Dual Optimization for Newsvendor-like Problem

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Primal problem

$$\min f(\delta, \epsilon) \tag{1}$$
$$\text{s.t.} \quad y + \delta - \epsilon = b \tag{2}$$
$$y \in \Omega_y \subseteq \mathbb{R}^n, \delta \in \mathbb{R}^n_+, \epsilon \in \mathbb{R}^n_+ \tag{3}$$

- Assume $f = p^\mathsf{T}\delta + h^\mathsf{T}\epsilon, p \geq 0, h \geq 0$, (2) expresses the newsvendor objective.
- Ω_y is a mixed integer set, and can be decomposed into small problems that are easier to solve.

Lagrangian relaxation

Relax the newsvendor equation: $y + \delta - \epsilon = b$

$$
\begin{aligned}
\phi(\lambda) &= \min_{\delta,\epsilon} (p+\lambda)^{\mathsf{T}}\delta + (h-\lambda)^{\mathsf{T}}\epsilon + \min_{y} \lambda^{\mathsf{T}} y - \lambda^{\mathsf{T}} b \\
&= \min_{y} \lambda^{\mathsf{T}} y - \lambda^{\mathsf{T}} b \\
\textbf{s.t.}& \\
&\quad y \in \Omega_y \\
&\quad \delta \in \mathbb{R}_{+}^{n}, \epsilon \in \mathbb{R}_{+}^{n}
\end{aligned}
\tag{4}
$$

(Since $\lambda \in [-p, h]$ and $\delta^{\star} = \epsilon^{\star} = 0$ else unbounded)

Subgradient method

We want to solve $\max_\lambda \phi(\lambda)$ by subgradient method:

$$
\begin{aligned}
g &= y - b \in \partial\phi \\
g_k &= y_k - b \\
\lambda_{k+1} &= \mathbf{P}(\lambda_k + s_k g_k) \\
s_k &= \gamma_k \frac{\phi^\star - \phi(\lambda_k)}{\|g_k\|^2}
\end{aligned}
\tag{5}
$$

P is the projection onto $[-p, h]$. Keep the averaged solution:

$$
\bar{y}_k = \frac{1}{k} \sum_i^k y_i \tag{6}
$$

Primal recovery

$(y_k, \epsilon_k = 0, \delta_k = 0)$ may not be feasible, use recovery:

$$\begin{aligned} \epsilon_k &= \max\{y_k - b, 0\} \\ \delta_k &= \max\{b - y_k, 0\} \\ \bar{\epsilon}_k &= \max\{\bar{y}_k - b, 0\} \\ \bar{\delta}_k &= \max\{b - \bar{y}_k, 0\} \end{aligned} \tag{7}$$

let corresponding primal value be $z(y_k) = f(\delta_k, \epsilon_k)$, $\bar{z}_k = z(\bar{y}_k)$

Motivation: fleet engine maintenance problem (FMP)

- engines: $i \in I$, time periods: $t = 1, 2, ..., n$, demand: $b = (b_1, ..., b_n)$.
- at each time we decide if engine i is working $u_{it} = 1$ or sent to maintenance $x_{it} = 1$ (and will be finished after τ periods)
- the lifespan of engine i decreases by α_i if working; increases by β_i if the maintenance is finished; the lifespan has a lower bound L.
- our goal is to satisfy demand b by minimizing the surplus ϵ and shortage δ: $f = p^\mathsf{T}\delta + h^\mathsf{T}\epsilon$
- let Ω_i be the mixed-integer set regarding the maintenance requirements individually, so we have Ω_i for each i

FMP

$$f = \min_{x_{it}, u_{it}, \delta_t, \epsilon_t} \sum_t (b \cdot \delta_t + h \cdot \epsilon_t) \tag{8}$$

s.t.

$$\sum_i u_{it} + \delta_t - \epsilon_t = d_t, \quad \forall t \in T \tag{9}$$

$$s_{i,t+1} = s_{it} - \alpha_i u_{it} + \beta_i x_{i,t-\tau}, \quad \forall i \in I, t \in T \tag{10}$$

$$x_{it} + u_{i,t} \leq 1, \quad \forall i \in I, t \in T \tag{11}$$

$$x_{it} + x_{i\rho} + u_{i,\rho} \leq 1, \quad \forall i \in I, t \in T, \rho = t+1, ..., t+\tau \tag{12}$$

$$s_{it} \geq L, \quad \forall i \in I, t \in T \tag{13}$$

(10) - (13) can be expressed as $\Omega_i, \forall i \in I$

FMP: continued

- (9) is the demand satisfaction constraint.
- (10) tracks the engine lifespan.
- (11) means an engine cannot work if sent to maintenance.
- (12) means the maintenance must be finished before an engine does anything else.
- (13) denotes the lower bound of lifespan.
- if relax (9) then we can solve for each i individually.

FMP: Lagrangian relaxation

For the FMP, dual function:

$$\phi(\lambda) = -\sum_t \lambda_t d_t + \sum_i \min_{\Omega_i} \sum_t \lambda_t u_{it}$$

It reduces to a set of low dimensional minimization problems for each $i \in I, \forall \lambda$
(Recall $\lambda_t \in [-b, h]$ and $\delta_t^\star = \epsilon_t^\star = 0$ else unbounded)

$$\min_{\Omega_i} \sum_t \lambda_t \cdot u_{i,t} \tag{14}$$

(14) is the subproblem to be solved by dynamic programming.
(states: lifespan, action: work or start maintenance)

FMP: subgradient method

At each iteration k:

- y_k solves $\phi(\lambda_k) = \min_y \lambda_k^\mathsf{T}(y - b)$
- for FMP: $y_k = U_k^\mathsf{T} 1 - d, U_k = (u_{it}^{(k)})$, solved from (14) by DP.
- update λ_k by (5)

Results

Tests are done on FMP with random instances: rand(low, high) means random integer in [low, high]

- L $= 2$
- $d_t = \text{rand}(\frac{|I|}{2}, |I|), \forall t$
- $\tau_i = \text{rand}(2,5), a_i = \text{rand}(2,5), b_i = \text{rand}(5,10), \forall i$
- $s_{i,0} = \text{rand}(5,8), \forall i$

We consider two cases for objective function

i , time-invariant: $h_t = 11, p_t = 18, \forall t$

ii , $h_t = \text{rand}(10,15), p_t = \text{rand}(10,16), \forall t$

Results

The subgradient method (sg) stops at (maximum) 400 iterations, then compared with benchmarks by Gurobi (grb).

- lb_grb, f_{grb} are lower bound, primal value from grb.
- t_grb, t_sg are runtime from grb and sg, respectively.
- ϕ_{sg} is the dual value (ϕ) by sg.
- z_{sg} is primal value in sg using averaged primal recovery, i.e.

$$z_{\mathrm{sg}} = z(\bar{y}_k) = f(\bar{\delta}_k, \bar{\epsilon}_k)$$

- ϕ_{gap}, z_gap are relative gap from sg to grb for dual and primal values.

FMP: invariant $h_t = h, b_t = b, \forall t = 1, \ldots, n$

	lb_grb	f_{grb}	t_grb	t_sg	ϕ_{sg}	ϕ_{gap}	z_{sg}	z_gap
0	936.00	936.00	23.93	11.46	936.00	-0.00%	952.02	1.71%
1	954.00	954.00	11.00	8.88	954.00	0.00%	958.78	0.50%
2	1224.00	1224.00	1.11	6.74	1224.00	0.00%	1230.12	0.50%
3	1026.00	1026.00	2.76	7.58	1026.00	0.00%	1031.13	0.50%
4	882.00	882.00	3.41	6.98	882.00	0.00%	886.39	0.50%
5	990.00	990.00	0.94	9.35	990.00	0.00%	994.95	0.50%
6	774.00	774.00	1.08	9.73	774.00	0.00%	778.58	0.59%
7	864.00	864.00	1.26	9.06	864.00	0.00%	868.34	0.50%
8	972.00	972.00	1.02	6.97	972.00	0.00%	976.85	0.50%
9	648.00	648.00	88.56	16.14	645.14	-0.44%	665.45	2.69%

FMP: variant h, b

	lb_grb	f_{grb}	t_grb	t_sg	ϕ_{sg}	ϕ_{gap}	z_{sg}	z_gap
0	608.00	608.00	8.34	12.76	607.57	-0.07%	650.63	7.01%
1	587.00	587.00	3.93	12.01	586.60	-0.07%	619.44	5.53%
2	538.00	538.00	22.73	12.33	537.36	-0.12%	574.96	6.87%
3	534.00	534.00	4.05	13.03	532.69	-0.24%	607.16	13.70%
4	684.00	684.00	19.14	7.61	684.00	0.00%	691.68	1.12%
5	620.00	620.00	4.88	8.06	619.98	-0.00%	625.29	0.85%
6	657.00	657.00	4.47	9.70	655.73	-0.19%	682.15	3.83%
7	625.00	625.00	8.25	12.04	622.30	-0.43%	670.87	7.34%
8	422.00	422.00	5.03	12.40	421.25	-0.18%	453.72	7.52%
9	485.00	485.00	4.32	13.59	484.87	-0.03%	517.35	6.67%

Conclusion

- zero duality gap: $\phi^\star = f^\star$, $\phi^\star$ is the best bound by ϕ and $f^\star$ is the best primal value.
- can we bound the quality of heuristic for averaged solution? $\bar{z}_k = z(\bar{y}_k)$ converges to $\phi^\star$:

$$|\bar{z}_k - \phi^\star|$$

- for variant case, can we improve $\bar{z} = z(\bar{y})$?