University of Birmingham
School of Mathematics

Real Analysis – Integration – Spring 2025

Problem Sheet 5
Issued Spring Week 1

Instructions: You are strongly encouraged to attempt all of the questions (Q) below, and as many of the extra questions (EQ) as you can, to help prepare for the final exam. Model solutions will only be released for questions Q1-Q4.

Q1. (a) Suppose that $X \subseteq \mathbb{R}$ is not empty and that $N \in \mathbb{R}$ is an upper bound for X. Prove that if for each $\epsilon > 0$, there exists x in X such that $N - \epsilon < x$, then $N = \sup X$.

(b) Find, with proof, the infimum of the set

$$A := \{x : 3 < |x - 2| < 5\}$$

(c) Suppose that $P = \{0, 4, 12, 17, 20\}$ and $g : [0, 20] \to \mathbb{R}$ is given by

$$g(x) := 3x + 7.$$

(i) Calculate the Riemann–Darboux sums $L(g, P)$ and $U(g, P)$.

(ii) For each $n \in \mathbb{N}$, let P_n denote the partition of $[0, 20]$ into n subintervals of equal width. Find a natural number N such that $U(g, P_N) - L(g, P_N) < \frac{1}{1{,}000}$.

The Riemann–Darboux sums calculator (Spring Week 1 Materials on Canvas) may be helpful in part (c).

Q2. (a) Calculate the Riemann–Darboux sums $L(f, P)$ and $U(f, P)$ for each of the following functions f and partitions P:

(i) $f : [0, 10] \to \mathbb{R},\ f(x) = x^2,\ P = \{0, 2, 7, 10\}$

(ii) $f : [0, 10] \to \mathbb{R},\ f(x) = e^{-x},\ P = \{0, 1, 5, 8, 9, 10\}$

(iii) $f : [0, \pi] \to \mathbb{R},\ f(x) = \sin(x),\ P = \{0, \frac{\pi}{2}, \frac{3\pi}{4}, \pi\}$

(b) Use the Riemann–Darboux sums calculator (Spring Week 1 Materials on Canvas) to visualize and check your answers above (note that $\frac{3\pi}{4}$ is entered as 3*Pi/4).

(c) For each of the functions f above, use the Riemann–Darboux sums calculator to calculate $L(f, P_n)$ and $U(f, P_n)$ when $n = 5$, $n = 10$ and $n = 100$, where P_n is the partition of the domain of f into n subintervals of equal width.

(d) For each $\delta \in (0, 1/10)$, find an expression for the sums $L(f, P_\delta)$ and $U(f, P_\delta)$ when $f : [-2, 2] \to \mathbb{R}$, $f(x) = \begin{cases} 5, & x \in \mathbb{N} \\ 3, & x \notin \mathbb{N} \end{cases}$ and $P_\delta = \{-2, 1 - \delta, 1 + \delta, 2 - \delta, 2\}$.

Q3. It is proved in Lecture 2.1 of the Integration Lecture Notes that if $f : [a, b] \to [0, \infty)$ is a bounded function, where $-\infty < a < b < \infty$, and P, Q are partitions of $[a, b]$ such that $P \subseteq Q$, then $L(f, P) \leq L(f, Q)$. List the changes that are needed to prove that $U(f, P) \geq U(f, Q)$, and in particular, explain how the inequalities $m_1 \leq m_1'$ and $m_1 \leq m_1''$ need to be modified.

Q4. Let $f : [0, b] \to [0, \infty)$ be defined by $f(x) = x^2$, where $b \in (0, \infty)$:

(a) For each $n \in \mathbb{N}$, let $P_n = \{x_i : i = 0, 1, \ldots, n\}$ denote the partition of $[0, b]$ into n subintervals of equal width. Express x_i in terms of i and b.

(b) Use the formula $\sum_{j=1}^{k} j^2 = \frac{1}{6}k(k+1)(2k+1)$ to prove that

$$L(f, P_n) = \frac{b^3}{6n^3}(n-1)n(2n-1) \quad \text{and} \quad U(f, P_n) = \frac{b^3}{6n^3}n(n+1)(2n+1).$$

(c) Find, with proof, the values $\sup\{L(f, P_n) : n \in \mathbb{N}\}$ and $\inf\{U(f, P_n) : n \in \mathbb{N}\}$.

(d) Find, with proof, the lower integral $\underline{\int_0^b} f$ and the upper integral $\overline{\int_0^b} f$.

(e) Prove that f is bounded and integrable, then find $\int_0^b f$ (without using calculus).

Extra Questions

EQ1. (a) Find, with proof, the infimum and supremum of each of the following sets:

(i) $A = (0, 1] \cup (2, 3]$

(ii) $B = \{x^2 - 4x + 5 : x \in (1, 3]\}$

(iii) $C = \{(2n + 3)/n : n \in \mathbb{N}\}$

(iv) $D = \{n^2 - 6n + 10 : n \in \mathbb{N}\}$

(b) Let X denote a nonempty bounded subset of $\mathbb{R}$. Prove that for each $\epsilon > 0$, there exists x in X such that $\inf X \le x < \inf X + \epsilon$.

EQ2. For each $n \in \mathbb{N}$, let P_n denote the partition of $[0, 1]$ into n subintervals of equal width. Find an expression, possibly involving summation notation, for the sums $L(f, P_n)$ and $U(f, P_n)$ for $f : [0, 1] \to \mathbb{R}$ in each of the following cases:

(a) $f(x) = x^2 + x$

(b) $f(x) = \cos(x)$

(c) $f(x) = \begin{cases} 5, & x \in \mathbb{Q} \\ 3, & x \notin \mathbb{Q} \end{cases}$

(d) $f(x) = \begin{cases} 1, & x \in \{0, 1\}, \\ 0, & x \notin \{0, 1\}. \end{cases}$

EQ3. Let P_1 denote a partition of $[a, b]$ and $P_2 = P_1 \cup \{c\}$, where $-\infty < a < c < b < \infty$. Use results from lectures to prove that $U(f, P_2) - L(f, P_2) \le U(f, P_1) - L(f, P_1)$.

EQ4. Let $f : [a, b] \to \mathbb{R}$ be defined by $f(x) = x^3$, where $0 \le a < b < \infty$. Use the procedure outlined in **Q4** and the formula $\sum_{j=1}^{k} j^3 = \frac{1}{4}k^4 + \frac{1}{2}k^3 + \frac{1}{4}k^2$ to prove that f is integrable with $\int_a^b f = \frac{1}{4}b^4 - \frac{1}{4}a^4$.