UNIVERSITY OF BIRMINGHAM

School of Mathematics

Programmes in the School of Mathematics First Examination
Programmes involving Mathematics First Examination

1VGLA 06 25664 Level C

LC Vectors, Geometry and Linear Algebra

May/June Examinations 2021-22

Three Hours

Full marks will be obtained with complete answers to all FOUR questions. Each question carries equal weight. You are advised to initially spend no more than 45 minutes on each question and then to return to any incomplete questions if you have time at the end. An indication of the number of marks allocated to parts of questions is shown in square brackets.

No calculator is permitted in this examination.

Section A

1. (a) For the vectors $\mathbf{a} = (1,2,4)$, $\mathbf{b} = (v_1, v_2, v_3)$ and $\mathbf{c} = (1,2,3)$ determine

(i) $\mathbf{a} + \mathbf{b}$;

(ii) $\mathbf{a} \cdot \mathbf{b}$;

(iii) $\mathbf{a} \times \mathbf{b}$; and

(iv) the angle θ between $\mathbf{a}$ and $\mathbf{c}$ (you may leave your answer in the form $\theta = \cos^{-1}(x)$ or $\sin^{-1}(x)$ where x is a number you have determined).

[8]

(b) Let $z_1 = -\frac{\sqrt{2}}{2}(1+i)$ and $z_2 = 2e^{i3\pi/4}$. Determine

(i) z_1 in modulus argument form, giving the principal value of the argument, and z_2 in the form $x + iy$;

(ii) $z_1 + z_2$ in the form $x + iy$;

(iii) $z_1 z_2^{-1}$ in the modulus argument form giving the principal value of the argument; and

(iv) z_1^{29} in exponential form giving the principal value of the argument.

[8]

(c) (i) Use the Gaussian elimination method to obtain a reduced echelon form and determine the solution set for the following system of simultaneous linear equations:

$$\begin{aligned} 3x + 2y - z &= 4 \\ 2x - y + 2z &= 10 \\ x - 3y - 4z &= 5. \end{aligned}$$

[7]

(ii) Writing the simultaneous linear equations in part (c)(i) as a matrix equation $\mathbf{Ax} = \mathbf{b}$, calculate the determinant of $\mathbf{A}$ by using details from the Gaussian elimination in part (c)(i) of the question.

[2]

2. (a) Which of the following subsets Y of V are subspaces of the given vector spaces V? Explain your answers by providing proofs of your claims.

(i) $V = \mathbb{R}^4$ and $Y = \{(x_1, x_2, x_3, x_4) \in V \mid x_1^2 + x_2 + x_3 = 0\}$; and

(ii) $V = \mathbb{R}^4$ and $Y = \{(0, 0, x, 2x) \in V \mid x \in \mathbb{R}\}$.

[6]

(b) Let $V = \mathbb{R}^4$ and $U = \langle (1,2,3,4), (0,1,2,3), (1,3,5,3), (2,5,8,7) \rangle$ (the span of the listed vectors).

(i) What are the defining properties for a basis of a finite-dimensional vector space?

(ii) Calculate a basis for U.

(iii) Determine the dimension of U, $\dim_{\mathbb{R}} U$.

[8]

(c) Let

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 1 & 2 & 1 \end{pmatrix}.$$

Find

(i) $\mathbf{A}^T$ (the transpose of $\mathbf{A}$);

(ii) $\det \mathbf{A}$ (the determinant of $\mathbf{A}$); and

(iii) $\mathbf{A}^{-1}$ (the inverse of $\mathbf{A}$).

[9]

(d) Suppose that V and W are vector spaces over a field $\mathbb{F}$. What is the definition of a linear transformation from V to W? [2]

Section B

3. (a) Show that the three vectors $\mathbf{a} = 3\mathbf{i}+2\mathbf{j}-\mathbf{k}$, $\mathbf{b} = 2\mathbf{i}-\mathbf{j}+3\mathbf{k}$ and $\mathbf{c} = \mathbf{i}-2\mathbf{j}+2\mathbf{k}$ are not coplanar. [5]

(b) Determine all the 7th roots of $128 = 2^7$. [5]

(c) Consider the locus of points on the Argand diagram given by the complex numbers $z = x+iy$ which satisfy the following equation

$$|z-3|+|z-4| = 6.$$

Say, giving a brief explanation but not an algebraic calculation, which type of conic the locus describes. [4]

(d) Consider the homogeneous system of linear equations

$$\begin{aligned} a_{11}x+a_{12}y+a_{13}z &= 0 \\ a_{21}x+a_{22}y+a_{23}z &= 0 \\ a_{31}x+a_{32}y+a_{33}z &= 0. \end{aligned}$$

Mark the following statements as true or false.

(i) The system of equations always has a solution.

(ii) There is not enough information to say if the system of equations has any solutions.

(iii) Regarding the solutions as vectors $(x,y,z) \in \mathbb{R}^3$, the solution set is a subspace of the vector space $\mathbb{R}^3$. [6]

(e) Suppose that $n \geq 2$ is a natural number and let

$$\mathbf{A}_n = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 & 1 & 1 \\ -1 & 1 & 1 & \dots & 1 & 1 & 1 \\ 0 & -1 & 1 & \dots & 1 & 1 & 1 \\ 0 & 0 & -1 & \dots & 1 & 1 & 1 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -1 & 1 & 1 \\ 0 & 0 & 0 & \dots & 0 & -1 & 1 \end{pmatrix} \in \mathscr{M}_{nn}(\mathbb{Q}).$$

That is $\mathbf{A}_n$ has entries -1 just below the diagonal, every entry on the diagonal and above the diagonal is 1 and all other entries are 0. Use mathematical induction to prove that $\det \mathbf{A}_n = 2^{n-1}$. [5]

4. (a) Suppose that V is a finite-dimensional vector space over the field $\mathbb{F}$. Assume that U and W are subspaces of V.

(i) Show that $U \cap W$ is a subspace of V.

(ii) State the definition of $U + W$ and show that it is a subspace of V.

(iii) Give an expression for $\dim_{\mathbb{F}}(U+W)$.

[9]

(b) Suppose that V and W are vector spaces over $\mathbb{F}$ and assume that T is a linear transformation from V to W.

(i) What is the kernel of T, $\ker(T)$? Prove that $\ker(T)$ is a subspace of V.

(ii) What is the image of T, $\mathrm{Im}(T)$? Prove that $\mathrm{Im}(T)$ is a subspace of W.

(iii) Prove that the dimension of V is greater than or equal to the dimension of the image of T.

[9]

(c) Let $V = \mathbb{R}^3$, $W = \mathbb{R}^3$ and $T : V \to W$ be defined by

$$T((v_1, v_2, v_3)) = (v_1 + 2v_2 - v_3, 2v_1 + v_3, v_1 - 2v_2 + 2v_3).$$

(i) Show that T is a linear transformation.

(ii) Determine a basis for the image of T, $\mathrm{Im}(T)$.

(iii) Determine the dimension of $\ker T$, $\dim_{\mathbb{R}} \ker(T)$.

[7]

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Do not complete the attendance slip, fill in the front of the answer book or turn over the question paper until you are told to do so.

Important Reminders

- Coats and outer-wear should be placed in the designated area.
- Unauthorised materials (e.g. notes or Tippex) **MUST** be placed in the designated area.
- Check that you **DO NOT** have any unauthorised materials with you (e.g. in your pockets, pencil case).
- Mobile phones and smart watches **MUST** be switched off and placed in the designated area or under your desk. They must not be left on your person or in your pockets.
- You are **NOT** permitted to use a mobile phone as a clock. If you have difficulty in seeing a clock, please alert an Invigilator.
- You are **NOT** permitted to have writing on your hand, arm or other body part.
- Check that you do not have writing on your hand, arm or other body part – if you do, you must inform an Invigilator immediately.
- Alert an Invigilator immediately if you find any unauthorised item upon you during the examination.

Any students found with non-permitted items upon their person during the examination, or who fail to comply with Examination rules may be subject to the Student Conduct procedures.