#9. $f(x,y,z) = x+y+z$

Constraints $\begin{array}{l} x^2-y^2=1 \\ 2x+z=1 \end{array}$ Extrema?

Sol: Let $g_1(x,y,z) = x^2-y^2$ and $S_1 = \{(x,y,z)\in\mathbb{R}^3 : g_1(x,y,z)=1\}$

$g_2(x,y,z) = 2x+z$. $S_2 = \{(x,y,z)\in\mathbb{R}^3 : g_2(x,y,z)=1\}$.

Observe that

$$\nabla g_1 = (2x, -2y, 0)$$
$$\nabla g_2 = (2, 0, 1)$$

∇g_1 and ∇g_2 are linearly independent forall $(x,y,z) \neq (0,0,0)$

Since $S_1 \cap S_2$ does not have the origin, we can apply the Lagrange Multiplier Theorem so that if $f|_{S_1\cap S_2}$ has a local extremum at $\vec{x}_0=(x_0, y_0, z_0)$,

then $\nabla f(\vec{x}_0) = \lambda_1 \nabla g_1(\vec{x}_0) + \lambda_2 \nabla g_2(\vec{x}_0)$.

$$\begin{array}{ccc} \| & & \| \\ (1,1,1) & & (2\lambda_1 x_0+2, -2\lambda_1 y_0, \lambda_2) \end{array}$$

So
$1 = 2\lambda_1 x_0 + 2 \quad \cdots ①$
$1 = -2\lambda_1 y_0 \quad \cdots ②$
$1 = \lambda_2$
$x_0^2 - y_0^2 = 1 \quad \cdots ③$
$2x_0 + z_0 = 1$

From ① and ② $\lambda_1 \neq 0$

$$x_0 = -\frac{1}{2\lambda_1} \quad y_0 = -\frac{1}{2\lambda_1}$$

Notice that, for no λ_1 the equation ③ is satisfied.

This means there is no $\vec{x}_0 \in S_1\cap S_2$ s.t. $f|_{S_1\cap S_2}$ is a local extremum.