Rather than starting with an historic introduction as it is usually done I will derive the central result of this theory, *Lorentz transformation*, by considering physical observation of Michelson-Morley experiment and its inconsistency with the Galilean invariance. For this we will need a short mathematical introduction.

Transformations of space and time.

Motion of a point-like mechanical object (a material point) can be described mathematically by specifying the dependence of its three Cartesian coordinates (x, y, z) on time t. We denote by

$$\boldsymbol{r}(t) = \big(x(t), y(t), z(t)\big)$$

the *trajectory of the object* (Fig. 1). Thus the full description requires *four* numbers: three Cartesian coordinates and time t. Special relativity can be viewed as geometry of such four-dimensional space. We shall see that this geometry is strikingly different from the one we are used to in three dimensions.

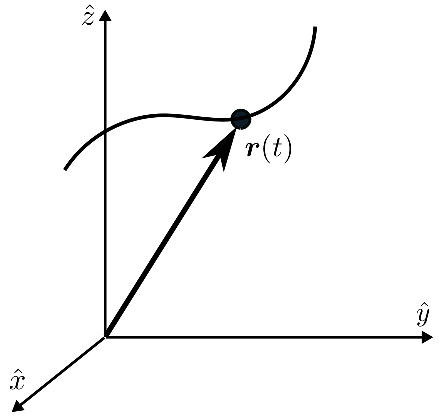

Figure 1: Trajectory of a moving object.

Let us review the idea of a transformation in three dimensions first. Consider a static object at $\boldsymbol{r} = (x, y, z)$. The coordinates can only be defined with respect to some fixed *coordinate system* consisting of three perpendicular axes as shown in Fig. 1. The choice of the axes is arbitrary (one can even use non-perpendicular axes, but this will complicate things a lot).

For example, one can rotate the coordinate system around $\hat{z}$-axis by angle α. The *same* object will have *different* coordinates (x', y', z') in the new coordinate system. To find them look at Fig. 2 which represents the $x - y$ plane (here $z = z' = 0$) with the new rotated axes.

To find the new coordinate x' we draw the line parallel to y'-axis passing through the object and register its intersection with x'-axis. Similarly the coordinate y' is obtained. If one compares the new coordinates to the old ones (also shown) it is not difficult to see (do it!) that the new coordinates x', y' are related to the old ones x, y by the mathematical relation

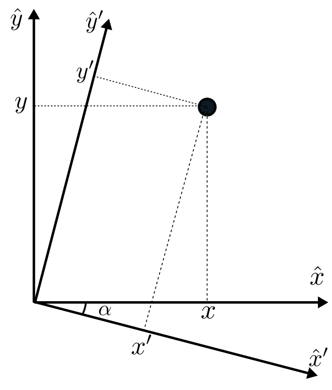

Figure 2: Rotation around $\hat{z}$.

$$\begin{aligned} x' &= x\cos\alpha - y\sin\alpha \\ y' &= x\sin\alpha + y\cos\alpha \\ z' &= z\,, \end{aligned}$$

or expressing thew old coordinates in terms of the new ones,

$$\begin{aligned} x &= x'\cos\alpha + y'\sin\alpha \\ y &= x'\sin\alpha - y'\cos\alpha \\ z &= z'\,. \end{aligned}$$

Despite the fact that the coordinates of the object have changed, there are some functions which will remain the same under such a transformation. One such function is the distance between the object and the origin $d = \sqrt{x^2 + y^2} = d' = \sqrt{x'^2 + y'^2}$, since

$$\begin{aligned} x^2 + y^2 &= \left(x' \cos\alpha + y' \sin\alpha\right)^2 + \left(x' \sin\alpha - y' \cos\alpha\right)^2 \\ &= x'^2 \cos^2\alpha + 2x'y' \cos\alpha \sin\alpha + y'^2 \sin^2\alpha \\ &+ x'^2 \sin^2\alpha - 2x'y' \sin\alpha \cos\alpha + y'^2 \cos^2\alpha \\ &= \left(x'^2 + y'^2\right)\left(\sin^2\alpha + \cos^2\alpha\right) = x'^2 + y'^2 \,. \end{aligned}$$

These functions are called *invariants* and play important role as they reflect symmetries and allow describing properties of objects irrespectively of a particular coordinate system.

Let us now turn to dynamics of objects and introduce time t into the picture. To make the graphical representation simple we will often be dealing with only one space coordinate, say x, measured along the horizontal axis $\hat{x}$. A typical trajectory $x(t)$ is shown in Fig. 3 where the vertical axis $\hat{t}$ represents time so that the time coordinate t can be obtained by projecting onto it. Of course, for each time t there only one corresponding coordinate $x(t)$.

Now consider a transformation (Galilean transformation) which involves *both space and time* ,

$$\begin{aligned} t' &= t \\ x' &= x - Vt \\ y' &= y \,, \qquad z' = z \,. \end{aligned}$$

This transformation involves a parameter V with dimensions of velocity (m/s). To see its physical meaning we can draw the line $x' = 0$ which is nothing but the position of the new time axis $\hat{t}'$. This line represents the trajectory of an object moving with constant velocity $x = Vt$ in the original coordinates and thus the Galilean transformation is the transformation into a reference frame moving with constant velocity V along x-axis.

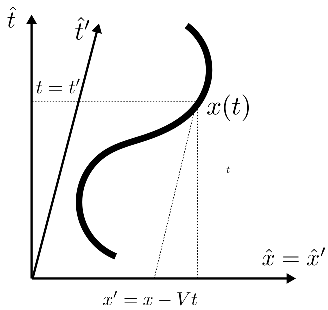

Figure 3: Trajectory and Galilean transformation into a moving frame.

Similarly we can write a more general Galilean transformation

$$t' = t \tag{1}$$
$$x' = x - V_x t \tag{2}$$
$$y' = y - V_y t \tag{3}$$
$$z' = z - V_z t \,, \tag{4}$$

or $\boldsymbol{r}' = \boldsymbol{r} - \boldsymbol{V}t$ depending on the velocity vector $\boldsymbol{V} = (V_x, V_y, V_z)$. Are there invariants in such a transformation? We can see that *acceleration*,

$$\boldsymbol{a} = (a_x, a_y, a_z) = \left(\frac{\mathrm{d}^2 x}{\mathrm{d}t^2}, \frac{\mathrm{d}^2 y}{\mathrm{d}t^2}, \frac{\mathrm{d}^2 z}{\mathrm{d}t^2}\right) = \frac{\mathrm{d}^2 \boldsymbol{r}}{\mathrm{d}t^2} = \ddot{\boldsymbol{r}}$$

will be the same in the moving frame, since

$$\boldsymbol{a}' = \frac{\mathrm{d}^2 \boldsymbol{r}'}{\mathrm{d}t^2} = \frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(\boldsymbol{r} - \boldsymbol{V}t\right) = \frac{\mathrm{d}^2 \boldsymbol{r}}{\mathrm{d}t^2} + 0 = \boldsymbol{a} \,.$$

Now we understand why the **2nd Newton's law**

$$\boldsymbol{F} = m\boldsymbol{a}$$

which is behind dynamics of physical objects contains acceleration: due to **Newtonian Principle of Relativity**:

> "no mechanical experiment can distinguish between frames moving with constant velocity with respect to each other"

We can also say that laws of Newtonian mechanics are invariant under Galilean transformation. In particular, if the force is absent $\boldsymbol{F} = 0$, we get in all frames

$$\boldsymbol{a} = \frac{\mathrm{d}^2\boldsymbol{r}}{\mathrm{d}t^2} = 0 \quad \Rightarrow \quad \boldsymbol{r}(t) = \boldsymbol{r}_0 + \boldsymbol{v}_0 t\,,$$

which is the mathematical formulation of the **1st Newton's law**

> "a body continues in its state of rest or uniform motion in a straight line unless external forces act on it"

Such frames are called inertial. Unlike acceleration velocity of an object is not an invariant and depends on the particular inertial frame.

Transformation of velocities under Galilean transformation

Suppose an object has velocity $\boldsymbol{v}$ in some frame. What is its velocity $\boldsymbol{v}'$ in the inertial frame described by transformation (1), *i.e.* in the frame moving with velocity $\boldsymbol{V}$?

Take a small time interval Δt. The displacement of the object during this interval is

$$\begin{aligned}\Delta x &= v_x \Delta t\\ \Delta y &= v_y \Delta t\\ \Delta z &= v_z \Delta t\,,\end{aligned}$$

or $\Delta\boldsymbol{r} = \boldsymbol{v}\Delta t$ in vector form. Applying the transform (1) to these formulae we have

$$\begin{aligned}\Delta x' &= \Delta x - V_x\Delta t = (v_x - V_x)\Delta t'\\ \Delta y' &= \Delta y - V_x\Delta t = (v_y - V_y)\Delta t'\\ \Delta z' &= \Delta z - V_x\Delta t = (v_z - V_z)\Delta t'\,,\end{aligned}$$

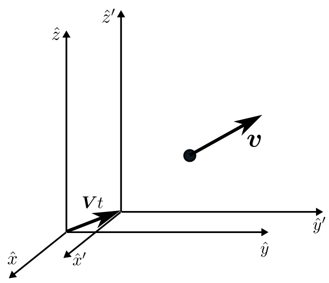

Figure 4: Trajectory and Galilean transformation into a moving frame.

as the time interval is the same in both frames $\Delta t = \Delta t'$. We see that $\Delta\boldsymbol{r}' = \boldsymbol{v}'\Delta t'$, or

$$\boldsymbol{v}' = \boldsymbol{v} - \boldsymbol{V}\,,$$

which constitutes the desired law of velocity transformation. This is consistent with everyday observation that looking from a moving car the other cars on the same motorway appear to move slower (if they move in the same direction).

In the next lecture we will apply these ideas to propagation of light. We will discuss the results of Michelson-Morley experiment which contradicted the above transformation of velocities and challenged the Galilean invariance behind Newton's laws.