MTH 9821 Numerical Methods for Finance

Fall 2023

Homework 7
Assigned: October 23; Due: October 30

This homework is to be done as a group. Each team will hand in one homework solution, and each member of the team should write at least one problem. On the cover page of the homework, please indicate the members of the team and who wrote each problem.

Pricing European Put Options Using Finite Differences on a Fixed Computational Domain

An underlying asset has lognormal distribution with volatility $\sigma = 0.32$, spot price $S_0 = 36$, and pays dividends continuously at the rate $q = 0.015$. Consider a put option with strike $K = 40$ and maturity $T = 0.75$, i.e., 9 months, The interest rate is assumed to be constant and equal to $r = 0.04$.

We will solve the diffusion equation (the heat equation) on a bounded domain as follows:

$$u_\tau \;=\; u_{xx}, \;\forall\, x_{left} \;<\; x \;<\; x_{right}, \;\forall\, 0 \;<\; \tau \;<\; \tau_{final},$$

with boundary conditions

$$\begin{aligned} u(x,0) &= f(x), \;\forall\, x_{left} \;\le\; x \;\le\; x_{right}; \\ u(x_{left},\tau) &= g_{left}(\tau), \;\forall\, 0 \;\le\; \tau \;\le\; \tau_{final}; \\ u(x_{right},\tau) &= g_{right}(\tau), \;\forall\, 0 \;\le\; \tau \;\le\; \tau_{final}. \end{aligned}$$

1. **Computational domain:**
   We will use a fixed computational domain, where the node $x_{compute} = \ln\left(\frac{S_0}{K}\right)$ will not be on the finite difference mesh.
   The upper bound τ_{final} for τ is

$$\tau_{final} \;=\; \frac{T\sigma^2}{2}.$$

The computational domain on the x–axis is

$$[x_{left},\, x_{right}] \;=\; \left[\ln\left(\frac{S_0}{K}\right) + \left(r - q - \frac{\sigma^2}{2}\right)T - 3\sigma\sqrt{T},\; \ln\left(\frac{S_0}{K}\right) + \left(r - q - \frac{\sigma^2}{2}\right)T + 3\sigma\sqrt{T}\right].$$

The finite difference discretization is built as follows:
Start with M and α_{temp} given (α will be slightly smaller than α_{temp} in the end). Then,

$$\delta\tau \;=\; \frac{\tau_{final}}{M},$$

and δx will be approximately $\sqrt{\delta\tau/\alpha_{temp}}$. Let

$$N \;=\; \text{floor}\left(\frac{x_{right} - x_{left}}{\sqrt{\delta\tau/\alpha_{temp}}}\right),$$

where floor(y) is the largest integer smaller than or equal to y.
Then,

$$\delta x \;=\; \frac{x_{right} - x_{left}}{N}$$

and α is defined as

$$\alpha \;=\; \frac{\delta\tau}{(\delta x)^2}.$$

Note that $\alpha < \alpha_{temp}$.

2. **Boundary conditions**
Recall that

$$V(S,t) \;=\; \exp(-ax - b\tau)u(x,\tau),$$

with $x = \ln\left(\frac{S}{K}\right)$ and $\tau = \frac{(T-t)\sigma^2}{2}$.
For a put option, the boundary conditions for $u(x,\tau)$ are:

$$\begin{aligned}
f(x) \;&=\; K\,\exp(ax)\;\max(1-\exp(x),0),\;\forall\, x_{left} \;\le\; x \;\le\; x_{right};\\
g_{left}(\tau) \;&=\; K\,\exp(ax_{left} + b\tau)\;\left(\exp\left(-\frac{2r\tau}{\sigma^2}\right) - \exp\left(x_{left} - \frac{2q\tau}{\sigma^2}\right)\right),\;\forall\, 0 \;\le\; \tau \;\le\; \tau_{final};\\
g_{right}(\tau) \;&=\; 0,\;\forall\, 0 \;\le\; \tau \;\le\; \tau_{final}.
\end{aligned}$$

3. **Finite difference scheme**
Use Forward Euler with $\alpha = 0.45$ with the initial value $M = 4$, and then quadruple the number of points on the τ-axis, i.e., choose $M \in \{4M, 16M, 64M\}$.

To check the numbers run Forward Euler with $\alpha = 0.45$, and $M = 4$. Run your codes and record the values of the finite difference approximations at each nodes, including at the boundary nodes. For $M = 4$ and $\alpha = 0.45$ the corresponding value of N is $N = 11$. Thus, for each method above, you will have to fill out a table with five rows (corresponding to time steps from time step 0 – the boundary conditions, to time step 4 – corresponding to τ_{final}) and 12 columns (the first and the last column correspond to the boundary conditions at x_{left} and x_{right}, respectively).

4. **Pointwise Convergence**
Identify the interval containing $x_{compute} = \ln\left(\frac{S_0}{K}\right)$, i.e., find i such that

$$x_i \;\le\; x_{compute} \;<\; x_{i+1}.$$

Let

$$S_i \;=\; Ke^{x_i}; \tag{1}$$

$$S_{i+1} \;=\; Ke^{x_{i+1}} \tag{2}$$

be the values of S corresponding to the nodes x_i and x_{i+1}. Let

$$\begin{aligned}
V_i \;&=\; exp(-ax_i - b\tau_{final})U^M(i);\\
V_{i+1} \;&=\; exp(-ax_{i+1} - b\tau_{final})U^M(i+1)
\end{aligned}$$

be the approximate values of the option at the nodes S_i and S_{i+1}, respectively, where $U^M(i)$ and $U^M(i+1)$ are the finite difference approximations of $u(x_i,\tau_{final})$ and $u(x_{i+1},\tau_{final})$, respectively.

The approximate value of the option at S_0 is computed from V_i and V_{i+1} as follows:

$$V_{approx}(S_0,0) \;=\; \frac{(S_{i+1}-S_0)V_i \;+\; (S_0-S_i)V_{i+1}}{S_{i+1}-S_i}.$$

Let $V_{exact}(S_0,0)$ be the Black–Scholes value of the option. The pointwise relative error of the finite difference solution is

$$error\_pointwise \;=\; |V_{approx}(S_0,0) - V_{exact}(S_0,0)|.$$

Another way of computing an approximate value for the option would be to use linear interpolation to find the value of $u(x_{compute}, \tau_{final})$, and then use the change of variables to obtain $V_{approx,2}(S_0, 0)$, i.e.,

$$u(x_{compute}, \tau_{final}) \;=\; \frac{(x_{i+1} - x_{compute})u(x_i, \tau_{final}) \;+\; (x_{compute} - x_i)u(x_{i+1}, \tau_{final})}{x_{i+1} - x_i};$$

$$V_{approx,2}(S_0, 0) \;=\; exp(-ax_{compute} - b\tau_{final})u(x_{compute}, \tau_{final}).$$

Let

$$error\_pointwise\_2 \;=\; |V_{approx,2}(S_0, 0) - V_{exact}(S_0, 0)|$$

be the pointwise relative error corresponding to this method.

For each finite difference method, compute and record $error\_pointwise$ and $error\_pointwise\_2$, as well as the ratio of the approximation errors from one discretization level to the next.

5. **Root-Mean-Squared (RMS) Error**

Let $x_k = x_{left} + k\delta x$, $k = 0 : N$, be a nodal point on the x-axis, and let $S_k = Ke^x_k$ be the corresponding asset value. The approximate value of a put option with spot price S_k obtained from the finite difference scheme is

$$V_{approx}(S_k, 0) \;=\; \exp(-ax_k - b\tau_{final})U^M(k),$$

where $U^M(k)$ is the finite difference approximation of $u(x_k, \tau_{final})$.

Let $V_{exact}(S_k, 0)$ be the Black-Scholes value of a put option with spot price S_k.

Denote by N_{RMS} the number of nodes k such that $V_{exact}(S_k, 0) > 0.00001 \cdot S_0$. The RMS error $error\_RMS$ is defined as

$$error\_RMS \;=\; \sqrt{\frac{1}{N_{RMS}} \sum_{0 \leq k \leq N \text{ with } \frac{V_{exact}(S_k,0)}{S_0} > 0.00001} \frac{|V_{approx}(S_k, 0) - V_{exact}(S_k, 0)|^2}{|V_{exact}(S_k, 0)|^2}}$$

For each finite difference method, compute and record $error\_RMS$, as well as the ratio of the RMS errors from one discretization level to the next.

6. **Finite Difference Approximation of Δ, Γ, and Θ:**

Recall that x_i and x_{i+1} are consecutive nodes such that $x_i \leq x_{compute} < x_{i+1}$.

Let

$$S_{i-1} \;=\; Ke^{x_{i-1}};$$
$$S_{i+2} \;=\; Ke^{x_{i+2}}$$

be the values of S corresponding to the nodes x_{i-1} and x_{i+2}, respectively. Let V_{i-1} and V_{i+2} be the approximate values of the option at the nodes S_{i-1} and S_{i+2}, i.e.,

$$V_{i-1} \;=\; exp(-ax_{i-1} - b\tau_{final})U^M(i-1);$$
$$V_{i+2} \;=\; exp(-ax_{i+2} - b\tau_{final})U^M(i+2),$$

where $U^M(i-1)$ and $U^M(i+2)$ are the finite difference approximations of $u(x_{i-1}, \tau_{final})$ and $u(x_{i+2}, \tau_{final})$, respectively.

Compute the following finite different approximations of the Delta and of the Gamma of the option:

$$\Delta_{fd} \;=\; \frac{V_{i+1} - V_i}{S_{i+1} - S_i};$$

$$\Gamma_{fd} \;=\; \frac{\frac{V_{i+2} - V_{i+1}}{S_{i+2} - S_{i+1}} - \frac{V_i - V_{i-1}}{S_i - S_{i-1}}}{\frac{S_{i+2} + S_{i+1}}{2} - \frac{S_i + S_{i-1}}{2}}.$$

A finite difference approximation for the Theta of the option, i.e., for $\Theta = \frac{\partial V}{\partial t}$, can be obtained as follows:

From the change of variables $\tau = \frac{(T-t)\sigma^2}{2}$, it follows that

$$t = T - \frac{2\tau}{\sigma^2}. \tag{3}$$

Recall that $\tau_{final} = \frac{T\sigma^2}{2}$. Thus, $T = \frac{2\tau_{final}}{\sigma^2}$, and, from (3), we find that

$$t = \frac{2(\tau_{final} - \tau)}{\sigma^2}.$$

Then, the next to last time step on the τ–axis, i.e., $\tau_{final} - \delta\tau$, corresponds to time

$$\frac{2(\tau_{final} - (\tau_{final} - \delta\tau))}{\sigma^2} = \frac{2\delta\tau}{\sigma^2}$$

on the t–axis, so

$$\delta t = \left| 0 - \frac{2\delta\tau}{\sigma^2} \right| = \frac{2\delta\tau}{\sigma^2}.$$

Let

$$\begin{aligned} V_{i,\delta t} &= exp(-ax_i - b(\tau_{final} - \delta\tau))U^{M-1}(i); \\ V_{i+1,\delta t} &= exp(-ax_{i+1} - b(\tau_{final} - \delta\tau))U^{M-1}(i+1), \end{aligned}$$

where $U^{M-1}(i)$ and $U^{M-1}(i+1)$ are the finite difference approximations of $u(x_i, \tau_{final} - \delta\tau)$ and $u(x_{i+1}, \tau_{final} - \delta\tau)$, respectively. Let

$$V_{approx}(S_0, \delta t) = \frac{(S_{i+1} - S_0)V_{i,\delta t} \; + \; (S_0 - S_i)V_{i+1,\delta t}}{S_{i+1} - S_i}.$$

Compute the following finite difference approximation for the Theta of the option:

$$\Theta_{fd} = \frac{V_{approx}(S_0, \delta t) - V_{approx}(S_0, 0)}{\delta t}.$$

Pricing American Put Options Using Finite Differences on a Fixed Computational Domain

We now want to price an American Put option with the same parameters, i.e., $S_0 = 42$, $K = 40$, $T = 0.75$, $\sigma = 0.32$, $q = 0.02$, and $r = 0.04$.

There is no closed formula for pricing American put options. To test convergence, use the following value, obtained from an average binomial tree method with 10,000 time steps as the exact value of the American put:

$$P\_amer\_bin \ = \ 3.3045362802172642.$$

To price the American Put, we solve the same diffusion equation as for the European Put, taking into account the fact that the value of the put option must be greater than the early exercise value, i.e., $V(S,t) \geq \max(K - S, 0)$.

1. **Computational domain:**

The computational domain is the same as the computational domain used for the European Put option. (Note that the domain is therefore a fixed computational domain and interpolation will be required to compute the finite difference approximation of the value of the American option.)

2. **Boundary conditions**

As $S \searrow 0$, it is optimal to exercise the American put, and therefore $V(S,t) = K - S$. Using the change of variables from (S,t) to (x,τ), this corresponds to the following boundary condition at x_{left}:

$$g_{left}(\tau) \ = \ K \ \exp(a x_{left} + b\tau) \ \left(1 - \exp\left(x_{left}\right)\right), \ \forall \ 0 \ \leq \ \tau \ \leq \ \tau_{final}.$$

3. **Finite difference schemes**

The explicit Forward Euler scheme is modified as follows:

For European options at each time step m corresponding to $\tau = m\delta\tau$, we computed U^m directly from U^{m-1}, without having to solve a linear system. For American Options, the value of the put must be greater than $\max(K - S, 0)$, i.e., i.e., $V(S,t) \geq \max(K - S, 0)$. Using the change of variables $V(S,t) = \exp(-ax - b\tau)u(x,\tau)$, we find the following lower bound for $u(x,\tau)$:

$$u(x,\tau) \ \geq \ K\exp(ax + b\tau) \ \max(1 - exp(x), 0).$$

The discretized version of this inequality is

$$U^m_{Amer}(i) \ \geq \ early\_ex\_premium(i,m), \ \forall \ i = 2 : N,$$

where $early\_ex\_premium(i,m)$ is computed as

for m=1:M
    for i=1:(N-1)
      $x = x_{left} + i\delta x; \tau = m\delta\tau;$
      $early\_ex\_premium(i,m) \ = \ Kexp(ax + b\tau) \ \max(1 - exp(x), 0);$
    end
end

The Forward Euler code for pricing European options needs to be modified in only one place, as follows:

for i=1:(N-1)
    $U^m_{Amer}(i) = \max(U^m_{Euro}(i),$ early_ex_premium(i,m));
end

Finite difference schemes

Use Forward Euler with $\alpha = 0.45$ with the initial value $M = 4$, and then quadruple the number of points on the τ-axis, i.e., choose $M \in \{4M, 16M, 64M\}$.

To check the numbers run Forward Euler with $\alpha = 0.45$, and $M = 4$. Run your codes and record the values of the finite difference approximations at each nodes, including at the boundary nodes. For $M = 4$ and $\alpha = 0.45$ the corresponding value of N is $N = 11$. Thus, for each method, you have to fill out a table with five rows (corresponding to time steps from time step 0 – the boundary conditions, to time step 4 – corresponding to τ_{final}) and 12 columns (the first and the last column correspond to the boundary conditions at x_{left} and x_{right}, respectively).

1. **Pointwise Convergence:**
Identify the interval containing $x_{compute} = \ln\left(\frac{S_0}{K}\right)$, i.e., find i such that

$$x_i \ \le \ x_{compute} \ < \ x_{i+1}.$$

Let

$$S_i \ = \ Ke^{x_i}; \tag{4}$$

$$S_{i+1} \ = \ Ke^{x_{i+1}} \tag{5}$$

be the values of S corresponding to the nodes x_i and x_{i+1}. Let

$$\begin{aligned} V_i &= exp(-ax_i - b\tau_{final})U^M(i); \\ V_{i+1} &= exp(-ax_{i+1} - b\tau_{final})U^M(i+1) \end{aligned}$$

be the approximate values of the option at the nodes S_i and S_{i+1}, respectively, where $U^M(i)$ and $U^M(i+1)$ are the finite difference approximations of $u(x_i, \tau_{final})$ and $u(x_{i+1}, \tau_{final})$, respectively.

The approximate value of the option at S_0 is computed from V_i and V_{i+1} as follows:

$$V_{approx}(S_0, 0) \ = \ \frac{(S_{i+1} - S_0)V_i \ + \ (S_0 - S_i)V_{i+1}}{S_{i+1} - S_i}. \tag{6}$$

Let $V_{exact}(S_0, 0) = P\_amer\_bin \ = \ 3.3045362802172642$ be the value computed using the average binomial tree method. The pointwise relative error is

$$error\_pointwise \ = \ |V_{approx}(S_0, 0) - V_{exact}(S_0, 0)|.$$

Another way of computing an approximate value for the option is to use linear interpolation to find the value of $u(x_{compute}, \tau_{final})$, and then use the change of variables to obtain $V_{approx,2}(S_0, 0)$, i.e.,

$$u(x_{compute}, \tau_{final}) \ = \ \frac{(x_{i+1} - x_{compute})u(x_i, \tau_{final}) \ + \ (x_{compute} - x_i)u(x_{i+1}, \tau_{final})}{x_{i+1} - x_i};$$

$$V_{approx,2}(S_0, 0) \ = \ exp(-ax_{compute} - b\tau_{final})u(x_{compute}, \tau_{final}). \tag{7}$$

Let

$$error\_pointwise\_2 \ = \ |V_{approx,2}(S_0, 0) - V_{exact}(S_0, 0)|$$

be the pointwise relative error corresponding to this method.

For each finite difference method, compute and record *error_pointwise* and *error_pointwise_2*, as well as the ratio of the approximation errors from one discretization level to the next.

2. **Finite Difference Approximation of Δ, Γ, and Θ:**
Identical to the European case.

3. **Variance Reduction for American Option pricing**
Any finite difference scheme used to price an American option can also be used to price the European version of the same option. If the exact value of the European option is known, as is the case for a plain vanilla option where the Black–Scholes value can be computed, then the following three pieces of information are available:

- $P^{Amer}_{approx}(S_0, 0)$, the finite difference approximation of the price of the American option;
- $P^{Eur}_{approx}(S_0, 0)$, the finite difference approximation of the price of the European option;
- $P_{BS}(S_0, 0)$, the Black–Scholes price of the European put option.

Use linear interpolation of P in S to approximate $P^{Amer/Eur}_{approx}(S_0, 0)$ as seen in (6). (As opposed to linear interpolation of u in τ).

The variance reduction technique generates a new approximation for the price of the American option, by adding the finite difference error corresponding to the European option to the finite difference approximation of the American option, i.e.,

$$P^{Amer}_{VarRed}(S_0, 0) \;=\; P^{Amer}_{approx}(S_0, 0) \;+\; \left(P_{BS}(S_0, 0) - P^{Eur}_{approx}(S_0, 0)\right).$$

The corresponding approximation error is

$$error\_pointwise_{VarRed} \;=\; |P^{Amer}_{VarRed}(S_0, 0) - P\_amer\_bin|.$$

For each finite difference method, compute and record:

(1) $P^{Amer}_{VarRed}(S_0, 0)$ as "Var Red"
(2) $error\_pointwise_{VarRed}$ as "Var Red Pointwise Error"

4. **Early exercise domain**
Look only at Forward Euler with $M = 16$ and $\alpha = 0.45$. For each time step m, identify the interval where early exercise becomes optimal, i.e., find N_{opt} such that

$$\begin{aligned} U^m(N_{opt}) &= early\_ex\_premium(N_{opt}); \\ U^m(N_{opt}+1) &> early\_ex\_premium(N_{opt}+1). \end{aligned}$$

Note that this corresponds to

$$\tau \;=\; m\delta\tau,$$

which, in t-space, is

$$t \;=\; T \;-\; \frac{2\tau}{\sigma^2} \;=\; T \;-\; \frac{2m\delta\tau}{\sigma^2}.$$

Compute the corresponding values for the spot price, i.e.,

$$\begin{aligned} S_{N_{opt}} &= Kexp(x_{N_{opt}}) \\ S_{N_{opt}+1} &= Kexp(x_{N_{opt}+1}) \end{aligned}$$

and compute their average

$$S_{opt}(t) \;=\; \frac{S_{N_{opt}} \;+\; S_{N_{opt}+1}}{2},$$

where t is given by

$$t \;=\; T \;-\; \frac{2m\delta\tau}{\sigma^2}.$$

Record $(t, S_{opt}(t))$ for all $m = 0 : 16$.