University of Birmingham

School of Mathematics

1RA Real Analysis and 1VGLA Vectors Geometry and Linear Algebra

Simulated Exam

Week 11 Semester 1, 2024–25

Two hours

Full marks will be obtained with complete answers to all TWO questions. Each question carries equal weight. You are advised to initially spend no more than 45 minutes on each question and then to return to any incomplete questions if you have time at the end. An indication of the number of marks allocated to parts of questions is shown in square brackets.

No calculator is permitted in this examination.

1RA Real Analysis

1. (a) Let $f : \mathbb{R} \to \mathbb{R}$ be a function and $\alpha, \ell \in \mathbb{R}$. [8]

(i) Define what it means that

$$\lim_{x \to \alpha} f(x) = \ell.$$

(ii) By directly using the definition of limit from (i), show that

$$\lim_{x \to 1} 3x + 2 = 5.$$

(b) Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function and $\alpha \in \mathbb{R}$. [5]

(i) Define $f'(\alpha)$, the derivative of f at α.

(ii) By directly using the definition of derivative from (i), find the derivative of the function $f(x) = x^3$ at the point $x = \pi$.

(c) Find the following limits. Show your detailed work. You can use any results discussed in lectures, provided you clearly state what you are using. [4]

(i) $\lim_{x \to 0} \dfrac{\sin 7x}{\sin 4x}$,

(ii) $\lim_{x \to \infty} \dfrac{6x^2 + 8x + 9}{x^2 + 2x - 6}$

(d) Find the derivatives of the following functions. Show your detailed work. You can use any results discussed in lectures, provided you clearly state what you are using. [4]

(i) $f(x) = \cos x + \ln x + x^4$,

(ii) $f(x) = \dfrac{\cos x - 1}{e^x + 1}$

(e) Let $f(x) = 2x^3 - 6x^2 - 18x + 7$. [4]

(i) Find the increasing and decreasing intervals of f.

(ii) Find all local maximum and local minimum points of f.

2. (a) For the vectors $\mathbf{a} = (1,2,4)$, $\mathbf{b} = (v_1, v_2, v_3)$ and $\mathbf{c} = (1,2,3)$ determine

(i) $\mathbf{a}+\mathbf{b}$,

(ii) $\mathbf{a}\cdot\mathbf{b}$,

(iii) $\mathbf{a}\times\mathbf{b}$, and

(iv) the angle θ between $\mathbf{a}$ and $\mathbf{c}$

You may leave your answer to (a)(iv) in the form $\theta = \cos^{-1}(x)$ or $\sin^{-1}(x)$, where x is a real number that you have determined. [8]

(b) Let $z_1 = -\frac{\sqrt{2}}{2}(1+i)$ and $z_2 = 2e^{i\frac{3\pi}{4}}$. Determine

(i) z_1 in modulus argument form giving the principal value of the argument, and z_2 in the form $x+iy$,

(ii) $z_1 + z_2$ in the form $x+iy$,

(iii) $z_1 z_2^{-1}$ in modulus argument form giving the principal value of the argument, and

(iv) z_1^{29} in exponential form giving the principal value of the argument. [8]

(c) (i) Use the Gaussian elimination method to obtain a reduced echelon form and determine the solution set for the following system of simultaneous linear equations:

$$\begin{aligned} 3x+2y-z &= 4 \\ 2x-y+2z &= 10 \\ x-3y-4z &= 5. \end{aligned}$$

(ii) Let a be a real number. Use your working from (c)(i) to find a solution of the following system of simultaneous linear equations:

$$\begin{aligned} 3x+2y-z &= 11a \\ 2x-y+2z &= 0 \\ x-3y-4z &= 0. \end{aligned}$$

[9]