THE UNIVERSITY OF NEW SOUTH WALES

DEPARTMENT OF STATISTICS

MID SESSION TEST - 2018 -Friday, 7th September (Week 7)

MATH5905

Time allowed: 75 minutes

1. In a sequence of consecutive years $1, 2, \ldots, n$, an annual number of high-risk events is recorded by a bank. The random counts $X_i, i = 1, 2, \ldots, n$ of high-risk events in a given year is modelled via Poisson(θ) distribution and can be assumed independent from year to year. Within the last eight years counts were $0, 3, 1, 1, 2, 2, 4, 1$.

   a) Given that $T = \sum_{i=1}^{n} X_i$ is sufficient and complete for θ, derive the UMVUE of $\tau(\theta) = \theta e^{-\theta}$, i.e., the probability that exactly one extremal event in a given year will emerge. Justify your answer and evaluate the probability using the given data.

   b) Calculate the Cramer-Rao bound for the minimal variance of an unbiased estimator of $\tau(\theta) = \theta e^{-\theta}$. Does the variance of the UMVUE of $\tau(\theta)$ attain this bound? Give reasons.

   c) Find the MLE $\hat{\tau}$ of $\tau(\theta)$. Compare the numerical values in a) and c) and comment.

   d) The prior on θ is Gamma$(2, 0.5)$. Determine the Bayesian estimator of θ w.r.t. quadratic loss. **Note:** You may use that for known $\alpha > 0, \beta > 0$, the $Gamma(\alpha, \beta)$ density is given by:

   $$f(x; \alpha, \beta) = \frac{e^{-\frac{x}{\beta}} x^{\alpha-1}}{\Gamma(\alpha)\beta^{\alpha}}, x > 0.$$

   Here $\Gamma(\alpha) = \int_0^{\infty} e^{-x} x^{\alpha-1} dx$ is the gamma function. If X is distributed $Gamma(\alpha, \beta)$ then $EX = \alpha\beta, V(X) = \alpha\beta^2$ holds. $\diamond$

   e) The bank claims that the intensity θ is less than 1.5. Test the bank's claim via Bayesian testing with a zero-one loss. You can use: $10^{16}/\Gamma(16) = 7647.164$, $\int_0^{1.5} exp(-10x) * x^{15} dx = 0.000056$.

2. Let $X_1, X_2, \ldots, X_n$ be independent random variables, with a density

   $$f(x; \theta) = \begin{cases} \frac{2x}{\theta^2}, 0 < x < \theta, \\ 0 \text{ else} \end{cases}$$

   where $\theta > 0$ is an unknown parameter. If $Z_n = X_{(n)}$, then

   a) Show that the density of Z_n is

   $$f_{Z_n}(z; \theta) = \begin{cases} \frac{2nz^{2n-1}}{\theta^{2n}}, 0 < z < \theta, \\ 0 \text{ else} \end{cases}$$

   (**Hint:** find the cdf $F_{Z_n}(z; \theta)$ of Z_n first).

   b) Argue that Z_n is a sufficient and complete statistic for θ.

   c) Find the UMVUE of the parameter θ as a function of Z_n.