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UNIVERSITY OF BIRMINGHAM

School of Mathematics

Programmes in the School of Mathematics First Examination

Programmes involving Mathematics First Examination

**1SAS 06 34047 Level C**

**LC Sequences and Series**

# Alternative Assessment

January Examinations 2021-22

One Hour and Thirty Minutes

Full marks will be obtained with complete answers to BOTH questions. Each question carries equal weight. You are advised to initially spend no more than 45 minutes on each question and then to return to any incomplete questions if you have time at the end. An indication of the number of marks allocated to parts of questions is shown in square brackets.

# Section A

**1.** (a) Define what it means for a sequence of real numbers $(a_n)$ to *tend to infinity*. Use the definition to prove that the sequence $(a_n)$ given by

$$a_n = n^{\frac{1}{3}}$$

tends to infinity. [5]

(b) Define what it means for a sequence of real numbers $(a_n)$ to *converge to a real number* $\ell$. Use the definition to prove that the sequence $(a_n)$ given by

$$a_n = \frac{n^2+3n+1}{2n^2+n+2}$$

converges to $\frac{1}{2}$. [6]

(c) Which of the following series converge? Justify any assertions that you make.

(i)
$$\sum_{n=1}^{\infty} \left(\frac{n+1}{3n}\right)^2$$

(ii)
$$\sum_{n=1}^{\infty} \left(\frac{n+1}{3n}\right)^n$$

(iii)
$$\sum_{n=1}^{\infty} \frac{3n^2+5n+1}{4n^3+8n^2-1}$$

(iv)
$$\sum_{n=1}^{\infty} 2^n \frac{(n!)^2}{(2n)!}$$

(v)
$$\sum_{n=1}^{\infty} (-1)^{n+1} \left(\sqrt{2n+1} - \sqrt{2n-1}\right)$$

[In this question you may appeal to standard limit theorems for sequences and convergence tests for series, provided you make it clear that you are doing so.] [14]

## Section B

**2.** In this question you may use standard results on the convergence of sequences and series, provided you make it clear that you are doing so.

(a) A sequence of nonnegative real numbers $(a_n)$ is defined recursively by

$$a_{n+1} = \frac{1}{3} + \frac{2a_n^2}{3}; \quad a_1 = 0.$$

(i) Using induction, or otherwise, show that $(a_n)$ is bounded above by $\frac{1}{2}$.

(ii) Show that $(a_n)$ is increasing.

(iii) Deduce that $(a_n)$ converges, and find its limit, justifying any assertions that you make.

[10]

(b) Define what it means for a series of real numbers

$$\sum_{n=1}^{\infty} a_n$$

to *converge*. Use the definition to show that the series

$$\sum_{n=1}^{\infty} \frac{1}{n^{\frac{2}{3}}}$$

does not converge. [5]

(c) (i) Prove that a sequence $(a_n)$ of real numbers converges if and only if the series

$$\sum_{n=1}^{\infty} (a_{n+1} - a_n)$$

converges.

(ii) Suppose that a sequence $(a_n)$ of real numbers satisfies

$$|a_{n+1} - a_n| \leq 2^{-n}$$

for all $n \in \mathbb{N}$. Prove that $(a_n)$ converges.

[10]