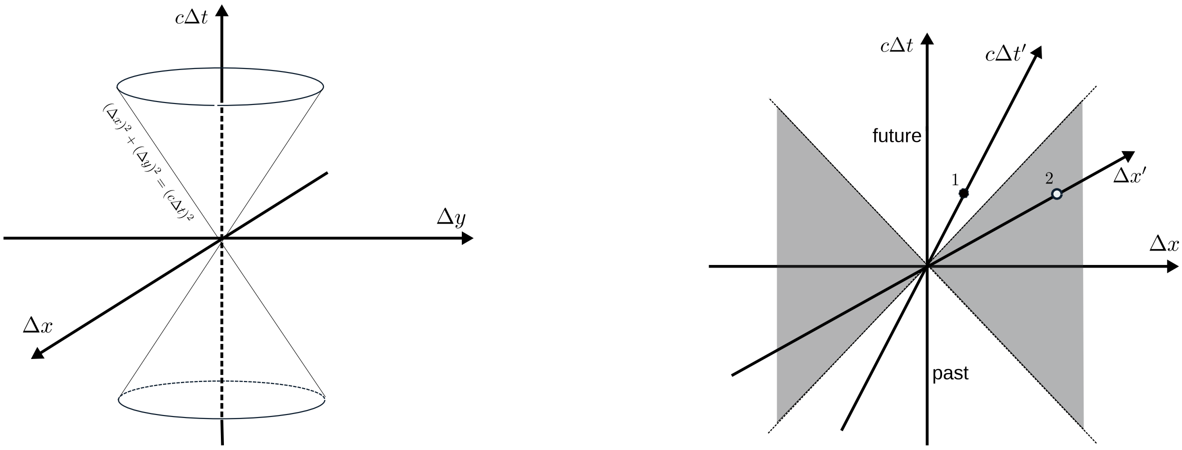

Figure 1: Left: light cone $(\Delta s)^2 = 0$ separating time-like $(\Delta s)^2 > 0$ and space-like $(\Delta s)^2 < 0$ intervals. Right: in one space dimension light cone divides the Δx - Δt plane into four regions. For time-like intervals events can be put into time order, so that future and past of an event can be defined: event 1 is in the future of the event $(0, 0)$. In the space-like region events cannot influence each other as they are too far apart so even light signal cannot link them, like the event $(0, 0)$ and event 2.

Space-time interval and proper time

We define now space-time interval. For two events happening in space positions $\boldsymbol{r}_1 = (x_1, y_1, z_1)$ and $\boldsymbol{r}_2 = (x_2, y_2, z_2)$ at time t_1 and t_2 respectively we have the invariant quantity,

$$(\Delta s)^2 = (c\Delta t)^2 - (|\Delta \boldsymbol{r}|)^2 = (c\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2 \,,$$

where $\Delta t = t_2 - t_1$, $\Delta \boldsymbol{r} = (\Delta x, \Delta y, \Delta z) = (x_2 - x_1, y_2 - y_1, z_2 - z_1)$. The quantity Δs with dimensionality of length, is called *space-time interval* between two events. It is also invariant under Lorentz transformation. In particular, its value is zero for events connected via a light signal. For two-dimensional space these events can be represented as a cone in the (three-dimensional) space-time is shown in the left panel of Fig. 1, hence the name "*light cone*" also in three dimensions. The light cone divides intervals between events into two categories:

- *time-like* for which $(\Delta s)^2 > 0$ or $|c\Delta t| > |\Delta \boldsymbol{r}|$
- *space-like* for which $(\Delta s)^2 < 0$ or $|c\Delta t| < |\Delta \boldsymbol{r}|$

Events connected by time-like intervals can influence each other, so an earlier event can be a cause for the later event. Similarly such events can be said to be in the *future* or in the *past* with respect to each other (see right panel in Fig. 1).

Events connected by space-like intervals are too far apart so even a light signal cannot link them, therefore they cannot influence each other. It is also meaningless to speak about which of these event happens first as they can be made to happen at the same time by appropriate choice of reference frame as the event 2 on the right panel of Fig. 1 .

Similarly events separated by a time-like interval are happening at the same point of space in a certain reference frame for which

$$(\Delta s)^2 = (c\Delta t)^2 - (\Delta x)^2 = (c^2 - v^2)(\Delta t)^2 = (c\Delta t')^2$$

thus the proper time interval $\Delta t' = \sqrt{1 - v^2/c^2}\Delta t$ as expected.

We can now reformulate the twin paradox by comparing the total space-time interval experienced by each of the twin. For the earth twin the space time interval is simply

$$\Delta s_{\mathrm{e}} = cT$$

as he does not move anywhere in the earth frame. For the travelling twin one must add the space-time intervals accumulated during each leg of his journey:

$$\Delta s_{\mathrm{t}} = \Delta s_{\mathrm{out}} + \Delta s_{\mathrm{back}} = \sqrt{(cT/2)^2 - (vT/2)^2} + \sqrt{(cT/2)^2 - (vT/2)^2} = \sqrt{c^2 - v^2}T\,,$$

which is clearly shorter than the space-time interval of the earth twin[1].

Relativistic velocity composition

Our knowledge of Lorentz transformations allows us to deduce the transformation law of velocities observed in different inertial frames. Imagine an object having velocity $\boldsymbol{u}$ in the frame K as shown in Fig. 2. What is the velocity $\boldsymbol{u}'$ observed in the inertial frame K' moving itself with velocity V_{rel} with respect to K. For simplicity we consider only two-dimensional space: x and x' axes are directed along the relative velocity between the frames, while y and y' axes are perpendicular to it. We can decompose the velocity $\boldsymbol{u} = u_x\hat{x} + u_y\hat{y}$ into the longitudinal, u_x and the transverse u_y components.

To find the corresponding components in the moving K' frame we write

$$u_x = \frac{\mathrm{d}x}{\mathrm{d}t}\,, \qquad u_y = \frac{\mathrm{d}y}{\mathrm{d}t}$$

and apply Lorentz transformation to the infinitesimal intervals

$$\begin{aligned}
\mathrm{d}t' &= \gamma(V_{\mathrm{rel}})\left(\mathrm{d}t - \left(V_{\mathrm{rel}}/c^2\right)\mathrm{d}x\right)\\
\mathrm{d}x' &= \gamma(V_{\mathrm{rel}})(\mathrm{d}x - V_{\mathrm{rel}}\mathrm{d}t)\\
\mathrm{d}y' &= \mathrm{d}y\,.
\end{aligned}$$

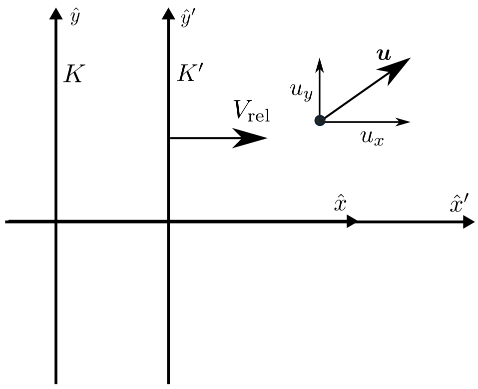

Figure 2: Velocity composition.

The transformed velocity component along the relative motion

$$u'_x = \frac{\mathrm{d}x'}{\mathrm{d}t'} = \frac{\mathrm{d}x - V_{\mathrm{rel}}\mathrm{d}t}{\mathrm{d}t - \frac{V_{\mathrm{rel}}}{c^2}\mathrm{d}x} = \frac{\frac{\mathrm{d}x}{\mathrm{d}t} - V_{\mathrm{rel}}}{1 - \frac{V_{\mathrm{rel}}}{c^2}\frac{\mathrm{d}x}{\mathrm{d}t}} = \frac{u_x - V_{\mathrm{rel}}}{1 - \frac{V_{\mathrm{rel}}u_x}{c^2}}\,.$$

Applying the same logic to the transverse component we get

$$u'_y = \frac{\mathrm{d}y'}{\mathrm{d}t'} = \frac{\mathrm{d}y}{\gamma(V_{\mathrm{rel}})\left(\mathrm{d}t - \frac{V_{\mathrm{rel}}}{c^2}\mathrm{d}x\right)} = \frac{1}{\gamma(V_{\mathrm{rel}})}\frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{1 - \frac{V_{\mathrm{rel}}}{c^2}\frac{\mathrm{d}x}{\mathrm{d}t}} = \frac{1}{\gamma(V_{\mathrm{rel}})}\frac{u_y}{1 - \frac{V_{\mathrm{rel}}u_x}{c^2}}$$

[1] One can even generalise this calculation to a case when the velocity of the travelling twin is not changing abruptly $v \to -v$ but depends smoothly on time. Let $v(t)$ be the instantaneous velocity of the travelling twin, so that the differential displacement dx and time interval dt in the earth frame are related by $dx = v(t)dt$. The total accumulated space-time interval is given by the integral along the world line of the travelling twin

$$\Delta s_{\mathrm{t}} = \int \sqrt{(cdt)^2 - (dx)^2} = \int_{-T/2}^{T/2} \sqrt{c^2 - v^2(t)}\,dt\,,$$

which is again smaller than $\Delta s_{\mathrm{e}} = cT$.

These formulae differ from the Galilean velocity transformation $u'_x = u_x - V_{\mathrm{rel}}$, $u'_y = u_y$ by the presence of a relativistic denominator, which becomes 1 in the non-relativistic limit $V_{\mathrm{rel}}/c \to 0$.

Using the inverse Lorentz transformation

$$\begin{aligned} \mathrm{d}t &= \gamma(V_{\mathrm{rel}}) \left(\mathrm{d}t' + \left(V_{\mathrm{rel}}/c^2\right) \mathrm{d}x'\right) \\ \mathrm{d}x &= \gamma(V_{\mathrm{rel}})(\mathrm{d}x' + V_{\mathrm{rel}}\mathrm{d}t') \\ \mathrm{d}y &= \mathrm{d}y' \,. \end{aligned}$$

we get the inverse velocity composition law

$$u_x = \frac{u'_x + V_{\mathrm{rel}}}{1 + \frac{V_{\mathrm{rel}} u'_x}{c^2}} \,, \qquad u_y = \frac{1}{\gamma(V_{\mathrm{rel}})} \frac{u'_y}{1 + \frac{V_{\mathrm{rel}} u'_x}{c^2}} \,.$$

and the rule "either prime or minus" works again!

Let us now demonstrate that the resulting velocity u_x cannot exceed the speed of light no matter how close the velocities u'_x and V_{rel} are to speed of light. Let us take

$$1 - \frac{u_x}{c} = 1 - \frac{\frac{u'_x}{c} + \frac{V}{c}}{1 + \frac{u'_x}{c}\frac{V}{c}} = \frac{\left(1 - \frac{u'_x}{c}\right)\left(1 - \frac{V}{c}\right)}{1 + \frac{u'_x}{c}\frac{V}{c}} \,.$$

The right side of this equation is positive for $0 < u'_x < c$, $0 < V < c$ which guarantees that $u_x/c < 1$. For the extreme case $u'_x = c$, $V = c$ the result of their composition is

$$u_x = \frac{c + c}{1 + \frac{c}{c^2} c} = \frac{2c}{2} = c \,.$$