Mastermind

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Mastermind

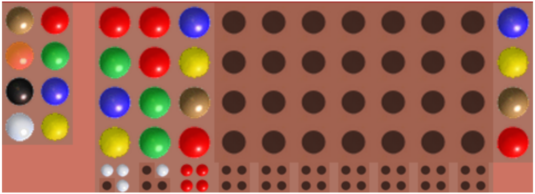
i. Codemaker vs. Codebreaker

ii. Queries: Guess a vector from $\{1, 2, \ldots, 6\}^4$

iii. Response

   i. Black (Red) hits

   ii. White hits



Knuth Paper – 1976

i. At most five turns needed

For each possible guess

    For each possible response to that guess

        Check how many possible solutions remain

    Let *score* be max. number solutions remaining

Make guess with minimum score

Extensions

i. Basic Extension: n spots, k colors
ii. Repeats vs. no repeats
iii. Non-adaptive vs. adaptive strategies

Trivial Lower Bound

1. One set of responses per solution
2. # of solutions $= k^n$
3. $\binom{n+2}{2} \approx n^2$ responses per turn

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   $\Rightarrow \binom{n+2}{2}^s \geq k^n$

(Somewhat) Trivial Upper Bound

1. Represent guesses and solutions as matrices ($Q_{ij} = 1$ iff the i-th spot is the j-th color)
2. $Q_{ij} \in \mathbb{R}^{nk}$
3. # of black hits of Q with hidden matrix X is $Q \cdot X$.
4. Dot products with basis $\Rightarrow$ orthogonal basis
   $\Rightarrow$ Projections onto orthogonal basis $\Rightarrow$ X

Coin-Weighing Problem

[Grebinski & Kucherov, 2000], [Bshouty, 2009]

i. Original Coin-Weighing algorithm by G&K, non-constructive (probabilistic method)

ii. Refined polynomial-time algorithm [Bshouty]

[Doerr et. al., 2013]

i. Split hidden vector into "coins" (subvectors).

ii. "Weight" of each "coin" is # of black hits.

iii. Use coin weighing algorithm to eliminate colors.

Entropy Method

Surprise Function: For an event x, we want

1. $S(x) = 0$ when $\mathbb{P}[x] = 1$
2. $S(x) = 1$ when $\mathbb{P}[x] = 1/2$
3. Decreasing function of $\mathbb{P}[x]$
4. $S(x \wedge y) = S(x) + S(y|x) \quad (= S(x) + S(y)$ if independent$)$

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$$\Rightarrow S(x) = -\log_2(\mathbb{P}[x]).$$

Entropy Method (cont'd)

Entropy is the expected surprise of a random variable.
Definition: Let X be a random variable with domain D.

$$H(X) = \sum_{x \in D} \mathbb{P}[X = x] \cdot (-\log_2(\mathbb{P}[X = x]))$$

Probabilistic Method

Non-Adaptive Game: Set of queries $Q = \{q_1, q_2, \dots, q_s\}$.

$$\mathbb{P}[Q \text{ is a winning set of guesses }] > 0$$

$$\Downarrow$$

$$\exists \text{ a winning set of } s \text{ guesses}$$