Optics and Waves

Lectures 5-6

-Superposition of waves
-Standing waves

Young and Freedman 15.6; 15.7; 15.8; 16.4; 16.5

The principle of superposition

When two waves y_1 and y_2 overlap, the displacement at any point on the string is:

$$y(x,t) = y_1(x,t) + y_2(x,t)$$

The wave equation is a linear equation, it contains the function $y(x, t)$ only to the first power. Hence, if any two functions, $y_1(x, t)$ and $y_2(x, t)$ are the solutions of the wave equation, their sum is also a solution.

$$\frac{\partial^2 y(x,t)}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y(x,t)}{\partial t^2}$$

superposition

Standing Waves

What happens if a sinusoidal wave is reflected at a fixed end of a string?:

standing wav
e

We can derive a wave function for the standing wave by adding $y_1(x, t)$ and $y_2(x, t)$ for two waves with equal amplitude, period and wavelength travelling in opposite directions. (page 493 Young)

Incident wave: $y_1(x,t) = -A\cos(kx + \omega t)$ travels from right to left

Reflected wave: $y_2(x,t) = A\cos(kx - \omega t)$ travels from left to right

Waves reflected from a fixed end is inverted, hence change of sign for amplitude.

We now add the two waves together.

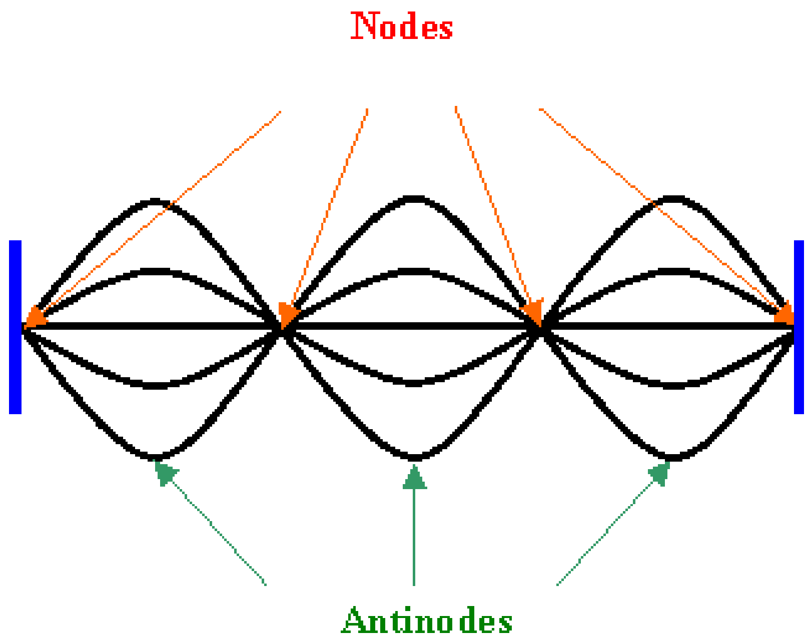
$$y_{total} = y_1 + y_2 = A[-\cos(\underset{A}{kx} + \underset{B}{\omega t}) + \cos(kx - \omega t)]$$

Remember: $\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$

$$y_{total} = A\left[\cancel{-\cos(kx)\cos(\omega t)} + \sin(kx)\sin(\omega t)\right] +$$
$$A\left[\cancel{\cos(kx)\cos(\omega t)} + \sin(kx)\sin(\omega t)\right]$$

$$y_{total} = 2A\sin(kx)\sin(\omega t)$$

$$y_{total} = 2A\sin(kx)\sin(\omega t) = A[-\cos(kx+\omega t)+\cos(kx-\omega t)]$$



Spatial nodes when:

$\sin(kx) = 0$

$kx = n\pi \qquad\qquad n = 0,1,2,3\ldots$

Nodes, independent of time

Each point moves up and down with amplitude $2A\sin(kx)$

$$y(x,t) = 2A\sin(kx)\sin(\omega t) = B(x)\sin(\omega t)$$

The wave does not propagate, it is stationary.

Note: standing waves are also solutions of the wave-equation

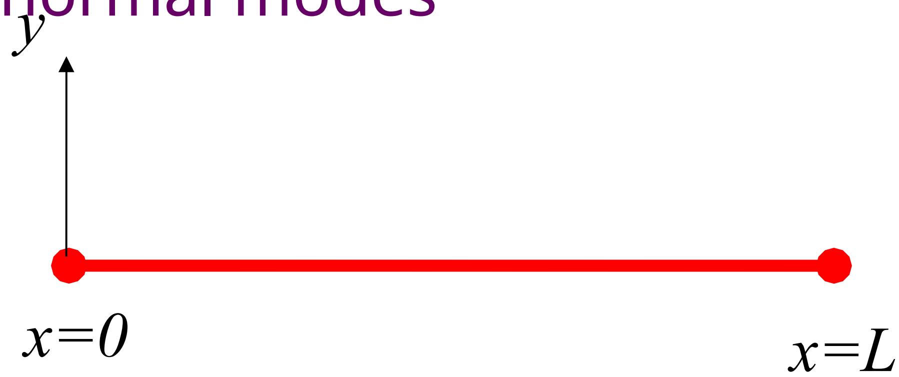
$$\frac{\partial y}{\partial x} = 2Ak\cos(kx)\sin(\omega t) \qquad \frac{\partial^2 y}{\partial x^2} = 2A(-k^2)\sin(kx)\sin(\omega t)$$

$$\frac{\partial y}{\partial t} = 2A\omega\sin(kx)\cos(\omega t) \qquad \frac{\partial^2 y}{\partial t^2} = 2A(-\omega^2)\sin(kx)\sin(\omega t)$$

$$\frac{\dfrac{\partial^2 y}{\partial t^2}}{\dfrac{\partial^2 y}{\partial x^2}} = \frac{2A(-\omega^2)\sin(kx)\sin(\omega t)}{2A(-k^2)\sin(kx)\sin(\omega t)} = \frac{\omega^2}{k^2} = v^2$$

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}$$

Standing waves; String with fixed ends. normal modes



What standing waves can we have?

$$y(x,t) = 2A\sin(kx)\sin(\omega t)$$

$$at\ x = 0;\ y = 0$$

$$at\quad x = L, y = 0 \text{ at all times.}$$

$$i.e.\quad kL = n\pi \quad (n = 1,2,3,...)$$

$$y(x,t) = 2A\sin\left(\frac{n\pi}{L}x\right)\sin\omega t$$

$$\lambda = \frac{2\pi}{k} = \frac{2L}{n}$$

superposition

$$\lambda = \frac{2L}{n}, \quad L = n\frac{\lambda}{2}$$

1st thru 5th harmonics of a vibrating string

$$\lambda = \frac{2L}{1} = 2L$$

$$\lambda = \frac{2L}{2} = L$$

$$\lambda = \frac{2L}{3}$$

$$\lambda = \frac{2L}{4} = \frac{L}{2}$$

Can you set up a standing wave on this string with $\lambda \neq \frac{2L}{n}$?

What conclusions can you make about the energy on the string?

Standing Wave Frequency

$$f_n = \frac{v}{\lambda_n} = \frac{v}{\frac{2L}{n}} = \frac{n}{2L}v$$

$f_1 = \dfrac{v}{2L}$ is the first harmonic or fundamental.

$f_2 = 2f_1$ is the second harmonic, 1st overtone.

$f_3 = 3f_1$ is the third harmonic, 2nd overtone.

Each of the frequencies corresponds to a **normal mode** of the system.

For each normal mode, the corresponding frequency is also called the resonant frequency.

What happens if you drive/excite a system at its resonant frequency?

Resonanceglass

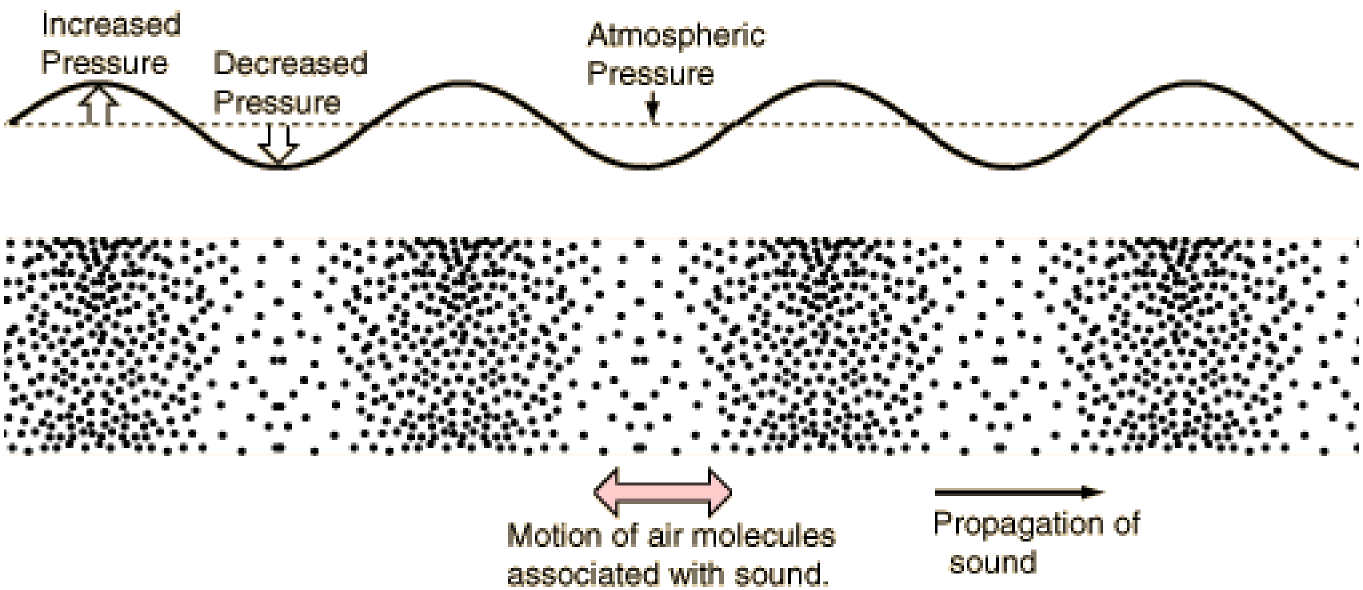
$$f_n = \frac{v}{\lambda_n} = \frac{n}{2L} v$$

To make high f, use shorter L, or high v.

High v is achieved with lighter strings or higher tension.

Basics of sound waves

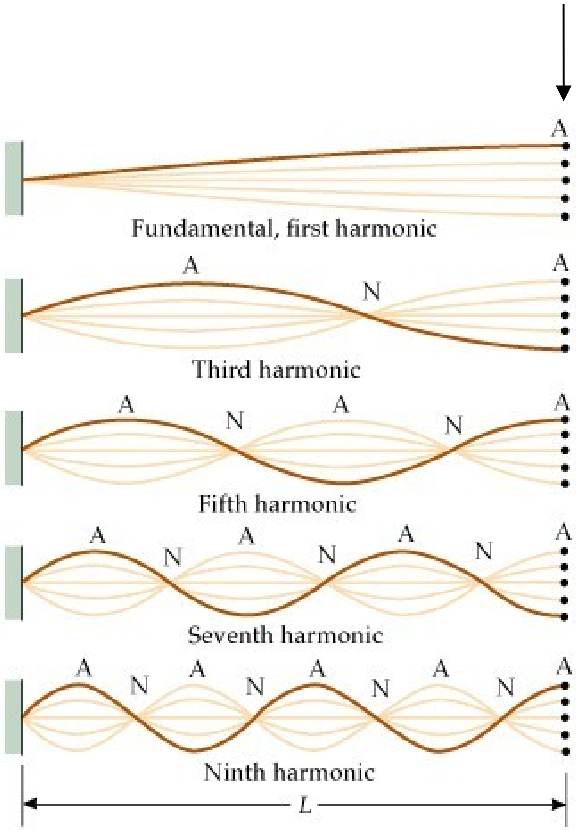
$$s(x,t) = S_m \cos(kx - \omega t)$$



$$\Delta P(x,t) = \Delta P_m \sin(kx - \omega t)$$

Standing waves of sound

Note: if the boundary conditions are different then we get a different solution for the standing waves. E.g. String with one end free.



First harmonic

$L=1/4\lambda_1$, $f_1=v/4L$

3^{rd} harmonic: $L=3/4\lambda_3$, $f_3=3v/4L=3f_1$

5^{th} harmonic: $L=5/4\lambda_5$, $f_5=5v/4L=5f_1$

Where are the 2^{nd}, 4^{th} harmonics?

Even harmonics do not exist!

Not allowed by boundary conditions

Not practical for string instruments:

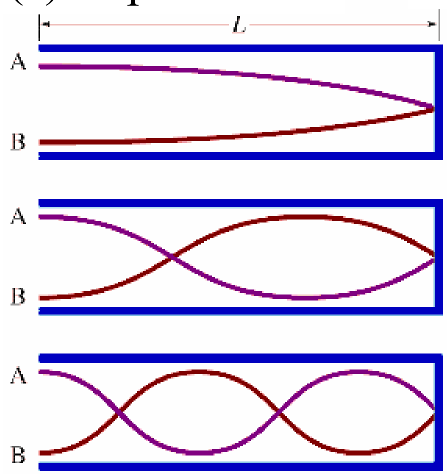
High Tension not possible, only low f.

Sound waves in a pipe/tube

(a)Displacement

Tube closed at one end

(b)Pressure



First harmonic

$L=1/4\lambda_1$, $f_1=v/4L$

Third harmonic

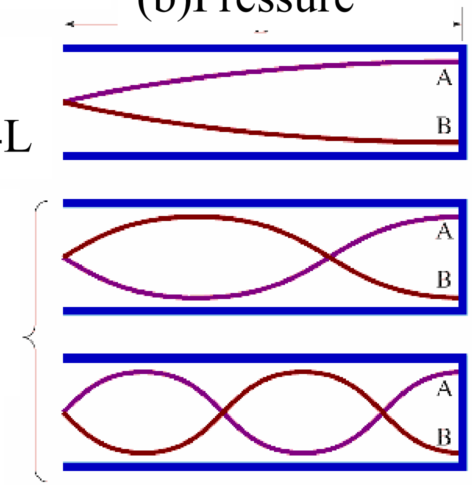
$L=3/4\lambda_1$,

$f_3=3v/4L=3f_1$

Fifth harmonic

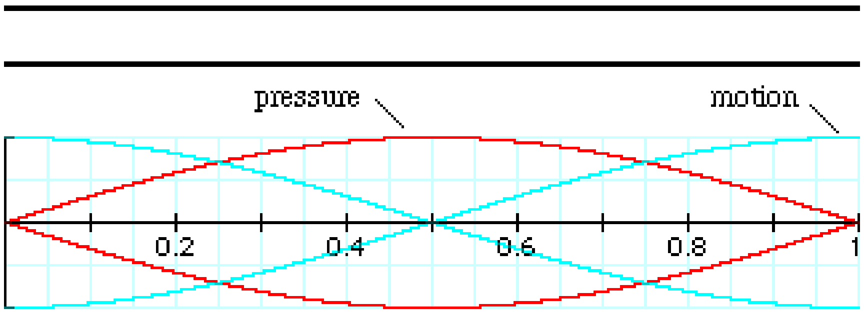
$L=5/4\lambda_1$,

$f_5=5v/4L=5f_1$

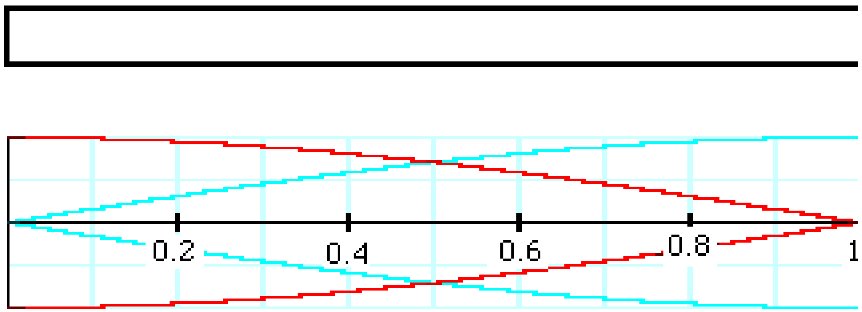


For pipes with one end closed, there are only odd harmonics(f_1, f_3, f_5, ..), no even harmonics (f_2, f_4, f_6…)

Flute



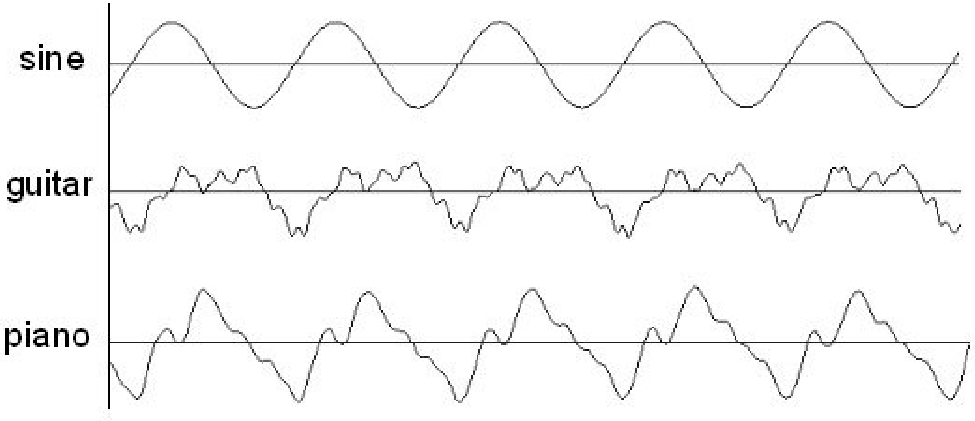
Clarinet



Estimate the frequency of the lowest note produced by a flute

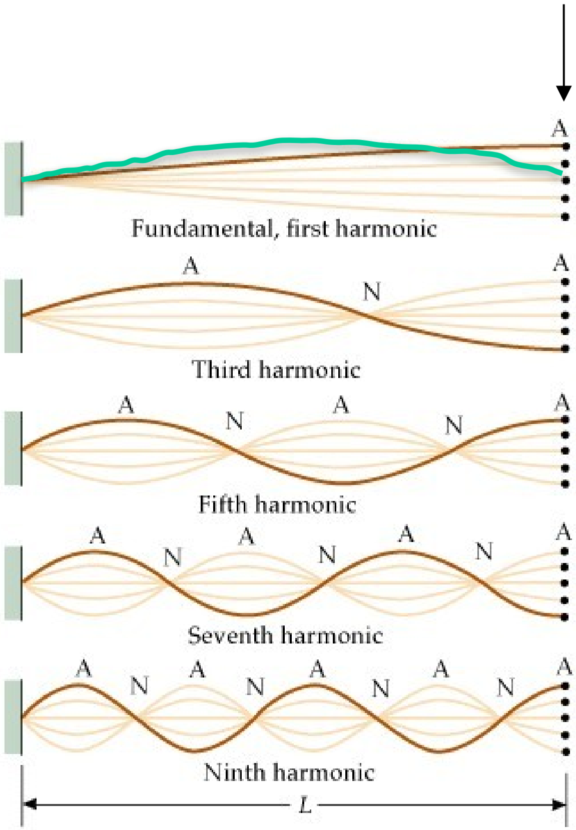
Waves in pipes

Real soundsnote



http://newt.phys.unsw.edu.au/jw/brassacoustics.html

String with one end free



First harmonic

$L=1/4\lambda_1$, $f_1=v/4L$

3rd harmonic: $L=3/4\lambda_3$, $f_3=3v/4L=3f_1$

Standing waves

1. Know how to add two travelling waves to make a standing wave.
2. Understand what is a normal mode and how to obtain the frequency of a normal mode.

3. Boundary conditions causing the missing harmonics.

4. Energy in a standing wave.

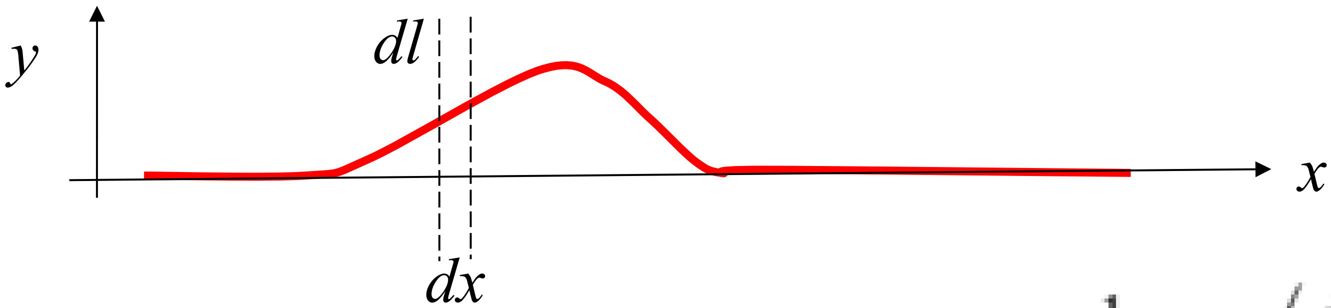
Optics and Waves

Lecture 7

-Energy carried by a wave
-Power of a wave

Energy in waves

Travelling waves (on a string): μ, T

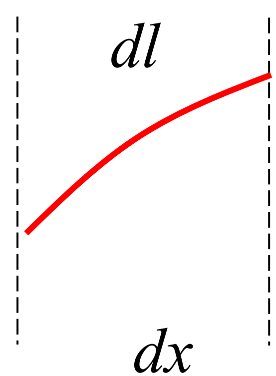


Kinetic energy of element, dx, of string: $\Delta KE = \frac{1}{2}(\mu dx)\left(\frac{dy}{dt}\right)^2$

$$\Delta KE = \frac{1}{2}(\mu dx)\left(A\omega \sin(kx - \omega t)\right)^2 = \frac{1}{2}\mu A^2\omega^2 \sin^2(kx - \omega t)dx$$

Strain energy (PE associated with stretching the string from dx to dl)

$$(dl)^2 = (dy)^2 + (dx)^2 = (dx)^2\left(1 + \left(\frac{dy}{dx}\right)^2\right)$$



$$dl = dx\left(1 + \left(\frac{dy}{dx}\right)^2\right)^{1/2} = dx\left(1 + \frac{1}{2}\left(\frac{dy}{dx}\right)^2\right)$$ if gradient is small

Extension of string:

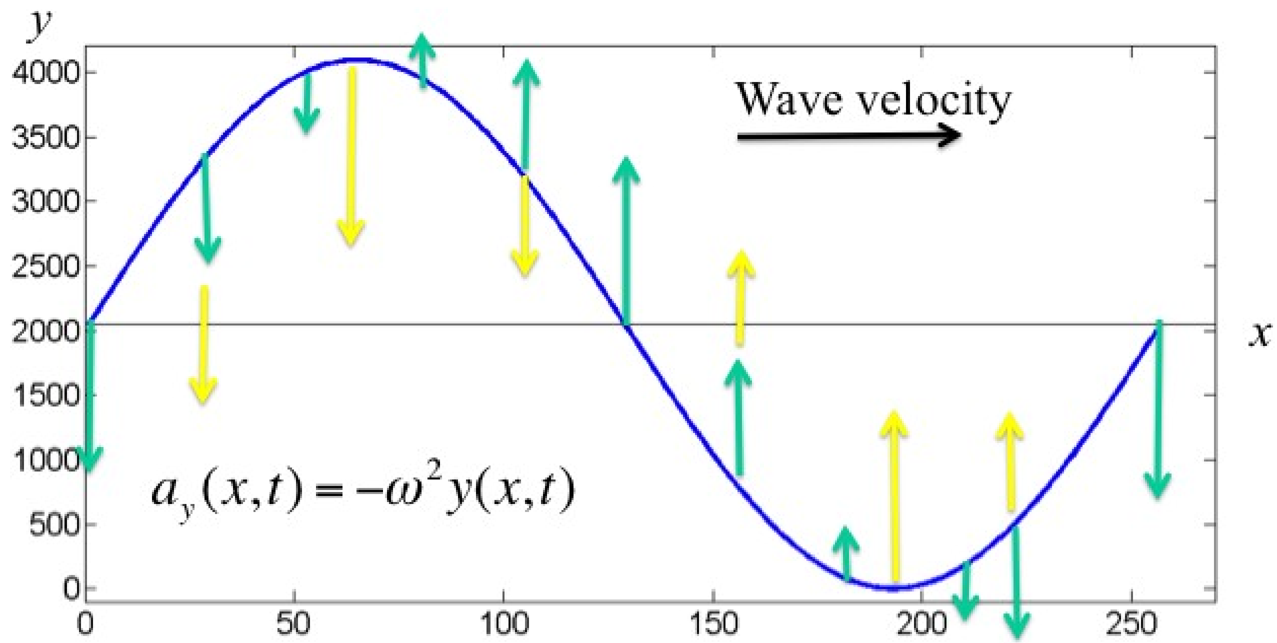
$$dl - dx \approx \frac{1}{2}dx\left(\frac{dy}{}\right)^2$$

Strain Energy in

$$= \frac{T}{2}dxA^2k^2\sin^2($$

$$\Delta U = \frac{1}{2}\mu v^2 dxA^2$$

Since: $\omega = vk$



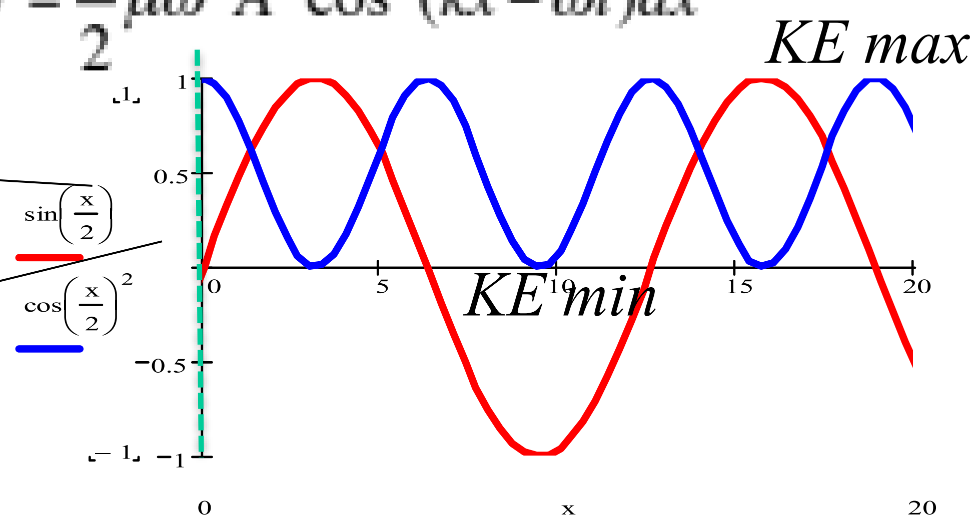
So the strain en

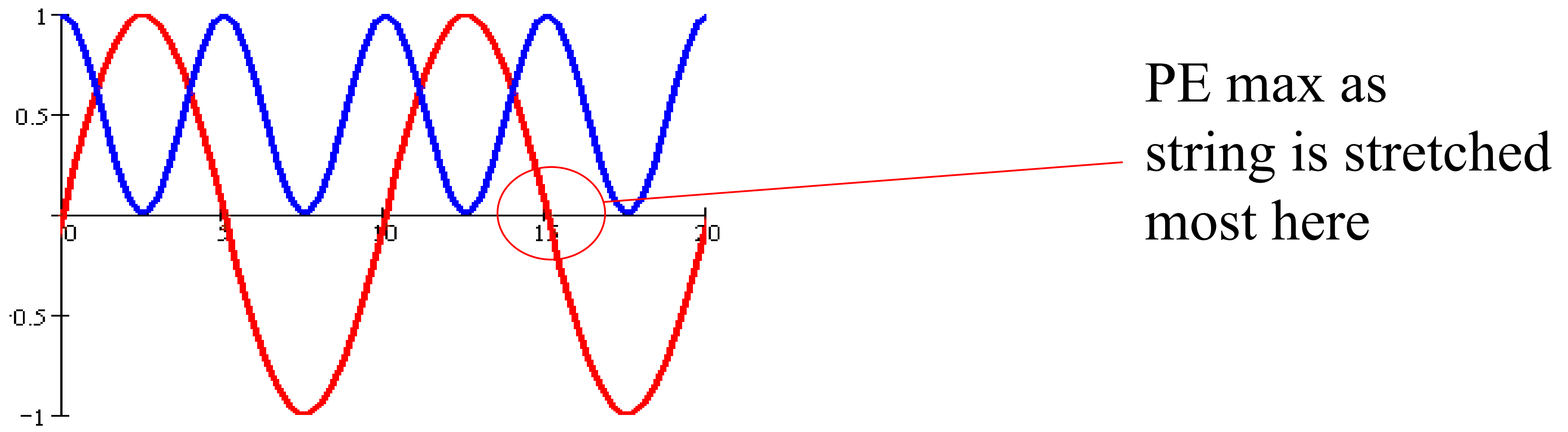
Diagram below

$$y(x,t) = A\sin(kx-\omega t), and\ \Delta U = \frac{1}{2}\mu\omega^2A^2\cos^2(kx-\omega t)dx$$

displacement

KE/Strain E





$$\Delta E = \Delta KE + \Delta U = \mu A^2 \omega^2 \sin^2(kx - \omega t)dx$$

Time average of energy for *dx*

$$\langle dE \rangle = \frac{1}{2}\mu A^2 \omega^2 dx$$

Power: energy flow per unit time:

$$P_{aver} = \frac{\langle dE \rangle}{dt} = \frac{1}{2}\mu A^2 \omega^2 \frac{dx}{dt} = \frac{1}{2}\mu A^2 \omega^2 v$$

Another way of working out the energy.
We can treat the wave as a large number of harmonic oscillators: each with a mass μdx, oscillating with amplitude A and angular frequency ω

The energy (KE+PE) for an oscillator is $$\frac{1}{2}(\mu dx)A^2\omega^2$$

So the energy contained in one wavelength is $$\frac{1}{2}\lambda\mu A^2\omega^2$$

This amount of energy is transported in time T(period), so the power is

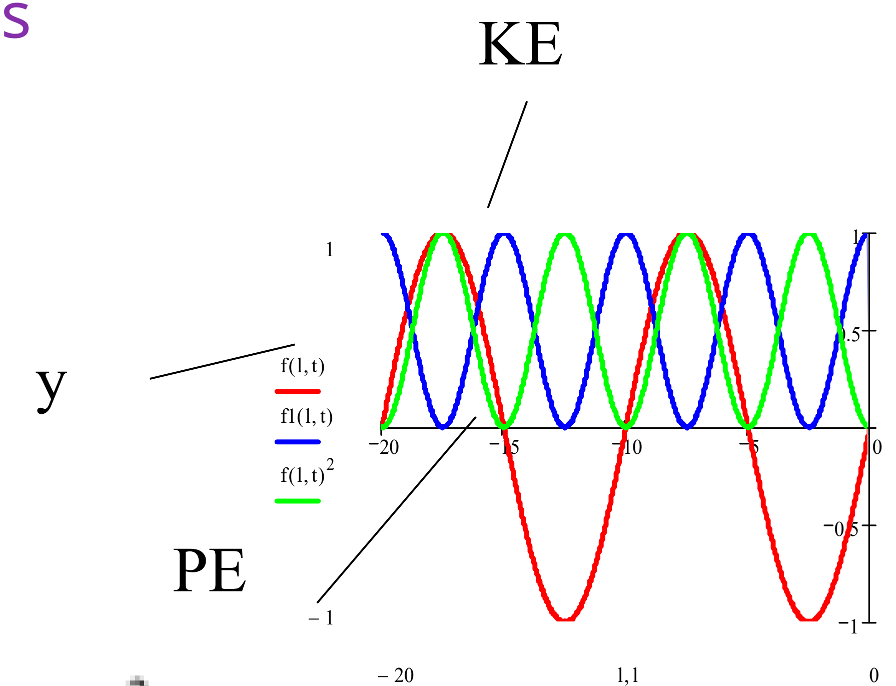
$$\frac{1}{2}\lambda\mu A^2\omega^2 / T = \frac{1}{2}\mu A^2\omega^2(\lambda / T) = \frac{1}{2}\mu A^2\omega^2 v$$

Energy in standing waves

$$\Delta KE = \frac{1}{2}\mu\left(\frac{dy}{dt}\right)^2 dx$$

$$\Delta U = \frac{1}{2}T\left(\frac{dy}{dx}\right)^2 dx$$

$$y = 2A\sin(kx)\sin(\omega t)$$



$$\Delta KE = \frac{1}{2}\mu(2A)^2(\omega)^2\sin^2(kx)\cos^2(\omega t)dx$$

$$\Delta U = \frac{1}{2}T(2A)^2(k)^2\cos^2(kx)\sin^2(\omega t)dx$$

$$= \frac{1}{2}\mu(2A)^2(\omega)^2\cos^2(kx)\sin^2(\omega t)dx$$

For an arbitrary point at *x*,

The energy (KE+PE) over a length *dx*

$$\Delta KE + \Delta U = \frac{1}{2}\mu(2A)^2(\omega)^2[\cos^2\omega t\sin^2(kx) + \sin^2\omega t\cos^2(kx)]dx$$

Averaging over one cycle/period, T:

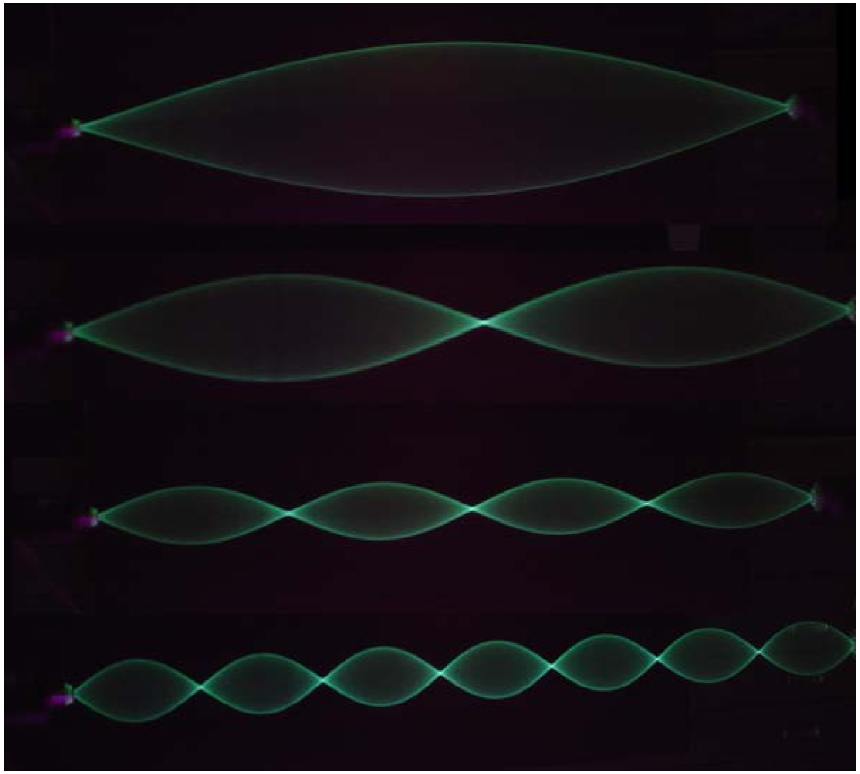
$$< \Delta KE + \Delta U > = \frac{1}{2}\mu(2A)^2(\omega)^2[\frac{1}{2}\sin^2(kx) + \frac{1}{2}\cos^2(kx)]dx$$

$$= \mu(A)^2(\omega)^2\, dx$$

Energy stored in one wavelength: $\mu A^2\omega^2\lambda$

This is twice of that a travelling wave. $\frac{1}{2}\lambda\mu A^2\omega^2$

Energy scales with ω^2



Wavelength, and thus frequency, is quantized! Energy is quantized.

Chladni Plates **normal modes**

Drumhttps://www.youtube.com/watch?v=v4ELxKKT5Rw

Another interesting site if you have ten minutes to spare:

http://www.youtube.com/watch?v=67NPGP5A2EI

Chladni: German Physicist and Musician