Dual Optimization for Newsvendor-like Problem

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Primal problem

$$\min f(\delta, \epsilon) \tag{1}$$
$$s.t. \quad y + \delta - \epsilon = b \tag{2}$$
$$y \in \Omega_y \subseteq \mathbb{R}^n, \delta \in \mathbb{R}^n_+, \epsilon \in \mathbb{R}^n_+ \tag{3}$$

- Assume $f = p^{\mathsf{T}}\delta + h^{\mathsf{T}}\epsilon, p \geq 0, h \geq 0$, (2) expresses the newsvendor objective.
- Ω_y is a mixed integer set, and can be decomposed into small problems that are easier to solve.

Lagrangian relaxation

Relax the newsvendor equation: $y + \delta - \epsilon = b$

$$
\begin{aligned}
\phi(\lambda) &= \min_{\delta,\epsilon} (p+\lambda)^\mathsf{T}\delta + (h-\lambda)^\mathsf{T}\epsilon + \min_y \lambda^\mathsf{T} y - \lambda^\mathsf{T} b \\
&= \min_y \lambda^\mathsf{T} y - \lambda^\mathsf{T} b \\
\textbf{s.t.}& \\
& \quad y \in \Omega_y \\
& \quad \delta \in \mathbb{R}^n_+, \epsilon \in \mathbb{R}^n_+
\end{aligned} \tag{4}
$$

(Since $\lambda \in [-p, h]$ and $\delta^\star = \epsilon^\star = 0$ else unbounded)

Subgradient method

We want to solve $\max_\lambda \phi(\lambda)$ by subgradient method:

$$
\begin{aligned}
g &= y - b \in \partial\phi \\
g_k &= y_k - b \\
\lambda_{k+1} &= \mathbf{P}(\lambda_k + s_k g_k) \\
s_k &= \gamma_k \frac{\phi^\star - \phi(\lambda_k)}{\|g_k\|^2}
\end{aligned}
\tag{5}
$$

$\mathbf{P}$ is the projection onto $[-p, h]$. Keep the averaged solution:

$$
\bar{y}_k = \frac{1}{k}\sum_i^k y_i \tag{6}
$$

Primal recovery

$(y_k, \epsilon_k = 0, \delta_k = 0)$ may not be feasible, use recovery:

$$\begin{aligned}
\epsilon_k &= \max\{y_k - b, 0\} \\
\delta_k &= \max\{b - y_k, 0\} \\
\bar{\epsilon}_k &= \max\{\bar{y}_k - b, 0\} \\
\bar{\delta}_k &= \max\{b - \bar{y}_k, 0\}
\end{aligned} \tag{7}$$

let corresponding primal value be $z(y_k) = f(\delta_k, \epsilon_k)$, $\bar{z}_k = z(\bar{y}_k)$

Motivation: fleet engine maintenance problem (FMP)

- engines: $i \in I$, time periods: $t = 1, 2, ..., n$, demand: $b = (b_1, ..., b_n)$.
- at each time we decide if engine i is working $u_{it} = 1$ or sent to maintenance $x_{it} = 1$ (and will be finished after τ periods)
- the lifespan of engine i decreases by α_i if working; increases by β_i if the maintenance is finished; the lifespan has a lower bound L.
- our goal is to satisfy demand b by minimizing the surplus ϵ and shortage δ: $f = p^\mathsf{T}\delta + h^\mathsf{T}\epsilon$
- let Ω_i be the mixed-integer set regarding the maintenance requirements individually, so we have Ω_i for each i

FMP

$$f = \min_{x_{it}, u_{it}, \delta_t, \epsilon_t} \sum_t (b \cdot \delta_t + h \cdot \epsilon_t) \tag{8}$$

s.t.

$$\sum_i u_{it} + \delta_t - \epsilon_t = d_t, \quad \forall t \in T \tag{9}$$

$$s_{i,t+1} = s_{it} - \alpha_i u_{it} + \beta_i x_{i,t-\tau}, \quad \forall i \in I, t \in T \tag{10}$$

$$x_{it} + u_{i,t} \leq 1, \quad \forall i \in I, t \in T \tag{11}$$

$$x_{it} + x_{i\rho} + u_{i,\rho} \leq 1, \quad \forall i \in I, t \in T, \rho = t+1, ..., t+\tau \tag{12}$$

$$s_{it} \geq L, \quad \forall i \in I, t \in T \tag{13}$$

(10) - (13) can be expressed as $\Omega_i, \forall i \in I$

FMP: continued

- (9) is the demand satisfaction constraint.
- (10) tracks the engine lifespan.
- (11) means an engine cannot work if sent to maintenance.
- (12) means the maintenance must be finished before an engine does anything else.
- (13) denotes the lower bound of lifespan.
- if relax (9) then we can solve for each i individually.

Lagrangian relaxation: FMP

For the FMP, dual function:

$$\phi(\lambda) = -\sum_t \lambda_t d_t + \sum_i \min_{\Omega_i} \sum_t \lambda_t u_{it}$$

It reduces to a set of low dimensional minimization problems for each $i \in I, \forall \lambda$
(Recall $\lambda_t \in [-b, h]$ and $\delta_t^\star = \epsilon_t^\star = 0$ else unbounded)

$$\min_{\Omega_i} \sum_t \lambda_t \cdot u_{i,t} \tag{14}$$

(14) is the subproblem to be solved by dynamic programming.
(states: lifespan, action: work or start maintenance)

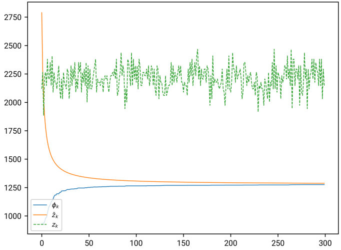
Results

Computational findings (from FMP)

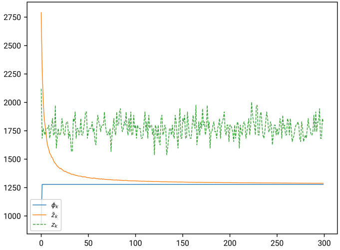
- zero duality gap: $\phi^\star = f^\star$, $\phi^\star$ is the best bound by ϕ and $f^\star$ is the best primal value.
- optimality of the heuristic for averaged solution: $\bar{z}_k = z(\bar{y}_k)$ converges to $\phi^\star$:

$$\bar{z}_k \to \phi^\star$$

Primal solution



(a) Normal subgradient method using g_k



(b) Convex subgradient method using d_k