Mechanics week 10: Energy (motion of a particle on a surface under gravity)

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1 Introduction

This week we will complete our study of conservation of energy, considering a particle moving in 3D, but constrained to a 2D curved surface, for example if it's sliding on the inside of a bowl or on the end of a spherical pendulum. We will consider the special case where the surface is axisymmetric, in particular where it is a surface of revolution, that is it can be constructed by rotating a curve in (e.g.) the $x - z$ plane about the z axis. At first glance this can seem quite complicated, but it's actually quite formulaic - focus on what you're trying to achieve at each step! This week's problem sheet questions are slightly longer, so the lecture videos are slightly lighter.

2 Motion of a particle on a surface under gravity

If a particle moves under the effect of gravity, on a surface in such a way that the reaction[1] between the particle and the surface is normal to the surface (and hence the motion which occurs on the surface), then energy is conserved since:

Newton's second law gives

$$m\ddot{\mathbf{r}} = m\mathbf{g} + \mathbf{R},$$

where $\mathbf{g}$ gives the acceleration due to gravity and $\mathbf{R}$ is the reaction force. We then take the dot product with $\dot{\mathbf{r}}$ to find

$$m\ddot{\mathbf{r}} \cdot \dot{\mathbf{r}} = m\mathbf{g} \cdot \dot{\mathbf{r}} + \mathbf{R} \cdot \dot{\mathbf{r}}.$$

Now, $\mathbf{R} \cdot \dot{\mathbf{r}} = 0$ since these are perpendicular vectors. We now integrate with respect to time to give

$$\begin{aligned} \frac{1}{2}m\dot{\mathbf{r}}^2 &= m\mathbf{g} \cdot \mathbf{r} + \text{const}, \\ \implies \frac{1}{2}m\dot{\mathbf{r}}^2 - m\mathbf{g} \cdot \mathbf{r} &= E, \end{aligned}$$

where E is the constant conserved energy.

Aside: Cylindrical polar coordinates For these types of problems we need to use cylindrical polar coordinates since the system is axisymmetric about the z axis (say). The surface can be considered to be made up of stacks of circles in the xy plane, with varying radii, piled up in the z direction, and hence we take cylindrical polar coordinates such that $x = \rho\cos\theta$, $y = \rho\sin\theta$, $z = z$ where ρ is the (varying) radius of the circle at a given z height, and θ measures the angle around circle; see Figure 1. Note that we've used ρ

[1] the reaction force is from Newton's third law - the surface pushes back on the particle

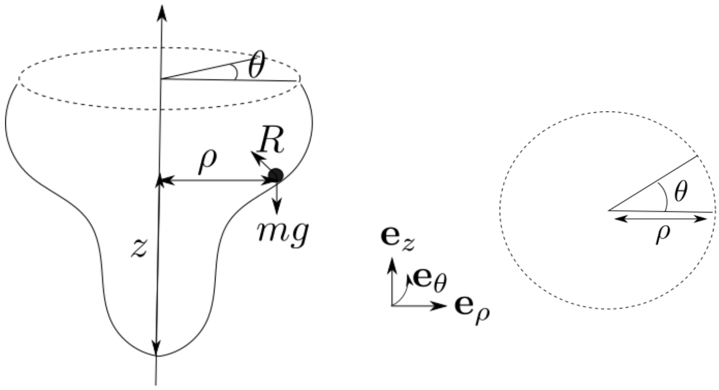

Figure 1: A particle on a surface of revolution from the "side" and the "top", showing cylindrical polar coordinates, and the reaction force and gravity acting on the particle.

for the plane polar coordinate (i.e. in the xy plane) to avoid confusion with $\mathbf{r}$, and $r = |\mathbf{r}|$ the position vector.

Hence the position of the particle is given by

$$\mathbf{r} = \rho \mathbf{e}_\rho + z \mathbf{e}_z,$$

where $\mathbf{e}_\rho$, $\mathbf{e}_\theta$ and $\mathbf{e}_z$ are unit vectors in the ρ, θ and z direction respectively, and $|\mathbf{r}| = \sqrt{\rho^2 + z^2}$. The velocity is then given by

$$\dot{\mathbf{r}} = \dot{\rho} \mathbf{e}_\rho + \rho \dot{\theta} \mathbf{e}_\theta + \dot{z} \mathbf{e}_z,$$

with the acceleration

$$\ddot{\mathbf{r}} = \left(\ddot{\rho} - \rho \dot{\theta}^2\right) \mathbf{e}_\rho + \frac{1}{\rho} \frac{d}{dt} \left(\rho^2 \dot{\theta}\right) \mathbf{e}_\theta + \ddot{z} \mathbf{e}_z,$$

i.e. the same as for plane polar coordinates but with an extra term in the z direction.

2.1 Surface of revolution

We are now going to consider motion on a surface of revolution i.e. we take a curve in ρz space and rotate is through 2π in θ to give an axisymmetric surface, with gravity pointing in the z direction. If we rotate a straight line going through the origin we'll get a cone; a semi circle will give a sphere etc (see e.g. https://www.geogebra.org/m/adb4bfbp, https://www.geogebra.org/m/PuBVTMep, or the picture at https://mathworld.wolfram.com/SurfaceofRevolution.html, but don't worry about the maths!).

If we have a particle on a surface of revolution it is subject to gravity and a reaction force $\mathbf{R}$. (See Figure 1). If the surface is smooth (i.e. no friction), then the (unknown) reaction is normal to the surface. Then:

- There is no component of $\mathbf{R}$ in the $\mathbf{e}_\theta$ direction, so the $\mathbf{e}_\theta$ component of $\mathbf{F} = m\ddot{\mathbf{r}}$ still gives

$$\begin{aligned} \frac{d}{dt}\left(\rho^2\dot{\theta}\right) &= 0, \\ \Longrightarrow \qquad \rho^2\dot{\theta} &= \text{const.} \end{aligned}$$

  This is conservation of angular momentum as before.

- The velocity $\dot{\mathbf{r}}$ is parallel to the surface (since the motion is **along** the surface), so $\mathbf{R} \cdot \dot{\mathbf{r}} = 0$ is also zero and hence energy is conserved. This gives

$$\frac{1}{2}m\dot{\mathbf{r}}^2 + mgz = \text{const},$$

  and hence

$$\frac{1}{2}m\left(\dot{\rho}^2 + \left(\rho\dot{\theta}\right)^2 + \dot{z}^2\right) + mgz = E.$$

We will focus on one extended example this week; in particular you should think about what we're trying to do at each stage.

Example 1: Particle motion on a cone Suppose a particle lies on a cone with equation $z = \rho$, with z pointing vertically upwards. Initially the particle is at height $z = a$ with initial velocity V horizontally. How does the particle move and what are the limits on the motion?

Solution. To find the motion we consider the following steps.

Model: We start with the equations of conservation of energy and angular momentum, that is:

$$\frac{1}{2}m\left(\dot{\rho}^2 + \left(\rho\dot{\theta}\right)^2 + \dot{z}^2\right) + mgz = E = \text{constant}$$

and

$$\rho^2\dot{\theta} = h = \text{constant},$$

(as explained above).

We first find the **initial conditions**. We know that $z = a$, $\dot{z} = 0$, $\rho\dot{\theta} = V$ at $t = 0$. We then need to calculate the initial values of ρ and $\dot{\rho}$ since we know the particle lies on the cone $z = \rho$. Note that

$$\dot{\rho} = \dot{z};$$

in this case this is very simple - however, you may have to differentiate implicitly and substitute to find this relationship, remembering that you know how z and ρ are related since the particle lies on the cone (see questions on the problem sheet). Thus

$$\begin{aligned} z = a \quad &\implies \quad \rho = a, \\ \dot{z} = 0 \quad &\implies \quad \dot{\rho} = 0, \end{aligned}$$

at $t = 0$.

Find the constants: We use these initial conditions to find the constants h, the angular momentum, and E, the energy. Firstly

$$\begin{aligned} h = \rho^2\dot{\theta} &= \rho \cdot \rho\dot{\theta}, \\ &= aV. \end{aligned}$$

Secondly

$$\begin{aligned} E = \frac{1}{2}m\left(\dot{\rho}^2 + \left(\rho\dot{\theta}\right)^2 + \dot{z}^2\right) + mgz &= \frac{1}{2}m\left(0 + V^2 + 0\right) + mga, \\ &= \frac{1}{2}mV^2 + mga. \end{aligned}$$

Eliminate variables: Equation (1) is one equation for the particle motion in terms of ρ, z and θ, whilst $\rho^2\dot{\theta} = h$ and the equation of the surface ($\rho = z$ in this case) give relationships between ρ and θ, and ρ and z. To be able to solve this we need to rewrite it in terms of one variable only; we do this by eliminating θ and one of ρ or z using conservation of angular momentum and the equation of the surface. It depends on what you're trying to find out, and what your equation looks like as to whether you want to write it in terms of ρ or z. In this case we will eliminate ρ and find an equation for the height of the particle z.

We know $\dot{\rho} = \dot{z}$ from the equation of the surface, and we can find

$$\begin{aligned} \rho^2\dot{\theta}^2 &= \left(\rho^2\dot{\theta}\right)^2/\rho^2, && \text{[rearrange]} \\ &= h^2/\rho^2, && \text{[cons. of angular momentum]} \\ &= h^2/z^2, && \text{[use equation of surface]} \\ &= a^2V^2/z^2. && \text{[sub in h]} \end{aligned}$$

Hence we find

$$\begin{aligned} \frac{1}{2}mV^2 + mga &= \frac{1}{2}m\left(\dot{\rho}^2 + \left(\rho\dot{\theta}\right)^2 + \dot{z}^2\right) + mgz, \\ \implies \frac{1}{2}V^2 + ga &= \frac{1}{2}\left(\dot{z}^2 + \frac{a^2V^2}{z^2} + \dot{z}^2\right) + gz, \end{aligned}$$

i.e. an equation just in terms of z.

What does this tell us? We now investigate what the model can tell us, this will depend on the question we're trying to answer. First we rearrange to give

$$\begin{aligned}\dot{z}^2 &= \frac{1}{2}V^2 + ga - \frac{1}{2}\frac{a^2V^2}{z^2} - gz, \\ &= \frac{1}{2}V^2\left(1 - \frac{a^2}{z^2}\right) + g\left(a - z\right),\end{aligned}$$

this gives the $\dot{z}^2$ term on the left hand side, and we know this is necessarily positive, and also it will be zero at the points where the particle changes the direction of motion.

The question asks for the limits on the motion, so we suspect we want to find the heights between which the particle moves (since this seems the most likely thing to happen!). Since $\dot{z}^2 > 0$, we want to try and factorise the right hand side if possible - this will easily tell us when $\dot{z}$ is zero and hence when the particle changes direction. Hence

$$\begin{aligned}\dot{z}^2 &= \frac{1}{2}\frac{V^2}{z^2}\left(z^2 - a^2\right) + g\left(a - z\right), \\ &= \frac{1}{2}\frac{V^2}{z^2}\left(z - a\right)\left(z + a\right) + g\left(a - z\right), \\ &= \frac{1}{2}\frac{V^2}{z^2}\left(z - a\right)\left(z + a\right) - g\left(z - a\right), \\ &= \frac{\left(z - a\right)g}{z^2}\left(\frac{V^2}{2g}\left(z + a\right) - z^2\right), \\ &= -\frac{\left(z - a\right)g}{z^2}\left(z^2 - \frac{V^2}{2g}z - \frac{V^2a}{2g}\right).\end{aligned}$$

Since we knew that $\dot{z} = 0$ at $z = a$ by the initial conditions we were expecting this to be one of the roots. This then leaves us with a quadratic equation $z^2 - \frac{V^2}{2g}z - \frac{V^2a}{2g}$ to factorise. We can write this as

$$z^2 - \frac{V^2}{2g}z - \frac{V^2a}{2g} = \left(z - z_1\right)\left(z - z_2\right),$$

where we need to find z_1, z_2 which are the roots of the quadratic equation

$$z^2 - \frac{V^2}{2g}z - \frac{V^2a}{2g} = 0.$$

Therefore we use the quadratic formula to find:

$$
\begin{aligned}
z_{1,2} &= \frac{\frac{V^2}{2g} \pm \sqrt{\left(\frac{V^2}{2g}\right)^2 + 4\frac{V^2 a}{2g}}}{2}, \\
&= \frac{V^2}{4g} \pm \sqrt{\frac{V^4}{16g^2} + \frac{V^2 a}{2g}},
\end{aligned}
$$

so

$$
\begin{aligned}
z_1 &= \frac{V^2}{4g} + \sqrt{\frac{V^4}{16g^2} + \frac{V^2 a}{2g}}, \\
z_2 &= \frac{V^2}{4g} - \sqrt{\frac{V^4}{16g^2} + \frac{V^2 a}{2g}}.
\end{aligned}
$$

We now want to think about whether these roots are physically realistic as the points at which the particle turns around. Note that z_2 must be negative, since the first term is smaller than the square root as

$$
\frac{V^4}{16g^2} + \frac{V^2 a}{2g} > \frac{V^4}{16g^2},
$$

as $\dfrac{V^2 a}{2g}$ is positive, so

$$
\begin{aligned}
\sqrt{\frac{V^4}{16g^2} + \frac{V^2 a}{2g}} &> \sqrt{\frac{V^4}{16g^2}}, \\
&= \frac{V^2}{4g}.
\end{aligned}
$$

But z can't be negative due to the geometry of the cone, so $z - z_2 > 0$.

Then since $\dot{z}^2 \geq 0$ we have

$$
\underbrace{\dot{z}^2}_{\geq 0} = \underbrace{-\frac{(z-a)\,g}{z^2}\,(z-z_1)}_{\geq 0}\underbrace{(z-z_2)}_{\geq 0},
$$

and hence

$$
(z-a)(z-z_1) \leq 0.
$$

This means the particle must lie between $z = a$ and $z = z_1$ (as a positive quadratic it'll be below the x axis (and hence negative) only between those point), since either

$$\begin{aligned} z-a\leq 0 \quad \text{and} \quad & z-z_1\geq 0,\\ \Longrightarrow \qquad & z_1\leq z\leq a, \end{aligned}$$

or

$$\begin{aligned} z-a\geq 0 \quad \text{and} \quad & z-z_1\leq 0,\\ \Longrightarrow \qquad & a\leq z\leq z_1, \end{aligned}$$

with $\dot{z}=0$ at the maximum/minimum values. The sign of $\ddot{z}$ will give whether the particle is rising or falling. ◀

This concludes the section on conservation of energy.

Activity: *You should now be able to tackle questions 1 and 2 on this week's problem sheet. You may find question 2 easier to tackle than question 1!*