Solutions to MST-2018-Inference

1)

a) First, we take an unbiased estimator of $\tau(\theta)$ as,

e.g. $W = I_{\{X_1=1\}}(\mathbf{X}) = \begin{cases} 1 & \text{if } X_1 = 1 \\ 0 & \text{if } X_1 \neq 1 \end{cases}$

$E_\theta W = 1 * P(X_1 = 1) = \theta e^{-\theta} = \tau(\theta)$ obviously holds.

The Lehmann-Scheffe theorem then tells us that

$\hat{\tau} = E(W / T = t)$ is the UMVUE of $\tau(\theta)$.

Now $\hat{\tau} = 1 * P(W = 1 / T = t) = \dfrac{P(W=1 \cap T=t)}{P(T=t)}$ and

we know that $\sum_{i=1}^{n} X_i \sim Po(n\theta)$. Hence we have

$$\hat{\tau} = \frac{P(X_1 = 1 \cap \sum_{i=2}^{n} X_i = t-1)}{P(\sum_{i=1}^{n} X_i = t)} = \frac{\theta e^{-\theta}\, e^{-(n-1)\theta}\, ((n-1)\theta)^{t-1}}{\cancel{(t-1)!}\, e^{-n\theta}\, n^t \theta^t / t!}$$

$$= \frac{t}{n}\left(\frac{n-1}{n}\right)^{t-1} = \bar{X}\left(1 - \frac{1}{n}\right)^{n\bar{X}-1}$$

Numerically: $\sum_{i=1}^{8} X_i = 14 \quad n = 8 \rightarrow 1.75\left(\frac{7}{8}\right)^{13} = 0.3084$

b) CR Bound is $\dfrac{(\tau'(\theta))^2}{I_{\mathbf{X}}(\theta)}$. Now $(\tau'(\theta))^2 = (\theta-1)^2 e^{-2\theta}$

For $I_{\mathbf{X}}(\theta)$: $\quad L(\mathbf{X}, \theta) = \dfrac{e^{-n\theta}\, \theta^{\sum_{i=1}^{n} X_i}}{\prod_{i=1}^{n} X_i!}$

$\ln L(\mathbf{X}, \theta) = -n\theta + \sum_{i=1}^{n} X_i \ln\theta + \text{const}$

$\frac{\partial}{\partial\theta} \ln L(\mathbf{X}, \theta) = V(\mathbf{X}, \theta) = -n + \frac{1}{\theta}\sum_{i=1}^{n} X_i$

$-\frac{\partial^2}{\partial\theta^2} \ln L(\mathbf{X}, \theta) = \frac{1}{\theta^2}\sum_{i=1}^{n} X_i \implies E\left[-\frac{\partial^2}{\partial\theta^2}\ln L(\mathbf{X}, \theta)\right] = \frac{n}{\theta}$

Hence CR Bound $= \dfrac{\theta(\theta-1)^2 e^{-2\theta}}{n}$

Bound is not attainable since the score

$V(\mathbf{X}, \theta) = -n + \frac{1}{\theta}\sum_{i=1}^{n} X_i$ cannot be factorized into $k(n,\theta)(\text{statistic} - \tau(\theta))$:

We have $V(\mathbf{X}, \theta) = \frac{n e^{\theta}}{\theta}(\bar{x} e^{-\theta} - \tau(\theta))$

c) From $V(\mathbb{X},\theta)=0$ we get $\hat{\theta}_{MLE}=\overline{X}$

Hence $\widehat{\tau(\theta)}_{MLE}=\overline{X}e^{-\overline{X}}$ Numerically this gives $\hat{\tau}(\theta)_{MLE}=0.3041 \to$ close to the UMVUE

d) $f(\mathbb{X}|\theta)\,prior(\theta) \propto \theta^{\sum_{i=1}^{n}X_i+1}e^{-(2+n)\theta}$

where $prior(\theta)\propto \theta^{2-1}e^{-2\theta}$

This reveals the posterior as $Gamma\left(\sum_{i=1}^{n}X_i+2, \frac{1}{2+n}\right)$

Hence the Bayes estimator is the mean of this posterior: $\hat{\theta}_{Bayes}=\frac{\sum_{i=1}^{n}X_i+2}{2+n}$. Numerically $\hat{\theta}_{Bayes}=1.6$

e) $P(H_0|\mathbb{X})=\frac{10^{16}}{\Gamma(16)}\int_0^{1.5}\exp(-10x)x^{15}dx=7647.164 * 0.000056$
$=0.4282<\frac{1}{2}$

Hence we reject H_0.

2) a) cdf of a single observation is $F_{X_1}(x)=\begin{cases}0 & x<0\\ \left(\frac{x}{\theta}\right)^2 & 0<x<\theta\\ 1 & x>\theta\end{cases}$

Then $F_{Z_n}(z)=P(X_{(n)}<z)=P(X_1<z\cap X_2<z\cap\dots\cap X_n<z)=$

$$=\begin{cases}0 & z<0\\ \left(\frac{z}{\theta}\right)^{2n} & 0<z<\theta\\ 1 & z>\theta\end{cases}$$

Hence $f_{Z_n}(z) = \begin{cases}\frac{2nz^{2n-1}}{\theta^{2n}} & 0<z<\theta\\ 0 & \text{else}\end{cases}$

b) - Sufficiency: write $f(x;\theta)=\frac{2x}{\theta^2}I_{(0,\theta)}(x)$. Then $L(\mathbb{X};\theta)=\frac{2^n\prod_{i=1}^{n}X_i}{\theta^{2n}}I_{(0,\theta)}(X_i)$

$=\frac{2^n}{\theta^{2n}}\prod_{i=1}^{n}X_i\, I_{(0,\theta)}(X_{(n)})$ which can be factorized in

$g(\theta,X_{(n)}).h(\mathbb{X})$ with $g(\theta,X_{(n)})=\frac{1}{\theta^{2n}}I_{(0,\theta)}(X_{(n)})$, $h(\mathbb{X})=2^n\prod_{i=1}^{n}X_i$

Hence $X_{(n)}$ is sufficient.

- Completenes: take $g(\cdot)$ with $E_\theta g(X_{(n)}) = 0 \ \forall \theta$

This implies $\frac{2n}{\theta^{2n}} \int_0^\theta g(t) t^{2n-1} dt = 0 \ \forall \theta$ and since $\frac{2n}{\theta^{2n}} \neq 0 \ \forall \theta > 0$

we have $\int_0^\theta g(t) t^{2n-1} dt = 0 \ \forall \theta > 0 \Rightarrow$ Take derivative

w.r. $\theta \Rightarrow$ $g(\theta)\theta^{2n-1} = 0 \ \forall \theta > 0 \Rightarrow g(\theta) = 0 \ \forall \theta > 0$

i.e. $P_\theta(g(t) = 0) = 1$ which implies completeness.

c) We first calculate $EZ_n = \int_0^\theta x \frac{2n x^{2n-1}}{\theta^{2n}} dx = \frac{2n}{2n+1}\theta \neq \theta$

i.e. Z_n is biased for θ but $\boxed{\frac{2n+1}{2n} Z_n}$ is unbiased

and is a function of complete and sufficient statistic.

Hence $E\left(\frac{2n+1}{2n} Z_n \middle| Z_n\right) = \boxed{\frac{2n+1}{2n} Z_n}$ is UMVUE

(this is Lehmann-Scheffé's theorem)

Total Marks 36 $= \underset{Q1}{24} + \underset{Q2}{12}$