LIE GROUPS, VECTOR FIELDS & BUNDLES: A BRIEF EXCURSION

BAILEE ZACOVIC

These partial notes created in Fall 2024 refer to the second edition of *Introduction to Smooth Manifolds* by John M. Lee, and aim to survey key objects and results contained in Chapters 7-10, with a view toward later applications.

1. Lie Groups

A Lie group is an object of differential topology which enjoys extra algebraic structure; this hybrid is an example of a *topological group*. We recall a few definitions before proceeding to central results.

Definition 1.1. A *Lie group* is a smooth manifold G (without boundary) with the property that multiplication $m : G \times G \to G$, $m(g, h) = gh$, and inversion $i : G \to G$, $i(g) = g^{-1}$, are smooth.

The multiplication map turns out to be a smooth submersion, as is evident via local sections.

Proposition 1.2. ([1, Exercise 7-1]) *For G a Lie group, the multiplication map $m : G \times G \to G$ is a smooth submersion.*

Proof. Recall that for $F : M \to N$ a smooth map, a local section is a map $\sigma : U \to M$ for some $U \subset N$ such that $F \circ \sigma = \mathrm{Id}_U$. By the local section theorem, it suffices to show that each $(g, h) \in G \times G$ lies in the image of a smooth local section of m. To this end, fix $h \in G$, and consider the smooth map $\sigma : G \to G \times G$ given by $\sigma(g') = (g'h^{-1}, h)$. Indeed, $(F \circ \sigma)(g') = g'h^{-1}h = g'$, and $(g, h) = \sigma(gh)$. □

Ubiquitous examples of Lie groups include the following.

Example 1.3. (A few Lie Groups)

- GL$(n, \mathbb{R})$: $n \times n$ real matrices with non-zero determinant (non-compact)
- GL$(n, \mathbb{C})$: $n \times n$ complex matrices with non-zero determinant (non-compact)
- SL$(n, \mathbb{R})$: $n \times n$ real matrices with determinant 1 (non-compact for $n \geq 2$)
- SL$(n, \mathbb{C})$: $n \times n$ complex matrices with determinant 1 (non-compact for $n \geq 2$)
- O(n): $n \times n$ real matrices satisfying $AA^T = I_n$ (compact)
- SO(n): $n \times n$ real determinant-1 matrices satisfying $AA^T = I_n$ (compact)
- U(n): $n \times n$ complex matrices satisfying $AA^* = I_n$ (compact)
- SU(n): $n \times n$ complex determinant-1 matrices satisfying $AA^* = I_n$ (compact)

We next define special maps between Lie groups, which are smooth in an analytic sense and simultaneously respect the group theoretic structure.

Definition 1.4. For G and H Lie groups, a *Lie group homomorphism from G to H* is a smooth map $F : G \to H$ that is also a group homomorphism.

Examples include $\det : \mathrm{GL}(n, \mathbb{R}) \to \mathbb{R}^*$ and conjugation $C_g : G \to G$, $C_g(h) = ghg^{-1}$. These homomorphisms are "rigid" in the following sense.

Theorem 1.5. *Every Lie group homomorphism has constant rank.*

Proof. Let $F : G \to H$ be a Lie group homomorphism. Via the Equivariant Rank Theorem for the action of left translation on G by G and on H by $F(G)$, we deduce F has constant rank. The commutative diagram to keep in mind is the following:

$$\begin{tikzcd} T_eG \arrow[r, "dF_e"] \arrow[d, "d(L_g)_e"'] & T_{\tilde{e}}H \arrow[d, "d(L_{F(g)})_{\tilde{e}}"] \\ T_gG \arrow[r, "dF_g"'] & T_{F(g)}H \end{tikzcd}.$$

□

Connected groups also admit especially nice universal covers, as we discover in the following result.

Theorem 1.6. *Let G be a connected Lie group. There exists a simply connected Lie group $\tilde{G}$, called the universal covering group of G, that admits a smooth covering map $\pi : \tilde{G} \to G$ that is also a Lie group homomorphism.*

For G a Lie group, we call the connected component of G containing the identity $e \in G$ the *identity component*. This component is distinguished by a few properties:

Proposition 1.7. *Let G be a Lie group and G_0 its identity component. Then G_0 is a normal subgroup of G, and is the only connected open subgroup. Every connected component of G is diffeomorphic to G_0.*

Proof. Normality follows from the fact that conjugation preserves connected components. Next, any open subgroup $H \subset G$ contains the identity, and is furthermore closed by $G \setminus H = \cup_{g \in G \setminus H} gH$ a union of open subsets which do not intersect H. It follows that H is a union of components – G_0 and others – so demanding that H be connected forces $H = G_0$. Every connected component of G is diffeomorphic to G_0 by smooth translation. □

Subgroups of Lie groups are sub-objects in both an algebraic and a topological sense.

Definition 1.8. A *Lie subgroup of G* is a subgroup of G endowed with a topology and smooth structure making it into a Lie group and an immersed submanifold of G.

As we might expect from group theory, Lie subgroups indeed arise as kernels of homomorphisms.

Proposition 1.9. *Let $F : G \to H$ be a Lie group homomorphism. The kernel of F is a properly embedded Lie subgroup of G, whose codimension equals the rank of F.*

Proof. By Thereoem 1.5, F has constant rank, and so the kernel of F is a properly embedded submanifold of G whose codimension equals the rank of F. It is further a Lie subgroup by the fact that multiplication and inversion restrict to smooth maps there (see [1, Proposition 7.11]). □

2. Vector Fields

If we imagine a particle moving along a manifold, a vector field is roughly a prescription of velocity at each point. For Lie groups, the collection of smooth vector fields invariant under left multiplication will comprise a special algebraic object known as the associated Lie algebra.

Definition 2.1. For M a smooth manifold with or without boundary, a *vector field on M* is a continuous map $X : M \to TM$ satisfying $\pi \circ X = \mathrm{Id}_M$.

We will shorthand $X_p := X(p)$. A choice of smooth chart renders a canonical local description of vector fields. For $X : M \to TM$ is a rough vector field and $(U, (x^i))$ is any smooth coordinate chart for M, we may write the value of X at $p \in U$ in terms of the coordinate basis:

$$X_p = X^i(p) \frac{\partial}{\partial x^i}\bigg|_p .$$

The X^i are called *component functions*.

A prosaic example of a vector field is supplied by *coordinate vector fields*.

Example 2.2 (Coordinate Vector Fields)**.** If $(U, (x^i))$ is any smooth chart on M, then the assignment $p \mapsto \frac{\partial}{\partial x^i}|_p$ is called the ith coordinate vector field on U. Its component functions are constant.

Theorem 2.3. *Let M be a smooth manifold with or without boundary, and let $X : M \to TM$ be a rough vector field. If $(U, (x^i))$ is any smooth coordinate chart on M, then the restriction fo X to U is smooth if and only if its component functions with respect to this chart are smooth.*

Occasionally, a vector field is only specified for an arbitrary subset $A \subset M$. It will be convenient to extend the vector field away from A.

Definition 2.4. A *smooth vector field along A* is a smooth map $X : A \to TM$ satisfying $\pi \circ X = \mathrm{Id}_A$ such that for all $p \in A$, there exists a neighborhood $V \ni p$ and a smooth vector field $\tilde{X}$ such that $\tilde{X}|_{V \cap A} = X|_{V \cap A}$.

Lemma 2.5 (Extension Lemma for Vector Fields)**.** *Let M be a smooth manifold with or without boundary, and let $A \subset M$ be a closed subset. Suppose X is a smooth vector field along A, then given any subset $U \supset A$, there exists a smooth global vector field $\tilde{X}$ on M such that $\tilde{X}|_A = X$ and supp$(\tilde{X}) \subset U$.*

We can gather a collection of vector fields on M and, at each point $p \in M$, query the usual linear algebraic notions like linear independence.

Definition 2.6. For M a smooth manifold with or without boundary, an ordered k-tuple $(X_1, \dots, X_k)$ of vector fields defined on $A \subset M$ is said to

- be *linearly independent* if $(X_1|_p, \dots, X_k|_p)$ is a linearly independent k-tuple in T_pM for each $p \in A$,
- *span the tangent bundle* if the k-tuple $(X_1|_p, \dots, X_k|_p)$ spans T_pM at each $p \in A$, and
- be *orthonormal* if for each $p \in A$, the vectors $(X_1|_p, \dots, X_k|_p)$ are orthonormal with respect to the Euclidean dot product.

Definition 2.7. A *local frame for M* is an ordered n-tuple of vector fields $(E_1, \dots, E_n)$ defined on an open subset $U \subset M$ which is linearly independent and spans the tangent bundle. We call it a *global frame* when $U = M$, and an *orthonormal frame* when $(E_1, \dots, E_n)$ is orthonormal.

Example 2.8. (Coordinate Frames) If $(U, (x^i))$ is any smooth coordinate chart for a smooth manifold M, then the coordinate vector fields form a smooth local frame $(\partial/\partial x^i)$ on U, called the *coordinate frame*.

Proposition 2.9. *Let M be a smooth n-manifold with or without boundary.*

(1) If $(X_1, \dots, X_k)$ is a linearly independent k-tuple of smooth vector fields on an open subset $U \subset M$, $1 \leq k < n$, then for each $p \in U$ there exist smooth vector fields $X_{k+1}, \dots, X_n$ in a neighborhood V of p such that $(X_1, \dots, X_n)$ is a smooth local frame for M on $V \cap U$.

(2) If $(v_1, \dots, v_k)$ is a linearly independent k-tuple of vectors in T_pM for some $p \in M$, $1 \leq k < n$, then there exists a smooth local frame (X_i) on a neighborhood of p such that $X_i|_p = v_i$ for $i = 1, \dots, k$.

(3) If $(X_1, \dots, X_n)$ is a linearly independent n-tuple of smooth vector fields along a closed subset $A \subset M$, then there exists a smooth local frame $(\tilde{X}_1, \dots, \tilde{X}_n)$ on some neighborhood of A such that $\tilde{X}_i|_A = X_i$ for $i = 1, \dots, n$.

Proof. The key ingredient in each part is that the determinant map and each X_i are continuous, hence the set of points where some $X_1|_p, \dots, X_n|_p$ are linearly independent is open in M. □

These notions will inspire a necessary and sufficient criterion for smooth subbundles later.

We use the notation $\mathscr{X}(M)$ to denote the vector space of all smooth vector fields on M. It is a module over the ring $C^\infty(M)$. Now, it is natural to ask for the relatedness of vector fields X and Y on M and N, respectively, wherever there exists a smooth map $F : M \to N$.

Definition 2.10. Suppose $F : M \to N$ is smooth and X is a vector field on M, and suppose there happens to be a vector field Y on N with the property that for each $p \in M$,

$$dF_p(X_p) = Y_{F(p)}.$$

Then X and Y are *F-related*.

Whenever F is a diffeomorphism, there is a distinguished pair of vector fields which are F-related, namely a vector field and its pushforward.

Definition 2.11. When $F : M \to N$ is a diffeomorphism and X is a vector field on M, define the *pushforward of X by F* F_*X given explicitly by,

$$(F_*X)_q = dF_{F^{-1}(q)}(X_{F^{-1}(q)}).$$

We might try computing the pushforward explicitly in an example.

Example 2.12. Consider M and N the following open submanifolds of $\mathbb{R}^2$:

$$M = \{(x, y) : y > 0 \text{ and } x + y > 0\},$$
$$N = \{(u, v) : u > 0 \text{ and } v > 0\}.$$

Define the smooth map $F : M \to N$ given by $F(x, y) = (x + y, x/y + 1)$. We can compute the smooth inverse: $F^{-1}(u, v) = (u - u/v, u/v)$. Let X be the smooth vector field on M given by

$$X_{(x,y)} = y^2 \frac{\partial}{\partial x}\bigg|_{(x,y)}.$$

Fix some point $(u, v) \in N$. Then the Jacobian of F at the point $(u - u/v, u/v)$ is given by

$$dF_{(u-u/v,u/v)} = \begin{pmatrix} 1 & 1 \\ \frac{v}{u} & \frac{v-v^2}{u} \end{pmatrix}.$$

Hence

$$(F_*X)_{(u,v)} = dF_{\left(u-\frac{u}{v},\frac{u}{v}\right)}\left(X_{\left(u-\frac{u}{v},\frac{u}{v}\right)}\right) = \frac{u^2}{v^2}\frac{\partial}{\partial u}\bigg|_{(u,v)} + \frac{u}{v}\frac{\partial}{\partial v}\bigg|_{(u,v)}.$$

Example 2.13. ([1, Exercise 8-10]) Consider again M the following open submanifold of $\mathbb{R}^2$:

$$M = \{(x, y) : y > 0 \text{ and } x + y > 0\},$$

and let $F : M \to M$ be the map $F(x, y) = (xy, y/x)$. We show that F is a diffeomorphism, and compute F_*X and F_*Y, where

$$X = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, \text{ and } Y = y\frac{\partial}{\partial x}.$$

Notice that the inverse F^{-1} can be explicitly computed as $F^{-1}(u, v) = \left(\sqrt{u/v}, \sqrt{uv}\right)$. We also compute the Jacobian of F at the point $\left(\sqrt{u/v}, \sqrt{uv}\right)$ is given by

$$dF_{\left(\sqrt{u/v},\sqrt{uv}\right)} = \begin{pmatrix} \sqrt{uv} & \sqrt{u/v} \\ -v\sqrt{\frac{v}{u}} & \sqrt{\frac{v}{u}} \end{pmatrix}.$$

Straightforward computation then yields

$$(F_*X)_{(u,v)} = dF_{\left(\sqrt{u/v},\sqrt{uv}\right)}\left(X_{\left(\sqrt{u/v},\sqrt{uv}\right)}\right) = 2u\frac{\partial}{\partial u},$$

$$(F_*Y)_{(u,v)} = dF_{\left(\sqrt{u/v},\sqrt{uv}\right)}\left(Y_{\left(\sqrt{u/v},\sqrt{uv}\right)}\right) = uv\frac{\partial}{\partial u} - v^2\frac{\partial}{\partial v}.$$

Next, we explore a composition of vector fields supplied by the Lie bracket. First, we define the notion of a derivation:

Definition 2.14. A map $X : C^\infty(M) \to C^\infty(M)$ is called a *derivation* if it is linear over $\mathbb{R}$ and satisfies

$$X(fg) = f \cdot Xg + g \cdot Xf, \tag{1}$$

where $Xf : M \to \mathbb{R}$ is given by $(Xf)(p) = X_p f$, $f \in C^\infty(M)$. (Here we are really invoking the point of view of X_p as a derivation.)

Proposition 2.15. *Let M be a smooth manifold with or without boundary. A map $D : C^\infty(M) \to C^\infty(M)$ is a derivation if and only if it is of the form $Df = Xf$ for some smooth vector field $X \in \mathscr{X}(M)$.*

We are now poised to define the Lie bracket.

Definition 2.16. For X and Y smooth vector fields on M, and $f : M \to \mathbb{R}$ a smooth function, the *Lie bracket of X and Y* is given by the operator $[X, Y] : C^\infty(M) \to C^\infty(M)$, defined by

$$[X, Y]f = XYf - YXf.$$

Lemma 2.17. *The Lie bracket of any pair of smooth vector fields X and Y is a smooth vector field.*

Proof. By Proposition 2.15, suffices to show that $[X, Y]$ satisfies 1. A straightforward check verifies that $[X, Y](fg) = f \cdot [X, Y]g + g \cdot [X, Y]f$ for $f, g \in C^\infty(M)$. ☐

For coordinate expressions, $X = X^i\partial/\partial x^i$ and $Y = Y^j\partial/\partial x^j$, $[X, Y]$ admits the coordinate expression given by

$$[X, Y] = \left(X^i\frac{\partial Y^j}{\partial x^i} - Y^i\frac{\partial X^j}{\partial x^i}\right)\frac{\partial}{\partial x^j}.$$

Example 2.18. Define smooth vector fields $X, Y \in \mathscr{X}(\mathbb{R}^3)$ by

$$X = x\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + x(y+1)\frac{\partial}{\partial z},$$

$$Y = \frac{\partial}{\partial x} + y\frac{\partial}{\partial z}.$$

We compute

$$[X, Y] = (-1)\frac{\partial}{\partial x} + (0)\frac{\partial}{\partial y} + (-(y+1)+1)\frac{\partial}{\partial z} = -\frac{\partial}{\partial x} - y\frac{\partial}{\partial z}.$$

Lie brackets are also naturally compatible with pushforward as per the following result:

Proposition 2.19. *Suppose $F : M \to N$ is a diffeomorphism and $X_1, X_2 \in \mathscr{X}(M)$. Then $F_*[X_1, X_2] = [F_*X_1, F_*X_2]$.*

Certain vector fields are closed under Lie brackets: left-invariant vector fields. Recall the smooth, transitive action by left translation L_g on a Lie group G. That is, $L_g(h) := gh$.

Definition 2.20. A vector field X on G is said to be *left-invariant* if it is L_g-related to itself for every $g \in G$, i.e.

$$d(L_g)_{g'}(X_{g'}) = X_{gg'} \text{ for all } g, g' \in G.$$

This means that the structure of the vector field internally reflects the action of the translation map. A nice feature of the Lie bracket is that it inherits the property of left-invariance whenever its constituent vector fields are left-invariant.

Proposition 2.21. *Let G be a Lie group, and suppose X and Y are smooth left-invariant vector fields on G. Then $[X, Y]$ is also left-invariant.*

Proof. Fix $g \in G$. Since $(L_g)_*X = X$ and $(L_g)_*Y = Y$, it follows by 2.19 that

$$(L_g)_*[X, Y] = [(L_g)_*X, (L_g)_*Y] = [X, Y].$$

□

We come to the definition of the Lie algebra associated to a Lie group G:

Definition 2.22. For G a Lie group, the *Lie algebra of G* denoted Lie(G) is given by the vector space of all smooth, left-invariant vector fields on G endowed with a multiplication given by the Lie bracket.

What is the dimension of Lie(G)?

Theorem 2.23. *For G a Lie group, the evaluation map ϵ : Lie$(G) \to T_eG$ given by $\epsilon(X) = X_e$ is a vector space isomorphism, hence dim(Lie(G)) = dim(G).*

Example 2.24. (A few Lie Algebras)

- Lie(GL$(n, \mathbb{R})$): $n \times n$ real matrices
- Lie(GL$(n, \mathbb{C})$): $n \times n$ complex matrices
- Lie(SL$(n, \mathbb{R})$): $n \times n$ real matrices with zero trace
- Lie(SL$(n, \mathbb{C})$): $n \times n$ complex matrices with trace zero
- Lie(O(n)): $n \times n$ real matrices satisfying $A^T + A = 0$
- Lie(SO(n)): $n \times n$ real matrices satisfying $A^T + A = 0$
- Lie(U(n)): $n \times n$ complex matrices satisfying $A^* + A = 0$
- Lie(SU(n)): $n \times n$ complex matrices satisfying $A^* + A = 0$ and trace zero

One important feature of Lie group homomorphisms is that they induce Lie algebra homormorphisms.

Theorem 2.25. *For G, H Lie groups, and $\mathfrak{g}, \mathfrak{h}$ their associated Lie algebras, if $F : G \to H$ is a Lie group homomorphism, then for every $X \in \mathfrak{g}$ there is a unique vector field $Y = F_*X$ in $\mathfrak{h}$ which is F-related to $\mathfrak{g}$. The map $F_* : \mathfrak{g} \to \mathfrak{h}$ is consequently a Lie algebra homomorphism.*

3. Flows

Flows are families of diffeomorphisms of smooth manifolds, which we will learn arise uniquely from smooth vector fields. These objects store the data of maximal integral curves on a manifold, which are smooth curves whose velocity at every point is prescribed by the associated vector field. It also happens that left-invariant vector fields on Lie groups yield particularly nice flows, which in turn furnish a coordinate-independent notion of a *Lie derivative*.

Definition 3.1. For V a vector field on a smooth manifold M, an *integral curve of* V is a differentiable curve $\gamma : J \to M$, $J \subset \mathbb{R}$ open, satisfying

$$\gamma'(t) = V_{\gamma(t)} \tag{2}$$

for all $t \in J$.

In practice, computing integral curves amounts to solving systems of ordinary differential equations.

Example 3.2. Let $W = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$ on $\mathbb{R}^2$. For $\gamma(t) = (x(t), y(t))$, then (2) stipulates

$$x'(t) = -y(t) \text{ and } y'(t) = x(t),$$

i.e. $x(t) = a\cos(t) - b\sin(t)$ and $y(t) = a\sin(t) + b\cos(t)$ for $a, b \in \mathbb{R}$.

The following lemma is a useful device which will guarantee periodicity for a particular family of integral curves, namely maximal and nonconstant.

Lemma 3.3. *(Translation Lemma) Let V, M, J, and γ be as above. For any $b \in \mathbb{R}$, the curve $\hat{\gamma} : \hat{J} \to M$ given by $\hat{\gamma}(t) = \gamma(t+b)$ is also an integral curve of V, where $\hat{J} = \{t : t + b \in J\}$.*

Proof. This follows from the observation that $\hat{\gamma}'(t) = \gamma'(t+b) = V_{\gamma(t+b)} = V_{\hat{\gamma}(t)}$. □

For a smooth map $F : M \to N$, integral curves of some vector field X over M may be transported to integral curves of a vector field Y over N whenever X and Y are F-related. It turns out that this F-relatedness is also necessary.

Proposition 3.4. *For M and N smooth manifolds and $F : M \to N$ a smooth map, $X \in \mathscr{X}(M)$ and $Y \in \mathscr{X}(N)$ are F-related if and only if F takes integral curves of X to integral curves of Y via composition.*

Proof. Both directions roughly follow from the computation,

$$Y_{F(\gamma(t))} = dF_{\gamma(t)}X_{\gamma(t)} = dF_{\gamma(t)} \cdot \gamma'(t) = (F \circ \gamma)'(t).$$

□

We can collect integral curves on smooth manifolds into global objects known as *flows*, which can be restricted to recover integral curves at every point. Special kinds of flows can be identified as continuous left $\mathbb{R}$-actions on smooth manifolds.

Definition 3.5. A *global flow* on a smooth manifold M is a continuous left $\mathbb{R}$-action on M, i.e. a continuous map $\theta : \mathbb{R} \times M \to M$ satisfying

$$\theta(t, \theta(s,p)) = \theta(t+s,p), \text{ and } \theta(0,p) = p \text{ for all } s,t \in \mathbb{R},\ p \in M.$$

We define, for each $p \in M$, the curve $\theta^{(p)} : \mathbb{R} \to M$ given by

$$\theta^{(p)}(t) = \theta(t,p) \text{ for all } t \in \mathbb{R}.$$

Global flows induce smooth vector fields in the following sense:

Proposition 3.6. *Let $\theta : \mathbb{R} \times M \to M$ be a smooth global flow on a smooth manifold M. The vector field V given by the assignment $p \mapsto (\theta^{(p)})'(0)$ is a smooth vector field on M, and each curve $\theta^{(p)}$ is an integral curve on M.*

Not all flows can be defined globally. For this reason, we define the following relaxed version:

Definition 3.7. A *flow* (or *local flow*) on M is a continuous map $\mathcal{D} \to M$ where $\mathcal{D} \subset \mathbb{R} \times M$ is such that $\mathcal{D}^{(p)} = \{t \in \mathbb{R} : (t,p) \in \mathcal{D}\}$ is an open interval containing 0 for all $p \in M$,

$$\theta(0,p) = p$$

for all $p \in M$, and

$$\theta\left(t, \theta(s,p)\right) = \theta(t+s,p)$$

for all $s \in \mathcal{D}^{(p)}, t \in \mathcal{D}^{(\theta(s,p)}$ such that $s + t \in \mathcal{D}^{(p)}$.

A *maximal integral curve* $\gamma : J \to M$ is one which admits no extension to an integral curve on a larger open interval $J' \supset J$, and analogously a *maximal flow* is a flow $\theta : \mathcal{D} \to M$ is one which cannot be extended to a flow on a larger domain $\mathcal{D}' \supset \mathcal{D}$.

We proceed to the central result of this chapter, a statement which establishes the unique association between smooth vector fields and flows.

Theorem 3.8. *Let V be a smooth vector field on a smooth manifold M. There is a unique smooth maximal flow $\theta : \mathcal{D} \to M$ such that $(\theta^{(p)})'(0) = V_p$. This flow satisfies:*

1. *for each $p \in M$, the curve $\theta^{(p)} : \mathcal{D}^{(p)} \to M$ is the unique maximal integral curve of V starting at p,*
2. *If $s \in \mathcal{D}^{(p)}$, then $\mathcal{D}^{(\theta(s,p))}$ is the interval $\mathcal{D}^{(p)} - s = \{t - s : t \in \mathcal{D}^{(p)}\}$,*
3. *for each $t \in \mathbb{R}$, the set $M_t = \{p \in M : (t,p) \in \mathbb{D}\} \subset M$ is open, and $\theta_t : M_t \to M_{-t}$ is a diffeomorphism with inverse θ_{-t}.*

For V and θ as above, the correct nomenclature for θ is the *flow generated by V* or the *flow of V*. While not all smooth vector fields generate global flows, the ones which do are deemed *complete*. Certain attributes of vector fields will guarantee them complete, as we discover in the subsequent lemma and theorem.

Lemma 3.9. *Let V be a smooth vector field on a smooth manifold M, and let θ be its flow. Suppose there is a positive number ϵ such that for every $p \in M$, the domain of $\theta^{(p)}$ contains $(-\epsilon, \epsilon)$. Then V is complete.*

Proof. By contradiction, and an application of the translation lemma. ☐

Theorem 3.10. *Every compactly supported smooth vector field on a smooth manifold is complete.*

Proof. Cover the support by $\theta^{(p)}\left((-\epsilon_p, \epsilon_p)\right)$ for all $p \in M$, extract a finite subcover, and take the minimum of the $\epsilon_p > 0$. ☐

Complete vector fields are particularly abundant for Lie groups by virtue of their algebraic structure, as we next discover.

Theorem 3.11. *Every left-invariant vector field on a Lie group is complete.*

Proof. Transfer the domain $(-\epsilon, \epsilon)$ of $\theta^{(e)}$ to $\theta^{(p)}$ for all $p \in M$ by left invariance, then apply Lemma 3.9. ☐

We distinguish two kinds of points of vector fields: singular and regular.

Definition 3.12. For V a vector field on M, a point $p \in M$ is *singular* if $V_p = 0$, and *regular* otherwise.

The behavior of a flow θ around these points is determined in the following sense:

Proposition 3.13. *Let V be a smooth vector field on a smooth manifold M, and let $\theta : \mathcal{D} \to M$ be the flow generated by V. If $p \in M$ is a singular point of V, then $\mathcal{D}^{(p)} = \mathbb{R}$ and $\theta^{(p)} \equiv p$. If p is a regular point, then $\theta^{(p)} : \mathcal{D}^{(p)} \to M$ is a smooth immersion.*

Proof. That $\theta^{(p)} : \mathbb{R} \to M$ is given by $\theta^{(p)}(t) = p$ follows from $\theta^{(p)}$ the unique maximal integral curve at p a singular point. That $\theta^{(p)}$ is an immersion for p regular follows by a contradiction argument and the previous case. ☐

Theorem 3.14. *Let V be a smooth vector field on a smooth manifold M, and let $p \in M$ be a regular point of V. There exist smooth coordinates (s^i) on some neighborhood of p in which V has the coordinate representation $\partial/\partial s^1$.*

Generalizing directional derivatives to arbitrary vector fields may at first seem too formidable a task. The traditional construction given by

$$D_v W(p) = \left.\frac{d}{dt}\right|_{t=0} W_{p+tv} = \lim_{t \to 0} \frac{W_{p+tv} - W_p}{t}$$

fails to generalize to arbitrary vector fields W, for one because W_{p+tv} and W_p lie in disparate spaces. To alleviate these impediments, we introduce the Lie derivative.

Definition 3.15. For any smooth vector field W on M, the *Lie derivative W with respect to V* is the rough vector field on M denoted $\mathcal{L}_V W$ and given by

$$(\mathcal{L}_V W)_p = \left.\frac{d}{dt}\right|_{t=0} d(\theta_{-t})_{\theta_t(p)}\left(W_{\theta_t(p)}\right) = \lim_{t \to 0} \frac{d(\theta_{-t})_{\theta_t(p)}\left(W_{\theta_t(p)}\right) - W_p}{t},$$

whenever the derivative exists.

The next result installs one particular utility of the Lie bracket in practice, while also providing an efficient computational method for $\mathcal{L}_V W$.

Theorem 3.16. *If M is a smooth manifold and $V, W \in \mathscr{X}(M)$, then $\mathcal{L}_V W = [V, W]$.*

Proof. Denote $\mathcal{R}(V) \subset M$ the collection of regular points of V, and let θ be the flow generated by V. We check the equality point-wise, taking cases on $p \in M$:

(1) $p \in \mathcal{R}(V)$: we choose some smooth coordinates (u^i) on a neighborhood of p in which V has coordinate representation $V = \partial/\partial u^1$ by Theorem 3.14, so the flow of V in these local coordinates is given by $\theta_t(u) = (u^1 + t, u^2, \dots, u^n)$. It follows that $d(\theta_{-t})_{\theta_t(u)}$ is the identity matrix, and

$$d(\theta_{-t})_{\theta_t(u)}(W_{\theta_t(u)}) = W^j(u^1 + t, u^2, \dots, u^n) \frac{\partial}{\partial u^j}\bigg|_u,$$

and so

$$\begin{aligned}(\mathcal{L}_V W)_u &= \frac{\partial}{\partial t}\bigg|_{t=0} W^j(u^1 + t, u^2, \dots, u^n) \frac{\partial}{\partial u^j}\bigg|_u \\ &= \frac{\partial W^j}{\partial u^1}(u^1, \dots, u^n) \frac{\partial}{\partial u^j}\bigg|_u \\ &= [V, W]_u.\end{aligned}$$

(2) $p \in \overline{\mathcal{R}(V)}$: $(\mathcal{L}_V W)_p = [V, W]_p$ follows by previous case and continuity.

(3) $p \in M \setminus \overline{\mathcal{R}(V)}$: then on some neighborhood $U \ni p$, $V|_U \equiv 0$, i.e. $\theta_t|_U = \mathrm{Id}_M$ for all $t \in \mathbb{R}$. It follows that $d(\theta_{-t})_{\theta_t(p)}(W_{\theta_t(p)}) = W_p$, which implies $(\mathcal{L}_V W)_p = 0 = [V, W]_p$.

□

Lie derivatives satisfy further relations, one of which is the following.

Corollary 3.17. *Suppose M is a smooth manifold with or without boundary, and $V, W, X \in \mathscr{X}(M)$. If $g \in C^\infty(M)$, then $\mathcal{L}_V(gW) = (Vg)W + g\mathcal{L}_V W$.*

Proof. Fix $f \in C^\infty(M)$. By Theorem 3.16, we obtain

$$\begin{aligned}\mathcal{L}_V(gW)(f) = [V, gW](f) &= V(gWf) - gW(Vf) \\ &= V(gWf) + \mathcal{L}_V W(f) - gV(Wf) \\ &= gV(Wf) + (Wf)Vg + \mathcal{L}_V W(f) - gV(Wf) \\ &= (Vg)Wf + \mathcal{L}_V W(f).\end{aligned}$$

□

We remark that this method can be upgraded further via *exterior derivatives* of smooth differential k-forms on M. These smooth k-forms are simply sections of the bundle $\Lambda^k T^* M$ consisting of alternating covariant k-tensors on M, but we will not discuss this further. (See [1, Chapter 14].)

4. Vector Bundles

The tangent bundle of a manifold is an example of an object which locally appears to be a Cartesian product, but may admit a global contortion. This kind of structure is more generally called a vector bundle, and arises extensively throughout differential geometry and topology.

Definition 4.1. A *(real) vector bundle of rank k over M* is the data of a topological space E ("total space") together with a continuous surjection $\pi : E \to M$ such that

(1) for each $p \in M$, the fiber $E_p := \pi^{-1}(p)$ over p is endowed with the structure of a k-dimensional real vector space, and
(2) for each $p \in M$, there exists a neighborhood $U \ni p$ in M and a homeomorphism $\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^k$ ("trivialization") which further satisfies:
   - $\pi_U \circ \Phi = \pi$, $\pi_U : U \times \mathbb{R}^k \to U$, and
   - for each $q \in U$, Φ restricts to a vector space isomorphism from $E_p \to \{q\} \times \mathbb{R}^k$.

I will henceforth use E and (E, π) interchangeably to refer to the vector bundle given by $\pi : E \to M$.

Whenever M and E are smooth manifolds with or without boundary, π is a smooth map, and the local trivializations can be chosen as diffeomorphisms, (E, π) is called a *smooth vector bundle*. If there exists a global trivialization of (E, π) over M, then (E, π) is called the *trivial bundle*. Vector bundles may be further distinguished by the *sections* which they may or may not admit: a bundle is trivial only when it admits a nowhere vanishing section.

Definition 4.2. Let $\pi : E \to M$ be a vector bundle. A *section of E* is a continuous map $\sigma : M \to E$ satisfying $\pi \circ \sigma = \mathrm{Id}_M$. This means $\sigma(p)$ is an element of the fiber over $p \in M$. A *local section of E* is a continous map $\sigma : U \to E$ satisfying $\pi \circ \sigma = \mathrm{Id}_U$ for some open $U \subset M$.

Example 4.3 (Zero section)**.** Let $\pi : E \to M$ be a vector bundle. The *zero section of E* is the (global) section $\xi : M \to E$ given by $\xi(p) = 0 \in E_p$ for each $p \in M$.

In [1, Chapter 12], sections of vector bundles will reincarnate as the covariant 1-tensors in the broader family of k-tensors on smooth manifolds.

Lemma 4.4 (Extension Lemma for Vector Bundles)**.** *Let $\pi : E \to M$ be a smooth vector bundle over a smooth manifold M with or without boundary. Suppose $A \subset M$ is closed, and $\sigma : A \to E$ is a section of $E|_A$ which extends to a smooth local section of E in some neighborhood V_p of each $p \in A$. For each open $A \subset U \subset M$, there exists a global smooth section $\tilde{\sigma}$ such that $\tilde{\sigma}|_A = \sigma$ and supp$(\tilde{\sigma}) \subset U$.*

Proof. The key ingredient is to consider a partition of unity subordinate to the open cover $\{V_p \cap U\}_{p \in A} \cup \{M \setminus A\}$. □

Example 4.5. Dual to the tangent bundle TM, another central example of a vector bundle over some smooth manifold M is supplied by the *cotangent bundle of M*, denoted

$$T^*M = \bigsqcup_{p \in M} T_p^*M,$$

where $T_p^*M = (T_pM)^*$ at every point. We call local (or global) sections of T^*M *covector fields* or *differential 1-forms*, and traditionally denote them $\omega : U \to T^*U$, $U \subset M$. It can be checked that T^*M is a smooth rank-n vector bundle by considering the standard coordinate functions (x^i) and their associated covector fields (λ^i) on some smooth chart (U, φ).

We can easily transport the notions of local and global frames for vector fields to vector bundles via local trivializations.

Definition 4.6. For M a smooth manifold with or without boundary, an ordered k-tuple $(\sigma_1, \dots, \sigma_k)$ of local sections of E defined over an open subset $U \subset M$ is said to

- be *linearly independent* if $(\sigma_1(p), \dots, \sigma_k(p))$ is a linearly independent k-tuple in E_p for each $p \in U$,
- *span* E if the k-tuple $(\sigma_1(p), \dots, \sigma_k(p))$ spans E_p at each $p \in U$.

Definition 4.7. A *local frame for E over U* is an ordered k-tuple of linearly independent local sections $(\sigma_1, \dots, \sigma_k)$ over $U \subset M$ which span E; i.e. $(\sigma_1(p), \dots, \sigma_k(p))$ is a basis for E_p for each $p \in U$. We call it a *global frame* when $U = M$, and a smooth frame when each σ_i is smooth.

We obtain the analogue of Proposition 2.9 for vector bundles.

Proposition 4.8. *Suppose $\pi : E \to M$ is a smooth vector bundle of rank k.*

(1) If $(\sigma_1, \dots, \sigma_m)$ is a linearly independent m-tuple of smooth local sections of E over an open subset $U \subset M$, $1 \leq m < k$, then for each $p \in U$ there exist smooth sections $\sigma_{m+1}, \dots, \sigma_k$ defined on some open neighborhood $V \ni p$ such that $(\sigma_1, \dots, \sigma_k)$ is a smooth local frame for E over $U \cap V$.

(2) If $(v_1, \dots, v_m)$ is a linearly independent m-tuple of elements of E_p for some $p \in M$, $1 \leq m \leq k$, then there exists a smooth local frame (σ_i) for E over some neighborhood of p such that $\sigma_i(p) = v_i$ for $i = 1, \dots, m$.

(3) If $A \subset M$ is a closed subset and $(\tau_1, \dots, \tau_k)$ is a linearly independent k-tuple of sections of $E|_A$ that are smooth (i.e. extend to smooth local sections of E in some neighborhood of each point), then there exists a smooth local frame $(\sigma_1, \dots, \sigma_k)$ for E over some neighborhood of A such that $\sigma_i|_A = \tau_i$ for $i = 1, \dots, k$.

Importantly, smooth local frames and smooth local trivializations appear together, as the following result asserts.

Proposition 4.9. *Every smooth local frame (σ_i) for a vector bundle $\pi : E \to M$ is associated with a smooth local trivialization $\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^k$.*

Recall Theorem 2.3, establishing a local criterion for smoothness of vector fields in terms of component functions. Analogously, smoothness of vector bundles can be characterized by smoothness of local frames in any smooth chart. Suppose (σ_i) is a smooth local frame for E over some open subset $U \subset M$. If $\tau : M \to E$ is a rough section, we may write

$$\tau(p) = (\tau^1(p), \ldots, \tau^k(p))$$

in the coordinates given by (σ_i). The *component functions of* τ are the functions $\tau^i : M \to \mathbb{R}$.

Proposition 4.10. *Let $\pi : E \to M$ be a smooth vector bundle, and let $\tau : M \to E$ be a rough section. If (σ_i) is a smooth local frame for E over an open subset $U \subset M$, then τ is smooth on U if and only if its component functions with respect to (σ_i) are smooth.*

Proof. Let $\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^k$ be the local trivialization (diffeomorophism) associated to the local frame (σ_i). The key observation is that $\Phi \circ \tau$ given by $\Phi \circ \tau(p) = (p, (\tau^1(p), \ldots, \tau^k(p)))$ is smooth if and only if τ is smooth. $\square$

Next, we study maps between vector bundles, which we require to be compatible with some map of base manifolds and induce linear maps on fibers.

Definition 4.11. For $\pi : E \to M$ and $\pi' : E' \to M'$ vector bundles, a continuous map $F : E \to E'$ is a *bundle homomorphism* if there exists a map $f : M \to M'$ satisfying $\pi' \circ F = f \circ \pi$ with the property that for each $p \in M$, $F|_{\pi^{-1}(p)} : E_p \to E'_{f(p)}$ is linear.

$$\begin{tikzcd} E \arrow[r, "F"] \arrow[d, "\pi"'] & E' \arrow[d, "\pi'"] \\ M \arrow[r, "f"'] & M' \end{tikzcd}$$

When F is a bijection whose inverse is also a bundle homomorphism, we call F a *bundle isomorphism*. In this setting, when F is further a diffeomorphism, we nominate it a *smooth bundle isomorphism*. Note that f is determined by $\pi' \circ F \circ \xi : M \to M$.

Example 4.12. If $F : M \to N$ is a smooth map, the global differential $dF : TM \to TN$ is a smooth bundle homomorphism covering F.

When $M = M'$, this amounts to demanding the following commutative diagram:

$$\begin{tikzcd} E \arrow[rr, "F"] \arrow[dr, "\pi"'] & & E' \arrow[dl, "\pi'"] \\ & M & \end{tikzcd} .$$

Definition 4.13. Given a vector bundle $\pi : E \to M$, a *subbundle of* E is a vector bundle denoted $\pi_D : D \to M$ in which D is a topological subspace of E and π_D is the restriction of π to D, such that for each $p \in M$, the restricted fiber $E_p \cap D$ is a linear subspace of the total fiber E_p.

A subbundle $\pi_D : D \to M$ is called a *smooth subbundle* of a smooth vector bundle $\pi : E \to M$ whenever it is an embedded submanifold with or without boundary of E.

Fact 4.14. *(*[1, Exercise 10.31]*) Given a smooth vector bundle $E \to M$ and a smooth subbundle $D \subset E$, the inclusion map $\iota : D \to E$ is a smooth bundle homomorphism over M.*

It is natural to wonder whether we can realize any arbitrary collection of linear subspaces $D_p \subset E_p$ at each $p \in M$ as a subbundle. As previously promised, we now provide a criterion based on local frames for subbundles. It turns out that an arbitrary prescription of subspaces $D_p \subset E_p$ for all $p \in M$ constitutes a smooth subbundle only when smooth local frames emerge over neighborhoods of each point.

Lemma 4.15. *Let $\pi : E \to M$ be a smooth vector bundle, and suppose that for each $p \in M$ we are given an m-dimensional linear subspace $D_p \subset E_p$. Then $D = \bigcup_{p\in M} D_p \subset E$ is a smooth subbundle of E if and only if the following holds:*

"Each $p \in M$ has a neighborhood $U \ni p$ on which there exist smooth local sections $\sigma_1, \dots, \sigma_m : U \to E$ such that $\sigma_1(q), \dots, \sigma_n(q)$ form a basis for D_q at each $q \in U$."

Proof. If D is a smooth subbundle, then for each $p \in M$ we obtain local trivializations $\Phi : \pi_D^{-1}(U) \to U \times \mathbb{R}^m$ for U a neighborhood of p. This trivialization yields the desired smooth local sections over U by pulling back $e_i \in \mathbb{R}^k$. Conversely, take any smooth local sections $(\sigma_1, \dots, \sigma_m)$ over $U \ni p$ and complete it to a smooth local frame $(\sigma_1, \dots, \sigma_k)$. finish □

Vector bundles may admit very interesting subbundles. Where do we find them? Under certain conditions, smooth bundle homomorphisms give rise to two canonical flavors:

Theorem 4.16. *Let E and E' be smooth vector bundles over a smooth manifold M, and let $F : E \to E'$ be a smooth subbundle homomorphism over M. Define subsets*

$$\mathit{Ker}(F) := \bigcup_{p\in M} \mathit{Ker}\left(F|_{E_p}\right), \text{ and } \mathit{Im}(F) := \bigcup_{p\in M} \mathit{Im}\left(F|_{E_p}\right).$$

Then Ker(F) and Im(F) are smooth subbundles of E and E', respectively, if and only if F has constant rank.

Proof. It is clearly necessary that F have constant rank. Conversely, let F be of constant rank, and fix some neighborhood U of a point $p \in M$. The goal is to construct smooth local frames for Ker(F) and Im(F) over U, by transporting a smooth local frame $(\sigma_1, \dots, \sigma_k)$ for E over U to a smooth local frame for E' over U via F. This can be done naturally by taking compositions $(F \circ \sigma_1, \dots, F \circ \sigma_k)$, which span $(\mathrm{Im}(F))|_U$. Since F has constant rank $r \leq k$, we assume without loss of generality that $(F \circ \sigma_1, \dots, F \circ \sigma_r)$ remain linearly independent over some $U_0 \subset U$, which together furnish a local frame for $(\mathrm{Im}(F))|_{U_0}$. The argument for Ker(F) can be done similarly by realizing Ker(F) as the image of an auxiliary map $\Psi : E_{U_0} \to E_{U_0}$. □

One method by which we extract an interesting subbundle of the tangent bundle is described in the following lemma.

Lemma 4.17. *Let M be an immersed submanifold with or without boundary in $\mathbb{R}^n$, and D be a smooth rank-k subbundle of $T\mathbb{R}^n|_M$. For each $p \in M$, let $D_p^\perp$ denote the orthogonal complement of D_p in $T_p\mathbb{R}^n$ with respect to the Euclidean dot product, and let $D^\perp \subset T\mathbb{R}^n|_M$ be the subset*

$$D^\perp = \{(p, v) \in T\mathbb{R}^n : p \in M, v \in D_p^\perp\}.$$

Then $D^\perp$ is a smooth rank-$(n-k)$ subbundle of $T\mathbb{R}^n|_M$. For each $p \in M$, there is a smooth orthonormal frame for $D^\perp$ on a neighborhood of p.

When $D = TM \subset T\mathbb{R}^n|_M$, $D^\perp =: NM$ is called the *normal bundle* to M. This bundle carries important information about the structure M. For M a properly embedded submanifold of codimension-k in $\mathbb{R}^n$, NM is trivial whenever there exists a smooth (global) defining function $\Phi : U \to \mathbb{R}^k$, $M \subset U$, for which M is a regular level set.

One important generalization of vector bundles is the notion of a fiber bundle, which loosens the restriction on the nature of the fibers over every point.

Definition 4.18. Let M and F be topological spaces. A *fiber bundle over M with model fiber F* is a topological space E together with a surjective continuous map $\pi : E \to M$ with the property that for each $x \in M$, there exists a neighborhood U of x in M and a homeomorphism $\Phi : \pi^{-1}(U) \to U \times F$ called a *local trivialization of E over U*, such that the following diagram commutes:

$$\begin{tikzcd} \pi^{-1}(U) \arrow[rr, "\Phi"] \arrow[dr, "\pi"'] & & U \times F \arrow[dl, "\pi_1"] \\ & U & \end{tikzcd}$$

For example, every covering map is a fiber bundle whose model fiber is discrete. Many fiber bundles are not vector bundles. Consider the *Hopf fibration* $S^1 \to S^3 \to S^2$, which arises by letting $U(1) \cong S^1$ act on S^3. It implies nontriviality of higher homotopy groups of spheres.

Proposition 4.19. *(*[1, Exercise 10-19]*) The following are a few further characteristics of a fiber bundle $\pi : E \to M$ with fiber F:*

- *π is an open quotient map,*
- *If the bundle is smooth (Φ chosen to be a diffeomorphism), then π is a smooth submersion,*
- *π is a proper map if and only if F is compact, and*
- *E is compact if and only if both M and F are compact.*

Vector bundles can articulate quite a bit information about their base manifolds, in particular when the manifold is paracompact. We can define a natural transformation from the category of rank-n vector bundles to mth cohomology groups. These natural transformations are called *charcteristic classes*, one flavor of which is given by *Stiefel-Whitney classes*. These classes determine when vector bundles embed into trivial bundles, meansure anticommutativity of certain cohomology operations, and inform *cobordism* of manifolds, among other applications.

5. A global vision

I would like to offer a final interpretation of the content covered in the first thirteen chapters of John Lee's *Introduction to Smooth Manifolds* as an amateur student of geometry and topology.

As I understand it, the project of smooth manifold theory is threefold: distill abstract spaces and maps between them down to real linear algebra, leverage the raw materials of vector spaces to assemble new objects, and then export this machinery carefully and meaningfully back to the abstract realm. The profit of these efforts is an injection of palpable geometry into even the most mystical spaces. We begin from *topological manifolds*, which locally appear to be n-dimensional real space.

We then refine our search to *smooth* manifolds, which inherit differential structure, and *smooth maps* between them. A crucial technology for smooth maps is a *partition of unity*, which informally facilitates a "gluing" of local data into a global picture. We then learn that smooth manifolds can be modeled locally by linear spaces. We can collect these "tangent spaces" over every point together to form the "tangent bundle," which itself admits a natural smooth structure. We can consider more general collections of linear spaces over a manifold called "vector bundles", which admit further generalizations given by "fiber bundles." Bundles are spaces which locally appear like a Cartesian product, but globally may be contorted.

Importantly, a smooth map between manifolds transports tangent spaces via its linear differential. This differential detects important properties of these smooth maps, and in turn the way in which their associated smooth manifolds lie with respect to one another (for example, one may locally look like a "slice" of another). One remarkable feature of smooth maps is that their differentials accurately capture the target tangent spaces almost everywhere. This fact confirms that there is a way in which any smooth manifold embeds smoothly into a Euclidean space, as we might naturally intuit.

Later, we discover a special algebraic collection of smooth manifolds called Lie groups, which act on other manifolds, and thereby reveal a great deal about those manifolds' geometries and symmetries. The structure of these special manifolds can be, to a greater extent as arbitrary smooth manifolds, reduced to its local linear-algebraic model via the group structure. A key ingredient in this procedure is the notion of a "smooth vector field," or a smooth attachment of a tangent direction at each point. Vector fields can be considered on arbitrary manifolds, forming a very large vector space with an additional multiplicative structure, which for Lie groups yields the "Lie algebra." This subspace reflects the algebraic structure of the global Lie group. Vector fields are one instance of a larger family of objects known as "tensor fields".

Though we do not provide a discussion in these notes, “alternating tensor fields” ultimately render a natural coordinate-independent theory of integration on smooth manifolds, where k-forms can be interpreted as assignments of volumes to submanifolds. This theory of integration culminates in *Stokes thoerem*–a generalization of the classical fundamental theorem of calculus. Finally, exterior differentiation, which produces $(k+1)$-forms from k-forms, will service the right kind of differential operator to study how and when *closed forms* fail to be *exact*. This question inspires the very rich subject of “de Rham cohomology”.

References

[1] J. M. Lee. *Introduction to smooth manifolds*, volume 218 of *Graduate Texts in Mathematics*. Springer, New York, second edition, 2013.