University of Birmingham
School of Mathematics

1SAS Sequences and Series Autumn 2024

Problem Sheet 5
(Issued Week 10)

Q1. Which of the following series converge? Justify your answers. You may use standard series convergence tests provided you make it clear that you are doing so.

(i)
$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{1+\log n}$$

(ii)
$$\sum_{n=3}^{\infty} \frac{1}{n(\log n)(\log\log n)}$$

(iii)
$$\sum_{n=1}^{\infty} \left(\frac{1}{n^3} + \frac{1}{n^3+1} + \cdots + \frac{1}{n^3+n} \right)$$

(iv)
$$\sum_{n=1}^{\infty} n^{-3/2} \sin\left(\frac{n\pi}{12}\right)$$

(v)
$$\sum_{n=1}^{\infty} \left(\frac{n^3+4n^2+17n+1}{2n^3-n^2+1} \right)^n \cos(n)$$

Q2. Let $\varepsilon > 0$. Use the Integral Test to show that the series
$$\sum_{n=2}^{\infty} \frac{1}{n(\log n)^{1+\varepsilon}}$$
converges. [Here you may use the fact that $(\log n)^{\varepsilon} \to \infty$ without proof.]

Q3. In this question you may appeal to results from the module, provided you make it clear when you are doing so.

(i) Does the series
$$\sum_{n=1}^{\infty} \frac{(-1)^n (n!)^2}{(2n)!}$$
converge? Justify your answer.

(ii) For which real numbers x does the series
$$\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n 5^n}$$
converge? Justify your answer.

(iii) Prove that the series
$$\sum_{n=1}^{\infty} \sin^2\left(\tfrac{1}{n}\right)$$
converges. (You may appeal to standard properties of the sine function provided you state them clearly.)

Q4. Which of the following series converge? Justify your answers carefully. You may appeal to Series Convergence Tests, along with limit theorems for sequences (such as the Algebra of Limits) where appropriate.

(i)
$$\sum_{n=1}^{\infty} \frac{n}{2n+1}$$

(ii)
$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}}$$

(iii)
$$\sum_{n=1}^{\infty} \frac{10^n (n!)^3}{(3n)!}$$

(iv)
$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1} \cos^2(n)}{n^2}$$

Q5. Give an example of:

(i) a convergent series $\sum_{n=1}^{\infty} a_n$ such that $\sum_{n=1}^{\infty} (-1)^n a_n$ diverges.

(ii) a convergent series $\sum_{n=1}^{\infty} a_n$ and a sequence $\lambda_n \to 0$ such that $\sum_{n=1}^{\infty} \lambda_n a_n$ diverges.

(iii) a convergent series $\sum_{n=1}^{\infty} a_n$ and a nonnegative sequence $\lambda_n \to 0$ such that $\sum_{n=1}^{\infty} \lambda_n a_n$ diverges.

Q6. Prove that if $\sum_{n=1}^{\infty} a_n$ is absolutely convergent then
$$\sum_{n=1}^{\infty} \frac{a_n}{1+a_n^2}$$
converges.

Q7. For which $x \in \mathbb{R}$ does the power series

$$\sum_{n=0}^{\infty} (-1)^n \frac{x^n}{2^n \sqrt{n+1}}$$

converge? For which x is it conditionally convergent? Justify your assertions.

Q8. Find the radius of convergence of the power series

$$\sum_{n=0}^{\infty} \frac{n^n}{7^n n!} x^n.$$

Q9. Find the convergence set of the power series

$$\sum_{n=0}^{\infty} \frac{(-1)^{\pi(n)}}{n^2} x^n,$$

where $\pi(n)$ denotes the number of primes not exceeding n. [Recall that the convergence set is the set of real numbers x for which the power series converges.]

Extra Questions

EQ1. Find the first four non-zero terms in the Maclaurin series of the function

$$f(x) = e^{-x^2} \sin x.$$

EQ2. Find an expression for the nth term in the Taylor series of the function

$$f(x) = \frac{1}{\sqrt{x}}$$

about the point $x = 1$.

EQ3. A function $f : (-R, R) \to \mathbb{R}$ is given by

$$f(x) = \sum_{n=0}^{\infty} \frac{1 + (-1)^n}{n+1} x^{n+1},$$

where R denotes the radius of convergence of the defining power series. Find R and show that

$$f'(x) = \frac{2}{1 - x^2}$$

on $(-R, R)$. Deduce a similarly explicit expression for $f(x)$.