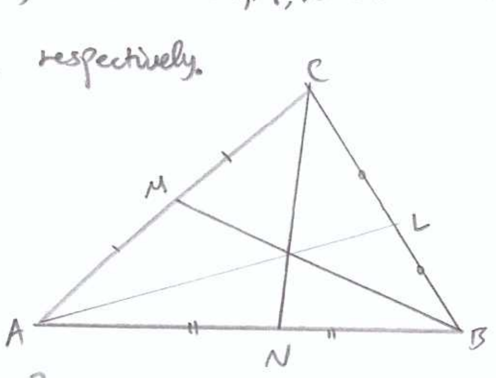
(5) F The proof of Desargue's theorem uses a point in the 3-space, and hence it needs Axioms 7, 8 as well.

#2. Suppose $\triangle ABC$ is given, and let L, M, N be medians eminating from A, B, C, respectively.

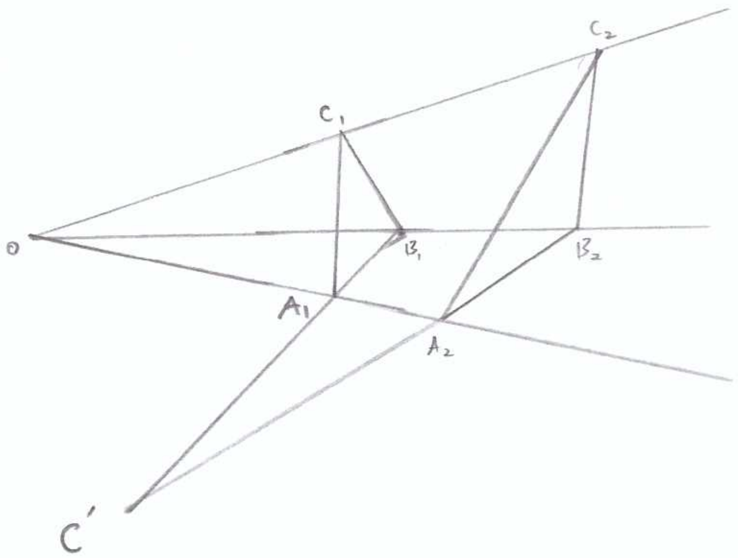
We observe that

$$\frac{AN}{NB} \cdot \frac{BL}{LC} \cdot \frac{CM}{MA} = 1.$$

Hence the result follows from the Ceva's theorem. □



#3. Consider the following figure:



We let C' be the point of intersection of A_1B_1 (extension) and A_2B_2 (extension) Then by Menelaus' theorem on ΔOA_2B_2

$$\frac{OA_1}{A_1A_2} \cdot \frac{A_2C'}{C'B_2} \cdot \frac{B_2B_1}{B_1O} = -1. \quad \cdots ①$$

Similarly, let A' be $B_1C_1 \cap B_2C_2$ and B' be $C_1A_1 \cap C_2A_2$

Applying Menelaus' theorem on ΔOB_2C_2 and ΔOC_2A_2 :

$$\frac{OB_1}{B_1B_2} \cdot \frac{B_2A'}{A'C_2} \cdot \frac{C_2C_1}{C_1O} = -1 \quad \cdots ②$$

$$\frac{OC_1}{C_1C_2} \cdot \frac{C_2B'}{B'A_2} \cdot \frac{A_2A_1}{A_1O} = -1 \quad \cdots ③$$

Multiplying ①, ②, ③ we get: $\frac{A_2 C'}{C' B_2} \cdot \frac{B_2 A'}{A' C_2} \cdot \frac{C_2 B'}{B' A_2} = -1$

By Menelaus' theorem C', A', B' are collinear. $\square$

#4. $H(AB, CD) \Leftrightarrow \frac{AC}{CB} = -\frac{AD}{DB}$.

$$\Leftrightarrow \frac{CB}{AC} = -\frac{DB}{AD}$$

$$\Leftrightarrow \frac{AC+CB}{AC} = -\frac{AD+DB}{AD} + 2$$

$$\Leftrightarrow AB\left(\frac{1}{AC} + \frac{1}{AD}\right) = 2$$

$$\Leftrightarrow \frac{2}{AB} = \frac{1}{AC} + \frac{1}{AD}. \quad \checkmark$$

#5. The cross ratio of points A, B, C, D: $\frac{AC/CB}{AD/DB} = \lambda$, where A, B, C, D are on a line L.

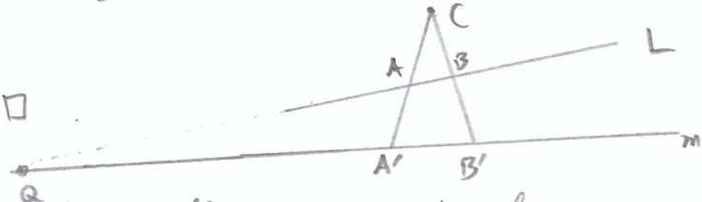
Suppose O is the center of the projection, and A', B', C', D' be images of A, B, C, D under the projection of L onto L', another straight line not passing through O.

Now let h be the distance from O to L.
and h' the distance from O to L'.

$$\lambda = \frac{AC}{CB}\,\frac{DB}{AD} = \frac{\frac{1}{2}h\cdot AC}{\frac{1}{2}h\cdot CB}\cdot\frac{\frac{1}{2}h\cdot DB}{\frac{1}{2}h\cdot AD} = \pm\frac{\frac{1}{2}OA\,OC\sin AOC}{\frac{1}{2}OC\,OB\sin COB}\cdot\frac{\frac{1}{2}OD\,OB\sin DOB}{\frac{1}{2}OA\,OD\sin AOD}$$

$$= \pm\frac{\frac{1}{2}OA'\,OC'\sin A'OC'}{\frac{1}{2}OC'\,OB'\sin C'OB'}\cdot\frac{\frac{1}{2}OD'\,OB'\sin D'OB'}{\frac{1}{2}OA'\,OD'\sin A'OD'} = \pm\frac{\frac{1}{2}h'\cdot A'C'}{\frac{1}{2}h'\cdot C'B'}\cdot\frac{\frac{1}{2}h'\cdot D'B'}{\frac{1}{2}h'\cdot A'D'}$$

$$= \pm\frac{A'C'/C'B'}{A'D'/D'B'}. \quad \square$$



#6. (1) Nothing to prove if L itself is a generating line, or

L passes through generating point C.

Suppose L is not m and L does not pass through C.

Take two points A', B' on m, and draw CA', CB'.

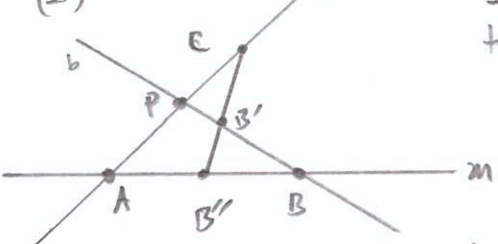
L has to meet CA' at A and CB' at B.

Now by Axiom 3, there exists Q on m

Such that Q, A, B are Collinear

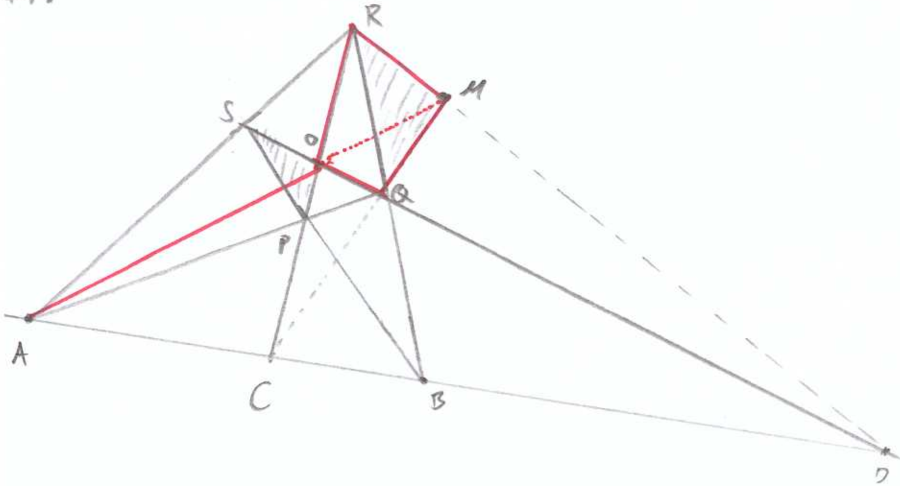
and Q, A', B' are collinear. □

(2)



Let a be a line in π passing through the generating point C, b a line in π.
b has to meet the generating line m ($\because$ (1))
Say at B. Let A be $a \cap m$.
Draw a line from C to any point on b which intersects m at B'' and b at B'.
Axiom 3 applied to $\Delta CAB''$ and points B', B guarantees that b meets a at □

#7.



H(AB, CD) <=> ∃ quadrangle PQRS such that

PR meets C and SQ meets D

Let M be the intersection point of DR and CQ.

We have H(CD, AB) with a quadrangle OQMR

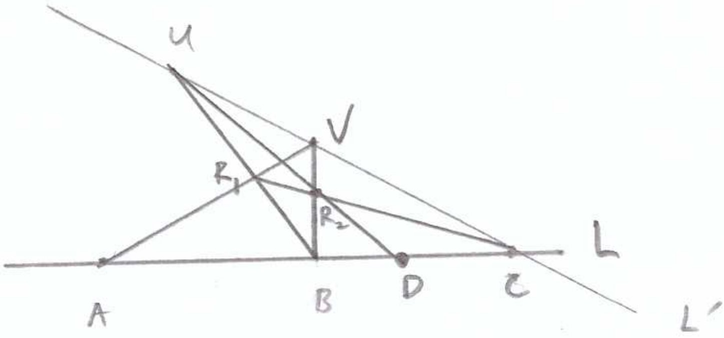
if A, O, M are collinear.

Since SP and RQ meet at B,
OP and MQ meet at C,
SO and RM meet at D,

By Desargue's theorem, A, O, M are collinear

Hence the result. □

#8.



At C draw any line L' and take two points U, V. Draw UB and VA

Let $R_1 = \binom{UB}{VA}$. Draw VB, let $R_2 = \binom{VB}{RC}$

Then the fourth point D is obtained by the intersection of L and UR_2. $\square$.