UNIVERSITY OF BIRMINGHAM

School of Mathematics

Programmes in the School of Mathematics First Examination

Programmes involving Mathematics First Examination

1VGLA 06 25664 Level C

LC Vectors, Geometry and Linear Algebra

May/June Examinations 2022-23

Three Hours

Full marks will be obtained with complete answers to all FOUR questions. Each question carries equal weight. You are advised to initially spend no more than 45 minutes on each question and then to return to any incomplete questions if you have time at the end. An indication of the number of marks allocated to parts of questions is shown in square brackets.

No calculator is permitted in this examination.

Section A

1. (a) For the vectors $\mathbf{a} = (1,2,4)$ and $\mathbf{b} = (3,5,7)$ calculate

(i) $\mathbf{a} \cdot \mathbf{b}$;

(ii) the line parallel to $\mathbf{a}$ through the point $(1,1,1)$;

(iii) $\mathbf{a} \times \mathbf{b}$; and

(iv) the plane containing the points $A = (1,2,4)$ and $B = (3,5,7)$ and $C = (1,1,1)$.

[8]

(b) Let $s = -\frac{\sqrt{2}}{2}(1+i)$ and $t = 16e^{-\frac{2}{3}\pi i}$. Determine

(i) s in modulus-argument form, giving the principal value of the argument;

(ii) $\overline{t^{-1}}$ and $s\overline{t^{-1}}$ (s multiplied by the complex conjugate of t^{-1}) giving your answers in exponential form with the principal value of the argument;

(iii) s^{29} in exponential form, and providing the principal value of the argument; and

(iv) the fourth roots of t in exponential form with principal value of the argument.

[8]

(c) (i) Use the Gaussian elimination method to determine the solution set to the following system of simultaneous linear equations:

$$\begin{aligned} 3x + 3y - 3z &= 1, \\ x - 2y + 3z &= 1, \\ 2x + 4y - 4z &= 1. \end{aligned}$$

[7]

(ii) Give an example of a system of three simultaneous linear equations in three variables which has a solution set with infinitely many elements. [2]

2. (a) Let

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 3 & 4 \end{pmatrix}$$

and

$$\mathbf{B} = \begin{pmatrix} 3 & 4 & 5 & 6 \\ 0 & 6 & 2 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

(i) Calculate $\det \mathbf{A}$.

(ii) Calculate $\mathbf{A}^{-1}$.

(iii) Calculate $\det(\mathbf{A}^{-1}\mathbf{B}^T)$.

[12]

(b) Suppose that $V = \mathbb{R}^4$ and

$$Y = \{(x_1, x_2, x_3, x_4) \in V \mid x_1 + x_2 + x_3 + x_4 = 0\}.$$

(i) Show that Y is a subspace of V.

(ii) Find a basis for Y and prove that it is a basis.

(iii) Determine $\dim_{\mathbb{R}} Y$, the dimension of Y.

[13]

Section B

3. (a) Let $\alpha \in \mathbb{C}$ have positive imaginary part $b > 0$. Show that the locus of points on the Argand diagram which satisfy

$$(\bar{z}+z)^2 = 2(\alpha\bar{z}+\overline{\alpha}z)$$

is a parabola. Determine its focal length and vertex.

[5]

(b) Assume the homogeneous system of simultaneous linear equations

$$\begin{aligned} a_{11}x+a_{12}y+a_{13}z &= 0, \\ a_{21}x+a_{22}y+a_{23}z &= 0, \\ a_{31}x+a_{32}y+a_{33}z &= 0, \end{aligned}$$

has solution $x=a, y=b, z=c$. Now assume that $d_1, d_2, d_3 \in \mathbb{R}$ are fixed and not all zero and consider the following system of linear equations:

$$\begin{aligned} a_{11}x+a_{12}y+a_{13}z = d_1, \\ a_{21}x+a_{22}y+a_{23}z = d_2, \\ a_{31}x+a_{32}y+a_{33}z = d_3. \end{aligned} \tag{1}$$

Let

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}.$$

Which of the following statements are true? Either give a proof or present a counterexample. [See next page.]

(i) $x = p, y = q, z = r$ is a solution to (1) if and only if $x = a + p, y = b + q, z = c + r$ is a solution to (1).

(ii) If $\det \mathbf{A} = 0$, then there is a unique solution to (1).

(iii) If $(a,b,c) \neq (0,0,0)$, then (1) has infinitely many solutions.

[9]

(c) Suppose that $\mathbf{A} \in \mathcal{M}_{4\times 4}(\mathbb{R})$. Write down the elementary matrix $\mathbf{E}$ which when we calculate $\mathbf{EA}$ results in a matrix in which

(i) row 1 of $\mathbf{A}$ is swapped with row 4 of $\mathbf{A}$;

(ii) row 3 of $\mathbf{A}$ is multiplied by 3;

(iii) row 2 of $\mathbf{A}$ is added to row 4 of $\mathbf{A}$.

[6]

(d) Suppose that $n \geq 2$ and $\mathbf{A} \in \mathcal{M}_{nn}(\mathbb{R})$ is an invertible $n \times n$ matrix. Prove that

$$\det(\operatorname{adj}(\mathbf{A})) = \det(\mathbf{A})^{n-1}$$

where $\operatorname{adj}(\mathbf{A})$ denotes the adjoint of $\mathbf{A}$. [You may use facts about the adjoint of $\mathbf{A}$ that are proved in the lecture notes so long as you explain carefully what you have used.] [5]

4. (a) Suppose that V and W are finite-dimensional vector spaces over the field $\mathbb{F}$. Assume that U is a subspace of V and $T : V \to W$ is a linear transformation. Let

$$Y = \{T(\mathbf{u}) \mid \mathbf{u} \in U\}$$

and

$$Z = \{\mathbf{z} \in V \mid T(\mathbf{z}) \in T(U)\}.$$

(i) Show that Y is a subspace of W.

(ii) Prove that $Z = \ker(T) + U$, where $\ker(T)$ is the kernel of T.

(iii) Show that $\dim_{\mathbb{F}} U = \dim_{\mathbb{F}} Y$ if and only if $\ker(T) \cap U = \{\mathbf{0}\}$.

[13]

(b) Let $V = \mathbb{R}^3$, $W = \mathbb{R}^3$ and $T : V \to W$ be defined by

$$T((v_1, v_2, v_3)) = (2v_1 + 2v_2 + 2v_3, 2v_1 + v_3, 4v_1 + 2v_2 + 3v_3).$$

(i) Show that T is a linear transformation.

(ii) Determine a basis for the image of T, and explain why it is a basis.

(iii) Determine the dimension of $\ker(T)$, $\dim_{\mathbb{R}} \ker(T)$.

(iv) Calculate a basis for $\ker(T)$.

[12]

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Do not complete the attendance slip, fill in the front of the answer book or turn over the question paper until you are told to do so.

Important Reminders

- Coats and outer-wear should be placed in the designated area.
- Unauthorised materials (e.g. notes or Tippex) **MUST** be placed in the designated area.
- Check that you **DO NOT** have any unauthorised materials with you (e.g. in your pockets, pencil case).
- Mobile phones and smart watches **MUST** be switched off and placed in the designated area or under your desk. They must not be left on your person or in your pockets.
- You are **NOT** permitted to use a mobile phone as a clock. If you have difficulty in seeing a clock, please alert an Invigilator.
- You are **NOT** permitted to have writing on your hand, arm or other body part.
- Check that you do not have writing on your hand, arm or other body part – if you do, you must inform an Invigilator immediately.
- Alert an Invigilator immediately if you find any unauthorised item upon you during the examination.

Any students found with non-permitted items upon their person during the examination, or who fail to comply with Examination rules may be subject to the Student Conduct procedures.