Final Exam

Fall 2019, Differential Geometry II

Mathematics Education, Chungbuk National University

16.12.2019 14:00–16:00

Instructions: On each page of your answer sheet, please write your name, page number, and total pages for example "홍길동 2/4면." Be sure to use your answer sheets as single-page. If you want some portion of your writings on your answer sheet not to be graded, just cross it out. You are not allowed to use your textbook or notes. You cannot use any electronic device in this exam. You are not allowed to talk to other students. Please write all details explicitly. Answers without justifications and/or calculation steps may receive no score.

1. Answer whether each of the following statements is true or false. No need to give reasons or details. **Just say true or false**. 2 points for each correct answer, 0 point for no answer, and -2 points for each incorrect answer.
(1) There exists a regular surface that has a geodesic which is not invariant under isometries.
(2) Every asymptotic directions are principal directions.
(3) A right cone and a 2-dimensional plane are locally isometric.
(4) The stereographic projection is an isometry.
(5) The Euler characteristic of a surface is invariant under diffeomorphisms.

2. (1) Prove that the directions of curvature bisect the asymptotic directions, when the Gauss curvature at the point of consideration is positive, zero, and negative. [10 points]

(2) Prove that on every compact surface $M \subset \mathbb{R}^3$, there is a point at which the Gauss curvature is strictly positive. [10 points]

3. In a 3-dimensional Euclidean space $\mathbb{R}^3$, a surface $\mathbf{x}(u,v) = (u^2+v, u-v^2, uv)$ is given. Let $P = \mathbf{x}(1,2)$. Find an equation of tangent plane at P. Also find the mean curvature H at P. [10 points]

4. In a Euclidean space $\mathbb{R}^3$, the surface S is obtained by rotating about the z-axis the curve $x = z^2+1$ on xz-plane. Let C_{z_0} be the intersection between the surface S and the plane $z = z_0$. Find the supremum (i.e. least upper bound) of $\left|\int_{C_z} \kappa_g ds\right|$ the absolute value of total geodesic curvature of the curve C_z in S. [10 points]

5. In a 3-dimensional Euclidean space $\mathbb{R}^3$, suppose that there is an isometry $F: S_1 \to S_2$ between two surfaces $S_1: z = xy, \quad S_2: z = ax^2 + by^2$. Find the value of ab for two real numbers a and b. [10 points]

6. Let

$$M: X(u,v)\left(u\cos v, u\sin v, \frac{1}{2}u^2\right) \quad (u \geq 0, 0 \leq v \leq 2\pi)$$

be a surface in a 3-dimensional Euclidean space $\mathbb{R}^3$ and $S = \{X(u,v) | 0 \leq u \leq 1, 0 \leq v \leq \pi\}$ be a region in the surface M. Find the absolute value of the total geodesic curvature $\left|\int_{\partial S} \kappa_g ds\right|$ of the geodesic curvature κ_g of the curve ∂S, the boundary of S, in M. Here the parameter s denotes the arc-length.[10 points]

7. (1) Prove the Cavalieri's theorem: Let $\triangle ABC$ be a spherical triangle on a sphere of radius R and A be the area of $\triangle ABC$. Then

$$\frac{A}{R^2} = (\angle A + \angle B + \angle C - \pi)$$

[10 points]

(2) Calculate the total curvature $\iint_{\mathcal{E}} K dA$ of the ellipsoid $\mathcal{E}$: $\frac{x^2}{1969^2} + \frac{y^2}{1978^2} + \frac{z^2}{2019^2} = 1$. [5 points]

8. Suppose that a compact, connected, and orientable surface M is not homeomorphic to a sphere. Prove that there are points on M satisfying that the Gauss curvature is positive, negative, and zero. [15 points]