Relativistic kinetic energy

Let us multiply the expression for relativistic mass by c^2. The resulting expression has dimensions of energy and for small velocities behaves as

$$m(v)c^2 = \frac{mc^2}{\sqrt{1 - v^2/c^2}} \simeq mc^2 \left(1 + \frac{1}{2}\frac{v^2}{c^2}\right) = mc^2 + \frac{mv^2}{2} \qquad \text{as} \qquad v/c \to 0\,.$$

One recognises the velocity-dependent term as simply the kinetic energy of a free particle. This non-relativistic expression comes as an addition to the constant mc^2 familiar from the popular science. Being independent of velocity this term does not play any role in non-relativistic physics and can be simply considered as the level from which kinetic energy is measured (*i.e.* set to zero). The fact that we found these two terms in the small velocity expansion of the relativistic mass (times c^2) is not a coincidence, but rather a deep statement of energy-mass equivalence in relativistic physics. The total kinetic energy, including the rest energy mc^2 is therefore

$$E(v) = m(v)c^2 = \gamma(v)mc^2 = \frac{mc^2}{\sqrt{1 - v^2/c^2}}\,.$$

The equivalence of mass and energy encoded in the expression $E = mc^2$ is used in physics to measure (rest) mass of elementary particles in energy units. Take, for example an electron with (rest) mass

$$m_\mathrm{e} \simeq 9 \times 10^{-31}\mathrm{kg}\,.$$

The corresponding energy is $m_\mathrm{e}c^2 \simeq 9 \times 3^2 \times 10^{-31} \times 10^{16}\mathrm{J} \simeq 8.1 \times 10^{-14}\mathrm{J}$. Joules are too big as an energy unit for elementary particles. Instead one measures energy in units of electron-volts (eV), *i.e.* the energy gained by an electron with charge $e = 1.6 \times 10^{-19}$Coulomb in a potential difference of one Volt. This corresponds to 1.6×10^{-19}J and the mass of electron expressed in these units is

$$m_\mathrm{e} = \frac{8 \times 10^{-14}\mathrm{J}}{1.6 \times 10^{-19}\mathrm{J/eV}} = 5 \times 10^5\mathrm{eV} = 0.5\mathrm{MeV}\,.$$

Similarly, a proton which is about 2000 times heavier than electron has its rest mass $m_\mathrm{p} = 939.6\mathrm{MeV}$.

Energy-momentum relation in relativistic mechanics

In many situations in physics kinetic energy should be considered as a function of momentum and not velocity. This is crucial, for example, in Quantum Mechanics, where momentum corresponds to de Broglie wavelength of a particle, $p = h/\lambda$, where $h = 6.6 \times 10^{-34}\mathrm{J \cdot s}$ is the Planck constant. In Newtonian mechanics the energy - momentum relation is obtained by expressing $v = p/m$ and substituting into $E = mv^2/2 = p^2/2m$. In relativistic mechanics one can do the same thing, but the relation between momentum and velocity is more complicated. Instead we can make a short cut by taking squares,

$$\frac{E^2}{c^2} = \frac{m^2c^2}{1 - v^2/c^2}\,, \qquad p^2 = \frac{m^2v^2}{1 - v^2/c^2} \quad \Rightarrow \quad \frac{E^2}{c^2} - p^2 = m^2c^2\frac{1 - v^2/c^2}{1 - v^2/c^2} = m^2c^2$$

and the energy-momentum relation is

$$E = \sqrt{m^2c^4 - p^2c^2}\,.$$

This relation is particularly useful for massless particles, like photon. Being a particle of light photon moves with speed of light and expressions for energy and momentum in terms of velocity are ill-defined as the denominator becomes zero. On the other hand, the photon mass is also zero, so its energy and momentum are still defined. Taking $m \to 0$ in the energy-momentum relation above is well defined as well and leads to

$$E = c|p|\,.$$

This expression is valid exactly for massless particles. For massive particles moving with velocities close to speed of light this is a 'ultra-relativistic approximation' valid for momenta $p \gg mc$ ('ultra-relativisitic limit). This is a good approximation for neutrinos which until recently were believed to have zero mass. In the opposite, non-relativistic limit $p \ll mc$ the energy momentum relation is $E \simeq mc^2 + p^2/2m$, as expected.

Using energy and momentum conservation for solving simple dynamical relativistic problems.

Consider decay of a particle with rest mass m_1 into another particle with mass $m_2 < m_1$ and a neutrino which can be assumed massless, see Fig . We want to calculate velocity of the particle after decay in the rest frame where the initial particle was at rest.

We work in the frame where the decaying particle is at rest, so the total energy is $E_{\rm i} = m_1c^2$ and the total momentum is zero, $p_{\rm i} = 0$. After the decay the total momentum is given by the momentum of the neutrino, p, and the momentum of the light particle moving with velocity $-v$,

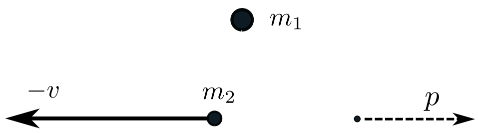

Figure 1: Decay of heavy particle.

$$p_{\rm f} = p - \frac{m_2 v}{\sqrt{1 - v^2/c^2}}\,.$$

The final energy is a sum of the energy of the moving particle and the energy cp of the neutrino,

$$E_{\rm f} = \frac{m_2c^2}{\sqrt{1 - v^2/c^2}} + cp\,.$$

From the conservation of momentum, $p_{\rm f} = p_{\rm i} = 0$ we obtain $p = m_2v/\sqrt{1 - v^2/c^2}$ and substituting this relation into conservation of energy, $E_{\rm f} = E_{\rm i}$ we get

$$m_1c^2 = \frac{m_2c^2}{\sqrt{1 - v^2/c^2}}(1 + v/c) = m_2\sqrt{\frac{1 + v/c}{1 - v/c}}\,.$$

Solving for v/c we obtain

$$v/c = \frac{m_1^2 - m_2^2}{m_1^2 + m_2^2}\,.$$

In this example a fraction of the rest energy m_1c^2 is transformed into kinetic energy of the moving particle and the neutrino.