4 Heat transfer

The problems are roughly in order of difficulty. The ones with ♣ are the hardest ones, which might only occur as a "sting in the tail" at the end of a long examination question.

(Gases at 0° C; others at about room temperature)

	kcal/s·m·C°	J/s·m·C°
Metals		
Aluminum	4.9×10^{-2}	20×10^{1}
Brass	2.6×10^{-2}	11×10^{1}
Copper	9.2×10^{-2}	39×10^{1}
Lead	8.3×10^{-3}	35
Silver	9.9×10^{-2}	41×10^{1}
Steel	1.1×10^{-2}	46
Gases		
Air	5.7×10^{-6}	2.4×10^{-2}
Hydrogen	3.3×10^{-5}	1.4×10^{-1}
Oxygen	5.6×10^{-6}	2.3×10^{-2}
Others		
Asbestos	2×10^{-5}	8×10^{-2}
Concrete	2×10^{-4}	8×10^{-1}
Cork	4×10^{-5}	17×10^{-2}
Glass	2×10^{-4}	8×10^{-1}
Ice	4×10^{-4}	17×10^{-1}
Wood	2×10^{-5}	8×10^{-2}

Figure 4.1: Table of thermal conductivities. Credit Resnick and Halliday *Physics* 1978.

Problem 4.1 **Simple conduction**
Consider a copper rod connecting two heat baths at $T_2 = 125$°C and $T_1 = 0$°C. It is thermally insulated (lagged) so there are no losses from its sides. The cross-sectional area is 1 cm^2 and its length is 25 cm.

Assuming a steady state has been achieved, find:

1. the temperature gradient;
2. the rate of heat transfer;
3. and the temperature of the rod 10 cm from the high temperature end.

Problem 4.2 **Melting ice**
A cylindrical copper rod of length 1.2 m and cross-sectional area 4.8 cm^2 is insulated on its curved surface to prevent heat loss laterally. The ends are maintained at an energy difference of 100°C by having one end in a water-ice mixture and the other end in boiling water and steam.

1. Find the rate at which heat is transferred along the rod.
2. Find the rate at which ice melts at the cold end. [You may need to look up one physical value.]

Problem 4.3 **Dry and damp skiers**

1. Calculate the rate at which body heat flows out through the clothing of a skier. Use the following data: the body surface area is 1.8 m^2; the clothing is 1 cm thick; the surface temperature of their body is 33°C and the outside temperature is -5°C; and the thermal conductivity of the clothing is $\kappa_{\text{clothing}} = 0.04$ $\text{Wm}^{-1}\text{K}^{-1}$.
2. If the skier falls over so their clothes are now wet (foolishly, they were wearing cotton clothing), how does the rate of heat loss change? [You may need to look up a physical quantity.]

Problem 4.4 **Cylinders and spheres**
The way in which we defined heat transfer was

$$\dot{Q} = -\kappa A \frac{\mathrm{d}T}{\mathrm{d}x} \ .$$

Let us deploy this in symmetric systems in two and three dimensions.

Consider a cylindrical shell of uniform κ lying between r and $r+\mathrm{d}r$. Assume the temperature is the same everywhere on that surface, but varies *radially*. Construct an equation for radial heat flow, and integrate it.

Perform the same exercise for a spherical shell lying between r and $r+\mathrm{d}r$.

Problem 4.5 **The heat equation**
Consider a "one-dimensional" thermal conductor of uniform cross-sectional area A and thermal conductivity κ. Let the specific heat per unit mass be c_v and the density be ρ. Now focus on a thin section through the conductor of length $\mathrm{d}x$. Let the temperature $T(x,t)$ now be a function of both position along the length of the thermal conductor, x and the time, t.

1. Determine the flux of heat through the two faces, using for each face:

$$\dot{Q} = -\kappa A \frac{\partial T}{\partial x} \ .$$

2. Assuming that the thickness, $\mathrm{d}x$ is small what is the net heat flux into the section?
3. How does the temperature, T, of the section change with time, t?

The resulting equation

$$\frac{\partial T}{\partial t} = \frac{\kappa}{\rho c_v}\frac{\partial^2 T}{\partial x^2} = \alpha \frac{\partial^2 T}{\partial x^2} \ ,$$

where α is the thermal diffusivity of the medium, is called the *heat equation*.

4. If the temperature is *linear* in position i.e. $T(x) = \tilde{T}(t)x$, what may be deduced about $\partial T/\partial t$?
5. Thus taking $\tilde{T}(t) = \tau$, a constant, what is the consequence on the variation of $\dot{Q}$ with x? Does this make physical sense?

Problem 4.6 **Series and parallel thermal conductors**
Consider two thermal conductances, with thermal conductivities κ_1 and κ_2, cross-sectional areas A_1 and A_2 and lengths L_1 and L_2.

Initially connect them in *series* between two heat baths at T_1 and T_2.

1. What is the condition that no heat builds up at the boundary between them and hence there is a steady state heat flow?
2. Deduce the temperature at the interface between the two materials.

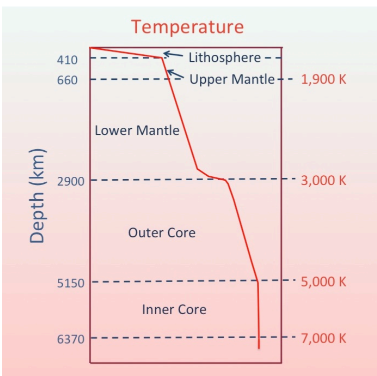
Now put them in *parallel* between the two heat baths, disregarding the inconvenient difference in lengths.

3. What is the heat flow through each conductor? And what is the total thermal conductor? Is this consistent, physically, with the factor of A in the expression for the thermal flux of a single conductor?

Problem 4.7 **Heat equation with internal heat generation**
Sometimes heat is produced within a body as well as conducted by it. Examples are:

1. electrical heating due to a current;
2. heating in the Earth is due to radioactive decays of predominantly ^{232}Th, ^{238}U and ^{40}K, with half lives of $10^{10} - 10^9$ years, which provide half of the heat flux from the interior of the Earth which is estimated to be 91.8 mW m^{-2} at the Earth's surface. This is small compared to the approximately 1 kW m^{-2} incoming from the Sun. But, together with the primordial heat from the collapse of the matter to form the Earth (the other half of the heat budget), it provides the energy source to drive the convection in the Earth's mantle which produces plate tectonics, for example.

Construct the counterpart to the heat equation for a "one-dimensional" body, of constant cross sectional area A, specific heat per unit mass c_V and density ρ. In the body there is such an internal heat source denoted by $\dot{q}$ per unit volume.



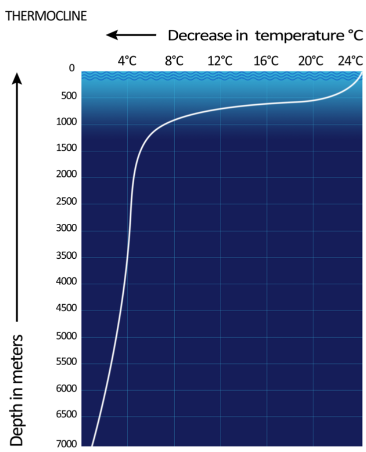


Figure 4.2: Comparison between temperature as a function of depth for the solid Earth (left), the geothermal gradient, and the ocean (right). Credit Wiki.

Problem 4.8 **The Great Ocean Conveyor**
It seems contradictory that the solid Earth gets *hotter* by 25-30 K km^{-1} with increasing depth below the surface, but the ocean gets *colder* by around 18 K in the first km from the surface, and then continues to get colder at a slower rate with increasing depth. After all, the bottom of the ocean should be benefiting from the heat flux from the rock below. See Fig. (4.2)

The heat current across the sea bed into the ocean is around 0.1 Wm^{-2}. However the water is moving at around 1 cm s^{-1} in the "Great Ocean Conveyor" which is a very large convection cell running across the equator with warm water, sinking in the Arctic Ocean and taking a tortuous route via the Antarctic to be heated and upwell in the Indian and Pacific Oceans and return via the Indian Ocean to the South Atlantic and finally pass the UK as the Gulf Stream.

Estimate the distance travelled at the sea bed. Then estimate the increase in temperature of the ocean during that distance due to the heat flux across the sea bed, given the speed is 1 cm s^{-1} and that the ocean from the sea bed upwards for 2 km is involved in the convection current. The specific heat of water is $c_{\text{water}} = 4\ 10^3$ J kg^{-1}K^{-1} and the density of water is 10^3 kg m^{-3}. The Great Ocean Conveyor is shown in Fig. (4.3)

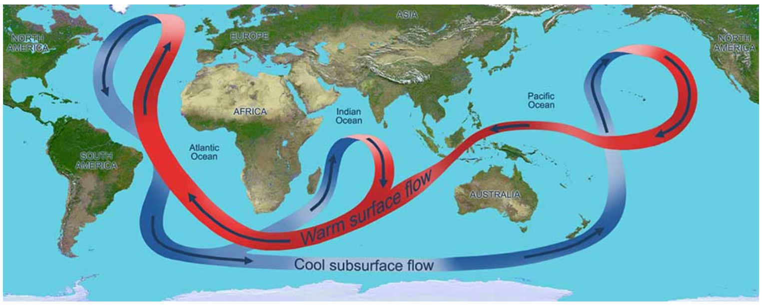


Figure 4.3: The Great Ocean Conveyor. Credit NOAA.

Problem 4.9 **Radiative transport in series with conduction and vacuum flasks**
Consider a cylindrical thermal conductance of thermal conductivity κ, of cross sectional area A and length L, attached to a heat bath at T_1. See Fig. (4.4).

The heat bath and the the side of length L of the conductor are coated with a material of *zero* emissivity, so the only radiative transport is from the end of the cylinder of area A. We assume the emissivity of the end is unity, and the system is in a vacuum black body cavity at $T = 0$.

Find an equation for the temperature of the free end of the conductor (the equation is not very easy to solve).

By algebraic dimensional analysis, or otherwise, show there is a hidden temperature scale, T_0, in the problem.

♣ By defining dimensionless temperatures $\tau_i = T_i/T_0$ with $i = 1, 2$, determine an approximate expression for T_2 when $\tau_i \ll 1$ and separate expression for $\tau_i \gg 1$. Sketch τ_2 as a function of τ_1. (You may disregard the melting temperature of Cu!)

Can you think of a domestic item which uses the disparity of efficiency of radiative and conductive heat transport?

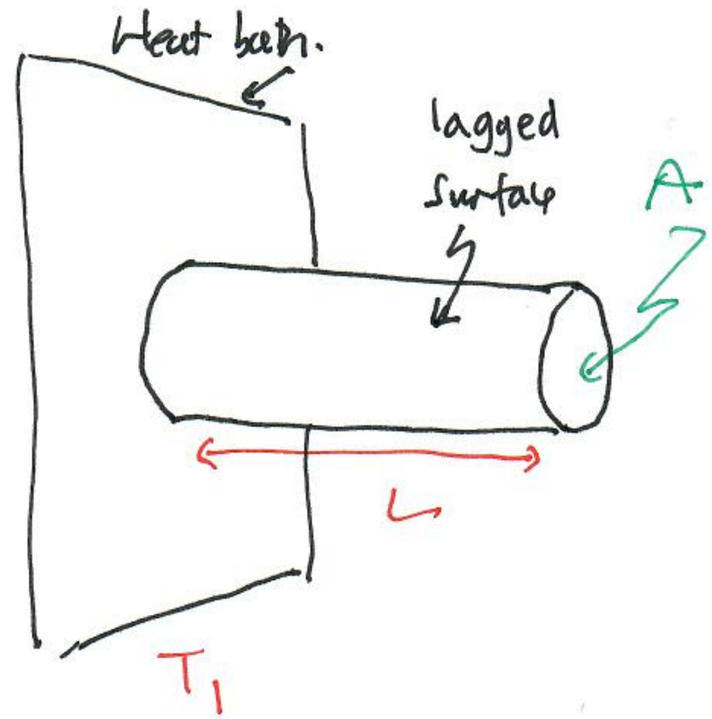


Figure 4.4: A copper rod connected to a heat bath at temperature T_1, cooling radiatively at the other end.

Problem 4.10 **Thermal capacitors and inductors?**
The problem earlier in this sheet about series and parallel conductors implies a similarity with Kirchhoff's laws for electrical (dc) circuits. What are the counterpart laws?

♣ Discuss what meaning might be given to the terms "thermal capacitor" and "thermal inductance" by analogy with *ac electrical circuits*. Are there physical ways to construct such objects?