Midterm Exam

Fall 2019, Differential Geometry II

Mathematics Education, Chungbuk National University

22.10.2019 14:00–16:00

Instructions: On each page of your answer sheet, please write your name, page number, and total pages for example "홍길순 2/4면." Be sure to use your answer sheets as single-page. If you want some portion of your writings on your answer sheet not to be graded, just cross it out. You are not allowed to use your textbook or notes. You cannot use any electronic device in this exam. You are not allowed to talk to other students. Please write all details explicitly. Answers without justifications and/or calculation steps may receive no score.

1. Answer whether each of the following statements is true or false. No need to give reasons or details. **Just say true or false**. 2 points for each correct answer, 0 point for no answer, and -2 points for each incorrect answer.

(1) Let S^2 be the unit sphere in $\mathbb{R}^3$ centered at $(0,0,0)$. The set of coordinate patches $\{\mathbf{x}^+, \mathbf{x}^-\}$ provided that $\mathbf{x}^{\pm}(u,v) = (u, v, \pm\sqrt{1-u^2-v^2})$ defined on $(u,v) \in \mathbb{R}^2$ satisfying $u^2+v^2 \leq 1$ constitutes a *basis* (or *a coordinate patch representation*) of S^2.

(2) The 2-dimensional cube $I \times I := \{(x,y,0) \in \mathbb{R}^3 : 0 \leq x, y \leq 1\}$ is a simple surface in $\mathbb{R}^3$.

(3) A parametrized surface $\mathbf{x} : (u,v) \mapsto \mathbf{x}(u,v)$ being regular at $p = \mathbf{x}(u,v)$ is equivalent to $\mathbf{x}_u \times \mathbf{x}_v \neq \mathbf{0}$ at p.

(4) The first fundamental coefficents are invariant under coordinate transformation, and the second fundamental coefficients don't vary under a orientation-preserving coordinate transformation.

(5) There exists a ruled surface whose Gauss curvature is not identically 0.

2. Consider $\mathbf{x}(u,v) = \frac{1}{2}(u+v)\mathbf{e}_1 + \frac{1}{2}(u-v)\mathbf{e}_2 + uv\mathbf{e}_3$.

(1) Show that $\mathbf{x}$ is a regular parametric representation. [4 points]

(2) Find the first and the second fundamental forms at $\mathbf{x}(0,0)$. [6 points]

(3) Determine whether the point $\mathbf{x}(0,0)$ is an elliptic, a hyperbolic, a parabolic, or a planar point. [5 points]

3. Let $g : \mathbb{R}^3 \to \mathbb{R}$ be a class C^m-function. Also let $c \in \mathbb{R}$ a constant. Show that the subset $M \subset \mathbb{R}^3$ defined by $M = \{(x,y,z) \in \mathbb{R}^3 : g(x,y,z) = c\}$ is a surface if one of $\frac{\partial g}{\partial x}$, $\frac{\partial g}{\partial y}$, or $\frac{\partial g}{\partial z}$ is nonvanishing at any point of M. [10 points]

4. Find the equation of the tangent plane and normal line to $\mathbf{x}(u,v) = (u+v)\mathbf{e}_1 + (u-v)\mathbf{e}_2 + uv\mathbf{e}_3$ at $\mathbf{x}(1,-1)$. [10 points]

5. Show that every point on the surface of revolution $\mathbf{x}(\theta,t) = (f(t)\cos\theta, f(t)\sin\theta, t)$, $f(t) > 0$ is a point with vanishing Gauss curvature if and only if the surface is a right cylinder ($f(t) = a$) or a cone ($f(t) = at + b$), where a is a nonzero constant, and b is a constant. [10 points]

6. Find the principal curvatures and principal directions on

$$\mathbf{x}(u,v) = (u, v, 4u^2 + v^2)$$

at $\mathbf{x}(0,0)$. [10 points]

7. Let $\mathbf{x}(u,v) = (u\cos v, u\sin v, v)$ be a parametrized surface with $u > 0$ and $v \in \mathbb{R}$. Also let $P = \mathbf{x}(1, 2\pi)$.

(1) Find both the Gauss curvature and the mean curvature at P. [10 points]

(2) Find principal curvatures at P. [5 points]

(3) Find principal directions at P. [5 points]

(4) Let $\mathbf{u} = (\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$ be a unit tangent vector at P. Calculate the normal curvature of the given surface in the $\mathbf{u}$-direction. [5 points]

8. Let

$$X(u,v) = \left(u\cos v, u\sin v, \frac{1}{u}\right) \quad (u > 0, -\pi < v < \pi)$$

be a surface and $p = (1,0,1)$ a point on the surface. Find principal curvatures κ_1, κ_2 ($\kappa_1 > \kappa_2$) and provide details for finding them. Also let the unit tangent vector at p be $\mathbf{w} = \frac{1}{\sqrt{3}}(1,1,-1)$. Find the normal curvature in the direction of $\mathbf{w}$ and provide necessary details for finding it. [10 points]