Contents

1 Progress summary

- Polynomial-time algorithm for deterministic problem:
  - The problem can be solved with subgradient method in polynomial time. If we allow tolerance $\epsilon \geq 0$ on subgradient method, the worst complexity is $O\left(\frac{1}{\epsilon^2} \cdot \tau \cdot |I| \cdot |T|^3\right)$.
  - Converge and rate:
    * we can only retrieve a feasible primal solution from the simplest subgradient method by Polyak (1967).
    * Good approximation to optimal solution can be found via the volume algorithm by Barahona and Anbil (2000).
    * The rate and convergence of such class of algorithm can be found in Nesterov (2009), Nedić and Ozdaglar (2009).
    * todo: try alternative formulation: $U^\top e - d + \epsilon_+ - \epsilon_- = 0$
  - Details:
    * The subproblem at each subgradient iteration on dual problem can be solved by dynamic programming, at the cost of $O\left(\tau \cdot |T|^3\right)$.
    * The dual variables can be solved **analytically**.
  - Computations has been done for DP, subgradient method (volume algorithm).

a quick look at the results ($|I| = 10, |T| = 20, 10$ random generated instances)

	sg_lb	sg_val	bench_lb	bench_sol	primal_gap
0	14.499735	17.533016	16.000000	16	0.095814
1	32.631830	35.403596	31.979908	34	0.041282
2	9.267048	12.402154	2.999999	10	0.240215
3	52.727507	55.299756	49.496651	54	0.024070
4	0.000000	4.745779	0.000000	0	inf
5	15.927653	16.950710	16.000000	16	0.059419
6	50.000000	52.318332	50.000000	50	0.046367
7	8.000000	10.682970	8.663961	10	0.068297
8	34.000000	35.615792	32.916046	34	0.047523
9	24.000000	25.193067	24.000000	24	0.049711

2 The air-repair model

> Notation

- I, T - set of plane, time periods, respectively
- b, h - demand withdraw and plane idle cost, respectively
- τ - lead time for maintenance

The demand is stochastic with some distribution $f \in \mathcal{F}$

- $\boldsymbol{d}_t$ - demand/number of planes needed at time t

> Decision

- x_{it} - 0 - 1 variable, 1 if plane i starts a maintenance at time t
- u_{it} - 0 - 1 variable, 1 if plane is working at time t
- $s_{it} \geq 0$ - the lifespan of plane i at time t

The DRO/SP model, the goal is to minimize unsatisfied demand and surplus (idle) flights, using a newsvendor-like objective function

$$q_t \equiv b \cdot (d_t - \sum_i u_{it})_+ + h \cdot (\sum_i u_{it} - d_t)_+$$

$$\inf \max_{f \in \mathcal{F}} \mathbb{E}_f \left[\sum_t q_t \right]$$

s.t.

$$q_t \geq b \cdot \left(d_t - \sum_i u_{it} \right) \qquad \forall t \in T$$

$$q_t \geq h \cdot \left(\sum_i u_{it} - d_t \right) \qquad \forall t \in T$$

$$s_{i,t+1} = s_{it} - \alpha_i u_{it} + \beta_i x_{i,t-\tau} \qquad \forall i \in I, t \in T$$

$$x_{it} + u_{i,t} \leq 1 \qquad \forall i \in I, t \in T$$

$$x_{it} + x_{i\rho} + u_{i,\rho} \leq 1 \qquad \forall i \in I, t \in T, \rho = t+1, ..., t+\tau$$

$$s_{it} \geq L \qquad \forall i \in I, t \in T$$

We define the last four sets of constraint as Ω_i, which describe the non-overlapping requirements during a maintenance.

3 Deterministic

We first consider the deterministic problem.

The problem can be solved with subgradient method in polynomial time and the solution is exact. If we allow tolerance $\epsilon \geq 0$ on subgradient method, the complexity is:

$$O\left(\frac{1}{\epsilon^2} \cdot \tau \cdot |I| \cdot |T|^3 \right)$$

3.1 Subgradient method

Relax binding constraints of q

$$z_{\mathsf{LD}}(\lambda, \mu) = \inf_{x,s,u} \left[\sum_t q_t(1 - \lambda_t - \theta_t) + d_t(b\lambda_t - h\theta_t) + \sum_i \sum_t u_{i,t}(h\theta_t - b\lambda_t) \right]$$

s.t.

$$x_{(i,\cdot)}, u_{(i,\cdot)}, s_{(i,\cdot)} \in \Omega_i$$

Ω_i defined as the region of $x_{(i,\cdot)}, u_{(i,\cdot)}, s_{(i,\cdot)}, \forall i \in I$.

we notice $z_{\mathsf{LD}}(\lambda, \theta)$ is unbounded unless:

$$\lambda, \theta \in \{\lambda, \theta \geq 0 \big| (1 - \lambda_t - \theta_t) \geq 0, \forall t \in T\}$$

we have:

$$z_{\mathsf{LD}}(\lambda, \theta) = \sum_t d_t(b\lambda_t - h\theta_t) + \inf_{x,s,u} \left[\sum_i \sum_t u_{i,t}(h\theta_t - b\lambda_t) \right]$$

We wish to solve:

$$\sup_{(\lambda,\theta)} z_{\mathsf{LD}}$$

We can solve for each $i \in I$ independently at each iteration of a subgradient method.

Notice:

- at iteration k, suppose multipliers $\lambda^k + \theta^k \leq 1$, if $(x^\star, s^\star, u^\star)$ solves the relaxed minimization problem, then it is also feasible for the original problem (compute q accordingly). The optimality gap can easily be calculated by simple evaluations.
- at each iteration k, the sub-gradients: $\mathcal{P}$ is the orthogonal projection onto $\mathcal{D} = \{(\lambda, \theta) \mid \lambda + \theta \leq 1\}$

$$\begin{aligned}
\nabla \lambda^k &= b \cdot \left(d - (U^k)^\top e\right) \\
\nabla \theta^k &= h \cdot \left((U^k)^\top e - d\right) \\
\theta^{k+1} &= \mathcal{P}(\theta^k + a^k \nabla \theta^k) \\
\lambda^{k+1} &= \mathcal{P}(\lambda^k + a^k \nabla \lambda^k)
\end{aligned}$$

- the projection $\mathcal{P}$ can be computed very easily. Since the projection $(\tilde{\lambda}, \tilde{\theta})$ onto $\mathcal{D}$ can be formulated as the model $\inf_{\lambda \geq 0, \theta \geq 0, \lambda + \theta \leq e} ||\tilde{\lambda} - \lambda||^2 + ||\tilde{\theta} - \theta||^2$, and solved analytically.

3.2 Subproblem for each plane

The subproblem $\forall i \in I$ is defined as follows:

$$\begin{gathered}
c_t \equiv (h\theta_t - b\lambda_t) \\
\inf_{\Omega_i} \sum_t c_t \cdot u_{i,t}
\end{gathered}$$

The model describes a problem to maximize total utility while keeping the lifespan safely away from the lower bound L. Define state: $y_t = [m_t, s_t]^\top$, where m_t **denotes the remaining time of the undergoing maintenance**. s_t is the remaining lifespan.

At each period t we decide whether the plane i is idle or waiting (for the maintenance), working, or starting a maintenance, i.e.:

$$(u_t, x_t) \in \{(1,0), (0,0), (0,1)\}$$

We have the optimal equation:

$$V_n(u_t, x_t | m_t, s_t) = c_t \cdot u_t + \inf_{u,x} V_{n-1}(...)$$

Complexity: let s_0 be the initial lifespan and finite time horizon be $|T|$, we notice the states for remaining maintenance waiting time is finite, $m_t \in \{0, 1, ..., \tau\}$.

Let total number of possible periods to initiate a maintenance be n_1, and working periods be n_2. If we ignore lower bound L on s, total number of possible values of s is bounded above: $|s| = \sum_i^{|T|} \sum_j^{|T|-i} 1 = (|T|+1)(\frac{1}{2}|T|+1)$ since $n_1 + n_2 \leq |T|$. For each subproblem we have at most 3 actions, thus we conclude this problem can be solved by dynamic programming in polynomial time, the complexity is: $O\left(\tau \cdot |T|^3\right)$

4 Stochastic

We use boldface notation to denote random variables. We let the vector $\boldsymbol{q} = [\boldsymbol{q}_1, ..., \boldsymbol{q}_{|T|}]^\top, \boldsymbol{d} = [\boldsymbol{d}_1, ..., \boldsymbol{d}_{|T|}]^\top$, Ξ_d be the support for $\boldsymbol{d}$. For simplicity, we let $\boldsymbol{e}$ be the vector of ones of corresponding dimension in matrix-vector calculations.

The DRO/SP model, the goal is to minimize worst-case expected unsatisfied demand and surplus (idle) flights

$$\begin{aligned}
& \inf \sup_{f \in \mathcal{F}} \mathbb{E}_f \left[\sum_t \boldsymbol{q}_t \right] \\
\textbf{s.t.} & \\
& \boldsymbol{q} \geq b \cdot (\boldsymbol{d} - \boldsymbol{U}^\top \boldsymbol{e}) \qquad \forall \boldsymbol{d} \in \Xi_d \\
& \boldsymbol{q} \geq h \cdot (\boldsymbol{U}^\top \boldsymbol{e} - \boldsymbol{d}) \qquad \forall \boldsymbol{d} \in \Xi_d \\
& \boldsymbol{x}_{(i,\cdot)}, \boldsymbol{u}_{(i,\cdot)}, \boldsymbol{s}_{(i,\cdot)} \in \Omega_i \qquad \forall i \in I
\end{aligned}$$

Same relaxation scheme can be used on the DRO models:

- Mean-variance, in Delage and Ye (2010).
- Likelihood, in Wang et al. (2016)

4.1 Mean-variance

With moment uncertainty for $\boldsymbol{d} : \mathbb{E}(\boldsymbol{d}) = \mu_0, ...,$ in Delage and Ye (2010). The DRO model is equivalent to the following problem:

$$
\begin{aligned}
\inf_{\boldsymbol{x},\boldsymbol{Q},\boldsymbol{\beta},r,s} \; & (\gamma_2 \boldsymbol{\Sigma}_0 - \boldsymbol{\mu}_0\boldsymbol{\mu}_0^\top) \bullet \boldsymbol{Q} + r + (\boldsymbol{\Sigma}_0 \bullet \boldsymbol{P}) - 2\boldsymbol{\mu}_0^\top \boldsymbol{p} + \gamma_1 s \\
\textbf{s.t.} \; & \\
& \boldsymbol{d}^\top \boldsymbol{Q} \boldsymbol{d} - \boldsymbol{d}^\top \boldsymbol{\beta} + r \geq \sum_t \boldsymbol{q}_t && \forall \boldsymbol{d} \in \Xi_d \\
& \boldsymbol{Q} \succeq 0, \boldsymbol{\beta} \in \mathbb{R}^{|T|}, \begin{bmatrix} \boldsymbol{P} & \boldsymbol{p} \\ \boldsymbol{p}^\top & s \end{bmatrix} \succeq 0, \; \boldsymbol{\beta} = 2(\boldsymbol{p} + \boldsymbol{Q}\boldsymbol{\mu}_0) \\
& \boldsymbol{q} \geq b \cdot (\boldsymbol{d} - \boldsymbol{U}^\top \boldsymbol{e}) && \forall \boldsymbol{d} \in \Xi_d \\
& \boldsymbol{q} \geq h \cdot (\boldsymbol{U}^\top \boldsymbol{e} - \boldsymbol{d}) && \forall \boldsymbol{d} \in \Xi_d \\
& \boldsymbol{x}_{(i,\cdot)}, \boldsymbol{u}_{(i,\cdot)}, \boldsymbol{s}_{(i,\cdot)} \in \Omega_i && \forall i \in I
\end{aligned}
$$

4.2 Finite support and likelihood robust

The problem is, let $\boldsymbol{Q} = [\boldsymbol{q}^1, ..., \boldsymbol{q}^N]$

$$
\begin{aligned}
\sup_{\beta,\theta,\Omega_i,\forall i} \; & \theta + \beta\gamma + \beta N - \underbrace{\beta \mathbf{N}^\top \log(\frac{\beta \mathbf{N}}{\boldsymbol{Q}e - \theta \mathbf{1}})}_{\mathcal{D}_{KL}(\beta \mathbf{N} | \boldsymbol{Q}e - \theta \mathbf{1})} \\
\textbf{s.t.} \; & \\
& \beta \geq 0 \\
& \boldsymbol{Q}e \geq \theta \mathbf{1} \\
& \boldsymbol{q}^n \geq b \cdot (\boldsymbol{d}^n - \boldsymbol{U}^\top e) && \forall n = 1, ..., N \\
& \boldsymbol{q}^n \geq h \cdot (\boldsymbol{U}^\top e - \boldsymbol{d}^n) && \forall n = 1, ..., N \\
& \boldsymbol{x}_{(i,\cdot)}, \boldsymbol{u}_{(i,\cdot)}, \boldsymbol{s}_{(i,\cdot)} \in \Omega_i && \forall i \in I
\end{aligned}
$$

Relax binding constraints for $\boldsymbol{Q}, \boldsymbol{U}$:

$$
\boldsymbol{z}_{\mathsf{LD}} = \sup_{\boldsymbol{Q}} ... + \sum_n (b\lambda^n - h\theta^n)^\top \boldsymbol{d}^n + \sum_n (\lambda^n + \theta^n)^\top q^n + \sup_{\boldsymbol{U}} \sum_i \sum_t (\sum_n h\theta_t^n - b\lambda_t^n) \boldsymbol{u}_{it}
$$

We can optimize for $\boldsymbol{Q}, \boldsymbol{U}$ separately, at each step, subproblems can be solved in polynomial time.

Reference

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