Dual Optimization for Newsvendor-like Problem

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FMP: variant case and weighted summation

$$f = \min_{x_{it}, u_{it}, \delta_t, \epsilon_t} b^\mathsf{T} \delta + h^\mathsf{T} \epsilon \tag{1}$$

s.t.

$$\sum_i c_i u_{it} + \delta_t - \epsilon_t = d_t, \quad \forall t \in T \tag{2}$$

$$U_{i,\cdot}, S_{i,\cdot}, X_{i,\cdot} \in \Omega_i, \quad \forall i \in I \tag{3}$$

- b, h are time-variant
- $c_i \neq 1$.

FMP: Lagrangian relaxation

For the FMP, dual function:

$$\phi(\lambda) = -\sum_t \lambda_t d_t + \sum_i \min_{\Omega_i} c_i \sum_t \lambda_t u_{it}$$

It reduces to a set of low dimensional minimization problems for each $i \in I, \forall \lambda$
(Recall $\lambda_t \in [-b, h]$ and $\delta_t^\star = \epsilon_t^\star = 0$ else unbounded)

$$\min_{\Omega_i} c_i \sum_t \lambda_t \cdot u_{i,t} \tag{4}$$

(4) is the subproblem to be solved by dynamic programming.
(states: lifespan, action: work or start maintence)

FMP: subgradient method

At each iteration k:

- y_k solves $\phi(\lambda_k) = \min_y \lambda_k^{\mathsf{T}}(y - b)$
- for FMP: $y_k = U_k^{\mathsf{T}} c$, $U_k = (u_{it}^{(k)})$, solved from (4) by DP.
- update λ_k.

Small case: $|I| = 1, t = 2$

$c = 1$

$$\begin{aligned} \min \quad & 2\delta_0 + 3\delta_1 + 4\epsilon_0 + \epsilon_1 \\ s.t. \quad & u_0 + \delta_0 - \epsilon_0 = 1 \\ & u_1 + \delta_1 - \epsilon_1 = 0 \\ & 2u_0 + s_0 = 6 \\ & -6x_0 + 2u_1 - s_0 + s_1 = 0 \\ & x_0 + u_0 \le 1 \\ & x_1 + u_1 \le 1 \\ & s_0, s_1 \ge 2 \end{aligned}$$

dual

$$-2 \le \lambda_0 \le 4$$

$$-3 \le \lambda_1 \le 1$$

$$\min_u \lambda_0(u_0 - 1) + \lambda_1(u_1 - 0)$$

$c = 2$

$$\begin{aligned} \min \quad & 2\delta_0 + 3\delta_1 + 4\epsilon_0 + \epsilon_1 \\ s.t. \quad & 2u_0 + \delta_0 - \epsilon_0 = 1 \\ & 2u_1 + \delta_1 - \epsilon_1 = 0 \\ & 2u_0 + s_0 = 6 \\ & -6x_0 + 2u_1 - s_0 + s_1 = 0 \\ & x_0 + u_0 \le 1 \\ & x_1 + u_1 \le 1 \\ & s_0, s_1 \ge 2 \end{aligned}$$

$$\min_u \lambda_0(2u_0 - 1) + \lambda_1(2u_1 - 0)$$

Small case: $|I| = 1, t = 2$

results:

$$
\begin{array}{rr}
\textit{gurobi}: & \\
c = 1 & c = 2 \\
u^\star = [1, 0] & u^\star = [0, 0] \\
f^\star = 0 & f^\star = 2 \\
\textit{subgradient}: & \\
\lambda_k = [-2, 0] & \lambda_k = [-3.5203e^{-5}, 0] \\
\phi_k = 0 & \phi_k = 0 \\
u_k = [1, 0] & u_k = [0, 0] \\
\phi_k = f^\star & \phi_k \neq f^\star
\end{array}
$$

Small case, relax

results:

$$\begin{aligned} \textit{gurobi} : u &\in [0,1] \\ c &= 1 & c &= 2 \\ u^\star &= [1,0] & u^\star &= [0.5,0] \\ f^\star &= 0 & f^\star &= 0 \\ \textit{gurobi} : x &\in [0,1] \\ c &= 1 & c &= 2 \\ u^\star &= [1,0] & u^\star &= [0,0] \\ f^\star &= 0 & f^\star &= 2 \end{aligned}$$

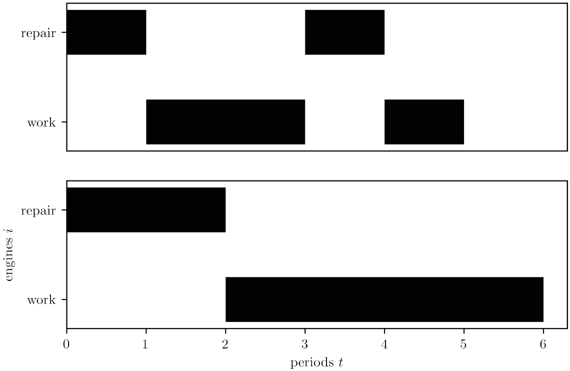
Small case relax on u, 2 engines, 6 periods, $u \in [0, 1]$

- d $=$ [2.0, 2.0, 1.0, 1.0, 2.0, 1.0],
- α: [5.0, 3.0],
- β: [8.0, 7.0],
- L: 2,
- c: array([1., 1.]),
- τ: [1, 1],
- s0: [5.0, 5.0],
- h: array([1, 2, 4, 4, 3, 1]),
- p: array([4, 5, 6, 6, 6, 4])

Small case, 2 engines, 6 periods, visualization

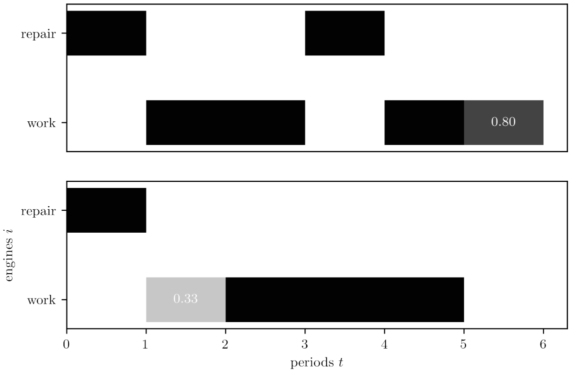
Below is an instance illustrating the gap from MILP to partial relaxation of integral constraints on u. The black and white cells represent the cases where corresponding binary variables take values in $0, 1$, respectively. The grey cells with attached values stand for fractional solutions in $(0, 1)$.

Small case, 2 engines, 6 periods, visualization



(a) Optimal solution from (1)

Small case, 2 engines, 6 periods, visualization



(b) Optimal solution from (1) with relaxed integral constraints on u

Small case

The subgradient method (sg) stops at (maximum) 400 iterations, then compared with benchmarks by Gurobi (grb).

- lb_grb, f_{grb} are lower bound, primal value from grb.
- $f^u_{\text{relax}}, f^x_{\text{relax}}$ primal value from grb for problem with relaxed u and x only, respectively: $f^x_{\text{relax}} = f^* $?
- t_grb, t_sg are runtime from grb and sg, respectively.
- ϕ_{sg} is the dual value (ϕ) by sg.
- z_{sg} is primal value in sg using averaged primal recovery, i.e.

$$z_{\text{sg}} = z(\bar{y}_k) = f(\bar{\delta}_k, \bar{\epsilon}_k)$$

- ϕ_{gap}, z_gap are relative gap from sg to grb for dual and primal values.

FMP: $c = 1$ small case 5×5

	lb_grb	f_{grb}	t_grb	f^{u}_{relax}	f^{x}_{relax}	t_sg	ϕ_{sg}	ϕ_{gap}
0	34.00	34.00	0.01	32.50	34.00	1.26	34.00	0.00%
1	42.00	42.00	0.01	40.83	42.00	0.01	42.00	0.00%
2	84.00	84.00	0.01	78.25	84.00	0.01	84.00	0.00%
3	49.00	49.00	0.01	45.50	49.00	0.04	48.96	-0.08%
4	50.00	50.00	0.01	50.00	50.00	0.00	50.00	0.00%
5	112.00	112.00	0.01	106.17	112.00	0.01	112.00	0.00%
6	47.00	47.00	0.02	47.00	47.00	0.02	47.00	0.00%
7	69.00	69.00	0.03	67.33	69.00	0.04	68.98	-0.04%
8	41.00	41.00	0.02	34.25	41.00	0.64	41.00	0.00%
9	91.00	91.00	0.00	89.67	91.00	0.01	91.00	0.00%

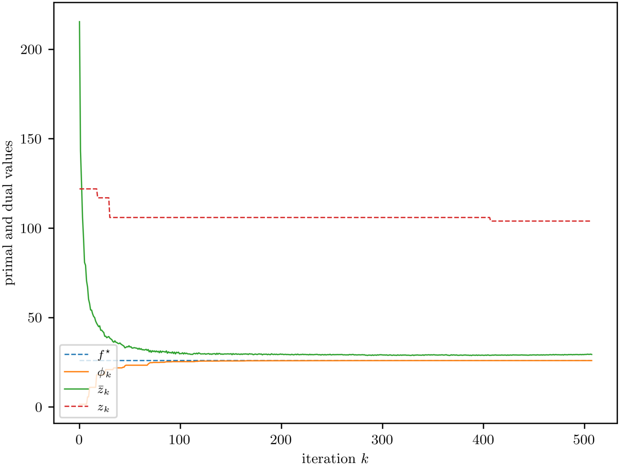
FMP: $c = 1$ large case 10×15

- $f^x_{\text{relax}} \neq f^\star$

	lb_grb	f_{grb}	t_grb	f^u_{relax}	f^x_{relax}	t_sg	ϕ_{sg}	ϕ_{gap}
0	467.00	467.00	0.59	460.25	455.00	3.97	467.00	0.00%
1	375.00	375.00	0.22	370.00	366.00	0.12	375.00	0.00%
2	158.00	158.00	0.67	155.92	158.00	4.73	158.00	0.00%
3	463.00	463.00	0.80	457.08	458.00	5.86	463.00	0.00%
4	370.00	370.00	0.11	367.17	363.00	4.46	370.00	0.00%
5	336.00	336.00	1.04	329.17	333.00	0.10	336.00	0.00%
6	350.00	350.00	0.74	344.42	348.00	3.93	350.00	0.00%
7	416.00	416.00	0.09	409.17	409.00	0.19	415.61	-0.09%
8	374.00	374.00	0.88	363.50	363.00	7.81	374.00	0.00%
9	343.00	343.00	1.30	330.67	336.00	0.24	342.67	-0.10%

FMP: $c = 1$ large case 10×15, primal solution

z_k is the value of best primal (recovery) solution, ϕ_k is the dual value, $f^\star$ is the benchmark optimal value by grb. there is a gap from z_k to $f^\star$



FMP: $c \neq 1$ small case 5×5

	lb_grb	f_{grb}	t_grb	f^{u}_{relax}	f^{x}_{relax}	t_sg	ϕ_{sg}	ϕ_{gap}
0	8.00	8.00	0.01	0.00	8.00	0.75	-0.00	-99.99%
1	9.00	9.00	0.01	0.00	9.00	0.68	8.00	-11.11%
2	55.00	55.00	0.02	52.50	55.00	1.33	54.98	-0.03%
3	11.00	11.00	0.03	5.00	11.00	1.06	10.00	-9.09%
4	7.00	7.00	0.02	0.00	7.00	0.85	4.00	-42.85%
5	20.00	20.00	0.03	12.33	20.00	1.08	20.00	-0.00%
6	22.00	22.00	0.01	20.00	22.00	0.01	22.00	0.00%
7	15.00	15.00	0.01	8.67	15.00	0.01	15.00	0.00%
8	14.00	14.00	0.03	6.00	14.00	1.01	10.00	-28.57%
9	18.00	18.00	0.02	13.00	18.00	0.68	18.00	-0.00%

Conclusion

- ? zero duality gap: $\phi^\star = f^\star$, $\phi^\star$ is the best bound by ϕ and $f^\star$ is the best primal value.
- ? can we bound the quality of heuristic for averaged solution? $\bar{z}_k = z(\bar{y}_k)$ converges to $\phi^\star$:

$$|\bar{z}_k - \phi^\star|$$

- ? can we improve $\bar{z} = z(\bar{y})$, round $\bar{z}$