Electromagnetism I – Solutions problem sheet 9

Problem 1.

1. The current is the integral of the current density

   $$j(r) = j_0 \frac{r}{a} ,$$

   therefore:

   $$\begin{aligned} I &= \int_a^b j(r) 2\pi r dr & \textbf{[1 mark]} \\ &= \frac{2\pi}{3a} j_0 \left(b^3 - a^3\right) . & \textbf{[1 mark]} \end{aligned}$$

2. We will use Ampere's law to derive the magnetic field:

   $$\oint \vec{B} \cdot d\vec{l} = \mu_0 I ,$$

   where I is the current linked to (enclosed by) the closed loop.

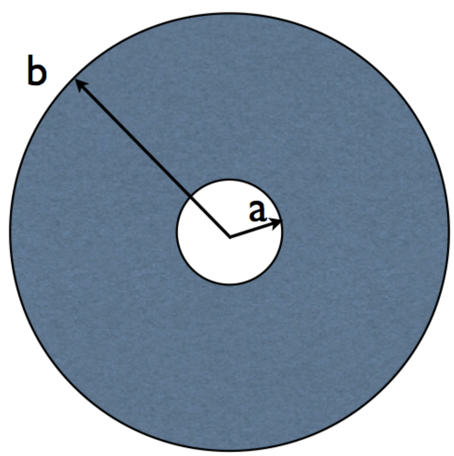
   (a) $r < a$: no current linked, therefore $B = 0$ **[1 mark]**

   (b) $a \leq r \leq b$:

   $$\begin{aligned} 2\pi r B(r) &= \int_a^r j(r) 2\pi r dr = \frac{2\pi}{3a} \mu_0 j_0 \left[r^3\right]_a^r \\ B(r) &= \frac{\mu_0 j_0}{3a} \frac{1}{r} \left(r^3 - a^3\right) . & \textbf{[1 mark]} \end{aligned}$$

   (c) $r > b$:

   $$B = \frac{\mu_0 I}{2\pi r} = \frac{\mu_0 j_0}{3a} \frac{1}{r} \left(b^3 - a^3\right) \qquad \textbf{[1 mark]}$$



Problem 2.

1. B and I are perpendicular to each other, therefore the force on an element dr of the rod is:

$$dF = IB\,dr\,.$$

The torque is:

$$d\vec{\tau}_B = \vec{r} \times d\vec{F}$$

and due to the fact that $\vec{r} \perp \vec{F}$:

$$d\tau_B = rdF = r(IBdr)\,.$$

Integrating, the torque is:

$$\tau_B = \int_0^L d\tau_B = IB \int_0^L rdr = \frac{1}{2} IBL^2$$

Putting in the numerical values:

$$\tau_b = \left[\frac{6.5 \times 0.34 \times 0.2^2}{2}\right] \mathrm{N\,m} = 0.044\,\mathrm{N\,m} \qquad \textbf{[1 mark]}$$

The torque tends to rotate the rod clockwise. **[1 mark]**

2. The rod is in equilibrium when the torque from B is balanced by the torque, τ_S, produced by the spring:

$$|F_S| = k\Delta x\,,$$
$$\tau_S = k\Delta x L \sin\alpha \qquad \alpha = 53^o \qquad \textbf{[1 mark]}$$

Therefore at equilibrium:

$$\tau_B = \tau_S\,,$$

hence:

$$\Delta x = \frac{\tau_B}{kL\sin\alpha} \qquad \textbf{[1 mark]}$$

The energy stored in the spring is:

$$\begin{aligned} U &= \frac{1}{2}k(\Delta x)^2 = \frac{1}{2}k\left(\frac{\tau_B}{kL\sin\alpha}\right)^2 = \frac{\tau_B^2}{2kL^2\sin^2\alpha} \\ &= \left[\frac{(0.044)^2}{2 \times 4.8 \times (0.2)^2\,[\sin(53\pi/180)]^2}\right] \mathrm{J} \\ &= 0.008\,\mathrm{J} \qquad \textbf{[1 mark]} \end{aligned}$$