Introduction to Probability

Lecture 6

Today

Ordered Events

Expectation Values

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Ordered Events and the PMF

Ordering

So far we have just had **events**

We will now look at when those events are **ordered**.

Examples:

The number of heads when tossing a coin

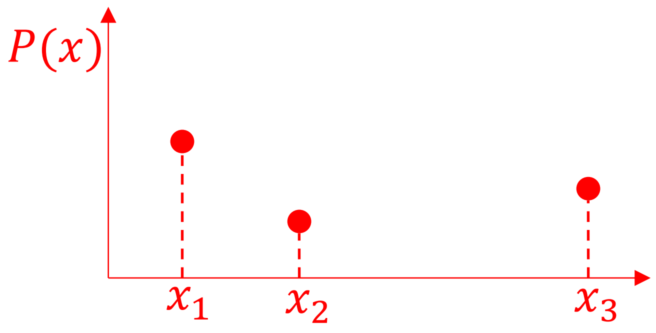
The outcome of a die

Probability Mass Functions (PMF)

If we have a variable x then the probability of observing x is written

$$P(x)$$

When the outcomes are numerical, we can place them on an axis and sketch out $P(x)$



Normalisation

We can normalise $P(x)$ through summation

$$\sum_x P(x) = 1$$

If $P(x) = \alpha f(x)$

Then

$$\sum_x P(x) = 1 = \alpha \sum_x f(x)$$

$$\alpha = \frac{1}{\sum_x f(x)}$$

In statistical physics α is called the partition function

Example

Normalise the following PMF

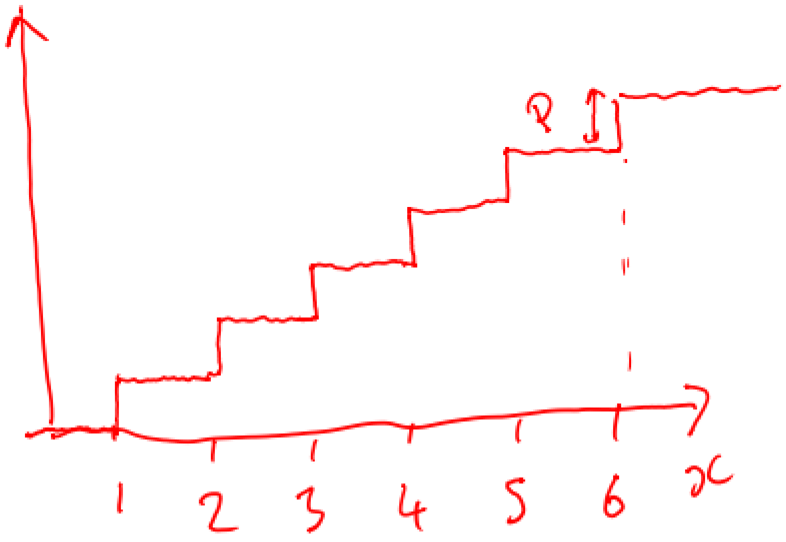
$$P(x) \propto \begin{cases} x \ \text{ if } x = \{1,2,3\} \\ 7 - x \ \text{ if } x = \{4,5,6\} \\ 0 \ \ \text{otherwise} \end{cases}$$

x	1	2	3	4	5	6	Total
F(x)	1	2	3	3	2	1	12
P(x)	1/12	1/6	1/4	1/4	1/6	1/12	1

Cumulative Distribution

Cumulative Distribution

Probability that a variable is **less than or equal to** a certain value.



$$C(x) \equiv \text{Prob}(X \leq x) = \sum_{X \leq x} P(x)$$

Die: $P(x) = \frac{1}{6}$ for $x = 1,2 \ldots 6$

x	1	2	3	4	5	6
P(x)	1/6	1/6	1/6	1/6	1/6	1/6
C(x)	1/6	2/6	3/6	4/6	5/6	1

Expectation Values

Expectation Values

The **expectation value** is defined by

$$\langle x \rangle \equiv \sum_x x\, P(x)$$

It is a weighted sum of all the outcomes and is a **measure of location**.

Warning

$$\langle x \rangle \equiv \sum_x x\, P(x)$$

Is confusing because x appears on both sides. You might prefer to write it has

$$\langle x \rangle \equiv \sum_y y\, P(y)$$

But somehow this is just as confusing.

Example

Calculate $\langle x \rangle$ for the following PMF

x	1	2	3	4	5
$P(x)$	0.1	0.2	0.4	0.2	0.1

$$\langle x \rangle \equiv \sum_x x\, P(x)$$

$$= 1 \times 0.1 + 2 \times 0.2 + 3 \times 0.4 + 4 \times 0.2 + 5 \times 0.1$$

$$= 3$$

Example

You play a game where you receive £x's for rolling an x on a die.

How much do you expect to win?

$$\langle x \rangle \equiv \sum_x x \, P(x)$$

$$= \frac{1}{6}(1 + 2 + 3 + 4 + 5 + 6)$$

$$= \frac{35}{6}$$

$$\rightarrow \text{£}3.50$$

Expectation value of function

In general:

$$\langle f \rangle \equiv \sum_x f(x) P(x)$$

Again: a weighted sum over the values of x.

$$\langle c \rangle = \sum_x c\, P(x) = c \sum_x P(x) = c$$

Note

$$\langle \langle x \rangle \rangle = \langle x \rangle$$

Linear Combinations

$$\langle ax + b \rangle = \sum_x (ax + b) P(x)$$

$$= \sum_x (ax) P(x) + \sum_x bP(x)$$

$$= a \sum_x x\, P(x) + b \sum_x P(x)$$

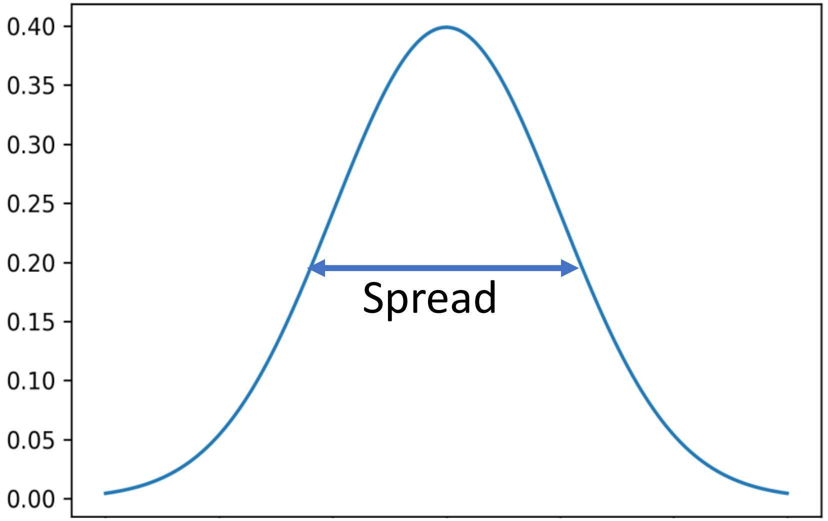
$$= a\langle x \rangle + b$$

Or we could have used

$$\langle ax + b \rangle = \langle ax \rangle + \langle b \rangle = a\langle x \rangle + b$$

Measure of Dispersion

How often is x far away from $\langle x \rangle$?



What about expected deviation from $\langle x \rangle$?

$$\langle x - \langle x \rangle \rangle = \langle x \rangle - \langle \langle x \rangle \rangle = 0$$

No good!

$$\langle | x - \langle x \rangle | \rangle = \mathrm{MAD}(x)$$

Is uncommon

$$\langle (x - \langle x \rangle)^2 \rangle = \mathrm{var}(x)$$

$$= \sum_x (x - \langle x \rangle)^2 \, P(x)$$

Variance and Standard Deviation

Show

$$\text{var}(x) = \langle x^2 \rangle - \langle x \rangle^2$$

Is a distance squared a good measure of spread?

$$\langle (x - \langle x \rangle)^2 \rangle = \langle (x^2 - 2\langle x \rangle x + \langle x \rangle^2) \rangle$$

$$= \langle x^2 \rangle - \langle 2\langle x \rangle x \rangle + \langle \langle x \rangle^2 \rangle$$

$$= \langle x^2 \rangle - 2\langle x \rangle \langle x \rangle + \langle x \rangle^2$$

$$\langle x^2 \rangle - 2\langle x \rangle^2 + \langle x \rangle^2 = \langle x^2 \rangle - \langle x \rangle^2$$

We define

$$\text{std}(x) = \sqrt{\text{var}(x)}$$

Example

A distribution is given by

$$P(x) = \frac{1}{2} \qquad x = 0,1$$

Calculate $\langle x \rangle$, $\mathrm{var}(x)$ and $\mathrm{std}(x)$.

$$\langle x \rangle \equiv \sum_x x\, P(x) = 0 \times \frac{1}{2} + 1 \times \frac{1}{2} = \frac{1}{2}$$

$$\mathrm{var}(x) = \langle x^2 \rangle - \langle x \rangle^2$$

$$\langle x^2 \rangle \equiv \sum_x x^2\, P(x) = 0^2 \times \frac{1}{2} + 1^2 \times \frac{1}{2} = \frac{1}{2}$$

$$\rightarrow \mathrm{var}(x) = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$$

$$\mathrm{std}(x) = \sqrt{\mathrm{var}(x)} = \frac{1}{2}$$

Class Example

A 3 sided fair dice has

$$P(x) = \frac{1}{3} \qquad x = 1,2,3$$

Calculate $\mathrm{var}(x)$.

$$\langle x \rangle = 2$$

$$\mathrm{var}(x) = \langle x^2 \rangle - \langle x \rangle^2$$

$$\langle x \rangle = \sum_x x\, P(x) = \frac{1}{3} \times (1 + 2 + 3) = 2$$

$$\langle x^2 \rangle = \sum_x x^2 P(x)$$

$$\langle x \rangle = 2$$

$$= \frac{1}{3}(1 + 4 + 9) = \frac{14}{3}$$

$$\langle x \rangle = 2$$

$$\langle x \rangle^2 = 2^2 = 4$$

$$\mathrm{var}(x) = \langle x^2 \rangle - \langle x \rangle^2 = \frac{14}{3} - 4 = \frac{2}{3}$$

Rules

Expectation

$$\langle ax + b \rangle = a\langle x \rangle + b$$

Variance

$$\mathrm{var}(ax + b) = a^2\,\mathrm{var}(x)$$

$$\mathrm{var}(ax) = \langle (ax)^2 \rangle - \langle ax \rangle^2 = a^2\langle x^2 \rangle - a^2\langle x \rangle^2$$

$$= a^2(\langle x^2 \rangle - \langle x \rangle^2)$$

Probability and Statistics

Probability

$$\langle x \rangle = \sum_x x\, P(x)$$

$$\mathrm{var}(x) = \sum_x (x - \langle x \rangle)^2\, P(x)$$

We have a distribution in probability

Statistics

$$\bar{x} = \frac{1}{N} \sum_n x_n$$

$$\sigma^2(x) = \frac{1}{N - 1} \sum_n (x_n - \bar{x})^2$$

We have a sample in statistics

Summary

Expectation

$$\langle x \rangle \equiv \sum_x x\, P(x) \qquad\qquad \langle f \rangle \equiv \sum_x f(x) P(x)$$

$$\langle ax + b \rangle = a\langle x \rangle + b$$

Variance

$$\mathrm{var}(x) \equiv \sum_x (x - \langle x \rangle)^2\, P(x) = \langle x^2 \rangle - \langle x \rangle^2$$

$$\mathrm{var}(ax + b) = a^2\, \mathrm{var}(x)$$

Examples

Example

You keep playing a game until you win. The probability you win at each stage is $1 - p$.

How many games do you expect to play until you have won?

Note:

$$\sum_{n=0}^{\infty} p^n = \frac{1}{1-p} \qquad \sum_{n=0}^{\infty} n\, p^n = \frac{p}{(1-p)^2}$$

We need to work out $P(x)$

$$P(0) = 1 - p$$
$$P(1) = p(1-p)$$
$$P(2) = p^2(1-p)$$
$$\rightarrow P(n) = p^n(1-p)$$

Check for normalisation

$$(1-p)\sum_n p^n = \frac{(1-p)}{(1-p)} = 1$$

Then

$$\langle n \rangle = (1-p)\sum_n np^n = \frac{(1-p)p}{(1-p)^2} = \frac{p}{1-p}$$