University of Birmingham
School of Mathematics

1SAS Sequences and Series Autumn 2024

Problem Sheet 4
(Issued Week 7)

Q1. Recall the following definition from the lectures:

Definition. *A series*

$$\sum_{n=1}^{\infty} a_n$$

converges to a real number s if its sequence of partial sums (s_N), given by

$$s_N = \sum_{n=1}^{N} a_n,$$

converges to s. In this case we write

$$\sum_{n=1}^{\infty} a_n = s,$$

and refer to s as the sum of the series. A series converges if it converges to s for some s.

Using the definition, prove that the following series converge, and find their sums.

$$\text{(i)} \quad \sum_{n=1}^{\infty} \frac{1}{(n+1)(n+3)}.$$

$$\text{(ii)} \quad \sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)}.$$

Q2. (a) Prove that if $|x| < 1$ then

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}.$$

(b) Prove that the series

$$\sum_{n=0}^{\infty} (-1)^n \frac{3^n}{2^{2n}}$$

converges, and find its sum.

Q3. (i) By appealing to the appropriate definition, prove that the series

$$\sum_{n=1}^{\infty} \frac{2n+1}{n^2(n+1)^2}$$

converges, and find its sum.

(ii) Use the Comparison Test to give an alternative proof that the series in Part (i) converges (you don't need to conclude anything about its sum).

(iii) Again by appealing again to the appropriate definition, prove that

$$\sum_{n=1}^{\infty} \frac{1}{n^2(n+1)} = \frac{\pi^2}{6} - 1.$$

In this question you may use the fact that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

without proof.

Q4. Study carefully the proof (from the notes/lectures) that

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

converges. For which values of $\alpha \in \mathbb{R}$ does this convergence proof adapt to the series

$$\sum_{n=1}^{\infty} \frac{1}{n^\alpha} \quad ?$$

Write down a complete proof for such values of α.

Q5. Which of the following series converge? Justify your answers. You may use standard series convergence tests provided you make it clear that you are doing so.

(i)

$$\sum_{n=1}^{\infty} \frac{n^4}{4^n}$$

(ii)

$$\sum_{n=1}^{\infty} \frac{1000^n}{n!}$$

(iii)

$$\sum_{n=1}^{\infty} \frac{3^n + 4^n}{3^n + 5^n}$$

(iv)

$$\sum_{n=1}^{\infty} \frac{2^n (n!)^2}{(2n)!}$$

Q6. Which of the following series converge? Justify any assertions that you make. You may use standard series convergence tests provided you make it clear that you are doing so.

(i)
$$\sum_{n=1}^{\infty}(-1)^n$$
(ii)
$$\sum_{n=1}^{\infty}\frac{n\sqrt{n}+9n}{n^2+4n+1}$$
(iii)
$$\sum_{n=1}^{\infty}\left(\sqrt{n+1}-\sqrt{n-1}\right)^3$$

$\boxed{\text{SUM}}$ **Q7**. Which of the following series converge? Justify any assertions that you make. You may appeal directly to the definition or use standard series convergence tests, provided you make it clear that you are doing so.

(i)
$$\sum_{n=1}^{\infty}\left((n+1)^{\frac{1}{4}}-n^{\frac{1}{4}}\right)$$
(ii)
$$\sum_{n=1}^{\infty}\frac{1+n+n^2}{1+n^2+3n^4}$$
(iii)
$$\sum_{n=1}^{\infty}\frac{n^n}{4^n n!}$$

Extra Questions

EQ1. Prove that
$$\sum_{n=1}^{N}\frac{n}{2^n}=2-2^{-N}(N+2)$$
for all $N\in\mathbb{N}$.
Deduce that
$$\sum_{n=1}^{\infty}\frac{n}{2^n}=2.$$
[In this question you may use without proof that $n2^{-n}\to 0$; indeed we proved this in Problem Sheet 1.]

EQ2. Suppose that $a_1, a_2, a_3, \ldots$ are positive real numbers for which

$$\frac{a_{n+1}}{a_n} \leq \frac{1}{2} \quad \text{for all} \quad n \in \mathbb{N}.$$

(i) Prove that $a_n \leq \left(\frac{1}{2}\right)^{n-1} a_1$ for all $n \in \mathbb{N}$.

(ii) Using the Comparison Test, or otherwise, deduce that

$$\sum_{n=1}^{\infty} a_n$$

converges.