Mechanics revision exercises

1. Consider the equation

$$U\frac{d\rho}{dx} = D\frac{d^2\rho}{dx^2} + \frac{F}{V},$$

where U is speed, ρ is density, x is location, D is diffusivity (SI units m^2 s^{-1}), F is force and V is volume. Determine whether the equation is dimensionally consistent.

2. If the acceleration of a particle is given by

$$\ddot{\mathbf{r}} = a\mathbf{i} + be^{-\omega t}\mathbf{j},$$

where a, b and ω are constants and t is time, find the velocity and position of a particle that starts from rest at $\mathbf{r} = 0$.

3. Consider a particle of mass m hanging vertically from a spring (with spring constant k) under its own weight.

   (a) If x gives the displacement of the particle measured downwards from the natural length of the spring, show that the equation of motion is

   $$\ddot{x} + \omega^2 x = g,$$

   where $\omega^2 = k/m$.

   (b) If the particle starts from rest at the point $x = h$, find $x(t)$ and describe the particle's motion.

4. A particle of mass m is thrown from ground level, with initial speed V at an angle α to the horizontal. The particle is subject to gravity, and an additional force of the form $mg(1 - \omega t)$ acting vertically upwards. Find the time at which the particle hits the ground, and the horizontal distance from its initial location.

5. A particle of mass m is attracted to the origin of an inertial frame by a central force c/r^3, where r gives the distance between the particle and the origin, and $c > 0$ is a constant. The particle is initially a distance d from the origin and moving with velocity $v\mathbf{e}_r + \sqrt{c/(md^2)}\mathbf{e}_\theta$.

   (a) Find the value of the constant $h = r^2\dot{\theta}$ and explain what it represents physically.

   (b) Show that the particle path satisfies $\dfrac{d^2u}{d\theta^2} = 0$, where $u = 1/r$.

   (c) Show that at $\theta = 0$, we have $u = 1/d$ and $\dfrac{du}{d\theta} = -v\sqrt{m/c}$.

   (d) Find the particle path, $r(\theta)$.

   (e) In each of the cases: $v = 0$, $v < 0$ and $v > 0$, describe the particle motion. You may assume that $\theta(t)$ strictly increases over time.

6. A particle of mass m is attracted to the origin of an inertial frame by a force mk/r^2, where r gives the distance between the particle and the origin, and $k > 0$ is a constant. The particle is initially a distance R from the origin and moving with velocity $V\mathbf{e}_\theta$.

   (a) Find a second-order ordinary differential equation for $u(\theta)$, where $u = 1/r$.

   (b) Find expressions for the initial conditions for the equation in part (a), and the value of the constant $h = r^2\dot{\theta}$.

   (c) Find the particle path $r(\theta)$. Show that the path is an circle when $V^2 = k/R$, and an ellipse when $V^2 < 2k/R$ and $V^2 \neq k/R$.

7. We will revisit question 6. A particle of mass m is attracted to the origin of an inertial frame by a force mk/r^2, where r gives the distance between the particle and the origin, and $k > 0$ is a constant. The particle is initially located a distance R from the origin and moving with velocity $\sqrt{k/R}\mathbf{e}_\theta$.

   (a) Briefly explain why the problem can be formulated as

$$\frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - \frac{mk}{r} = \text{constant}, \tag{1}$$

$$r^2\dot{\theta} = \text{constant}. \tag{2}$$

   Write down the physical interpretation of each equation.

   (b) Using the initial values for location and velocities, find the constants in the expressions from part (a).

   (c) Using (2), find an expression for $\dot{\theta}$ in terms of k, R and r.

   (d) Hence, rewrite (1) as an expression for $\dot{r}^2$ in terms of k, R and r.

   (e) By considering the signs of each side of the resulting expression in (d), show that the particle moves in a circle. Determine the period of the circular motion.

8. A smooth sphere of radius $2a$ has its centre at the origin. If ρ, θ, z give cylindrical polar coordinates, with the z axis pointing vertically downwards, the surface of the sphere is given by $\rho^2 + z^2 = 4a^2$. The particle starts at $\rho = 2a$, moving with horizontal velocity V.

   (a) Briefly explain why

$$\rho^2\dot{\theta} = h, \tag{3}$$

$$\frac{1}{2}m\left(\dot{\rho}^2 + \rho^2\dot{\theta}^2 + \dot{z}^2\right) - mgz = E, \tag{4}$$

   where h and E are constants. What are the physical interpretations of each term?

   (b) Using the initial conditions, find the values of E and h.

(c) Hence show that the motion satisfies

$$2a^2\dot{z}^2 = -gz\,(z - z_1)\,(z - z_2)\,,$$

for some $z_{1,2}$ you should define. What does this tell you about the particle's motion?

(d) Does the particle rise or fall initially? Justify your answer.

9. A raindrop falls through a cloud while accumulating mass at a rate λr^2 where r is its radius (assume that the raindrop remains spherical) and $\lambda > 0$ is a constant. Find its velocity v at time t if it starts from rest with radius a. (You should take the direction of positive v to be downwards.)

   (a) If ρ is the density of rainwater (assumed constant), show that the mass of the raindrop is $m = \frac{4}{3}\pi r^3 \rho$ and hence find an expression for $r(t)$. You may find it useful to define $\mu = \frac{\lambda}{4\rho\pi}$.

   (b) Show that

$$\frac{dv}{dt} + \frac{3\mu}{\mu t + a} v = g.$$

   (c) Find $v(t)$ and explains what happens in the limit $t \to \infty$.