Solution Sheet 4: Diagonalisation and phase space portraits

1. Diagonalise the matrix

$$\begin{bmatrix} -3 & 4 \\ 4 & 3 \end{bmatrix}$$

finding the eigenvalues and the right eigenvectors.

Answer 1. The equation that we need to solve is

$$\begin{bmatrix} -3-\lambda & 4 \\ 4 & 3-\lambda \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad (\lambda+3)(\lambda-3) - 4 \times 4 = \lambda^2 - 25 = 0$$

For the two eigenvalues $\lambda = \pm 5$ we need

$$\begin{bmatrix} -8 & 4 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} 2 & 4 \\ 4 & 8 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

to provide the required solution.

2. Diagonalise the matrix

$$\begin{bmatrix} 0 & 12 \\ 12 & 10 \end{bmatrix}$$

finding the eigenvalues and the right eigenvectors.

Answer 2. The equation that we need to solve is

$$\begin{bmatrix} -\lambda & 12 \\ 12 & 10-\lambda \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad \lambda(\lambda-10) - 12 \times 12 = (\lambda-5)^2 - 25 - 144 = 0$$

For the two eigenvalues $\lambda = 5 \pm 13$=18, -8 we need

$$\begin{bmatrix} -18 & 12 \\ 12 & -8 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} 8 & 12 \\ 12 & 18 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

to provide the required solution.

3. Diagonalise the matrix

$$\begin{bmatrix} -4 & 5 \\ -5 & 2 \end{bmatrix}$$

finding the eigenvalues and both types of eigenvector.

Answer 3. The equations that we need to solve are

$$\begin{bmatrix} -4-\lambda & 5 \\ -5 & 2-\lambda \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad (\lambda+4)(\lambda-2) + 5 \times 5 = (\lambda+1)^2 - 1 - 8 + 25 = 0$$

$$[\tilde{\alpha} \quad \tilde{\beta}] \begin{bmatrix} -4-\lambda & 5 \\ -5 & 2-\lambda \end{bmatrix} = [0 \quad 0]$$

For the two eigenvalues $\lambda = -1 \pm 4i$ we need

$$\begin{bmatrix} -3 \mp 4i & 5 \\ -5 & 3 \mp 4i \end{bmatrix} \begin{bmatrix} 5 \\ 3 \pm 4i \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad [-5 \quad 3 \pm 4i] \begin{bmatrix} -3 \mp 4i & 5 \\ -5 & 3 \mp 4i \end{bmatrix} = [0 \quad 0]$$

to provide the required solution.

4. Two energy surfaces take the forms

$$E + d = \frac{1}{2} a \left(\frac{d\theta_1}{dt} \right)^2 + \frac{1}{2} b \left(\frac{d\theta_2}{dt} \right)^2 + c \frac{d\theta_1}{dt} \frac{d\theta_2}{dt}$$

and

$$E - d = \frac{1}{2} a \left(\frac{d\theta_1}{dt} \right)^2 + \frac{1}{2} b \left(\frac{d\theta_2}{dt} \right)^2 - c \frac{d\theta_1}{dt} \frac{d\theta_2}{dt}$$

Diagonalise the two matrices

$$M_+ \equiv \begin{bmatrix} a & c \\ c & b \end{bmatrix} \qquad M_- \equiv \begin{bmatrix} a & -c \\ -c & b \end{bmatrix}$$

where $a > b$ and $c > 0$ and normalise the eigenvectors with $\mathbf{v}^T \mathbf{v}$=1. Employ

$$\begin{bmatrix} \frac{d\theta_1}{dt} \\ \frac{d\theta_2}{dt} \end{bmatrix} \equiv x_1 \mathbf{v}_1 + x_2 \mathbf{v}_2$$

to recognise the energy surfaces.

Answer 4. To diagonalise the first matrix we need

$$\begin{bmatrix} a - \lambda & c \\ c & b - \lambda \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow (\lambda - a)(\lambda - b) - c^2 = \left(\lambda - \frac{a+b}{2} \right)^2 - \left(\frac{a-b}{2} \right)^2 - c^2 = 0$$

and for the two eigenvalues, $\lambda = \frac{a+b}{2} \pm \Delta$, with $\Delta^2 = \left(\frac{a-b}{2}\right)^2 + c^2$ we need

$$\begin{bmatrix} \frac{a-b}{2} \mp \Delta & c \\ c & \frac{b-a}{2} \mp \Delta \end{bmatrix} \mathbf{v}_\pm = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

a particularly pretty solution is

$$\mathbf{v}_+ = \frac{1}{\sqrt{2}} \begin{bmatrix} \left[1 + \frac{a-b}{2\Delta}\right]^{\frac{1}{2}} \\ \left[1 - \frac{a-b}{2\Delta}\right]^{\frac{1}{2}} \end{bmatrix} \qquad \mathbf{v}_- = \frac{1}{\sqrt{2}} \begin{bmatrix} \left[1 - \frac{a-b}{2\Delta}\right]^{\frac{1}{2}} \\ -\left[1 + \frac{a-b}{2\Delta}\right]^{\frac{1}{2}} \end{bmatrix}$$

which can be proved using the identities

$$\frac{a-b}{2} \mp \Delta = \mp \Delta \left[1 \mp \frac{a-b}{2\Delta} \right] \qquad \left[1 + \frac{a-b}{2\Delta} \right] \left[1 - \frac{a-b}{2\Delta} \right] = \frac{c^2}{\Delta^2}$$

For the second matrix we do not need to redo the analysis, only appreciate that the relative sign of the two components must change

$$\mathbf{v}_+ = \frac{1}{\sqrt{2}} \begin{bmatrix} \left[1 + \frac{a-b}{2\Delta}\right]^{\frac{1}{2}} \\ -\left[1 - \frac{a-b}{2\Delta}\right]^{\frac{1}{2}} \end{bmatrix} \qquad \mathbf{v}_- = \frac{1}{\sqrt{2}} \begin{bmatrix} \left[1 - \frac{a-b}{2\Delta}\right]^{\frac{1}{2}} \\ \left[1 + \frac{a-b}{2\Delta}\right]^{\frac{1}{2}} \end{bmatrix}$$

For the final part

$$\left[x_1 \mathbf{v}_1^T + x_2 \mathbf{v}_2^T\right] M \left[x_1 \mathbf{v}_1 + x_2 \mathbf{v}_2\right] = \left[x_1 \mathbf{v}_1^T + x_2 \mathbf{v}_2^T\right] \left[x_1 \lambda_1 \mathbf{v}_1 + x_2 \lambda_2 \mathbf{v}_2\right] = x_1^2 \lambda_1 + x_2^2 \lambda_2$$

which is a constant. Ellipses with the minor axis along $\mathbf{v}_+$ and the major axis along $\mathbf{v}_-$.

5. Find the fixed points and stability matrix for the non-linear oscillator

$$\frac{d^2 x}{dt^2} = x - x^3$$

where

$$x_1 = x \qquad x_2 = \frac{dx}{dt}$$

Diagonalise the stability matrix at the fixed points and depict the local trajectories in their vicinity. Can you find a conservation law? Depict the full phase space portrait.

Answer 5. The fundamental equation is

$$\frac{dx_1}{dt} = x_2 \equiv f_1[x_1, x_2] \qquad \frac{dx_2}{dt} = x_1 - x_1^3 \equiv f_2[x_1, x_2]$$

The fixed points satisfy f_1=0 and f_2=0 so we find x_2=0 but we have x_1=0, ± 1

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

The stability matrix is

$$M \equiv \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 - 3x_1^2 & 0 \end{bmatrix}$$

Close to the origin $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ we find

$$M = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \Rightarrow \quad \lambda = \pm 1 \qquad \begin{bmatrix} \mp 1 & 1 \\ 1 & \mp 1 \end{bmatrix} \begin{bmatrix} 1 \\ \pm 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

We find a general solution

$$\begin{bmatrix} \delta x_1 \\ \delta x_2 \end{bmatrix} = A e^t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + B e^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

with two straight line trajectories; x_2=x_1 which moves away from the origin and x_2=-x_1 which moves towards the origin. The general curves are hyperbolae

$$(\delta x_1 + \delta x_2)(\delta x_1 - \delta x_2) = 4AB$$

Close to $\begin{bmatrix} \pm 1 \\ 0 \end{bmatrix}$ we find

$$M = \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix} \quad \Rightarrow \quad \lambda = \pm\sqrt{2}i \qquad \begin{bmatrix} \mp\sqrt{2}i & 1 \\ -2 & \mp\sqrt{2}i \end{bmatrix} \begin{bmatrix} 1 \\ \pm\sqrt{2}i \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

We find a general solution

$$\begin{bmatrix} \delta x_1 \\ \delta x_2 \end{bmatrix} = Ae^{i\sqrt{2}t} \begin{bmatrix} 1 \\ \sqrt{2}i \end{bmatrix} + \bar{A}e^{-i\sqrt{2}t} \begin{bmatrix} 1 \\ -\sqrt{2}i \end{bmatrix} = 2 \mid A \mid \begin{bmatrix} \cos\sqrt{2}(t-t_0) \\ -\sqrt{2}\sin\sqrt{2}(t-t_0) \end{bmatrix}$$

with ellipses that have a major x_2-axis

$$x_1^2 + \left(\frac{x_2}{\sqrt{2}}\right)^2 = 4 \mid A \mid^2$$

To find the conservation law we can divide the two equations and separate the variables to get

$$\frac{dx_1}{dx_2} = \frac{x_2}{x_1(1-x_1^2)} \quad \Rightarrow \quad dx_1 x_1(1-x_1^2) = dx_2 x_2 \quad \Rightarrow \quad \frac{1}{2}x_1^2 - \frac{1}{4}x_1^4 = \frac{1}{2}x_2^2 - E$$

and so

$$E = \frac{1}{2}x_2^2 - \frac{1}{2}x_1^2 + \frac{1}{4}x_1^4 = \frac{1}{2}\left(\frac{dx}{dt}\right)^2 - \frac{1}{2}x^2 + \frac{1}{4}x^4$$

the same as we would get from a classical mechanics treatment. The phase space portrait has an "infinity sign" around the two sets of ellipses, connected at the origin with dumbbell shaped curves surrounding this.

6. Find the fixed points and stability matrix for the damped oscillator

$$\frac{d^2\theta}{dt^2} + 2\frac{d\theta}{dt} + 5\theta = 0$$

with the choice

$$x_1 = \theta \qquad x_2 = \frac{d\theta}{dt} + \theta$$

Diagonalise the stability matrix and find the general solution. Depict the phase space portrait.

Answer 6. The fundamental equation is

$$\frac{dx_1}{dt} = x_2 - x_1 \equiv f_1[x_1, x_2] \qquad \frac{dx_2}{dt} = \frac{d^2\theta}{dt^2} + \frac{d\theta}{dt} = -\frac{d\theta}{dt} - 5\theta = -4x_1 - x_2 \equiv f_2[x_1, x_2]$$

The only fixed point is the origin and the stability matrix is

$$M = \begin{bmatrix} -1 & 1 \\ -4 & -1 \end{bmatrix} \quad \Rightarrow \quad (\lambda+1)^2 + 4 = 0 \quad \Rightarrow \quad \lambda = -1 \pm 2i$$

although this provides the exact solution. The eigenvectors satisfy

$$\begin{bmatrix} \mp 2i & 1 \\ -4 & \mp 2i \end{bmatrix} \begin{bmatrix} 1 \\ \pm 2i \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

and so the solution is

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = Ae^{-t+2it} \begin{bmatrix} 1 \\ 2i \end{bmatrix} + \bar{A}e^{-t-2it} \begin{bmatrix} 1 \\ -2i \end{bmatrix} = 2 \mid A \mid e^{-t} \begin{bmatrix} \cos 2(t-t_0) \\ -2\sin 2(t-t_0) \end{bmatrix}$$

and so we have a clock-wise spiral on ellipses with a major x_2-axis

$$x_1^2 + \left(\frac{x_2}{2}\right)^2 = 4 \mid A \mid^2 e^{-2t}$$

7. Find the fixed points of the system

$$\frac{dx_1}{dt} = x_1(1-x_1)(2-x_2) \qquad \frac{dx_2}{dt} = x_2(1-x_2)(2-x_1)$$

and depict the local trajectories in the vicinity of each point. Can you find a conservation law? Depict the phase space portrait.

Answer 7. The fundamental equation is

$$\frac{dx_1}{dt} = x_1(1-x_1)(2-x_2) \equiv f_1[x_1,x_2] \qquad \frac{dx_2}{dt} = x_2(1-x_2)(2-x_1) \equiv f_1[x_1,x_2]$$

To find the fixed points we need f_1=0 and f_2=0 so

$$x_1 = 0 \quad 2x_2(1-x_2) = 0 \quad or \quad x_1 = 1 \quad x_2(1-x_2) = 0 \quad or \quad x_2 = 2 \quad 2(-1)(2-x_1) = 0$$

which gives us five fixed points

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

The stability matrix is

$$M \equiv \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} (1-2x_1)(2-x_2) & -x_1(1-x_1) \\ -x_2(1-x_2) & (1-2x_2)(2-x_1) \end{bmatrix}$$

Close to the origin $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ we find

$$M = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

and so

$$\begin{bmatrix} \delta x_1 \\ \delta x_2 \end{bmatrix} = e^{2t} \begin{bmatrix} A \\ B \end{bmatrix}$$

and an unstable fixed point with straight lines locally. Close to the point $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ we find

$$M = \begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix}$$

and so

$$\begin{bmatrix} \delta x_1 \\ \delta x_2 \end{bmatrix} = Ae^{-2t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + Be^{t} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

an unstable fixed point that collapses quickly along the x_1-axis and expands along the x_2-axis. Close to the point $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ we find

$$M = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}$$

and so

$$\begin{bmatrix} \delta x_1 \\ \delta x_2 \end{bmatrix} = Ae^{-2t} \begin{bmatrix} 0 \\ 1 \end{bmatrix} + Be^{t} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

an unstable fixed point that collapses quickly along the x_2-axis and expands along the x_1-axis. Close to the point $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ we find

$$M = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

and so

$$\begin{bmatrix} \delta x_1 \\ \delta x_2 \end{bmatrix} = e^{-t} \begin{bmatrix} A \\ B \end{bmatrix}$$

and a stable fixed point with straight lines locally. Close to the point $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$ we find

$$M = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix} \quad \Rightarrow \quad \lambda = \pm 2 \quad \Rightarrow \quad \begin{bmatrix} \mp 2 & 2 \\ 2 & \mp 2 \end{bmatrix} \begin{bmatrix} 1 \\ \pm 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

and so

$$\begin{bmatrix} \delta x_1 \\ \delta x_2 \end{bmatrix} = Ae^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + Be^{-2t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

and an unstable fixed point that collapses along the line x_2=-x_1 and expands along x_2=x_1 with local hyperbolae as the trajectories

$$(\delta x_1 + \delta x_2)(\delta x_1 - \delta x_2) = 4AB$$

We may separate the variables

$$\frac{dx_1}{dx_2} = \frac{x_1(1-x_1)(2-x_2)}{x_2(1-x_2)(2-x_1)} \qquad \Rightarrow \qquad \frac{dx_1(2-x_1)}{x_1(1-x_1} = \frac{dx_2(2-x_2)}{x_2(1-x_2}$$

which integrates to provide

$$dx_1\left[\frac{2}{x_1}+\frac{1}{1-x_1}\right] = dx_2\left[\frac{2}{x_2}+\frac{1}{1-x_2}\right] \qquad \Rightarrow \qquad 2\ln \mid x_1 \mid - \ln \mid 1-x_1 \mid$$

$$= 2\ln \mid x_2 \mid - \ln \mid 1-x_2 \mid + A$$

and finally

$$\ln\left[\frac{x_1^2 \mid 1-x_2 \mid}{x_2^2 \mid 1-x_1 \mid}\right] = A$$

One might worry about the modulus signs, but x_1=1 and x_2=1 are both trajectories and because trajectories cannot cross there is never any change along a trajectory which are all smooth and well behaved. The overall phase space portrait has two important straight line trajectories that start at the points $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and end up at the point $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and extensions that follow these lines from infinity back to this fixed point. These trajectories split phase space up into pieces. The little square has trajectories that start at the origin and end up at the stable fixed point. The two infinite rectangles have trajectories that start at infinity and end up at the stable fixed point. The lines x_1=2 and x_2=2 mark minima for the trajectories as a function of x_2 and x_1 respectively. The final infinite square also has two critical trajectories that start at infinity and end up at the unstable fixed point $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$. These trajectories split up phase space into two regions, the piece closer to the origin ends up at the stable fixed point but the piece further away ends up at infinity. The attractor is the stable fixed point together with infinity, along the line x_2=x_1.

8. For a fairly general 2×2 matrix, $b > 0$ and $c > 0$,

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

show that the eigenvalues and eignvectors

$$M\mathbf{v} = \lambda\mathbf{v} \qquad \tilde{\mathbf{v}}^T M = \lambda\tilde{\mathbf{v}}^T$$

satisfy

$$\lambda_+ = \frac{a+d}{2} + \left[\left(\frac{a-d}{2}\right)^2 + bc\right]^{\frac{1}{2}} \equiv \frac{a+d}{2} + \Delta \quad \mathbf{v}_+ = \begin{bmatrix} \left[b\left(\Delta + \frac{a-d}{2}\right)\right]^{\frac{1}{2}} \\ \left[c\left(\Delta - \frac{a-d}{2}\right)\right]^{\frac{1}{2}} \end{bmatrix}$$

$$\tilde{\mathbf{v}}_+^T = \begin{bmatrix} \left[c\left(\Delta + \frac{a-d}{2}\right)\right]^{\frac{1}{2}} & \left[b\left(\Delta - \frac{a-d}{2}\right)\right]^{\frac{1}{2}} \end{bmatrix}$$

$$\lambda_- = \frac{a+d}{2} - \left[\left(\frac{a-d}{2}\right)^2 + bc\right]^{\frac{1}{2}} \equiv \frac{a+d}{2} - \Delta \quad \mathbf{v}_- = \begin{bmatrix} -\left[b\left(\Delta - \frac{a-d}{2}\right)\right]^{\frac{1}{2}} \\ \left[c\left(\Delta + \frac{a-d}{2}\right)\right]^{\frac{1}{2}} \end{bmatrix}$$

$$\tilde{\mathbf{v}}_-^T = \begin{bmatrix} -\left[c\left(\Delta - \frac{a-d}{2}\right)\right]^{\frac{1}{2}} & \left[b\left(\Delta + \frac{a-d}{2}\right)\right]^{\frac{1}{2}} \end{bmatrix}$$

and find $\tilde{\mathbf{v}}_+^T\mathbf{v}_+$, $\tilde{\mathbf{v}}_-^T\mathbf{v}_-$ and $\tilde{\mathbf{v}}_+^T\mathbf{v}_-$, $\tilde{\mathbf{v}}_-^T\mathbf{v}_+$.

Answer 8. We need to check that

$$[M - \lambda_+ I]\,\mathbf{v}_+ = \begin{bmatrix} \frac{a-d}{2} - \Delta & b \\ c & -\frac{a-d}{2} - \Delta \end{bmatrix} \begin{bmatrix} \left[b\left(\Delta + \frac{a-d}{2}\right)\right]^{\frac{1}{2}} \\ \left[c\left(\Delta - \frac{a-d}{2}\right)\right]^{\frac{1}{2}} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

and

$$\left(\Delta - \frac{a-d}{2}\right)\left[b\left(\Delta + \frac{a-d}{2}\right)\right]^{\frac{1}{2}} = \left[\Delta - \frac{a-d}{2}\right]^{\frac{1}{2}}\left[\Delta - \frac{a-d}{2}\right]^{\frac{1}{2}}\left[\Delta + \frac{a-d}{2}\right]^{\frac{1}{2}}\sqrt{b}$$

$$= \left[c\left(\Delta - \frac{a-d}{2}\right)\right]^{\frac{1}{2}} b$$

and analogues does the job. We can immediately calculate that

$$\tilde{\mathbf{v}}_+^T\mathbf{v}_+ = \tilde{\mathbf{v}}_-^T\mathbf{v}_- = 2\Delta\sqrt{(bc)} \qquad \tilde{\mathbf{v}}_+^T\mathbf{v}_- = \tilde{\mathbf{v}}_-^T\mathbf{v}_+ = 0$$