Lecture 10

Filter circuits

1. Introduction

In the last 2 lectures, I introduced the concept of complex impedance and showed how the magnitude and phase of the current and voltage in a circuit is determined by the magnitude and argument of the complex impedance. In this final lecture, I will use complex impedances to study the application of RC and RL circuits as filters.

2. Filter circuits

One of the first things we learned about capacitors and inductors in a.c. circuits, is that behave like frequency dependent resistors. This is an extremely useful property, which we can exploit to build a filter: a circuit that transmits signals (voltages) at frequencies of interest, while blocking other frequencies. One example of such a circuit is a RLC resonant circuit. Only near the resonant frequency does current flow easily through the circuit. Tuneable LCR circuits are found in radio receivers. In fact, such circuits are known as band pass circuits, because they can be made sensitive to a narrow range (or band) of frequencies. RC and RL circuits behave either like low pass or high pass filters. These are circuits that transmit mainly low frequency or high frequency signals.

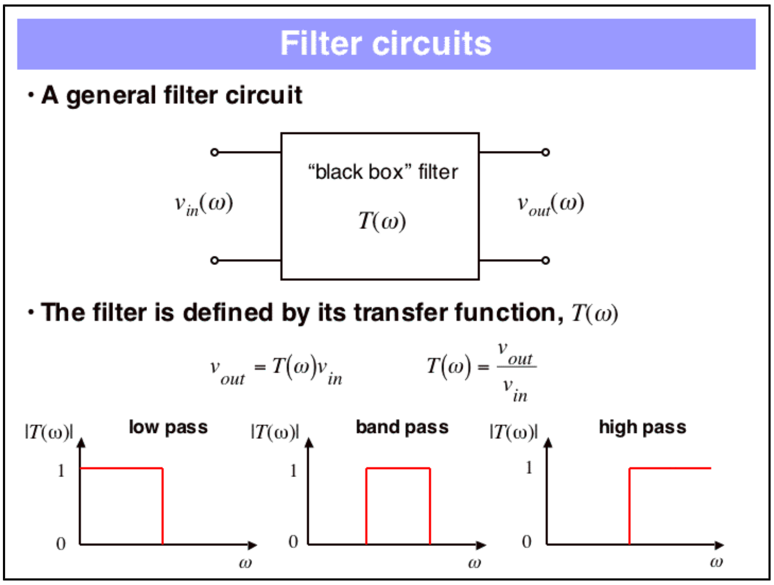


Figure 10.1: Types of filter circuit.

A schematic diagram of a filter circuit is shown in Figure 10.1. It introduces the idea of a two-port network, which may be connected between two parts of a circuit. We are interested in the transfer characteristics of the network. This is defined by the transfer function, $T(\omega)$, which will be a function of the angular frequency. In our case, the black box will contain some network involving resistors, capacitors and inductors. Since it will not contain any active devices like a battery or an amplifier, these are known as passive filters. A passive filter cannot make the input signal bigger; it can only attenuate the signal or leave the input unmodified. The output voltage is related to the input voltage at some frequency by the equation

$$v_{out} = T(\omega) v_{in} \tag{1}$$

where v_{in} and v_{out} represent complex voltage phasors. The magnitude of the transfer function determines the ratio of the output magnitude and the input magnitude.

$$|T(\omega)| = \left| \frac{v_{out}}{v_{in}} \right| \tag{2}$$

The input signal may contain a range of frequencies. For example, the audio band is the range of frequencies 20 Hz – 20 kHz. This would be the range of frequencies present if the voltage signal was being used to drive a loudspeaker. The transfer function tells us what fraction of the signal is passed at a particular frequency.

Figure 10.1 shows what the magnitude of the transfer function would look like for three types of filter. These are idealised transfer characteristics, showing a sharp drop in the transfer function at a particular frequency. It is actually quite difficult to build a filter with this kind of "brick wall" behaviour. However, these sketches show what we are trying to achieve. The low pass filter passes signals with frequencies from as low as zero (d.c.) up to some cut-off frequency. A high pass filter only passes signals above some cut-off frequency. A band pass filter, passes a range of frequencies between two cut-off frequencies. (There is also such a thing as a band stop filter, sometimes called a notch filter. What do you think the transfer characteristics of such a filter would look like?)

3. The decibel scale

Filters are designed to attenuate signals in one part of the frequency spectrum, while passing signals in another. Because the signals may be attenuated by a large factor (many orders of magnitude) it is convenient to plot the magnitude of the transfer function on a log scale. A standard way of doing this, is to express the transfer function in terms of decibels.

The decibel scale is defined by the logarithm of a ratio of powers.

$$\mathrm{dB} = 10\log_{10}\left(\frac{P_2}{P_1}\right) \tag{3}$$

If P_2 would be ten times greater than P_1, then equation (3) would give a ratio of 10 dB. P_2 is said to be 10 dB up on P_1. If P_2 would be one hundred times greater than P_1, then P_2 would be 20 dB up. If P_2 would be ten times smaller than P_1, then equation (3) would give a ratio of –10 dB. In this case, P_2 would be 10 dB down on P_1.

The transfer function is a ratio of two voltages, not a ratio of powers. However, we can relate the voltage to the power that would be dissipated in a resistor using the well-known relationship P = V^2/R. Hence, for a ratio of voltages the decibel equivalent becomes.

$$\mathrm{dB} = 20\log_{10}\left(\frac{V_2}{V_1}\right) \tag{4}$$

Figure 10.2 shows the value in decibels (dB for short) that correspond to various voltage ratios. If the output voltage (V_2) is 1000 times larger than the input (V_1), this corresponds to a gain of 60 dB. If the output is 1000 times smaller than the input, this corresponds to a loss of –60 dB. The key values that are really worth remembering are: ±20 dB corresponds to a factor of 10 gain or loss, and ±3 dB corresponds to a gain or loss by a factor of the square-root of 2. In particular, a loss of –3 dB represents a decrease in the power by a factor of two.

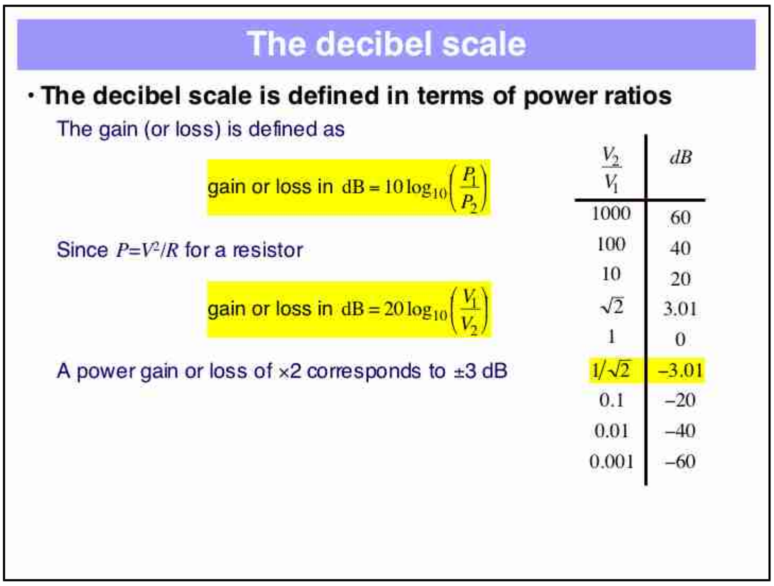


Figure 10.2: The decibel scale.

4. Filter circuits

We are now ready to consider what might put inside the “black box” in Figure 10.1.

4.1 RC low pass filter

It turns out that we can make a useful low pass filter using just a resistor and a capacitor. The required circuit is shown in Figure 10.3.

To begin with, let’s just think about how the circuit works. The circuit’s frequency dependent behaviour will be determined by the behaviour of the capacitor (the only reactive element in the circuit). If the input voltage varies very slowly (low frequencies) then the capacitor will have time to charge up and most of the input voltage will be measured across it. If the input voltage varies very quickly (high frequencies) then we know that since the voltage across the capacitor cannot change instantaneously, most of the input voltage will appear across the resistor. If we take the output across the capacitor, then we will find that high frequency signals are attenuated, while low frequency signal will be passed. Exactly what we mean by “high” and “low” frequency will depend on the time constant of the circuit.

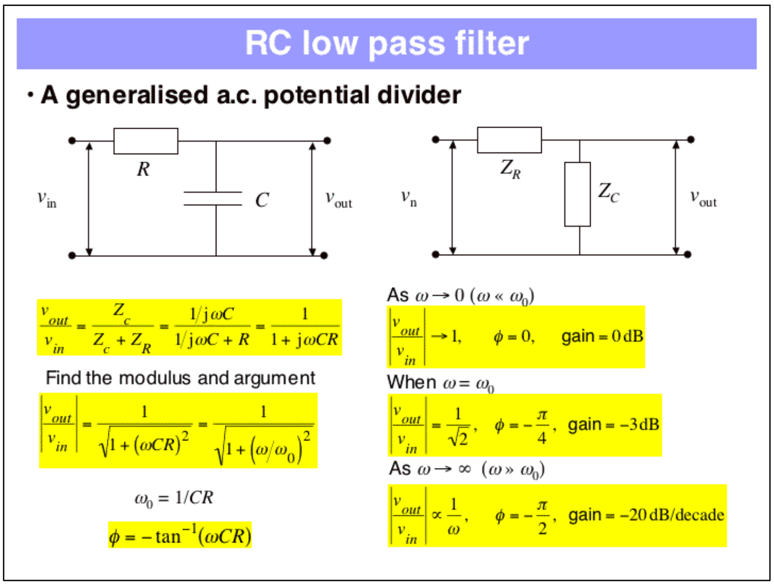


Figure 10.3: An RC low pass filter.

Great! So, how do we analyse this circuit? Well, the most convenient way is to use complex impedances. This allows us to treat the circuit as a potential divider, just as we would if this were a d.c. problem and the network consisted of two resistors. The potential divider rule tells me that the output voltage will be proportional to the ratio of the impedance of the capacitor to the total impedance of the circuit.

$$\frac{v_{out}}{v_{in}} = \frac{Z_c}{Z_c + Z_R} = \frac{1/\mathrm{j}\omega C}{1/\mathrm{j}\omega C + R} = \frac{1}{1 + \mathrm{j}\omega CR} \tag{5}$$

Notice that I have multiplied top and bottom by $\mathrm{j}\omega C$ to simplify this expression. We now need to find the magnitude of the transfer function. We can also determine argument of the transfer function, which represents the phase difference between the output and the input voltage. You should be able to show that

$$|T(w)| = |\frac{v_{out}}{v_{in}}| = \frac{1}{\sqrt{1+(wCR)^2}} = \frac{1}{\sqrt{1+(w/w_0)^2}} \text{ where } w_0 = \frac{1}{CR}$$

$$\phi = \tan^{-1}(-wCR) = \tan^{-1}(-w/w_0) \tag{6}$$

In the last step, I have made the substitution $\omega_0 = 1/CR$. This can be thought of as the natural frequency of the circuit and is determined by the reciprocal of the time constant. We can now explore the frequency dependence of the transfer function and the phase, by examining the expressions shown in equation (6) in the limits of very low and very high frequency, and when the frequency is equal to ω_0.

(a) At low frequency ($\omega \to 0$), the magnitude of the transfer function is $|T(\omega)| \approx 1$, so that $|v_{out}| = |v_{in}|$. The phase angle is zero, $\phi \approx 0$, so v_{out} is in phase with v_{in}. Thus we have demonstrated that low frequency signals are passed unattenuated and with no change of phase.

(b) At high frequency ($\omega \to \infty$), the magnitude of the transfer function tends to zero, and v_{out} lags v_{in} by 90 degrees. At frequencies much larger than ω_0, the transfer function is proportional to the reciprocal of the angular frequency.

$$\left|T(\omega \gg \omega_0)\right| \to \frac{\omega_0}{\omega} \tag{7}$$

This means that or every factor of 10 (or decade) increase in frequency, the ratio $|v_{out}|/|v_{in}|$ drops by another factor of 10 (or –20 dB). In short, we say that the transfer function falls off at –20 dB per decade.

(c) When $\omega = \omega_0$, $|v_{out}|/|v_{in}| = 1/\sqrt{2}$ (or –3 dB) and $\phi = \tan^{-1}(1) = 45$ degrees.

4.2 Bode plots

The best way to visualise this response is by means of a Bode plot. There are two kinds of Bode plot: a plot of the magnitude of the transfer function versus frequency and a plot of the phase versus frequency. The Bode magnitude and phase plots of a typical low pass filter are shown in Figure 10.4. I've arbitrarily chosen values for R (1 kΩ) and C (1 μF), which gives a value of $\omega_0 = 10^3$ rad/s.

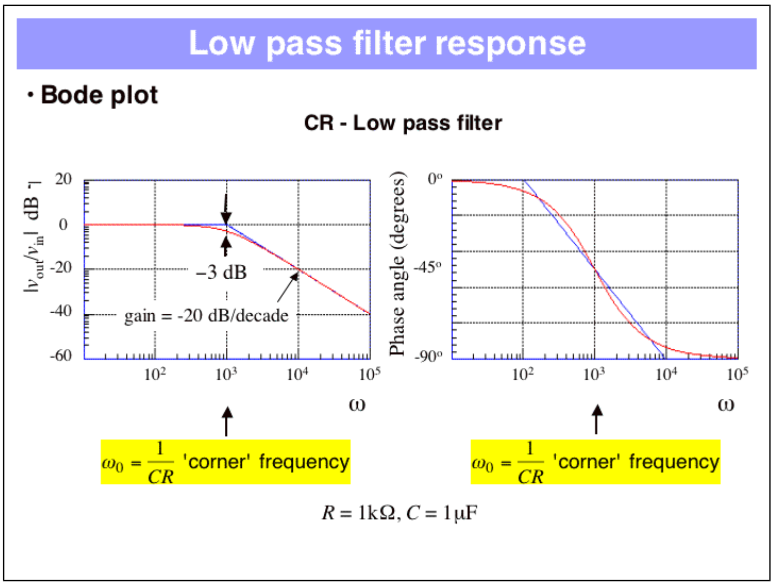


Figure 10.4: The low pass filter response.

The Bode magnitude plot is the log of the ratio $|v_{out}|/|v_{in}|$, usually expressed in dB, plotted against the log of the angular frequency. The Bode phase plot is the phase angle (linear scale) plotted against the log of the angular frequency. Whilst both the transfer function and the phase vary smoothly as a function of frequency, Bode plots can be approximated using straight line segments. These are indicated in Figure 10.4 by the blue lines. The actual response in indicated by the red curve.

The transfer function can be approximated by a straight line at 0 dB ($|v_{out}| = |v_{in}|$) from low frequencies up to the frequency ω_0. Above, ω_0, the transfer function can be approximated by a –20 dB/decade drop ($|v_{out}|$ falls by a factor of 10 (or -20 dB) for every factor of 10 (or decade) increase in the frequency). It can be seen from Figure 10.4 that the straight line approximations are actually a rather good representation of the actual response. The frequency ω_0, which is determined by the components R and C, is the frequency at which the two straight line segments meet. Due to the change in gradient at this frequency, ω_0 is known as the "break" or "corner" frequency. From the preceding analysis we know that the actual response is –3 dB down at the break frequency. This allows us to draw the smooth curve between the two lines.

The Bode phase plot can also be approximated by straight line segments (also shown in Figure 10.4). Here, three line segments are needed. The phase shift can be approximated as zero from low frequencies until within a factor of 10 of the break frequency. The phase shift then drops at 45 degrees per decade until a factor of 10 above the break frequency, whereupon the phase is approximated by a constant 90 degree shift (lagging). At the break frequency the phase shift is exactly –45 degrees.

The fact that the Bode plots can be approximated by straight line segments make them extremely easy to draw. You are expected to learn how to sketch the response of a low pass filter (Figure 10.4) and that of a high pass filter, which we will discuss next. Before leaving Bode plots, it is important to recognise the importance of the break (or corner) frequency. This frequency can be chosen using appropriate values of R and C and determines where the transfer function begins to attenuate the signal at loss of –20 dB/decade. At this frequency the transfer function is down by –3 dB on its low frequency value and the phase is –45 degrees.

4.3 RC high pass filter

It turns out that the combination of R and C can also be used to build a high pass filter. This shouldn't come as a great surprise if you ask yourself where does the input voltage appear at high frequency in the previous example? If the input voltage does not fall across the capacitor, it must fall across the resistor. This suggest that a high pass filter can be made by interchanging the positions of R and C in the two-port network, as shown in Figure 10.5.

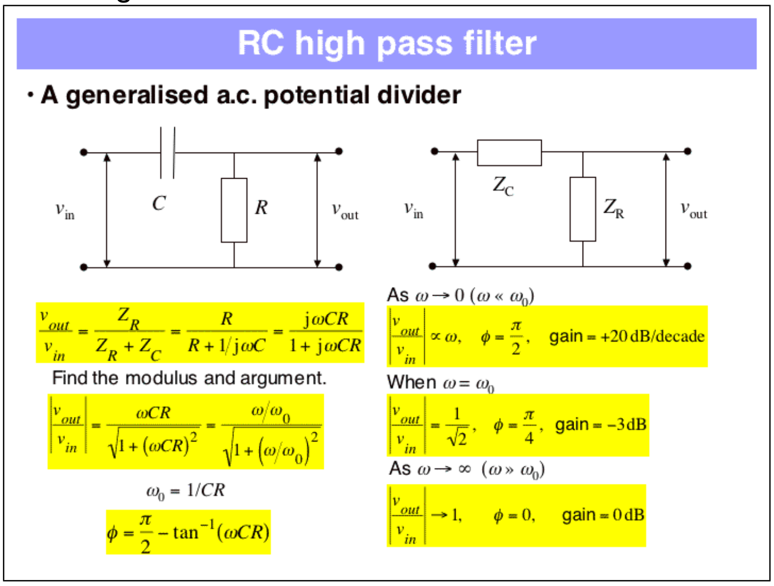


Figure 10.5: An RC high pass filter.

The analysis of this configuration of R and C is the same as for the low pass filter. The transfer function can be derived by using complex impedances and by treating the circuit as a simple potential divider. Once again, I've multiplied the top and bottom of the transfer function by $j\omega C$ to simplify the expression. In this case, the result is that the transfer function is given by

$$T(\omega) = \frac{v_{out}}{v_{in}} = \frac{\mathrm{j}\,\omega CR}{1 + \mathrm{j}\,\omega CR} \tag{8}$$

This expression involves a complex number in the numerator and the denominator. To find the magnitude of the transfer function, just find the magnitude of the numerator and the denominator individually and form the ratio. This is simpler and easier than expressing the transfer function in the form $z = a + \mathrm{j}b$, and then finding the magnitude of a single complex number. Try it for yourself and see. With a little bit of rearranging you should be able to show that

$$|T(\omega)| = \left|\frac{v_{out}}{v_{in}}\right| = \frac{\omega CR}{\sqrt{1 + (\omega CR)^2}} = \frac{\omega/\omega_0}{\sqrt{1 + (\omega/\omega_0)^2}} \tag{9}$$

where, once again, I have made the substitution $\omega_0 = 1/CR$. It gets slightly more complicated to calculate the phase angle. Recall that the phase angle is the argument of transfer function

$$\phi = \arg(T) = \tan^{-1}\left(\frac{\operatorname{Im} T}{\operatorname{Re} T}\right) \tag{10}$$

but, we can only do this if we express the transfer function in the form $z = a + \mathrm{j}b$. However, if we were to write the transfer function in terms of two complex exponentials, z_1 and z_2, we can easily see how we should combine the argument of the numerator and the denominator.

$$T = \frac{z_1}{z_2} = \frac{|z_1| e^{j\phi_1}}{|z_1| e^{j\phi_2}} = \frac{|z_1|}{|z_1|} e^{j(\phi_1 - \phi_2)} \tag{11}$$

The argument transfer function, T, is simply the argument of the numerator minus the argument of the denominator. In our case this is found to be

$$\phi = \phi_1 - \phi_2 = \tan^{-1}\left(\frac{w\,CR}{0}\right) - \tan^{-1}\left(\frac{w\,CR}{1}\right) = \frac{p}{2} - \tan^{-1}(w\,CR) \tag{12}$$

Once again, we can investigate the response of the circuit in the limits of very low and very high frequency, and when $\omega_0 = 1/CR$. This time, I will leave it to you to verify the results summarised in Figure 10.5. (Notice that at very low frequency, ω/ω_0 « 1, the denominator in equation (9) is approximately equal to 1. This means that the transfer function is proportional to ω and increases at a rate of +20 dB/decade.)

It is quite straightforward to show that this configuration of R and C behaves like a high pass filter. The Bode magnitude and phase plots for this circuit as shown in Figure 10.6, where, again, I've chosen values for R (1 kΩ) and C (1 μF) to give $\omega_0 = 10^3$ rad/s.

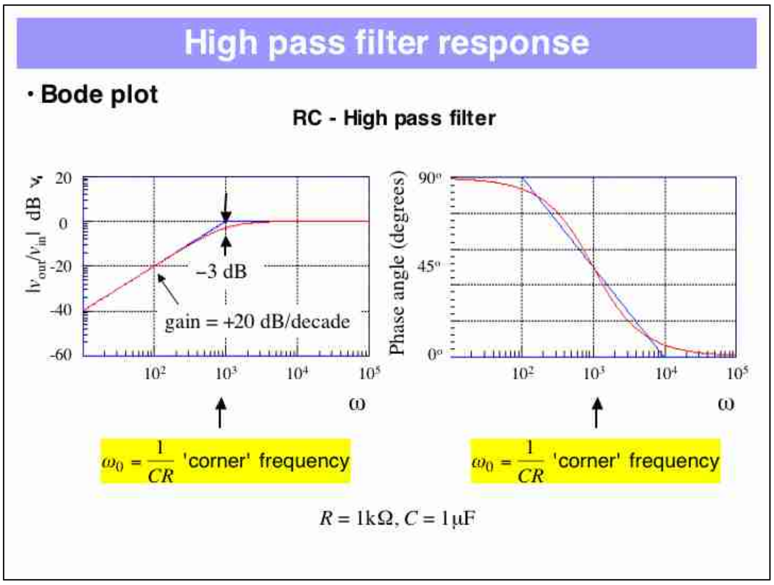


Figure 10.6: The high pass filter response.

Notice how the response can be approximated using straight line segments. In this case, the transfer function increases by a factor of 10 (20 dB) for an increase in frequency by a factor of ten (per decade) at low frequency. At the break or corner frequency, the transfer function reaches a value of unity (0 dB), and stays constant at higher frequency. At low frequency, the output leads the input by 90 degrees until a decade below the break frequency. The phase then drops by 45 degrees per decade, reaching zero a decade above the break frequency, whereupon the phase stays constant. The location of the break or corner frequency is determined by the choice of R and C, as was the case for the low pass filter.

5. Example problem

By the end of this lecture, I hope you will be able to use complex impedances to determine the transfer function of simple circuits involving R, C and L. Because the impedance of a capacitor and an inductor is complex, the transfer function will be complex. You should be able to determine the magnitude and argument of the transfer function and be able to identify the response of the circuit in the limits of low and high frequency. You should also be able to sketch the magnitude of the transfer function and the phase difference between the output and input signals versus frequency using Bode plots.

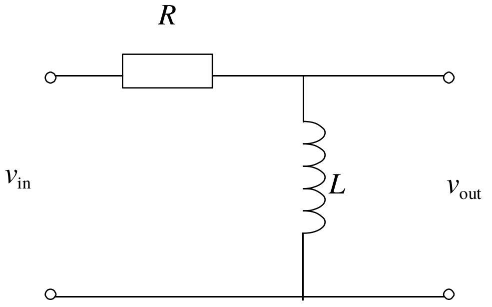
You should be able to quickly identify the type of filter using your understanding of how capacitors and inductors behave in the limits of low and high frequency. At **low** frequency, the **capacitor** behaves like an **open-circuit**; the **inductor** behaves like a **short-circuit**. At **high** frequencies, the **capacitor** behaves like a **short-circuit**; the **inductor** behaves like an **open-circuit**. In a series circuit all the input voltage will appear across the open-circuit. This should help you to decide whether you have a low or a high pass filter.

In numerical examples one may ask you to design a filter based upon certain criteria. For example, let's say that you are asked to design a high pass RC filter with a break frequency of 80 Hz (approximately 500 rad/s) and having an input impedance of 1 kΩ at high frequency. The input impedance is that seen by v_{in} with no load attached. In this case, the impedance of the network is $Z = R + 1/(j\omega C)$ (a combination of a resistor and a capacitor in series). At high frequency, the impedance is approximately given by $Z = R$. So, when we are told what the input impedance is 1 kΩ, this tells us the value of the resistance. Since the break frequency is given by $\omega = 1/(RC)$ and we now know ω and R, we can find C = 2 μF.

6. Try it for yourself

In order to help you get to grips with this week's material, and get used to working with complex numbers, I would like you to repeat this exercise replacing the capacitor in Figures 10.3 and 10.5 with an inductor. You should be able to derive the expressions summarised below. Notice that when you make the substitution $\omega_0 = R/L$ you are left with exactly the same expression for the low and high pass RL filter as we found for RC filters.

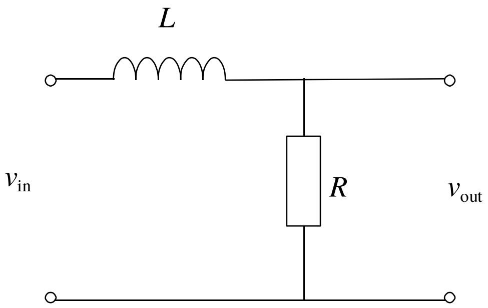
Once again, ω_0 is just the reciprocal of the time constant of the circuit. You can therefore build an equivalent filter using either an RC circuit or an RL circuit. However, the filters will have different input impedances in the limit of low and high frequency. The choice of circuit is usually determined by which configuration best matches the input, based upon a consideration of the maximum power transfer.



High pass

$$|T(w)| = \frac{wL}{\sqrt{R^2 + (wL)^2}} = \frac{w/w_0}{\sqrt{1 + (w/w_0)^2}}$$

$$\phi = \frac{\pi}{2} - \tan^{-1}\left(\frac{wL}{R}\right) = -\frac{\pi}{2} - \tan^{-1}\left(\frac{w}{w_0}\right)$$

	**Low pass** $\lvert T(w)\rvert=\frac{R}{\sqrt{R^2+(wL)^2}}=\frac{1}{\sqrt{1+(w/w_0)^2}}$ $\phi=\tan^{-1}(-w\,CR)=\tan^{-1}(-w/w_0)$

8. Further reading

Unfortunately, Tipler does not cover complex impedances. You can find more information in Powell's "Introduction to electric circuits" and Silvester's "Electric circuits". Silvester's book gives a particularly good introduction to the concept of complex impedances.