Query Complexity of Mastermind Variants

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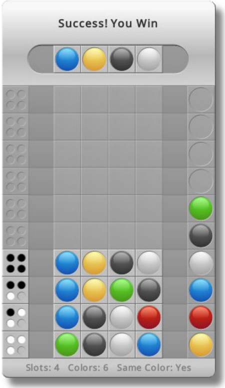
Mastermind

i. Codemaker vs. Codebreaker
ii. Queries: Guess a vector from $\{1, 2, \ldots, 6\}^4$.
iii. Response
    i. Black (Red) hits
    ii. White hits



Knuth Paper – 1976

At most five turns needed to guarantee a victory

Minimax

For each possible guess

 For each possible response to that guess

  Check how many possible solutions remain

 Let *score* be min. number solutions eliminated

Make guess with maximum score

Optimality of the Minimax Algorithm

Lemma (Pigeonhole Principle)

For a query with r possible responses, there exists a response that would leave at leasr $1/r$ of possible solutions remaining.

Analyze the worst-case performance of any algorithm.

i. Worst case response to first guess $\Rightarrow$ At least 256 solutions remain.
ii. Second guess: At least $\lceil 256/14 \rceil = 19$ solutions remain.
iii. Third guess: At least $\lceil 19/14 \rceil = 2$ solutions remain
iv. Fourth guess: At least two possible solutions left $\Rightarrow$ cannot guarantee to guess the solution on the fourth turn.

Extensions

i. Basic Extension: n spots, k colors
ii. Repeats vs. no repeats
iii. Non-adaptive vs. adaptive strategies

Trivial Lower Bound

Theorem

(Game: n spots, k colors, repeats, adaptive allowed). For any strategy that guarantees a win in s turns,

$$s \geq O\left(\frac{n \log k}{\log n^2}\right)$$

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2. $\binom{n+2}{2}$ possible responses (cf. stars and bars), giving $\binom{n+2}{2}^s$ possible response sequences.
3. Therefore, s must satisfy

$$\binom{n+2}{2}^s \geq k^n$$

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1. Represent guesses and solutions as matrices ($Q_{ij} = 1$ iff the i-th spot is the j-th color)

Example

The guess $(1, 1, 3)$ would become:

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5. Dot products with basis $\Rightarrow$ Uniquely determine X

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Theorem

There exists a set of at most nk guesses such that any hidden vector is uniquely determined by the responses to those guesses.

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Surprise Function: For an event x, we want

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Unique Surprise Function

$S(x) = -\log_2(\mathbb{P}[x])$.

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Definition (Entropy)

Let X be a random variable with domain D.

$$H(X) = \sum_{x \in D} \mathbb{P}[X = x] \cdot (-\log_2 (\mathbb{P}[X = x]))$$

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Theorem

The number of queries submitted by any winning non-adaptive strategy is at least

$$\frac{1}{4}\log_2\left(\frac{k!}{(k-n)!}\right).$$

Probabilistic Method

Choose a random set of queries $Q = \{q_1, q_2, \ldots, q_s\}$.

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$$\mathbb{P}[Q \text{ is a winning set of guesses }] > 0$$
$$\Downarrow$$
$$\exists \text{ a winning set of } 4k \log k \text{ guesses}$$

Thanks!

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