

# Differential Cohomology and gerbes:

An introduction to higher differential geometry

## 1. Čech cohomology and characteristic classes

Def: Let  $G$  be a Lie group,  $M$ : a smooth mfld.

A Principal  $G$ -bundle  $P$  is a smooth map

$\pi: P \rightarrow M$  and a right  $G$ -action on  $P$  s.t.

(1)  $\pi$  is  $G$ -invariant.

(2) On every fiber,  $G$  acts on  $P$  freely and transitively from the right.

(3)  $P$  is locally trivial via  $G$ -equivariant trivialization

Examples (or a prop) when  $G = GL_n(\mathbb{C})$

"associated fiber bundle"

A  $G$ -bundle  $P \xrightarrow{\pi} M$



A complex vector bundle  
 $E \rightarrow M$ .

frame bundle  
(or bundle of bases)

rank  $n$

So we don't really distinguish a  $U(1)$ -bundle and

$\mathbb{C}^*$ -  
 $S^1$

complex line bundles.

Def: Let  $G$  be an abelian group,  $M$ : a top. space.

$\mathcal{U} = \{U_i\}_{i \in I}$ : an open cover of  $M$ .

$$\check{C}^p(U; G) := \left\{ f_{i_0 \dots i_p} : U_{i_0 \dots i_p} \rightarrow G \right\}_{i_0, \dots, i_p \in I}$$

Notation:  $U_{i_1, i_2, \dots, i_n} := U_{i_1} \cap \dots \cap U_{i_n}$

$$\delta : \check{C}^p(U; G) \rightarrow \check{C}^{p+1}(U; G)$$

$$f \longmapsto (\delta f)_{i_0 \dots i_p i_{p+1}} := f_{i_0, \dots, i_p, i_{p+1}} - f_{i_0, \dots, \overset{\textcolor{red}{i_{p+1}}}{i_{p+1}}, \dots, i_{p+1}}$$

So  $(\check{C}^\bullet(U; G), \delta_\cdot)$  is a complex

Def:  $\check{H}^p(U; G) := \ker \delta / \operatorname{Im} \delta$  Cech cohomology  
of  $M$  defined on  $\mathcal{U}$   
of degree  $p$

Question What goes wrong if  $G$  is not abelian.

- $p=0$ ; No prob! ( $\because \delta f = 0 \Leftrightarrow f_i = f_j$ , so  $\check{H}^0(U; G) = \operatorname{Map}(M, G)$ )

- $p=1$ ;  $\check{H}^1(U; G)$  is not a group

It is a pointed set,  $*$  = constant map to  $1_G$

Neither  $\check{Z}^1(U; G)$  nor  $\check{B}^1(U; G)$   
form a subgroup.

But we can use an equivalence rel.

$$g'_{ij} \sim g_{ij} : U_{ij} \rightarrow G \iff g'_{ij} = f_j^{-1} g_{ij} f_i$$

and define  $\check{H}^1(U; G) = \ker \delta / \sim$

$\bullet p \geq 2 \quad \check{H}^p(U; G) : \text{no hope.}$

Prop Let  $1 \rightarrow k \xrightarrow{i} \tilde{G} \xrightarrow{j} G \rightarrow 1$  be a <sup>(\*)</sup> S.G.S.

We have a L.G.S.

$$1 \rightarrow \check{H}^0(U; k) \rightarrow \check{H}^0(U; \tilde{G}) \rightarrow \check{H}^0(U; G)$$

$$\hookrightarrow \check{H}^1(U; k) \rightarrow \check{H}^1(U; \tilde{G}) \rightarrow \check{H}^1(U; G)$$

Prop: If the S.G.S. (\*) has  $k$ : abelian,  
and  $i(k) \subseteq \text{Center}(\tilde{G})$ , then the LGS(\*\*)  
extends to  $\check{H}^2(U; k)$ .

$$1 \rightarrow \check{H}^0(U; k) \rightarrow \check{H}^0(U; \tilde{G}) \rightarrow \check{H}^0(U; G)$$

$$\hookrightarrow \check{H}^1(U; k) \rightarrow \check{H}^1(U; \tilde{G}) \rightarrow \check{H}^1(U; G)$$

$$2 \hookrightarrow \check{H}^2(U; k)$$

Furthermore,

Thm (Dixmier - Douady) If the sheaf  $\tilde{G}_U$  is soft

( i.e.  $\Gamma(M, \mathcal{A}) \xrightarrow{\text{re}} \Gamma(C, \mathcal{A})$  : onto )  
 q  
 any closed set

then  $\delta: \check{H}^1(U; G) \rightarrow \check{H}^2(U; K)$  is a bijection.

Note:  $\check{H}^1(M; G)$  classifies principal  $G$ -bundles over  $M$  up to isomorphism

Having  $G \rightarrow P \downarrow M$   $\leftrightarrow$  Having  $g_{ij}: U_{ij} \rightarrow G$   $\forall_{ij \in I}$   
 s.t.  $g_{kj}(x) \circ g_{jk}(x) \circ g_{ij}(x) = 1_G \quad \forall x \in U_{ijk}$   
 (cocycle condition)

Having  $P \rightarrow P' \downarrow M$   $\leftrightarrow$  Having  $\{f_i: U_i \rightarrow G\}$   
 s.t.  $g_{ij}'(x) = f_j^{-1}(x) f_i(x) g_{ij}(x)$ .

In general  $\check{H}^1(U; G) \xrightarrow{\text{if } U \text{ is good}} \check{H}^1(M; G) := \varinjlim_U \check{H}^1(U; G)$

If  $U$  is good.

Def: A cover  $U$  of  $M$  is good if all open sets and their intersections are contractible.

Example: (1)  $1 \rightarrow SO(n) \rightarrow O(n) \rightarrow \mathbb{Z}_2 \rightarrow 1$

Given  $[P] \in \text{Bun}_{O(n)}(M) / \cong \mapsto w_1([P])$  : the first Stiefel-Whitney class.  
 $w_1: \check{H}^1(M; O(n)) \longrightarrow \check{H}^1(M; \mathbb{Z}_2)$

Note  $w_1(P) = 0 \Leftrightarrow P$  comes from an  
SO(n)-bundle  
 $P$ : orientable

$$(2) \quad 1 \rightarrow \mathbb{Z}_2 \rightarrow \text{Spin}(n) \rightarrow \text{SO}(n) \rightarrow 1.$$

$$\check{H}^1(M; \text{SO}(n)) \longrightarrow \check{H}^2(M; \mathbb{Z}_2)$$

$[P] \longmapsto w_2(P)$  : the second  
Stiefel-Whitney  
class

Note :  $w_2(P) = 0 \Leftrightarrow P$  comes from a  
Spin(n)-bundle.

Remark The Whitehead tower of  $O(n)$

$$\dots \xrightarrow{\text{Fivebrane}} \text{String}(n) \rightarrow \text{Spin}(n) \rightarrow \text{SO}(n) \rightarrow O(n)$$

$$(3) \quad 1 \rightarrow \mathbb{Z} \rightarrow \text{(R)} \xrightarrow{\text{soft} (\text{: Tietze extension})} S^1 \rightarrow 1$$

$$c_1 : \check{H}^1(M; S^1) \xrightarrow{\cong} \check{H}^2(M; \mathbb{Z}) \cong H_{\text{sing}}^2(M; \mathbb{Z})$$

$$[L] \longmapsto c_1(L) : \text{the first Chern class}$$

$$(4) \quad \text{Thm (Dixmier - Donadhy)} \quad \underbrace{U(H)}_M : \text{soft}.$$

$$\text{Consider } 1 \rightarrow U(1) \rightarrow U(H) \rightarrow PU(H) \rightarrow 1$$

$$\text{DD} : \check{H}^1(M; PU(H)) \xrightarrow{\cong} \check{H}^2(M; S^1) \cong H_{\text{sing}}^3(M; \mathbb{Z})$$

$$[P] \longmapsto \text{DD}(P)$$

Def A characteristic class  $c$  of a  $G$ -bundle  $P \rightarrow M$

is an assignment

$$c : \text{Bun}_G(M) / \sim \longrightarrow H^*(M; A)$$

Ab. gp.

$[P] \xrightarrow{\quad} c(P)$

that is natural (i.e. for  $\begin{array}{ccc} \bar{P} & \xrightarrow{f} & P \\ \downarrow & & \downarrow f^* \\ M & \xrightarrow{f} & M \end{array}$ )

$$= c(\bar{f}^* P)$$

Remark (Yoneda Lemma)

$$\text{Bun}_G(-) \cong \text{Maps}(-, BG)$$

An assignment

$$\left\{ \begin{array}{l} \text{Characteristic} \\ \text{class of } P \in \text{Bun}_G(M) \end{array} \right\} \rightarrow H^*(BG; A)$$

is 1-1 and onto.

Remark There's an alternative way to define characteristic classes using form. data.

"Chern - Weil"

$$\text{E.g. } L, \nabla \rightsquigarrow \text{Chern-Weil form}$$

$$c(\nabla) = \frac{i}{2\pi} \text{Curv}(\nabla)$$

$$\text{Thm } [c_1(\nabla)]$$

is indep. of  
choice of  $\nabla$ .

$H^2(M; \mathbb{Z})$  classifies Line bundles  
 $H^3(M; \mathbb{Z})$  classifies "Gerbés"

What classifies line bundles w/ conn? ( $L, D$ )  
 $H^2(M)$   
 Gerbes w/ conn?  $H^3(M)$   
 higher gerbes w/ conn?  $H^{\bullet}(M)$

2023. 12. 18. 21:30  
 IUT Math & Stat Research Seminar

## 2. Cheeger - Simons differential characters

"Diff.-characters and geom. invariants"

Notations :  $M$  : Smooth manifold

$C^k(M; R)$  : Smooth singular  $k$ -cochains in  $M$  with coeff. in  $R$   
 $R$ : Comm. ring with 1.

$Z^k(M; R)$  : Smooth singular  $k$ -cocycles in  $M$  w/ coeff. in  $\mathbb{Z}$ .

$\Omega^k(M)$  : diff.  $k$ -forms on  $M$ .

$\Omega_{cl}^k(M)_{\mathbb{Z}}$  : closed  $k$ -forms with integral periods  
 i.e.  $\omega \in \Omega_{cl}^k(M)_{\mathbb{Z}} \Leftrightarrow \begin{cases} d\omega = 0 \\ \int \omega \Big|_{Z_k(M)} \in \mathbb{Z} \end{cases}$

Notation  $\sim : \mathbb{R} \rightarrow \mathbb{R}/\mathbb{Z}$

\* Fact: Let  $1 \subsetneq \mathbb{R}$  : proper subring. Then a nonvanishing

diff. form does not take values in  $\mathbb{Z}$ .

$$\text{Prop : The map } \int : \Omega^k(M) \rightarrow \widetilde{\mathcal{C}^k}(M; \mathbb{R}/\mathbb{Z})$$

$$\omega \longmapsto \int \omega$$

is one-to-one.

Def (Cheeger-Simons differential characters)

$$\widehat{H}^k(M) := \left\{ (\chi, \omega) : \begin{array}{l} \chi \in \text{Hom}_{\mathbb{Z}}(Z_{k-1}(M), \mathbb{R}/\mathbb{Z}) \\ \omega \in \Omega^k(M) \\ \text{s.t. , for } \forall D \in C_k(M; \mathbb{Z}) \end{array} \right\}$$

$$\chi \circ \partial D = \int_D \omega \text{ mod } \mathbb{Z}$$

with the componentwise addition is the differential characters (or differential cohomology) of  $M$  with degree  $k$ .

Prop: (Differential cohomology hexagon diagram)

With all triangles and Squares commute,  
 diagonal) are SESs, and the upper and lower  
 Sequences are LESs.

Proof: (1)  $I, R$  maps

Algebra facts we need:

- A1: Any subgroup of a free abelian group is free.
- A2: An ab.gp.  $G$  is divisible if  $\forall x \in G$  and  $\forall n \in \mathbb{Z}$   
 $\exists y$  s.t.  $x = ny$ .
- A3: Prop:  $G$ : divisible  $\Leftrightarrow$   $G$  is an injective obj.  
 in the cat. of ab.gps

Take  $(\mathbf{1}, \omega) \in \widehat{H}^k(M)$ .

$$\begin{array}{ccc}
 & \textcircled{0} & \\
 & \downarrow & \\
 \mathbb{Z}_{k+1}(M) & \xrightarrow{\bar{\pi}} & \mathbb{R} \\
 \downarrow & & \text{divisible} \\
 C_{k+1}(M) & \xrightarrow{\exists T} & \mathbb{R}
 \end{array}$$

$\exists \bar{x} \dashrightarrow \mathbb{R}$   
 $\xrightarrow{\pi} \mathbb{R}/\mathbb{Z}$   
 $\xrightarrow{\mathbb{R}/\mathbb{Z}} \mathbb{R}$   
 $\xrightarrow{\exists T}$

$\sum_{k+1}(M)$  free  $\Rightarrow$  projective  
 $(\because \text{A1})$

So  $T|_{\mathbb{Z}_{k+1}(M)} = x$ .

$$\tilde{\delta T} = \int \tilde{T} = \tilde{T} \circ \delta = \int \omega \bmod \mathbb{Z}$$

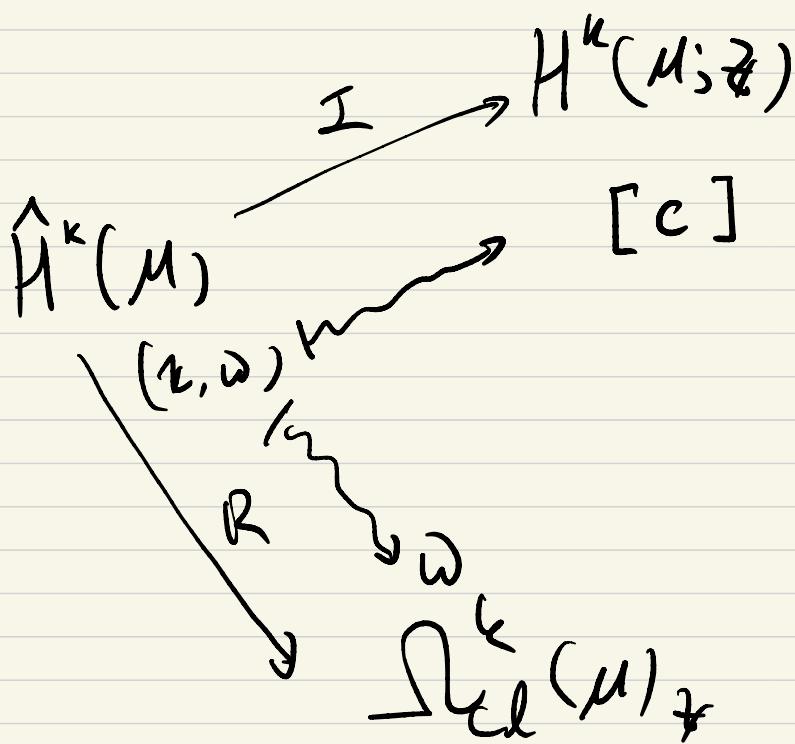
$$\text{So } \delta T = \int \omega - c \quad \exists c \in C^k(M; \mathbb{R})$$

$$\underline{\text{Note:}} \quad 0 = \delta^2 T = \int d\omega - \delta c$$

$$\text{So } \int d\omega = \delta c \quad \xrightarrow{\text{★}} \quad d\omega \equiv 0 = \delta c$$

Exercise (easy) Verify that  $\omega \in \Omega_{cl}^k(M)_{\mathbb{R}}$ .

$$(\because \int \omega \Big|_{Z_k(M)} = c + \delta T \Big|_{Z_k(M)} \xrightarrow{?} )$$



Exercise: Verify the well-definedness of  $I, R$  maps.  
(i.e. indep. of choice of lifts)

Pf.: Let  $T'$ : another lift, satisfying  $\delta T' = \int \omega' - c'$ .

$$\text{Then } \overbrace{T' - T \Big|_{Z_k(M)}}^{d\omega} = 0 \quad \text{so } T' = T + \delta s + d \underbrace{s \in C^{k-1}_{cl}(M; \mathbb{R})}_{\text{and } c' = c + \delta s}$$

$$\delta T' = \delta T + 0 + \delta d$$

$$\Leftrightarrow \int \omega' - c' = \int \omega - c + \delta d$$

$$\Leftrightarrow \int (\omega' - \omega) = c' - c + \delta d$$

~~\*~~

$$\Rightarrow \omega' = \omega \quad \text{and} \quad [c'] = [c]. \quad \checkmark$$

(2) Surjectivity of  $I, R$

Prop (Exercise) Given  $\omega \in \Omega_{cl}^k(M; \mathbb{Z})$ ,  $\exists u \in H^k(M; \mathbb{Z})$   
 s.t.  $r(u) = [\int \omega]$ .

Let  $u = [c]$ . Then  $c - \int \omega = \underbrace{\lambda}_{\lambda \in C^{k-1}(M; \mathbb{R})}$  for some

Define  $\chi := \overbrace{\lambda|_{Z_{k-1}(M)}}^{} \quad \text{i.e. } R: \text{onto.}$

Given any  $[c] \in H^k(M; \mathbb{Z})$

$\delta c = 0$  as real cochains. By de Rham theorem,

$\exists \omega \in \Omega_{cl}^k(M)$  s.t.  $\int \omega - c = \delta u$ .

Define  $\chi := \overbrace{u|_{Z_{k-1}(M)}}^{} \quad$

So  $I: \text{onto.}$

Exercise  $e: H^{k+1}(M; \mathbb{R}/\mathbb{Z}) \rightarrow \widehat{H}^k(M)$   
 $[x] \longmapsto (x|_{Z_{k-1}(M)}, 0)$

Show that  $\varphi$  is well-defined, one-to-one (univalent  
coeff. thm)

$$\textcircled{2} \quad \text{Im } \varphi = \ker R$$

Exercise:

$$\begin{aligned} \mathcal{L}^{k-1}(M) &\longrightarrow \widehat{\mathcal{H}}^k(M) \\ d &\longmapsto \left( \int_M d\alpha \right)_{\mathcal{Z}_k(M)}, d\alpha \end{aligned}$$

Show that \textcircled{1} kernel is  $\Omega_{cl}^{k-1}(M)$ .

$$\textcircled{2} \quad a: \mathcal{L}^{k-1}(M) / \ker \xrightarrow{\cong} \widehat{\mathcal{H}}^k(M).$$

$$\text{Im } a = \ker I.$$

Exercise

$$\begin{array}{ccc} H^{k+1}(M; \mathbb{R}/\mathbb{Z}) & \xrightarrow{-B} & H^k(M; \mathbb{Z}) \\ \downarrow e & \curvearrowright & \uparrow I \\ \widehat{\mathcal{H}}^k(M) & & \\ \xrightarrow{a} \mathcal{L}^{k-1}(M) & \xrightarrow{d} & \mathcal{Z}_k(M) \\ \downarrow \mathcal{N}_{cl}^{k-1}(M) & & \end{array}$$

Exercise: upper/lower seg. are LFSs.

Examples  $\widehat{H}^0(M) = 0$

$$\widehat{H}^1(M) = C^0(M, \mathbb{R}/\mathbb{Z})$$

Prop: The assignment

$$\pi_0 \text{Prim}_{S^1, D}(M) \xrightarrow{\cong} \widehat{H}^2(M)$$

$$(P, \theta) \mapsto (\kappa, d\alpha)$$

where  $\kappa(r) := \text{Hol}(r)$  r: any loop in M

Extend it to all  $Z_1(M)$  by

$$\kappa(x) := \kappa(r) + \frac{1}{2\pi} \int d\alpha(y) \quad \text{if } x = r + dy$$

Given  $d\alpha \in \Omega_{cl}^2(M; \mathbb{Z})$ , by prop(Exercise),

$$\exists c \in H^2(M; \mathbb{Z}) \text{ s.t. } [\int d\alpha] = r(c).$$

So  $c$  here classifies  $P$ , which is the 1<sup>st</sup> chern class.

Question What's the analogue of "Prop" in  $k=3$ , higher?

We need gerbes w/ connection, holonomy of a gerbe

See: Grzedzki's work on TI and gerbes

(Simons-Sullivan)

Remark:  $H^k(M)$  is determined uniquely by the hexagon diagram.

(Bunke-Schick) uniqueness.

2023.12.25. 21:30  
IUT Math & Stat Research Seminar

### 3. Higher line bundles with Connection

Recall:  $H^2(M; \mathbb{Z})$  is classified by line bundles  
 $\widehat{H}^2(M)$  is classified by line bundles w/  $\nabla$ .

Q: What are corresponding geometric objects representing  
 $H^n(M; \mathbb{Z})$ ,  $\widehat{H}^n(M)$ ?

A:  $(n-2)$ -gerbes,  $(n-2)$ -gerbes w/ connection.

Remark:  $E^\bullet : \text{HoTop}^{\text{op}} \xrightarrow{\text{homotopy invariant}} \text{GrAb}$  satisfying  
(1) Wedge axiom:  $\text{connected CW complexes}$   $E^\bullet(X_1 \wedge X_2) = E^\bullet(X_1) \oplus E^\bullet(X_2)$

(2) Mayer-Vietoris axiom.

Brown representability theorem says  $E^n$  is representable

i.e.  $E^n(-) \simeq \pi_0 \text{Map}(-, E_n)$

If  $E^*$  is a generalized cohomology theory,  
i.e.  $E^*$  satisfies all Eilenberg - Steenrod axioms  
except the dimension axiom.

then there is a 1-1 correspondence with  
S<sub>L</sub>-Spectra

$$E^* \xrightarrow{\text{Brown's thm}} E_* \\ \pi_0 \text{Maps}(-, E_m) \xleftarrow{} E^*$$

So there are at least as many generalised cohomology theories as the number of Spectra.

Examples :  $kU^*$ ,  $KO^*$ , ...  
 $MU$ ,  $MO$ , ...  
 $TMF$ ,  $Ell$ , ...

Question Given  $E^n(X)$ , can we have a  
geometric cocycle representing each class in  $E^n(X)$ ?

→ The Stolz - Teichner Program (Conjecture)

$$\left\{ \begin{array}{l} \text{Supersymmetric field theories} \\ \text{degree } n \text{ over } M \end{array} \right\} / \text{Concordance} \cong E^n(X)$$

How do gerbes arise?

Consider  $1 \rightarrow U(1) \rightarrow \tilde{G} \rightarrow G \rightarrow 1$

Prop: A principal  $G$ -bundle lifts to a principal  $\tilde{G}$ -bundle iff the cocycle representing  $DD(P)$  is trivializable.

Proof: On  $M$ : Smooth mfld, choose a good cover  $\mathcal{U}$  by HLP

(Note:  $U(1) \rightarrow \tilde{G}$  : fibration)  
 $\downarrow$   
 $G$

satisfies HLP

So we have  $\tilde{g}_{ij} \cdot \tilde{g}_{jk} \cdot \tilde{g}_{ki} = \lambda_{ijk} \cdot 1_{\tilde{G}}$   
 where  $\lambda_{ijk} \in C^2(\mathcal{U}; U(1))$ .

Exercise : ①  $\lambda$  is a degree 2 cocycle:  $(\delta \lambda)_{ijkl} = 0$   
 ②  $[\lambda]$  does not depend on the choice of lifting.

$DD(P) = [\lambda] \in H^2(M; U(1)) \cong H^3(M; \mathbb{Z})$

is an obstruction for the existence of lifting. ✓

Let  $Y \xrightarrow{\pi} M$  be a Surjective Submersion.

A fiber product of  $Y \xrightarrow{\pi} M \xleftarrow{\pi} Y$  is

$$Y^{[2]} = Y \times_M Y := \{(y_1, y_2) \in Y^2 : \pi(y_1) = \pi(y_2)\}$$

cf. A fiber product is a Smooth manifold if  $\pi$ : Submersion

Notation:  $Y^{[P]} = \{(y_1, \dots, y_p) \in Y^p : \pi(y_1) = \dots = \pi(y_p)\}$

$$\pi_{i_1, \dots, i_k} : Y^{[P]} \rightarrow Y^{[k]}$$

$$(y_1, \dots, y_p) \mapsto (y_{i_1}, \dots, y_{i_k})$$

Def (Murray) A bundle gerbe is a triple  $\mathcal{L} = (L, \pi, \mu)$

where (1)  $\pi : Y \rightarrow M$ : Surjective Submersion.

$$(2) L \in \text{Prin}_{S^1}(Y^{[2]})$$

$$(3) \mu : \pi_{12}^* L \otimes \pi_{23}^* L \rightarrow \pi_{13}^* L \quad S^1\text{-bundle iso.}$$

$$(4) \mu : \text{associative over } Y^{[4]}$$

Note:  $Y_U := \{(x, i) \in M \times N : x \in U_i\} \subset M \times N$ ,  $i$ : index set of  $U_i$

$$\pi : Y_U \rightarrow M : \text{Surjective Submersion (open cover)}$$

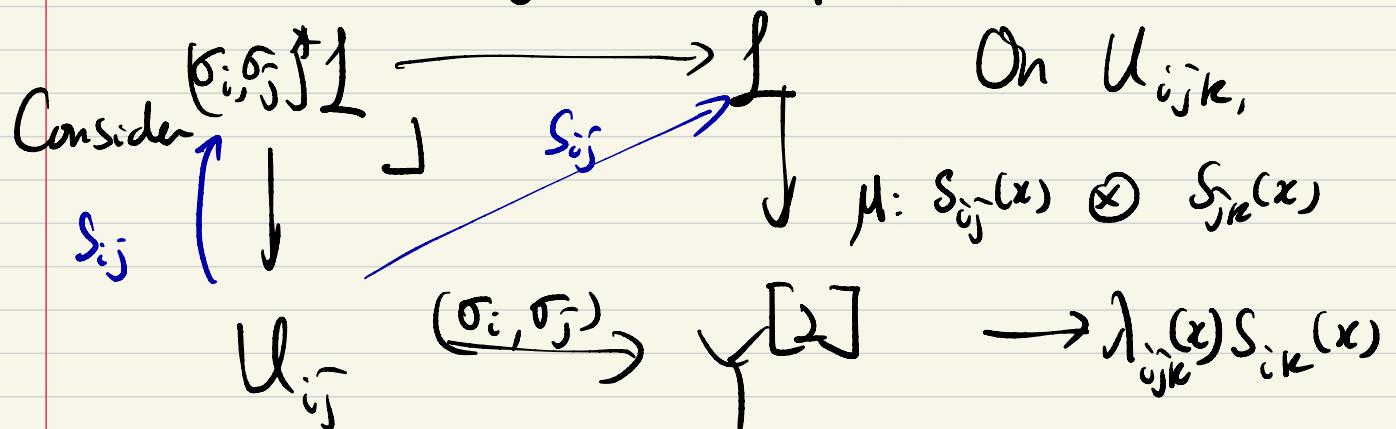
Construction of  $\text{DD}(\mathcal{L})$  (Dixmier-Douady class of  $\mathcal{L}$ )

Let  $\mathcal{L} = (L, \pi: L \rightarrow M, \mu)$   $\in \text{Grb}(M)$

Choose an open cover  $\mathcal{U}$  on  $M$   $= \{U_i\}$ .

We may take local sections  $\sigma_i: U_i \rightarrow Y$

and  $(\sigma_i, \sigma_j): U_{ij} \rightarrow Y^{[2]}$



Exercise (1) Show that the associativity of  $\mu$  implies that  $\{\lambda_{ijk}\}$  is a degree 2 Čech cocycle on  $\mathcal{U}$ .

(2)  $[\lambda]$  does not depend on the choices made.

Define  $\text{DD}(\mathcal{L}) = [\lambda] \in H^3(M; \mathbb{Z})$

Def: A connection on  $\mathcal{L} = (L, \pi, \mu)$  is a connection  $\nabla$  on  $L$  compatible with  $\mu$  i.e.

$$\pi_{12}^*(L, \nabla) \otimes \pi_{23}^*(L, \nabla) \xrightarrow{\mu} \pi_{13}^*(L, \nabla)$$

$\mu: S^1$ -  
 bundle  
 $\hookrightarrow$   
 preserv.  
 connection

Def: A curving  $B$  of  $\mathcal{L} = (L, \pi, \mu)$  with connection  $\nabla$  is a diff. 2-form on  $Y$  s.t.

$$\text{Curv}(\nabla) = \pi_2^* B - \pi_1^* B$$

A bundle gerbe with connection would mean

$$\widehat{\mathcal{L}} = (L, \pi, \mu, \nabla, B) \in \text{Grb}_\nabla(M)$$

Prop: The following sequence is a L.E.S.

$$0 \rightarrow \Omega^k(M) \xrightarrow{\pi^*} \Omega^k(Y) \xrightarrow{\delta} \Omega^k(Y^{[2]}) \rightarrow \dots$$

when  $f := \sum_{k=1}^p (-1)^{k-1} \pi_{i_1 \dots i_k}^*$

Pf Murray. Bott-Tu.

Note  $0 = d \text{Curv}(\nabla) = d \delta B = \delta d B$

i.e.  $\exists H \in \Omega^3_c(M)$  s.t.  $\pi^* H = dB$ .

Def: Let  $\widehat{L} = (L, \pi, \mu, \sigma, B) \in \text{Grb}(M)$

The 3-curvature of  $\widehat{L}$  is  $H$ .

(a.k.a. 3-form flux, H-flux, ...)

Thm: Let  $\widehat{L}$  be as above.

$$r(DD(\widehat{L})) = [H]$$

Example (Lifting bundle gerbe)

Let  $\pi: Y \rightarrow M$ : principal  $\text{PU}(1)$ -bundle.

Then  $\exists$  a natural map  $Y^{[2]} \xrightarrow{\delta} \text{PU}(1)$   
 $(y_1, y_2) \mapsto \delta(y_1, y_2)$

Pull-back  $U(H) \rightarrow \text{PU}(H)$  to obtain

$$\begin{array}{ccc} L & \longrightarrow & U(H) \\ \downarrow \dashv & & \downarrow \\ Y^{[2]} & \longrightarrow & \Phi_{\text{PU}(H)} \end{array} \quad (L, \pi, \mu)$$

$\mu$  is defined by the right coset mult.

Note

Deligne Complex

$$\check{C}^2(U; \underline{U(1)})$$

$$\check{C}^1(U; \underline{U(1)})$$

$$\check{C}^0(U; \underline{U(1)})$$

$$D = (-1)\delta + d$$

$$H_D^k(U)$$

$$\text{form degree}$$

$$\text{cohomology}$$

$$S_{ij}^* \triangleright \sigma_i^* B.$$

$$D \hat{\lambda} = 0$$

$$\text{Indep. of choice.}$$

(2)  $\overset{\text{!}}{\uparrow}_1$ ,  $\overset{\text{!}}{\circ}_1 \equiv \overset{\text{!}}{\circ}_2$   $\Leftrightarrow$  They have the same  
Deligne cohomology  
class

$$\text{Thm : } H_D^P(M) \cong \overset{\wedge}{H}^{P+1}(M)$$

$\xrightarrow{\text{Brylinski}}$   
 Loop spaces  
 $\xrightarrow{\text{Chen class}}$

$$\text{So } \pi_0 \text{Grb}_D(M) \cong \overset{\wedge}{H}^3(M)$$

Remark About  $\overset{\text{!}}{\circ}$  If you define naively like

$$1 \cong 1'$$

$$Y \xrightarrow{\cong} Y'$$

$$\downarrow \cong$$

$$M$$

$$DD(1) = DD(1')$$

but  $\exists 1 \not\cong 1'$

$\Rightarrow$  We need "Stable isomorphism" of bundle gerbes

Gf. Waldorf's More morphisms of b.g...

$\text{Grb}_D(M)$  : 2-groupoid.