A34930

No Calculator

UNIVERSITY OF BIRMINGHAM

School of Mathematics

Programmes in the School of Mathematics — First Examination

Programmes involving Mathematics — First Examination

1RA 06 34051 Level C

LC Real Analysis

May/June Examinations 2023-24

Three Hours

Full marks will be obtained with complete answers to all FOUR questions. Each question carries equal weight. You are advised to initially spend no more than 45 minutes on each question and then to return to any incomplete questions if you have time at the end. An indication of the number of marks allocated to parts of questions is shown in square brackets.

No calculator is permitted in this examination.

Section A

1. (a) Let $f : \mathbb{R} \to \mathbb{R}$ be a function and let $\alpha, \ell \in \mathbb{R}$.

(i) Give the definition of the limit

$$\lim_{x \to \alpha} f(x) = \ell.$$

(ii) By directly using the definition of limit from (i), show that

$$\lim_{x \to 1} 4x + 96 = 100.$$

[8]

(b) Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function and let $\alpha \in \mathbb{R}$.

(i) Define $f'(\alpha)$, the derivative of f at α.

(ii) By directly using the definition of derivative from (i), find the derivative of the function $f(x) = x^3$ at the point $x = 2$.

[5]

(c) Find the following limits and justify your answers. You can use any of the results discussed in lectures, provided you clearly state what you are using.

(i) $\displaystyle\lim_{x \to 0} \frac{\sin(2x)}{3x}$;

(ii) $\displaystyle\lim_{x \to 1} \frac{x}{3x^2 - 2x + 4}$.

[6]

(d) Find the derivatives of the following functions.

(i) $f(x) = 3x^3 + 2x^2 - 6x + 1$;

(ii) $f(x) = \ln(2x + 3)$ for $x > 0$.

[6]

2. Suppose that $-\infty < a < b < \infty$.

(a) Is the function $f : [a,b] \to \mathbb{R}$ given by

$$f(x) := \begin{cases} 5, & \text{if } x \in \mathbb{Q}; \\ 9, & \text{if } x \notin \mathbb{Q}, \end{cases}$$

integrable? Prove your answer. [5]

(b) State Riemann's Integrability Criterion for a bounded function $g : [a,b] \to [0,\infty)$. [2]

(c) Use the Fundamental Theorem of Calculus, or otherwise, to prove that any continuous function $h : [a,b] \to \mathbb{R}$ has an antiderivative. [3]

(d) Find $\int \sin^6(x)\cos^3(x)\,\mathrm{d}x$ in terms of elementary functions. [3]

(e) Compute $\displaystyle\int_1^2 \frac{1}{x^2\sqrt{x^2+9}}\,\mathrm{d}x$. [4]

(f) Find a solution $y : [1,\infty) \to \mathbb{R}$ of the initial value problem

$$y' + \frac{9y}{x} = 4x^3, \quad y(1) = 2.$$ [3]

(g) Find the general solution $y : \mathbb{R} \to \mathbb{R}$ of the differential equation $y'' + 2y' + 5y = 4x$. [5]

Section B

3. (a) Use L'Hôpital's rule to find the following limits when it applies:

(i) $\displaystyle\lim_{x\to 0} \frac{1-\cos x}{x^2}$;

(ii) $\displaystyle\lim_{x\to 0^+} x^{0.001} \ln x$;

(iii) $\displaystyle\lim_{x\to\infty} \left(\frac{4^{1/x}+9^{1/x}}{2}\right)^x$.

[12]

(b) Let $f : [1,3] \to \mathbb{R}$ be a function such that

$$|f(x)-f(y)| \leq 100|x-y|^{1.001}$$

for all $x, y \in [1,3]$. Prove that f is a constant function. (Hint: first show $f'(x) = 0$.) [5]

(c) Let $f : \mathbb{R}\setminus\{0\} \to \mathbb{R}$ be the function defined by

$$f(x) = \frac{1-2x}{x^2} + 1$$

for all $x \in \mathbb{R}\setminus\{0\}$. Find and determine the nature (i.e. local minima or local maxima) of the stationary points of f. Justify any assertions that you make. [5]

(d) Given $c \in \mathbb{R}$, prove that there is at most one solution to the equation

$$x^3 - 4x + c = 0$$

on the interval $[0,1]$. [3]

4. (a) Find $\displaystyle\int \frac{4x^2+x+10}{x^3-x^2+4x-4}\,\mathrm{d}x$ in terms of elementary functions. [6]

(b) Compute $\displaystyle\int_0^\infty x^2 e^{-3x}\,\mathrm{d}x$ with appropriate justification or prove that it diverges. [6]

(c) Suppose that $f : [a,b] \to [0,\infty)$ is a bounded function, where $-\infty < a < b < \infty$, with the property that for each $n \in \mathbb{N}$ there exist two partitions P_n and Q_n of $[a,b]$ such that

$$L(f,Q_n) \geq (b-a)^2 - \frac{5}{n^3+n^2} \quad \text{and} \quad U(f,P_n) \leq (b-a)^2 + \frac{10}{n}.$$

(i) Prove that for each $n \in \mathbb{N}$ there exists a partition R_n of $[a,b]$ such that

$$U(f,R_n) - L(f,R_n) \leq \frac{10}{n} + \frac{5}{n^3+n^2}.$$

(ii) Use (i) to prove that f is integrable.

[6]

(d) Suppose that $g : [0,\pi] \to \mathbb{R}$ is given by

$$g(x) := \int_0^{2x\cos^2(3x)} \sin(t^2)\,\mathrm{d}t \quad \text{for all} \quad x \in [0,\pi].$$

Prove that g is differentiable and find its derivative g' in terms of elementary functions. [7]

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Do not complete the attendance slip, fill in the front of the answer book or turn over the question paper until you are told to do so.

Important Reminders

- Coats and outer-wear should be placed in the designated area.
- Unauthorised materials (e.g. notes or Tippex) **MUST** be placed in the designated area.
- Check that you **DO NOT** have any unauthorised materials with you (e.g. in your pockets, pencil case).
- Mobile phones and smart watches **MUST** be switched off and placed in the designated area or under your desk. They must not be left on your person or in your pockets.
- You are **NOT** permitted to use a mobile phone as a clock. If you have difficulty in seeing a clock, please alert an Invigilator.
- You are **NOT** permitted to have writing on your hand, arm or other body part.
- Check that you do not have writing on your hand, arm or other body part – if you do, you must inform an Invigilator immediately.
- Alert an Invigilator immediately if you find any unauthorised item upon you during the examination.

Any students found with non-permitted items upon their person during the examination, or who fail to comply with Examination rules may be subject to the Student Conduct procedures.