COMP9313: Big Data Management

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Course web site: http://www.cse.unsw.edu.au/~cs9313/

Chapter 9: Finding Similar Items: Locality Sensitive Hashing

A Common Metaphor

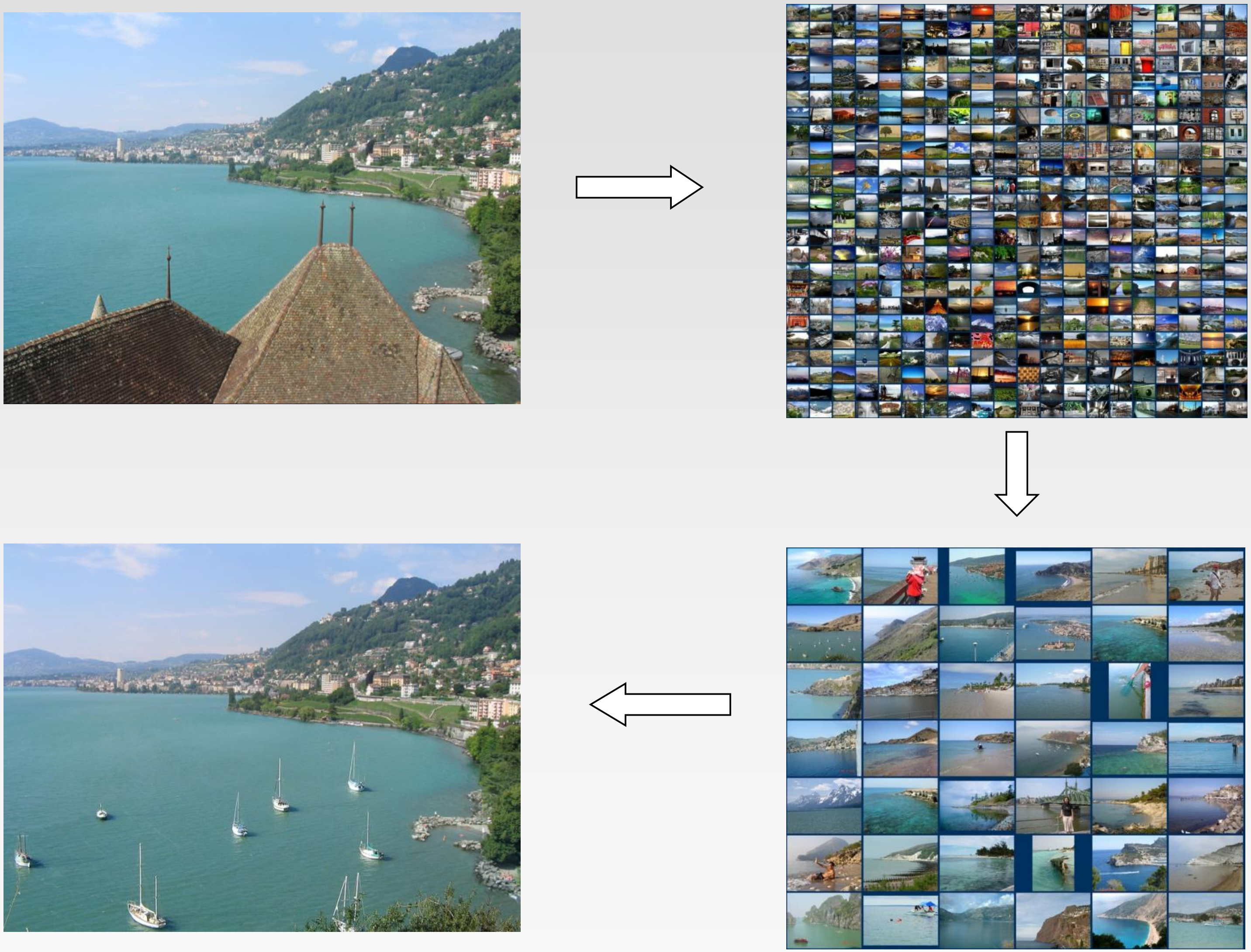
Many problems can be expressed as finding "similar" sets:

- **Find near-neighbors in high-dimensional space**

Examples:

- **Pages with similar words**
  - For duplicate detection, classification by topic
- **Customers who purchased similar products**
  - Products with similar customer sets
- **Images with similar features**
  - Google image search

Images with Similar Features



Problem for Today's Lecture

Given: High dimensional data points $x_1, x_2, \ldots$

For example: Image is a long vector of pixel colors

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 1 \\ 0 & 1 & 0 \end{bmatrix} \rightarrow [1\ 2\ 1\ 0\ 2\ 1\ 0\ 1\ 0]$$

And some distance function $d(x_1, x_2)$

Which quantifies the "distance" between x_1 and x_2

Goal: Find **all pairs of data points** (x_i, x_j) that are within some distance threshold $d(x_i, x_j) \leq s$

Note: Naïve solution would take $O(N^2)$ ☹

where N is the number of data points

MAGIC: This can be done in $O(N)$!! How?

Distance Measures

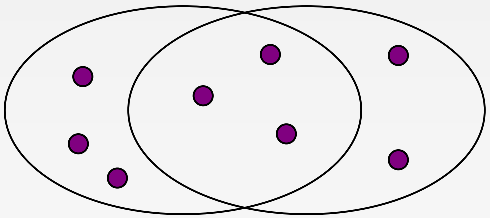
Goal: Find near-neighbors in high-dim. space

- We formally define "near neighbors" as points that are a "small distance" apart

For each application, we first need to define what "**distance**" means

Today: Jaccard distance/similarity

- The **Jaccard similarity** of two **sets** is the size of their intersection divided by the size of their union:
  $\boldsymbol{sim(C_1, C_2) = |C_1 \cap C_2| / |C_1 \cup C_2|}$
- **Jaccard distance:** $\boldsymbol{d(C_1, C_2) = 1 - |C_1 \cap C_2| / |C_1 \cup C_2|}$



3 in intersection
8 in union
Jaccard similarity= 3/8
Jaccard distance = 5/8

Task: Finding Similar Documents

Goal: **Given a large number (N in the millions or billions) of documents, find "near duplicate" pairs·**

Applications:

Mirror websites, or approximate mirrors

- Don't want to show both in search results

Similar news articles at many news sites

- Cluster articles by "same story"

Problems:

Many small pieces of one document can appear out of order in another

Too many documents to compare all pairs

Documents are so large or so many that they cannot fit in main memory

3 Essential Steps for Similar Docs

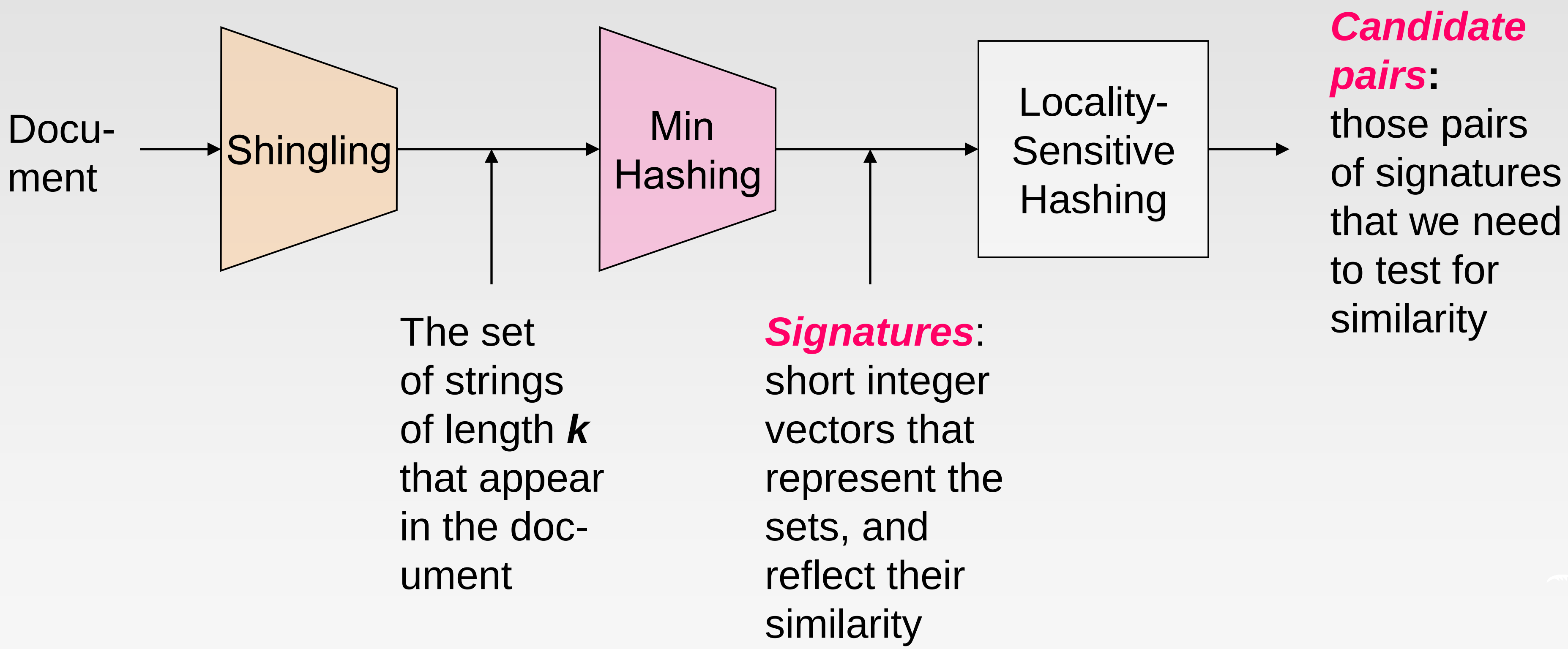
1. ***Shingling:*** Convert documents to sets

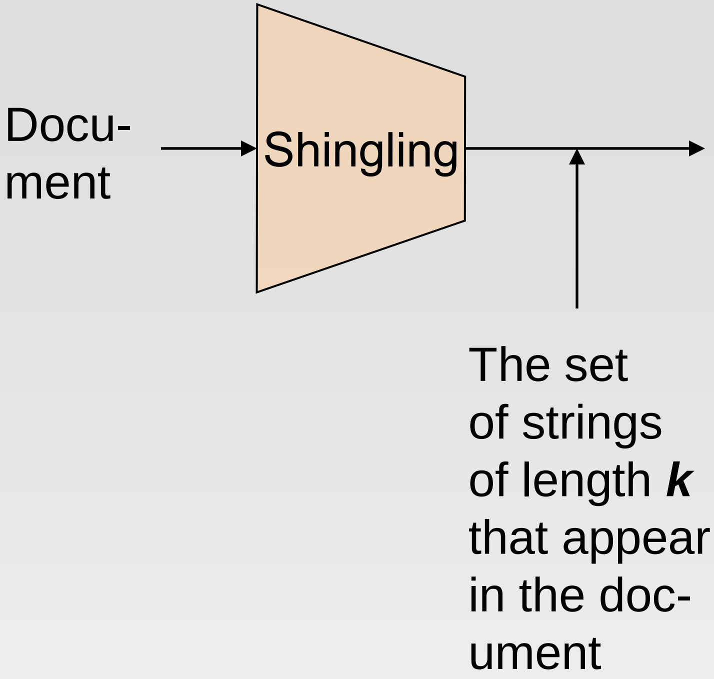
2. ***Min-Hashing:*** Convert large sets to short signatures, while preserving similarity

3. ***Locality-Sensitive Hashing:*** Focus on pairs of signatures likely to be from similar documents

   Candidate pairs!

The Big Picture





Step 1: ***Shingling:*** Convert documents to sets

Documents as High-Dim Data

Step 1: ***Shingling:*** **Convert documents to sets**

Simple approaches:

- Document = set of words appearing in document
- Document = set of "important" words
- Don't work well for this application. Why?

Need to account for ordering of words!

A different way: **Shingles!**

Define: Shingles

A *k*-shingle (or *k*-gram) for a document is a sequence of *k* tokens that appears in the doc

- Tokens can be characters, words or something else, depending on the application
- Assume tokens = characters for examples

Example: **k=2**; document $\mathbf{D_1}$ = abcab
Set of 2-shingles: $\mathbf{S(D_1)}$ = {ab, bc, ca}

- **Option:** Shingles as a bag (multiset), count ab twice: $\mathbf{S'(D_1)}$ **=** {ab, bc, ca, ab}

Shingles and Similarity

Documents that are intuitively similar will have many shingles in common.

Changing a word only affects k-shingles within distance k-1 from the word.

Reordering paragraphs only affects the 2k shingles that cross paragraph boundaries.

Example: k=3, “The dog which chased the cat” versus “The dog that chased the cat”.

Only 3-shingles replaced are g_w, _wh, whi, hic, ich, ch_, and h_c.

Compressing Shingles

To **compress long shingles**, we can **hash** them to (say) 4 bytes

Represent a document by the set of hash values of its *k*-shingles

> **Idea:** Two documents could (rarely) appear to have shingles in common, when in fact only the hash-values were shared

Example: **k=2**; document $\mathbf{D_1}$= abcab
Set of 2-shingles: $\mathbf{S(D_1)}$ = {ab, bc, ca}
Hash the singles: $\mathbf{h(D_1)}$ = {1, 5, 7}

Similarity Metric for Shingles

Document D_1 is a set of its k-shingles C_1=S(D_1)

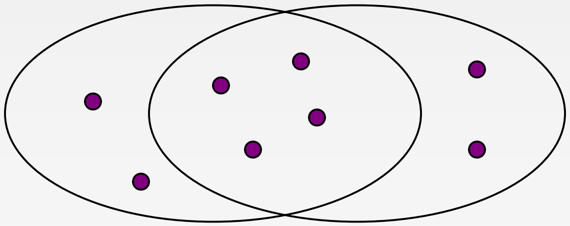
Equivalently, each document is a
0/1 vector in the space of *k*-shingles

Each unique shingle is a dimension

Vectors are very sparse

A natural similarity measure is the Jaccard similarity:

$$sim(D_1, D_2) = |C_1 \cap C_2| / |C_1 \cup C_2|$$



Working Assumption

Documents that have lots of shingles in common have similar text, even if the text appears in different order

If we pick ***k*** too small, then we would expect most sequences of ***k*** characters to appear in most documents

- We could have documents whose shingle-sets had high Jaccard similarity, yet the documents had none of the same sentences or even phrases
- Extreme case: when we use ***k*** = 1, almost all Web pages will have high similarity.

Caveat: You must pick ***k*** large enough, or most documents will have most shingles

- ***k*** = 5 is OK for short documents
- ***k*** = 10 is better for long documents

Motivation for Minhash/LSH

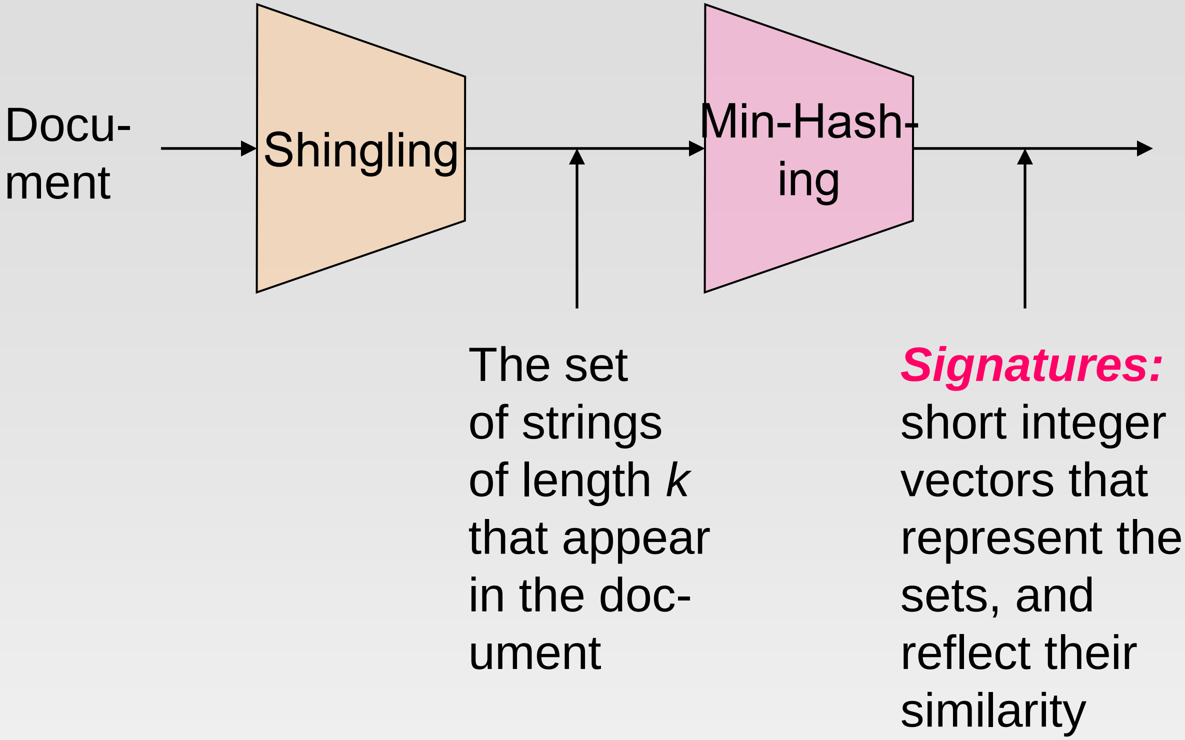
Suppose we need to find near-duplicate documents among $N = 1$ million documents

Naïvely, we would have to compute **pairwise Jaccard similarities** for **every pair of docs**

$N(N-1)/2 \approx 5*10^{11}$ comparisons

At 10^5 secs/day and 10^6 comparisons/sec, it would take **5 days**

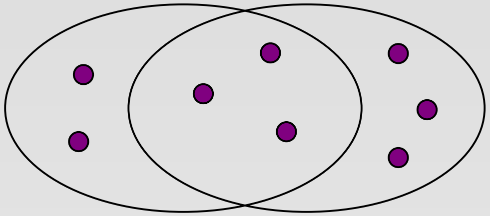
For $N = 10$ million, it takes more than a year…



Step 2: ***Minhashing:*** Convert **large sets** to **short signatures**, while **<u>preserving similarity</u>**

Encoding Sets as Bit Vectors

Many similarity problems can be formalized as **finding subsets that have significant intersection**



Encode sets using 0/1 (bit, boolean) vectors

- One dimension per element in the universal set

Interpret set intersection as bitwise **AND**, and set union as bitwise **OR**

Example: $\mathbf{C_1}$ = 10111; $\mathbf{C_2}$ = 10011

- Size of intersection **= 3**; size of union **= 4**,
- **Jaccard similarity** (not distance) **= 3/4**
- **Distance: $d(C_1,C_2)$ = 1 – (Jaccard similarity) = 1/4**

From Sets to Boolean Matrices

Rows = elements (shingles)

Columns = sets (documents)

- 1 in row ***e*** and column ***s*** if and only if ***e*** is a member of ***s***
- Column similarity is the Jaccard similarity of the corresponding sets (rows with value *1)*
- **Typical matrix is sparse!**

Each document is a column:

Documents

Shingles

1	1	1	0
1	1	0	1
0	1	0	1
0	0	0	1
1	0	0	1
1	1	1	0
1	0	1	0

From Sets to Boolean Matrices

Example: S_1 = {a, d}, S_2 = {c}, S_3 = {b, d, e}, and S_4 = {a, c, d}

$Element$	S_1	S_2	S_3	S_4
a	1	0	0	1
b	0	0	1	0
c	0	1	0	1
d	1	0	1	1
e	0	0	1	0

sim(S_1, S_3) = ?

- Size of intersection = 1; size of union = 4, Jaccard similarity (not distance) = 1/4
- **d(S_1, S_2) = 1 – (Jaccard similarity) = 3/4**

Outline: Finding Similar Columns

So far:

Documents $\rightarrow$ Sets of shingles

Represent sets as boolean vectors in a matrix

Next goal: **Find similar columns while computing small signatures**

Similarity of columns == similarity of signatures

Outline: Finding Similar Columns

Next Goal: **Find similar columns, Small signatures**

Naïve approach:

- **1) Signatures of columns:** small summaries of columns
- **2) Examine pairs of signatures** to find similar columns
  - **Essential:** Similarities of signatures and columns are related
- **3) Optional:** Check that columns with similar signatures are really similar

Warnings:

- Comparing all pairs may take too much time: **Job for LSH**
  - These methods can produce false negatives, and even false positives (if the optional check is not made)

Hashing Columns (Signatures)

Key idea: "hash" each column C to a small ***signature*** $h(C)$, such that:

(1) $h(C)$ is small enough that the signature fits in RAM

(2) $sim(C_1, C_2)$ is the same as the "similarity" of signatures $h(C_1)$ and $h(C_2)$

Goal: Find a hash function $h(\cdot)$ such that:

If $sim(C_1,C_2)$ is high, then with high prob. $h(C_1) = h(C_2)$

If $sim(C_1,C_2)$ is low, then with high prob. $h(C_1) \neq h(C_2)$

Hash docs into buckets. Expect that "most" pairs of near duplicate docs hash into the same bucket!

Min-Hashing

Goal: Find a hash function *h(·)* such that:

if $sim(C_1,C_2)$ is high, then with high prob. $h(C_1) = h(C_2)$

if $sim(C_1,C_2)$ is low, then with high prob. $h(C_1) \neq h(C_2)$

Clearly, the hash function depends on the similarity metric:

Not all similarity metrics have a suitable hash function

There is a suitable hash function for the Jaccard similarity: Min-Hashing

Min-Hashing

Imagine the rows of the boolean matrix permuted under **random permutation** π

Define a **"hash" function** $h_\pi(C)$ = the index of the **first** (in the permuted order π) row in which column ***C*** has value **1**:

$$h_\pi(C) = \min_\pi \pi(C)$$

Use several (e.g., 100) independent hash functions (that is, permutations) to create a signature of a column

Min-Hashing Example

1	0	1	1	0
2	0	0	1	1
3	1	0	0	0
4	0	1	0	1
5	0	0	0	1
6	1	1	0	0
7	0	0	1	0

Input Matrix

3	1	1	2

Signature Matrix

Min-Hashing Example

7	1	0	1	1	0
6	2	0	0	1	1
5	3	1	0	0	0
4	4	0	1	0	1
3	5	0	0	0	1
2	6	1	1	0	0
1	7	0	0	1	0

Input Matrix

3	1	1	2
2	2	1	3

Signature Matrix

Min-Hashing Example

6	7	1		0	1	1	0
3	6	2		0	0	1	1
1	5	3		1	0	0	0
7	4	4		0	1	0	1
2	3	5		0	0	0	1
5	2	6		1	1	0	0
4	1	7		0	0	1	0

Input Matrix

3	1	1	2
2	2	1	3
1	5	3	2

Signature Matrix

Min-Hashing Exa

Note: Another (equivalent) way is to store row indexes:

1	5	1	5
2	3	1	3
6	4	6	4

2nd element of the permutation is the first to map to a 1

Permutation π

2	4	3
3	2	4
7	1	7
6	3	2
1	6	6
5	7	1
4	5	5

Input matrix (Shingles x Documents)

1	0	1	0
1	0	0	1
0	1	0	1
0	1	0	1
0	1	0	1
1	0	1	0
1	0	1	0

Signature matrix *M*

2	1	2	1
2	1	4	1
1	2	1	2

4th element of the permutation is the first to map to a 1

The Min-Hash Property

0	0
0	0
1	**1**
0	0
0	1
1	0

Choose a random permutation π

Claim: **$\Pr[h_\pi(C_1) = h_\pi(C_2)] = sim(C_1, C_2)$**

Why?

Let **X** be a doc (set of shingles), ***y*** $\in$ ***X*** is a shingle

Then: **$\Pr[\pi(y) = \min(\pi(X))] = 1/|X|$**

- It is equally likely that any ***y*** $\in$ ***X*** is mapped to the ***min*** element

Let ***y*** be s.t. $\pi(y) = \min(\pi(C_1 \cup C_2))$

Then either: $\pi(y) = \min(\pi(C_1))$ if $y \in C_1$, **or**

$\pi(y) = \min(\pi(C_2))$ if $y \in C_2$

So the prob. that **both** are true is the prob. ***y*** $\in C_1 \cap C_2$

$\Pr[\min(\pi(C_1))=\min(\pi(C_2))]=|C_1 \cap C_2|/|C_1 \cup C_2|= sim(C_1, C_2)$

One of the two cols had to have 1 at position ***y***

Four Types of Rows

Given cols C_1 and C_2, rows may be classified as:

	C_1	C_2
A	1	1
B	1	0
C	0	1
D	0	0

a = # rows of type A, etc.

Note: **$sim(C_1, C_2) = a/(a + b + c)$**

Then: **Pr**$[h(C_1) = h(C_2)] = Sim(C_1, C_2)$

Look down the cols C_1 and C_2 until we see a 1

If it's a type-*A* row, then $h(C_1) = h(C_2)$
If a type-*B* or type-*C* row, then not

Similarity for Signatures

Permutation π

2	4	3
3	2	4
7	1	7
6	3	2
1	6	6
5	7	1
4	5	5

Input matrix (Shingles x Documents)

1	0	1	0
1	0	0	1
0	1	0	1
0	1	0	1
0	1	0	1
1	0	1	0
1	0	1	0

Signature matrix *M*

2	1	2	1
2	1	4	1
1	2	1	2

Similarities:

	1-3	2-4	1-2	3-4
Col/Col	0.75	0.75	0	0
Sig/Sig	0.67	1.00	0	0

Similarity for Signatures

We know: **$\Pr[h_\pi(C_1) = h_\pi(C_2)] = sim(C_1, C_2)$**

Now generalize to multiple hash functions

The *similarity of two signatures* is the fraction of the hash functions in which they agree

Note: Because of the Min-Hash property, the similarity of columns is the same as the expected similarity of their signatures

Min-Hash Signatures

Pick K=100 random permutations of the rows

Think of ***sig*(C)** as a column vector

s*ig*(C)[i] = according to the *i*-th permutation, the index of the first row that has a 1 in column *C*

$$sig(C)[i] = \min(\pi_i(C))$$

Note: The sketch (signature) of document *C* is small **~100 bytes!**

We achieved our goal! We "compressed" long bit vectors into short signatures

Implementation Trick

Permuting rows even once is prohibitive

Row hashing!

Pick **K = 100** hash functions k_i

Ordering under k_i gives a random row permutation!

One-pass implementation

For each column C and hash-func. k_i keep a "slot" for the min-hash value

Initialize all $sig(C)[i] = \infty$

Scan rows looking for 1s

- Suppose row j has 1 in column C
- Then for each k_i :
  - If $k_i(j) < sig(C)[i]$, then $sig(C)[i] \leftarrow k_i(j)$

How to pick a random hash function h(x)?

Universal hashing:

$h_{a,b}(x)=((a \cdot x+b) \bmod p) \bmod N$

where:

a,b … random integers

p … prime number (p > N)

Implementation Example

Row	S_1	S_2	S_3	S_4	$x+1 \mod 5$	$3x+1 \mod 5$
0	1	0	0	1	1	1
1	0	0	1	0	2	4
2	0	1	0	1	3	2
3	1	0	1	1	4	0
4	0	0	1	0	0	3

0. Initialize all ***sig(C)[i] = ∞***

	S_1	S_2	S_3	S_4
h_1	∞	∞	∞	∞
h_2	∞	∞	∞	∞

Row 0: we see that the values of $h_1(0)$ and $h_2(0)$ are both 1, thus $sig(S_1)[0] = 1$, $sig(S_1)[1] = 1$, $sig(S_4)[0] = 1$, $sig(S_4)[1] = 1$,

	S_1	S_2	S_3	S_4
h_1	1	∞	∞	1
h_2	1	∞	∞	1

Row 1, we see $h_1(1) = 2$ and $h_2(1) = 4$, thus $sig(S_3)[0] = 2$, $sig(S_3)[1] = 4$

	S_1	S_2	S_3	S_4
h_1	1	∞	2	1
h_2	1	∞	4	1

Implementation Example

Row	S_1	S_2	S_3	S_4	$x+1 \mod 5$	$3x+1 \mod 5$
0	1	0	0	1	1	1
1	0	0	1	0	2	4
2	0	1	0	1	3	2
3	1	0	1	1	4	0
4	0	0	1	0	0	3

Row 2: $h_1(2) = 3$ and $h_2(2) = 2$, thus

$sig(S_2)[0] = 3$, $sig(S_2)[1] = 2$, no update for S_4

	S_1	S_2	S_3	S_4
h_1	1	3	2	1
h_2	1	2	4	1

Row 3: $h_1(2) = 4$ and $h_2(2) = 0$, update

$sig(S_1)[1] = 0$, $sig(S_3)[1] = 0$, $sig(S_4)[1] = 0$,

	S_1	S_2	S_3	S_4
h_1	1	3	2	1
h_2	0	2	0	0

Row 4: $h_1(2) = 0$ and $h_2(2) = 3$, update

$sig(S_3)[0] = 0$,

	S_1	S_2	S_3	S_4
h_1	1	3	0	1
h_2	0	2	0	0

Implementation Example

Row	S_1	S_2	S_3	S_4	$x+1 \mod 5$	$3x+1 \mod 5$
0	1	0	0	1	1	1
1	0	0	1	0	2	4
2	0	1	0	1	3	2
3	1	0	1	1	4	0
4	0	0	1	0	0	3

$\Downarrow$

	S_1	S_2	S_3	S_4
h_1	1	3	0	1
h_2	0	2	0	0

We can estimate the Jaccard similarities of the underlying sets from this signature matrix.

Signature matrix: SIM(S_1, S_4) = 1.0

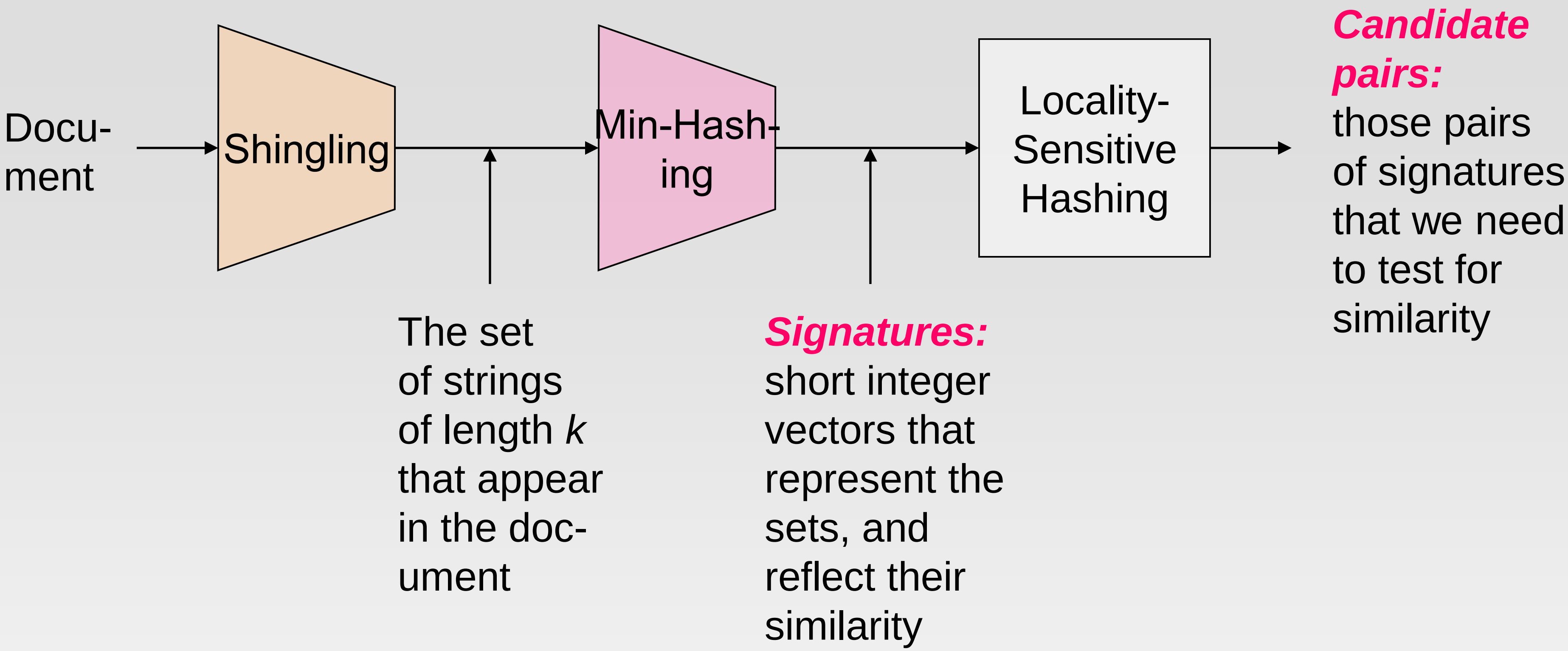
Jaccard Similarity: SIM(S_1, S_4) = 2/3

Implementation Practice

Row	C1	C2
1	1	0
2	0	1
3	1	1
4	1	0
5	0	1

$h(x) = x \bmod 5$
$g(x) = (2x+1) \bmod 5$

	Sig1	Sig2
$h(1) = 1$	∞	∞
$g(1) = 3$	∞	∞
$h(1) = 1$	1	∞
$g(1) = 3$	3	∞
$h(2) = 2$	1	2
$g(2) = 0$	3	0
$h(3) = 3$	1	2
$g(3) = 2$	2	0
$h(4) = 4$	1	2
$g(4) = 4$	2	0
$h(5) = 0$	1	0
$g(5) = 1$	2	0



Step 3: ***Locality-Sensitive Hashing:***

Focus on pairs of signatures likely to be from similar documents

LSH: First Cut

2	1	4	1
1	2	1	2
2	1	2	1

Goal: Find documents with Jaccard similarity at least ***s*** (for some similarity threshold, e.g., ***s***=0.8)

LSH – General idea: Use a function ***f(x,y)*** that tells whether ***x*** and ***y*** is a ***candidate pair****:* a pair of elements whose similarity must be evaluated

For Min-Hash matrices:

- Hash columns of signature matrix ***M*** to many buckets
- Each pair of documents that hashes into the same bucket is a **candidate pair**

Candidates from Min-Hash

2	1	4	1
1	2	1	2
2	1	2	1

Pick a similarity threshold *s* (0 < s < 1)

Columns ***x*** and ***y*** of ***M*** are a **candidate pair** if their signatures agree on at least fraction ***s*** of their rows:
M* (*i*, *x*) = *M* (*i*, *y*)** for at least frac. ***s values of ***i***

- We expect documents ***x*** and ***y*** to have the same (Jaccard) similarity as their signatures

LSH for Min-Hash

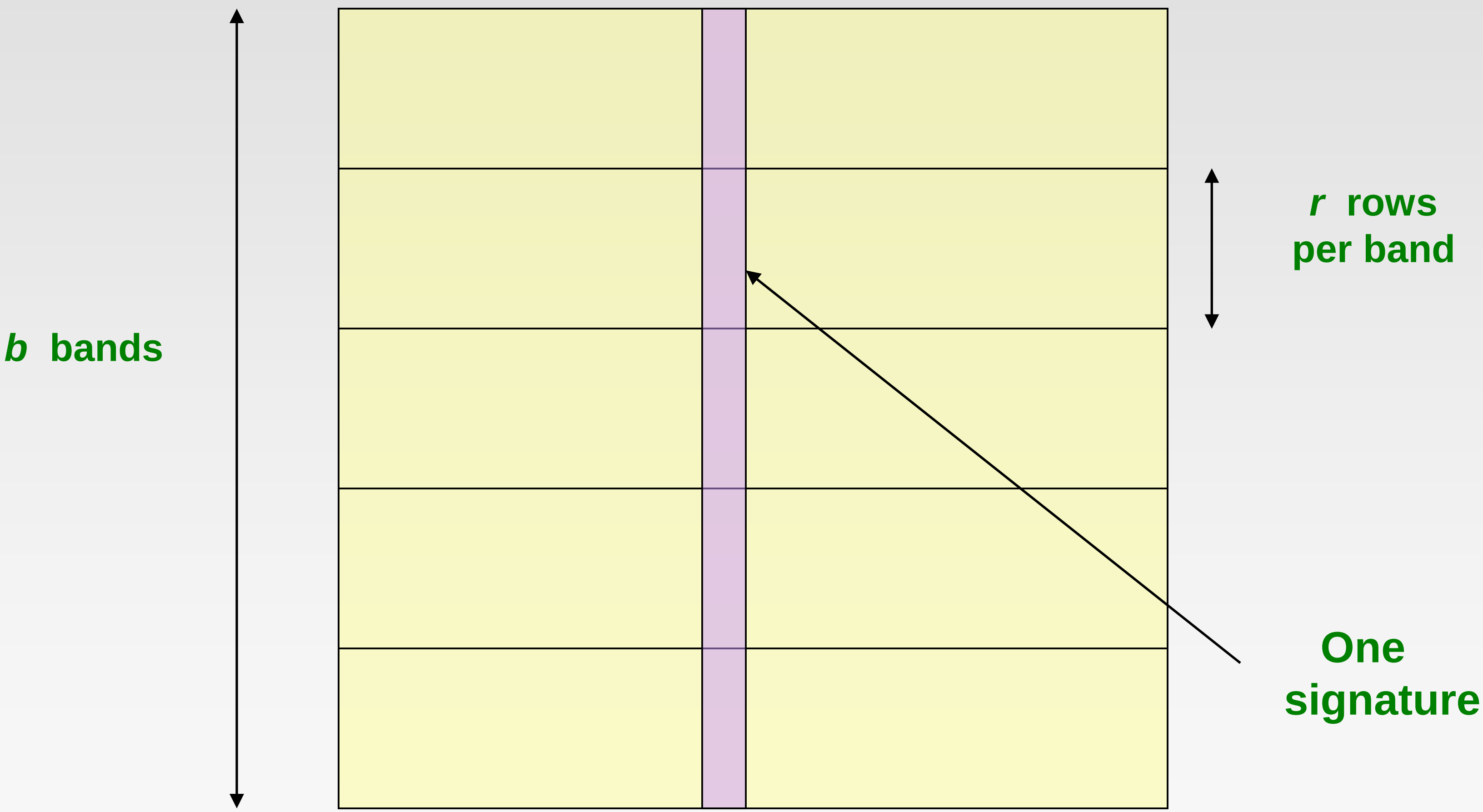
2	1	4	1
1	2	1	2
2	1	2	1

Big idea: Hash columns of signature matrix *M* several times

Arrange that (only) **similar columns** are likely to **hash to the same bucket**, with high probability

Candidate pairs are those that hash to the same bucket

Partition *M* into *b* Bands



Signature matrix *M*

Partition M into Bands

Divide matrix ***M*** into ***b*** bands of ***r*** rows

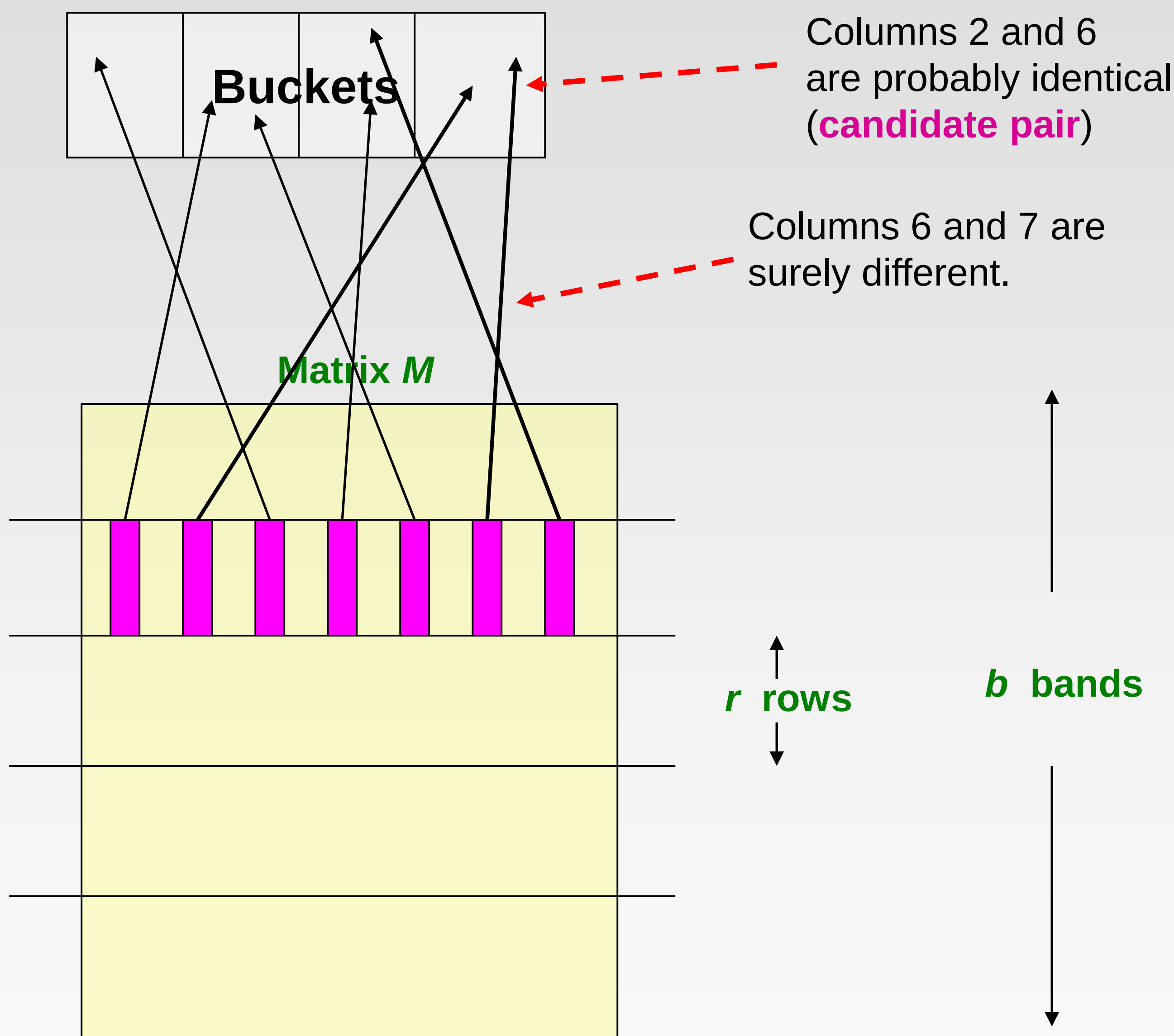
For each band, hash its portion of each column to a hash table with ***k*** buckets

- Make ***k*** as large as possible

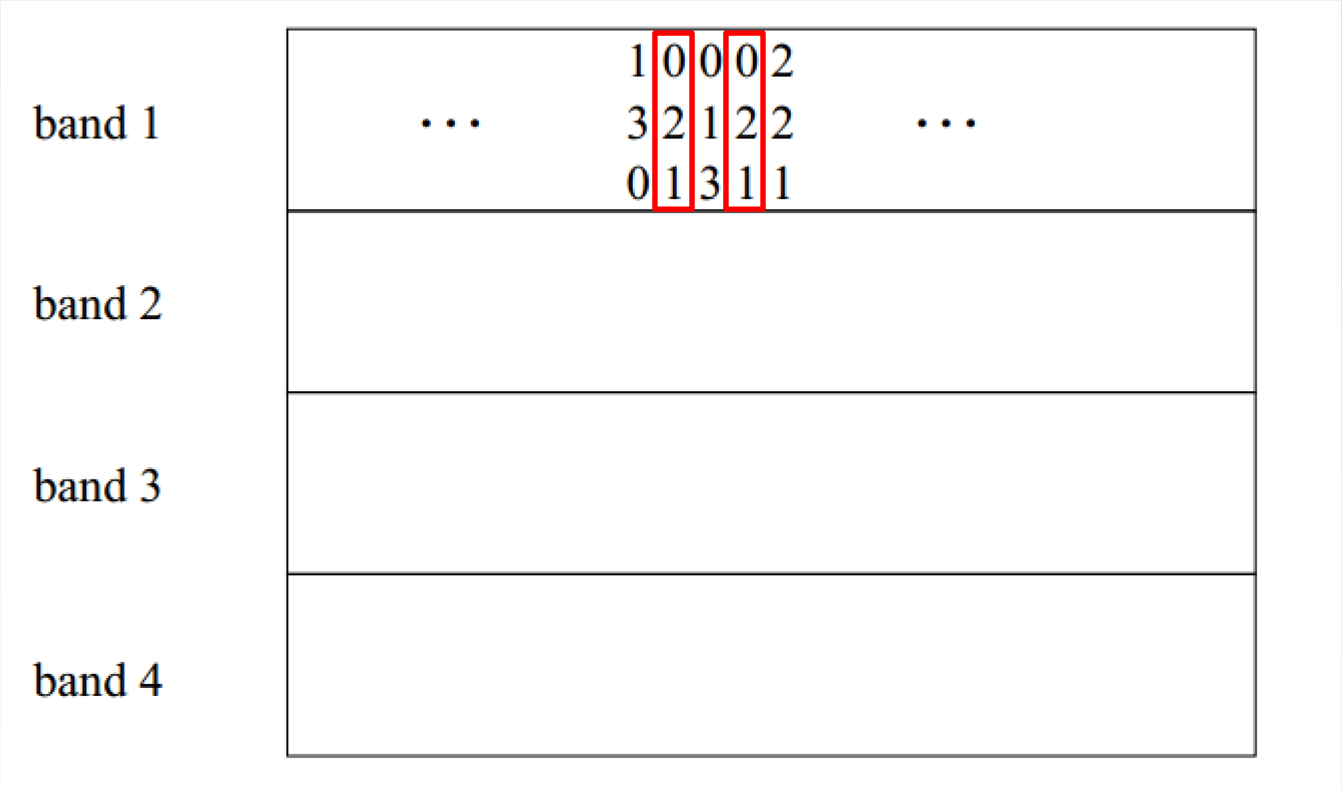
Candidate column pairs are those that hash to the same bucket for ≥ **1** band

Tune ***b*** and ***r*** to catch most similar pairs, but few non-similar pairs

Hashing Bands



Hashing Bands



Regardless of what those columns look like in the other three bands, this pair of columns will be a candidate pair

Two columns that do not agree in band 1 have three other chances to become a candidate pair; they might be identical in any one of these other bands.

Simplifying Assumption

There are **enough buckets** that columns are unlikely to hash to the same bucket unless they are **identical** in a particular band

Hereafter, we assume that “**same bucket**” means “**identical in that band**”

Assumption needed only to simplify analysis, not for correctness of algorithm

Example of Bands

Assume the following case:

Suppose 100,000 columns of ***M*** (100k docs)

Signatures of 100 integers (rows)

Therefore, signatures take 40Mb

Choose ***b*** = 20 bands of ***r*** = 5 integers/band

Goal: Find pairs of documents that are at least ***s*** = *0.8* similar

C_1, C_2 are 80% Similar

Find pairs of $\geq$ ***s***=0.8 similarity, set **b**=20, **r**=5

Assume: sim(C_1, C_2) = 0.8

Since sim(C_1, C_2) $\geq$ **s**, we want C_1, C_2 to be a **candidate pair**: We want them to hash to at **least 1 common bucket** (at least one band is identical)

Probability C_1, C_2 identical in one particular band: $(0.8)^5 = 0.328$

Probability C_1, C_2 are ***not*** similar in all of the 20 bands: $(1-0.328)^{20} = 0.00035$

i.e., about 1/3000th of the 80%-similar column pairs are **false negatives** (we miss them)

We would find 99.965% pairs of truly similar documents

C_1, C_2 are 30% Similar

Find pairs of $\geq$ ***s***=0.8 similarity, set **b**=20, **r**=5

Assume: sim(C_1, C_2) = 0.3

Since sim(C_1, C_2) < **s** we want C_1, C_2 to hash to **NO common buckets** (all bands should be different)

Probability C_1, C_2 identical in one particular band: $(0.3)^5 = 0.00243$

Probability C_1, C_2 identical in at least 1 of 20 bands: $1 - (1 - 0.00243)^{20} = 0.0474$

In other words, approximately 4.74% pairs of docs with similarity 0.3% end up becoming **candidate pairs**

- They are **false positives** since we will have to examine them (they are candidate pairs) but then it will turn out their similarity is below threshold **s**

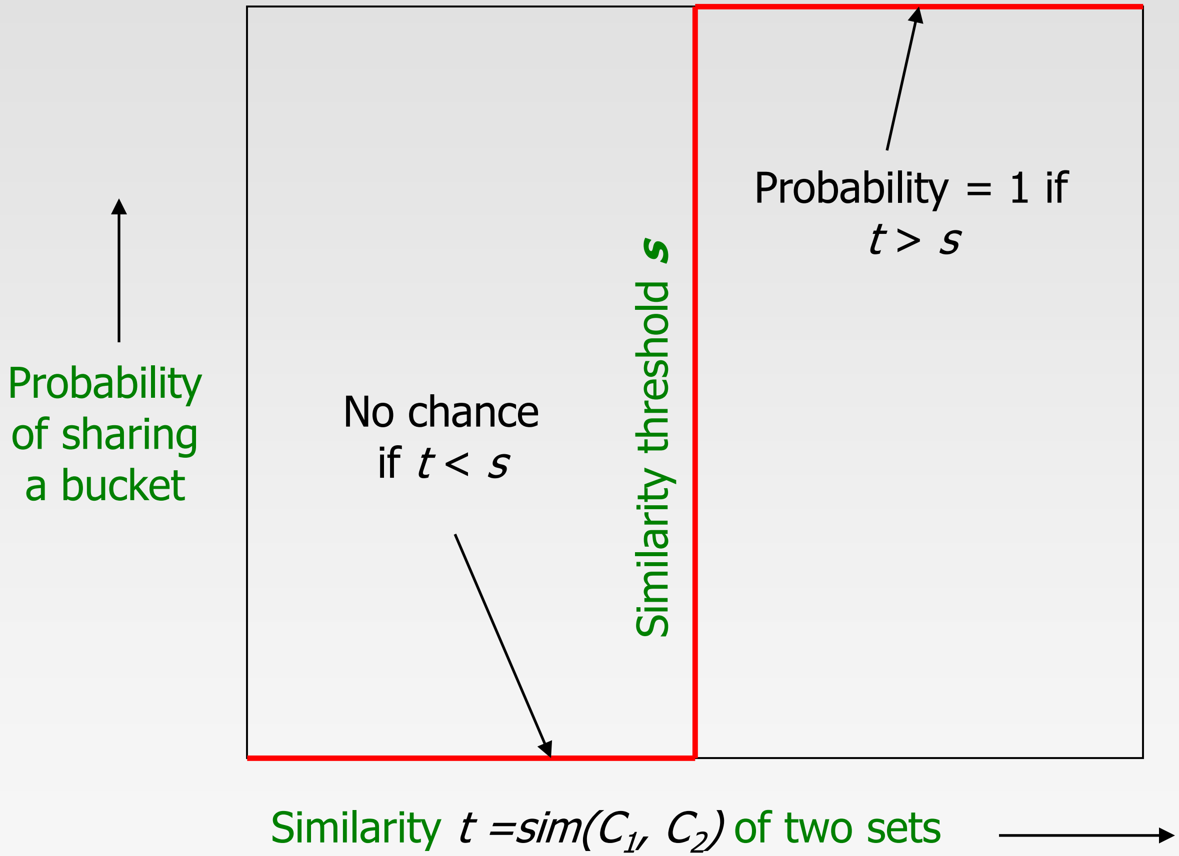
LSH Involves a Tradeoff

Pick:

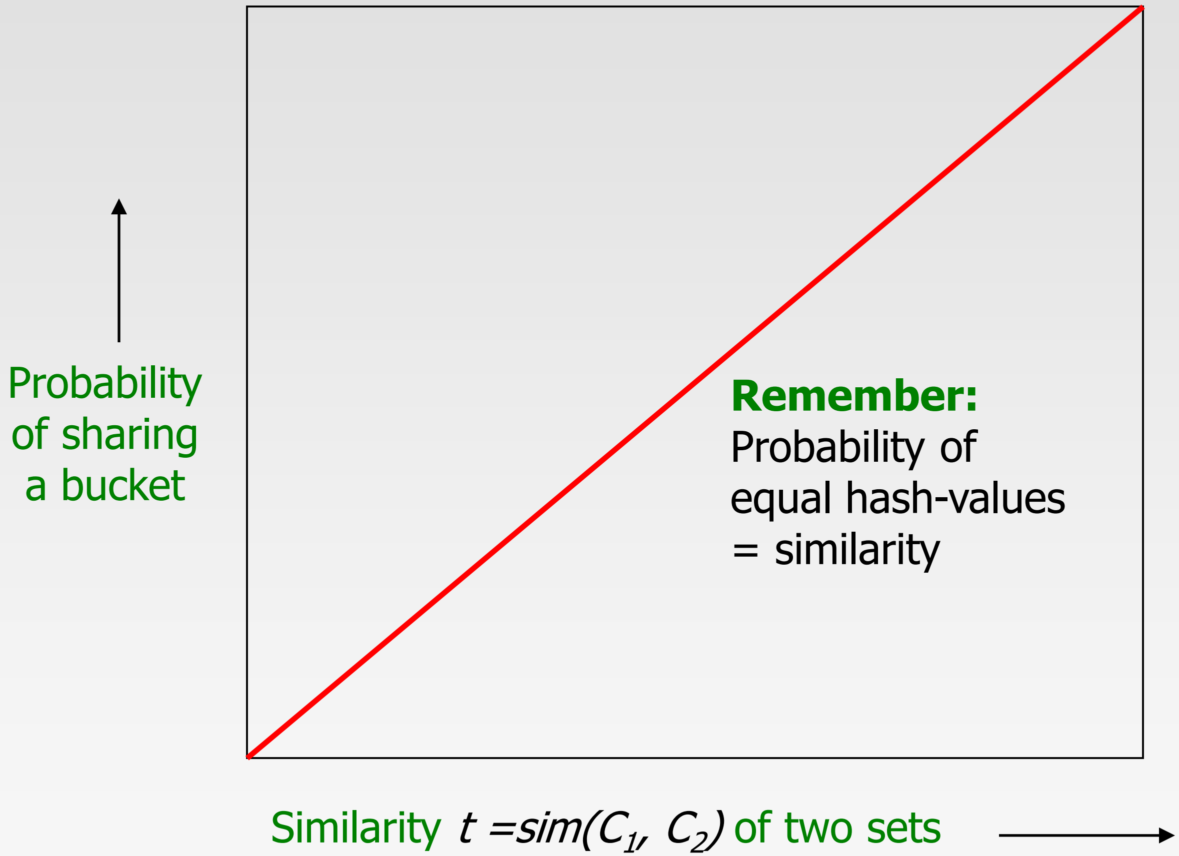
- The number of Min-Hashes (rows of ***M***)
- The number of bands ***b***, and
- The number of rows ***r*** per band to balance false positives/negatives

Example: If we had only 15 bands of 5 rows, the number of false positives would go down, but the number of false negatives would go up

Analysis of LSH – What We Want



What 1 Band of 1 Row Gives You



b bands, *r* rows/band

The probability that the minhash signatures for the documents agree in any one particular row of the signature matrix is $\mathbf{t}$ ($sim(C_1, C_2)$)

Pick any band (***r*** rows)

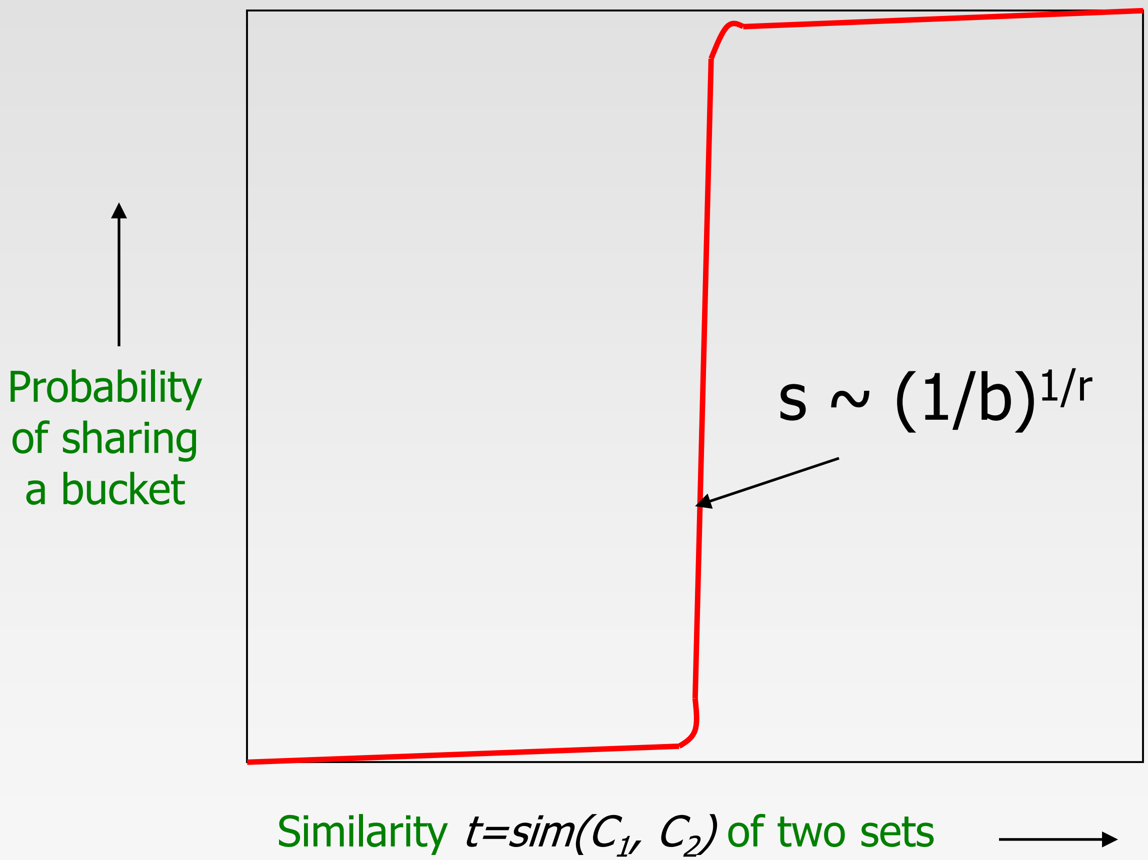
Prob. that all rows in band equal = t^r

Prob. that some row in band unequal = $1 - t^r$

Prob. that no band identical = $(1 - t^r)^b$

Prob. that at least 1 band identical = $1 - (1 - t^r)^b$

What *b* Bands of *r* Rows Gives You



At least
one band
identical

No bands
identical

$$1 - (1 - t^r)^b$$

Some row
of a band
unequal

All rows
of a band
are equal

Example: $b = 20$, $r = 5$

Similarity threshold s

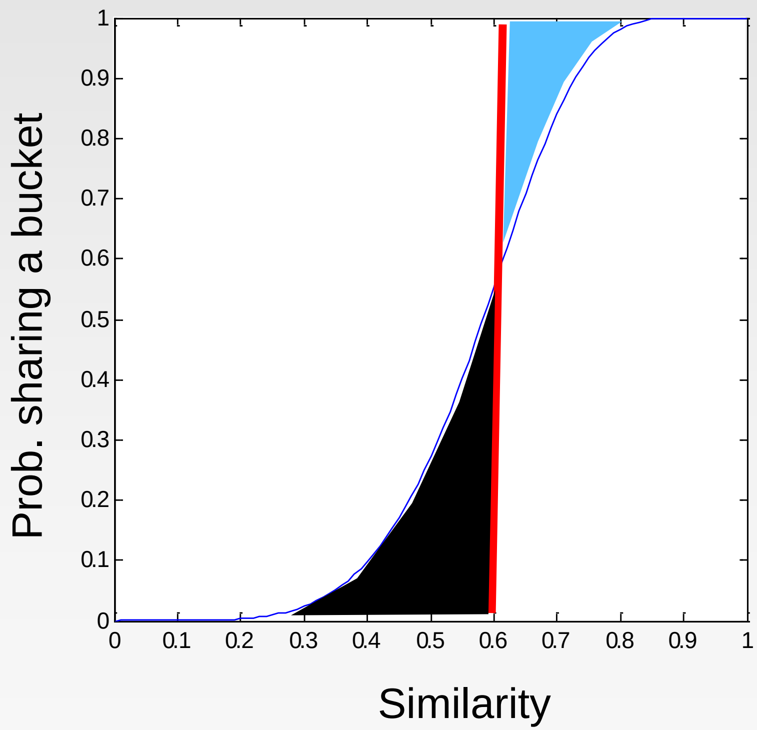
Prob. that at least 1 band is identical:

s	$1-(1-s^r)^b$
.2	.006
.3	.047
.4	.186
.5	.470
.6	.802
.7	.975
.8	.9996

Picking *r* and *b*: The S-curve

Picking *r* and *b* to get the best S-curve

50 hash-functions (r=5, b=10)



Blue area: False Negative rate
Black area: False Positive rate

LSH Summary

Tune ***M, b, r*** to get almost all pairs with similar signatures, but eliminate most pairs that do not have similar signatures

Check in main memory that **candidate pairs** really do have **similar signatures**

Optional: In another pass through data, check that the remaining candidate pairs really represent similar documents

Summary: 3 Steps

Shingling: Convert documents to sets

- We used hashing to assign each shingle an ID

Min-Hashing: Convert large sets to short signatures, while preserving similarity

- We used **similarity preserving hashing** to generate signatures with property $\mathbf{Pr}[h_\pi(C_1) = h_\pi(C_2)] = sim(C_1, C_2)$
- We used hashing to get around generating random permutations

Locality-Sensitive Hashing: Focus on pairs of signatures likely to be from similar documents

- We used hashing to find **candidate pairs** of similarity $\geq$ **s**

Distance Measures

Generalized LSH is based on some kind of “distance” between points.

Similar points are “close.”

Example: Jaccard similarity is not a distance; 1 minus Jaccard similarity is.

d is a *distance measure* if it is a function from pairs of points to real numbers such that:

1. $d(x,y) \geq 0$.
2. $d(x,y) = 0$ iff $x = y$.
3. $d(x,y) = d(y,x)$.
4. $d(x,y) \leq d(x,z) + d(z,y)$ (*triangle inequality*).

Some Euclidean Distances

L_2 norm: d(x,y) = square root of the sum of the squares of the differences between *x* and *y* in each dimension.

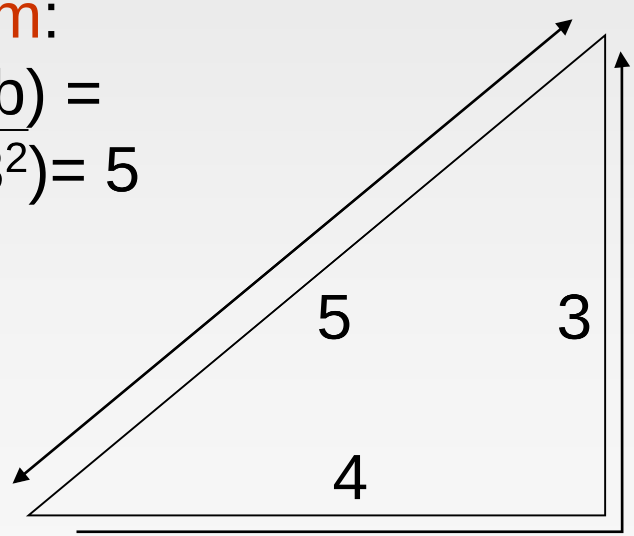
The most common notion of "distance."

L_1 norm: sum of the differences in each dimension.

Manhattan distance = distance if you had to travel along coordinates only.

b = (9,8)

L_2-norm:
dist(a,b) = $\sqrt{(4^2+3^2)} = 5$



a = (5,5)

L_1-norm:
dist(a,b) =4+3 = 7

Some Non-Euclidean Distances

Jaccard distance for sets = 1 minus Jaccard similarity.

Cosine distance for vectors = angle between the vectors.

Edit distance for strings = number of inserts and deletes to change one string into another.

Cosine Distance

Think of a point as a vector from the origin [0,0,…,0] to its location.

Two points' vectors make an angle, whose cosine is the normalized dot-product of the vectors: $p_1.p_2/|p_2||p_1|$.

Example: $p_1 = [1,0,2,-2,0]$; $p_2 = [0,0,3,0,0]$.

$p_1.p_2 = 6$; $|p_1| = |p_2| = \sqrt{9} = 3$.

$\cos(\theta) = 6/9$; θ is about 48 degrees.

Edit Distance

The *edit distance* of two strings is the number of inserts and deletes of characters needed to turn one into the other.

An equivalent definition: d(x,y) = |x| + |y| - 2|LCS(x,y)|.

LCS = *longest common subsequence* = any longest string obtained both by deleting from *x* and deleting from *y*.

Example:

x = *abcde* ; *y* = *bcduve*.

Turn *x* into *y* by deleting *a*, then inserting *u* and *v* after *d*.

- Edit distance = 3.

Or, computing edit distance through the LCS, note that LCS(x,y) = *bcde*.

Then:|x| + |y| - 2|LCS(x,y)| = 5 + 6 –2*4 = 3 = edit distance.

Hash Functions Decide Equality

There is a subtlety about what a “hash function” is, in the context of LSH families.

A hash function h really takes two elements x and y, and returns a decision whether x and y are candidates for comparison.

Example: the family of minhash functions computes minhash values and says “yes” iff they are the same.

Shorthand: “h(x) = h(y)” means h says “yes” for pair of elements x and y.

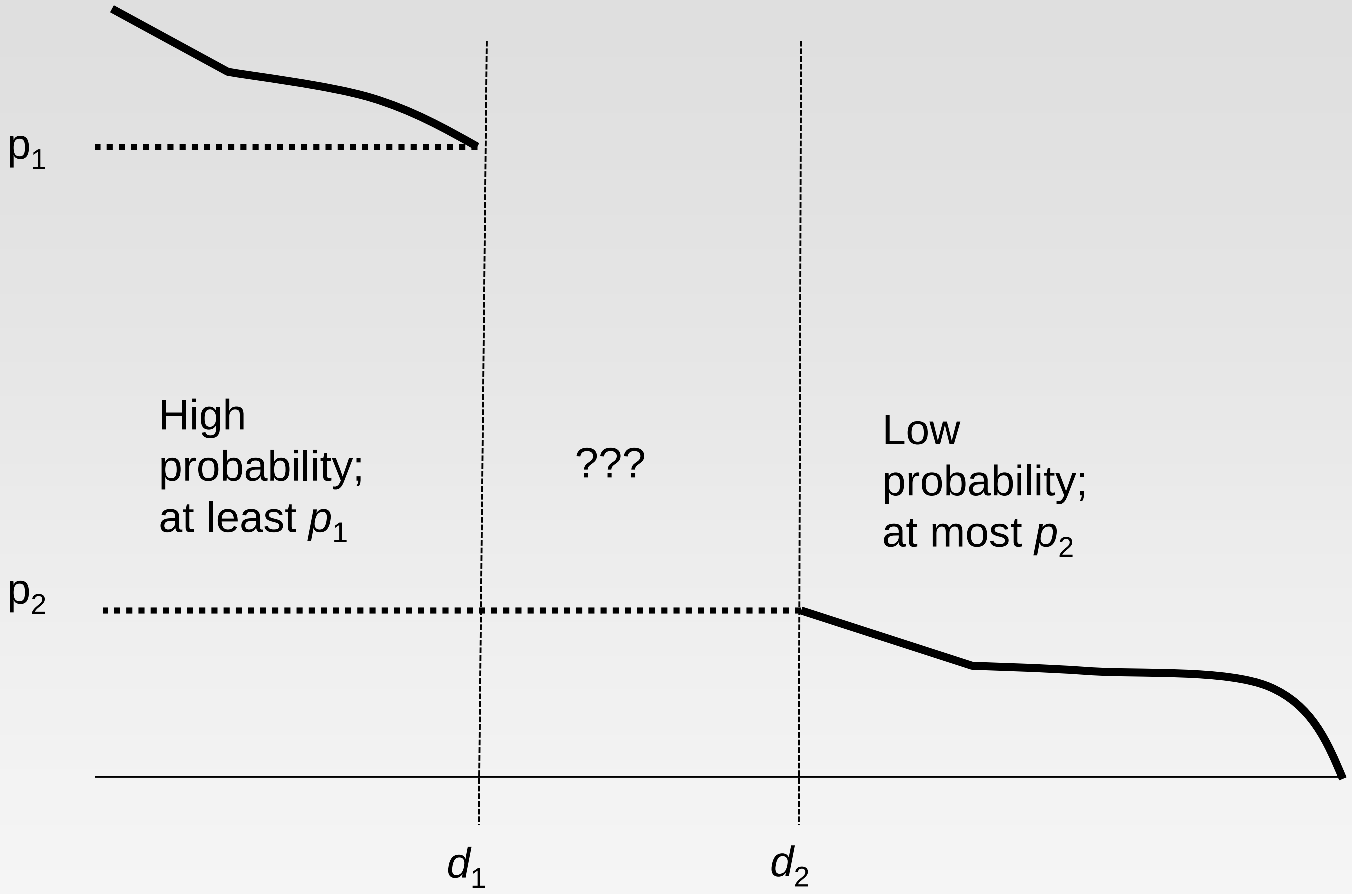
LSH Families Defined

Suppose we have a space *S* of points with a distance measure *d*.

A family **H** of hash functions is said to be *(d_1,d_2,p_1,p_2)-sensitive* if for any *x* and *y* in *S*:

1. If $d(x,y) \leq d_1$, then the probability over all *h* in **H**, that $h(x) = h(y)$ is at least p_1.
2. If $d(x,y) \geq d_2$, then the probability over all *h* in **H**, that $h(x) = h(y)$ is at most p_2.

LS Families: Illustration



Example: LS Family – (2)

Claim: **H** is a ($\boxed{1/3}$, $\boxed{3/4}$, $\boxed{2/3}$, $\boxed{1/4}$)-sensitive family for *S* and *d*.

If distance $\geq$ 3/4
(so similarity $\leq$ 1/4)

Then probability
that minhash values
agree is $\leq$ 1/4

If distance $\leq$ 1/3
(so similarity $\geq$ 2/3)

Then probability
that minhash values
agree is $\geq$ 2/3

For Jaccard similarity, minhashing gives us a
$(d_1,d_2,(1-d_1),(1-d_2))$-sensitive family for any $d_1 < d_2$.

References

Chapter 3 of Mining of Massive Datasets.

End of Chapter 9