1VGLA Vectors, Geometry and Linear Algebra

2024/2025 Semester 2

Problem Sheet 4 (summative)

This Problem Sheet will be marked.

Inconsistent notations and incoherent or incomplete reasoning will be penalised.

When using a theorem or general result from the lecture notes, state the result, instead of quoting the theorem number.

1. Consider the vector space $\mathbb{R}^3$ and

$$u_1 = (2, -1, 0) \in \mathbb{R}^3, \quad u_2 = (0, -1, 2) \in \mathbb{R}^3,$$
$$U = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : 3x_1 + 2x_2 + x_3 = 0\} \leq \mathbb{R}^3.$$

   (a) Show that $\operatorname{span}\{u_1, u_2\}$ is the plane through the origin with normal vector $(1, 2, 1)$.

   (b) Find a spanning set for U.

2. Let $\mathcal{M}_{22}(\mathbb{R})$ be the vector space of 2×2 real matrices, with the standard matrix addition and scalar multiplication. Let

$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{C} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

   Show that $\{\mathbf{A}, \mathbf{B}, \mathbf{C}\}$ is a linearly independent set.

3. Let $\mathcal{P}_2(\mathbb{R})$ be the vector space of real polynomials of order at most 2:

$$\mathcal{P}_2(\mathbb{R}) = \{a_2x^2 + a_1x + a_0 \ : \ a_0, a_1, a_2 \in \mathbb{R}\},$$

   with vector addition defined by

$$(a_2x^2 + a_1x + a_0) + (b_2x^2 + b_1x + b_0) = (a_2 + b_2)x^2 + (a_1 + b_1)x + (a_0 + b_0),$$

   and scalar multiplication defined for all $\lambda \in \mathbb{R}$ by

$$\lambda(a_2x^2 + a_1x + a_0) = \lambda a_2x^2 + \lambda a_1x + \lambda a_0.$$

   (a) Show that

$$U = \{a_2x^2 + a_1x + a_0 \ : \ a_0, a_1, a_2 \in \mathbb{R}, \ a_0 + a_1 + a_2 = 0\}$$

   is a subspace of $\mathcal{P}_2(\mathbb{R})$.

   (b) Show that

$$U = \operatorname{span}\{x^2 - 1, x - 1\}.$$

   Hence, show that $\dim(U) = 2$.

4. By finding the rank of a suitably defined matrix, show that the following set of vectors in $\mathbb{R}^4$ is linearly independent:

$$U = \{(1, 0, 0, 0), (1, 1, 0, 0), (1, 1, 1, 0)\}.$$

   Extend U to a basis B of $\mathbb{R}^4$ such that $U \subseteq B$.