2023. 12. 11. 21:30
IUT Math & Stat Research Seminar

Differential Cohomology and gerbes:

An introduction to higher differential geometry

1. Čech cohomology and characteristic classes

<u>Def</u>: Let G be a Lie group, M: a smooth mfld.
A <u>principal G-bundle</u> P is a smooth map $\pi : P \to M$ and a right G-action on P s.t.

(1) π is G-invariant.

(2) On every fiber, G acts on P freely and transitively from the right.

(3) P is locally trivial via G-equivariant trivializations

<u>Examples</u> (or a <u>prop</u>) when $G = GL_n(\mathbb{C})$

$$\text{A } G\text{-bundle } P \xrightarrow{\pi} M \quad \underset{\text{frame bundle (or bundle of bases)}}{\overset{\text{"associated fiber bundle"}}{\rightleftarrows}} \quad \text{A complex vector bundle } E \to M, \text{ rank } n$$

So we don't really distinguish a $U(1)$-bundle and complex line bundles.
$\mathbb{C}^\times$ -
S^1

Def: Let G be an ~~abelian~~ group, M: a top. space.

$\mathcal{U} = \{U_i\}_{i \in \Lambda}$: an open cover of M.

$$\check{C}^p(\mathcal{U}; G) := \{ f_{i_0 \cdots i_p} : U_{i_0 \cdots i_p} \to G \}_{i_0, \cdots i_p \in \Lambda}$$

Notation: $U_{i_1 i_2 \cdots i_n} := U_{i_1} \cap \cdots \cap U_{i_n}$

$$\delta : \check{C}^p(\mathcal{U}; G) \longrightarrow \check{C}^{p+1}(\mathcal{U}; G)$$

$$f \longmapsto (\delta f)_{i_0 \cdots i_p i_{p+1}} := f_{\hat{i}_0 i_1 \cdots i_p i_{p+1}} - f_{i_0 \hat{i}_1 \cdots i_{p+1}}$$

So $(\check{C}^\bullet(\mathcal{U}; G), \delta_\bullet)$ is a complex

Def: $\check{H}^p(\mathcal{U}; G) := \ker \delta / \operatorname{Im} \delta$ Čech cohomology of M defined on $\mathcal{U}$ of degree p

Question what goes wrong if G is not abelian.

- $p = 0$; No prob! ($\because \delta f = 0 \Leftrightarrow f_i = f_j$, so $\check{H}^0(\mathcal{U}; G) = Map(M, G)$)
- $p = 1$; $\check{H}^1(\mathcal{U}; G)$ is not a group

  It is a pointed set, $*$ = Constant map to 1_G

  Neither $\check{Z}^1(\mathcal{U}; G)$ nor $\check{B}^1(\mathcal{U}; G)$ form a subgroup.

But we can use an equivalence rel.

$$g'_{ij} \sim g_{ij} : U_{ij} \to G \iff g'_{ij} = f_j^{-1} g_{ij} f_i$$

and define $\check{H}^1(\mathcal{U}; G) = \ker \delta / \sim$.

- $p \geq 2$ $\check{H}^p(\mathcal{U}; G)$: no hope.

<u>Prop</u> Let $1 \to K \xrightarrow{i} \tilde{G} \xrightarrow{j} G \to 1$ ... (*) be a S.E.S.

We have a L.E.S.

$$\begin{aligned} 1 &\to \check{H}^0(\mathcal{U}; K) \to \check{H}^0(\mathcal{U}; \tilde{G}) \to \check{H}^0(\mathcal{U}; G) \quad (**) \\ &\to \check{H}^1(\mathcal{U}; K) \to \check{H}^1(\mathcal{U}; \tilde{G}) \to \check{H}^1(\mathcal{U}; G). \end{aligned}$$

<u>Prop</u>: If the S.E.S. (*) has K : abelian, and $i(K) \subseteq \mathrm{Center}(\tilde{G})$, then the LES (**) extends to $\check{H}^2(\mathcal{U}; K)$.

$$\begin{aligned} 1 &\to \check{H}^0(\mathcal{U}; K) \to \check{H}^0(\mathcal{U}; \tilde{G}) \to \check{H}^0(\mathcal{U}; G) \\ &\to \check{H}^1(\mathcal{U}; K) \to \check{H}^1(\mathcal{U}; \tilde{G}) \to \check{H}^1(\mathcal{U}; G) \\ &\xrightarrow{\ d\ } \check{H}^2(\mathcal{U}; K) \end{aligned}$$

Furthermore,

<u>Thm</u> (Dixmier-Douady) If the sheaf $\underline{\tilde{G}}_M$ is soft

(i.e. $\Gamma(M, A) \xrightarrow{\text{res}} \Gamma(C, A)$: onto)

C any closed set

then $\alpha : \check{H}^1(\mathcal{U}; G) \longrightarrow \check{H}^2(\mathcal{U}; K)$ is a bijection.

Note: $\check{H}^1(M; G)$ classifies principal G-bundles over M up to isomorphism

$$\text{Having } \begin{array}{c} G \to P \\ \downarrow \\ M \end{array} \longleftrightarrow \text{Having } g_{ij} : U_{ij} \to G \quad U_{ij} \in \mathcal{U}$$

s.t. $g_{ki}(x) \cdot g_{jk}(x) \cdot g_{ij}(x) = \mathbb{1}_G \quad \forall \ x \in U_{ijk}$

(cocycle condition)

$$\text{Having } \begin{tikzcd} P \arrow[r] \arrow[dr] & P' \arrow[d] \\ & M \end{tikzcd} \longleftrightarrow \text{Having } \{ f_i : U_i \to G \}$$

s.t. $g_{ij}'(x) = f_j^{-1}(x) g_{ij}(x) f_i(x)$.

In general $\check{H}^1(\mathcal{U}; G) \xrightarrow{\cong ?} \check{H}^1(M; G) := \operatorname{colim}_{\mathcal{U}} \check{H}^1(\mathcal{U}; G)$

If $\mathcal{U}$ is good.

Def: A cover $\mathcal{U}$ of M is good if all open sets and their intersections are contractible.

Examples:

(1) $1 \to SO(n) \to O(n) \to \mathbb{Z}_2 \to 1$

Given $[P] \in \mathrm{Bun}_{O(n)}(M)/\cong \ \longmapsto w_1([P])$: the first Stiefel-Whitney class.

$w_1 : \check{H}^1(M; O(n)) \longrightarrow \check{H}^1(M; \mathbb{Z}_2)$

<u>Note</u> $w_1(P) = 0 \iff P$ comes from an $SO(n)$-bundle
P: orientable

(2) $1 \to \mathbb{Z}_2 \to Spin(n) \to SO(n) \to 1$

$$\check{H}^1(M; SO(n)) \longrightarrow \check{H}^2(M; \mathbb{Z}_2)$$
$$[P] \longmapsto w_2(P)$$: the second Stiefel–Whitney class

<u>Note</u>: $w_2(P) = 0 \iff P$ comes from a $Spin(n)$-bundle.

<u>Remark</u> The Whitehead tower of $O(n)$

$\cdots$ FiveBrane $\to String(n) \to Spin(n) \to SO(n) \to O(n)$

(3) $1 \to \mathbb{Z} \to \mathbb{R} \to S^1 \to 1$

$\mathbb{R}$: soft ($\because$ Tietze extension)

$$c_1 : \check{H}^1(M; S^1) \xrightarrow{\cong} \check{H}^2(M; \mathbb{Z}) \cong H^2_{sing}(M; \mathbb{Z})$$
$$[L] \longmapsto c_1(L)$$: the first Chern class

(4) <u>Thm</u> (Dixmier–Douady) $\underline{U(\mathcal{H})}_M$: soft.

Consider $1 \to U(1) \to U(\mathcal{H}) \to PU(\mathcal{H}) \to 1$

$$DD : \check{H}^1(M; PU(\mathcal{H})) \xrightarrow{\cong} \check{H}^2(M; S^1) \xrightarrow{\cong} H^3_{sing}(M; \mathbb{Z})$$
$$[P] \longmapsto DD(P)$$

Dixmier – Douady class

Def A <u>characteristic class</u> c of a G-bundle $P \to M$ is an assignment

$$c : \mathrm{Bun}_G(M)/\cong \longrightarrow H^*(M; \underset{\text{Ab. gp.}}{A})$$
$$[P] \longmapsto c(P)$$

that is natural $\left(\text{i.e. for } \begin{array}{ccc} \bar{P} & \xrightarrow{\bar{f}} & P \\ \downarrow & & \downarrow \\ M' & \xrightarrow{f} & M \end{array} \quad f^* c(P) = c(\bar{f}^* P)\right)$

Remark (Yoneda Lemma) $\mathrm{Bun}_G(-) \cong \mathrm{Maps}(-, BG)$

An assignment

$$\left\{\begin{array}{l}\text{Characteristic} \\ \text{class of } P \in \mathrm{Bun}_G M\end{array}\right\} \longrightarrow H^*(BG; A)$$

is 1-1 and onto.

Remark There's an alternative way to define characteristic classes using geom. data.

"Chern – Weil"

E.g. $L, \nabla \rightsquigarrow$ Chern-Weil form $c(\nabla) = \frac{i}{2\pi} \mathrm{Curv}(\nabla)$

Thm (Chern-Weil) $[c_1(\nabla)]$ is indep. of choice of ∇.

$H^2(M;\mathbb{Z})$ classifies Line bundles
$H^3(M;\mathbb{Z})$ classifies "Gerbes"

What classifies line bundles w/ conn? (L,∇) $\hat{H}^2(M)$

gerbes w/ conn? $\hat{H}^3(M)$

higher gerbes w/ conn? $\hat{H}^\bullet(M)$

2023.12.18. 21:30
IUT Math & Stat Research Seminar

2. Cheeger-Simons differential characters

"Diff. characters and geom. invariants"

Notations : M : smooth manifold

$C^k(M;R)$: smooth singular k-cochains in M with coeff. in R

R: comm. ring with 1

$Z^k(M;R)$: smooth singular k-cocycles in M w/ coeff. in R.

$\Omega^k(M)$: diff. k-forms on M.

$\Omega^k_{cl}(M)_{\mathbb{Z}}$: closed k-forms with integral periods

i.e. $\omega \in \Omega^k_{cl}(M)_{\mathbb{Z}} \iff \begin{cases} d\omega = 0 \\ \int \omega \Big|_{Z_k(M)} \in \mathbb{Z}. \end{cases}$

Notation $\sim$: $\mathbb{R} \to \mathbb{R}/\mathbb{Z}$

Fact: Let $\Lambda \subsetneq \mathbb{R}$: proper subring. Then a nonvanishing diff. form does not take values in Λ.

Prop: The map $\displaystyle \int : \Omega^k(M) \longrightarrow \underset{\sim}{C^k}(M; \mathbb{R}/\mathbb{Z})$

$$\omega \longmapsto \int \omega$$

is one-to-one.

Def (Cheeger–Simons differential characters)

$$\widehat{H}^k(M) := \left\{ (\chi, \omega) : \begin{array}{l} \chi \in \mathrm{Hom}_{\mathbb{Z}}(Z_{k-1}(M), \mathbb{R}/\mathbb{Z}) \\ \omega \in \Omega^k(M) \\ \text{s.t., for } \forall\, D \in C_k(M;\mathbb{Z}) \\ \qquad \chi \circ \partial D = \int_D \omega \mod \mathbb{Z} \end{array} \right\}$$

with the componentwise addition is the group of differential characters (or differential cohomology) of M with degree k.

Prop: (Differential cohomology hexagon diagram)

$$\begin{tikzcd}
0 \arrow[dr] & & & & 0 \\
 & H^{k-1}(M;\mathbb{R}/\mathbb{Z}) \arrow[rr, "-B\text{: Bockstein}"] \arrow[rr, phantom, "\circlearrowright"'] \arrow[dr] & & H^k(M;\mathbb{Z}) \arrow[ur] \arrow[dr, "r"] & \\
H^{k-1}(M;\mathbb{R}) \arrow[ur] \arrow[dr, "rep"'] \arrow[rr, phantom, "\circlearrowright"] & & \widehat{H}^k(M) \arrow[ur, "I"] \arrow[dr, "R"] \arrow[rr, phantom, "\circlearrowright"] & & H^k(M;\mathbb{R}) \\
 & \dfrac{\Omega^{k-1}(M)}{\Omega^{k-1}_{cl}(M)_{\mathbb{Z}}} \arrow[ur, "a"] \arrow[rr, "d"] \arrow[rr, phantom, "\circlearrowright"] & & \Omega^k_{cl}(M)_{\mathbb{Z}} \arrow[ur, "\int \circ\, dR"'] \arrow[dr] & \\
0 \arrow[ur] & & & & 0
\end{tikzcd}$$

with all triangles and squares commute, diagonals are SESs, and the upper and lower sequences are LESs.

Proof: (1) I, R maps

Algebra facts we need:

- A1: Any subgroup of a free abelian group is free.
- A2: An ab.gp. G is divisible if $\forall x \in G$ and $\forall n \in \mathbb{Z}^+$ $\exists y$ s.t. $x = ny$.
- A3: Prop: G: divisible $\iff$ G is an injective obj. in the cat. of ab.gps

Take $(\chi, \omega) \in \hat{H}^k(M)$.

$$
\begin{tikzcd}
 & \mathbb{R} \arrow[d] \\
Z_{k-1}(M) \arrow[ur, dashed, "\exists \bar{\chi}"] \arrow[r, "\chi"] & \mathbb{R}/\mathbb{Z} \arrow[d] \\
 & 0
\end{tikzcd}
\qquad
\begin{tikzcd}
0 \arrow[d] & \\
Z_{k-1}(M) \arrow[r, "\bar{\chi}"] \arrow[d, hook] & \mathbb{R} \\
C_{k-1}(M) \arrow[ur, dashed, "\exists T"] &
\end{tikzcd}
$$

2

$Z_{k-1}(M)$ free $\Rightarrow$ projective ($\because$ A1)

$\mathbb{R}$ divisible

So $\widetilde{T}\big|_{Z_{k-1}(M)} = \chi$.

$$\widetilde{\delta T} = \delta \widetilde{T} = \widetilde{T} \circ \partial = \int \omega \mod \mathbb{Z}.$$

So $\delta T = \int \omega - c \quad \exists\, c \in C^k(M;\mathbb{Z})$

<u>Note</u>: $0 = \delta^2 T = \int d\omega - \delta c$

So $\int d\omega = \delta c \overset{(*)}{\Longrightarrow} d\omega \equiv 0 = \delta c$

<u>Exercise</u> (easy) Verify that $\omega \in \Omega^k_{cl}(M)_{\mathbb{Z}}$.

$\left(\because \int \omega \Big|_{Z_k(M)} = c + \cancel{\delta T \Big|_{Z_k(M)}}^{\to 0} \right)$

$$
\begin{tikzcd}
 & H^k(M;\mathbb{Z}) \\
\hat{H}^k(M) \arrow[ur, "I"] \arrow[dr, "R"'] & \\
 & \Omega^k_{cl}(M)_{\mathbb{Z}}
\end{tikzcd}
\qquad
(z,\omega) \mapsto [c], \qquad (z,\omega) \mapsto \omega
$$

<u>Exercise</u>: Verify the well-definedness of I, R maps.
(i.e. indep. of choice of lifts)

<u>Pf</u>: Let T': another lift, satisfying $\delta T' = \int \omega' - c'$.

Then $\widetilde{T' - T}\Big|_{Z_k(M)} = 0$ So $T' = T + \delta s + d$ $\quad d \in C^{k-1}(M;\mathbb{Z})$, $s \in C^{k-2}(M;\mathbb{R})$

$$\delta T' = \delta T + 0 + \delta d$$

$$\Leftrightarrow \int \omega' - c' = \int \omega - c + \delta d$$

$$\Leftrightarrow \int (\omega' - \omega) = c' - c + \delta d$$

$$\Rightarrow \quad \omega' \equiv \omega \quad \text{and } [c'] = [c]. \quad \checkmark$$

(2) Surjectivity of I, R

Prop (Exercise) Given $\omega \in \Omega^k_{cl}(M)_{\mathbb{Z}}$, $\exists\, u \in H^k(M;\mathbb{Z})$ s.t. $r(u) = [\int \omega]$.

Let $u = [c]$. Then $c - \int \omega = \delta \lambda$ for some $\lambda \in C^{k-1}(M;\mathbb{R})$

Define $\chi := \widetilde{\lambda}\big|_{Z_{k-1}(M)}$ i.e. R: onto.

Given any $[c] \in H^k(M;\mathbb{Z})$

$\delta c = 0$ as real cochains. By de Rham thm,

$\exists\, \omega \in \Omega^k_{cl}(M)$ s.t. $\int \omega - c = \delta \mu$.

Define $\chi := \widetilde{\mu}\big|_{Z_{k-1}(M)}$ So I: onto.

Exercise $e: H^{k-1}(M;\mathbb{R}/\mathbb{Z}) \longrightarrow \hat{H}^k(M)$

$[x] \longmapsto (x|_{Z_{k-1}(M)}, 0)$

Show that ① e is well-defined, one-to-one (universal coeff. thm)

② $\operatorname{Im} e = \ker R$

Exercise:

$$\Omega^{k-1}(M) \longrightarrow \hat{H}^k(M)$$

$$\alpha \longmapsto \left(\int \alpha \Big|_{Z_{k-1}(M)}, \ d\alpha\right)$$

Show that ① kernel is $\Omega^{k-1}_{cl}(M)_{\mathbb{Z}}$.

② $a : \Omega^{k-1}(M)/\ker \longrightarrow \hat{H}^k(M)$

$\operatorname{Im} a = \ker I$.

Exercise

$$\begin{tikzcd}
H^{k-1}(M;\mathbb{R}/\mathbb{Z}) \arrow[rr, "-B"] \arrow[dr, "e"'] \arrow[rr, phantom, shift right=4ex, "\circlearrowright"] & & H^k(M;\mathbb{Z}) \\
& \hat{H}^k(M) \arrow[ur, "I"'] \arrow[dr, "R"] & \\
\dfrac{\Omega^{k-1}(M)}{\Omega^{k-1}_{cl}(M)_{\mathbb{Z}}} \arrow[ur, "a"] \arrow[rr, "d"] \arrow[rr, phantom, shift left=4ex, "\circlearrowright"] & & \Omega^k_{cl}(M)_{\mathbb{Z}}
\end{tikzcd}$$

Exercise: upper/lower seq. are LESs.

Examples $\hat{H}^0(M) = 0$

$\hat{H}^1(M) = C^\infty(M, \mathbb{R}/\mathbb{Z})$

Prop: The assignment

$$\pi_0 \operatorname{Prin}_{S^1, \nabla}(M) \xrightarrow{\cong} \hat{H}^2(M)$$

$$(P, \theta) \longmapsto (\chi, d\theta)$$

where $\chi(\gamma) := \operatorname{Hol}(\gamma)$ $\quad$ γ: any loop in M

Extend it to all $Z_1(M)$ by

$$\chi(x) := \chi(\gamma) + \frac{1}{2\pi}\int d\theta(y) \quad \text{if } x = \gamma + \delta y$$

Given $d\theta \in \Omega^2_{cl}(M)_{\mathbb{Z}}$, by Prop (Exercise),

$\exists\ c \in H^2(M; \mathbb{Z})$ s.t. $\left[\int d\theta\right] = r(c)$.

So c here classifies P, which is the 1^{st} chern class.

Question What's the analogue of "Prop in $k=3$, higher?

We need gerbes w/ connection, holonomy of a gerbe

See: Gawedzki's work on TI and gerbes

(Simons – Sullivan)

<u>Remark</u>: $\hat{H}^k(M)$ is determined uniquely by the hexagon diagram.

(Bunke – Schick) uniqueness.

2023. 12. 25. 21:30
IUT Math & Stat Research Seminar

3. Higher line bundles with Connection

<u>Recall</u>: $H^2(M;\mathbb{Z})$ is classified by line bundles

$\hat{H}^2(M)$ is classified by line bundles w/ ∇.

<u>Q</u>: What are corresponding geometric objects representing $H^n(M;\mathbb{Z})$, $\hat{H}^n(M)$?

<u>A</u>: $(n-2)$-gerbes, $(n-2)$-gerbes w/ connection.

<u>Remark</u>: $E^\bullet : \mathrm{Ho}\,\underbrace{\mathrm{Top}}_{\text{connected CW complexes}}{}^{op} \to \mathrm{GrAb}$ satisfying

homotopy invariant

(1) Wedge axiom: $E^\bullet(X_1 \wedge X_2) = E^\bullet(X_1) \oplus E^\bullet(X_2)$

(2) Mayer – Vietoris axiom.

Brown representability theorem says E^n is representable

i.e. $E^n(-) \cong \pi_0 \mathrm{Map}(-, E_n)$

If $E^\bullet$ is a generalised cohomology theory,

i.e. $E^\bullet$ satisfies all Eilenberg – Steenrod axioms except the dimension axiom.

then there is a 1-1 correspondence with Ω-spectra

$$\begin{tikzcd} E^\bullet \arrow[r, bend left, "\text{Brown's thm}"] & E_\bullet \arrow[l, bend left, "\pi_0 \mathrm{Maps}(-, E_m)"] \end{tikzcd}$$

So there are at least as many generalised cohomology theorys as the number of spectra.

Examples: $KU^\bullet$, $KO^\bullet$, ...

MU, MO, ...

TMF, Ell, ...

Question Given $E^n(X)$, can we have a geometric cocycle representing each class in $E^n(X)$?

$\rightarrow$ The Stolz – Teichner Program (conjecture)

$$\left\{ \begin{matrix} \text{Supersymmetric field theories} \\ \text{degree } n \text{ over } M \end{matrix} \right\} \Big/ \text{concordance} \cong E^n(X)$$

How do gerbes arise?

Consider $1 \to U(1) \to \tilde{G} \to G \to 1$.

<u>Prop</u>: A principal G-bundle lifts to a principal $\tilde{G}$-bundle iff the cocycle representing $DD(P)$ is trivializable.

<u>Proof</u>: On M: smooth mfld, choose a good cover $\mathcal{U}$ on M

by HLP

$$\begin{tikzcd} & \tilde{G} \arrow[d] \\ U_{ij} \arrow[ur, "\tilde{g}_{ij}"] \arrow[r, "g_{ij}"'] & G \end{tikzcd}$$

$\therefore$ contractible

(<u>Note</u>: $U(1) \to \tilde{G} \to G$: fibration) $\therefore$ satisfies HLP

So we have $\tilde{g}_{ij} \cdot \tilde{g}_{jk} \cdot \tilde{g}_{ki} = \lambda_{ijk} \cdot 1_{\tilde{G}}$

where $\lambda_{ijk} \in \check{C}^2(\mathcal{U}; U(1))$.

<u>Exercise</u>: ① λ is a degree 2 cocycle: $(\delta\lambda)_{ijkl} = 0$

② $[\lambda]$ does not depend on the choice of lifting.

$$DD(P) := [\lambda] \in \check{H}^2(M; U(1)) \cong H^3(M; \mathbb{Z})$$

is an obstruction for the existence of lifting.

✓

Let $Y \xrightarrow{\pi} M$ be a surjective submersion.

A fiber product of $Y \xrightarrow{\pi} M \xleftarrow{\pi} Y$ is

$$Y^{[2]} = Y \times_M Y := \{ (y_1, y_2) \in Y^2 : \pi(y_1) = \pi(y_2) \}$$

cf. A fiber product is a smooth manifold if π : submersion.

Notation: $Y^{[p]} = \{ (y_1, \cdots, y_p) \in Y^p : \pi(y_1) = \cdots \pi(y_p) \}$

$$\pi_{i_1 \cdots i_k} : Y^{[p]} \longrightarrow Y^{[k]}$$
$$(y_1, \cdots, y_p) \longmapsto (y_{i_1}, \cdots, y_{i_k})$$

Def (Murray) A bundle gerbe is a triple $\mathcal{L} = (L, \pi, \mu)$

where (1) $\pi : Y \to M$: surj. submersion.

(2) $L \in \mathrm{Prin}_{S^1}(Y^{[2]})$

(3) $\mu : \pi_{12}^* L \otimes \pi_{23}^* L \to \pi_{13}^* L$ S^1-bundle iso.

(4) μ : associative over $Y^{[4]}$

Note: $Y_{\mathcal{U}} := \{ (x, i) \in M \times \Lambda : x \in U_i \} \subset M \times \Lambda$, Λ: index set of $\mathcal{U}$

$\pi : Y_{\mathcal{U}} \longrightarrow M$: surjective submersion (open cover)

Construction of $DD(\mathcal{L})$ (Dixmaier-Douady class of $\mathcal{L}$)

Let $\mathcal{L} = (L, \overset{\pi: Y \to M}{\pi}, \mu) \in \mathrm{Grb}(M)$

Choose an open cover $\mathcal{U} = \{U_i\}$ on M.

We may take local sections $\sigma_i : U_i \to Y$

and $(\sigma_i, \sigma_j) : U_{ij} \to Y^{[2]}$

Consider

$$\begin{tikzcd}
(\sigma_i,\sigma_j)^* L \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & L \arrow[d] \\
U_{ij} \arrow[r, "{(\sigma_i,\sigma_j)}"'] \arrow[u, bend left, "s_{ij}"] \arrow[ur, "s_{ij}"] & Y^{[2]}
\end{tikzcd}$$

On U_{ijk}, we have

$$s_{ij}(x) \otimes s_{jk}(x) = \lambda_{ijk}(x) s_{ik}(x)$$

Exercise (1) Show that $\{\lambda_{ijk}\}$ is a degree 2 Čech cocycle on $\mathcal{U}$.

(2) $[\lambda]$ does not depend on the choices made.

Define $DD(\mathcal{L}) = [\lambda] \in H^3(M; \mathbb{Z})$.

Def: A **connection** on $\mathcal{L} = (L, \pi, \mu)$ is a connection ∇ on L compatible with μ i.e.

$$\pi_{12}^*(L, \nabla) \otimes \pi_{23}^*(L, \nabla) \xrightarrow{\mu} \pi_{13}^*(L, \nabla)$$

μ: S^1-bundle iso preserving connection.

Def: A **curving** B of $\mathcal{L} = (L, \pi, \mu)$ with connection ∇ is a diff. 2-form on Y s.t.

$$\mathrm{Curv}(\nabla) = \pi_2^* B - \pi_1^* B$$

A **bundle gerbe with connection** would mean

$$\widehat{\mathcal{L}} = (L, \pi, \mu, \nabla, B) \in \mathrm{Grb}_\nabla(M)$$

Prop: The following sequence is a L.E.S.

$$0 \to \Omega^k(M) \xrightarrow{\pi^*} \underset{dB}{\Omega^k(Y)} \xrightarrow{\delta} \Omega^k(Y^{[2]}) \to \cdots$$

where $\delta := \sum_{k=1}^{p} (-1)^{k-1} \pi^*_{i_1 \cdots \hat{i_k} \cdots i_p}$

Pf Murray. Bott–Tu.

Note $0 = d\,\mathrm{Curv}(\nabla) = d\delta B = \delta d B$

i.e. $\exists!\, H \in \Omega^3_{cl}(M)$ s.t. $\pi^* H = dB$.

Def: Let $\hat{L} = (L, \pi, \mu, \nabla, B) \in \mathrm{Grb}_\nabla(M)$

The <u>3-curvature</u> of $\hat{L}$ is H.

(a.k.a. 3-form flux, H-flux, ...)

Thm: Let $\hat{L}$ be as above.

$$r(DD(L)) = [H]$$

Example (Lifting bundle gerbe)

Let $\pi : Y \to M$: principal $PU(\mathcal{H})$-bundle.

Then $\exists$ a natural map $Y^{[2]} \xrightarrow{g} PU(\mathcal{H})$

$$(y_1, y_2) \longmapsto g(y_1, y_2)$$

Pull-back $U(\mathcal{H}) \to PU(\mathcal{H})$ to obtain

$$\begin{tikzcd} L \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & U(\mathcal{H}) \arrow[d] \\ Y^{[2]} \arrow[r] & PU(\mathcal{H}) \end{tikzcd}$$

$(L, \pi, \overset{?}{\mu})$

μ is defined by the right coset mult.

Note Deligne Complex

$$\begin{tikzcd}
\check{C}^2(\mathcal{U};\underline{U(1)}) \arrow[r, "d\log"] & \check{C}^2(\mathcal{U};\Omega^1) \arrow[r, "d"] & \check{C}^2(\mathcal{U};\Omega^2) \\
\check{C}^1(\mathcal{U};\underline{U(1)}) \arrow[r, "d\log"] \arrow[u] & \check{C}^1(\mathcal{U};\Omega^1) \arrow[r, "d"] \arrow[u, "\delta"] & \check{C}^1(\mathcal{U};\Omega^2) \arrow[u, "\delta"] \arrow[r, "d"] & {} \\
\check{C}^0(\mathcal{U};\underline{U(1)}) \arrow[r, "d\log"] \arrow[u, "\delta"] & \check{C}^0(\mathcal{U};\Omega^1) \arrow[r, "d"] \arrow[u, "\delta"] & \check{C}^0(\mathcal{U};\Omega^2) \arrow[u, "\delta"] \arrow[r, "d"] & {}
\end{tikzcd}$$

0 1 2

$$D = (-1)^{\text{formdegree}}\delta + d$$

$\downarrow$ cohomology

$H^k_D(\mathcal{U})$: Deligne cohomology

Prop: (1) $\hat{\mathcal{J}} \in \mathrm{Grb}_\nabla(M)$ determines a Deligne 2-cocycle.

Pf: Take $(\lambda_{ijk}, A_{ij}, B_i) =: \hat{\lambda}$

$A_{ij} = s_{ij}^*\nabla$, $B_i = \sigma_i^* B$.

Exercise: ① Show that it satisfies $D\hat{\lambda} = 0$

② Indep. of choice.

(2) $\hat{\mathcal{L}}_1 \overset{?}{\cong} \hat{\mathcal{L}}_2 \iff$ They have the same Deligne cohomology class.

<u>Thm</u> : $H_D^p(M) \cong \hat{H}^{p+1}(M)$ Brylinski's Loop spaces ... Chern class

So $\pi_0 \, Grb_D(M) \cong \hat{H}^3(M)$

<u>Remark</u> About ! If you define naively like

$$\mathcal{L} \cong \mathcal{L}'$$

$$\begin{array}{ccc} Y & \xrightarrow{\cong} & Y' \\ & \searrow \; \swarrow & \\ & M & \end{array}$$

$DD(\mathcal{L}) = DD(\mathcal{L}')$

but $\exists \; \mathcal{L} \not\cong_{iso} \mathcal{L}'$

$\Rightarrow$ We need "Stable isomorphism" of bundle gerbes

cf. Waldorf's More morphisms of b.g. ...

$Grb_D(M)$: 2-groupoid.