Appendix: Solving Algebraic Equations

To find the low order n-cycles exactly, we need to deal with algebraic equations. For the logistic map the 1-cycles are controlled by

$$x_1 = rx_1(1 - x_1)$$

the 2-cycles by

$$x_2 = rx_1(1 - x_1) \qquad x_1 = rx_2(1 - x_2)$$

the 3-cycles by

$$x_2 = rx_1(1 - x_1) \qquad x_3 = rx_2(1 - x_2) \qquad x_1 = rx_3(1 - x_3)$$

and the four cycles by

$$x_2 = rx_1(1 - x_1) \qquad x_3 = rx_2(1 - x_2) \qquad x_4 = rx_3(1 - x_3) \qquad x_1 = rx_4(1 - x_4)$$

Obviously the general n-cycle will involve n similar equations. The first crucial piece of understanding is that the problem that we are trying to solve cannot distinguish between the points in the cycle, only the order of the points. If we were to start from x_2, instead of x_1, we would get the same system of equations. This means that we cannot target x_1 above any of the other possible points and we need an approach that cannot distinguish between the points. The approach can distinguish the order however.

The key area of mathematics is algebraic equations and in particular polynomial equations. The low order equations in terms of their roots are

$$(z - x_1) \equiv z - b_1$$

$$(z - x_1)(z - x_2) = z^2 - (x_1 + x_2)z + x_1x_2 \equiv z^2 - b_1z + b_2$$

$$(z - x_1)(z - x_2)(z - x_3) = z^3 - (x_1 + x_2 + x_3)z^2 + (x_2x_3 + x_3x_1 + x_1x_2)z - x_1x_2x_3$$
$$\equiv z^3 - b_1z^2 + b_2z - b_3$$

$$(z - x_1)(z - x_2)(z - x_3)(z - x_4) = z^4 - (x_1 + x_2 + x_3 + x_4)z^3$$
$$+(x_2x_3 + x_3x_4 + x_4x_1 + x_1x_2 + x_2x_4 + x_3x_1)z^2 - (x_2x_3x_4 + x_3x_4x_1 + x_4x_1x_2 + x_1x_2x_3)z$$
$$+x_1x_2x_3x_4 \equiv z^4 - b_1z^3 + b_2z^2 - b_3z + b_4$$

These equations cannot distinguish which root is which and so we target the finding of these equations rather than trying to find the roots themselves. If we can then solve these equations then we can find the roots. The focus is now on the b_m coefficients. b_1 is just the sum of the roots. b_2 is the sum of the products of any and all pairs of roots. b_3 is the sum of the products of any and all triples of roots...and so on.

The details of the logistic map now take centre stage. For 1-cycles we have

$$b_1 = rb_1(1 - b_1)$$

a quadratic equation for b_1 which we can solve to provide

$$b_1 = 0,\ 1 - \frac{1}{r}$$

which are the desired roots. For 2-cycles we add the two equations to get

$$x_1 + x_2 = r\left[x_1 + x_2 - x_1^2 - x_2^2\right] \quad \Rightarrow \quad b_1 = r\left[b_1 - b_1^2 + 2b_2\right]$$

where we used

$$(x_1 + x_2)^2 = x_1^2 + x_2^2 + 2x_1x_2 \quad \Rightarrow \quad x_1^2 + x_2^2 = b_1^2 - 2b_2$$

to obtain a relationship between b_1 and b_2. Now we have a choice however. We can product the two equations together to obtain

$$x_1x_2 = r^2x_1x_2(1 - x_1)(1 - x_2) \quad \Rightarrow \quad b_2 = r^2b_2(1 - b_1 + b_2)$$

which provides a second equation relating b_1 and b_2. Alternatively we can subtract the two equations to provide

$$x_2 - x_1 = r\left[x_1 - x_2 - x_1^2 + x_2^2\right] \quad \Rightarrow \quad (x_2 - x_1)\left[1 - r(1 - b_1)\right] = 0$$

If we employ the product approach then we need to avoid one of the x_n vanishing, which is the 1-cycle where they all vanish, and obtain the two equations

$$b_2 = \frac{1}{2}\left[b_1^2 - b_1 + \frac{1}{r}b_1\right] \qquad b_2 = b_1 - 1 + \frac{1}{r^2}$$

Eliminating b_2 then provides

$$b_1^2 - b_1 + \frac{1}{r}b_1 = 2b_1 - 2 + \frac{2}{r^2} \quad \Rightarrow \quad \left[b_1 - \frac{3}{2} + \frac{1}{2r}\right]^2 = \left[\frac{3}{2} - \frac{1}{2r}\right]^2 - 2 + \frac{2}{r^2}$$

$$= \frac{1}{4} - \frac{3}{2r} + \frac{9}{4r^2} = \left[\frac{1}{2} - \frac{3}{2r}\right]^2$$

where we completed the square to solve the quadratic. We did not need to do this, however, because we know of the existence of the previous 1-cycle, so

$$b_1^2 - 3b_1 + \frac{1}{r}b_1 + 2 - \frac{2}{r^2} = \left[b_1 - 2 + \frac{2}{r}\right]\left[b_1 - 1 - \frac{1}{r}\right]$$

and we have the previous 1-cycle together with a new solution

$$b_1 = 1 + \frac{1}{r} \quad \Rightarrow \quad b_2 = b_1 - 1 + \frac{1}{r^2} = \frac{1}{r} + \frac{1}{r^2}$$

and we can construct the quadratic equation that the two points in the 2-cycle satisfy

$$z^2 - \left(1 + \frac{1}{r}\right)z + \frac{1}{r} + \frac{1}{r^2} = 0$$

Before we solve this equation and analyse the 2-cycles, let us look at the second approach taking the difference of the two equations. Here we get

$$b_1 = 1 + \frac{1}{r}$$

immediately, because we eliminated the 1-cycles by assuming $x_1 \neq x_2$. For this approach calculating b_2 is more tricky because we need the more complicated initial equation. A hybrid approach appears best, finding all three equations and using the simplest for each task.

Interestingly, the general solution to polynomial equations also centres on the current ideas. One needs to find useful quantities that cannot tell the difference between the roots and then we can find them in terms of the b_n. For a quadratic equation it is

$$(x_1 - x_2)^2 = x_1^2 + x_2^2 - 2x_1x_2 = (x_1 + x_2)^2 - 4x_1x_2 = b_1^2 - 4b_2$$

and we can find the difference between the roots with the aid of a square-root. The final solutions are then

$$x_1 = \frac{x_1 + x_2}{2} + \frac{x_1 - x_2}{2} = \frac{1}{2}\left[b_1 + \left(b_1^2 - 4b_2\right)^{\frac{1}{2}}\right]$$

Ignoring the quadratic equation and plugging in our b_1 and b_2 we find

$$x_1 = \frac{1}{2}\left[1 + \frac{1}{r} + \left(\left[1 + \frac{1}{r}\right]^2 - 4\left[\frac{1}{r} + \frac{1}{r^2}\right]\right)^{\frac{1}{2}}\right] = \frac{1}{2r}\left[r + 1 + \left(r^2 - 2r - 3\right)^{\frac{1}{2}}\right]$$

Obviously, the other sign to the square-root provides the other point. These points are only real for $r \geq 3$ and $r \leq -1$ and in the region between these two values of r there are no 2-cycles.

To investigate 3-cycles we can employ a completely analogous approach. Firstly we sum the three equations to get

$$x_1 + x_2 + x_3 = r\left[x_1 + x_2 + x_3 - x_1^2 - x_2^2 - x_3^2\right]$$

$$= r\left[x_1 + x_2 + x_3 - (x_1 + x_2 + x_3)^2 + 2(x_2x_3 + x_3x_1 + x_1x_2)\right]$$

$$\Rightarrow \quad b_1 = r(b_1 - b_1^2 + 2b_2)$$

Note that this equation is identical to the previous case for 2-cycles and remains true in all cases. Next we product the three equations to get

$$x_1x_2x_3 = r^3x_1x_2x_3(1 - x_1)(1 - x_2)(1 - x_3) \quad \Rightarrow \quad b_3\left[1 - r^3(1 - b_1 + b_2 - b_3)\right] = 0$$

and now we can find b_2 and b_3 in terms of b_1. We need yet another equation to close the system and there are many possible choices. Another quite general idea is to multiply each equation by the sum of the roots not found on the right-hand side and sum. This automatically provides an equation between up to triples of roots and so involves b_1, b_2 and b_3. We get

$$x_2(x_2+x_3)+x_3(x_3+x_1)+x_1(x_1+x_2)=$$
$$r\left[(x_2+x_3)x_1(1-x_1)+(x_3+x_1)x_2(1-x_2)+(x_1+x_2)x_3(1-x_3)\right]$$
$$\Rightarrow \quad b_1^2-2b_2+b_2=r\left[2b_2-b_1b_2+3b_3\right]$$

This idea would have worked for the previous 2-cycles, giving

$$x_2^2+x_1^2=r\left[x_2x_1(1-x_1)+x_1x_2(1-x_2)\right] \quad \Rightarrow \quad b_1^2-2b_2=r\left[2b_2-b_2b_1\right]$$

and providing yet another way to solve the problem. For the 3-cycles we now have three equations in three unknowns and we can eliminate b_2 and b_3 using the first two equations in the third

$$-b_1^2+b_2(1+2r-rb_1)+3rb_3=-b_1^2-3rb_1+b_2(1+5r-rb_1)+3r(b_3-b_2+b_1)=$$

$$-b_1^2-3rb_1+\frac{1}{2r}\left[b_1-rb_1+rb_1^2\right](1+5r-rb_1)+3r\left[1-\frac{1}{r^3}\right]=0$$

and then there is a clear factor coming from the 1-cycle to give

$$\left[b_1-3+\frac{3}{r}\right]\left[b_1^2-\left(3+\frac{1}{r}\right)b_1+2+\frac{2}{r}+\frac{2}{r^2}\right]=0$$

Using the previous analysis we can deduce that

$$b_1=\frac{1}{2}\left(3+\frac{1}{r}\right)+\frac{1}{2r}\left[(r-1)^2-8\right]^{\frac{1}{2}}$$

as one choice with the other sign for the square-root giving a second. We can immediately deduce that there are two 3-cycle solutions, provided that either $r\geq 1+\sqrt{8}$ or $r\leq 1-\sqrt{8}$, and the two three cycles merge at these end-points.

We close with an analysis of 4-cycles. This is much more sophisticated than the previous calculations and the logic of how to choose the approach is much more subtle. The order of the roots is important for this case. We look to every second root and add and subtract the two equations

$$x_2+x_4=r\left[x_1+x_3-(x_1+x_3)^2+2x_1x_3\right]$$

$$x_2-x_4=r(x_1-x_3)\left[1-x_1-x_3\right]$$

together with

$$x_1+x_3=r\left[x_2+x_4-(x_2+x_4)^2+2x_2x_4\right]$$
$$x_3-x_1=r(x_2-x_4)\left[1-x_2-x_4\right]$$

The first equation provides x_1x_3 and the third equation provides x_2x_4. Taking the product of the second and fourth gives

$$(x_1-x_3)(x_2-x_4)\left(r^2\left[1-x_1-x_3\right]\left[1-x_2-x_4\right]+1\right)=0$$

If we assume that we do not have any 2-cycles but only 4-cycles then we have

$$[x_1 + x_3 - 1]\,[x_2 + x_4 - 1] = -\frac{1}{r^2}$$

To obtain the final equation we multiply the second by $(x_1 - x_3)$ and the fourth by $(x_2 - x_4)$ and add

$$(x_1 - x_3)^2\,[1 - x_1 - x_3] + (x_2 - x_4)^2\,[1 - x_2 - x_4] = 0$$

$$= \left[(x_1 + x_3)^2 - 2x_1x_3\right][1 - x_1 - x_3] + \left[(x_2 + x_4)^2 - 2x_2x_4\right][1 - x_2 - x_4]$$

$$= \left[2(x_1 + x_3) - (x_1 + x_3)^2 - \frac{2}{r}(x_2 + x_4)\right][1 - x_1 - x_3]$$

$$+ \left[2(x_2 + x_4) - (x_2 + x_4)^2 - \frac{2}{r}(x_1 + x_3)\right][1 - x_2 - x_4]$$

Now we can observe

$$y_1 \equiv x_1 + x_3 - 1 \qquad y_2 \equiv x_2 + x_4 - 1 \qquad \Rightarrow \qquad y_1 + y_2 = b_1 - 1$$

as natural and then

$$y_1 y_2 = -\frac{1}{r^2}$$

$$\left[y_1^2 - 1 + \frac{2}{r}\,(y_2 + 1)\right] y_1 + \left[y_2^2 - 1 + \frac{2}{r}\,(y_1 + 1)\right] y_2 = 0$$

together with

$$(y_1 + y_2)^3 = y_1^3 + y_2^3 + 3y_1y_2(y_1 + y_2) = y_1^3 + y_2^3 - \frac{3}{r^2}(y_1 + y_2)$$

provides

$$(y_1 + y_2)^3 + \frac{3}{r^2}(y_1 + y_2) - (y_1 + y_2) + \frac{2}{r}(y_1 + y_2) - \frac{4}{r^3} = 0$$

and a cubic equation for b_1

$$(b_1 - 2)^3 + \left[\frac{3}{r^2} + \frac{2}{r} - 1\right](b_1 - 2) - \frac{4}{r^3} = 0$$

or

$$b_1^3 - 6b_1^2 + \left[11 + \frac{2}{r} + \frac{3}{r^2}\right] b_1 - 6 - \frac{4}{r} - \frac{6}{r^2} - \frac{4}{r^3} = 0$$

This is a cubic equation and so there can be up to three sets of 4-cycles. To assess how many exist and where, we need another important fundamental idea; the discriminant.

Solutions to polynomial equations with real coefficients are either real or come in complex-congugate pairs, since

$$z^n - b_1 z^{n-1} + b_2 z^{n-2} + ... \pm b_n = 0 \qquad \Rightarrow \qquad \bar{z}^n - \bar{b}_1 \bar{z}^{n-1} + \bar{b}_2 \bar{z}^{n-2} + ... \pm \bar{b}_n = 0$$

$$\Rightarrow \qquad \bar{z}^n - b_1\bar{z}^{n-1} + b_2\bar{z}^{n-2} + ... \pm b_n = 0$$

where $\bar{a}$ is the complex-conjugate of a. Clearly if z is a solution then so is $\bar{z}$. If $z=\bar{z}$ we have a real solution otherwise we have a complex-conjugate pair. The only way for a pair of complex-conjugate solutions to smoothly become real solutions, as a parameter is varied, is for them to merge providing a double root which can then separate into two distinct real roots. The quantity that vanishes when two roots merge is known as the discriminant. To find the discriminant we may use the idea that

$$P(z) = (z - \lambda)^2 Q(z)$$

where λ is the double root and then

$$\frac{dP}{dz}(z) = (z - \lambda)^2 \frac{dQ}{dz}(z) + 2(z - \lambda)Q(z)$$

leads to the defining equations that

$$P(z) = 0 \qquad \frac{dP}{dz}(z) = 0$$

We can then eliminate z to find the discriminant. At quadratic order

$$P(z) = z^2 - b_1 z + b_2 = 0$$

$$\frac{dP}{dz}(z) = 2z - b_1 = 0$$

We may multiply the first by 2 and the second by z and subtract to get

$$b_1 z - 2b_2 = 0$$

and then multiply the second by b_1 and the third by 2 and subtract to find that

$$b_1^2 - 4b_2 = 0$$

Recall that

$$D_2 = (x_1 - x_2)^2 = b_1^2 - 4b_2$$

which provides a more explicit formula for the discriminant. We may employ this idea to find the discriminant directly, but the derivative idea is more simple to use and algebraically less intensive. The idea for cancelling powers of z^n, one at a time, is quite general. For the cubic equation

$$z^3 - b_1 z^2 + b_2 z - b_3 = 0$$

$$3z^2 - 2b_1 z + b_2 = 0$$

$$b_1 z^2 - 2b_2 z + 3b_3 = 0$$

$$(2b_1^2 - 6b_2)z - b_1 b_2 + 9b_3 = 0$$

$$\left[2b_2 2(b_1^2 - 3b_2) - b_1(b_1 b_2 - 9b_3)\right] z - 3b_3 2(b_1^2 - 3b_2) = 0$$

$$\left[b_1^2b_2 + 3b_1b_3 - 4b_2^2\right] z - 2b_3(b_1^2 - 3b_2) = 0$$

$$\left[b_1^2b_2 + 3b_1b_3 - 4b_2^2\right](b_1b_2 - 9b_3) - 2(b_1^2 - 3b_2)2b_3(b_1^2 - 3b_2) = 0$$

$$b_1^3b_2^2 - 9b_1^2b_2b_3 + 3b_1^2b_2b_3 - 27b_1b_2^3 - 4b_1b_2^3 + 36b_2^2b_3 - 4b_1^4b_3 + 24b_1^2b_2b_3 - 36b_2^2b_3 = 0$$

and so

$$D_3 \equiv b_1^2b_2^2 + 18b_1b_2b_3 - 27b_3^2 - 4b_2^3 - 4b_1^3b_3$$

This calculation only needs to be done once and we could have calculated directly using

$$D_3 \equiv (x_2 - x_3)^2(x_3 - x_1)^2(x_1 - x_2)^2 = \left[x_1^2x_3 + x_2^2x_1 + x_3^2x_2 - x_1^2x_2 - x_2^2x_3 - x_3^2x_1\right]^2$$

$$= \left[x_1^2x_3 + x_2^2x_1 + x_3^2x_2 + x_1^2x_2 + x_2^2x_3 + x_3^2x_1\right]^2$$

$$-4\left(x_1^4x_2x_3 + x_2^4x_3x_1 + x_3^4x_1x_2 + x_2^3x_3^3 + x_3^3x_1^3 + x_1^3x_2^3 + 3x_1^2x_2^2x_3^2\right)$$

The sequence of identities

$$(x_2x_3 + x_3x_1 + x_1x_2)^3 = x_2^3x_3^3 + x_3^3x_1^3 + x_1^3x_2^3 + 6x_1^2x_2^2x_3^2$$

$$+3\left[x_2^2x_3^2(x_2 + x_3)x_1 + x_3^2x_1^2(x_3 + x_1)x_2 + x_1^2x_2^2(x_1 + x_2)x_3\right]$$

which leads to

$$x_2^3x_3^3 + x_3^3x_1^3 + x_1^3x_2^3 = b_2^3 - 6b_3^2 - 3b_3\left[b_1b_2 - 3b_3\right]$$

$$x_1^3 + x_2^3 + x_3^3 = b_1(x_1^2 + x_2^2 + x_3^2) - b_2(x_1 + x_2 + x_3) + 3b_3 = b_1^3 - 3b_1b_2 + 3b_3$$

leads to

$$D_3 = \left[b_1b_2 - 3b_3\right]^2 - 4\left(b_3\left[b_1^3 - 3b_1b_2 + 3b_3\right] + b_2^3 - 3b_1b_2b_3 + 9b_3^2 - 6b_3^2 + 3b_3^2\right)$$

$$= b_1^2b_2^2 + 18b_1b_2b_3 - 27b_3^2 - 4b_1^3b_3 - 4b_2^3$$

the same as the derivative method.

We can now employ this quantity to determine when two of the 4-cycles become complex

$$b_1 = 0 \qquad b_2 = \frac{3}{r^2} + \frac{2}{r} - 1 \qquad b_3 = \frac{4}{r^3}$$

and so

$$D_3 = -4\left[\left(\frac{3}{r^2} + \frac{2}{r} - 1\right)^3 + \frac{108}{r^6}\right]$$

which vanishes when

$$r^2 - 2r - 3 = 3 \times 2^{\frac{2}{3}} \qquad \Rightarrow \qquad (r-1)^2 = 4\left[1 + \frac{3}{2^{\frac{4}{3}}}\right]$$

and so

$$r_4 = 1 + 2\left[1 + \frac{3}{2^{\frac{4}{3}}}\right]^{\frac{1}{2}}$$

marks the appearence of a pair of 4-cycles which only exist when $r \geq r_4$ and when $r \leq 2 - r_4$.

Once we have solved the cubic, and found values for b_1, then we can solve the quadratic

$$(y - y_1)(y - y_2) = y^2 - (b_1 - 2)y - \frac{1}{r^2} = 0$$

to find y_1 and y_2. We can then construct

$$(x - x_1)(x - x_3) = x^2 - (y_1 + 1)x + \frac{y_2 + 1}{2r} + \frac{y_1(y_1 + 1)}{2}$$

$$(x - x_2)(x - x_4) = x^2 - (y_2 + 1)x + \frac{y_1 + 1}{2r} + \frac{y_2(y_2 + 1)}{2}$$

and solve these quadratic equations to find the points which make up each 4-cycle. We may also assess when the 4-cycle merges to form a 2-cycle using the discriminant of these quadratics

$$D_2 = (y_1 + 1)^2 - 2y_1(y_1 + 1) - \frac{2}{r}(y_2 + 1) = 1 - y_1^2 - \frac{2}{r}(y_2 + 1)$$

$$D_2 = (y_2 + 1)^2 - 2y_2(y_2 + 1) - \frac{2}{r}(y_1 + 1) = 1 - y_2^2 - \frac{2}{r}(y_1 + 1)$$

Since both vanish at the same time we may use the sum and difference to provide

$$2 - y_1^2 - y_2^2 - \frac{2}{r}(y_1 + y_2 + 2) = 2 - \frac{4}{r} - (y_1 + y_2)^2 + 2y_1 y_2 - \frac{2}{r}(y_1 + y_2) = 0$$

$$(y_2 - y_1)\left[y_1 + y_2 - \frac{2}{r}\right] = 0$$

and since $y_1 y_2 = -\frac{1}{r^2}$ we find that $y_1 + y_2 = \frac{2}{r}$ and

$$2 - \frac{4}{r} - \frac{4}{r^2} - \frac{2}{r^2} - \frac{4}{r^2} = 2\left[1 - \frac{2}{r} - \frac{5}{r^2}\right] = \frac{2}{r^2}\left[(r-1)^2 - 6\right] = 0$$

so the 4-cycle merges into a 2-cycle at $r_c = 1 + \sqrt{6}$ and this 4-cycle only exists when $r > r_c$ and $r < 2 - r_c$.

We close this appendix with an analysis of the stability of the 3-cycles and 4-cycles that we have just found. For the logistic map, stability of the 3-cycle is controlled by

$$P_3 \equiv r^3(1 - 2x_1)(1 - 2x_2)(1 - 2x_3) = r^3\left[1 - 2b_2 + 4b_2 - 8b_3\right]$$

and in this appendix

$$1 - b_1 + b_2 - b_3 = \frac{1}{r^3} \qquad b_2 = \frac{1}{2r}b_1 - \frac{1}{2}\left[b_1 - b_1^2\right]$$

so

$$P_3 = r^3\left[\frac{8}{r^3} - 7 + 6b_1 - 4b_2\right] = 8 - 7r^3 - 2r^2 b_1 + 8r^3 b_1 - 2r^3 b_1^2$$

together with the quadratic equation which controls b_1

$$b_1^2 - \left(3 + \frac{1}{r}\right) b_1 + 2 + \frac{2}{r} + \frac{2}{r^2} = 0 \qquad \Rightarrow \qquad 2rb_1 = 3r + 1 + \left[(r-1)^2 - 8\right]^{\frac{1}{2}} \equiv 3r + 1 + \Delta$$

provides the stability for the 3-cycle. When $P_3 = 1$ the 3-cycle first appears and when P_3=-1 the 3-cycle becomes unstable to a 6-cycle. The algebra is elementary but irritating

$$P_3 = 7 - 7r^3 - 2r^2 b_1 - 6r^3 b_1 - 2r^2 b_1 + 4r^3 + 4r^2 + 4r = 8 + 4r + 4r^2 - 3r^3 - 2(2r^2 - r^3)b_1$$

$$= 8 + 4r + 4r^2 - 3r^3 - r(2-r)(3r+1+\Delta) = 8 + 2r - r^2 - r(2-r)\Delta = 9 - (r-1)^2 - r(2-r)\Delta$$

$$= 1 - \Delta^2 + \left[\Delta^2 + 7\right]\Delta$$

and so the 3-cycle first appears when Δ=0 and then loses stability when

$$\Delta^3 - \Delta^2 + 7\Delta + 2 = \left(\Delta - \frac{1}{3}\right)^3 + \frac{20}{3}\left(\Delta - \frac{1}{3}\right) + \frac{115}{27} = 0$$

The stability of the 4-cycles is controlled by

$$P_4 = r^4(1-2x_1)(1-2x_2)(1-2x_3)(1-2x_4)$$

which takes the value 1 when they first appear and takes the value -1 when the 4-cycle becomes unstable to an 8-cycle. We can expand out using every second solution to obtain

$$P_4 = r^4\left[1 - 2(x_1+x_3) + 4x_1x_3\right]\left[1 - 2(x_2+x_4) + 4x_2x_4\right]$$

We may employ the previous analysis where

$$x_1x_3 = \frac{1}{2}\left[(x_1+x_3)^2 - x_1 - x_3 + \frac{1}{2r}(x_2+x_4)\right]$$

$$x_2x_4 = \frac{1}{2}\left[(x_2+x_4)^2 - x_2 - x_4 + \frac{1}{2r}(x_1+x_3)\right]$$

to give

$$P_4 = r^4\left[1 - 2(x_1+x_3) + 2(x_1+x_3)^2 - 2(x_1+x_3) + \frac{2}{r}(x_2+x_4)\right]$$

$$\times\left[1 - 2(x_2+x_4) + 2(x_2+x_4)^2 - 2(x_2+x_4) + \frac{2}{r}(x_1+x_3)\right]$$

and then using the natural quantities $y_1 = x_1 + x_3 - 1$ and $y_2 = x_2 + x_4 - 1$ we can deduce that

$$P_4 = r^4\left[2y_1^2 - 1 + \frac{2}{r}y_2 + \frac{2}{r}\right]\left[2y_2^2 - 1 + \frac{2}{r}y_1 + \frac{2}{r}\right] = r^4\left[4y_1^2y_2^2 - 2\left(1 - \frac{2}{r}\right)(y_1^2 + y_2^2)\right.$$

$$+\frac{4}{r}(y_1^3+y_2^3)-\left(1-\frac{2}{r}\right)\frac{2}{r}(y_1+y_2)+\frac{4}{r^2}y_1y_2+\left(1-\frac{2}{r}\right)^2\Bigg]$$

The product formula

$$y_1y_2=-\frac{1}{r^2}\qquad\Rightarrow\qquad (y_1+y_2)^3=y_1^3+y_2^3+3y_1y_2(y_1+y_2)=y_1^3+y_2^3-\frac{1}{r^2}(y_1+y_2)$$

allows us to rewrite in terms of y_1+y_2

$$P_4=r^4\Bigg[\frac{4}{r^4}+\frac{4}{r}\left((y_1+y_2)^3+\frac{3}{r^2}(y_1+y_2)\right)-2\left(1-\frac{2}{r}\right)\left((y_1+y_2)^2+\frac{2}{r^2}\right)$$

$$-\left(1-\frac{2}{r}\right)\frac{2}{r}(y_1+y_2)-\frac{4}{r^4}+\left(1-\frac{2}{r}\right)^2\Bigg]$$

and the governing cubic

$$(y_1+y_2)^3+\left(\frac{3}{r^2}+\frac{2}{r}-1\right)(y_1+y_2)-\frac{4}{r^3}=0$$

provides the mathematical problem to be tackled. We need to eliminate the variable y_1+y_2 from these two equations to generate the required relationship between P_4 and r.