Recap from last time

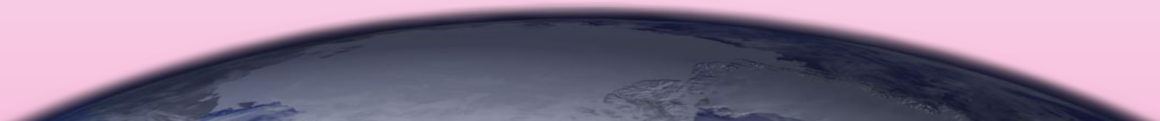
Assumptions:

1) Atmosphere is isothermal
(in reality, $dT/dh \sim 10\ K\ km^{-1}$) [5.8]

2) Treat the atmosphere as consisting of an ideal gas
(good assumption for majority of component gases at low pressure – not water vapour)

3) The Earth is flat
(~15 km << 6400 km)

4) The atmosphere is stationary and thus in mechanical equilibrium



Recap from last time

Probability exists on a number line between 0 and 1

e.g. throwing a 7 on a 6 sided die or throwing either a 1,2,3,4,5,6?

0
(will never happen)

1
(will happen)

$$P = \rho_N k_B T$$

Number density (N/V)

Forces: forces due to pressure and gravity (in equilibrium)

$$0 = P(h)A - P(h + \mathrm{d}h)A - m(\rho_N\,\mathrm{d}h)Ag$$

$$0 = dPA - m(\rho_N\,\mathrm{d}h)Ag$$

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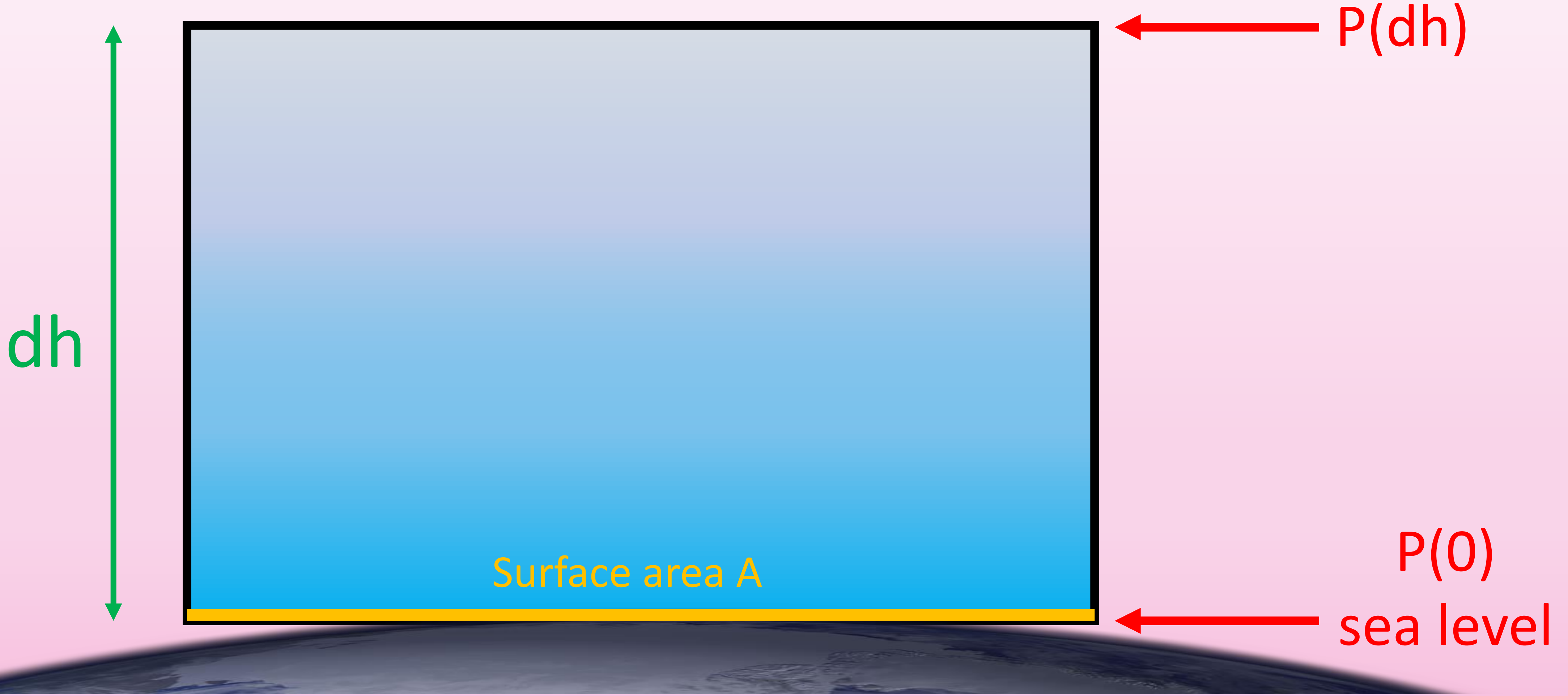


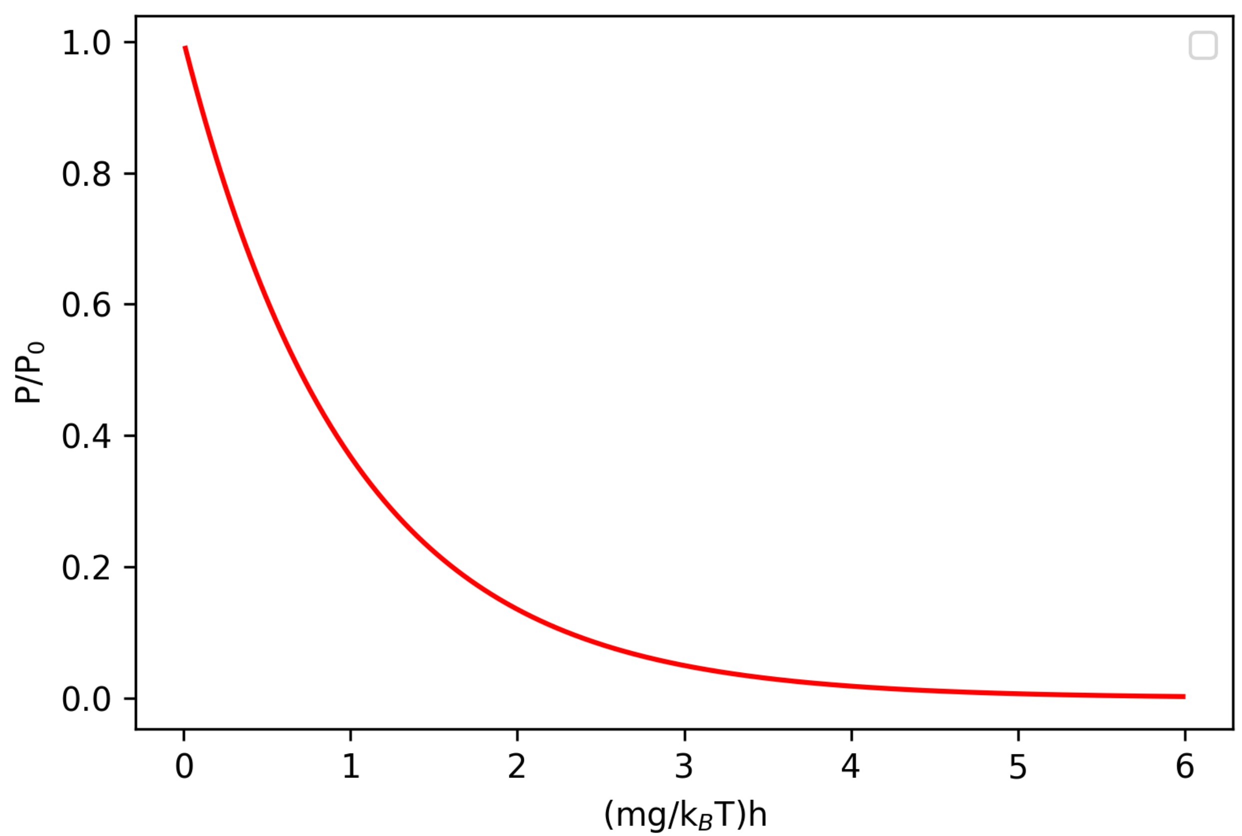
Isothermal model of the atmosphere

$$\rho_N(h) = \rho_N(0)\mathrm{e}^{-\left(\frac{mgh}{k_B T}\right)}$$

$$P(h) = P(0)\mathrm{e}^{-\left(\frac{mgh}{k_B T}\right)}$$

$\frac{mgh}{k_B T}$ must be dimensionless, and so $\frac{k_B T}{mg}$ must have dimensions of h -> scale height, h_0

(at which $P(h_0) = P(0)e^{-1}$)



Scale heights

Particle	Mass	Scale height
O_2 gas	32 amu = 5.3137e-26 kg	~10 km
N_2 gas	28 amu = 4.6495e-26 kg	~10 km
Covid virus	~10^{-18} kg	~0.5 mm
Smallest known virus	~10^{-21} kg	~50 cm
Bacteria	~10^{-15} kg	~5 μm
Dr Stuart Pirrie	~10^{2} kg	~50 ym (10^{-24} m is 1 ym)

Isothermal model meets probability

We have already established that the number density (number of fluid molecules per unit volume), $\rho_N(0)$, can be related to atmospheric pressure, $P_{at}(= P(0))$, by

$$\rho_N(0) = \frac{P_{at}}{k_B T}$$

If we wanted to determine the total number of gas molecules, N, in our slab of atmosphere (with surface area A), we can integrate across all possible heights, $h = 0 \rightarrow h = \infty$,

$$A \int_0^\infty \rho_N(h) \, \mathrm{d}h = N$$

Isothermal model meets probability

We can then, through solving the integral, relate the total number, N, to the number density $\rho_N(0)$:

$$\rho_N(0) = \frac{N}{Ah_0} \qquad \text{With } h_0 = \frac{k_B T}{mg}$$

Here, we can see that $\rho_N(0)$ is in fact the average number density of the slab of atmosphere between $h = 0$ and $h = h_0$

We can then define a new quantity, $Pr(h) = \frac{A\rho_N(h)}{N}$, which is therefore the contribution from one molecule

Isothermal model meets probability

As $A \int_0^\infty \rho_N(h)\,\mathrm{d}h = N$, we can show that

$$\int_0^\infty Pr(h)\,\mathrm{d}h = \frac{A}{N}\int_0^\infty \rho_N(h)\,\mathrm{d}h = \frac{N}{N} = 1$$

Thus, it is clear that the quantity $Pr(h)$ is in some way a probability (normalised to be equal to 1 between 0 and infinity) – and the quantity $Pr(h)\,\mathrm{d}h$ gives the probability of finding a given molecule between h and $h + \mathrm{d}h$

Probability density functions

We can also work out the function form of the probability by just looking at the expression

$$Pr(h) = C\rho_N(0)\mathrm{e}^{-\left(\frac{mgh}{k_B T}\right)}$$

If we include some constant (C), we can make sure the integral of the function is equal to 1 and hence represents a probability to make a probability density function – in our case, this takes the form

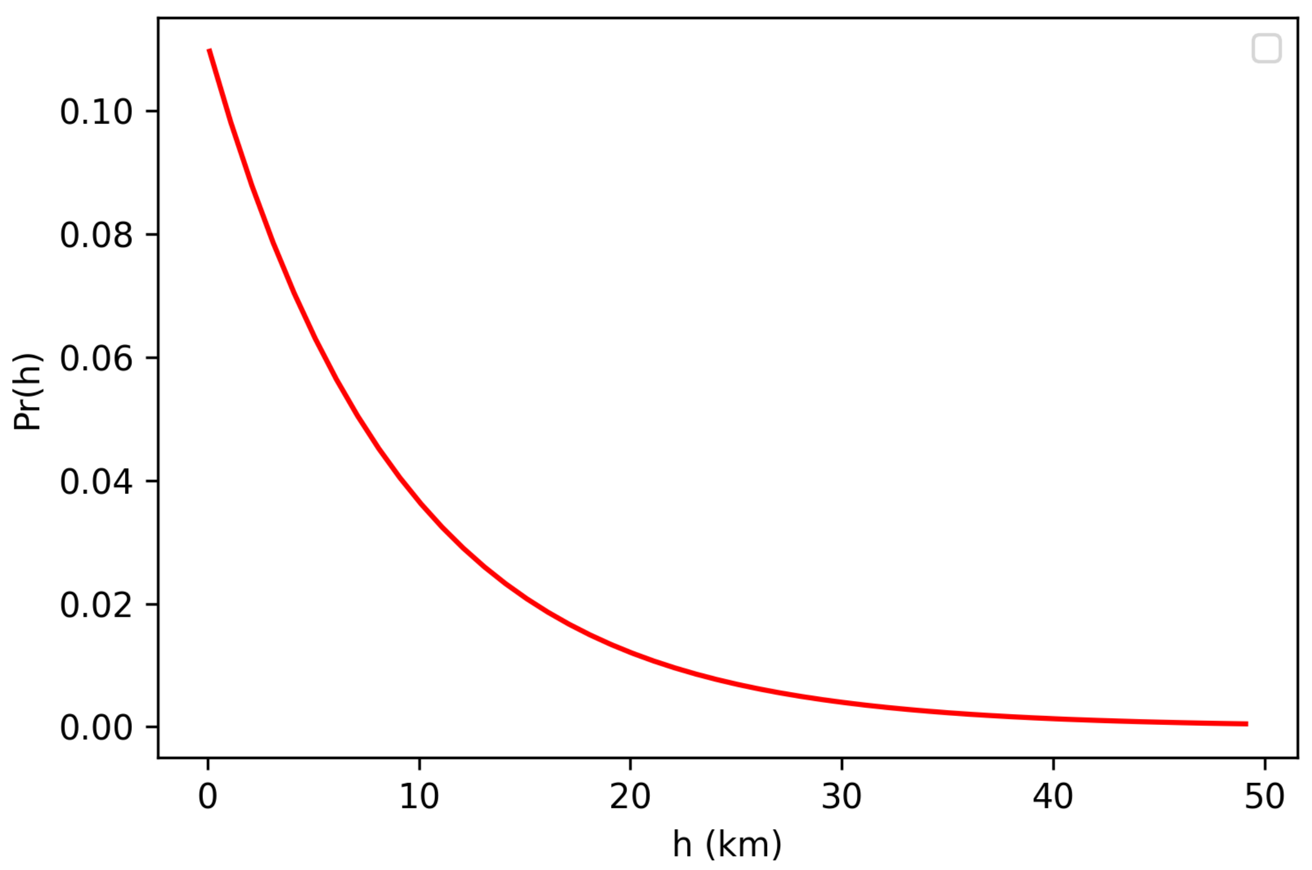
$$Pr(h) = \frac{mg}{k_B T}\mathrm{e}^{-\left(\frac{mgh}{k_B T}\right)} = \frac{1}{9020}\mathrm{e}^{-\left(\frac{h}{9020}\right)}$$

m = 28 amu
g = 9.81 ms^{-1}
T = 298 K
k_B = 1.38 x 10^{-23} $\mathrm{m}^2\ \mathrm{kg}\ \mathrm{s}^{-2}\ \mathrm{K}^{-1}$

Probability density functions

$$Pr(h) = \frac{1}{9020} e^{-\left(\frac{h}{9020}\right)}$$

Without any microscopic information regarding the motion of these molecules, we can gather the information we're interested in just from this distribution



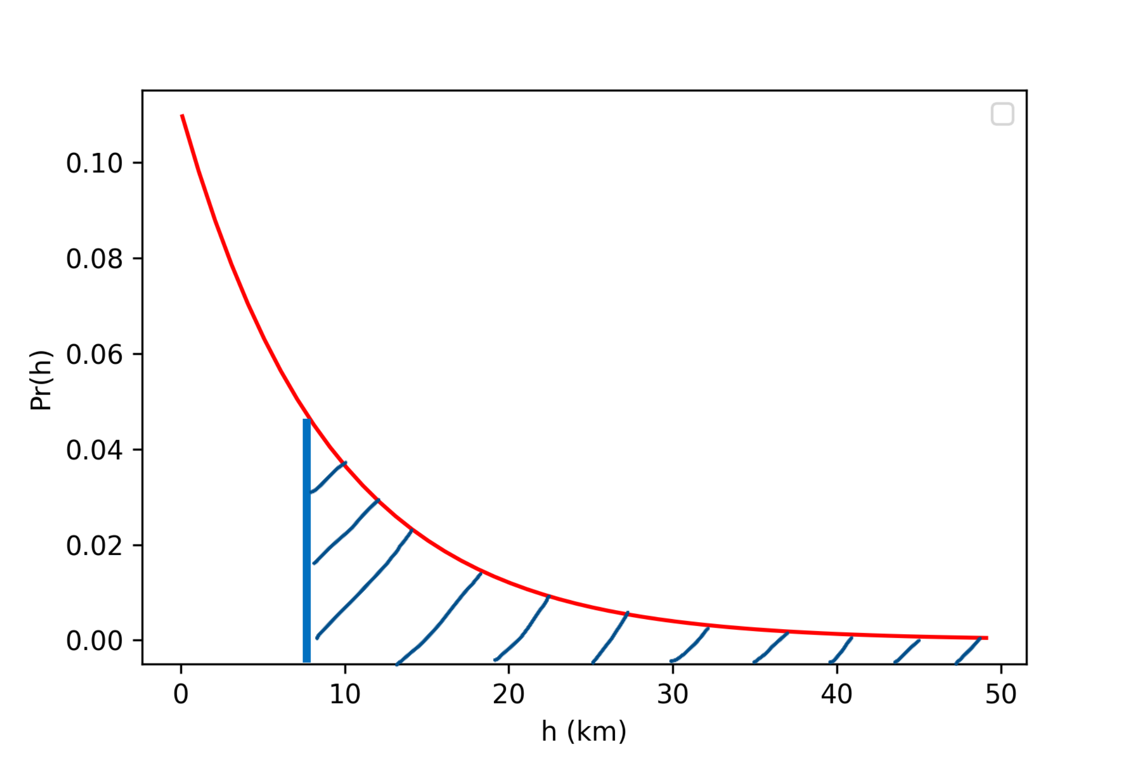
Probability density functions

$$Pr(h) = \frac{1}{9020} e^{-\left(\frac{h}{9020}\right)}$$

Q1: For a random molecule in the atmosphere, what is the probability that it can be found above 8 km?

Q2: At any given time, what proportion of molecules in the atmosphere have a height greater than 8 km?

Q3: Averaged over a long timescale, what fraction of the time does a particular molecule spend at an altitude > 8 km?



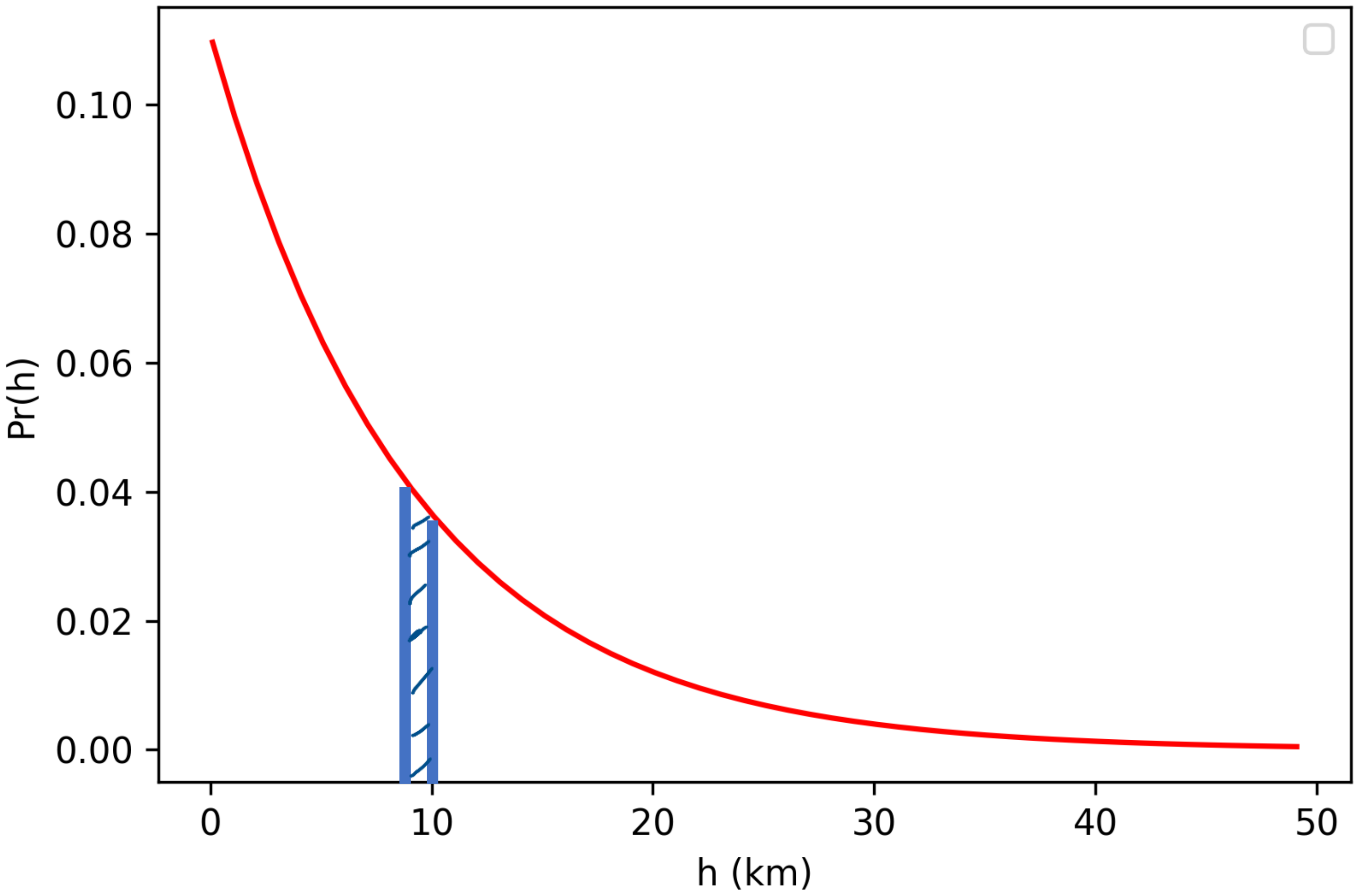
Identical questions!

A: $Pr(h > 8\ km) = \int_{8\ \mathrm{km}}^{\infty} Pr(h)\, \mathrm{d}h = 0.41$

More PDF examples

Roughly, what is the probability of finding a particle between 9 km and 10 km?

A: ~ 0.04



Boltzmann factors

Remember that the quantity $\frac{mgh}{k_B T}$ must be dimensionless… what physically does it mean?

mgh = (gravitational) potential energy

$k_B T$ = thermal energy

$$Pr(h) = \frac{A\rho_N(h)}{N} = \frac{mg}{k_B T} e^{-\left(\frac{mgh}{k_B T}\right)}$$

$$Pr(h) \propto e^{-\frac{E}{k_B T}}$$

This is the Boltzmann factor – gives the probability of measuring a certain energy state at a given temperature