# Contents

# 1 计算几何

## 1.1 二维计算几何基本操作

```
const double PI = 3.141592653589793238462643383279502288;
double arcSin(const double &a) {
  return (a <= -1.0) ? (-PI / 2) : ((a >= 1.0) ? (PI / 2) : (asin(a))); }
double arcCos(const double &a) {
  return (a <= -1.0) ? (PI) : ((a >= 1.0) ? (0) : (acos(a))); }
struct point { double x, y; // something omitted
  point rot(const double &a) const { // counter-clockwise
    return point(x * cos(a) - y * sin(a), x * sin(a) + y * cos(a)); }
  point rot90()  const { return point(-y,  x); } // counter-clockwise
  point project(const point &p1, const point &p2) const {
    const point &q = *this; return p1 + (p2 - p1) * (dot(p2 - p1, q - p1) / (p2 - p1).norm()); }
  bool onSeg(const point &a, const point &b) const { // a, b inclusive
    const point &c = *this; return sign(dot(a - c, b - c)) <= 0 && sign(det(b - a, c - a)) == 0; }
  double distLP(const point &p1, const point &p2) const { // dist from *this to line p1->p2
    const point &q = *this; return fabs(det(p2 - p1, q - p1)) / (p2 - p1).len(); }
  double distSP(const point &p1, const point &p2) const { // dist from *this to segment [p1, p2]
    const point &q = *this;
    if (dot(p2 - p1, q - p1) < EPS) return (q - p1).len();
    if (dot(p1 - p2, q - p2) < EPS) return (q - p2).len();
    return distLP(p1, p2);
  }
  bool inAngle(const point &p1, const point &p2) const { // det(p1, p2) ≥ 0
    const point &q = *this; return det(p1, q) > -EPS && det(p2, q) < EPS;
  }
};
bool lineIntersect(const point &a, const point &b, const point &c, const point &d, point &e) {
  double s1 = det(c - a, d - a), s2 = det(d - b, c - b);
  if (!sign(s1 + s2)) return false; e = (b - a) * (s1 / (s1 + s2)) + a; return true;
}
int segIntersectCheck(const point &a, const point &b, const point &c, const point &d, point &o) {
  static double s1, s2, s3, s4;
  static int iCnt;
  int d1 = sign(s1 = det(b - a, c - a)), d2 = sign(s2 = det(b - a, d - a));
  int d3 = sign(s3 = det(d - c, a - c)), d4 = sign(s4 = det(d - c, b - c));
  if ((d1 ^ d2) == -2 && (d3 ^ d4) == -2) {
    o = (c * s2 - d * s1) / (s2 - s1); return true;
  } iCnt = 0;
  if (d1 == 0 && c.onSeg(a, b)) o = c, ++iCnt;
  if (d2 == 0 && d.onSeg(a, b)) o = d, ++iCnt;
  if (d3 == 0 && a.onSeg(c, d)) o = a, ++iCnt;
  if (d4 == 0 && b.onSeg(c, d)) o = b, ++iCnt;
  return iCnt ? 2 : 0; // 不相交返回 0, 严格相交返回 1, 非严格相交返回 2
}
struct circle {
  point o; double r, rSquare;
  bool inside(const point &a) { return (a - o).len() < r + EPS; } // 非严格
  bool contain(const circle &b) const { return sign(b.r + (o - b.o).len() - r) <= 0; } // 非严格
  bool disjunct(const circle &b) const { return sign(b.r + r - (o - b.o).len()) <= 0; } // 非严格
  int isCL(const point &p1, const point &p2, point &a, point &b) const {
    double x = dot(p1 - o, p2 - p1), y = (p2 - p1).norm();
    double d = x * x - y * ((p1 - o).norm() - rSquare);
    if (d < -EPS) return 0; if (d < 0) d = 0;
    point q1 = p1 - (p2 - p1) * (x / y);
    point q2 = (p2 - p1) * (sqrt(d) / y);
    a = q1 - q2; b = q1 + q2; return q2.len() < EPS ? 1 : 2;
  }
  int tanCP(const point &p, point &a, point &b) const { // 返回切点, 注意可能与 p 重合
    double x = (p - o).norm(), d = x - rSquare;
    if (d < -EPS) return 0; if (d < 0) d = 0;
    point q1 = (p - o) * (rSquare / x), q2 = ((p - o) * (-r * sqrt(d) / x)).rot90();
    a = o + (q1 - q2); b = o + (q1 + q2); return q2.len() < EPS ? 1 : 2;
  }
};
bool checkCrossCS(const circle &cir, const point &p1, const point &p2) { // 非严格
  const point &c = cir.o; const double &r = cir.r;
  return c.distSP(p1, p2) < r + EPS && (r < (c - p1).len() + EPS || r < (c - p2).len() + EPS);
}
bool checkCrossCC(const circle &cir1, const circle &cir2) { // 非严格
  const double &r1 = cir1.r, &r2 = cir2.r, d = (cir1.o - cir2.o).len();
  return d < r1 + r2 + EPS && fabs(r1 - r2) < d + EPS;
}
int isCC(const circle &cir1, const circle &cir2, point &a, point &b) {
  const point &c1 = cir1.o, &c2 = cir2.o;
  double x = (c1 - c2).norm(), y = ((cir1.rSquare - cir2.rSquare) / x + 1) / 2;
  double d = cir1.rSquare / x - y * y;
  if (d < -EPS) return 0; if (d < 0) d = 0;
  point q1 = c1 + (c2 - c1) * y, q2 = ((c2 - c1) * sqrt(d)).rot90();
  a = q1 - q2; b = q1 + q2; return q2.len() < EPS ? 1 : 2;
}
vector<pair<point, point> > tanCC(const circle &cir1, const circle &cir2) {
```

```
// 注意: 如果只有三条切线, 即 s1 = 1, s2 = 1, 返回的切线可能重复, 切点没有问题
  vector<pair<point, point> > list;
  if (cir1.contain(cir2) || cir2.contain(cir1)) return list;
  const point &c1 = cir1.o, &c2 = cir2.o;
  double r1 = cir1.r, r2 = cir2.r; point p, a1, b1, a2, b2; int s1, s2;
  if (sign(r1 - r2) == 0) {
    p = c2 - c1; p = (p * (r1 / p.len())).rot90();
    list.push_back(make_pair(c1 + p, c2 + p)); list.push_back(make_pair(c1 - p, c2 - p));
  } else {
    p = (c2 * r1 - c1 * r2) / (r1 - r2);
    s1 = cir1.tanCP(p, a1, b1); s2 = cir2.tanCP(p, a2, b2);
    if (s1 >= 1 && s2 >= 1)
      list.push_back(make_pair(a1, a2)), list.push_back(make_pair(b1, b2));
  } p = (c1 * r2 + c2 * r1) / (r1 + r2);
  s1 = cir1.tanCP(p, a1, b1); s2 = cir2.tanCP(p, a2, b2);
  if (s1 >= 1 && s2 >= 1)
    list.push_back(make_pair(a1, a2)), list.push_back(make_pair(b1, b2));
  return list;
}
bool distConvexPIn(const point &p1, const point &p2, const point &p3, const point &p4, const point &q) {
  point o12 = (p1 - p2).rot90(), o23 = (p2 - p3).rot90(), o34 = (p3 - p4).rot90();
  return (q - p2).inAngle(o12, o23) || (q - p3).inAngle(o23, o34)
    || ((q - p2).inAngle(o23, p3 - p2) && (q - p3).inAngle(p2 - p3, o23));
}
double distConvexP(int n, point ps[], const point &q) { // 外部点到多边形的距离
  int left = 0, right = n; while (right - left > 1) { int mid = (left + right) / 2;
    if (distConvexPIn(ps[(left + n - 1) % n], ps[left], ps[mid], ps[(mid + 1) % n], q))
      right = mid; else left = mid;
  } return q.distSP(ps[left], ps[right % n]);
}
double areaCT(const circle &cir, point pa, point pb) {
  pa = pa - cir.o; pb = pb - cir.o; double R = cir.r;
  if (pa.len() < pb.len()) swap(pa, pb); if (pb.len() < EPS) return 0;
  point pc = pb - pa; double a = pa.len(), b = pb.len(), c = pc.len(), S, h, theta;
  double cosB = dot(pb, pc) / b / c, B = acos(cosB);
  double cosC = dot(pa, pb) / a / b, C = acos(cosC);
  if (b > R) {
    S = C * 0.5 * R * R; h = b * a * sin(C) / c;
    if (h < R && B < PI * 0.5) S -= acos(h / R) * R * R - h * sqrt(R * R - h * h);
  } else if (a > R) {
    theta = PI - B - asin(sin(B) / R * b);
    S = 0.5 * b * R * sin(theta) + (C - theta) * 0.5 * R * R;
  } else S = 0.5 * sin(C) * b * a;
  return S;
}
circle minCircle(const point &a, const point &b) {
  return circle((a + b) * 0.5, (b - a).len() * 0.5);
}
circle minCircle(const point &a, const point &b, const point &c) { // 钝角三角形没有被考虑
  double a2( (b - c).norm() ), b2( (a - c).norm() ), c2( (a - b).norm() );
  if (b2 + c2 <= a2 + EPS) return minCircle(b, c);
  if (a2 + c2 <= b2 + EPS) return minCircle(a, c);
  if (a2 + b2 <= c2 + EPS) return minCircle(a, b);
  double A = 2.0 * (a.x - b.x), B = 2.0 * (a.y - b.y);
  double D = 2.0 * (a.x - c.x), E = 2.0 * (a.y - c.y);
  double C = a.norm() - b.norm(), F = a.norm() - c.norm();
  point p((C * E - B * F) / (A * E - B * D), (A * F - C * D) / (A * E - B * D));
  return circle(p, (p - a).len());
}
circle minCircle(point P[], int N) { // 1-based
  if (N == 1) return circle(P[1], 0.0);
  random_shuffle(P + 1, P + N + 1); circle O = minCircle(P[1], P[2]);
  Rep(i, 1, N) if (!O.inside(P[i])) { O = minCircle(P[1], P[i]);
    Foru(j, 1, i) if (!O.inside(P[j])) { O = minCircle(P[i], P[j]);
      Foru(k, 1, j) if (!O.inside(P[k])) O = minCircle(P[i], P[j], P[k]); }
  } return O;
}
```

## 1.2 圆的面积模板

```
struct Event { point p; double alpha; int add; // 构造函数省略
  bool operator < (const Event &other) const { return alpha < other.alpha; } };
void circleKCover(circle *c, int N, double *area) { // area[k] : 至少被覆盖 k 次
  static bool overlap[MAXN][MAXN], g[MAXN][MAXN];
  Rep(i, 0, N + 1) area[i] = 0.0; Rep(i, 1, N) Rep(j, 1, N) overlap[i][j] = c[i].contain(c[j]);
  Rep(i, 1, N) Rep(j, 1, N) g[i][j] = !(overlap[i][j] || overlap[j][i] || c[i].disjunct(c[j]));
  Rep(i, 1, N) { static Event events[MAXN * 2 + 1]; int totE = 0, cnt = 1;
    Rep(j, 1, N) if (j != i && overlap[j][i]) ++cnt;
    Rep(j, 1, N) if (j != i && g[i][j]) {
      circle &a = c[i], &b = c[j]; double l = (a.o - b.o).len();
      double s = ((a.r - b.r) * (a.r + b.r) / l + l) * 0.5;
      double t = sqrt(-(l - sqr(a.r - b.r)) * (l - sqr(a.r + b.r)) / (l * l * 4.0));
```

```
13          point dir = b.o - a.o, nDir = point(-dir.y, dir.x);
14          point aa = a.o + dir * s + nDir * t;
15          point bb = a.o + dir * s - nDir * t;
16          double A = atan2(aa.y - a.o.y, aa.x - a.o.x);
17          double B = atan2(bb.y - a.o.y, bb.x - a.o.x);
18          events[totE++] = Event(bb, B, 1); events[totE++] = Event(aa, A, -1); if (B > A) ++cnt;
19        } if (totE == 0) { area[cnt] += PI * c[i].rSquare; continue; }
20        sort(events, events + totE); events[totE] = events[0];
21        Foru(j, 0, totE) {
22          cnt += events[j].add; area[cnt] += 0.5 * det(events[j].p, events[j + 1].p);
23          double theta = events[j + 1].alpha - events[j].alpha; if (theta < 0) theta += 2.0 * PI;
24          area[cnt] += 0.5 * c[i].rSquare * (theta - sin(theta));
25 }}}
```

## 1.3   多边形相关

```
1   struct Polygon { // stored in [0, n)
2     int n; point ps[MAXN];
3     Polygon cut(const point &a, const point &b) {
4       static Polygon res; static point o; res.n = 0;
5       for (int i = 0; i < n; ++i) {
6         int s1 = sign(det(ps[i] - a, b - a));
7         int s2 = sign(det(ps[(i + 1) % n] - a, b - a));
8         if (s1 <= 0) res.ps[res.n++] = ps[i];
9         if (s1 * s2 < 0) {
10          lineIntersect(a, b, ps[i], ps[(i + 1) % n], o);
11          res.ps[res.n++] = o;
12        }
13      } return res;
14    }
15    bool contain(const point &p) const { // 1 if on border or inner, 0 if outter
16      static point A, B; int res = 0;
17      for (int i = 0; i < n; ++i) {
18        A = ps[i]; B = ps[(i + 1) % n];
19        if (p.onSeg(A, B)) return 1;
20        if (sign(A.y - B.y) <= 0) swap(A, B);
21        if (sign(p.y - A.y) > 0) continue;
22        if (sign(p.y - B.y) <= 0) continue;
23        res += (int)(sign(det(B - p, A - p)) > 0);
24      } return res & 1;
25    }
26    #define qs(x) (ps[x] - ps[0])
27    bool convexContain(point p) const { // counter-clockwise
28      point q = qs(n - 1); p = p - ps[0];
29      if (!p.inAngle(qs(1), q)) return false;
30      int L = 0, R = n - 1;
31      while (L + 1 < R) { int M((L + R) >> 1);
32        if (p.inAngle(qs(M), q)) L = M; else R = M;
33      } if (L == 0) return false; point l(qs(L)), r(qs(R));
34      return sign( fabs(det(l, p)) + fabs(det(p, r)) + fabs(det(r - l, p - l)) - det(l, r) ) == 0;
35    }
36    #undef qs
37    double isPLAtan2(const point &a, const point &b) {
38      double k = (b - a).alpha(); if (k < 0) k += 2 * PI;
39      return k;
40    }
41    point isPL_Get(const point &a, const point &b, const point &s1, const point &s2) {
42      double k1 = det(b - a, s1 - a), k2 = det(b - a, s2 - a);
43      if (sign(k1) == 0) return s1;
44      if (sign(k2) == 0) return s2;
45      return (s1 * k2 - s2 * k1) / (k2 - k1);
46    }
47    int isPL_Dic(const point &a, const point &b, int l, int r) {
48      int s = (det(b - a, ps[l] - a) < 0) ? -1 : 1;
49      while (l <= r) {
50        int mid = (l + r) / 2;
51        if (det(b - a, ps[mid] - a) * s <= 0) r = mid - 1;
52        else l = mid + 1;
53      }
54      return r + 1;
55    }
56    int isPL_Find(double k, double w[]) {
57      if (k <= w[0] || k > w[n - 1]) return 0;
58      int l = 0, r = n - 1, mid;
59      while (l <= r) {
60        mid = (l + r) / 2;
61        if (w[mid] >= k) r = mid - 1;
62        else l = mid + 1;
63      } return r + 1;
64    }
65    bool isPL(const point &a, const point &b, point &cp1, point &cp2) {  // O(logN)
66      static double w[MAXN * 2]; // pay attention to the array size
```

```
67      for (int i = 0; i <= n; ++i) ps[i + n] = ps[i];
68      for (int i = 0; i < n; ++i) w[i + n] = isPLAtan2(ps[i], ps[i + 1]);
69      int i = isPL_Find(isPLAtan2(a, b), w);
70      int j = isPL_Find(isPLAtan2(b, a), w);
71      double k1 = det(b - a, ps[i] - a), k2 = det(b - a, ps[j] - a);
72      if (sign(k1) * sign(k2) > 0) return false; // no intersection
73      if (sign(k1) == 0 || sign(k2) == 0) { // intersect with a point or a line in the convex
74        if (sign(k1) == 0) {
75          if (sign(det(b - a, ps[i + 1] - a)) == 0) cp1 = ps[i], cp2 = ps[i + 1];
76          else cp1 = cp2 = ps[i];
77          return true;
78        }
79        if (sign(k2) == 0) {
80          if (sign(det(b - a, ps[j + 1] - a)) == 0) cp1 = ps[j], cp2 = ps[j + 1];
81          else cp1 = cp2 = ps[j];
82        }
83        return true;
84      }
85      if (i > j) swap(i, j);
86      int x = isPL_Dic(a, b, i, j), y = isPL_Dic(a, b, j, i + n);
87      cp1 = isPL_Get(a, b, ps[x - 1], ps[x]);
88      cp2 = isPL_Get(a, b, ps[y - 1], ps[y]);
89      return true;
90    }
91    double getI(const point &O) const {
92      if (n <= 2) return 0;
93      point G(0.0, 0.0);
94      double S = 0.0, I = 0.0;
95      for (int i = 0; i < n; ++i) {
96        const point &x = ps[i], &y = ps[(i + 1) % n];
97        double d = det(x, y);
98        G = G + (x + y) * d / 3.0;
99        S += d;
100     } G = G / S;
101     for (int i = 0; i < n; ++i) {
102       point x = ps[i] - G, y = ps[(i + 1) % n] - G;
103       I += fabs(det(x, y)) * (x.norm() + dot(x, y) + y.norm());
104     }
105     return I = I / 12.0 + fabs(S * 0.5) * (O - G).norm();
106   }
107 };
```

## 1.4   直线与凸包求交点

```
1   int isPL(point a, point b, vector<point> &res) { // μ   ³   μ 猙
2     static double theta[MAXN];
3     for (int i = 0; i < n; ++i) theta[i] = (list[(i + 1) % n] - list[i]).atan2();
4     double delta = theta[0];
5     for (int i = 0; i < n; ++i) theta[i] = normalize(theta[i] - delta);
6     int x = lower_bound(theta, theta + n, normalize((b - a).atan2() - delta)) - theta;
7     int y = lower_bound(theta, theta + n, normalize((a - b).atan2() - delta)) - theta;
8     for (int k = 0; k <= 1; ++k, swap(a, b), swap(x, y)) {
9       if (y < x) y += n;
10      int l = x, r = y, m;
11      while (l + 1 < r) {
12        if (sign(det(b - a, list[(m = (l + r) / 2) % n] - a)) < 0) l = m;
13        else r = m;
14      }
15      l %= n, r %= n;
16      if (sign(det(b - a, list[r] - list[l])) == 0) {
17        if (sign(det(b - a, list[l] - a)) == 0)
18          return -1; //   (list[l], list[r])
19      {
20        else {
21          point p; lineIntersect(list[l], list[r], a, b, p);
22          if (p.onSeg(list[l], list[r]))
23            res.push_back(p);
24      }
25    }
26    return res.size();
27 }
```

## 1.5   半平面交

```
1   struct Border {
2     point p1, p2; double alpha;
3     Border() : p1(), p2(), alpha(0.0) {}
```

```
 4    Border(const point &a, const point &b): p1(a), p2(b), alpha( atan2(p2.y - p1.y, p2.x - p1.x) ) {}
 5    bool operator == (const Border &b) const { return sign(alpha - b.alpha) == 0; }
 6    bool operator < (const Border &b) const {
 7      int c = sign(alpha - b.alpha); if (c != 0) return c > 0;
 8      return sign(det(b.p2 - b.p1, p1 - b.p1)) >= 0;
 9    }
10  };
11  point isBorder(const Border &a, const Border &b) { // a and b should not be parallel
12    point is; lineIntersect(a.p1, a.p2, b.p1, b.p2, is); return is;
13  }
14  bool checkBorder(const Border &a, const Border &b, const Border &me) {
15    point is; lineIntersect(a.p1, a.p2, b.p1, b.p2, is);
16    return sign(det(me.p2 - me.p1, is - me.p1)) > 0;
17  }
18  double HPI(int N, Border border[]) {
19    static Border que[MAXN * 2 + 1]; static point ps[MAXN];
20    int head = 0, tail = 0, cnt = 0; // [head, tail)
21    sort(border, border + N); N = unique(border, border + N) - border;
22    for (int i = 0; i < N; ++i) {
23      Border &cur = border[i];
24      while (head + 1 < tail && !checkBorder(que[tail - 2], que[tail - 1], cur)) --tail;
25      while (head + 1 < tail && !checkBorder(que[head], que[head + 1], cur)) ++head;
26      que[tail++] = cur;
27    } while (head + 1 < tail && !checkBorder(que[tail - 2], que[tail - 1], que[head])) --tail;
28    while (head + 1 < tail && !checkBorder(que[head], que[head + 1], que[tail - 1])) ++head;
29    if (tail - head <= 2) return 0.0;
30    Foru(i, head, tail) ps[cnt++] = isBorder(que[i], que[(i + 1 == tail) ? (head) : (i + 1)]);
31    double area = 0; Foru(i, 0, cnt) area += det(ps[i], ps[(i + 1) % cnt]);
32    return fabs(area * 0.5); // or (-area * 0.5)
33  }
```

## 1.6  最大面积空凸包

```
 1  inline bool toUpRight(const point &a, const point &b) {
 2    int c = sign(b.y - a.y); if (c > 0) return true;
 3    return c == 0 && sign(b.x - a.x) > 0;
 4  }
 5  inline bool cmpByPolarAngle(const point &a, const point &b) { // counter-clockwise, shorter first if they
        share the same polar angle
 6    int c = sign(det(a, b)); if (c != 0) return c > 0;
 7    return sign(b.len() - a.len()) > 0;
 8  }
 9  double maxEmptyConvexHull(int N, point p[]) {
10    static double dp[MAXN][MAXN];
11    static point vec[MAXN];
12    static int seq[MAXN]; // empty triangles formed with (0,0), vec[o], vec[seq[i]]
13    double ans = 0.0;
14    Rep(o, 1, N) {
15      int totVec = 0;
16      Rep(i, 1, N) if (toUpRight(p[o], p[i])) vec[++totVec] = p[i] - p[o];
17      sort(vec + 1, vec + totVec + 1, cmpByPolarAngle);
18      Rep(i, 1, totVec) Rep(j, 1, totVec) dp[i][j] = 0.0;
19      Rep(k, 2, totVec) {
20        int i = k - 1;
21        while (i > 0 && sign( det(vec[k], vec[i]) ) == 0) --i;
22        int totSeq = 0;
23        for (int j; i > 0; i = j) {
24          seq[++totSeq] = i;
25          for (j = i - 1; j > 0 && sign(det(vec[i] - vec[k], vec[j] - vec[k])) > 0; --j);
26          double v = det(vec[i], vec[k]) * 0.5;
27          if (j > 0) v += dp[i][j];
28          dp[k][i] = v;
29          cMax(ans, v);
30        } for (int i = totSeq - 1; i >= 1; --i) cMax( dp[k][ seq[i] ], dp[k][seq[i + 1]] );
31      }
32    } return ans;
33  }
```

## 1.7  最近点对

```
 1  int N; point p[maxn];
 2  bool cmpByX(const point &a, const point &b) { return sign(a.x - b.x) < 0;  }
 3  bool cmpByY(const int &a, const int &b) { return p[a].y < p[b].y;  }
 4  double minimalDistance(point *c, int n, int *ys) {
 5    double ret = 1e+20;
 6    if (n < 20) {
 7      Foru(i, 0, n) Foru(j, i + 1, n) cMin(ret, (c[i] - c[j]).len() );
 8      sort(ys, ys + n, cmpByY); return ret;
```

```
 9    } static int mergeTo[maxn];
10    int mid = n / 2; double xmid = c[mid].x;
11    ret = min(minimalDistance(c, mid, ys), minimalDistance(c + mid, n - mid, ys + mid));
12    merge(ys, ys + mid, ys + mid, ys + n, mergeTo, cmpByY);
13    copy(mergeTo, mergeTo + n, ys);
14    Foru(i, 0, n) {
15      while (i < n && sign(fabs(p[ys[i]].x - xmid) - ret) > 0) ++i;
16      int cnt = 0;
17      Foru(j, i + 1, n)
18        if (sign(p[ys[j]].y - p[ys[i]].y - ret) > 0) break;
19        else if (sign(fabs(p[ys[j]].x - xmid) - ret) <= 0) {
20          ret = min(ret, (p[ys[i]] - p[ys[j]]).len());
21          if (++cnt >= 10) break;
22        }
23    } return ret;
24  }
25  double work() {
26    sort(p, p + n, cmpByX); Foru(i, 0, n) ys[i] = i; return minimalDistance(p, n, ys);
27  }
```

## 1.8  凸包与点集直径

```
 1  vector<point> convexHull(int n, point ps[]) { // counter-clockwise, strict
 2    static point qs[MAXN * 2];
 3    sort(ps, ps + n, cmpByXY);
 4    if (n <= 2) return vector(ps, ps + n);
 5    int k = 0;
 6    for (int i = 0; i < n; qs[k++] = ps[i++])
 7      while (k > 1 && det(qs[k - 1] - qs[k - 2], ps[i] - qs[k - 1]) < EPS) --k;
 8    for (int i = n - 2, t = k; i >= 0; qs[k++] = ps[i--])
 9      while (k > t && det(qs[k - 1] - qs[k - 2], ps[i] - qs[k - 1]) < EPS) --k;
10    return vector<point>(qs, qs + k);
11  }
12  double convexDiameter(int n, point ps[]) {
13    if (n < 2) return 0; if (n == 2) return (ps[1] - ps[0]).len();
14    double k, ans = 0;
15    for (int x = 0, y = 1, nx, ny; x < n; ++x) {
16      for(nx = (x == n - 1) ? (0) : (x + 1); ; y = ny) {
17        ny = (y == n - 1) ? (0) : (y + 1);
18        if ( sign(k = det(ps[nx] - ps[x], ps[ny] - ps[y])) <= 0) break;
19      } ans = max(ans, (ps[x] - ps[y]).len());
20      if (sign(k) == 0) ans = max(ans, (ps[x] - ps[ny]).len());
21    } return ans;
22  }
```

## 1.9  Farmland

```
 1  struct node { int begin[MAXN], *end; } a[MAXN]; // 按对 p[i] 的极角的 atan2 值排序
 2  bool check(int n, point p[], int b1, int b2, bool vis[MAXN][MAXN]) {
 3    static pii l[MAXN * 2 + 1]; static bool used[MAXN];
 4    int tp(0), *k, p, p1, p2; double area(0.0);
 5    for (l[0] = pii(b1, b2); ; ) {
 6      vis[p1 = l[tp].first][p2 = l[tp].second] = true;
 7      area += det(p[p1], p[p2]);
 8      for (k = a[p2].begin; k != a[p2].end; ++k) if (*k == p1) break;
 9      k = (k == a[p2].begin) ? (a[p2].end - 1) : (k - 1);
10      if ((l[++tp] = pii(p2, *k)) == l[0]) break;
11    } if (sign(area) < 0 || tp < 3) return false;
12    Rep(i, 1, n) used[i] = false;
13    for (int i = 0; i < tp; ++i) if (used[p = l[i].first]) return false; else used[p] = true;
14    return true; // a face with tp vertices
15  }
16  int countFaces(int n, point p[]) {
17    static bool vis[MAXN][MAXN]; int ans = 0;
18    Rep(x, 1, n) Rep(y, 1, n) vis[x][y] = false;
19    Rep(x, 1, n) for (int *itr = a[x].begin; itr != a[x].end; ++itr) if (!vis[x][*itr])
20      if (check(n, p, x, *itr, vis)) ++ans;
21    return ans;
22  }
```

## 1.10  Voronoi 图

不能有重点, 点数应当不小于 2

```
1   #define Oi(e) ((e)->oi)
2   #define Dt(e) ((e)->dt)
3   #define On(e) ((e)->on)
4   #define Op(e) ((e)->op)
5   #define Dn(e) ((e)->dn)
6   #define Dp(e) ((e)->dp)
7   #define Other(e, p) ((e)->oi == p ? (e)->dt : (e)->oi)
8   #define Next(e, p) ((e)->oi == p ? (e)->on : (e)->dn)
9   #define Prev(e, p) ((e)->oi == p ? (e)->op : (e)->dp)
10  #define V(p1, p2, u, v) (u = p2->x - p1->x, v = p2->y - p1->y)
11  #define C2(u1, v1, u2, v2) (u1 * v2 - v1 * u2)
12  #define C3(p1, p2, p3) ((p2->x - p1->x) * (p3->y - p1->y) - (p2->y - p1->y) * (p3->x - p1->x))
13  #define Dot(u1, v1, u2, v2) (u1 * u2 + v1 * v2)
14  #define dis(a,b) (sqrt( (a->x - b->x) * (a->x - b->x) + (a->y - b->y) * (a->y - b->y) ))
15  const int maxn = 110024;
16  const int aix = 4;
17  const double eps = 1e-7;
18  int n, M, k;
19  struct gEdge {
20    int u, v; double w;
21    bool operator <(const gEdge &e1) const { return w < e1.w - eps; }
22  } E[aix * maxn], MST[maxn];
23  struct point {
24    double x, y; int index; edge *in;
25    bool operator <(const point &p1) const { return x < p1.x - eps || (abs(x - p1.x) <= eps && y < p1.y -
      eps); }
26  };
27  struct edge { point *oi, *dt; edge *on, *op, *dn, *dp; };
28
29  point p[maxn], *Q[maxn];
30  edge mem[aix * maxn], *elist[aix * maxn];
31  int nfree;
32  void Alloc_memory() { nfree = aix * n; edge *e = mem; for (int i = 0; i < nfree; i++) elist[i] = e++; }
33  void Splice(edge *a, edge *b, point *v) {
34    edge *next;
35    if (Oi(a) == v) next = On(a), On(a) = b; else next = Dn(a), Dn(a) = b;
36    if (Oi(next) == v) Op(next) = b; else Dp(next) = b;
37    if (Oi(b) == v) On(b) = next, Op(b) = a; else Dn(b) = next, Dp(b) = a;
38  }
39  edge *Make_edge(point *u, point *v) {
40    edge *e = elist[--nfree];
41    e->on = e->op = e->dn = e->dp = e; e->oi = u; e->dt = v;
42    if (!u->in) u->in = e;
43    if (!v->in) v->in = e;
44    return e;
45  }
46  edge *Join(edge *a, point *u, edge *b, point *v, int side) {
47    edge *e = Make_edge(u, v);
48    if (side == 1) {
49      if (Oi(a) == u) Splice(Op(a), e, u);
50      else Splice(Dp(a), e, u);
51      Splice(b, e, v);
52    } else {
53      Splice(a, e, u);
54      if (Oi(b) == v) Splice(Op(b), e, v);
55      else Splice(Dp(b), e, v);
56    } return e;
57  }
58  void Remove(edge *e) {
59    point *u = Oi(e), *v = Dt(e);
60    if (u->in == e) u->in = e->on;
61    if (v->in == e) v->in = e->dn;
62    if (Oi(e->on) == u) e->on->op = e->op; else e->on->dp = e->op;
63    if (Oi(e->op) == u) e->op->on = e->on; else e->op->dn = e->on;
64    if (Oi(e->dn) == v) e->dn->op = e->dp; else e->dn->dp = e->dp;
65    if (Oi(e->dp) == v) e->dp->on = e->dn; else e->dp->dn = e->dn;
66    elist[nfree++] = e;
67  }
68  void Low_tangent(edge *e_l, point *o_l, edge *e_r, point *o_r, edge **l_low, point **OL, edge **r_low,
      point **OR) {
69    for (point *d_l = Other(e_l, o_l), *d_r = Other(e_r, o_r); ; )
70      if (C3(o_l, o_r, d_l) < -eps)      e_l = Prev(e_l, d_l), o_l = d_l, d_l = Other(e_l, o_l);
71      else if (C3(o_l, o_r, d_r) < -eps) e_r = Next(e_r, d_r), o_r = d_r, d_r = Other(e_r, o_r);
72      else break;
73    *OL = o_l, *OR = o_r; *l_low = e_l, *r_low = e_r;
74  }
75  void Merge(edge *lr, point *s, edge *rl, point *u, edge **tangent) {
76    double l1, l2, l3, l4, r1, r2, r3, r4, cot_L, cot_R, u1, v1, u2, v2, n1, cot_n, P1, cot_P;
77    point *O, *D, *OR, *OL; edge *B, *L, *R;
78    Low_tangent(lr, s, rl, u, &L, &OL, &R, &OR);
79    for (*tangent = B = Join(L, OL, R, OR, 0), O = OL, D = OR; ; ) {
80      edge *El = Next(B, O), *Er = Prev(B, D), *next, *prev;
81      point *l = Other(El, O), *r = Other(Er, D);
82      V(l, O, l1, l2); V(l, D, l3, l4); V(r, O, r1, r2); V(r, D, r3, r4);
83      double cl = C2(l1, l2, l3, l4), cr = C2(r1, r2, r3, r4);
84      bool BL = cl > eps, BR = cr > eps;
85      if (!BL && !BR) break;
86      if (BL) {
87        double dl = Dot(l1, l2, l3, l4);
88        for (cot_L = dl / cl; ; Remove(El), El = next, cot_L = cot_n) {
89          next = Next(El, O); V(Other(next, O), O, u1, v1); V(Other(next, O), D, u2, v2);
90          n1 = C2(u1, v1, u2, v2); if (!(n1 > eps)) break;
91          cot_n = Dot(u1, v1, u2, v2) / n1;
92          if (cot_n > cot_L) break;
93        }
94      } if (BR) {
95        double dr = Dot(r1, r2, r3, r4);
96        for (cot_R = dr / cr; ; Remove(Er), Er = prev, cot_R = cot_P) {
97          prev = Prev(Er, D); V(Other(prev, D), O, u1, v1); V(Other(prev, D), D, u2, v2);
98          P1 = C2(u1, v1, u2, v2); if (!(P1 > eps)) break;
99          cot_P = Dot(u1, v1, u2, v2) / P1;
100         if (cot_P > cot_R) break;
101       }
102     } l = Other(El, O); r = Other(Er, D);
103     if (!BL || (BL && BR && cot_R < cot_L)) B = Join(B, O, Er, r, 0), D = r;
104     else B = Join(El, l, B, D, 0), O = l;
105   }
106 }
107 void Divide(int s, int t, edge **L, edge **R) {
108   edge *a, *b, *c, *ll, *lr, *rl, *rr, *tangent;
109   int n = t - s + 1;
110   if (n == 2) *L = *R = Make_edge(Q[s], Q[t]);
111   else if (n == 3) {
112     a = Make_edge(Q[s], Q[s + 1]), b = Make_edge(Q[s + 1], Q[t]);
113     Splice(a, b, Q[s + 1]);
114     double v = C3(Q[s], Q[s + 1], Q[t]);
115     if (v > eps)       c = Join(a, Q[s], b, Q[t], 0), *L = a, *R = b;
116     else if (v < -eps) c = Join(a, Q[s], b, Q[t], 1), *L = c, *R = c;
117     else *L = a, *R = b;
118   } else if (n > 3) {
119     int split = (s + t) / 2;
120     Divide(s, split, &ll, &lr); Divide(split + 1, t, &rl, &rr);
121     Merge(lr, Q[split], rl, Q[split + 1], &tangent);
122     if (Oi(tangent) == Q[s]) ll = tangent;
123     if (Dt(tangent) == Q[t]) rr = tangent;
124     *L = ll; *R = rr;
125   }
126 }
127 void Make_Graph() {
128   edge *start, *e; point *u, *v;
129   for (int i = 0; i < n; i++) {
130     start = e = (u = &p[i])->in;
131     do{ v = Other(e, u);
132       if (u < v) E[M++].u = (u - p, v - p, dis(u, v)); // M < aix * maxn
133     } while ((e = Next(e, u)) != start);
134   }
135 }
136 int b[maxn];
137 int Find(int x) { while (x != b[x]) { b[x] = b[b[x]]; x = b[x]; } return x; }
138 void Kruskal() {
139   memset(b, 0, sizeof(b)); sort(E, E + M);
140   for (int i = 0; i < n; i++) b[i] = i;
141   for (int i = 0, kk = 0; i < M && kk < n - 1; i++) {
142     int m1 = Find(E[i].u), m2 = Find(E[i].v);
143     if (m1 != m2) b[m1] = m2, MST[kk++] = E[i];
144   }
145 }
146 void solve() {
147   scanf("%d", &n);
148   for (int i = 0; i < n; i++) scanf("%lf%lf", &p[i].x, &p[i].y), p[i].index = i, p[i].in = NULL;
149   Alloc_memory(); sort(p, p + n);
150   for (int i = 0; i < n; i++) Q[i] = p + i;
151   edge *L, *R; Divide(0, n - 1, &L, &R);
152   M = 0; Make_Graph(); Kruskal();
153 }
154 int main() { solve(); return 0; }
```

## 1.11 四边形双费马点

```
1   typedef complex<double> Tpoint;
2   const double eps = 1e-8;
3   const double sqrt3 = sqrt(3.0);
4   bool cmp(const Tpoint &a, const Tpoint &b) {
5     return a.real() < b.real() - eps || (a.real() < b.real() + eps && a.imag() < b.imag());
6   }
7   Tpoint rotate(const Tpoint &a, const Tpoint &b, const Tpoint &c) {
8     Tpoint d = b - a; d = Tpoint(-d.imag(), d.real());
9     if (Sign(cross(a, b, c)) == Sign(cross(a, b, a + d))) d *= -1.0;
```

```
10      return unit(d);
11  }
12  Tpoint p[10], a[10], b[10];
13  int N, T;
14  double totlen(const Tpoint &p, const Tpoint &a, const Tpoint &b, const Tpoint &c) {
15      return abs(p - a) + abs(p - b) + abs(p - c);
16  }
17  double fermat(const Tpoint &x, const Tpoint &y, const Tpoint &z, Tpoint &cp) {
18      a[0] = a[3] = x; a[1] = a[4] = y; a[2] = a[5] = z;
19      double len = 1e100, len2;
20      for (int i = 0; i < 3; i++) {
21          len2 = totlen(a[i], x, y, z);
22          if (len2 < len) len = len2, cp = a[i];
23      }
24      for (int i = 0; i < 3; i++) {
25          b[i] = rotate(a[i + 1], a[i], a[i + 2]);
26          b[i] = (a[i + 1] + a[i]) / 2.0 + b[i] * (abs(a[i + 1] - a[i]) * sqrt3 / 2.0);
27      }
28      b[3] = b[0];
29      Tpoint cp2 = intersect(b[0], a[2], b[1], a[3]);
30      len2 = totlen(cp2, x, y, z);
31      if (len2 < len) len = len2, cp = cp2;
32      return len;
33  }
34  double getans(const Tpoint &a) {
35      double len = 0; for (int i = 0; i < N; i++) len += abs(a - p[i]);
36      return len;
37  }
38  double mindist(const Tpoint &p, const Tpoint &a, const Tpoint &b, const Tpoint &c, const Tpoint &d) {
39      return min( min(abs(p - a), abs(p - b)), min(abs(p - c), abs(p - d)));
40  }
41  int main() {
42      N = 4;
43      for (cin >> T; T; T--) {
44          double ret = 1e100, len_cur, len_before, len1, len2, len;
45          Tpoint cp, cp1, cp2;
46          Foru(i, 0, N) cin >> p[i];
47          Foru(i, 0, N) ret = min(ret, getans(p[i]));
48          Foru(i, 1, N) Foru(j, 1, N) if (j != i) Foru(k, 1, N) if (k != i && k != j) {
49              cMin(ret, abs(p[0] - p[i]) + abs(p[j] - p[k])
50                  + min( min(abs(p[0] - p[j]), abs(p[0] - p[k])),
51                      min(abs(p[i] - p[j]), abs(p[i] - p[k]))
52              ));
53              ret = min(ret, getans(intersect(p[0], p[i], p[j], p[k])));
54          }
55          Foru(i, 0, N) Foru(j, i + 1, N) Foru(k, j + 1, N) {
56              double len = fermat(p[i], p[j], p[k], cp);
57              ret = min(ret, len + mindist(p[6 - i - j - k], p[i], p[j], p[k], cp));
58          }
59          sort(p, p + N, cmp);
60          for (int i = 1; i < N; i++) {
61              cp1 = (p[0] + p[i]) / 2.0;
62              int j, k;
63              for (j = 1; j < N && j == i; j++);
64              for (k = 6 - i - j, len_before = 1e100; ; ) {
65                  len1 = fermat(cp1, p[j], p[k], cp2);
66                  len1 = fermat(cp2, p[0], p[i], cp1);
67                  len = len1 + abs(cp2 - p[j]) + abs(cp2 - p[k]);
68                  if (len < len_before - (1e-6)) len_before = len;
69                  else break;
70              } ret = min(ret, len_before);
71          } printf("%.4f\n", ret);
72      }
73      return 0;
74  }
```

## 1.12 三维计算几何基本操作

```
1   struct point { double x, y, z; // something omitted
2       friend point det(const point &a, const point &b) {
3           return point(a.y * b.z - a.z * b.y, a.z * b.x - a.x * b.z, a.x * b.y - a.y * b.x);
4       }
5       friend double mix(const point &a, const point &b, const point &c) {
6           return a.x * b.y * c.z + a.y * b.z * c.x + a.z * b.x * c.y - a.z * b.y * c.x - a.x * b.z * c.y - a.y
            * b.x * c.z;
7       }
8       double distLP(const point &p1, const point &p2) const {
9           return det(p2 - p1, *this - p1).len() / (p2 - p1).len();
10      }
11      double distFP(const point &p1, const point &p2, const point &p3) const {
12          point n = det(p2 - p1, p3 - p1); return fabs( dot(n, *this - p1) / n.len() );
13      }
```

```
14  };
15  double distLL(const point &p1, const point &p2, const point &q1, const point &q2) {
16      point p = q1 - p1, u = p2 - p1, v = q2 - q1;
17      double d = u.norm() * v.norm() - dot(u, v) * dot(u, v);
18      if (sign(d) == 0) return p1.distLP(q1, q2);
19      double s = (dot(p, u) * v.norm() - dot(p, v) * dot(u, v)) / d;
20      return (p1 + u * s).distLP(q1, q2);
21  }
22  double distSS(const point &p1, const point &p2, const point &q1, const point &q2) {
23      point p = q1 - p1, u = p2 - p1, v = q2 - q1;
24      double d = u.norm() * v.norm() - dot(u, v) * dot(u, v);
25      if (sign(d) == 0) return min( min((p1 - q1).len(), (p1 - q2).len()),
26                              min((p2 - q1).len(), (p2 - q2).len()));
27      double s1 = (dot(p, u) * v.norm() - dot(p, v) * dot(u, v)) / d;
28      double s2 = (dot(p, v) * u.norm() - dot(p, u) * dot(u, v)) / d;
29      if (s1 < 0.0) s1 = 0.0; if (s1 > 1.0) s1 = 1.0;
30      if (s2 < 0.0) s2 = 0.0; if (s2 > 1.0) s2 = 1.0;
31      point r1 = p1 + u * s1; point r2 = q1 + v * s2;
32      return (r1 - r2).len();
33  }
34  bool isFL(const point &p, const point &o, const point &q1, const point &q2, point &res) {
35      double a = dot(o, q2 - p), b = dot(o, q1 - p), d = a - b;
36      if (sign(d) == 0) return false;
37      res = (q1 * a - q2 * b) / d;
38      return true;
39  }
40  bool isFF(const point &p1, const point &o1, const point &p2, const point &o2, point &a, point &b) {
41      point e = det(o1, o2), v = det(o1, e);
42      double d = dot(o2, v); if (sign(d) == 0) return false;
43      point q = p1 + v * (dot(o2, p2 - p1) / d);
44      a = q; b = q + e;
45      return true;
46  }
```

## 1.13 凸多面体切割

```
1   vector<vector<point> > convexCut(const vector<vector<point> > &pss, const point &p, const point &o) {
2       vector<vector<point> > res;
3       vector<point> sec;
4       for (unsigned itr = 0, size = pss.size(); itr < size; ++itr) {
5           const vector<point> &ps = pss[itr];
6           int n = ps.size();
7           vector<point> qs;
8           bool dif = false;
9           for (int i = 0; i < n; ++i) {
10              int d1 = sign( dot(o, ps[i] - p) );
11              int d2 = sign( dot(o, ps[(i + 1) % n] - p) );
12              if (d1 <= 0) qs.push_back(ps[i]);
13              if (d1 * d2 < 0) {
14                  point q;
15                  isFL(p, o, ps[i], ps[(i + 1) % n], q); // must return true
16                  qs.push_back(q);
17                  sec.push_back(q);
18              }
19              if (d1 == 0) sec.push_back(ps[i]);
20              else dif = true;
21              dif |= dot(o, det(ps[(i + 1) % n] - ps[i], ps[(i + 2) % n] - ps[i])) < -EPS;
22          }
23          if (!qs.empty() && dif)
24              res.insert(res.end(), qs.begin(), qs.end());
25      }
26      if (!sec.empty()) {
27          vector<point> tmp( convexHull2D(sec, o) );
28          res.insert(res.end(), tmp.begin(), tmp.end());
29      }
30      return res;
31  }
32
33  vector<vector<point> > initConvex() {
34      vector<vector<point> > pss(6, vector<point>(4));
35      pss[0][0] = pss[1][0] = pss[2][0] = point(-INF, -INF, -INF);
36      pss[0][3] = pss[1][1] = pss[5][2] = point(-INF, -INF,  INF);
37      pss[0][1] = pss[2][3] = pss[4][2] = point(-INF,  INF, -INF);
38      pss[0][2] = pss[5][3] = pss[4][1] = point(-INF,  INF,  INF);
39      pss[1][3] = pss[2][1] = pss[3][2] = point( INF, -INF, -INF);
40      pss[1][2] = pss[5][1] = pss[3][3] = point( INF, -INF,  INF);
41      pss[2][2] = pss[4][3] = pss[3][1] = point( INF,  INF, -INF);
42      pss[5][0] = pss[4][0] = pss[3][0] = point( INF,  INF,  INF);
43      return pss;
44  }
```

## 1.14 三维凸包

不能有重点

```
namespace ConvexHull3D {
  #define volume(a, b, c, d) (mix(ps[b] - ps[a], ps[c] - ps[a], ps[d] - ps[a]))
  vector<Facet> getHull(int n, point ps[]) {
    static int mark[MAXN][MAXN], a, b, c; int stamp = 0; bool exist = false;
    vector<Facet> facet; random_shuffle(ps, ps + n);
    for (int i = 2; i < n && !exist; i++) {
      point ndir = det(ps[0] - ps[i], ps[1] - ps[i]);
      if (ndir.len() < EPS) continue;
      swap(ps[i], ps[2]); for (int j = i + 1; j < n && !exist; j++)
        if (sign(volume(0, 1, 2, j)) != 0) {
          exist = true; swap(ps[j], ps[3]);
          facet.push_back(Facet(0, 1, 2)); facet.push_back(Facet(0, 2, 1));
        }
    } if (!exist) return ConvexHull2D(n, ps);
    for (int i = 0; i < n; ++i) for (int j = 0; j < n; ++j) mark[i][j] = 0;
    stamp = 0; for (int v = 3; v < n; ++v) {
      vector<Facet> tmp; ++stamp;
      for (unsigned i = 0; i < facet.size(); i++) {
        a = facet[i].a; b = facet[i].b; c = facet[i].c;
        if (sign(volume(v, a, b, c)) < 0)
          mark[a][b] = mark[a][c] = mark[b][a] = mark[b][c] = mark[c][a] = mark[c][b] = stamp;
        else tmp.push_back(facet[i]);
      } facet = tmp;
      for (unsigned i = 0; i < tmp.size(); i++) {
        a = facet[i].a; b = facet[i].b; c = facet[i].c;
        if (mark[a][b] == stamp) facet.push_back(Facet(b, a, v));
        if (mark[b][c] == stamp) facet.push_back(Facet(c, b, v));
        if (mark[c][a] == stamp) facet.push_back(Facet(a, c, v));
      }
    } return facet;
  }
  #undef volume
}
namespace Gravity {
  using ConvexHull3D::Facet;
  point findG(point ps[], const vector<Facet> &facet) {
    double ws = 0; point res(0.0, 0.0, 0.0), o = ps[ facet[0].a ];
    for (int i = 0, size = facet.size(); i < size; ++i) {
      const point &a = ps[ facet[i].a ], &b = ps[ facet[i].b ], &c = ps[ facet[i].c ];
      point p = (a + b + c + o) * 0.25; double w = mix(a - o, b - o, c - o);
      ws += w; res = res + p * w;
    } res = res / ws;
    return res;
  }
}
```

## 1.15 球面点表面点距离

```
double distOnBall(double lati1, double longi1, double lati2, double longi2, double R) {
  lati1 *= PI / 180; longi1 *= PI / 180;
  lati2 *= PI / 180; longi2 *= PI / 180;
  double x1 = cos(lati1) * sin(longi1);
  double y1 = cos(lati1) * cos(longi1);
  double z1 = sin(lati1);
  double x2 = cos(lati2) * sin(longi2);
  double y2 = cos(lati2) * cos(longi2);
  double z2 = sin(lati2);
  double theta = acos(x1 * x2 + y1 * y2 + z1 * z2);
  return R * theta;
}
```

## 1.16 长方体表面点距离

```
int r;
void turn(int i, int j, int x, int y, int z, int x0, int y0, int L, int W, int H) {
  if (z == 0) r = min(r, x * x + y * y);
  else {
    if (i >= 0 && i < 2)  turn(i + 1, j, x0 + L + z, y, x0 + L - x, x0 + L, y0, H, W, L);
    if (j >= 0 && j < 2)  turn(i, j + 1, x, y0 + W + z, y0 + W - y, x0, y0 + W, L, H, W);
    if (i <= 0 && i > -2) turn(i - 1, j, x0 - z, y, x - x0, x0 - H, y0, H, W, L);
    if (j <= 0 && j > -2) turn(i, j - 1, x, y0 - z, y - y0, x0, y0 - H, L, H, W);
  }
}
```

```
int calc(int L, int H, int W, int x1, int y1, int z1, int x2, int y2, int z2) {
  if (z1 != 0 && z1 != H)
    if (y1 == 0 || y1 == W) swap(y1, z1), swap(y2, z2), swap(W, H);
    else                    swap(x1, z1), swap(x2, z2), swap(L, H);
  if (z1 == H) z1 = 0, z2 = H - z2;
  r = INF; turn(0, 0, x2 - x1, y2 - y1, z2, -x1, -y1, L, W, H);
  return r;
}
```

## 1.17 最小覆盖球

```
int outCnt; point out[4], res; double radius;
void ball() {
  static point q[3];
  static double m[3][3], sol[3], L[3], det;
  int i, j; res = point(0.0, 0.0, 0.0); radius = 0.0;
  switch (outCnt) {
  case 1: res = out[0]; break;
  case 2: res = (out[0] + out[1]) * 0.5; radius = (res - out[0]).norm();
    break;
  case 3:
    q[0] = out[1] - out[0]; q[1] = out[2] - out[0];
    for (i = 0; i < 2; ++i) for (j = 0; j < 2; ++j)
      m[i][j] = dot(q[i], q[j]) * 2.0;
    for (i = 0; i < 2; ++i) sol[i] = dot(q[i], q[i]);
    det = m[0][0] * m[1][1] - m[0][1] * m[1][0];
    if (sign(det) == 0) return;
    L[0] = (sol[0] * m[1][1] - sol[1] * m[0][1]) / det;
    L[1] = (sol[1] * m[0][0] - sol[0] * m[1][0]) / det;
    res = out[0] + q[0] * L[0] + q[1] * L[1];
    radius = (res - out[0]).norm();
    break;
  case 4:
    q[0] = out[1] - out[0]; q[1] = out[2] - out[0]; q[2] = out[3] - out[0];
    for (i = 0; i < 3; ++i) for (j = 0; j < 3; ++j) m[i][j] = dot(q[i], q[j]) * 2;
    for (i = 0; i < 3; ++i) sol[i] = dot(q[i], q[i]);
    det = m[0][0] * m[1][1] * m[2][2] + m[0][1] * m[1][2] * m[2][0]
      + m[0][2] * m[2][1] * m[1][0] - m[0][2] * m[1][1] * m[2][0]
      - m[0][1] * m[1][0] * m[2][2] - m[0][0] * m[1][2] * m[2][1];
    if (sign(det) == 0) return;
    for (j = 0; j < 3; ++j) { for (i = 0; i < 3; ++i) m[i][j] = sol[i];
      L[j] = (m[0][0] * m[1][1] * m[2][2] + m[0][1] * m[1][2] * m[2][0]
        + m[0][2] * m[2][1] * m[1][0] - m[0][2] * m[1][1] * m[2][0]
        - m[0][1] * m[1][0] * m[2][2] - m[0][0] * m[1][2] * m[2][1]) / det;
      for (i = 0; i < 3; ++i) m[i][j] = dot(q[i], q[j]) * 2;
    } res = out[0];
    for (i = 0; i < 3; ++i) res += q[i] * L[i]; radius = (res - out[0]).norm();
  }
}
void minball(int n, point pt[]) {
  ball();
  if (outCnt < 4) for (int i = 0; i < n; ++i)
    if ((res - pt[i]).norm() > +radius + EPS) {
      out[outCnt] = pt[i]; ++outCnt; minball(i, pt); --outCnt;
      if (i > 0) {
        point Tt = pt[i];
        memmove(&pt[1], &pt[0], sizeof(point) * i);
        pt[0] = Tt;
      }
    }
}
pair<point, double> main(int npoint, point pt[]) { // 0-based
  random_shuffle(pt, pt + npoint); radius = -1;
  for (int i = 0; i < npoint; i++) { if ((res - pt[i]).norm() > EPS + radius) {
    outCnt = 1; out[0] = pt[i]; minball(i, pt); } }
  return make_pair(res, sqrt(radius));
}
```

## 1.18 三维向量操作矩阵

- 绕单位向量 $u = (u_x, u_y, u_z)$ 右手方向旋转 $\theta$ 度的矩阵:

$$\begin{bmatrix} \cos\theta + u_x^2(1 - \cos\theta) & u_x u_y(1 - \cos\theta) - u_z\sin\theta & u_x u_z(1 - \cos\theta) + u_y\sin\theta \\ u_y u_x(1 - \cos\theta) + u_z\sin\theta & \cos\theta + u_y^2(1 - \cos\theta) & u_y u_z(1 - \cos\theta) - u_x\sin\theta \\ u_z u_x(1 - \cos\theta) - u_y\sin\theta & u_z u_y(1 - \cos\theta) + u_x\sin\theta & \cos\theta + u_z^2(1 - \cos\theta) \end{bmatrix}$$

$$= \cos\theta I + \sin\theta \begin{bmatrix} 0 & -u_z & u_y \\ u_z & 0 & -u_x \\ -u_y & u_x & 0 \end{bmatrix} + (1 - \cos\theta) \begin{bmatrix} u_x^2 & u_x u_y & u_x u_z \\ u_y u_x & u_y^2 & u_y u_z \\ u_z u_x & u_z u_y & u_z^2 \end{bmatrix}$$

- 点 $a$ 绕单位向量 $u = (u_x, u_y, u_z)$ 右手方向旋转 $\theta$ 度的对应点为 $a' = a\cos\theta + (u \times a)\sin\theta + (u \otimes u)a(1 - \cos\theta)$

- 关于向量 $v$ 作对称变换的矩阵 $H = I - 2\frac{vv^T}{v^T v}$,

- 点 $a$ 对称点: $a' = a - 2\frac{v^T a}{v^T v} \cdot v$

### 1.19 立体角

对于任意一个四面体 $OABC$, 从 $O$ 点观察 $\triangle ABC$ 的立体角 $\tan\frac{\Omega}{2} = \frac{\text{mix}(\vec{a}, \vec{b}, \vec{c})}{|a||b||c| + (\vec{a} \cdot \vec{b})|c| + (\vec{a} \cdot \vec{c})|b| + (\vec{b} \cdot \vec{c})|a|}$.

## 2 数据结构

### 2.1 动态凸包 (只支持插入)

```
1  #define x first    // upperHull ← (x, y)
2  #define y second   // lowerHull ← (x, -y)
3  typedef map<int, int> mii;
4  typedef map<int, int>::iterator mit;
5  struct point { point(const mit &p): x(p->first), y(p->second) {} };
6  inline bool checkInside(mii &a, const point &p) { // border inclusive
7    int x = p.x, y = p.y; mit p1 = a.lower_bound(x);
8    if (p1 == a.end()) return false; if (p1->x == x) return y <= p1->y;
9    if (p1 == a.begin()) return false; mit p2(p1--);
10   return sign(det(p - point(p1), point(p2) - p)) >= 0;
11 } inline void addPoint(mii &a, const point &p) { // no collinear points
12   int x = p.x, y = p.y; mit pnt = a.insert(make_pair(x, y)).first, p1, p2;
13   for (pnt->y = y; ; a.erase(p2)) {
14     p1 = pnt; if (++p1 == a.end()) break;
15     p2 = p1;  if (++p1 == a.end()) break;
16     if (det(point(p2) - p, point(p1) - p) < 0) break;
17   } for ( ; ; a.erase(p2)) {
18     if ((p1 = pnt) == a.begin()) break;
19     if (--p1 == a.begin()) break; p2 = p1--;
20     if (det(point(p2) - p, point(p1) - p) > 0) break;
21   }
22 }
```

### 2.2 Rope 用法

```
1  #include <ext/rope>
2  using __gnu_cxx::crope; using __gnu_cxx::rope;
3  a = b.substr(from, len);      // [from, from + len)
4  a = b.substr(from);           // [from, from]
5  b.c_str();                    // might lead to memory leaks
6  b.delete_c_str();             // delete the c_str that created before
7  a.insert(p, str);             // insert str before position p
8  a.erase(i, n);                // erase [i, i + n)
```

### 2.3 Treap

```
1  struct node { int key, prio, size; node *ch[2]; } base[MAXN], *top, *root, *null, nil;
2  typedef node *tree;
3  tree newNode(int key) {
4    static int seed = 3312;
5    top->key = key; top->prio = seed = int(seed * 48271LL % 2147483647);
6    top->size = 1; top->ch[0] = top->ch[1] = null; return top++;
7  }
8  void Rotate(tree &x, int d) {
9    tree y = x->ch[!d]; x->ch[!d] = y->ch[d]; y->ch[d] = x; y->size = x->size;
10   x->size = x->ch[0]->size + 1 + x->ch[1]->size; x = y;
```

```
11 }
12 void Insert(tree &t, int key) {
13   if (t == null) t = newNode(key);
14   else { int d = t->key < key; Insert(t->ch[d], key); ++t->size;
15     if (t->ch[d]->prio < t->prio) Rotate(t, !d); }
16   }
17 }
18 void Delete(tree &t, int key) {
19   if (t->key != key) { Delete(t->ch[t->key < key], key); --t->size; }
20   else if (t->ch[0] == null) t = t->ch[1];
21   else if (t->ch[1] == null) t = t->ch[0];
22   else { int d = t->ch[0]->prio < t->ch[1]->prio;
23     Rotate(t, d); Delete(t->ch[d], key); --t->size;
24   }
25 }
```

### 2.4 可持久化 Treap

```
1  inline bool randomBySize(int a, int b) {
2    static long long seed = 1;
3    return (seed = seed * 48271 % 2147483647) * (a + b) < 2147483647LL * a;
4  }
5  tree merge(tree x, tree y) {
6    if (x == null) return y; if (y == null) return x;
7    tree t = NULL;
8    if (randomBySize(x->size, y->size)) t = newNode(x), t->r = merge(x->r, y);
9    else t = newNode(y), t->l = merge(x, y->l);
10   update(t); return t;
11 }
12 void splitByKey(tree t, int k, tree &l, tree &r) { // [-∞, k)[k, +infty)
13   if (t == null) l = r = null;
14   else if (t->key < k) l = newNode(t), splitByKey(t->r, k, l->r, r), update(l);
15   else                 r = newNode(t), splitByKey(t->l, k, l, r->l), update(r);
16 }
17 void splitBySize(tree t, int k, tree &l, tree &r) { // [1, k)[k, +∞)
18   static int s; if (t == null) l = r = null;
19   else if ((s = t->l->size + 1) < k) l = newNode(t), splitBySize(t->r, k - s, l->r, r), update(l);
20   else                               r = newNode(t), splitBySize(t->l, k, l, r->l), update(r);
21 }
```

### 2.5 左偏树

```
1  tree merge(tree a, tree b) {
2    if (a == null) return b;
3    if (b == null) return a;
4    if (a->key > b->key) swap(a, b);
5    a->rc = merge(a->rc, b);
6    a->rc->fa = a;
7    if (a->lc->dist < a->rc->dist) swap(a->lc, a->rc);
8    a->dist = a->rc->dist + 1;
9    return a;
10 }
11 void erase(tree t) {
12   tree x = t->fa, y = merge(t->lc, t->rc);
13   if (y != null) y->fa = x;
14   if (x == null) root = y;
15   else
16     for ((x->lc == t ? x->lc : x->rc) = y; x != null; y = x, x = x->fa) {
17       if (x->lc->dist < x->rc->dist) swap(x->lc, x->rc);
18       if (x->rc->dist + 1 == x->dist) return;
19       x->dist = x->rc->dist + 1;
20     }
21 }
```

### 2.6 Link-Cut Tree

```
1  struct node { int rev; node *pre, *ch[2]; } base[MAXN], nil, *null;
2  typedef node *tree;
3  #define isRoot(x) (x->pre->ch[0] != x && x->pre->ch[1] != x)
4  #define isRight(x) (x->pre->ch[1] == x)
5  inline void MakeRev(tree t) { if (t != null) { t->rev ^= 1; swap(t->ch[0], t->ch[1]); } }
6  inline void PushDown(tree t) { if (t->rev) { MakeRev(t->ch[0]); MakeRev(t->ch[1]); t->rev = 0; } }
7  inline void Rotate(tree x) {
```

```
 8    tree y = x->pre; PushDown(y); PushDown(x);
 9    int d = isRight(x);
10    if (!isRoot(y)) y->pre->ch[isRight(y)] = x; x->pre = y->pre;
11    if ((y->ch[d] = x->ch[!d]) != null) y->ch[d]->pre = y;
12    x->ch[!d] = y; y->pre = x; Update(y);
13  }
14  inline void Splay(tree x) {
15    PushDown(x); for (tree y; !isRoot(x); Rotate(x)) {
16      y = x->pre; if (!isRoot(y)) Rotate(isRight(x) != isRight(y) ? x : y);
17    } Update(x);
18  }
19  inline void Splay(tree x, tree to) {
20    PushDown(x); for (tree y; (y = x->pre) != to; Rotate(x)) if (y->pre != to)
21      Rotate(isRight(x) != isRight(y) ? x : y);
22    Update(x);
23  }
24  inline tree Access(tree t) {
25    tree last = null; for (; t != null; last = t, t = t->pre) Splay(t),t->ch[1] = last, Update(t);
26    return last;
27  }
28  inline void MakeRoot(tree t) { Access(t); Splay(t); MakeRev(t); }
29  inline tree FindRoot(tree t) { Access(t); Splay(t); tree last = null;
30    for ( ; t != null; last = t, t = t->ch[0]) PushDown(t); Splay(last); return last;
31  }
32  inline void Join(tree x, tree y) { MakeRoot(y); y->pre = x; }
33  inline void Cut(tree t) {Access(t); Splay(t); t->ch[0]->pre = null; t->ch[0] = null; Update(t);}
34  inline void Cut(tree x, tree y) {
35    tree upper = (Access(x), Access(y));
36    if (upper == x) { Splay(x); y->pre = null; x->ch[1] = null; Update(x); }
37    else if (upper == y) { Access(x); Splay(y); x->pre = null; y->ch[1] = null; Update(y); }
38    else assert(0); // impossible to happen
39  }
40  inline int Query(tree a, tree b) { // query the cost in path a <-> b, lca inclusive
41    Access(a); tree c = Access(b); // c is lca
42    int v1 = c->ch[1]->maxCost; Access(a);
43    int v2 = c->ch[1]->maxCost;
44    return max(max(v1, v2), c->cost);
45  }
46  void Init() {
47    null = &nil; null->ch[0] = null->ch[1] = null->pre = null; null->rev = 0;
48    Rep(i, 1, N) { node &n = base[i]; n.rev = 0; n.pre = n.ch[0] = n.ch[1] = null; }
49  }
```

## 2.7   K-D Tree Nearest

```
 1  struct Point { int x, y; };
 2  struct Rectangle {
 3    int lx , rx , ly , ry;
 4    void set(const Point &p) { lx = rx = p.x; ly = ry = p.y; }
 5    void merge(const Point &o) {
 6      lx = min(lx, o.x); rx = max(rx, o.x); ly = min(ly, o.y); ry = max(ry, o.y);
 7    } void merge(const Rectangle &o) {
 8      lx = min(lx , o.lx); rx = max(rx , o.rx); ly = min(ly , o.ly); ry = max(ry , o.ry);
 9    } LL dist(const Point &p) {
10      LL res = 0;
11      if (p.x < lx) res += sqr(lx - p.x); else if (p.x > rx) res += sqr(p.x - rx);
12      if (p.y < ly) res += sqr(ly - p.y); else if (p.y > ry) res += sqr(p.y - ry);
13      return res;
14    }
15  };
16  struct Node { int child[2]; Point p; Rectangle rect; };
17  const int MAX_N = 1111111;
18  const LL INF = 100000000;
19  int n, m, tot, root; LL result;
20  Point a[MAX_N], p; Node tree[MAX_N];
21  int build(int s, int t, bool d) {
22    int k = ++tot, mid = (s + t) >> 1;
23    nth_element(a + s, a + mid , a + t, d ? cmpXY : cmpYX);
24    tree[k].p = a[mid]; tree[k].rect.set(a[mid]); tree[k].child[0] = tree[k].child[1] = 0;
25    if (s < mid)
26      tree[k].child[0] = build(s, mid , d ^ 1), tree[k].rect.merge(tree[tree[k].child[0]].rect);
27    if (mid + 1 < t)
28      tree[k].child[1] = build(mid + 1, t, d ^ 1), tree[k].rect.merge(tree[tree[k].child[1]].rect);
29    return k;
30  }
31  int insert(int root, bool d) {
32    if (root == 0) {
33      tree[++tot].p = p; tree[tot].rect.set(p); tree[tot].child[0] = tree[tot].child[1] = 0;
34      return tot;
35    } tree[root].rect.merge(p);
36    if ((d && cmpXY(p, tree[root].p)) || (!d && cmpYX(p, tree[root].p)))
37      tree[root].child[0] = insert(tree[root].child[0], d ^ 1);
38    else tree[root].child[1] = insert(tree[root].child[1], d ^ 1);
```

```
39    return root;
40  }
41  void query(int k, bool d) {
42    if (tree[k].rect.dist(p) >= result) return;
43    cMin(result, dist(tree[k].p, p));
44    if ((d && cmpXY(p, tree[k].p)) || (!d && cmpYX(p, tree[k].p))) {
45      if (tree[k].child[0]) query(tree[k].child[0], d ^ 1);
46      if (tree[k].child[1]) query(tree[k].child[1], d ^ 1);
47    } else {
48      if (tree[k].child[1]) query(tree[k].child[1], d ^ 1);
49      if (tree[k].child[0]) query(tree[k].child[0], d ^ 1);
50    }
51  }
52  void example(int n) {
53    root = tot = 0; scan(a); root = build(0, n, 0); // init, a[0 . . . n − 1]
54    scan(p); root = insert(root, 0); // insert
55    scan(p); result = INF; ans = query(root, 0); // query
56  }
```

## 2.8   K-D Tree Farthest

输入 $n$ 个点, 对每个询问 $px, py, k$, 输出 $k$ 远点的编号

```
 1  struct Point { int x, y, id; };
 2  struct Rectangle {
 3    int lx, rx, ly, ry;
 4    void set(const Point &p) { lx = rx = p.x; ly = ry = p.y; }
 5    void merge(const Rectangle &o) {
 6      lx = min(lx, o.lx); rx = max(rx, o.rx); ly = min(ly, o.ly); ry = max(ry, o.ry);
 7    }
 8    LL dist(const Point &p) { LL res = 0;
 9      res += max(sqr(rx - p.x), sqr(lx - p.x));
10      res += max(sqr(ry - p.y), sqr(ly - p.y));
11      return res;
12    }
13  }; struct Node { Point p; Rectangle rect; };
14  const int MAX_N = 111111;
15  const LL INF = 1LL << 60;
16  int n, m;
17  Point a[MAX_N], b[MAX_N ];
18  Node tree[MAX_N * 3];
19  Point p; // p is the query point
20  pair<LL, int> result[22];
21  void build(int k, int s, int t, bool d) {
22    int mid = (s + t) >> 1;
23    nth_element(a + s, a + mid , a + t, d ? cmpX : cmpY);
24    tree[k].p = a[mid];
25    tree[k].rect.set(a[mid]);
26    if (s < mid)
27      build(k << 1, s, mid , d ^ 1), tree[k].rect.merge(tree[k << 1]. rect);
28    if (mid + 1 < t)
29      build(k << 1 | 1, mid + 1, t, d ^ 1), tree[k].rect.merge(tree[k << 1 | 1]. rect);
30  }
31  void query(int k, int s, int t, bool d, int kth) {
32    if (tree[k].rect.dist(p) < result[kth].first) return;
33    pair<LL, int> tmp(dist(tree[k].p, p), -tree[k].p.id);
34    for (int i = 1; i <= kth; i++) if (tmp > result[i]) {
35      for (int j = kth + 1; j > i; j--) result[j] = result[j - 1]; result[i] = tmp;
36      break;
37    }
38    int mid = (s + t) >> 1;
39    if ((d && cmpX(p, tree[k].p)) || (!d && cmpY(p, tree[k].p))) {
40      if (mid + 1 < t) query(k << 1 | 1, mid + 1, t, d ^ 1, kth);
41      if (s < mid)     query(k << 1, s, mid , d ^ 1, kth);
42    } else {
43      if (s < mid)     query(k << 1, s, mid , d ^ 1, kth);
44      if (mid + 1 < t) query(k << 1 | 1, mid + 1, t, d ^ 1, kth);
45    }
46  }
47  void example(int n) {
48    scan(a); build(1, 0, n, 0); // init, a[0 . . . n − 1]
49    scan(p, k);  // query
50    Rep(j, 1, k) result[j].first = -1;
51    query(1, 0, n, 0, k); ans = -result[k].second + 1;
52  }
```

## 2.9   树链剖分

```
1  int N, fa[MAXN], dep[MAXN], que[MAXN], size[MAXN], own[MAXN];
2  int LCA(int x, int y) { if (x == y) return x;
3    for ( ; ; x = fa[own[x]]) {
4      if (dep[x] < dep[y]) swap(x, y); if (own[x] == own[y]) return y;
5      if (dep[own[x]] < dep[own[y]]) swap(x, y);
6    } return -1;
7  }
8  void Decomposion() {
9    static int path[MAXN]; int x, y, a, next, head = 0, tail = 0, cnt; // BFS omitted
10   for (int i = 1; i <= N; ++i) if (own[a = que[i]] == -1)
11     for (x = a, cnt = 0; ; x = next) { next = -1; own[x] = a; path[++cnt] = x;
12       for (edge e(fir[x]); e; e = e->next) if ( (y = e->to) != fa[x] )
13         if (next == -1 || size[y] > size[next]) next = y;
14       if (next == -1) { tree[a].init(cnt, path); break; }
15     }
16 }
```

## 3  字符串相关

### 3.1  Manacher

```
1  // len[i] : the max length of palindrome whose mid point is (i / 2)
2  void Manacher(int n, char cs[], int len[]) { // 0-based, len[] must be double sized
3    for (int i = 0; i < n + n; ++i) len[i] = 0;
4    for (int i = 0, j = 0, k; i < n * 2; i += k, j = max(j - k, 0)) {
5      while (i - j >= 0 && i + j + 1 < n * 2 && cs[(i - j) / 2] == cs[(i + j + 1) / 2]) j++;
6      len[i] = j; for (k = 1; i - k >= 0 && j - k >= 0 && len[i - k] != j - k; k++)
7        len[i + k] = min(len[i - k], j - k);
8    }
9  }
```

### 3.2  KMP

$$next[i] = \max\{len|A[0\ldots len-1] = A的第\ i\ 位向前或后的长度为\ len\ 的串\}$$
$$ext[i] = \max\{len|A[0\ldots len-1] = B的第\ i\ 位向前或后的长度为\ len\ 的串\}$$

```
1  void KMP(char *a, int la, char *b, int lb, int *next, int *ext) {
2    --a; --b; --next; --ext;
3    for (int i = 2, j = next[1] = 0; i <= la; i++) {
4      while (j && a[j + 1] != a[i]) j = next[j]; if (a[j + 1] == a[i]) ++j; next[i] = j;
5    } for (int i = 1, j = 0; i <= lb; ++i) {
6      while (j && a[j + 1] != b[i]) j = next[j]; if (a[j + 1] == b[i]) ++j; ext[i] = j;
7      if (j == la) j = next[j];
8    }
9  } void ExKMP(char *a, int la, char *b, int lb, int *next, int *ext) {
10   next[0] = la; for (int &j = next[1] = 0; j + 1 < la && a[j] == a[j + 1]; ++j);
11   for (int i = 2, k = 1; i < la; ++i) {
12     int p = k + next[k], l = next[i - k]; if (l < p - i) next[i] = l;
13     else for (int &j = next[k = i] = max(0, p - i); i + j < la && a[j] == a[i + j]; ++j);
14   } for (int &j = ext[0] = 0; j < la && a[j] == b[j]; ++j);
15   for (int i = 1, k = 0; i < lb; ++i) {
16     int p = k + ext[k], l = next[i - k]; if (l < p - i) ext[i] = l;
17     else for (int &j = ext[k = i] = max(0, p - i); j < la && i + j < lb && a[j] == b[i + j]; ++j);
18   }
19 }
```

### 3.3  后缀自动机

```
1  struct node { int len; node *fa, *go[26]; } base[MAXNODE], *top = base, *root, *que[MAXNODE];
2  typedef node *tree;
3  inline tree newNode(int len) {
4    top->len = len; top->fa = NULL; memset(top->go, 0, sizeof(top->go)); return top++;
5  } inline tree newNode(int len, tree fa, tree *go) {
6    top->len = len; top->fa = fa; memcpy(top->go, go, sizeof(top->go)); return top++;
7  } void construct(char *A, int N) {
8    tree p = root = newNode(0), q, up, fa;
9    for (int i = 0; i < N; ++i) {
10     int w = A[i] - 'a'; up = p; p = newNode(i + 1);
11     for ( ; up && !up->go[w]; up = up->fa) up->go[w] = p;
12     if (!up) p->fa = root;
13     else { q = up->go[w];
```

```
14       if (up->len + 1 == q->len) p->fa = q;
15       else { fa = newNode(up->len + 1, q->fa, q->go);
16         for (p->fa = q->fa = fa; up && up->go[w] == q; up = up->fa) up->go[w] = fa;
17       }
18     }
19   } static int cnt[MAXLEN]; memset(cnt, 0, sizeof(int) * (N + 1));
20   for (tree i(base); i != top; ++i) ++cnt[i->len];
21   Rep(i, 1, N) cnt[i] += cnt[i - 1];
22   for (tree i(base); i != top; ++i) Q[ cnt[i->len]-- ] = i;
23 }
```

### 3.4  后缀数组

待排序的字符串放在 $r[0\ldots n-1]$ 中，最大值小于 $m$.
$r[0\ldots n-2] > 0, r[n-1] = 0$.
结果放在 $sa[0\ldots n-1]$.

```
1  namespace SuffixArrayDoubling {
2    int wa[MAXN], wb[MAXN], wv[MAXN], ws[MAXN];
3    int cmp(int *r, int a, int b, int l) {
4      return r[a] == r[b] && r[a + l] == r[b + l];
5    }
6    void da(int *r, int *sa, int n, int m) {
7      int i, j, p, *x = wa, *y = wb, *t;
8      for (i = 0; i < m; i++) ws[i] = 0;
9      for (i = 0; i < n; i++) ws[x[i] = r[i]]++;
10     for (i = 1; i < m; i++) ws[i] += ws[i - 1];
11     for (i = n - 1; i >= 0; i--) sa[--ws[x[i]]] = i;
12     for (j = 1, p = 1; p < n; j *= 2, m = p) {
13       for (p = 0, i = n - j; i < n; i++) y[p++] = i;
14       for (i = 0; i < n; i++) if (sa[i] >= j) y[p++] = sa[i] - j;
15       for (i = 0; i < n; i++) wv[i] = x[y[i]];
16       for (i = 0; i < m; i++) ws[i] = 0;
17       for (i = 0; i < n; i++) ws[wv[i]]++;
18       for (i = 1; i < m; i++) ws[i] += ws[i - 1];
19       for (i = n - 1; i >= 0; i--) sa[--ws[wv[i]]] = y[i];
20       for (t = x, x = y, y = t, p = 1, x[sa[0]] = 0, i = 1; i < n; i++)
21         x[sa[i]] = cmp(y, sa[i - 1], sa[i], j) ? p - 1 : p++;
22     }
23   }
24 }
25 namespace SuffixArrayDC3 { // r 与 sa 大小需 3 倍
26   #define F(x) ((x) / 3 + ((x) % 3 == 1 ? 0 : tb))
27   #define G(x) ((x) < tb ? (x) * 3 + 1 : ((x) - tb) * 3 + 2)
28   int wa[MAXN], wb[MAXN], wv[MAXN], ws[MAXN];
29   int c0(int *r, int a, int b) {
30     return r[a] == r[b] && r[a + 1] == r[b + 1] && r[a + 2] == r[b + 2];
31   }
32   int c12(int k, int *r, int a, int b) {
33     if (k == 2) return r[a] < r[b] || (r[a] == r[b] && c12(1, r, a + 1, b + 1));
34     else       return r[a] < r[b] || (r[a] == r[b] && wv[a + 1] < wv[b + 1]);
35   }
36   void sort(int *r, int *a, int *b, int n, int m) {
37     for (int i = 0; i < n; i++) wv[i] = r[a[i]];
38     for (int i = 0; i < m; i++) ws[i] = 0;
39     for (int i = 0; i < n; i++) ws[wv[i]]++;
40     for (int i = 1; i < m; i++) ws[i] += ws[i - 1];
41     for (int i = n - 1; i >= 0; i--) b[--ws[wv[i]]] = a[i];
42   }
43   void dc3(int *r, int *sa, int n, int m) {
44     int i, j, *rn = r + n, *san = sa + n, ta = 0, tb = (n + 1) / 3, tbc = 0, p;
45     r[n] = r[n + 1] = 0;
46     for (i = 0; i < n; i++) if (i % 3 != 0) wa[tbc++] = i;
47     sort(r + 2, wa, wb, tbc, m);
48     sort(r + 1, wb, wa, tbc, m);
49     sort(r, wa, wb, tbc, m);
50     for (p = 1, rn[F(wb[0])] = 0, i = 1; i < tbc; i++)
51       rn[F(wb[i])] = c0(r, wb[i - 1], wb[i]) ? p - 1 : p++;
52     if (p < tbc) dc3(rn, san, tbc, p);
53     else for (i = 0; i < tbc; i++) san[rn[i]] = i;
54     for (i = 0; i < tbc; i++) if (san[i] < tb) wb[ta++] = san[i] * 3;
55     if (n % 3 == 1) wb[ta++] = n - 1;
56     sort(r, wb, wa, ta, m);
57     for (i = 0; i < tbc; i++) wv[wb[i] = G(san[i])] = i;
58     for (i = 0, j = 0, p = 0; i < ta && j < tbc; p++)
59       sa[p] = c12(wb[j] % 3, r, wa[i], wb[j]) ? wa[i++] : wb[j++];
60     for ( ; i < ta; p++) sa[p] = wa[i++];
61     for ( ; j < tbc; p++) sa[p] = wb[j++];
62   }
63   #undef F
64   #undef G
```

```
65  }
66  namespace CalcHeight {
67    int rank[MAXN], height[MAXN];
68    void calheight(int *r, int *sa, int n) {
69      int i, j, k = 0;
70      for (i = 1; i <= n; i++) rank[sa[i]] = i;
71      for (i = 0; i < n; height[rank[i++]] = k)
72        for (k ? k-- : 0, j = sa[rank[i] - 1]; r[i + k] == r[j + k]; k++);
73    }
74  }
```

## 3.5 环串最小表示

```
1  int minimalRepresentation(int N, char *s) { // s must be double-sized and 0-based
2    int i, j, k, l; for (i = 0; i < N; ++i) s[i + N] = s[i]; s[N + N] = 0;
3    for (i = 0, j = 1; j < N; ) {
4      for (k = 0; k < N && s[i + k] == s[j + k]; ++k);
5      if (k >= N) break; if (s[i + k] < s[j + k]) j += k + 1;
6      else l = i + k, i = j, j = max(l, j) + 1;
7    } return i; // [i, i + N) is the minimal representation
8  }
```

# 4 图论

## 4.1 带花树

```
1   namespace Blossom {
2     int n, head, tail, S, T, lca;
3     int match[MAXN], Q[MAXN], pred[MAXN], label[MAXN], inq[MAXN], inb[MAXN];
4     vector<int> link[MAXN];
5     inline void push(int x) { Q[tail++] = x; inq[x] = true; }
6     int findCommonAncestor(int x, int y) {
7       static bool inPath[MAXN]; for (int i = 0; i < n; ++i) inPath[i] = 0;
8       for ( ; ; x = pred[ match[x] ]) { x = label[x]; inPath[x] = true; if (x == S) break; }
9       for ( ; ; y = pred[ match[y] ]) { y = label[y]; if (inPath[y]) break; } return y;
10    }
11    void resetTrace(int x, int lca) {
12      while (label[x] != lca) { int y = match[x]; inb[ label[x] ] = inb[ label[y] ] = true;
13        x = pred[y]; if (label[x] != lca) pred[x] = y; }}
14    void blossomContract(int x, int y) {
15      lca = findCommonAncestor(x, y);
16      Foru(i, 0, n) inb[i] = 0; resetTrace(x, lca); resetTrace(y, lca);
17      if (label[x] != lca) pred[x] = y; if (label[y] != lca) pred[y] = x;
18      Foru(i, 0, n) if (inb[ label[i] ]) { label[i] = lca; if (!inq[i]) push(i); }
19    }
20    bool findAugmentingPath() {
21      Foru(i, 0, n) pred[i] = -1, label[i] = i, inq[i] = 0;
22      int x, y, z; head = tail = 0;
23      for (push(S); head < tail; ) for (int i = (int)link[x = Q[head++]].size() - 1; i >= 0; --i) {
24        y = link[x][i]; if (label[x] == label[y] || x == match[y]) continue;
25        if (y == S || (match[y] >= 0 && pred[ match[y] ] >= 0)) blossomContract(x, y);
26        else if (pred[y] == -1) {
27          pred[y] = x; if (match[y] >= 0) push(match[y]);
28          else {
29            for (x = y; x >= 0; x = z) {
30              y = pred[x], z = match[y]; match[x] = y, match[y] = x;
31            } return true; }}} return false;
32    }
33    int findMaxMatching() {
34      int ans = 0; Foru(i, 0, n) match[i] = -1;
35      for (S = 0; S < n; ++S) if (match[S] == -1) if (findAugmentingPath()) ++ans;
36      return ans;
37    }
38  }
```

## 4.2 最大流

```
1  namespace Maxflow {
2    int h[MAXNODE], vh[MAXNODE], S, T, Ncnt; edge cur[MAXNODE], pe[MAXNODE];
3    void init(int _S, int _T, int _Ncnt) { S = _S; T = _T; Ncnt = _Ncnt; }
4    int maxflow() {
```

```
5      static int Q[MAXNODE]; int x, y, augc, flow = 0, head = 0, tail = 0; edge e;
6      Rep(i, 0, Ncnt) cur[i] = fir[i]; Rep(i, 0, Ncnt) h[i] = INF; Rep(i, 0, Ncnt) vh[i] = 0;
7      for (Q[++tail] = T, h[T] = 0; head < tail; ) {
8        x = Q[++head]; ++vh[ h[x] ];
9        for (e = fir[x]; e; e = e->next) if (e->op->c)
10         if (h[y = e->to] >= INF) h[y] = h[x] + 1, Q[++tail] = y;
11     } for (x = S; h[S] < Ncnt; ) {
12       for (e = cur[x]; e; e = e->next) if (e->c)
13         if (h[y = e->to] + 1 == h[x]) { cur[x] = e; pred[y] = e; x = y; break; }
14       if (!e) {
15         if (--vh[ h[x] ] == 0) break; h[x] = Ncnt; cur[x] = NULL;
16         for (e = fir[x]; e; e = e->next) if (e->c)
17           if ( cMin( h[x], h[e->to] + 1  ) ) cur[x] = e;
18         ++vh[ h[x] ];
19         if (x != S) x = pe[x]->op->to;
20       } else if (x == T) { augc = INF;
21         for (x = T; x != S; x = pe[x]->op->to) cMin(augc, pe[x]->c);
22         for (x = T; x != S; x = pe[x]->op->to)
23           pe[x]->c -= augc; pe[x]->op->c += augc;
24       } flow += augc;
25     }
26   } return flow;
27 }
28 }
```

## 4.3 最高标号预流推进

```
1   namespace Network {
2     int S, T, Ncnt, hsize, heap[MAXN], h[MAXN], inq[MAXN], Q[MAXN], vh[MAXN * 2 + 1];
3     LL E[MAXN]; edge cur[MAXN];
4     inline void pushFlow(int x, int y, edge e) {
5       int d = (int)min(E[x], (LL)e->c);
6       E[x] -= d; e->c -= d; E[y] += d; e->op->c += d;
7     } inline bool heapCmp(int x, int y) { return h[x] < h[y]; }
8     inline void hpush(int x) {
9       inq[x] = true; heap[++hsize] = x; push_heap(heap + 1, heap + hsize + 1, heapCmp);
10    } inline void hpop(int x) {
11      inq[x] = false; pop_heap(heap + 1, heap + hsize + 1, heapCmp); --hsize;
12    } LL maxFlow() {
13      int head = 0, tail = 0, x, y, h0;
14      memset(h, 63, sizeof(int) * (Ncnt + 1));
15      memset(vh, 0, sizeof(int) * (2 * Ncnt + 2));
16      memset(E, 0, sizeof(LL) * (Ncnt + 1));
17      memset(inq, 0, sizeof(int) * (Ncnt + 1));
18      memcpy(cur, fir, sizeof(edge) * (Ncnt + 1));
19      for (Q[++tail] = T, h[T] = 0; head < tail; )
20        for (edge e(fir[x = Q[++head]]); e; e = e->next) if (e->op->c)
21          if (h[y = e->to] >= INF) h[y] = h[x] + 1, Q[++tail] = y;
22      if (h[S] >= Ncnt) return 0;
23      h[S] = Ncnt; E[S] = LL_INF;
24      for (int i = 1; i <= Ncnt; ++i) if (h[i] <= Ncnt) ++vh[ h[i] ];
25      hsize = 0;
26      for (edge e(fir[S]); e; e = e->next) if (e->c && h[y = e->to] < Ncnt) {
27        pushFlow(S, y, e); if (!inq[y] && y != S && y != T) hpush(y);
28      } while (hsize) {
29        bool good = false;
30        for (edge &e(cur[x = heap[1]]); e; e = e->next) if (e->c)
31          if (h[x] == h[y = e->to] + 1) {
32            good = true; pushFlow(x, y, e); if (E[x] == 0) hpop(x);
33            if (inq[y] == false && y != S && y != T) hpush(y);
34            break;
35          }
36        if (!good) { // relabel
37          hpop(x); --vh[ h0 = h[x] ];
38          int &minH = h[x] = INF; cur[x] = NULL;
39          for (edge e(fir[x]); e; e = e->next) if (e->c)
40            if ( cMin(minH, h[e->to] + 1) ) cur[x] = fir[x];
41          hpush(x); ++vh[ h[x] ];
42          if (vh[h0] == 0 && h0 < Ncnt) {
43            hsize = 0;
44            for (int i = 1; i <= Ncnt; ++i) {
45              if (h[i] > h0 && h[i] < Ncnt) --vh[ h[i] ], ++vh[ h[i] = Ncnt + 1 ];
46              if (i != S && i != T && E[i]) heap[++hsize] = i;
47            } make_heap(heap + 1, heap + hsize + 1, heapCmp);
48          }
49        }
50      } return E[T];
51    }
52  }
```

## 4.4 KM

```
1  int N, Tcnt, w[MAXN][MAXN], slack[MAXN];
2  int lx[MAXN], linkx[MAXN], visy[MAXN], ly[MAXN], linky[MAXN], visx[MAXN]; // 初值全为 0
3  bool DFS(int x) { visx[x] = Tcnt;
4    Rep(y, 1, N) if (visy[y] != Tcnt) { int t = lx[x] + ly[y] - w[x][y];
5      if (t == 0) { visy[y] = Tcnt;
6        if (!linky[y] || DFS(linky[y])) { linkx[x] = y; linky[y] = x; return true; }
7      } else cMin(slack[y], t);
8    } return false;
9  } void KM() {
10   Tcnt = 0; Rep(x, 1, N) Rep(y, 1, N) cMax(lx[x], w[x][y]);
11   Rep(S, 1, N) { Rep(i, 1, N) slack[i] = INF;
12     for (++Tcnt; !DFS(S); ++Tcnt) { int d = INF;
13       Rep(y, 1, N) if (visy[y] != Tcnt) cMin(d, slack[y]);
14       Rep(x, 1, N) if (visx[x] == Tcnt) lx[x] -= d;
15       Rep(y, 1, N) if (visy[y] == Tcnt) ly[y] += d; else slack[y] -= d;
16     }
17   }
18 }
```

## 4.5 2-SAT 与 Kosaraju

注意 Kosaraju 需要建反图

```
1  namespace SCC {
2    int code[MAXN * 2], seq[MAXN * 2], sCnt;
3    void DFS_1(int x) { code[x] = 1;
4      for (edge e(fir[x]); e; e = e->next) if (code[e->to] == -1) DFS_1(e->to);
5      seq[++sCnt] = x;
6    } void DFS_2(int x) { code[x] = sCnt;
7      for (edge e(fir2[x]); e; e = e->next) if (code[e->to] == -1) DFS_2(e->to); }
8    void SCC(int N) {
9      sCnt = 0; for (int i = 1; i <= N; ++i) code[i] = -1;
10     for (int i = 1; i <= N; ++i) if (code[i] == -1) DFS_1(i);
11     sCnt = 0; for (int i = 1; i <= N; ++i) code[i] = -1;
12     for (int i = N; i >= 1; --i) if (code[seq[i]] == -1) {
13       ++sCnt; DFS_2(seq[i]); }
14   }
15 }// true  - 2i - 1
16 //  false  - 2i
17 bool TwoSat() { SCC::SCC(N + N);
18   // if code[2i - 1] = code[2i]: no solution
19   // if code[2i - 1] > code[2i]: i selected. else i not selected
20 }
```

## 4.6 全局最小割 Stoer-Wagner

```
1  int minCut(int N, int G[MAXN][MAXN]) { // 0-based
2    static int weight[MAXN], used[MAXN]; int ans = INT_MAX;
3    while (N > 1) {
4      for (int i = 0; i < N; ++i) used[i] = false; used[0] = true;
5      for (int i = 0; i < N; ++i) weight[i] = G[i][0];
6      int S = -1, T = 0;
7      for (int _r = 2; _r <= N; ++_r) { // N - 1 selections
8        int x = -1;
9        for (int i = 0; i < N; ++i) if (!used[i])
10         if (x == -1 || weight[i] > weight[x]) x = i;
11       for (int i = 0; i < N; ++i) weight[i] += G[x][i];
12       S = T; T = x; used[x] = true;
13     } ans = min(ans, weight[T]);
14     for (int i = 0; i < N; ++i) G[i][S] += G[i][T], G[S][i] += G[i][T];
15     G[S][S] = 0; --N;
16     for (int i = 0; i <= N; ++i) swap(G[i][T], G[i][N]);
17     for (int i = 0; i < N; ++i) swap(G[T][i], G[N][i]);
18   } return ans;
19 }
```

## 4.7 Hopcroft-Karp

```
1  int N, M, level[MAXN], matchX[MAXN], matchY[MAXN];
2  bool used[MAXN];
3  bool DFS(int x) {
4    used[x] = true; for (edge e(fir[x]); e; e = e->next) {
5      int y = e->to, z = matchY[y];
6      if (z == -1 || (!used[z] && level[x] < level[z] && DFS(z))) {
7        matchX[x] = y; matchY[y] = x; return true;
8      }
9    } return false;
10 }
11 int maxMatch() {
12   for (int i = 0; i < N; ++i) used[i] = false;
13   for (int i = 0; i < N; ++i) matchX[i] = -1;
14   for (int i = 0; i < N; ++i) matchY[i] = -1;
15   for (int i = 0; i < N; ++i) level[i] = -1;
16   int match = 0, d;
17   for ( ; ; match += d) {
18     static int Q[MAXN * 2 + 1];
19     int head = 0, tail = d = 0;
20     for (int x = 0; x < N; ++x) level[x] = -1;
21     for (int x = 0; x < N; ++x) if (matchX[x] == -1)
22       level[x] = 0, Q[++tail] = x;
23     while (head < tail)
24       for (edge e(fir[x = Q[++head]]); e; e = e->next) {
25         int y = e->to, z = matchY[y];
26         if (z != -1 && level[z] < 0) level[z] = level[x] + 1, Q[++tail] = z;
27       }
28     for (int x = 0; x < N; ++x) used[x] = false;
29     for (int x = 0; x < N; ++x) if (matchX[x] == -1) if (DFS(x)) ++d;
30     if (d == 0) break;
31   } return match;
32 }
```

## 4.8 欧拉路

```
1  vector<int> eulerianWalk(int N, int S) {
2    static int res[MAXM], stack[MAXN]; static edge cur[MAXN];
3    int rcnt = 0, top = 0, x; for (int i = 1; i <= N; ++i) cur[i] = fir[i];
4    for (stack[top++] = S; top; ) {
5      for (x = stack[--top]; ; ) {
6        edge &e = cur[x]; if (e == NULL) break;
7        stack[top++] = x; x = e->to; e = e->next;
8        // 对于无向图需要删掉反向边
9      } res[rcnt++] = x;
10   } reverse(res, res + rcnt); return vector<int>(res, res + rcnt);
11 }
```

## 4.9 稳定婚姻

```
1  namespace StableMatching {
2    int pairM[MAXN], pairW[MAXN], p[MAXN];
3    // init: pairM[0...n - 1] = pairW[0...n - 1] = -1, p[0...n - 1] = 0
4    void stableMatching(int n, int orderM[MAXN][MAXN], int preferW[MAXN][MAXN]) {
5      for (int i = 0; i < n; i++) while (pairM[i] < 0) {
6        int w = orderM[i][p[i]++], m = pairW[w];
7        if (m == -1) pairM[i] = w, pairW[w] = i;
8        else if (preferW[w][i] < preferW[w][m])
9          pairM[m] = -1, pairM[i] = w, pairW[w] = i, i = m;
10     }
11   }
12 }
```

## 4.10 最大团搜索

```
1  namespace MaxClique { // 1-based
2    int g[MAXN][MAXN], len[MAXN], list[MAXN][MAXN], mc[MAXN], ans, found;
3    void DFS(int size) {
4      if (len[size] == 0) { if (size > ans) ans = size, found = true; return; }
5      for (int k = 0; k < len[size] && !found; ++k) {
6        if (size + len[size] - k <= ans) break;
7        int i = list[size][k]; if (size + mc[i] <= ans) break;
8        for (int j = k + 1, len[size + 1] = 0; j < len[size]; ++j) if (g[i][list[size][j]])
```

```
9          list[size + 1][len[size + 1]++] = list[size][j];
10       DFS(size + 1);
11     }
12   }
13   int work(int n) {
14     mc[n] = ans = 1; for (int i = n - 1; i; --i) { found = false; len[1] = 0;
15       for (int j = i + 1; j <= n; ++j) if (g[i][j]) list[1][len[1]++] = j;
16       DFS(1); mc[i] = ans;
17     } return ans;
18   }
19 }
```

## 4.11  最小树形图

```
1  namespace EdmondsAlgorithm { // O(ElogE + V^2) !!! 0-based !!!
2    struct enode { int from, c, key, delta, dep; enode *ch[2], *next;
3    } ebase[maxm], *etop, *fir[maxn], nil, *null, *inEdge[maxn], *chs[maxn];
4    typedef enode *edge; typedef enode *tree;
5    int n, m, setFa[maxn], deg[maxn], que[maxn];
6    inline void pushDown(tree x) { if (x->delta) {
7      x->ch[0]->key += x->delta; x->ch[0]->delta += x->delta;
8      x->ch[1]->key += x->delta; x->ch[1]->delta += x->delta; x->delta = 0;
9    }}
10   tree merge(tree x, tree y) {
11     if (x == null) return y; if (y == null) return x;
12     if (x->key > y->key) swap(x, y); pushDown(x); x->ch[1] = merge(x->ch[1], y);
13     if (x->ch[0]->dep < x->ch[1]->dep) swap(x->ch[0], x->ch[1]);
14     x->dep = x->ch[1]->dep + 1; return x;
15   }
16   void addEdge(int u, int v, int w) {
17     etop->from = u; etop->c = etop->key = w; etop->delta = etop->dep = 0;
18     etop->next = fir[v]; etop->ch[0] = etop->ch[1] = null;
19     fir[v] = etop; inEdge[v] = merge(inEdge[v], etop++);
20   }
21   void deleteMin(tree &r) { pushDown(r); r = merge(r->ch[0], r->ch[1]); }
22   int findSet(int x) { return setFa[x] == x ? x : setFa[x] = findSet(setFa[x]); }
23   void clear(int V, int E) {
24     null = &nil; null->ch[0] = null->ch[1] = null; null->dep = -1;
25     n = V; m = E; etop = ebase; Foru(i, 0, V) fir[i] = NULL; Foru(i, 0, V) inEdge[i] = null;
26   }
27   int solve(int root) { int res = 0, head, tail;
28     for (int i = 0; i < n; ++i) setFa[i] = i;
29     for ( ; ; ) { memset(deg, 0, sizeof(int) * n); chs[root] = inEdge[root];
30       for (int i = 0; i < n; ++i) if (i != root && setFa[i] == i) {
31         while (findSet(inEdge[i]->from) == findSet(i)) deleteMin(inEdge[i]);
32         ++deg[ findSet((chs[i] = inEdge[i])->from) ];
33       }
34       for (int i = head = tail = 0; i < n; ++i)
35         if (i != root && setFa[i] == i && deg[i] == 0) que[tail++] = i;
36       while (head < tail) {
37         int x = findSet(chs[que[head++]]->from);
38         if (--deg[x] == 0) que[tail++] = x;
39       } bool found = false;
40       for (int i = 0; i < n; ++i) if (i != root && setFa[i] == i && deg[i] > 0) {
41         int j = i; tree temp = null; found = true;
42         do {setFa[j = findSet(chs[j]->from)] = i;
43           deleteMin(inEdge[j]); res += chs[j]->key;
44           inEdge[j]->key -= chs[j]->key; inEdge[j]->delta -= chs[j]->key;
45           temp = merge(temp, inEdge[j]);
46         } while (j != i); inEdge[i] = temp;
47       } if (!found) break;
48     } for (int i = 0; i < n; ++ i) if (i != root && setFa[i] == i) res += chs[i]->key;
49     return res;
50   }
51 }
52 namespace ChuLiu { // O(V ^ 3) !!! 1-based !!!
53   int n, used[maxn], pass[maxn], eg[maxn], more, que[maxn], g[maxn][maxn];
54   void combine(int id, int &sum) { int tot = 0, from, i, j, k;
55     for ( ; id != 0 && !pass[id]; id = eg[id]) que[tot++] = id, pass[id] = 1;
56     for (from = 0; from < tot && que[from] != id; from++);
57     if (from == tot) return; more = 1;
58     for (i = from; i < tot; i++) {
59       sum += g[eg[que[i]]][que[i]]; if (i == from) continue;
60       for (j = used[que[i]] = 1; j <= n; j++) if (!used[j])
61         if (g[que[i]][j] < g[id][j]) g[id][j] = g[que[i]][j];
62     }
63     for (i = 1; i <= n; i++) if (!used[i] && i != id)
64       for (j = from; j < tot; j++) {
65         k = que[j]; if (g[i][id] > g[i][k] - g[eg[k]][k])
66         g[i][id] = g[i][k] - g[eg[k]][k];
67       }
68   }
```

```
69   void clear(int V) { n = V; Rep(i, 1, V) Rep(j, 1, V) g[i][j] = inf; }
70   int solve(int root) {
71     int i, j, k, sum = 0; memset(used, 0, sizeof(int) * (n + 1));
72     for (more = 1; more; ) {
73       more = 0; memset(eg, 0, sizeof(int) * (n + 1));
74       for (i = 1; i <= n; i++) if (!used[i] && i != root) {
75         for (j = 1, k = 0; j <= n; j++) if (!used[j] && i != j)
76           if (k == 0 || g[j][i] < g[k][i]) k = j;
77         eg[i] = k;
78       } memset(pass, 0, sizeof(int) * (n + 1));
79       for (i = 1; i <= n; i++) if (!used[i] && !pass[i] && i != root)
80         combine(i, sum);
81     } for (i = 1; i <= n; i++) if (!used[i] && i != root) sum += g[eg[i]][i];
82     return sum;
83   }
84 }
```

## 4.12  离线动态最小生成树

$O(Qlog^2Q)$. $(qx[i], qy[i])$ 表示将编号为 $qx[i]$ 的边的权值改为 $qy[i]$, 删除一条边相当于将其权值改为 $\infty$, 加入一条边相当于将其权值从 $\infty$ 变成某个值.

```
1  const int maxn = 100000 + 5;
2  const int maxm = 1000000 + 5;
3  const int maxq = 1000000 + 5;
4  const int qsize = maxm + 3 * maxq;
5  int n, m, Q, x[qsize], y[qsize], z[qsize], qx[maxq], qy[maxq], a[maxn], *tz;
6  int kx[maxn], ky[maxn], kt, vd[maxn], id[maxm], app[maxm];
7  bool extra[maxm];
8  void init() {
9    scanf("%d%d", &n, &m); for (int i = 0; i < m; i++) scanf("%d%d%d", x + i, y + i, z + i);
10   scanf("%d", &Q); for (int i = 0; i < Q; i++) { scanf("%d%d", qx + i, qy + i); qx[i]--; }
11 }
12 int find(int x) {
13   int root = x, next; while (a[root]) root = a[root];
14   while ((next = a[x]) != 0) a[x] = root, x = next; return root;
15 }
16 inline bool cmp(const int &a, const int &b) { return tz[a] < tz[b]; }
17 void solve(int *qx, int *qy, int Q, int n, int *x, int *y, int *z, int m, long long ans) {
18   int ri, rj;
19   if (Q == 1) {
20     for (int i = 1; i <= n; i++) a[i] = 0; z[qx[0]] = qy[0];
21     for (int i = 0; i < m; i++) id[i] = i;
22     tz = z; sort(id, id + m, cmp);
23     for (int i = 0; i < m; i++) {
24       ri = find(x[id[i]]); rj = find(y[id[i]]);
25       if (ri != rj) ans += z[id[i]], a[ri] = rj;
26     } printf("%I64d\n", ans);
27     return;
28   } int tm = kt = 0, n2 = 0, m2 = 0;
29   for (int i = 1; i <= n; i++) a[i] = 0;
30   for (int i = 0; i < Q; i++) {
31     ri = find(x[qx[i]]); rj = find(y[qx[i]]); if (ri != rj) a[ri] = rj;
32   }
33   for (int i = 0; i < m; i++) extra[i] = true;
34   for (int i = 0; i < Q; i++) extra[qx[i]] = false;
35   for (int i = 0; i < m; i++) if (extra[i]) id[tm++] = i;
36   tz = z; sort(id, id + tm, cmp);
37   for (int i = 0; i < tm; i++) {
38     ri = find(x[id[i]]); rj = find(y[id[i]]);
39     if (ri != rj)
40       a[ri] = rj, ans += z[id[i]], kx[kt] = x[id[i]], ky[kt] = y[id[i]], kt++;
41   }
42   for (int i = 1; i <= n; i++) a[i] = 0;
43   for (int i = 0; i < kt; i++) a[find(kx[i])] = find(ky[i]);
44   for (int i = 1; i <= n; i++) if (a[i] == 0) vd[i] = ++n2;
45   for (int i = 1; i <= n; i++) if (a[i] != 0) vd[i] = vd[find(i)];
46   int *Nx = x + m, *Ny = y + m, *Nz = z + m;
47   for (int i = 0; i < m; i++) app[i] = -1;
48   for (int i = 0; i < Q; i++)
49     if (app[qx[i]] == -1)
50       Nx[m2] = vd[x[qx[i]]], Ny[m2] = vd[y[qx[i]]], Nz[m2] = z[qx[i]], app[qx[i]] = m2, m2++;
51   for (int i = 0; i < Q; i++) {
52     z[qx[i]] = qy[i];
53     qx[i] = app[qx[i]];
54   }
55   for (int i = 1; i <= n2; i++) a[i] = 0;
56   for (int i = 0; i < tm; i++) {
57     ri = find(vd[x[id[i]]]); rj = find(vd[y[id[i]]]);
58     if (ri != rj)
59       a[ri] = rj, Nx[m2] = vd[x[id[i]]], Ny[m2] = vd[y[id[i]]], Nz[m2] = z[id[i]], m2++;
60   }
```

```
61      int mid = Q / 2;
62      solve(qx, qy, mid, n2, Nx, Ny, Nz, m2, ans);
63      solve(qx + mid, qy + mid, Q - mid, n2, Nx, Ny, Nz, m2, ans);
64  }
65  void work() { if (Q) solve(qx, qy, Q, n, x, y, z, m, 0); }
66  int main() { init(); work(); return 0; }
```

## 4.13 弦图

- 任何一个弦图都至少有一个单纯点, 不是完全图的弦图至少有两个不相邻的单纯点.

- 设第 i 个点在弦图的完美消除序列第 $p(i)$ 个. 令 $N(v) = \{w|w$ 与 $v$ 相邻且 $p(w) > p(v)\}$ 弦图的极大团一定是 $v \cup N(v)$ 的形式.

- 弦图最多有 $n$ 个极大团.

- 设 $next(v)$ 表示 $N(v)$ 中最前的点. 令 $w*$ 表示所有满足 $A \in B$ 的 $w$ 中最后的一个点. 判断 $v \cup N(v)$ 是否为极大团, 只需判断是否存在一个 $w$, 满足 $Next(w) = v$ 且 $|N(v)| + 1 \leq |N(w)|$ 即可.

- 最小染色: 完美消除序列从后往前依次给每个点染色, 给每个点染上可以染的最小的颜色. (团数 = 色数)

- 最大独立集: 完美消除序列从前往后能选就选.

- 最小团覆盖: 设最大独立集为 $\{p_1, p_2, \ldots, p_t\}$, 则 $\{p_1 \cup N(p_1), \ldots, p_t \cup N(p_t)\}$ 为最小团覆盖. (最大独立集数 = 最小团覆盖数)

```
1   class Chordal { // 1-Based, G is the Graph, must be sorted before call Check_Chordal
2   public: // Construct will sort it automatically
3       int v[Maxn], id[Maxn];  bool inseq[Maxn]; priority_queue<pair<int, int> > pq;
4       vector<int> Construct_Perfect_Elimination_Sequence(vector<int> *G, int n) { // O(m + nlogn)
5           vector<int> seq(n + 1, 0);
6           for (int i = 0; i <= n; ++i) inseq[i] = false, sort(G[i].begin(), G[i].end()), v[i] = 0;
7           int cur = n; pair<int, int> Mx; while(!pq.empty()) pq.pop(); pq.push(make_pair(0, 1));
8           for (int i = n; i >= 1; --i) {
9               while (!pq.empty() && (Mx = pq.top(), inseq[Mx.second] || Mx.first != v[Mx.second])) pq.pop();
10              id[Mx.second] = cur;
11              int x = seq[cur--] = Mx.second, sz = (int)G[Mx.second].size(); inseq[x] = true;
12              for (int j = 0; j < sz; ++j) {
13                  int y = G[x][j]; if(!inseq[y]) pq.push(make_pair(++v[y], y));
14              }
15          } return seq;
16      }
17      bool Check_Chordal(vector<int> *G, vector<int> &seq, int n) { // O(n + mlogn), plz gen seq first
18          bool isChordal = true;
19          for (int i = n - 1; i >= 1 && isChordal; --i) {
20              int x = seq[i], sz, y = -1;
21              if ((sz = (int)G[x].size()) == 0) continue;
22                  for(int j = 0; j < sz; ++j) {
23                      if (id[G[x][j]] < i) continue;
24                      if (y == -1 || id[y] > id[G[x][j]]) y = G[x][j];
25                  } if (y == -1) continue;
26                  for (int j = 0; j < sz; ++j) {
27                      int y1 = G[x][j]; if (id[y1] < i) continue;
28                      if (y1 == y || binary_search(G[y].begin(), G[y].end(), y1)) continue;
29                      isChordal = false; break;
30                  }
31          } return isChordal;
32      }
33  };
```

## 4.14 K 短路 (允许重复)

```
1   #define for_each(it, v) for (vector<Edge*>::iterator it = (v).begin(); it != (v).end(); ++it)
2   const int MAX_N = 10000, MAX_M = 50000, MAX_K = 10000, INF = 1000000000;
3   struct Edge { int from, to, weight; };
4   struct HeapNode { Edge* edge; int depth; HeapNode* child[4]; }; // child[0..1] for heap G, child[2..3]
        for heap out edge
5
6   int n, m, k, s, t; Edge* edge[MAX_M];
7   int dist[MAX_N]; Edge* prev[MAX_N];
```

```
8   vector<Edge*> graph[MAX_N]; vector<Edge*> graphR[MAX_N];
9   HeapNode* nullNode; HeapNode* heapTop[MAX_N];
10
11  HeapNode* createHeap(HeapNode* curNode, HeapNode* newNode) {
12      if (curNode == nullNode) return newNode; HeapNode* rootNode = new HeapNode;
13      memcpy(rootNode, curNode, sizeof(rootNode));
14      if (newNode->edge->weight < curNode->edge->weight) {
15          rootNode->edge = newNode->edge; rootNode->child[2] = newNode->child[2]; rootNode->child[3] = newNode
                ->child[3];
16          newNode->edge = curNode->edge; newNode->child[2] = curNode->child[2]; newNode->child[3] = curNode->
                child[3];
17      } if (rootNode->child[0]->depth < rootNode->child[1]->depth) rootNode->child[0] = createHeap(rootNode->
            child[0], newNode);
18      else rootNode->child[1] = createHeap(rootNode->child[1], newNode);
19      rootNode->depth = max(rootNode->child[0]->depth, rootNode->child[1]->depth) + 1;
20      return rootNode;
21  }
22  bool heapNodeMoreThan(HeapNode* node1, HeapNode* node2) { return node1->edge->weight > node2->edge->
        weight; }
23
24  int main() {
25      scanf("%d%d%d", &n, &m, &k); scanf("%d%d", &s, &t); s--, t--;
26      while (m--) { Edge* newEdge = new Edge;
27          int i, j, w; scanf("%d%d%d", &i, &j, &w);
28          i--, j--; newEdge->from = i; newEdge->to = j; newEdge->weight = w;
29          graph[i].push_back(newEdge); graphR[j].push_back(newEdge);
30      }
31      //Dijkstra
32      queue<int> dfsOrder; memset(dist, -1, sizeof(dist));
33      typedef pair<int, pair<int, Edge*> > DijkstraQueueItem;
34      priority_queue<DijkstraQueueItem, vector<DijkstraQueueItem>, greater<DijkstraQueueItem> > dq;
35      dq.push(make_pair(0, make_pair(t, (Edge*) NULL)));
36      while (!dq.empty()) {
37          int d = dq.top().first; int i = dq.top().second.first;
38          Edge* edge = dq.top().second.second; dq.pop();
39          if (dist[i] != -1) continue;
40          dist[i] = d; prev[i] = edge; dfsOrder.push(i);
41          for_each(it, graphR[i]) dq.push(make_pair(d + (*it)->weight, make_pair((*it)->from, *it)));
42      }
43      //Create edge heap
44      nullNode = new HeapNode; nullNode->depth = 0; nullNode->edge = new Edge; nullNode->edge->weight = INF;
45      fill(nullNode->child, nullNode->child + 4, nullNode);
46      while (!dfsOrder.empty()) {
47          int i = dfsOrder.front(); dfsOrder.pop();
48          if (prev[i] == NULL) heapTop[i] = nullNode;
49          else heapTop[i] = heapTop[prev[i]->to];
50          vector<HeapNode*> heapNodeList;
51          for_each(it, graph[i]) { int j = (*it)->to; if (dist[j] == -1) continue;
52              (*it)->weight += dist[j] - dist[i]; if (prev[i] != *it) {
53                  HeapNode* curNode = new HeapNode;
54                  fill(curNode->child, curNode->child + 4, nullNode);
55                  curNode->depth = 1; curNode->edge = *it;
56                  heapNodeList.push_back(curNode);
57              }
58          } if (!heapNodeList.empty()) { //Create heap out
59              make_heap(heapNodeList.begin(), heapNodeList.end(), heapNodeMoreThan);
60              int size = heapNodeList.size();
61              for (int p = 0; p < size; p++) {
62                  heapNodeList[p]->child[2] = 2 * p + 1 < size ? heapNodeList[2 * p + 1] : nullNode;
63                  heapNodeList[p]->child[3] = 2 * p + 2 < size ? heapNodeList[2 * p + 2] : nullNode;
64              } heapTop[i] = createHeap(heapTop[i], heapNodeList.front());
65          }
66      } //Walk on DAG
67      typedef pair<long long, HeapNode*> DAGQueueItem;
68      priority_queue<DAGQueueItem, vector<DAGQueueItem>, greater<DAGQueueItem> > aq;
69      if (dist[s] == -1) printf("NO\n");
70      else { printf("%d\n", dist[s]);
71          if (heapTop[s] != nullNode) aq.push(make_pair(dist[s] + heapTop[s]->edge->weight, heapTop[s]));
72      } k--; while (k--) {
73          if (aq.empty()) { printf("NO\n"); continue; }
74          long long d = aq.top().first; HeapNode* curNode = aq.top().second; aq.pop();
75          printf("%I64d\n", d);
76          if (heapTop[curNode->edge->to] != nullNode)
77              aq.push(make_pair(d + heapTop[curNode->edge->to]->edge->weight, heapTop[curNode->edge->to]));
78          for (int i = 0; i < 4; i++) if (curNode->child[i] != nullNode)
79              aq.push(make_pair(d - curNode->edge->weight + curNode->child[i]->edge->weight, curNode->child[i]));
80      } return 0;
81  }
```

## 4.15 K 短路 (不允许重复)

```
1   int Num[10005][205], Path[10005][205], dev[10005], from[10005], value[10005], dist[205], Next[205], Graph
        [205][205];
```

```
2    int N, M, K, s, t, tot, cnt; bool forbid[205], hasNext[10005][205];
3    struct cmp {
4      bool operator()(const int &a, const int &b) {
5        int *i, *j; if (value[a] != value[b]) return value[a] > value[b];
6        for (i = Path[a], j = Path[b]; (*i) == (*j); i++, j++);
7        return (*i) > (*j);
8      }
9    };
10   void Check(int idx, int st, int *path, int &res) {
11     int i, j; for (i = 0; i < N; i++) dist[i] = 1000000000, Next[i] = t;
12     dist[t] = 0; forbid[t] = true; j = t;
13     for ( ; ; ) {
14       for (i = 0; i < N; i++) if (!forbid[i] && (i != st || !hasNext[idx][j]) && (dist[j] + Graph[i][j] <
                dist[i] || (dist[j] + Graph[i][j] == dist[i] && j < Next[i])))
15         Next[i] = j, dist[i] = dist[j] + Graph[i][j];
16       j = -1; for (i = 0; i < N; i++) if (!forbid[i] && (j == -1 || dist[i] < dist[j])) j = i;
17       if (j == -1) break; forbid[j] = 1; if (j == st) break;
18     } res += dist[st]; for (i = st; i != t; i = Next[i], path++) (*path) = i; (*path) = i;
19   }
20   int main() {
21     int i, j, k, l;
22     while (scanf("%d%d%d%d", &N, &M, &K, &s, &t) && N) {
23       priority_queue<int, vector<int>, cmp> Q;
24       for (i = 0; i < N; i++) for (j = 0; j < N; j++) Graph[i][j] = 1000000000;
25       for (i = 0; i < M; i++) { scanf("%d%d%d", &j, &k, &l); Graph[j - 1][k - 1] = l; }
26       s--; t--;
27       memset(forbid, false, sizeof(forbid)); memset(hasNext[0], false, sizeof(hasNext[0]));
28       Check(0, s, Path[0], value[0]); dev[0] = 0; from[0] = 0; Num[0][0] = 0; Q.push(0);
29       cnt = 1; tot = 1;
30       for (i = 0; i < K; i++) {
31         if (Q.empty()) break; l = Q.top(); Q.pop();
32         for (j = 0; j <= dev[l]; j++) Num[l][j] = Num[from[l]][j];
33         for (; Path[l][j] != t; j++) {
34           memset(hasNext[tot], false, sizeof(hasNext[tot])); Num[l][j] = tot++;
35         } for (j = 0; Path[l][j] != t; j++) hasNext[Num[l][j]][Path[l][j + 1]] = true;
36         for (j = dev[l]; Path[l][j] != t; j++) {
37           memset(forbid, false, sizeof(forbid)); value[cnt] = 0;
38           for (k = 0; k < j; k++) {
39             forbid[Path[l][k]] = true;
40             Path[cnt][k] = Path[l][k];
41             value[cnt] += Graph[Path[l][k]][Path[l][k + 1]];
42           } Check(Num[l][j], Path[l][j], &Path[cnt][j], value[cnt]);
43           if (value[cnt] > 2000000) continue;
44           dev[cnt] = j; from[cnt] = l; Q.push(cnt); cnt++;
45         }
46       }
47       if (i < K || value[l] > 2000000) printf("None\n");
48       else {
49         for (i = 0; Path[l][i] != t; i++) printf("%d-", Path[l][i] + 1);
50         printf("%d\n", t + 1);
51       }
52     } return 0;
53   }
```

## 4.16 小知识

- 平面图: 一定存在一个度小于等于 5 的点. $E \leq 3V - 6$. 欧拉公式: $V + F - E = 1 +$ 连通块数

- 图连通度:

  1. $k-$ 连通 ($k$-connected): 对于任意一对结点都至少存在结点各不相同的 $k$ 条路

  2. 点连通度 (vertex connectivity): 把图变成非连通图所需删除的最少点数

  3. Whitney 定理: 一个图是 $k-$ 连通的当且仅当它的点连通度至少为 $k$

- Lindstroem-Gessel-Viennot Lemma: 给定一个图的 $n$ 个起点和 $n$ 个终点, 令 $A_{ij} =$ 第 $i$ 个起点到第 $j$ 个终点的路径条数, 则从起点到终点的不相交路径条数为 $det(A)$

- 欧拉回路与树形图的联系: 对于出度等于入度的连通图 $s(G) = t_i(G) \prod_{j=1}^{n} (d^+(v_j) - 1)!$

- 密度子图: 给定无向图, 选取点集及其导出子图, 最大化 $W_e + P_v$ (点权可负).

  - $(S, u) = U$, $(u, T) = U - 2P_u - D_u$, $(u, v) = (v, u) = W_e$

  - ans $= \frac{Un - C[S,T]}{2}$, 解集为 $S - \{s\}$

- 最大权闭合图: 选 $a$ 则 $a$ 的后继必须被选

  - $P_u > 0$, $(S, u) = P_u$, $P_u < 0$, $(u, T) = -P_u$

  - ans $= \sum_{P_u > 0} P_u - C[S, T]$, 解集为 $S - \{s\}$

- 判定边是否属于最小割:

  - 可能属于最小割: $(u, v)$ 不属于同一 SCC

  - 一定在所有最小割中: $(u, v)$ 不属于同一 SCC, 且 $S, u$ 在同一 SCC, $u, T$ 在同一 SCC

- 图同构 Hash: $F_t(i) = (F_{t-1}(i) \times A + \sum_{i \to j} F_{t-1}(j) \times B + \sum_{j \leftarrow i} F_{t-1}(j) \times C + D \times (i = a)) \pmod P$, 枚举点 $a$, 迭代 $K$ 次后求得的 $F_k(a)$ 就是 $a$ 点所对应的 Hash 值.

# 5 数学

## 5.1 单纯形 Cpp

max $\{cx | Ax \leq b, x \geq 0\}$

```
1    const int MAXN = 11000, MAXM = 1100;
2    // here MAXN is the MAX number of conditions, MAXM is the MAX number of vars
3
4    int avali[MAXM], avacnt;
5    double A[MAXN][MAXM];
6    double b[MAXN], c[MAXM];
7    double* simplex(int n, int m) {
8    // here n is the number of conditions, m is the number of vars
9      m++;
10     int r = n, s = m - 1;
11     static double D[MAXN + 2][MAXM + 1];
12     static int ix[MAXN + MAXM];
13     for (int i = 0; i < n + m; i++) ix[i] = i;
14     for (int i = 0; i < n; i++) {
15       for (int j = 0; j < m - 1; j++) D[i][j] = -A[i][j];
16       D[i][m - 1] = 1;
17       D[i][m] = b[i];
18       if (D[r][m] > D[i][m]) r = i;
19     }
20     for (int j = 0; j < m - 1; j++) D[n][j] = c[j];
21     D[n + 1][m - 1] = -1;
22     for (double d; ; ) {
23       if (r < n) {
24         int t = ix[s]; ix[s] = ix[r + m]; ix[r + m] = t;
25         D[r][s] = 1.0 / D[r][s];
26         for (int j = 0; j <= m; j++) if (j != s) D[r][j] *= -D[r][s];
27         avacnt = 0;
28         for (int i = 0; i <= m; ++i)
29           if(fabs(D[r][i]) > EPS)
30             avali[avacnt++] = i;
31         for (int i = 0; i <= n + 1; i++) if (i != r) {
32           if(fabs(D[i][s]) < EPS) continue;
33           double *cur1 = D[i], *cur2 = D[r], tmp = D[i][s];
34           //for (int j = 0; j <= m; j++) if (j != s) cur1[j] += cur2[j] * tmp;
35           for(int j = 0; j < avacnt; ++j) if(avali[j] != s) cur1[avali[j]] += cur2[avali[j]] * tmp;
36           D[i][s] *= D[r][s];
37         }
38       }
39       r = -1; s = -1;
40       for (int j = 0; j < m; j++) if (s < 0 || ix[s] > ix[j]) {
41         if (D[n + 1][j] > EPS || D[n + 1][j] > -EPS && D[n][j] > EPS) s = j;
42       }
43       if (s < 0) break;
44       for (int i = 0; i < n; i++) if (D[i][s] < -EPS) {
45         if (r < 0 || (d = D[r][m] / D[r][s] - D[i][m] / D[i][s]) < -EPS
46             || d < EPS && ix[r + m] > ix[i + m])
47           r = i;
48       }
49       if (r < 0) return null; // 非有界
50     }
51     if (D[n + 1][m] < -EPS) return null; // 无法执行
52     static double x[MAXM - 1];
53     for (int i = m; i < n + m; i++) if (ix[i] < m - 1) x[ix[i]] = D[i - m][m];
54     return x; // 值为 D[n][m]
55   }
```

## 5.2 单纯形 Java

```
1  double[] simplex(double[][] A, double[] b, double[] c) {
2    int n = A.length, m = A[0].length + 1, r = n, s = m - 1;
3    double[][] D = new double[n + 2][m + 1];
4    int[] ix = new int[n + m];
5    for (int i = 0; i < n + m; i++) ix[i] = i;
6    for (int i = 0; i < n; i++) {
7      for (int j = 0; j < m - 1; j++) D[i][j] = -A[i][j];
8      D[i][m - 1] = 1; D[i][m] = b[i]; if (D[r][m] > D[i][m]) r = i;
9    }
10   for (int j = 0; j < m - 1; j++) D[n][j] = c[j];
11   D[n + 1][m - 1] = -1;
12   for (double d; ; ) {
13     if (r < n) {
14       int t = ix[s]; ix[s] = ix[r + m]; ix[r + m] = t; D[r][s] = 1.0 / D[r][s];
15       for (int j = 0; j <= m; j++) if (j != s) D[r][j] *= -D[r][s];
16       for (int i = 0; i <= n + 1; i++) if (i != r) {
17         for (int j = 0; j <= m; j++) if (j != s) D[i][j] += D[r][j] * D[i][s];
18         D[i][s] *= D[r][s];
19       }
20     } r = -1; s = -1;
21     for (int j = 0; j < m; j++) if (s < 0 || ix[s] > ix[j]) {
22       if (D[n + 1][j] > EPS || D[n + 1][j] > -EPS && D[n][j] > EPS) s = j;
23     }
24     if (s < 0) break;
25     for (int i = 0; i < n; i++) if (D[i][s] < -EPS) {
26       if (r < 0 || (d = D[r][m] / D[r][s] - D[i][m] / D[i][s]) < -EPS
27         || d < EPS && ix[r + m] > ix[i + m])
28         r = i;
29     }
30     if (r < 0) return null; // 非有界
31   } if (D[n + 1][m] < -EPS) return null; // 无法执行
32   double[] x = new double[m - 1];
33   for (int i = m; i < n + m; i++) if (ix[i] < m - 1) x[ix[i]] = D[i - m][m];
34   return x; // 值为 D[n][m]
35 }
```

## 5.3 高斯消元

```
1  #define Zero(x) (fabs(x) <= EPS)
2  bool GaussElimination(double G[MAXN][MAXM], int N, int M) {
3    int rb = 1; memset(res, 0, sizeof(res));
4    Rep(i_th, 1, N) { int maxRow = 0;
5      Rep(row, rb, N) if (!Zero(G[row][i_th]))
6        if (!maxRow || fabs(G[row][i_th]) > fabs(G[maxRow][i_th]))
7          maxRow = row;
8      if (!maxRow) continue;
9      swapRow(G[rb], G[maxRow]);
10     maxRow = rb++;
11     Rep(row, 1, N) if (row != maxRow && !Zero(G[row][i_th])) {
12       double coef = G[row][i_th] / G[maxRow][i_th];
13       Rep(col, 0, M) G[row][col] -= coef * G[maxRow][col];
14     }
15   }
16   Rep(row, 1, N) if (!Zero(G[row][0])) {
17     int i_th = 1;
18     for ( ; i_th <= M; ++i_th) if (!Zero(G[row][i_th])) break;
19     if (i_th > N) return false;
20     res[i_th] = G[row][0] / G[row][i_th];
21   }
22   return true;
23 }
```

## 5.4 FFT

```
1  namespace FFT {
2    #define mul(a, b) (Complex(a.x * b.x - a.y * b.y, a.x * b.y + a.y * b.x))
3    struct Complex {}; // something omitted
4    void FFT(Complex P[], int n, int oper) {
5      for (int i = 1, j = 0; i < n - 1; i++) {
6        for (int s = n; j ^= s >>= 1, ~j & s; );
7        if (i < j) swap(P[i], P[j]);
8      }
9      for (int d = 0; (1 << d) < n; d++) {
10       int m = 1 << d, m2 = m * 2;
```

```
11       double p0 = PI / m * oper;
12       Complex unit_p0(cos(p0), sin(p0));
13       for (int i = 0; i < n; i += m2) {
14         Complex unit(1.0, 0.0);
15         for (int j = 0; j < m; j++) {
16           Complex &P1 = P[i + j + m], &P2 = P[i + j];
17           Complex t = mul(unit, P1);
18           P1 = Complex(P2.x - t.x, P2.y - t.y);
19           P2 = Complex(P2.x + t.x, P2.y + t.y);
20           unit = mul(unit, unit_p0);
21   }}}}
22   vector<int> doFFT(const vector<int>& a, const vector<int>& b) {
23     vector<int> ret(max(0, (int) a.size() + (int) b.size() - 1), 0);
24     static Complex A[MAXB], B[MAXB], C[MAXB];
25     int len = 1; while (len < (int)ret.size()) len *= 2;
26     for (int i = 0; i < len; i++) A[i] = i < (int)a.size() ? a[i] : 0;
27     for (int i = 0; i < len; i++) B[i] = i < (int)b.size() ? b[i] : 0;
28     FFT(A, len, 1); FFT(B, len, 1);
29     for (int i = 0; i < len; i++) C[i] = mul(A[i], B[i]);
30     FFT(C, len, -1);
31     for (int i = 0; i < (int)ret.size(); i++)
32       ret[i] = (int) (C[i].x / len + 0.5);
33     return ret;
34   }
35 }
```

## 5.5 整数 FFT

```
1  namespace FFT {
2  // 替代方案: 23068673(= 11 * 2^21 + 1), 原根为 3
3    const int MOD = 786433, PRIMITIVE_ROOT = 10; // 3 * 2^18 + 1
4    const int MAXB = 1 << 20;
5    int getMod(int downLimit) { // 或者现场自己找一个 MOD
6      for (int c = 3; ; ++c) { int t = (c << 21) | 1;
7        if (t >= downLimit && isPrime(t)) return t;
8    }}
9    int modInv(int a) { return a <= 1 ? a : (long long) (MOD - MOD / a) * modInv(MOD % a) % MOD; }
10   void NTT(int P[], int n, int oper) {
11     for (int i = 1, j = 0; i < n - 1; i++) {
12       for (int s = n; j ^= s >>= 1, ~j & s;);
13       if (i < j) swap(P[i], P[j]);
14     }
15     for (int d = 0; (1 << d) < n; d++) {
16       int m = 1 << d, m2 = m * 2;
17       long long unit_p0 = powMod(PRIMITIVE_ROOT, (MOD - 1) / m2);
18       if (oper < 0) unit_p0 = modInv(unit_p0);
19       for (int i = 0; i < n; i += m2) {
20         long long unit = 1;
21         for (int j = 0; j < m; j++) {
22           int &P1 = P[i + j + m], &P2 = P[i + j];
23           int t = unit * P1 % MOD;
24           P1 = (P2 - t + MOD) % MOD; P2 = (P2 + t) % MOD;
25           unit = unit * unit_p0 % MOD;
26   }}}}
27   vector<int> mul(const vector<int> &a, const vector<int> &b) {
28     vector<int> ret(max(0, (int) a.size() + (int) b.size() - 1), 0);
29     static int A[MAXB], B[MAXB], C[MAXB];
30     int len = 1; while (len < (int)ret.size()) len <<= 1;
31     for (int i = 0; i < len; i++) A[i] = i < (int)a.size() ? a[i] : 0;
32     for (int i = 0; i < len; i++) B[i] = i < (int)b.size() ? b[i] : 0;
33     NTT(A, len, 1); NTT(B, len, 1);
34     for (int i = 0; i < len; i++) C[i] = (long long) A[i] * B[i] % MOD;
35     NTT(C, len, -1); for (int i = 0, inv = modInv(len); i < (int)ret.size(); i++) ret[i] = (long long) C[
       i] * inv % MOD;
36     return ret;
37   }
38 }
```

## 5.6 扩展欧几里得

$$ax + by = g = gcd(x, y)$$

```
1  void exgcd(LL x, LL y, LL &a0, LL &b0, LL &g) {
2    LL a1 = b0 = 0, b1 = a0 = 1, t;
3    while (y != 0) {
4      t = a0 - x / y * a1, a0 = a1, a1 = t;
5      t = b0 - x / y * b1, b0 = b1, b1 = t;
6      t = x % y, x = y, y = t;
```

```
7    } if (x < 0) a0 = -a0, b0 = -b0, x = -x;
8    g = x;
9  }
```

## 5.7 线性同余方程

- 中国剩余定理: 设 $m_1, m_2, \cdots, m_k$ 两两互素, 则同余方程组 $x \equiv a_i \pmod{m_i}$ for $i = 1, 2, \cdots, k$ 在 $[0, M = m_1 m_2 \cdots m_k)$ 内有唯一解. 记 $M_i = M/m_i$, 找出 $p_i$ 使得 $M_i p_i \equiv 1 \pmod{m_i}$, 记 $e_i = M_i p_i$, 则 $x \equiv e_1 a_1 + e_2 a_2 + \cdots + e_k a_k \pmod{M}$

- 多变元线性同余方程组: 方程的形式为 $a_1 x_1 + a_2 x_2 + \cdots + a_n x_n + b \equiv 0 \pmod{m}$, 令 $d = (a_1, a_2, \cdots, a_n, m)$, 有解的充要条件是 $d|b$, 解的个数为 $m^{n-1}d$

## 5.8 Miller-Rabin 素性测试

```
1  bool test(LL n, int base) {
2    LL m = n - 1, ret = 0; int s = 0;
3    for ( ; m % 2 == 0; ++s) m >>= 1; ret = pow_mod(base, m, n);
4    if (ret == 1 || ret == n - 1) return true;
5    for (--s; s >= 0; --s) {
6      ret = multiply_mod(ret, ret, n); if (ret == n - 1) return true;
7    } return false;
8  }
9  LL special[7] = {
10   1373653LL,           25326001LL,
11   3215031751LL,        25000000000LL,
12   2152302898747LL,   3474749660383LL, 341550071728321LL};
13 /*
14  * n < 2047                        test[] = {2}
15  * n < 1,373,653                   test[] = {2, 3}
16  * n < 9,080,191                   test[] = {31, 73}
17  * n < 25,326,001                  test[] = {2, 3, 5}
18  * n < 4,759,123,141               test[] = {2, 7, 61}
19  * n < 1,122,004,669,633           test[] = {2, 13, 23, 1662803}
20  * n < 2,152,302,898,747           test[] = {2, 3, 5, 7, 11}
21  * n < 3,474,749,660,383           test[] = {2, 3, 5, 7, 11, 13}
22  * n < 341,550,071,728,321         test[] = {2, 3, 5, 7, 11, 13, 17}
23  * n < 3,825,123,056,546,413,051   test[] = {2, 3, 5, 7, 11, 13, 17, 19, 23}
24  */
25 bool is_prime(LL n) {
26   if (n < 2) return false;
27   if (n < 4) return true;
28   if (!test(n, 2) || !test(n, 3)) return false;
29   if (n < special[0]) return true;
30   if (!test(n, 5)) return false;
31   if (n < special[1]) return true;
32   if (!test(n, 7)) return false;
33   if (n == special[2]) return false;
34   if (n < special[3]) return true;
35   if (!test(n, 11)) return false;
36   if (n < special[4]) return true;
37   if (!test(n, 13)) return false;
38   if (n < special[5]) return true;
39   if (!test(n, 17)) return false;
40   if (n < special[6]) return true;
41   return test(n, 19) && test(n, 23) && test(n, 29) && test(n, 31) && test(n, 37);
42 }
```

## 5.9 PollardRho

```
1  LL pollardRho(LL n, LL seed) {
2    LL x, y, head = 1, tail = 2; x = y = random() % (n - 1) + 1;
3    for ( ; ; ) {
4      x = addMod(multiplyMod(x, x, n), seed, n);
5      if (x == y) return n; LL d = gcd(myAbs(x - y), n);
6      if (1 < d && d < n) return d;
7      if (++head == tail) y = x, tail <<= 1;
8  }} vector<LL> divisors;
9  void factorize(LL n) { // 需要保证 n > 1
10   if (isPrime(n)) divisors.push_back(n);
11   else { LL d = n;
12     while (d >= n) d = pollardRho(n, random() % (n - 1) + 1);
13     factorize(n / d); factorize(d);
14 }}
```

## 5.10 多项式求根

```
1  const double error = 1e-12;
2  const double infi = 1e+12;
3  int n; double a[10], x[10];
4  double f(double a[], int n, double x) {
5    double tmp = 1, sum = 0;
6    for (int i = 0; i <= n; i++) sum = sum + a[i] * tmp, tmp = tmp * x;
7    return sum;
8  }
9  double binary(double l, double r, double a[], int n) {
10   int sl = sign(f(a, n, l)), sr = sign(f(a, n, r));
11   if (sl == 0) return l; if (sr == 0) return r;
12   if (sl * sr > 0) return infi;
13   while (r - l > error) {
14     double mid = (l + r) / 2;
15     int ss = sign(f(a, n, mid));
16     if (ss == 0) return mid;
17     if (ss * sl > 0) l = mid; else r = mid;
18   } return l;
19 }
20 void solve(int n, double a[], double x[], int &nx) {
21   if (n == 1) { x[1] = -a[0] / a[1]; nx = 1; return; }
22   double da[10], dx[10]; int ndx;
23   for (int i = n; i >= 1; i--) da[i - 1] = a[i] * i;
24   solve(n - 1, da, dx, ndx); nx = 0;
25   if (ndx == 0) {
26     double tmp = binary(-infi, infi, a, n);
27     if (tmp < infi) x[++nx] = tmp; return;
28   } double tmp = binary(-infi, dx[1], a, n);
29   if (tmp < infi) x[++nx] = tmp;
30   for (int i = 1; i <= ndx - 1; i++) {
31     tmp = binary(dx[i], dx[i + 1], a, n);
32     if (tmp < infi) x[++nx] = tmp;
33   } tmp = binary(dx[ndx], infi, a, n);
34   if (tmp < infi) x[++nx] = tmp;
35 }
36 int main() {
37   scanf("%d", &n);
38   for (int i = n; i >= 0; i--) scanf("%lf", &a[i]);
39   int nx; solve(n, a, x, nx);
40   for (int i = 1; i <= nx; i++) printf("%0.6f\n", x[i]);
41   return 0;
42 }
```

## 5.11 线性递推

$$\text{for } a_{i+n} = \left(\sum_{i=0}^{n-1} k_j a_{i+j}\right) + d, \ a_m = \left(\sum_{i=0}^{n-1} c_i a_i\right) + c_n d$$

```
1  vector<int> recFormula(int n, int k[], int m) {
2    vector<int> c(n + 1, 0);
3    if (m < n) c[m] = 1;
4    else {
5      static int a[MAX_K * 2 + 1];
6      vector<int> b = recFormula(n, k, m >> 1);
7      for (int i = 0; i < n + n; ++i) a[i] = 0;
8      int s = m & 1;
9      for (int i = 0; i < n; i++) {
10       for (int j = 0; j < n; j++) a[i + j + s] += b[i] * b[j];
11       c[n] += b[i];
12     } c[n] = (c[n] + 1) * b[n];
13     for (int i = n * 2 - 1; i >= n; i--) {
14       int add = a[i]; if (add == 0) continue;
15       for (int j = 0; j < n; j++) a[i - n + j] += k[j] * add;
16       c[n] += add;
17     } for (int i = 0; i < n; ++i) c[i] = a[i];
18   } return c;
19 }
```

## 5.12 原根

原根 $g$: $g$ 是模 $n$ 简化剩余系构成的乘法群的生成元. 模 $n$ 有原根的充要条件是 $n = 2, 4, p^n, 2p^n$, 其中 $p$ 是奇质数, $n$ 是正整数

```
1  vector<int> findPrimitiveRoot(int N) {
2    if (N <= 4) return vector<int>(1, max(1, N - 1));
```

```
3    static int factor[100];
4    int phi = N, totF = 0;
5    { // check no solution and calculate phi
6      int M = N, k = 0;
7      if (-M & 1) M >>= 1, phi >>= 1;
8      if (-M & 1) return vector<int>(0);
9      for (int d = 3; d * d <= M; ++d) if (M % d == 0) {
10       if (++k > 1) return vector<int>(0);
11       for (phi -= phi / d; M % d == 0; M /= d);
12     } if (M > 1) {
13       if (++k > 1) return vector<int>(0); phi -= phi / M;
14     }
15   } { // factorize phi
16     int M = phi;
17     for (int d = 2; d * d <= M; ++d) if (M % d == 0) {
18       for ( ; M % d == 0; M /= d); factor[++totF] = d;
19     } if (M > 1) factor[++totF] = M;
20   } vector<int> ans;
21   for (int g = 2; g <= N; ++g) if (Gcd(g, N) == 1) {
22     bool good = true;
23     for (int i = 1; i <= totF && good; ++i)
24       if (powMod(g, phi / factor[i], N) == 1) good = false;
25     if (!good) continue;
26     for (int i = 1, gp = g; i <= phi; ++i, gp = (LL)gp * g % N)
27       if (Gcd(i, phi) == 1) ans.push_back(gp);
28     break;
29   } sort(ans.begin(), ans.end());
30   return ans;
31 }
```

## 5.13 离散对数

$A^x \equiv B \pmod{C}$, 对非质数 $C$ 也适用.

```
1  int modLog(int A, int B, int C) {
2    static pii baby[MAX_SQRT_C + 1];
3    int d = 0; LL k = 1, D = 1; B %= C;
4    for (int i = 0; i < 100; ++i, k = k * A % C) // [0, log C]
5      if (k == B) return i;
6    for (int g; ; ++d) {
7      g = gcd(A, C); if (g == 1) break;
8      if (B % g != 0) return -1;
9      B /= g; C /= g; D = (A / g * D) % C;
10   } int m = (int) ceil(sqrt((double) C)); k = 1;
11   for (int i = 0; i <= m; ++i, k = k * A % C) baby[i] = pii(k, i);
12   sort(baby, baby + m + 1); // [0, m]
13   int n = unique(baby, baby + m + 1, equalFirst) - baby, am = powMod(A, m, C);
14   for (int i = 0; i <= m; ++i) {
15     LL e, x, y; exgcd(D, C, x, y, e); e = x * B % C;
16     if (e < 0) e += C;
17     if (e >= 0) {
18       int k = lower_bound(baby, baby + n, pii(e, -1)) - baby;
19       if (baby[k].first == e) return i * m + baby[k].second + d;
20     } D = D * am % C;
21   } return -1;
22 }
```

## 5.14 平方剩余

- Legrendre Symbol: 对奇质数 $p$, $(\frac{a}{p}) = \begin{cases} 1 & \text{是平方剩余} \\ -1 & \text{是非平方剩余} = a^{\frac{p-1}{2}} \bmod p \\ 0 & a \equiv 0 \pmod{p} \end{cases}$

- 若 $p$ 是奇质数, $(\frac{-1}{p}) = 1$ 当且仅当 $p \equiv 1 \pmod 4$

- 若 $p$ 是奇质数, $(\frac{2}{p}) = 1$ 当且仅当 $p \equiv \pm 1 \pmod 8$

- 若 $p, q$ 是奇素数且互质, $(\frac{p}{q})(\frac{q}{p}) = (-1)^{\frac{p-1}{2} \times \frac{q-1}{2}}$

- Jacobi Symbol: 对奇数 $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$, $(\frac{a}{n}) = (\frac{a}{p_1})^{\alpha_1} (\frac{a}{p_2})^{\alpha_2} \cdots (\frac{a}{p_k})^{\alpha_k}$

- Jacobi Symbol 为 $-1$ 则一定不是平方剩余, 所有平方剩余的 Jacobi Symbol 都是 1, 但 1 不一定是平方剩余

$$ax^2 + bx + c \equiv 0 \pmod{p}, \text{ 其中 } a \neq 0 \pmod{p}, \text{ 且 } p \text{ 是质数}$$

```
1  inline int normalize(LL a, int P) { a %= P; return a < 0 ? a + P : a; }
2  vector<int> QuadraticResidue(LL a, LL b, LL c, int P) {
3    int h, t; LL r1, r2, delta, pb = 0;
4    a = normalize(a, P); b = normalize(b, P); c = normalize(c, P);
5    if (P == 2) { vector<int> res;
6      if (c % P == 0) res.push_back(0);
7      if ((a + b + c) % P == 0) res.push_back(1);
8      return res;
9    } delta = b * rev(a + a, P) % P;
10   a = normalize(-c * rev(a, P) + delta * delta, P);
11   if (powMod(a, P / 2, P) + 1 == P) return vector<int>(0);
12   for (t = 0, h = P / 2; h % 2 == 0; ++t, h /= 2);
13   r1 = powMod(a, h / 2, P);
14   if (t > 0) { do b = random() % (P - 2) + 2;
15     while (powMod(b, P / 2, P) + 1 != P); }
16   for (int i = 1; i <= t; ++i) {
17     LL d = r1 * r1 % P * a % P;
18     for (int j = 1; j <= t - i; ++j) d = d * d % P;
19     if (d + 1 == P) r1 = r1 * pb % P; pb = pb * pb % P;
20   } r1 = a * r1 % P; r2 = P - r1;
21   r1 = normalize(r1 - delta, P); r2 = normalize(r2 - delta, P);
22   if (r1 > r2) swap(r1, r2); vector<int> res(1, r1);
23   if (r1 != r2) res.push_back(r2);
24   return res;
25 }
```

## 5.15 N 次剩余

- 若 $p$ 为奇质数, $a$ 为 $p$ 的 $n$ 次剩余的充要条件是 $a^{\frac{p-1}{(a, p-1)}} \equiv 1 \pmod{p}$.

$$x^N \equiv a \pmod{p}, \text{ 其中 } p \text{ 是质数}$$

```
1  vector<int> solve(int p, int N, int a) {
2    if ((a %= p) == 0) return vector<int>(1, 0);
3    int g = findPrimitiveRoot(p), m = modLog(g, a, p); // g ^ m = a (mod p)
4    if (m == -1) return vector<int>(0);
5    LL B = p - 1, x, y, d; exgcd(N, B, x, y, d);
6    if (m % d != 0) return vector<int>(0);
7    vector<int> ret; x = (x * (m / d) % B + B) % B; // g ^ B mod p = g ^ (p - 1) mod p = 1
8    for (int i = 0, delta = B / d; i < d; ++i) {
9      x = (x + delta) % B; ret.push_back((int)powMod(g, x, p));
10   } sort(ret.begin(), ret.end());
11   ret.resize(unique(ret.begin(), ret.end()) - ret.begin());
12   return ret;
13 }
```

## 5.16 Pell 方程

$$\begin{pmatrix} x_k \\ y_k \end{pmatrix} = \begin{pmatrix} x_1 & dy_1 \\ y_1 & x_1 \end{pmatrix}^{k-1} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$$

```
1  pair<ULL, ULL> Pell(int n) {
2    static ULL p[50] = {0, 1}, q[50] = {1, 0}, g[50] = {0, 0}, h[50] = {0, 1}, a[50];
3    ULL t = a[2] = Sqrt(n);
4    for (int i = 2; ; ++i) {
5      g[i] = -g[i - 1] + a[i] * h[i - 1];
6      h[i] = (n - g[i] * g[i]) / h[i - 1];
7      a[i + 1] = (g[i] + t) / h[i];
8      p[i] = a[i] * p[i - 1] + p[i - 2];
9      q[i] = a[i] * q[i - 1] + q[i - 2];
10     if (p[i] * p[i] - n * q[i] * q[i] == 1) return make_pair(p[i], q[i]);
11   } return make_pair(-1, -1);
12 }
```

## 5.17 Romberg 积分

```cpp
template <class T> double Romberg(const T&f, double a, double b, double eps = 1e-8) {
  vector<double> t; double h = b - a, last, now; int k = 1, i = 1;
  t.push_back(h * (f(a) + f(b)) / 2); // 梯形
  do {
    last = t.back(); now = 0; double x = a + h / 2;
    for (int j = 0; j < k; ++j, x += h) now += f(x);
    now = (t[0] + h * now) / 2; double k1 = 4.0 / 3.0, k2 = 1.0 / 3.0;
    for (int j = 0; j < i; ++j, k1 = k2 + 1) {
      double tmp = k1 * now - k2 * t[j];
      t[j] = now; now = tmp; k2 /= 4 * k1 - k2; // 防止溢出
    } t.push_back(now); k *= 2; h /= 2; ++i;
  } while (fabs(last - now) > eps);
  return t.back();
}
```

## 5.18 公式

### 5.18.1 级数与三角

- $\sum_{k=1}^{n} k^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$

- $\sum_{k=1}^{n} k^5 = \frac{n^2(n+1)^2(2n^2+2n-1)}{12}$

- 错排: $D_n = n!(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^n}{n!}) = (n-1)(D_{n-2} - D_{n-1})$

- $\tan\alpha \pm \tan\beta = \frac{\sin(\alpha \pm \beta)}{\cos\alpha\cos\beta}$

- $\cos n\alpha = \binom{n}{0}\cos^n\alpha - \binom{n}{2}\cos^{n-2}\alpha\sin^2\alpha + \binom{n}{4}\cos^{n-4}\alpha\sin^4\alpha\cdots$

- $\sin n\alpha = \binom{n}{1}\cos^{n-1}\alpha\sin\alpha - \binom{n}{3}\cos^{n-3}\alpha\sin^3\alpha + \binom{n}{5}\cos^{n-5}\alpha\sin^5\alpha\cdots$

- $\sum_{n=1}^{N}\cos nx = \frac{\sin(N+\frac{1}{2})x - \sin\frac{x}{2}}{2\sin\frac{x}{2}}$

- $\sum_{n=1}^{N}\sin nx = \frac{-\cos(N+\frac{1}{2})x + \cos\frac{x}{2}}{2\sin\frac{x}{2}}$

- $\int_0^{\frac{\pi}{2}}\sin^n x\,\mathrm{d}x = \begin{cases} \dfrac{(n-1)!!}{n!!} \times \dfrac{\pi}{2} & n是偶数 \\ \dfrac{(n-1)!!}{n!!} & n是奇数 \end{cases}$

- $\int_0^{+\infty}\frac{\sin x}{x}\mathrm{d}x = \frac{\pi}{2}$

- $\int_0^{+\infty} e^{-x^2}\,\mathrm{d}x = \frac{\sqrt{\pi}}{2}$

- 傅里叶级数: 设周期为 $2T$. 函数分段连续. 在不连续点的值为左右极限的平均数.

  - $a_n = \frac{1}{T}\int_{-T}^{T} f(x)\cos\frac{n\pi}{T}x\,\mathrm{d}x$

  - $b_n = \frac{1}{T}\int_{-T}^{T} f(x)\sin\frac{n\pi}{T}x\,\mathrm{d}x$

  - $f(x) = \frac{a_0}{2} + \sum_{n=1}^{+\infty}(a_n\cos\frac{n\pi}{T}x + b_n\sin\frac{n\pi}{T}x)$

- Beta 函数: $B(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1}\mathrm{d}x$

  - 定义域 $(0,+\infty) \times (0,+\infty)$, 在定义域上连续

  - $B(p,q) = B(q,p) = \frac{q-1}{p+q-1}B(p,q-1) = 2\int_0^{\frac{\pi}{2}}\cos^{2p-1}\phi\sin^{2p-1}\phi\,\mathrm{d}\phi = \int_0^{+\infty}\frac{t^{q-1}}{(1+t)^{p+q}}\mathrm{d}t = \int_0^1\frac{t^{p-1}+t^{q-1}}{(1+t)^{(p+q)}}$

  - $B(\frac{1}{2},\frac{1}{2}) = \pi$

- Gamma 函数: $\Gamma = \int_0^{+\infty} x^{s-1}e^{-x}\mathrm{d}x$

  - 定义域 $(0,+\infty)$, 在定义域上连续
  - $\Gamma(1) = 1$, $\Gamma(\frac{1}{2}) = \sqrt{\pi}$
  - $\Gamma(s) = (s-1)\Gamma(s-1)$
  - $B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$
  - $\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin\pi s}$ for $s > 0$
  - $\Gamma(s)\Gamma(s+\frac{1}{2}) = 2\sqrt{\pi}\frac{\Gamma(s)}{2^{2s-1}}$ for $0 < s < 1$

- 积分：平面图形面积、曲线弧长、旋转体体积、旋转曲面面积 $y = f(x)$, $\int_a^b f(x)\mathrm{d}x$, $\int_a^b\sqrt{1+f'^2(x)}\mathrm{d}x$, $\pi\int_a^b f^2(x)\mathrm{d}x$, $2\pi\int_a^b |f(x)|\sqrt{1+f'^2(x)}\mathrm{d}x$

  $x = x(t), y = y(t), t \in [T_1, T_2]$, $\int_{T_1}^{T_2}|y(t)x'(t)|\mathrm{d}t$, $\int_{T_1}^{T_2}\sqrt{x'^2(t)+y'^2(t)}\mathrm{d}t$, $\pi\int_{T_1}^{T_2}|x'(t)|y^2(t)\mathrm{d}t$, $2\pi\int_{T_1}^{T_2}|y(t)|\sqrt{x'^2(t)+y'^2(t)}\mathrm{d}t$,

  $r = r(\theta), \theta \in [\alpha,\beta]$, $\frac{1}{2}\int_\alpha^\beta r^2(\theta)\mathrm{d}\theta$, $\int_\alpha^\beta\sqrt{r^2(\theta)+r'^2(\theta)}\mathrm{d}\theta$, $\frac{2}{3}\pi\int_\alpha^\beta r^3(\theta)\sin\theta\mathrm{d}\theta$, $2\pi\int_\alpha^\beta r(\theta)\sin\theta\sqrt{r^2(\theta)+r'^2(\theta)}\mathrm{d}\theta$

### 5.18.2 三次方程求根公式

对一元三次方程 $x^3 + px + q = 0$, 令

$$A = \sqrt[3]{-\frac{q}{2} + \sqrt{(\frac{q}{2})^2 + (\frac{p}{3})^3}}$$

$$B = \sqrt[3]{-\frac{q}{2} - \sqrt{(\frac{q}{2})^2 + (\frac{p}{3})^3}}$$

$$\omega = \frac{(-1 + \mathrm{i}\sqrt{3})}{2}$$

则 $x_j = A\omega^j + B\omega^{2j}$ (j = 0, 1, 2).

当求解 $ax^3 + bx^2 + cx + d = 0$ 时, 令 $x = y - \frac{b}{3a}$, 再求解 $y$, 即转化为 $y^3 + py + q = 0$ 的形式. 其中,

$$p = \frac{b^2 - 3ac}{3a^2}$$

$$q = \frac{2b^3 - 9abc + 27a^2d}{27a^3}$$

卡尔丹判别法: 令 $\Delta = (\frac{q}{2})^2 + (\frac{p}{3})^3$. 当 $\Delta > 0$ 时, 有一个实根和一对个共轭虚根; 当 $\Delta = 0$ 时, 有三个实根, 其中两个相等; 当 $\Delta < 0$ 时, 有三个不相等的实根.

### 5.18.3 椭圆

- 椭圆 $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, 其中离心率 $e = \frac{c}{a}, c = \sqrt{a^2 - b^2}$; 焦点参数 $p = \frac{b^2}{a}$

- 椭圆上 $(x,y)$ 点处的曲率半径为 $R = a^2b^2(\frac{x^2}{a^4} + \frac{y^2}{b^4})^{\frac{3}{2}} = \frac{(r_1r_2)^{\frac{3}{2}}}{ab}$, 其中 $r_1$ 和 $r_2$ 分别为 $(x,y)$ 与两焦点 $F_1$ 和 $F_2$ 的距离.

- 椭圆的周长 $L = 4a \int_0^{\frac{\pi}{2}} \sqrt{1 - e^2\sin^2 t}\,\mathrm{d}t = 4aE(e, \frac{\pi}{2})$, 其中

$$E(e, \frac{\pi}{2}) = \frac{\pi}{2}[1 - (\frac{1}{2})^2 e^2 - (\frac{1 \times 3}{2 \times 4})^2 \frac{e^4}{3} - (\frac{1 \times 3 \times 5}{2 \times 4 \times 6})^2 \frac{e^6}{5} - \cdots$$

- 设椭圆上点 $M(x,y), N(x,-y), x, y > 0, A(a,0)$, 原点 $O(0,0)$, 扇形 $OAM$ 的面积 $S_{OAM} = \frac{1}{2}ab\arccos\frac{x}{a}$, 弓形 $MAN$ 的面积 $S_{MAN} = ab\arccos\frac{x}{a} - xy$.

- 设 $\theta$ 为 $(x,y)$ 点关于椭圆中心的极角, $r$ 为 $(x,y)$ 到椭圆中心的距离, 椭圆极坐标方程:

$$x = r\cos\theta, y = r\sin\theta, r^2 = \frac{b^2a^2}{b^2\cos^2\theta + a^2\sin^2\theta}$$

### 5.18.4 抛物线

- 标准方程 $y^2 = 2px$, 曲率半径 $R = \frac{(p+2x)^{\frac{3}{2}}}{\sqrt{p}}$

- 弧长: 设 $M(x,y)$ 是抛物线上一点, 则 $L_{OM} = \frac{p}{2}[\sqrt{\frac{2x}{p}(1 + \frac{2x}{p})} + \ln(\sqrt{\frac{2x}{p}} + \sqrt{1 + \frac{2x}{p}})]$

- 弓形面积: 设 $M, D$ 是抛物线上两点, 且分居一, 四象限. 做一条平行于 $MD$ 且与抛物线相切的直线 $L$. 若 $M$ 到 $L$ 的距离为 $h$. 则有 $S_{MOD} = \frac{2}{3}MD \cdot h$.

### 5.18.5 重心

- 半径 $r$, 圆心角为 $\theta$ 的扇形的重心与圆心的距离为 $\frac{4r\sin\frac{\theta}{2}}{3\theta}$

- 半径 $r$, 圆心角为 $\theta$ 的圆弧的重心与圆心的距离为 $\frac{4r\sin^3\frac{\theta}{2}}{3(\theta - \sin\theta)}$

- 椭圆上半部分的重心与圆心的距离为 $\frac{4b}{3\pi}$

- 抛物线中弓形 $MOD$ 的重心满足 $CQ = \frac{2}{5}PQ$, $P$ 是直线 $L$ 与抛物线的切点, $Q$ 在 $MD$ 上且 $PQ$ 平行 $x$ 轴, $C$ 是重心

### 5.18.6 向量恒等式

- $\vec{a} \cdot (\vec{b} \times \vec{c}) = \vec{b} \cdot (\vec{c} \times \vec{a}) = \vec{c} \cdot (\vec{a} \times \vec{b})$

- $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{c} \times \vec{b}) \times \vec{a} = \vec{b}(\vec{a} \cdot \vec{c}) - \vec{c}(\vec{a} \cdot \vec{b})$

### 5.18.7 常用几何公式

- 三角形的五心

  - 重心 $\vec{G} = \frac{\vec{A} + \vec{B} + \vec{C}}{3}$

  - 内心 $\vec{I} = \frac{a\vec{A} + b\vec{B} + c\vec{C}}{a+b+c}$, $R = \frac{2S}{a+b+c}$

  - 外心 $x = \frac{\vec{A} + \vec{B} - \frac{\vec{BC} \cdot \vec{AC}}{\vec{AB} \times \vec{BC}}\vec{AB}^T}{2}$, $y = \frac{\vec{A} + \vec{B} + \frac{\vec{BC} \cdot \vec{AC}}{\vec{AB} \times \vec{BC}}\vec{AB}^T}{2}$, $R = \frac{abc}{4S}$

  - 垂心 $\vec{H} = 3\vec{G} - 2\vec{O}$

  - 旁心 (三个) $\frac{-a\vec{A} + b\vec{B} + c\vec{C}}{-a+b+c}$

- 四边形: 设 $D_1, D_2$ 为对角线, $M$ 为对角线中点连线, $A$ 为对角线夹角

  - $a^2 + b^2 + c^2 + d^2 = D_1^2 + D_2^2 + 4M^2$

  - $S = \frac{1}{2}D_1D_2\sin A$

  - $ac + bd = D_1D_2$ (内接四边形适用)

  - Bretschneider 公式: $S = \sqrt{(p-a)(p-b)(p-c)(p-d) - abcd\cos^2(\frac{\theta}{2})}$, 其中 $\theta$ 为对角和

### 5.18.8 树的计数

- 有根数计数: 令 $S_{n,j} = \sum_{1 \le i \le n/j} a_{n+1-ij} = S_{n-j,j} + a_{n+1-j}$

  于是, $n + 1$ 个结点的有根数的总数为 $a_{n+1} = \dfrac{\sum_{1 \le j \le n} j \cdot a_j \cdot S_{n,j}}{n}$

  附: $a_1 = 1, a_2 = 1, a_3 = 2, a_4 = 4, a_5 = 9, a_6 = 20, a_9 = 286, a_{11} = 1842$

- 无根树计数: 当 $n$ 是奇数时, 则有 $a_n - \sum_{1 \le i \le \frac{n}{2}} a_i a_{n-i}$ 种不同的无根树

  当 $n$ 是偶数时, 则有 $a_n - \sum_{1 \le i \le \frac{n}{2}} a_i a_{n-i} + \frac{1}{2}a_{\frac{n}{2}}(a_{\frac{n}{2}} + 1)$ 种不同的无根树

- Matrix-Tree 定理: 对任意图 $G$, 设 $\mathrm{mat}[i][i] = i$ 的度数, $\mathrm{mat}[i][j] = i$ 与 $j$ 之间边数的相反数, 则 $\mathrm{mat}[i][j]$ 的任意余子式的行列式就是该图的生成树个数

## 5.19 小知识

- 勾股数: 设正整数 $n$ 的质因数分解为 $n = \prod p_i^{a_i}$, 则 $x^2 + y^2 = n$ 有整数解的充要条件是 $n$ 中不存在形如 $p_i \equiv 3 \pmod 4$ 且指数 $a_i$ 为奇数的质因数 $p_i$. $(\frac{a-b}{2})^2 + ab = (\frac{a+b}{2})^2$.

- 素勾股数: 若 $m$ 和 $n$ 互质, 而且 $m$ 和 $n$ 中有一个是偶数, 则 $a = m^2 - n^2, b = 2mn, c = m^2 + n^2$, 则 $a$、$b$、$c$ 是素勾股数.

- Stirling 公式: $n! \approx \sqrt{2\pi n}(\frac{n}{e})^n$

- Pick 定理: 简单多边形, 不自交, 顶点如果全是整点. 则: 严格在多边形内部的整点数 $+\frac{1}{2}$ 在边上的整点数 $-1=$ 面积

- Mersenne 素数: $p$ 是素数且 $2^p - 1$ 的数是素数. (10000 以内的 $p$ 有: 2, 3, 5, 7, 13, 17, 19, 31, 61, 89, 107, 127, 521, 607, 1279, 2203, 2281, 3217, 4253, 4423, 9689, 9941)

- Fermat 分解算法: 从 $t = \sqrt{n}$ 开始, 依次检查 $t^2 - n, (t+1)^2 - n, (t+2)^2 - n, \ldots$, 直到出现一个平方数 $y$, 由于 $t^2 - y^2 = n$, 因此分解得 $n = (t-y)(t+y)$. 显然, 当两个因数很接近时这个方法能很快找到结果, 但如果遇到一个素数, 则需要检查 $\frac{n+1}{2} - \sqrt{n}$ 个整数

- 牛顿迭代: $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$

- 球与盒子的动人故事: ($n$ 个球, $m$ 个盒子, $S$ 为第二类斯特林数)

  1. 球同, 盒同, 无空: dp
  2. 球同, 盒同, 可空: dp
  3. 球同, 盒不同, 无空: $\binom{n-1}{m-1}$
  4. 球同, 盒不同, 可空: $\binom{n+m-1}{n-1}$
  5. 球不同, 盒同, 无空: $S(n, m)$
  6. 球不同, 盒同, 可空: $\sum_{k=1}^{m} S(n, k)$
  7. 球不同, 盒不同, 无空: $m! S(n, m)$
  8. 球不同, 盒不同, 可空: $m^n$

- 组合数奇偶性: 若 $(n \& m) = m$, 则 $\binom{n}{m}$ 为奇数, 否则为偶数

- 格雷码 $G(x) = x \otimes (x >> 1)$

- Fibonacci 数:

  - $F_0 = F_1 = 1, F_i = F_{i-1} + F_{i-2}, F_{-i} = (-1)^{i-1} F_i$
  - $F_i = \frac{1}{\sqrt{5}}((\frac{1+\sqrt{5}}{2})^n - (\frac{1-\sqrt{5}}{2})^n)$
  - $\gcd(F_n, F_m) = F_{\gcd(n,m)}$
  - $F_{i+1} F_i - F_i^2 = (-1)^i$
  - $F_{n+k} = F_k F_{n+1} + F_{k-1} F_n$

- 第一类 Stirling 数: $\begin{bmatrix} n \\ k \end{bmatrix}$ 代表第一类无符号 Stirling 数, 代表将 $n$ 阶置换群中有 $k$ 个环的置换个数; $s(n, k)$ 代表有符号型, $s(n, k) = (-1)^{n-k} \begin{bmatrix} n \\ k \end{bmatrix}$.

  - $(x)^{(n)} = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} x^k, (x)_n = \sum_{k=0}^{n} s(n, k) x^k$
  - $\begin{bmatrix} n \\ k \end{bmatrix} = n \begin{bmatrix} n-1 \\ k \end{bmatrix} + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 1, \begin{bmatrix} 0 \\ n \end{bmatrix} = \begin{bmatrix} 0 \\ n \end{bmatrix} = 0$
  - $\begin{bmatrix} n \\ n-2 \end{bmatrix} = \frac{1}{4}(3n-1)\binom{n}{3}, \begin{bmatrix} n \\ n-3 \end{bmatrix} = \binom{n}{2}\binom{n}{4}$
  - $\sum_{k=0}^{a} \begin{bmatrix} n \\ k \end{bmatrix} = n! - \sum_{k=0}^{n} \begin{bmatrix} n \\ k+a+1 \end{bmatrix}$
  - $\sum_{p=k}^{n} \begin{bmatrix} n \\ p \end{bmatrix} \binom{p}{k} = \begin{bmatrix} n+1 \\ k+1 \end{bmatrix}$

- 第二类 Stirling 数: $\begin{Bmatrix} n \\ k \end{Bmatrix} = S(n, k)$ 代表 $n$ 个不同的球, 放到 $k$ 个相同的盒子里, 盒子非空.

  - $\begin{Bmatrix} n \\ k \end{Bmatrix} = \frac{1}{k!} \sum_{j=0}^{k} (-1)^j \binom{k}{j} (k-j)^n$
  - $\begin{Bmatrix} n+1 \\ k \end{Bmatrix} = k \begin{Bmatrix} n \\ k \end{Bmatrix} + \begin{Bmatrix} n \\ k-1 \end{Bmatrix}, \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} = 1, \begin{Bmatrix} n \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ n \end{Bmatrix} = 0$
  - 奇偶性: $(n-k) \& \frac{k-1}{2} == 0$

- Bell 数: $B_n$ 代表将 $n$ 个元素划分成若干个非空集合的方案数

  - $B_0 = B_1 = 1, B_n = \sum_{k=0}^{n-1} \binom{n-1}{k} B_k$
  - $B_n = \sum_{k=0}^{n} \begin{Bmatrix} n \\ k \end{Bmatrix}$
  - Bell 三角形: $a_{1,1} = 1, a_{n,1} = a_{n-1,n-1}, a_{n,m} = a_{n,m-1} + a_{n-1,m-1}, B_n = a_{n,1}$
  - 对质数 $p$, $B_{n+p} \equiv B_n + B_{n+1} \pmod{p}$
  - 对质数 $p$, $B_{n+p^m} \equiv m B_n + B_{n+1} \pmod{p}$
  - 对质数 $p$, 模的周期一定是 $\frac{p^p - 1}{p-1}$ 的约数, $p \leq 101$ 时就是这个值
  - 从 $B_0$ 开始, 前几项是 $1, 1, 2, 5, 15, 52, 203, 877, 4140, 21147, 115975 \cdots$

- Bernoulli 数

  - $B_0 = 1, B_1 = \frac{1}{2}, B_2 = \frac{1}{6}, B_4 = -\frac{1}{30}, B_6 = \frac{1}{42}, B_8 = B_4, B_{10} = \frac{5}{66}$
  - $\sum_{k=1}^{n} k^m = \frac{1}{m+1} \sum_{k=0}^{m} \binom{m+1}{k} B_k n^{m+1-k}$
  - $B_m = 1 - \sum_{k=0}^{m-1} \binom{m}{k} \frac{B_k}{m-k+1}$

- 完全数: $x$ 是偶完全数等价于 $x = 2^{n-1}(2^n - 1)$, 且 $2^n - 1$ 是质数.

# 6 其他

## 6.1 Extended LIS

```
int G[MAXN][MAXN];
void insertYoung(int v) {
  for (int x = 1, y = INT_MAX; ; ++x) {
    Down(y, *G[x]); while (y > 0 && G[x][y] >= v) --y;
    if (++y > *G[x]) { ++*G[x]; G[x][y] = v; break; }
    else swap(G[x][y], v);
  }
}
int solve(int N, int seq[]) {
  Rep(i, 1, N) *G[i] = 0;
  Rep(i, 1, N) insertYoung(seq[i]);
  printf("%d\n", *G[1] + *G[2]);
  return 0;
}
```

## 6.2 生成 nCk

```
void nCk(int n, int k) {
  for (int comb = (1 << k) - 1; comb < (1 << n); ) {
    int x = comb & -comb, y = comb + x;
    comb = (((comb & ~y) / x) >> 1) | y;
  }
}
```

## 6.3 nextPermutation

```
boolean nextPermutation(int[] is) {
  int n = is.length;
  for (int i = n - 1; i > 0; i--) {
    if (is[i - 1] < is[i]) {
      int j = n; while (is[i - 1] >= is[--j]);
      swap(is, i - 1, j); // swap is[i - 1], is[j]
      rev(is, i, n); // reverse is[i, n)
      return true;
    }
  } rev(is, 0, n);
  return false;
}
```

## 6.4 Josephus 数与逆 Josephus 数

```
int josephus(int n, int m, int k) { int x = -1;
  for (int i = n - k + 1; i <= n; i++) x = (x + m) % i; return x;
}
int invJosephus(int n, int m, int x) {
  for (int i = n; ; i--) { if (x == i) return n - i; x = (x - m % i + i) % i; }
}
```

## 6.5 表达式求值

```
inline int getLevel(char ch) {
  switch (ch) { case '+': case '-': return 0; case '*': return 1; } return -1;
}
int evaluate(char *&p, int level) {
  int res;
  if (level == 2) {
    if (*p == '(') ++p, res = evaluate(p, 0);
    else res = isdigit(*p) ? *p - '0' : value[*p - 'a'];
    ++p; return res;
  } res = evaluate(p, level + 1);
  for (int next; *p && getLevel(*p) == level; ) {
    char op = *p++; next = evaluate(p, level + 1);
    switch (op) {
      case '+': res += next; break;
      case '-': res -= next; break;
      case '*': res *= next; break;
    }
  } return res;
}
int makeEvaluation(char *str) { char *p = str; return evaluate(p, 0); }
```

## 6.6 直线下的整点个数

求 $\sum_{i=0}^{n-1} \lfloor \frac{a+bi}{m} \rfloor$

```
LL count(LL n, LL a, LL b, LL m) {
  if (b == 0) return n * (a / m);
  if (a >= m) return n * (a / m) + count(n, a % m, b, m);
  if (b >= m) return (n - 1) * n / 2 * (b / m) + count(n, a, b % m, m);
  return count((a + b * n) / m, (a + b * n) % m, m, b);
}
```

## 6.7 Java 多项式

```
class Polynomial {
  final static Polynomial ZERO = new Polynomial(new int[] { 0 });
  final static Polynomial ONE = new Polynomial(new int[] { 1 });
  final static Polynomial X = new Polynomial(new int[] { 0, 1 });
  int[] coef;
  static Polynomial valueOf(int val) { return new Polynomial(new int[] { val }); }
  Polynomial(int[] coef) { this.coef = Arrays.copyOf(coef, coef.length); }
  Polynomial add(Polynomial o, int mod); // omitted
  Polynomial subtract(Polynomial o, int mod); // omitted
  Polynomial multiply(Polynomial o, int mod); // omitted
  Polynomial scale(int o, int mod); // omitted
  public String toString() {
    int n = coef.length; String ret = "";
    for (int i = n - 1; i > 0; --i) if (coef[i] != 0)
      ret += coef[i] + "x^" + i + "+";
    return ret + coef[0];
  }
  static Polynomial lagrangeInterpolation(int[] x, int[] y, int mod) {
    int n = x.length; Polynomial ret = Polynomial.ZERO;
    for (int i = 0; i < n; ++i) {
      Polynomial poly = Polynomial.valueOf(y[i]);
      for (int j = 0; j < n; ++j) if (i != j) {
        poly = poly.multiply(
          Polynomial.X.subtract(Polynomial.valueOf(x[j]), mod), mod);
        poly = poly.scale(powMod(x[i] - x[j] + mod, mod - 2, mod), mod);
      } ret = ret.add(poly, mod);
    } return ret;
  }
}
```

## 6.8 long long 乘法取模

```
LL multiplyMod(LL a, LL b, LL P) { // 需要保证 a 和 b 非负
  LL t = (a * b - LL((long double)a / P * b + 1e-3) * P) % P;
  return t < 0 : t + P : t;
}
```

## 6.9 重复覆盖

```
namespace DLX {
  struct node { int x, y; node *l, *r, *u, *d; } base[MAX * MAX], *top, *head;
  typedef node *link;
  int row, col, nGE, ans, stamp, cntc[MAX], vis[MAX];
  vector<link> eachRow[MAX], eachCol[MAX];
  inline void addElement(int x, int y) {
    top->x = x; top->y = y; top->l = top->r = top->u = top->d = NULL;
    eachRow[x].push_back(top); eachCol[y].push_back(top++);
  }
  void init(int _row, int _col, int _nGE) {
    row = _row; col = _col; nGE = _nGE; top = base; stamp = 0;
    for (int i = 0; i <= col; ++i) vis[i] = 0;
    for (int i = 0; i <= row; ++i) eachRow[i].clear();
    for (int i = 0; i <= col; ++i) eachCol[i].clear();
    for (int i = 0; i <= col; ++i) addElement(0, i);
    head = eachCol[0].front();
  }
  void build() {
    for (int i = 0; i <= row; ++i) {
      vector<link> &v = eachRow[i];
      sort(v.begin(), v.end(), cmpByY);
      int s = v.size();
      for (int j = 0; j < s; ++j) {
        link l = v[j], r = v[(j + 1) % s]; l->r = r; r->l = l;
      }
    }
    for (int i = 0; i <= col; ++i) {
      vector<link> &v = eachCol[i];
      sort(v.begin(), v.end(), cmpByX);
      int s = v.size();
      for (int j = 0; j < s; ++j) {
        link u = v[j], d = v[(j + 1) % s]; u->d = d; d->u = u;
      }
    } for (int i = 0; i <= col; ++i) cntc[i] = (int) eachCol[i].size() - 1;
  }
```

```
36    void removeExact(link c) {
37      c->l->r = c->r; c->r->l = c->l;
38      for (link i = c->d; i != c; i = i->d)
39        for (link j = i->r; j != i; j = j->r) {
40          j->d->u = j->u; j->u->d = j->d; --cntc[j->y];
41        }
42    }
43    void resumeExact(link c) {
44      for (link i = c->u; i != c; i = i->u)
45        for (link j = i->l; j != i; j = j->l) {
46          j->d->u = j; j->u->d = j; ++cntc[j->y];
47        }
48      c->l->r = c; c->r->l = c;
49    }
50    void removeRepeat(link c) {
51      for (link i = c->d; i != c; i = i->d) {
52        i->l->r = i->r; i->r->l = i->l;
53      }
54    }
55    void resumeRepeat(link c) {
56      for (link i = c->u; i != c; i = i->u) {
57        i->l->r = i; i->r->l = i;
58      }
59    }
60    int calcH() {
61      int y, res = 0; ++stamp;
62      for (link c = head->r; (y = c->y) <= row && c != head; c = c->r) {
63        if (vis[y] != stamp) {
64          vis[y] = stamp; ++res;
65          for (link i = c->d; i != c; i = i->d)
66            for (link j = i->r; j != i; j = j->r) vis[j->y] = stamp;
67        }
68      } return res;
69    }
70    void DFS(int dep) { if (dep + calcH() >= ans) return;
71      if (head->r->y > nGE || head->r == head) {
72        if (ans > dep) ans = dep; return;
73      } link c = NULL;
74      for (link i = head->r; i->y <= nGE && i != head; i = i->r)
75        if (!c || cntc[i->y] < cntc[c->y]) c = i;
76      for (link i = c->d; i != c; i = i->d) {
77        removeRepeat(i);
78        for (link j = i->r; j != i; j = j->r) if (j->y <= nGE) removeRepeat(j);
79        for (link j = i->r; j != i; j = j->r) if (j->y  > nGE) removeExact(base + j->y);
80        DFS(dep + 1);
81        for (link j = i->l; j != i; j = j->l) if (j->y  > nGE) resumeExact(base + j->y);
82        for (link j = i->l; j != i; j = j->l) if (j->y <= nGE) resumeRepeat(j);
83        resumeRepeat(i);
84      }
85    }
86    int solve() { build(); ans = INF; DFS(0); return ans; }
87  }
```

## 6.10 星期几判定

```
1  int getDay(int y, int m, int d) {
2    if (m <= 2) m += 12, y--;
3    if (y < 1752 || (y == 1752 && m < 9) || (y == 1752 && m == 9 && d < 3))
4      return (d + 2 * m + 3 * (m + 1) / 5 + y + y / 4 + 5) % 7 + 1;
5    return (d + 2 * m + 3 * (m + 1) / 5 + y + y / 4 - y / 100 + y / 400) % 7 + 1;
6  }
```

## 6.11 LCSequence Fast

```
1  ULL *a, *b, *s, c, d;
2  for (i = 0, a = appear[(int)B[k]], b = row[max(k - 1, 0)], s = X; i < bitSetLen; ++i)
3    *s++ = *a++ | *b++;                      // X = row[i - 1] or appear[ B[i] ]
4  for (i = 0, a = dp, c = d = 0; i < bitSetLen; ++a, c = d, ++i)
5    d = *a >> 63, *a = ~((*a << 1) + c);     // row[i] = ~((row[i] << 1) + 1)
6  for (i = 0, a = dp, b = X, c = 0; i < bitSetLen; ++a, ++b, ++i)
7    d = *b + c, c = (*a >= ~d), *a += d;     // row[i] += X
8  for (i = 0, a = dp, b = X; i < bitSetLen; ++a, ++b, ++i)
9    *a = (*a ^ *b) & *b;                     // row[i] = X and (row[i] xor X)
```

# 7 Templates

## 7.1 vim 配置

在.bashrc 中加入 export CXXFLAGS="-Wall -Wconversion -Wextra -g3"

```
1   set nu ru nobk cindent si
2   set mouse=a sw=4 sts=4 ts=4
3   set hlsearch incsearch
4   set whichwrap=b,s,<,>,[,]
5   syntax on
6
7   nmap <C-A> ggVG
8   vmap <C-C> "+y
9
10  autocmd␣BufNewFile␣*.cpp␣0r␣~/Templates/cpp.cpp
11  map<F9>␣:!g++␣%␣-o␣%<␣-Wall␣-Wconversion␣-Wextra␣-g3␣<CR>
12  map<F5>␣:!./%<␣<CR>
13  map<F8>␣:!./%<␣<␣%<.in␣<CR>
14
15  map<F3>␣:vnew␣%<.in␣<CR>
16  map<F4>␣:!(gedit␣%␣&)<CR>
```

## 7.2 C++

```
1   #pragma comment(linker, "/STACK:10240000")
2   #include <cstdio>
3   #include <cstdlib>
4   #include <cstring>
5   #include <iostream>
6   #include <algorithm>
7   #define Rep(i, a, b) for(int i = (a); i <= (b); ++i)
8   #define Foru(i, a, b) for(int i = (a); i < (b); ++i)
9   using namespace std;
10  typedef long long LL;
11  typedef pair<int, int> pii;
12  namespace BufferedReader {
13    char buff[MAX_BUFFER + 5], *ptr = buff, c; bool flag;
14    bool nextChar(char &c) {
15      if ( (c = *ptr++) == 0 ) {
16        int tmp = fread(buff, 1, MAX_BUFFER, stdin);
17        buff[tmp] = 0; if (tmp == 0) return false;
18        ptr = buff; c = *ptr++;
19      } return true;
20    }
21    bool nextUnsignedInt(unsigned int &x) {
22      for (;;){if (!nextChar(c)) return false; if ('0'<=c && c<='9') break;}
23      for (x=c-'0'; nextChar(c); x = x * 10 + c - '0') if (c < '0' || c > '9') break;
24      return true;
25    }
26    bool nextInt(int &x) {
27      for (;;) { if (!nextChar(c)) return false; if (c=='-' || ('0'<=c && c<='9')) break; }
28      for ((c=='-') ? (x=0,flag=true) : (x=c-'0',flag=false); nextChar(c); x=x*10+c-'0')
29        if (c<'0' || c>'9') break;
30      if (flag) x=-x; return true;
31    }
32  };
33  #endif
```

## 7.3 Java

```
1   import java.io.*;
2   import java.util.*;
3   import java.math.*;
4
5   public class Main {
6     public void solve() {}
7     public void run() {
8       tokenizer = null; out = new PrintWriter(System.out);
9       in = new BufferedReader(new InputStreamReader(System.in));
10      solve();
11      out.close();
12    }
13    public static void main(String[] args) {
14      new Main().run();
15    }
16    public StringTokenizer tokenizer;
```

```
17      public BufferedReader in;
18      public PrintWriter out;
19      public String next() {
20          while (tokenizer == null || !tokenizer.hasMoreTokens()) {
21              try { tokenizer = new StringTokenizer(in.readLine()); }
22              catch (IOException e) { throw new RuntimeException(e); }
23          } return tokenizer.nextToken();
24      }
25  }
```

## 7.4 Eclipse 配置

Exec=env UBUNTU_MENUPROXY= /opt/eclipse/eclipse
preference general keys 把 word completion 设置成 alt+c, 把 content assistant 设置成 alt + /

## 7.5 泰勒级数

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots \qquad = \sum_{i=0}^{\infty} x^i$$

$$\frac{1}{1-cx} = 1 + cx + c^2x^2 + c^3x^3 + \cdots \qquad = \sum_{i=0}^{\infty} c^i x^i$$

$$\frac{1}{1-x^n} = 1 + x^n + x^{2n} + x^{3n} + \cdots \qquad = \sum_{i=0}^{\infty} x^{ni}$$

$$\frac{x}{(1-x)^2} = x + 2x^2 + 3x^3 + 4x^4 + \cdots \qquad = \sum_{i=0}^{\infty} i x^i$$

$$\sum_{k=0}^{n} \left\{{n \atop k}\right\} \frac{k! z^k}{(1-z)^{k+1}} = x + 2^n x^2 + 3^n x^3 + 4^n x^4 + \cdots \qquad = \sum_{i=0}^{\infty} i^n x^i$$

$$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \cdots \qquad = \sum_{i=0}^{\infty} \frac{x^i}{i!}$$

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 - \cdots \qquad = \sum_{i=1}^{\infty} (-1)^{i+1} \frac{x^i}{i}$$

$$\ln \frac{1}{1-x} = x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4 + \cdots \qquad = \sum_{i=1}^{\infty} \frac{x^i}{i}$$

$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots \qquad = \sum_{i=0}^{\infty} (-1)^i \frac{x^{2i+1}}{(2i+1)!}$$

$$\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots \qquad = \sum_{i=0}^{\infty} (-1)^i \frac{x^{2i}}{(2i)!}$$

$$\tan^{-1} x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \cdots \qquad = \sum_{i=0}^{\infty} (-1)^i \frac{x^{2i+1}}{(2i+1)}$$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2}x^2 + \cdots \qquad = \sum_{i=0}^{\infty} \binom{n}{i} x^i$$

$$\frac{1}{(1-x)^{n+1}} = 1 + (n+1)x + \binom{n+2}{2}x^2 + \cdots \qquad = \sum_{i=0}^{\infty} \binom{i+n}{i} x^i$$

$$\frac{x}{e^x - 1} = 1 - \frac{1}{2}x + \frac{1}{12}x^2 - \frac{1}{720}x^4 + \cdots \qquad = \sum_{i=0}^{\infty} \frac{B_i x^i}{i!}$$

$$\frac{1}{2x}(1 - \sqrt{1-4x}) = 1 + x + 2x^2 + 5x^3 + \cdots \qquad = \sum_{i=0}^{\infty} \frac{1}{i+1}\binom{2i}{i} x^i$$

$$\frac{1}{\sqrt{1-4x}} = 1 + 2x + 6x^2 + 20x^3 + \cdots \qquad = \sum_{i=0}^{\infty} \binom{2i}{i} x^i$$

$$\frac{1}{\sqrt{1-4x}} \left(\frac{1 - \sqrt{1-4x}}{2x}\right)^n = 1 + (2+n)x + \binom{4+n}{2}x^2 + \cdots \qquad = \sum_{i=0}^{\infty} \binom{2i+n}{i} x^i$$

$$\frac{1}{1-x} \ln \frac{1}{1-x} = x + \frac{3}{2}x^2 + \frac{11}{6}x^3 + \frac{25}{12}x^4 + \cdots \qquad = \sum_{i=1}^{\infty} H_i x^i$$

$$\frac{1}{2}\left(\ln \frac{1}{1-x}\right)^2 = \frac{1}{2}x^2 + \frac{3}{4}x^3 + \frac{11}{24}x^4 + \cdots \qquad = \sum_{i=2}^{\infty} \frac{H_{i-1} x^i}{i}$$

$$\frac{x}{1-x-x^2} = x + x^2 + 2x^3 + 3x^4 + \cdots \qquad = \sum_{i=0}^{\infty} F_i x^i$$

$$\frac{F_n x}{1 - (F_{n-1} + F_{n+1})x - (-1)^n x^2} = F_n x + F_{2n} x^2 + F_{3n} x^3 + \cdots \qquad = \sum_{i=0}^{\infty} F_{ni} x^i$$

## 7.6 积分表

- $d(\tan x) = \sec^2 x dx$

- $d(\cot x) = \csc^2 x dx$

- $d(\sec x) = \tan x \sec x dx$

- $d(\csc x) = -\cot x \csc x dx$

- $d(\arcsin x) = \frac{1}{\sqrt{1-x^2}} dx$

- $d(\arccos x) = \frac{-1}{\sqrt{1-x^2}} dx$

- $d(\arctan x) = \frac{1}{1+x^2} dx$

- $d(\text{arccot} x) = \frac{-1}{1+x^2} dx$

- $d(\text{arcsec} x) = \frac{1}{x\sqrt{1-x^2}} dx$

- $d(\text{arccsc} x) = \frac{-1}{u\sqrt{1-x^2}} dx$

- $\int cu \, dx = c \int u \, dx$

- $\int (u + v) \, dx = \int u \, dx + \int v \, dx$

- $\int x^n \, \mathrm{d}x = \dfrac{1}{n+1}x^{n+1}, \quad n \neq -1$

- $\int \dfrac{1}{x}\mathrm{d}x = \ln x$

- $\int e^x \, \mathrm{d}x = e^x$

- $\int \dfrac{\mathrm{d}x}{1+x^2} = \arctan x$

- $\int u\dfrac{\mathrm{d}v}{\mathrm{d}x}\mathrm{d}x = uv - \int v\dfrac{\mathrm{d}u}{\mathrm{d}x}\mathrm{d}x$

- $\int \sin x \, \mathrm{d}x = -\cos x$

- $\int \cos x \, \mathrm{d}x = \sin x$

- $\int \tan x \, \mathrm{d}x = -\ln|\cos x|$

- $\int \cot x \, \mathrm{d}x = \ln|\cos x|$

- $\int \sec x \, \mathrm{d}x = \ln|\sec x + \tan x|$

- $\int \csc x \, \mathrm{d}x = \ln|\csc x + \cot x|$

- $\int \arcsin\dfrac{x}{a}\mathrm{d}x = \arcsin\dfrac{x}{a} + \sqrt{a^2 - x^2}, \quad a > 0$

- $\int \arccos\dfrac{x}{a}\mathrm{d}x = \arccos\dfrac{x}{a} - \sqrt{a^2 - x^2}, \quad a > 0$

- $\int \arctan\dfrac{x}{a}\mathrm{d}x = x\arctan\dfrac{x}{a} - \dfrac{a}{2}\ln(a^2 + x^2), \quad a > 0$

- $\int \sin^2(ax)\mathrm{d}x = \dfrac{1}{2a}\big(ax - \sin(ax)\cos(ax)\big)$

- $\int \cos^2(ax)\mathrm{d}x = \dfrac{1}{2a}\big(ax + \sin(ax)\cos(ax)\big)$

- $\int \sec^2 x \, \mathrm{d}x = \tan x$

- $\int \csc^2 x \, \mathrm{d}x = -\cot x$

- $\int \sin^n x \, \mathrm{d}x = -\dfrac{\sin^{n-1}x\cos x}{n} + \dfrac{n-1}{n}\int \sin^{n-2}x \, \mathrm{d}x$

- $\int \cos^n x \, \mathrm{d}x = \dfrac{\cos^{n-1}x\sin x}{n} + \dfrac{n-1}{n}\int \cos^{n-2}x \, \mathrm{d}x$

- $\int \tan^n x \, \mathrm{d}x = \dfrac{\tan^{n-1}x}{n-1} - \int \tan^{n-2}x \, \mathrm{d}x, \quad n \neq 1$

- $\int \cot^n x \, \mathrm{d}x = -\dfrac{\cot^{n-1}x}{n-1} - \int \cot^{n-2}x \, \mathrm{d}x, \quad n \neq 1$

- $\int \sec^n x \, \mathrm{d}x = \dfrac{\tan x\sec^{n-1}x}{n-1} + \dfrac{n-2}{n-1}\int \sec^{n-2}x \, \mathrm{d}x, \quad n \neq 1$

- $\int \csc^n x \, \mathrm{d}x = -\dfrac{\cot x\csc^{n-1}x}{n-1} + \dfrac{n-2}{n-1}\int \csc^{n-2}x \, \mathrm{d}x, \quad n \neq 1$

- $\int \sinh x \, \mathrm{d}x = \cosh x$

- $\int \cosh x \, \mathrm{d}x = \sinh x$

- $\int \tanh x \, \mathrm{d}x = \ln|\cosh x|$

- $\int \coth x \, \mathrm{d}x = \ln|\sinh x|$

- $\int \mathrm{sech}x \, \mathrm{d}x = \arctan \sinh x$

- $\int \mathrm{csch}x \, \mathrm{d}x = \ln\left|\tanh\dfrac{x}{2}\right|$

- $\int \sinh^2 x \, \mathrm{d}x = \dfrac{1}{4}\sinh(2x) - \dfrac{1}{2}x$

- $\int \cosh^2 x \, \mathrm{d}x = \dfrac{1}{4}\sinh(2x) + \dfrac{1}{2}x$

- $\int \mathrm{sech}^2 x \, \mathrm{d}x = \tanh x$

- $\int \mathrm{arcsinh}\dfrac{x}{a}\mathrm{d}x = x\,\mathrm{arcsinh}\dfrac{x}{a} - \sqrt{x^2 + a^2}, \quad a > 0$

- $\int \mathrm{arctanh}\dfrac{x}{a}\mathrm{d}x = x\,\mathrm{arctanh}\dfrac{x}{a} + \dfrac{a}{2}\ln|a^2 - x^2|$

- $\int \mathrm{arccosh}\dfrac{x}{a}\mathrm{d}x = \begin{cases} x\,\mathrm{arccosh}\dfrac{x}{a} - \sqrt{x^2 + a^2}, \text{if } \mathrm{arccosh}\dfrac{x}{a} > 0 \text{ and } a > 0 \\ x\,\mathrm{arccosh}\dfrac{x}{a} + \sqrt{x^2 + a^2}, \text{if } \mathrm{arccosh}\dfrac{x}{a} < 0 \text{ and } a > 0 \end{cases}$

- $\int \dfrac{\mathrm{d}x}{\sqrt{a^2 + x^2}} = \ln\left(x + \sqrt{a^2 + x^2}\right), \quad a > 0$

- $\int \dfrac{\mathrm{d}x}{a^2 + x^2} = \dfrac{1}{a}\arctan\dfrac{x}{a}, \quad a > 0$

- $\int \sqrt{a^2 - x^2} \, \mathrm{d}x = \dfrac{x}{2}\sqrt{a^2 - x^2} + \dfrac{a^2}{2}\arcsin\dfrac{x}{a}, \quad a > 0$

- $\int (a^2 - x^2)^{3/2}\mathrm{d}x = \dfrac{x}{8}(5a^2 - 2x^2)\sqrt{a^2 - x^2} + \dfrac{3a^4}{8}\arcsin\dfrac{x}{a}, \quad a > 0$

- $\int \dfrac{\mathrm{d}x}{\sqrt{a^2 - x^2}} = \arcsin\dfrac{x}{a}, \quad a > 0$

- $\int \dfrac{\mathrm{d}x}{a^2 - x^2} = \dfrac{1}{2a}\ln\left|\dfrac{a+x}{a-x}\right|$

- $\int \dfrac{\mathrm{d}x}{(a^2 - x^2)^{3/2}} = \dfrac{x}{a^2\sqrt{a^2 - x^2}}$

- $\int \sqrt{a^2 \pm x^2} \, \mathrm{d}x = \dfrac{x}{2}\sqrt{a^2 \pm x^2} \pm \dfrac{a^2}{2}\ln\left|x + \sqrt{a^2 \pm x^2}\right|$

- $\int \dfrac{\mathrm{d}x}{\sqrt{x^2 - a^2}} = \ln\left|x + \sqrt{x^2 - a^2}\right|, \quad a > 0$

- $\int \dfrac{\mathrm{d}x}{ax^2 + bx} = \dfrac{1}{a}\ln\left|\dfrac{x}{a + bx}\right|$

- $\int x\sqrt{a + bx}\,\mathrm{d}x = \dfrac{2(3bx - 2a)(a + bx)^{3/2}}{15b^2}$

- $\int \dfrac{\sqrt{a + bx}}{x}\,\mathrm{d}x = 2\sqrt{a + bx} + a\int \dfrac{1}{x\sqrt{a + bx}}\mathrm{d}x$

- $\int \dfrac{x}{\sqrt{a + bx}}\,\mathrm{d}x = \dfrac{1}{\sqrt{2}}\ln\left|\dfrac{\sqrt{a + bx} - \sqrt{a}}{\sqrt{a + bx} + \sqrt{a}}\right|, \quad a > 0$

- $\int \dfrac{\sqrt{a^2 - x^2}}{x}\,\mathrm{d}x = \sqrt{a^2 - x^2} - a\ln\left|\dfrac{a + \sqrt{a^2 - x^2}}{x}\right|$

- $\int x\sqrt{a^2 - x^2}\,\mathrm{d}x = -\dfrac{1}{3}(a^2 - x^2)^{3/2}$

- $\int x^2\sqrt{a^2 - x^2}\,\mathrm{d}x = \dfrac{x}{8}(2x^2 - a^2)\sqrt{a^2 - x^2} + \dfrac{a^4}{8}\arcsin\dfrac{x}{a}, \quad a > 0$

- $\int \dfrac{\mathrm{d}x}{\sqrt{a^2 - x^2}} = -\dfrac{1}{a}\ln\left|\dfrac{a + \sqrt{a^2 - x^2}}{x}\right|$

- $\int \dfrac{x\,\mathrm{d}x}{\sqrt{a^2 - x^2}} = -\sqrt{a^2 - x^2}$

- $\int \dfrac{x^2\,\mathrm{d}x}{\sqrt{a^2 - x^2}} = -\dfrac{x}{2}\sqrt{a^2 - x^2} + \dfrac{a^2}{2}\arcsin\dfrac{x}{a}, \quad a > 0$

- $\int \dfrac{\sqrt{a^2 + x^2}}{x}\,\mathrm{d}x = \sqrt{a^2 + x^2} - a\ln\left|\dfrac{a + \sqrt{a^2 + x^2}}{x}\right|$

- $\int \dfrac{\sqrt{x^2 - a^2}}{x}\,\mathrm{d}x = \sqrt{x^2 - a^2} - a\arccos\dfrac{a}{|x|}, \quad a > 0$

- $\int x\sqrt{x^2 \pm a^2}\,\mathrm{d}x = \dfrac{1}{3}(x^2 \pm a^2)^{3/2}$

- $\int \dfrac{\mathrm{d}x}{x\sqrt{x^2 + a^2}} = \dfrac{1}{a}\ln\left|\dfrac{x}{a + \sqrt{a^2 + x^2}}\right|$

- $\int \dfrac{\mathrm{d}x}{x\sqrt{x^2 - a^2}} = \dfrac{1}{a}\arccos\dfrac{a}{|x|}, \quad a > 0$

- $\int \dfrac{\mathrm{d}x}{x^2\sqrt{x^2 \pm a^2}} = \mp\dfrac{\sqrt{x^2 \pm a^2}}{a^2 x}$

- $\int \dfrac{x\,\mathrm{d}x}{\sqrt{x^2 \pm a^2}} = \sqrt{x^2 \pm a^2}$

- $\int \dfrac{\sqrt{x^2 \pm a^2}}{x^4}\,\mathrm{d}x = \mp\dfrac{(x^2 + a^2)^{3/2}}{3a^2 x^3}$

- $\int \dfrac{\mathrm{d}x}{ax^2 + bx + c} = \begin{cases} \dfrac{1}{\sqrt{b^2 - 4ac}}\ln\left|\dfrac{2ax + b - \sqrt{b^2 - 4ac}}{2ax + b + \sqrt{b^2 - 4ac}}\right|, \text{if } b^2 > 4ac \\ \dfrac{2}{\sqrt{4ac - b^2}}\arctan\dfrac{2ax + b}{\sqrt{4ac - b^2}}, \text{if } b^2 < 4ac \end{cases}$

- $\int \dfrac{\mathrm{d}x}{\sqrt{ax^2 + bx + c}} = \begin{cases} \dfrac{1}{\sqrt{a}}\ln\left|2ax + b + 2\sqrt{a}\sqrt{ax^2 + bx + c}\right|, \text{if } a > 0 \\ \dfrac{1}{\sqrt{-a}}\arcsin\dfrac{-2ax - b}{\sqrt{b^2 - 4ac}}, \text{if } a < 0 \end{cases}$

- $\int \sqrt{ax^2 + bx + c}\,\mathrm{d}x = \dfrac{2ax + b}{4a}\sqrt{ax^2 + bx + c} + \dfrac{4ax - b^2}{8a}\int \dfrac{\mathrm{d}x}{\sqrt{ax^2 + bx + c}}$

- $\int \dfrac{x\,\mathrm{d}x}{\sqrt{ax^2 + bx + c}} = \dfrac{\sqrt{ax^2 + bx + c}}{a} - \dfrac{b}{2a}\int \dfrac{\mathrm{d}x}{\sqrt{ax^2 + bx + c}}$

- $\int \dfrac{\mathrm{d}x}{x\sqrt{ax^2 + bx + c}} = \begin{cases} \dfrac{-1}{\sqrt{c}}\ln\left|\dfrac{2\sqrt{c}\sqrt{ax^2 + bx + c} + bx + 2c}{x}\right|, \text{if } c > 0 \\ \dfrac{1}{\sqrt{-c}}\arcsin\dfrac{bx + 2c}{|x|\sqrt{b^2 - 4ac}}, \text{if } c < 0 \end{cases}$

- $\int x^3\sqrt{x^2 + a^2}\,\mathrm{d}x = (\dfrac{1}{3}x^2 - \dfrac{2}{15}a^2)(x^2 + a^2)^{3/2}$

- $\int x^n \sin(ax)\,\mathrm{d}x = -\dfrac{1}{a}x^n\cos(ax) + \dfrac{n}{a}\int x^{n-1}\cos(ax)\,\mathrm{d}x$

- $\int x^n \cos(ax)\,\mathrm{d}x = \dfrac{1}{a}x^n\sin(ax) - \dfrac{n}{a}\int x^{n-1}\sin(ax)\,\mathrm{d}x$

- $\int x^n e^{ax}\,\mathrm{d}x = \dfrac{x^n e^{ax}}{a} - \dfrac{n}{a}\int x^{n-1}e^{ax}\,\mathrm{d}x$

- $\int x^n \ln(ax)\,\mathrm{d}x = x^{n+1}\left(\dfrac{\ln(ax)}{n + 1} - \dfrac{1}{(n + 1)^2}\right)$

- $\int x^n (\ln ax)^m\,\mathrm{d}x = \dfrac{x^{n+1}}{n + 1}(\ln ax)^m - \dfrac{m}{n + 1}\int x^n(\ln ax)^{m-1}\,\mathrm{d}x$