Example Sheet 2: Maps

1. Find all the 1-cycles for the map

$$x_{n+1} = a - x_n^2$$

2. Find the stability region of a for all the 1-cycles of the map

$$x_{n+1} = a - x_n^2$$

3. Find all the 2-cycles for the map

$$x_{n+1} = a - x_n^2$$

4. Find the stability region of a for all the 2-cycles of the map

$$x_{n+1} = a - x_n^2$$

5*. Find all the 3-cycles for the map

$$x_{n+1} = a - x_n^2$$

6*. Find the stability region of a for all the 3-cycles of the map

$$x_{n+1} = a - x_n^2$$

7*. Find all the 4-cycles for the map

$$x_{n+1} = a - x_n^2$$

8*. Find the stability region of a for all the 4-cycles of the map

$$x_{n+1} = a - x_n^2$$

9. Verify that the mapping

$$r\left[x_n - \frac{1}{2}\right] = R\left[X_n - \frac{1}{2}\right] \equiv z_n$$

with $r+R=2$ maps the logistic map, $x_{n+1} = rx_n(1 - x_n)$, onto itself. Show that z_n satisfies

$$z_{n+1} = a - z_n^2$$

and find the relationship between a and r. What interval of a, for the new map, corresponds to the original logistic map? Use the sepcial cases, r=2 and r=4, to find exact solutons to the map

$$z_{n+1} = a - z_n^2$$

10. Find all the 1-cycles for the map

$$x_{n+1} = rx_n(1 - x_n)(1 - 2x_n)$$

11. Find the stability region of r for all the 1-cycles of the map

$$x_{n+1} = rx_n(1 - x_n)(1 - 2x_n)$$

12. Consider the map

$$x_{n+1} = r(2x_n^2 - 1)$$

Find all the 1-cycles and establish the regions of control parameter for which they are stable. Find all the 2-cycles and establish the regions of control parameter for which they are stable. Employ $x_n = \cos \pi y_n$ to establish an exact solution which is controlled by the tent map and find the value of r for which the exact solution exists.

13. Consider the map

$$x_{n+1} = rx_n(3 - 4x_n^2)$$

Find all the 1-cycles and establish the regions of control parameter for which they are stable. Find all the 2-cycles and establish the regions of control parameter for which they are stable. Employ $x_n = \sin \frac{\pi y_n}{2}$ to establish an exact solution. Find the value of r for which the exact solution exists and find the map that controls it.

14. Consider the mapping

$$x_{n+1} = rx_n(x_n^2 - 1)$$

where r is a control parameter. Employ the transformation

$$x_n = \frac{2}{\sqrt{3}} \cos \pi y_n$$

to provide an exact solution for a value of r that you should determine. Using $y_{n+1} = M[y_n]$ for this solution find the map $M[y]$ and depict it. Find the 1-cycles. Find the 2-cycles.

15. Consider the mapping

$$x_{n+1} = x_n \left[f(x_n)\right]^2$$

and employ the transformation $x_n = z_n^2$ to provide a second representation. Use this theory to deduce the 1-cycles and 2-cycles for the map

$$x_{n+1} = r^2 x_n (1 - x_n)^2$$

from the solution to the map

$$z_{n+1} = rz_n(1 - z_n^2)$$

16. Consider the map

$$\begin{aligned} M[x] =& -2 - 3x \qquad && x \in \left[-1, -\frac{1}{3}\right] \\ =& 3x && x \in \left[-\frac{1}{3}, \frac{1}{3}\right] \\ =& 2 - 3x && x \in \left[\frac{1}{3}, 1\right] \end{aligned}$$

Depict $M[x]$, $M^{(2)}[x]$ and $M^{(3)}[x]$. Determine the number of solutions to $x = M^{(N)}[x]$ and use this result to calculate the number of n-cycles for $n \leq 10$.

17. Consider the map

$$x_{n+1} = rx_n \left[1 - x_n^2 + \frac{x_n^4}{5}\right]$$

Find all the 1-cycles and establish the region of r for which each is stable. Employ the transformation

$$x_n = 2 \sin \frac{\pi y_n}{2}$$

to find an exact solution and determine the particular value of r associated with this solution. Using $y_{n+1} = M[y_n]$ determine the map $M[y]$.

18*. Consider the map

$$x_{n+1} = a - x_n^3$$

Show that there is only one 1-cycle and find the region of stability in a. Show that there is always at least one 2-cycle and find the region of a for which there are three 2-cycles. Show that these 2-cycles are never unstable to 4-cycles and find the region of stability in a. Describe the attractor as a function of a.

19. Demonstrate explicitely that the transformation

$$X_n = \frac{1}{R}\left[R - 1 + (2 - R)x_n\right] \qquad r + R = 2$$

maps the 2-cycle of the logistic map for $r \geq 1$ onto itself for $R \leq 1$.

20.* Show that the map

$$x_{n+1} = r^3 x_n (1 - x_n)^3$$

can be transformed into the map

$$z_{n+1} = rz_n(1 - z_n^3)$$

using the transformation $x_n = z_n^3$ and use this idea to find the 1-cycles and 2-cycles for the original map.