THE UNIVERSITY OF NEW SOUTH WALES

DEPARTMENT OF STATISTICS

Solutions to selected exercises for MATH5905, Statistical Inference

Part four: Multinomial distribution. Order statistics. Robustness

1). This is just substitution in the formula.

a) $P(X_1 = 2, X_2 = 2, X_3 = 4) = \frac{8!}{2!2!4!}(.2)^2(.3)^2(.5)^4 = 0.0945$ The marginal distributions are Binomial which means that $X_2 \sim Bin(8, 0.3)$ and therefore $E(X_2) = 8 * .3 = 2.4, Var(X_2) = 8 * .3 * .7 = 1.68$. $Cov(X_1, X_3) = -8 * (0.2) * (0.5) = -0.8$.

b) $P(X_1 = 3, X_2 = 1, X_3 = 2) = \frac{6!}{3!1!2!}(0.5)^3(0.2)^1(0.3)^2 = 0.135$.

A little "trick" helps to do calculations quicker: we notice that $P(X_1 + X_2) = 2 = P(X_3 = 4)$. Since $X_3 \sim Bin(6, 0.3)$ we get $P(X_1 + X_2 = 2) = \frac{6!}{4!2!}(0.3)^4(0.7)^2 = 0.059535$.

2).The general formula is

$$f_{X_{(i)}}(x) = \frac{n!}{(i-1)!(n-i)!}[F(x)]^{i-1}[1 - F(x)]^{n-i}f(x)$$

Here we have $n = 4, i = 2, f(x) = e^{1-x}, x > 0$. We get from here $F(x) = \int_1^x e^{1-y}dy = 1 - e^{1-x}, x > 1$. Then

$$f_{X_{(2)}}(x) = \frac{4!}{1!2!}(1 - e^{1-x})e^{2(1-x)}e^{1-x} = 12e^{3(1-x)}(1 - e^{1-x}), x > 1.$$

3). We use the general formula from Problem 4. Here we have $n = 5, i = 4, f(x) = \frac{1}{x^2}, x > 1 \to F(x) = \int_1^x y^{-2}dy = 1 - \frac{1}{x}, x > 1$. Hence

$$f_{X_{(4)}}(x) = \frac{5!}{1!3!}(1 - \frac{1}{x})^3\frac{1}{x}\frac{1}{x^2} = \frac{20}{x^3}(1 - \frac{1}{x})^3, x > 1.$$

4). We use the general formula: $n = 2, f(y) = \frac{1}{2}e^{-\frac{y-4}{2}}, y \geq 4 \to F(y) = 1 - e^{-\frac{1}{2}(y-4)}, y > 4$. Hence

$$f_{y_{(1)}}(y) = n[1 - F(y)]^{n-1}f(y) = 2e^{-\frac{1}{2}(y-4)}\frac{1}{2}e^{-\frac{y-4}{2}} = e^{-(y-4)}, y > 4.$$

Then

$$E(Y_{(1)}) = \int_4^\infty ye^{-(y-4)}dy = \int_4^\infty (y-4)e^{-(y-4)}d(y-4) + 4\int_4^\infty e^{-(y-4)}d(y-4) = \Gamma(2) + 4 = 5.$$

5). The general formula gives for the density of the largest order statistic: $g_{X_{(n)}}(x) = nF^{n-1}(x)f(x)$.

a) Here $f(x) = e^{-x}, x > 0 \to F(x) = 1 = e^{-x}$. We get: $f_{X_{(3)}}(x) = 3(1 - e^{-x})^2e^{-x}$. Then we can get the expected value:

$$EX_{(3)} = 3\int_0^\infty xe^{-x}(1 - e^{-x})^2dx = 3\int_0^\infty xe^{-x}(1 - 2e^{-x} + e^{-2x})dx =$$

$$3[\Gamma(2) - \frac{2}{4}\int_0^\infty 2xe^{-2x}d(2x) + \frac{1}{9}\int_0^\infty 3xe^{-3x}d(3x)] = 3\Gamma(2)(1 - \frac{1}{2} + \frac{1}{9}) = \frac{11}{6}.$$

b) This part is more "tricky" and uses some specific properties of the CDF and the density of the standard normal distribution. We start with the general statement:

$EX_{(3)} = 3\int_{-\infty}^\infty xF^2(x)f(x)dx$ where $f(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}, F(x) = \int_{-\infty}^x f(u)du$. We note that $f(-x) = f(x), F(-x) = 1 - F(x)$ holds. Hence

$$EX_{(3)} = 3[\int_{-\infty}^0 .. + \int_0^\infty ...] = 3\int_0^\infty (-u)F^2(-u)f(u)du + 3\int_0^\infty uF^2(u)f(u)du =$$

$$3\int_0^\infty u[F^2(u)-(1-F(u))^2]f(u)du = 3\int_0^\infty u(2F(u)-1)f(u)du = 6\int_0^\infty uF(u)f(u)du - 3\int_0^\infty uf(u)du.$$

Now we note that $f'(u) = -uf(u)$ holds and therefore $\int uf(u)du = -f(u)$. We get then:

$$EX_{(3)} = -6\int_0^\infty F(u)df(u) + 3[f(\infty)-f(0)] = 6\frac{1}{2}\frac{1}{\sqrt{2\pi}} + 6\int_0^\infty \frac{e^{-u^2}}{2\pi}du - \frac{3}{\sqrt{2\pi}} =$$

$$\frac{6\sqrt{\frac{1}{2}}}{\sqrt{2\pi}}\int_0^\infty \frac{e - \frac{u^2}{2\frac{1}{2}}}{\sqrt{2\pi}\sqrt{\frac{1}{2}}}du$$

But the integral is equal to 1/2 (WHY!), hence $EX_{(3)} = \frac{3}{2\sqrt{\pi}}$.
6). a) $X = min(Y_1, Y_2)$. Hence $f_X(x) = 2e^{-\frac{x}{100}}\frac{1}{100}e^{-\frac{x}{100}} = \frac{1}{50}e^{-\frac{x}{50}}$.
b) $X = max(Y_1, Y_2)$. Hence $f_X(x) = 2(1-e^{-\frac{x}{100}})\frac{1}{100}e^{-\frac{x}{100}}$.

7). a) It holds $g(x_{(1)}, x_{(n)}) = n(n-1)(F(x_{(n)}-F(x_{(1)})^{n-2}f(x_{(1)})f(x_{(n)}),\ x_{(1)} < x_{(n)}$. In the particular case considered here, we have $f(x) = e^{-x}, F(x) = 1-e^{-x}, x > 0$. Since $n = 3$, we get:

$$g(x_{(1)}, x_{(3)}) = 6(e^{-x_{(1)}} - e^{-x_{(3)}})e^{-x_{(1)}}e^{-x_{(3)}}$$

for $0 < x_{(1)} < x_{(3)} < \infty$.
b) We can integrate out the unwanted variable $x_{(3)}$ in the joint density from a) to get the marginal of $x_{(1)}$:

$$g(x_{(1)}) = \int_{x_{(1)}}^\infty 6(e^{-x_{(1)}} - e^{-x_{(3)}})e^{-x_{(1)}}e^{-x_{(3)}}dx_{(3)} = \dots$$

but, of course, we could also use directly the formula for the marginal density:

$$g(x_{(1)}) = 3[1-(1-e^{-x_{(1)}})]^2e^{-x_{(1)}} = 3e^{-3x_{(1)}}, x_{(1)} > 0.$$

Similarly, we could integrate out the $x_{(1)}$ variable and get the marginal of $x_{(3)}$. But, of course, we could directly use the formula

$$g(x_{(3)}) = 3[1-e^{-x_{(3)}})]^2e^{-x_{(3)}}, 0 < x_{(3)} < \infty.$$

c) $EX_{(1)} = \int_0^\infty x * 3e^{-3x}dx = 1/3$,

$$EX_{(3)} = \int_0^\infty x * 3e^{-3x}(1-2e^{-x}+e^{-2x})dx = 3(\Gamma(2) - \frac{1}{2}\Gamma(2) + \frac{1}{9}\Gamma(2)) = \frac{11}{6}.$$

d) We define the transform:

$$U = X_{(3)} - X_{(1)}, V = X_{(1)}.$$

The joint density of $X_{(1)}$ and $X_{(3)}$ is (see a)):

$$g(x_{(1)}, x_{(3)}) = 6(e^{-x_{(1)}} - e^{-x_{(3)}})e^{-x_{(1)}}e^{-x_{(3)}}, 0 < x_{(1)} < x_{(3)} < \infty.$$

We get: $x_{(3)} = u+v, x_{(1)} = v$ with a Jacobian of the transformation equal to (-1). We get the joint density

$$f_{(U,V)}(u,v) = 6(e^{-u} - e^{-(u+v)})e^{-v}e^{-(u+v)} * |-1|$$

The relationship $0 < x_{(1)} < x_{(3)} < \infty$ transfers into $0 < v < u+v < \infty$ which is equivalent to $0 < u < \infty, 0 < v < \infty$. Hence the density of the range $R = U = X_{(3)} - X_{(1)}$ is

$$f_R(u) = \int_0^\infty (6e^{-3v-u} - 6e^{-2u-3v})dv = 2(1-e^{-u})e^{-u}, u > 0.$$

8). The transformation $V = X_{(3)}, U = X_{(3)} - X_{(1)}$ has an inverse defined as $x_{(1)} = v - u, x_{(3)} = v$. The absolute value of the Jacobian is 1. Hence

$$f_{(U,V)}(u,v) = n(n-1)[F(v) - F(v-u)]^{n-2} f(v-u) f(v)$$

where $F(.)$ denotes the cdf of a single observation. The region $0 < x_{(1)} < x_{(3)} < 1$ is transformed into $0 < u < v < 1$ for the new variables. Hence we get for the density $f_R(.)$ of the range:

$$f_R(u) = \int_u^1 6[v^2 - (v-u)^2]2(v-u)2v dv = 24 \int_u^1 (-u^2 + 2uv)(v^2 - uv) dv = \ldots = 12u(1-u)^2, 0 < u < 1.$$

Try to get the same result by using the transform $V = X_{(1)}, U = X_{(3)} - X_{(1)}$.

9). This problem is a bit more technical. Let $\phi(.)$ denote the standard normal density. We transform: $-\infty < X_{(1)} < X_{(2)} < \infty$ into $X_{(2)} - X_{(1)} = U, X_{(1)} = V$. The region for U and V becomes: $-\infty < v < \infty, 0 < u < \infty$. The joint density $h(u,v)$ of (U,V) becomes:

$$h(u,v) = 2\phi(v)\phi(u+v) = \frac{1}{\pi} e^{-\frac{v^2}{2} - \frac{(u+v)^2}{2}}$$

Hence, for the density $f_R(u)$ of the range we get:

$$f_R(u) = \frac{1}{\pi} e^{-\frac{u^2}{2}} \int_{-\infty}^{\infty} e^{-v^2 - uv} dv$$

Completing the square, we get finally:

$$f_R(u) = \frac{1}{\sqrt{2}} \sqrt{2\pi} \frac{1}{\pi} e^{-\frac{u^2}{2} + \frac{u^2}{4}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}/\sqrt{2}} e^{-\frac{1}{2*\frac{1}{2}}(v+\frac{u}{2})^2} dv = \frac{1}{\sqrt{\pi}} e^{-u^2/4}, u > 0.$$

(The integral above is equal to one since it is if fact the integral of the $N(-\frac{u}{2}, \frac{1}{2})$ density .)
10) a) If X is a random variable with density $f(x)$ we have to show that

$$\frac{E\Psi^2(X)}{[E\Psi'(X)]^2} \geq \frac{1}{E[\frac{f'}{f}(X)]^2}$$

holds. Since

$$E\Psi'(X) = \int_{-\infty}^{\infty} \Psi'(x) f(x) dx = \int_{-\infty}^{\infty} f(x) d\Psi(x) = -\int_{-\infty}^{\infty} \Psi(x) f'(x) dx$$

we see that we need to show that

$$[\int_{-\infty}^{\infty} \Psi(x) f'(x) dx]^2 \leq E\Psi^2(X) E[[\frac{f'}{f}(X)]^2$$

holds. But indeed, by applying Cauchy-Schwartz Inequality, we have

$$[\int_{-\infty}^{\infty} \Psi(x) f'(x) dx]^2 = [\int_{-\infty}^{\infty} \Psi(x) \frac{f'(x)}{f(x)} f(x) dx]^2 = [E\Psi(X) \frac{f'}{f}(X)]^2 \leq E\Psi^2(X) E[\frac{f'}{f}(X)]^2.$$

b) The equality means that equality in the Cauchy-Schwartz must hold and this means that we must have $\Psi(x) = c\frac{f'(x)}{f(x)}$ with certain constant c (which constant, without loss of generality, can also be set to one). Then $\Psi(x;\theta) = \frac{f'(x-\theta)}{f(x-\theta)} = \frac{\partial}{\partial\theta} lnf(x;\theta)$. Hence the equation $\sum_{i=1}^n \Psi(x_i;\theta) = 0$ that defines the M-estimator is the same as the equation for the score $V(\mathbf{X}, \theta) = 0$ that defines the MLE.

11) Start with a Taylor expansion along θ_0 :

$$0 = \sum_{i=1}^n \psi(x_i - \hat{\theta}_M) = \sum_{i=1}^n \psi(x_i - \theta_0) - (\hat{\theta}_M - \theta_0) \sum_{i=1}^n \psi'(x_i - \theta_0) + \ldots$$

where θ_0 is defined as in the formulation of the problem, $\hat{\theta}_M$ is the solution of the M-estimator equation and we ignore the higher order terms. We can now rearrange terms, divide through $\sqrt{n}$ both sides and ignore remainder terms (HOT) of higher order to get

$$\sqrt{n}(\hat{\theta}_M - \theta_0) = \frac{\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \psi(x_i - \theta_0)}{\frac{1}{n} \sum_{i=1}^{n} \psi'(x_i - \theta_0)} + \text{HOT}$$

For the expression on top of the RHS the central limit theorem can be applied to show it converging in law to $N(0, E_{\theta_0}(\psi(X_1 - \theta_0)^2)$.

For the expression on bottom of the RHS, the Law of Large Numbers can be applied to show convergence to $E_{\theta_0}\psi'(X_1 - \theta_0)$ in probability. Hence

$$\sqrt{n}(\hat{\theta}_M - \theta_0) \to^d N(0, \frac{E_{\theta_0}\psi(X_1 - \theta_0)^2}{[E_{\theta_0}\psi'(X_1 - \theta_0)]^2}).$$

Since we are dealing with a location family, the expression about the variance on the RHS is the same for all values of the location parameter hence it is equal to $\frac{\int \psi^2(x) f(x) dx}{(\int \psi'(x) f(x) dx)^2}$.