Introduction to Probability

Lecture 4

Today

Conditional Probability

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Summary

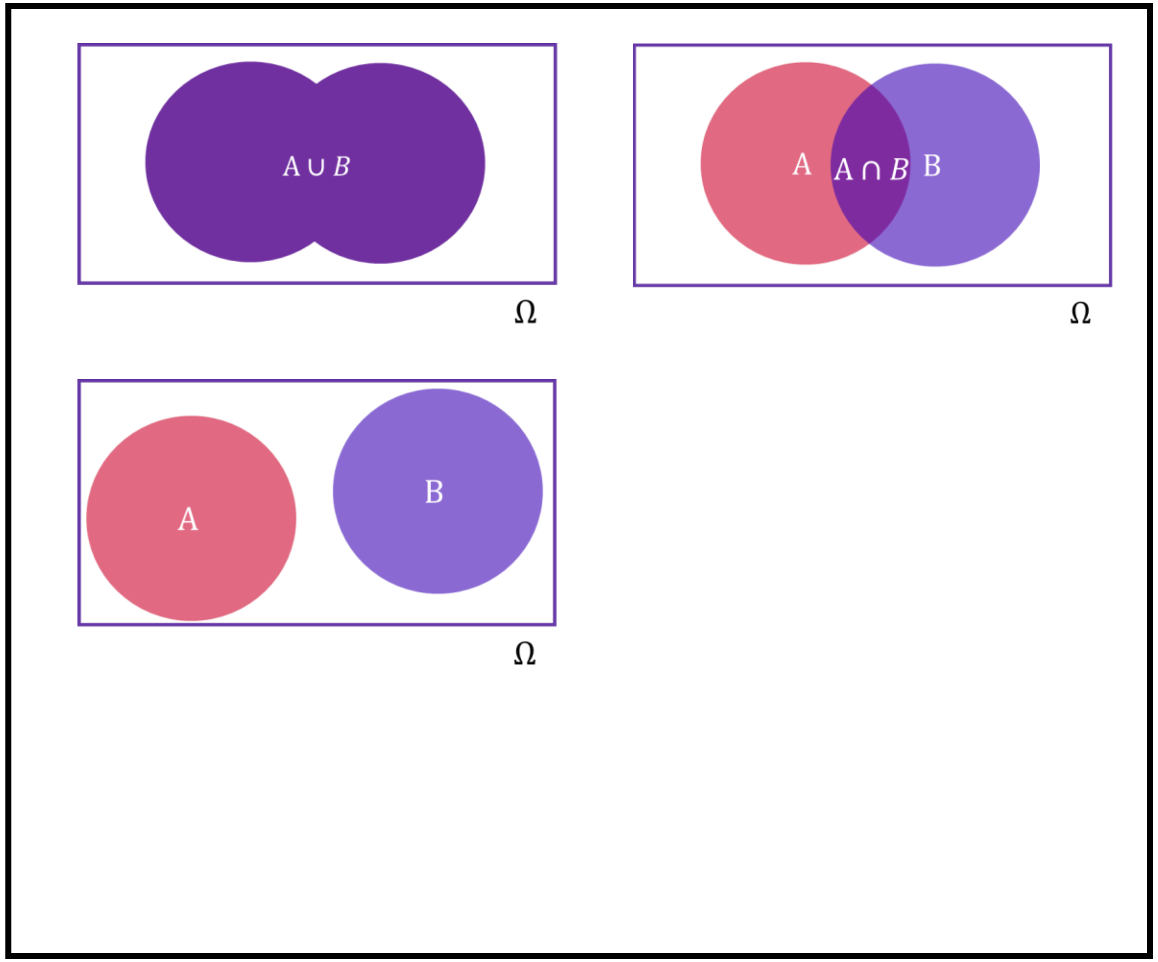
From sets:

Union: $A \cup B$

Intersection: $A \cap B$

Pairwise disjoint $\mathrm{A} \cap A^C = \emptyset$

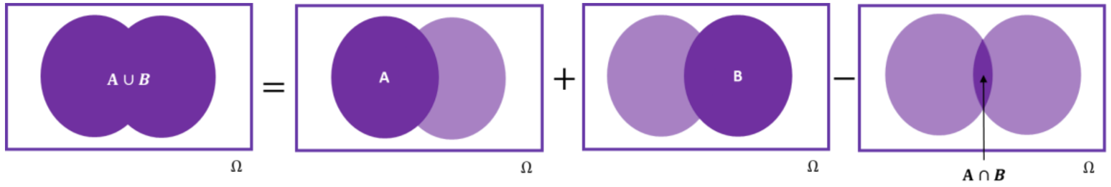
Sample space $\mathrm{A} \cup A^C = \Omega$



Summary (2)

$$P(A \cap B)$$
$$= P(A) + P(B) - P(A \cup B)$$

$$P(e_1 \cup e_2 \cup \cdots \cup e_N)$$
$$= P(e_1) + P(e_2) + \cdots P(e_N)$$



$$|A \cup B| = |A| + |B| - |A \cap B|$$
$$\rightarrow P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

If

$$A \cap B = \emptyset$$

$$P(A \cup B) = P(A) + P(B)$$

Axioms of Probability

Axioms

Given Ω, the probability function $P(x)$ satisfies:

1. $P(x) \geq 0$ for any subset of Ω
2. $P(\Omega) = 1$
3. $P(e_1 \cup e_2 \cup \cdots e_N) = P(e_1) + P(e_2) + \cdots P(e_N)$ provided $e_i \cap e_j = \emptyset$

Conditional Probability

Example

Throw two dice. What is the probability that we see a 4, **given that** the total was 6?

$$\Omega = \begin{matrix} (1,1) & (2,1) & (3,1) & (4,1) & (5,1) & (6,1) \\ (1,2) & (2,2) & (3,2) & (4,2) & (5,2) & (6,2) \\ (1,3) & (2,3) & (3,3) & (4,3) & (5,3) & (6,3) \\ (1,4) & (2,4) & (3,4) & (4,4) & (5,4) & (6,4) \\ (1,5) & (2,5) & (3,5) & (4,5) & (5,5) & (6,5) \\ (1,6) & (2,6) & (3,6) & (4,6) & (5,6) & (6,6) \end{matrix}$$

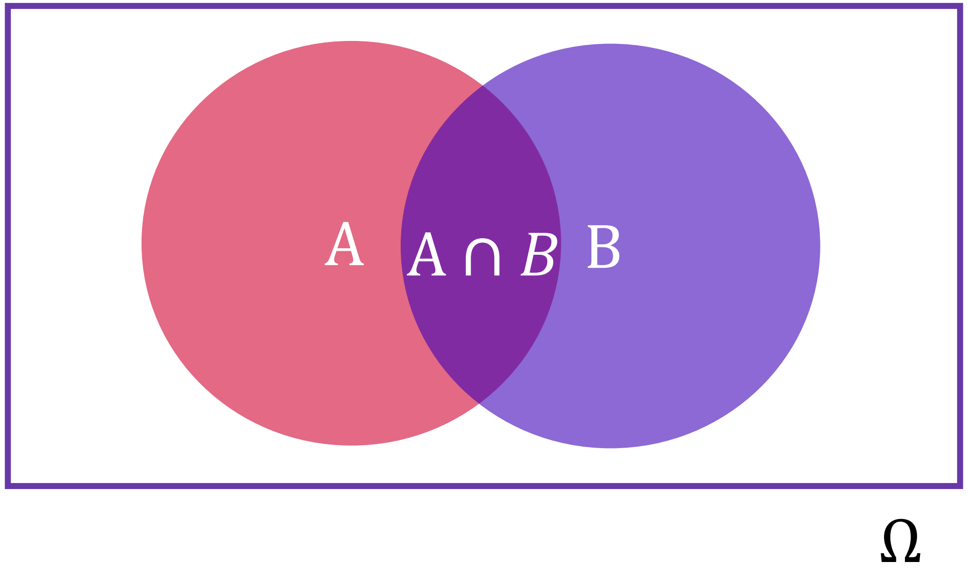
There are two out of five that have a 4.

Definition

The conditional probability of A given B is written $\mathrm{P}(A|B)$ and **defined** by

$$P(A|B) \equiv \frac{P(A \cap B)}{P(B)} \qquad P(B) \neq 0$$

The fraction of B where both A and B happen.



Given that B occurred, what is the probability that A occurred?

$$P(A|B) = \frac{\text{Number of events in } A \text{ and } B}{\text{Number of events in } B}$$

Is it a probability?

Assume P is valid, and define

$$Q(A|B) \equiv \frac{P(A \cap B)}{P(B)}$$

1. $Q(\mathrm{a}|B) = \frac{P(\mathrm{a} \cap B)}{P(B)} \geq 0$ and only 0 $\mathrm{a} \cap B = \emptyset$

2. $Q(\Omega|B) = \frac{P(\Omega \cap B)}{P(B)} = \frac{P(B)}{P(B)} = 1$

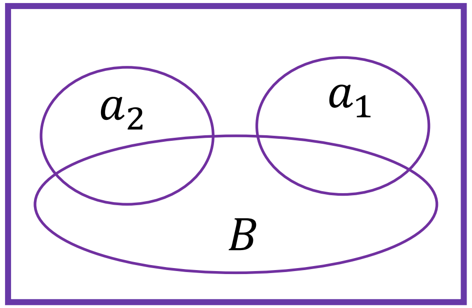
3. Finally is $Q(\mathrm{a}_1 \cup a_2|B) = Q(\mathrm{a}_1|B) + Q(a_2|B)$ if $\mathrm{a}_1 \cap a_2 = \emptyset$?

Is it a probability (2)?

$$Q(a_1 \cup a_2|B) = \frac{P\left((a_1 \cup a_2) \cap B\right)}{P(B)}$$

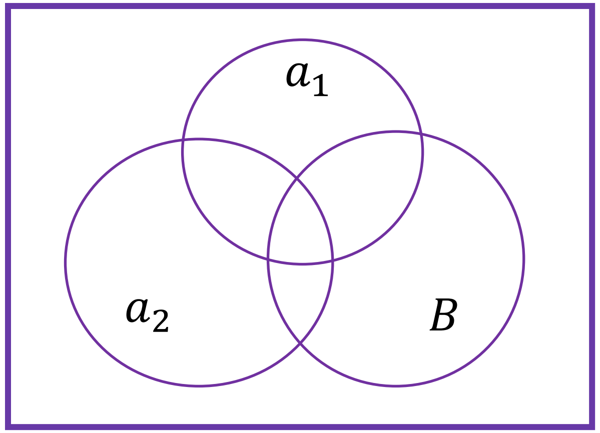
$$= \frac{P\left((a_1 \cap B) \cup (a_2 \cap B)\right)}{P(B)}$$

$(a_1 \cap B)$ and $(a_2 \cap B)$ are themselves disjoint

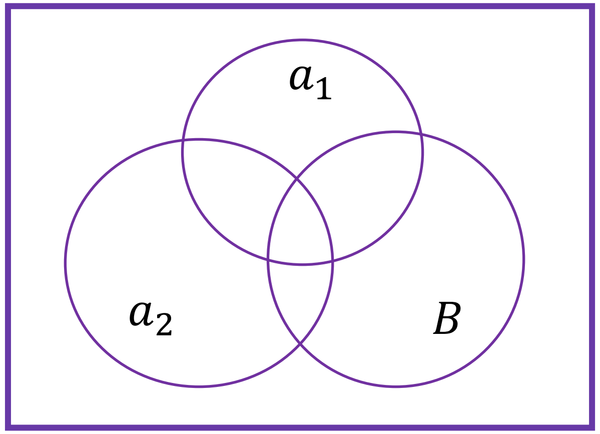


$$\rightarrow \frac{P(a_1 \cap B\,)}{P(B)} + \frac{P(a_2 \cap B\,)}{P(B)}$$

So $Q(a_1 \cup a_2|B) = Q(a_1|B) + Q(a_2|B)$ if $a_1 \cap a_2 = \emptyset$



$(a_1 \cup a_2) \cap B$



$(a_1 \cap B) \cup (a_2 \cap B)$

Example

If $P(A \cup B) = 0.4$, $\mathrm{P(A)} = 0.2$ and $\mathrm{P(B)} = 0.3$, what is $\mathrm{P}(A|\mathrm{B})$ and $\mathrm{P}(\mathrm{B}|A)$?

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

$$P(A \cap B) = P(A) + P(B) - P(A \cup B)$$
$$= 0.2 + 0.3 - 0.4 = 0.1$$

$$\rightarrow P(A|B) = \frac{0.1}{0.3} = \frac{1}{3}$$

$$\rightarrow P(B|A) = \frac{0.1}{0.2} = \frac{1}{2}$$

Example

You toss a coin 7 times. Given that at least 6 heads were observed, what is the probability that the total number of heads was 7?

Use that the probability is proportional to the size of the set.

Guesses:

$1 \qquad \frac{1}{2} \qquad \frac{1}{6} \qquad \frac{1}{7} \qquad \frac{1}{8}$ other?

Let A_n be the even that n heads observed.

$$A_7 = \{HHHHHHH\}$$
$$A_6 = \{THHHHHH, HTHHHHH \dots\}$$

We have that either A_6 or A_7 occurred and we want

$$P(A_7 | A_6 \cup A_7) = \frac{|A_7|}{|A_6| + |A_7|} = \frac{1}{8}$$

Reconditioning

Reconditioning

We can use $P(A|B)$ to work out $P(B|A)$.

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \rightarrow P(A \cap B) = P(A|B)P(B)$$

$$P(B|A) = \frac{P(B \cap A)}{P(A)} \rightarrow P(B \cap A) = P(B|A)P(A)$$

$$B \cap A = A \cap B$$

$$\rightarrow P(B|A) = \frac{P(A|B)P(B)}{P(A)}$$

Example

A rare cold-like disease has symptoms with probability 0.95.

The probability of having the disease is 0.0001.

The probability of having the cold-like symptoms is 0.4.

Given that someone has the symptoms, what is the probability they have the disease?

Let s be symptoms and d be has disease

$$P(s|d) = 0.95$$
$$P(d) = 0.0001$$
$$P(s) = 0.4$$

$$P(d|s) = \frac{P(s|d)P(d)}{P(s)} = \frac{0.95 \times 0.0001}{0.4} \approx 0.0002$$

Class Example

An email spam filter works by looking for keywords.

If:

1. 50% of emails are spam
2. 10% of spam contain “refinance”
3. 6% of all emails contain “refinance”

What is the probability of an email being spam if it contains “refinance”?

$$P(\text{Spam}) = \frac{1}{2}$$

$$P(\text{"refinance"}|\text{Spam}) = \frac{1}{10}$$

$$P(\text{"refinance"}) = \frac{6}{100}$$

$$\rightarrow P(\text{Spam}|\text{"refinance"}) = \frac{P(\text{"refinance"}|\text{Spam})P(\text{Spam})}{P(\text{"refinance"})}$$

$$= \frac{\frac{1}{10} \times \frac{1}{2}}{\frac{6}{100}} = \frac{5}{6}$$

Statistical Independence

Independence

If knowing that B happened has no impact on A then

$$P(A|B) = P(A)$$

In which case

$$P(A \cap B) = P(A|B)P(B) = P(A)P(B)$$

This is called **statistical independence**.

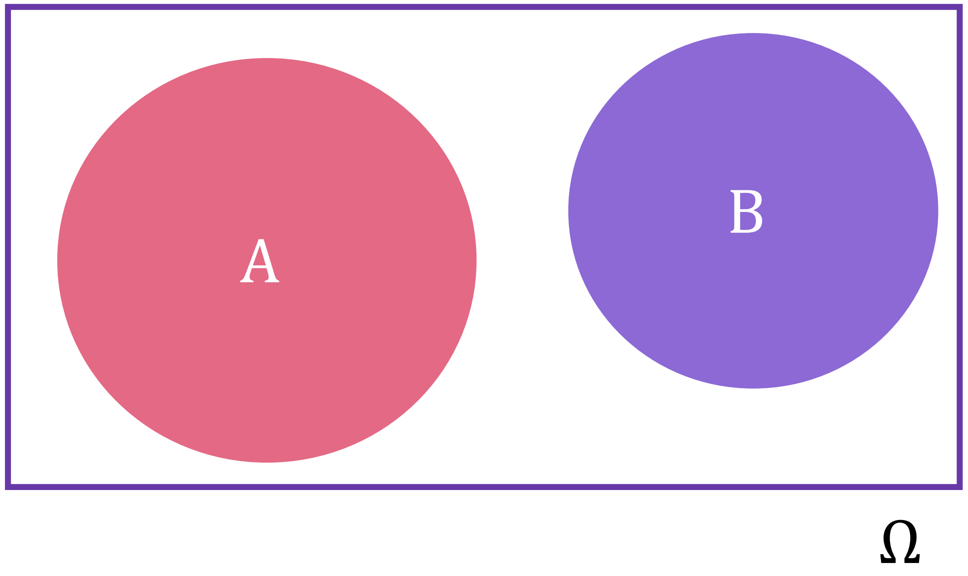
$$P(x_1 \cap x_2 \cap \cdots \cap x_N) = P(x_1)P(x_2) \dots P(x_N)$$

Independence and Mutually Exclusive

Mutually Exclusive

$$P(A \cup B) = P(A) + P(B)$$

Comes from set theory



Statistical Independence

$$P(A \cap B) = P(A)P(B)$$

Comes from conditional probability

Example

The probability of rain at the weekend is 25% for Saturday and 25% for Sunday.

If the rain is assumed to be independent on each day, is the probability of rain over the weekend 50%?

Remember

$$P(e_1 \cup e_2) = P(e_1) + P(e_2) - P(e_1 \cap e_2)$$

$$P(e_1 \cap e_2) = P(e_1)P(e_2)$$

$$P(\text{Rain}) = \frac{1}{4} + \frac{1}{4} - \frac{1}{4} \times \frac{1}{4} = \frac{7}{16}$$

Class Example

In a garden there are 100 flowers. 40 of them are pretty, and 10 are rare. 4 are both rare and pretty.

Is rare and pretty independent in the garden?

$$P(\text{pretty}) = \frac{40}{100}$$

$$P(\text{rare}) = \frac{10}{100}$$

$$P(\text{pretty} \cap \text{rare}) = \frac{40}{100} \times \frac{10}{100} = \frac{4}{100}$$

Yes!

Summary

$$P(A|B) \equiv \frac{P(A \cap B)}{P(B)}$$

If events are **independent** then

$$P(A \cap B) = P(A)P(B)$$

And for many events:

$$P(e_1 \cap e_2 \cap \cdots e_N) = P(e_1)P(e_2) \ldots P(e_N)$$

Examples

Examples

The probability of rain on Wednesday is 0.4

The probability it rains on Thursday is 0.6

The probability that both days are not raining is 0.2

What is the probability it rains on Thursday given that it rained on Wednesday?

Note: $P(A^c \cap B^c) = 1 - P(A \cup B)$

Let W be rain on Wednesday and T for Thursday.

$$P(W) = 0.4, \qquad P(T) = 0.6,$$
$$P(\bar{W} \cap \bar{T}) = 0.2$$

We want $P(T|W)$

$$P(\bar{W} \cap \bar{T}) = 1 - P(W \cup T) \rightarrow P(W \cup T) = 0.8$$

Then

$$P(W \cap T) = P(W) + P(T) - P(W \cup T) = 0.2$$

And so

$$P(T|W) = \frac{P(W \cap T)}{P(W)} = \frac{0.2}{0.4} = \frac{1}{2}$$

Class Example

Polygraphs are used to screen people. Each individual either tells the truth or lies. The (imperfect) polygraph returns 0 if it thinks the person is lying, or 1 for truthfulness, with the following probability:

$$P(1|\text{Truth}) = 0.86$$

People don't lie commonly, approximately 1% of the time and the probability the machine returns 0 is 0.15.

What is the probability that the polygraph wrongly accuses someone of lying?

We have

$$\mathrm{P(Lies)} = 0.01; \mathrm{P}(0) = 0.15; \mathrm{P}(1|\text{Truth}) = 0.86$$

$$\mathrm{P}(\text{Truth}|0) = \frac{\mathrm{P}(0|\text{Truth})P(\text{Truth})}{P(0)}$$

$$= \frac{(1-0.86)\times(1-0.01)}{0.15}$$

$$\approx 0.92$$

Example

An insurance company has the following data about accidents

	Number Of Accidents	Number of Journeys
Young	10	1000
Old	1	500

If an accident happens, what is the probability (according to this data) of the accident involving a young person?

Let A be accident and Y be young.

We want $\mathrm{P(Y}|A) = \frac{P(Y \cap A)}{P(A)}$

The probability of an accident is

$$P(A) = \frac{11}{1500}$$

$$P(Y \cap A) = \frac{10}{1500}$$

So

$$\mathrm{P(Y}|A) = \frac{10}{11}$$