University of Birmingham
School of Mathematics

1SAS | Sequences and Series | Semester 1 2022/23

Main Exam
Generic Feedback

General remarks

While most were able to recall the required definitions accurately, not all were. For example, in Part (a)(i) of Question 1 some wrote "There exists $\epsilon > 0$ such that..." or "$|a_n - \ell| < \epsilon$ for some $n > N$, where $\epsilon > 0$". This lost marks. **Please have an accurate form of this definition at your fingertips:** Given $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $|a_n - \ell| < \epsilon$ for all $n > N$. Similar comments apply to Part (b)(i) of Question 1 and Part (b)(i) of Question 2.

As usual a good performance also required students to **carefully apply theorems from the module**, such as the Algebra of Limits, the various Series Convergence Tests and the Monotone Convergence Theorem. As I describe below, marks were often lost here. There are plenty of practise questions available in the notes, problem sheets and past exam papers.

Question 1

Part (a). As mentioned above, not all were able to recall the definition of a convergent sequence accurately. Most were able to apply the definition to the given example, although sometimes marks were lost for accuracy in manipulating and bounding $|a_n - 1/2|$.

Part (b). Similar comments apply here.

Part (c). This was an exercise in selecting and applying Series Convergence Tests. A common issue came from students attempting to apply the converse of a test, which is typically false. For example, the Null Sequence Test cannot be used to show that a series converges, and the Absolute Convergence Test cannot be used to show that a series diverges. Please review the statements of the tests with this in mind. Some confused a series $\sum_{n=1}^{\infty} a_n$ with the sequence (a_n) of its terms, which sometimes led to confusion.

For Part (i),

$$\sum_{n=1}^{\infty} \frac{2^n}{3^n},$$

while most scored well, many failed to notice that this is a geometric series. The Ratio and Root Tests can also be used here, and many did. For Part (ii),

$$\sum_{n=1}^{\infty} \frac{(2n)!}{3^n (n!)^2},$$

most successfully applied the Ratio Test to prove divergence. However, some lost marks for making mistakes manipulating $\frac{a_{n+1}}{a_n}$. For Part (iii),

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}},$$

most successfully applied the Alternating Series Test. However, some failed to refer to the monotonicity condition in that test. Others tried to use the Absolute

Convergence Test, which doesn't work as this series is not absolutely convergent. For Part (iv),

$$\sum_{n=1}^{\infty} \frac{2n + (-1)^n}{n^3 + 2},$$

most accurately applied the Comparison Test. Some made mistakes with their inequalities, either using inequalities that go the wrong way, or simply claiming false inequalities. Many confused sequences and series in their arguments here, stating "a_n converges" when they meant "$\sum a_n$ converges". Please be careful. For Part (v),

$$\sum_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)^n,$$

many correctly appealed to the Null Sequence Test to deduce that there is divergence. Some confused the convergence of the sequence $(1 + \frac{1}{n})^n$ (Euler's sequence) with the convergence of the series.

Question 2

Part (a). The identity in Part (i) was established successfully by most. Most correctly identified that this identity can be used to show that the sequence is decreasing. It was pertinent to refer to the fact that $(2 + a_n)(3 + a_{n+1})$ is positive in completing the inductive step, and not all did this. Part (iii) was an application of three results from the module: the Monotone Convergence Theorem, the Algebra of Limits, and the Theorem on Limits and Order. To apply the Monotone Convergence Theorem it was enough to simply observe that (a_n) is bounded below by zero – something given in the question. Some argued too informally, writing, for example, "if (a_n) converges then we can write $a_{n+1} = a_n$ in the recursive formula" (the Algebra of Limits is needed to make sense of this, as can be seen in examples from the notes and problem sheets). Part (iv) was true rather vacuously as many noticed (the left hand side is nonpositive by Part (ii)). This was not intended, so the grading scheme was adapted to make sure that no student was disadvantaged.

Part (b). This was done well by most, although some were unable to recall the definition accurately. Most observed that the example is a telescoping series, and made appropriate reference to the definition.

Part (c). This was an exercise in applying the Absolute Convergence Test and the Root Test. Some tried to draw incorrect conclusions, such as "since $|a_n|^{1/n} \to 1$ it follows that $a_n \to 1$". The pivotal observation to make is that $|a_n x^n|^{1/n} = |a_n|^{1/n}|x|$, and those that did this generally scored well.