University of Birmingham
School of Mathematics

1RA Real Analysis January 2023

Final Exam
Generic Feedback

The performance of the students in the final exam (January exam 2023, Questions 1 and 3) was satisfactory overall. They demonstrated a good understanding of some fundamental concepts in calculus, such as function, convergence, and derivative, and were able to use them in simple calculations. However, they struggled with applying the concept of differentiation to more complex scenarios, such as optimization and L'Hospital's rule. Question 3 (c) was especially challenging, as it involved several techniques, such as limits at infinity, monotonicity and derivatives, continuity, and the intermediate value theorem. Many students failed to provide a complete and rigorous proof of this problem.

Q1.
(a). This is bookwork, and was done well. However, some students lost points because they did not mention that "one element of Y".

(b). This is also standard bookwork, and was done well. A good understanding of the definition of convergence is needed to apply it in the proof of (ii).

(c). This is also standard bookwork, and was done well. Some students lost points because they did carry the limit sign $\lim_{x\to 5}$ all the way in their computation.

(d). This question is in general done well. Please note that the increeaxing/decreasing intervals may include endpoints, i.e. they may be closed intervals.

(d.i) To solve this problem, the first step is to calculate the derivative of f, which is denoted by f'. The sign of the derivative gives us the increasing/decreasing intervals. The next step is to use sign analysis to determine where f' is negative or positive. If f' is negative, then f is decreasing. If f' is positive, then f is increasing.

(d.ii) This is just a consequence of (c.i). The points where f' changes sign are called critical points, and they are candidates for local minima or maxima of f. By comparing the values of f at the critical points and the the nearby domain, we can identify the local extrema of f.

(d.iii) The second derivative of f tells us how the graph of the function curves. If the second derivative is positive, the function is convex. If the second derivative is negative, the function is concave. To find the convex and concave intervals of f, we need to compute the second derivative of f and find where it changes sign.

Q2.
2(a). Either the definition of integrabilty, or the equivalent Riemann's Criterion for integratbility, could be used.

2(b). Marks were lost for not using the correct logical order and quantifiers in the statement of this theorem.

2(c). A common mistake was "$U(f,P) = 62$," but the correct answer is $U(f,P) = 68$, since $M_2 := \sup\{f(x) : x \in [4,10]\} = 10$ (i.e. $M_2 \neq 9$).

2(d). This part asks for an *antiderivative* of g (not the derivative g'). This means that the answer must be a differentiable *function* f such that $f' = g$. Marks were lost for computing the integral $\int_{-2\pi}^{3\pi} g$, which is a number, not a function.

2(e). This is an improper integral because the denominator $x^2 - 4x + 3$ is not defined at $x = 1$ and it must be calculated as $\lim_{\delta \to 0^+} \int_{1+\delta}^{2}$. Marks were lost for not showing the limit computations and for substituting the $x = 1$ endpoint without justification.

2(f). This can be solved by a direct integration using Integration-by-Parts. Marks were lost for not using the initial data to determine the constant of integration.

2(g). This is a second-order differential equation with constant coefficients, so the Characteristic Polynomial method can be used to determine its general solution.

Q3.
(a) This is a typical optimization queation using calculus. To solve this problem, we need to identify the function that we want to optimize, which is the surface area of the can, and the constraint that limits the possible values of the variables, which comes from the fixed volume of the can. Then, we use techniques such as differentiation to find the optimal solution. Also note that the can has a bottom but no top, so the surface area of such a can is

$$A = 2\pi r h + \pi r^2$$

where r is the radius and h is the hight of the cylinder. As the volume of the can is 27, which gives the constraint

$$r^2 h = 27 \text{ or } h = \frac{27}{r^2}.$$

The key step, that many got wrong, is plugging the constraint to the function of A to get

$$A(r) = \frac{54\pi}{r} + \pi r^2.$$

Thus the question reduces to minimize $A(r)$, which is a routine computation.

(b.1) To solve this problem, we need to apply the concept of continuity. Note that all the functions involved in the problem are continuous at the limit point, and that the denominator of the fraction does not equal zero at the point where we want to find the limit. Then we just need to plug $x = 0$ to the function and get its limit.

(b.2) To solve this question, we need to apply the L'Hospital rule twice. This rule allows us to find the limit of a function that has an indeterminate form, such as $0/0$ or ∞/∞. The first step is to rewrite the given limit as a fraction and check if it has an indeterminate form, i.e.

$$\lim_{x\to 0}\left(\frac{\pi}{2} - \arctan x\right)^{\frac{1}{\ln x}} = e^{\lim\limits_{x\to 0}\frac{\ln\left(\frac{\pi}{2} - \arctan x\right)}{\ln x}}.$$

We note that

$$\frac{\ln\left(\frac{\pi}{2} - \arctan x\right)}{\ln x}$$

is an indeterminate form of ∞/∞ when $x \to \infty$.

(c) This is a challenging question, as it involved several techniques, such as limits at infinity, monotonicity and derivatives, continuity, and the intermediate value theorem. Many students failed to provide a complete and rigorous proof of this problem.

One first check the limit of $f(x)$ at ∞ and $-\infty$, which are ∞ and $-\infty$, respectively. Then one finds M and M' such that $f(M) > 0 > f(M')$. This, together with the continuity of f, enables us to apply the intermediate value theorem and find a point x_0 such that $f(x_0) = 0$.

To show that this is the only solution to $f(x) = 0$, we need to work further. Note the derivative of f is strictly positive on $\mathbb{R}$, which implies that f is strictly increasing on $\mathbb{R}$. Thus the solution must be unique.

Q4.
4(a). The function f has two points of discontinuity, so a suitable partition to use is $P_\delta := \{-6, -6+\delta, -5-\delta, -5\}$ for $\delta \in (0, \frac{1}{2})$. This provides a δ-interval around *each* of the three points ($x = -6, -5$) where f is discontinuous. Marks were lost for not introducing δ when it is first used (e.g. "Let $\delta \in (0, \frac{1}{2})$") and for incorrect computations of $L(f, P_\delta)$ and $U(f, P_\delta)$. The proof required introducing a parameter $\epsilon > 0$, making an appropriate choice of δ, and then concluding by referring back to Riemann's Criterion.

4(b). The key here, as in the proof of the Partition Lemma in the Lecture Notes, was to observe that $m_2 \leq m_2'$ and $m_2 \leq m_2''$, where $m_2 = \inf\{f(x) : x \in [1,3]\}$, $m_2' = \inf\{f(x) : x \in [1,2]\}$, and $m_2'' = \inf\{f(x) : x \in [2,3]\}$.

4(c). This is an application of the First Fundamental Theorem of Calculus. The function underneath the integral, i.e. $t \mapsto \sqrt{100 - t^4}$, is continuous because it is the composition of two continuous functions (the square-root function on $\mathbb{R}_+$ and a polynomial). Marks were lost for not observing and also not justifying this continuity, as it is essential for the application of the First Fundamental Theorem of Calculus. It was then necessary to correctly define an auxiliary function, e.g. $F(x) := \int_0^x \sqrt{100 - t^4}\, \mathrm{d}t$, on which the First Fundamental Theorem of Calculus could be applied to deduce that $F'(x) = \sqrt{100 - x^4}$. The Chain Rule could then be used to conclude that h is differentiable, and to compute $h(x)$, since $h(x) = F(2x+1)$ and so $h'(x) = F'(2x+1)(2x+1)'$. Marks were lost here for not defining an auxiliary function and not correctly applying the Chain Rule.

4(d). The key here was to recall the flow equation $y' = \text{Rate In} - \text{Rate Out}$, and to note that $y' = \frac{\mathrm{d}y}{\mathrm{d}t}$ is measured in kg min^{-1}. The result is a linear first-order differential equation, so the Integration Factor method can be applied to solve for $y(t)$.