# VGLA: Vectors Practice Questions

The following questions relate to Chapter 1, Vectors. Questions are ranked in difficulty from A (basic) to C (challenging).

**(A) Question 1.** For each of the following sets of points $U$, $V$ and $W$ calculate $\vec{UV}$, $\vec{UW}$ and hence determine whether $U$, $V$ and $W$ are co-linear:

(a) $U = (1, 3, -1)$, $V = (5, 1, -2)$ and $W = (3, 2, -3)$;

(b) $U = (2, 1, 4)$, $V = (1, 4, 2)$ and $W = (4, -5, 8)$.

**(A) Question 2.** In each of the following cases, find numbers $s$ and $t$, if they exist, such that $\mathbf{w} = s\mathbf{u}+t\mathbf{v}$:

(a) $\mathbf{u} = (3, -1, 1)$, $\mathbf{v} = (4, 3, -3)$ and $\mathbf{w} = (17, 3, -3)$;

(b) $\mathbf{u} = (2, -3, 5)$, $\mathbf{v} = (-1, 4, 6)$ and $\mathbf{w} = (8, -17, 2)$.

(If $\mathbf{w} = s\mathbf{u} + t\mathbf{v}$, $\mathbf{w}$ is said to be a *linear combination* of $\mathbf{u}$ and $\mathbf{v}$.)

**(A) Question 3.** Find the vector $\mathbf{u}$ which has magnitude 6 and has the same direction as the vector $\mathbf{v} = (1, -2, 1)$.

**(A) Question 4.** In each of the following cases, find the unit vector with the same direction as $\mathbf{v}$ and the unit vector with the opposite direction to $\mathbf{v}$:

(a) $\mathbf{v} = (3, -2, -6)$;

(b) $\mathbf{v} = (0, -3, 5)$.

**(A) Question 5.** For each of the following pairs of vectors $\mathbf{u}$ and $\mathbf{v}$, determine whether they are perpendicular:

(a) $\mathbf{u} = (2, -3, 1)$ and $\mathbf{v} = (5, 4, 2)$;

(b) $\mathbf{u} = (\cos\theta, \sin\theta, 1)$ and $\mathbf{v} = (\sin\theta, -\cos\theta, 1)$;

(c) $\mathbf{u} = (\cos\theta, \sin\theta, 1)$ and $\mathbf{v} = (\cos\theta, \sin\theta, -1)$.

**(A) Question 6.** Find all values of $\lambda$, if any, for which the vectors $\mathbf{u}$ and $\mathbf{v}$ are perpendicular:

(a) $\mathbf{u} = (3, -2, 1)$ and $\mathbf{v} = (4, \lambda, -2)$;

(b) $\mathbf{u} = (\lambda, 2, 7)$ and $\mathbf{v} = (\lambda, -3, 1)$;

(c) $\mathbf{u} = (1, \lambda, \lambda)$ and $\mathbf{v} = (-2, \lambda, 1)$.

**(A) Question 7.** If $\mathbf{u} = (2, 3, -1)$, $\mathbf{v} = (-2, -1, 2)$ and $\mathbf{w} = (1, 2, 1)$, calculate:

(a) $\mathbf{u} \times \mathbf{v}$;

(b) $(\mathbf{u} \times \mathbf{v}) \times \mathbf{w}$;

(c) $\mathbf{v} \times \mathbf{w}$;

(d) $\mathbf{u} \times (\mathbf{v} \times \mathbf{w})$;

(e) $\mathbf{u}.(\mathbf{v} \times \mathbf{w})$.

(f) Deduce from (a), the two unit vectors that are perpendicular to both $\mathbf{u}$ and $\mathbf{v}$.

**(A) Question 8.** In each of the following cases, find an equation for the plane $\pi$:

(a) $\pi$ is parallel to the $yz$-plane and intersects the point $(1, 2, 3)$;

(b) $\pi$ is parallel to the $zx$-plane and intersects the point $(3, -1, 4)$;

(c) $\pi$ intersects the point $(2, -1, -4)$ and has normal vector $(2, 1, 0)$;

(d) $\pi$ intersects the points $U = (1, 1, 3)$, $V = (-1, 3, 2)$ and $W = (1, -2, 5)$.

**(A) Question 9.** Obtain a set of parametric equations for the straight line $L$ that intersects the points $P = (3, 1, 4)$ and $Q = (-1, -2, 8)$. Find the coordinates of the points of intersection of $L$ and the plane $\pi$ in part (A) Question 8, part (d).

**(A) Question 10.** In each of the following cases, find the volume of the parallelepiped with vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ as adjacent edges:

(a) $\mathbf{u} = (0, 2, 2)$, $\mathbf{v} = (3, 1, 1)$ and $\mathbf{w} = (3, -5, 1)$;

(b) $\mathbf{u} = (1, 0, 2)$, $\mathbf{v} = (1, 1, 0)$ and $\mathbf{w} = (0, 1, 1)$.

**(A) Question 11.** For the vectors $\mathbf{u} = (1, -1, 0)$, $\mathbf{v} = (0, 1, 1)$ and $\mathbf{w} = (2, 0, -1)$ calculate:

(a) $\mathbf{u} \times \mathbf{v}$;

(b) $(\mathbf{u} \times \mathbf{v}) \times \mathbf{w}$;

(c) $\mathbf{u}.\mathbf{w}$;

(d) $\mathbf{v}.\mathbf{w}$.

Verify directly for vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ that

$$(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = (\mathbf{u}.\mathbf{w})\mathbf{v} - (\mathbf{v}.\mathbf{w})\mathbf{u}.$$

**(A) Question 12.** For the vectors $\mathbf{a} = (3, 0, 0)$ and $\mathbf{b} = (1, -2, 2)$, find the following:

(a) $|\mathbf{a}|$ and $|\mathbf{b}|$;

(b) $\mathbf{a} \cdot \mathbf{b}$;

(c) unit vectors parallel to $\mathbf{b}$ in the same/opposite direction;

(d) the projection of the vector $\mathbf{a}$ unto the direction of vector $\mathbf{b}$, i.e. $\text{proj}_{\mathbf{b}}(\mathbf{a})$;

(e) the vectors $\mathbf{u}$ and $\mathbf{w}$ such that $\mathbf{a} = \mathbf{u} + \mathbf{w}$ with $\mathbf{u} \parallel \mathbf{b}$ and $\mathbf{u} \perp \mathbf{w}$.

**(A) Question 13.** Consider the vectors $\mathbf{a} = (3, -2, 0)$ and $\mathbf{b} = (1, -2, 2)$, and the points $P(1, -1, 2)$, $Q(1, 0, 0)$ and $R(-3, 0, 1)$.

(a) Determine $\mathbf{a} \times \mathbf{b}$;

(b) Find two distinct unit vectors perpendicular to the plane containing the vectors $\mathbf{a}$ and $\mathbf{b}$;

(c) Find an equation of the plane $\Pi_1$ that intersects the point $P$ and is perpendicular to the vector $\mathbf{b}$;

(d) Find an equation of the plane $\Pi_2$ parallel to the plane $\Pi_1$ that intersects the origin;

(e) Find an equation of the plane $\Pi_3$ that intersects the points $P$, $Q$ and $R$;

(f) Find an equation of the line $L_1$ that intersects the point $A$ with position vector $\mathbf{a}$ and is parallel to the vector $\mathbf{b}$;

(g) Find an equation of the plane $\Pi_4$ which contains the line $L_1$ and the point $P$;

(h) Find the point of intersection of the line $L_1$ and the plane $\Pi_5$ with equation $2x - 3y + z = -3$.

**(B) Question 14.** The mid-points of the sides $AB$, $BC$, $CD$ and $DA$ of a quadrilateral $ABCD$ are $E$, $F$, $G$ and $H$ respectively. Show, **using vectors**, that $EFGH$ is a parallelogram. *Hint: Take position vectors relative to an origin O and use the result that a quadrilateral with a pair of opposite sides which are parallel and equal in length is a parallelogram.*

**(B) Question 15.** Let $\mathbf{u}$ and $\mathbf{v}$ be the distinct position vectors relative to an origin $O$ of the points $U$ and $V$ respectively. Additionally, let $W$ be a point on either $UV$ (outside the closed line segment [UV]) or $VU$ (outside the closed line segment [VU]) such that $UW : WV = s : t$.

Establish that the position vector $\mathbf{w}$ of $W$ relative to $O$ is given by

$$\mathbf{w} = \left(\frac{s}{s-t}\right)\mathbf{v} - \left(\frac{t}{s-t}\right)\mathbf{u}.$$

If $U = (2, -4, 3)$ and $V = (-3, 1, -2)$, find the coordinates of $W$ when

(a) $W$ is on $VU$ and $UW : WV = 2 : 7$;

(b) $W$ is on $UV$ and $UW : WV = 7 : 2$.

**(B) Question 16.** If $D$, $E$ and $F$ are the mid-points of the sides $BC$, $CA$ and $AB$ respectively of $\triangle ABC$, the line segments $AD$, $BE$ and $CF$ are known as the **medians** of $\triangle ABC$. If $A$, $B$ and $C$ have position vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ relative to an origin $O$, find the position vector of the point $G$ on $AD$ s.t. $AG : GD = 2 : 1$ and establish that $G$ also lies on $BE$ and $CF$. Hence prove that the medians of a triangle are concurrent (they all intersect).

**(B) Question 17.** The triangle $OAB$ has vertices at $O$ (the origin) and at points $A$ and $B$ with position vectors $\mathbf{a}$ and $\mathbf{b}$ respectively. The point $P$ is the midpoint of $OA$ and $Q$ is the midpoint of $PB$. The line $OQ$ (extended) intersects $AB$ at $R$.

(a) Sketch a diagram of the information above.

(b) Write down, in terms of $\mathbf{a}$ and $\mathbf{b}$, the position vectors $\mathbf{p}$ and $\mathbf{q}$ of the points $P$ and $Q$.

(c) Find the vector form of the lines $AB$ and $OQ$ and hence find the position vector $\mathbf{r}$ of $R$. In what ratio does $R$ split $AB$?

(d) If $A$ has coordinates $(4, -3)$ and $B$ has coordinates $(1, 3)$ show that $OR$ is perpendicular to $AB$ and find the cosine of the angle $A\hat{O}R$.

**(B) Question 18.** Find the (non-reflex) angle between the vectors $\mathbf{u} = (1, 1, 0)$ and $\mathbf{v} = (0, -1, 1)$.

**(B) Question 19.** If $U = (6, -2, 1)$, $V = (5, 4, 2)$ and $W = (6, -3, 4)$ respectively, determine:

(i) the length of the sides of $\triangle UVW$;

(ii) if $\angle VUW$ is acute or obtuse.

**(B) Question 20.** Find the components of the two **unit** vectors $\mathbf{w} = (w_1, w_2, w_3)$ which make an angle of $\pi/3$ rad. with the vector $\mathbf{u} = (1, 0, -1)$ and an angle of $\pi/4$ rad with the vector $\mathbf{v} = (1, -2, -2)$. Show that these two unit vectors are perpendicular.

**(B) Question 21.** If $\mathbf{u}$ is a non-zero vector and $\mathbf{u} \times \mathbf{v} = \mathbf{u} \times \mathbf{w}$, is it necessarily true that $\mathbf{v} = \mathbf{w}$? Justify your answer. When $\mathbf{u} = (2, 3, -4)$ and $\mathbf{v} = (-3, -1, 2)$, find a vector $\mathbf{w}$ ($\neq \mathbf{v}$), if one exists, such that $\mathbf{u} \times \mathbf{v} = \mathbf{u} \times \mathbf{w}$.

**(B) Question 22.** Determine all values of $\alpha$, if any, for which $\mathbf{u} \times \mathbf{v}$ is perpendicular to $\mathbf{u} \times \mathbf{w}$ when

(a) $\mathbf{u} = (1, 2, -1)$, $\mathbf{v} = (1, \alpha, 1)$ and $\mathbf{w} = (2, -1, \alpha)$;

(b) $\mathbf{u} = (1, -1, 1)$, $\mathbf{v} = (1, 3, \alpha)$ and $\mathbf{w} = (1, \alpha, -1)$.

**(B) Question 23.** Using the definition given in lectures for $\mathbf{u} \times \mathbf{v}$ in the case where $\mathbf{u}$ and $\mathbf{v}$ are two non-zero vectors which are not parallel, establish that if $\mathbf{u}$ is a non-zero vector and $\mathbf{v}$ is a vector such that

$$\mathbf{u} = \mathbf{v} \times (\mathbf{u} \times \mathbf{v}),$$

then $\mathbf{v}$ is a unit vector which is perpendicular to $\mathbf{u}$.

**(B) Question 24.** Obtain the components of a vector which is perpendicular to the vectors represented by $\vec{AB}$ and $\vec{CD}$ where

$$A = (1, -2, -3), \quad B = (4, 1, 1), \quad C = (0, 1, -1) \text{ and } D = (-6, 1, 3).$$

Hence obtain scalar equations for two parallel planes $\pi$ and $\pi'$ where $\pi$ contains $A$ and $B$, and $\pi'$ contains $C$ and $D$. Calculate the distance between $\Pi$ and $\Pi'$.

**(B) Question 25.** Find an equation of the plane $\Pi$ which contains the point $U = (5, 0, 2)$ and the line $L$ given by the parametric equations $x = 1 + 3t$, $y = 4 - 2t$, $z = -3 + t$.

**(B) Question 26.** Prove that if $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are distinct non-zero vectors such that

$$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = \mathbf{0} \qquad \text{and} \qquad (\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = \mathbf{0},$$

then either $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are parallel or $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are mutually perpendicular.