THE UNIVERSITY OF NEW SOUTH WALES

DEPARTMENT OF STATISTICS

Additional Exercises for MATH5905, Statistical Inference

Part one: Decision theory. Bayes and minimax rules

1. Suppose d_1, d_2, d_3 and d_4 are nonrandomized decision rules with risks as given in the following table:

i	1	2	3	4
$R(\theta_1, d_i)$	0	1	2	3
$R(\theta_2, d_i)$	6	5	3	5

   a) Find the minimax rule(s) amongst the **nonrandomized** rules $D = \{d_1, d_2, d_3, d_4\}$;

   b) Obtain the minimax rule in the set of randomized rules $\mathcal{D}$ generated by the set of rules in D. State the minimax risk of this rule.

   c) Find the Bayes rule and the Bayes risk for the prior $(\frac{1}{3}, \frac{2}{3})$ on (θ_1, θ_2).

   d) Express the randomized decision rule with risk point (2,5) using the given non-randomized decision rules.

   e) Calculate all priors for which d_1 is a Bayes rule.

2. A decision rule d is called admissible in a class of rules if there is no other decision rule d^* in the class such that $R(\theta, d^*) \leq R(\theta, d)$ for all $\theta \in \Theta$ and $R(\theta, d^*) < R(\theta, d)$ for at least one value of $\theta \in \Theta$. Let X be uniformly distributed on $[0, \theta]$ where $\theta \in (0, \infty)$ is an unknown parameter (i.e., $\Theta = [0, \infty)$). Let the action space be $[0, \infty)$ and the loss function $L(\theta, a) = (\theta - a)^2$ where a is the chosen action (the action now is estimation so $a = d(X)$ for given observation X and decision d). Consider the set of decision rules $d_\mu(x) = \mu x, \mu \geq 0$. For what value of μ is d_μ unbiased? Show that $\mu = 3/2$ is necessary condition for d_μ to be admissible.

3. Suppose $X_1, X_2, \ldots, X_n$ have conditional joint density

$$f_{X_1,X_2,\ldots,X_n|\Theta}(x_1, x_2, \ldots, x_n|\theta) = \theta^n e^{-\theta \sum_{i=1}^n x_i}, x_i > 0 \text{ for } i = 1, \ldots, n; \theta > 0$$

   and a prior density is given by $\tau(\theta) = ke^{-k\theta}, \theta > 0$, where k is a known constant.

   i) Calculate the posterior density of Θ given $X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n$.

   ii) Find the Bayesian estimator of θ with respect to squared error loss.

4. Suppose a **single** observation x is available from the uniform distribution with a density $f(x|\theta) = \frac{1}{\theta} I_{(x,\infty)}(\theta), \theta > 0$. The prior on θ is with a density $\tau(\theta) = \theta \exp(-\theta), \theta > 0$. Find the Bayes estimator of θ:

   i) with respect to quadratic loss;

   ii) with respect to absolute value loss $L(\theta, a) = |\theta - a|$.

   iii)(*) with respect to the loss $L_\eta(\theta, a) = (\theta - a)(\eta - I(\theta - a < 0))$ where $\eta \in (0, 1)$ is a fixed weight.

5. Let $X_1, X_2, \ldots, X_n$ be a random sample from the normal density with mean μ and variance 1. Consider estimating μ with a squared-error loss. Assume that the prior $\tau(\mu)$ is a normal density with mean μ_0 and variance 1. Show that the Bayes estimator of μ is $\frac{\mu_0 + \sum_{i=1}^n X_i}{n+1}$.

6. As part of a quality inspection program, five components are selected at random from a batch of components to be tested. From past experience, the parameter θ (the probability of failure), has a beta distribution with density

$$\tau(\theta) = 30\theta(1 - \theta)^4, 0 \leq \theta \leq 1.$$

We wish to test the hypothesis $H_0 : \theta \leq 0.2$ against $H_1 : \theta > 0.2$ using Bayesian hypothesis testing with a 0-1 loss. What is your decision if:

i) In a batch of five, no failures were found

ii) In a batch of five, one failure was found.