Fall 2019. Midterm exam Solutions

#1. (1) False ($\because$ $\mathbf{x}^+, \mathbf{x}^-$ are not coordinate patches)

(2) False ($\because$ I^2 is not a simple surface. See those points in $\partial I^2 = \{(x,y,0) : x \in [0,1], y \in \{0,1\}$ or $x \in \{0,1\}; y \in [0,1]\}$

(3) True.

(4) False. ($\because$ First and Second fundamental coefficients are not independent of coordinate change transformation. There are tons of examples)

(5) True. ($\because$ If $\mathbf{x}(u,v) = v\gamma(u) + \alpha(u)$, $\mathbf{x}_{vv} = 0$ and hence $K \le 0$ but K does not have to be identically zero. See for example hyperboloid with one-sheet.

#2. (1) $\mathbf{x}_u = (\frac{1}{2}, \frac{1}{2}, v)$
$\mathbf{x}_v = (\frac{1}{2}, -\frac{1}{2}, u)$

$\mathbf{x}_u \times \mathbf{x}_v = (\frac{1}{2}(u+v), \frac{1}{2}(v-u), -\frac{1}{2})$
$\neq 0$ because its z-coordinate is a constant.

(2) $\vec{n}(\mathbf{x}(0,0)) = \frac{\mathbf{x}_u \times \mathbf{x}_v}{\|\mathbf{x}_u \times \mathbf{x}_v\|}(\mathbf{x}(0,0)) = (0,0,-1)$

$\mathbf{x}_{uu} = (0,0,0)$ $\quad L = 0$ $\quad E(\mathbf{x}(0,0)) = \frac{1}{2}$
$\mathbf{x}_{uv} = (0,0,1)$ $\quad M = -1$ $\quad F(\mathbf{x}(0,0)) = 0$
$\mathbf{x}_{vv} = (0,0,0)$ $\quad N = 0$ $\quad G(\mathbf{x}(0,0)) = \frac{1}{2}$.

$\mathrm{I}(d\mathbf{x}) = \frac{1}{2}du^2 + \frac{1}{2}dv^2$
$\mathrm{II}(d\mathbf{x}) = -2\,du\,dv$.

(3) From $LN - M^2(\mathbf{x}(0,0)) = 0 \cdot 0 - (-1)^2 = -1 < 0$. It is a hyperbolic point

#3. Suppose $\frac{\partial g}{\partial z}(p) \neq 0$. By the Implicit function theorem, there is an open neighborhood U of $(x_0, y_0) \in \mathbb{R}^2$ and $f: U \to \mathbb{R}$ such that $g(x, y, f(x,y)) = c$ for all $(x,y) \in U$.

Accordingly, points of the form $(x, y, f(x,y))$ $(x,y) \in U$ fill a neighborhood of p in M, which constitutes a Monge patch

$$\mathbf{x} : U \to M$$
$$(x,y) \mapsto \mathbf{x}(x,y) = (x, y, f(x,y)).$$

Since the choice of p was arbitrary, we are done. □

#4. $\mathbf{x}_u = (1, 1, v)$, $\mathbf{x}_u(\mathbf{x}(1,-1)) = (1, 1, -1)$ $\mathbf{x}_u \times \mathbf{x}_v(\mathbf{x}(1,-1)) = (0, -2, -2)$

$\mathbf{x}_v = (1, -1, u)$, $\mathbf{x}_v(\mathbf{x}(1,-1)) = (1, -1, 1)$

$\vec{n}(\mathbf{x}(1,-1)) = \frac{1}{\sqrt{2}}(0, -1, -1)$

Tangent plane $y + z = d$. From $\mathbf{x}(1,-1) = (0, 2, -1)$, $y + z = 1$

Hence $\boxed{y + z = 1}$

Normal line : $\mathbf{x}(1,-1) + t\,\vec{n}_{\mathbf{x}(1,-1)} = (0, 2, -1) + t \cdot \frac{1}{\sqrt{2}}(0, -1, -1)$

$$= \underline{\left(0, 2 - \frac{t}{\sqrt{2}}, -1 - \frac{t}{\sqrt{2}}\right)}.$$

#5. $\mathbf{x}_\theta = (-f\sin\theta, f\cos\theta, 0)$ $\quad \mathbf{x}_\theta \times \mathbf{x}_t = (f\cos\theta, f\sin\theta, -ff')$

$\mathbf{x}_t = (f'\cos\theta, f'\sin\theta, 1)$ $\quad \|\mathbf{x}_\theta \times \mathbf{x}_t\| = (f^2 + f^2 f'^2)^{1/2}$

$\mathbf{x}_{\theta\theta} = (-f\cos\theta, -f\sin\theta, 0)$ $\quad \vec{n} = \dfrac{\mathbf{x}_\theta \times \mathbf{x}_t}{\|\mathbf{x}_\theta \times \mathbf{x}_t\|} = \dfrac{1}{\sqrt{1+f'^2}}(\cos\theta, \sin\theta, -f$

$\mathbf{x}_{\theta t} = (-f'\sin\theta, f'\cos\theta, 0)$

$\mathbf{x}_{tt} = (f''\cos\theta, f''\sin\theta, 0)$

$E = f^2$ $\quad L = \dfrac{-f}{\sqrt{1+f'^2}}$

$F = 0$ $\quad M = 0$ $\quad \therefore k = \dfrac{LN - M^2}{EG - F^2} = \dfrac{-ff''}{\sqrt{1+f'^2}}$

$G = f'^2 + 1$ $\quad N = \dfrac{f''}{\sqrt{1+f'^2}}$

Now $k = 0$ if and only if $f'' = 0$ $\quad (\because f > 0, \forall t)$

which is possible when $f(t) = a$ (constant)

or $f(t) = at + b$

#6. $\mathbf{x}_u = (1, 0, 8u)$ $\quad \mathbf{x}_u \times \mathbf{x}_v = (-8u, -2v, 1)$

$\mathbf{x}_v = (0, 1, 2v)$ $\quad \|\mathbf{x}_u \times \mathbf{x}_v\| = (64u^2 + 4v^2 + 1)^{1/2}$

$\vec{n}_{\mathbf{x}(0,0)} = (0, 0, 1)$ $\quad$ At $\mathbf{x}(0,0)$,

$E = 1 \quad F = 0, \quad G = 1$

$L = 8, \quad M = 0, \quad N = 2$

$\mathbf{x}_{uu} = (0,0,8)$

$\mathbf{x}_{uv} = (0,0,0)$

$\mathbf{x}_{vv} = (0,0,2)$

Because $F = M = 0$, $\mathbf{x}_u, \mathbf{x}_v$ are principal directions.

Principal Curvatures corresponding to $\mathbf{x}_u$- and $\mathbf{x}_v$- directions are 8 and 2, respectively.

#7. (1) $x_u = (\cos v, \sin v, 0)$ $\quad x_u(p) = (1, 0, 0)$

$x_v = (-u\sin v, u\cos v, 1)$ $\quad x_v(p) = (0, 1, 1)$

$x_u \times x_v(p) = (0, -1, 1)$

$E(p) = 1, \quad F(p) = 0, \quad G(p) = 2$ $\quad \|x_u \times x_v(p)\| = \sqrt{2}$

$x_{uu} = (0, 0, 0)$ $\quad x_{uu}(p) = (0, 0, 0)$

$x_{uv} = (-\sin v, \cos v, 0)$ $\quad x_{uv}(p) = (0, 1, 0)$

$x_{vv} = (-u\cos v, -u\sin v, 0)$ $\quad x_{vv}(p) = (-1, 0, 0)$

$L(p) = 0 \quad M(p) = -\frac{1}{\sqrt{2}} \quad N(p) = 0$

Hence from $K = \frac{LN - M^2}{EG - F^2} = \frac{-\frac{1}{2}}{2} = \underline{\underline{-\frac{1}{4}}}$

$H = \frac{EN + LG - 2FM}{2(EG - F^2)} = \underline{\underline{0}}$.

(2) From $k^2 - 2Hk + K = 0$

$\Leftrightarrow k^2 - \frac{1}{4} = 0$

$\Leftrightarrow \underline{k = \frac{1}{2}, -\frac{1}{2}}$.

(3) Let $S = \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} \begin{pmatrix} L & M \\ M & N \end{pmatrix}$. At p, $S(p) = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}^{-1} \begin{pmatrix} 0 & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & 0 \end{pmatrix}$

$= \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 0 & -1/\sqrt{2} \\ -1/\sqrt{2} & 0 \end{pmatrix}$

$= \begin{pmatrix} 0 & -1/\sqrt{2} \\ -1/2\sqrt{2} & 0 \end{pmatrix}$.

$\begin{pmatrix} 0 & -1/\sqrt{2} \\ -1/2\sqrt{2} & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \Leftrightarrow \begin{aligned} -\tfrac{1}{\sqrt{2}} v_2 &= \tfrac{1}{2} v_1 \\ -\tfrac{1}{2\sqrt{2}} v_1 &= \tfrac{1}{2} v_2 \end{aligned} \Leftrightarrow v_1 : v_2 = 1 : -1/\sqrt{2}$

$\begin{pmatrix} 0 & -1/\sqrt{2} \\ -1/2\sqrt{2} & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = -\frac{1}{2} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \Leftrightarrow \begin{aligned} -\tfrac{1}{\sqrt{2}} v_2 &= -\tfrac{1}{2} v_1 \\ -\tfrac{1}{2\sqrt{2}} v_1 &= -\tfrac{1}{2} v_2 \end{aligned} \Leftrightarrow v_1 : v_2 = 1 : \frac{1}{\sqrt{2}}$

Hence principal directions corresponding to principal curvatures $+\frac{1}{2}$, $-\frac{1}{2}$ are $1 : -\frac{1}{\sqrt{2}}$ and $1 : \frac{1}{\sqrt{2}}$, respectively.

(4) Let $\vec{e}_1 = X_u - \frac{1}{\sqrt{2}} X_v, \quad = (1,0,0) - \frac{1}{\sqrt{2}}(0,1,1) = (1, -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}$

$\vec{e}_2 = X_u + \frac{1}{\sqrt{2}} X_v$

$\|\vec{e}_1\| = \sqrt{2}$ Let $\angle \vec{e}_1, \vec{w} = \theta$.

$$\cos\theta = \frac{\vec{e}_1}{\|\vec{e}_1\|} \cdot \vec{w} = \left(\frac{1}{\sqrt{2}}, -\frac{1}{2}, -\frac{1}{2}\right) \cdot \frac{1}{\sqrt{3}}(1,1,1)$$

$$= \frac{1}{\sqrt{6}} - \frac{1}{\sqrt{3}} = \frac{\sqrt{6} - 2\sqrt{3}}{6}.$$

$$\cos^2\theta = \frac{6 + 12 - 4\sqrt{18}}{36} = \frac{18 - 12\sqrt{2}}{36} = \frac{1}{2} - \frac{\sqrt{2}}{3}$$

$$\sin^2\theta = \frac{1}{2} + \frac{\sqrt{2}}{3}$$

$$k_n(\vec{w}) = \frac{1}{2} \cdot \left(\frac{1}{2} - \frac{\sqrt{2}}{3}\right) - \frac{1}{2}\left(\frac{1}{2} + \frac{\sqrt{2}}{3}\right) = -\frac{\sqrt{2}}{3}.$$

#8. $\mathbf{x}_u = (\cos v, \sin v, -\frac{1}{u^2})$ $\quad$ $\mathbf{x}_u(p) = (1, 0, -1)$

$\mathbf{x}_v = (-u\sin v, u\cos v, 0)$ $\quad$ $\mathbf{x}_v(p) = (0, 1, 0)$

$\mathbf{x}_u \times \mathbf{x}_v(p) = (1, 0, 1)$

$\mathbf{x}(u,v) = (1,0,1)$ when $u=1, v=0$

$= \mathbf{x}(1,0).$

$$\vec{n}(p) = \frac{\mathbf{x}_u \times \mathbf{x}_v(p)}{\|\mathbf{x}_u \times \mathbf{x}_v(p)\|} = \frac{1}{\sqrt{2}}(1, 0, 1)$$

$\mathbf{x}_{uu} = (0, 0, \frac{2}{u^3})$ $\quad$ $\mathbf{x}_{uu}(p) = (0, 0, 2)$

$\mathbf{x}_{uv} = (-\sin v, \cos v, 0)$ $\quad$ $\mathbf{x}_{uv}(p) = (0, 1, 0)$

$\mathbf{x}_{vv} = (-u\cos v, -u\sin v, 0)$ $\quad$ $\mathbf{x}_{vv}(p) = (-1, 0, 0)$

At p,

$E = 2$ $\quad$ $L = \sqrt{2}$

$F = 0$ $\quad$ $M = 0$

$G = 1$ $\quad$ $N = -1/\sqrt{2}$

principal directions are $du:dv = 1:0$ with principal curvature $k_1 = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$

and $du:dv = 0:1$ with $k_2 = \frac{-1/\sqrt{2}}{1} = -\frac{1}{\sqrt{2}}$

Since $\vec{\omega}$ is in the direction of $\mathbf{x}_u + \mathbf{x}_v$

i.e. $du:dv = 1:1$,

$$k_n(\vec{\omega}) = \frac{II_p(d\mathbf{x})}{I_p(d\mathbf{x})} = \frac{L\,du^2 + 2M\,du\,dv + N\,dv^2}{E\,du^2 + 2F\,du\,dv + G\,dv^2} = \frac{\sqrt{2} - \frac{1}{\sqrt{2}}}{2+1}$$

$$= \frac{2-1}{3\sqrt{2}}$$

$$= \frac{\sqrt{2}}{6}$$