# Dual Optimization for Newsvendor-like Problem

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February 22, 2021

# Primal problem

$$\min f(\delta, \epsilon) \tag{1}$$
$$\text{s.t.} \quad y + \delta - \epsilon = b \tag{2}$$
$$y \in \Omega_y \subseteq \mathbb{R}^n, \delta \in \mathbb{R}^n_+, \epsilon \in \mathbb{R}^n_+ \tag{3}$$

- Assume $f = p^\mathsf{T}\delta + h^\mathsf{T}\epsilon, p \geq 0, h \geq 0$, (2) expresses the newsvendor objective.
- $\Omega_y$ is a mixed integer set, and can be decomposed into small problems that are easier to solve.

# Lagrangian relaxation

Relax the newsvendor equation: $y + \delta - \epsilon = b$

$$\begin{aligned}
\phi(\lambda) &= \min_{\delta,\epsilon} (p + \lambda)^\mathsf{T} \delta + (h - \lambda)^\mathsf{T} \epsilon + \min_y \lambda^\mathsf{T} y - \lambda^\mathsf{T} b \\
&= \min_y \lambda^\mathsf{T} y - \lambda^\mathsf{T} b \\
&\textbf{s.t.} \\
&\qquad y \in \Omega_y \\
&\qquad \delta \in \mathbb{R}^n_+, \epsilon \in \mathbb{R}^n_+
\end{aligned} \tag{4}$$

(Since $\lambda \in [-p, h]$ and $\delta^\star = \epsilon^\star = 0$ else unbounded)

# Subgradient method

We want to solve $\max_\lambda \phi(\lambda)$ by subgradient method:

$$\begin{aligned} g &= y - b \in \partial\phi \\ g_k &= y_k - b \\ \lambda_{k+1} &= \mathbf{P}(\lambda_k + s_k g_k) \\ s_k &= \gamma_k \frac{\phi^\star - \phi(\lambda_k)}{\|g_k\|^2} \end{aligned} \tag{5}$$

$\mathbf{P}$ is the projection onto $[-p, h]$. Keep the averaged solution:

$$\bar{y}_k = \frac{1}{k}\sum_i^k y_i \tag{6}$$

# Primal recovery

$(y_k, \epsilon_k = 0, \delta_k = 0)$ may not be feasible, use recovery:

$$
\begin{aligned}
\epsilon_k &= \max\{y_k - b, 0\} \\
\delta_k &= \max\{b - y_k, 0\} \\
\bar{\epsilon}_k &= \max\{\bar{y}_k - b, 0\} \\
\bar{\delta}_k &= \max\{b - \bar{y}_k, 0\}
\end{aligned} \tag{7}
$$

let corresponding primal value be $z(y_k) = f(\delta_k, \epsilon_k)$, $\bar{z}_k = z(\bar{y}_k)$

# Motivation: fleet engine maintenance problem (FMP)

- engines: $i \in I$, time periods: $t = 1, 2, ..., n$, demand: $b = (b_1, ..., b_n)$.
- at each time we decide if engine $i$ is working $u_{it} = 1$ or sent to maintenance $x_{it} = 1$ (and will be finished after $\tau$ periods)
- the lifespan of engine $i$ decreases by $\alpha_i$ if working; increases by $\beta_i$ if the maintenance is finished; the lifespan has a lower bound $L$.
- our goal is to satisfy demand $b$ by minimizing the surplus $\epsilon$ and shortage $\delta$: $f = p^\mathsf{T}\delta + h^\mathsf{T}\epsilon$
- let $\Omega_i$ be the mixed-integer set regarding the maintenance requirements individually, so we have $\Omega_i$ for each $i$

# FMP

$$f = \min_{x_{it}, u_{it}, \delta_t, \epsilon_t} \sum_t (b \cdot \delta_t + h \cdot \epsilon_t) \tag{8}$$

**s.t.**

$$\sum_i u_{it} + \delta_t - \epsilon_t = d_t, \quad \forall t \in T \tag{9}$$

$$s_{i,t+1} = s_{it} - \alpha_i u_{it} + \beta_i x_{i,t-\tau}, \quad \forall i \in I, t \in T \tag{10}$$

$$x_{it} + u_{i,t} \leq 1, \quad \forall i \in I, t \in T \tag{11}$$

$$x_{it} + x_{i\rho} + u_{i,\rho} \leq 1, \quad \forall i \in I, t \in T, \rho = t+1, ..., t+\tau \tag{12}$$

$$s_{it} \geq L, \quad \forall i \in I, t \in T \tag{13}$$

(10) - (13) can be expressed as $\Omega_i, \forall i \in I$

# FMP: continued

- (9) is the demand satisfaction constraint.
- (10) tracks the engine lifespan.
- (11) means an engine cannot work if sent to maintenance.
- (12) means the maintenance must be finished before an engine does anything else.
- (13) denotes the lower bound of lifespan.
- if relax (9) then we can solve for each $i$ individually.

# FMP: Lagrangian relaxation

For the FMP, dual function:

$$\phi(\lambda) = -\sum_t \lambda_t d_t + \sum_i \min_{\Omega_i} \sum_t \lambda_t u_{it}$$

It reduces to a set of low dimensional minimization problems for each $i \in I, \forall \lambda$
(Recall $\lambda_t \in [-b, h]$ and $\delta_t^\star = \epsilon_t^\star = 0$ else unbounded)

$$\min_{\Omega_i} \sum_t \lambda_t \cdot u_{i,t} \tag{14}$$

(14) is the subproblem to be solved by dynamic programming.
(states: lifespan, action: work or start maintenance)

# FMP: subgradient method

At each iteration $k$:

- $y_k$ solves $\phi(\lambda_k) = \min_y \lambda_k^\top (y - b)$
- for FMP: $y_k = U_k^\top 1 - d, U_k = (u_{it}^{(k)})$, solved from (14) by DP.
- update $\lambda_k$ by (5)

# Results

Tests are done on FMP with random instances: rand(low, high) means random integer in [low, high]

- L = 2
- $d_t = \text{rand}(\frac{|I|}{2}, |I|), \forall t$
- $\tau_i = \text{rand}(2, 5), a_i = \text{rand}(2, 5), b_i = \text{rand}(5, 10), \forall i$
- $s_{i,0} = \text{rand}(5, 8), \forall i$

We consider two cases for objective function

i , time-invariant: $h_t = 11, p_t = 18, \forall t$

ii , $h_t = \text{rand}(10, 15), p_t = \text{rand}(10, 16), \forall t$

# Results

The subgradient method (sg) stops at (maximum) 400 iterations, then compared with benchmarks by Gurobi (grb).

- lb_grb, $f_{\text{grb}}$ are lower bound, primal value from grb.
- t_grb, t_sg are runtime from grb and sg, respectively.
- $\phi_{\text{sg}}$ is the dual value ($\phi$) by sg.
- $z_{\text{sg}}$ is primal value in sg using averaged primal recovery, i.e.

$$z_{\text{sg}} = z(\bar{y}_k) = f(\bar{\delta}_k, \bar{\epsilon}_k)$$

- $\phi_{\text{gap}}$, z_gap are relative gap from sg to grb for dual and primal values.

# FMP: invariant $h_t = h, b_t = b, \forall t = 1, \ldots, n$

| | lb_grb | $f_{\text{grb}}$ | t_grb | t_sg | $\phi_{\text{sg}}$ | $\phi_{\text{gap}}$ | $z_{\text{sg}}$ | z_gap |
|---|---|---|---|---|---|---|---|---|
| 0 | 936.00 | 936.00 | 23.93 | 11.46 | 936.00 | -0.00% | 952.02 | 1.71% |
| 1 | 954.00 | 954.00 | 11.00 | 8.88 | 954.00 | 0.00% | 958.78 | 0.50% |
| 2 | 1224.00 | 1224.00 | 1.11 | 6.74 | 1224.00 | 0.00% | 1230.12 | 0.50% |
| 3 | 1026.00 | 1026.00 | 2.76 | 7.58 | 1026.00 | 0.00% | 1031.13 | 0.50% |
| 4 | 882.00 | 882.00 | 3.41 | 6.98 | 882.00 | 0.00% | 886.39 | 0.50% |
| 5 | 990.00 | 990.00 | 0.94 | 9.35 | 990.00 | 0.00% | 994.95 | 0.50% |
| 6 | 774.00 | 774.00 | 1.08 | 9.73 | 774.00 | 0.00% | 778.58 | 0.59% |
| 7 | 864.00 | 864.00 | 1.26 | 9.06 | 864.00 | 0.00% | 868.34 | 0.50% |
| 8 | 972.00 | 972.00 | 1.02 | 6.97 | 972.00 | 0.00% | 976.85 | 0.50% |
| 9 | 648.00 | 648.00 | 88.56 | 16.14 | 645.14 | -0.44% | 665.45 | 2.69% |

# FMP: variant $h$, $b$

| | lb_grb | $f_{grb}$ | t_grb | t_sg | $\phi_{sg}$ | $\phi_{gap}$ | $z_{sg}$ | z_gap |
|---|---|---|---|---|---|---|---|---|
| 0 | 608.00 | 608.00 | 8.34 | 12.76 | 607.57 | -0.07% | 650.63 | 7.01% |
| 1 | 587.00 | 587.00 | 3.93 | 12.01 | 586.60 | -0.07% | 619.44 | 5.53% |
| 2 | 538.00 | 538.00 | 22.73 | 12.33 | 537.36 | -0.12% | 574.96 | 6.87% |
| 3 | 534.00 | 534.00 | 4.05 | 13.03 | 532.69 | -0.24% | 607.16 | 13.70% |
| 4 | 684.00 | 684.00 | 19.14 | 7.61 | 684.00 | 0.00% | 691.68 | 1.12% |
| 5 | 620.00 | 620.00 | 4.88 | 8.06 | 619.98 | -0.00% | 625.29 | 0.85% |
| 6 | 657.00 | 657.00 | 4.47 | 9.70 | 655.73 | -0.19% | 682.15 | 3.83% |
| 7 | 625.00 | 625.00 | 8.25 | 12.04 | 622.30 | -0.43% | 670.87 | 7.34% |
| 8 | 422.00 | 422.00 | 5.03 | 12.40 | 421.25 | -0.18% | 453.72 | 7.52% |
| 9 | 485.00 | 485.00 | 4.32 | 13.59 | 484.87 | -0.03% | 517.35 | 6.67% |

# Conclusion

- zero duality gap: $\phi^\star = f^\star$, $\phi^\star$ is the best bound by $\phi$ and $f^\star$ is the best primal value.
- can we bound the quality of heuristic for averaged solution? $\bar{z}_k = z(\bar{y}_k)$ converges to $\phi^\star$:

$$|\bar{z}_k - \phi^\star|$$

- for variant case, can we improve $\bar{z} = z(\bar{y})$?