Lecture 4. The Classification Theorem.

1. Listing normal forms

Following the theorem above, any conic is congruent to the following:

$$Q = Ax^2 + By^2 + 2Gx + 2Fy + C = 0.$$

Let's find its centers (since the geometry of conics largely depend on centers)

$$\frac{\partial Q}{\partial x} = 2Ax + 2G$$

$$\frac{\partial Q}{\partial y} = 2By + 2F$$

The equations of center

$$\begin{cases} Ax + G = 0 \\ By + F = 0 \end{cases}.$$

Theorem: Any nondegenerate conic $Q = ax^2 + 2hxy + by^2 + 2gx + 2fy + c$ with a unique center congruent to one of the following:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = -1, \quad \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

Proof: Given $Q \xrightarrow[\text{rotate}]{\text{"Theorem"}} AX^2 + BY^2 + 2GX + 2FY + C = 0$

Translate so that the center gets placed at $(0,0)$

$$AX^2 + BY^2 - k = 0.$$

Note that Neither A nor B ~~Cannot be both~~ Can be zero. (Otherwise, Contradicts to the premise of unique center)

Now Suppose $A > 0$ (multiply -1 if necessary)

Also, $k \neq 0$ (Otherwise, $\Delta = 0$: degenerate. Contradiction)

We may suppose $k = 1$ or $k = -1$.

Also we can find a, b such that

$$A a^2 = 1 \qquad B b^2 = 1 \text{ if } B > 0$$

$$B b^2 = -1 \text{ if } B < 0$$

Now if $B < 0$ and $k = 1$

(III) $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$

$B < 0$ and $k = -1$

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1 \quad \text{(nothing new)}$$

$90°$ rotation.

if $B > 0$, $k = 1$

(I) $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

if $B < 0$, $k = -1$.

(II) $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = -1$

따름정리

Corollary Any real circle, virtual circle, or rectangular hyperbola is congruent to one of the following:

$$x^2+y^2=a^2, \qquad x^2+y^2=-a^2, \qquad x^2-y^2=a^2.$$

proof This is the theorem for the case $A=B$.

Theorem: Any parabola $Q = ax^2+2hxy+by^2+2gx+2fy+c$ is congruent to a standard parabola $y^2=4ax$, $a>0$.

proof: By the "Theorem" we may write Q as

$$Ax^2+By^2+2Gx+2Fy+C=0 \qquad \cdots \quad (*)$$

And that $(*)$ does not have a center means

$$\begin{cases} Ax+G=0 \\ By+F=0 \end{cases} \quad \text{does not have a solution.}$$

This is when (i) $A=0$, $G\neq 0$, $B\neq 0$.

(ii) $B=0$, $F\neq 0$, $A\neq 0$.

~~(iii)~~ $A=0$, $B=0$, $G\neq 0$, $F\neq 0$.

not a conic

$\begin{pmatrix} -y \\ x \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

Among (i), (ii), one of them reduces to the other (Consider a rotation with $\theta = \frac{\pi}{2}$)

We consider only (i). We may suppose $B=1$ (Divide by B)

$$y^2+2Gx+2Fy+C=0.$$

Translate through $(0,F)$, $(y-F)^2+2Gx+2F(y-F)+C=0.$

$$y^2-\cancel{2Fy}+F^2+2Gx+\cancel{2Fy}-2F^2+C=0.$$

$$\Rightarrow \quad y^2 + 2G\underset{(x-\alpha)}{x} + K = 0.$$

$$y^2 + 2Gx \boxed{-2G\alpha + K} = 0$$

Set α so that this vanishes.

If $G<0$ we're done. If $G>0$, rotate about $\theta = \pi$.

$(x,y) \mapsto (-x,-y)$

Set $a>0$ so that $G = -2a$. ▨

So we can conclude that non-degenerate conics are the standard ellipses, parabolas, and hyperbolas up to congruence.

Now how about degenerate conics?

Thm: Any degenerate conic with unique center is congruent to a real line pair $y^2 = c^2x^2$ or a virtual line pair $y^2 = -c^2x^2$ with $0 < c \le 1$.

Proof: We may assume that the conic is $Q = Ax^2 + By^2 - K$.

($\because$ Similarly as in ellipses)

Since Q must have a unique center, both A, B are nonzero.

Hence, $K=0$. Dividing by B, we get

$$Q = C\,x^2 + y^2 . \qquad C \neq 0.$$

$C < 0$, real line pair

$C_1 = c^2$.

$C > 0$, virtual line pair

$C_1 = -c^2$. ☒

Thm: Any conic having a line of centers is congruent to the real parallel lines $y^2 = k^2$, the virtual parallel lines $y^2 = -k^2$ or the repeated line $y^2 = 0$ where $k > 0$.

Proof: Consider (up to rotation)

$$Q = Ax^2 + By^2 + 2Gx + 2Fy + C = 0 .$$

Equations of centers $\begin{cases} Ax + G = 0 \\ By + F = 0 . \end{cases}$

Q has lines of centers if (i) $A = 0$, $B \neq 0$, $G = 0$

(ii) $A \neq 0$, $B = 0$, $F = 0$.

~~(iii)~~ $A = 0$, $B = 0$, $G = 0$, $F = 0$

not a conic.

In this case (ii) reduces to (i)

$By^2 + 2Fy + C = 0$.

Divide by $B \neq 0$ and translate along the y-axis yields $y^2+k=0$. The rest follows depending on $k>0$, $k=0$, $k<0$. $\boxtimes$

What about the case Q: degenerate with no centers

$$\begin{cases} Ax+G=0 \\ By+F=0 \end{cases}$$

$A=0$, $G\neq 0$, $B\neq 0$

$B=0$, $F\neq 0$, $A\neq 0$

$$Ax^2+By^2+2Gx+2Fy+C=0.$$

$\delta=0 \iff A=0$ or $B=0$.

$$\Delta=0 \iff \begin{vmatrix} A & 0 & G \\ 0 & B & F \\ G & F & C \end{vmatrix} = \underbrace{AB(C-F^2)}_{\neq 0,\ =0} + \underbrace{G(-BG)}_{=0,\ \neq 0}$$

Is this possible?

If $A=0$. $G\neq 0$. $B\neq 0$.

If $A\neq 0$ $F\neq 0$, $B=0$.

Hence degenerate comes with no center do not exist.

2. Proof of the invariance theorem

<u>Aside</u> $L = \lambda L'$ $\quad Z(L) = Z(L')$

$Z(L) = \{(x,y) : L(x,y) = 0\}$

$Q_1 = x^2+y^2+1$

$Z(Q_1) = \phi$?

$Q_2 = (x-1)^2+y^2+1$

$Z(Q_2) = \phi$

<u>Theorem</u> (Uniqueness)

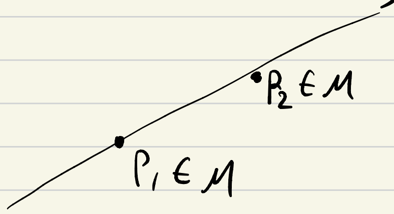
Let Q, Q' be conics having the same zero sets. If the common zero set is infinite, then Q, Q' coincide.

<u>Sketch.</u> (Case I) Q : reducible

$Q = LM$.

$$\begin{array}{ccc} Z(Q) & = & Z(Q') \\ \cup & & \cup \\ Z(L) & & Z(L) \end{array}$$

Component Lemma $\quad Q' = LM'$



$P_2 \in Q'$,

$P_1 \in Q'$

$Z(M') = \overline{P_1 P_2} \subset Z(Q')$ $\quad \therefore M' = \lambda M$.

M

(Case II) irreducible.

$$Q = ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0.$$

$P_i = (x_i, y_i) \quad i = 1, 2, 3, 4, 5.$

$$\begin{cases} a x_1^2 + \cdots + 2 f y_1 + c = 0 \\ \vdots \\ a x_5^2 + \cdots + 2 f y_5 + c = 0 \end{cases}$$

5개 방정식

Since $Z(Q)$ is infinite, hence $z_i = (x_i, y_i)$ $(i = 1, \cdots, 5)$ can be found in $Z(Q)$. Any conic Q' having the same zero set (i.e. $Z(Q') = Z(Q)$) is equivalent to Q if the above 5 equations are "independent". Actual proof: "Suppose not"

$$z_5 \text{ depends on } z_1, z_2, z_3, z_4.$$

$$L_{ij} = \overline{z_i z_j}$$

L_{13}, L_{24} must pass through z_5.

z_1, z_3, z_5 : on the same line.

$\Rightarrow$ (Component lemma) $Q = L_{13} M$. : Contradiction.

Thm (Invariance; Classification thm) If Q, Q' be strictly congruent (i.e. $Q \underset{\text{Cong}}{\sim} Q' \Leftrightarrow Q' = \underset{\substack{\| \\ 1}}{\mu}(Q \circ \phi)$),

then $\tau = \tau'$
$\delta = \delta'$
$\Delta = \Delta'$.

Proof $Q = ax^2 + 2hxy + by^2 + 2gx + 2fy + c$

$$= \underset{z}{\begin{pmatrix} x & y & 1 \end{pmatrix}} \underset{A}{\begin{pmatrix} a & h & g \\ h & b & f \\ g & f & c \end{pmatrix}} \overset{z^T}{\begin{pmatrix} x \\ y \\ 1 \end{pmatrix}}$$

$A = (a_{ij})_{\substack{i=1,\cdots,m \\ j=1,\cdots,n}}$

$A^T = (a_{ji})_{\substack{j=1,\cdots,n \\ i=1,\cdots,m}}$

ϕ: Congruence on $\mathbb{R}^2$

$(\bar{X}, \bar{Y}) \longmapsto (x, y)$

$\det P = 1$

$$P = \begin{pmatrix} \cos\alpha & -\sin\alpha & u \\ \sin\alpha & \cos\alpha & v \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} x\cos\alpha - y\sin\alpha + u \\ x\sin\alpha + y\cos\alpha + v \\ 1 \end{pmatrix}$$

Realizing congruence transformation.

$$\boxed{z^T = P Z^T}$$

$Z = (\bar{X}, \bar{Y}, 1)$

$$Q'(X,Y) = (x\,y\,1)\,A\begin{pmatrix}x\\y\\1\end{pmatrix} = z A z^T \overset{*}{=} Z P^T A P Z^T$$

$(AB)^T \overset{*}{=} B^T A^T.$

$$\Delta' = \det(P^T A P) \underset{\uparrow}{=} \det P^T \det A \det P$$

$\det(AB) = \det A \det B$

$$\underset{\uparrow}{=} \det A\,(\underset{\uparrow 1}{\det P})^2 = \Delta$$

$\det A^T = \det A$

Now $B, B', S =$

$$S = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \leftarrow P$$

$$B = \begin{pmatrix} a & h \\ h & b \end{pmatrix}, \quad A = \begin{pmatrix} a & h & g \\ h & b & f \\ g & f & c \end{pmatrix}$$

$$B' = \begin{pmatrix} a' & h' \\ h' & b' \end{pmatrix} \leftarrow A'$$

$$\delta' = \det B' = \det(S^T B S) \underset{\uparrow}{=} \det B = \delta.$$

$\because B' = S^T B S.$ $\quad \det S = 1 = \det S^T$

$$\tau' = \operatorname{tr}(B') = \operatorname{tr}(S^T B S) = \operatorname{tr}(\underbrace{S S^T}_{= I} B) = \operatorname{tr} B = \tau$$

$\operatorname{tr}\begin{pmatrix} a & b \\ c & d\end{pmatrix} = a+d.$ $\quad \operatorname{tr}(AB) = \operatorname{tr}(BA)$ $\quad \square$

Question for you (during the rest of summer)

Is the converse of the invariance theorem true?

I.e. Given Q, Q'

$$\begin{array}{lccc} & Q & & Q' \\ & \downarrow & & \downarrow \\ \text{가정:} & \tau & = & \tau' \\ & \delta & = & \delta' \\ & \Delta & = & \Delta' \end{array}$$

결론: $Q' = Q \circ \phi$

$\phi \in \mathrm{Cong}(\mathbb{R}^2)$

"folklore fact"