THE UNIVERSITY OF NEW SOUTH WALES

DEPARTMENT OF STATISTICS

Additional exercises for MATH5905, Statistical Inference

Part three: Hypothesis testing

1. For each of the families $L(\mathbf{x}, \theta) = \prod_{i=1}^{n} f(x_i, \theta)$ below suggest a statistic $T(\mathbf{X})$ with respect to which the family has the MLR property:
   a) $f(x;\theta)$ is $N(\theta, 1)$
   b) $f(x;\theta) = \frac{1}{\theta} e^{-\frac{x}{\theta}}, x > 0; \theta > 0.$
   c) $f(x;\theta) = \theta e^{-x\theta}, x > 0; \theta > 0.$
   d) $f(x;\theta)$ is $N(0, \theta^2)$
   e) $f(x;\lambda)$ is Poisson $(\lambda), \lambda > 0.$
   f) $f(x;p)$ is Bernoulli $(p), p \in (0, 1).$
   g) $f(x;\theta)$ is Uniform $(0, \theta).$

2. Find the *ump* size α test of $H_0 : \sigma \leq \sigma_0$ versus $H_1 : \sigma > \sigma_0$ based on n i.i.d. observations from $N(0, \sigma^2)$ population. Sketch a graph of its power function. Answer the same question in case that $H_0 : \sigma \geq \sigma_0$ versus $H_1 : \sigma < \sigma_0$ was to be tested.

3. Let X be a single observation from the density
$$f(x;\theta) = \theta x^{\theta-1}, 0 \leq x \leq 1, \ \theta > 0 \ .$$
   a) For testing $H_0 : \theta \leq 1$ versus $H_1 : \theta > 1$, find the power function and size of the test with rejection region $x \geq \frac{1}{2}$.
   b) Find the most powerful test of size $\alpha = 0.05$ of $H_0 : \theta = 2$ versus $H_1 : \theta = 1$.
   c) Find the *ump* size α test of $H_0 : \theta \geq 2$ versus $H_1 : \theta < 2$ and calculate its power function.
   d) Find the generalized likelihood-ratio test of $H_0 : \theta = 1$ versus $H_1 : \theta \neq 1$ with size $\alpha = 0.1$.

4. Suppose X_1 and X_2 are independent random variables, each with density

   $f(x;\theta) = \theta x^{\theta-1}$, $0 \leq x \leq 1$. For testing $H_0 : \theta \leq 1$ versus $H_1 : \theta > 1$, find the size and the power function of the test with rejection region $3x_1 \leq 4x_2$. Would you use this test? What alternative test could you suggest?

5. For a random sample of size n from the density $f(x;\theta) = e^{-(x-\theta)}, x \geq \theta$, construct the *ump* size α test of $H_0 : \theta \leq \theta_0$ versus $H_1 : \theta > \theta_0$. Calculate the power function of the test and sketch its graph.

6. For a sample of size $n = 10$ from a Poisson (λ) family construct the *ump* $\alpha = .10$ size test of $H_0 : \lambda \leq 1$ versus $H_1 : \lambda > 1$. You may utilize the following extract of a table of Poisson (10) probabilities:

x	12	13	14	15	16
$P(X \leq x)$	0.7915	0.8644	0.9165	0.9512	0.9729

7. Find the form of the rejection region of the *ump* test of $H_0 : \theta \leq \theta_0$ versus $H_1 : \theta > \theta_0$ based on independent random variables $X_1, X_2, \ldots, X_n$ each with density
$$f(x;\theta) = \frac{1}{\theta} e^{-x/\theta}, x > 0, \theta > 0 \ .$$
   Use the Central Limit Theorem to determine approximately the constant specifying the rejection region for a size α test. Hence find an appropriate expression for the power function.

8. Let $X_1, X_2, \ldots, X_n$ be a random sample from $N(\mu, \sigma^2)$ where σ^2 is known. Let Λ denote the generalized likelihood ratio for testing $H_0 : \mu = \mu_0$ versus $H_1 : \mu \neq \mu_0$. Find the exact distribution of $-2 \log \Lambda$, and compare it with the corresponding asymptotic distribution when H_0 is true.

9. Suppose $X_1, X_2, \ldots, X_m$ are independent $N(\mu_1, 1)$ random variables and $Y_1, Y_2, \ldots, Y_n$ is an independent set of independent $N(\mu_2, 1)$ random variables.

   a) Show that when $\mu_1 = \mu_2 = \mu$, say, the MLE of μ is $\tilde{\mu} = \frac{m\bar{X} + n\bar{Y}}{m+n}$.

   b) Prove that when the MLE's of μ_1 and μ_2 are $\hat{\mu}_1 = \bar{X}$ and $\hat{\mu}_2 = \bar{Y}$.

   c) Derive the generalized likelihood ratio statistic $\Lambda_{m,n}$ for testing $H_0 : \mu_1 = \mu_2$ versus $H_1 : \mu_1 \neq \mu_2$ and show that, when H_0 is true, $-2 \log \Lambda_{m,n}$ has precisely χ^2_1 distribution for every m, n.

Answers:

1) MLR in: a) $T(\mathbf{X}) = \sum_{i=1}^n X_i$, b) $T(\mathbf{X}) = \sum_{i=1}^n X_i$, c) $T(\mathbf{X}) = -\sum_{i=1}^n X_i$, d) $T(\mathbf{X}) = \sum_{i=1}^n X_i{}^2$, e) $T(\mathbf{X}) = \sum_{i=1}^n X_i$, f) $T(\mathbf{X}) = \sum_{i=1}^n X_i$, g) $T(\mathbf{X}) = X_{(n)}$.

2) $\varphi^*(\mathbf{X}) = \begin{cases} 1 \text{ if } \sum_{i=1}^n X_i^2 \geq \sigma_0^2 \chi^2_{n,\alpha} \\ 0 \text{ if } \sum_{i=1}^n X_i^2 < \sigma_0^2 \chi^2_{n,\alpha}. \end{cases}$ with a power function $p(t) = P(\chi^2_n \geq \frac{\sigma_0^2}{t^2} \chi^2_{n,\alpha})$. But in the case of $H_0 : \sigma \geq \sigma_0$ versus $H_1 : \sigma < \sigma_0$, the ump $\alpha-$ test changes to

$$\varphi^*(\mathbf{X}) = \begin{cases} 1 \text{ if } \sum_{i=1}^n X_i^2 \leq \sigma_0^2 \chi^2_{n,1-\alpha} \\ 0 \text{ if } \sum_{i=1}^n X_i^2 > \sigma_0^2 \chi^2_{n,1-\alpha}. \end{cases}$$

with a power function $p(t) = P(\chi^2_n \leq \frac{\sigma_0^2}{t^2} \chi^2_{n,1-\alpha})$.

3) a) $E_\theta \varphi = 1 - \frac{1}{2^\theta}, \theta > 0$, size at $\theta_0 = 1 : E_{\theta_0} \varphi = 0.5$.

b) $\varphi^*(\mathbf{X}) = \begin{cases} 1 \text{ if } X \leq 0.2236 \\ 0 \text{ if } X > 0.2236 \end{cases}$

c) same test as in b)

d) $\varphi^*(\mathbf{X}) = \begin{cases} 1 \text{ if } X \leq 0.05 \text{ or } X > 0.95 \\ 0 \text{ else.} \end{cases}$

4) Too high size, test is not to be recommended. Test based on $T = \log(X_1) + \log(X_2)$ should be used instead.

5) $\varphi^*(\mathbf{X}) = \begin{cases} 1 \text{ if } X_{(1)} > k \\ 0 \text{ if } X_{(1)} \leq k \end{cases}$ where $k = \theta_0 - \frac{ln(\alpha)}{n}$.

6) $\varphi^*(\mathbf{X}) = \begin{cases} 1 \text{ if } T = \sum_{i=1}^{10} X_i > 14 \\ 0.317 \text{ if } T = 14 \\ 0 \text{ if } T < 14 \end{cases}$

7) Rejection region: $\{\mathbf{X} : \sum_{i=1}^n X_i \geq k = \theta_0 \gamma_{n,\alpha}\}$ where $\gamma_{n,\alpha}$ is the upper $\alpha * 100\%$ point of the gamma (n) distribution (exact result). *Asymptotically,* it holds $k \approx n\theta_0 + \sqrt{n}\theta_0 z_\alpha$.

8) See your lecture.

9) c) If $T = \frac{mn}{m+n}(\bar{X} - \bar{Y})^2$ then the generalized likelihood ratio test is

$\varphi^*(\mathbf{X}) = \begin{cases} 1 \text{ if } T \geq \chi^2_{1,\alpha} \\ 0 \text{ if } T < \chi^2_{1,\alpha} \end{cases}$