Lecture 8:
Bézier curves and surfaces

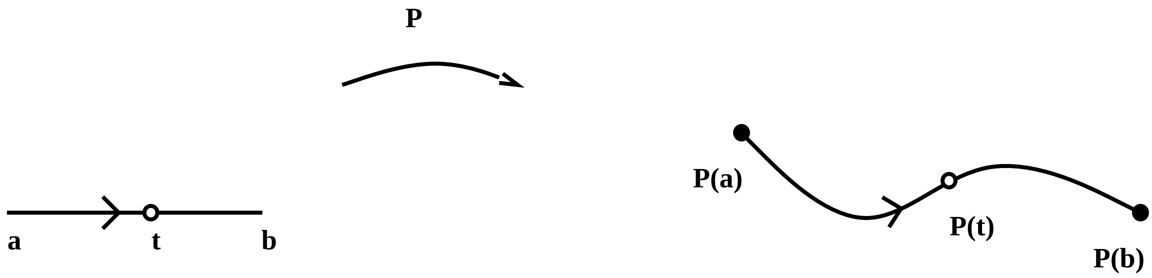
Topics:

1. Parametric curves
2. Bezier curves
3. Shape preservation
2. Derivatives
2. Evaluation: de Casteljau
5. Tensor-product surfaces

Michael S. Floater, Oslo, Oct. 2003

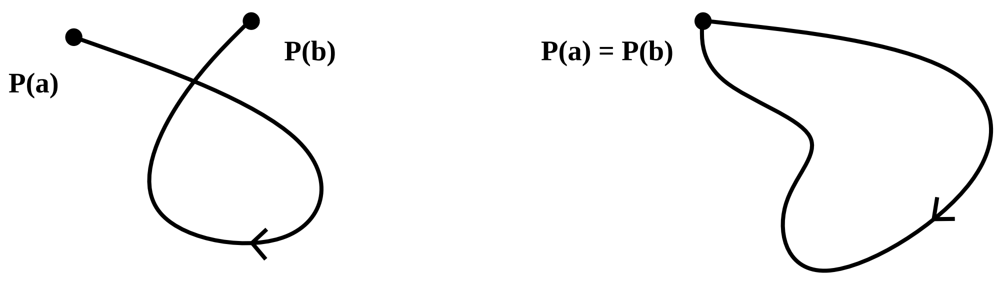
Parametric Curves

A **parametric curve** is defined to be a function $P : [a, b] \to \mathbb{R}^n$, where usually $n = 2$ or $n = 3$.



The function P maps each **parameter value** t between a and b to a corresponding point $P(t)$ in $\mathbb{R}^n$. We think of the curve as having the direction corresponding to increasing t and in this sense the curve begins at the point $P(a)$ and ends at $P(b)$.

Normally, we assume that the curve P is at least continuous. The curve is said to have a **self-intersection** if $P(t_1) = P(t_2)$ for some distinct parameter values t_1 and t_2 in $[a, b]$. In the special case that $t_1 = a$ and $t_2 = b$, so that $P(a) = P(b)$, the curve is said to be **closed**. If on the other hand $P(a) \neq P(b)$ we say that the curve is **open**.



Differential Geometry

If the curve $P : [a,b] \to \mathbb{R}^n$ is differentiable at a point $P(t)$, we write its (vector) derivative as

$$\frac{d}{dt}P(t), \qquad \text{or} \qquad P'(t).$$

If the derivative $P'(t)$ is non-zero at every parameter value t, we say that the parameterization of the curve is **regular**. Provided it is non-zero, the vector $P'(t)$ is the tangent direction to the curve at the point $P(t)$.

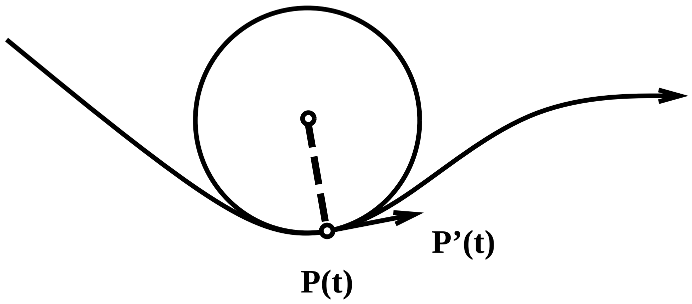
If the curve is twice differentiable, we can use the second derivative $P''(t)$ to calculate the **curvature** of the curve, which in $\mathbb{R}^3$ is

$$\kappa(t) = \frac{\|P'(t) \times P''(t)\|}{\|P'(t)\|^3}.$$

Here $u \times v$ is the **cross product** of two vectors $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$,

$$u \times v = (u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1).$$

Curvature is a measure of how much the curve bends and in fact $\kappa(t) = 1/\rho(t)$, where $\rho(t)$ is the radius of the osculating circle at $P(t)$.



If the third derivative exists at a point, we may also calculate the **torsion** of the curve in $\mathbb{R}^3$,

$$\tau(t) = \frac{\det[P'(t), P''(t), P'''(t)]}{\|P'(t) \times P''(t)\|^2}.$$

Here, $\det[u, v, w]$ is the determinant

$$\begin{vmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{vmatrix},$$

for vectors u, v, w in $\mathbb{R}^3$. Torsion is a measure of how much the curve twists.

Both curvature and torsion are independent of the parameterization of the curve and are Euclidean **invariants**, i.e. they do not change under rigid body motions of the curve.

Polynomial curves

We often use polynomials or piecewise polynomials to represent curves. A polynomial of degree d is defined as

$$p(t) = \sum_{i=0}^{d} c_i t^i = c_0 + c_1 t + c_2 t^2 + \ldots + c_d t^d,$$

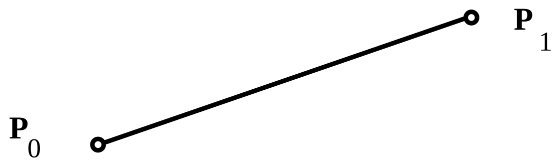
for real coefficients c_i. This can be generalized to curves but there are other ways of representing the same polynomial curve, which are better suited to modelling. The most popular method is the **Bézier** representation.

Bézier Curves

Bézier curves are parametric polynomial curves, usually parameterized from 0 to 1. A linear Bézier curve has the form

$$P(t) = (1 - t)P_0 + tP_1,$$

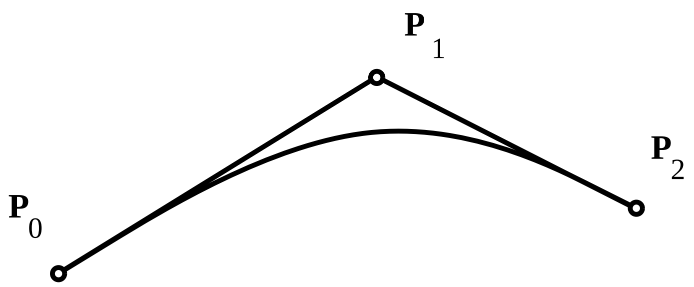
and represents the line segment between two points P_0 and P_1 in $\mathbb{R}^n$.



A quadratic Bézier curve has the form

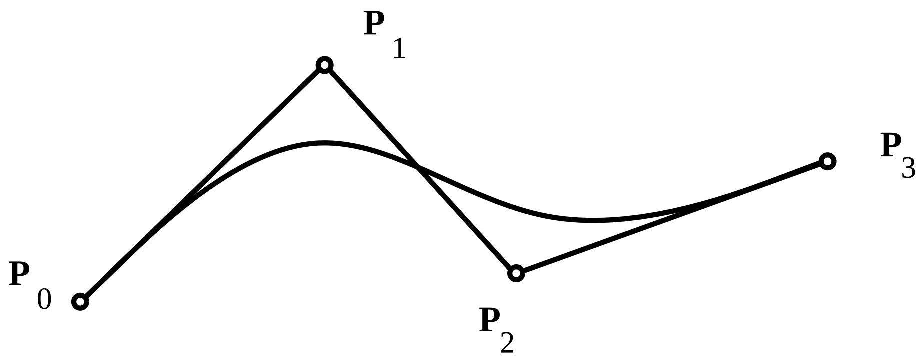
$$P(t) = (1 - t)^2 P_0 + 2t(1 - t)P_1 + t^2 P_2,$$

and describes a segment of a parabola.

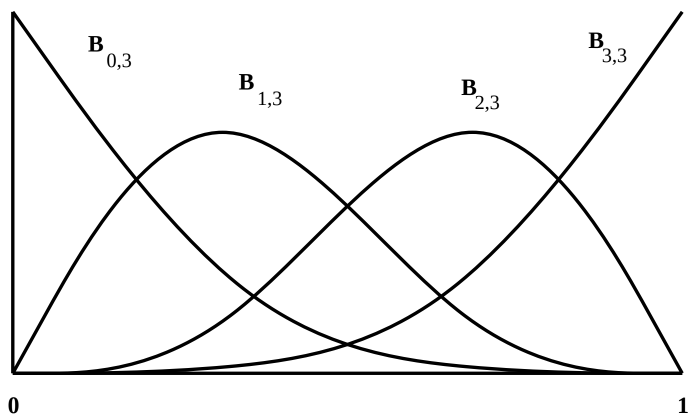


In general, a Bézier curve of degree d is written

$$
\begin{aligned}
P(t) &= (1-t)^d P_0 + dt(1-t)^{d-1} P_1 + \ldots + t^d P_d \\
&= \sum_{i=0}^{d} \frac{d!}{i!(d-i)!} t^i (1-t)^{d-i} P_i \\
&= \sum_{i=0}^{d} P_i B_{i,d}(t).
\end{aligned}
$$



The polynomials $B_{i,d}$ are called *Bernstein* polynomials and together they form a basis of the space of all polynomials of degree $\leq d$.



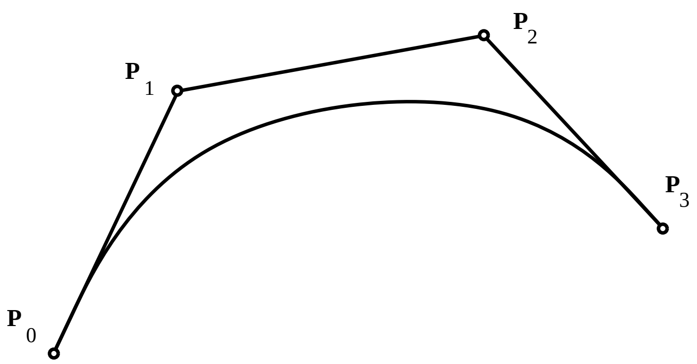
Shape Preservation

Bézier curves have the *convex hull* property: the curve is contained in the convex hull of its control points. The shape of the Bézier curve

$$P(t) = \sum_{i=0}^{d} P_i B_{i,d}(t)$$

tends to mimic the shape of the polygon with vertices $P_0, \ldots, P_d$. For this reason, we call this polygon the *control polygon* of the curve and the points P_i the *control points* of the curve.

More concretely, Bézier curves have the **variation diminishing property**: the curve has no more intersection with a plane (or line in $\mathbb{R}^2$) than its control polygon. Thus the curve tends to **reduce** the shape variations in the control polygon. As an example, if the control polygon is convex then so is the curve itself.



Derivatives

The derivative of the Bernstein polynomial $B_{i,d}$ can be found to be

$$B'_{i,d}(t) = d(B_{i-1,d-1}(t) - B_{i,d-1}(t)),$$

and it follows that

$$\begin{aligned} P'(t) &= d\sum_{i=0}^{d} P_i(B_{i-1,d-1}(t) - B_{i,d-1}(t)) \\ &= d\sum_{i=0}^{d-1}(P_{i+1} - P_i)B_{i,d-1}(t). \end{aligned}$$

For higher derivatives we define the r-th forward difference by

$$\Delta P_i = P_{i+1} - P_i$$

and

$$\Delta^r P_i = \Delta^{r-1}P_{i+1} - \Delta^{r-1}P_i$$

so that

$$\Delta^2 P_i = P_{i+2} - 2P_{i+1} + P_i$$

and

$$\Delta^3 P_i = P_{i+3} - 3P_{i+2} + 3P_{i+1} - P_i,$$

and so on. The r-th derivative of a Bézier curve is then

$$P^{(r)}(t) = \frac{d!}{(d-r)!}\sum_{i=0}^{d-r}\Delta^r P_i B_{i,d-r}(t).$$

Evaluation

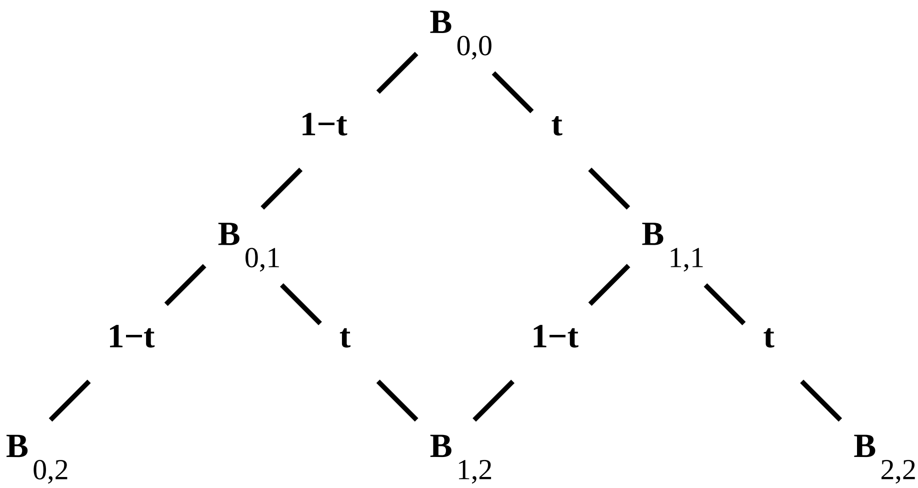
There are two recursive approaches to evaluating Bézier curves.

Method 1. Recursion on basis functions

This method first evaluates the Bernstein basis functions recursively and then multiplies them by the control points and sums them together. The basis function evaluation uses the identity

$$B_{i,d}(t) = (1-t)B_{i,d-1}(t) + tB_{i-1,d-1},$$

where we define $B_{0,0} = 1$ and $B_{i,d} = 0$ for $i < 0$ and $i > d$. First we compute $B_{0,1}$ and $B_{1,1}$ from $B_{0,0}$, then $B_{0,2}, B_{1,2}, B_{2,2}$ from $B_{0,1}$ and $B_{1,1}$ and so on, giving a triangular scheme.



Method 2. Recursion on control points

This method is known as the de Casteljau algorithm. In this method we begin by relabelling the control points $P_0, \dots, P_d$ as

$$P_0^0, P_1^0, \dots, P_d^0,$$

From this sequence of points we then calculate a new (shorter) sequence

$$P_0^1, P_1^1, \dots, P_{d-1}^1,$$

and thereafter a further sequence

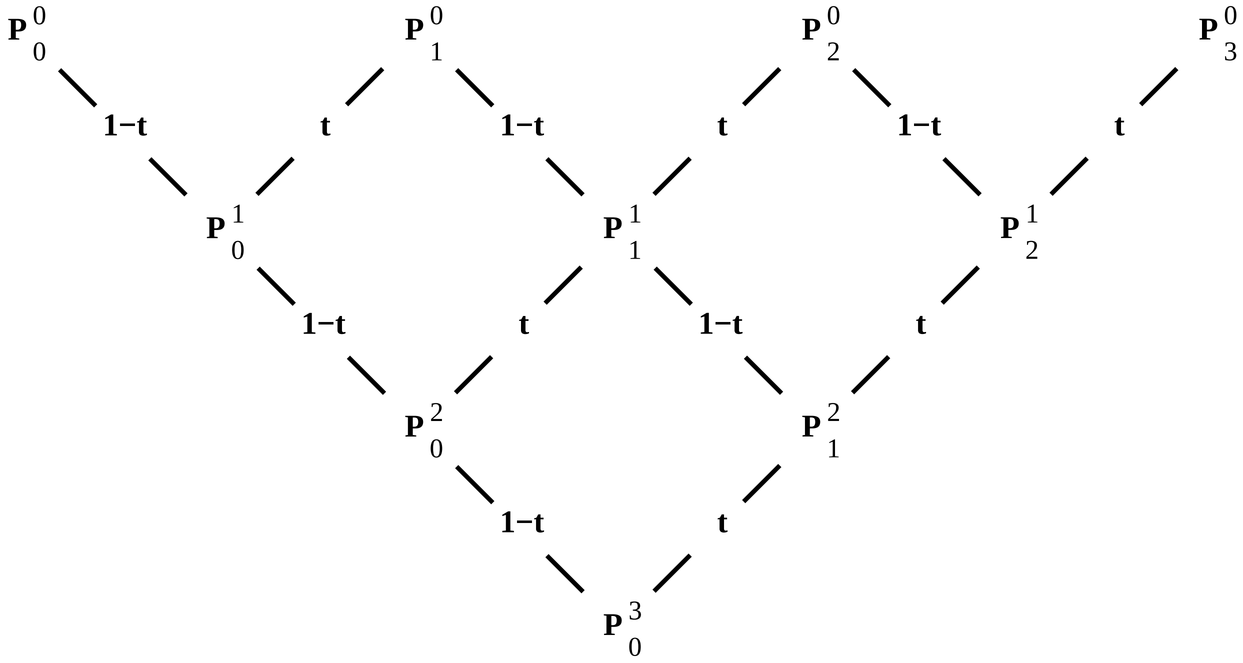
$$P_0^2, P_1^2, \dots, P_{d-2}^2,$$

and so on until we have just one point P_0^d. It can be shown that if the formula for calculating one sequence from the previous one is

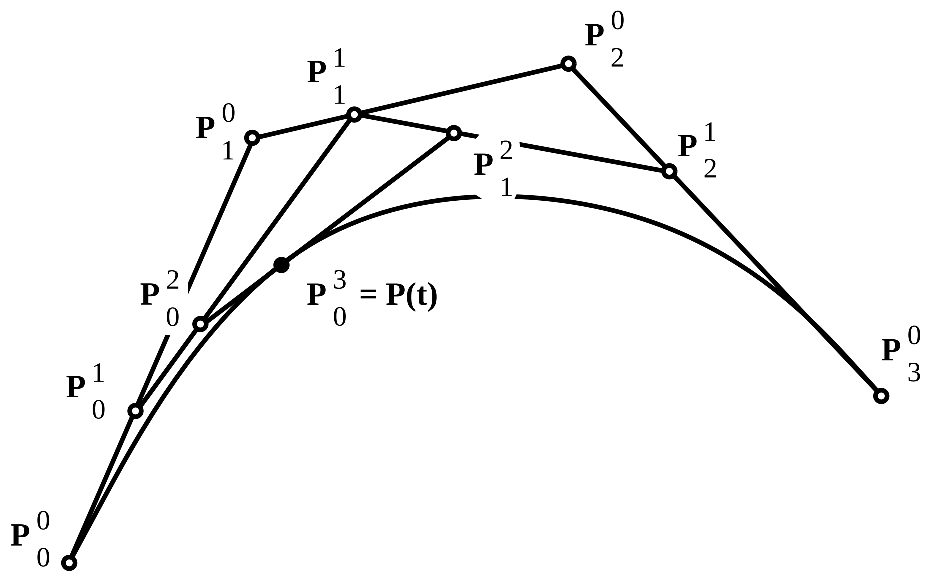
$$P_i^{r+1} = (1-t)P_i^r + tP_{i+1}^r.$$

then we obtain a point on the Bézier curve,

$$P_0^d = P(t) = \sum_{i=0}^{d} P_i B_{i,d}(t),$$



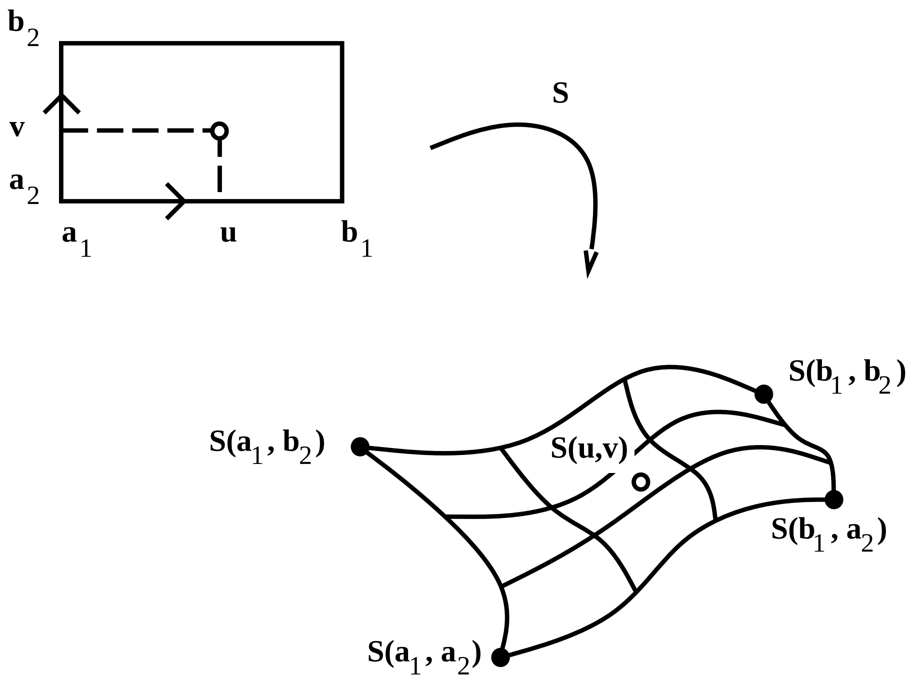
P_0^0
P_1^0
P_2^0
P_3^0
1−t
t
1−t
t
1−t
t
P_0^1
P_1^1
P_2^1
1−t
t
1−t
t
P_0^2
P_1^2
1−t
t
P_0^3



P_2^0
P_1^1
P_1^0
P_1^2
P_2^1
P_0^2
$P_0^3 = P(t)$
P_3^0
P_0^1
P_0^0

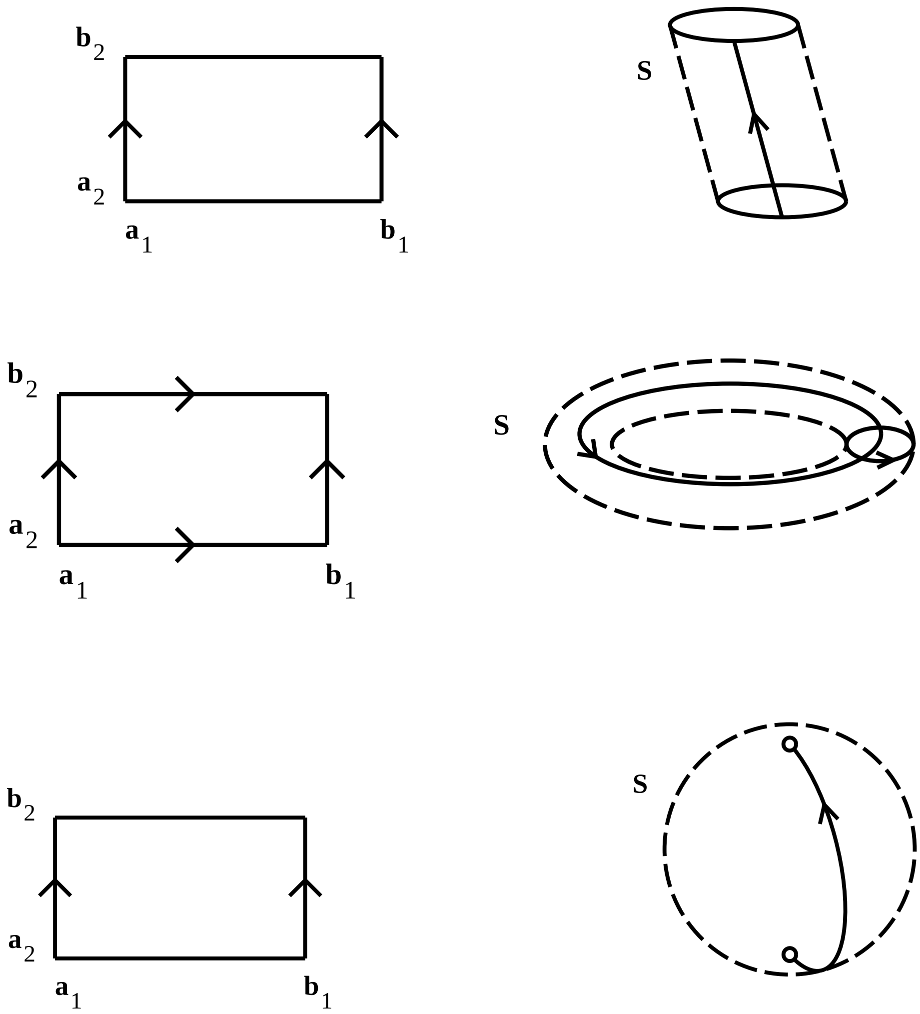
Parametric Surfaces

A (rectangular) **parametric surface** in $\mathbb{R}^3$ is a mapping S from a rectangular domain $[a_1, b_1] \times [a_2, b_2]$ to $\mathbb{R}^3$.



The function S maps each **parameter point** (u, v) to a corresponding point $S(u, v)$ in $\mathbb{R}^3$.

The surface can be open or can be closed in various ways. If, for example, $S(a_1, v) = S(b_1, v)$ for all $a_2 \le v \le b_2$, the two edges of constant u are glued together, and the surface takes on cylindrical topology. If both the u and v edges are stuck together, the surface is toroidal. By sticking together the u edges together and degenerating the two v edges to points, the surface can become spherical.



Differential Geometry

If the surface $[a_1, b_1] \times [a_2, b_2] \to \mathbb{R}^3$ is differentiable at a point $S(u, v)$, we write its partial (vector) derivatives as

$$\frac{\partial}{\partial u} S(u, v) \quad \text{and} \quad \frac{\partial}{\partial u} S(u, v), \qquad \text{or} \qquad S_u(u, v) \quad \text{and} \quad S_v(u, v).$$

The direction of the (oriented) normal to the surface is the cross product

$$S_u(u, v) \times S_v(u, v).$$

If this vector is non-zero at every parameter pair (u, v), we say the surface is **regular**, and

$$n(u, v) = \frac{S_u(u, v) \times S_v(u, v)}{\|S_u(u, v) \times S_v(u, v)\|}$$

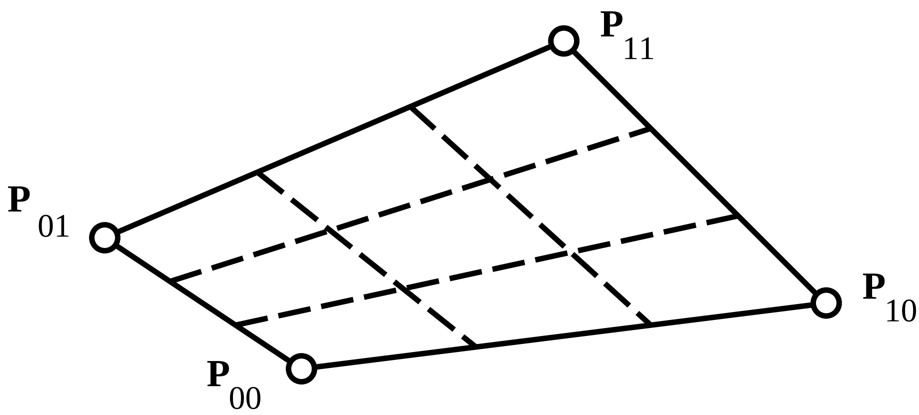
is the unit normal to S.

If the curve is twice differentiable, we can use the three second derivatives $S_{uu}(u, v)$, $S_{uv}(u, v)$, $S_{vv}(u, v)$ to calculate various **curvatures**, such as the normal curvature in a given direction, the Gaussian curvature and mean curvature.

Tensor-product Bézier surfaces

Tensor-product Bézier surfaces are a natural generalization of Bézier curves. A bilinear Bézier surface has the form

$$S(u,v) = (1-u)(1-v)P_{00} + u(1-v)P_{10} + (1-u)vP_{01} + uvP_{11}.$$



The iso-curves (surface curves in which one of the parameters is fixed) are straight lines, because

$$\begin{aligned} S(u,v) &= (1-u)\big((1-v)P_{00} + vP_{01}\big) + u\big((1-v)P_{10} + vP_{11}\big) \\ &= (1-v)\big((1-u)P_{00} + uP_{10}\big) + v\big((1-u)P_{01} + uP_{11}\big) \end{aligned}$$

In general, a tensor-product Bézier surface of degree $d_1 \times d_2$ is written

$$S(u,v) = \sum_{i=0}^{d_1} \sum_{j=0}^{d_2} P_{ij} B_{i,d_1}(u) B_{j,d_2}(v).$$

The basis functions are products of Bernstein polynomials, $B_{i,d_1}(u)B_{j,d_2}(v)$. They form a a basis of the space of all tensor-product polynomials of degree $d_1 \times d_2$. The figure below shows a bicubic Bézier surface.