9.15 Theorem *Suppose E is an open set in R^n, $\mathbf{f}$ maps E into R^m, $\mathbf{f}$ is differentiable at $\mathbf{x}_0 \in E$, $\mathbf{g}$ maps an open set containing $\mathbf{f}(E)$ into R^k, and $\mathbf{g}$ is differentiable at $\mathbf{f}(\mathbf{x}_0)$. Then the mapping $\mathbf{F}$ of E into R^k defined by*

$$\mathbf{F}(\mathbf{x}) = \mathbf{g}(\mathbf{f}(\mathbf{x}))$$

is differentiable at $\mathbf{x}_0$, and

$$\mathbf{F}'(\mathbf{x}_0) = \mathbf{g}'(\mathbf{f}(\mathbf{x}_0))\mathbf{f}'(\mathbf{x}_0). \tag{21}$$

On the right side of (21), we have the product of two linear transformations, as defined in Sec. 9.6.

Proof Put $\mathbf{y}_0 = \mathbf{f}(\mathbf{x}_0)$, $A = \mathbf{f}'(\mathbf{x}_0)$, $B = \mathbf{g}'(\mathbf{y}_0)$, and define

$$\mathbf{u}(\mathbf{h}) = \mathbf{f}(\mathbf{x}_0 + \mathbf{h}) - \mathbf{f}(\mathbf{x}_0) - A\mathbf{h},$$
$$\mathbf{v}(\mathbf{k}) = \mathbf{g}(\mathbf{y}_0 + \mathbf{k}) - \mathbf{g}(\mathbf{y}_0) - B\mathbf{k},$$

for all $\mathbf{h} \in R^n$ and $\mathbf{k} \in R^m$ for which $\mathbf{f}(\mathbf{x}_0 + \mathbf{h})$ and $\mathbf{g}(\mathbf{y}_0 + \mathbf{k})$ are defined. Then

$$|\mathbf{u}(\mathbf{h})| = \varepsilon(\mathbf{h})|\mathbf{h}|, \qquad |\mathbf{v}(\mathbf{k})| = \eta(\mathbf{k})|\mathbf{k}|, \tag{22}$$

where $\varepsilon(\mathbf{h}) \to 0$ as $\mathbf{h} \to \mathbf{0}$ and $\eta(\mathbf{k}) \to 0$ as $\mathbf{k} \to \mathbf{0}$.

Given $\mathbf{h}$, put $\mathbf{k} = \mathbf{f}(\mathbf{x}_0 + \mathbf{h}) - \mathbf{f}(\mathbf{x}_0)$. Then

$$|\mathbf{k}| = |A\mathbf{h} + \mathbf{u}(\mathbf{h})| \leq [\|A\| + \varepsilon(\mathbf{h})]\,|\mathbf{h}|, \tag{23}$$

and

$$\begin{aligned}\mathbf{F}(\mathbf{x}_0 + \mathbf{h}) - \mathbf{F}(\mathbf{x}_0) - BA\mathbf{h} &= \mathbf{g}(\mathbf{y}_0 + \mathbf{k}) - \mathbf{g}(\mathbf{y}_0) - BA\mathbf{h} \\ &= B(\mathbf{k} - A\mathbf{h}) + \mathbf{v}(\mathbf{k}) \\ &= B\mathbf{u}(\mathbf{h}) + \mathbf{v}(\mathbf{k}).\end{aligned}$$

Hence (22) and (23) imply, for $\mathbf{h} \neq \mathbf{0}$, that

$$\frac{|\mathbf{F}(\mathbf{x}_0 + \mathbf{h}) - \mathbf{F}(\mathbf{x}_0) - BA\mathbf{h}|}{|\mathbf{h}|} \leq \|B\|\,\varepsilon(\mathbf{h}) + [\|A\| + \varepsilon(\mathbf{h})]\eta(\mathbf{k}).$$

Let $\mathbf{h} \to \mathbf{0}$. Then $\varepsilon(\mathbf{h}) \to 0$. Also, $\mathbf{k} \to \mathbf{0}$, by (23), so that $\eta(\mathbf{k}) \to 0$. It follows that $\mathbf{F}'(\mathbf{x}_0) = BA$, which is what (21) asserts.