Task 2

a) Microstates

n_0	n_1	Energy
N	0	0
$N-1$	1	ϵ
$N-2$	2	2ϵ
$\vdots$	$\vdots$	$\vdots$
0	N	$N\epsilon$

b)

$$Z_c = \sum_{\substack{\text{all} \\ \text{microstates}}} e^{-\beta E}$$

where E is the total energy

For a system with n bosons on the first excited state, $E = n\epsilon$.

Also, there are $\binom{N}{n}$ ways to choose n bosons on the excited state

$$\therefore \quad Z_c = \sum_{n=0}^{N} \binom{N}{n} e^{-\beta n \epsilon}$$

The probability of find a specific particle at energy E is

$$\frac{\binom{N}{E/\epsilon} e^{-\beta E}}{Z_c}$$

c)

$$\langle n_\epsilon \rangle_c = \sum_{n_\epsilon = 0}^{N} n_\epsilon P(n_\epsilon)$$

$$= \sum_{n_\epsilon = 0}^{N} n_\epsilon \frac{\binom{N}{N_\epsilon} e^{-\beta n_\epsilon \epsilon}}{Z_c}$$

$$\langle n_0 \rangle_c = N - \langle n_\epsilon \rangle_c$$

d) Since quantum particles are indiztinguisteble

$$Z_c = \sum_{n=0}^{N} e^{-\beta n \epsilon}$$

$$P(E) = \frac{e^{-\beta E}}{Z_c}$$

e)

$$\langle n_\epsilon \rangle = \sum_{n_\epsilon=0}^{N} n_\epsilon \, P(n_\epsilon)$$

$$= \sum_{n_\epsilon=0}^{N} n_\epsilon \frac{e^{-\beta n_\epsilon \epsilon}}{Z_c}$$

$$= \frac{1 - e^{-\beta(N+1)\epsilon}}{1 - e^{-\beta\epsilon}}$$

$$\langle n_0 \rangle = 1 - \langle n_\epsilon \rangle$$

f)

$$\Omega_G = \sum_{N=0}^{\infty} e^{\beta\mu N} Z$$

$$= \frac{1}{(e^{\beta(\mu-\epsilon)} - 1)(e^{\beta\mu} - 1)}$$

$\mu - \epsilon < 0$ for the system to be normalizable g)

$$\langle N \rangle = k_B T \frac{d}{d\mu} \ln(\Omega_G)$$

$$= \frac{e^{\beta(\epsilon-\mu)}\left(1+e^{\beta\epsilon}-2e^{\beta\mu}\right)}{\left(e^{\beta(\mu-\epsilon)}-1\right)^2\left(e^{\beta\mu}-1\right)^2}$$

h)

$$\langle n_0 \rangle = \frac{1}{\Omega_G} \sum_{N=0}^{\infty} \sum_{k=0}^{N} (N-k)\, e^{\beta\mu N} e^{-\beta k \epsilon}$$

$$= \frac{1}{e^{-\beta\mu}-1}$$