Contents

1 Lagrangian relaxation

Consider the following newsvendor-like problem

$$\begin{aligned} & \min f(\delta, \epsilon) \\ \textbf{s.t.} & \\ & y + \delta - \epsilon = b \\ & y \in \Omega_y \subseteq \mathbb{R}^n, \delta \in \mathbb{R}^n_+, \epsilon \in \mathbb{R}^n_+ \end{aligned}$$

where f is a convex function of δ, ϵ. The right-hand-side on the binding constraints is in the positive orthant: $b \in \mathbb{R}^n_+$. This problem widely appears in applications of device maintenance, inventory management, and so on. In the basic settings, let y be the ordering quantity quantities in a multi-item newsvendor problem, one minimizes the total expected cost:

$$\min_{y \in \mathbb{R}_+} \mathbf{E} \left(h \cdot e^{\mathsf{T}} \max\{y - b, 0\} + p \cdot e^{\mathsf{T}} \max\{b - y, 0\} \right)$$

It is easy to verify its equivalence to the problem above.

Let $\lambda \in \mathbb{R}^n$ be the Lagrangian multiplier, the dual function is:

$$
\begin{aligned}
\phi(\lambda) = &\min_{\delta,\epsilon} f(\delta,\epsilon) + \lambda^\mathsf{T}\delta - \lambda^\mathsf{T}\epsilon + \min_y \lambda^\mathsf{T} y - \lambda^\mathsf{T} b \\
&\textbf{s.t.} \\
&\quad y \in \Omega_y \\
&\quad \delta \in \mathbb{R}^n_+, \epsilon \in \mathbb{R}^n_+
\end{aligned}
$$

We assume the resulting two subproblems for δ, ϵ and y are easy.

1.1 Affine case

The case for repair problem

Let $f = p^\mathsf{T}\delta + h^\mathsf{T}\epsilon$, we have

$$
\phi(\lambda) = \min_{\delta,\epsilon}(p+\lambda)^\mathsf{T}\delta + (h-\lambda)^\mathsf{T}\epsilon + \min_y \lambda^\mathsf{T} y - \lambda^\mathsf{T} b
$$

Then ϕ is unbounded unless $\lambda \in \Lambda$ where $\Lambda = \{\lambda : \lambda \in [-p, h]\}$, in which case

$$
\phi(\lambda) = \min_{y\in\Omega_y} \lambda^\mathsf{T} y - \lambda^\mathsf{T} b,\ \lambda \in \Lambda
$$

and $\delta^\star, \epsilon^\star = 0$ are corresponding optimizers for any $\lambda \in \Lambda$

1.2 Conditions for strong duality

It's well known that strong duality does not hold in general. We review some of the cases here. The Lagrangian duality theory can be found in any standard text.

(a) if Ω_y is convex then the strong duality holds ..., i.e. $\phi^\star = f^\star$

... add justifications here (slater, ...)

A more interesting result is devoted to mixed integer problems. **(Review Here)**.

(b) if $\Omega_y = \{y \in \mathbb{R}^n : y \in \Omega, y \in \mathbb{Z}^n\}$. Then we have the following relation for dual function,

$$
\phi^\star = \min_{\delta,\epsilon} f(\delta,\epsilon) \quad \textbf{s.t. } y + \delta - \epsilon = b,\ y \in \operatorname{conv}(\Omega_y)
$$

We conclude the strong duality holds since $Y = \{(y,\delta,\epsilon) : y + \delta - \epsilon = b,\ y \in \mathsf{conv}(\Omega_y)\}$ is already *a perfect formulation* in the sense that $Y = \mathsf{conv}(Y)$

add a proposition to show this or add more conditions to justify

2 Subgradient method

To solve the reduced problem, we consider a variant of subgradient method:

$$\lambda^{k+1} = \mathbf{P}(\hat{\lambda}^k + s^k \bar{g}^k)$$

where $\mathbf{P}$ is the projection onto dual space $\{\lambda : \lambda \in [-c, d]\}$. $\hat{\lambda}^k$ is the multiplier associated with the best dual bound:

$$\hat{\phi}^k = \max_{t=1,\dots,k} \phi(\lambda^t)$$

$\bar{g}^k$ is the update direction for current iteration and s^k is the step size using target-based rules:

$$s^k = \frac{\phi^\star - \phi(\lambda^k)}{||\bar{g}^k||^2}$$

Note the direction $\bar{g}^k$ computed by $\bar{g}^k = \bar{y}^k - d$, where $\bar{y}^k$ is the convex combination of previous iterations $y^1, ..., y^k$ such that y^k solves $\phi^k = \phi(\lambda^k)$.

$$\bar{y}^k = (1-\alpha) \cdot \bar{y}^{k-1} + \alpha \cdot y^k$$

For simplicity we let $g^k = y^k - d$, then g^k is a subgradient of ϕ at $\lambda^k : g^k \in \partial\phi^k$, we can write the direction as the combination of the subgradient and previous directions:

$$\bar{g}^k = (1-\alpha)\bar{g}^{k-1} + \alpha \cdot g^k$$

The target based rule are well-known as the Polyak rule [4]. The idea of using previous searching directions is introduced to accelerate the subgradient method and provide a better stopping criterion, see [3], [2], [1]. Brannlund [2] showed that with convex combinations the optimal choice of stepsize is equivalent to the Camerini-Fratta-Maffioli modification, it also provides an analysis on its linear convergence rate. Barahona and Anbil [1] uses a slight different modification to approximate the primal feasible solutions in the dual searching process.

It is obvious to see the solutions during dual optimization $(y, \epsilon, \delta) = (y^k, 0, 0)$ are feasible if and only if we can find $y^k = d$, which in general will not hold. This motivates the following heuristic.

Algorithm 1: Recovery Heuristic

$$\epsilon^k = \max\{y^k - d, 0\}$$
$$\delta^k = \max\{d - y^k, 0\}$$

Let the primal objective value be z^k. We apply same averaging scheme to produce $(\bar{\delta}^k, \bar{\epsilon}^k)$, and record the corresponding primal objective value as $\bar{z}^k$ such that $\bar{z}^k = f(\bar{\delta}^k, \bar{\epsilon}^k)$. The dual subgradient algorithm can be summarized as follows.

Algorithm 2: The Subgradient Algorithm (Volume)

... **while** ... **do**
...
end

Computational results The typical convergence of $\bar{z}^k$ and $\hat{\phi}^k$ computed from the repair model can be shown: (green - z^k, orange - $\bar{z}^k$, blue - $\hat{\phi}^k$)

and test on a group of examples (the repair model)

3 Convergence

(We wish to show the convergence): $|\bar{z}^k - \hat{\phi}^k|$

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We first review several properties for the subgradient method outlined in 3.1. We adopt some of the results in Camerini-Fratta-Maffioli modification [3] and the analysis of using convex combination of previous iterations in [2]

Lemma 3.1. *Properties for subgradient method*

1.
$$2\langle d^k, \lambda^\star - \hat{\lambda}^k \rangle \geq \rho(\phi^\star - \hat{\phi}^k) = s^k ||d^k||^2 \tag{3.1}$$

2. by (3.1), hopefully we have:
$$||\hat{\lambda}^{k+1} - \lambda^\star|| \leq ||\hat{\lambda}^k - \lambda^\star|| \tag{3.2}$$

Proposition 2.

(a) For fixed $y = y^k$, (ϵ^k, δ^k) is the optimal solution for the restricted primal problem.

$$f(\epsilon^k, \delta^k) \leq f(\epsilon, \delta), \quad \forall \delta \geq 0, \epsilon \geq 0, y = y^k$$

(b) $\bar{z}^k \leq \bar{f}(\delta^k, \epsilon^k)$

PF. By convexity.

3.1 Computational results

3.1.1 Repair problem

3.1.2 General case

References

[1] F. Barahona and R. Anbil, *The volume algorithm: producing primal solutions with a subgradient method*, Mathematical Programming, 87 (2000), pp. 385–399. Publisher: Springer.

[2] U. Brännlund, *A generalized subgradient method with relaxation step*, Mathematical Programming, 71 (1995), pp. 207–219.

[3] P. M. Camerini, L. Fratta, and F. Maffioli, *On improving relaxation methods by modified gradient techniques*, in Nondifferentiable optimization, Springer, 1975, pp. 26–34.

[4] B. T. Polyak, *A general method for solving extremal problems*, Soviet Mathematics Doklady, (1967), p. 5.