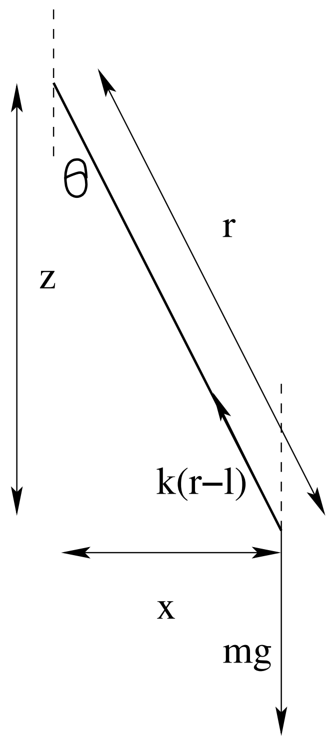
Appendix: Equations of motion

The sprung pendulum

Here we derive the equations of motion for the sprung pendulum using Newton's laws. The most useful picture is



where we can read off the equations of motion in the Cartesian directions using Newton's laws and trigonometry

$$m\frac{d^2x}{dt^2} = -k(r-l)\sin\theta$$

$$m\frac{d^2z}{dt^2} = -k(r-l)\cos\theta + mg$$

It is more natural to use r and θ as the variables and so we can observe that

$$x = r\sin\theta \qquad z = r\cos\theta$$

and then determine the velocities to be

$$\frac{dx}{dt} = \frac{dr}{dt}\sin\theta + r\cos\theta\frac{d\theta}{dt}$$

$$\frac{dz}{dt} = \frac{dr}{dt}\cos\theta - r\sin\theta\frac{d\theta}{dt}$$

using function of a function. A second derivative gives the acceleration

$$\frac{d^2x}{dt^2} = \frac{d^2r}{dt^2}\sin\theta + 2\frac{dr}{dt}\cos\theta\frac{d\theta}{dt} + r\cos\theta\frac{d^2\theta}{dt^2} - r\sin\theta\left(\frac{d\theta}{dt}\right)^2$$

$$\frac{d^2z}{dt^2} = \frac{d^2r}{dt^2}\cos\theta - 2\frac{dr}{dt}\sin\theta\frac{d\theta}{dt} - r\sin\theta\frac{d^2\theta}{dt^2} - r\cos\theta\left(\frac{d\theta}{dt}\right)^2$$

The natural combinations of these quantities are

$$\sin\theta\frac{d^2x}{dt^2} + \cos\theta\frac{d^2z}{dt^2} = \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2$$

$$\cos\theta\frac{d^2x}{dt^2} - \sin\theta\frac{d^2z}{dt^2} = r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt} = \frac{1}{r}\frac{d}{dt}\left[r^2\frac{d\theta}{dt}\right]$$

where we used an identity for the second equation. We can then substitute in the equations of motion to verify that

$$m\frac{d^2r}{dt^2} = mr\left(\frac{d\theta}{dt}\right)^2 - k(r-l) + mg\cos\theta$$

$$m\frac{d}{dt}\left[r^2\frac{d\theta}{dt}\right] = -mgr\sin\theta$$

In the second year you will be taught an easier method. In terms of the kinetic energy, T, and the potential energy, V, we define a Lagrangian, $L \equiv T - V$. There are then momenta

$$p_r = \frac{\partial L}{\partial\left[\frac{dr}{dt}\right]} \qquad p_\theta = \frac{\partial L}{\partial\left[\frac{d\theta}{dt}\right]}$$

and equations of motion

$$\frac{dp_r}{dt} = \frac{\partial L}{\partial r} \qquad \frac{dp_\theta}{dt} = \frac{\partial L}{\partial \theta}$$

and the energy is $E = T + V$.

For our problem

$$T = \frac{m}{2}\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2\right] = \frac{m}{2}\left[\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2\right]$$

$$V = -mgr\cos\theta + \frac{1}{2}k(r-l)^2$$

and so

$$p_r = m\frac{dr}{dt} \qquad p_\theta = mr^2\frac{d\theta}{dt}$$

and the equations of motion are

$$m\frac{d^2r}{dt^2} = mr\left(\frac{d\theta}{dt}\right)^2 + mg\cos\theta - k(r-l)$$

$$m\frac{d}{dt}\left[r^2\frac{d\theta}{dt}\right] = -mgr\sin\theta$$

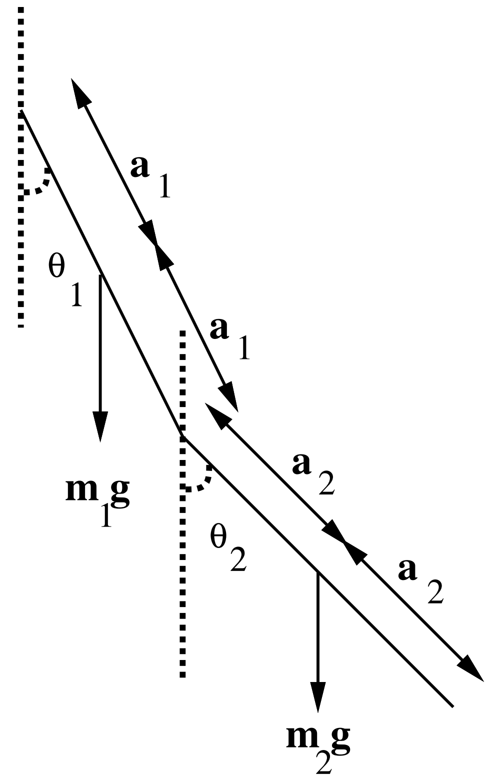
in agreement with Newton's laws. The energy is

$$E = \frac{m}{2}\left[\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2\right] - mgr\cos\theta + \frac{1}{2}k(r-l)^2$$

This derivation is much simpler and we will use it for the next example.

The double pendulum

For this problem the most useful picture is



and for this problem we only use the Lagrangian. The motion of the centre of mass of each of the rods is controlled by the velocities

$$\frac{dx_1}{dt} = a_1\sin\theta_1\frac{d\theta_1}{dt} \qquad \frac{dz_1}{dt} = -a_1\cos\theta_1\frac{d\theta_1}{dt}$$

$$\frac{dx_2}{dt} = 2a_1\sin\theta_1\frac{d\theta_1}{dt} + a_2\sin\theta_2\frac{d\theta_2}{dt} \qquad \frac{dz_2}{dt} = -2a_1\cos\theta_1\frac{d\theta_1}{dt} - a_2\cos\theta_2\frac{d\theta_2}{dt}$$

and so the kinetic energy of the centre of mass is

$$T = \frac{1}{2}m_1a_1^2\left(\frac{d\theta_1}{dt}\right)^2 + \frac{1}{2}m_2\left[4a_1^2\left(\frac{d\theta_1}{dt}\right)^2 + a_2^2\left(\frac{d\theta_2}{dt}\right)^2\right.$$

$$\left. +4a_1a_2\frac{d\theta_1}{dt}\frac{d\theta_2}{dt}\left(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2\right)\right]$$

and one should note that

$$\cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2 = \cos(\theta_1 - \theta_2)$$

The potential energy is

$$V = -mg_1 a_1 \cos\theta_1 - m_2 g\left[2a_1 \cos\theta_1 + a_2 \cos\theta_2\right]$$

The rotational kinetic energy is

$$T = \frac{1}{2} I_1 \left(\frac{d\theta_1}{dt}\right)^2 + \frac{1}{2} I_2 \left(\frac{d\theta_2}{dt}\right)^2$$

where

$$I_1 = \frac{1}{3} m_1 a_1^2 \qquad\qquad I_2 = \frac{1}{3} m_2 a_2^2$$

are the appropriate moments of inertia for the rods. This gives

$$L \equiv T - V = \frac{1}{2} a \left(\frac{d\theta_1}{dt}\right)^2 + \frac{1}{2} b \left(\frac{d\theta_2}{dt}\right)^2 + c \frac{d\theta_1}{dt}\frac{d\theta_2}{dt} \cos(\theta_1 - \theta_2) + d\cos\theta_1 + e\cos\theta_2$$

with

$$a = \frac{1}{3} m_1 a_1^2 + m_1 a_1^2 + 4m_2 a_1^2 \qquad b = \frac{1}{3} m_2 a_2^2 + m_2 a_2^2$$

$$c = 2m_2 a_1 a_2 \qquad d = (m_1 + 2m_2) a_1 g \qquad e = m_2 a_2 g$$

The momenta are

$$p_{\theta_1} = \frac{\partial L}{\partial\left[\frac{d\theta_1}{dt}\right]} = a\frac{d\theta_1}{dt} + c\frac{d\theta_2}{dt}\cos(\theta_1 - \theta_2)$$

$$p_{\theta_2} = \frac{\partial L}{\partial\left[\frac{d\theta_2}{dt}\right]} = b\frac{d\theta_2}{dt} + c\frac{d\theta_1}{dt}\cos(\theta_1 - \theta_2)$$

and the equations of motion are

$$\frac{dp_{\theta_1}}{dt} = -c\frac{d\theta_1}{dt}\frac{d\theta_2}{dt}\sin(\theta_1 - \theta_2) - d\sin\theta_1$$

$$\frac{dp_{\theta_2}}{dt} = c\frac{d\theta_1}{dt}\frac{d\theta_2}{dt}\sin(\theta_1 - \theta_2) - e\sin\theta_2$$

and the energy is

$$E = \frac{1}{2} a \left(\frac{d\theta_1}{dt}\right)^2 + \frac{1}{2} b \left(\frac{d\theta_2}{dt}\right)^2 + c\frac{d\theta_1}{dt}\frac{d\theta_2}{dt}\cos(\theta_1 - \theta_2) - d\cos\theta_1 - e\cos\theta_2$$