THE UNIVERSITY OF NEW SOUTH WALES

DEPARTMENT OF STATISTICS

Solutions to selected exercises for MATH5905, Statistical Inference

Part three: Hypothesis Testing

1) Check the answers and try to explain to yourself why the MLR property holds. Study the examples from lectures first.

2) Using 1d), we know that the rejection region of the ump-α test of $H_0 : \sigma \le \sigma_0$ versus $H_1 : \sigma > \sigma_0$ is in the form $\{\sum_{i=1}^n X_i^2 \ge k\}$. To determine k, we have to "exhaust the level", that is:

$$\alpha = P(\sum_{i=1}^n X_i^2 \ge k|\sigma = \sigma_0) = P(\frac{\sum_{i=1}^n X_i^2}{\sigma_0^2} \ge \frac{k}{\sigma_0^2}) = P(\chi_n^2 \ge \frac{k}{\sigma_0^2}).$$

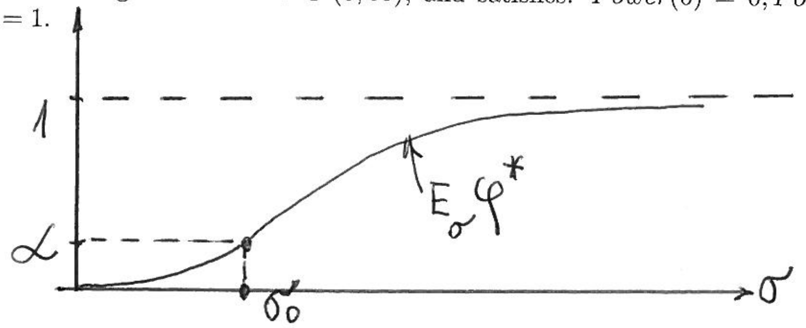
Hence $k/\sigma_0^2 = \chi^2_{n,\alpha} \Rightarrow k = \sigma_0^2\chi^2_{n,\alpha}$ and the ump-α test is:

$$\varphi^*(\mathbf{X}) = \begin{cases} 1 \text{ if } \sum_{i=1}^n X_i^2 \ge \sigma_0^2\chi^2_{n,\alpha} \\ 0 \text{ if } \sum_{i=1}^n X_i^2 < \sigma_0^2\chi^2_{n,\alpha} \end{cases}$$

The power function:

$$Power(t) = P(\sum_{i=1}^n X_i^2 > \sigma_0^2\chi^2_{n,\alpha}|\sigma = t) = P(\chi_n^2 \ge (\frac{\sigma_0}{t})^2\chi^2_{n,\alpha}).$$

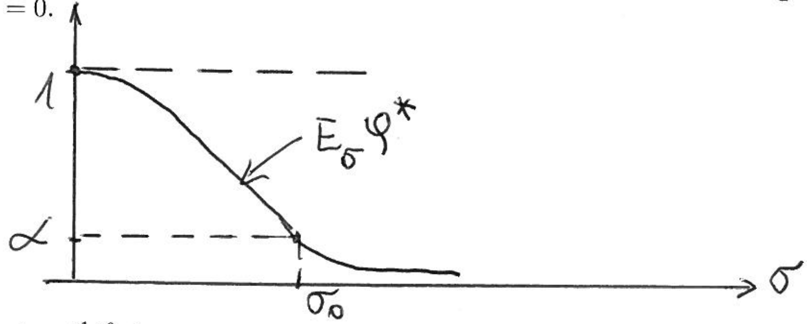
Thus $Power(t)$ is increasing function of $t \in (0, \infty)$, and satisfies: $Power(0) = 0, Power(\sigma_0) = \alpha$, and $lim_{t\to\infty}Power(t) = 1$.



If $H_0 : \sigma \ge \sigma_0$ versus $H_1 : \sigma < \sigma_0$ was to be tested, then using the **Note** after the Blackwell-Girshick theorem, we know that the ump-α test exists and that now the rejection region is $\{\sum_{i=1}^n X_i^2 < \sigma_0^2\chi^2_{n,1-\alpha}\}$. The power function is

$$Power(t) = P(\chi_n^2 \le (\frac{\sigma_0}{t})^2\chi^2_{n,1-\alpha}).$$

The graph will be "reversed" now since the hypothetical and the alternative region have been changed. We will have: $Power(0) = 1, Power(\sigma_0) = \alpha$, and $Power(t)$ is monotonically decreasing when t ranges from 0 to ∞ with $lim_{t\to\infty}Power(t) = 0$.



3) a) $E_\theta\varphi = \int_{1/2}^1 \theta x^{\theta-1}dx = 1 - (\frac{1}{2})^\theta, \theta > 0$ is the power function. The size is obtained at $\theta_0 = 1$, so $E_{\theta_0}\varphi = \frac{1}{2}$.

b) By the Neyman-Pearson lemma, for $H_0 : \theta = 2$ versus $H_1 : \theta = 1$, the best α-test is the one with a rejection region in the form $\{\frac{L(\mathbf{X};1)}{L(\mathbf{X};2)} \geq k\}$. Here, the sample size is $n = 1$ and we have $\frac{L(\mathbf{X};1)}{L(\mathbf{X};2)} = \frac{1x^0}{2x} \geq k$. Equivalently: $x \leq \frac{1}{2k} = k'$. To make it an α-test, we need

$$\alpha = 0.05 = P(X \leq k'|\theta = 2) = \int_0^{k'} 2xdx = (k')^2.$$

This implies that $k' = \sqrt{0.05} \approx .2236$. Hence the best 0.05-size test of $H_0 : \theta = 2$ versus $H_1 : \theta = 1$ is

$$\varphi^* = \begin{cases} 1 \text{ if } x \leq .2236 \\ 0 \text{ if } x > .2236 \end{cases}$$

c) $f(x;\theta) = \theta e^{(\theta-1)lnx}$ is obviously a member of one-parameter exponential family with $d(x) = lnx$ and a monotonically increasing $c(\theta) = \theta - 1$. Hence, according to the **Note** to the Blackwell-Girshick theorem, the ump-α test exists and has a rejection region $S = \{x : lnx \leq k\}$. But $lnx \leq k \iff x \leq k'$. For $\alpha = 0.05$ we get: $\alpha = 0.05 = P(x \leq k'|\theta = 2) = (k')^2$ and we again get $k' = 0.2236$. This means that the same test as in b) is ump-α size test of $H_0 : \theta \geq 2$ versus $H_1 : \theta < 2$. (We could have also argued about this by noticing that in b) the rejection region **did not** depend on the θ value under the alternative, hence the same test as in b) will be an **ump**-0.05 test.

d)

$$\lambda = \frac{max_{\theta\in\Theta_0} L(x;\theta)}{max_{\theta\in\Theta} L(x;\theta)} = \frac{1}{(-\frac{1}{lnx})x^{-\frac{1}{lnx}-1}}$$

(on the bottom we have replaced the argument that gives rise to the maximum by the value of the MLE $\hat{\theta}_{mle} = -\frac{1}{lnx}$. We now observe that $g(x) = \frac{1}{(-\frac{1}{lnx})x^{-\frac{1}{lnx}-1}} = -exlnx$ tends to zero when $x \to 0$ or $x \to 1$. Therefore, $\lambda \leq$ constant is equivalent to $\{x \leq k_1 \text{ or } x \geq k_2\}$. The values of k_1 and k_2 must be such that

$$0.1 = \alpha = P(X \leq k_1 \text{ or } x \geq k_2|\theta = 1) = k_1 + 1 - k_2.$$

Several choices are possible for k_1 and k_2. If we want an **equal-tailed** test then $k_1 = 0.05$ and $k_2 = 1 - 0.05 = 0.95$ should be chosen, that is:

$$\varphi = \begin{cases} 1 \text{ if } x \leq 0.05 \text{ or } x > 0.95 \\ 0 \text{ else} \end{cases}$$

4) Draw a diagram of the first quadrant with axis OX_1 and OX_2 and try to represent the rejection region S as a subset of the unit square. This helps to understand the calculations below.
The joint density, because of the assumed independence, is given by

$$f_{X_1,X_2}(x_1, x_2) = \theta^2 x_1^{\theta-1} x_2^{\theta-1}.$$

Hence

$$E_\theta\varphi = \int\int_S f_{X_1,X_2}(x_1, x_2)dx_1dx_2 = \int_0^1 (\int_{3x_1/4}^1 \theta^2(x_1x_2)^{\theta-1}dx_2)dx_1 = \ldots = 1 - \frac{1}{2}(\frac{3}{4})^\theta.$$

The size of this test is $E_\theta\varphi_{|\theta=1} = \frac{5}{8}$. Note that the test is **not good** since the size is too high and $E_\theta\varphi_{|\theta=0} = \frac{1}{2}$ which is also very high. Test based on the statistic $T = lnX_1 + lnX_2$ should be used instead.

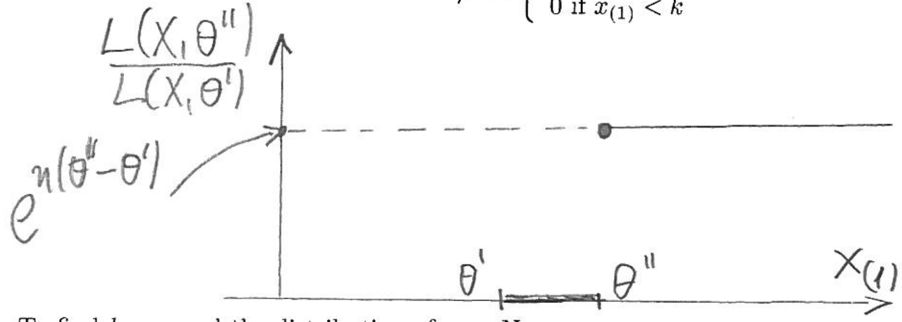
5) The likelihood function is

$$L(\mathbf{x}, \theta) = I_{(\theta,\infty)}(x_{(1)})e^{[-\sum_{i=1}^n x_i + n\theta]}$$

The family has a **monotone likelihood ratio in** $X_{(1)}$ (show this by examining the behaviour of $\frac{L(\mathbf{X},\theta'')}{L(\mathbf{X},\theta')}, \theta'' > \theta'$ as a function of $x_{(1)}$ and convince yourself that the ratio is zero when $x_{(1)} \in (\theta', \theta'')$ but is equal to a positive constant $e^{n(\theta''-\theta')}$ when $x_{(1)} > \theta''$).

This means that a ump-α test exists and has the form

$$\varphi^* = \begin{cases} 1 \text{ if } x_{(1)} > k \\ 0 \text{ if } x_{(1)} < k \end{cases}$$



To find k we need the distribution of $x_{(1)}$. Now:

$$P_\theta(X_{(1)} > k) = (P_\theta(X_1 > k))^n$$

But

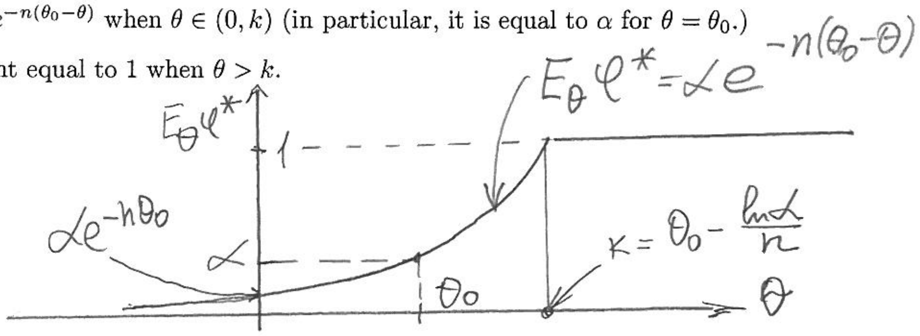
$$P_\theta(X_1 > k) = \begin{cases} 1 \text{ if } k \le \theta \\ 1 - \int_\theta^k e^{-(t-\theta)}dt = e^{-(k-\theta)} \text{ if } \theta < k \end{cases}$$

Hence

$$E_\theta\varphi^* = P_\theta(X_{(1)} \ge k) = \begin{cases} 1 \text{ if } k \le \theta \\ e^{-n(k-\theta)}, \text{ if } \theta < k \end{cases}$$

To find k we solve the equation $E_{\theta_0}\varphi^* = e^{-n(k-\theta)} = \alpha$. This gives $k = \theta_0 - \frac{ln\alpha}{n}$.
The powerfunction is defined on the positive half axis and is:

- equal to $\alpha e^{-n\theta_0}$ for $\theta = 0$.
- equal to $\alpha e^{-n(\theta_0-\theta)}$ when $\theta \in (0, k)$ (in particular, it is equal to α for $\theta = \theta_0$.)
- is a constant equal to 1 when $\theta > k$.



6) According to 1e) we have a MLR property in $T = \sum_{i=1}^{10} X_i$. Moreover, $T \sim Po(10\lambda)$. For $\lambda_0 = 1$ this is the Poisson distribution with parameter 10. Blackwell-Girshick theorem tells us that a ump-α test ($\alpha = 0.1$) exists and is in the form

$$\varphi^* = \begin{cases} 1 \text{ if } T > 14 \\ \gamma \text{ if } T = 14 \\ 0 \text{ if } T < 14 \end{cases}$$

The value of γ is

$$\gamma = \frac{.1 - .0835}{.9165 - .8644} = \frac{.0165}{.0521} = .317$$

7) We use the density transformation formula:

$$f_Y(y) = f_X(w^{-1}(y))|\frac{dw^{-1}(y)}{dy}|$$

Since $X \sim f(x;\theta) = \frac{1}{\theta}e^{-\frac{x}{\theta}}, x > 0, \theta > 0$ we get: $Y = \frac{X}{\theta}$ has a standard exponential density $f_Y(y) = e^{-y}, y > 0$ and then, using the properties of Gamma distribution,

$$\sum_{i=1}^{n} X_i/\theta \sim gamma(n)$$

with the density $f_{gamma(n)}(x) = \frac{e^{-x}x^{n-1}}{\Gamma(n)}, x > 0$. Hence, the ump-$\alpha$ test is given by

$$\varphi^* = \begin{cases} 1 \text{ if } \sum_{i=1}^n X_i \geq k \\ 0 \text{ if } \sum_{i=1}^n X_i < k \end{cases}$$

and

$$P(\frac{1}{\theta_0}\sum_{i=1}^n X_i \geq \frac{k}{\theta_0}|\theta = \theta_0) = \int_{k/\theta_0}^{\infty} \frac{e^{-x}x^{n-1}}{\Gamma(n)} dx.$$

Hence, the threshold is $k = \theta_0 \gamma_{n,\alpha}$ where $\gamma_{n,\alpha}$ is the upper $\alpha * 100\%$ point of the $gamma(n)$ density. This is an **exact** result.

On the other hand, asymptotically (for large n), by using the Central Limit Theorem (CLT) and the fct that $EX_i = \theta_0, VarX_i = \theta_0^2$ we can get the following approximate value for the threshold:

$$\alpha = P(\sum_{i=1}^n X_i \geq k|\theta = \theta_0) = P(\frac{\sqrt{n}(\bar{X} - \theta_0)}{\theta_0} \geq \frac{\sqrt{n}(\frac{k}{n} - \theta_0)}{\theta_0}) \approx 1 - \Phi(\frac{\sqrt{n}(\frac{k}{n} - \theta_0)}{\theta_0})$$

which implies that $(\frac{k}{n} - \theta_0)\sqrt{n} = \theta_0 z_\alpha$. Hence $k \approx n\theta_0 + \sqrt{n}\theta_0 z_\alpha$ should be chosen in order to have the size asymptotically equal to α.

8) Discussed at lectures.

9)

$$L(\mathbf{X,Y}; \mu_1, \mu_2) = (2\pi)^{-\frac{m+n}{2}} e^{-\frac{1}{2}[\sum_{i=1}^m (x_i - \mu_1)^2 + \sum_{i=1}^n (y_i - \mu_2)^2]}.$$

Unrestricted maximisation with respect to both μ_1 and μ_2 leads to $\bar{X}, \bar{Y}$ as solutions. Now, **restricted** maximisation under the restriction $\mu_1 = \mu_2 = \mu$ leads to $\hat{\mu}_{mle(restricted)} = \frac{\sum_{i=1}^m x_i + \sum_{i=1}^n y_i}{m+n}$. Therefore,

$$2ln\Lambda_{m,n} = \sum_{i=1}^m (x_i - \bar{x})^2 + \sum_{i=1}^n (y_i - \bar{y})^2 - \sum_{i=1}^m (x_i - \frac{m\bar{x} + n\bar{y}}{m+n})^2 - \sum_{i=1}^n (y_i - \frac{m\bar{x} + n\bar{y}}{m+n})^2 =$$

$$-m\bar{x}^2 - n\bar{y}^2 + 2\frac{m\bar{x} + n\bar{y}}{m+n}(\sum_{i=1}^m x_i + \sum_{i=1}^n y_i) - (m+n)(\frac{m\bar{x} + n\bar{y}}{m+n})^2 = \ldots = -\frac{mn(\bar{x} - \bar{y})^2}{m+n}$$

Hence

$$-2ln\Lambda_{m,n} = \frac{mn(\bar{x} - \bar{y})^2}{m+n} = T$$

(which can be seen directly to be distributed as chi-square with one degree of freedom under the hypothesis of equal means. Our Generalised LRT test is then:

$$\varphi = \begin{cases} 1 \text{ if } T \geq \chi^2_{1,\alpha} \\ 0 \text{ if } T < \chi^2_{1,\alpha} \end{cases}$$