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IUT Math & Stat Research Seminar

Differential Cohomology and gerbes:

An introduction to higher differential geometry

1. Čech cohomology and characteristic classes

<u>Def</u>: Let G be a Lie group, M: a smooth mfld.
A <u>principal G-bundle</u> P is a smooth map
$\pi : P \to M$ and a right G-action on P s.t.

(1) π is G-invariant.

(2) On every fiber, G acts on P freely and transitively from the right.

(3) P is locally trivial via G-equivariant trivializ[ation]

<u>Examples</u> (or a <u>prop</u>) when $G = GL_n(\mathbb{C})$

$$\begin{tikzcd}[column sep=huge] \text{A } G\text{-bundle } P \xrightarrow{\pi} M \arrow[r, bend left, "\text{``associated fiber bundle''}"] & \text{A complex vector bundle } E \to M, \text{ rank } n \arrow[l, bend left, "\text{frame bundle (or bundle of bases)}"] \end{tikzcd}$$

So we don't really distinguish a $U(1)$-bundle and
$\mathbb{C}^\times$ -
S^1
complex line bundles.

Def: Let G be an ~~abelian~~ group, M: a top. space.

$\mathcal{U} = \{U_i\}_{i \in \Lambda}$: an open cover of M.

$$\check{C}^p(\mathcal{U}; G) := \{ f_{i_0 \cdots i_p} : U_{i_0 \cdots i_p} \to G \}_{i_0, \cdots, i_p \in \Lambda}$$

Notation: $U_{i_1 i_2 \cdots i_n} := U_{i_1} \cap \cdots \cap U_{i_n}$

$$\delta : \check{C}^p(\mathcal{U}; G) \longrightarrow \check{C}^{p+1}(\mathcal{U}; G)$$

$$f \longmapsto (\delta f)_{i_0 \cdots i_p i_{p+1}} := f_{\hat{i}_0 i_1 \cdots i_p i_{p+1}} - f_{i_0 \hat{i}_1 \cdots i_p i_{p+1}}$$

So $(\check{C}^\bullet(\mathcal{U}; G), \delta_\bullet)$ is a complex

Def: $\check{H}^p(\mathcal{U}; G) := \ker \delta / \operatorname{Im} \delta$ Čech cohomology of M defined on $\mathcal{U}$ of degree p

Question what goes wrong if G is not abelian.

- $p = 0$; No prob! ($\because \delta f = 0 \iff f_i = f_j$, so
$$\check{H}^0(\mathcal{U}; G) = \mathrm{Map}(M, G))$$

- $p = 1$; $\check{H}^1(\mathcal{U}; G)$ is not a group

  It is a pointed set, $*$ = constant map to 1_G

  Neither $\check{Z}^1(\mathcal{U}; G)$ nor $\check{B}^1(\mathcal{U}; G)$ form a subgroup.

But we can use an equivalence rel.

$$g'_{ij} \sim g_{ij} : U_{ij} \to G \iff g'_{ij} = f_j^{-1} g_{ij} f_i$$

and define $\check{H}^1(\mathcal{U}; G) = \ker \delta / \sim$.

- $p \geq 2$ $\check{H}^p(\mathcal{U}; G)$: no hope.

<u>Prop</u> Let $1 \to K \xrightarrow{i} \tilde{G} \xrightarrow{j} G \to 1$ ... (*) be a S.E.S.

We have a L.E.S.

$$\begin{tikzcd} 1 \arrow[r] & \check{H}^0(\mathcal{U}; K) \arrow[r] & \check{H}^0(\mathcal{U}; \tilde{G}) \arrow[r] & \check{H}^0(\mathcal{U}; G) \arrow[dll, out=-10, in=170] \\ & \check{H}^1(\mathcal{U}; K) \arrow[r] & \check{H}^1(\mathcal{U}; \tilde{G}) \arrow[r] & \check{H}^1(\mathcal{U}; G) \end{tikzcd} \quad (**)$$

<u>Prop</u>: If the S.E.S. (*) has K: abelian, and $i(K) \subseteq \mathrm{Center}(\tilde{G})$, then the LES (**) extends to $\check{H}^2(\mathcal{U}; K)$.

$$\begin{tikzcd} 1 \arrow[r] & \check{H}^0(\mathcal{U}; K) \arrow[r] & \check{H}^0(\mathcal{U}; \tilde{G}) \arrow[r] & \check{H}^0(\mathcal{U}; G) \arrow[dll, out=-10, in=170] \\ & \check{H}^1(\mathcal{U}; K) \arrow[r] & \check{H}^1(\mathcal{U}; \tilde{G}) \arrow[r] & \check{H}^1(\mathcal{U}; G) \arrow[dll, out=-10, in=170, "\alpha"'] \\ & \check{H}^2(\mathcal{U}; K) & & \end{tikzcd}$$

Furthermore,

<u>Thm</u>(Dixmier–Douady) If the sheaf $\underline{\tilde{G}}_M$ is soft

(i.e. $\Gamma(M, A) \xrightarrow{\text{res}} \Gamma(\underset{\substack{\uparrow \\ \text{any closed set}}}{C}, A)$: onto)

then $\alpha : \check{H}^1(\mathcal{U}; G) \longrightarrow \check{H}^2(\mathcal{U}; K)$ is a bijection.

<u>Note</u>: $\check{H}^1(M; G)$ classifies principal G-bundles over M up to isomorphism

Having $G \to P$, $P \downarrow M$ $\longleftrightarrow$ Having $g_{ij} : U_{ij} \to G$ $U_{ij} \in \mathcal{U}$ s.t. $g_{ki}(x) \cdot g_{jk}(x) \cdot g_{ij}(x) = \mathbb{1}_G \quad \forall x \in U_{ijk}$ (cocycle condition)

$$\begin{tikzcd} P \arrow[rr] \arrow[dr] & & P' \arrow[dl] \\ & M & \end{tikzcd}$$

Having the above $\longleftrightarrow$ Having $\{ f_i : U_i \to G \}$ s.t. $g_{ij}'(x) = f_j^{-1}(x) g_{ij}(x) f_i(x)$.

In general $\check{H}^1(\mathcal{U}; G) \xrightarrow{\cong\,?} \check{H}^1(M; G) := \underset{\mathcal{U}}{\mathrm{colim}}\, \check{H}^1(\mathcal{U}; G)$

If $\mathcal{U}$ is good.

<u>Def</u>: A cover $\mathcal{U}$ of M is <u>good</u> if all open sets and their intersections are contractible.

<u>Examples</u>:

(1) $1 \to SO(n) \to O(n) \to \mathbb{Z}_2 \to 1$

Given $[P] \in \mathrm{Bun}_{O(n)}(M)/\cong \;\longmapsto w_1([P])$: the first Stiefel-Whitney class.

$w_1 : \check{H}^1(M; O(n)) \longrightarrow \check{H}^1(M; \mathbb{Z}_2)$

Note $w_1(P) = 0 \iff P$ comes from an $SO(n)$-bundle

P: orientable

(2) $1 \to \mathbb{Z}_2 \to Spin(n) \to SO(n) \to 1$

$$\check{H}^1(M; SO(n)) \longrightarrow \check{H}^2(M; \mathbb{Z}_2)$$

$$[P] \longmapsto w_2(P)$$: the second Stiefel-Whitney class

Note : $w_2(P) = 0 \iff P$ comes from a $Spin(n)$-bundle.

Remark The Whitehead tower of $O(n)$

$$\cdots \text{FiveBrane} \to String(n) \to Spin(n) \to SO(n) \to O(n)$$

(3) $1 \to \mathbb{Z} \to \mathbb{R} \to S^1 \to 1$: soft ($\because$ Tietze extension)

$$c_1 : \check{H}^1(M; S^1) \xrightarrow{\cong} \check{H}^2(M; \mathbb{Z}) \cong H^2_{sing}(M; \mathbb{Z})$$

$$[L] \longmapsto c_1(L)$$: the first Chern class

(4) Thm (Dixmier–Douady) $\underline{U(\mathcal{H})}_M$: soft.

Consider $1 \to U(1) \to U(\mathcal{H}) \to PU(\mathcal{H}) \to 1$

$$DD : \check{H}^1(M; PU(\mathcal{H})) \xrightarrow{\cong} \check{H}^2(M; S^1) \xrightarrow{\cong} H^3_{sing}(M; \mathbb{Z})$$

$$[P] \longmapsto DD(P)$$

Dixmier - Douady class

Def A <u>characteristic class</u> c of a G-bundle $P \to M$ is an assignment

$$c : \mathrm{Bun}_G(M)/\cong \longrightarrow H^*(M; \overset{\text{Ab. gp.}}{A})$$
$$[P] \longmapsto c(P)$$

that is natural $\left(\text{i.e. for } \begin{array}{ccc} \bar{P} & \xrightarrow{\bar{f}} & P \\ \downarrow & & \downarrow \\ M' & \xrightarrow{f} & M \end{array} \quad f^* c(P) = c(\bar{f}^* P) \right)$

Remark (Yoneda Lemma) $\mathrm{Bun}_G(-) \cong \mathrm{Maps}(-, BG)$

An assignment

$$\left\{ \begin{array}{l} \text{Characteristic} \\ \text{class of } P \in \mathrm{Bun}_G M \end{array} \right\} \longrightarrow H^*(BG; A)$$

is 1-1 and onto.

Remark There's an alternative way to define characteristic classes using geom. data.

"Chern - Weil"

E.g. $L, \nabla \rightsquigarrow$ Chern-Weil form $c(\nabla) = \frac{i}{2\pi} \mathrm{Curv}(\nabla)$

Thm (Chern-Weil) $[c_1(\nabla)]$ is indep. of choice of ∇.

$H^2(M;\mathbb{Z})$ classifies Line bundles
$H^3(M;\mathbb{Z})$ classifies "Gerbes"

What classifies line bundles w/ conn ? (L,∇) $\hat{H}^2(M)$

gerbes w/ conn ? $\hat{H}^3(M)$

higher gerbes w/ conn ? $\hat{H}^{\bullet}(M)$

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2. Cheeger - Simons differential characters

"Diff. characters and geom. invariants"

<u>Notations</u> : M : smooth manifold

$C^k(M;R)$: smooth singular k-cochains in M with coeff. in R

R: comm. ring with 1

$Z^k(M;R)$: smooth singular k-cocycles in M w/ coeff. in R.

$\Omega^k(M)$: diff. k-forms on M.

$\Omega^k_{cl}(M)_{\mathbb{Z}}$: closed k-forms with integral periods

i.e. $\omega \in \Omega^k_{cl}(M)_{\mathbb{Z}} \Leftrightarrow \begin{cases} d\omega = 0 \\ \int \omega \big|_{Z_k(M)} \in \mathbb{Z}. \end{cases}$

<u>Notation</u> $\sim$: $\mathbb{R} \to \mathbb{R}/\mathbb{Z}$

<u>Fact</u>: Let $\Lambda \subsetneq \mathbb{R}$: proper subring. Then a nonvanishing

diff. form does not take values in Λ.

<u>Prop</u>: The map $\int : \Omega^k(M) \longrightarrow \widetilde{C^k}(M; \mathbb{R}/\mathbb{Z})$
$$\omega \longmapsto \int \omega$$

is one-to-one.

<u>Def</u> (Cheeger-Simons differential characters)

$$\hat{H}^k(M) := \left\{ (\chi, \omega) : \begin{array}{l} \chi \in \mathrm{Hom}_{\mathbb{Z}}(Z_{k-1}(M), \mathbb{R}/\mathbb{Z}) \\ \omega \in \Omega^k(M) \\ \text{s.t., for } \forall D \in C_k(M; \mathbb{Z}) \\ \chi \circ \partial D = \int_D \omega \mod \mathbb{Z}. \end{array} \right\}$$

with the componentwise addition is the group of <u>differential characters</u> (or <u>differential cohomology</u>) of M with degree k.

<u>Prop</u>: (Differential cohomology hexagon diagram)

$$\begin{tikzcd}
0 \arrow[dr] & & & & 0 \\
& H^{k-1}(M; \mathbb{R}/\mathbb{Z}) \arrow[rr, "-B: \text{Bockstein}"] \arrow[dr] & & H^k(M; \mathbb{Z}) \arrow[ur] \arrow[dr, "r"] & \\
H^{k-1}(M; \mathbb{R}) \arrow[ur] \arrow[dr, "\mathrm{rep}"'] \arrow[rr, phantom, "\circlearrowright"] & & \hat{H}^k(M) \arrow[ur, "I"] \arrow[dr, "R"] \arrow[rr, phantom, "\circlearrowright"] & & H^k(M; \mathbb{R}) \\
& \dfrac{\Omega^{k-1}(M)}{\Omega^{k-1}_{cl}(M)_{\mathbb{Z}}} \arrow[ur, "a"] \arrow[rr, "d"] & & \Omega^k_{cl}(M)_{\mathbb{Z}} \arrow[ur, "\int \circ\, dR"'] \arrow[dr] & \\
0 \arrow[ur] & & & & 0
\end{tikzcd}$$

with all triangles and squares commute, diagonals are SESs, and the upper and lower sequences are LESs.

<u>Proof</u>: (1) I, R maps

Algebra facts we need:

- A1: Any subgroup of a free abelian group is free.
- A2: An ab. gp. G is <u>divisible</u> if $\forall x \in G$ and $\forall n \in \mathbb{Z}^+$ $\exists y$ s.t. $x = ny$.
- A3: <u>Prop</u>: G: divisible $\Leftrightarrow$ G is an injective obj. in the cat. of ab. gps

Take $(\chi, \omega) \in \hat{H}^k(M)$.

$$\begin{tikzcd}
 & \mathbb{R} \arrow[d] \\
Z_{k-1}(M) \arrow[r, "\chi"] \arrow[ur, dashed, "\exists \bar{\chi}"] & \mathbb{R}/\mathbb{Z} \arrow[d] \\
 & 0
\end{tikzcd}$$

free $\Rightarrow$ projective $(\because \text{A1})$

$$\begin{tikzcd}
0 \arrow[d] & \\
Z_{k-1}(M) \arrow[r, "\bar{\chi}"] \arrow[d, hook] & \mathbb{R} \\
C_{k-1}(M) \arrow[ur, dashed, "\exists T"'] &
\end{tikzcd}$$

divisible

So $\widetilde{T\big|_{Z_{k-1}(M)}} = \chi$.

$$\widetilde{\delta T} = \delta \tilde{T} = \tilde{T} \circ \partial = \int \omega \mod \mathbb{Z}.$$

So $\delta T = \int \omega - c \quad \exists\, c \in C^k(M;\mathbb{Z})$

<u>Note</u>: $0 = \delta^2 T = \int d\omega - \delta c$

So $\int d\omega = \delta c \overset{(*)}{\Longrightarrow} d\omega \equiv 0 = \delta c$

<u>Exercise (easy)</u> Verify that $\omega \in \Omega^k_{cl}(M)_{\mathbb{Z}}$.

$$\left(\because \int \omega \Big|_{Z_k(M)} = c + \cancel{\delta T}^{\to 0}\Big|_{Z_k(M)} \right)$$

$$\begin{tikzcd}
 & H^k(M;\mathbb{Z}) \\
 & {[c]} \\
\hat{H}^k(M) \arrow[uur, "I"] \arrow[ddr, "R"'] & \\
(z,\omega) \arrow[uur, mapsto, squiggly] \arrow[dr, mapsto, squiggly] & \\
 & \omega \\
 & \Omega^k_{cl}(M)_{\mathbb{Z}}
\end{tikzcd}$$

<u>Exercise</u>: Verify the well-definedness of I, R maps.
(i.e. indep. of choice of lifts)

<u>Pf</u>: Let T': another lift, satisfying $\delta T' = \int \omega' - c'$.

Then $\widetilde{T' - T\big|_{Z_k(M)}} = 0$ So $T' = T + \delta s + d\sigma$ $\quad d\sigma \in C^{k-1}(M;\mathbb{Z})$, $s \in C^{k-2}(M;\mathbb{R})$

$$\delta T' = \delta T + 0 + \delta d$$

$$\Leftrightarrow \int \omega' - c' = \int \omega - c + \delta d$$

$$\Leftrightarrow \int (\omega' - \omega) = c' - c + \delta d$$

$$\overset{(*)}{\Rightarrow} \quad \omega' \equiv \omega \quad \text{and } [c'] = [c]. \quad \checkmark$$

(2) Surjectivity of I, R

Prop (Exercise) Given $\omega \in \Omega^k_{cl}(M)_{\mathbb{Z}}$, $\exists\, u \in H^k(M;\mathbb{Z})$
s.t. $r(u) = [\int \omega]$.

Let $u = [c]$. Then $c - \int \omega = \delta \lambda$ for some $\lambda \in C^{k-1}(M;\mathbb{R})$

Define $\chi := \widetilde{\lambda}\big|_{Z_{k-1}(M)}$ i.e. R: onto.

Given any $[c] \in H^k(M;\mathbb{Z})$
$\delta c = 0$ as real cochains. By de Rham thm,
$\exists\, \omega \in \Omega^k_{cl}(M)$ s.t. $\int \omega - c = \delta \mu$.

Define $\chi := \widetilde{\mu}\big|_{Z_{k-1}(M)}$ So I: onto.

Exercise $e: H^{k-1}(M;\mathbb{R}/\mathbb{Z}) \longrightarrow \hat{H}^k(M)$
$[x] \longmapsto (x|_{Z_{k-1}(M)}, 0)$

Show that ① e is well-defined, one-to-one (universal coeff. thm)

② $\operatorname{Im} e = \ker R$

Exercise:

$$\Omega^{k-1}(M) \longrightarrow \hat{H}^k(M)$$

$$\alpha \longmapsto \left(\int \alpha \Big|_{Z_{k-1}(M)}, \, d\alpha \right)$$

Show that ① kernel is $\Omega^{k-1}_{cl}(M)_{\mathbb{Z}}$.

② $a : \Omega^{k-1}(M) / \ker \longrightarrow \hat{H}^k(M)$.

$\operatorname{Im} a = \ker I$.

Exercise

$$\begin{tikzcd}
H^{k-1}(M;\mathbb{R}/\mathbb{Z}) \arrow[rr, "-B"] \arrow[dr, "e"'] & \arrow[d, phantom, "\circlearrowright"] & H^k(M;\mathbb{Z}) \\
& \hat{H}^k(M) \arrow[ur, "I"'] \arrow[dr, "R"] \arrow[d, phantom, "\circlearrowright"] & \\
\dfrac{\Omega^{k-1}(M)}{\Omega^{k-1}_{cl}(M)_{\mathbb{Z}}} \arrow[ur, "a"] \arrow[rr, "d"] & \, & \Omega^k_{cl}(M)_{\mathbb{Z}}
\end{tikzcd}$$

Exercise: upper/lower seq. are LESs.

Examples $\hat{H}^0(M) = 0$

$\hat{H}^1(M) = C^\infty(M, \mathbb{R}/\mathbb{Z})$

Prop: The assignment

$$\pi_0 \mathrm{Prin}_{S^1, \nabla}(M) \xrightarrow{\cong} \hat{H}^2(M)$$

$$(P, \theta) \longmapsto (\chi, d\theta)$$

where $\chi(\gamma) := \mathrm{Hol}(\gamma)$ $\quad\gamma$: any loop in M

Extend it to all $Z_1(M)$ by

$$\chi(x) := \chi(\gamma) + \frac{1}{2\pi}\int d\theta(y) \quad \text{if } x = \gamma + \delta y$$

Given $d\theta \in \Omega^2_{cl}(M)_{\mathbb{Z}}$, by Prop (Exercise),

$\exists\ c \in H^2(M; \mathbb{Z})$ s.t $[\int d\theta] = r(c)$.

So c here classifies P, which is the 1st chern class.

Question What's the analogue of "Prop in $k=3$, higher?

We need gerbes w/ connection, holonomy of a gerbe

See: Gawedzki's work on TI and gerbes

(Simons – Sullivan)

Remark: $\hat{H}^k(M)$ is determined uniquely by the hexagon diagram.

(Bunke – Schick) uniqueness.