THE HEAT EQUATION METHOD IN INDEX THEORY

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Abstract. In this note, we give a brief exposition of the idea of the heat equation method in classical proofs of the Atiyah-Singer index theorem. We begin with a heuristic introduction to the notion of Sobolev spaces to provide an idea about how a finiteness condition for the index theory is achieved. This allows us to provide a definition of the heat operator and the heat kernel of a positive elliptic operator and define the trace of a heat operator. We then exhibit the time-invariance of the index of a differential operator whose composition with a formal adjoint is a Laplace operator. Finally we remark on proving local Atiyah-Singer index theorems.

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1. Consequences from the theory of Sobolev spaces([4] p.170-177)

After defining the notion of differential operators between spaces of smooth sections, we may expect bringing operator theoretic consequences into the index theory. Albeit very natural, we simply cannot do so without building a careful setup, because a space of smooth sections is not necessarily complete. Among the simplest, one may consider a trivial line bundle over the reals. Even though the absolute value function is not smooth, one can find a sequence of smooth functions that converges to the absolute value function. One reason we are concerned about non-completeness of the space of smooth sections in understanding index theory is that we wish to secure a sufficient condition for finiteness. i.e. the fact that every elliptic operator on a compact Riemannian manifold extends to a Fredholm map, and the index of the original operator is the same as the index of its Fredholm extension, independent from the choice of an extension.

Let $E \to X$ be a complex vector bundle over a real compact Riemannian manifold endowed with a positive-definite Hermitian metric. We also put a connection ∇. Then the following defines a norm on $\Gamma(E)$:

$$\|u\|_k^2 := \sum_{j=0}^{k} \int_X |\overbrace{d^\nabla \circ \ldots \circ d^\nabla}^{j} u|^2 dV_g < +\infty$$

where d^∇ is the covariant exterior differential. This is called the **basic Sobolev k-norm** on $\Gamma(E)$. One can prove that the equivalence class of this norm is independent of the choice

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of the Hermitian metric and the connection. The completion of $\Gamma(E)$ in this norm is the **Sobolev space** $L^2_k(E)$. We leave the proof of following proposition as an exercise.

Proposition 1. A differential operator $P : \Gamma(E) \to \Gamma(F)$ of order m extends to to a bounded linear map $P : L^2_k(E) \to L^2_{k-m}(F)$ for all $k \geq m$, where $k, m \in \mathbb{Z}^+$.

A goal of this thread of analytic discussion is, as mentioned above, establishing a finiteness condition that if $P : \Gamma(E) \to \Gamma(E)$ is an elliptic operator on a compact Riemannian manifold, then P extends to a Fredholm operator $P : L^2_k(E) \to L^2_{k-m}(E)$ whose kernel and cokernel have dimensions independent of k and are composed of smooth sections. This is discussed in Lawson and Michelsohn [4] p.170–198, and in particular Theorem 5.2 in p.193.

To establish smoothness results, we need to use the Sobolev embedding theorem, and this requires more than Sobolev spaces obtained by basic Sobolev norms. We need to bring the notion which allows us to have the embedding theorem, and for this we locally bring the notion of the Sobolev space L^2_s defined by a completion of the Schwartz space $\mathcal{S}$ using the Sobolev s-norm given by the formula:

$$\|u\|^2_s = \int (1 + |\xi|)^{2s} |\hat{u}(\xi)|^2 d\xi \qquad u \in \mathcal{S},\ s \in \mathbb{R}.$$

To make sense this local construction globally on a manifold, one should carefully choose a local trivialization and a partition of unity satisfying certain conditions(called good presentations) and prove that the local Sobolev s-norm defined under these choices are independent of the choice of a local trivialization or a partition of unity. Furthermore, one can also prove that if $s = k \in \mathbb{Z}^+$, the equivalence class of a globalized Sobolev s-norm matches up with that of basic Sobolev k-norm defined above, for any choice of Hermitian metric or connection on E. i.e. the finiteness and smoothness conditions are not relying on additional structures like metrics and connections on a bundle. Consequently, we obtain the following global statements from corresponding local results. For details, see Lawson and Michelsohn [4] p.170–177.

Proposition 2. Let E and F be smooth complex vector bundles over a compact Riemannian n-manifold X.
(1) For each integer $k \geq 0$ and each $s > (n/2) + k$, there is a continuous inclusion $L^2_s(E) \subset C^k(E)$.
(2) For any Riemannian volume measure μ on X, the bilinear map on $\Gamma(E) \times \Gamma(E^*)$ given by setting

$$(u, u^*) = \int_X u^*(u) d\mu$$

extends to a pairing $L^2_s(E) \times L^2_{-s}(E^*)$ for all $s \in \mathbb{R}$, where L^2_{-s} is identified with $(L^2_s(E))^*$ for all $s \in \mathbb{R}$.
(3) Multiplication $T_A u := Au$ by any $A \in \Gamma(\mathrm{Hom}(E, F))$ extends to a bounded linear map $T_A : L^2_s(E) \to L^2_s(F)$ for all $s \in \mathbb{R}$
(4) Any differential operator $P : \Gamma(E) \to \Gamma(F)$ of order m extends to a bounded linear map $P : L^2_s(E) \to L^2_{s-m}(F)$ for all $s \in \mathbb{R}$.

By the results on elliptic operators, given an elliptic operator $P : \Gamma(E) \to \Gamma(E)$ on a compact Riemannian manifold, we may consider its Fredholm extension $P : L^2_s(E) \to L^2_{s-m}(E)$ for all $s \in \mathbb{R}$, and in particular we may set $s = m$, in which case the codomain becomes the usual L^2-completion of $\Gamma(E)$. As in operator theory, we say that the operator P is **positive** if P is self-adjoint and $\langle Pu, u\rangle_0 \geq 0$ for all $u \in \Gamma(E)$. The self-adjointness in defining a positive operator is essential, since any self-adjoint operator can only have real eigenvalues, the former guarantees that the inner product can have its value only from reals.

2. The heat operator and the heat kernel of a differential operator ([4] p.198-199)

Let $E \to X$ be a complex vector bundle over a compact Riemannian n-manifold, and $P : \Gamma(E) \to \Gamma(E)$ be a positive self-adjoint elliptic differential operator of order m. We shall construct the *heat operator* $e^{-tP} : \Gamma(E) \to \Gamma(E)$ for $t > 0$, so that it becomes an infinitely smoothing operator such that $u_t := e^{-tP}u$ for some $u \in \Gamma(E)$ solves the equation

$$\frac{\partial u_t}{\partial t} + Pu_t = 0.$$

We define the **heat operator** $e^{-tP} : \Gamma(E) \to \Gamma(E)$ as an integral operator

$$e^{-tP}u(x) = \int_X K_t(x, y)u(y)dy \qquad x, y \in X$$

with

$$K_t(x, y) := \sum_{k=1}^{\infty} e^{-\lambda_k t} u_k(x) \otimes u_k^*(y) \tag{1}$$

where $\{u_k\}_{k=1}^{\infty}$ is a complete orthonormal basis of $L^2_0(E)$ consisting of eigensections of P with $Pu_k = \lambda_k u_k$, and which is varying smoothly with respect to t, x and y. K is a section of a bundle $p_0^*(p_1^*E \otimes p_2^*E^*) \to \mathbb{R}^+ \times X \times X$ that satisfies (1), where $p_i : X \times X \to X$ is the canonical projection to the i-th factor, and p_0 is a slice-preserving projection map $\mathbb{R}^+ \times X \times X \to X \times X$. This section $K \in \Gamma(p_0^*(p_1^*E \otimes p_2^*E^*))$ is called the **heat kernel** for P.

As it appears in the formulation (1) of the heat kernel, it is not at all obvious whether this definition even makes sense. Specifically, it is not clear if eigensections of P can constitute an orthonormal basis of $L^2_0(E)$, and the summation converges. In what follows, we shall prove that this summation actually converges and the heat kernel is smooth. To proceed, we need the following key fact.

Proposition 3. Let $P : \Gamma(E) \to \Gamma(E)$ be a positive self-adjoint elliptic differential operator of order $m > 0$ over a compact Riemannian n-manifold. Then there is a complete orthonormal basis $\{u_k\}_{k=1}^{\infty}$ of $L^2_0(E)$ such that

$$Pu_k = \lambda_k u_k \qquad \text{for all } k$$

where

$$0 \leq \lambda_1 \leq \lambda_2 \leq \ldots \to \infty.$$

In fact, for some constant $c > 0$,

$$\lambda_k \geq c k^{\frac{2m}{n(n+2m+2)}} \qquad \text{for all } k.$$

A proof of this statement can be found in Lawson and Michelsohn [4] p.196–198, in particular Corollary 5.9 in p.198.

Recall that the **uniform C^r-norm** on $C^r(E)$, the set of r-differentiable sections of E, is defined by $\|\cdot\|_{C^r} : C^r(E) \to \mathbb{R}$ such that

$$\|u\|^2_{C^r} = \sup_{x\in X} \sum_{k\leq r} \|d^{\nabla^k} u\|^2$$

where

$$\|u\|^2 = \langle u, u^* \rangle := \int_X u^* u dV \qquad \text{for } u \in C^r(E).$$

One has to make the choice of a Hermitian metric and a connection on E to define a norm on $C^r(E)$ as above. However, as we discussed in section 1, the globalized Sobolev norm $\|\cdot\|_s$ coincides with the Sobolev norm $\|\cdot\|_k$ for any positive integer k which defined by using a given Hermitian metric and a connection, and the equivalence class of $\|\cdot\|_k$ is independent of the choice of a Hermitian metric and a connection. Thus we do not need to specify a Hermitian metric and a connection to define the uniform C^r-norm on $C^r(E)$.

Lemma 4. For any $r \geq 0$, and any closed interval $I \in (0, \infty)$, the series

$$K_t(x, y) = \sum_{k=1}^{\infty} e^{-\lambda_k t} u_k(x) \otimes u^*_k(y)$$

converges uniformly in $(C^r(E), \|\cdot\|_{C^r})$ on $I \times X \times X$.

Proof. We shall use the following two estimates:

- For each real number $s > \frac{n}{2} + k$, there exists a constant K_s such that

  $$\|u\|_{C^r} \leq K_s \|u\|_s \qquad \text{for all } u \in \Gamma(E).$$

  This is the *Sobolev embedding theorem*: there is a continuous embedding $L^2_s(E) \hookrightarrow C^r(E)$. See Lawson and Michelsohn [4] p.172 Theorem 2.5 and p.176 Theorem 2.15.
- Let $P : \Gamma(E) \to \Gamma(E)$ be an elliptic operator of order m on a compact Riemannian manifold X. For each $s \in \mathbb{R}$ there is a constant C_s such that

  $$\|u\|_s \leq C_s(\|u\|_{s-m} + \|Pu\|_{s-m}) \qquad \text{for all } u \in L^2_s(E).$$

  Hence the norms $\|\cdot\|_s$ and $\|\cdot\|_{s-m} + \|P\cdot\|_{s-m}$ on $L^2_s(E)$ are equivalent. This is the *fundamental elliptic estimates*. See Lawson and Michelsohn [4] p.193 Theorem 5.2(iii).

From the above two estimates, there exists constants c and c' such that

$$\|u_k\|_{C^r} \leq c' \|u_k\|_{ms} \leq c(\|u_k\|_0 + \|P^s u_k\|_0) = c(1 + \lambda^s_k).$$

By Proposition 3, λ_k satisfies the inequality

$$\lambda_k \geq k^\gamma \qquad \text{where } \gamma = \frac{2m}{n(n+2m+2)}.$$

Now we observe the tail of the series:

$$K_t(x,y) - \sum_{k=1}^{N-1} e^{-\lambda_k t} u_k(x) \otimes u_k^*(y) = \sum_{k=N}^{\infty} e^{-\lambda_k t} u_k(x) \otimes u_k^*(y).$$

It suffices to show that the convergence of $K_t(x,y)u_k(y)$ for an eigensection u_k:

$$\begin{aligned} \| \sum_{k=N}^{\infty} e^{-\lambda_k t} u_k(x) \|_{C^r} \leq \sum_{k=N}^{\infty} e^{-\lambda_k t} \|u_k(x)\|_{C^r} &\leq \sum_{k=N}^{\infty} e^{-\lambda_k t} c(1+\lambda_k^s) \\ &= c\sum_{k=N}^{\infty} e^{-\lambda_k t} + c\sum_{k=N}^{\infty} e^{-\lambda_k t}\lambda_k^s. \end{aligned} \tag{2}$$

Hence the tail is uniformly bounded. Observe that

$$\lambda_k \geq k^\gamma \Rightarrow e^{-\lambda_k t} \leq e^{-k^\gamma t}.$$

This proves that the first summand in the far RHS of (2) converges to 0 as $N \to \infty$. We may assume k is sufficient large. Then, even if $k^{\gamma^s} \leq \lambda_k^s$, after some k, we have

$$\lambda_k^s e^{-k^\gamma t} \leq k^{\gamma^s} e^{-\lambda_k t} \leq k^{\gamma^s} e^{-k^\gamma t}.$$

This shows that the second summand of (2) converges to 0 as $N \to \infty$, by the following comparison:

$$\sum_{k=N}^{\infty} k^{\gamma^s} e^{-k^\gamma t} \leq \int_{N-1}^{\infty} x^s e^{-tx} dx \to 0 \text{ as } N \to \infty.$$

$\square$

A linear operator $P : \Gamma(E) \to \Gamma(F)$ over a compact Riemannian manifold X is called **infinitely smoothing** if it is an integral operator whose kernel is a C^∞-section over a bundle $X \times X$. The following proposition immediately follows from the above lemma.

Proposition 5. (1) For each $t > 0$, the operator $e^{-tP} : \Gamma(E) \to \Gamma(E)$ is infinitely smoothing.
(2) Given $u \in L_s^2(E)$ for any $s \in \mathbb{R}$, the section $e^{-tP}u(x)$ is a smooth section over $\mathbb{R}^+ \times X$.
(3) $e^{-tP}u(x)$ solves the equation

$$\frac{\partial u}{\partial t} + Pu = 0.$$

Proof. (1) and (2) are consequences of the lemma. We check (3).

$$\begin{aligned}
\frac{\partial}{\partial t}(e^{-tP}u) + P(e^{-tP}u) &= \int_X \frac{\partial}{\partial t}K_t(x,y)u(y)dy + P(e^{-tP}u) \\
&= \int_X \sum_{k=1}^{\infty} -\lambda_k e^{-\lambda_k t} u_k(x) \otimes u_k^*(y)\big(u(y)\big)dy + P(e^{-tP}u) \\
&= -\sum_{k=1}^{\infty} \lambda_k \int_X e^{-\lambda_k t} u_k(x) \otimes u_k^*(y)\big(u(y)\big)dy + P(e^{-tP}u) \\
&= -P(e^{-tP}u) + P(e^{-tP}u) = 0.
\end{aligned}$$

□

3. The trace of a heat operator([4] p.199-201)

We define the **trace** of e^{-tP} by

$$\mathrm{tr} e^{-tP} := \sum_{k=1}^{\infty} \langle e^{-tP}u_k, u_k \rangle$$

where $\{u_k\}_{k=1}^{\infty}$ is a complete orthonormal basis of $L^2(E)$ consisting eigensections of P. Observe that if λ_j is an eigenvalue of P, $e^{-t\lambda_j}$ is an eigenvalue of e^{-tP}:

$$\begin{aligned}
e^{-tP}u_j(x) &= \int_X K_t(x,y)u_j(y)dy \\
&= \int_X \sum_{k=1}^{\infty} e^{-t\lambda_k} u_k(x) \otimes u_k^*(y)\big(u_j(y)\big)dy \\
&= e^{-t\lambda_j}u_j(x) \int_X dy = e^{-t\lambda_j}u_j(x).
\end{aligned}$$

Accordingly we may write

$$\mathrm{tr} e^{-tP} := \sum_{k=1}^{\infty} e^{-t\lambda_k}$$

where $\{\lambda_k\}_{k=1}^{\infty}$ is the set of eigenvalues of P. By Proposition 3, this summation converges. Note that this convergence is not necessarily the case for an arbitrary bounded linear operator on a Hilbert space. This is a special property arising from ellipticity and positivity of a self-adjoint differential operator.

In applications, the operator P is often in the form of positive self-adjoint operators Q^*Q or QQ^*. These are called **Laplace operators**. Many examples of such operators arise from the Dirac operator on spinors:

$$\mathcal{D} : S^+ \oplus S^- \to S^- \oplus S^+$$

with the property that $\mathcal{D}$ is self-adjoint. Being self-adjoint, Ind $\mathcal{D}$ is always zero, however each restricted operators

$$\mathcal{D}_+ : S^+ \to S^- \qquad \mathcal{D}_- : S^- \to S^+$$

may have nonzero index. Among these two, we only need to calculate the index of $\mathcal{D}_+$ since Ind $\mathcal{D}_+ = -$Ind ${\mathcal{D}_+}^*$ from ellipticity of $\mathcal{D}_+$, and Ind $\mathcal{D}_- =$ Ind ${\mathcal{D}_+}^*$. The latter is because of the self-adjointness of $\mathcal{D}$: From $\mathcal{D}_+ + \mathcal{D}_- = \mathcal{D} = \mathcal{D}^* = {\mathcal{D}_+}^* + {\mathcal{D}_-}^*$, we must have $\mathcal{D}_+ = {\mathcal{D}_-}^*$ and $\mathcal{D}_- = {\mathcal{D}_+}^*$. Recall that, for a Dirac operator $D_+ = d + d^* : \Omega^{\text{even}}(X) \to \Omega^{\text{odd}}(X)$, we obtained Ind $D_+ = \chi(X)$, and for another Dirac operator $D_+ : \Omega^\bullet_+(X) \to \Omega^\bullet_-(X)$, we obtained Ind $D_+ = \text{Sign}(X)$.

Let $P : \Gamma(E) \to \Gamma(F)$ be an elliptic differential operator over a compact Riemannian manifold X, and consider the following Laplace operators

$$P^*P : \Gamma(E) \to \Gamma(E) \qquad PP^* : \Gamma(F) \to \Gamma(F).$$

Assume that a Hermitian metric is defined on E and F so that the formal adjoint P^* makes sense. Observe that PP^* and P^*P are self-adjoint, and since $\langle P^*Pu, u\rangle_0 = \|Pu\|_0^2$ and $\langle PP^*v, v\rangle_0 = \|Pv\|_0^2$, these are positive operators. Also, if we send $t \to \infty$ for a heat operator e^{-tP}, all eigenvalues $e^{-t\lambda_k} \to 0$ for $\lambda_k \neq 0$, and $e^{-t\lambda_k} \to 1$ for $\lambda_k = 0$. Hence $\lim_{t\to\infty} e^{-tP}$ is a projection of $L^2_0(E)$ onto Ker P. It follows that $\operatorname{tr} \lim_{t\to\infty} e^{-tP}$ is the dimension of the eigenspace of P corresponding to the zero-eigenvalue, which is the dimension of Ker P. We thus have

$$\begin{aligned}
\text{Ind } P &= \dim \text{Ker } P - \dim \text{Ker } P^* \\
&= \dim \text{Ker } P^*P - \dim \text{Ker } PP^* \\
&= \operatorname{tr}\Big(L^2_0(E) \xrightarrow{\text{proj}} \text{Ker } P^*P\Big) - \operatorname{tr}\Big(L^2_0(F) \xrightarrow{\text{proj}} \text{Ker } PP^*\Big) \\
&= \operatorname{tr}\Big(\lim_{t\to\infty} e^{-tP^*P}\Big) - \operatorname{tr}\Big(\lim_{t\to\infty} e^{-tPP^*}\Big) \\
&= \lim_{t\to\infty} \Big(\sum_{k=1}^{\infty} e^{-t\lambda_k^{P^*P}} - \sum_{l=1}^{\infty} e^{-t\lambda_l^{PP^*}}\Big) \\
&\overset{(\dagger)}{=} \dim E_0 - \dim F_0
\end{aligned}$$

where E_0 and F_0 are eigenspaces of P^*P and PP^*, respectively, corresponding to the zero-eigenvalue. From this, we see that we can calculate the index of P by letting $t \to \infty$. We further claim that, even if eigenspaces E_0 and F_0 are not necessarily related, operators P^*P and PP^* should have exactly the same positive eigenvalues with the same multiplicities(hence their eigenspaces are isomorphic for each positive eigenvalue). This shows that the cancellation in (†) above happens for any $t > 0$ without limit. We can prove this claim as follows. Let $E_\lambda := \{u \in \Gamma(E) : P^*Pu = \lambda u\}$ and $F_\lambda := \{v \in \Gamma(F) : PP^*v = \lambda v\}$, where $\lambda \neq 0$. From

$$\begin{aligned}
\lambda Pu &= P(P^*Pu) = (PP^*)Pu \\
\lambda P^*v &= P^*(PP^*v) = (P^*P)P^*v,
\end{aligned}$$

if λ is an eigenvalue of P^*P, then it is an eigenvalue of PP^*(and vice versa). Furthermore, P restricted to E_λ is an isomorphism onto F_λ with its inverse $(1/\lambda)P^*$.

By virtue of the above argument, we have

$$\begin{aligned}
\mathrm{tr}e^{-tP^*P} - \mathrm{tr}e^{-tPP^*} &= \dim E_0 + \sum_{\lambda_k \neq 0}^{\infty} e^{-t\lambda_k^{P^*P}} - \Big(\dim F_0 + \sum_{\lambda_l \neq 0}^{\infty} e^{-t\lambda_l^{PP^*}}\Big) \\
&= \dim E_0 - \dim F_0 \\
&= \mathrm{Ind}\ P \qquad \text{for all } t > 0.
\end{aligned}$$

Since we can think of the heat kernel $K_t(x, y)$ as an $\infty \times \infty$-matrix and the integration

$$\int_X K_t(x, y)u(y)dy$$

as a multiplication of a matrix to a column vector, it is expectable that

$$\mathrm{tr}e^{-tP^*P} = \int_X \mathrm{tr}_x K_t(x, x)dx := \int_X \mathrm{ev}\big(K_t(x, x)\big)dx \tag{3}$$

where $\mathrm{ev}(u_k(x) \otimes u_k^*(x)) = |u_k(x)|^2$ so that $\int_X |u_k(x)|^2 dx = \|u_k\|_0^2 = 1$.

When $\mathrm{deg}P = 1$, it turns out that, as $t \to 0$, the heat kernel for P^*P has an asymptotic expansion

$$\mathrm{tr}_x K_t(x, x) \sim \sum_{k=0}^{\infty} \rho_k(x) t^{\frac{k-n}{2}} \tag{4}$$

where $\rho_k(x)$ are densities on X which are locally and explicitly computable in terms of the geometry of X and P. Since our observation showed in the above that Ind P is independent of t, it is the coefficient $\rho_k(x)$ that determines Ind P. In this context, proving the local Atiyah-Singer index theorem means a careful writing of this term $\rho_k(x)$ for the operators P^*P and PP^*, and as a consequence one obtains

$$\mathrm{Ind}\ P = \int_X \mathrm{tr}_x\big(K_t^{P^*P}(x, x) - K_t^{PP^*}(x, x)\big)dx.$$

The integrand is appearing as characteristic classes. For example, the signature operator yields the Hirzebruch L-polynomials, and the Dirac operator on spinors yields $\widehat{A}$-genus as its integrand.

Remark. See Lawson and Michelsohn [4] p.277 for further remarks about the heat equation proof. They mention Atiyah, Bott and Patodi [1] and Gilkey [3] as sources one can find the calculation for the local expansion of the heat kernel. See also Bismut [2] for history and overview of the index theory.

References

[1] M. Atiyah, R. Bott and V. Patodi, *On the heat equation and the index theorem*, Invent. Math., 19:279–330, 1973. See also the Errata, same journal 28:277–280, 1975

[2] J.-M. Bismut, *Index Theorem and the Heat Equation*, Proceedings of the International Congress of Mathematicians, 491–504, Berkeley CA, 1986

[3] P. Gilkey, *Invariance Theory, the Heat Equation and the Atiyah-Singer Index Theorem*, Mathematical Lecture Series 4, Publish or Perish Press, Berkeley CA, 1984

[4] B. Lawson and M.-L. Michelsohn, *Spin Geometry*, Princeton University Press, Princeton NJ, 1989.

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