University of Birmingham
School of Mathematics

1SAS Sequences and Series Autumn 2024

Problem Sheet 1
Model Solutions

Q1. Recall the following definition from the lectures:

Definition. *A sequence of real numbers* (a_n) tends to infinity *if given any real number* $A > 0$ *there exists* $N \in \mathbb{N}$ *such that*

$$a_n > A \quad \text{for all } n > N.$$

(i) Using the definition prove that the sequence (a_n) given by

$$a_n = \sqrt{n}$$

tends to infinity.

(ii) Using the definition prove that the sequence (b_n) given by

$$b_n = \frac{n^4 + n^2 + 4}{3n^3 + 2n + 1}$$

tends to infinity.

Solution. (i) Let $A > 0$. Observe that

$$\sqrt{n} > A \text{ whenever } n > A^2.$$

Choosing $N \in \mathbb{N}$ such that $N \geq A^2$ we have

$$a_n = \sqrt{n} > \sqrt{N} \geq \sqrt{A^2} = A$$

whenever $n > N$.

(ii) We observe [1] first that

$$b_n = \frac{n^4 + n^2 + 4}{3n^3 + 2n + 1} \geq \frac{n^4}{3n^3 + 2n + 1} \geq \frac{n^4}{3n^3 + 2n^3 + n^3} = \frac{n^4}{6n^3} = \frac{n}{6}$$

for all $n \in \mathbb{N}$. Now let $A > 0$ and choose $N \in \mathbb{N}$ such that $N \geq 6A$. With this choice of N,

$$b_n \geq \frac{n}{6} > \frac{N}{6} \geq \frac{6A}{6} = A$$

whenever $n > N$. □

Q2. Recall the following definition from the lectures:

Definition. *A sequence* (a_n) *of real numbers* converges to a real number ℓ *if given any* $\varepsilon > 0$ *there exists* $N \in \mathbb{N}$ *such that*

$$|a_n - \ell| < \varepsilon \quad \text{for all} \quad n > N.$$

A sequence (a_n) converges *if it converges to* ℓ *for some real number* ℓ.

[1] As is very often the case, there are many similar observations that one could make here. It doesn't matter what the choice of N is as long as it works (and is justified properly). Here we are invoking elementary inequalities on the numerators and denominators separately based on the fact that $n \in \mathbb{N}$ – typically that n is positive, or that $n \geq 1$ specifically. What guides us is the desire to reach an expression that's simple enough for a suitable N to be chosen by inspection.

(i) Using the definition prove that the sequence (a_n) given by

$$a_n = \frac{n+2}{n+1}$$

converges to 1.

(ii) Using the definition prove that the sequence (b_n) given by

$$b_n = \frac{n+(-1)^n}{2n+(-1)^n}$$

converges.

Solution. (i) Observe that

$$|a_n - 1| = \left|\frac{n+2}{n+1} - 1\right| = \frac{1}{n+1} \le \frac{1}{n}$$

for all $n \in \mathbb{N}$. Now let $\varepsilon > 0$ and choose $N \in \mathbb{N}$ such that $N \ge \frac{1}{\varepsilon}$. With this choice of N,

$$|a_n - 1| \le \frac{1}{n} < \frac{1}{N} \le \frac{1}{1/\varepsilon} = \varepsilon$$

whenever $n > N$.

(ii) Observe that

$$\left|b_n - \frac{1}{2}\right| = \left|\frac{n+(-1)^n}{2n+(-1)^n} - \frac{1}{2}\right| = \left|\frac{(-1)^n}{2(2n+(-1)^n)}\right| \le \frac{1}{2(2n-1)} \le \frac{1}{2(2n-n)} = \frac{1}{2n}$$

for all $n \in \mathbb{N}$. Now let $\varepsilon > 0$ and choose $N \in \mathbb{N}$ such that $N \ge \frac{1}{2\varepsilon}$. With this choice of N,

$$\left|b_n - \frac{1}{2}\right| \le \frac{1}{2n} < \frac{1}{2N} \le \frac{1}{2(\frac{1}{2\varepsilon})} = \varepsilon$$

whenever $n > N$.

[*Remark.* I could have left my intial observation above as $\left|b_n - \frac{1}{2}\right| \le \frac{1}{2(2n-1)}$, and taken $N \in \mathbb{N}$ such that $N \ge \frac{1}{2}\left(1 + \frac{1}{2\varepsilon}\right)$. I decided to go a bit further with my estimations in order to make the expressions a little simpler.] □

Q3. Prove (using the definition) that the sequence (a_n) tends to infinity in each of the following cases:
(i) $a_n = n^{1/4}$; (ii) $a_n = \frac{n^4+4n^3+1}{n^3+2}$; (iii) $a_n = (2+(-1)^n)n$; (iv) $a_n = \left(n + \frac{1}{n}\right)^3 - n^3$.

Solution. (i) Let $A > 0$ and choose N to be any natural number at least as large as A^4. With this choice of N,

$$a_n = n^{1/4} > N^{1/4} \ge (A^4)^{1/4} = A$$

whenever $n > N$.

(ii) We begin by observing that

$$\frac{n^4+4n^3+1}{n^3+2} \ge \frac{n^4}{n^3+2} \ge \frac{n^4}{n^3+2n^3} = \frac{n}{3}$$

for all $n \in \mathbb{N}$. Let $A > 0$ and choose N to be any natural number at least as large as $3A$. With this choice of N,

$$a_n = \frac{n^4+4n^3+1}{n^3+2} \ge \frac{n}{3} > \frac{N}{3} \ge \frac{3A}{3} = A,$$

whenever $n > N$.

(iii) Observe that

$$a_n = (2+(-1)^n)n \ge (2-1)n = n$$

for all $n \in \mathbb{N}$. Let $A > 0$ and choose N to be any natural number at least as large as A. With this choice of N,

$$a_n \geq n > N \geq A$$

whenever $n > N$.

(iv) Observe that

$$a_n = n^3 + 3n + \frac{3}{n} + \frac{1}{n^3} - n^3 \geq 3n$$

for all $n \in \mathbb{N}$. Let $A > 0$ and choose N to be any natural number at least as large as $\frac{A}{3}$. With this choice of N,

$$a_n \geq 3n > 3N \geq 3\left(\frac{A}{3}\right) = A$$

whenever $n > N$.

$\square$

Q4. For each of the following sequences (a_n) and values of ℓ, prove (using the definition) that $a_n \to \ell$:

(i) $a_n = \frac{1+(-1)^n}{n}$, $\quad \ell = 0$;

(ii) $a_n = \frac{n^2+4\sin(n)}{2n^2+3}$, $\quad \ell = \frac{1}{2}$.

Solution. (i) We begin by observing that

$$|a_n - 0| = \frac{1+(-1)^n}{n} \leq \frac{2}{n}$$

for all $n \in \mathbb{N}$. Now let $\varepsilon > 0$ and choose $N \in \mathbb{N}$ to be at least as large as $2/\varepsilon$. With this choice of N,

$$|a_n - 0| \leq \frac{2}{n} < \frac{2}{N} \leq \frac{2}{2/\varepsilon} = \varepsilon$$

whenever $n > N$.

(ii) We begin by observing that

$$\left|a_n - \frac{1}{2}\right| = \left|\frac{n^2 + 4\sin(n)}{2n^2+3} - \frac{1}{2}\right| = \left|\frac{8\sin(n) - 3}{2(2n^2+3)}\right| \leq \frac{11}{2(2n^2+3)} \leq \frac{11}{4n^2}$$

for all $n \in \mathbb{N}$. Here we have used the fact that $|\sin x| \leq 1$ for real numbers x, along with the triangle inequality to establish that $|8\sin(n) - 3| \leq 8|\sin(n)| + |-3| \leq 11$. Now let $\varepsilon > 0$ and choose $N \in \mathbb{N}$ to be at least as large as $\sqrt{\frac{11}{4\varepsilon}}$. With this choice of N,

$$|a_n - \frac{1}{2}| \leq \frac{11}{4n^2} < \frac{11}{4N^2} < \varepsilon$$

whenever $n > N$.

$\square$

Q5. (i) Prove that if $x, y > 0$ then

$$\sqrt{x} - \sqrt{y} = \frac{x-y}{\sqrt{x}+\sqrt{y}}.$$

(ii) Using the definition of convergence, prove that the sequence (a_n) given by

$$a_n = \sqrt{n+1} - \sqrt{n}$$

converges to 0.

(iii) Does the sequence (b_n) given by

$$b_n = \sqrt{n^2+n} - n$$

converge? If so, what does it converge to? Justify your assertions.

Solution. (i) Observe that

$$(\sqrt{x}-\sqrt{y})(\sqrt{x}+\sqrt{y}) = x - y$$

(here we're using the elementary factorisation of the "difference of two squares" $a^2 - b^2 = (a-b)(a+b)$). Dividing through by the non-zero quantity $\sqrt{x}+\sqrt{y}$ leads to the required identity.

(ii) Applying the identity in Part (i), we have

$$a_n = \frac{1}{\sqrt{n+1}+\sqrt{n}},$$

so that

$$|a_n - 0| = \frac{1}{\sqrt{n+1}+\sqrt{n}} \leq \frac{1}{\sqrt{n}}$$

for all $n \in \mathbb{N}$. Now let $\varepsilon > 0$ and choose $N \in \mathbb{N}$ to be at least as large as $1/\varepsilon^2$. With this choice of N,

$$|a_n - 0| \leq \frac{1}{\sqrt{n}} < \frac{1}{\sqrt{N}} \leq \frac{1}{\sqrt{1/\varepsilon^2}} = \varepsilon$$

whenever $n > N$.

(iii) By Part (i),

$$\begin{aligned} b_n &= \sqrt{n^2+n} - n \\ &= \sqrt{n^2+n} - \sqrt{n^2} \\ &= \frac{n^2+n-n^2}{\sqrt{n^2+n}+n} \\ &= \frac{n}{\sqrt{n^2+n}+n}. \end{aligned}$$

We claim that $b_n \to \frac{1}{2}$. To show this we write

$$\begin{aligned} \left|b_n - \frac{1}{2}\right| &= \left|\frac{n}{\sqrt{n^2+n}+n} - \frac{1}{2}\right| \\ &= \left|\frac{n-\sqrt{n^2+n}}{2(\sqrt{n^2+n}+n)}\right| \\ &= \frac{\sqrt{n^2+n}-n}{2(\sqrt{n^2+n}+n)}, \end{aligned}$$

where we recognise b_n appearing as the numerator. Substituting our first observation into this last expression we obtain

$$\left|b_n - \frac{1}{2}\right| = \frac{n}{2(\sqrt{n^2+n}+n)^2},$$

and so

$$\left|b_n - \frac{1}{2}\right| = \frac{n}{2(\sqrt{n^2+n}+n)^2} \leq \frac{n}{2n^2} = \frac{1}{2n}$$

for all $n \in \mathbb{N}$. Now let $\varepsilon > 0$ and choose $N \in \mathbb{N}$ to be at least as large as $\frac{1}{2\varepsilon}$. With this choice of N we have

$$\left|b_n - \frac{1}{2}\right| \leq \frac{1}{2n} < \frac{1}{2N} \leq \varepsilon$$

whenever $n > N$. This proves our claim. $\square$

Extra Questions

These are some additional questions that you may find helpful, either now or at a later date.

EQ1. Consider the sequence
$$1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \frac{1}{5}, \cdots,$$
which has nth term $a_n = \frac{(-1)^{n+1}}{n}$.

(i) Find a natural number N for which $a_n \in (-\frac{1}{10}, \frac{1}{10})$ for all $n > N$.
[Here $(-\frac{1}{10}, \frac{1}{10})$ denotes the interval $\{x \in \mathbb{R} : -\frac{1}{10} < x < \frac{1}{10}\}$.]
(ii) Find a natural number N for which $a_n \in (-\frac{1}{1000}, \frac{1}{1000})$ for all $n > N$.
(iii) Let ε be *any* positive real number. Find a natural number N for which $a_n \in (-\varepsilon, \varepsilon)$ for all $n > N$. What does this prove?

Solution. Omitted. $\square$

EQ2. Let the sequence (a_n) be given by
$$a_n = \sum_{k=1}^{n} \frac{1}{\sqrt{n+k}}.$$

(i) Show that $a_n \geq \sqrt{\frac{n}{2}}$ for all $n \in \mathbb{N}$.
(ii) By appealing to the appropriate definition, deduce that $a_n \to \infty$.

Solution. (i) One might reasonably try to argue by induction here, although I expect it is simpler to argue as follows: Since
$$\frac{1}{\sqrt{n+k}} \geq \frac{1}{\sqrt{n+n}} = \frac{1}{\sqrt{2n}}$$
for all $1 \leq k \leq n$,
$$a_n = \sum_{k=1}^{n} \frac{1}{\sqrt{n+k}} \geq \sum_{k=1}^{n} \frac{1}{\sqrt{2n}} = \frac{n}{\sqrt{2n}} = \sqrt{\frac{n}{2}}.$$

(ii) Let $A > 0$ and choose $N \in \mathbb{N}$ such that $N \geq 2A^2$. With this choice of N,
$$a_n \geq \sqrt{\frac{n}{2}} > \sqrt{\frac{N}{2}} \geq \sqrt{\frac{2A^2}{2}} = A$$
whenever $n > N$. $\square$

EQ3. Suppose that $\ell > 0$ and that $a_n \to \ell$. Prove that there exists $N \in \mathbb{N}$ such that $a_n > 0$ for all $n > N$.

Solution. Since $a_n \to \ell$, by the definition applied with $\varepsilon = \ell/2 > 0$, there exists $N \in \mathbb{N}$ such that $|a_n - \ell| < \ell/2$ whenever $n > N$. In other words, $-\ell/2 < a_n - \ell < \ell/2$ whenever $n > N$. In particular, by the first of these two inequalities, we have that $a_n > \ell/2$ whenever $n > N$. Since $\ell > 0$, the desired conclusion follows. $\square$

EQ4. Prove that if $a_n \to \infty$ and $b_n \to \infty$ then (a) $a_n + b_n \to \infty$, and (b) $a_n b_n \to \infty$.

Solution. (i) Let $A > 0$. Since $a_n \to \infty$ there exists $N_1 \in \mathbb{N}$ such that

$$(\dagger) \qquad a_n > \frac{A}{2} \quad \text{for all} \quad n > N_1.$$

Since $b_n \to \infty$ there exists $N_2 \in \mathbb{N}$ such that

$$(\dagger\dagger) \qquad b_n > \frac{A}{2} \quad \text{for all} \quad n > N_2.$$

Let $N = \max\{N_1, N_2\}$. Since $N \geq N_1$ and $N \geq N_2$, on combining (†) and (††) we obtain

$$a_n + b_n > \frac{A}{2} + \frac{A}{2} = A \quad \text{for all} \quad n > N.$$

Since A was arbitrary, we conclude that $a_n + b_n \to \infty$ as $n \to \infty$.

(ii) Let $A > 0$. Since $a_n \to \infty$ there exists $N_1 \in \mathbb{N}$ such that

$$(\dagger) \qquad a_n > \sqrt{A}; \quad \text{for all} \quad n > N_1.$$

Since $b_n \to \infty$ there exists $N_2 \in \mathbb{N}$ such that

$$(\dagger\dagger) \qquad b_n > \sqrt{A} \quad \text{for all} \quad n > N_2.$$

Let $N = \max\{N_1, N_2\}$. Since $N \geq N_1$ and $N \geq N_2$, on combining (†) and (††) we obtain

$$a_n b_n > \sqrt{A}\sqrt{A} = A \quad \text{for all} \quad n > N.$$

Since A was arbitrary, we conclude that $a_n b_n \to \infty$ as $n \to \infty$. □

EQ5. Suppose that $N_0 \in \mathbb{N}$ and that $a_n \geq b_n$ for all $n > N_0$. Prove that if $b_n \to \infty$ then $a_n \to \infty$.

Solution. Let $A > 0$. Since $b_n \to \infty$ there exists $N' \in \mathbb{N}$ such that

$$b_n > A \quad \text{for all } n > N'.$$

Now let $N = \max\{N', N_0\}$ and observe that by the hypotheses of the theorem,

$$a_n \geq b_n > A \quad \text{for all } n > N.$$

Hence $a_n \to \infty$. □

EQ6. Give explicit examples of sequences (a_n) and (b_n), satisfying $a_n \to \infty$ and $b_n \to 0$, for which

(i) $a_n b_n \to 1$.
(ii) $a_n b_n \to 0$.
(iii) $a_n b_n \to \infty$.
(iv) $a_n b_n \to -\infty$.
(v) the sequence $(a_n b_n)$ neither converges nor tends to $\pm\infty$.
(vi) $b_n > 0$ for all $n \in \mathbb{N}$, and the sequence $(a_n b_n)$ neither converges nor tends to $\pm\infty$.

[Here $(a_n b_n)$ is the sequence whose nth term is the product $a_n b_n$.]

Solution. Omitted. □

EQ7. Give explicit examples of sequences (a_n) and (b_n), satisfying $a_n \to \infty$ and $b_n \to \infty$, for which

(i) $a_n - b_n \to 0$,
(ii) $a_n - b_n \to 1$,
(iii) $a_n - b_n \to \infty$,
(iv) $a_n - b_n \to -\infty$,

(v) the sequence $(a_n - b_n)$ neither converges nor tends to $\pm\infty$.

[Here $(a_n - b_n)$ is the sequence whose nth term is the difference $a_n - b_n$.]

Solution. (i) For example, take $a_n = n$ and $b_n = n$.
(ii) For example, take $a_n = n + 1$ and $b_n = n$.
(iii) For example, take $a_n = 2n$ and $b_n = n$.
(iv) For example, take $a_n = n$ and $b_n = 2n$.
(v) For example, take $a_n = n + (-1)^n$ and $b_n = n$.

□

EQ8. Suppose that (a_n) is a sequence of positive real numbers converging to $\ell > 0$.

(i) Prove that

$$|\sqrt{a_n} - \sqrt{\ell}| \leq \frac{|a_n - \ell|}{\sqrt{\ell}}$$

for all $n \in \mathbb{N}$.

(ii) Using Part (i), or otherwise, prove that $\sqrt{a_n} \to \sqrt{\ell}$ as $n \to \infty$.

(iii) Deduce that

$$\frac{\sqrt{n^4 + 4n}}{n^2 + 1} \to 1,$$

justifying any assertions that you make.

Solution. (i) We first observe the identity

$$\sqrt{a_n} - \sqrt{\ell} = \frac{a_n - \ell}{\sqrt{a_n} + \sqrt{\ell}}.$$

(This is a direct application of the formula $(x - y)(x + y) = x^2 - y^2$ where $x = \sqrt{a_n}$ and $y = \sqrt{\ell}$.) Hence, in particular

$$|\sqrt{a_n} - \sqrt{\ell}| = \frac{|a_n - \ell|}{\sqrt{a_n} + \sqrt{\ell}}.$$

Since $\sqrt{a_n} + \sqrt{\ell} \geq \sqrt{\ell}$, we have that

$$|\sqrt{a_n} - \sqrt{\ell}| \leq \frac{|a_n - \ell|}{\sqrt{\ell}}$$

for all $n \in \mathbb{N}$.

(ii) Let $\varepsilon > 0$ be arbitrary.

Since $a_n \to \ell$ and $\ell > 0$, there exists $N \in \mathbb{N}$ such that

$$|a_n - \ell| < \varepsilon\sqrt{\ell} \quad \text{for all} \quad n > N.$$

Hence by Part (i),

$$|\sqrt{a_n} - \sqrt{\ell}| < \frac{\varepsilon\sqrt{\ell}}{\sqrt{\ell}} = \varepsilon \quad \text{for all} \quad n > N.$$

Hence $\sqrt{a_n} \to \ell$, as required.

(iii) Observe that

$$\frac{\sqrt{n^4 + 4n}}{n^2 + 1} = \sqrt{\frac{n^4 + 4n}{n^4 + 2n^2 + 1}}.$$

By Part (ii), it will suffice to show that

$$\frac{n^4 + 4n}{n^4 + 2n^2 + 1} \to 1.$$

We leave the details of this to the reader, being similar to some of the earlier exercises.

$\square$

EQ9. (i) For which values of $n \in \mathbb{N}$ is it true that $2^n \geq n$? Prove your assertion.
Using your result, or otherwise, prove that

$$2^n \to \infty.$$

(ii) For which values of $n \in \mathbb{N}$ is it true that $2^n \geq n^2$? Prove your assertion.
Using your result, or otherwise, prove that

$$\frac{2^n}{n} \to \infty.$$

(iii) Outline a potential strategy for proving that

$$\frac{2^n}{n^k} \to \infty$$

for all $k = 0, 1, 2, 3, \ldots$.

Solution. (i) *Claim:* $2^n \geq n$ for all $n \in \mathbb{N}$. To see this we argue by induction.

Since $2^1 = 2 \geq 1$, the claimed inequality is true for $n = 1$ (this is the base case for the induction).

Suppose now that $2^k \geq k$ holds for *some* $k \in \mathbb{N}$ (this is the inductive hypothesis). Now,

$$2^{k+1} = 2.2^k \geq 2k,$$

by the inductive hypothesis. However, since $k \geq 1$ we have that $2k \geq k+1$, and so

$$2^{k+1} \geq k+1.$$

Hence $2^n \geq n$ holds for $n = k+1$.

Our claim now follows by the principle of mathematical induction.

It just remains to show that $2^n \to \infty$. To this end let $A > 0$ and choose N to be any natural number greater than A. With this choice of N,

$$2^n \geq n > N > A$$

whenever $n > N$. Since A was arbitrary, we conclude that $2^n \to \infty$.

(ii) *Claim:* $2^n \geq n^2$ for all natural numbers $n \neq 3$. For $n = 1$ and $n = 2$ we may verify the inequality directly since $2^1 = 2 \geq 1 = 1^2$ and $2^2 = 4 \geq 4 = 2^2$. For $n = 3$ it fails since $2^3 = 8 < 9 = 3^2$.

In order to prove that $2^n \geq n^2$ holds for $n \geq 4$, we argue by induction.

Since $2^4 = 16 \geq 16 = 4^2$, the claimed inequality is true for $n = 4$ (this is the base case for the induction).

Suppose now that $2^k \geq k^2$ holds for *some* $k \geq 4$ (this is the inductive hypothesis). Now,

$$2^{k+1} = 2.2^k \geq 2k^2,$$

by the inductive hypothesis. In order to complete the inductive step, it remains to show that $2k^2 \geq (k+1)^2$. To this end consider

$$2k^2 - (k+1)^2 = k^2 - 2k - 1.$$

Since $k \geq 4$ we have $k^2 = k.k \geq 4k$, so that

$$2k^2 - (k+1)^2 \geq 4k - 2k - 1 = 2k - 1 \geq 0,$$

as required. Hence $2^{k+1} \geq (k+1)^2$, completing the inductive step.

Therefore $2^n \geq n^2$ for all $n \geq 4$ by the principle of mathematical induction.

Finally let's use this inequality to show that $\frac{2^n}{n} \to \infty$. Let $A > 0$ and choose N to be any natural number greater than $\max\{4, A\}$. With this choice of N we have

$$\frac{2^n}{n} \geq \frac{n^2}{n} = n > N > A$$

whenever $n > N$. Since A was arbitrary, we conclude that $\frac{2^n}{n} \to \infty$.

(iii) We do not provide solutions to this part. $\square$