

Some of my white board writing from week 10

I discussed the generalized likelihood ratio test for two examples related to the normal distribution:

(a) Testing  $H_0: \mu = \mu_0$  vs  $H_1: \mu \neq \mu_0$  for a sample of  $n$  i.i.d.  $N(\mu, \sigma^2)$  ( $\sigma^2$  assumed known)

In this case

$$-2 [\ln L(\mathbf{X}, H_0) - \ln L(\mathbf{X}, \bar{x})] = -2 \left[ -\frac{\sum_{i=1}^n (x_i - \mu_0)^2}{2\sigma^2} + \frac{\sum_{i=1}^n (x_i - \bar{x})^2}{2\sigma^2} \right] \\ = \frac{1}{\sigma^2} \left[ \sum_{i=1}^n (x_i - \mu_0)^2 - \sum_{i=1}^n (x_i - \bar{x})^2 \right] = \frac{n(\bar{x} - \mu_0)^2}{\sigma^2}$$

This should be  $\sim \chi_1^2$  asymptotically but in this case, because of dealing with normal, the result is precise (not only asymptotic). Indeed, we know that under

$$H_0: \bar{x} \sim N(\mu_0, \frac{\sigma^2}{n}) \Rightarrow \sqrt{n}(\bar{x} - \mu_0) \sim N(0, 1) \Rightarrow \frac{n(\bar{x} - \mu_0)^2}{\sigma^2} \sim \chi_1^2$$

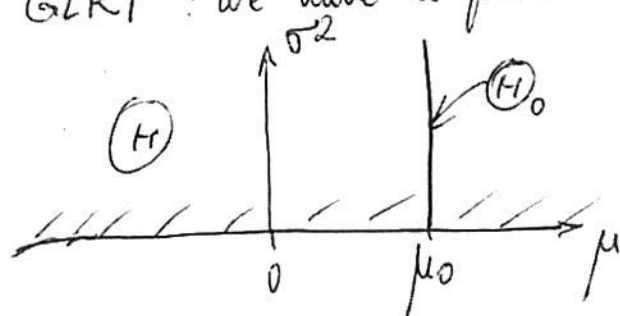
$$\text{The GLRT is: } \varphi^* = \begin{cases} 1 & \text{if } \frac{n(\bar{x} - \mu_0)^2}{\sigma^2} > \chi_{\alpha, 1}^2 \\ 0 & \text{if } \frac{n(\bar{x} - \mu_0)^2}{\sigma^2} \leq \chi_{\alpha, 1}^2 \end{cases} \text{ and}$$

is equivalent to the standard Z-test in this case.

(b) Testing  $H_0: \mu = \mu_0$  vs  $H_1: \mu \neq \mu_0$  again but when  $\sigma^2$  is unknown. Hence we are testing in effect:

$$H_0: \begin{cases} \mu = \mu_0 \\ \sigma^2 > 0 \end{cases} \text{ vs } H_1: \begin{cases} \mu \neq \mu_0 \\ \sigma^2 > 0 \end{cases} \text{ In terms of the notation of}$$

GLRT: we have a parameter vector  $\theta = \begin{pmatrix} \mu \\ \sigma^2 \end{pmatrix}$  and



$H_0$  is one-dimensional subspace as sketched, whereas  $H_1$  is "anything" above the  $\mu$  axis.

The dimensions  $K, r, S$ , as discussed in Section 6.10.1 of ②  
Lecture 6, p. 56, are  $K=2, r=1, S=1$ .

To perform the GLRT we need to maximize  $L(X, \theta)$   
under the Hypothesis and under the alternative.

i) under the hypothesis:  $\mu = \mu_0$ , so we need to optimize  
w.r. to  $\sigma$  only.  $\ln L = -\frac{n}{2} \ln \sigma^2 - \frac{1}{2\sigma^2} \sum_{i=1}^n (X_i - \mu_0)^2 + \text{const}$

$$\frac{\partial}{\partial \sigma^2} \ln L = 0 = -\frac{n}{2\sigma^2} + \frac{1}{2\sigma^4} \sum_{i=1}^n (X_i - \mu_0)^2 \quad \text{implies}$$

$$\hat{\sigma}_{\mu_0}^2 = \frac{1}{n} \sum_{i=1}^n (X_i - \mu_0)^2$$

$$\text{and } \sup L | H_0 = \frac{1}{(\sqrt{2\pi})^n (\hat{\sigma}_{\mu_0}^2)^{n/2}} \exp\left(-\frac{n}{2}\right)$$

ii) without the restriction of  $H_0$ , we have to  
maximise  $\ln L$  w.r. to both  $\mu$  and  $\sigma^2$ , i.e.

$$\text{Solve the system } \begin{cases} \frac{\partial}{\partial \mu} \ln L = 0 \\ \frac{\partial}{\partial \sigma^2} \ln L = 0 \end{cases} \quad \left| \quad \begin{array}{l} \text{This leads to} \\ \hat{\mu} = \bar{X} \\ \hat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^n (X_i - \bar{X})^2 \end{array} \right.$$

When plugging in, we get the  $\sup L$  without

$$\text{any restriction and it is } \sup L = \frac{1}{(\sqrt{2\pi})^n (\hat{\sigma}^2)^{n/2}} \exp\left(-\frac{n}{2}\right)$$

$$\text{Hence } -2 \log \Lambda = -2 \log \left( \frac{\hat{\sigma}_{\mu_0}^2}{\hat{\sigma}^2} \right)^{n/2} = n \log \left( \frac{\hat{\sigma}_{\mu_0}^2}{\hat{\sigma}^2} \right) \text{ and the}$$

$$\text{GLRT is } \varphi^* = \begin{cases} 1 & \text{if } n \log \left( \frac{\hat{\sigma}_{\mu_0}^2}{\hat{\sigma}^2} \right) > \chi_{\alpha, 1}^2 \\ 0 & \text{if } n \log \left( \frac{\hat{\sigma}_{\mu_0}^2}{\hat{\sigma}^2} \right) \leq \chi_{\alpha, 1}^2 \end{cases}$$

(the degrees of freedom are  $=1$  since in this case

$$r=1 = K-S \quad (K=2, S=1).$$

Note that now the convergence of  $-2 \log \Lambda$  to the limi-  
ting  $\chi_1^2$  is only asymptotic (not precise as in case a)).

But  $-2 \log \Lambda = n \log \left( 1 + \frac{(\bar{X} - \mu_0)^2}{\hat{\sigma}^2} \right) \approx \frac{n(\bar{X} - \mu_0)^2}{\hat{\sigma}^2}$ , so it is "almost"  
equivalent to the standard  $t$ -test for  $\mu = \mu_0$  when  
 $\sigma^2$  is unknown.



Continuation:  
Some of my white board writing in week 10 (3)

1.) Regarding the multinomial distribution:

I explained the formula

$$P(X_1=x_1, X_2=x_2, \dots, X_k=x_k) = \frac{n!}{x_1! x_2! \dots x_k!} w_1^{x_1} w_2^{x_2} \dots w_k^{x_k},$$

$0 < w_i < 1$ ,  $\sum_{i=1}^k w_i = 1$  for calculating the probability of a particular outcome  $\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \end{pmatrix}$  with  $x_1 + x_2 + \dots + x_k = n$ ,  $n$  being the number of independent trials.

I also discussed two examples:

i) If a die is tossed 6 times, what is the probability that each number (1, 2, 3, 4, 5, 6) turns up once.  
Applying the above formula with  $k=6$ ,  $x_i=1$ ,  $i=1, 2, \dots, 6$ , and  $w_i = \frac{1}{6}$ ,  $i=1, 2, \dots, 6$  we get  $6! \left(\frac{1}{6}\right)^6 = \frac{5}{324}$

ii) Out of 7 tosses, what is the probability that each number (1, 2, 3, 4, 5, 6) turns up at least once.

Answer:  $6 \cdot \frac{7!}{2! (1!)^6} \cdot \left(\frac{1}{6}\right)^7 = \frac{35}{648}$

2.) I discussed in detail the proof of Theorem 7.2. (p. 60) but I see that all details are presented in the lecture note so I will abstain from reproducing them again here.

3.) I discussed a simple method to derive the density of the  $r$ -th order statistic as stated in Theorem 7.3, p. 61

④

Now we first derive  $F_{X_{(n)}}(x)$  (the cdf of  $X_{(n)}$ ) and then differentiate it to find the density. The main observation we make is that

$$F_{X_{(r)}}(x) = P(X_{(r)} \leq x) = P(Y \geq r)$$

Hence we can state that

$$F_{X(n)}(x) = \sum_{k=r}^n \binom{n}{k} (F_X(x))^k (1 - F_X(x))^{n-k}$$

Now to get the density we need to differentiate each of the summands in  $\sum_{k=r}^n$  by applying the  $(uv)' = u'v + v'u$  formula each time.

We get:

We get:

$$f_{X(r)}(x) = \binom{n}{r} r f_X(x) \bar{F}_X(x)^{r-1} (1 - \bar{F}_X(x))^{n-r} - \binom{n}{r} (n-r) \bar{F}_X(x)^r (1 - \bar{F}_X(x))^{n-r-1} f_X(x)$$

$$+ \binom{n}{r+1} (r+1) f_X(x) \bar{F}_X(x)^r (1 - \bar{F}_X(x))^{n-r-1} - \binom{n}{r+1} (n-r) \bar{F}_X(x)^{r+1} (1 - \bar{F}_X(x))^{n-r-2} f_X(x)$$

$$+ \binom{n}{r+2} (r+2) f_X(x) \bar{F}_X(x)^{r+1} (1 - \bar{F}_X(x))^{n-r-2} - \binom{n}{r+2} (n-r-1) \bar{F}_X(x)^{r+2} (1 - \bar{F}_X(x))^{n-r-3} f_X(x)$$

$$+ \binom{n}{r+3} (r+3) f_X(x) \bar{F}_X(x)^{r+2} (1 - \bar{F}_X(x))^{n-r-3} - \binom{n}{r+3} (n-r-2) \bar{F}_X(x)^{r+3} (1 - \bar{F}_X(x))^{n-r-4} f_X(x)$$

$$+ \binom{n}{r+4} (r+4) f_X(x) \bar{F}_X(x)^{r+3} (1 - \bar{F}_X(x))^{n-r-4} - \binom{n}{r+4} (n-r-3) \bar{F}_X(x)^{r+4} (1 - \bar{F}_X(x))^{n-r-5} f_X(x)$$

$$+ \binom{n}{r+5} (r+5) f_X(x) \bar{F}_X(x)^{r+4} (1 - \bar{F}_X(x))^{n-r-5} - \binom{n}{r+5} (n-r-4) \bar{F}_X(x)^{r+5} (1 - \bar{F}_X(x))^{n-r-6} f_X(x)$$

$$+ \binom{n}{r+6} (r+6) f_X(x) \bar{F}_X(x)^{r+5} (1 - \bar{F}_X(x))^{n-r-6} - \binom{n}{r+6} (n-r-5) \bar{F}_X(x)^{r+6} (1 - \bar{F}_X(x))^{n-r-7} f_X(x)$$

$$+ \binom{n}{r+7} (r+7) f_X(x) \bar{F}_X(x)^{r+6} (1 - \bar{F}_X(x))^{n-r-7} - \binom{n}{r+7} (n-r-6) \bar{F}_X(x)^{r+7} (1 - \bar{F}_X(x))^{n-r-8} f_X(x)$$

$$+ \binom{n}{r+8} (r+8) f_X(x) \bar{F}_X(x)^{r+7} (1 - \bar{F}_X(x))^{n-r-8} - \binom{n}{r+8} (n-r-7) \bar{F}_X(x)^{r+8} (1 - \bar{F}_X(x))^{n-r-9} f_X(x)$$

$$+ \binom{n}{r+9} (r+9) f_X(x) \bar{F}_X(x)^{r+8} (1 - \bar{F}_X(x))^{n-r-9} - \binom{n}{r+9} (n-r-8) \bar{F}_X(x)^{r+9} (1 - \bar{F}_X(x))^{n-r-10} f_X(x)$$

$$+ \binom{n}{r+10} (r+10) f_X(x) \bar{F}_X(x)^{r+9} (1 - \bar{F}_X(x))^{n-r-10} - \binom{n}{r+10} (n-r-9) \bar{F}_X(x)^{r+10} (1 - \bar{F}_X(x))^{n-r-11} f_X(x)$$

$$+ \binom{n}{r+11} (r+11) f_X(x) \bar{F}_X(x)^{r+10} (1 - \bar{F}_X(x))^{n-r-11} - \binom{n}{r+11} (n-r-10) \bar{F}_X(x)^{r+11} (1 - \bar{F}_X(x))^{n-r-12} f_X(x)$$

$$+ \binom{n}{r+12} (r+12) f_X(x) \bar{F}_X(x)^{r+11} (1 - \bar{F}_X(x))^{n-r-12} - \binom{n}{r+12} (n-r-11) \bar{F}_X(x)^{r+12} (1 - \bar{F}_X(x))^{n-r-13} f_X(x)$$

$$+ \binom{n}{r+13} (r+13) f_X(x) \bar{F}_X(x)^{r+12} (1 - \bar{F}_X(x))^{n-r-13} - \binom{n}{r+13} (n-r-12) \bar{F}_X(x)^{r+13} (1 - \bar{F}_X(x))^{n-r-14} f_X(x)$$

$$+ \binom{n}{r+14} (r+14) f_X(x) \bar{F}_X(x)^{r+13} (1 - \bar{F}_X(x))^{n-r-14} - \binom{n}{r+14} (n-r-13) \bar{F}_X(x)^{r+14} (1 - \bar{F}_X(x))^{n-r-15} f_X(x)$$

$$+ \binom{n}{r+15} (r+15) f_X(x) \bar{F}_X(x)^{r+14} (1 - \bar{F}_X(x))^{n-r-15} - \binom{n}{r+15} (n-r-14) \bar{F}_X(x)^{r+15} (1 - \bar{F}_X(x))^{n-r-16} f_X(x)$$

$$+ \binom{n}{r+16} (r+16) f_X(x) \bar{F}_X(x)^{r+15} (1 - \bar{F}_X(x))^{n-r-16} - \binom{n}{r+16} (n-r-15) \bar{F}_X(x)^{r+16} (1 - \bar{F}_X(x))^{n-r-17} f_X(x)$$

$$+ \binom{n}{r+17} (r+17) f_X(x) \bar{F}_X(x)^{r+16} (1 - \bar{F}_X(x))^{n-r-17} - \binom{n}{r+17} (n-r-16) \bar{F}_X(x)^{r+17} (1 - \bar{F}_X(x))^{n-r-18} f_X(x)$$

$$+ \binom{n}{r+18} (r+18) f_X(x) \bar{F}_X(x)^{r+17} (1 - \bar{F}_X(x))^{n-r-18} - \binom{n}{r+18} (n-r-17) \bar{F}_X(x)^{r+18} (1 - \bar{F}_X(x))^{n-r-19} f_X(x)$$

$$+ \binom{n}{r+19} (r+19) f_X(x) \bar{F}_X(x)^{r+18} (1 - \bar{F}_X(x))^{n-r-19} - \binom{n}{r+19} (n-r-18) \bar{F}_X(x)^{r+19} (1 - \bar{F}_X(x))^{n-r-20} f_X(x)$$

$$+ \binom{n}{r+20} (r+20) f_X(x) \bar{F}_X(x)^{r+19} (1 - \bar{F}_X(x))^{n-r-20} - \binom{n}{r+20} (n-r-19) \bar{F}_X(x)^{r+20} (1 - \bar{F}_X(x))^{n-r-21} f_X(x)$$

$$+ \binom{n}{r+21} (r+21) f_X(x) \bar{F}_X(x)^{r+20} (1 - \bar{F}_X(x))^{n-r-21} - \binom{n}{r+21} (n-r-20) \bar{F}_X(x)^{r+21} (1 - \bar{F}_X(x))^{n-r-22} f_X(x)$$

$$+ \binom{n}{r+22} (r+22) f_X(x) \bar{F}_X(x)^{r+21} (1 - \bar{F}_X(x))^{n-r-22} - \binom{n}{r+22} (n-r-21) \bar{F}_X(x)^{r+22} (1 - \bar{F}_X(x))^{n-r-23} f_X(x)$$

$$+ \binom{n}{r+23} (r+23) f_X(x) \bar{F}_X(x)^{r+22} (1 - \bar{F}_X(x))^{n-r-23} - \binom{n}{r+23} (n-r-22) \bar{F}_X(x)^{r+23} (1 - \bar{F}_X(x))^{n-r-24} f_X(x)$$

$$+ \binom{n}{r+24} (r+24) f_X(x) \bar{F}_X(x)^{r+23} (1 - \bar{F}_X(x))^{n-r-24} - \binom{n}{r+24} (n-r-23) \bar{F}_X(x)^{r+24} (1 - \bar{F}_X(x))^{n-r-25} f_X(x)$$

$$+ \binom{n}{r+25} (r+25) f_X(x) \bar{F}_X(x)^{r+24} (1 - \bar{F}_X(x))^{n-r-25} - \binom{n}{r+25} (n-r-24) \bar{F}_X(x)^{r+25} (1 - \bar{F}_X(x))^{n-r-26} f_X(x)$$

$$+ \binom{n}{r+26} (r+26) f_X(x) \bar{F}_X(x)^{r+25} (1 - \bar{F}_X(x))^{n-r-26} - \binom{n}{r+26} (n-r-25) \bar{F}_X(x)^{r+26} (1 - \bar{F}_X(x))^{n-r-27} f_X(x)$$

$$+ \binom{n}{r+27} (r+27) f_X(x) \bar{F}_X(x)^{r+26} (1 - \bar{F}_X(x))^{n-r-27} - \binom{n}{r+27} (n-r-26) \bar{F}_X(x)^{r+27} (1 - \bar{F}_X(x))^{n-r-28} f_X(x)$$

$$+ \binom{n}{r+28} (r+28) f_X(x) \bar{F}_X(x)^{r+27} (1 - \bar{F}_X(x))^{n-r-28} - \binom{n}{r+28} (n-r-27) \bar{F}_X(x)^{r+28} (1 - \bar{F}_X(x))^{n-r-29} f_X(x)$$

$$+ \binom{n}{r+29} (r+29) f_X(x) \bar{F}_X(x)^{r+28} (1 - \bar{F}_X(x))^{n-r-29} - \binom{n}{r+29} (n-r-28) \bar{F}_X(x)^{r+29} (1 - \bar{F}_X(x))^{n-r-30} f_X(x)$$

$$+ \binom{n}{r+30} (r+30) f_X(x) \bar{F}_X(x)^{r+29} (1 - \bar{F}_X(x))^{n-r-30} - \binom{n}{r+30} (n-r-29) \bar{F}_X(x)^{r+30} (1 - \bar{F}_X(x))^{n-r-31} f_X(x)$$

$$+ \binom{n}{r+31} (r+31) f_X(x) \bar{F}_X(x)^{r+30} (1 - \bar{F}_X(x))^{n-r-31} - \binom{n}{r+31} (n-r-30) \bar{F}_X(x)^{r+31} (1 - \bar{F}_X(x))^{n-r-32} f_X(x)$$

$$+ \binom{n}{r+32} (r+32) f_X(x) \bar{F}_X(x)^{r+31} (1 - \bar{F}_X(x))^{n-r-32} - \binom{n}{r+32} (n-r-31) \bar{F}_X(x)^{r+32} (1 - \bar{F}_X(x))^{n-r-33} f_X(x)$$

$$+ \binom{n}{r+33} (r+33) f_X(x) \bar{F}_X(x)^{r+32} (1 - \bar{F}_X(x))^{n-r-33} - \binom{n}{r+33} (n-r-32) \bar{F}_X(x)^{r+33} (1 - \bar{F}_X(x))^{n-r-34} f_X(x)$$

$$+ \binom{n}{r+34} (r+34) f_X(x) \bar{F}_X(x)^{r+33} (1 - \bar{F}_X(x))^{n-r-34} - \binom{n}{r+34} (n-r-33) \bar{F}_X(x)^{r+34} (1 - \bar{F}_X(x))^{n-r-35} f_X(x)$$

$$+ \binom{n}{r+35} (r+35) f_X(x) \bar{F}_X(x)^{r+34} (1 - \bar{F}_X(x))^{n-r-35} - \binom{n}{r+35} (n-r-34) \bar{F}_X(x)^{r+35} (1 - \bar{F}_X(x))^{n-r-36} f_X(x)$$

$$+ \$$

Huge cancellation happens (!) and, because of the equality  $\binom{n}{r}(n-r) = \binom{n}{r+1}(r+1)$  each of the summands after the first one disappears. Hence

$$f_{X(r)}(x) = \frac{n!}{(r-1)!(n-r)!} f_X(x) F_X(x)^{r-1} (1-F_X(x))^{n-r} \text{ holds}$$



(5)

4) I also discussed the idea of the proof of Theorem 7.4 on p. 62. Again, we first get the cdf and then find the mixed partial derivative

$$\frac{\partial^2}{\partial u \partial v} F_{X(i), X(j)}(u, v) \text{ to calculate the density } f_{X(i), X(j)}(u, v).$$

With the discrete variables  $U$  and  $V$  as introduced on p. 62 we see that

$$(U, V, n - U - V) \sim \text{Multinomial}(n; \frac{F(u)}{X}, \frac{F(v) - F(u)}{X}, 1 - \frac{F(v)}{X})$$

Then we observe that

$$F_{X(i), X(j)}(u, v) = P(U \geq i \cap U + V \geq j) =$$

$$= \sum_{k=i}^{j-1} \sum_{m=j-k}^{n-k} P(U=k, V=m) + P(U \geq j)$$

Since the second summand does not involve  $v$ , its mixed partial derivative w.r.  $u$  and  $v$  will be zero and hence

$$f_{X(i), X(j)}(u, v) = \frac{\partial^2}{\partial u \partial v} \sum_{k=i}^{j-1} \sum_{m=j-k}^{n-k} \frac{n!}{k! m! (n-k-m)!} \left( \frac{F(u)}{X} \right)^k \left( \frac{F(v) - F(u)}{X} \right)^m \left( 1 - \frac{F(v)}{X} \right)^{n-k-m}$$

Again, a huge cancellation happens when we calculate the partial derivatives by using the product rule for differentiation and we end up with

$$f_{X(i), X(j)}(u, v) = \frac{n!}{(i-1)!(j-1-i)!(n-j)!} f_X(u) f_X(v) \left( \frac{F(u)}{X} \right)^{i-1} \left( \frac{F(v) - F(u)}{X} \right)^{j-1-i} \left( 1 - \frac{F(v)}{X} \right)^{n-j}$$

for  $n \geq j \geq i \geq 1$ ,  $-\infty < u < v < \infty$  (and = 0 else)

⑤ I also discussed in detail the example stating that <sup>⑥</sup>  
 for the range  $R = X_{(n)} - X_{(1)}$  for order statistic  
 $X_{(1)} < X_{(2)} < \dots < X_{(n)}$  from the uniform  $(0,1)$   
 distribution, it holds  $f_R(u) = \begin{cases} n(n-1)u^{n-2}(1-u), & 0 < u < 1 \\ 0 & \text{else} \end{cases}$

Proof:

To this end, first we note that by using the formula  
 from Theorem 7.4 we have (with  $i=1$  and  $j=n$ ):

$$f_{X_{(1)}, X_{(n)}}(x, y) = n(n-1)(F_X(y) - F_X(x))^{n-2} f_X(x) f_X(y)$$

We introduce the variable of interest

$U = X_{(n)} - X_{(1)}$  and one auxiliary variable

$V = X_{(n)}$  so that we could apply

the density transformation formula

$$f_{\begin{pmatrix} U \\ V \end{pmatrix}}(u, v) = f_{\begin{pmatrix} X_{(1)} \\ X_{(n)} \end{pmatrix}}(x(u, v), y(u, v)) \left| \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \right|$$

Note:  $\begin{cases} X_{(1)} = V - U =: x \\ X_{(n)} = V =: y \end{cases}$  and  $\left| \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \right| = \left| \begin{vmatrix} -1 & 1 \\ 0 & 1 \end{vmatrix} \right| = 1$

Hence  $f_{\begin{pmatrix} U \\ V \end{pmatrix}}(u, v) = n(n-1)(F_X(v) - F_X(v-u))^{n-2} f_X(v-u) f_X(v) \times 1$

To get  $f_U(u)$  (which we are interested in) we need to  
 integrate out the unwanted variable  $V$  from  
 the joint density  $f_{\begin{pmatrix} U \\ V \end{pmatrix}}(u, v)$ . We need to be care-  
 ful with the integration range when doing this:

(7)

Since  $0 < X_{(1)} < X_{(n)} < 1$  we get

$$0 < V - U < V < 1$$

$$0 < U < V < 1. \text{ This means that for}$$

a fixed  $U$ ,  $V$  ranges in the interval  $(U, 1)$ .

Therefore:

$$f_R(u) = \int_U^1 f(u, v) dv = \int_U^1 n(n-1) (1 - (1-u))^{n-2} dv$$

$$= \begin{cases} n(n-1)u^{n-2}(1-u) & \text{if } 0 < u < 1 \\ 0 & \text{else} \end{cases}$$

I also advised you to repeat this exercise by using

$X_{(1)} = V$  as an auxiliary variable. The intermediate calculations will be slightly different but at the end after integrating out  $V$  again (BUT THIS TIME in the range  $(0, 1-u)$  (!)) you will get the same final result for the density  $f_R(u)$ .

6.) I also discussed one more problem in class

(7/d) from tutorial sheet 4) but because it is completely solved in the solutions to tutorial set 4, I abstain from reproducing the derivation here.