UNIVERSITY OF BIRMINGHAM

School of Mathematics

Programmes in the School of Mathematics First Examination

Programmes involving Mathematics First Examination

1VGLA 06 25664 Level C

LC Vectors, Geometry and Linear Algebra

May/June Examinations 2023-24

Three Hours

Full marks will be obtained with complete answers to all FOUR questions. Each question carries equal weight. You are advised to initially spend no more than 45 minutes on each question and then to return to any incomplete questions if you have time at the end. An indication of the number of marks allocated to parts of questions is shown in square brackets.

No calculator is permitted in this examination.

Section A

1. (a) Given vectors **a** $= (1,2,3)$, **b** $= (-3,0,1)$ and **c** $= (1,2,2)$, compute the following:

(i) $5\mathbf{a} - \mathbf{c}$;

(ii) $(\mathbf{a} \cdot \mathbf{c})\mathbf{a}$;

(iii) $(\mathbf{a} \times \mathbf{b}) + \mathbf{c}$;

(iv) the non-reflex angle between **a** and **b**.

[9]

(b) Suppose that **a** $= (1,2,3)$ is a vector. Determine the scalar equation of the plane Π perpendicular to **a** containing the point $P = (2,1,-2)$. [3]

(c) Which of the following statements are true and which are false? For any statements that are **false**, provide a brief explanation as to why this is the case.

(i) Given any complex number $z \in \mathbb{C}$, $\mathrm{Re}(z) = \mathrm{Re}(\bar{z})$.

(ii) Given any complex number $z \in \mathbb{C}$, $z\bar{z}$ is an integer.

(iii) Given any complex number $z \in \mathbb{C}$, the principal value $\mathrm{Arg}(z)$ of $\arg(z)$ satisfies $\mathrm{Arg}(z) \in [0, 2\pi]$.

[3]

(d) Give an example of a system of two real simultaneous linear equations in two variables that has no solutions. (You do not need to justify your answer.) [2]

(e) Using the augmented matrix and row operations, find the solution set of the following system of real simultaneous linear equations:

$$\begin{array}{rcrcrcr} x & + & 3y & + & 2z & = & 1, \\ & & 5y & + & 3z & = & -3, \\ 3x & + & 4y & + & 3z & = & 6. \end{array}$$

[8]

2. (a) **A**, **B** and **C** are matrices in $\mathcal{M}_{nn}(\mathbb{R})$ such that $\det(\mathbf{A}) = 2$, $\det(\mathbf{B}) = -\frac{1}{2}$ and $\det(\mathbf{C}) = \frac{1}{4}$. Determine the value of $\det(\mathbf{C}^3 \cdot \mathbf{B}^{-1} \cdot \mathbf{A}^2 \cdot (\mathbf{B}^T)^2 \cdot \mathbf{C}^T \cdot (\mathbf{A}^{-1})^3)$. [2]

(b) Use Cramer's rule to solve the system of simultaneous linear equations:

$$\begin{array}{rcrcl} 4x_1 & - & 6x_2 & = & 2, \\ -5x_1 & + & 9x_2 & = & 5. \end{array}$$

[4]

(c) Let $V = \mathcal{M}_{nn}(\mathbb{R})$ be the vector space of $n \times n$ matrices. Prove or disprove that the set $W = \{\mathbf{A} \in \mathcal{M}_{nn}(\mathbb{R}) : \mathbf{A} = \mathbf{A}^2\}$ is a subspace of V. [4]

(d) Let $V = \mathbb{R}^3$ and U be a subspace of V defined by

$$U = \{(x,y,z) \in \mathbb{R}^3 : 2x - 3y + z = 0\}.$$

Determine a basis for U. [4]

(e) Prove or disprove whether the following mappings are linear:

(i) $T : \mathbb{R}^3 \to \mathbb{R}^2$ defined by $T((x,y,z)) = (x, x+y)$,

(ii) $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T((x,y)) = (y^2, x^2)$. [6]

(f) Sketch the conic section defined by

$$y^2 + 8x - 6y + 8 = 0$$

and determine its type and eccentricity. [5]

Section B

3. (a) Suppose that $z = 1 - i$.

(i) Calculate z^6 in both modulus-argument form and exponential form, giving the principal value of the argument.

(ii) What is the smallest choice of $n \in \mathbb{N}$ so that $z^n \in \mathbb{N}$?

(iii) Find all the fourth roots of z. Present your answers in exponential form giving the principal value of the argument.

[9]

(b) Suppose that

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 5 \\ 2 & 2 & 3 \\ 0 & 0 & 1 \end{pmatrix}.$$

Use the Gaussian Elimination Algorithm to determine whether **A** is invertible or not. In particular, if **A** is invertible then determine $\mathbf{A}^{-1}$. [7]

(c) Suppose that **A**, **B** and **C** are matrices so that

(1) $\mathbf{A} \cdot \mathbf{B}$ is defined and is a 5×3 matrix;

(2) $\mathbf{C} \cdot \mathbf{A}$ is defined and is a 3×3 matrix.

Determine the dimensions of each of the matrices **A**, **B** and **C**. Justify your answer. [5]

(d) Suppose that $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a 2×2 matrix such that a, b, c, d are positive real numbers. Prove that $\mathbf{A} \neq \mathbf{A}^{-1}$. [4]

4. (a) Determine the eigenvalues of the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 6 & -6 \\ 1 & 2 & -1 \end{pmatrix}.$$ [4]

(b) Let U and W be subspaces of a vector space V.

(i) Prove that $U \cap W$ is a subspace of V.

(ii) Prove that $U + W$ is a subspace of V where $U + W = \{\mathbf{u} + \mathbf{w} : \mathbf{u} \in U \text{ and } \mathbf{w} \in W\}$. [4]

(c) Let P_2 be the vector space of real polynomials of order 2 defined by

$$P_2 = \{a_0 + a_1 x + a_2 x^2 : a_0, a_1, a_2 \in \mathbb{R}\}.$$

What are the coordinates of $\mathbf{v} \in P_2$ given by $\mathbf{v} = x^2 + 4x - 3$ with respect to the ordered basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ given by

$$\mathbf{e}_1 = x^2 - 2x + 5, \quad \mathbf{e}_2 = 2x^2 - 3x, \quad \mathbf{e}_3 = x + 3.$$ [4]

(d) Consider the vector space of 2×2 real matrices $V = \mathcal{M}_{22}(\mathbb{R})$.

(i) What is the dimension of V? Write down the standard basis for V.

(ii) Consider the linear transformation $T : V \to V$ defined by $T(\mathbf{A}) = \mathbf{M} \cdot \mathbf{A}$ where $\mathbf{M} = \begin{pmatrix} 1 & -1 \\ -2 & 2 \end{pmatrix}$. Find a basis and dimension of the kernel of T and a basis and dimension of the image of T.

[8]

(e) This question involves the rotation transform given by

$$x = \tilde{x}\cos(\alpha) - \tilde{y}\sin(\alpha)$$

and

$$y = \tilde{x}\sin(\alpha) + \tilde{y}\cos(\alpha)$$

where the value of the rotation angle α is chosen to eliminate the cross term xy. Determine an expression for

$$-2x^2 + 2\sqrt{3}xy - 4 = 0$$

in $(\tilde{x}, \tilde{y})$ coordinates and hence identify the type and eccentricity of this conic section. [5]

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Do not complete the attendance slip, fill in the front of the answer book or turn over the question paper until you are told to do so.

Important Reminders

- Coats and outer-wear should be placed in the designated area.
- Unauthorised materials (e.g. notes or Tippex) **MUST** be placed in the designated area.
- Check that you **DO NOT** have any unauthorised materials with you (e.g. in your pockets, pencil case).
- Mobile phones and smart watches **MUST** be switched off and placed in the designated area or under your desk. They must not be left on your person or in your pockets.
- You are **NOT** permitted to use a mobile phone as a clock. If you have difficulty in seeing a clock, please alert an Invigilator.
- You are **NOT** permitted to have writing on your hand, arm or other body part.
- Check that you do not have writing on your hand, arm or other body part – if you do, you must inform an Invigilator immediately.
- Alert an Invigilator immediately if you find any unauthorised item upon you during the examination.

Any students found with non-permitted items upon their person during the examination, or who fail to comply with Examination rules may be subject to the Student Conduct procedures.