MTH 9821 Numerical Methods for Finance

Fall 2023

Homework 10
Assigned: November 13; Due: November 20

This homework is to be done as a group. Each team will hand in one homework solution, using the blueprint solution provided by our teaching assistant.

In this homework you will be pricing American options using finite differences

Pricing American Put Options Using Finite Differences on a Fixed Computational Domain

We want to price an American Put option with the same parameters: An underlying asset has lognormal distribution with volatility $\sigma = 0.28$, spot price $S_0 = 37$, and pays dividends continuously at the rate $q = 0.02$. Consider an American put option with strike $K = 40$ and maturity $T = 0.75$, i.e., 9 months, The interest rate is assumed to be constant and equal to $r = 0.04$.

There is no closed formula for pricing American put options. To test convergence, use the following value, obtained from an average binomial tree method with 10,000 time steps as the exact value of the American put:

$$P\_amer\_bin \ = \ 5.0455539623.$$

To price the American Put, we solve the same diffusion equation as for the European Put, taking into account the fact that the value of the put option must be greater than the early exercise value, i.e., $V(S,t) \geq \max(K-S,0)$.

1. **Computational domain:**

The computational domain is the same as the computational domain used for the European Put option. (Note that the domain is therefore a fixed computational domain and interpolation will be required to compute the finite difference approximation of the value of the American option.)

2. **Boundary conditions**

As $S \searrow 0$, it is optimal to exercise the American put, and therefore $V(S,t) = K - S$. Using the change of variables from (S,t) to (x,τ), this corresponds to the following boundary condition at x_{left}:

$$g_{left}(\tau) \ = \ K \, \exp(a x_{left} + b\tau) \ \left(1 - \exp\left(x_{left}\right)\right), \ \forall \ 0 \ \leq \ \tau \ \leq \ \tau_{final}.$$

3. **Finite difference schemes**

The implicit Crank-Nicolson scheme using at iterative SOR solver to price American put options is modified as follows:

For European options, a linear system of the form

$$A U^m_{Euro} \ = \ b^m,$$

where b^m is computed using U^{m-1}, is solved at each time step using the SOR iteration as follows:

for j = 1:(N-1)

$$x_{n+1}(j) \ = \ (1-\omega)x_n(j) \ + \ \frac{\omega\alpha}{2(1+\alpha)}\left(x_{n+1}(j-1) + x_n(j+1)\right) \ + \ \frac{\omega}{1+\alpha} b(j);$$

end

until convergence is achieved, and then we set U^m_{Euro} to be x_n.

For American options, we use Projected SOR, i.e., we modify the iteration above as follows:

for j = 1:(N-1)

$$
\begin{aligned}
x_{n+1}(j) &= max(early\_ex\_premium(j,m), \\
& (1-\omega)x_n(j) \; + \; \frac{\omega\alpha}{2(1+\alpha)}\left(x_{n+1}(j-1)+x_n(j+1)\right) \; + \; \frac{\omega}{1+\alpha}b(j) \;);
\end{aligned}
$$

end

As initial guess, use the vector x_0 given by the early exercise premium, i.e.,

$$x_0(j) \;=\; early\_ex\_premium(j,m), \;\forall\; i = 1:(N-1).$$

Finite difference schemes

Use Crank-Nicolson with $\alpha \in \{0.45, 5\}$, to solve the diffusion equation for $u(x,\tau)$. For the implicit methods, use Projected SOR with relaxation parameter $\omega = 1.2$. The stopping criterion for Projected SOR is that the norm of the difference between two consecutive approximations is less than $tol = 10^{-6}$.

Run each finite difference method for the initial value $M = 4$, and then quadruple the number of points on the τ-axis, i.e., choose $M \in \{4M, 16M, 64M\}$.

To check the numbers, include the following: for Crank-Nicolson with $\alpha = 0.45$, let $M = 4$. Run your codes and record the values of the finite difference approximations at each nodes, including at the boundary nodes. For $M = 4$ and $\alpha = 0.45$ the corresponding value of N is $N = 11$. Thus, for each method, you have to fill out a table with five rows (corresponding to time steps from time step 0 – the boundary conditions, to time step 4 – corresponding to τ_{final}) and 12 columns (the first and the last column correspond to the boundary conditions at x_{left} and x_{right}, respectively).

1. **Pointwise Convergence:**

Identify the interval containing $x_{compute} = \ln\left(\frac{S_0}{K}\right)$, i.e., find i such that

$$x_i \;\le\; x_{compute} \;<\; x_{i+1}.$$

Let

$$S_i \;=\; Ke^{x_i}; \tag{1}$$

$$S_{i+1} \;=\; Ke^{x_{i+1}} \tag{2}$$

be the values of S corresponding to the nodes x_i and x_{i+1}. Let

$$
\begin{aligned}
V_i &= exp(-ax_i - b\tau_{final})U^M(i); \\
V_{i+1} &= exp(-ax_{i+1} - b\tau_{final})U^M(i+1)
\end{aligned}
$$

be the approximate values of the option at the nodes S_i and S_{i+1}, respectively, where $U^M(i)$ and $U^M(i+1)$ are the finite difference approximations of $u(x_i,\tau_{final})$ and $u(x_{i+1},\tau_{final})$, respectively.

The approximate value of the option at S_0 is computed from V_i and V_{i+1} as follows:

$$V_{approx}(S_0,0) \;=\; \frac{(S_{i+1}-S_0)V_i \;+\; (S_0-S_i)V_{i+1}}{S_{i+1}-S_i}. \tag{3}$$

Let $V_{exact}(S_0,0) = P\_amer\_bin \;=\; 3.3045362802172642$ be the value computed using the average binomial tree method. The pointwise relative error is

$$error\_pointwise \;=\; |V_{approx}(S_0,0) - V_{exact}(S_0,0)|.$$

Another way of computing an approximate value for the option is to use linear interpolation to find the value of $u(x_{compute},\tau_{final})$, and then use the change of variables to obtain $V_{approx,2}(S_0,0)$, i.e.,

$$u(x_{compute},\tau_{final}) \;=\; \frac{(x_{i+1}-x_{compute})u(x_i,\tau_{final}) \;+\; (x_{compute}-x_i)u(x_{i+1},\tau_{final})}{x_{i+1}-x_i};$$

$$V_{approx,2}(S_0,0) \;=\; exp(-ax_{compute} - b\tau_{final})u(x_{compute},\tau_{final}). \tag{4}$$

Let

$$error\_pointwise\_2 \;=\; |V_{approx,2}(S_0,0) - V_{exact}(S_0,0)|$$

be the pointwise relative error corresponding to this method.

For each finite difference method, compute and record *error_pointwise* and *error_pointwise_2*, as well as the ratio of the approximation errors from one discretization level to the next.

2. **Finite Difference Approximation of Δ, Γ, and Θ:**
   Recall that x_i and x_{i+1} are consecutive nodes such that $x_i \leq x_{compute} < x_{i+1}$.
   Let

$$\begin{aligned} S_{i-1} &= Ke^{x_{i-1}}; \\ S_{i+2} &= Ke^{x_{i+2}} \end{aligned}$$

be the values of S corresponding to the nodes x_{i-1} and x_{i+2}, respectively. Let V_{i-1} and V_{i+2} be the approximate values of the option at the nodes S_{i-1} and S_{i+2}, i.e.,

$$\begin{aligned} V_{i-1} &= exp(-ax_{i-1} - b\tau_{final})U^M(i-1); \\ V_{i+2} &= exp(-ax_{i+2} - b\tau_{final})U^M(i+2), \end{aligned}$$

where $U^M(i-1)$ and $U^M(i+2)$ are the finite difference approximations of $u(x_{i-1}, \tau_{final})$ and $u(x_{i+2}, \tau_{final})$, respectively.

Compute the following finite different approximations of the Delta and of the Gamma of the option:

$$\begin{aligned} \Delta_{fd} &= \frac{V_{i+1} - V_i}{S_{i+1} - S_i}; \\ \Gamma_{fd} &= \frac{\frac{V_{i+2}-V_{i+1}}{S_{i+2}-S_{i+1}} - \frac{V_i - V_{i-1}}{S_i - S_{i-1}}}{\frac{S_{i+2}+S_{i+1}}{2} - \frac{S_i + S_{i-1}}{2}}. \end{aligned}$$

A finite difference approximation for the Theta of the option, i.e., for $\Theta = \frac{\partial V}{\partial t}$, can be obtained as follows:

From the change of variables $\tau = \frac{(T-t)\sigma^2}{2}$, it follows that

$$t = T - \frac{2\tau}{\sigma^2}. \tag{5}$$

Recall that $\tau_{final} = \frac{T\sigma^2}{2}$. Thus, $T = \frac{2\tau_{final}}{\sigma^2}$, and, from (5), we find that

$$t = \frac{2(\tau_{final} - \tau)}{\sigma^2}.$$

Then, the next to last time step on the τ–axis, i.e., $\tau_{final} - \delta\tau$, corresponds to time

$$\delta t = \frac{2(\tau_{final} - (\tau_{final} - \delta\tau))}{\sigma^2} = \frac{2\delta\tau}{\sigma^2}$$

on the t–axis.

Let

$$\begin{aligned} V_{i,\delta t} &= exp(-ax_i - b(\tau_{final} - \delta\tau))U^{M-1}(i); \\ V_{i+1,\delta t} &= exp(-ax_{i+1} - b(\tau_{final} - \delta\tau))U^{M-1}(i+1), \end{aligned}$$

where $U^{M-1}(i)$ and $U^{M-1}(i+1)$ are the finite difference approximations of $u(x_i, \tau_{final} - \delta\tau)$ and $u(x_{i+1}, \tau_{final} - \delta\tau)$, respectively. Let

$$V_{approx}(S_0, \delta t) = \frac{(S_{i+1} - S_0)V_{i,\delta t} + (S_0 - S_i)V_{i+1,\delta t}}{S_{i+1} - S_i}.$$

Compute the following finite difference approximation for the Theta of the option:

$$\Theta_{fd} = \frac{V_{approx}(S_0, \delta t) - V_{approx}(S_0, 0)}{\delta t}.$$

3. **Variance Reduction for American Option pricing**

Any finite difference scheme used to price an American option can also be used to price the European version of the same option. If the exact value of the European option is known, as is the case for a plain vanilla option where the Black–Scholes value can be computed, then the following three pieces of information are available:

- $P_{approx}^{Amer}(S_0, 0)$, the finite difference approximation of the price of the American option;

- $P^{Eur}_{approx}(S_0, 0)$, the finite difference approximation of the price of the European option;
- $P_{BS}(S_0, 0)$, the Black–Scholes price of the European put option.

Use linear interpolation of P in S to approximate $P^{Amer/Eur}_{approx}(S_0, 0)$ as seen in (3). (As opposed to linear interpolation of u in τ).

The variance reduction technique generates a new approximation for the price of the American option, by adding the finite difference error corresponding to the European option to the finite difference approximation of the American option, i.e.,

$$P^{Amer}_{VarRed}(S_0, 0) \;=\; P^{Amer}_{approx}(S_0, 0) \;+\; \Big(P_{BS}(S_0, 0) - P^{Eur}_{approx}(S_0, 0)\Big).$$

The corresponding approximation error is

$$error\_pointwise_{VarRed} \;=\; |P^{Amer}_{VarRed}(S_0, 0) - P\_amer\_bin|.$$

For each finite difference method, compute and record:

(1) $P^{Amer}_{VarRed}(S_0, 0)$ as “Var Red”
(2) $error\_pointwise_{VarRed}$ as “Var Red Pointwise Error”