1Mech — Mechanics

Mechanics exercises 5 (weeks 9 and 10)

This week's problem sheet is formative only (i.e. you don't have to hand it in and it does not contribute to your grade). You should still attempt all questions!

1. A smooth sphere of radius a has its centre at the origin. If (ρ, θ, z) give cylindrical polar coordinates, with the z axis pointing vertically upwards, the surface of the sphere is given by $\rho^2 + z^2 = a^2$. A particle of mass m moves on the interior surface of the sphere under the action of gravity, with position vector $\mathbf{r} = \rho\mathbf{e}_\rho + z\mathbf{e}_z$, where $\mathbf{e}_\rho$, $\mathbf{e}_z$ are basis vectors pointing in the $\rho(t)$ and $z(t)$ direction respectively, with normal reaction $\mathbf{R}$ acting purely perpendicular to the surface.

   (a) Show that the particle velocity satisfies $|\dot{\mathbf{r}}|^2 = \dot{\rho}^2 + \rho^2\dot{\theta}^2 + \dot{z}^2$.

   (b) Hence show that

   $$\begin{aligned} \rho^2\dot{\theta} &= h, \\ \frac{1}{2}m\left(\dot{\rho}^2 + \rho^2\dot{\theta}^2 + \dot{z}^2\right) + mgz &= E, \end{aligned}$$

   where h and E are constants. What do these expressions represent physically?

   (c) Show that

   $$2\rho\dot{\rho} + 2z\dot{z} = 0,$$

   and hence

   $$\dot{\rho}^2 = \frac{z^2\dot{z}^2}{a^2 - z^2}.$$

   (d) If the particle is initially located at $\rho = a/\sqrt{2}$, $z = -a/\sqrt{2}$, moving with speed V in the direction of the horizontal tangent of the surface, find the values of h and E.

   (e) Hence show that the motion satisfies

   $$a^2\dot{z}^2 = \left(z + a/\sqrt{2}\right)\left(2g\left(z^2 - a^2\right) - V^2\left(z - a/\sqrt{2}\right)\right).$$

   (f) By differentiating to find an expression for $\ddot{z}$, show that the particle will initially rise if $V^2 > ag/\sqrt{2}$.

   (g) **Optional and not assessed:** Suppose instead that the particle is initially located at $\rho = a$, $z = 0$, and is projected vertically with speed V. Find the smallest value of V that will allow the particle to reach $z = a$.

2. A smooth surface of revolution has equation $z = a^2/\rho$ where ρ, z, θ give cylindrical polar coordinates, with z pointing downwards. A small particle of mass m slides on the interior of the surface.

(a) **Briefly** explain why

$$
\begin{aligned}
\rho^2\dot{\theta} &= h, \\
\frac{1}{2}m\left(\dot{\rho}^2 + \rho^2\dot{\theta}^2 + \dot{z}^2\right) - mgz &= E,
\end{aligned}
$$

are both constants.

(b) If the particle initially moves horizontally with velocity $\rho\dot{\theta} = a\omega$ at depth $z = a$ below the origin, show that

$$
\begin{aligned}
\rho^2\dot{\theta} &= a^2\omega, && (1) \\
\frac{1}{2}m\left(\dot{\rho}^2 + \rho^2\dot{\theta}^2 + \dot{z}^2\right) - mgz &= \frac{1}{2}m\omega^2a^2 - mga. && (2)
\end{aligned}
$$

(c) By rewriting (2) in terms of z, $\dot{z}$, show that the particle moves between $z = a$ and $z = 2g/\omega^2 - a$. Hence determine the particle moves in a circle when $g = a\omega^2$.

3. A rocket of mass m launches from rest and expels exhaust gases at speed u relative to the rocket's motion. The rocket burns fuel such that the total mass of the rocket over time is given by $m = m_0e^{-bt}$ where $b > g/u$ is a constant, and moves under the action of gravity g and air resistance assumed to be of the form kmv^2 where v is the velocity of the rocket.

   (a) Find the velocity of the rocket as a function of time.

   (b) Hence find the limiting velocity of the rocket as time tends to infinity.

4. Two railway workers, each of mass m, are standing on a frictionless railway cart of mass M which is on a horizontal track and is initially stationary. The railway workers run from the front of the cart and jump off of the rear of the cart with a speed u relative to the cart.

   (a) What is the final speed of the cart if both workers jump simultaneously?

   (b) What is the final speed of the cart if the workers do not jump at the same time (i.e. the second worker only jumps after the first worker has already jumped off the cart).

   (c) If instead there are N workers, what is the final speed if they all jump sequentially?