DIFFERENTIAL COHOMOLOGY AND GERBES: AN INTRODUCTION TO HIGHER DIFFERENTIAL GEOMETRY 1/3

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Abstract. This is a lecture note for a minicourse given at the IUT Mathematics and Statistics Research Seminar. This version contains the notes for the lecture given on December 11th, 2023.

1. Čech cohomology and characteristic classes

Definition 1.1. Let G be a Lie group. A **principal G-bundle** over a smooth manifold M is a smooth map $\pi\colon P \to M$ and a right G-action on P satisfying

(1) π is G-invariant; i.e., $\pi(p \cdot g) = \pi(p)$ for all $p \in P$ and $g \in G$.
(2) On each fiber G acts freely and transitively from the right.
(3) P is locally trivial via G-equivariant trivialization; i.e., at every $m \in M$ there exists an open subset $U \subset M$ and a diffeomorphism $\varphi\colon \pi^{-1}(m) \to U \times G$ such that $p \mapsto (\pi(p), \phi(p))$ satisfying $p \cdot g \mapsto (\pi(p), \phi(p) \cdot g)$.

The conditions (1) and (2) means that the G-orbits are fibers of π. This is equivalent to saying $P \times G \to P \times_M P$, $(p, g) \mapsto (p, p \cdot g)$ is a diffeomorphism; i.e., P is a G-*torsor*.

Definition 1.2. A **bundle map** of principal G-bundles from $\pi_1\colon P_1 \to M$ to $\pi_2\colon P_2 \to M$ is a diffeomorphism $f\colon P_1 \to P_2$ that preserves the fiber and G-equivariant; i.e., $f(p \cdot g) = f(p) \cdot g$ and $\pi_2 \circ f = \pi_1$.

Principal G-bundles over M with maps form a groupoid (a category whose morphisms are invertible) and it is denoted by $\mathrm{Prin}_G(M)$. We will also use the notation $\mathrm{Bun}_{\mathbb{C}^n}(M)$ to denote the groupoid of rank n complex vector bundles over M.

Example 1.3. Let $G = GL_n(\mathbb{C})$. Consider $\pi\colon P \to M$ and take an associated fiber bundle $E(P) \to M$ with a fiber $\mathbb{C}^n$ defined by $E(P) := (P \times \mathbb{C}^n)/G$ with a diagonal G-action: $(p, v) \mapsto (pg, g^{-1}v)$. The bundle $E(P)$ is a complex vector bundle over M of rank n. On the other hand, let $E \in \mathrm{Bun}_{\mathbb{C}^n}(M)$. At each $x \in M$, consider the set $\mathrm{Fr}(E)_x$ of all bases of the vector space E_x; equivalently the set of all $\mathbb{C}$-linear maps $p\colon \mathbb{C}^n \to E_x$. Then the smooth map $\pi\colon \mathrm{Fr}(E) \to M$ with $\pi^{-1}(x) = P(E)_x$ and a right G-action on $\mathrm{Fr}(E)$ defined by $p \mapsto p \circ g$ is a principal G-bundle over

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M. It leads to the following equivalence of categories.

$$\mathrm{Prin}_{GL_n(\mathbb{C})}(M) \underset{\mathrm{Fr}}{\overset{E}{\rightleftarrows}} \mathrm{Bun}_{\mathbb{C}^n}(M)$$

For this reason, in what follows, we don't distinguish a $\mathbb{C}^\times$-, S^1-, or a $U(1)$-bundle and a complex line bundle.

Notation 1.4. We shall use the notation $U_{i_1\cdots i_n}$ to denote the n-fold intersection $U_{i_1}\cap\cdots\cap U_{i_n}$.

Definition 1.5. Let G be an abelian group, M a topological space, and $\mathcal{U}=\{U_i\}_{i\in\Lambda}$ an open cover of M. The set $\check{C}^p(\mathcal{U};G)=\{f_{i_0\cdots i_p}\colon U_{i_0\cdots i_p}\to G\}_{i_0,\cdots,i_p\in\Lambda}$ inherited the operation of the group G is degree p **Čech cochain group**. Together with the map $\delta_p:\check{C}^p(\mathcal{U};G)\to\check{C}^{p+1}(\mathcal{U};G)$, $(f)_{i_0\cdots i_p}\mapsto(\delta f)_{i_0\cdots i_{p+1}}:=f_{\widehat{i_0}i_1\cdots i_{p+1}}-f_{i_0\widehat{i_1}\cdots i_{p+1}}+\cdots+(-1)^{p+1}_{i_0i_1\cdots i_p\widehat{i_{p+1}}}$, the sequence of groups $(\check{C}^\bullet(\mathcal{U};G),\delta_\bullet)$ is the **Čech cochain complex**. (It is easy to verify that $\delta^2=0$. Here the hat means an omission). The cohomology of this complex $\check{H}^\bullet(\mathcal{U};G):=\ker(\delta_\bullet)/\mathrm{Im}(\delta_{\bullet-1})$ is the Čech cohomology of M defined on an open cover $\mathcal{U}$.

Now if the group G in the definition above is not abelian, in general, the coboundary maps δ are not group homomorphisms, neither $\ker\delta$ nor $\mathrm{Im}\delta$ form a group, and if we apply δ to a cocycle, we do not get $\delta^2=1$. We shall see below what goes on starting from the lowest degree.

- $p=0$: There is no problem. $\check{H}^0(\mathcal{U};G)=\{f\in\check{C}^0(\mathcal{U};G):\delta(f)_{ij}=0\}=\mathrm{Map}(M,G)$. This is a group under a pointwise group multiplication.
- $p=1$: Neither $\ker\delta_1$ nor $\mathrm{Im}\delta_0$ form a group. On the set $\ker\delta_1$, we may impose an equivalence relation defined by the action of 0-cochains
$$g_{ij}\sim g'_{ij}\quad\text{if and only if}\quad g'_{ij}=f_i^{-1}g_{ij}f_j.$$
So we may define $\check{H}^1(\mathcal{U};G)$ as the pointed set $\ker\delta_1/\sim$ with a distinguished element the constant map $g_{ij}\equiv1$. Notice that the set $\check{H}^1(\mathcal{U};G)$ is precisely the set of isomorphism classes of principal G-bundles over M defined on the open cover U (see Remark below). For this reason, principal G-bundles are geometric models of a degree 1 nonabelian cohomology of M with coefficients in a group G.
- $p\geq2$: There is no reasonable way to make sense of $\check{H}^p(\mathcal{U};G)$.

Remark 1.6. We shall closely look into how the set $\check{H}^1(M;G)$ classifies principal G-bundle over M up to isomorphism. Recall that every principal G-bundle is locally trivial and diffeomorphic to $U\times G$ for some open $U\subset M$. That means if we are given a family of *transition functions* on every double overlap $U_{ij}\in\mathcal{U}=\{U_{ij}\}_{i,j\in\Lambda}$, i.e., $\{g_{ij}\colon U_{ij}\to G:i,j\in\Lambda\}$, we can rebuild the principal G-bundle. Since the transition functions satisfy

$$g_{ij}(x)\cdot g_{jk}(x)\cdot g_{ki}(x)=1,\ \text{for all } x\in U_{ijk} \tag{1.1}$$

The equation (1.1) is called the *cocycle condition* of a principal G-bundle. So if we have a principal bundle P over M, we have a family of transition functions $\{g_{ij}\}_{i,j\in\Lambda}$ satisfying the condition (1.1) and vice versa (under a mild condition). Likewise, if we have a bundle map $f\colon P \to P'$ covering M, we have a family of functions on open sets in the cover $\{f_i\}_{i\in\Lambda}$ satisfying that $g'_{ij}(x) = f_j^{-1}(x)g_{ij}(x)$ for all $x \in U_{ij}$ and vice versa (under the same mild condition). Here the mild condition is that the open cover $\mathcal{U}$ has to be a good cover. A **good cover** (a.k.a. Leray's covering) is an open cover of M if all open sets and their intersections are contractible. Such a covering always exists (see [4, Prop. A.1] and references therein). An open cover $(\mathcal{V}, \imath)$ is a **refinement** of $\mathcal{U}$ if $\imath\colon \mathcal{V} \to \mathcal{U}$ such that $V \subseteq \imath(V)$ for all $V \in \mathcal{V}$. A refinement induces a map $\mathrm{res}_{\mathcal{V},\mathcal{U}}\colon \check{H}^1(\mathcal{U}; G) \to \check{H}^1(\mathcal{V}; G)$, and it satisfies $\mathrm{res}_{\mathcal{W},\mathcal{U}} = \mathrm{res}_{\mathcal{W},\mathcal{V}} \circ \mathrm{res}_{\mathcal{V},\mathcal{U}}$. So we can define the set $\check{H}^1(M; G)$ a direct limit over refinements of open cover; i.e.,

$$\check{H}^1(M; G) = \varinjlim_{\mathcal{U}} \check{H}^1(\mathcal{U}; G).$$

If the cover $\mathcal{U}$ is good, the restriction map $\check{H}^1(\mathcal{U}; G) \xrightarrow{\cong} \check{H}^1(M; G)$ is an isomorphism. Therefore, we conclude that

$$\begin{aligned} \pi_0\mathrm{Prin}_G(\mathcal{U}) &\to \check{H}^1(\mathcal{U}; G) \\ [P] &\mapsto (g_{ij}). \end{aligned} \tag{1.2}$$

If we remove the abelian assumption of groups, the long exact sequence induced by a short exact sequence of groups cannot go any further than the degree $p = 1$.

Proposition 1.7. Let

$$1 \longrightarrow K \xrightarrow{i} \widetilde{G} \xrightarrow{j} G \longrightarrow 1 \tag{1.3}$$

be a short exact sequence of groups. We have the following long exact sequence of groups and pointed sets

$$\begin{tikzcd} 1 \arrow[r] & \check{H}^0(\mathcal{U}; K) \arrow[r, "i_*"] & \check{H}^0(\mathcal{U}; \widetilde{G}) \arrow[r, "j_*"] & \check{H}^0(\mathcal{U}; G) \arrow[dll, out=0, in=180] \\ & \check{H}^1(\mathcal{U}; K) \arrow[r, "i_*"] & \check{H}^1(\mathcal{U}; \widetilde{G}) \arrow[r, "j_*"] & \check{H}^1(\mathcal{U}; G) \end{tikzcd}$$

However, in a special case that the second term in the sequence is an abelian group whose image is in the center of the third, we can extend the long exact sequence just one term further. We have the following propositions.

Proposition 1.8. If the group K in the short exact sequence (1.3) is abelian and $i(K)$ belongs to the center of $\widetilde{G}$, then the long exact sequence in the Proposition 1.7 extends to $\check{H}^2(\mathcal{U}; K)$:

$$\begin{tikzcd} 1 \arrow[r] & \check{H}^0(\mathcal{U}; K) \arrow[r, "i_*"] & \check{H}^0(\mathcal{U}; \widetilde{G}) \arrow[r, "j_*"] & \check{H}^0(\mathcal{U}; G) \arrow[dll, out=0, in=180] \\ & \check{H}^1(\mathcal{U}; K) \arrow[r, "i_*"] & \check{H}^1(\mathcal{U}; \widetilde{G}) \arrow[r, "j_*"] & \check{H}^1(\mathcal{U}; G) \arrow[dll, dashed, out=0, in=180, "\alpha"] \\ & \check{H}^2(\mathcal{U}; K) & & \end{tikzcd}$$

Proposition 1.9 (Dixmier–Douady). If the sheaf $\underline{\widetilde{G}}_M$ is soft, then

$$\alpha : \check{H}^1(\mathcal{U}; G) \to \check{H}^2(\mathcal{U}; K)$$

is a bijection.

Proof. See Dixmier–Douady [2, Lemme 22, p.278] or Brylinski [1, Prop. 4.1.8, p.162] $\square$

In the above, $\underline{G}_M$ is a sheaf such that $\underline{G}_M(U)$ is a group of smooth functions $f\colon U \to G$ for each open $U \subseteq M$. A sheaf $\underline{G}_M$ is **soft** if $\underline{G}_M(M) \to \underline{G}_M(C)$ is onto for every closed $C \subset M$. Here, we can think of $\underline{G}_M(C) = \lim_U \underline{G}_M(U)$ (since M is paracompact) where the direct limit is taken over all open neighborhoods of C.

Example 1.10. (1) Consider a short exact sequence

$$1 \longrightarrow SO_n \xrightarrow{\ i\ } O_n \xrightarrow{\det} \mathbb{Z}_2 \longrightarrow 1.$$

The induced map $w_1\colon \check{H}^1(M; O_n) \to \check{H}^1(M; \mathbb{Z}_2)$ is a correspondence $[P] \in \pi_0 \mathrm{Prin}_{O_n}(M) \mapsto w_1([P])$ which is the *first Stifel–Whitney class*. So $w_1([P]) = 0$ if and only if P comes from an SO_n-bundle; i.e., P is orientable. Equivalently the obstruction for transition maps of a Euclidean vector bundle lift to SO_n is the first Stifel–Whitney class.

(2) Consider a short exact sequence

$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathrm{Spin}_n \longrightarrow SO_n \longrightarrow 1.$$

The induced map $w_2\colon \check{H}^1(M; SO_n) \to \check{H}^2(M; \mathbb{Z}_2)$ is a correspondence $[P] \in \pi_0 \mathrm{Prin}_{SO_n}(M) \mapsto w_2([P])$ which is the *second Stifel–Whitney class*. So $w_2([P]) = 0$ if and only if P comes from a Spin_n-bundle. Equivalently the obstruction for transition maps of an oriented Euclidean vector bundle lift to Spin_n is the second Stifel–Whitney class. Here one can think of Spin_n as a double cover of SO_n, which is also a universal cover. For a construction of Spin_n in terms of Clifford algebras, see [5, Section 1.2].

Remark 1.11. The Whitehead tower of O_n is of particular interest. The **Whitehead tower** of a space X is a factorization of the point inclusion $\mathrm{pt} \to X$

$$\mathrm{pt} \simeq \lim_{n\to\infty} X_n \longrightarrow \cdots \longrightarrow X_2 \longrightarrow X_1 \longrightarrow X_0 \simeq X$$

such that each X_n is $(n-1)$-connected (i.e., all homotopy groups π_k vanish for $k \le n-1$) and each map $X_n \to X_{n-1}$ is a fibration which is an isomorphism on all π_k for $k \ge n$. For the space O_n, we have a Whitehead tower as follows:

$$\mathrm{pt} \longrightarrow \cdots \longrightarrow \mathrm{FiveBrane}_n \longrightarrow \mathrm{String}_n \longrightarrow \mathrm{Spin}_n \longrightarrow SO_n \longrightarrow O_n$$

Here String_n is a 6-connected cover of Spin_n

$$1 \longrightarrow K(\mathbb{Z}, 2) \longrightarrow \mathrm{String}_n \longrightarrow \mathrm{Spin}_n \longrightarrow 1.$$

and $\mathrm{FiveBrane}_n$ is a 7-connected cover of String_n

$$1 \longrightarrow K(\mathbb{Z},6) \longrightarrow \mathrm{FiveBrane}_n \longrightarrow \mathrm{String}_n \longrightarrow 1.$$

It is known that the obstruction to lift a Spin_n-bundle to a String_n-bundle is the *first fractional Pontryagin class* $\frac{1}{2}p_1$ and a String_n-bundle to a $\mathrm{FiveBrane}_n$-bundle the *second fractional Pontryagin class* $\frac{1}{6}p_2$ and so on. See [3] for more details.

Example 1.12. (3) Consider a short exact sequence

$$1 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{R} \longrightarrow S^1 \longrightarrow 1.$$

Note that $\underline{\mathbb{R}}_M$ is a soft sheaf (recall Tietze extension theorem). The induced map $c_1\colon \check{H}^1(M;S^1) \overset{\cong}{\to} \check{H}^2(M;\mathbb{Z})$ is a correspondence $[L] \in \pi_0\mathrm{Prin}_{S^1}(M) \mapsto c_1([L])$ which is the *first Chern class*. Note that if the group G is abelian and $\mathcal{G}$ is a sheaf of locally constant functions in G, $\check{H}^p(M;\mathcal{G})$ and $H^p(M;G)$ the degree p singular cohomology with coefficients in G are the same. Since the group $\mathbb{Z}$ is discrete, we can identify $\check{H}^p(M;\mathbb{Z})$ and $H^p(M;\mathbb{Z})$ for any degree p.

Proposition 1.13. (Dixmier–Douady) Let $\mathcal{H}$ be a complex separable Hilbert space. The sheaf $\underline{U(\mathcal{H})}_M$ is soft.

Proof. See Dixmier–Douady [2, Lemme 4, p.252] or Brylinski [1, Cor. 4.1.6, p.162] $\square$

Example 1.14. (4) Consider a short exact sequence

$$1 \longrightarrow U_1 \longrightarrow U(\mathcal{H}) \longrightarrow PU(\mathcal{H}) \longrightarrow 1.$$

Since $U(\mathcal{H})$ is a soft sheaf, the induced map $DD\colon \check{H}^1(M;PU(\mathcal{H})) \overset{\cong}{\to} \check{H}^2(M;S^1) \overset{\cong}{\to} H^3(M;\mathbb{Z})$ is a correspondence $[P] \in \pi_0\mathrm{Prin}_{PU(\mathcal{H})}(M) \mapsto DD([P])$ which is the *Dixmier–Douady class* of a gerbe.

Definition 1.15. A **characteristic class** of a principal G-bundle P over M is an assignment

$$\begin{aligned} c\colon \pi_0\mathrm{Prin}_G(M) &\to H^\bullet(M;A) \\ [P] &\mapsto c(P) \end{aligned}$$

that is natural; i.e. $f^*c(P) = c(\overline{f}^*P)$ for

$$\begin{tikzcd} P' \arrow[r, "\overline{f}"] \arrow[d, "\pi'"'] \arrow[dr, phantom, "\circlearrowleft"] & P \arrow[d, "\pi"] \\ M' \arrow[r, "f"] & M \end{tikzcd}$$

Here A is an abelian group.

Since $\mathrm{Prin}_G(-) : \mathbf{Man}^{\mathrm{op}} \to \mathbf{Sets}$ is representable by BG, by Yoneda Lemma (See MacLane [6]) we have the following proposition

Proposition 1.16. An assignment

$$\{\text{Characteristic class of principal } G\text{-bundles}\} \longrightarrow H^\bullet(BG; A)$$

is one-to-one and onto.

Remark 1.17. There is an alternative way to define characteristic classes using a "geometric datum ," i.e. a connection ∇ on $P \in \mathrm{Prin}_G(M)$. This is the *Chern–Weil theory*. For example, given a line bundle with connection (L, ∇), the first Chern class of ∇ is defined by a Chern–Weil form $\frac{i}{2\pi}\mathrm{curv}(\nabla)$. Here $\mathrm{curv}(\nabla)$ is the curvature 2-form of the connection ∇. Chern–Weil theorem shows that the cohomology class of a Chern–Weil form does not depend on the choice of connection. So $\left[\frac{i}{2\pi}\mathrm{curv}(\nabla)\right] \in H^2(M; \mathbb{R})$ is a topological invariant of a line bundle. A priori the class $\left[\frac{i}{2\pi}\mathrm{curv}(\nabla)\right]$ is a class in $H^2(M; \mathbb{C})$, but it can be shown that it is actually a class in $H^2(M; \mathbb{R})$. The realification of the first Chern class Example 1.12 above is equal to the first Chern class $\left[\frac{i}{2\pi}\mathrm{curv}(\nabla)\right]$ from the Chern–Weil theory. See Morita [7, Chapter 5] to learn more about Chern–Weil theory of characteristic classes.

We have seen that, up to isomorphism, complex line bundles are classified by $H^2(M; \mathbb{Z})$ via the first Chern class (Example 1.12) and principal $PU(\mathcal{H})$-bundles are by $H^3(M; \mathbb{Z})$ via the Dixmier–Douady class (Example 1.14). We can ask the following question: What classifies (higher) line bundles with connection? For example, if we consider a groupoid $\mathrm{Bun}_{\mathbb{C}}^{\nabla}(M)$ whose objects are line bundles with connection (L, ∇) and morphisms are bundle isomorphism preserving the connection, what classifies the isomorphism classes of $\mathrm{Bun}_{\mathbb{C}}^{\nabla}(M)$? This question leads us to "differential cohomology." Up to isomorphism, line bundles with connection are classified by the degree 2 differential cohomology $\widehat{H}^2(M)$, gerbes with connection by $\widehat{H}^3(M)$, 2-gerbes with connection by $\widehat{H}^4(M)$, and so on.

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