미분기하학 II 기말고사

#1. (1) F: Geodesics are invariant under isometries

(2) F: Consider any hyperbolic point

(3) T

(4) F: Stereographic projection is an example of Conformal (angle-preserving) mapping which is not an isometry.

(5) T: It is invariant under homeomorphisms so, obviously.

#2. (1) Suppose $K < 0$.
Let $\vec{e}_1, \vec{e}_2$ be principal directions with principal curvatures k_1, k_2 and
let $\vec{v}_1$ be one of asymptotic directions

Then $k_n(\vec{v}_1) = k_1 \cos^2\theta + k_2 \sin^2\theta$ where $\theta = \cos\angle\vec{v}_1, \vec{e}_1$

$= 0.$

Hence $\dfrac{\sin^2\theta}{\cos^2\theta} = -\dfrac{k_1}{k_2}$ ($k_2 \neq 0$ by not being one of asymptotic directions)

$\tan^2\theta =$

Since the angle $-\theta$ also satisfies the above equations

$\vec{v}_2$ (another asymptotic direction) has $\angle e_1, v_2 = -\theta$

and hence the result. When $K = 0$ either all directions are asymptotic directions or there's only one asymptotic dir.
When $K > 0$, there is no asymptotic directions.

#2(2) On every compact surface $M \subset \mathbb{R}^3$, there is a point at which the Gaussian curvature is strictly positive.

proof: Let $f: M \to \mathbb{R}$ defined by

$$\vec{p} = (x_1, x_2, x_3) \mapsto f(\vec{p}) = \| \vec{p} \| = x_1^2 + x_2^2 + x_3^2.$$

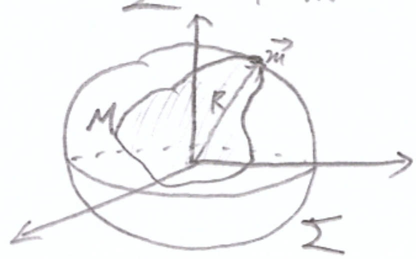
which is clearly smooth and hence continuous.

Recall Lemma: Let $f: M \to \mathbb{R}$ be a continuous function on a compact surface M. Then there exists $\vec{x}_{max}, \vec{x}_{min} \in M$ such that $f(\vec{x}_{max}), f(\vec{x}_{min})$ are maximum and minimum, respectively.

Since M is compact and f is continuous, there exists $\vec{m} \in M$ such that $f(\vec{m})$ is the maximum.

Intuitively, if we consider a sphere of radius R satisfying $R^2 = f(\vec{m})$, at $\vec{m}$, M has to be tangent to a sphere Σ centered at $\vec{0}$ with radius R.

Hence M has to be curved more than Σ at $\vec{m}$ and expect $\underbrace{K(\vec{m}) \overset{?}{=} \frac{1}{R^2}}_{(*)} > 0$.



We prove the inequality (*).

Let α be a unit-speed curve in M s.t. $\alpha(0) = \vec{m}$ and $\alpha'(0) = \vec{u}$.
Here $\vec{u}$ could be any unit tangent vector at $\vec{m} \in M$.
Clearly, $f(\alpha(t))$ attains maximum when $t=0$. So note that

$$\left.\frac{d}{dt}\right|_{t=0} f(\alpha(t)) = 0$$

$$\left.\frac{d^2}{dt^2}\right|_{t=0} f(\alpha(t)) \leq 0.$$

Now $0 = \left.\frac{d}{dt}\right|_{t=0} f(\alpha(t)) = \left.\frac{d}{dt}\right|_{t=0} \left[\alpha(t) \cdot \alpha(t) \right] = 2\alpha'(0) \cdot \alpha(0)$

$$= 2\vec{u} \cdot \vec{m}. \quad \cdots (1)$$

$$0 \geq \left.\frac{d^2}{dt^2}\right|_{t=0} f(\alpha(t)) = 2\alpha''(0) \cdot \alpha(0) + 2\alpha'(0) \cdot \alpha'(0)$$

$$= 2\alpha''(0) \cdot \vec{m} + 2\vec{u} \cdot \vec{u}$$

$$\Leftrightarrow \quad \alpha''(0) \cdot \vec{m} + 1 \leq 0 \quad \cdots (2)$$

The equation (1) means $\vec{m}$ is normal to M.

In equation (2), $\alpha''(0) \cdot \frac{\vec{m}}{R}$ is the normal curvature of M at $\vec{m}$ in the $\vec{u}$-direction ($\because R = |\vec{m}|$).

Hence $k(\vec{u}) \leq -\frac{1}{R}$.

$$\therefore K(\vec{m}) \geq \frac{1}{R^2} > 0.$$

#3 In $\mathbb{R}^3$, $\mathbf{x}(u,v) = (u^2+v,\ u-v^2,\ uv)$

$P = \mathbf{x}(1,2)$. Find an equation of tangent plane at P. Also find the mean curvature $H(p)$.

$\boxed{\text{Intent:}}$ (1) Equation of tangent plane

(2) Mean curvature.

Solution: Note that $\mathbf{x}(1,2) = (3,-3,2)$

$\mathbf{x}_u = (2u, 1, v)$ $\qquad \mathbf{x}_u(1,2) = (2,1,2)$

$\mathbf{x}_v = (1, -2v, u)$ $\qquad \mathbf{x}_v(1,2) = (1,-4,1)$

$\mathbf{x}_u \times \mathbf{x}_v\,(1,2) = (9,0,-9)$

$\|\mathbf{x}_u \times \mathbf{x}_v\| = 9\sqrt{2}$.

$\vec{n}(p) = \dfrac{\mathbf{x}_u \times \mathbf{x}_v}{\|\mathbf{x}_u \times \mathbf{x}_v\|}(1,2) = \left(\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right)$

An equation of tangent plane : $\frac{1}{\sqrt{2}}x - \frac{1}{\sqrt{2}}z = d$

At P, $\frac{1}{\sqrt{2}}\cdot 3 - \frac{1}{\sqrt{2}}\cdot 2 = 1/\sqrt{2}$.

$\therefore x - z = 1$.

$\mathbf{x}_{uu} = (2,0,0)$ $\qquad L = \sqrt{2}$ $\qquad E(p) = 9$

$\mathbf{x}_{uv} = (0,0,1)$ $\qquad M = -\frac{1}{\sqrt{2}}$ $\qquad F(p) = 0$

$\mathbf{x}_{vv} = (0,-2,0)$ $\qquad N = 0$ $\qquad G(p) = 18$

$$H = \frac{EN + LG - 2FM}{2(EG - F^2)} = \frac{\sqrt{2}}{18}$$

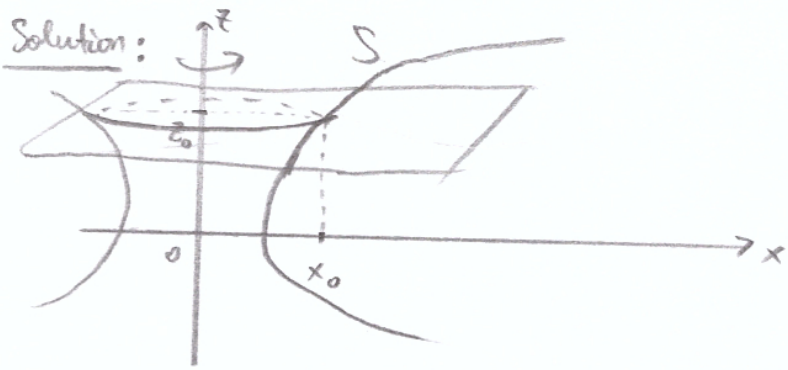
#4 In $\mathbb{R}^3$, S be a surface obtained by rotating about the z-axis a curve $x = z^2+1$. Let C_{z_0} be the intersection curve $S \cap \{(x,y,z) : z = z_0\}$. On the surface, find the supremum of $\left| \int_{C_z} \kappa_g \, ds \right|$.

$\boxed{\text{Intent}}$ (1) Geometric intuition

(2) Geodesic curvature

(3) ML inequality of line integral.

Solution:



Let $x_0 = z_0^2+1$

$$C_{z_0}(t) = (x_0 \cos t, x_0 \sin t, z_0)$$

$$C_{z_0}'(t) = (-x_0 \sin t, x_0 \cos t, 0)$$

$$C_{z_0}''(t) = (-x_0 \cos t, -x_0 \sin t, 0)$$

$$C_{z_0}' \times C_{z_0}'' = (0, 0, x_0^2)$$

$$\kappa_g = \frac{2z_0}{(z_0^2+1)\sqrt{1+4z_0^2}}$$

Parametrization of S:

$$X(u,v) = ((u^2+1)\cos v, (u^2+1)\sin v, u)$$

$$X_u = (2u\cos v, 2u \sin v, 1)$$

$$X_v = (-(u^2+1)\sin v, (u^2+1)\cos v, 0)$$

$$X_u \times X_v = (-(u^2+1)\cos v, -(u^2+1)\sin v, 2u(u^2+1))$$

$$\|X_u \times X_v\| = \left[(u^2+1)^2 + 4u^2(u^2+1)^2\right]^{1/2} = (u^2+1)\sqrt{1+4u^2}$$

$$\vec{n} = \left(-\frac{\cos v}{\sqrt{1+4u^2}}, -\frac{\sin v}{\sqrt{1+4u^2}}, \frac{2u}{\sqrt{1+4u^2}} \right)$$

$$\vec{n}\Big|_{C_{z_0}} = \left(-\frac{\cos t}{\sqrt{1+4z_0^2}}, -\frac{\sin t}{\sqrt{1+4z_0^2}}, \frac{2z_0}{\sqrt{1+4z_0^2}} \right)$$

Hence $\left| \int_{C_z} \kappa_g \, ds \right| \le \sup \frac{2|z|}{(z^2+1)\sqrt{1+4z^2}} \cdot 2\pi(z^2+1) = \frac{4\pi|z|}{\sqrt{1+4z^2}} = \frac{4\pi}{\sqrt{\frac{1}{z^2}+4}} \xrightarrow{|z|\to\infty} 2\pi.$

#5

In $\mathbb{R}^3$ two surfaces

$S_1: z = xy$

$S_2: z = ax^2+by^2$ are isometric under $F: S_1 \to S_2$.

Find ab, where $a, b \in \mathbb{R}$.

$\boxed{\text{Intent:}}$ (1) Gauss Theorema Egregium

(2) Gauss Curvature Calculation.

Solution: We shall compare their Gauss curvature.

Let

$S_1 = X(u,v) = (u, v, uv)$

$S_2 = Y(x,y) = (x, y, ax^2+by^2)$

$X_u = (1, 0, v)$ | $Y_x = (1, 0, 2ax)$

$X_v = (0, 1, u)$ | $Y_y = (0, 1, 2by)$

$X_u \times X_v = (-v, -u, 1)$ | $Y_x \times Y_y = (-2ax, -2by, 1)$

$\vec{n}_X = (u^2+v^2+1)^{-1/2}(-v, -u, 1)$ | $\vec{n}_Y = (-2ax, -2by, 1)\cdot(4a^2x^2+4b^2y^2+1)^{-1/2}$

$X_{uu} = (0,0,0)$ | $Y_{xx} = (0, 0, 2a)$

$X_{uv} = (0,0,1)$ | $Y_{xy} = (0,0,0)$

$X_{vv} = (0,0,0)$ | $Y_{yy} = (0,0,2b)$

$E = 1+v^2$ $\quad L = 0$

$F = uv$ $\quad M = \frac{1}{(u^2+v^2+1)^{1/2}}$

$G = 1+u^2$ $\quad N = 0$

$e = 1+4a^2x^2$ $\quad l = \frac{2a}{(4a^2x^2+4b^2y^2+1)^{1/2}}$

$f = 4abxy$ $\quad m = 0$

$g = 1+4b^2y^2$ $\quad n = \frac{2b}{(4a^2x^2+4b^2y^2+1)^{1/2}}$

$$K_X = -\frac{1}{(u^2+v^2+1)^{3/2}} \qquad K_Y = \frac{4ab}{(4a^2x^2+4b^2y^2+1)^2}$$

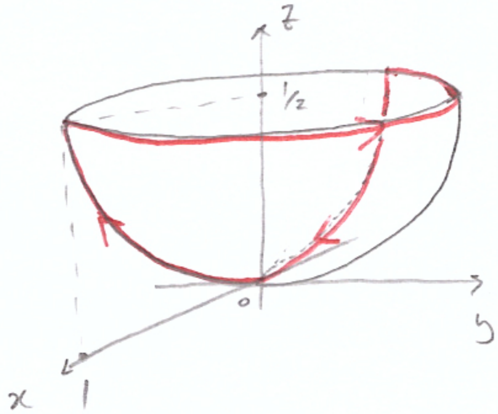
Now by Gauss Theorema Egregium, $K(Y(x,y)) = K(X(u,v))$ if $Y(x,y) = F(X(u,v))$. That said at (u,v) where K_X attains the unique extremum, K_Y is the unique extremum at $Y(x,y) = X(u,v)$. That is when $(u,v)=(0,0)=(x,y)$. Hence $-1 = 4ab$. $ab = -1/4$. □

#6

$M: \mathbf{x}(u,v) = (u\cos v,\ u\sin v,\ \frac{1}{2}u^2)$ $(u \geq 0,\ 0 \leq v \leq 2\pi)$

$P := \{\mathbf{x}(u,v) : 0 \leq u \leq 1,\ 0 \leq v \leq \pi\}$.

Find $\left|\int_{\partial P} k_g \, ds\right|$ where ∂P: boundary of P.



<u>Solution</u>: We Consider ① $\mathbf{x}(u,0) = (u, 0, \frac{1}{2}u^2)$, $u \in [0,1]$.

② $\mathbf{x}(u,\pi) = (-u, 0, \frac{1}{2}u^2)$, $u \in [0,1]$

③ $\mathbf{x}(1,v) = (\cos v, \sin v, \frac{1}{2})$, $v \in [0,\pi]$.

$\partial P = ① + ③ - ②$.

Case ③ $\alpha(v) := \mathbf{x}(1,v)$

$$\vec{n}_\alpha = \frac{\mathbf{x}_u \times \mathbf{x}_v}{\|\mathbf{x}_u \times \mathbf{x}_v\|} = \left(\frac{\cos v}{\sqrt{2}}, -\frac{\sin v}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

$\mathbf{x}_u = (\cos v, \sin v, u)$ $\qquad \alpha'(v) = (-\sin v, \cos v, 0)$

$\mathbf{x}_v = (-u\sin v, u\cos v, 0)$ $\qquad \alpha''(v) = (-\cos v, -\sin v, 0)$

$\mathbf{x}_u \times \mathbf{x}_v = (u^2\cos v, -u^2\sin v, u)$ $\qquad \alpha' \times \alpha'' = (0, 0, 1)$

$\|\mathbf{x}_u \times \mathbf{x}_v\| = \sqrt{u^4 + u^2} = u\sqrt{u^2+1}$

$k_g = \vec{n}_\alpha \cdot (\alpha' \times \alpha'') = \frac{1}{\sqrt{2}}$ $\qquad$ Hence $\left|\int_\alpha k_g \, ds\right| = \left|\int_\alpha ds\right|$

$= \frac{\pi}{\sqrt{2}}$.

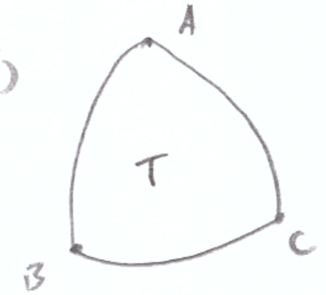
Since k_g for Case ①, ② are 0, $\left|\int_{\partial P} k_g \, ds\right| = \frac{\pi}{\sqrt{2}}$.

($\because$ curves in Case ①, ② are geodesics)

#7.

(1)

A spherical triangle



By the Gauss–Bonnet formula,

$$\underbrace{\int_{\partial T} k_g \, ds}_{(1)} + \underbrace{\iint_T K \, dS}_{(2)} = 2\pi - \sum \text{exterior angles} = \angle A + \angle B + \angle C - \pi$$

Note that AB, BC, CA : pieces of great circles.

Hence $k_g \equiv 0$, thus (1) $= 0$.

Also on a sphere $K \equiv \frac{1}{R^2}$. Hence (2) $= \frac{1}{R^2}$ Area of T

Therefore $\frac{A}{R^2} = (\angle A + \angle B + \angle C - \pi)$

(2) An ellipsoid is a compact orientable surface with genus 0. Hence

$$\iint_{\varepsilon} K \, dA = 2\pi \chi(\varepsilon) = 2\pi(2 - 2\cdot g^{=0}) = 4\pi.$$

#8.

Because M is compact, by problem #2(2), there is a point p in M such that $K(p) > 0$

Now from $g > 0$, by Gauss - Bonnet theorem

$\iint_M K \, dA \leq 0$. Because the Gauss curvature K is a function

on M into $\mathbb{R}$ which is continuous, if we assume

there is no $q \in M$ such that $K(q) < 0$, that would

imply $\iint_M K \, dA > 0$, because we already know the

fact that $\exists p \in M$ s.t. $K(p) > 0$. However $\iint_M K \, dA > 0$ contradict to $g > 0$.

Hence there exists $q \in M$ such that $K(q) < 0$

Now because M is connected, by intermediate Value theorem, $\exists r \in M$ s.t. $K(r) = 0$

$\square$