

# Dual Optimization for Newsvendor-like Problem

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# FMP: variant case and weighted summation

$$f = \min_{x_{it}, u_{it}, \delta_t, \epsilon_t} \mathbf{b}^T \delta + \mathbf{h}^T \epsilon \quad (1)$$

**s.t.**

$$\sum_i \mathbf{c}_i u_{it} + \delta_t - \epsilon_t = d_t, \quad \forall t \in T \quad (2)$$

$$U_{i,\cdot}, S_{i,\cdot}, X_{i,\cdot} \in \Omega_i, \quad \forall i \in I \quad (3)$$

- ▶  $b, h$  are time-variant
- ▶  $c_i \neq 1$ .

# FMP: Lagrangian relaxation

For the FMP, dual function:

$$\phi(\lambda) = - \sum_t \lambda_t d_t + \sum_i \min_{\Omega_i} c_i \sum_t \lambda_t u_{it}$$

It reduces to a set of low dimensional minimization problems for each  $i \in I, \forall \lambda$

(Recall  $\lambda_t \in [-b, h]$  and  $\delta_t^* = \epsilon_t^* = 0$  else unbounded)

$$\min_{\Omega_i} c_i \sum_t \lambda_t \cdot u_{i,t} \tag{4}$$

(4) is the subproblem to be solved by dynamic programming.  
(states: lifespan, action: work or start maintenance)

# FMP: subgradient method

At each iteration  $k$ :

- ▶  $y_k$  solves  $\phi(\lambda_k) = \min_y \lambda_k^T (y - b)$
- ▶ for FMP:  $y_k = U_k^T \mathbf{c}$ ,  $U_k = (u_{it}^{(k)})$ , solved from (4) by DP.
- ▶ update  $\lambda_k$ .

Small case:  $|I| = 1, t = 2$

$c = 1$

$$\begin{aligned} \min \quad & 2\delta_0 + 3\delta_1 + 4\epsilon_0 + \epsilon_1 \\ \text{s.t.} \quad & u_0 + \delta_0 - \epsilon_0 = 1 \\ & u_1 + \delta_1 - \epsilon_1 = 0 \\ & 2u_0 + s_0 = 6 \\ & -6x_0 + 2u_1 - s_0 + s_1 = 0 \\ & x_0 + u_0 \leq 1 \\ & x_1 + u_1 \leq 1 \\ & s_0, s_1 \geq 2 \end{aligned}$$

dual

$$-2 \leq \lambda_0 \leq 4$$

$$-3 \leq \lambda_1 \leq 1$$

$$\min_u \lambda_0(u_0 - 1) + \lambda_1(u_1 - 0)$$

$c = 2$

$$\begin{aligned} \min \quad & 2\delta_0 + 3\delta_1 + 4\epsilon_0 + \epsilon_1 \\ \text{s.t.} \quad & 2u_0 + \delta_0 - \epsilon_0 = 1 \\ & 2u_1 + \delta_1 - \epsilon_1 = 0 \\ & 2u_0 + s_0 = 6 \\ & -6x_0 + 2u_1 - s_0 + s_1 = 0 \\ & x_0 + u_0 \leq 1 \\ & x_1 + u_1 \leq 1 \\ & s_0, s_1 \geq 2 \end{aligned}$$

$$\min_u \lambda_0(2u_0 - 1) + \lambda_1(2u_1 - 0)$$

Small case:  $|I| = 1, t = 2$

results:

*gurobi :*

$$c = 1$$

$$u^* = [1, 0]$$

$$f^* = 0$$

*subgradient :*

$$\lambda_k = [-2, 0]$$

$$\phi_k = 0$$

$$u_k = [1, 0]$$

$$\phi_k = f^*$$

$$c = 2$$

$$u^* = [0, 0]$$

$$f^* = 2$$

$$\lambda_k = [-3.5203e^{-5}, 0]$$

$$\phi_k = 0$$

$$u_k = [0, 0]$$

$$\phi_k \neq f^*$$

## Small case, relax

results:

*gurobi* :  $u \in [0, 1]$

$$c = 1$$

$$u^* = [1, 0]$$

$$f^* = 0$$

*gurobi* :  $x \in [0, 1]$

$$c = 1$$

$$u^* = [1, 0]$$

$$f^* = 0$$

$$c = 2$$

$$u^* = [0.5, 0]$$

$$f^* = 0$$

$$c = 2$$

$$u^* = [0, 0]$$

$$f^* = 2$$

## Small case relax on $u$ , 2 engines, 5 periods, $u \in [0, 1]$

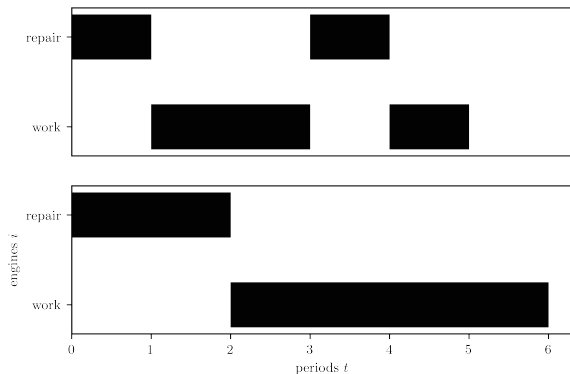
- ▶  $d = [2.0, 2.0, 1.0, 1.0, 2.0, 1.0]$ ,
- ▶  $\alpha: [5.0, 3.0]$ ,
- ▶  $\beta: [8.0, 7.0]$ ,
- ▶  $L: 2$ ,
- ▶  $c: \text{array}([1., 1.])$ ,
- ▶  $\tau: [1, 1]$ ,
- ▶  $s0: [5.0, 5.0]$ ,
- ▶  $h: \text{array}([1, 2, 4, 4, 3, 1])$ ,
- ▶  $p: \text{array}([4, 5, 6, 6, 6, 4])$



## Small case, 2 engines, 5 periods, visualization

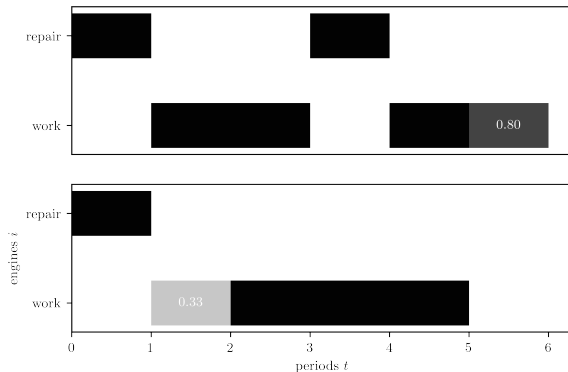
Below is an instance illustrating the gap from MILP to partial relaxation of integral constraints on  $u$ . The black and white cells represent the cases where corresponding binary variables take values in  $\{0, 1\}$ , respectively. The grey cells with attached values stand for fractional solutions in  $(0, 1)$ .

## Small case, 2 engines, 5 periods, visualization



(a) Optimal solution from (1)

## Small case, 2 engines, 5 periods, visualization



**(b)** Optimal solution from (1) with relaxed integral constraints on  $u$

## Small case

The subgradient method (sg) stops at (maximum) 400 iterations, then compared with benchmarks by Gurobi (grb).

- ▶  $lb\_grb$ ,  $f_{grb}$  are lower bound, primal value from grb.
- ▶  $f_{relax}^u$ ,  $f_{relax}^x$  primal value from grb for problem with relaxed  $u$  and  $x$  only, respectively:  $f_{relax}^x = f^*$  ?
- ▶  $t\_grb$ ,  $t\_sg$  are runtime from grb and sg, respectively.
- ▶  $\phi_{sg}$  is the dual value ( $\phi$ ) by sg.
- ▶  $z_{sg}$  is primal value in sg using averaged primal recovery, i.e.

$$z_{sg} = z(\bar{y}_k) = f(\bar{\delta}_k, \bar{\epsilon}_k)$$

- ▶  $\phi_{gap}$ ,  $z_{gap}$  are relative gap from sg to grb for dual and primal values.

# FMP: $c = 1$ small case $5 \times 5$

	lb_grb	$f_{\text{grb}}$	t_grb	$f_{\text{relax}}^u$	$f_{\text{relax}}^x$	t_sg	$\phi_{\text{sg}}$	$\phi_{\text{gap}}$
0	34.00	34.00	0.01	32.50	34.00	1.26	34.00	0.00%
1	42.00	42.00	0.01	40.83	42.00	0.01	42.00	0.00%
2	84.00	84.00	0.01	78.25	84.00	0.01	84.00	0.00%
3	49.00	49.00	0.01	45.50	49.00	0.04	48.96	-0.08%
4	50.00	50.00	0.01	50.00	50.00	0.00	50.00	0.00%
5	112.00	112.00	0.01	106.17	112.00	0.01	112.00	0.00%
6	47.00	47.00	0.02	47.00	47.00	0.02	47.00	0.00%
7	69.00	69.00	0.03	67.33	69.00	0.04	68.98	-0.04%
8	41.00	41.00	0.02	34.25	41.00	0.64	41.00	0.00%
9	91.00	91.00	0.00	89.67	91.00	0.01	91.00	0.00%

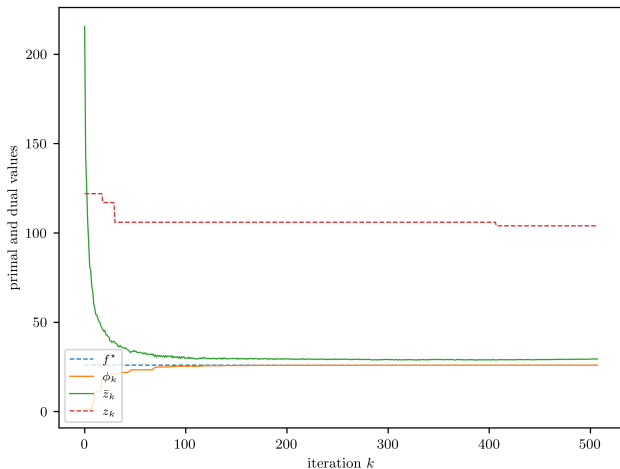
# FMP: $c = 1$ large case $10 \times 15$

►  $f_{\text{relax}}^x \neq f^x$

	lb_grb	$f_{\text{grb}}$	t_grb	$f_{\text{relax}}^u$	$f_{\text{relax}}^x$	t_sg	$\phi_{\text{sg}}$	$\phi_{\text{gap}}$
0	467.00	467.00	0.59	460.25	455.00	3.97	467.00	0.00%
1	375.00	375.00	0.22	370.00	366.00	0.12	375.00	0.00%
2	158.00	158.00	0.67	155.92	158.00	4.73	158.00	0.00%
3	463.00	463.00	0.80	457.08	458.00	5.86	463.00	0.00%
4	370.00	370.00	0.11	367.17	363.00	4.46	370.00	0.00%
5	336.00	336.00	1.04	329.17	333.00	0.10	336.00	0.00%
6	350.00	350.00	0.74	344.42	348.00	3.93	350.00	0.00%
7	416.00	416.00	0.09	409.17	409.00	0.19	415.61	-0.09%
8	374.00	374.00	0.88	363.50	363.00	7.81	374.00	0.00%
9	343.00	343.00	1.30	330.67	336.00	0.24	342.67	-0.10%

## FMP: $c = 1$ large case $10 \times 15$ , primal solution

$z_k$  is the value of best primal (recovery) solution,  $\phi_k$  is the dual value,  $f^*$  is the benchmark optimal value by grb. **there is a gap from  $z_k$  to  $f^*$**



# FMP: $c \neq 1$ small case $5 \times 5$

	lb_grb	$f_{\text{grb}}$	t_grb	$f_{\text{relax}}^u$	$f_{\text{relax}}^x$	t_sg	$\phi_{\text{sg}}$	$\phi_{\text{gap}}$
0	8.00	8.00	0.01	0.00	8.00	0.75	-0.00	-99.99%
1	9.00	9.00	0.01	0.00	9.00	0.68	8.00	-11.11%
2	55.00	55.00	0.02	52.50	55.00	1.33	54.98	-0.03%
3	11.00	11.00	0.03	5.00	11.00	1.06	10.00	-9.09%
4	7.00	7.00	0.02	0.00	7.00	0.85	4.00	-42.85%
5	20.00	20.00	0.03	12.33	20.00	1.08	20.00	-0.00%
6	22.00	22.00	0.01	20.00	22.00	0.01	22.00	0.00%
7	15.00	15.00	0.01	8.67	15.00	0.01	15.00	0.00%
8	14.00	14.00	0.03	6.00	14.00	1.01	10.00	-28.57%
9	18.00	18.00	0.02	13.00	18.00	0.68	18.00	-0.00%



# Conclusion

- ▶ ? zero duality gap:  $\phi^* = f^*$ ,  $\phi^*$  is the best bound by  $\phi$  and  $f^*$  is the best primal value.
- ▶ ? can we bound the quality of heuristic for averaged solution?  
 $\bar{z}_k = z(\bar{y}_k)$  converges to  $\phi^*$ :

$$|\bar{z}_k - \phi^*|$$

- ▶ ? can we improve  $\bar{z} = z(\bar{y})$ , round  $\bar{z}$