




# Iteratively Regularized Gradient Tracking Methods for Optimal Equilibrium Seeking

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**Abstract**—In noncooperative Nash games, equilibria are often inefficient. This is exemplified by the Prisoner’s Dilemma and was first provably shown in the 1980s. Since then, understanding the quality of Nash equilibrium (NE) received considerable attention, leading to the emergence of inefficiency measures characterized by the best or the worst equilibrium. Traditionally, computing an optimal NE in monotone regimes is done through two-loop schemes which lack scalability and provable performance guarantees. The goal in this work lies in the development of among the first single-timescale distributed gradient tracking optimization methods for optimal NE seeking over networks. Our main contributions are as follows. By employing a regularization-based relaxation approach within two existing distributed gradient tracking methods, namely Push-Pull and DSGT, we devise and analyze two single-timescale iteratively regularized gradient tracking algorithms. The first method addresses computing the optimal NE over directed networks, while the second method addresses a stochastic variant of this problem over undirected networks. For both methods, we establish the convergence to the optimal NE and derive new convergence rate statements for the consensus error of the generated iterates. We provide preliminary numerical results on a Nash-Cournot game.

**Index Terms**—Distributed optimization, gradient tracking, iterative regularization, optimal equilibrium seeking.

## I. INTRODUCTION

NONCOOPERATIVE game theory [26] offers a rigorous mathematical framework for the modeling and analysis of multi-agent decision-making systems. It has been applied in a myriad of applications including transportation systems [37], communication [24], telecommunication networks [1], smart grids [10], social networks [16], economics [11], and robotics [7], among others. These applications share the following key characteristic in common; a collection of independent and self-interested agents (i.e., entities such as sensors, people, robots, for example) compete with each other where each agent seeks to optimize an individual objective function. The result of this noncooperative rationality is mathematically captured by the concept of Nash equilibrium

(NE). In view of the agents’ selfish behavior, it is natural to expect the underlying system to reach a non-optimal global performance. In fact, equilibria are known to be inefficient. This is exemplified by the Prisoner’s Dilemma and was first provably shown by Dubey [12]. Since then, understanding the quality of NE received considerable attention in addressing questions such as: *Is there a unique equilibrium? If not, how many equilibria exist? What is the most desirable equilibrium of a game?* Addressing these questions is critically important from the standpoint of the game’s protocol designer in areas such as routing [31] and load balancing [9]. Indeed, in many cases a multitude of equilibria exist [27], and as such, finding a single equilibrium could be far from satisfactory. Research efforts in tackling these questions led to the emergence of inefficiency measures characterized by the best or the worst equilibrium with respect to a global welfare function. In the game theory literature, the problem of computing an optimal equilibrium has been regarded as a computationally challenging task [8], [17]. This is primarily because the optimal NE seeking problem is cast as an optimization problem with equilibrium constraints. Naturally in optimization, the presence of constraints is tackled by leveraging a handful of avenues including the celebrated Lagrangian duality theory, projected schemes, and penalty techniques (cf. [6]). However, when the constraint set is characterized as the set of equilibria, which is indeed the case in optimal NE seeking, these standard approaches may not be applicable. Our focus in this paper lies in addressing the following key research question: *Can we devise single-timescale distributed first-order methods (over both undirected and directed networks) for optimal NE seeking, in settings possibly afflicted with the presence of uncertainty, with provable global guarantees?* Here, the term *single-timescale* refers to a method that does not involve solving an inner-level optimization/variational problem and the term *global* refers to computing a global optimal solution to the NE selection optimization problem. To this end, in this work, we aim to address a distributed optimization problem with a distributed variational inequality (VI) constraint. This is of the form

$$\min_{x \in \mathbb{R}^n} \sum_{i=1}^m f_i(x) \quad \text{s.t.} \quad x \in \text{SOL}(X, \sum_{i=1}^m F_i), \quad (1)$$

where there are  $m$  agents, each agent is associated with a local function  $f_i$  and a mapping  $F_i : \mathbb{R}^n \rightarrow \mathbb{R}^n$ . Agents may communicate over a directed or an undirected network. Here, for a given nonempty convex set  $X \subseteq \mathbb{R}^n$  and mapping  $F$ ,  $\text{SOL}(X, F) \triangleq \{x \in X \mid F(x)^\top(y - x) \geq 0, \forall y \in X\}$ ,

This work is supported in part by the National Science Foundation under CAREER Grant ECCS-1944500, in part by the Office of Naval Research under Grant N00014-22-1-2757, and in part by the Department of Energy under Grant DE-SC0023303.

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denotes the solution set of  $\text{VI}(X, F)$ . The VI problem [13] is an immensely powerful mathematical framework for capturing several important problem classes. In particular, under standard differentiability and convexity assumptions [13, Prop. 1.4.2], the set of all NEs among a group of players is equal to the solution set of a Cartesian VI. In Section IV, we provide a detailed example illustrating how a Nash game can be reformulated as a distributed VI problem. Notably, solving problem (1) is challenging, because of the following reasons: (i) The feasible solution set in (1) is itself the solution set of a VI problem and is often unavailable. In fact, there does not exist an algorithm that can compute all NEs to a game [15]; (ii) The set  $\text{SOL}(X, F)$  is characterized by infinitely many, and possibly nonconvex, inequality constraints. Therefore, traditional Lagrangian duality theory may not be applicable in this setting.

### A. Related work

The classical notion of duality for VIs finds its origin in the work by Mosco [25] and its extensions were studied in [2]. Such duality frameworks are focused on the resolution of VIs in the presence of standard inequalities within the set  $X$ , which is substantially different than the setting considered in this work. The traditional approach for computing an optimal NE lies in *sequential regularization (SR)*, described as follows. At time step  $k$ , given a strictly positive regularization parameter  $\lambda_k$ , convex set  $X$ , strongly convex function  $f$ , and monotone mapping  $F$ , the regularized problem  $\text{VI}(X, F + \lambda_k \nabla f)$  is solved. Let  $x_{\lambda_k}^*$  denote the unique solution of this problem. When  $\lambda_k \rightarrow 0$ , it can be shown that the solution trajectory of  $\{x_{\lambda_k}^*\}$ , known as *Tikhonov trajectory*, converges to the unique optimal NE. Although it is theoretically appealing, a key shortcoming of the SR scheme is that it is a two-loop scheme where at each iteration, an increasingly more difficult VI problem needs to be solved. This computational burden is exacerbated further by stochasticity in the objective function and the equilibrium constraints where it may be impossible to compute an exact solution to a regularized VI problem. Further, the computational complexity of the SR scheme appears to be unknown. To address these shortcomings, inexact SR schemes [14] and iteratively regularized (IR) gradient methods [19], [22], [23], [35], [41], [42] have been studied more recently. In prior work by the authors [21], [33], both asymptotic and non-asymptotic guarantees of IR schemes for addressing centralized VI-constrained optimization problems are studied. Notably, in [33] it is shown that IR schemes may admit the fastest known convergence speed in addressing VI-constrained optimization problems and their subclasses [32]. In [18], an iteratively penalized scheme is devised to address stochastic settings. In [20], an IR incremental gradient method is devised for addressing problem (1), where it is assumed that agents communicate over a directed cycle graph.

Another avenue for addressing the optimal NE selection problem lies in leveraging the fixed-point selection theory that has been employed in centralized [40] and distributed settings [4], [5]. Of these, the work in [5] proposes a distributed method with asymptotic convergence guarantees for tracking the optimal NE in deterministic settings, over undirected and time-varying networks.

### B. Research gap and contributions

Despite recent progress in addressing optimal NE seeking problems, it appears that problem (1) over networks has not been addressed. More precisely, we are unaware of IR gradient tracking (GT) schemes for optimal NE seeking in that the welfare loss objective is distributed among the agents, as in (1). In recent years, distributed GT methods have been designed and analyzed for addressing the canonical problem of the form  $\min_{x \in X} \sum_{i=1}^m f_i(x)$  in convex [29], [34], nonconvex [39], and stochastic settings [28], among others. Motivated by the extensive recent work on GT methods for addressing standard distributed optimization problems, we devise and analyze two new iteratively regularized distributed GT methods equipped with provable guarantees. Our main contributions are as follows. (i) In the first part of the paper, we consider problem (1) over directed networks and assume that the global mapping is monotone and the global objective is strongly convex. We consider the Push-Pull method in [29] that addresses unconstrained distributed optimization problems over directed networks. By leveraging the IR framework within the Push-Pull method, we devise an iteratively regularized method called IR-Push-Pull for addressing (1), where at each iteration, the information of iteratively regularized local mappings is pushed to the neighbors, while the information about the local copies of vector  $x$  is pulled from the neighbors. In Theorem 1, we establish the convergence to the unique optimal NE and derive new convergence rate statements for consensus errors of the generated iterates.

(ii) In the second part of the paper, we consider a stochastic variant of problem (1) over undirected networks given as

$$\begin{aligned} \min_{x \in \mathbb{R}^n} \quad & \sum_{i=1}^m \mathbb{E}[f_i(x, \xi_i)] \\ \text{s.t.} \quad & x \in \text{SOL}(\mathbb{R}^n, \sum_{i=1}^m \mathbb{E}[F_i(x, \xi_i)]), \end{aligned} \quad (2)$$

where  $\xi_i \in \mathbb{R}^d$  is a local random variable associated with agent  $i$ . We consider the DSGT method in [28] that addresses unconstrained distributed stochastic optimization problems over undirected networks. Again, by leveraging the IR framework, we devise a single-timescale method called IR-DSGT for addressing (2). In Theorem 2, we establish the convergence to the optimal NE in a mean-squared sense and derive new non-asymptotic bounds for mean-squared of consensus errors.

**Outline of the paper.** We organize the remaining sections as follows. In Section II, we present and analyze the IR-Push-Pull method for addressing problem (1) over directed networks and provide convergence guarantees for computing the optimal NE in Theorem 1. In Section III, we present and analyze the IR-DSGT method for addressing the stochastic problem (2), over undirected networks and provide the convergence guarantees in Theorem 2. In Section IV, we validate our theoretical findings through numerical experiments on a Nash-Cournot game. Concluding remarks are presented in Section V.

**Notation.** For an integer  $m$ , the set  $\{1, \dots, m\}$  is denoted as  $[m]$ . A vector  $x$  is assumed to be a column vector (unless otherwise noted) and  $x^\top$  denotes its transpose. We use  $\|x\|_2$  to denote the Euclidean vector norm of  $x$ . A mapping  $F: \mathbb{R}^n \rightarrow \mathbb{R}^n$  is  $L_F$ -Lipschitz continuous if for any  $x, y \in \mathbb{R}^n$ ,  $\|F(x) - F(y)\|_2 \leq L_F \|x - y\|_2$ ;  $F$  is monotone if

$(F(x) - F(y))^T (x - y) \geq 0$  for any  $x, y \in \mathbb{R}^n$ . A continuously differentiable function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  is said to be  $\mu_f$ -strongly convex if its gradient mapping is  $\mu_f$ -strongly monotone, i.e.,  $(\nabla f(x) - \nabla f(y))^T (x - y) \geq \mu_f \|x - y\|_2^2$  for any  $x, y \in \mathbb{R}^n$ ;  $f$  is called  $L_f$ -smooth if its gradient mapping is  $L_f$ -Lipschitz continuous. We use the following definitions:

$$\mathbf{x} \triangleq [x_1, \dots, x_m]^T, \quad \mathbf{y} \triangleq [y_1, \dots, y_m]^T \in \mathbb{R}^{m \times n} \quad (3)$$

$$f(\mathbf{x}) \triangleq \sum_{i=1}^m f_i(x_i), \quad \mathbf{f}(\mathbf{x}) \triangleq \sum_{i=1}^m f_i(x_i), \quad (4)$$

$$\nabla \mathbf{f}(\mathbf{x}) \triangleq [\nabla f_1(x_1), \dots, \nabla f_m(x_m)]^T \in \mathbb{R}^{m \times n}, \quad (5)$$

$$F(\mathbf{x}) \triangleq \sum_{i=1}^m F_i(x_i), \quad (6)$$

$$\mathbf{F}(\mathbf{x}) \triangleq [F_1(x_1), \dots, F_m(x_m)]^T \in \mathbb{R}^{m \times n}. \quad (7)$$

Here,  $x_i$  is the local copy of agent  $i$ 's decision vector,  $\mathbf{x}$  includes the local copies of all agents, and  $y_i$  is an auxiliary variable for agent  $i$  to track the average of regularized mappings.

## II. DETERMINISTIC SETTING OVER DIRECTED NETWORKS

In this section, we consider problem (1) over directed networks, under the following main assumption.

*Assumption 1:* (a) Function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  is  $\mu_f$ -strongly convex. (b) Functions  $f_i : \mathbb{R}^n \rightarrow \mathbb{R}$  are  $L_f$ -smooth. (c) Mapping  $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is real-valued and (merely) monotone. (d) Mappings  $F_i : \mathbb{R}^n \rightarrow \mathbb{R}^n$  are  $L_F$ -Lipschitz continuous.

Throughout this section, we utilize the following notation and preliminaries. Given a set of nodes  $\mathcal{N}$ , a directed graph (digraph) is denoted by  $\mathcal{G} = (\mathcal{N}, \mathcal{E})$  where  $\mathcal{E} \subseteq \mathcal{N} \times \mathcal{N}$  is the set of ordered pairs of vertices. For any edge  $(i, j) \in \mathcal{E}$ ,  $i$  and  $j$  are called parent node and child node, respectively. Graph  $\mathcal{G}$  is called *strongly connected* if there is a path between the pair of any two different vertices. The digraph induced by a given nonnegative matrix  $\mathbf{B} \in \mathbb{R}^{m \times m}$  is denoted by  $\mathcal{G}_{\mathbf{B}} \triangleq (\mathcal{N}_{\mathbf{B}}, \mathcal{E}_{\mathbf{B}})$ , where  $\mathcal{N}_{\mathbf{B}} \triangleq [m]$  and  $(j, i) \in \mathcal{E}_{\mathbf{B}}$  if and only if  $B_{ij} > 0$ . We let  $\mathcal{N}_{\mathbf{B}}^{\text{in}}(i)$  and  $\mathcal{N}_{\mathbf{B}}^{\text{out}}(i)$  denote the set of parents (in-neighbors) and the set of children (out-neighbors) of vertex  $i$ , respectively. Also,  $\mathcal{R}_{\mathbf{B}}$  denotes the set of roots of all possible spanning trees in  $\mathcal{G}_{\mathbf{B}}$ . Throughout this section, we use the following definition of a matrix norm: Given an arbitrary vector norm  $\|\cdot\|$ , the induced norm of a matrix  $\mathbf{W} \in \mathbb{R}^{m \times n}$  is defined as  $\|\mathbf{W}\| \triangleq \|\begin{bmatrix} \|\mathbf{W}_{\bullet,1}\| \\ \vdots \\ \|\mathbf{W}_{\bullet,n}\| \end{bmatrix}\|_2$ .

*Remark 1:* Under the above definition of matrix norm, it can be shown that  $\|\mathbf{A}\mathbf{x}\| \leq \|\mathbf{A}\| \|\mathbf{x}\|$  for any  $\mathbf{A} \in \mathbb{R}^{m \times m}$  and  $\mathbf{x} \in \mathbb{R}^{m \times p}$ . Also, for any  $a \in \mathbb{R}^{m \times 1}$  and  $x \in \mathbb{R}^{1 \times n}$ , we have  $\|ax\| = \|a\| \|x\|_2$  (cf. [29, Lemma 5]).

*Remark 2:* Throughout this work, we assume that the set  $X$  in problem (1) is equal to  $\mathbb{R}^n$ , as our focus in this work primarily lies in addressing the equilibrium constraint. Notably, the methods and results in this work can be extended to address the more general setting when  $X \subset \mathbb{R}^n$ . In particular, in our prior work [30], we employ Moreau smoothing to address the constraint set and show that the solution to the approximate smoothed problem is an approximate solution to the original problem. In the numerical experiments section, we employ this technique to address the presence of the constraint set  $X$ . In resolving problem (1), in view of the presence of the VI constraint, Lagrangian duality is generally not applicable. In fact, even in the special case where the VI constraint captures

the solution set of an optimization problem, standard constraint qualifications (e.g., Slater condition) fail to hold. Overcoming this challenge calls for new relaxation rules that can tackle the VI constraint. To this end, motivated by the recent success of so-called *iteratively regularized (IR)* algorithms [19]–[22], [33], we develop Algorithm 1. Central to the IR framework is the principle that the regularization parameter  $\lambda_k$  is updated after every step within the algorithm. Algorithm 1 is an iteratively regularized variant of the Push-Pull method in [29]. The novelty in the design of Algorithm 1 lies in how we address the presence of the VI constraint through the IR approach. Here, each agent holds a local copy of the global variable  $x$ , denoted by  $x_{i,k}$ , and an auxiliary variable  $y_{i,k}$  is used to track the average of a regularized gradient. At each iteration, each agent  $i$  uses the  $i$ th row of matrix  $\mathbf{R} = [R_{ij}] \in \mathbb{R}^{m \times m}$  and the  $i$ th column of matrix  $\mathbf{C} = [C_{ij}] \in \mathbb{R}^{m \times m}$  to update vectors  $x_{i,k}$  and  $y_{i,k}$ , respectively. Below, we state the main assumptions on these two *weight mixing* matrices.

*Assumption 2:* (a) The matrix  $\mathbf{R}$  is nonnegative, with a strictly positive diagonal, and is row-stochastic, i.e.,  $\mathbf{R}\mathbf{1} = \mathbf{1}$ . (b) The matrix  $\mathbf{C}$  is nonnegative, with a strictly positive diagonal, and is column-stochastic, i.e.,  $\mathbf{1}^T \mathbf{C} = \mathbf{1}^T$ . (c) The induced digraphs  $\mathcal{G}_{\mathbf{R}}$  and  $\mathcal{G}_{\mathbf{C}^T}$  satisfy  $\mathcal{R}_{\mathbf{R}} \cap \mathcal{R}_{\mathbf{C}^T} \neq \emptyset$ .

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### Algorithm 1 IR-Push-Pull

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- 1: **Input:** For all  $i \in [m]$ , agent  $i$  sets stepsize  $\gamma_{i,0} \geq 0$ , pulling weights  $R_{ij} \geq 0$  for all  $j \in \mathcal{N}_{\mathbf{R}}^{\text{in}}(i)$ , pushing weights  $C_{ij} \geq 0$  for all  $j \in \mathcal{N}_{\mathbf{C}}^{\text{out}}(i)$ , an arbitrary initial point  $x_{i,0} \in \mathbb{R}^n$  and  $y_{i,0} := F_i(x_{i,0}) + \lambda_0 \nabla f_i(x_{i,0})$ ;
  - 2: **for**  $k = 0, 1, \dots$  **do**
  - 3:   For all  $i \in [m]$ , agent  $i$  receives (pulls) the vector  $x_{j,k} - \gamma_{j,k} y_{j,k}$  from each agent  $j \in \mathcal{N}_{\mathbf{R}}^{\text{in}}(i)$ , sends (pushes)  $C_{\ell i} y_{i,k}$  to each agent  $\ell \in \mathcal{N}_{\mathbf{C}}^{\text{out}}(i)$ , and does the following updates:
  - 4:    $x_{i,k+1} := \sum_{j=1}^m R_{ij} (x_{j,k} - \gamma_{j,k} y_{j,k})$ ;
  - 5:    $y_{i,k+1} := \sum_{j=1}^m C_{ij} y_{j,k} + F_i(x_{i,k+1}) + \lambda_{k+1} \nabla f_i(x_{i,k+1}) - F_i(x_{i,k}) - \lambda_k \nabla f_i(x_{i,k})$ ;
  - 6: **end for**
- 

Assumption 2 does not require the strong condition of a doubly stochastic matrix for communication in a directed network. In turn, utilizing a push-pull protocol and in a similar fashion to [29], it only entails a row stochastic  $\mathbf{R}$  and a column stochastic matrix  $\mathbf{C}$ . An example is as follows where agent  $i$  chooses scalars  $r_i, c_i > 0$  and sets  $R_{i,j} := 1/(|\mathcal{N}_{\mathbf{R}}^{\text{in}}(i)| + r_i)$  for  $j \in \mathcal{N}_{\mathbf{R}}^{\text{in}}(i)$ ,  $R_{i,i} := r_i/(|\mathcal{N}_{\mathbf{R}}^{\text{in}}(i)| + r_i)$ ,  $C_{\ell,i} := 1/(|\mathcal{N}_{\mathbf{C}}^{\text{out}}(i)| + c_i)$  for  $\ell \in \mathcal{N}_{\mathbf{C}}^{\text{out}}(i)$ ,  $C_{i,i} := c_i/(|\mathcal{N}_{\mathbf{C}}^{\text{out}}(i)| + c_i)$ , and 0 otherwise. Note that Assumption 2(c) is weaker than imposing strong connectivity on  $\mathcal{G}_{\mathbf{R}}$  and  $\mathcal{G}_{\mathbf{C}}$ . The update rules in Algorithm 1 can be compactly represented as the following:

$$\mathbf{x}_{k+1} := \mathbf{R}(\mathbf{x}_k - \boldsymbol{\gamma}_k \mathbf{y}_k), \quad (8)$$

$$\mathbf{y}_{k+1} := \mathbf{C} \mathbf{y}_k + \mathbf{F}(\mathbf{x}_{k+1}) + \lambda_{k+1} \nabla \mathbf{f}(\mathbf{x}_{k+1}) - \mathbf{F}(\mathbf{x}_k) - \lambda_k \nabla \mathbf{f}(\mathbf{x}_k), \quad (9)$$

where  $\boldsymbol{\gamma}_k \geq 0$  is defined as  $\boldsymbol{\gamma}_k \triangleq \text{diag}(\gamma_{1,k}, \dots, \gamma_{m,k})$ .

### A. Preliminaries of convergence analysis of Algorithm 1

Under Assumption 2, there exists a unique nonnegative left eigenvector  $u \in \mathbb{R}^m$  such that  $u^\top \mathbf{R} = u^\top$  and  $u^\top \mathbf{1} = m$ . Similarly, there exists a unique nonnegative right eigenvector  $v \in \mathbb{R}^m$  such that  $\mathbf{C}v = v$  and  $\mathbf{1}^\top v = m$  (cf. Lemma 1 in [29]). We utilize the following in the analysis of Algorithm 1.

**Definition 1:** For  $k \geq 0$  and the regularization parameter  $\lambda_k > 0$ , let  $x^* \triangleq \operatorname{argmin}_{x \in \operatorname{SOL}(\mathbb{R}^n, F)} \{f(x)\} \in \mathbb{R}^{1 \times n}$ ,  $x_{\lambda_k}^* \triangleq \operatorname{SOL}(\mathbb{R}^n, F + \lambda_k \nabla f)$ . We define the mapping  $\mathbf{G}_k(\mathbf{x}) \triangleq \mathbf{F}(\mathbf{x}) + \lambda_k \nabla \mathbf{f}(\mathbf{x}) \in \mathbb{R}^{m \times n}$ , and functions  $G_k(\mathbf{x}) \triangleq \frac{1}{m} \mathbf{1}^\top \mathbf{G}_k(\mathbf{x}) \in \mathbb{R}^{1 \times n}$ ,  $\mathcal{G}_k(x) \triangleq G_k(\mathbf{1}x^\top) \in \mathbb{R}^{1 \times n}$ ,  $\bar{g}_k \triangleq \mathcal{G}_k(\bar{x}_k) \in \mathbb{R}^{1 \times n}$ . We let  $L_k \triangleq L_F + \lambda_k L_f$ , and define vectors  $\bar{x}_k \triangleq \frac{1}{m} u^\top \mathbf{x}_k \in \mathbb{R}^{1 \times n}$ , and  $\bar{y}_k \triangleq \frac{1}{m} \mathbf{1}^\top \mathbf{y}_k \in \mathbb{R}^{1 \times n}$ . Lastly,  $\Lambda_k \triangleq \left|1 - \frac{\lambda_{k+1}}{\lambda_k}\right|$ . Here,  $x^*$  denotes the optimal solution of problem (1) and  $x_{\lambda_k}^*$  is defined as the unique solution to a regularized problem. Note that the strong monotonicity of  $F(x) + \lambda_k \nabla f(x)$  implies that  $x_{\lambda_k}^*$  exists and is a unique vector. Also, under Assumption 1, the set  $\operatorname{SOL}(\mathbb{R}^n, \sum_{i=1}^m F_i)$  is closed and convex. As such, from the strong convexity of  $f$  and invoking [6, Prop. 1.1.2] again, we conclude that  $x^*$  also exists and is a unique vector. The sequence  $\{x_{\lambda_k}^*\}$  is the so-called *Tikhonov trajectory* and plays a key role in the convergence analysis (cf. [13, Ch. 12]). The mapping  $\mathbf{G}_k(\mathbf{x})$  denotes the regularized gradient matrix. The vector  $\bar{x}_k$  holds a weighted average of local copies of agents' iterates. Next, we consider a family of update rules for the sequences of the stepsize and the regularization parameter under which the convergence and rate analysis can be performed.

**Assumption 3 (Update rules):** Let  $\hat{\gamma}_k := \frac{\hat{\gamma}}{(k+\Gamma)^a}$  and  $\lambda_k := \frac{\lambda}{(k+\Gamma)^b}$  for all  $k \geq 0$  where  $\hat{\gamma}, \lambda, \Gamma, a$  and  $b$  are strictly positive scalars and  $\hat{\gamma}_k \triangleq \max_{j \in [m]} \gamma_{j,k}$ . Let  $a > b > 0$ ,  $a+b < 1$ , and  $2a+3b < 2$ . Assume that  $\Gamma \geq 1$  and  $\Gamma \geq \hat{\Gamma}_1 \triangleq \frac{1-a-b}{1-a-b} \sqrt{\frac{4}{\hat{\gamma} \lambda \mu_f \tau}}$ , for some  $\tau > 0$ . Also, let  $\alpha_k \geq \theta \hat{\gamma}_k$  for  $k \geq 0$  for some  $\theta > 0$ , where  $\alpha_k \triangleq \frac{1}{m} u^\top \gamma_k v$ .

The scalars  $a$  and  $b$  prescribe the tuning rules for the stepsize and iterative regularization. The specified assumptions on these scalars play a key role in establishing the convergence of the method. The constant  $\theta$  in Assumption 3 measures the size of the range within which the agents in  $\mathcal{R}_R \cap \mathcal{R}_{C^\top}$  select their stepsizes. The condition  $\alpha_k \geq \theta \hat{\gamma}_k$  is satisfied in many cases including the case where all the agents choose strictly positive stepsizes (see [29, Remark 4]). In the following lemma, we list some of the main properties of the update rules in Assumption 3 that will be used in the analysis.

**Lemma 1 (Properties of the update rules):** Let Assumption 3 hold. Then, the following results hold.

- (i)  $\{\lambda_k\}_{k=0}^\infty$  is a decreasing strictly positive sequence satisfying  $\lambda_k \rightarrow 0$ ;  $\{\hat{\gamma}_k\}_{k=0}^\infty$  is a decreasing strictly positive sequence such that  $\hat{\gamma}_k \rightarrow 0$  and  $\frac{\hat{\gamma}_k}{\lambda_k} \rightarrow 0$ .
- (ii)  $\frac{\Lambda_k}{\lambda_k} \rightarrow 0$ ,  $\Lambda_{k+1} \leq \Lambda_k$  for all  $k \geq 0$ ,  $\Lambda_{k-1} \leq \frac{1}{k+\Gamma}$  for  $k \geq 1$ , where  $\Lambda_k$  is given by Def. 1.
- (iii)  $\frac{(k+\Gamma)\hat{\gamma}_k \lambda_k}{(k+\Gamma-1)\hat{\gamma}_{k-1} \lambda_{k-1}} \leq 1 + 0.5\mu_f \hat{\gamma}_k \lambda_k \tau$  for all  $k \geq 1$  and  $\tau > 0$ .

**Proof:** (i) Recall that  $\hat{\gamma}_k = \frac{\hat{\gamma}}{(k+\Gamma)^a}$  and  $\lambda_k = \frac{\lambda}{(k+\Gamma)^b}$  where  $0 < b < a < 1$ ,  $\Gamma \geq 1$  and  $a+b < 1$ . Consequently,  $\{\hat{\gamma}_k\}_{k=0}^\infty$  and  $\{\lambda_k\}_{k=0}^\infty$  are strictly positive decreasing sequences and  $\hat{\gamma}_k \rightarrow 0$ ,  $\lambda_k \rightarrow 0$ , and  $\frac{\hat{\gamma}_k}{\lambda_k} \rightarrow 0$ .

- (ii) First, we show that  $\Lambda_{k-1} \leq \frac{1}{k+\Gamma}$  for  $k \geq 1$ . From Def. 1 and that  $\lambda_k \leq \lambda_{k-1}$ , for any  $k \geq 1$  we have

$$\begin{aligned} \Lambda_{k-1} &= 1 - \frac{\lambda_k}{\lambda_{k-1}} = 1 - \frac{\lambda(k+\Gamma)^{-b}}{\lambda(k-1+\Gamma)^{-b}} \\ &= 1 - \left(\frac{k+\Gamma-1}{k+\Gamma}\right)^b = 1 - \left(1 - \frac{1}{k+\Gamma}\right)^b. \end{aligned} \quad (10)$$

From  $0 < b < a$  and  $a+b < 1$ , we have  $b < 0.5$ . This implies that  $(1 - \frac{1}{k+\Gamma})^b \geq (1 - \frac{1}{k+\Gamma})^{0.5}$ . We obtain

$$\Lambda_{k-1} \leq 1 - \sqrt{1 - \frac{1}{k+\Gamma}} = \left(1 + \sqrt{1 - \frac{1}{k+\Gamma}}\right)^{-1} \frac{1}{k+\Gamma} \leq \frac{1}{k+\Gamma},$$

where the last inequality is implied from  $k \geq 1$ . Next, we show  $\Lambda_{k+1} \leq \Lambda_k$  for all  $k \geq 0$ . From (10), we have

$$\Lambda_{k+1} = 1 - \left(1 - \frac{1}{k+2+\Gamma}\right)^b \leq 1 - \left(1 - \frac{1}{k+1+\Gamma}\right)^b = \Lambda_k.$$

- (iii) From the update rules of  $\hat{\gamma}_k$  and  $\lambda_k$ , we have

$$\begin{aligned} &\left(\frac{(k+\Gamma)\hat{\gamma}_k \lambda_k}{(k+\Gamma-1)\hat{\gamma}_{k-1} \lambda_{k-1}} - 1\right) \frac{1}{\hat{\gamma}_k \lambda_k \mu_f \tau} \\ &= \left(\frac{(k+\Gamma)}{(k+\Gamma-1)} \left(\frac{k+\Gamma-1}{k+\Gamma}\right)^{a+b} - 1\right) \frac{(k+\Gamma)^{a+b}}{\hat{\gamma} \lambda \mu_f \tau} \\ &= \left(\left(1 + \frac{1}{k+\Gamma-1}\right)^{1-a-b} - 1\right) \frac{(k+\Gamma)^{a+b}}{\hat{\gamma} \lambda \mu_f \tau} \leq \frac{(k+\Gamma)^{a+b}}{(k+\Gamma-1)\hat{\gamma} \lambda \mu_f \tau} \\ &\leq \frac{1+\frac{1}{\Gamma}}{(k+\Gamma)^{1-a-b}\hat{\gamma} \lambda \mu_f \tau} \leq \frac{2}{\Gamma^{1-a-b}\hat{\gamma} \lambda \mu_f \tau} \leq \frac{1}{2}, \end{aligned}$$

where the last two inequalities are implied by  $\Gamma \geq 1$  and  $\Gamma^{1-a-b} \geq \frac{4}{\hat{\gamma} \lambda \mu_f \tau}$ . Then, the result in (iii) follows. ■

Next, we present some key properties of the regularized sequence  $\{x_{\lambda_k}^*\}$  that will be used in the rate analysis.

**Lemma 2 (Properties of Tikhonov trajectory):** Let Assumptions 1 and 3 hold and  $x_{\lambda_k}^*$  be given by Def. 1. Then, we have: (i) The sequence  $\{x_{\lambda_k}^*\}$  converges to the unique solution of problem (1), i.e.,  $x^*$ . (ii) There exists a scalar  $M > 0$  such that for any  $k \geq 1$ , we have  $\|x_{\lambda_k}^* - x_{\lambda_{k-1}}^*\|_2 \leq \frac{M}{\mu_f} \Lambda_{k-1}$ .

**Proof:** The proof can be done in a similar vein to that of [21, Lemma 4.5]. ■

In the following, we state some properties of the regularized maps to be used in obtaining error bounds in the next section.

**Lemma 3:** Consider Algorithm 1. Let Assumptions 1 and 2 hold. For any  $k \geq 0$ , mappings  $G_k$ ,  $\mathcal{G}_k$ , and  $\bar{g}_k$  given by Def. 1 satisfy the following relations: (i) We have that  $\bar{y}_k = G_k(\mathbf{x}_k)$ . (ii) We have  $\mathcal{G}_k(x_{\lambda_k}^*) = 0$ . (iii) The mapping  $\mathcal{G}_k(x)$  is  $(\lambda_k \mu_f)$ -strongly monotone and Lipschitz continuous with parameter  $L_k$ . (iv) We have  $\|\bar{y}_k - \bar{g}_k\|_2 \leq \frac{L_k}{\sqrt{m}} \|\mathbf{x}_k - \mathbf{1}\bar{x}_k\|_2$  and  $\|\bar{g}_k\|_2 \leq L_k \|\bar{x}_k - x_{\lambda_k}^*\|_2$ .

**Proof:** (i) Multiplying both sides of (9) by  $\frac{1}{m} \mathbf{1}^\top$  and from the definitions of  $\mathbf{G}_k$  and  $G_k$  in Def. 1, we obtain

$$\begin{aligned} \bar{y}_{k+1} &= \frac{1}{m} \mathbf{1}^\top \mathbf{y}_k + \frac{1}{m} \mathbf{1}^\top \mathbf{G}_{k+1}(\mathbf{x}_{k+1}) - \frac{1}{m} \mathbf{1}^\top \mathbf{G}_k(\mathbf{x}_k) \\ &= \bar{y}_k + G_{k+1}(\mathbf{x}_{k+1}) - G_k(\mathbf{x}_k), \end{aligned}$$

where we used  $\mathbf{1}^\top \mathbf{C} = \mathbf{1}^\top$ . From Algorithm 1, we have  $\mathbf{y}_0 := \mathbf{F}(\mathbf{x}_0) + \lambda_0 \nabla \mathbf{f}(\mathbf{x}_0) = \mathbf{G}_0(\mathbf{x}_0)$ , implying that  $\bar{y}_0 = G_0(\mathbf{x}_0)$ . From the two preceding relations, we obtain that  $\bar{y}_k = G_k(\mathbf{x}_k)$ . (ii, iii) From Def. 1, we have that  $G_k(\mathbf{x}) = \frac{1}{m} \sum_{i=1}^m (F_i(x_{i,k}) + \lambda_k \nabla f_i(x_{i,k}))$ . Thus, from the definition of  $\mathcal{G}_k$  we obtain that  $\mathcal{G}_k(x) = G_k(\mathbf{1}x^\top) = \frac{1}{m} \sum_{i=1}^m (F_i(x) + \lambda_k \nabla f_i(x)) = \frac{1}{m} (F(x) + \lambda_k \nabla f(x))$ . Thus, from the definition of  $x_{\lambda_k}^*$  in Def. 1, we obtain  $\mathcal{G}_k(x_{\lambda_k}^*) = 0$ . Also, from Assumption 1, we conclude that

$\mathcal{G}_k(x)$  is a  $(\lambda_k \mu_f)$ -strongly monotone mapping and Lipschitz continuous with parameter  $L_k \triangleq L_F + \lambda_k L_f$  for  $k \geq 0$ .

(iv) For any  $\mathbf{u}, \mathbf{v} \in \mathbb{R}^{m \times n}$ , with  $u_i, v_i \in \mathbb{R}^n$  denoting the  $i$ th row of  $\mathbf{u}, \mathbf{v}$ , respectively, we have

$$\begin{aligned} & \|G_k(\mathbf{u}) - G_k(\mathbf{v})\|_2 \\ &= \left\| \frac{1}{m} \mathbf{1}^\top (\mathbf{F}(\mathbf{u}) + \lambda_k \nabla \mathbf{f}(\mathbf{u})) - \frac{1}{m} \mathbf{1}^\top (\mathbf{F}(\mathbf{v}) + \lambda_k \nabla \mathbf{f}(\mathbf{v})) \right\|_2 \\ &\leq \frac{1}{m} \left\| \sum_{i=1}^m F_i(u_i) - \sum_{i=1}^m F_i(v_i) \right\|_2 \\ &+ \frac{\lambda_k}{m} \left\| \sum_{i=1}^m \nabla f_i(u_i) - \sum_{i=1}^m \nabla f_i(v_i) \right\|_2 \\ &\leq \frac{1}{m} \sum_{i=1}^m (\|F_i(u_i) - F_i(v_i)\|_2 + \lambda_k \|\nabla f_i(u_i) - \nabla f_i(v_i)\|_2) \\ &\leq \frac{1}{m} \sum_{i=1}^m (L_F \|u_i - v_i\|_2 + \lambda_k L_f \|u_i - v_i\|_2) \\ &\leq \frac{L_k}{m} \sum_{i=1}^m \|u_i - v_i\|_2 \leq \frac{L_k}{\sqrt{m}} \|\mathbf{u} - \mathbf{v}\|_2. \end{aligned}$$

Consequently, we obtain  $\|\bar{y}_k - \bar{g}_k\|_2 = \|G_k(\mathbf{x}_k) - G_k(\mathbf{1}\bar{x}_k)\|_2 \leq \frac{L_k}{\sqrt{m}} \|\mathbf{x}_k - \mathbf{1}\bar{x}_k\|_2$ . Also, using the Lipschitzian property of  $\mathcal{G}_k$  in part (ii) and  $\mathcal{G}_k(x_{\lambda_k}^*) = 0$ , we obtain  $\|\bar{g}_k\|_2 = \|\mathcal{G}_k(\bar{x}_k)\|_2 = \|\mathcal{G}_k(\bar{x}_k) - \mathcal{G}_k(x_{\lambda_k}^*)\|_2 \leq L_k \|\bar{x}_k - x_{\lambda_k}^*\|_2$ . ■

Next, we derive a bound on the squared distance between the average iterate and the Tikhonov trajectory.

**Lemma 4:** Let  $\alpha_k \leq \frac{\lambda_k \mu_f}{L_0^2}$  for  $k \geq 0$ . We have

$$\|\bar{x}_k - \alpha_k \bar{g}_k - x_{\lambda_k}^*\|_2^2 \leq (1 - 0.5\alpha_k \lambda_k \mu_f)^2 \|\bar{x}_k - x_{\lambda_k}^*\|_2^2.$$

*Proof:* We have

$$\begin{aligned} & \|\bar{x}_k - \alpha_k \bar{g}_k - x_{\lambda_k}^*\|_2^2 \\ &= \|\bar{x}_k - x_{\lambda_k}^*\|_2^2 + \alpha_k^2 \|\bar{g}_k\|_2^2 - 2\alpha_k \bar{g}_k^\top (\bar{x}_k - x_{\lambda_k}^*) \\ &\stackrel{\text{Lemma 3(iii)}}{\leq} \|\bar{x}_k - x_{\lambda_k}^*\|_2^2 + \alpha_k^2 L_k^2 \|\bar{x}_k - x_{\lambda_k}^*\|_2^2 \\ &\quad - 2\alpha_k \lambda_k \mu_f \|\bar{x}_k - x_{\lambda_k}^*\|_2^2 \\ &= (1 + \alpha_k^2 L_k^2 - 2\alpha_k \lambda_k \mu_f) \|\bar{x}_k - x_{\lambda_k}^*\|_2^2. \end{aligned}$$

Let  $\alpha_k \leq \frac{\lambda_k \mu_f}{L_0^2}$  and recall that  $L_k \leq L_0$ . We obtain

$$\|\bar{x}_k - \alpha_k \bar{g}_k - x_{\lambda_k}^*\|_2^2 \leq (1 - \alpha_k \lambda_k \mu_f) \|\bar{x}_k - x_{\lambda_k}^*\|_2^2.$$

Then, the desired relation is obtained by noting that  $1 - \alpha_k \lambda_k \mu_f \leq (1 - 0.5\alpha_k \lambda_k \mu_f)^2$ . ■

We state the following result from [29] introducing two matrix norms induced by matrices  $\mathbf{R}$  and  $\mathbf{C}$ .

**Lemma 5** (cf. Lemma 4 and Lemma 6 in [29]): Let Assumption 2 hold. Then: (i) There exist matrix norms  $\|\cdot\|_{\mathbf{R}}$  and  $\|\cdot\|_{\mathbf{C}}$  such that for  $\sigma_{\mathbf{R}} \triangleq \|\mathbf{R} - \frac{1}{m} \mathbf{1}\mathbf{1}^\top\|_{\mathbf{R}}$  and  $\sigma_{\mathbf{C}} \triangleq \|\mathbf{C} - \frac{1}{m} \mathbf{v}\mathbf{v}^\top\|_{\mathbf{C}}$  we have that  $\sigma_{\mathbf{R}} < 1$  and  $\sigma_{\mathbf{C}} < 1$ . (ii) There exist scalars  $\delta_{\mathbf{R},2}, \delta_{\mathbf{C},2}, \delta_{\mathbf{R},\mathbf{C}}, \delta_{\mathbf{C},\mathbf{R}} > 0$  such that for any  $W \in \mathbb{R}^{m \times n}$ , we have  $\|W\|_{\mathbf{R}} \leq \delta_{\mathbf{R},2} \|W\|_2$ ,  $\|W\|_{\mathbf{C}} \leq \delta_{\mathbf{C},2} \|W\|_2$ ,  $\|W\|_{\mathbf{R}} \leq \delta_{\mathbf{R},\mathbf{C}} \|W\|_{\mathbf{C}}$ ,  $\|W\|_{\mathbf{C}} \leq \delta_{\mathbf{C},\mathbf{R}} \|W\|_{\mathbf{R}}$ ,  $\|W\|_2 \leq \|W\|_{\mathbf{R}}$ , and  $\|W\|_2 \leq \|W\|_{\mathbf{C}}$ . The following result will be employed in the rate analysis.

**Lemma 6:** Let  $\alpha_k$  be given by Assumption 3. Suppose  $\Gamma \geq \hat{\Gamma}_2 \triangleq \max \left\{ \sqrt{\frac{L_0 \hat{\gamma}_0 u^\top v}{m}}, \sqrt{\frac{\gamma u^\top v L_0^2}{m \mu_f \lambda}} \right\}$ . Then, we have  $\alpha_k \leq \min \left\{ \frac{1}{L_0}, \frac{\lambda_k \mu_f}{L_0^2} \right\}$  for all  $k \geq 0$ .

*Proof:* First, we show that  $\alpha_k \leq \frac{1}{L_0}$ . Recall the definition of  $\alpha_k$  and  $\hat{\gamma}_k$ . We may write  $\alpha_k \triangleq \frac{1}{m} u^\top \gamma_k v = \frac{1}{m} \sum_{i=1}^m \gamma_{i,k} u_i v_i \leq \frac{1}{m} \sum_{i=1}^m \hat{\gamma}_k u_i v_i$ , where  $u_i$  and  $v_i$  are the  $i$ th element of  $u$  and  $v$ , respectively. Then, we have

$\alpha_k \leq \frac{1}{m} \sum_{i=1}^m \frac{\hat{\gamma}_k u_i v_i}{(k+\Gamma)^a} = \frac{\hat{\gamma}_k u^\top v}{m(k+\Gamma)^a}$ , for all  $k \geq 0$ . From  $\Gamma \geq \sqrt{\frac{L_0 \hat{\gamma}_0 u^\top v}{m}}$  and rearranging the terms, we obtain  $\frac{1}{L_0} \geq \frac{\hat{\gamma}_k u^\top v}{m \Gamma^a}$ . Therefore, we have  $\alpha_k \leq \frac{\hat{\gamma}_k u^\top v}{m(k+\Gamma)^a} \leq \frac{\hat{\gamma}_k u^\top v}{m \Gamma^a} \leq \frac{1}{L_0}$ . The second bound on  $\alpha_k$  can be established in a similar manner. ■

## B. Convergence and rate analysis of Algorithm 1

We analyze the convergence of Algorithm 1 by introducing the error metrics  $\|\bar{x}_{k+1} - x_{\lambda_k}^*\|_2$ ,  $\|\mathbf{x}_{k+1} - \mathbf{1}\bar{x}_{k+1}\|_{\mathbf{R}}$ ,  $\|\mathbf{y}_{k+1} - v\bar{y}_{k+1}\|_{\mathbf{C}}$ . Of these, the first term relates the averaged iterate with the Tikhonov trajectory, the second term measures the consensus error for the decision matrix, and the third term measures the consensus error for the matrix of the regularized gradients. For  $k \geq 1$ , let us define  $\Delta_k$  as

$$\Delta_k \triangleq [\|\bar{x}_k - x_{\lambda_{k-1}}^*\|_2, \|\mathbf{x}_k - \mathbf{1}\bar{x}_k\|_{\mathbf{R}}, \|\mathbf{y}_k - v\bar{y}_k\|_{\mathbf{C}}]^\top.$$

**Proposition 1:** Consider Algorithm 1 under Assumptions 1, 2, and 3. Let  $\alpha_k$  and  $\hat{\gamma}_k$  be given by Assumption 3,  $c_0 \triangleq \delta_{\mathbf{C},2} \|\mathbf{I} - \frac{1}{m} \mathbf{v}\mathbf{v}^\top\|_{\mathbf{C}}$ , and  $\Gamma \geq \hat{\Gamma}_2$ . Then, there exist scalars  $M > 0$  and  $B_F > 0$  such that for any  $k \geq 0$ , we have  $\Delta_{k+1} \leq H_k \Delta_k + h_k$  where  $H_k = [H_{i,j,k}]_{3 \times 3}$  and  $h_k = [h_{i,k}]_{3 \times 1}$  are given as follows:

$$\begin{aligned} H_{11,k} &:= 1 - 0.5\alpha_k \lambda_k \mu_f, \quad H_{12,k} := \frac{\alpha_k L_k}{\sqrt{m}}, \quad H_{13,k} := \frac{\hat{\gamma}_k \|u\|_2}{m}, \\ H_{21,k} &:= \sigma_{\mathbf{R}} \hat{\gamma}_k L_k \|\mathbf{v}\|_{\mathbf{R}}, \quad H_{22,k} := \sigma_{\mathbf{R}} (1 + \hat{\gamma}_k \|\mathbf{v}\|_{\mathbf{R}} \frac{L_k}{\sqrt{m}}), \\ H_{23,k} &:= \sigma_{\mathbf{R}} \hat{\gamma}_k \delta_{\mathbf{R},\mathbf{C}}, \quad H_{33,k} := \sigma_{\mathbf{C}} + c_0 L_k \hat{\gamma}_k \|\mathbf{R}\|_2, \\ H_{31,k} &:= c_0 L_k (\hat{\gamma}_k \|\mathbf{R}\|_2 \|\mathbf{v}\|_2 L_k + 2\sqrt{m} \Lambda_k), \\ H_{32,k} &:= c_0 L_k (\|\mathbf{R} - \mathbf{I}\|_2 + \hat{\gamma}_k \|\mathbf{R}\|_2 \|\mathbf{v}\|_2 \frac{L_k}{\sqrt{m}} + 2\Lambda_k), \\ h_{1,k} &:= \frac{M \Lambda_{k-1}}{\mu_f}, \quad h_{2,k} := \frac{M \sigma_{\mathbf{R}} \hat{\gamma}_k L_k \|\mathbf{v}\|_{\mathbf{R}} \Lambda_{k-1}}{\mu_f}, \\ h_{3,k} &:= c_0 L_k (\hat{\gamma}_k \|\mathbf{R}\|_2 \|\mathbf{v}\|_2 L_k + \sqrt{m} \Lambda_k + \frac{\mu_f c_0 B_F}{M}) \frac{M \Lambda_{k-1}}{\mu_f}. \end{aligned}$$

*Proof:* First, we show  $\Delta_{1,k+1} \leq \sum_{j=1}^3 H_{1j,k} \Delta_{j,k} + h_{1,k}$ . From (8) and Def. 1, we obtain

$$\bar{x}_{k+1} = u^\top \mathbf{R} (\mathbf{x}_k - \gamma_k \mathbf{y}_k) / m = \bar{x}_k - u^\top \gamma_k \mathbf{y}_k / m.$$

Thus, we have

$$\begin{aligned} \bar{x}_{k+1} &= \bar{x}_k - u^\top \gamma_k (\mathbf{y}_k - v\bar{y}_k + v\bar{y}_k) / m \\ &= \bar{x}_k - \alpha_k \bar{g}_k - \alpha_k (\bar{y}_k - \bar{g}_k) - u^\top \gamma_k (\mathbf{y}_k - v\bar{y}_k) / m. \end{aligned}$$

From Lemma 6,  $\alpha_k \leq \frac{1}{L_0} < \frac{2}{L_0} \leq \frac{2}{L_k}$  for all  $k \geq 0$  for  $\Gamma \geq \sqrt{\frac{L_0 \hat{\gamma}_0 u^\top v}{m}}$ . From Lemma 3(iii),  $\mathcal{G}_k(x)$  is  $(\mu_f \lambda_k)$ -strongly convex and  $L_k$ -smooth. Invoking Lemma 4, we obtain

$$\begin{aligned} \|\bar{x}_{k+1} - x_{\lambda_k}^*\|_2 &= \|\bar{x}_k - x_{\lambda_k}^* - \alpha_k \bar{g}_k - \alpha_k (\bar{y}_k - \bar{g}_k) \\ &\quad - \frac{1}{m} u^\top \gamma_k (\mathbf{y}_k - v\bar{y}_k)\|_2 \leq (1 - 0.5\alpha_k \lambda_k \mu_f) \|\bar{x}_k - x_{\lambda_k}^*\|_2 \\ &\quad + \alpha_k \|\bar{y}_k - \bar{g}_k\|_2 + \frac{1}{m} \|u^\top \gamma_k (\mathbf{y}_k - v\bar{y}_k)\|_2. \end{aligned}$$

Adding and subtracting  $x_{\lambda_{k-1}}^*$  and using Lemmas 2 and 3(iv), we obtain

$$\begin{aligned} \|\bar{x}_{k+1} - x_{\lambda_k}^*\|_2 &\leq (1 - 0.5\alpha_k \lambda_k \mu_f) \|\bar{x}_k - x_{\lambda_{k-1}}^*\|_2 \\ &\quad + \frac{M \Lambda_{k-1}}{\mu_f} + \frac{\alpha_k L_k}{\sqrt{m}} \|\mathbf{x}_k - \mathbf{1}\bar{x}_k\|_2 + \frac{\|u\|_2 \|\gamma_k\|_2}{m} \|\mathbf{y}_k - v\bar{y}_k\|_2. \end{aligned}$$

Then, the desired inequality is obtained by invoking Lemma 5(ii), Remark 1, and definition of  $\hat{\gamma}_k$ .

Second, we show  $\Delta_{2,k+1} \leq \sum_{j=1}^3 H_{2j,k} \Delta_{j,k} + h_{2,k}$ . From (8) and Def. 1 and that  $\mathbf{R}\mathbf{1} = \mathbf{1}$ , we have

$$\begin{aligned} \mathbf{x}_{k+1} - \mathbf{1}\bar{x}_{k+1} &= \mathbf{R}(\mathbf{x}_k - \gamma_k \mathbf{y}_k) - \mathbf{1}\bar{x}_k + \frac{1}{m} \mathbf{1} u^\top \gamma_k \mathbf{y}_k \\ &= (\mathbf{R} - \mathbf{1} u^\top / m) ((\mathbf{x}_k - \mathbf{1}\bar{x}_k) - \gamma_k \mathbf{y}_k). \end{aligned}$$

Applying Lemma 5, Remark 1, and Lemma 3, we obtain

$$\begin{aligned} \|\mathbf{x}_{k+1} - \mathbf{1}\bar{x}_{k+1}\|_{\mathbf{R}} &\leq \sigma_{\mathbf{R}} \|\mathbf{x}_k - \mathbf{1}\bar{x}_k\|_{\mathbf{R}} + \sigma_{\mathbf{R}} \|\gamma_k\|_{\mathbf{R}} \|\mathbf{y}_k\|_{\mathbf{R}} \\ &\leq \sigma_{\mathbf{R}} \|\mathbf{x}_k - \mathbf{1}\bar{x}_k\|_{\mathbf{R}} + \sigma_{\mathbf{R}} \|\gamma_k\|_2 \|\mathbf{y}_k - v\bar{y}_k\|_{\mathbf{R}} \\ &\quad + \sigma_{\mathbf{R}} \|\gamma_k\|_2 \|v\|_{\mathbf{R}} \|\bar{y}_k\|_2 \\ &\leq \sigma_{\mathbf{R}} (1 + \hat{\gamma}_k \|v\|_{\mathbf{R}} L_k / \sqrt{m}) \|\mathbf{x}_k - \mathbf{1}\bar{x}_k\|_{\mathbf{R}} \\ &\quad + \sigma_{\mathbf{R}} \hat{\gamma}_k \delta_{\mathbf{R},\mathbf{C}} \|\mathbf{y}_k - v\bar{y}_k\|_{\mathbf{C}} + \sigma_{\mathbf{R}} \hat{\gamma}_k L_k \|v\|_{\mathbf{R}} \|\bar{x}_k - x_{\lambda_k}^*\|_2. \end{aligned}$$

Adding and subtracting  $x_{\lambda_{k-1}}^*$  and using Lemma 2, we obtain the desired inequality.

Third, we show  $\Delta_{3,k+1} \leq \sum_{j=1}^3 H_{3j,k} \Delta_{j,k} + h_{3,k}$ . From (9) and the definition of  $\mathbf{G}_k(\mathbf{x})$  in Def. 1, we obtain  $\mathbf{y}_{k+1} = \mathbf{C}\mathbf{y}_k + \mathbf{G}_{k+1}(\mathbf{x}_{k+1}) - \mathbf{G}_k(\mathbf{x}_k)$ . Multiplying both sides of the preceding relation by  $\frac{1}{m} \mathbf{1}^\top$  and using the definition of  $\bar{y}_k$  in Def. 1, we obtain that  $\bar{y}_{k+1} = \bar{y}_k + \frac{1}{m} \mathbf{1}^\top \mathbf{G}_{k+1}(\mathbf{x}_{k+1}) - \frac{1}{m} \mathbf{1}^\top \mathbf{G}_k(\mathbf{x}_k)$ . From the last two relations, we have

$$\begin{aligned} \mathbf{y}_{k+1} - v\bar{y}_{k+1} &= (\mathbf{C} - v\mathbf{1}^\top / m) (\mathbf{y}_k - v\bar{y}_k) \\ &\quad + (\mathbf{I} - v\mathbf{1}^\top / m) (\mathbf{G}_{k+1}(\mathbf{x}_{k+1}) - \mathbf{G}_k(\mathbf{x}_k)). \end{aligned}$$

Invoking Lemma 5,  $\mathbf{G}_k(\mathbf{x})$  in Def. 1 and  $c_0$ , and we obtain

$$\begin{aligned} \|\mathbf{y}_{k+1} - v\bar{y}_{k+1}\|_{\mathbf{C}} &\leq \sigma_{\mathbf{C}} \|\mathbf{y}_k - v\bar{y}_k\|_{\mathbf{C}} \\ &\quad + c_0 \|\mathbf{G}_{k+1}(\mathbf{x}_{k+1}) - \mathbf{G}_k(\mathbf{x}_k)\|_2 \\ &\leq \sigma_{\mathbf{C}} \|\mathbf{y}_k - v\bar{y}_k\|_{\mathbf{C}} + c_0 \|\lambda_{k+1} \nabla \mathbf{f}(\mathbf{x}_k) - \lambda_k \nabla \mathbf{f}(\mathbf{x}_k)\|_2 \\ &\quad + c_0 \|\mathbf{G}_{k+1}(\mathbf{x}_{k+1}) - \mathbf{F}(\mathbf{x}_k) - \lambda_{k+1} \nabla \mathbf{f}(\mathbf{x}_k)\|_2 \\ &\leq \sigma_{\mathbf{C}} \|\mathbf{y}_k - v\bar{y}_k\|_{\mathbf{C}} + c_0 |1 - \lambda_{k+1} / \lambda_k| \|\lambda_k \nabla \mathbf{f}(\mathbf{x}_k)\|_2 \\ &\quad + c_0 L_k \|\mathbf{x}_{k+1} - \mathbf{x}_k\|_2. \end{aligned} \quad (11)$$

From Lemma 2, there exists a scalar  $B_F < \infty$  such that  $L_F \|\mathbf{1}x_{\lambda_k}^* - \mathbf{1}x^*\|_2 \leq B_F$  for all  $k \geq 0$ . Given that  $F(x^*) = 0$ , we have

$$\begin{aligned} \|\lambda_k \nabla \mathbf{f}(\mathbf{x}_k)\|_2 &\leq \|\mathbf{F}(\mathbf{x}_k) + \lambda_k \nabla \mathbf{f}(\mathbf{x}_k)\|_2 \\ &\quad + \|\mathbf{F}(\mathbf{x}_k) - \mathbf{F}(\mathbf{1}x^*)\|_2 \\ &\leq \|\mathbf{F}(\mathbf{x}_k) + \lambda_k \nabla \mathbf{f}(\mathbf{x}_k) - \mathbf{F}(\mathbf{1}x_{\lambda_k}^*) - \lambda_k \nabla \mathbf{f}(\mathbf{1}x_{\lambda_k}^*)\|_2 \\ &\quad + L_F \|\mathbf{x}_k - \mathbf{1}x^*\|_2 \\ &\leq (L_k + L_F) \|\mathbf{x}_k - \mathbf{1}x_{\lambda_k}^*\|_2 + L_F \|\mathbf{1}x_{\lambda_k}^* - \mathbf{1}x^*\|_2 \\ &\leq 2L_k (\|\mathbf{x}_k - \mathbf{1}\bar{x}_k\|_2 + \|\mathbf{1}\bar{x}_k - \mathbf{1}x_{\lambda_k}^*\|_2) + B_F \\ &\leq 2L_k \|\mathbf{x}_k - \mathbf{1}\bar{x}_k\|_2 + 2\sqrt{m} L_k \|\bar{x}_k - x_{\lambda_k}^*\|_2 + B_F. \end{aligned}$$

From row-stochasticity of  $\mathbf{R}$ , we have  $(\mathbf{R} - \mathbf{I})\mathbf{1}\bar{x}_k = 0$ . Thus, from Lemma 3 we have

$$\begin{aligned} \|\mathbf{x}_{k+1} - \mathbf{x}_k\|_2 &= \|\mathbf{R}(\mathbf{x}_k - \gamma_k \mathbf{y}_k) - \mathbf{x}_k\|_2 \\ &= \|(\mathbf{R} - \mathbf{I})(\mathbf{x}_k - \mathbf{1}\bar{x}_k) - \mathbf{R}\gamma_k \mathbf{y}_k\|_2 \\ &\leq \|\mathbf{R} - \mathbf{I}\|_2 \|\mathbf{x}_k - \mathbf{1}\bar{x}_k\|_2 + \|\mathbf{R}\|_2 \|\gamma_k\|_2 (\|\mathbf{y}_k - v\bar{y}_k\|_2 \\ &\quad + \|v\|_2 \|\bar{y}_k - \bar{g}_k\|_2 + \|v\|_2 \|\bar{g}_k\|_2) \leq \|\mathbf{R} - \mathbf{I}\|_2 \|\mathbf{x}_k - \mathbf{1}\bar{x}_k\|_2 \\ &\quad + \hat{\gamma}_k \|\mathbf{R}\|_2 (\|\mathbf{y}_k - v\bar{y}_k\|_2 \\ &\quad + L_k \|v\|_2 (\|\mathbf{x}_k - \mathbf{1}\bar{x}_k\|_2 / \sqrt{m} + \|\bar{x}_k - x_{\lambda_k}^*\|_2)). \end{aligned}$$

It suffices to find a recursive bound for the term  $\|\mathbf{x}_{k+1} - \mathbf{x}_k\|_2$ . From Lemma 2, we may write

$$\begin{aligned} \|\bar{x}_k - x_{\lambda_k}^*\|_2 &\leq \|\bar{x}_k - x_{\lambda_{k-1}}^*\|_2 + \|x_{\lambda_{k-1}}^* - x_{\lambda_k}^*\|_2 \\ &\leq \|\bar{x}_k - x_{\lambda_{k-1}}^*\|_2 + M / \mu_f \Lambda_{k-1}. \end{aligned}$$

From (11) and the preceding three relations, we can obtain the desired inequality. ■

Next, we derive a unifying recursive bound for the three error bounds introduced earlier.

**Proposition 2:** Consider Algorithm 1. Let Assumptions 1, 2, and 3 hold where  $\tau := 0.5\theta$ . Suppose we have  $\Gamma \geq \hat{\Gamma}_3 \triangleq \max\{\hat{\Gamma}_2, \sqrt{\frac{\mu_f \lambda \hat{\gamma} u^\top v + 2\hat{\gamma} \|v\|_{\mathbf{R}} L_0}{(1-\sigma_{\mathbf{R}})\sqrt{m}}}, \sqrt{\frac{\mu_f \lambda \hat{\gamma} u^\top v + 2\hat{\gamma} c_0 L_0 \|\mathbf{R}\|_2}{1-\sigma_{\mathbf{C}}}}, \sqrt{\frac{0.5\mu_f \lambda \hat{\gamma} u^\top v}{m}}, \sqrt{\frac{3c_1 \hat{\gamma}^2}{c_3 \lambda}}, \sqrt{\frac{3c_2 \hat{\gamma}}{c_3 \lambda}}, \sqrt{\frac{3c_4}{c_3 \lambda}}\}$ . Then, for any  $k \geq 1$ , the following holds:

(i)  $\|\Delta_{k+1}\|_2 \leq (1 - 0.5\mu_f \alpha_k \lambda_k) \|\Delta_k\|_2 + \Theta \Lambda_{k-1}$ , where

$$\begin{aligned} \Theta &\triangleq \max\{1, \sigma_{\mathbf{R}} \hat{\gamma} L_0 \|v\|_{\mathbf{R}}, c_0 L_0 (\hat{\gamma} \|\mathbf{R}\|_2 \|v\|_2 L_0 \\ &\quad + \sqrt{m} \Lambda_0 + \mu_f c_0 B_F / M)\} \sqrt{3M} / \mu_f. \end{aligned}$$

(ii)  $\|\Delta_k\|_2 \leq \frac{\mathcal{B}}{(k+\Gamma-1)^{1-a-b}}$  where

$$\mathcal{B} \triangleq \max\{(\Gamma+1)^{1-a-b} \|\Delta_1\|_2, \frac{4\Theta}{\mu_f \lambda \hat{\gamma} \theta}\}.$$

*Proof:* (i) In the first step, we consider Proposition 1. Let us define the sequence  $\{\rho_k\}$  as  $\rho_k \triangleq 1 - 0.5\mu_f \alpha_k \lambda_k$  for  $k \geq 0$ . Next, we utilize our assumptions to find suitable upper bounds for some of the above terms. We define  $\hat{H}_k = [\hat{H}_{ij,k}]_{3 \times 3}$  and  $\hat{h}_k = [\hat{h}_{i,k}]_{3 \times 1}$  as follows:

$$\begin{aligned} \hat{H}_{11,k} &:= H_{11,k}, \quad \hat{H}_{12,k} := \frac{\alpha_k L_0}{\sqrt{m}}, \quad \hat{H}_{13,k} := H_{13,k}, \\ \hat{H}_{21,k} &:= \sigma_{\mathbf{R}} \hat{\gamma}_k L_0 \|v\|_{\mathbf{R}}, \quad \hat{H}_{22,k} := \rho_k - \frac{1-\sigma_{\mathbf{R}}}{2}, \\ \hat{H}_{23,k} &:= H_{23,k}, \quad \hat{H}_{31,k} := c_0 L_0 (\hat{\gamma}_k \|\mathbf{R}\|_2 \|v\|_2 L_0 + 2\sqrt{m} \Lambda_k), \\ \hat{H}_{32,k} &:= c_0 L_0 (\|\mathbf{R} - \mathbf{I}\|_2 + \hat{\gamma}_k \|\mathbf{R}\|_2 \|v\|_2 \frac{L_0}{\sqrt{m}} + 2\Lambda_0), \\ \hat{H}_{33,k} &:= \rho_k - \frac{1-\sigma_{\mathbf{C}}}{2}, \\ \hat{h}_{1,k} &:= \frac{\Theta}{\sqrt{3}} \Lambda_{k-1}, \quad \hat{h}_{2,k} := \frac{\Theta}{\sqrt{3}} \Lambda_{k-1}, \quad \hat{h}_{3,k} := \frac{\Theta}{\sqrt{3}} \Lambda_{k-1}. \end{aligned}$$

Note that we have

$$\hat{H}_{22,k} - H_{22,k} = \frac{1-\sigma_{\mathbf{R}}}{2} - 0.5\mu_f \alpha_k \lambda_k - \hat{\gamma}_k \|v\|_{\mathbf{R}} \frac{L_k}{\sqrt{m}}.$$

Next we show that for  $\Gamma \geq \sqrt{\frac{\mu_f \lambda \hat{\gamma} u^\top v + 2\hat{\gamma} \|v\|_{\mathbf{R}} L_0}{(1-\sigma_{\mathbf{R}})\sqrt{m}}}$ , we have  $H_{22,k} \leq \hat{H}_{22,k}$  for all  $k \geq 1$ . By using Lemma 6, it suffices to show that

$$\frac{\frac{1-\sigma_{\mathbf{R}}}{2} - \frac{\hat{\gamma}}{(k+\Gamma)^a} \|v\|_{\mathbf{R}} \frac{L_k}{\sqrt{m}}}{0.5\mu_f \frac{\lambda}{(k+\Gamma)^b}} \geq \frac{\hat{\gamma} u^\top v}{m(k+\Gamma)^a}. \quad (12)$$

Consider  $\Gamma \geq \sqrt{\frac{\mu_f \lambda \hat{\gamma} u^\top v + 2\hat{\gamma} \|v\|_{\mathbf{R}} L_0}{(1-\sigma_{\mathbf{R}})\sqrt{m}}}$ . By rearranging terms and invoking  $L_0 \geq L_k$ , we obtain

$$\begin{aligned} \frac{1-\sigma_{\mathbf{R}}}{2} &\geq \frac{0.5\mu_f \lambda \hat{\gamma} u^\top v + \hat{\gamma} \|v\|_{\mathbf{R}} L_0}{\Gamma^a \sqrt{m}} \\ &\geq \frac{0.5\mu_f \lambda \hat{\gamma} u^\top v + \hat{\gamma} \|v\|_{\mathbf{R}} L_k}{(k+\Gamma)^a \sqrt{m}}. \end{aligned}$$

This relation can be further written as  $\frac{1-\sigma_{\mathbf{R}}}{2} \geq \frac{0.5\mu_f \lambda \hat{\gamma} u^\top v}{(k+\Gamma)^{a+b} m} + \frac{\hat{\gamma} \|v\|_{\mathbf{R}} L_k}{(k+\Gamma)^a \sqrt{m}}$ . By rearranging terms and dividing both sides



by  $\frac{1}{(k+\Gamma)^b}$ , we obtain  $\frac{\frac{1-\sigma_{\mathbf{R}}}{2} - \frac{\hat{\gamma}\|\mathbf{v}\|\mathbf{R}_{L_k}}{(k+\Gamma)^a\sqrt{m}}}{\frac{1}{(k+\Gamma)^b}} \geq \frac{0.5\mu_f\lambda\hat{\gamma}u^\top v}{(k+\Gamma)^a m}$ , which implies (12). Similarly, we can prove that for  $\Gamma \geq \frac{a\sqrt{\mu_f\lambda\hat{\gamma}u^\top v + 2\hat{\gamma}c_0L_0\|\mathbf{R}\|_2}}{1-\sigma_{\mathbf{C}}}$ , we have  $H_{33,k} \leq \hat{H}_{33,k}$  for all  $k \geq 1$ . Recall that  $H_{33,k} = \sigma_{\mathbf{C}} + c_0L_k\hat{\gamma}_k\|\mathbf{R}\|_2$  and  $\hat{H}_{33,k} = 1 - 0.5\mu_f\alpha_k\lambda_k - \frac{1-\sigma_{\mathbf{C}}}{2}$ . Then, by invoking Lemma 6, it suffices to show that

$$\frac{\frac{1-\sigma_{\mathbf{C}}}{2} - \frac{\hat{\gamma}}{(k+\Gamma)^a} c_0L_k\|\mathbf{R}\|_2}{0.5\mu_f\frac{\lambda}{(k+\Gamma)^b}} \geq \frac{\hat{\gamma}u^\top v}{m(k+\Gamma)^a}. \quad (13)$$

Recall that  $\Gamma \geq \frac{a\sqrt{\mu_f\lambda\hat{\gamma}u^\top v + 2\hat{\gamma}c_0L_0\|\mathbf{R}\|_2}}{1-\sigma_{\mathbf{C}}}$ . By rearranging terms and using  $L_0 \geq L_k$  once again, we obtain  $\frac{1-\sigma_{\mathbf{C}}}{2} \geq \frac{0.5\mu_f\lambda\hat{\gamma}u^\top v + \hat{\gamma}c_0L_0\|\mathbf{R}\|_2}{\Gamma^a} \geq \frac{0.5\mu_f\lambda\hat{\gamma}u^\top v + \hat{\gamma}c_0L_k\|\mathbf{R}\|_2}{(k+\Gamma)^a}$ . This relation can be further written as  $\frac{1-\sigma_{\mathbf{C}}}{2} \geq \frac{0.5\mu_f\lambda\hat{\gamma}u^\top v}{(k+\Gamma)^{a+b}m} + \frac{\hat{\gamma}c_0L_k\|\mathbf{R}\|_2}{(k+\Gamma)^a}$ . By rearranging terms and dividing both sides by  $\frac{1}{(k+\Gamma)^b}$ , we

obtain  $\frac{\frac{1-\sigma_{\mathbf{C}}}{2} - \frac{\hat{\gamma}c_0L_k\|\mathbf{R}\|_2}{(k+\Gamma)^a}}{\frac{1}{(k+\Gamma)^b}} \geq \frac{0.5\mu_f\lambda\hat{\gamma}u^\top v}{(k+\Gamma)^a m}$ , which implies (13).

Thus, by taking to account that  $\Lambda_k$  is a nonincreasing sequence and invoking the definition of  $\Theta$ , we have  $H_k \leq \hat{H}_k$  and  $h_k \leq \hat{h}_k$ . Then by using the conclusion from Proposition 1, for  $\Gamma \geq \hat{\Gamma}_3$ , we have  $\Delta_{k+1} \leq \hat{H}_k\Delta_k + \hat{h}_k$  for all  $k \geq 1$ . Consequently, we obtain

$$\|\Delta_{k+1}\|_2 \leq \rho(\hat{H}_k)\|\Delta_k\|_2 + \Theta\Lambda_{k-1}, \quad (14)$$

where  $\rho(\hat{H}_k)$  denotes the spectral norm of  $\hat{H}_k$ . Next, we show that for a suitable choice of  $\Gamma$ , we have  $\rho(\hat{H}_k) \leq \rho_k$  for all  $k \geq 1$ . To show this relation, employing Lemma 5 in [29], it suffices to show that  $0 \leq \hat{H}_{ii,k} < \rho_k$  for  $i \in \{1, 2, 3\}$  and  $\det(\rho_k\mathbf{I} - \hat{H}_k) > 0$ . Among these, it can be easily seen that  $\hat{H}_{ii,k} < \rho_k$  holds for all  $i \in \{1, 2, 3\}$ . We then show that for  $\Gamma \geq \frac{a+b\sqrt{0.5\mu_f\lambda\hat{\gamma}u^\top v}}{m}$ , we have  $\hat{H}_{11,k} = 1 - 0.5\mu_f\alpha_k\lambda_k \geq 0$  for all  $k \geq 1$ . From  $\Gamma \geq \frac{a+b\sqrt{0.5\mu_f\lambda\hat{\gamma}u^\top v}}{m}$ , we obtain  $1 \geq \frac{0.5\mu_f\lambda\hat{\gamma}u^\top v}{m\Gamma^{a+b}}$ , implying that  $1 - \frac{0.5\mu_f\lambda\hat{\gamma}u^\top v}{m(k+\Gamma)^{a+b}} \geq 0$ . In view of the definition of  $\lambda_k$ , we have  $1 - \frac{0.5\mu_f\lambda\hat{\gamma}u^\top v}{m(k+\Gamma)^a}\lambda_k \geq 0$ . Then, invoking Lemma 6, we obtain  $1 - 0.5\mu_f\alpha_k\lambda_k \geq 0$ . Also,  $\hat{H}_{22,k} = 1 - 0.5\mu_f\alpha_k\lambda_k - \frac{1-\sigma_{\mathbf{R}}}{2} = \frac{1}{2} - 0.5\mu_f\alpha_k\lambda_k + \frac{\sigma_{\mathbf{R}}}{2} = \frac{\hat{H}_{11,k}}{2} + \frac{\sigma_{\mathbf{R}}}{2} \geq 0$ . Similarly,  $\hat{H}_{33,k} = 1 - 0.5\mu_f\alpha_k\lambda_k - \frac{1-\sigma_{\mathbf{C}}}{2} = \frac{\hat{H}_{11,k}}{2} + \frac{\sigma_{\mathbf{C}}}{2} \geq 0$ . Note that  $\hat{H}_{22,k}$  and  $\hat{H}_{33,k}$  are nonnegative due to  $\Gamma \geq \frac{a+b\sqrt{0.5\mu_f\lambda\hat{\gamma}u^\top v}}{m}$ . In the following, we show that  $\det(\rho_k\mathbf{I} - \hat{H}_k) > 0$ . We have

$$\begin{aligned} \det(\rho_k\mathbf{I} - \hat{H}_k) &= (0.5\mu_f\alpha_k\lambda_k)\left(\frac{1-\sigma_{\mathbf{R}}}{2}\right)\left(\frac{1-\sigma_{\mathbf{C}}}{2}\right) \\ &\quad - (0.5\mu_f\alpha_k\lambda_k)(\sigma_{\mathbf{R}}\hat{\gamma}_k\delta_{\mathbf{R},\mathbf{C}})c_0L_0(\|\mathbf{R} - \mathbf{I}\|_2 + 2\Lambda_0) \\ &\quad + \hat{\gamma}_k\|\mathbf{R}\|_2\|v\|_2\frac{L_0}{\sqrt{m}} - \left(\frac{1-\sigma_{\mathbf{C}}}{2}\right)\left(\frac{\alpha_kL_0}{\sqrt{m}}\right)(\sigma_{\mathbf{R}}\hat{\gamma}_kL_0\|v\|_{\mathbf{R}}) \\ &\quad - \left(\frac{\alpha_kL_0}{\sqrt{m}}\right)(\sigma_{\mathbf{R}}\hat{\gamma}_k\delta_{\mathbf{R},\mathbf{C}})(c_0L_0(\hat{\gamma}_k\|\mathbf{R}\|_2\|v\|_2L_0 + 2\sqrt{m}\Lambda_k)) \\ &\quad - \left(\frac{\hat{\gamma}_k\|u\|_2}{m}\right)(\sigma_{\mathbf{R}}\hat{\gamma}_kL_0\|v\|_{\mathbf{R}})(c_0L_0(\|\mathbf{R} - \mathbf{I}\|_2 \\ &\quad + \hat{\gamma}_k\|\mathbf{R}\|_2\|v\|_2\frac{L_0}{\sqrt{m}} + 2\Lambda_0)) \\ &\quad - \left(\frac{1-\sigma_{\mathbf{R}}}{2}\right)(c_0L_0(\hat{\gamma}_k\|\mathbf{R}\|_2\|v\|_2L_0 + 2\sqrt{m}\Lambda_k))\left(\frac{\hat{\gamma}_k\|u\|_2}{m}\right). \end{aligned}$$

Next, we find lower and upper bounds on  $\alpha_k$  in terms of  $\hat{\gamma}_k$ . Assumption 3 provides  $\theta\hat{\gamma}_k$  as a lower bound for  $\alpha_k$  while

Lemma 6 provides its upper bound. Let us define  $\bar{\theta} = \frac{1}{m}u^\top v$ . Thus, we have  $\theta\hat{\gamma}_k \leq \alpha_k \leq \bar{\theta}\hat{\gamma}_k$  for all  $k \geq 0$ . Using these bounds and rearranging the terms, we can obtain

$$\det(\rho_k\mathbf{I} - \hat{H}_k) \geq -c_1\hat{\gamma}_k^3 - c_2\hat{\gamma}_k^2 + c_3\hat{\gamma}_k\lambda_k - c_4\hat{\gamma}_k\Lambda_k,$$

where the scalars  $c_1, c_2, c_3$  are defined as below:

$$\begin{aligned} c_1 &\triangleq (0.5\mu_f\bar{\theta}\lambda)(\sigma_{\mathbf{R}}\delta_{\mathbf{R},\mathbf{C}})c_0L_0(\|\mathbf{R}\|_2\|v\|_2\frac{L_0}{\sqrt{m}}) \\ &\quad + \left(\frac{\bar{\theta}L_0}{\sqrt{m}}\right)(\sigma_{\mathbf{R}}\delta_{\mathbf{R},\mathbf{C}})(c_0L_0(\|\mathbf{R}\|_2\|v\|_2L_0)) \\ &\quad + \left(\frac{\|u\|_2}{m}\right)(\sigma_{\mathbf{R}}L_0\|v\|_{\mathbf{R}})(c_0L_0\|\mathbf{R}\|_2\|v\|_2\frac{L_0}{\sqrt{m}}), \\ c_2 &\triangleq (0.5\mu_f\bar{\theta}\lambda)(\sigma_{\mathbf{R}}\delta_{\mathbf{R},\mathbf{C}})c_0L_0(\|\mathbf{R} - \mathbf{I}\|_2 + 2\Lambda_0) \\ &\quad + \left(\frac{1-\sigma_{\mathbf{C}}}{2}\right)\left(\frac{\bar{\theta}L_0}{\sqrt{m}}\right)(\sigma_{\mathbf{R}}L_0\|v\|_{\mathbf{R}}) \\ &\quad + \left(\frac{\bar{\theta}L_0}{\sqrt{m}}\right)(\sigma_{\mathbf{R}}\delta_{\mathbf{R},\mathbf{C}})(c_0L_02\sqrt{m}\Lambda_0) \\ &\quad + \left(\frac{\|u\|_2}{m}\right)(\sigma_{\mathbf{R}}L_0\|v\|_{\mathbf{R}})(c_0L_0(\|\mathbf{R} - \mathbf{I}\|_2 + 2\Lambda_0)) \\ &\quad + \left(\frac{1-\sigma_{\mathbf{R}}}{2}\right)(c_0L_0(\|\mathbf{R}\|_2\|v\|_2L_0))\left(\frac{\|u\|_2}{m}\right), \\ c_3 &\triangleq (0.5)^3\mu_f\bar{\theta}(1 - \sigma_{\mathbf{R}})(1 - \sigma_{\mathbf{C}}), \\ c_4 &\triangleq \left(\frac{1-\sigma_{\mathbf{R}}}{2}\right)(c_0L_0\sqrt{m})\left(\frac{\|u\|_2}{m}\right). \end{aligned}$$

It suffices to show that  $-c_1\hat{\gamma}_k^3 - c_2\hat{\gamma}_k^2 + c_3\hat{\gamma}_k\lambda_k - c_4\hat{\gamma}_k\Lambda_k > 0$  for any  $k \geq 1$ . Or equivalently, we show that  $\frac{1}{3}c_3\frac{\lambda}{(k+\Gamma)^b} > c_1\frac{\hat{\gamma}^2}{(k+\Gamma)^{2a}} + \frac{1}{3}c_2\frac{\lambda}{(k+\Gamma)^b} > c_2\frac{\hat{\gamma}}{(k+\Gamma)^a}$  and  $\frac{1}{3}c_3\frac{\lambda}{(k+\Gamma)^b} > c_4\Lambda_k$  for  $k \geq 1$ .

Let  $\Gamma > \frac{2a-b\sqrt{3c_1\hat{\gamma}^2}}{c_3\lambda}$ , we obtain  $(k+\Gamma)^{2a-b} > \frac{3c_1\hat{\gamma}^2}{c_3\lambda}$ , then by rearranging terms, we obtain  $\frac{c_3\lambda}{3(k+\Gamma)^b} > \frac{c_1\hat{\gamma}^2}{(k+\Gamma)^{2a}}$ . Similarly, we can obtain  $\frac{c_3\lambda}{3(k+\Gamma)^b} > \frac{c_2\hat{\gamma}}{(k+\Gamma)^a}$  by letting  $\Gamma > \frac{a-b\sqrt{3c_2\hat{\gamma}}}{c_3\lambda}$ . Next, we choose  $\Gamma > \frac{1-b\sqrt{3c_4}}{c_3\lambda}$ , we can obtain  $(k+\Gamma)^{1-b} > \frac{3c_4}{c_3\lambda}$ , then by rearranging terms, we obtain  $\frac{c_3\lambda}{3(k+\Gamma)^b} > \frac{c_4}{k+\Gamma}$ , then utilize Lemma 1(ii), we obtain  $\frac{c_3\lambda}{3(k+\Gamma)^b} > \frac{c_4}{k+\Gamma} \geq c_4\Lambda_{k-1} \geq c_4\Lambda_k$  for all  $k \geq 1$ . We conclude that for  $\Gamma \geq \hat{\Gamma}_3$ , we have  $\det(\rho_k\mathbf{I} - \hat{H}_k) > 0$  for any  $k \geq 1$ . Therefore, we have  $\rho(\hat{H}_k) \leq 1 - 0.5\mu_f\alpha_k\lambda_k$  for all  $k \geq 1$ . The desired inequality is obtained from this relation and the relation (14).

(ii) From Lemma 1, we have that  $\Lambda_{k-1} \leq \frac{1}{k+\Gamma}$ . From part (i) and invoking  $\alpha_k \geq \theta\hat{\gamma}_k$ , we obtain for all  $k \geq 1$ ,

$$\|\Delta_{k+1}\|_2 \leq (1 - 0.5\mu_f\lambda_k\hat{\gamma}_k\theta)\|\Delta_k\|_2 + \frac{\Theta}{k+\Gamma}. \quad (15)$$

We use induction to show that the desired relation holds for  $\mathcal{B}$ . First, it can be easily verified that the inequality holds for  $k = 1$ . Let us assume that  $\|\Delta_k\|_2 \leq \frac{\mathcal{B}}{(k+\Gamma-1)^{1-a-b}}$  holds for some  $k \geq 2$ . Next, we show that this relation also holds for  $k + 1$ . Consider Lemma 1. Let  $\tau = 0.5\theta$ . We obtain  $\frac{(k+\Gamma)\hat{\gamma}_k\lambda_k}{(k+\Gamma-1)\hat{\gamma}_{k-1}\lambda_{k-1}} \leq 1 + 0.25\mu_f\hat{\gamma}_k\lambda_k\theta$  for all  $k \geq 1$ . Using the definition of  $\hat{\gamma}_k$  and  $\lambda_k$  and rearranging terms, we obtain

$$\frac{1}{(k+\Gamma-1)^{1-a-b}} \leq \frac{1}{(k+\Gamma)^{1-a-b}}(1 + 0.25\mu_f\lambda\hat{\gamma}\theta). \quad (16)$$

Consider  $\Gamma > \frac{a+b\sqrt{0.5\mu_f\lambda\hat{\gamma}\theta}}{m}$ . We obtain  $1 > 0.5\mu_f\lambda_k\hat{\gamma}_k\theta$ . From (15) and the induction hypothesis, we obtain

$$\|\Delta_{k+1}\|_2 \leq (1 - 0.5\mu_f\lambda_k\hat{\gamma}_k\theta)\frac{\mathcal{B}}{(k+\Gamma-1)^{1-a-b}} + \frac{\Theta}{k+\Gamma}.$$

From the preceding relation and (16), we obtain

$$\|\Delta_{k+1}\|_2 \leq \frac{\mathcal{B}(1-0.5\mu_f\lambda_k\hat{\gamma}_k\theta)(1+0.25\mu_f\lambda_k\hat{\gamma}_k\theta)}{(k+\Gamma)^{1-a-b}} + \frac{\Theta}{k+\Gamma}.$$

From the definition of  $\mathcal{B}$ , we have  $\Theta \leq 0.25\mu_f\lambda\hat{\gamma}\theta\mathcal{B}$ . Therefore, we obtain

$$\begin{aligned}\|\Delta_{k+1}\|_2 &\leq \frac{\mathcal{B}(1-0.25\mu_f\lambda\hat{\gamma}\theta-0.125(\mu_f\lambda\hat{\gamma}\theta)^2)}{(k+\Gamma)^{1-a-b}} + \frac{0.25\mu_f\lambda\hat{\gamma}\theta\mathcal{B}}{(k+\Gamma)^{1-a-b}} \\ &= \frac{\mathcal{B}(1-0.125(\mu_f\lambda\hat{\gamma}\theta)^2)}{(k+\Gamma)^{1-a-b}} \leq \frac{\mathcal{B}}{(k+\Gamma)^{1-a-b}}.\end{aligned}$$

where we used  $0.5\mu_f\lambda_k\hat{\gamma}_k\theta < 1$ . Thus, the induction statement holds for  $k+1$  and hence, the proof is completed. ■

Our first main result is provided below where we provide convergence guarantees for addressing problem (1).

**Theorem 1 (Convergence statements for (1) over digraphs):** Consider problem (1) and Algorithm 1. Let Assumptions 1, 2, and 3 hold. Suppose  $\Gamma \geq \hat{\Gamma}_3$ , where  $\hat{\Gamma}_3$  is given by Prop. 2. Then, the following results hold:

(i) [Consensus error bound for  $x$ ] For any  $k \geq 1$ , we have

$$\|\mathbf{x}_{k+1} - \mathbf{1}\bar{x}_{k+1}\|_{\mathbf{R}} \leq \frac{\max\{(\Gamma+1)^{1-a-b}\|\Delta_1\|_2, \frac{4\Theta}{\mu_f\lambda\hat{\gamma}\theta}\}}{(k+\Gamma-1)^{1-a-b}}.$$

(ii) [Consensus error bound for  $y$ ] For any  $k \geq 1$ , we have

$$\|\mathbf{y}_{k+1} - v\bar{y}_{k+1}\|_{\mathbf{C}} \leq \frac{\max\{(\Gamma+1)^{1-a-b}\|\Delta_1\|_2, \frac{4\Theta}{\mu_f\lambda\hat{\gamma}\theta}\}}{(k+\Gamma-1)^{1-a-b}}.$$

(iii) [Asymptotic convergence] The sequence of the averaged iterate,  $\{\bar{x}_k\}$ , admits a limit point. It converges to the unique optimal solution of problem (1), i.e.,  $\lim_{k \rightarrow \infty} \bar{x}_k = x^*$ .

*Proof:* (i, ii) The bound in (i) holds immediately from Prop. 2(ii), in view of  $\|\mathbf{x}_k - \mathbf{1}\bar{x}_k\|_{\mathbf{R}} \leq \|\Delta_k\|_2$ . The bound in part (ii) can be shown in a similar vein.

(iii) From the triangle inequality, for any  $k \geq 1$  we may write

$$\|\bar{x}_k - x^*\|_2 \leq \|\bar{x}_k - x_{\lambda_{k-1}}^*\|_2 + \|x_{\lambda_{k-1}}^* - x^*\|_2. \quad (17)$$

Recall from Lemma 2(i) that  $\lim_{k \rightarrow \infty} x_{\lambda_k}^* = x^*$ . From Proposition 2(ii), we have for any  $k \geq 1$ ,

$$\|\bar{x}_k - x_{\lambda_{k-1}}^*\|_2 \leq \frac{\max\{(\Gamma+1)^{1-a-b}\|\Delta_1\|_2, \frac{4\Theta}{\mu_f\lambda\hat{\gamma}\theta}\}}{(k+\Gamma-1)^{1-a-b}}.$$

Taking the limit on both sides of (17) and invoking the preceding results, we obtain  $\lim_{k \rightarrow \infty} \bar{x}_k = x^*$ . ■

**Remark 3:** Importantly, Theorem 1 provides convergence guarantees for computing the unique optimal NE. We observe that the choice of the parameters  $a$  and  $b$  plays an important role in the convergence speed. Recall that  $a$  and  $b$  need to satisfy  $0 < b < a < 1$  and  $2a + 3b < 2$ . This implies that the term  $1 - a - b$  in Theorem 1(i,ii) ranges between  $(0, 1)$ . In particular, with suitable choices of  $a \in (0, 1)$  and  $b \in (0, \min\{a, 0.4\})$ , the consensus error bounds can achieve a nearly sublinear rate (e.g.,  $a = 0.01$  and  $b = 0.001$ ). We should note, however, that a too small  $b$  would slow down the convergence speed of  $\|x_{\lambda_{k-1}}^* - x^*\|_2$  in (17) to zero. This indeed shows that in implementations,  $b$  should be chosen within the range  $(0, \min\{a, 0.4\})$  but not too small. This trade-off will be numerically studied in Section IV as well.

### III. STOCHASTIC SETTING OVER UNDIRECTED NETWORKS

The goal in this section is to devise a gradient tracking method for addressing the stochastic problem (2) over undirected networks. To this end, we consider the DSGT method in [28] that addresses unconstrained distributed stochastic optimization problems. By leveraging the IR framework, we

devise a single-timescale method called IR-DSGT, presented in Algorithm 2. In this section, an (undirected) graph is denoted by  $\mathcal{G} = (\mathcal{N}, \mathcal{E})$  where  $\mathcal{N}$  is a set of nodes and  $\mathcal{E} \subseteq \mathcal{N} \times \mathcal{N}$  is the set of ordered pairs of vertices. We let  $\mathcal{N}(i)$  denote the set of neighbors of agent  $i$ , i.e.,  $\mathcal{N}(i) \triangleq \{j \mid (i, j) \in \mathcal{E}\}$ . Throughout, we consider the definitions given by (3), (4), and (5). Additionally, we define the following terms:

$$\begin{aligned}\xi &\triangleq [\xi_1, \dots, \xi_m]^\top \in \mathbb{R}^{m \times d}, \quad f_i(x) \triangleq \mathbb{E}[f_i(x, \xi_i) \mid x], \\ \mathbf{f}(\mathbf{x}, \xi) &\triangleq \sum_{i=1}^m f_i(x_i, \xi_i), \\ \nabla \mathbf{f}(\mathbf{x}, \xi) &\triangleq [\nabla f(x_1, \xi_1), \dots, \nabla f(x_m, \xi_m)]^\top \in \mathbb{R}^{m \times n}, \\ F(\mathbf{x}, \xi) &\triangleq \sum_{i=1}^m F_i(x_i, \xi_i), \\ \mathbf{F}(\mathbf{x}, \xi) &\triangleq [F_1(x_1, \xi_1), \dots, F_m(x_m, \xi_m)]^\top \in \mathbb{R}^{m \times n}.\end{aligned}$$

Here, we assume that each agent  $i$  has access to a stochastic oracle to obtain sampled mappings  $\nabla f_i(x, \xi_i)$  and  $F_i(x, \xi_i)$ . Throughout this section, we use  $\|\cdot\|$  to denote the Euclidean norm and the Frobenius norm of a vector and a matrix, respectively. We let  $\langle \cdot, \cdot \rangle$  denote the Frobenius inner product. The update rules in Algorithm 2 can be compactly represented

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#### Algorithm 2 Iteratively Regularized distributed stochastic gradient tracking (IR-DSGT)

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- 1: **Input:** Choose  $\gamma_0 > 0$ ,  $\lambda_0 > 0$ , the weight matrix  $\mathbf{W}$ . For all  $i \in [m]$ , agents set an arbitrary initial point  $x_{i,0} \in \mathbb{R}^n$  and  $y_{i,0} := F_i(x_{i,0}, \xi_{i,0}) + \lambda_0 \nabla f_i(x_{i,0}, \xi_{i,0})$ ;
  - 2: **for**  $k = 0, 1, \dots$  **do**
  - 3:   For all  $i \in [m]$ , agent  $i$  receives the vector  $x_{j,k} - \gamma_{j,k} y_{j,k}$  from each agent  $j \in \mathcal{N}_{\mathbf{W}}(i)$ , sends  $W_{\ell i} y_{i,k}$  to each agent  $\ell \in \mathcal{N}_{\mathbf{W}}(i)$ , and does the following updates:
  - 4:    $x_{i,k+1} := \sum_{j=1}^m W_{ij} (x_{j,k} - \gamma_{j,k} y_{j,k})$   
 $y_{i,k+1} := \sum_{j=1}^m W_{ij} y_{j,k} + F_i(x_{i,k+1}, \xi_{i,k+1})$
  - 5:    $+ \lambda_{k+1} \nabla f_i(x_{i,k+1}, \xi_{i,k+1}) - F_i(x_{i,k}, \xi_{i,k})$   
 $- \lambda_k \nabla f_i(x_{i,k}, \xi_{i,k})$
  - 6: **end for**
- 

as the following:

$$\mathbf{x}_{k+1} := \mathbf{W}(\mathbf{x}_k - \gamma_k \mathbf{y}_k), \quad (18)$$

$$\begin{aligned}\mathbf{y}_{k+1} &:= \mathbf{W} \mathbf{y}_k + \mathbf{F}(\mathbf{x}_{k+1}, \xi_{k+1}) + \lambda_{k+1} \nabla \mathbf{f}(\mathbf{x}_{k+1}, \xi_{k+1}) \\ &\quad - \mathbf{F}(\mathbf{x}_k, \xi_k) - \lambda_k \nabla \mathbf{f}(\mathbf{x}_k, \xi_k),\end{aligned} \quad (19)$$

where  $\gamma_k > 0$  denotes the stepsize at the iteration  $k \geq 0$ . Throughout, we define the history of the method as  $\mathcal{F}_k \triangleq \cup_{i=1}^m \{x_{i,0}, \xi_{i,1}, \dots, \xi_{i,k-1}\}$  for  $k \geq 1$ , and  $\mathcal{F}_0 \triangleq \cup_{i=1}^m \{x_{i,0}\}$ . Next, we present key assumptions regarding Algorithm 2.

**Assumption 4:** The weight matrix  $\mathbf{W}$  is doubly stochastic and  $w_{i,i} > 0$  for all  $i \in [m]$ . Also, the graph  $\mathcal{G}$  corresponding to the communication network is undirected and connected.

**Assumption 5:** For all  $i \in [m]$  and  $x \in \mathbb{R}^n$ , random vectors  $\xi_i \in \mathbb{R}^d$  are independent and satisfy the following conditions:

- (a)  $\mathbb{E}[\nabla f_i(x, \xi_i) \mid x] = \nabla f_i(x)$ ,  $\mathbb{E}[F_i(x, \xi_i) \mid x] = F_i(x)$ .
- (b)  $\mathbb{E}[\|\nabla f_i(x, \xi_i) - \nabla f_i(x)\|^2 \mid x] \leq \nu_f^2$  for some  $\sigma_f > 0$  and  $\mathbb{E}[\|F_i(x, \xi_i) - F_i(x)\|^2 \mid x] \leq \nu_F^2$  for some  $\nu_F > 0$ .

We utilize the following definitions for analyzing Algorithm 2.

**Definition 2:** Let  $x^*$ ,  $x_{\lambda_k}^*$ ,  $L_k$ , and  $\Lambda_k$  be defined as in Definition 1. Also, let us define  $\nu_k^2 \triangleq \nu_F^2 + \lambda_k^2 \nu_f^2$ ,



$\mathbf{G}_k(\mathbf{x}, \boldsymbol{\xi}) \triangleq \mathbf{F}(\mathbf{x}, \boldsymbol{\xi}) + \lambda_k \nabla f(\mathbf{x}, \boldsymbol{\xi}) \in \mathbb{R}^{m \times n}$ ,  $G_k(\mathbf{x}, \boldsymbol{\xi}) \triangleq \frac{1}{m} \mathbf{1}^\top \mathbf{G}_k(\mathbf{x}, \boldsymbol{\xi}) \in \mathbb{R}^{1 \times n}$ . We let  $\mathbf{G}_k(\mathbf{x})$ ,  $G_k(\mathbf{x})$  denote the expected values of  $\mathbf{G}_k(\mathbf{x}, \boldsymbol{\xi})$  and  $G_k(\mathbf{x}, \boldsymbol{\xi})$ , respectively. Also, we define,  $\mathcal{G}_k(x) \triangleq G_k(\mathbf{1} \mathbf{x}^\top) \in \mathbb{R}^{1 \times n}$ ,  $\bar{g}_k \triangleq \mathcal{G}_k(\bar{x}_k) \in \mathbb{R}^{1 \times n}$ ,  $\bar{x}_k \triangleq \frac{1}{m} \mathbf{1}^\top \mathbf{x}_k \in \mathbb{R}^{1 \times n}$ , and  $\bar{y}_k \triangleq \frac{1}{m} \mathbf{1}^\top \mathbf{y}_k \in \mathbb{R}^{1 \times n}$ .

The following result will be employed in the analysis.

**Lemma 7** ([28, Lemma 1]): Let Assumption 4 hold. Let  $\rho_W$  denote the spectral norm of the matrix  $\mathbf{W} - \frac{1}{m} \mathbf{1} \mathbf{1}^\top$ . Then,  $\rho_W < 1$  and  $\|\mathbf{W}\mathbf{u} - \mathbf{1}\bar{u}\| \leq \rho_W \|\mathbf{u} - \mathbf{1}\bar{u}\|$  for all  $\mathbf{u} \in \mathbb{R}^{m \times n}$ , where  $\bar{u} \triangleq \frac{1}{m} \mathbf{1}^\top \mathbf{u}$ .

**Lemma 8:** Consider Algorithm 2 and Definition 2. Let Assumptions 1, 4, and 5 hold. Then: (i) We have that  $\bar{y}_k = G_k(\mathbf{x}_k, \boldsymbol{\xi}_k)$ . (ii) We have  $\mathcal{G}_k(x_{\lambda_k}^*) = 0$ . (iii) The mapping  $\mathcal{G}_k(x)$  is  $(\lambda_k \mu_f)$ -strongly monotone and Lipschitz continuous with parameter  $L_k$ . (iv)  $\mathbb{E}[\|\bar{y}_k - G_k(\mathbf{x}_k)\|^2 | \mathcal{F}_k] \leq \frac{2\nu_k^2}{m}$ . (v)  $\|G_k(\mathbf{x}_k) - \bar{g}_k\| \leq \frac{L_k}{\sqrt{m}} \|\mathbf{x}_k - \mathbf{1}\bar{x}_k\|$ . (vi)  $\|\bar{g}_k\| \leq L_k \|\bar{x}_k - x_{\lambda_k}^*\|$ .

**Proof:** (i) This can be shown in a similar way to the proof of Lemma 3(i) using the update rule (19) and the definition of  $G_k(\mathbf{x}_k, \boldsymbol{\xi}_k)$  in Definition 2.

(ii–iii) See the proof of Lemma 3(ii–iii).

(iv) From part (i), we may write

$$\begin{aligned} \mathbb{E}[\|\bar{y}_k - G_k(\mathbf{x}_k)\|^2 | \mathcal{F}_k] &= \mathbb{E}[\|G_k(\mathbf{x}_k, \boldsymbol{\xi}_k) - G_k(\mathbf{x}_k)\|^2 | \mathcal{F}_k] \\ &\leq \frac{2}{m^2} \mathbb{E}[\|\sum_{i=1}^m (F_i(x_{i,k}, \xi_{i,k}) - F_i(x_{i,k}))\|^2 | \mathcal{F}_k] \\ &+ \lambda_k^2 \frac{2}{m^2} \mathbb{E}[\|\sum_{i=1}^m (\nabla f_i(x_{i,k}, \xi_{i,k}) - \nabla f_i(x_{i,k}))\|^2 | \mathcal{F}_k] \\ &= \frac{2}{m^2} \mathbb{E}[\sum_{i=1}^m \|F_i(x_{i,k}, \xi_{i,k}) - F_i(x_{i,k})\|^2 | \mathcal{F}_k] \\ &+ \frac{2\lambda_k^2}{m^2} \mathbb{E}[\sum_{i=1}^m \|\nabla f_i(x_{i,k}, \xi_{i,k}) - \nabla f_i(x_{i,k})\|^2 | \mathcal{F}_k] \leq \frac{2\nu_k^2}{m}, \end{aligned}$$

where the equation is implied by Assumption 5(a) and that vectors  $\xi_{i,k}$  are independent across the agents for any  $k \geq 0$ . (iv–v) See the proof of Lemma 3(iv–v). ■

As in Proposition 1, we introduce three error metrics  $\mathbb{E}[\|\bar{x}_{k+1} - x_{\lambda_k}^*\|^2 | \mathcal{F}_k]$ ,  $\mathbb{E}[\|\mathbf{x}_{k+1} - \mathbf{1}\bar{x}_{k+1}\|^2 | \mathcal{F}_k]$  and  $\mathbb{E}[\|\mathbf{y}_{k+1} - \mathbf{1}\bar{y}_{k+1}\|^2 | \mathcal{F}_k]$  to analyze the convergence of Algorithm 2. In the following, we provide three recursive inequalities for these error metrics.

**Proposition 3:** Consider Algorithm 2. Let Assumptions 1, 4, and 5 hold. Let  $\gamma_k \leq \frac{1}{L_k}$  for  $k \geq 0$ . Then, there exist  $M > 0$  and  $B_F > 0$  such that for all  $k \geq 1$ , the following hold.

$$\begin{aligned} (a) \quad &\mathbb{E}[\|\bar{x}_{k+1} - x_{\lambda_k}^*\|^2 | \mathcal{F}_k] \leq (1 - \frac{\mu_f \lambda_k \gamma_k}{2}) \|\bar{x}_k - x_{\lambda_{k-1}}^*\|^2 \\ &+ \frac{L_k^2 \gamma_k (1 + \mu_f \lambda_k \gamma_k)}{m \lambda_k \mu_f} \|\mathbf{x}_k - \mathbf{1}\bar{x}_k\|^2 + \frac{2\nu_k^2 \gamma_k^2}{m} + \frac{2M^2 \Lambda_{k-1}^2}{\mu_f^3 \lambda_k \gamma_k}. \\ (b) \quad &\|\mathbf{x}_{k+1} - \mathbf{1}\bar{x}_{k+1}\|^2 \leq \frac{1 + \rho_W^2}{2} \|\mathbf{x}_k - \mathbf{1}\bar{x}_k\|^2 \\ &+ \frac{\gamma_k^2 (1 + \rho_W^2) \rho_W^2}{1 - \rho_W^2} \|\mathbf{y}_k - \mathbf{1}\bar{y}_k\|^2. \\ (c) \quad &\mathbb{E}[\|\mathbf{y}_{k+1} - \mathbf{1}\bar{y}_{k+1}\|^2 | \mathcal{F}_k] \\ &\leq ((1 + \zeta^{-1}) (4L_k^2 \gamma_k^2) + \zeta + 1) \rho_W^2 \mathbb{E}[\|\mathbf{y}_k - \mathbf{1}\bar{y}_k\|^2 | \mathcal{F}_k] \\ &+ (1 + \zeta^{-1}) 4L_k^2 (6\Lambda_k^2 L_k^2 + \|\mathbf{W} - \mathbf{I}\|^2) \\ &+ 1.5\gamma_k^2 \|\mathbf{x}_k - \mathbf{1}\bar{x}_k\|^2 \\ &+ (1 + \zeta^{-1}) 12mL_k^2 (4\Lambda_k^2 + L_k^2 \gamma_k^2) \|\bar{x}_k - x_{\lambda_{k-1}}^*\|^2 \\ &+ (1 + \zeta^{-1}) 12L_k^2 \gamma_k^2 \nu_k^2 + (1 + \zeta^{-1}) 6\Lambda_k^2 B_F^2 \\ &+ 2m(\nu_{k+1}^2 + \nu_k^2) + 4m\gamma_k L_{k+1} \nu_k^2 + 4\nu_k^2 \\ &+ (1 + \zeta^{-1}) 12mL_k^2 (4\Lambda_k^2 + L_k^2 \gamma_k^2) \frac{M^2}{\mu_f^2} \Lambda_{k-1}^2. \end{aligned}$$

**Proof:** (a) Applying  $\frac{1}{m} \mathbf{1}^\top$  to (18) and using  $\mathbf{1}^\top \mathbf{W} = \mathbf{1}^\top$  implies that  $\bar{x}_{k+1} = \bar{x}_k - \gamma_k \bar{y}_k$ . Thus, we can write

$$\begin{aligned} \|\bar{x}_{k+1} - x_{\lambda_k}^*\|^2 &= \|\bar{x}_k - x_{\lambda_k}^*\|^2 - 2\gamma_k \bar{y}_k^\top (\bar{x}_k - x_{\lambda_k}^*) \\ &+ \gamma_k^2 \|\bar{y}_k\|^2. \end{aligned}$$

Taking conditional expectation on the preceding relation and adding and subtracting  $G_k(\mathbf{x}_k)$  in the last term, we obtain

$$\begin{aligned} \mathbb{E}[\|\bar{x}_{k+1} - x_{\lambda_k}^*\|^2 | \mathcal{F}_k] &= \|\bar{x}_k - x_{\lambda_k}^*\|^2 + \gamma_k^2 \|G_k(\mathbf{x}_k)\|^2 \\ &+ \gamma_k^2 \mathbb{E}[\|\bar{y}_k - G_k(\mathbf{x}_k)\|^2 | \mathcal{F}_k] - 2\gamma_k G_k(\mathbf{x}_k)^\top (\bar{x}_k - x_{\lambda_k}^*), \end{aligned}$$

where we used  $\mathbb{E}[\bar{y}_k | \mathcal{F}_k] = G_k(\mathbf{x}_k)$ . Adding and subtracting  $\bar{g}_k$  and using Lemma 8(iv), we have

$$\begin{aligned} \mathbb{E}[\|\bar{x}_{k+1} - x_{\lambda_k}^*\|^2 | \mathcal{F}_k] &= \|\bar{x}_k - x_{\lambda_k}^*\|^2 + \gamma_k^2 \|G_k(\mathbf{x}_k) - \bar{g}_k\|^2 \\ &+ \gamma_k^2 \|\bar{g}_k\|^2 + 2\gamma_k^2 \bar{g}_k^\top (G_k(\mathbf{x}_k) - \bar{g}_k) + \frac{2\nu_k^2}{m} \gamma_k^2 \\ &- 2\gamma_k (G_k(\mathbf{x}_k) - \bar{g}_k)^\top (\bar{x}_k - x_{\lambda_k}^*) - 2\gamma_k \bar{g}_k^\top (\bar{x}_k - x_{\lambda_k}^*) \\ &\leq \|\bar{x}_k - \gamma_k \bar{g}_k - x_{\lambda_k}^*\|^2 + \gamma_k^2 \|G_k(\mathbf{x}_k) - \bar{g}_k\|^2 + \frac{2\nu_k^2}{m} \gamma_k^2 \\ &- 2\gamma_k (G_k(\mathbf{x}_k) - \bar{g}_k)^\top (\bar{x}_k - x_{\lambda_k}^* - \gamma_k \bar{g}_k). \end{aligned}$$

Let  $\rho_k \triangleq 1 - \mu_f \lambda_k \gamma_k$ . From Lemmas 4 and 8(v) we obtain

$$\begin{aligned} \mathbb{E}[\|\bar{x}_{k+1} - x_{\lambda_k}^*\|^2 | \mathcal{F}_k] &\leq \rho_k^2 \|\bar{x}_k - x_{\lambda_k}^*\|^2 + \frac{L_k^2 \gamma_k^2}{m} \|\mathbf{x}_k - \mathbf{1}\bar{x}_k\|^2 + \frac{2\nu_k^2}{m} \gamma_k^2 \\ &+ 2 \frac{L_k \gamma_k \rho_k}{\sqrt{m}} \|\mathbf{x}_k - \mathbf{1}\bar{x}_k\| \|\bar{x}_k - x_{\lambda_k}^*\|. \end{aligned}$$

We write

$$\begin{aligned} &2 \frac{L_k \gamma_k \rho_k}{\sqrt{m}} \|\mathbf{x}_k - \mathbf{1}\bar{x}_k\| \|\bar{x}_k - x_{\lambda_k}^*\| \\ &\leq \gamma_k \left( \frac{L_k^2}{\lambda_k \mu_f m} \|\mathbf{x}_k - \mathbf{1}\bar{x}_k\|^2 + \lambda_k \mu_f \rho_k^2 \|\bar{x}_k - x_{\lambda_k}^*\|^2 \right). \end{aligned}$$

From the preceding two relations, we obtain

$$\begin{aligned} \mathbb{E}[\|\bar{x}_{k+1} - x_{\lambda_k}^*\|^2 | \mathcal{F}_k] &\leq \rho_k^2 (1 + \mu_f \lambda_k \gamma_k) \|\bar{x}_k - x_{\lambda_k}^*\|^2 \\ &+ \frac{L_k^2 \gamma_k}{m \mu_f \lambda_k} (1 + \mu_f \lambda_k \gamma_k) \|\mathbf{x}_k - \mathbf{1}\bar{x}_k\|^2 + \frac{2\nu_k^2}{m} \gamma_k^2. \end{aligned}$$

Note that we have  $\rho_k^2 (1 + \lambda_k \mu_f \gamma_k) \leq \rho_k$ . For any  $u, v \in \mathbb{R}^n$  and  $\theta > 0$  we have  $\|u + v\|^2 \leq (1 + \theta) \|u\|^2 + (1 + \frac{1}{\theta}) \|v\|^2$ . Using this and invoking Lemma 2, we have

$$\begin{aligned} \|\bar{x}_k - x_{\lambda_k}^*\|^2 &\leq (1 + \frac{\mu_f \lambda_k \gamma_k}{2}) \|\bar{x}_k - x_{\lambda_{k-1}}^*\|^2 \\ &+ (1 + \frac{2}{\mu_f \lambda_k \gamma_k}) \|x_{\lambda_{k-1}}^* - x_{\lambda_k}^*\|^2 \\ &\leq (1 + \frac{\mu_f \lambda_k \gamma_k}{2}) \|\bar{x}_k - x_{\lambda_{k-1}}^*\|^2 + (1 + \frac{2}{\mu_f \lambda_k \gamma_k}) \frac{M^2}{\mu_f^2} \Lambda_{k-1}^2. \end{aligned}$$

We obtain the result by noting that  $\rho_k (1 + \frac{\mu_f \lambda_k \gamma_k}{2}) \leq 1 - \frac{\mu_f \lambda_k \gamma_k}{2}$  and that  $\rho_k (1 + \frac{2}{\mu_f \lambda_k \gamma_k}) \leq \frac{2}{\mu_f \lambda_k \gamma_k}$ .

(b) Next, we show the second recursive relation. We write

$$\begin{aligned} \|\mathbf{x}_{k+1} - \mathbf{1}\bar{x}_{k+1}\|^2 &= \|\mathbf{W}\mathbf{x}_k - \gamma_k \mathbf{W}\mathbf{y}_k - \mathbf{1}(\bar{x}_k - \gamma_k \bar{y}_k)\|^2 \\ &= \|\mathbf{W}\mathbf{x}_k - \mathbf{1}\bar{x}_k\|^2 - 2\gamma_k \langle \mathbf{W}\mathbf{x}_k - \mathbf{1}\bar{x}_k, \mathbf{W}\mathbf{y}_k - \mathbf{1}\bar{y}_k \rangle \\ &+ \gamma_k^2 \|\mathbf{W}\mathbf{y}_k - \mathbf{1}\bar{y}_k\|^2. \end{aligned}$$

Invoking Lemma 7, we obtain

$$\begin{aligned} \|\mathbf{x}_{k+1} - \mathbf{1}\bar{x}_{k+1}\|^2 &= \rho_W^2 \|\mathbf{x}_k - \mathbf{1}\bar{x}_k\|^2 + \rho_W^2 \gamma_k^2 \|\mathbf{y}_k - \mathbf{1}\bar{y}_k\|^2 \\ &\quad + 2\gamma_k \|\mathbf{W}\mathbf{x}_k - \mathbf{1}\bar{x}_k\| \|\mathbf{W}\mathbf{y}_k - \mathbf{1}\bar{y}_k\| \\ &\leq \rho_W^2 \|\mathbf{x}_k - \mathbf{1}\bar{x}_k\|^2 + \rho_W^2 \gamma_k^2 \|\mathbf{y}_k - \mathbf{1}\bar{y}_k\|^2 \\ &\quad + \rho_W^2 \gamma_k \left( \frac{1-\rho_W^2}{2\gamma_k \rho_W^2} \|\mathbf{x}_k - \mathbf{1}\bar{x}_k\|^2 + \frac{2\gamma_k \rho_W^2}{1-\rho_W^2} \|\mathbf{y}_k - \mathbf{1}\bar{y}_k\|^2 \right) \\ &= \frac{1+\rho_W^2}{2} \|\mathbf{x}_k - \mathbf{1}\bar{x}_k\|^2 + \frac{\gamma_k^2 (1+\rho_W^2) \rho_W^2}{1-\rho_W^2} \|\mathbf{y}_k - \mathbf{1}\bar{y}_k\|^2. \end{aligned}$$

(c) Next we obtain the third recursive relation. To utilize the space, we use the abstract notation  $\mathbf{G}_k \triangleq \mathbf{G}_k(\mathbf{x}_k)$ ,  $\tilde{\mathbf{G}}_k \triangleq \mathbf{G}_k(\mathbf{x}_k, \xi_k)$ ,  $\nabla_{i,k}^F \triangleq F_i(x_{i,k})$ ,  $\tilde{\nabla}_{i,k}^F \triangleq F_i(x_{i,k}, \xi_{i,k})$ ,  $\nabla_{i,k}^f \triangleq \nabla f_i(x_{i,k})$ ,  $\tilde{\nabla}_{i,k}^f \triangleq \nabla f_i(x_{i,k}, \xi_{i,k})$ ,  $\nabla_{i,k}^\lambda \triangleq \nabla_{i,k}^F + \lambda \nabla_{i,k}^f$ ,  $\tilde{\nabla}_{i,k}^\lambda \triangleq \tilde{\nabla}_{i,k}^F + \lambda \tilde{\nabla}_{i,k}^f$  for some  $\lambda > 0$ . From (19) we have

$$\begin{aligned} \|\mathbf{y}_{k+1} - \mathbf{1}\bar{y}_{k+1}\|^2 &\leq \|\mathbf{W}\mathbf{y}_k + \tilde{\mathbf{G}}_{k+1} - \tilde{\mathbf{G}}_k - \mathbf{1}\bar{y}_k + \mathbf{1}\bar{y}_{k+1} - \mathbf{1}\bar{y}_{k+1}\|^2 \\ &= \|\mathbf{W}\mathbf{y}_k - \mathbf{1}\bar{y}_k\|^2 + \|\tilde{\mathbf{G}}_{k+1} - \tilde{\mathbf{G}}_k\|^2 \\ &\quad + 2\langle \mathbf{W}\mathbf{y}_k - \mathbf{1}\bar{y}_k, \tilde{\mathbf{G}}_{k+1} - \tilde{\mathbf{G}}_k \rangle \\ &\quad + 2\langle \mathbf{y}_{k+1} - \mathbf{1}\bar{y}_k, \mathbf{1}(\bar{y}_k - \bar{y}_{k+1}) \rangle + m\|\bar{y}_k - \bar{y}_{k+1}\|^2 \\ &= \rho_W^2 \|\mathbf{y}_k - \mathbf{1}\bar{y}_k\|^2 + \|\tilde{\mathbf{G}}_{k+1} - \tilde{\mathbf{G}}_k\|^2 \\ &\quad + 2\langle \mathbf{W}\mathbf{y}_k - \mathbf{1}\bar{y}_k, \tilde{\mathbf{G}}_{k+1} - \tilde{\mathbf{G}}_k \rangle - m\|\bar{y}_k - \bar{y}_{k+1}\|^2 \\ &\leq \rho_W^2 \|\mathbf{y}_k - \mathbf{1}\bar{y}_k\|^2 + \|\tilde{\mathbf{G}}_{k+1} - \tilde{\mathbf{G}}_k\|^2 \\ &\quad + 2\langle \mathbf{W}\mathbf{y}_k - \mathbf{1}\bar{y}_k, \tilde{\mathbf{G}}_{k+1} - \tilde{\mathbf{G}}_k \rangle. \end{aligned} \quad (20)$$

In the following, we present some intermediary results that will be used to derive the third recursive inequality.

**Claim 1.** The following holds

$$\begin{aligned} \mathbb{E}[\|\tilde{\mathbf{G}}_{k+1} - \tilde{\mathbf{G}}_k\|^2 | \mathcal{F}_k] &\leq \mathbb{E}[\|\mathbf{G}_{k+1} - \mathbf{G}_k\|^2 | \mathcal{F}_k] \\ &\quad + 2\mathbb{E}[\langle \mathbf{G}_{k+1}, -\tilde{\mathbf{G}}_k + \mathbf{G}_k \rangle | \mathcal{F}_k] + 2m(\nu_{k+1}^2 + \nu_k^2). \end{aligned} \quad (21)$$

**Proof of Claim 1.** We may write

$$\begin{aligned} \mathbb{E}[\|\tilde{\mathbf{G}}_{k+1} - \tilde{\mathbf{G}}_k\|^2 | \mathcal{F}_k] &= \mathbb{E}[\|\mathbf{G}_{k+1} - \mathbf{G}_k\|^2 | \mathcal{F}_k] \\ &\quad + 2\mathbb{E}[\langle \mathbf{G}_{k+1}, \tilde{\mathbf{G}}_{k+1} - \tilde{\mathbf{G}}_k - \mathbf{G}_{k+1} + \mathbf{G}_k \rangle | \mathcal{F}_k] \\ &\quad - 2\mathbb{E}[\langle \mathbf{G}_k, \tilde{\mathbf{G}}_{k+1} - \tilde{\mathbf{G}}_k - \mathbf{G}_{k+1} + \mathbf{G}_k \rangle | \mathcal{F}_k] \\ &\quad + \mathbb{E}[\|\tilde{\mathbf{G}}_{k+1} - \tilde{\mathbf{G}}_k - \mathbf{G}_{k+1} + \mathbf{G}_k\|^2 | \mathcal{F}_k]. \end{aligned} \quad (22)$$

Note that since  $\mathbf{x}_{k+1}$  is characterized in terms of  $\xi_k$ , we have

$$\begin{aligned} \mathbb{E}[\tilde{\mathbf{G}}_{k+1} - \mathbf{G}_{k+1} | \mathcal{F}_k] &= \mathbb{E}_{\xi_k} \left[ \mathbb{E}_{\xi_{k+1}} [\tilde{\mathbf{G}}_{k+1} - \mathbf{G}_{k+1} | \mathcal{F}_{k+1}] \right] = \mathbb{E}_{\xi_k} [0] = 0. \end{aligned}$$

Also, we have  $\mathbb{E}[\tilde{\mathbf{G}}_k - \mathbf{G}_k | \mathcal{F}_k] = 0$ . Thus, we obtain

$$\mathbb{E}[\langle \mathbf{G}_k, \tilde{\mathbf{G}}_{k+1} - \tilde{\mathbf{G}}_k - \mathbf{G}_{k+1} + \mathbf{G}_k \rangle | \mathcal{F}_k] = 0.$$

We may also write

$$\begin{aligned} \mathbb{E}[\langle \mathbf{G}_{k+1}, \tilde{\mathbf{G}}_{k+1} - \tilde{\mathbf{G}}_k - \mathbf{G}_{k+1} + \mathbf{G}_k \rangle | \mathcal{F}_k] &= \mathbb{E}_{\xi_k} \left[ \mathbb{E}_{\xi_{k+1}} [\langle \mathbf{G}_{k+1}, \tilde{\mathbf{G}}_{k+1} - \tilde{\mathbf{G}}_k - \mathbf{G}_{k+1} + \mathbf{G}_k \rangle | \mathcal{F}_{k+1}] \right] \\ &= \mathbb{E}_{\xi_k} [\langle \mathbf{G}_{k+1}, \mathbf{G}_{k+1} - \tilde{\mathbf{G}}_k - \mathbf{G}_{k+1} + \mathbf{G}_k \rangle | \mathcal{F}_k] \\ &= \mathbb{E}[\langle \mathbf{G}_{k+1}, -\tilde{\mathbf{G}}_k + \mathbf{G}_k \rangle | \mathcal{F}_k]. \end{aligned}$$

From the preceding relations and (22) we have

$$\begin{aligned} \mathbb{E}[\|\tilde{\mathbf{G}}_{k+1} - \tilde{\mathbf{G}}_k\|^2 | \mathcal{F}_k] &\leq \mathbb{E}[\|\mathbf{G}_{k+1} - \mathbf{G}_k\|^2 | \mathcal{F}_k] \\ &\quad + 2\mathbb{E}[\langle \mathbf{G}_{k+1}, -\tilde{\mathbf{G}}_k + \mathbf{G}_k \rangle | \mathcal{F}_k] \\ &\quad + \mathbb{E}[\|\tilde{\mathbf{G}}_{k+1} - \tilde{\mathbf{G}}_k - \mathbf{G}_{k+1} + \mathbf{G}_k\|^2 | \mathcal{F}_k] \\ &\leq \mathbb{E}[\|\mathbf{G}_{k+1} - \mathbf{G}_k\|^2 | \mathcal{F}_k] + 2\mathbb{E}[\langle \mathbf{G}_{k+1}, -\tilde{\mathbf{G}}_k + \mathbf{G}_k \rangle | \mathcal{F}_k] \\ &\quad + \mathbb{E}[\|\tilde{\mathbf{G}}_{k+1} - \mathbf{G}_{k+1}\|^2 | \mathcal{F}_k] + \mathbb{E}[\|\tilde{\mathbf{G}}_k - \mathbf{G}_k\|^2 | \mathcal{F}_k]. \end{aligned}$$

From Lemma 8(iv), we obtain Claim 1.

**Claim 2.** The following holds

$$\mathbb{E}[\langle \mathbf{G}_{k+1}, -\tilde{\mathbf{G}}_k + \mathbf{G}_k \rangle | \mathcal{F}_k] \leq 2m\gamma_k L_{k+1} \nu_k^2. \quad (23)$$

**Proof of Claim 2.** Let us define for all  $i \in [m]$

$$\hat{x}_{i,k+1} \triangleq x_{i,k+1} + \gamma_k W_{ii} (\tilde{\nabla}_{i,k}^{\lambda_k} - \nabla_{i,k}^{\lambda_k}).$$

We may write

$$\begin{aligned} \mathbb{E}[\langle \nabla_{i,k+1}^{\lambda_{k+1}}, -\tilde{\nabla}_{i,k}^{\lambda_k} - \nabla_{i,k}^{\lambda_k} \rangle | \mathcal{F}_k] &= \mathbb{E}[\langle \nabla_{i,k+1}^{\lambda_{k+1}} - F_i(\hat{x}_{i,k+1}) - \lambda_{k+1} \nabla f_i(\hat{x}_{i,k+1}), \\ &\quad -\tilde{\nabla}_{i,k}^{\lambda_k} + \nabla_{i,k}^{\lambda_k} \rangle | \mathcal{F}_k] \\ &\quad + \mathbb{E}[\langle F_i(\hat{x}_{i,k+1}) + \lambda_{k+1} \nabla f_i(\hat{x}_{i,k+1}), -\tilde{\nabla}_{i,k}^{\lambda_k} + \nabla_{i,k}^{\lambda_k} \rangle | \mathcal{F}_k]. \end{aligned}$$

It can be seen from the update rules of Algorithm 2 that  $\hat{x}_{i,k+1}$  is independent of  $\xi_{i,k}$ . Thus, we may write

$$\mathbb{E}[\langle F_i(\hat{x}_{i,k+1}) + \lambda_{k+1} \nabla f_i(\hat{x}_{i,k+1}), -\tilde{\nabla}_{i,k}^{\lambda_k} + \nabla_{i,k}^{\lambda_k} \rangle | \mathcal{F}_k] = 0.$$

Also, using Assumption 1, we have

$$\begin{aligned} \|\nabla_{i,k+1}^{\lambda_{k+1}} - F_i(\hat{x}_{i,k+1}) - \lambda_{k+1} \nabla f_i(\hat{x}_{i,k+1})\| &= L_{k+1} \|x_{i,k+1} - \hat{x}_{i,k+1}\| \leq \gamma_k L_{k+1} \|\tilde{\nabla}_{i,k}^{\lambda_k} - \nabla_{i,k}^{\lambda_k}\|. \end{aligned}$$

From the preceding relations, we have

$$\begin{aligned} \mathbb{E}[\langle \nabla_{i,k+1}^{\lambda_{k+1}}, -\tilde{\nabla}_{i,k}^{\lambda_k} + \nabla_{i,k}^{\lambda_k} \rangle | \mathcal{F}_k] &\leq \mathbb{E}[\|\nabla_{i,k+1}^{\lambda_{k+1}} - F_i(\hat{x}_{i,k+1}) - \lambda_{k+1} \nabla f_i(\hat{x}_{i,k+1})\| \\ &\quad \times \|\tilde{\nabla}_{i,k}^{\lambda_k} - \nabla_{i,k}^{\lambda_k}\| | \mathcal{F}_k] \\ &\leq \gamma_k L_{k+1} \mathbb{E}[\|\tilde{\nabla}_{i,k}^{\lambda_k} - \nabla_{i,k}^{\lambda_k}\|^2 | \mathcal{F}_k] \\ &\leq 2\gamma_k L_{k+1} \mathbb{E}[\|\tilde{\nabla}_{i,k}^F - \nabla_{i,k}^F\|^2 + \lambda_k^2 \|\tilde{\nabla}_{i,k}^f - \nabla_{i,k}^f\|^2 | \mathcal{F}_k] \\ &\leq 2\gamma_k L_{k+1} \nu_k^2. \end{aligned}$$

Summing the preceding relation over  $i$ , we obtain (23).

**Claim 3.** The following holds

$$\begin{aligned} \|\mathbf{G}_{k+1} - \mathbf{G}_k\|^2 &\leq 4L_k^2 (6\Lambda_k^2 L_k^2 + \|\mathbf{W} - \mathbf{I}\|^2 + 1.5\gamma_k^2) \|\mathbf{x}_k - \mathbf{1}\bar{x}_k\|^2 \\ &\quad + 4L_k^2 \rho_W^2 \gamma_k^2 \|\mathbf{y}_k - \mathbf{1}\bar{y}_k\|^2 \\ &\quad + 6mL_k^2 (4\Lambda_k^2 + L_k^2 \gamma_k^2) \|\bar{x}_k - x_{\lambda_k}^*\|^2 \\ &\quad + 12L_k^2 \gamma_k^2 \nu_k^2 + 6\Lambda_k^2 B_F^2. \end{aligned} \quad (24)$$

**Proof of Claim 3.** From Definition 2 we have

$$\begin{aligned} \|\mathbf{G}_{k+1} - \mathbf{G}_k\|^2 &\leq 2\|\mathbf{G}_{k+1} - \mathbf{F}(\mathbf{x}_k) - \lambda_{k+1}\nabla\mathbf{f}(\mathbf{x}_k)\|^2 \\ &\quad + 2\|\lambda_{k+1}\nabla\mathbf{f}(\mathbf{x}_k) - \lambda_k\nabla\mathbf{f}(\mathbf{x}_k)\|^2 \\ &\leq 2\Lambda_k^2\|\lambda_k\nabla\mathbf{f}(\mathbf{x}_k)\|^2 + 2L_k^2\|\mathbf{x}_{k+1} - \mathbf{x}_k\|^2. \end{aligned}$$

From Lemma 2, there exists a scalar  $B_F < \infty$  such that  $L_F\|\mathbf{1}x_{\lambda_k}^* - \mathbf{1}x^*\|_2 \leq B_F$ . Invoking  $F(x^*) = 0$ , we have

$$\begin{aligned} \|\lambda_k\nabla\mathbf{f}(\mathbf{x}_k)\| &\leq \|\mathbf{F}(\mathbf{x}_k) + \lambda_k\nabla\mathbf{f}(\mathbf{x}_k)\| \\ &\quad + \|\mathbf{F}(\mathbf{x}_k) - \mathbf{F}(\mathbf{1}x^*)\| \\ &\leq \|\mathbf{F}(\mathbf{x}_k) + \lambda_k\nabla\mathbf{f}(\mathbf{x}_k) - \mathbf{F}(\mathbf{1}x_{\lambda_k}^*) - \lambda_k\nabla\mathbf{f}(\mathbf{1}x_{\lambda_k}^*)\| \\ &\quad + L_F\|\mathbf{x}_k - \mathbf{1}x^*\| \\ &\leq (L_k + L_F)\|\mathbf{x}_k - \mathbf{1}x_{\lambda_k}^*\| + L_F\|\mathbf{1}x_{\lambda_k}^* - \mathbf{1}x^*\| \\ &\leq 2L_k(\|\mathbf{x}_k - \mathbf{1}\bar{x}_k\| + \|\mathbf{1}\bar{x}_k - \mathbf{1}x_{\lambda_k}^*\|_2) + B_F \\ &\leq 2L_k\|\mathbf{x}_k - \mathbf{1}\bar{x}_k\| + 2\sqrt{m}L_k\|\bar{x}_k - x_{\lambda_k}^*\| + B_F. \end{aligned}$$

This implies that

$$\|\lambda_k\nabla\mathbf{f}(\mathbf{x}_k)\|^2 \leq 12L_k^2\|\mathbf{x}_k - \mathbf{1}\bar{x}_k\|^2 + 12mL_k^2\|\bar{x}_k - x_{\lambda_k}^*\|^2 + 3B_F^2.$$

We also have

$$\begin{aligned} \|\mathbf{x}_{k+1} - \mathbf{x}_k\|^2 &= \|\mathbf{W}\mathbf{x}_k - \gamma_k\mathbf{W}\mathbf{y}_k - \mathbf{x}_k\|^2 \\ &= \|(\mathbf{W} - \mathbf{I})(\mathbf{x}_k - \mathbf{1}\bar{x}_k) - \gamma_k\mathbf{W}\mathbf{y}_k - \mathbf{x}_k\|^2 \\ &\leq \|\mathbf{W} - \mathbf{I}\|^2\|\mathbf{x}_k - \mathbf{1}\bar{x}_k\|^2 + \gamma_k^2\|\mathbf{W}\mathbf{y}_k\|^2 \\ &\quad - 2\gamma_k\langle(\mathbf{W} - \mathbf{I})(\mathbf{x}_k - \mathbf{1}\bar{x}_k), \mathbf{W}\mathbf{y}_k\rangle \\ &= \|\mathbf{W} - \mathbf{I}\|^2\|\mathbf{x}_k - \mathbf{1}\bar{x}_k\|^2 + \gamma_k^2\|\mathbf{W}\mathbf{y}_k - \mathbf{1}\bar{y}_k\|^2 \\ &\quad + m\gamma_k^2\|\bar{y}_k\|^2 - 2\gamma_k\langle(\mathbf{W} - \mathbf{I})(\mathbf{x}_k - \mathbf{1}\bar{x}_k), \mathbf{W}\mathbf{y}_k - \mathbf{1}\bar{y}_k\rangle \\ &\leq \|\mathbf{W} - \mathbf{I}\|^2\|\mathbf{x}_k - \mathbf{1}\bar{x}_k\|^2 + \rho_W^2\gamma_k^2\|\mathbf{y}_k - \mathbf{1}\bar{y}_k\|^2 \\ &\quad + m\gamma_k^2\|\bar{y}_k\|^2 + 2\rho_W\gamma_k\|\mathbf{W} - \mathbf{I}\|\|\mathbf{x}_k - \mathbf{1}\bar{x}_k\|\|\mathbf{y}_k - \mathbf{1}\bar{y}_k\| \\ &\leq 2\|\mathbf{W} - \mathbf{I}\|^2\|\mathbf{x}_k - \mathbf{1}\bar{x}_k\|^2 + 2\rho_W^2\gamma_k^2\|\mathbf{y}_k - \mathbf{1}\bar{y}_k\|^2 \\ &\quad + m\gamma_k^2\|\bar{y}_k\|^2. \end{aligned}$$

From Lemma 3, we have

$$\begin{aligned} \mathbb{E}[\|\bar{y}_k\|^2 | \mathcal{F}_k] &\leq 3\mathbb{E}[\|\bar{y}_k - G_k(\mathbf{x}_k)\|^2 | \mathcal{F}_k] \\ &\quad + 3\|G_k(\mathbf{x}_k) - \bar{g}_k\|^2 + 3\|\bar{g}_k\|^2 \\ &\leq \frac{6\nu_k^2}{m} + \frac{3L_k^2}{m}\|\mathbf{x}_k - \mathbf{1}\bar{x}_k\|^2 + 3L_k^2\|\bar{x}_k - x_{\lambda_k}^*\|^2. \end{aligned}$$

From the preceding relations, we obtain (24).

**Claim 4.** The following holds

$$\begin{aligned} &\mathbb{E}[\langle \mathbf{W}\mathbf{y}_k - \mathbf{1}\bar{y}_k, \tilde{\mathbf{G}}_{k+1} - \tilde{\mathbf{G}}_k \rangle | \mathcal{F}_k] \\ &= \mathbb{E}[\langle \mathbf{W}\mathbf{y}_k - \mathbf{1}\bar{y}_k, \mathbf{G}_{k+1} - \mathbf{G}_k \rangle | \mathcal{F}_k] \\ &\quad + \mathbb{E}[\langle \mathbf{W}\mathbf{y}_k - \mathbf{1}\bar{y}_k, -\tilde{\mathbf{G}}_k + \mathbf{G}_k \rangle | \mathcal{F}_k]. \end{aligned} \quad (25)$$

**Proof of Claim 4.** We have

$$\begin{aligned} &\mathbb{E}[\langle \mathbf{W}\mathbf{y}_k - \mathbf{1}\bar{y}_k, \mathbf{G}_{k+1} - \tilde{\mathbf{G}}_{k+1} \rangle | \mathcal{F}_k] \\ &= \mathbb{E}_{\xi_k} \left[ \mathbb{E}_{\xi_{k+1}} [\langle \mathbf{W}\mathbf{y}_k - \mathbf{1}\bar{y}_k, \mathbf{G}_{k+1} - \tilde{\mathbf{G}}_{k+1} \rangle | \mathcal{F}_{k+1}] \right] = 0. \end{aligned}$$

The result follows by adding the above expectation to the left-hand side of (25).

**Claim 5.** The following holds

$$\mathbb{E}[\langle \mathbf{W}\mathbf{y}_k - \mathbf{1}\bar{y}_k, -\tilde{\mathbf{G}}_k + \mathbf{G}_k \rangle | \mathcal{F}_k] \leq 2\nu_k^2. \quad (26)$$

**Proof of Claim 5.** First, note that from Algorithm 2, we have for any  $i, j \in [m]$

$$\begin{aligned} &\mathbb{E}[\langle y_{j,k}, -\tilde{\nabla}_{i,k}^{\lambda_k} + \nabla_{i,k}^{\lambda_k} \rangle | \mathcal{F}_k] \\ &= \mathbb{E}[\langle \sum_{\ell=1}^m W_{j\ell} y_{\ell,k-1} + \tilde{\nabla}_{j,k}^{\lambda_k} - \tilde{\nabla}_{j,k-1}^{\lambda_{k-1}}, -\tilde{\nabla}_{i,k}^{\lambda_k} + \nabla_{i,k}^{\lambda_k} \rangle | \mathcal{F}_k] \\ &= \mathbb{E}[\langle \tilde{\nabla}_{j,k}^{\lambda_k}, -\tilde{\nabla}_{i,k}^{\lambda_k} + \nabla_{i,k}^{\lambda_k} \rangle | \mathcal{F}_k]. \end{aligned}$$

Multiplying by  $W_{ij}$  and summing over  $j \in [m]$ , we have

$$\begin{aligned} &\mathbb{E}[\langle \sum_{j=1}^m W_{ij} y_{j,k}, -\tilde{\nabla}_{i,k}^{\lambda_k} + \nabla_{i,k}^{\lambda_k} \rangle | \mathcal{F}_k] \\ &= \mathbb{E}[\langle W_{ii} \tilde{\nabla}_{i,k}^{\lambda_k}, -\tilde{\nabla}_{i,k}^{\lambda_k} + \nabla_{i,k}^{\lambda_k} \rangle | \mathcal{F}_k] \\ &= W_{ii} \mathbb{E}[\langle \tilde{\nabla}_{i,k}^{\lambda_k} - \nabla_{i,k}^{\lambda_k}, -\tilde{\nabla}_{i,k}^{\lambda_k} + \nabla_{i,k}^{\lambda_k} \rangle | \mathcal{F}_k] \leq 0. \end{aligned} \quad (27)$$

Second, from Lemma 8(i) we have

$$\begin{aligned} &-\mathbb{E}[\langle \bar{y}_k, -\tilde{\nabla}_{i,k}^{\lambda_k} + \nabla_{i,k}^{\lambda_k} \rangle | \mathcal{F}_k] \\ &= -\mathbb{E}[\langle \frac{1}{m} \sum_{j=1}^m \tilde{\nabla}_{j,k}^{\lambda_k}, -\tilde{\nabla}_{i,k}^{\lambda_k} + \nabla_{i,k}^{\lambda_k} \rangle | \mathcal{F}_k] \\ &= \frac{1}{m} \mathbb{E}[\langle -\tilde{\nabla}_{i,k}^{\lambda_k}, -\tilde{\nabla}_{i,k}^{\lambda_k} + \nabla_{i,k}^{\lambda_k} \rangle | \mathcal{F}_k] \\ &= \frac{1}{m} \mathbb{E}[\langle -\tilde{\nabla}_{i,k}^{\lambda_k} + \nabla_{i,k}^{\lambda_k}, -\tilde{\nabla}_{i,k}^{\lambda_k} + \nabla_{i,k}^{\lambda_k} \rangle | \mathcal{F}_k] \\ &= \frac{1}{m} \mathbb{E}[\|\nabla_{i,k}^{\lambda_k} - \tilde{\nabla}_{i,k}^{\lambda_k}\|^2 | \mathcal{F}_k] \\ &\leq \frac{2}{m} \mathbb{E}[\|\tilde{\nabla}_{i,k}^F - \nabla_{i,k}^F\|^2 + \lambda_k^2 \|\tilde{\nabla}_{i,k}^f - \nabla_{i,k}^f\|^2 | \mathcal{F}_k] \leq \frac{2\nu_k^2}{m}. \end{aligned} \quad (28)$$

Employing (27) and (28), we have

$$\begin{aligned} &\mathbb{E}[\langle \mathbf{W}\mathbf{y}_k - \mathbf{1}\bar{y}_k, -\tilde{\mathbf{G}}_k + \mathbf{G}_k \rangle | \mathcal{F}_k] \\ &= \sum_{i=1}^m \mathbb{E}[\langle \sum_{j=1}^m W_{ij} y_{j,k} - \bar{y}_k, -\tilde{\nabla}_{i,k}^{\lambda_k} + \nabla_{i,k}^{\lambda_k} \rangle | \mathcal{F}_k] \\ &\leq \sum_{i=1}^m \frac{2\nu_k^2}{m} = 2\nu_k^2. \end{aligned}$$

This implies that (26) holds.

**Claim 6.** The following holds for any  $\zeta > 0$

$$\begin{aligned} &2\langle \mathbf{W}\mathbf{y}_k - \mathbf{1}\bar{y}_k, \mathbf{G}_{k+1} - \mathbf{G}_k \rangle \\ &\leq \zeta \rho_W^2 \|\mathbf{y}_k - \mathbf{1}\bar{y}_k\|^2 + \zeta^{-1} \|\mathbf{G}_{k+1} - \mathbf{G}_k\|^2. \end{aligned} \quad (29)$$

**Proof of Claim 6.** This relation is directly implied by the properties of the Frobenius inner product and using Lemma 7.

From Claim 1 to 6 and relation (20), we have

$$\begin{aligned} &\mathbb{E}[\|\mathbf{y}_{k+1} - \mathbf{1}\bar{y}_{k+1}\|^2 | \mathcal{F}_k] \\ &\leq \rho_W^2 \mathbb{E}[\|\mathbf{y}_k - \mathbf{1}\bar{y}_k\|^2 | \mathcal{F}_k] \\ &\quad + (1 + \zeta^{-1}) 4L_k^2 (6\Lambda_k^2 L_k^2 + \|\mathbf{W} - \mathbf{I}\|^2 + 1.5\gamma_k^2) \|\mathbf{x}_k - \mathbf{1}\bar{x}_k\|^2 \\ &\quad + ((1 + \zeta^{-1}) (4L_k^2 \gamma_k^2) + \zeta) \rho_W^2 \mathbb{E}[\|\mathbf{y}_k - \mathbf{1}\bar{y}_k\|^2 | \mathcal{F}_k] \\ &\quad + (1 + \zeta^{-1}) 6mL_k^2 (4\Lambda_k^2 + L_k^2 \gamma_k^2) \|\bar{x}_k - x_{\lambda_k}^*\|^2 \\ &\quad + (1 + \zeta^{-1}) 12L_k^2 \gamma_k^2 \nu_k^2 + (1 + \zeta^{-1}) 6\Lambda_k^2 B_F^2 \\ &\quad + 2m(\nu_{k+1}^2 + \nu_k^2) + 4m\gamma_k L_{k+1} \nu_k^2 + 4\nu_k^2. \end{aligned}$$

Also, in view of Lemma 2, we have

$$\begin{aligned} \|\bar{x}_k - x_{\lambda_k}^*\|^2 &\leq 2\|\bar{x}_k - x_{\lambda_{k-1}}^*\|^2 + 2\|x_{\lambda_{k-1}}^* - x_{\lambda_k}^*\|^2 \\ &\leq 2\|\bar{x}_k - x_{\lambda_{k-1}}^*\|^2 + \frac{2M^2}{\mu_f^2} \Lambda_{k-1}^2. \end{aligned}$$

The preceding two relations give the third recursive inequality. ■

Next, we present convergence guarantees for Algorithm 2 in resolving the stochastic NE seeking problem (2).

**Theorem 2 (Convergence statements for (2)):** Consider Algorithm 2. Let Assumptions 1, 4, and 5 hold. Let us define  $e_{1,k} \triangleq \mathbb{E}[\|\bar{x}_k - x_{\lambda_{k-1}}^*\|^2]$ ,  $e_{2,k} \triangleq \mathbb{E}[\|\mathbf{x}_k - \mathbf{1}\bar{x}_k\|^2]$ , and  $e_{3,k} \triangleq \mathbb{E}[\|\mathbf{y}_k - \mathbf{1}\bar{y}_k\|^2]$  for  $k \geq 0$ . Suppose  $\gamma_k := \frac{\gamma}{(k+\Gamma)^a}$  and  $\lambda_k := \frac{\lambda}{(k+\Gamma)^b}$  with  $\gamma, \lambda, a, b > 0$ ,  $\Gamma \geq 1$ ,  $3a + b < 2$ , and

$$\Gamma \geq \max \left\{ \frac{a+b}{2} \sqrt{\frac{\mu_f \gamma \lambda}{2}}, \sqrt{\frac{\lambda L_f}{L_F}}, \sqrt{2\gamma L_F}, \sqrt{4\gamma L_F \zeta^{-1} \sqrt{1+\zeta}} \right\},$$

where  $0 < \zeta \leq \frac{1-\rho_W^2}{4\rho_W^2}$ . The following results hold.

(a) [Recursive error bounds] There exist scalars  $\theta_t > 0$  for  $t = 1, \dots, 9$  with  $\theta_4 < 1$  and  $\theta_6 < 1$  such that for all  $k \geq 0$ ,

$$\begin{aligned} e_{1,k+1} &\leq (1 - \theta_1 \gamma_k \lambda_k) e_{1,k} + \theta_2 e_{2,k} + \theta_3 \gamma_k^2, \\ e_{2,k+1} &\leq (1 - \theta_4) e_{2,k} + \theta_5 \gamma_k^2 e_{3,k}, \\ e_{3,k+1} &\leq (1 - \theta_6) e_{3,k} + \theta_7 e_{1,k} + \theta_8 e_{2,k} + \theta_9, \end{aligned}$$

where

$$\begin{aligned} \theta_1 &:= 0.5\mu_f, \quad \theta_2 := \frac{L_0^2 \gamma (1+\mu_f \lambda \gamma)}{m \lambda_k \mu_f}, \\ \theta_3 &:= \frac{2(\nu_F^2 \Gamma^{2b} + \lambda^2 \nu_f^2)}{m \Gamma^{2b}} + \frac{2M^2}{\mu_f^3 \lambda \gamma^3 \Gamma^{2-3a-b}}, \\ \theta_4 &:= \frac{1-\rho_W^2}{2}, \quad \theta_5 := \frac{(1+\rho_W^2)\rho_W^2}{1-\rho_W^2}, \\ \theta_6 &:= 1 - ((1+\zeta^{-1})(4L_k^2 \gamma_k^2) + \zeta + 1) \rho_W^2, \\ \theta_7 &:= (1+\zeta^{-1})12mL_0^2 \left( \frac{4}{\Gamma^2} + 1 \right), \\ \theta_8 &:= (1+\zeta^{-1})4L_0^2 \left( 6\frac{1}{\Gamma^2} L_0^2 + \|\mathbf{W} - \mathbf{I}\|^2 + 1.5\frac{\gamma^2}{\Gamma^2} \right), \\ \theta_9 &:= (1+\zeta^{-1})12(\nu_F^2 + \frac{\lambda^2}{\Gamma^2} \nu_f^2) + (1+\zeta^{-1})6\frac{1}{\Gamma^2} B_F^2 \\ &\quad + 4m(\nu_F^2 + \frac{\lambda^2}{\Gamma^2} \nu_f^2) + 4m(\nu_F^2 + \frac{\lambda^2}{\Gamma^2} \nu_f^2) \\ &\quad + 4(\nu_F^2 + \frac{\lambda^2}{\Gamma^2} \nu_f^2) + (1+\zeta^{-1})12mL_0^2 \frac{M^2}{\mu_f^2} \left( \frac{4}{\Gamma^2} + 1 \right) \frac{1}{\Gamma^2}. \end{aligned}$$

(b) [Non-asymptotic error bounds] Let  $\gamma\lambda > \frac{1}{\theta_1}$ . Let us define

$$\hat{e}_1 := \Gamma^{a-b} e_{1,0} n_1, \quad \hat{e}_2 := \Gamma^{2a} e_{2,0} n_2, \quad \hat{e}_3 := \Gamma^{2a} \frac{3\theta_9}{\theta_6} n_3. \quad (30)$$

where  $n_1, n_2, n_3 > 0$  are given as

$$n_1 := \frac{2C_2}{C_3 \Gamma^{a-b} - 2C_1 C_4 C_5}, \quad n_2 := \frac{2C_4 C_5}{\Gamma^{2a}} n_1, \quad n_3 := \frac{C_5}{\Gamma^{2a}} n_1.$$

where

$$\begin{aligned} C_1 &\triangleq \theta_2 e_{2,0}, \quad C_2 \triangleq \theta_3 \gamma^2, \quad C_3 \triangleq (\gamma\lambda\theta_1 - 1) e_{1,0} \\ C_4 &\triangleq \frac{6\theta_9 \theta_5 \gamma^2}{e_{2,0} \theta_4 \theta_6}, \quad C_5 \triangleq \frac{\theta_7 e_{1,0}}{\theta_9}, \quad C_6 \triangleq \frac{\theta_8 e_{2,0}}{\theta_9}. \end{aligned}$$

Then, if

$$\Gamma \geq \max \left\{ \sqrt[2a]{2C_4 C_6}, \sqrt[2a]{\frac{2C_1 C_4 C_5}{C_3}}, \frac{1}{1 - \sqrt[2a]{1-0.5\theta_4}} - 1 \right\},$$

the following results hold for all  $k \geq 0$ .

$$e_{1,k} \leq \frac{\hat{e}_1}{(k+\Gamma)^{a-b}}, \quad e_{2,k} \leq \frac{\hat{e}_2}{(k+\Gamma)^{2a}}, \quad e_{3,k} \leq \hat{e}_3. \quad (31)$$

(c) [Asymptotic convergence] The sequence of the averaged iterate,  $\{\bar{x}_k\}$ , converges to the unique optimal solution of problem (2) in a mean-squared sense, i.e., we have

$$\lim_{k \rightarrow \infty} \mathbb{E}[\|\bar{x}_{k+1} - x^*\|^2] = 0.$$

*Proof:* (a) The three inequalities in part (a) follow from Proposition 3 after invoking Lemma 1. To complete the proof, we elaborate on the condition required for  $\Gamma$ . Note that  $\Gamma \geq \max \left\{ \sqrt[2a]{\frac{\lambda L_f}{L_F}}, \sqrt[2a]{2\gamma L_F} \right\}$  implies that  $\gamma L_0 \leq 1$  and thus,  $\gamma_k \leq \frac{1}{L_k}$  for all  $k \geq 0$  implying that the condition in Proposition 3 holds. Also, the condition  $\Gamma \geq \max \left\{ \sqrt[2a]{\frac{\lambda L_f}{L_F}}, \sqrt[2a]{4\gamma L_F \zeta^{-1} \sqrt{1+\zeta}} \right\}$  together with  $0 < \zeta \leq \frac{1-\rho_W^2}{4\rho_W^2}$  imply that  $((1+\zeta^{-1})(4L_k^2 \gamma_k^2) + \zeta + 1) \rho_W^2 < 1$  for all  $k \geq 0$ . Therefore,  $\theta_6 < 1$ . We note that the condition  $3a+b < 2$  is used in obtaining the first inequality, where we bound  $\frac{2M^2 \Lambda_{k-1}^2}{\mu_f^3 \lambda_k \gamma_k}$  by  $\frac{2M^2 \gamma_k^2}{\mu_f^3 \lambda \gamma^3 \Gamma^{2-3a-b}}$ , where we invoke  $\Lambda_{k-1} \leq \frac{1}{k+\Gamma}$  given in Lemma 1.

(b) First, we observe that each one of the scalars  $n_1, n_2$ , and  $n_3$  is proportional to the value of  $\theta_3$ . We also observe that  $C_i$  for  $i \neq 2$  are independent of the value of  $\theta_3$ . In view of these remarks, without loss of generality, let us assume that  $\theta_3$  is sufficiently large such that  $n_i \geq 1$  for  $i = 1, 2, 3$  and that  $e_{3,0} \leq \Gamma^{2a} \frac{3\theta_9}{\theta_6} n_3$ . To show the three inequalities in part (b), we use induction on  $k \geq 0$ . For  $k := 0$ , from (30), we have

$$\begin{aligned} e_{1,0} &= \frac{\hat{e}_1}{\Gamma^{a-b} n_1} \leq \frac{\hat{e}_1}{\Gamma^{a-b}} = \frac{\hat{e}_1}{(0+\Gamma)^{a-b}}, \\ e_{2,0} &= \frac{\hat{e}_2}{\Gamma^{2a} n_2} \leq \frac{\hat{e}_2}{\Gamma^{2a}} = \frac{\hat{e}_2}{(0+\Gamma)^{2a}}, \\ e_{3,0} &\leq \Gamma^{2a} \frac{3\theta_9}{\theta_6} n_3 = \hat{e}_3. \end{aligned}$$

These imply that the hypothesis statement holds for  $k := 0$ . Suppose the three inequalities in (31) hold for some  $k \geq 0$ . We show that they hold for  $k+1$  in three steps as follows.

*Step 1:* From the definition of  $n_1$  we have  $C_2 \leq (C_3 \Gamma^{a-b} - 2C_1 C_4 C_5) n_1$ . Rearranging the terms and invoking the definition of  $n_2$ , we have  $C_1 \Gamma^{2a} n_2 + C_2 \leq C_3 \Gamma^{a-b} n_1$ . Substituting the values of  $C_1, C_2$ , and  $C_3$ , we obtain

$$\theta_2 e_{2,0} \Gamma^{2a} n_2 + \theta_3 \gamma^2 \leq (\gamma\lambda\theta_1 - 1) e_{1,0} \Gamma^{a-b} n_1.$$

Using the definition of  $\hat{e}_1$  and  $\hat{e}_2$  we have  $\theta_2 \hat{e}_2 + \theta_3 \gamma^2 \leq (\gamma\lambda\theta_1 - 1) \hat{e}_1$ . This implies that  $\hat{e}_1 \leq \gamma\lambda\theta_1 \hat{e}_1 - \theta_2 \hat{e}_2 - \theta_3 \gamma^2$ . We have

$$\frac{\hat{e}_1}{(k+\Gamma)^{2a}} \leq \frac{\gamma\lambda\theta_1 \hat{e}_1}{(k+\Gamma)^{2a}} - \frac{\theta_2 \hat{e}_2}{(k+\Gamma)^{2a}} - \frac{\theta_3 \gamma^2}{(k+\Gamma)^{2a}}.$$

Note that from  $\Gamma \geq 1$  we have that for all  $k \geq 0$

$$\begin{aligned} \frac{1}{(k+\Gamma)^{2a}} &= \frac{1}{(k+\Gamma)^{a-b} (k+\Gamma)^{a+b}} \geq \frac{1}{(k+\Gamma+1)^{a-b} (k+\Gamma)} \\ &\geq \frac{1}{(k+\Gamma+1)^{a-b}} \left( \left(1 + \frac{1}{k+\Gamma}\right)^{a-b} - 1 \right) \\ &= \frac{1}{(k+\Gamma)^{a-b}} - \frac{1}{(k+\Gamma+1)^{a-b}}. \end{aligned}$$

From the preceding relations we have

$$\begin{aligned} \frac{\hat{e}_1}{(k+\Gamma)^{a-b}} - \frac{\hat{e}_1}{(k+\Gamma+1)^{a-b}} &\leq \frac{\gamma\lambda\theta_1\hat{e}_1}{(k+\Gamma)^{a+b}(k+\Gamma)^{a-b}} - \frac{\theta_2\hat{e}_2}{(k+\Gamma)^{2a}} \\ &- \frac{\theta_3\gamma^2}{(k+\Gamma)^{2a}} \leq \frac{\gamma_k\lambda_k\theta_1\hat{e}_1}{(k+\Gamma)^{a-b}} - \theta_2\hat{e}_{2,k} - \theta_3\gamma_k^2, \end{aligned}$$

where we used the hypothesis statement and the update rules of  $\gamma_k$  and  $\lambda_k$ . We obtain

$$(1 - \theta_1\gamma_k\lambda_k) \frac{\hat{e}_1}{(k+\Gamma)^{a-b}} + \theta_2\hat{e}_{2,k} + \theta_3\gamma_k^2 \leq \frac{\hat{e}_1}{(k+\Gamma+1)^{a-b}}.$$

From  $\Gamma \geq \sqrt[a+b]{\theta_1\gamma\lambda}$ , we have  $1 - \theta_1\gamma_k\lambda_k \geq 0$ . Thus we have

$$\mathbf{e}_{1,k+1} \leq (1 - \theta_1\gamma_k\lambda_k)\mathbf{e}_{1,k} + \theta_2\mathbf{e}_{2,k} + \theta_3\gamma_k^2 \leq \frac{\hat{e}_1}{(k+\Gamma+1)^{a-b}}.$$

This shows that the first inequality in (31) holds for  $k+1$ .

*Step 2:* From the definition of  $n_1$  and  $n_2$  we have  $C_4n_3 \leq n_2$ .

We obtain  $(\frac{6\theta_9\theta_5\gamma^2}{\mathbf{e}_{2,0}\theta_4\theta_6})n_3 \leq n_2$ . From (30) we have

$$\begin{aligned} \left(\frac{6\theta_9\theta_5\gamma^2}{\mathbf{e}_{2,0}\theta_4\theta_6}\right) \frac{\theta_6\hat{e}_3}{\Gamma^{2a}3\theta_9} &\leq \frac{\hat{e}_2}{\Gamma^{2a}\mathbf{e}_{2,0}} \Rightarrow \hat{e}_3 \leq \frac{\theta_4}{2\theta_5\gamma^2}\hat{e}_2 \\ &\Rightarrow \frac{\theta_4}{2}\hat{e}_2 \leq \theta_4\hat{e}_2 - \theta_5\gamma^2\hat{e}_3. \end{aligned}$$

From  $\Gamma \geq \frac{1}{1-2\sqrt[2]{1-0.5\theta_4}} - 1$  we have for all  $k \geq 0$

$$\begin{aligned} \frac{\theta_4}{2} &\geq 1 - \left(1 - \frac{1}{\Gamma+1}\right)^{2a} \geq 1 - \left(1 - \frac{1}{k+\Gamma+1}\right)^{2a} \\ \Rightarrow \frac{\theta_4}{2} &\geq (k+\Gamma)^{2a} \left(\frac{1}{(k+\Gamma)^{2a}} - \frac{1}{(k+\Gamma+1)^{2a}}\right). \end{aligned}$$

From the preceding two relations we obtain

$$\left(\frac{1}{(k+\Gamma)^{2a}} - \frac{1}{(k+\Gamma+1)^{2a}}\right)\hat{e}_2 \leq \frac{\theta_4\hat{e}_2}{(k+\Gamma)^{2a}} - \frac{\theta_5\gamma^2\hat{e}_3}{(k+\Gamma)^{2a}}.$$

This implies that  $(1 - \theta_4)\mathbf{e}_{2,k} + \theta_5\gamma_k^2\mathbf{e}_{3,k} \leq \frac{\hat{e}_2}{(k+\Gamma+1)^{2a}}$ . Thus, the second inequality in (31) holds for  $k+1$ .

*Step 3:* From the definition of  $\hat{e}_1$ ,  $\hat{e}_3$ ,  $n_1$ ,  $n_3$ , and  $C_5$  we have

$$\frac{\hat{e}_3}{\hat{e}_1} = \frac{\Gamma^{2a} \frac{3\theta_9}{\theta_6} n_3}{\Gamma^{a-b}\mathbf{e}_{1,0}n_1} = \frac{\Gamma^{2a} \frac{3\theta_9}{\theta_6} C_5}{\Gamma^{a-b}\mathbf{e}_{1,0}\Gamma^{2a}} = \frac{3\theta_7}{\theta_6\Gamma^{a-b}}. \quad (32)$$

From the definition of  $\hat{e}_2$ ,  $\hat{e}_3$ ,  $n_2$ ,  $n_3$ , and  $C_6$  we have

$$\frac{\hat{e}_3}{\hat{e}_2} = \frac{\Gamma^{2a} \frac{3\theta_9}{\theta_6} n_3}{\Gamma^{2a}\mathbf{e}_{2,0}n_2} = \frac{\Gamma^{2a} \frac{3\theta_9}{\theta_6}}{2\Gamma^{2a}\mathbf{e}_{2,0}C_4} = \frac{3\theta_8}{\theta_6(2C_4C_6)} \geq \frac{3\theta_8}{\theta_6\Gamma^{2a}}, \quad (33)$$

where the last inequality is due to  $\Gamma \geq \sqrt[2]{2C_4C_6}$ . Also recalling  $n_3 \geq 1$  and  $\Gamma \geq 1$ , from the definition of  $\hat{e}_3$  we have  $\hat{e}_3 \geq \frac{3\theta_9}{\theta_6}$ . From this relation, (32), and (33) we have  $\frac{\theta_7}{\Gamma^{a-b}}\hat{e}_1 + \frac{\theta_8}{\Gamma^{2a}}\hat{e}_2 + \theta_9 \leq \theta_6\hat{e}_3$ . This implies that

$$(1 - \theta_6)\hat{e}_3 + \theta_7 \frac{\hat{e}_1}{(k+\Gamma)^{a-b}} + \theta_8 \frac{\hat{e}_2}{(k+\Gamma)^{2a}} + \theta_9 \leq \hat{e}_3.$$

From the hypothesis statement we conclude that  $\mathbf{e}_{3,k+1} \leq \hat{e}_3$ . Therefore, from the preceding three steps, the three inequalities in (31) hold for  $k+1$ . Hence the proof is completed.

(c) Consider the term  $\mathbb{E}[\|\bar{x}_{k+1} - x^*\|^2]$ . Adding and subtracting  $x_{\lambda_k}^*$  inside the norm, we have

$$\begin{aligned} \mathbb{E}[\|\bar{x}_{k+1} - x^*\|^2] &= \mathbb{E}[\|\bar{x}_{k+1} - x_{\lambda_k}^* + x_{\lambda_k}^* - x^*\|^2] \\ &\leq 2\mathbb{E}[\|\bar{x}_{k+1} - x_{\lambda_k}^*\|^2] + 2\|x_{\lambda_k}^* - x^*\|^2 \\ &= 2\mathbf{e}_{1,k+1} + 2\|x_{\lambda_k}^* - x^*\|^2, \end{aligned} \quad (34)$$

noting that  $x_{\lambda_k}^*$  is a deterministic term. Invoking (31), we get

$$\mathbb{E}[\|\bar{x}_{k+1} - x^*\|^2] \leq \frac{2\hat{e}_1}{(k+1+\Gamma)^{a-b}} + 2\|x_{\lambda_k}^* - x^*\|^2.$$

Taking limits on both sides when  $k \rightarrow \infty$ , and invoking  $a > b$  and Lemma 2, we obtain  $\lim_{k \rightarrow \infty} \mathbb{E}[\|\bar{x}_{k+1} - x^*\|^2] = 0$ . ■

*Remark 4:* Note that Theorem 2(c) provides an asymptotic convergence guarantee for the averaged iterate generated by Algorithm 2 for addressing problem (2). Further, Theorem 2(b) provides a non-asymptotic bound for the consensus error  $\mathbb{E}[\|\mathbf{x}_k - 1\bar{x}_k\|^2]$ , which is of the order  $1/k^{2a}$ . Notably, the given conditions on parameters  $a$  and  $b$  in Theorem 2 imply that  $a \in (0, \frac{2}{3})$  and  $b \in (0, \min\{0.5, a\})$ . Consequently, with a sufficiently small  $b$ ,  $\mathbb{E}[\|\mathbf{x}_k - 1\bar{x}_k\|^2]$  may converge nearly as fast as  $1/k^{4/3}$ . However, note that a value of  $b$  that is too small will cause  $\lambda_k$  to diminish too slowly. Therefore, the convergence of the Tikhonov trajectory  $x_{\lambda_k}^*$  to the unique solution  $x^*$  will be too slow. This, in turn, will result in a slow global convergence of the term  $\mathbb{E}[\|\bar{x}_{k+1} - x^*\|^2]$  to zero. These observations indeed suggest that in implementations,  $b$  should be chosen within the range  $(0, \min\{a, 0.5\})$  but not too small. This trade-off will be numerically explored in Section IV.

#### IV. NUMERICAL RESULTS

We present preliminary numerical experiments for computing the optimal NE of a Cournot game considered in [36]. To this end, we demonstrate the performance of Algorithms 1 and 2 under different algorithm parameters, network sizes, and network topologies. Consider a game among  $m$  players such that each player  $i \in [m]$  seeks to minimize the function  $J_i(x_i, x_{-i}) := h_i(x_i) + l_i(x_{-i})x_i$ , where  $h_i(x_i) := 0.5a_ix_i^2 + b_ix_i$ ,  $l_i(x_{-i}) := \sum_{j \neq i} c_{ij}x_j$ , and  $a_i \geq 0$ . Each player  $i$  is associated with a box constraint set  $X_i := [0, c_i^{\text{up}}]$ , where  $c_i^{\text{up}} > 0$ . Succinctly, the game is captured by the collection of optimization problems  $\min_{x_i \in \mathbb{R}} h_i(x_i) + l_i(x_{-i})x_i + \mathbb{I}_{X_i}(x_i)$ , for  $i \in [m]$ , where  $\mathbb{I}_{X_i}(x)$  denotes the indicator function of the set  $X_i$ . To contend with the nonsmoothness, we consider a Moreau smoothed [30] variant of each player's optimization problem. Consider Moreau smoothing of the indicator function of a nonempty, closed, and convex set  $Y$ . Recall that  $\mathbb{I}_Y^{\eta}(z) = \frac{1}{2\eta} \text{dist}^2(z, Y)$  (cf. [3]). Notably,  $\mathbb{I}_Y^{\eta}(z)$  admits a gradient, defined as  $\nabla_z \mathbb{I}_Y^{\eta}(z) = \frac{1}{\eta}(z - \Pi_Y(z))$ . Consequently, player  $i$ 's smoothed problem is

$$\min_{x_i \in \mathbb{R}} f_i(x) \triangleq h_i(x_i) + l_i(x_{-i})x_i + \frac{\text{dist}^2(x_i, X_i)}{2\eta}, \quad (\mathbf{P}_i(x_{-i}))$$

where  $\eta > 0$  is the smoothing parameter and  $\Pi_{X_i}(\bullet)$  denotes the Euclidean projection onto the set  $X_i$ . Next, we reformulate the smoothed game  $(\mathbf{P}_i(x_{-i}))$  as a distributed VI problem. We let  $x = [x_i]_{i=1}^m$  denote the tuple of players decisions.

*Definition 3:* Let  $\underline{C} \in \mathbb{R}^{m \times m}$  be defined as  $\underline{C}_{ii} = 0.5a_i$  and  $\underline{C}_{ij} = c_{ij}$  for all  $i \neq j$ . Let us also define  $\bar{C} \triangleq \underline{C} + 0.5 \text{diag}(\bar{a})$ ,  $\bar{a} = [a_i]_{i=1}^m$ , and  $\bar{b} = [b_i]_{i=1}^m$ .

*Lemma 9:* Let us define  $F(x) \triangleq \sum_{i=1}^m F_i(x)$ , where  $F_i(x) \triangleq \left(a_ix_i + b_i + \sum_{j \neq i} c_{ij}x_j + \frac{1}{\eta}(x_i - \Pi_{X_i}(x_i))\right) \mathbf{e}_i$ , where  $\mathbf{e}_i \in \mathbb{R}^m$  denotes a unit vector where the  $i$ th element is 1, and all other elements are 0. Then, the following hold.

(i) The set of all Nash equilibria to  $(\mathbf{P}_i(x_{-i}))$ , for  $i \in [m]$ , is equal to  $\text{SOL}(\mathbb{R}^m, F)$ .

(ii) If  $\frac{1}{2}(\bar{C} + \bar{C}^{\top})$  is positive semidefinite, then  $F$  is monotone.

*Proof:* (i) To show this result we invoke [13, Prop. 1.4.2]. First, note that  $f_i$  is continuously differentiable and convex in  $x_i$ , for all  $i \in [m]$ . This is because the smoothness of

$f_i$  follows from its definition, and by invoking the smoothness of the Moreau smoothed component  $\frac{1}{2\eta}\text{dist}^2(x_i, X_i)$ , as mentioned earlier. Also, the convexity of  $f_i(\bullet, x_{-i})$ , given any  $x_{-i}$ , follows from the convexity of the Moreau smoothed component (cf. [3]). To complete the proof, we need to show that  $F(x) = [\nabla_{x_1} f_1(x); \dots; \nabla_{x_m} f_m(x)]$ . Note that we have  $F_i(x) = \nabla_{x_i} f_i(x) \mathbf{e}_i$ . Thus, we obtain  $F(x) = \sum_{i=1}^m F_i(x) = \sum_{i=1}^m \nabla_{x_i} f_i(x) \mathbf{e}_i = [\nabla_{x_i} f_i(x)]_{i=1}^m$ . Hence, the statement in part (i) follows from [13, Prop. 1.4.2].

(ii) From the proof in part (ii) and the definition of  $\bar{C}$ , we have  $F(x) = [\nabla_{x_i} f_i(x)]_{i=1}^m = \bar{C}x + \bar{b} + \frac{1}{\eta}[x_i - \Pi_{X_i}(x_i)]_{i=1}^m$ . Thus, for any  $x, y \in \mathbb{R}^m$ , we may write

$$\begin{aligned} (F(x) - F(y))^\top (x - y) &= (x - y)^\top \bar{C}(x - y) \\ &+ \frac{1}{\eta} \sum_{i=1}^m (x_i - y_i)^2 - \frac{1}{\eta} \sum_{i=1}^m (\Pi_{X_i}(x_i) - \Pi_{X_i}(y_i))(x_i - y_i) \\ &\geq \frac{1}{2}(x - y)^\top (\bar{C} + \bar{C}^\top)(x - y) \\ &+ \frac{1}{\eta} \sum_{i=1}^m (x_i - y_i)^2 - \frac{1}{\eta} \sum_{i=1}^m |\Pi_{X_i}(x_i) - \Pi_{X_i}(y_i)| |x_i - y_i|, \end{aligned}$$

where we used  $(x - y)^\top \bar{C}(x - y) = (x - y)^\top \bar{C}^\top(x - y)$  and the Cauchy-Schwarz inequality. From the nonexpansivity of the Euclidean projection, we have  $|\Pi_{X_i}(x_i) - \Pi_{X_i}(y_i)| \leq |x_i - y_i|$ . From the two preceding equations and  $\frac{1}{2}(\bar{C} + \bar{C}^\top) \succeq \mathbf{0}_{m \times m}$ , for any  $x, y \in \mathbb{R}^m$  we have  $(F(x) - F(y))^\top (x - y) \geq 0$ . This implies that  $F$  is a monotone mapping.  $\blacksquare$

**Remark 5:** Lemma 9 provides us with a monotone VI problem,  $\text{VI}(\mathbb{R}^m, F)$ , whose solution set captures all the NEs that the game  $(P_i(x_{-i}))$ , for  $i \in [m]$ , may admit. Importantly, this is a distributed VI problem, of the form  $\text{VI}(\mathbb{R}^m, \sum_{i=1}^m F_i)$ , in that  $F_i$  is known locally by player  $i$ .

To select among the NEs of the game  $(P_i(x_{-i}))$ , we consider a *utilitarian* approach, where players seek to find an NE that minimizes  $f(x) \triangleq \sum_{i=1}^m f_i(x)$ , where  $f_i$  denotes the local loss function in  $(P_i(x_{-i}))$ . A question is whether  $f$  is a convex function. This is addressed next.

**Lemma 10:** Consider the global welfare loss function  $f$ . If  $\frac{1}{2}(\underline{C} + \underline{C}^\top)$  is positive semidefinite, then  $f$  is convex in  $x$ .

*Proof:* From the definition of  $f$ , we obtain

$$\begin{aligned} f(x) &= \sum_{i=1}^m \left( \frac{a_i x_i^2}{2} + b_i x_i + \sum_{j \neq i} c_{ij} x_j x_i + \frac{1}{2\eta} \text{dist}^2(x_i, X_i) \right) \\ &= x^\top \underline{C}x + \bar{b}^\top x + \frac{1}{2\eta} \sum_{i=1}^m \text{dist}^2(x_i, X_i) \\ &= \frac{1}{2} x^\top (\underline{C} + \underline{C}^\top) x + \bar{b}^\top x + \frac{1}{2\eta} \sum_{i=1}^m \text{dist}^2(x_i, X_i). \end{aligned}$$

Notably,  $\text{dist}^2(x_i, X_i)$  is convex in  $x_i$ , for all  $i \in [m]$ . Thus,  $\frac{1}{2\eta} \sum_{i=1}^m \text{dist}^2(x_i, X_i)$  is convex in  $x$ . The convexity of  $f$  follows from the assumption that  $\frac{1}{2}(\underline{C} + \underline{C}^\top) \succeq \mathbf{0}_{m \times m}$ .  $\blacksquare$

**Parameter settings.** To ensure that  $F$  is monotone, for each setting of the implementations, we randomly generate a rank-deficient positive semidefinite matrix  $C$ , where we set  $a_i := C_{ii}$  for all  $i \in [m]$ , and  $\bar{C}_{ij} := C_{ij}$  for  $j \neq i$ . In terms of the convexity of  $f$ , we note from Lemmas 9 and 10 that the sufficient condition to guarantee convexity of the global function  $f$  is different than that to guarantee monotonicity of the mapping  $F$ . Indeed, even if  $\frac{1}{2}(\bar{C} + \bar{C}^\top) \succeq \mathbf{0}$ , it is not necessarily guaranteed to have  $\frac{1}{2}(\underline{C} + \underline{C}^\top) \succeq \mathbf{0}$ . To validate the theoretical results, we employ a regularization for the welfare loss function  $f$  as  $\sum_{i=1}^m f_i(x) + \frac{\theta}{2} \|x\|_2^2$ , where  $\theta > 0$ , where

$\theta := 10^{-5} + \max\{0, -\lambda_{\min}\}$  such that  $\lambda_{\min}$  is the minimum eigenvalue of  $\frac{1}{2}(\underline{C} + \underline{C}^\top)$ , where following Definition 3, we set  $\underline{C} := \bar{C} - 0.5\text{diag}(\bar{a})$ . Thus, in view of Lemma 9, the regularized global function is strongly convex and the optimal NE is unique. Throughout, we use  $\eta := 0.1$  and generate each of the parameters  $c_i^{\text{up}}$  for the box constraints, for  $i \in [m]$ , uniformly at random from the interval  $[50, 100]$ .

### A. Optimal NE seeking over directed networks

We generate parameters  $b_i$  for  $i \in [m]$  normally at random with a mean of 0 and a variance of 10. We test the performance of Algorithm 1 on two network settings: (i) directed star graphs with  $m = 10$  nodes; (ii) random digraphs with  $m = 100$  nodes. In (ii), this is done by generating a random tree and then adding edges at random until we reach the desired number of edges, that is  $\lfloor 100 \ln 100 \rfloor = 460$  edges in our setting. We then use a modification of the *max-degree weights* heuristic in [38] by setting  $\alpha := 1/(2d_{\max})$ . Notably, this approach satisfies  $\mathcal{R}_{\mathbf{R}} \cap \mathcal{R}_{\mathbf{C}^\top} \neq \emptyset$ . For each setting, we compare the effect of three choices of  $(a, b) \in \{(0.5, 0.3), (0.6, 0.25), (0.675, 0.2)\}$  satisfying Assumption 3. We use  $\|F(\bar{x}_k)\|_2$ ,  $\|\bar{x}_{k+1} - \bar{x}_k\|_2$  and  $\|\bar{x}_k - \mathbf{1}\bar{x}_k\|$  as metric of the lower-level error, the upper-level error, and the consensus error, respectively. The implementation results are shown in Figure 1.

**Insights.** We observe in both the star and random graph settings that all the three metrics appear to be converging. The performance of the lower-level error metric for Algorithm 1 is best for  $(a, b) = (0.5, 0.3)$  and worst for  $(a, b) = (0.675, 0.2)$ . This is reversed for the upper-level error metric. These observations appear to be reasonable and are consistent with our theoretical findings. This is mainly because a larger  $b$  implies that the regularization parameter  $\lambda_k = \frac{\lambda}{(k+\Gamma)^b}$  decreases faster. As a result, the information of  $\nabla f_i$  is multiplied by smaller values, emphasizing less on the upper-level local objectives and more on the information of lower-level mappings  $F_i$ .

### B. Stochastic setting over undirected networks

We implement Algorithm 2 for solving problem (2), where we consider a stochastic setting of  $(P_i(x_{-i}))$  as follows. Let us define  $f_i(x, \xi_i) = \frac{1}{2} a_i x_i^2 + b_i(\xi_i) x_i + (\sum_{j \neq i} c_{ij} x_j) x_i + \frac{1}{2\eta} \|x_i - \Pi_{X_i}(x_i)\|^2$  and  $F_i(x, \xi_i) = \left( a_i x_i + b_i(\xi_i) + \sum_{j \neq i} c_{ij} x_j + \frac{1}{\eta} (x_i - \Pi_{X_i}(x_i)) \right) \mathbf{e}_i$ . Here,  $b_i(\xi_i)$  is sampled iteratively in the method by each player, uniformly at random from  $[1, 10]$ , for all players. For the communication network among the players, we consider the Petersen graph with 10 nodes and an undirected random graph with 100 nodes. We compare the effect of  $a$  and  $b$  with three choices:  $(a, b) \in \{(0.5, 0.4), (0.55, 0.3), (0.6, 0.175)\}$ , satisfying the conditions in Theorem 2. We run each experiment for 10 sample paths and report the mean of the errors. The results are shown in Figure 2.

**Insights.** In both the network settings, we again observe that all the three metrics appear to be converging in a mean sense. Further, in a similar fashion to what we observed in the previous experiment, we again see that the setting with the largest value of  $b$  performs the best in terms of the lower-level metric, while



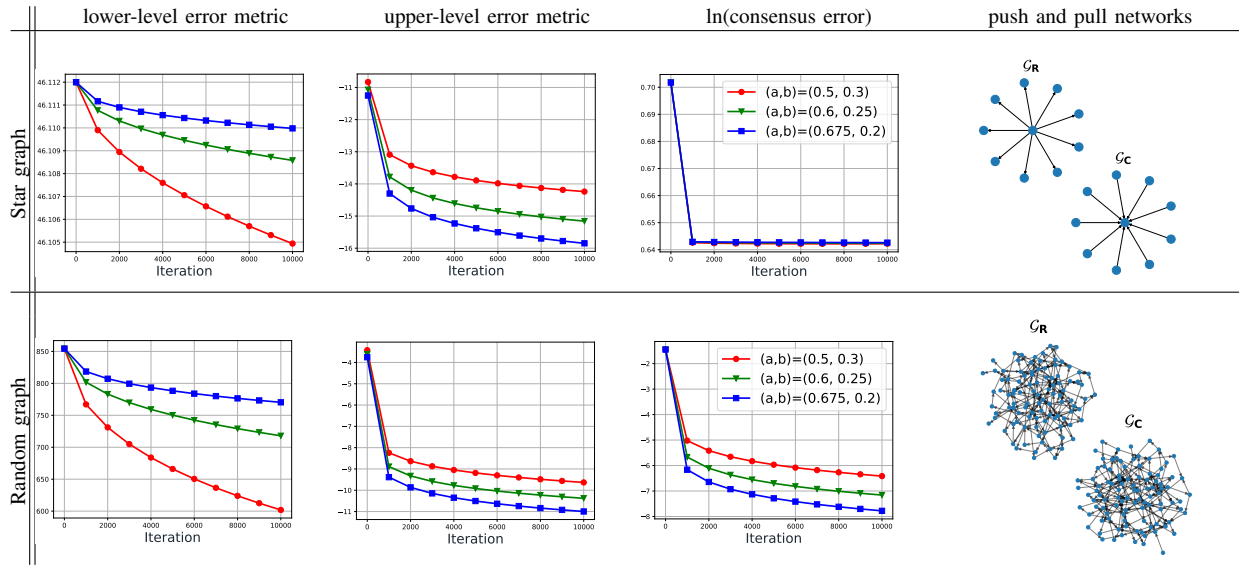


Fig. 1: Performance of Algorithm 1 on directed star graphs with 10 nodes and random digraphs with 100 nodes.

the setting with the smallest value of  $b$  performs the best in terms of the upper-level metric.

## V. CONCLUDING REMARKS

Traditional approaches to optimal equilibrium-seeking problems are computationally inefficient as they are often executed through two-loop schemes and lack provable guarantees. In this paper, by leveraging iterative regularization within existing gradient tracking methods Push-Pull and DSGT, we develop two single-timescale methods tailored for optimal equilibrium selection problems. These methods include the Iteratively Regularized Push-Pull (IR-Push-Pull) for directed networks and the Iteratively Regularized Distributed Stochastic Gradient Tracking (IR-DSGT) for undirected networks. Under some standard assumptions, we establish the global convergence to the unique optimal equilibrium and derive provable consensus guarantees. Preliminary numerical experiments on a Cournot game validate our theoretical findings and demonstrate robustness across two network settings with different node sizes.

## REFERENCES

- [1] E. ALTMAN AND T. BASAR, *Multiuser rate-based flow control*, IEEE Transactions on Communications, 46 (1998), pp. 940–949.
- [2] A. AUSLENDER AND M. TEBoulLE, *Lagrangian duality and related multiplier methods for variational inequality problems*, SIAM Journal on Optimization, 10 (2000), pp. 1097–1115.
- [3] A. BECK, *First-order methods in optimization*, SIAM, 2017.
- [4] E. BENENATI, W. ANANDUTA, AND S. GRAMMATICO, *On the optimal selection of generalized Nash equilibria in linearly coupled aggregative games*, in 2022 IEEE 61st Conference on Decision and Control (CDC), IEEE, 2022, pp. 6389–6394.
- [5] —, *Optimal selection and tracking of generalized Nash equilibria in monotone games*, IEEE Transactions on Automatic Control, 68 (2023), pp. 7644–7659.
- [6] D. P. BERTSEKAS, *Nonlinear Programming: 3rd Edition*, Athena Scientific, Belmont, MA, 2016.
- [7] T. H. CHUNG, G. A. HOLLINGER, AND V. ISLER, *Search and pursuit-evasion in mobile robotics*, Autonomous robots, 31 (2011), pp. 299–316.
- [8] V. CONITZER AND T. SANDHOLM, *New complexity results about Nash equilibria*, Games and Economic Behavior, 63 (2008), pp. 621–641. Second World Congress of the Game Theory Society.
- [9] J. R. CORREA, A. S. SCHULZ, AND N. E. STIER-MOSES, *Selfish routing in capacitated networks*, Mathematics of Operations Research, 29 (2004), pp. 961–976.
- [10] Z. DENG, *Distributed algorithm design for aggregative games of euler-lagrange systems and its application to smart grids*, IEEE Transactions on Cybernetics, (2021), pp. 1–11.
- [11] E. J. DOCKNER, S. JORGENSEN, N. VAN LONG, AND G. SORGER, *Differential games in economics and management science*, Cambridge University Press, 2000.
- [12] P. DUBEY, *Inefficiency of Nash equilibria*, Mathematics of Operations Research, 11 (1986), pp. 1–8.
- [13] F. FACCHINEI AND J.-S. PANG, *Finite-dimensional Variational Inequalities and Complementarity Problems. Vols. I,II*, Springer Series in Operations Research, Springer-Verlag, New York, 2003.
- [14] F. FACCHINEI, J.-S. PANG, G. SCUTARI, AND L. LAMPARIELLO, *VI-constrained hemivariational inequalities: distributed algorithms and power control in ad-hoc networks*, Mathematical Programming, 145 (2014), pp. 59–96.
- [15] Z. FEINSTEIN AND B. RUDLOFF, *Characterizing and computing the set of Nash equilibria via vector optimization*, Operations Research, (2023).
- [16] J. GHADERI AND R. SRIKANT, *Opinion dynamics in social networks with stubborn agents: Equilibrium and convergence rate*, Automatica, 50 (2014), pp. 3209–3215.
- [17] E. HAZAN AND R. KRAUTHGAMER, *How hard is it to approximate the best Nash equilibrium?*, SIAM Journal on Computing, 40 (2011), pp. 79–91.
- [18] A. JALILZADEH, F. YOUSEFIAN, AND M. EBRAHIMI, *Stochastic approximation for estimating the price of stability in stochastic Nash games*, ACM Transactions on Modeling and Computer Simulation, 34 (2024), pp. 1–24.
- [19] A. KANNAN AND U. V. SHANBHAG, *Distributed computation of equilibria in monotone Nash games via iterative regularization techniques*, SIAM Journal on Optimization, 22 (2012), pp. 1177–1205.
- [20] H. D. KAUSHIK, S. SAMADI, AND F. YOUSEFIAN, *An incremental gradient method for optimization problems with variational inequality constraints*, IEEE Transactions on Automatic Control, (2023).
- [21] H. D. KAUSHIK AND F. YOUSEFIAN, *A method with convergence rates for optimization problems with variational inequality constraints*, SIAM Journal on Optimization, 31 (2021), pp. 2171–2198.
- [22] J. KOSHAL, A. NEDIĆ, AND U. V. SHANBHAG, *Regularized iterative stochastic approximation methods for variational inequality problems*, IEEE Transactions on Automatic Control, 58(3) (2013), pp. 594–609.
- [23] J. LEI, U. V. SHANBHAG, AND J. CHEN, *A distributed iterative tikhonov method for networked monotone aggregative hierarchical stochastic games*, arXiv preprint arXiv:2304.03651, (2023).
- [24] Y. LOU, Y. HONG, L. XIE, G. SHI, AND K. H. JOHANSSON, *Nash equilibrium computation in subnetwork zero-sum games with switching communications*, IEEE Transactions on Automatic Control, 61 (2016), pp. 2920–2935.

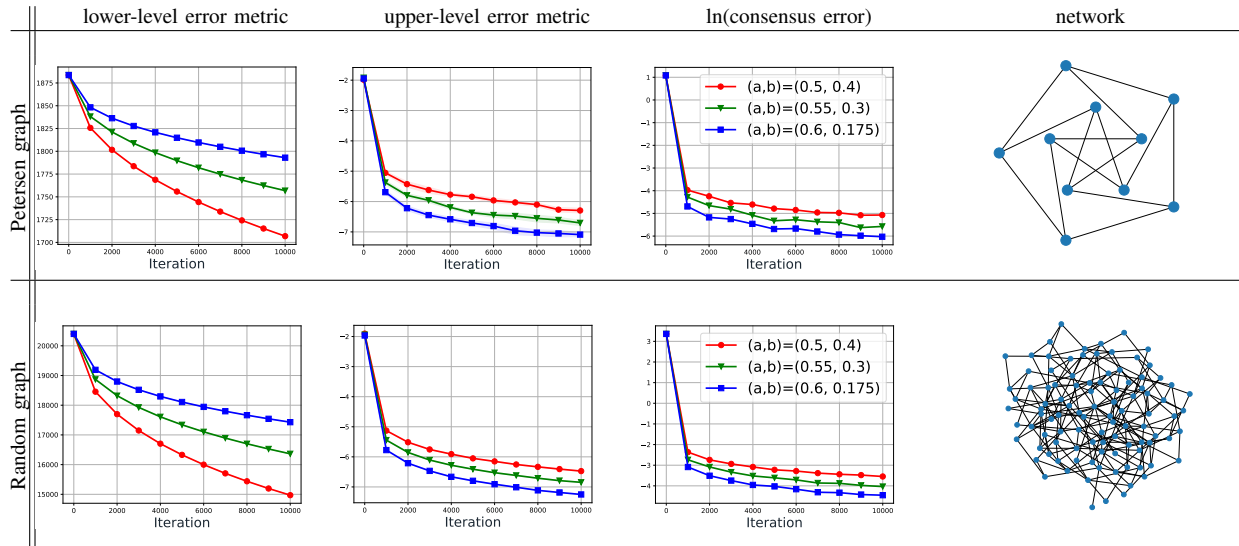


Fig. 2: Performance of Algorithm 2 on Petersen graph with 10 nodes and an undirected random graph with 100 nodes.

- [25] U. MOSCO, *Dual variational inequalities*, Journal of Mathematical Analysis and Applications, 40 (1972), pp. 202–206.
- [26] J. NASH, *Non-cooperative games*, Annals of mathematics, (1951), pp. 286–295.
- [27] C. PAPADIMITRIOU, *Algorithms, games, and the internet*, in Proceedings of the thirty-third annual ACM symposium on Theory of computing, 2001, pp. 749–753.
- [28] S. PU AND A. NEDIĆ, *Distributed stochastic gradient tracking methods*, Mathematical Programming, 187 (2021), pp. 409–457.
- [29] S. PU, W. SHI, J. XU, AND A. NEDIĆ, *Push–pull gradient methods for distributed optimization in networks*, IEEE Transactions on Automatic Control, 66 (2020), pp. 1–16.
- [30] Y. QIU, U. SHANBHAG, AND F. YOUSEFIAN, *Zeroth-order methods for nondifferentiable, nonconvex, and hierarchical federated optimization*, Advances in Neural Information Processing Systems, 36 (2023).
- [31] T. ROUGHGARDEN AND É. TARDOS, *How bad is selfish routing?*, Journal of the ACM (JACM), 49 (2002), pp. 236–259.
- [32] S. SAMADI, D. BURBANO, AND F. YOUSEFIAN, *Achieving optimal complexity guarantees for a class of bilevel convex optimization problems*, in 2024 American Control Conference (ACC), IEEE, 2024, pp. 2206–2211.
- [33] S. SAMADI AND F. YOUSEFIAN, *Improved guarantees for optimal Nash equilibrium seeking and bilevel variational inequalities*, SIAM Journal on Optimization (accepted), arXiv preprint arXiv:2307.12511, (2024).
- [34] Z. SONG, L. SHI, S. PU, AND M. YAN, *Optimal gradient tracking for decentralized optimization*, Mathematical Programming, (2023), pp. 1–53.
- [35] M. STAUDIGL AND S. QU, *Tikhonov regularized exterior penalty dynamics for constrained variational inequalities*, IEEE Control Systems Letters, (2024).
- [36] T. TATARENKO, W. SHI, AND A. NEDIĆ, *Geometric convergence of gradient play algorithms for distributed Nash equilibrium seeking*, IEEE Transactions on Automatic Control, 66 (2020), pp. 5342–5353.
- [37] J. G. WARDROP, *Road paper. some theoretical aspects of road traffic research.*, Proceedings of the institution of civil engineers, 1 (1952), pp. 325–362.
- [38] L. XIAO AND S. BOYD, *Fast linear iterations for distributed averaging*, Systems & Control Letters, 53 (2004), pp. 65–78.
- [39] R. XIN, U. A. KHAN, AND S. KAR, *An improved convergence analysis for decentralized online stochastic non-convex optimization*, IEEE Transactions on Signal Processing, 69 (2021), pp. 1842–1858.
- [40] I. YAMADA AND N. OGURA, *Hybrid steepest descent method for variational inequality problem over the fixed point set of certain quasi-nonexpansive mappings*, (2005).
- [41] F. YOUSEFIAN, *Bilevel distributed optimization in directed networks*, in 2021 American Control Conference (ACC), IEEE, 2021, pp. 2230–2235.
- [42] F. YOUSEFIAN, A. NEDIĆ, AND U. V. SHANBHAG, *On smoothing, regularization, and averaging in stochastic approximation methods for stochastic variational inequality problems*, Mathematical Programming, 165 (2017), pp. 391–431.



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