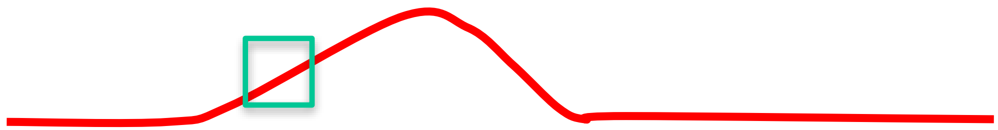
$$\frac{\partial^2 y(x,t)}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y(x,t)}{\partial t^2}$$

Optics and Waves

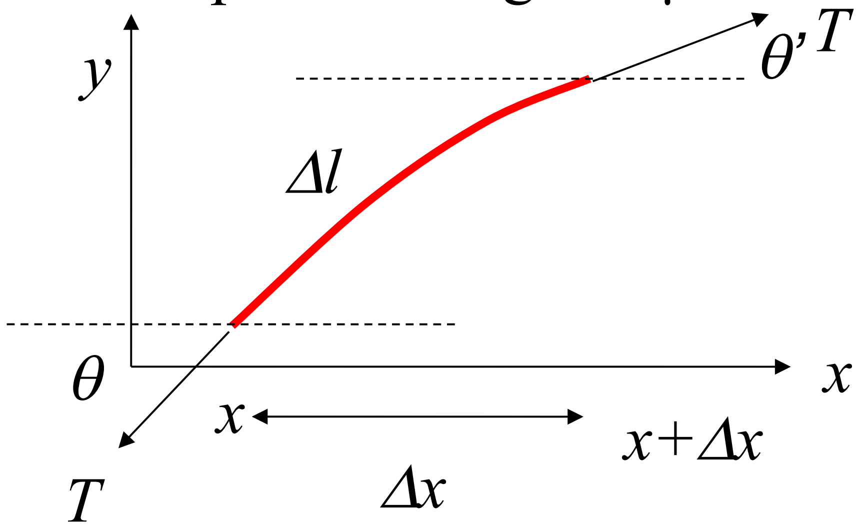
Lecture 4

-Wave equation for a string
-Reflection and transmission at a boundary

Wave-equation for a string



Assume that the string is under tension T (constant throughout) and the mass per unit length is μ

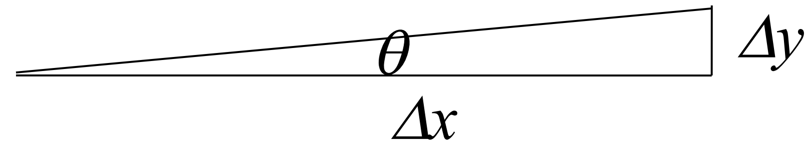


A snapshot!

The transverse force (in y-direction) $F_y = T\sin\theta' - T\sin\theta$

For small angles (easier to analyse)

$$\sin\theta \sim \tan\theta = \frac{\Delta y}{\Delta x} = \frac{dy}{dx}$$



$$\therefore F_y = T\frac{dy}{dx}(measured\ at\ x+\Delta x) - T\frac{dy}{dx}(measured\ at\ x) = T\left.\frac{dy}{dx}\right|_{x+\Delta x} - T\left.\frac{dy}{dx}\right|_{x}$$

$$\therefore F_y = T\left(\left.\frac{dy}{dx}\right|_{x+\Delta x} - \left.\frac{dy}{dx}\right|_{x}\right)$$

Remember:

$$\frac{dy}{dx} = \frac{y(x+\Delta x) - y(x)}{\Delta x}$$

$$\Delta x \rightarrow 0$$

$$\frac{\left.\frac{dy}{dx}\right|_{x+\Delta x} - \left.\frac{dy}{dx}\right|_{x}}{\Delta x} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$$

$$So \qquad F_y = T\frac{d^2y}{dx^2}\Delta x$$

The mass of the section of string is: $\mu\Delta x$

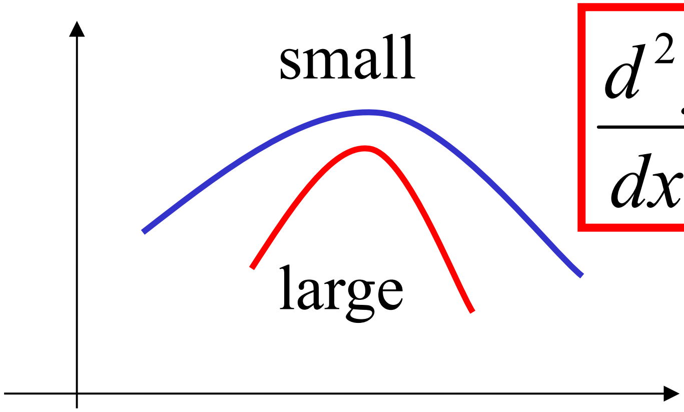
Apply *F=ma, consider acceleration in y-direction*

$$T\frac{d^2y}{dx^2}\Delta x = \mu\Delta x\frac{d^2y}{dt^2}$$

$$\boxed{\frac{d^2y}{dx^2} = \frac{\mu}{T}\frac{d^2y}{dt^2}}$$

What does it all mean? $\boxed{\dfrac{d^2y}{dx^2} = \dfrac{\mu}{T}\dfrac{d^2y}{dt^2}}$

LHS $\boxed{\dfrac{d^2y}{dx^2}}$ is the curvature of the string (rate of change of grad.)



$\boxed{\dfrac{d^2y}{dx^2}}$

RHS $\boxed{\dfrac{d^2y}{dt^2}}$ is the transverse acceleration of the string

So acceleration is proportional to curvature

Note that we calculate the curvature at a fixed time and the acceleration at a fixed position.
So we should write

$$\left.\frac{d^2y}{dx^2}\right|_{t\ fixed} = \frac{\mu}{T}\left.\frac{d^2y}{dt^2}\right|_{x\ fixed}$$

or

$$\frac{\partial^2 y}{\partial x^2} = \frac{\mu}{T}\frac{\partial^2 y}{\partial t^2}$$

Note before we had $$\boxed{\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}}$$

$$\frac{\partial^2 y}{\partial x^2} = \frac{\mu}{T}\frac{\partial^2 y}{\partial t^2}$$

So $$v = \sqrt{\frac{T}{\mu}}$$

Wave on a string travels faster at higher T, lower μ.

The speed of mechanical waves has a general form:

$$v = \sqrt{\frac{\text{Restoring force returning the system to equilibrium}}{\text{Inertia resisting the return to equilibrium}}}$$

Speed of sound in a gas: $v = \sqrt{\frac{B}{\rho}} = \sqrt{\frac{\gamma RT}{M}}$

B is the bulk modulus

ρ is the density of the gas

T is the absolute temperature

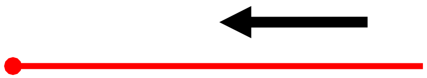
R is the ideal gas constant= 8.31J/mol. K

γ is the ratio of heat capacities, equals C_p/C_v

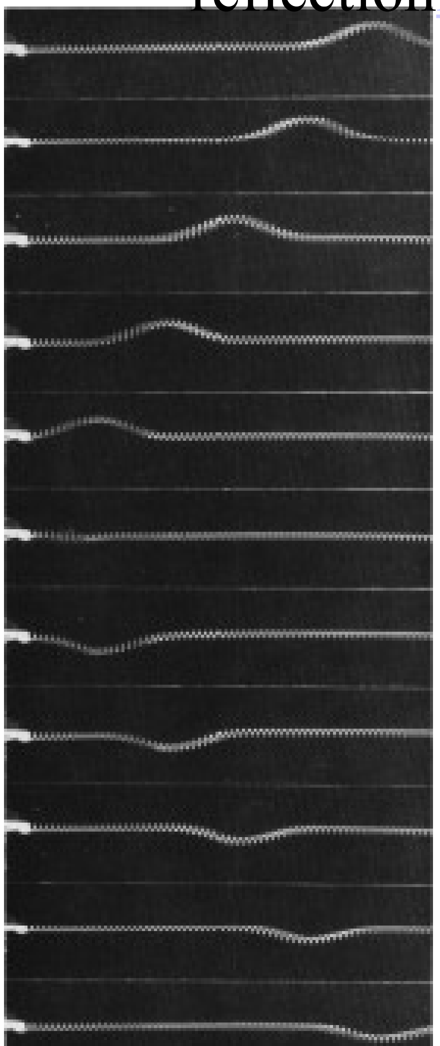
M is the molar mass, in kg/mol.

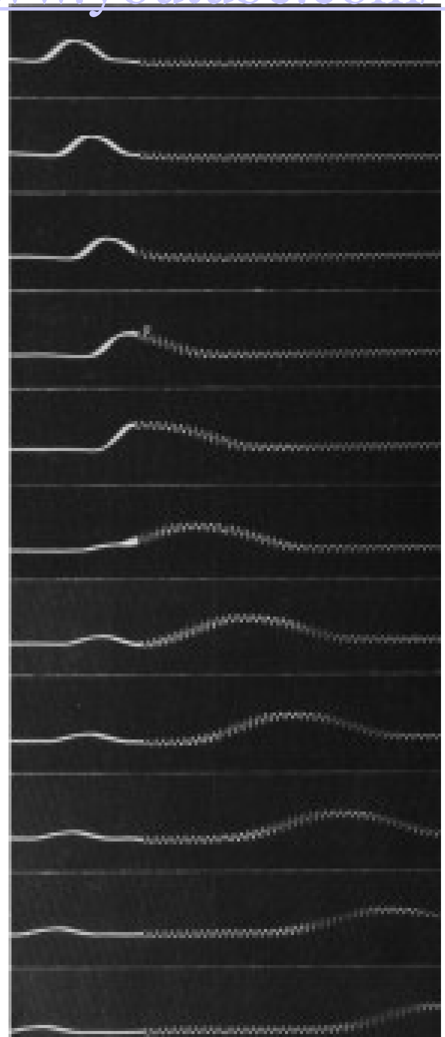
Find the relevant values for air molecules and thus verify that the speed of sound in air at room temperature (293 K) is about 344 m/s

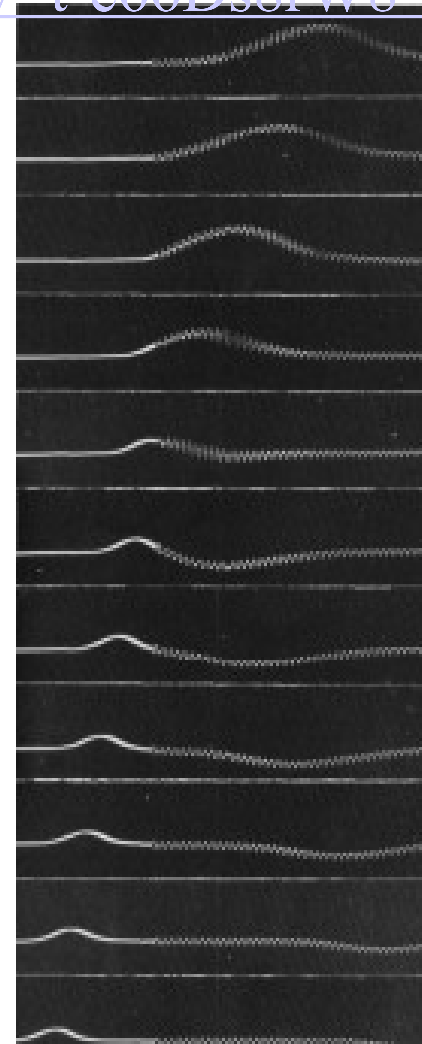
Reflection and transmission of waves



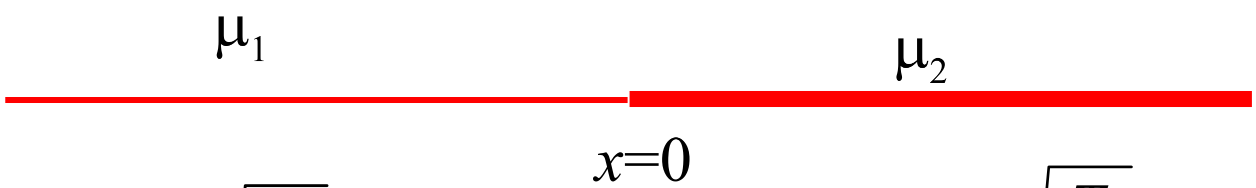
reflection http://www.youtube.com/watch?v=t-e66Ds8rW8 - re







What happens when light and heavy strings are joined?

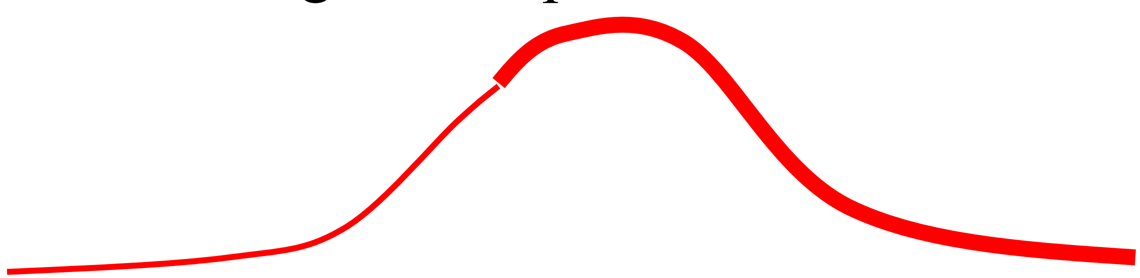


$$v_1 = \sqrt{\frac{T}{\mu_1}} \qquad v_2 = \sqrt{\frac{T}{\mu_2}}$$

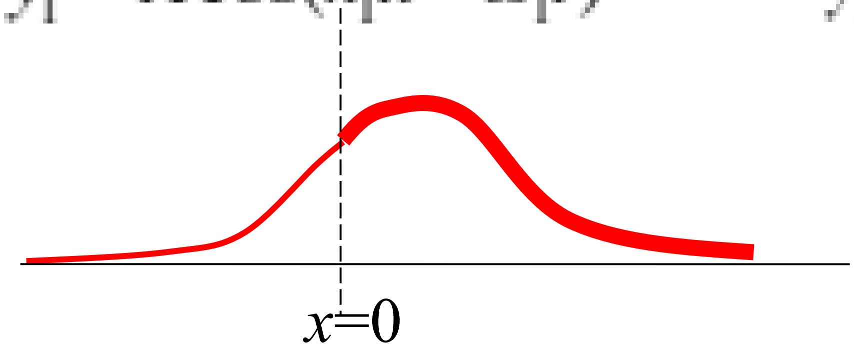
Consider a travelling wave is incident from the left

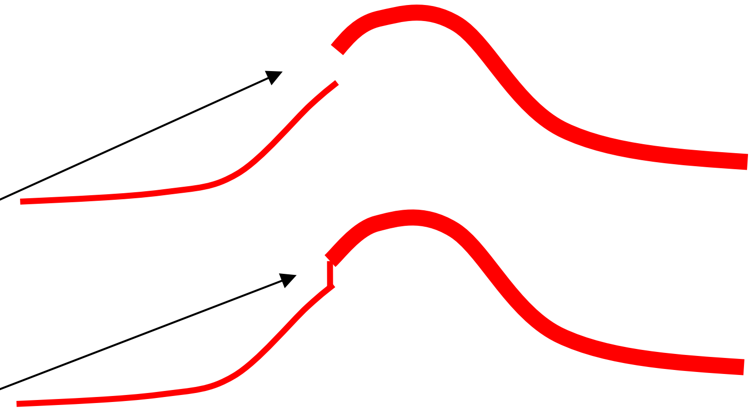
$$y = A\cos(k_1 x - \omega t)$$

At the junction of the strings the displacement must be continuous



$y_1 = A\cos(k_1 x - \omega_1 t)$ $\qquad$ $y_2 = B\cos(k_2 x - \omega_2 t)$





At $x = 0$,

$$y_1 = y_2 \qquad (1)$$

$$\frac{\partial y_1}{\partial x} = \frac{\partial y_2}{\partial x} \qquad (2)$$

A sharp corner means large curvature. Force is proportional to curvature, so curvature must be finite because force cannot be infinite.

Mini summary

Any wave can be described by a wave function.
For sinusoidal waves:

$$y(x,t) = A\cos(kx - \omega t)$$

ALL wave functions, including transverse, longitudinal, acoustic, EM, satisfy the wave Eq.

$$\frac{\partial^2 y(x,t)}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y(x,t)}{\partial t^2}$$

$$v = \frac{\omega}{k}$$

For waves on a string $v = \sqrt{\dfrac{T}{\mu}} = \dfrac{\omega}{k}$

The frequency with which the waves travel down the string is the same for both parts (depends on the frequency with which the waves are generated)

$$\omega_1 = \omega_2 = \omega$$

$$From(1)\ [x=0] \qquad A\cos(-\omega t) = B\cos(-\omega t) \qquad (3)$$

$$From(2)\ [x=0] \qquad -k_1 A\sin(-\omega t) = -k_2 B\sin(-\omega t) \qquad (4)$$

i.e.

$$A = B \qquad and \qquad k_1 A = k_2 B$$

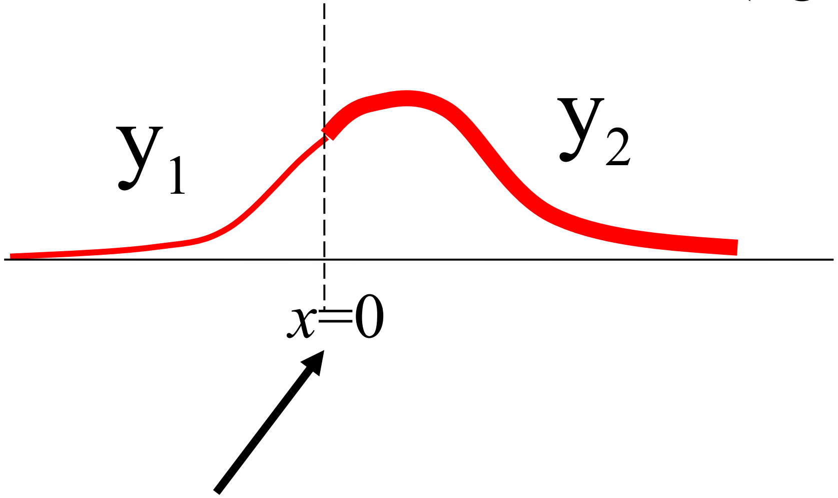
Only solution is

$$A = 0;\ \ B = 0$$

k_1 and k_2 are different and non zero.

So this is not really a solution! i.e. cannot satisfy the boundary conditions in this way.

Need to consider a transmitted wave in medium 2 (heavy string)
a reflected wave in medium 1 (light string)



$$y_1 = A\cos(k_1 x - \omega t) + C\cos(k_1 x + \omega t), \quad y_2 = B\cos(k_2 x - \omega t)$$

incident reflected transmitted

From $y_1 = y_2$ at $x = 0$, we get: $A\cos(-\omega t) + C\cos(\omega t) = B\cos(-\omega t)$

$$i.e. \qquad A\cos(-\omega t) + C\cos(-\omega t) = B\cos(-\omega t)$$

$$\boxed{A + C = B}$$

From $\dfrac{\partial y_1}{\partial x} = \dfrac{\partial y_2}{\partial x}$, we get: $-k_1 A\sin(-\omega t) - k_1 C\sin(\omega t) = -k_2 B\sin(-\omega t)$

$$-k_1 A\sin(-\omega t) + k_1 C\sin(-\omega t) = -k_2 B\sin(-\omega t)$$

$$\boxed{k_1(A - C) = k_2 B}$$

Solution is $\boxed{B = \dfrac{2k_1}{k_1 + k_2} A,}$ $\boxed{C = \dfrac{k_1 - k_2}{k_1 + k_2} A}$

Incident Wave:

$$y_1 = A\cos(k_1 x - \omega t)$$

Transmitted Wave:

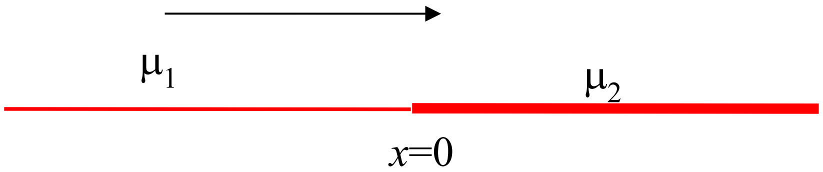
$$y_2 = B\cos(k_2 x - \omega t) = \frac{2k_1}{k_1 + k_2} A\cos(k_2 x - \omega t)$$

Reflected Wave:

$$y_3 = C\cos(k_1 x + \omega t) = \frac{k_1 - k_2}{k_1 + k_2} A\cos(k_1 x - \omega t)$$

k_1: wave number in the medium where the incident wave comes from;

k_2: wave number in the medium where the transmitted wave goes into.

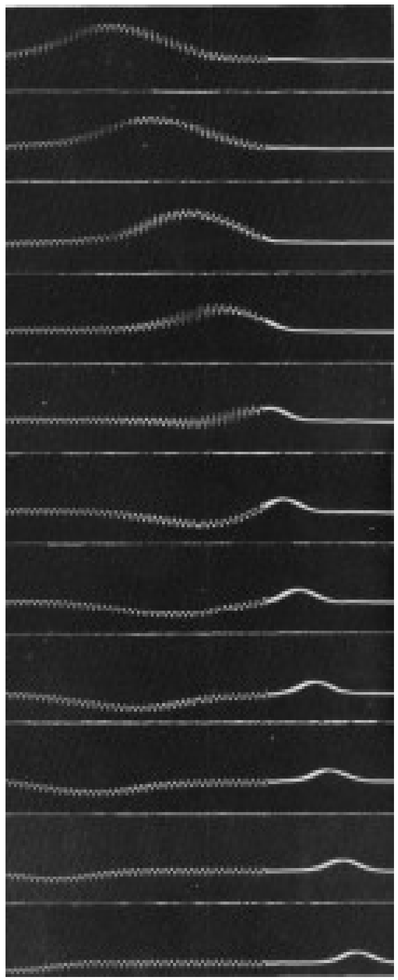


$\text{Note: if the wave comes from the left}$

and $\mu_2 > \mu_1$

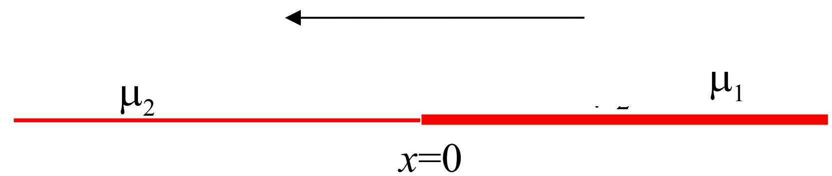
then $k_2 > k_1$

C is negative



$$\boxed{v = \frac{\omega}{k} = \sqrt{\frac{T}{\mu}} \qquad k = \omega\sqrt{\frac{\mu}{T}}}$$

$$\boxed{C = \frac{k_1 - k_2}{k_1 + k_2} A}$$

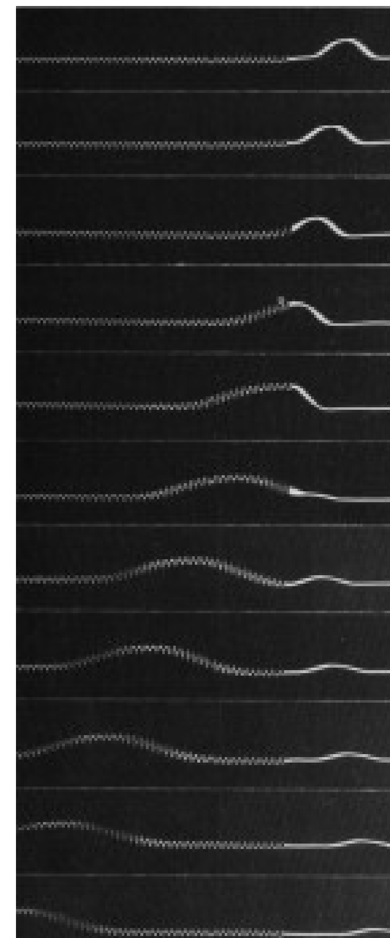


If the wave comes from the right

and $\mu_1 > \mu_2$

then $k_1 > k_2$

C is positive



$$B = \frac{2k_1}{k_1 + k_2} A = \frac{2}{1 + \dfrac{k_2}{k_1}}$$

What happens at the ends of the string? $B = \dfrac{2k_1}{k_1 + k_2} A, \quad C = \dfrac{k_1 - k_2}{k_1 + k_2} A$

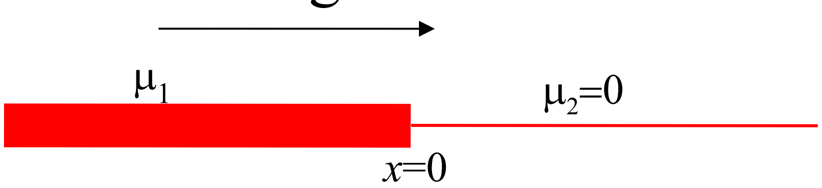
$$\boxed{\begin{aligned} v &= \frac{\omega}{k} = \sqrt{\frac{T}{\mu}} \\ k &= \omega\sqrt{\frac{\mu}{T}} \end{aligned}}$$

If the second string is massless then $k_2 = 0$ and $C = A$.

"massless" is another way to say that the second string does not exist, so the first string has a "free" end.

if the second string has infinite mass then $k_2 = \infty$ and $C = -A$.

Infinite mass is another way to say that the end of the first string is fixed and hence unable to move.



reflection of waves

If k1 and k2 are the same, then there is no reflection. Different k means different velocity because frequency is the same.

When light is incident onto an air-glass interface, reflection takes place in a similar manner.