1VGLA: DETERMINANTS PRACTICE QUESTIONS

The following questions relate to Chapter 7, Determinants. Questions are ranked in difficulty from A (basic) to C (challenging). For questions with multiple cases a fully justified solution will be given for at least 1 case.

(A) Question 1. Find the number of inversions in the following permutations:
(a) $\sigma = [5,4,3,2,1]$, (b) $\sigma = [1,2,3,4,5]$,
(c) $\sigma = [5,3,2,4,6,1]$, (d) $\sigma = [2,1,3,4]$.

(A) Question 2. A matrix $\mathbf{A} = [a_{ij}]$ in $\mathcal{M}_{22}(\mathbb{R})$ is such that $a_{ij} = 2i - 5j$ for $i, j = 1, 2$. Find $\det(\mathbf{A})$.

(A) Question 3. A matrix $\mathbf{A} = [a_{ij}]$ in $\mathcal{M}_{22}(\mathbb{R})$ is such that $a_{ii} = 1$ for $i = 1, 2$ and $a_{ij} = -3$ if $i \neq j$ and $i, j = 1, 2$. Find $\det(\mathbf{A})$.

(A) Question 4. Justify if you can/cannot compute the determinant of $\mathbf{A} \in \mathcal{M}_{23}(\mathbb{R})$.

(A) Question 5. Evaluate $|\mathbf{A}|$ for each of the following matrices $\mathbf{A}$:

(a) $\mathbf{A} = \begin{pmatrix} 5 & 7 & 9 \\ 1 & 3 & 2 \\ 3 & 9 & 6 \end{pmatrix}$, (b) $\mathbf{A} = \begin{pmatrix} 5 & 3 & 5 \\ 2 & -7 & 3 \\ 5 & 3 & 5 \end{pmatrix}$,

(c) $\mathbf{A} = \begin{pmatrix} 9 & 5 & 0 \\ 4 & -3 & -7 \\ 19 & 11 & 0 \end{pmatrix}$, (d) $\mathbf{A} = \begin{pmatrix} 3 & 0 & 7 \\ 5 & -3 & 2 \\ -2 & 0 & 3 \end{pmatrix}$,

(e) $\mathbf{A} = \begin{pmatrix} 3 & 5 & -7 & 11 \\ -4 & -3 & 4 & -13 \\ 0 & -2 & 0 & 0 \\ 0 & -5 & 3 & 0 \end{pmatrix}$, (f) $\mathbf{A} = \begin{pmatrix} 2 & -1 & 0 & 3 \\ 5 & 0 & 7 & 0 \\ 0 & 0 & 1 & 0 \\ -2 & -1 & -11 & 5 \end{pmatrix}$,

(A) Question 6. Calculate the determinant, $|\mathbf{A}|$, of the matrices $\mathbf{A}$ given below and explain why the calculation is true. *Refer to the property of determinants you are using (no need to reference the exact number of the result in the notes).*

(a) $\mathbf{A} = \begin{pmatrix} 2 & -3 & 5 \\ 0 & 1 & -7 \\ 0 & 0 & -4 \end{pmatrix}$, (b) $\mathbf{A} = \begin{pmatrix} 3 & -5 & 3 \\ 4 & -7 & 4 \\ 3 & 8 & 3 \end{pmatrix}$.

(A) Question 7. Calculate the determinant, $|\mathbf{A}|$ of the matrix $\mathbf{A}$ given below by an expansion along a row or column of your choice, giving details of your calculation.

$$\mathbf{A} = \begin{pmatrix} -2 & -3 & 0 \\ 3 & -5 & -4 \\ -3 & -2 & 1 \end{pmatrix}.$$

(A) Question 8. A diagonal matrix $\mathbf{D} = [d_{ij}]$ in $\mathcal{M}_{44}(\mathbb{R})$ has integer entries and $|\mathbf{D}| = 12$. Find all possible values of d_{22} and d_{44} if $d_{11} = 3$ and $d_{33} = 1$.

(A) Question 9. Let $\mathbf{A}, \mathbf{B} \in \mathcal{M}_{nn}(\mathbb{R})$. Prove, using determinants, that $\mathbf{A} \cdot \mathbf{B}$ is invertible iff $\mathbf{A}$ and $\mathbf{B}$ are invertible.

(A) Question 10. Let $\mathbf{A}$ and $\mathbf{B}$ be invertible matrices in $\mathcal{M}_{nn}(\mathbb{R})$. Determine the value of

$$\det(\mathbf{A} \cdot \mathbf{B} \cdot \mathbf{A}^{-1} \cdot \mathbf{B}^{-1}).$$

Recall, matrix multiplication in $\mathcal{M}_{nn}(\mathbb{R})$ is not commutative.

(A) Question 11. $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are matrices in $\mathcal{M}_{nn}(\mathbb{R})$ such that $\det(\mathbf{A}) = 2$, $\det(\mathbf{B}) = -\frac{1}{2}$ and $\det(\mathbf{C}) = \frac{1}{4}$. Determine the value of

(a) $\det(\mathbf{C}^3 \cdot \mathbf{B}^{-1} \cdot \mathbf{A}^2 \cdot (\mathbf{B}^T)^2 \cdot \mathbf{C}^T \cdot (\mathbf{A}^{-1})^3)$,
(b) $\det((\mathbf{A}^{-1})^2 \cdot (\mathbf{B}^{-1})^2 \cdot \mathbf{C}^T \cdot (\mathbf{A}^T)^2 \cdot \mathbf{B}^T \cdot (\mathbf{C}^{-1})^2)$.

(A) Question 12. One expression for a certain invertible matrix $\mathbf{A}$ in $\mathcal{M}_{33}(\mathbb{R})$ as a product of elementary matrices is given below. Find the value of $|\mathbf{A}|$ without calculating the elements of $\mathbf{A}$.

(a) $\mathbf{A} = \mathbf{E}_{[2,-4]} \cdot \mathbf{E}_{(2,3,-3)} \cdot \mathbf{E}_{(1,3,2)} \cdot \mathbf{E}_{[3,\frac{1}{2}]} \cdot \mathbf{E}_{(2,3)} \cdot \mathbf{E}_{(2,1,-4)} \cdot \mathbf{E}_{[1,-1]} \cdot \mathbf{E}_{(1,3)}$,
(b) $\mathbf{A} = \mathbf{E}_{(2,1,\sqrt{5})} \cdot \mathbf{E}_{(2,3)} \cdot \mathbf{E}_{[3,\sqrt{3}]} \cdot \mathbf{E}_{(2,3,\frac{5}{11})} \cdot \mathbf{E}_{[2,\frac{7}{\sqrt{3}}]} \cdot \mathbf{E}_{(2,11,-\sqrt{11})} \cdot \mathbf{E}_{[3,-2]} \cdot \mathbf{E}_{(1,3,11)}$.

Recall that

- $\boldsymbol{E}_{[i,\lambda]}$ *is the elementary matrix which multiplies row i by λ;*
- $\boldsymbol{E}_{(i,j)}$ *is the elementary matrix which swaps rows i and j; and*
- $\boldsymbol{E}_{(i,j,\lambda)}$ *is the elementary matrix which adds λ times row j to row i.*

(A) Question 13. A matrix $\mathbf{A} \in \mathcal{M}_{nn}(\mathbb{R})$ is called an *orthogonal* matrix if

$$\mathbf{A} \cdot \mathbf{A}^T = \mathbf{A}^T \cdot \mathbf{A} = \mathbf{I}_n.$$

List all elements of the set

$$\mathcal{S} = \{x \in \mathbb{R} \ : \ n \in \mathbb{N},\ \mathbf{A} \in \mathcal{M}_{nn}(\mathbb{R}) \text{ is an orthogonal matrix and } |\mathbf{A}| = x\}.$$

(A) Question 14. For each of the following invertible matrices $\mathbf{A} \in \mathcal{M}_{33}(\mathbb{R})$, find

(i) the cofactor matrix $\mathbf{C}(\mathbf{A})$ of $\mathbf{A}$,
(ii) the adjoint matrix $\operatorname{adj}(\mathbf{A})$ of $\mathbf{A}$,
(iii) $\det(\mathbf{A})$,
(iv) the inverse $\mathbf{A}^{-1}$ of $\mathbf{A}$,
(v) the solution of the system of linear equations $\mathbf{A} \cdot \mathbf{x} = \mathbf{b}$, for $\mathbf{x} = (x_1\ x_2\ x_3)^T$.

(a) $\mathbf{A} = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 4 \\ 4 \\ 7 \end{pmatrix}$,

(b) $\mathbf{A} = \begin{pmatrix} 1 & -2 & -1 \\ 1 & -5 & -6 \\ 5 & -4 & 6 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 3 \\ 7 \\ 8 \end{pmatrix}$,

(c) $\mathbf{A} = \begin{pmatrix} -9 & -1 & -9 \\ -10 & -8 & -4 \\ 3 & 4 & 1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$.

(A) Question 15. Use Cramer's rule to solve the systems of simultaneous linear equations

(a) $\begin{cases} 11x_1 + 61x_2 = 2 \\ 19x_1 - 31x_2 = -3, \end{cases}$ (b) $\begin{cases} 4x_1 - 6x_2 = 2 \\ -5x_1 + 9x_2 = 5, \end{cases}$

(c) $\begin{cases} -x_1 - x_2 = -2 \\ -2x_1 = 3, \end{cases}$ (d) $\begin{cases} -5x_1 - x_2 = -5 \\ 5x_1 - 3x_2 = 0. \end{cases}$

(A) Question 16. Use Cramer's rule to find the coordinates of the point of intersection of the planes with equations

$$\text{(a)}\ \begin{cases} -2x \phantom{{}-y} + 2z = 2 \\ -2x - y + 3z = 1 \\ - 4z = -1, \end{cases} \qquad \text{(b)}\ \begin{cases} 4x - 2y + 2z = 1 \\ 3x - 3y - 5z = -1 \\ 3x - 2y + z = 0. \end{cases}$$

(A) Question 17. Determine whether or not the following homogeneous systems have non-trivial solutions. *You do not need to find these solutions if they exist!*

$$\text{(a)}\ \begin{cases} -3x + 3y - 4z = 0 \\ - 3y = 0 \\ -3x + 2y - 4z = 0, \end{cases} \qquad \text{(b)}\ \begin{cases} 5x - 4y - z = 0 \\ -5x + 6y - z = 0 \\ -4x + 9y - 5z = 0. \end{cases}$$

(A) Question 18. Is it true of false that a homogeneous system of n linear equations, say, $\mathbf{A} \cdot \mathbf{x} = \mathbf{0}$, has non-trivial solutions if and only if $|\mathbf{A}| = 0$?

(B) Question 19. For $\sigma \in S_n$ with $N(\sigma) > 0$, establish that there exists $i \in \Sigma_n$ such that $\sigma(i) > \sigma(i+1)$.

(B) Question 20. Evaluate $\det(\mathbf{A})$ for each of the following matrices $\mathbf{A}$:

$$\text{(a)}\ \mathbf{A} = \begin{pmatrix} 1 & 2 & -1 & 5 \\ -2 & -2 & 5 & -3 \\ 3 & 0 & -11 & -2 \\ -1 & 4 & 3 & 2 \end{pmatrix}, \quad \text{(b)}\ \mathbf{A} = \begin{pmatrix} 3 & 1 & 0 & 0 \\ 1 & 3 & 1 & 0 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 1 & 3 \end{pmatrix}.$$

(B) Question 21. For each of the following matrices $\mathbf{A} \in \mathcal{M}_{22}(\mathbb{R})$, find all real values of x, if any, for which $|\mathbf{A}| = 0$:

$$\text{(a)}\ \mathbf{A} = \begin{pmatrix} 3x+4 & -3 \\ 1 & 2x-1 \end{pmatrix}, \qquad \text{(b)}\ \mathbf{A} = \begin{pmatrix} x & x-1 \\ 1 & x^2+1 \end{pmatrix},$$

(B) Question 22. If a, b, c are distinct real numbers, show that

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = (a-b)(b-c)(c-a).$$

This matrix is called a Vandermonde matrix and can be used to prove results related to polynomial interpolation.

(B) Question 23. For each of the following matrices $\mathbf{A} \in \mathcal{M}_{33}(\mathbb{R})$, find all real numbers $\lambda \in \mathbb{C}$ such that $|\mathbf{A} - \lambda \mathbf{I}_3| = 0$.

$$\text{(a)}\ \mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 6 & -6 \\ 1 & 2 & -1 \end{pmatrix}, \qquad \text{(b)}\ \mathbf{A} = \begin{pmatrix} 1 & 2 & -3 \\ 0 & -4 & 0 \\ 7 & 8 & 9 \end{pmatrix},$$

(B) Question 24. Let $\mathbf{A}, \mathbf{P} \in \mathcal{M}_{nn}(\mathbb{R})$ with $\mathbf{P}$ invertible. Prove that

(a) $\det(\mathbf{P}^{-1} \cdot \mathbf{A} \cdot \mathbf{P}) = \det(\mathbf{A})$,

(b) $\det(\mathbf{P}^{-1} \cdot \mathbf{A} \cdot \mathbf{P} - \lambda \mathbf{I}_n) = \det(\mathbf{A} - \lambda \mathbf{I}_n)$ for any $\lambda (\neq 0) \in \mathbb{R}$. *It may be helpful to write $\lambda \boldsymbol{I}_n$ as $\boldsymbol{P}^{-1} \cdot (\lambda \boldsymbol{I}_n) \cdot \boldsymbol{P}$.*

(c) If

$$\mathbf{A} = \begin{pmatrix} b+c & c-a & b-a \\ c-b & c+a & a-b \\ b-c & a-c & a+b \end{pmatrix} \text{ and } \mathbf{P} = \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}$$

find $\mathbf{P}^{-1}$ and then directly calculate $\mathbf{P}^{-1} \cdot \mathbf{A} \cdot \mathbf{P}$. Hence find $|\mathbf{A}|$.

(B) Question 25. (a) If $\mathbf{A} \in \mathcal{M}_{nn}(\mathbb{R})$, find a set $\mathcal{S}$ of cardinality less than 4 which contains the possible real numbers $\det(\mathbf{A})$ in each of the following cases?

(i) $\mathbf{A}^4 = \mathbf{I}_n$,
(ii) $\mathbf{A}^3 = \mathbf{A}$,
(iii) $\mathbf{A}^2 = -\mathbf{A}$ with n odd.
(iv) $\mathbf{A}^2 = -\mathbf{A}$ with n even.

(b) Does there exist a matrix $\mathbf{A} \in \mathcal{M}_{33}(\mathbb{R})$ such that $\mathbf{A}^2 = -\mathbf{I}_3$? Justify your answer.

(B) Question 26. Let $\mathbf{A}, \mathbf{B} \in \mathcal{M}_{nn}(\mathbb{R})$. Prove that

(a) $\det(\mathbf{A} \cdot \mathbf{B}) = \det(\mathbf{B} \cdot \mathbf{A})$,
(b) $\det(\mathbf{A}^T + \mathbf{B}) = \det(\mathbf{A} + \mathbf{B}^T)$.

Recall that matrix multiplication in $\mathcal{M}_{nn}(\mathbb{R})$ is not commutative and $(\boldsymbol{P}+\boldsymbol{Q})^T = \boldsymbol{P}^T + \boldsymbol{Q}^T$.

(B) Question 27. Is the following argument acceptable as an answer to the following question which was (apparently) set on an examination paper?

Question: "Prove that if $\mathbf{A}$ is a skew-symmetric matrix in $\mathcal{M}_{nn}(\mathbb{R})$ with n an *odd* positive integer, then $\det(\mathbf{A}) = 0$."

(Recall that $\mathbf{A}$ is skew-symmetric if $\mathbf{A}^T = -\mathbf{A}$.)

Answer: "Let

$$\mathbf{A} = \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix}$$

where $a, b, c \in \mathbb{R}$. Then by expanding elements of row 1 with cofactors,

$$\begin{aligned} \det(\mathbf{A}) &= \begin{vmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{vmatrix} = (-a)\begin{vmatrix} -a & c \\ -b & 0 \end{vmatrix} + b\begin{vmatrix} -a & 0 \\ -b & -c \end{vmatrix} \\ &= -abc + abc \\ &= 0.'' \end{aligned}$$

If you decide that the argument above is not acceptable as an answer, state why not and then give an argument which proves that the statement in the question is true.

(B) Question 28. Using Cramer's rule, find

(a) x such that $\begin{cases} w + x + y + z = 3 \\ 7w + 3x - y + z = 1 \\ 2w - 2x - 3y + 3z = 4 \\ w + x + y + 8z = 7 \end{cases}$

(b) z such that $\begin{cases} -w - 2x + y + z = 3 \\ w - x + 2y - 2z = -1 \\ -2w - x + \quad + z = 1 \\ 2w + 2x + \quad - 3z = 0 \end{cases}$

(B) Question 29. Read the question below and then study the given 'suggested' answer with a view to deciding if this answer is correct. If you conclude that the 'suggested' answer is wrong, then produce a correct answer.

Question: Decide whether the following statement is true or false. "If $\mathbf{A}$ and $\mathbf{B}$ are any two matrices in $\mathcal{M}_{22}(\mathbb{R})$, then

$$\det(\mathbf{A} + \mathbf{B}) = \det(\mathbf{A}) + \det(\mathbf{B}).''$$

Give a proof if you conclude that the statement is true or a counterexample if you think it is false.

Answer: The statement is true. Let

$$\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \text{ and } \mathbf{B} = \begin{pmatrix} 1 & 3 \\ 3 & 4 \end{pmatrix}.$$

Then $\det(\mathbf{A}) = 2 - 1 = 1$, $\det(\mathbf{B}) = 4 - 9 = -5$ and hence, $\det(\mathbf{A}) + \det(\mathbf{B}) = -4$. Also

$$\mathbf{A} + \mathbf{B} = \begin{pmatrix} 2 & 4 \\ 4 & 6 \end{pmatrix} \Longrightarrow \det(\mathbf{A} + \mathbf{B}) = 12 - 16 = -4.$$

Hence $\det(\mathbf{A} + \mathbf{B}) = \det(\mathbf{A}) + \det(\mathbf{B})$ for all $\mathbf{A}, \mathbf{B} \in \mathcal{M}_{22}(\mathbb{R})$.

(C) Question 30. A matrix $\mathbf{A} = [a_{ij}] \in \mathcal{M}_{nn}(\mathbb{R})$ with $n \geq 2$ is such that $a_{ij} = 0$ if $i + j < n + 1$. Establish that $|\mathbf{A}| = (-1)^k a_{1,n} a_{2,n-1} \ldots a_{n-1,2} a_{n,1}$ and $k = \frac{n(n-1)}{2}$.

(C) Question 31. Establish that if $\mathbf{A} \in \mathcal{M}_{33}(\mathbb{R})$ is the matrix

$$\mathbf{A} = \begin{pmatrix} b + c & a^2 & a \\ c + a & b^2 & b \\ a + b & c^2 & c \end{pmatrix}$$

then $|\mathbf{A}| = -(a - b)(b - c)(c - a)(a + b + c)$. Deduce the conditions on a, b and c under which the system

$$\begin{array}{ccccccc} (b + c)x & + & a^2 y & + & az & = & a^3, \\ (c + a)x & + & b^2 y & + & bz & = & b^3, \\ (a + b)x & + & c^2 y & + & cz & = & c^3 \end{array}$$

of three linear equations in three unknowns possesses a unique solution.

(C) Question 32. For some fixed $k \in \{1, \ldots, n\}$, let $\mathbf{A} = [a_{ij}], \mathbf{B} = [b_{ij}], \mathbf{C} = [c_{ij}] \in \mathcal{M}_{nn}(\mathbb{R})$ be given by

$$b_{ij} = \begin{cases} a_{ij}, & j \neq k \\ b_{ik}, & j = k, \end{cases} \qquad c_{ij} = \begin{cases} a_{ij} & j \neq k \\ c_{ik} & j = k, \end{cases}$$

and $a_{ik} = b_{ik} + c_{ik}$, for $i = 1, \ldots, n$. Prove that $|\mathbf{A}| = |\mathbf{B}| + |\mathbf{C}|$.