MATH 156 LAB 13

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Topic 1: The integral test.

We can compare the partial sums of certain series with Riemann sums (left-hand sums and right-hand sums) of certain improper integrals. We will only work with continuous functions, decreasing and positive. The integral test says that, if $f(x)$ satisfies these conditions, then the series $\sum_{n=1}^{infinity} f(n)$ and the improper integral $\int_1^{infinity} f(x)\,dx$ either both converge or both diverge.

Example: The series $\sum_{n=1}^{infinity} \frac{1}{n^2}$. Since $f(x) = \frac{1}{x^2}$ is positive, decreasing and continuous we have to decide whether the improper integral $\int_1^{infinity} \frac{1}{x^2}\,dx$ converges or not.

We define the function, the integral and we ask Maple the value of the improper integral.

```
> f:=x->1/x^2;
```

$$f := x \rightarrow \frac{1}{x^2}$$

```
> integral:=Int(f(x), x=1..infinity);
```

$$integral := \int_1^{\infty} \frac{1}{x^2}\,dx$$

```
> value(%);
```

$$1$$

Since the integral converges, the series converges as well. We can see it by defining the sequence of partial sums, plot it and find its limit.

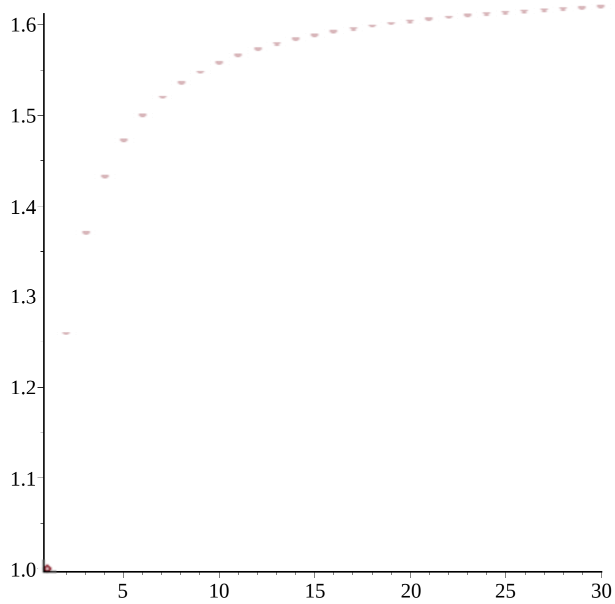
```
> s:=N->sum(f(n), n=1..N);
```

$$s := N \to \sum_{n=1}^{N} f(n)$$

```
> graph:=[seq([k,s(k)], k=1..30)];
```

$$graph := \left[[1, 1], \left[2, \frac{5}{4}\right], \left[3, \frac{49}{36}\right], \left[4, \frac{205}{144}\right], \left[5, \frac{5269}{3600}\right], \left[6, \frac{5369}{3600}\right], \left[7, \frac{266681}{176400}\right], \left[8, \frac{1077749}{705600}\right], \left[9, \frac{9778141}{6350400}\right], \left[10, \frac{1968329}{1270080}\right], \left[11, \frac{239437889}{153679680}\right], \left[12, \frac{240505109}{153679680}\right], \left[13, \frac{40799043101}{25971865920}\right], \left[14, \frac{40931552621}{25971865920}\right], \left[15, \frac{205234915681}{129859329600}\right], \left[16, \frac{822968714749}{519437318400}\right], \left[17, \frac{238357395880861}{150117385017600}\right], \left[18, \frac{238820721143261}{150117385017600}\right], \left[19, \frac{86364397717734821}{54192375991353600}\right], \left[20, \frac{17299975731542641}{10838475198270720}\right], \left[21, \frac{353562301485889}{221193371393280}\right], \left[22, \frac{354019312583809}{221193371393280}\right], \left[23, \frac{187497409728228241}{117011293467045120}\right], \left[24, \frac{187700554334941861}{117011293467045120}\right], \left[25, \frac{2348597155056114l649}{146264116833806400000}\right], \left[26, \frac{23507608254234781649}{14626411683380640000}\right], \left[27, \frac{21174904727185847484l}{131637705150425760000}\right], \left[28, \frac{1038393067289296687720 9}{64502475523708622400000}\right], \left[29, \frac{8739335943455356005972769}{5424658191543895143840000}\right], \left[30, \frac{8745363341445960333910369}{5424658191543895143840000}\right] \right]$$

```
> plot(graph, style=point);
```



```
> limit(s(k), k=infinity);
```

$$\frac{1}{6}\pi^2$$

We can also create a table of values of the sequence of partial sums:

```
> for N from 1 to 30 do evalf(s(N)); od;
```

1.

1.250000000

1.361111111

1.423611111

1.463611111

1.491388889

1.511797052

1.527422052

1.539767731

1.549767731

1.558032194

1.564976638

1.570893798

1.575995839

1.580440283

1.584346533

1.587806741

1.590893161

1.593663244

1.596163244

1.598430818

1.600496933

1.602387292

1.604123404

1.605723404

1.607202694

1.608574436

1.609849946

1.611039006

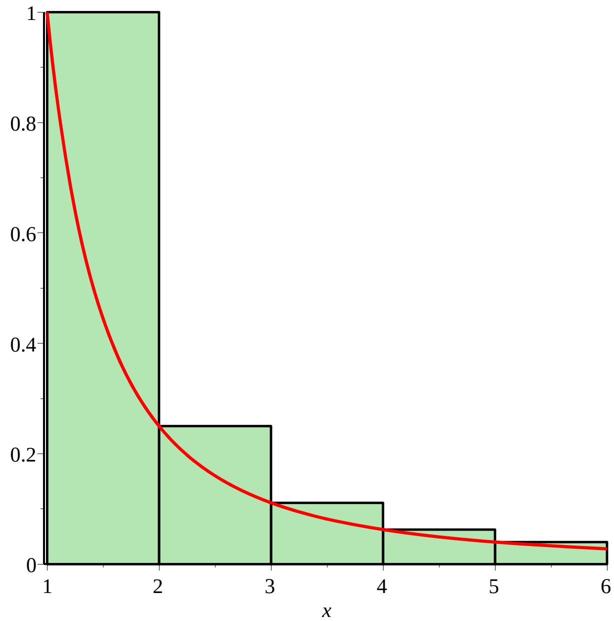
1.612150118

```
> evalf(Pi^2/6);
```

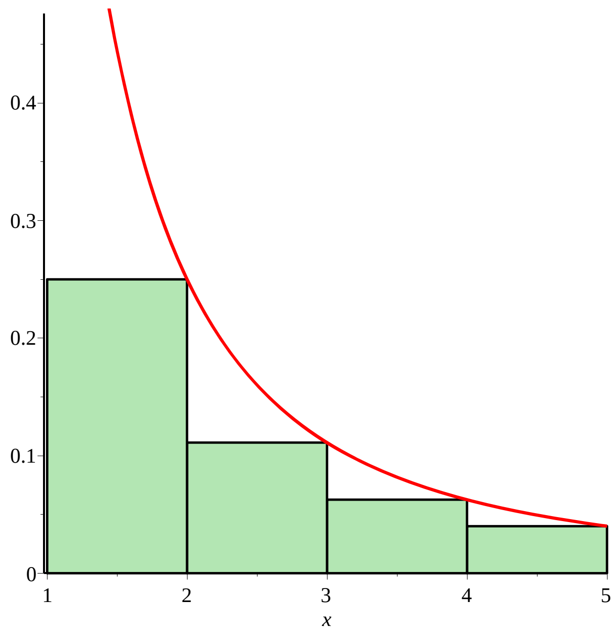
1.644934068

As you see the series converges and its sum it $\frac{1\ \mathrm{Pi}^2}{6}$. This is not the same number as the area represented by the improper integral. To understand a bit better what is going on, we plot the function $f(x)$ and its left-hand sum and right-hand sum on the intervals [1, N+1] and [1, N] respectively with subintervals of length 1 . To see a good graph, however, we take N=5.

```
> with(student):leftbox(f(x), x=1..6, 5);
```




```
> rightbox(f(x), x=1..5, 4);
```



We see that s(5)=LHS(5)>$\int_1^6 f(x)\,dx$>$\int_1^5 f(x)\,dx$>RHS(4)=s(5)-1. In general s(N)<$\int_1^N f(x)\,dx$+1. Since the improper integral converges, i.e. the area under the curve all the way to infinity is finite, we get that the series converges.

Plot the partial sums for the series $\sum_{n=1}^{infinity} \frac{1}{n}$**. and the series**

$$\sum_{n=1}^{infinity} \frac{1}{n^2+1}.$$

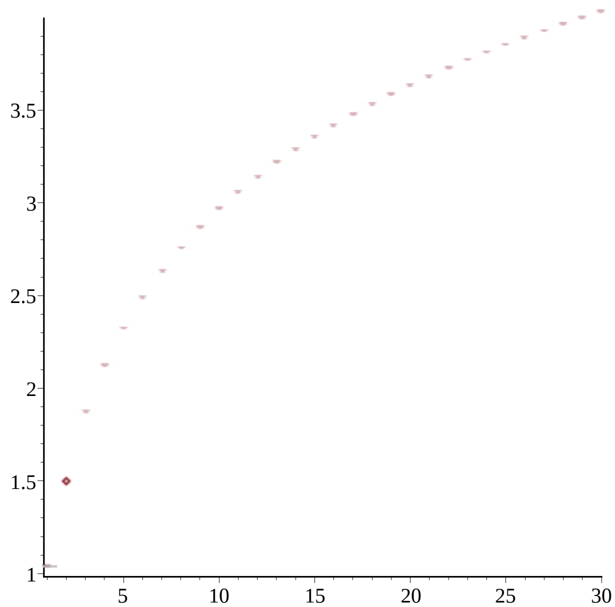
```
> s:=k->sum(1/n,n=1..k);
```

$$s := k \to \sum_{n=1}^{k} \frac{1}{n}$$

```
> graph:=[seq([k,s(k)], k=1..30)];
```

$$graph := \left[[1, 1], \left[2, \frac{3}{2}\right], \left[3, \frac{11}{6}\right], \left[4, \frac{25}{12}\right], \left[5, \frac{137}{60}\right], \left[6, \frac{49}{20}\right], \left[7, \frac{363}{140}\right], \left[8, \frac{761}{280}\right], \left[9, \frac{7129}{2520}\right], \left[10, \frac{7381}{2520}\right], \left[11, \frac{83711}{27720}\right], \left[12, \frac{86021}{27720}\right], \left[13, \frac{1145993}{360360}\right], \left[14, \frac{1171733}{360360}\right], \left[15, \frac{1195757}{360360}\right], \left[16, \frac{2436559}{720720}\right], \left[17, \frac{42142223}{12252240}\right], \left[18, \frac{14274301}{4084080}\right], \left[19, \frac{275295799}{77597520}\right], \left[20, \frac{55835135}{15519504}\right], \left[21, \frac{18858053}{5173168}\right], \left[22, \frac{19093197}{5173168}\right], \left[23, \frac{444316699}{118982864}\right], \left[24, \frac{1347822955}{356948592}\right], \left[25, \frac{34052522467}{8923714800}\right], \left[26, \frac{34395742267}{8923714800}\right], \left[27, \frac{312536252003}{80313433200}\right], \left[28, \frac{315404588903}{80313433200}\right], \left[29, \frac{9227046511387}{2329089562800}\right], \left[30, \frac{9304682830147}{2329089562800}\right]\right]$$

```
> plot(graph,style=point); for N from 1 to 10 do evalf(s(N)); od;
```



$$1.$$
$$1.500000000$$
$$1.833333333$$
$$2.083333333$$
$$2.283333333$$
$$2.450000000$$
$$2.592857143$$
$$2.717857143$$
$$2.828968254$$
$$2.928968254$$

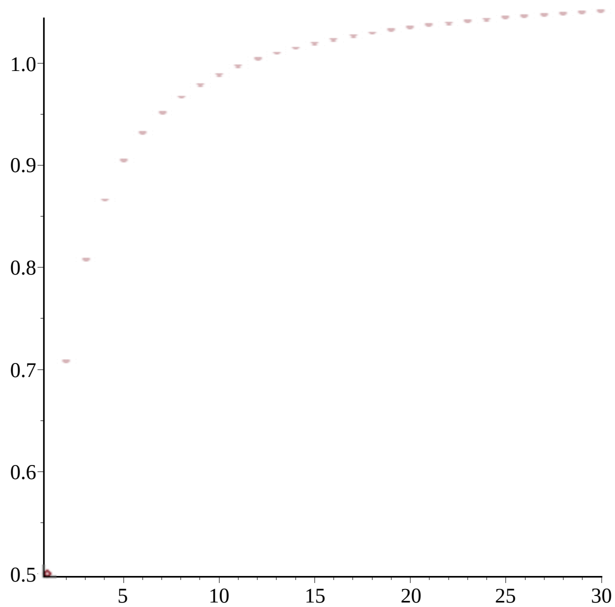
```
> s1:=k->sum(1/(n^2+1),n=1..k);
```

$$s1 := k \rightarrow \sum_{n=1}^{k} \frac{1}{n^2+1}$$

```
> graph:=[seq([k,s1(k)], k=1..30)];
```

$$
\mathit{graph} := \left[\left[1, \frac{1}{2}\right], \left[2, \frac{7}{10}\right], \left[3, \frac{4}{5}\right], \left[4, \frac{73}{85}\right], \left[5, \frac{1983}{2210}\right], \left[6, \frac{75581}{81770}\right], \left[7, \frac{193041}{204425}\right], \left[8, \frac{196186}{204425}\right], \left[9, \frac{16291677}{16762850}\right], \left[10, \frac{1662222227}{1693047850}\right], \left[11, \frac{51121039886}{51637959425}\right], \left[12, \frac{1492837748579}{1497500823325}\right], \left[13, \frac{3003293153903}{2995001646650}\right], \left[14, \frac{594643752965541}{590015324390050}\right], \left[15, \frac{33744875873650579}{33335865828037825}\right], \left[16, \frac{870576896535623662 8}{856731751780572102 5}\right], \left[17, \frac{174706228791111334 01}{171346350356114420 50}\right], \left[18, \frac{3504668966613372183}{3426927007122288410}\right], \left[19, \frac{318029273230290754664}{310136894144567101105}\right], \left[20, \frac{1278398754594911597 21369}{1243648945519714075 43105}\right], \left[21, \frac{25624248799840300454 4743}{24872978910394281508 6210}\right], \left[22, \frac{2490526729366588000 3857313}{2412678954308245306 3362370}\right], \left[23, \frac{661195922759299942755386913}{639359922891685006179102805}\right], \left[24, \frac{38214940735500775197 6037351606}{36891067550850224856 5342318485}\right], \left[25, \frac{2395944396797433549855647244 23841}{2309380828683224076019042913 71610}\right], \left[26, \frac{162436373746054573732829222726311967}{156345082101854269946489205258579970}\right], \left[27, \frac{5936744895836084654745591089773315794}{5706595496717680853046855991938168905}\right], \left[28, \frac{93321026774560882696566717229 2798213439}{89593549298467589392835639073 4292518085}\right], \left[29, \frac{786658980934787308199020115461270388233723}{754377685093097102687676080998274300227570}\right], \left[30, \frac{787496248065856228401781132199337285348293}{754377685093097102687676080998274300227570}\right]\right]
$$

```
> plot(graph,style=point); for N from 1 to 10 do evalf(s1(N)); od;
```



0.5000000000

0.7000000000

0.8000000000

0.8588235294

0.8972850679

0.9243120949

0.9443120949

0.9596967103

0.9718918322

0.9817928223

From this graph it is not clear that the sequence of partial sums converges or diverges. **Make a list of values for N=1..10**.

Even with this table we are not sure, the values we get are rather small. **Do a loop, showing every other 10th term up to 200**.

```
> for N from 1 to 20 do evalf(s(10*N)); od; for N from 1 to 20 do
  evalf(s1(10*N)); od;
```

2.928968254

3.597739657

3.994987131

4.278543039

4.499205338

4.679870413

4.832836758

4.965479279

5.082570603

5.187377518

5.282234598

5.368868287

5.448591338

5.522425264

5.591180589

5.655511225

5.715952395

5.772947722

5.826869008

5.878030948

0.9817928223

1.027941815

1.043901833

1.051988958

1.056875301

1.060147003

1.062490839

1.064252486

1.065624886

1.066724209

1.067624583

1.068375530

$$1.069011400$$
$$1.069556760$$
$$1.070029651$$
$$1.070443619$$
$$1.070809029$$
$$1.071133952$$
$$1.071424764$$
$$1.071686568$$

We see that the sequence of partial sums increases rather slowly.

Compare it with Riemann sums of the function $f(x) = \frac{1}{x}$**:**

```
> with(student): f:=x->1/x; evalf(leftsum(f(x),x=1..200,200));
```

$$f := x \to \frac{1}{x}$$
$$5.869820228$$

Repeat the work for the series $\sum_{n=1}^{infinity} \frac{1}{n^2+1}$

```
> h:=x->1/(x^2+1);
```

$$h := x \to \frac{1}{x^2+1}$$

```
> evalf(leftsum(h(x),x=1..200,200));
```

$$1.070006397$$

Topic 2: Other tests: Ratio test.

The ratio test for the series $\sum_{n=1}^{infinity} a(n)$ with a_n positive ask us to compute the limit $\lim_{n \to infinity} \frac{a_{n+1}}{a_n}$ = r. If $r < 1$ the series converges, if $1 < r$ the series diverges and if $r = 1$ the test is inconclusive.

Example: Use the ratio test on the series $\sum_{n=1}^{infinity} \frac{1}{n}$ and on the series $\sum_{n=1}^{infinity} \frac{1}{n^2+1}$.

```
> g:=n->1/(n^2+1); Limit(g(n+1)/g(n), n=infinity);
```

$$g := n \to \frac{1}{n^2+1}$$

$$\lim_{n \to \infty} \frac{n^2+1}{(n+1)^2+1}$$

```
> value(%);
```

$$1$$

The test is inconclusive. However, with the integral test we decided that the series diverges.

```
> Limit(h(n+1)/h(n), n=infinity);
```

$$\lim_{n \to \infty} \frac{n^2+1}{(n+1)^2+1}$$

```
> value(%);
```

$$1$$

The test is inconclusive. However, with the integral test we decided that the series converges.

Examine the following two series for convergence using the ratio test.

$$\sum_{n=1}^{infinity} \frac{n!}{3^n}$$ **and** $$\sum_{n=1}^{infinity} \frac{3^n}{n!}$$. Recall the definition of factorials: 0!=1, 1!=1, 2!=1·2, 3!=1·2·3. And in general n!=1·2·3...n.

```
> f1:=n->n!/3^n;
```

$$f1 := n \rightarrow \frac{n!}{3^n}$$

```
> Limit(f1(n+1)/f1(n),n=infinity);
```

$$\lim_{n \rightarrow \infty} \frac{(n+1)!\, 3^n}{3^{n+1}\, n!}$$

```
> value(%);
```

$$\infty$$

```
> f2:=n->3^n/n!;
```

$$f2 := n \rightarrow \frac{3^n}{n!}$$

```
> Limit(f2(n+1)/f2(n),n=infinity);
```

$$\lim_{n \rightarrow \infty} \frac{3^{n+1}\, n!}{(n+1)!\, 3^n}$$

```
> value(%);
```

$$0$$