University of Birmingham
School of Mathematics

Real Analysis – Integration – Spring 2025

Problem Sheet 6 and Assignment 3

Model Solutions

Instructions: Your solution to the Assignment Question (**AQ1**) must be submitted via Assignments on the LC Real Analysis Canvas page before the following time:

Due 17:00 Wednesday 26 February 2025

You are strongly encouraged to attempt all of the remaining formative questions, and as many of the extra questions as you can, to prepare for the final exam, but only the Assignment Question should be submitted to Canvas. Model solutions will only be released for the Assignment Question AQ1 and Questions Q1-Q4.

Important: Late submissions will be penalised at a rate of 5% per day late up until exactly two days after the submission deadline, at which point the model solutions will be released and the Assignment will be closed to further submissions on Canvas. Your Assignment Question solutions must be submitted as a single PDF file. You may upload newer versions, BUT only the most recent upload will be viewed and graded. In particular, this means that subsequent uploads will need to contain ALL of your work, not just the parts which have changed. Moreover, if you upload a new version after the deadline, then your submission will be counted as late and the late penalty will be applied REGARDLESS of whether an older version was submitted before the deadline. In the interest of fairness to all students and staff, there will be no exceptions to these rules. All of this and more is explained in detail on the Submission of Continuous Assessment page in the Student Handbook on Canvas.

Assignment Question

AQ1. (a) Let X denote a subset of $\mathbb{R}$ and suppose that $-\infty < a < b < \infty$:

(i) Define what it means for X to be a bounded set.

(ii) Define what it means for $f : X \to \mathbb{R}$ to be a bounded function.

(iii) Define what it means for P to be a partition of the interval $[a, b]$.

(iv) Define what it means for a bounded function $f : [a, b] \to [0, \infty)$ to be integrable.

(v) State Riemann's Criterion for Integrability.

(b) Consider the function $f : [1, 6\pi] \to [-9, 7000]$ defined by[1]

$$f(x) = \begin{cases} x^2, & x < 2\pi; \\ 3 + \sin(x), & x \in [2\pi, 4\pi]; \\ e^{-x}, & x > 4\pi. \end{cases}$$

(i) Suppose that $x_0 = 1$, $x_1 = 2$ and $x_2 = 4\pi$. For each $i \in \{1, 2\}$, find the interval I_i such that $\{f(x) : x \in [x_{i-1}, x_i]\} = I_i$. Note that here an interval refers to any of $[a, b]$, $(a, b]$, $[a, b)$, or (a, b), where $-\infty < a < b < \infty$.

(ii) Suppose that $P = \{1, 2, 4\pi, 6\pi\}$. Calculate the Riemann–Darboux sum $L(f, P)$.

(iii) Suppose that $Q = \{1, 3, 5, 5\pi, 6\pi\}$ and $R = \{1, 3, 2\pi, 6\pi\}$. Calculate $U(f, Q \cup R)$.

(c) Use Riemann's Criterion to prove that $f : [12, 22] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 7, & 12 \leq x \leq 15; \\ 4, & 15 < x < 22; \\ 15, & x = 22, \end{cases}$$

is integrable.

The Riemann–Darboux sums calculator (Spring Week 1 Materials on Canvas) can be used for this question, although it is not needed, and you must include a screenshot of any calculation used to obtain your answers.

[1] Correction (updated at 18:00 on 7 February 2025): The defintion of f was updated to require $x \in [2\pi, 4\pi]$ and $x > 4\pi$ (instead of $x \in [2\pi, 4\pi)$ and $x \geq 4\pi$).

Questions

Q1. (a) Sketch the graph of $f : [0,4] \to \mathbb{R}$, $f(x) = \sqrt{x}$ and highlight the area covered by the difference $U(f,P) - L(f,P)$ for the partition $P = \{0,1,2,3,4\}$.

(b) Use Riemann's Criterion to prove each of the functions below are integrable:

(i) $f : [0,3] \to [0,\infty)$, $f(x) = x^2$

(ii) $f : [2,4] \to [0,100]$, $f(x) = \begin{cases} 5, & x < 3; \\ 100, & x = 3; \\ 3, & x > 3. \end{cases}$

You may use the results stated in **Q4**(b) on Problem Sheet 5 to answer (i).

Solution. (a) Figure 1 shows the graph of $f : [0,4] \to \mathbb{R}$, $f(x) = \sqrt{x}$ with the area covered by the difference $U(f,P) - L(f,P)$ for the partition $P = \{0,1,2,3,4\}$ highlighted in dark blue. This was created using the Riemann–Darboux Sums Calculator.

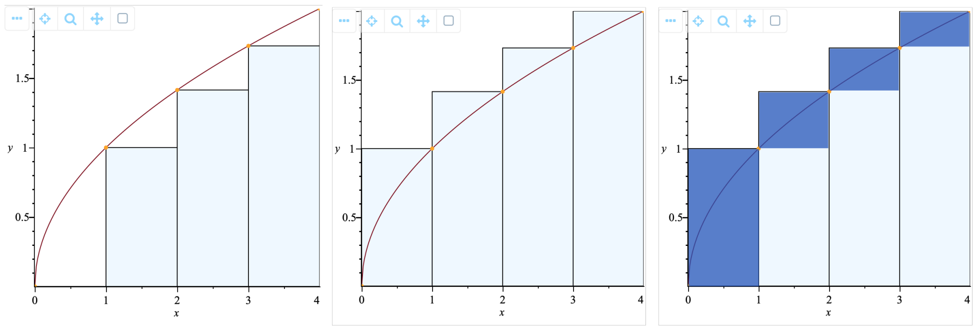

Figure 1. Three graphs of $f : [0,4] \to \mathbb{R}$, $f(x) = \sqrt{x}$ showing the areas covered by $L(f,P)$ in light blue, $U(f,P)$ in light blue, and $U(f,P) - L(f,P)$ in dark blue.

(b)(i) $f : [0,3] \to [0,\infty)$, $f(x) = x^2$: Let $\epsilon > 0$, and for each $n \in \mathbb{N}$, consider the partition P_n of $[0,3]$ into n subintervals of equal width. Using the values for $L(f,P_n)$ and $U(f,P_n)$ in **Q4**(b), we have

$$\begin{aligned} U(f,P_n) - L(f,P_n) &= \frac{3^3}{6n^3} n(n+1)(2n+1) - \frac{3^3}{6n^3}(n-1)n(2n-1) \\ &= \frac{9}{2n^2}[(n+1)(2n+1) - (n-1)(2n-1)] \\ &= \frac{9}{2n^2}[(2n^2+3n+1) - (2n^2-3n+1)] \\ &= \frac{9}{2n^2} 6n \\ &= \frac{27}{n}. \end{aligned}$$

To prove Riemann's Criterion, we need to have $\frac{27}{n} < \epsilon$ or equivalently $n > \frac{27}{\epsilon}$. Therefore, we now choose $N \in \mathbb{N}$ with $N > \frac{27}{\epsilon}$, so the above calculations give

$$U(f,P_N) - L(f,P_N) = \tfrac{27}{N} < \epsilon.$$

This proves that f is integrable by Riemann's Criterion.

(b)(ii) $f : [2,4] \to [0,100]$, $f(x) = \begin{cases} 5, & x < 3 \\ 100, & x = 3 \\ 3, & x > 3 \end{cases}$

Let $\epsilon > 0$, and for each $\delta \in (0, \frac{1}{10})$, consider the partition $P_\delta := \{2, 3-\delta, 3+\delta, 4\}$. In the notation $m_i := \inf\{f(x) : x \in [x_{i-1}, x_i]\}$ and $M_i := \inf\{f(x) : x \in [x_{i-1}, x_i]\}$, where $\{x_0, x_1, x_2, x_3\} := \{2, 3-\delta, 3+\delta, 4\}$, we find $m_1 = 5$, $m_2 = 3$, $m_3 = 3$, whilst $M_1 = 5$, $M_2 = 100$, $M_3 = 3$, so we have

$$\begin{aligned} L(f, P_\delta) &= \sum_{i=1}^{3} m_i(x_i - x_{i-1}) \\ &= 5(1-\delta) + 3(2\delta) + 3(1-\delta) = 8 - 2\delta \end{aligned}$$

and

$$\begin{aligned} U(f, P_\delta) &= \sum_{i=1}^{3} M_i(x_i - x_{i-1}) \\ &= 5(1-\delta) + 100(2\delta) + 3(1-\delta) = 8 + 192\delta. \end{aligned}$$

This gives

$$U(f, P_\delta) - L(f, P_\delta) = (8 + 192\delta) - (8 - 2\delta) = 194\delta.$$

To prove Riemann's Criterion, we need to have $194\delta < \epsilon$ or equivalently $\delta < \frac{\epsilon}{194}$. Therefore, we now choose $\delta_0 \in (0, \frac{1}{10})$ with $\delta_0 < \frac{\epsilon}{194}$, so the above calculations give

$$U(f, P_{\delta_0}) - L(f, P_{\delta_0}) = 194\delta_0 < \epsilon.$$

This proves that f is integrable by Riemann's Criterion. $\square$

Q2. (a) State Riemann's Criterion for integrability.

(b) For each $n \in \mathbb{N}$, let P_n denote the partition of $[a, b]$ into n subintervals of equal width. If $f : [a, b] \to [0, \infty)$ is monotonic increasing, then we have seen that

$$U(f, P_n) - L(f, P_n) = (f(b) - f(a))(b-a)\tfrac{1}{n}. \tag{1}$$

(i) Now prove that if $f : [a, b] \to [0, \infty)$ is monotonic decreasing, then

$$U(f, P_n) - L(f, P_n) = (f(a) - f(b))(b-a)\tfrac{1}{n}. \tag{2}$$

(ii) Use Riemann's Criterion to prove that $f : [-2, 5] \to \mathbb{R}$ given by $f(x) = x^2 - 4x + 9$ for all $x \in [-2, 5]$ is integrable. Here you may combine (1) and (2) with results from the Lecture Notes, but you are NOT allowed to use that monotonic functions and continuous functions are integrable.

(c) Use Riemann's Criterion to prove directly that $g : [2, 4] \to \mathbb{R}$ given by

$$g(x) := \lceil x \rceil := \min\{n \in \mathbb{N} : n \geq x\} \text{ for all } x \in [2, 4]$$

is integrable. Here you are NOT allowed to use any results from (b) and you are NOT allowed to use that monotonic functions and continuous functions are integrable.

Solution. (a) It is enough here to *either* quote the following theorem, which states the equivalence of Riemann's Criterion with integrability, *or* to state property (2) from the theorem, which is Riemann's Criterion specifically.

Theorem (Riemann's Criterion)**.** The following properties are equivalent for any bounded function $f : [a, b] \to \mathbb{R}$:

(1) The function f is integrable.
(2) For each $\epsilon > 0$, there exists a partition P of $[a, b]$ such that $U(f, P) - L(f, P) < \epsilon$.

(b)(i) For each $n \in \mathbb{N}$, consider the partition $P_n := \{x_0, x_1, \ldots, x_n\}$ of $[a, b]$ into n subintervals of width equal to $(b-a)/n$. For each $i \in \{1, \ldots, n\}$, since f is monotonic decreasing, we have

$$\begin{aligned} m_i &:= \inf\{f(x) : x \in [x_{i-1}, x_i]\} = f(x_i), \\ M_i &:= \sup\{f(x) : x \in [x_{i-1}, x_i]\} = f(x_{i-1}). \end{aligned}$$

Together, we have

$$
\begin{aligned}
U(f,P_n) - L(f,P_n) &= \sum_{i=1}^{n}(M_i - m_i)(x_i - x_{i-1}) \\
&= \sum_{i=1}^{n}(f(x_{i-1}) - f(x_i))\tfrac{b-a}{n} \\
&= [(f(x_0) - f(x_1)) + (f(x_1) - f(x_2)) + \ldots + (f(x_{n-1}) - f(x_n))]\tfrac{b-a}{n} \\
&= [f(x_0) + (f(x_1) - f(x_1)) + \ldots + (f(x_{n-1}) - f(x_{n-1})) + -f(x_n)]\tfrac{b-a}{n} \\
&= (f(x_0) - f(x_n))\tfrac{b-a}{n} \\
&= (f(a) - f(b))(b-a)\tfrac{1}{n},
\end{aligned}
$$

where in the fourth line we have simply reordered the terms in the sum to show how they can be grouped into pairs that cancel. Alternatively, the telescoping sum can also be simplified using summation notation, setting $j = i - 1$, and writing

$$\sum_{i=1}^{n}(f(x_{i-1}) - f(x_i))\tfrac{b-a}{n} = \left(\sum_{j=0}^{n-1} f(x_j) - \sum_{i=1}^{n} f(x_i)\right)\tfrac{b-a}{n}.$$

(b)(ii) The function f has a stationary point at $x = 2$ because $f'(x) = 0$ implies that $2x - 4 = 0$, hence $x = 2$. Now define the restricted functions

$$f_1 : [-2,2] \to [0,\infty),\ f_1(x) := f(x),\ x \in [-2,2]$$

and

$$f_2 : [2,5] \to [0,\infty),\ f_2(x) := f(x),\ x \in [-2,2].$$

Observe that f_1 is monotonic decreasing, as $f_1'(x) = 2x - 4 \leq 0$ for all $x \in [-2,2]$. For each $n \in \mathbb{N}$, let $P_n^{(1)}$ denote the partition of $[-2,2]$ into n subintervals of equal width, so (2) gives

$$U(f_1,P_n^{(1)}) - L(f_1,P_n^{(1)}) = (f(-2) - f(2))(2 - -2)\tfrac{1}{n} = (21-5)(4)\tfrac{1}{n} = \tfrac{64}{n}.$$

Observe that f_2 is monotonic increasing, as $f_2'(x) = 2x - 4 \geq 0$ for all $x \in [2,5]$. For each $n \in \mathbb{N}$, let $P_n^{(2)}$ denote the partition of $[2,5]$ into n subintervals of equal width, so (1) gives

$$U(f_2,P_n^{(2)}) - L(f_2,P_n^{(2)}) = (f(5) - f(2))(5-2)\tfrac{1}{n} = (14-5)(3)\tfrac{1}{n} = \tfrac{27}{n}.$$

Now consider the merged partition $Q_n := P_n^{(1)} \cup P_n^{(2)}$ and observe that

$$
\begin{aligned}
U(f,Q_n) - L(f,Q_n) &= [U(f_1,P_n^{(1)}) + U(f_2,P_n^{(2)})] - [L(f_1,P_n^{(1)}) + L(f_2,P_n^{(2)})] \\
&= [U(f_1,P_n^{(1)}) - L(f_1,P_n^{(1)})] + [U(f_2,P_n^{(2)}) - L(f_2,P_n^{(2)})] \\
&= \tfrac{64}{n} + \tfrac{27}{n} = \tfrac{91}{n}.
\end{aligned}
$$

Finally, let $\epsilon > 0$. To prove Riemann's Criterion, we need $\frac{91}{n} < \epsilon$, or equivalently $n > \frac{91}{\epsilon}$. Therefore, we now choose an *integer* N such that $N > \frac{91}{\epsilon}$, so the above calculations give

$$U(f,Q_N) - L(f,Q_N) = \tfrac{91}{N} < \epsilon.$$

Hence f is integrable by Riemann's Criterion.

Alternatively, we can choose integers $N_1 > \frac{64}{\epsilon}$ and $N_2 > \frac{27}{\epsilon}$ to prove separately that f_1 and f_2 are integrable by Riemann's Criterion. This means that f is integrable on $[-2,2]$ and on $[2,5]$, so the restriction and extension property of the integral in Theorem 4.2.1(1) of the Lecture Notes implies that f is integrable (on $[-2,5] = [-2,2] \cup [2,5]$).

(c) Observe that

$$g(x) = \begin{cases} 2, & x = 2; \\ 3, & x \in (2,3]; \\ 4, & x \in (3,4]. \end{cases}$$

This function is not continuous at $x = 2$ and $x = 3$, which suggests we consider the following P_δ-type partition of $[2, 4]$. Let $\epsilon > 0$, and for each $\delta \in (0, \frac{1}{10})$, consider the partition

$$P_\delta := \{2, 2 + \delta, 3, 3 + \delta, 4\} = \{x_0, x_1, x_2, x_3, x_4\}.$$

(There are many choices of partition which work equally well here, e.g. setting $x_2 = 3 - \delta$ works, but a δ-window to the left of 3 is not actually needed as g is continuous from the left.) Observe that

$$\begin{aligned}
U(g, P_\delta) - L(g, P_\delta) &= \sum_{i=1}^{4} (M_i - m_i)(x_i - x_{i-1}) \\
&= (M_1 - m_1)((2 + \delta) - 2) + (M_2 - m_2)(3 - (2 + \delta)) \\
&\quad + (M_3 - m_3)((3 + \delta) - 3) + (M_4 - m_4)(4 - (3 + \delta)) \\
&= (3 - 2)(\delta) + (3 - 3)(1 - \delta) + (4 - 3)(\delta) + (4 - 4)(1 - \delta)) \\
&= 2\delta.
\end{aligned}$$

To prove Riemann's Criterion, we need to have $2\delta < \epsilon$, or equivalently $\delta < \frac{\epsilon}{2}$. Therefore, we now choose $\delta_0 \in (0, \frac{1}{10})$ with $\delta_0 < \frac{\epsilon}{2}$, so the above calculations give

$$U(g, P_{\delta_0}) - L(g, P_{\delta_0}) = 2\delta_0 < \epsilon.$$

This proves that g is integrable by Riemann's Criterion. □

Q3. Suppose that $f : [a, b] \to [0, \infty)$ is bounded and integrable, where $-\infty < a < b < \infty$:

(a) Prove that if $\alpha \geq 0$ and P is a partition of $[a, b]$, then

$$U(\alpha f, P) = \alpha U(f, P) \quad \text{and} \quad L(\alpha f, P) = \alpha L(f, P).$$

(b) Use Riemann's Criterion and (a) to prove that αf is integrable.

(c) Explain, without working hard, why the integrability of αf implies that

$$\int_a^b (\alpha f) = \sup\{L(\alpha f, P) : P \text{ is a partition of } [a, b]\}.$$

(d) Use (a) and (c) to conclude that $\int_a^b (\alpha f) = \alpha(\int_a^b f)$.

Solution. (a) Let $P = \{x_0, \ldots, x_n\}$ denote a partition of $[a, b]$ and define

$$\begin{aligned}
M_i &:= \sup\{f(x) : x \in [x_{i-1}, x_i]\} \\
M_{i,\alpha} &:= \sup\{(\alpha f)(x) : x \in [x_{i-1}, x_i]\} \\
m_i &:= \inf\{f(x) : x \in [x_{i-1}, x_i]\} \\
m_{i,\alpha} &:= \inf\{(\alpha f)(x) : x \in [x_{i-1}, x_i]\}
\end{aligned}$$

for each $i \in \{1, \ldots, n\}$. Observe that $M_{i,\alpha} = \alpha M_i$ and $m_{i,\alpha} = \alpha m_i$ for each $i \in \{1, \ldots, n\}$, hence

$$\begin{aligned}
U(\alpha f, P) &= \sum_{i=1}^{n} M_{i,\alpha}(x_i - x_{i-1}) \\
&= \alpha \sum_{i=1}^{n} M_i(x_i - x_{i-1}) = \alpha U(f, P)
\end{aligned}$$

and

$$\begin{aligned}
L(\alpha f, P) &= \sum_{i=1}^{n} m_{i,\alpha}(x_i - x_{i-1}) \\
&= \alpha \sum_{i=1}^{n} m_i(x_i - x_{i-1}) = \alpha L(f, P),
\end{aligned}$$

as required.

(b) The function f is integrable, so Riemann's Criterion implies that for each $\delta > 0$, there exists a partition P_δ of $[a,b]$ such that $U(f,P_\delta) - L(f,P_\delta) < \delta$. Moreover, by (a), we then have

$$U(\alpha f, P_\delta) - L(\alpha f, P_\delta) = \alpha U(f,P_\delta) - \alpha L(f,P_\delta) < \alpha\delta.$$

We can now use Riemann's Criterion to prove that αf is integrable. Let $\epsilon > 0$ and choose $\delta_0 > 0$ such that $\alpha\delta_0 = \epsilon$ (i.e. $\delta_0 := \epsilon/\alpha$) so that

$$U(\alpha f, P_{\delta_0}) - L(\alpha f, P_{\delta_0}) < \alpha\delta_0 = \epsilon.$$

This proves that αf is integrable by Riemann's Criterion.

(c) The function αf is integrable, which means $\underline{\int_a^b}(\alpha f) = \overline{\int_a^b}(\alpha f)$ by *definition*. If we write out the definition of the lower and upper integrals, then this becomes

$$\begin{aligned}&\sup\{L(\alpha f, P) : P \text{ is a partition of } [a,b]\}\\ &\qquad = \inf\{U(\alpha f, P) : P \text{ is a partition of } [a,b]\}\end{aligned}$$

and the integral $\int_a^b(\alpha f)$ is *defined* to equal this common value.

(d) It follows from (a) that $\sup_P L(\alpha f, P) = \alpha \sup_P L(f,P)$ whilst (c) shows that $\int_a^b(\alpha f) = \sup_P L(\alpha f, P)$ and $\int_a^b f = \sup_P L(f,P)$, hence

$$\int_a^b(\alpha f) = \sup_P L(\alpha f, P) = \alpha \sup_P L(f,P) = \alpha\int_a^b f,$$

as required. □

Q4. Suppose that $f : [a,b] \to [0,\infty)$ is bounded and integrable, where $-\infty < a < b < \infty$, and that $0 \le f(x) \le M$ for all $x \in [a,b]$ and some $M > 0$:

(a) Prove that $0 \le \underline{\int_a^b} f$ and that $\overline{\int_a^b} f \le M(b-a)$.

(b) Use (a) and the definition of the integral to prove that $0 \le \int_a^b f \le M(b-a)$.

(c) Use (b) to prove that $\lim_{h\to 0^+}\left(\int_a^{a+h} f\right) = 0$ and $\lim_{h\to 0^+}\left(\int_{b-h}^b f\right) = 0$.

Solution. (a) Let $P = \{x_0, x_1, \dots, x_n\}$ denote a partition of $[a,b]$. Observe that $m_i := \inf\{f(x) : x \in [x_{i-1}, x_i]\} \ge 0$ for each $i \in \{0,1,\dots,n\}$, hence

$$L(f,P) := \sum_{i=1}^n m_i(x_i - x_{i-1}) \ge 0.$$

We combine this with the definition of the lower integral to obtain

$$\underline{\int_a^b} f := \sup_P L(f,P) \ge 0.$$

Next, observe that $M_i := \sup\{f(x) : x \in [x_{i-1}, x_i]\} \le M$ for each $i \in \{0,1,\dots,n\}$, hence

$$U(f,P) := \sum_{i=1}^n M_i(x_i - x_{i-1}) \le M\sum_{i=1}^n (x_i - x_{i-1}) = M(b-a).$$

We combine this with the definition of the upper integral to obtain

$$\overline{\int_a^b} f := \inf_P U(f,P) \le M(b-a).$$

(b) The function f is integrable, so (a) implies that

$$0 \le \underline{\int_a^b} f = \int_a^b f = \overline{\int_a^b} f \le M(b-a),$$

as required

(c) If $h \in (0, b-a)$, then (b) implies that

$$0 = 0((a+h)-a) \leq \int_a^{a+h} f \leq M((a+h)-a) = Mh$$

and

$$0 = 0(b-(b-h)) \leq \int_{b-h}^{b} f \leq M(b-(b-h)) = Mh.$$

Now $\lim_{h\to 0^+}\left(\int_a^{a+h} f\right) = 0$ and $\lim_{h\to 0^+}\left(\int_{b-h}^{b} f\right) = 0$ follow from the Sandwich Theorem, since $\lim_{h\to 0^+} Mh = 0$. $\square$

Extra Questions

EQ1. Suppose that $f : [a,b] \to [0,\infty)$ is bounded and integrable, where $-\infty < a < b < \infty$. A *tagged partition* (P,T) of $[a,b]$ consists of a partition $P = \{x_0, x_1, \ldots, x_n\}$ of $[a,b]$ and a collection $T = \{t_1, \ldots, t_n\}$ of *tags* satisfying $t_1 \in [x_0, x_1], \ldots, t_n \in [x_{n-1}, x_n]$. The corresponding *Riemann Sum* is defined by

$$R(f,P,T) := \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1}).$$

(a) Prove that $L(f,P) \le R(f,P,T) \le U(f,P)$ for any tagged partition (P,T).

(b) Prove that $L(f,P) \le \int_a^b f \le U(f,P)$ for any partition P.

(c) Use Riemann's Criterion to prove that for each $\epsilon > 0$, there exists a partition P such that $|R(f,P,T) - \int_a^b f| < \epsilon$ whenever T is a collection of tags for P.

EQ2. (a) For each function defined below, use the properties of the function and results from Lectures/Lectures Notes to prove that it is integrable:

(i) $f : [0,10] \to \mathbb{R}$, $f(x) = 3x^2 + 5x + 9$

(ii) $g : [1,100] \to \mathbb{R}$, $g(x) = \lfloor x \rfloor := \max\{n \in \mathbb{N} : n \le x\}$

(b) Use Riemann's Criterion to prove that each function in part (a) is integrable. You may use the results stated in **Q4**(b) on Problem Sheet 5 (but you are not required to do so).

EQ3. (a) Prove that if $f : [a,b] \to \mathbb{R}$ and $g : [a,b] \to \mathbb{R}$ are both uniformly continuous, then $f+g$ is uniformly continuous.

(b) Suppose that $f : [a,\infty) \to \mathbb{R}$ is a continuous function, where $-\infty < a < b < \infty$. Prove that if f is uniformly continuous on both $[a,b]$ and $[b,\infty)$, then f is uniformly continuous on $[a,\infty)$.

EQ4. Suppose that $f : [1,5] \to \mathbb{R}$ and $g : [2,6] \to [-2,10]$ are both integrable functions, whilst $10 \le f(x) \le 1000$ for all $x \in [2,4]$. For each of the integrals below, use the properties of integrable functions to prove that the integral exists, and then find an upper bound and a lower bound for the value of the integral:

(a) $\int_2^4 f$

(b) $\int_2^5 (f-g)$

(c) $\int_3^4 6fg$