Example related to LMP tests:

$X = (X_1, X_2, \dots, X_n)$ i.i.d. Cauchy$(\theta, 1)$ $H_0: \theta \le 0$ vs $H_1: \theta > 0$

$$L(X, \theta) = \frac{1}{\pi^n} \prod_{i=1}^{n} \frac{1}{1 + (x_i - \theta)^2}$$

$V(X, \theta)\big|_{\theta = 0}$ is proportional to $\displaystyle \sum_{i=1}^{n} \frac{2(X_i - \theta)}{1 + (x_i - \theta)^2}\Bigg|_{\theta = 0} = \sum_{i=1}^{n} \frac{2X_i}{1 + X_i^2}$

Hence LMP test:

$$\varphi^* = \begin{cases} 1 & T = \displaystyle\sum_{i=1}^{n} \frac{2X_i}{1 + X_i^2} > K \\ 0 & \quad -\,\|\,- \le K \end{cases}$$

How to choose K:

If $U = \frac{2x}{1 + x^2}$, $x \sim C(0,1) \Rightarrow EU = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{2x\,dx}{(1 + x^2)^2} = 0$

$$\operatorname{Var} U = EU^2 = \frac{4}{\pi} \int_{-\infty}^{+\infty} \frac{x^2\,dx}{(1 + x^2)^3} = \frac{2}{\pi} \int_{-\infty}^{+\infty} \frac{x\,d(1 + x^2)}{(1 + x^2)^3} =$$

$$= -\frac{1}{\pi} \int_{-\infty}^{+\infty} x\,d\frac{1}{(1 + x^2)^2} = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{1 \pm x^2}{(1 + x^2)^2}\,dx = \dots = \frac{1}{2}$$

Hence $U_i = \frac{2X_i}{1 + X_i^2}$ have mean 0 and $\operatorname{Var} U_i = \frac{1}{2} < \infty$

Hence $\frac{\sqrt{2}\,T}{\sqrt{n}} \xrightarrow{L} N(0, 1)$ (CLT)

$$T > K \Leftrightarrow \frac{\sqrt{2}\,T}{\sqrt{n}} > \frac{\sqrt{2}\,K}{\sqrt{n}} \approx Z_\alpha \longrightarrow K = \frac{Z_\alpha \cdot \sqrt{n}}{\sqrt{2}}$$