Task 1

The system has energy levels

$$\epsilon, 2\epsilon, 3\epsilon, \ldots, M\epsilon$$

The occupation number at each level can only be 0 or 1

For each level, its contribution for the grand partition function is

$$\mathcal{Z}_i = \sum_{n_i=0,1} e^{-\beta(\epsilon_i-\mu)n_i} = 1 + e^{-\beta(\epsilon_i-\mu)}$$

Total partition function is

$$\mathcal{Z}_G = \prod_{k=1}^{M} \mathcal{Z}_i = \prod_{k=1}^{M} \left(1 + e^{-\beta(\epsilon_i-\mu)}\right)$$

Task 2

a) Microstates

n_0	n_1	Energy
N	0	0
$N-1$	1	ϵ
$N-2$	2	2ϵ
$\vdots$	$\vdots$	$\vdots$
0	N	$N\epsilon$

b)

$$Z_c = \sum_{\substack{\text{all} \\ \text{microstates}}} e^{-\beta E}$$

where E is the total energy

For a system with n bosons on the first excited state, $E = n\epsilon$

Also, there are $\binom{N}{n}$ ways to choose n bosons on the excited state

$$\therefore \quad Z_c = \sum_{n=0}^{N} \binom{N}{n} e^{-\beta n \epsilon}$$

The probability of find a specific particle at energy E is

$$\frac{\binom{N}{E/\epsilon} e^{-\beta E}}{Z_c}$$

c)

$$\langle n_\epsilon \rangle_c = \sum_{n_\epsilon = 0}^{N} n_\epsilon P(n_\epsilon)$$

$$= \sum_{n_\epsilon = 0}^{N} n_\epsilon \frac{\binom{N}{N_\epsilon} e^{-\beta n_\epsilon \epsilon}}{Z_c}$$

$$\langle n_0 \rangle_c = N - \langle n_\epsilon \rangle_c$$

d) Since quantum particles are indistinguishable

$$Z_c = \sum_{n=0}^{N} e^{-\beta n \epsilon}$$

$$P(E) = \frac{e^{-\beta E}}{Z_c}$$

e)

$$\langle n_\epsilon \rangle = \sum_{n_\epsilon=0}^{N} n_\epsilon \, P(n_\epsilon)$$

$$= \sum_{n_\epsilon=0}^{N} n_\epsilon \frac{e^{-\beta n_\epsilon \epsilon}}{Z_C}$$

$$= \frac{1 - e^{-\beta(N+1)\epsilon}}{1 - e^{-\beta\epsilon}}$$

$$\langle n_0 \rangle = 1 - \langle n_\epsilon \rangle$$

f)

$$\Omega_G = \sum_{N=0}^{\infty} e^{\beta\mu N} Z$$

$$= \frac{1}{(e^{\beta(\mu-\epsilon)} - 1)(e^{\beta\mu} - 1)}$$

$\mu - \epsilon < 0$ for the system to be normalizable

g)

$$\langle N \rangle = k_B T \frac{\partial}{\partial \mu} \ln(\Omega_G)$$

$$= \frac{e^{\beta(\epsilon-\mu)}\left(1 + e^{\beta\epsilon} - 2e^{\beta\mu}\right)}{\left(e^{\beta(\mu-\epsilon)} - 1\right)^2 \left(e^{\beta\mu} - 1\right)^2}$$

h)

$$\langle n_0 \rangle = \frac{1}{\Omega_G} \sum_{N=0}^{\infty} \sum_{k=0}^{N} (N-k)\, e^{\beta\mu N}\, e^{-\beta k \epsilon}$$

$$= \frac{1}{e^{-\beta\mu} - 1}$$