Mechanics week 11: Variable mass and conservation of momentum

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1 Introduction

Everything we've studied so far has assumed that the particle has constant mass. This is clearly not realistic in many cases. This week we will relax this assumption, and reconsider some simple examples, in particular using Conservation of Momentum.

2 Conservation of Momentum and Newton's Second Law

We are used to Newton's Second Law in the form "Force equals mass times acceleration". This, however, only works when mass is constant. For examples with variable mass we need the more general law of "Conservation of Momentum", where force is equal to the rate of change of (linear) momentum. Recalling that momentum $\mathbf{p}$ is defined via $\mathbf{p} = m\mathbf{v}$ where $\mathbf{v}$ is velocity (and $\mathbf{v} = \dot{\mathbf{r}}$ for a particle with position vector $\mathbf{r}$), we therefore have

$$\mathbf{F} = \frac{d\mathbf{p}}{dt}.$$

Note that when mass is constant this reduces to

$$\begin{aligned} \mathbf{F} &= \frac{d\mathbf{p}}{dt}, \\ &= \frac{d(m\mathbf{v})}{dt}, \\ &= m\frac{d\mathbf{v}}{dt}, \\ &= m\mathbf{a}, \end{aligned}$$

where $\mathbf{a}$ gives the acceleration, as before.

This concept (along with a similar equation representing conservation of mass, saying that the amount of "stuff" you have remains the same) forms the basis for most mechanical models used in a wide variety of applications, from understanding how planes fly, to how cancer tumours grow, to how a chocolate fountain works, and will be used extensively in year 3 modules such as Continuum Mechanics and Advanced Mathematical Modelling.

3 No net force

Even if there is no net force acting on the particle, allowing for variable mass can allow changes in motion. For example consider standing on ice (or some other approximation of a zero friction surface e.g. standing on a skateboard or floating in space) and throwing a heavy item away from you. Since momentum over the whole system will be conserved, you will start to move in the opposite direction, despite the fact that no external force has acted on you. Rocket ships use this principle to generate thrust by burning fuel and expelling exhaust gas at high velocity to push the rocket forwards.

3.1 Instantaneous mass ejection

When all the mass that is going to ejected is thrown out instantaneously, conservation of momentum can be easily applied to work out the final velocity.

Example 1: Movement on ice

If a person of mass m_p is standing on smooth ice (so there is no friction) holding a heavy object of mass m_o which they push away from them with velocity v_o *relative to their new motion*, what is the resulting motion of the person?

Solution. We assume all motion happens in 1D, with the object thrown in the positive x direction. Since there are no net forces acting on the person, and the person (and hence also the object) is initially at rest, conservation of momentum across the whole system gives

$$m_p v_p + m_o \left(v_o + v_p\right) \quad = \quad 0,$$

where v_p is the velocity of the person, by summing the momentum of both the object and the person. Hence rearranging gives the velocity of the person as

$$v_p \quad = \quad -\frac{m_o}{m_p + m_o} v_o,$$

where the minus sign shows they are moving in the opposite direction to that which they threw the object. ◀

3.2 Continuous mass ejection

In contrast, when the mass is being ejected continuously (for example a rocket burning fuel), more care is needed as the ejected materal will have a different velocity depending on the time at which it was expelled. In such cases we consider what's happening over a small time δt, and then take the limit as δt tends to zero to find the governing equation. We will consider examples where the motion occurs only in one dimension, but things generalise to more than one dimension in the obvious way. It is also important to be very clear about what frame of reference velocities are being measured in.

Derivation We first derive the **Rocket Equation** for a rocket generating forward thrust by expelling exhaust gases. This equation has wider applicability, but it's easiest to be more specific in the derivation.

At time t let the rocket ship have mass $m(t)$ and be moving with velocity $v(t)$ relative to a fixed inertial frame (i.e. as a fixed observed would see it). The total momentum of the rocket at time t is then given by $P(t) = mv$.

We consider what happens over a short time interval δt, during which the rocket expels a small amount of exhaust gas backwards at a velocity v_{ex} relative to the rocket, or at a velocity $v + \delta v - v_{\text{ex}}$ in the fixed inertial reference frame, where δv gives the change in the velocity of the rocket. This leads to a change in the mass of the rocket which we denote δm (which we know will be negative), so the mass at time $t + \delta t$ will be $m + \delta m$ (see footnote for an alternative method [1]). This will also lead to a change in the velocity of the rocket $v + \delta v$, and hence the momentum of the rocket at time $t + \delta t$ is $P_{rocket}(t + \delta t) = (m + \delta m)(v + \delta v)$.

The expelled fuel also has momentum which is contributing to the overall momentum of the system. The expelled fuel has mass $-\delta m$ and velocity $v + \delta v - v_{\text{ex}}$ so that the fuel momentum at time $t + \delta t$ is given by $P_{fuel}(t + \delta t) = -\delta m(v - v_{\text{ex}})$.

Thus the total momentum at time $t + \delta t$ is

$$\begin{aligned} P(t + \delta t) &= P_{rocket}(t + \delta t) + P_{fuel}(t + \delta t), \\ &= (m + \delta m)(v + \delta v) - \delta m(v + \delta v - v_{\text{ex}}). \end{aligned}$$

Since no forces are acting on the rocket, there is no change in momentum over the time δt, and hence

$$\begin{aligned} 0 &= P(t + \delta t) - P(t), \\ &= (m + \delta m)(v + \delta v) - \delta m(v + \delta v - v_{\text{ex}}) - mv. \end{aligned}$$

Now, since δt is small, we also expect δm and δv to be small. Therefore $\delta m \delta v$ is very small, and these terms can be neglected (in fact they cancel in this case anyway). Hence

$$0 = m\delta v + v_{\text{ex}}\delta m$$

since the leading order "mv" terms cancel.

Dividing through by δt and taking the limit as $\delta t \to 0$ we have

$$0 = m\frac{dv}{dt} + v_{\text{ex}}\frac{dm}{dt}.$$

The quantity $-v_{\text{ex}}\frac{dm}{dt}$ is often referred to as the thrust of the rocket.

Example 2: Rocket with no external force

A rocket of mass M_r neglecting fuel blasts off from rest and expels fuel at a constant rate k and velocity v_{ex} relative to the ship. Ignoring the effect of gravity, if the total mass of fuel initially is M_f, what will be the final velocity of the rocket?

Solution. The Rocket equation

$$0 = m\frac{dv}{dt} + v_{\text{ex}}\frac{dm}{dt},$$

will hold as derived above. If the fuel is burnt at a constant rate k, the total mass of fuel m_f will evolve according to

$$\frac{dm_f}{dt} = -k,$$

[1] This could also be written as $m(t + \delta t)$ etc. The derivation can then be replicated by Taylor expanding all functions for small δt.

which we integrate to give

$$m_f = M_f - kt,$$

where M_f gives the initial mass of fuel. The total mass of the rocket is therefore

$$\begin{aligned} m(t) &= M_r + m_f(t), \\ &= M_r + M_f - kt, \end{aligned}$$

by summing the mass of the rocket and the mass of the fuel. This will hold until all the fuel has been burnt such that $m_f = 0$ at $t = M_f/k$.

Thus the rocket equation gives

$$\begin{aligned} 0 &= m\frac{dv}{dt} + v_{\text{ex}}\frac{dm}{dt}, \\ (M_r + M_f - kt)\frac{dv}{dt} &= kv_{\text{ex}}, \end{aligned}$$

since $dm/dt = -k$. We rearrange and integrate with respect to t to solve:

$$\begin{aligned} \int \frac{dv}{dt}dt &= \int \frac{kv_{\text{ex}}}{M_r + M_f - kt}dt, \\ \implies \int dv &= -v_{\text{ex}} \ln(M_r + M_f - kt) + \text{constant}, \\ \implies v &= -v_{\text{ex}} \ln(M_r + M_f - kt) + v_{\text{ex}} \ln(M_r + M_f), \\ &= v_{\text{ex}} \ln\left(\frac{M_r + M_f}{M_r + M_f - kt}\right), \end{aligned}$$

using the initial condition that $v = 0$ at $t = 0$.

Now, we run out of fuel when $t = M_f/k$ at which point the velocity is

$$v = v_{\text{ex}} \ln\left(\frac{M_r + M_f}{M_r}\right),$$

giving us the final velocity of the rocket. We see this is controlled by varying either the expulsion speed v_{ex}, or the rocket-to-fuel mass ratio M_f/M_r. The final velocity will be much more sensitive to the expulsion speed than the mass ratio due to the logarithm, so this should be targeted if you want to increase the rocket's final speed. ◀

4 Including a net force

If the rocket (for example) is also subject to a net force, most of the derivation above continues to hold. However the change in momentum of the system $P(t + \delta t) - P(t)$ no longer equals zero, but rather changes in momentum will be related to the net force applied $F(t)$ and the length of time over which it is applied δt. Hence instead we have

$$\begin{aligned} F\delta t &= P(t + \delta t) - P(t), \\ &= m\delta v + v_{\text{ex}}\delta m, \end{aligned}$$

again neglecting higher order terms of the form $\delta m \delta v$. Dividing by δt we find

$$\begin{aligned} F &= \frac{P(t+\delta t) - P(t)}{\delta t}, \\ &= m\frac{\delta v}{\delta t} + v_{\text{ex}}\frac{\delta m}{\delta t}, \end{aligned}$$

and letting δt tend to zero gives

$$\begin{aligned} F &= \frac{dP}{dt}, \\ &= m\frac{dv}{dt} + v_{\text{ex}}\frac{dm}{dt}. \end{aligned}$$

This is very similar to before, but where the force F can effect changes in the overall momentum of the system.

Example 3: Rocket acted upon by gravity

In the early stages of a rocket flight gravity cannot be neglected. If we take a coordinate system in which gravity points downwards, where a rocket of mass M_r neglecting fuel blasts off from rest and expels fuel at a constant rate k and velocity v_{ex} relative to the ship, the governing equation becomes

$$-mg = m\frac{dv}{dt} + v_{ex}\frac{dm}{dt}.$$

As before $m(t) = M_r + M_f - kt$, and hence

$$-(M_r + M_f - kt)g = (M_r + M_f - kt)\frac{dv}{dt} - kv_{ex}.$$

Rearranging this we get

$$\frac{dv}{dt} = -g + \frac{kv_{ex}}{M_r + M_f - kt}.$$

Integrating this equation and using $v = 0$ at $t = 0$ we find

$$v = -gt + v_{ex}\ln\left(\frac{M_r + M_f}{M_r + M_f - kt}\right).$$

The final velocity is again when $kt = M_f$ and all the fuel has been burnt. This gives

$$v = v_{ex}\ln\left(\frac{M_r + M_f}{M_r}\right) - \frac{gM_f}{k}.$$

This shows that the final velocity in the presence of gravity is always less than the velocity in the absence of gravity. Also, for fixed values of M_r, M_f and v_{ex}, the influence of gravity can be reduced by increasing the value of k (i.e. burning the rocket fuel at a faster rate).

Example 4: Raindrop accumulating mass as it falls through a cloud

Suppose that a raindrop falls through an (assumed stationary) cloud, and accumulates mass at a rate kmv for some constant k, where m is the mass of the raindrop and v is its velocity. If the raindrop starts from rest with initial mass m_0, find its mass and velocity at time t.

Solution. Since the mass being added to the raindrop has zero initial velocity it also has zero initial momentum (note that this is not the case if the added mass is moving with an initial velocity when more care is needed!). Taking gravity to act in the positive direction conservation of momentum gives

$$\begin{aligned} mg &= \frac{d}{dt}(mv), \\ &= m\frac{dv}{dt} + v\frac{dm}{dt}. \end{aligned}$$

Now we know that the mass evolves according to

$$\frac{dm}{dt} = kmv,$$

giving

$$mg = m\frac{dv}{dt} + kmv^2.$$

To solve this we cancel through by m and rearrange to find

$$\frac{dv}{dt} = g - kv^2.$$

This is a separable equation and hence

$$\begin{aligned} \frac{1}{g-kv^2}\frac{dv}{dt} &= 1, \\ \implies \int \frac{1}{g-kv^2}\frac{dv}{dt}dt &= \int dt. \end{aligned}$$

We solve this using partial fractions, first noticing this will be easier is we define $\gamma^2 = g/k(>0)$. Then

$$\begin{aligned} \frac{1}{g-kv^2} &= \frac{1}{k}\frac{1}{g/k - v^2}, \\ &= \frac{1}{k}\frac{1}{\gamma^2 - v^2}, \\ &= \frac{1}{k}\frac{1}{(\gamma - v)(\gamma + v)}, \\ &= \frac{1}{2k\gamma}\left(\frac{1}{(\gamma - v)} + \frac{1}{(\gamma + v)}\right). \end{aligned}$$

Hence

$$\begin{aligned} \int \frac{1}{g-kv^2}\frac{dv}{dt}dt &= \int dt, \\ \implies \int \frac{1}{2k\gamma}\left(\frac{1}{(\gamma - v)} + \frac{1}{(\gamma + v)}\right)dv &= \int dt, \\ \implies \frac{1}{2k\gamma}\left(-\ln(\gamma - v) + \ln(\gamma + v)\right) &= t + \text{constant}. \end{aligned}$$

Using $v = 0$ at $t = 0$ we find

$$t = \frac{1}{2k\gamma} \ln\left(\frac{\gamma + v}{\gamma - v}\right).$$

This is rearranged to give

$$\begin{aligned}
\frac{\gamma + v}{\gamma - v} &= e^{2k\gamma t}, \\
\implies \gamma + v &= (\gamma - v)\, e^{2k\gamma t}, \\
\implies v\left(1 + e^{2k\gamma t}\right) &= \gamma\left(e^{2k\gamma t} - 1\right), \\
\implies v &= \gamma \frac{e^{2k\gamma t} - 1}{1 + e^{2k\gamma t}}, \\
&= \gamma \tanh(\gamma k t), \\
&= \sqrt{\frac{g}{k}} \tanh\left(\sqrt{kgt}\right).
\end{aligned}$$

This gives the velocity of the raindrop.

We can now find the mass, since

$$\begin{aligned}
\frac{dm}{dt} &= kmv, \\
&= km\sqrt{\frac{g}{k}} \tanh\left(\sqrt{kgt}\right), \\
&= m\sqrt{kg} \tanh\left(\sqrt{kgt}\right).
\end{aligned}$$

This is again separable, so we find

$$\begin{aligned}
\int \frac{1}{m}\frac{dm}{dt} dt &= \sqrt{kg} \int \tanh\left(\sqrt{kgt}\right) dt, \\
\implies \ln m &= \ln \cosh\left(\sqrt{kgt}\right) + \text{constant}, \\
\implies \ln m &= \ln \cosh\left(\sqrt{kgt}\right) + \ln m_0,
\end{aligned}$$

where $m = m_0$ at $t = 0$ is the initial mass. Hence

$$m = m_0 \cosh\left(\sqrt{kgt}\right),$$

gives the evolving mass of the raindrop.

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This concludes the section on variable mass, and hence the content of the module!

Activity: *You should now be able to tackle questions 3 and 4 on this week's problem sheet.*

5 Summary

In this module we have covered some of the fundamental building blocks and concepts of applied mathematics. We've focussed on Newtonian mechanics, but the techniques and concepts of forming and solving mathematical models are more broadly applicable to a wide range of topics.

In particular, we have covered:

- Units and dimensions.
- Kinematics (position, velocity and acceleration).
- Newton's laws of motion.
- Solving second order linear ODEs with constant coefficients.
- Formulating and solving problems using Newton's second law in one or more dimensions.
- Talked briefly about moving frames of reference.
- Central forces.
- Kepler's laws.
- Formulating problems using polar coordinates, including calculating and using the components of velocity and acceleration in radial and transverse directions.
- Finding the equation of the particle path in terms of u as a function of θ.
- Showing that the angular momentum $h = r^2\dot{\theta}$ is constant.
- Formulating and solving central force problems, in particular using initial conditions to find the constant angular momentum h.
- Bounded orbits and precession (briefly touched on).
- Angular momentum/moment of momentum.
- Conservation of energy in one or more dimensions including:
  - kinetic energy.
  - potential energy for different forces.
- Formulating conservation of energy equations (including using initial conditions and conservation of angular momentum) and using them to find bounds on the motion of a particle.
- Using conservation of energy to describe the motion of a particle on surface under gravity, including on a surface of revolution.
- Using conservation of momentum to consider variable mass problems, including deriving evolution equations by considering the dynamics over a small time frame.

I hope you've enjoyed this module - relevant applied mathematics techniques will be taught in further modules including 2MVA and 2DE, with more modelling content in third year modules such as Advanced Mathematical Modelling and Continuum Mechanics.