Review Problems for Chapters 1 and 2
Spring 2016 MATH 250 Section 02

Review Problems

1. Prove the following result: $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (a-b)(b-c)(c-a).$

Remark: This is called a *Vandermonde determinant* that has a general form

$$\begin{vmatrix} 1 & a_1 & a_1^2 & \cdots & a_1^{n-1} \\ 1 & a_2 & a_2^2 & \cdots & a_2^{n-1} \\ 1 & a_3 & a_3^2 & \cdots & a_3^{n-1} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 1 & a_n & a_n^2 & \cdots & a_n^{n-1} \end{vmatrix} = \prod_{1 \le q \le p \le n} (a_p - a_q).$$

2. Prove that any open ball in $\mathbb{R}^n$ is an open set.

3. Prove by using $\epsilon - \delta$ method that $\lim_{(x,y)\to(0,0)} \frac{x^2}{\sqrt{x^2+y^2}}$ is 0.

4. Find the equation of the plane tangent to the surface $z = x^2 + y^3$ at $(3, 1, 10)$.

5. Decide if each of the given functions is differentiable.

$$(1) \quad f(x,y) = \frac{xy}{\sqrt{x^2+y^2}}$$

$$(2) \quad f(x,y) = \frac{2xy}{(x^2+y^2)^2}$$

6. [One of these will be asked in exam identically] (1) Prove if the following statement is true, or disprove by giving an example if it is false:
Let $f : A \subset \mathbb{R}^n \to \mathbb{R}$ be a continuous function on A whose all first partial derivatives $\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}$ exist at $\overrightarrow{x}_0 \in A$. Then the function f is differentiable.

(2) Give an example of a continuous function in more than one variable that is not differentiable but whose partial derivatives exist.

(3) Prove if the following statement is true, or disprove by giving an example if it is false:
Let $f : A \subset \mathbb{R}^n \to \mathbb{R}$ be a function on A whose all first partial derivatives $\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}$ exist at $\overrightarrow{x}_0 \in A$. Then the function f is continuous.

(4) Give an example of a differentiable function that has a partial derivative which is not continuous.

7. Find a unit vector normal to the surface S given by $z = x^2y^2 + y + 1$ at $(0, 0, 1)$.

8. Suppose $f : \mathbb{R}^3 \to \mathbb{R}$ is differentiable, and $\nabla f(\overrightarrow{x}) \neq 0$. Show that the direction of $\nabla f(\overrightarrow{x})$ is that f is increasing the fastest.

9. Let $f : \mathbb{R}^3 \to \mathbb{R}$ be differentiable at $\overrightarrow{x}_0 \in \mathbb{R}^3$. Prove that

$$\frac{|f(\overrightarrow{x}) - f(\overrightarrow{x}_0)|}{\|\overrightarrow{x} - \overrightarrow{x}_0\|}$$

is bounded by a positive constant. (Hint: Use the triangle inequality and the Cauchy-Schwarz inequality)

10. Let j be the coordinate change map from the spherical coordinate to the cartesian coordinate defined by

$$\begin{aligned} x &= r\cos\theta\sin\phi \\ y &= r\sin\theta\sin\phi \\ z &= r\cos\phi \end{aligned}$$

Also let $f : \mathbb{R}^3 \to \mathbb{R}$ be a differentiable map. Calculate $D(f \circ j)$.

11. Given $g(x, y) = (x - y, x + y)$ and $f(u, v) = (u - v, u + v, uv)$, compute the derivative of $f \circ g$ at the point $(1, 1)$ using the chain rule.

If you need any help, please feel free to send an email (or multiple emails) to `byungdpark@gmail.com`.