SEQUENCES AND SERIES

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Contents

These notes consist of two chapters, the first entitled "Sequences" and the second "Series". As will become very apparent, these chapters are intimately connected, and in particular Chapter II will draw heavily on the theory developed in Chapter I.

Before Chapter I begins we devote a few pages to some motivation for the course, along with some preliminary material. Some of this preliminary material is also covered in Yuzhao Wang's notes for the module 1RA "Real Analysis", and we include it here with a view to making these notes as self-contained as possible.

Introduction and Motivation

For many of us, our first encounters with *sequences* and *series* were somewhat recreational. For example, you may have been asked to spot patterns in sequences of numbers such as

$$1, 1, 2, 3, 5, 8, 13, \ldots,$$

or

$$\frac{1}{2}, -\frac{1}{5}, \frac{1}{10}, -\frac{1}{17}, \frac{1}{26}, -\frac{1}{37}, \frac{1}{50}, \ldots.$$

In the context of series of numbers (or "infinite sums of numbers") you may have been encouraged to marvel at famous formulae like

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \cdots = \frac{\pi}{4},$$

or

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{6}.$$

The mathematical theory of sequences and series is much more than recreational, and is at the foundations of the branch of mathematics known as *Analysis*.

Very *informally*:

(1) a *sequence* is an ordered list of numbers

$$a_1, a_2, a_3, \ldots,$$

where the three dots mean that the list never terminates.

(2) a *series* is an expression of the form

$$a_1 + a_2 + a_3 + a_4 + a_5 + \cdots,$$

where the three dots indicate that the summation never terminates.

The purpose of this course is to give a *rigorous* (i.e. *proof-based*) treatment of this subject that will provide you with a deeper understanding and a broad array of skills for solving problems involving sequences and series.

Why is this rigour necessary?

We illustrate this with an example involving sums.

When computing a *finite* sum of numbers

$$a_1 + a_2 + \cdots + a_n,$$

it doesn't matter in which order one does the sum, as we will probably all agree. For example $a_1 + a_2 + a_3 = a_2 + a_3 + a_1$ (here we are considering $n = 3$ merely for the sake of simplicity). However, when summing *infinitely many* numbers, this general rule ceases to apply. For instance, it can be shown that

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} + \frac{1}{11} - \cdots = \log 2,$$

which is about 0.69, but the rearranged sum

$$1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \frac{1}{11} - \frac{1}{6} + \cdots = \frac{3}{2}\log 2. \qquad (!)$$

The point that this example makes is that when we work with *infinite* sums, we cannot simply rely on the rules that govern finite sums. Great care has to be taken when *defining* infinite sums, and there is a need to *prove* any claims we make about them.

What will I be able to do by the end of the module? This module consists of two main chapters, Chapter I entitled *Sequences*, and Chapter II entitled *Series*. At its heart is the notion of *convergence*, and you will learn how to

(i) establish whether a given sequence or series converges, and
(ii) prove general statements involving convergent sequences and series.

Before we get started

In this section we will cover a few basic mathematical notions that we will need along the way. Some of these may be found in Dr Wang's lecture notes, and so are only included here to make the notes suitably self-contained (in particular, I will keep duplication in my lectures to a minimum).

0.1. **Sets and sets of real numbers.** Very often in this module (and in mathematics in general) we will be dealing with *Sets*.

What is a set?

A *set* is a collection of objects. These objects are called *elements* or *members* of the set.

How do we describe a set mathematically?

Often when we are dealing with *finite* sets (sets that have finitely many elements) we simply describe them by listing their elements inside "curly brackets". For example, the set whose elements are the numbers 1, 6 and 2 is written

$$\{1, 6, 2\}.$$

It is not important how we order this list, indeed

$$\{1, 6, 2\} = \{6, 1, 2\} = \cdots \text{ etc..}$$

For a set A we write

$$x \in A$$

if x is an element of A; e.g. $1 \in \{1, 6, 2\}$. By $x \notin A$ we mean that x is not an element of A; e.g $3 \notin \{1, 6, 2\}$.

For more complicated sets than this one we need some more sophisticated notation.

We typically denote a set whose members have a certain property P by

$$\{x : x \text{ has property } P\}.$$

This notation is sometimes called *set-builder notation*.

Examples:

- $\{1, 2, 3, \ldots\}$, or $\{n : n \text{ is a positive whole number}\}$ is called the set of *Natural Numbers* and is denoted $\mathbb{N}$.

- $\{\ldots,-3,-2,-1,0,1,2,3,\ldots\}$, or $\{n : n \text{ is a whole number}\}$ is called the set of *Integers* and is denoted $\mathbb{Z}$.
- $\{m/n : m \in \mathbb{Z}, n \in \mathbb{N}\}$ is called the set of *Rational Numbers* (or *Rationals*) and is denoted $\mathbb{Q}$.

We denote by $\mathbb{R}$ the set of *Real Numbers*. Informally, the set of real numbers is the "completion" of the rationals, or "the rationals with the gaps filled in". The proper definition of the set of real numbers is quite subtle[1].

There are many real numbers that are known not to be rational. For example, the real numbers $\sqrt{2}, \pi \notin \mathbb{Q}$. *All of the sets that feature in this lecture series will be sets of real numbers (sometimes referred to as "subsets of* $\mathbb{R}$*").*

If A and B are sets with the property that every element of A belongs to B; i.e.

$$x \in A \implies x \in B,$$

then we say that A is a *subset* of B (or A is *contained in* B). If A is a subset of B we often write $A \subseteq B$. Notice that

$$\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}.$$

Set-builder notation may be used to neatly describe quite complicated sets. For example, the set of real roots of the quartic polynomial $8x^4 - 3x^3 - x - 54$ may be written as

$$\{x \in \mathbb{R} : 8x^4 - 3x^3 - x - 54 = 0\}.$$

I haven't checked to see whether this particular polynomial has any real roots[2]. If this were the case, we would refer to the set above as the *empty set*. The empty set (the set with no elements) is often denoted $\emptyset$.

There are certain special sets of real numbers that crop up particularly often. These are called *intervals*, and take one of the following forms where a and b denote fixed real numbers:

$$(a,b) = \{x \in \mathbb{R} : a < x < b\}$$
$$[a,b) = \{x \in \mathbb{R} : a \leq x < b\}$$
$$(a,b] = \{x \in \mathbb{R} : a < x \leq b\}$$
$$[a,b] = \{x \in \mathbb{R} : a \leq x \leq b\}$$
$$(a,\infty) = \{x \in \mathbb{R} : a < x\}$$
$$(-\infty,b) = \{x \in \mathbb{R} : x < b\}$$
$$[a,\infty) = \{x \in \mathbb{R} : a \leq x\}$$
$$(-\infty,b] = \{x \in \mathbb{R} : x \leq b\}$$

The sets with only curved brackets are called *open intervals*, and the sets with only square brackets are called *closed intervals*. We don't write $(a,\infty]$, for example, because ∞ is not a real number!

Of particular importance throughout our treatment of sequences and series will be the *Modulus Function.*[3]

[1] If you wish to explore this further you might like to have a look at "Principles of Mathematical Analysis" by Walter Rudin.

[2] It is conceivable that it doesn't. E.g. the quartic polynomial $x^4 + 1$ does not have any real roots.

[3] For the definition of a function see the lectures of Dr. Wang.

0.2. **The Modulus Function and the Triangle Inequality.** For a real number x we define $|x|$ ("the modulus of x") by

$$|x| := \begin{cases} x & \text{if} \quad x \geq 0 \\ -x & \text{if} \quad x < 0. \end{cases}$$

The real function $x \mapsto |x|$ is called *the modulus function*. The following elementary properties are straightforward consequences of the definition.

(i) $|x| \geq 0$ for all x.
$|x| = 0$ (that is to say "equality holds") if and only if $x = 0$.
(ii) Since $|x|^2 = x^2$ for all x it follows that $\sqrt{x^2} = |x|$.
(iii) $-|x| \leq x \leq |x|$ for all x.
(iv) If $\epsilon > 0$ then $|x| < \epsilon$ if and only if $-\epsilon < x < \epsilon$.
(v) $|xy| = |x|\,|y|$ for all x, y.

Distance. From the point of view of this course (and Analysis in general) the most important feature of the modulus function is that it provides us with a very useful formula for the *distance between two real numbers.* If $x, y \in \mathbb{R}$, the *distance between* x and y is simply

$$|x - y|.$$

You should convince yourself of this before continuing too far in these notes.

There is a simple inequality involving the modulus function that we will need to use many times.

Theorem 0.1 (The Triangle Inequality)**.**
For any $a, b \in \mathbb{R}$,

$$|a + b| \leq |a| + |b|.$$

Proof. This is really careful algebra as follows: for $a, b \in \mathbb{R}$

$$ab \leq |ab|, \text{ so that } ab \leq |a||b|.$$

Thus

$$a^2 + 2ab + b^2 \leq a^2 + 2|a||b| + b^2 = |a|^2 + 2|a||b| + |b|^2.$$

Factoring this gives

$$|a + b|^2 \leq (|a| + |b|)^2$$

Taking the positive square root on each side we obtain

$$|a + b| \leq |a| + |b|.$$

$\square$

CHAPTER I: SEQUENCES

1. What is a Sequence?

Recall the *informal* description of a real sequence[4] from above:

A (real) *sequence* is an ordered list of real numbers

$$a_1, a_2, a_3, \dots,$$

where the three dots mean that the list never terminates.

It is not too difficult to formalise this. Notice that our informal description of a sequence involves the assignment of a real number a_n to each natural number n. In other words, a sequence corresponds to a mapping (or *function*) from the natural numbers $\mathbb{N}$ to the real numbers $\mathbb{R}$. This is the function

$$n \mapsto a_n.$$

(Here the symbol "$\mapsto$" should be read "maps to".)

Definition 1.1 (A Sequence)**.** A sequence of real numbers is a function from $\mathbb{N}$ to $\mathbb{R}$.

Although the formal definition of a sequence is a *function*, it is usually convenient to keep the familiar notation

$$a_1, a_2, a_3, \dots,$$

or the more compact (a_n), or $(a_n)_{n=1}^{\infty}$.

This last expression $(a_n)_{n=1}^{\infty}$ is used when there is some room for ambiguity about where the sequence starts. For instance, sometimes it is convenient to start a sequence with a term a_0; if we wished to emphasise this feature we might write $(a_n)_{n=0}^{\infty}$ instead of (a_n).[5] Similarly it might be natural to start with $n = 5$, and in such a situation we might write $(a_n)_{n=5}^{\infty}$.

Since sequences are just functions from $\mathbb{N}$ to $\mathbb{R}$, we can represent them, at least for finitely many values of n, in the form of a graph.

Let's begin with some informal discussion. The most significant feature of a sequence (from the point of view of most mathematical problems involving sequences), is its long-term, or "asymptotic" behaviour; i.e. "how the numbers a_n behave for large n".

What sort of asymptotic behaviour can we imagine?

Well, let's look at some examples of sequences for inspiration.

(1) Consider the sequence

$$1, 2, 3, 4, 5, \dots,$$

[4] In this course all sequences will be *real* sequences, i.e. sequences whose terms consist of real numbers. When we use the term "sequence", it should be understood from the context that we mean "real sequence".

[5] Strictly speaking this involves us changing the definition of a sequence from "a function from $\mathbb{N}$ to $\mathbb{R}$" to "a function from $\mathbb{N}_0$ to $\mathbb{R}$". We shall simply gloss over this sort of technicality as they rarely cause us any difficulties.

that is[6], the sequence (a_n) given by $a_n = n$. This sequence is certainly growing. Moreover, it is growing "without bound"; i.e. the terms in the sequence will eventually exceed any number (however large) that we care to choose. This is an *informal description* of what we will come to term "tending to infinity".

(2) Consider the sequence
$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots,$$
that is, the sequence (a_n) given by $a_n = 1/n$. The terms in this sequence "get closer and closer to some fixed number as n gets larger and larger" – the fixed number being 0 in this case.

Similarly, consider the sequence (a_n) given by $a_n = 1 + (-1)^n 2^{-n}$. The terms in this sequence "get closer and closer to 1 as n gets larger and larger".

These are *informal descriptions* of what we will come to term "converging" or "tending to a limit".

As we will be interested in *proving theorems* about sequences exhibiting such types of asymptotic behaviour, it is vital that we provide *proper mathematical definitions* of them. (You'll probably agree that the informal descriptions of "converging" and "tending to infinity" are a bit hand-wavy.)

2. Sequences Tending to Infinity

Recall the discussion of the sequence (a_n) given by $a_n = n$ above. We noted there that the terms a_n "grow without bound", in the sense that the terms a_n exceed any number we care to choose (however large), provided n is taken large enough. In mathematical terms, the correct way to capture this general idea is as follows.

Definition 2.1 (Tends to infinity)**.** A sequence (a_n) of real numbers *tends to infinity* if given any real number $A > 0$ there exists $N \in \mathbb{N}$ such that
$$a_n > A \quad \text{for all } n > N.$$

Note: We sometimes write "$a_n \to \infty$" or "a_n tends to infinity" instead of "(a_n) tends to infinity".

Remarks. There are some important aspects of the definition that we should pay particular attention to:

- The number N will in general depend on A, as we might expect – the larger the A we choose, the further along the sequence we should expect to have to go in order to ensure that a_n exceeds A.
- We should emphasise the "for all" requirement in the definition. If we were to change this to "for some" then it would change the definition completely. Why?
- We should also emphasise the "any" aspect. Could we change this to "some" without fundamentally changing the definition?

Thinking about these points will help you understand what this definition really means.

[6]If one wishes to describe a sequence, it is not enough to list just the first few terms. One must describe *every* term. In this case there is a nice formula for the nth term of the sequence we have in mind, and so it suffices to give this formula. As we shall see, sometimes sequences are defined rather more implicitly, such as via a recursive formula for example.

You may prefer to write the above definition in a more symbolic manner. For instance, we could have written

Definition (Tends to infinity)**.** A sequence (a_n) of real numbers *tends to infinity* if

$$\forall A > 0 \; \exists N \in \mathbb{N} \; s.t. \; a_n > A \; \forall n > N.$$

Whether you choose to write this in words or symbolically is up to you. However, whichever way you choose, it is vital that you are precise!

Examples.

(1) Prove that the sequence (a_n) given by $a_n = \sqrt{n}$ tends to infinity.

Proof. Let $A > 0$ be given. We are required to find an $N \in \mathbb{N}$ such that

$$\sqrt{n} > A \quad \text{for all } n > N.$$

Notice that if $n > A^2$ then $\sqrt{n} > A$. This suggests choosing N to be any[7] natural number greater than or equal to A^2. With this choice of N we have

$$n > N \implies n > A^2 \implies \sqrt{n} > A,$$

as required. □

You'll notice that our choice of N above followed merely by inspection. This is because we were dealing with a very simple example. Our next example looks a little more complicated, but as we'll see, even here there is a simple way to come up with an N that works.

(2) Prove that the sequence (a_n), given by

$$a_n = \frac{n^2+n}{n+5}$$

tends to infinity. Let us start with a very simple observation:

$$(\dagger) \qquad a_n = \frac{n^2+n}{n+5} \geq \frac{n^2}{n+5} \geq \frac{n^2}{n+5n} = \frac{n}{6}$$

for all $n \in \mathbb{N}$.[8]

Let $A > 0$ be given, and choose N to be any natural number greater than or equal to $6A$. With this choice of N we have

$$n > N \implies n > 6A \implies \frac{n}{6} > A \implies a_n > A,$$

as required. Here in the last implication we have used (†).

Remark. We'll see the use of elementary inequalities such as those in (†) many times in this module. As with (†), these are often based on the simple fact that, for a, b positive real numbers, the quotient $\frac{a}{b}$ shrinks if a shrinks or b grows (or similarly, $\frac{a}{b}$ grows if a grows or b shrinks). Often there are many different inequalities that serve our purpose, indeed in the example above it is also true that $a_n \geq \frac{n}{50}$ for all n. This would have led us to choose N to be a natural number greater than or equal to $50A$ instead of $6A$, but otherwise would have had no effect.

Our proof of the second example above hints at the following general theorem – this is also presented as an "Extra Question" on Problem Sheet 1. This is perhaps not the most exciting

[7] The definition only requires the *existence* of such an N. There is no need for us to come up with the smallest (or "best") N that works.

[8] The point is that, while a_n is a (relatively) complicated expression, it is larger than a very simple expression which is also growing with n. As we will see now, this will allow us to "spot" an N that works.

theorem in the world, but it is a nice illustration of a general statement that we are already in a position to prove.

Theorem 2.2. *Suppose $N_0 \in \mathbb{N}$ and that (a_n) and (b_n) are sequences of real numbers satisfying*

$$a_n \geq b_n \quad \text{for all } n \geq N_0.$$

We may then conclude that if $b_n \to \infty$ then $a_n \to \infty$.

Proof. Let $A > 0$. Since $b_n \to \infty$ there exists $N' \in \mathbb{N}$ such that

$$b_n > A \quad \text{for all } n > N'.$$

Now let $N = \max\{N', N_0\}$ and observe that by the hypotheses of the theorem,

$$a_n \geq b_n > A \quad \text{for all } n > N.$$

Hence $a_n \to \infty$. □

Example. Use the above theorem to prove that the sequence (a_n), given by

$$a_n = 2\sqrt{n} + 5(-1)^n$$

tends to infinity.

Proof. Observe that

$$5(-1)^n \geq -5 \geq -\sqrt{n} \quad \text{for all} \quad n \geq 25.$$

Hence,

$$a_n = 2\sqrt{n} + 5(-1)^n \geq \sqrt{n} \quad \text{for all} \quad n \geq 25.$$

Now, we have already seen that $\sqrt{n} \to \infty$, and so we may apply the above theorem with $b_n = \sqrt{n}$ and $N_0 = 25$, and conclude that $a_n \to \infty$. □

As you may expect, there is a similar definition of what it means for a sequence to "tend to minus infinity".

Definition 2.3 (Tends to minus infinity)**.** A sequence (a_n) of real numbers *tends to minus infinity* if given any real number $A > 0$ there exists $N \in \mathbb{N}$ such that

$$a_n < -A \quad \text{for all } n > N.$$

Note: We sometimes write "$a_n \to -\infty$" or "a_n tends to $-\infty$" instead of "(a_n) tends to $-\infty$".

Remark. A direct comparison of the definitions reveals that $a_n \to -\infty$ if and only if $-a_n \to \infty$ (as you might hope!).

Exercise. Formulate and prove an analogue of Theorem 2.2 involving sequences that tend to $-\infty$.

3. Convergent sequences

Recall the informal and rather crude description: "A sequence (a_n) of real numbers converges to ℓ if a_n gets closer and closer to ℓ as n gets larger".

Let's try to come up with an appropriate formal mathematical definition for this.

First things first – how do we capture "closeness" in mathematical terms?

Well, thanks to the modulus function we have a very nice formula for the distance between two real numbers – if $x, y \in \mathbb{R}$ then the distance between x and y is simply $|x-y|$. So, we can interpret "a_n close to ℓ" as $|a_n - \ell|$ (the distance between a_n and ℓ) being "small".

That's all very well, but what should we *mean* by "a_n gets closer and closer to ℓ"? Well, we ought to mean that the numbers a_n approximate ℓ for large n in some sense. What is an appropriate interpretation of this in mathematical terms?

To begin to answer this it is helpful to think in terms of *errors*. If the terms a_n are to approximate the real number ℓ, then we would like them to satisfy the following reasonable property: If ϵ is any positive quantity (think of ϵ as a small "allowed error"), then provided one goes far enough along the sequence (a_n) (that is, provided $n > N$ for some sufficiently large $N \in \mathbb{N}$), the error in approximating ℓ by a_n (that is, $|a_n - \ell|$) will always be less that ϵ.

The precise definition is the following.

Definition 3.1 (Convergent sequence)**.** A sequence (a_n) of real numbers *converges to a real number* ℓ if given any $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$|a_n - \ell| < \epsilon \quad \text{for all} \quad n > N.$$

We say that a sequence (a_n) *converges* if it converges to ℓ for some[9] $\ell \in \mathbb{R}$.

Note. If a sequence (a_n) converges to a real number ℓ, then we refer to ℓ as the limit of the sequence. We sometimes write $a_n \to \ell$, or

$$\lim_{n\to\infty} a_n = \ell,$$

or "a_n tends to ℓ", instead of "(a_n) converges to ℓ".

Remarks. There are some important aspects of the definition that we should pay particular attention to:

- The number N will in general depend on ϵ, as we might expect – the smaller the $\epsilon > 0$ we choose (the "allowed error"), the further along the sequence we should expect to have to go in order to ensure that the distance between a_n and ℓ is less than ϵ.
- We should emphasise the "for all" requirement in the definition. If we were to change this to "for some" then it would change the definition completely. Why?
- We should also emphasise the "any" aspect. Could we change this to "some" without fundamentally changing the definition?
- By the properties of the modulus function we see that $|a_n - \ell| < \epsilon$ if and only if

  $$\ell - \epsilon < a_n < \ell + \epsilon.$$

  This is of course entirely consistent with our interpretation of $|a_n - \ell|$ as the distance between a_n and ℓ.

Thinking about these points will help you understand what this definition really means.

You may prefer to write the above definition in a more symbolic manner. For instance, we could have written

[9] Remember that $\pm\infty$ should not be considered as elements of $\mathbb{R}$. Therefore, in particular, ℓ cannot be $\pm\infty$ here.

Definition (Convergent sequence)**.** A sequence (a_n) of real numbers *converges to a real number* ℓ if

$$\forall \epsilon > 0 \; \exists N \in \mathbb{N} \; s.t. \; |a_n - \ell| < \epsilon \; \forall n > N.$$

Again, whether you choose to write this in words or symbolically is up to you. However, whichever way you choose, it is vital that you are precise!

Examples.

(1) Prove that the sequence (a_n) given by $a_n = \frac{n+1}{n}$ converges to 1.

Proof. Since the definition of convergence makes reference to the quantity $|a_n - \ell|$, it will be helpful to begin by inspecting this in our situation (that is, with $a_n = \frac{n+1}{n}$ and $\ell = 1$). Observe that

$$(*) \qquad |a_n - 1| = \left|\frac{n+1}{n} - 1\right| = \left|1 + \frac{1}{n} - 1\right| = \left|\frac{1}{n}\right| = \frac{1}{n}.$$

Let $\epsilon > 0$ be arbitrary.[10]
Let N be any natural number greater than $1/\epsilon$.
Now, if $n > N$ then by $(*)$,

$$|a_n - 1| = \frac{1}{n} < \frac{1}{N} < \epsilon.$$

Since ϵ was chosen arbitrarily we conclude that $a_n \to 1$. □

In the example above it was relatively straightforward to choose a natural number N with the required property. This was because of the simplicity of the expression $(*)$. Let us consider a more complicated sequence, and see how we might find an appropriate N there.

(2) Prove that the sequence (a_n), given by

$$a_n = \frac{n^2 + n + 1}{2n^2 + 1}$$

converges to $1/2$.

Proof. Let us begin by making some observations about a_n. Now,

$$a_n - \frac{1}{2} = \frac{n^2 + n + 1}{2n^2 + 1} - \frac{1}{2} = \frac{2n + 1}{2(2n^2 + 1)} > 0,$$

and so

$$\left|a_n - \frac{1}{2}\right| = \frac{2n + 1}{2(2n^2 + 1)}.$$

Next we observe an elementary inequality:

$$(**) \qquad \left|a_n - \frac{1}{2}\right| = \frac{2n + 1}{2(2n^2 + 1)} < \frac{2n + 1}{4n^2} < \frac{2n + n}{4n^2} = \frac{3}{4n},$$

which holds for all $n \in \mathbb{N}$.
We now begin the proof proper.
Let $\epsilon > 0$ be given.
Let N be any natural number greater than $\frac{3}{4\epsilon}$.

[10] We are required to show that the sequence (a_n) given by $a_n = \frac{n+1}{n}$ satisfies the definition of "converging to 1". In order to proceed we look carefully at the definition, and *let it tell us how to structure our argument.* Since it asks that something be true for *any* $\epsilon > 0$, we should begin by writing "Let $\epsilon > 0$ be any positive real number", or equivalently "Let $\epsilon > 0$ be given", or "Let $\epsilon > 0$ be arbitrary". What does the definition ask we do next?

Now, if $n \in \mathbb{N}$ is such that $n > N$, then by $(**)$,

$$\left| a_n - \frac{1}{2} \right| < \frac{3}{4n} < \frac{3}{4N} < \epsilon.$$

Since ϵ was chosen arbitrarily we conclude that $a_n \to 1/2$. □

Some remarks about the inequality $(**)$ are in order. Here the idea was to bound the rather complicated-looking $\frac{2n+1}{2(2n^2+1)}$ by a somewhat simpler-looking quantity which also tends to 0 as $n \to \infty$. The purpose of this was to facilitate our forthcoming choice of N.

The main purpose of the formal definition of convergence (for us at least) is to allow us to prove general theorems involving convergent sequences. A simple, yet rather crucial one is the following.

Theorem 3.2 (Uniqueness of limits)**.** *If a sequence (a_n) converges then it has a unique limit.*

Proof. Suppose for a contradiction that (a_n) has two distinct limits ℓ and m. Let $\epsilon = |\ell - m|/4 > 0$ (that is, one quarter the distance between ℓ and m).

The idea of the proof is as follows: since (a_n) converges both to ℓ and m, a_n is contained *both* in $(\ell - \epsilon, \ell + \epsilon)$ and $(m - \epsilon, m + \epsilon)$ for sufficiently large n. This is a contradiction because the two intervals are disjoint.

Let us formalise this argument.

Since (a_n) converges to ℓ, there exists some $N_1 \in \mathbb{N}$ such that

$$|a_n - l| < \epsilon \quad \text{for all} \quad n \geq N_1.$$

Since (a_n) converges to m, there exists some $N_2 \in \mathbb{N}$ such that

$$|a_n - m| < \epsilon \quad \text{for all} \quad n \geq N_2.$$

Now, if $n > \max\{N_1, N_2\}$ then by the Triangle Inequality,

$$\begin{aligned} |\ell - m| &= |\ell - a_n + a_n - m| \\ &\leq |a_n - \ell| + |a_n - m| \\ &< 2\epsilon \\ &= |\ell - m|/2. \end{aligned}$$

The inequality $|\ell - m| < |\ell - m|/2$ provides the contradiction we were seeking. □

The Triangle Inequality will crop up in proofs like this time after time.

4. Bounded Sequences

Definition 4.1 (Bounded sequence)**.** Let (a_n) be a sequence of real numbers. We say that

(1) (a_n) is *bounded above* if there exists some $M \in \mathbb{R}$ such that $a_n \leq M$ for all $n \in \mathbb{N}$.
(2) (a_n) is *bounded below* if there exists some $M \in \mathbb{R}$ such that $a_n \geq M$ for all $n \in \mathbb{N}$.
(3) (a_n) is *bounded* if it is both bounded above and below; i.e. there exist $M_1, M_2 \in \mathbb{R}$ such that $M_1 \leq a_n \leq M_2$ for all $n \in \mathbb{N}$.

Remarks.

A sequence (a_n) is bounded if and only if there exists some M such that $|a_n| \leq M$ for all $n > N$. This follows directly from the definition and the properties of the modulus function.

Question: Does a bounded sequence necessarily converge?

Simple examples show that the answer to this question is NO. Take for example the sequence (a_n) given by

$$a_n = (-1)^n.$$

This is a bounded sequence, since

$$|a_n| = 1 \leq 1 \quad \text{for all } n \in \mathbb{N}.$$

However, (a_n) does *not* converge.[11]

Although it is not true that every bounded sequence converges, the converse *is* true. This is our next result.

Lemma 4.2 (Convergent sequences are bounded)**.** *Every convergent sequence of real numbers is a bounded sequence.*

Proof. Suppose (a_n) converges to $\ell \in \mathbb{R}$, say. By the definition, there exists $N \in \mathbb{N}$ such that

$$|a_n - \ell| < 1 \quad \text{for all} \quad n > N.$$

(Here we have applied the definition with $\epsilon = 1$.)[12] Now, by the triangle inequality we have

$$|a_n| = |a_n - \ell + \ell| \leq |a_n - \ell| + |\ell| < 1 + |\ell| \quad \text{for all} \quad n > N.$$

Hence

$$|a_n| \leq M \quad \text{for all} \quad n \in \mathbb{N},$$

where $M = \max\{|a_1|, |a_2|, \ldots, |a_N|, 1 + |\ell|\}$; i.e. (a_n) is a bounded sequence. □

5. The Algebra of Limits

As we have seen, a direct use of the definition of convergence to prove that a given sequence converges can be quite laborious and cumbersome. Thankfully there are some theorems which often allow us to establish convergence (and evaluate limits) in a much more routine manner, and without direct recourse to the definition. Our main theorem of this type is the following.

Theorem 5.1 (The Algebra of Limits)**.** *Let $\ell, m \in \mathbb{R}$, and suppose that $\lim a_n = \ell$ and $\lim b_n = m$. Then*

(i) $\lim(a_n + b_n) = \ell + m$,
(ii) $\lim \lambda a_n = \lambda \ell$ *for all* $\lambda \in \mathbb{R}$,
(iii) $\lim a_n b_n = \ell m$,
(iv) *if* $m \neq 0$, $\lim \frac{a_n}{b_n} = \frac{\ell}{m}$.

We shall only prove (i) and (iii) here.

[11] We have not actually *proved* that this sequence does not converge yet. This we shall do before too long.

[12] There is nothing special about this choice of ϵ. For the purposes of this proof we could have chosen any ϵ we liked here.

Proof. (i) Let $\epsilon > 0$ be arbitrary.

We are required to show that there exists $N \in \mathbb{N}$ such that

$$|a_n + b_n - (\ell + m)| < \epsilon \quad \text{for all} \quad n > N.$$

Since $a_n \to \ell$ there exists $N_1 \in \mathbb{N}$ such that

$$|a_n - \ell| < \frac{\epsilon}{2} \quad \text{for all} \quad n > N_1.$$

Similarly, since $b_n \to m$ there exists $N_2 \in \mathbb{N}$ such that

$$|b_n - m| < \frac{\epsilon}{2} \quad \text{for all} \quad n > N_2.$$

Setting $N = \max\{N_1, N_2\}$ we then have that

$$|a_n - \ell| < \frac{\epsilon}{2} \quad \text{and} \quad |b_n - m| < \frac{\epsilon}{2} \quad \text{for all} \quad n > N.$$

Hence by the triangle inequality,

$$|a_n + b_n - (\ell + m)| = |(a_n - \ell) + (b_n - m)| \leq |a_n - \ell| + |b_n - m| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \quad \text{for all} \quad n > N,$$

as required.[13]

(iii) We begin by noting that since (b_n) converges, it is bounded, and so there exists $M > 0$ such that $|b_n| \leq M$ for all $n \in \mathbb{N}$.

Let $\epsilon > 0$ be arbitrary.

We are required to show that there exists $N \in \mathbb{N}$ such that

$$|a_n b_n - \ell m| < \epsilon \quad \text{for all} \quad n > N.$$

Now, since (a_n) converges to ℓ there exists $N_1 \in \mathbb{N}$ such that

$$|a_n - \ell| < \frac{\epsilon}{2M} \quad \text{for all} \quad n > N_1.$$

Similarly, since $b_n \to m$ there exists $N_2 \in \mathbb{N}$ such that

$$|b_n - m| < \frac{\epsilon}{2(1 + |\ell|)} \quad \text{for all} \quad n > N_2.$$

Setting $N = \max\{N_1, N_2\}$ we then have that

$$|a_n - \ell| < \frac{\epsilon}{2M} \quad \text{and} \quad |b_n - m| < \frac{\epsilon}{2(1 + |\ell|)} \quad \text{for all} \quad n > N.$$

[13]This last line, where we express $a_n + b_n - (\ell + m)$ in terms of $a_n - \ell$ and $b_n - m$, explains our use of "$\epsilon/2$" in this argument. We chose to express $a_n + b_n - (\ell + m)$ as $a_n - \ell + b_n - m$ because $|a_n - \ell|$ and $|b_n - m|$ are the *only* quantities that we know something about. In the forthcoming proofs of Parts (iii) and (iv), we will need to do something similar; for example, for Part (iii) we will need to find a way to express $a_n b_n - \ell m$ in terms of $a_n - \ell$ and $b_n - m$. This will be somewhat more complicated, and as a result the "$\epsilon/2$" will be replaced by more complicated expressions.

Hence by the triangle inequality,

$$\begin{aligned}
|a_n b_n - \ell m| &= |a_n b_n - \ell b_n + \ell b_n - \ell m| \\
&= |(a_n - \ell)b_n + (b_n - m)\ell| \\
&\leq |a_n - \ell||b_n| + |b_n - m||\ell| \\
&\leq |a_n - \ell| M + |b_n - m||\ell| \\
&< \frac{\epsilon}{2M} M + \frac{\epsilon}{2(1+|\ell|)}|\ell| \\
&\leq \epsilon
\end{aligned}$$

for all $n > N$, as required. □

Questions.

Why did we omit the proof of Part (ii)?

Why did we use the quantity $\frac{\epsilon}{2(1+|\ell|)}$, rather than simply $\frac{\epsilon}{2|\ell|}$ in the proof of Part (iii)?

Using the algebra of limits. The algebra of limits allows us to reduce the calculation of complicated limits to simpler limits that we have already established, such as $1/n \to 0$ for example. Let us use the algebra of limits to give an alternative proof of

$$\frac{n^2+n+1}{2n^2+1} \to \frac{1}{2}.$$

(We established this limit directly from the definition of convergence in Lecture 5.)

Proof. Observe that

$$\frac{n^2+n+1}{2n^2+1} = \frac{1+\frac{1}{n}+\frac{1}{n^2}}{2+\frac{1}{n^2}} \quad \text{for all} \quad n \in \mathbb{N}.$$

Since $\frac{1}{n} \to 0$, by the algebra of limits we have that

$$\begin{aligned}
\lim_{n\to\infty} \frac{n^2+n+1}{2n^2+1} &= \lim_{n\to\infty} \frac{1+\frac{1}{n}+\frac{1}{n^2}}{2+\frac{1}{n^2}} \\
&= \lim_{n\to\infty} \frac{1+\frac{1}{n}+\frac{1}{n}\cdot\frac{1}{n}}{2+\frac{1}{n}\cdot\frac{1}{n}} \\
&= \frac{\lim_{n\to\infty}\left(1+\frac{1}{n}+\frac{1}{n}\cdot\frac{1}{n}\right)}{\lim_{n\to\infty}\left(2+\frac{1}{n}\cdot\frac{1}{n}\right)} \quad \text{(by the AOL for quotients)} \\
&= \frac{\lim_{n\to\infty} 1 + \lim_{n\to\infty}\frac{1}{n} + \lim_{n\to\infty}\frac{1}{n}\cdot\frac{1}{n}}{\lim_{n\to\infty} 2 + \lim_{n\to\infty}\frac{1}{n}\cdot\frac{1}{n}} \quad \text{(by the AOL for sums)} \\
&= \frac{\lim_{n\to\infty} 1 + \lim_{n\to\infty}\frac{1}{n} + \left(\lim_{n\to\infty}\frac{1}{n}\right)^2}{\lim_{n\to\infty} 2 + \left(\lim_{n\to\infty}\frac{1}{n}\right)^2} \quad \text{(by the AOL for products)} \\
&= \frac{1}{2}.
\end{aligned}$$

□

6. Limits and order

This section considers how limits and inequalities interact – as you will see, all of the statements that we'll prove involve convergent sequences and inequalities. The main theorem that we'll prove is called the Sandwich Theorem (or Squeeze Rule), and this will turn out to be useful sometimes when proving that a given sequence converges (very much as the Algebra of Limits is useful for proving that sequences converge).

Before we get to the Sandwich Theorem, we present another natural and useful theorem of this flavour.

Theorem 6.1 (Limits and order). *Let $N_0 \in \mathbb{N}$. Suppose that $a_n \to \ell$ and $b_n \to m$, and that*

$$a_n \le b_n \quad \text{for all} \quad n \ge N_0.$$

Then $\ell \le m$.

Theorem 6.1 is typically used via the following corollary.[14]

Corollary 6.2. *Suppose that (a_n) is a sequence with upper bound M; i.e. that $a_n \le M$ for all $n \in \mathbb{N}$. If $a_n \to \ell$ then $\ell \le M$.*

Remark. Similarly, if M is a *lower* bound for (a_n), and $a_n \to \ell$, then $\ell \ge M$. So, Corollary 6.2 tells us that the limit of a convergent sequence satisfies any upper or lower bounds that you might have been able to establish for the sequence.

Remark. We cannot replace the inequalities in the statement of the above proposition with strict inequalities (that is, replace $\le$ with $<$ everywhere they occur), and still expect it to be true. For example, if $a_n = -\frac{1}{n}$ and $b_n = \frac{1}{n}$ we have $a_n < b_n$ for all $n \in \mathbb{N}$. However, in this case the limits $\ell = 0$ and $m = 0$ of (a_n) and (b_n), respectively, do *not* satisfy $\ell < m$.

In order to prove Theorem 6.1 we'll first need to establish the following lemma, which you have already seen on Problem Sheet 1 (Question 5).

Lemma 6.3. *Suppose that (a_n) converges to $\ell > 0$ ($\ell < 0$). Then, there exists $N \in \mathbb{N}$ such that $a_n > 0$ ($a_n < 0$) for all $n > N$.*

Let us prove the lemma now, and then proceed to the proof of Theorem 6.1.

Proof. Suppose that $\ell > 0$, as the case where $\ell < 0$ may be argued similarly. Since $a_n \to \ell$, by the definition applied with $\epsilon = \ell/2$, there exists $N \in \mathbb{N}$ such that $|a_n - \ell| < \ell/2$ whenever $n > N$. In other words, $-\ell/2 < a_n - \ell < \ell/2$ whenever $n > N$. In particular, by the first of these two inequalities, we have that $a_n > \ell/2$ whenever $n > N$. Since $\ell > 0$ the lemma follows. $\square$

It is worth thinking about the rationale behind our choice $\epsilon = \ell/2$ above – the idea is that if a_n is within a distance $\ell/2$ of $\ell > 0$, then a_n must be larger than $\ell/2$, which is positive.

We now turn to the proof of Theorem 6.1.

[14] A *corollary* is a statement that follows quickly from a theorem, sometimes so quickly that an explicit proof is taken as self-evident. In this case the corollary follows from the theorem by applying the theorem to the constant sequence $b_n = M$.

Proof. Suppose, for a contradiction, that $\ell > m$.

Let $c_n = a_n - b_n$. By the algebra of limits we have that

$$\lim_{n\to\infty} c_n = \ell - m > 0.$$

Hence by Lemma 6.3 there exists $N \in \mathbb{N}$ such that $c_n > 0$ for all $n > N$. This is a contradiction since $c_n = a_n - b_n \leq 0$ for all $n \geq N_0$, by hypothesis.

Hence we must have that $\ell \leq m$. □

We now come to the main result of this section – the Sandwich Theorem.

Theorem 6.4 (Sandwich Theorem/Squeeze Rule)**.** *Let $N_0 \in \mathbb{N}$ and $\ell \in \mathbb{R}$. Suppose (a_n), (b_n) and (c_n) are sequences satisfying*

$$a_n \leq b_n \leq c_n \quad \textit{for all} \quad n \geq N_0.$$

If $a_n \to \ell$ and $c_n \to \ell$, then $b_n \to \ell$.

Proof. Let $\epsilon > 0$ be arbitrary.

Since $a_n \to \ell$ there exists $N_1 \in \mathbb{N}$ such that

$$\ell - \epsilon < a_n < \ell + \epsilon \quad \text{for all} \quad n > N_1.$$

Similarly, since $c_n \to \ell$ there exists $N_2 \in \mathbb{N}$ such that

$$\ell - \epsilon < c_n < \ell + \epsilon \quad \text{for all} \quad n > N_2.$$

Hence if $N = \max\{N_0, N_1, N_2\}$, then by the hypotheses of the theorem we have both

$$\ell - \epsilon < b_n \quad \text{for all} \quad n > N$$

and

$$b_n < \ell + \epsilon \quad \text{for all} \quad n > N;$$

i.e.,

$$\ell - \epsilon < b_n < \ell + \epsilon \quad \text{for all} \quad n > N.$$

□

Examples.

(i) Prove that $\frac{\sin n}{n} \to 0$.

Proof. This may be done directly from the definition, although we shall use the Sandwich Theorem here.

Observe first that since $-1 \leq \sin x \leq 1$ for all $x \in \mathbb{R}$, we have that

$$-\frac{1}{n} \leq \frac{\sin n}{n} \leq \frac{1}{n} \quad \text{for all} \quad n \in \mathbb{N}.$$

Since

$$-\frac{1}{n} \to 0 \qquad \text{and} \qquad \frac{1}{n} \to 0,$$

we have that

$$\frac{\sin n}{n} \to 0$$

by the Sandwich Theorem. □

Note: We often use the fact that $1/n \to 0$ in this way – that is, without proof. We will permit this assumption when calculating limits *via limit theorems* (such as the *algebra of limits*, or the *Sandwich Theorem*). Similarly we will permit the assumption that $1/n^2$, $1/n^3$ etc. tend to zero in such contexts.

(ii) Prove that

$$\frac{n^6 + n^3 \cos n + 1}{3n^6 + n^4 + \sin n} \to \frac{1}{3}.$$

Proof. Observe first that

$$\frac{n^6 + n^3 \cos n + 1}{3n^6 + n^4 + \sin n} = \frac{1 + \frac{\cos n}{n^3} + \frac{1}{n^6}}{3 + \frac{1}{n^2} + \frac{\sin n}{n^6}}.$$

In order to use the algebra of limits here we need to first consider the behaviour of $\frac{\cos n}{n^3}$ and $\frac{\sin n}{n^6}$ as $n \to \infty$. For these we may appeal to the Sandwich Theorem, as above: Since $-1 \leq \cos x \leq 1$ for all $x \in \mathbb{R}$, we have that

$$-\frac{1}{n^3} \leq \frac{\cos n}{n^3} \leq \frac{1}{n^3} \quad \text{for all} \quad n \in \mathbb{N}.$$

Since

$$-\frac{1}{n^3} \to 0 \quad \text{and} \quad \frac{1}{n^3} \to 0,$$

we have that

$$\frac{\cos n}{n^3} \to 0$$

by the Sandwich Theorem. Arguing similarly we obtain that $\frac{\sin n}{n^6} \to 0$.

Hence by the algebra of limits for sums we have

$$1 + \frac{\cos n}{n^3} + \frac{1}{n^6} \to 1$$

and

$$3 + \frac{1}{n^2} + \frac{\sin n}{n^6} \to 3,$$

and finally by the algebra of limits for quotients we conclude that

$$\frac{n^6 + n^3 \cos n + 1}{3n^6 + n^4 + \sin n} = \frac{1 + \frac{\cos n}{n^3} + \frac{1}{n^6}}{3 + \frac{1}{n^2} + \frac{\sin n}{n^6}} \to \frac{1}{3},$$

as required. □

(iii) Prove that

$$a_n := \frac{1}{n^2 + n + 1} + \frac{1}{n^2 + n + 2} + \cdots + \frac{1}{n^2 + n + n} \to 0.$$

Proof. Note that there are n terms in the sum defining a_n. Furthermore, for each $n \in \mathbb{N}$, a_n is nonnegative and the greatest term in the sum is

$$\frac{1}{n^2 + n + 1}.$$

Hence

$$0 \leq a_n \leq \frac{n}{n^2 + n + 1} < \frac{n}{n^2} = \frac{1}{n} \quad \text{for all} \quad n \in \mathbb{N}.$$

Since $\frac{1}{n} \to 0$, we have that

$$\lim_{n \to \infty} a_n = 0,$$

by the Sandwich Theorem. □

7. Some frequently occurring null sequences

Sequences which converge to zero (such as the one in the last example) are often referred to as *null sequences.* Before we continue with the general theory of sequences, we take a brief look at some important and frequently occurring examples of such sequences.

Theorem 7.1 (Common null sequences)**.** (i) *If* $s > 0$ *then* $\frac{1}{n^s} \to 0$ *as* $n \to \infty$.
(ii) *If* $|\lambda| < 1$ *then* $\lambda^n \to 0$ *as* $n \to \infty$.
(iii) *If* $s \in \mathbb{R}$ *and* $|\lambda| < 1$ *then* $n^s \lambda^n \to 0$ *as* $n \to \infty$.
(iv) *If* $s \in \mathbb{R}$ *then* $\frac{n^s}{n!} \to 0$ *as* $n \to \infty$.
(v) *If* $\lambda \in \mathbb{R}$ *then* $\frac{\lambda^n}{n!} \to 0$ *as* $n \to \infty$.

Due to time constraints we omit the proof of this. Detailed proofs may be found in Hart.

Informal remark. The above theorem tells us that there is a definite hierarchy on rate of growth of certain functions of n. This can be expressed as follows from the most rapid to the slowest growing:

(i)	Factorial growth,	$n!$,	
(ii)	Exponential growth,	λ^n, $(\lambda > 1)$	e.g. 2^n
(iii)	Polynomial growth,	n^s, $(s > 0)$	e.g. n^3
(iv)	Constants,	c	

Factorial terms will always "eventually beat" a power or exponent. As a specific example, eventually $n! > 100^n$. Similarly, powers of n, e.g. (n^3), never grow as rapidly as exponents of n, e.g. (2^n). [15]

However, although the sequences in the above theorem converge to zero, there is nothing to say that they are *decreasing.*

8. The Monotone Convergence Theorem (MCT)

We begin with some simple definitions.

Definition 8.1 (Monotone sequence)**.** A sequence (a_n) of real numbers is

(i) *increasing* if $a_{n+1} \geq a_n$ for all $n \in \mathbb{N}$.
(ii) *strictly increasing* if $a_{n+1} > a_n$ for all $n \in \mathbb{N}$.
(iii) *decreasing* if $a_{n+1} \leq a_n$ for all $n \in \mathbb{N}$.
(iv) *strictly decreasing* if $a_{n+1} < a_n$ for all $n \in \mathbb{N}$.

A sequence that is either increasing or decreasing is said to be *monotone.*

Remark. One slightly curious feature of the definition above is that constant sequences, such as

$$1, 1, 1, 1, 1, 1, \ldots, ; \quad \text{i.e. } a_n = 1 \quad \text{for all} \quad n \in \mathbb{N},$$

are *both* increasing and decreasing.

It is very useful to know whether a particular sequence is monotone because of the following fundamental theorem.

[15] These observations are useful as a *guide to your intuition* about what a given sequence might behave like for large n.

Theorem 8.2 (Monotone Convergence Theorem (MCT)). *Let (a_n) be a sequence of real numbers.*

(i) *If (a_n) is increasing and bounded above, then (a_n) converges.*
(ii) *If (a_n) is decreasing and bounded below, then (a_n) converges.*

Remark. In short, the Monotone Convergence Theorem says that *every bounded monotone sequence converges.*

Proof. We'll prove Part (i) only, as Part (ii) is similar.[16]

Let (a_n) be increasing and bounded above, and consider the set

$$E = \{a_n : n \in \mathbb{N}\}.$$

This is a nonempty subset of $\mathbb{R}$; for example, $a_1 \in E$. Furthermore, the set E is bounded above, as the sequence (a_n) is bounded above. Hence by the Completeness Axiom (or Least Upper Bound Property) – see Dr Wang's 1RA lectures – the set E has a supremum[17], which we denote by ℓ.

We claim that (a_n) converges to ℓ, and we establish this by directly verifying the definition.

To this end, let $\varepsilon > 0$. Since ℓ is the *least* upper bound for E, $\ell - \varepsilon$ is not an upper bound for E, and so there exists an element x_N of E such that

$$x_N > \ell - \varepsilon.$$

Since (a_n) is increasing, it follows that

$$x_n > \ell - \varepsilon \ \text{ whenever } \ n > N.$$

Recalling that ℓ is an upper bound for E we deduce further that

$$\ell - \varepsilon < a_n \leq \ell \ \text{ whenever } \ n > N.$$

In particular,

$$|a_n - \ell| < \varepsilon \ \text{ whenever } \ n > N,$$

as required. $\square$

9. Applying the MCT

There are lots of applications of the Monotone Convergence Theorem. Here we'll demonstrate how it may be used to establish the convergence of some *recursively-defined* sequences.

Example. A sequence (a_n) is defined recursively by the formula

$$a_{n+1} = \frac{1}{2}\left(a_n^2 + \frac{1}{2}\right),$$

with $a_1 = 0$. Prove that:

(i) the sequence (a_n) is bounded above (by 1 for example);
(ii) the sequence (a_n) is increasing;
(iii) the sequence (a_n) converges to $1 - \frac{\sqrt{2}}{2}$.

[16] Actually Part (ii) follows from Part (i) on observing that if (a_n) is decreasing and bounded below, then $(-a_n)$ is increasing and bounded above.

[17] Also known as the Least Upper Bound.

Proof. (i) This we do by induction.
First of all we observe that $a_1 = 0 \leq 1$.
Next suppose that $a_k \leq 1$ for some $k \in \mathbb{N}$, and observe that

$$a_{k+1} = \frac{1}{2}\left(a_k^2 + \frac{1}{2}\right) \leq \frac{1}{2}\left(1^2 + \frac{1}{2}\right) = \frac{3}{4} \leq 1.$$

Hence $a_n \leq 1$ for all $n \in \mathbb{N}$ by the principle of mathematical induction. Hence 1 is an upper bound for (a_n).

(ii) To prove Part (ii) we argue by induction again. Now, by definition, (a_n) is increasing if $a_{n+1} \geq a_n$ for all n.
Observing that $a_2 = \frac{1}{2}\left(a_1^2 + \frac{1}{2}\right) = \frac{1}{2}\left(0^2 + \frac{1}{2}\right) = \frac{1}{4}$, we have that $a_2 \geq a_1$. This serves as the base case for our inductive argument.
Next suppose that $a_{k+1} \geq a_k$ for some $k \in \mathbb{N}$. Now, by the recursive formula,

$$a_{k+2} - a_{k+1} = \frac{1}{2}\left(a_{k+1}^2 + \frac{1}{2}\right) - \frac{1}{2}\left(a_k^2 + \frac{1}{2}\right) = \frac{1}{2}(a_{k+1}^2 - a_k^2).$$

Since $a_n \geq 0$ for all n, we have that $a_{k+1}^2 \geq a_k^2$, and so $a_{k+1} \geq a_{k+1}$, completing the inductive step.
Hence (a_n) is increasing by the principle of mathematical induction.

(iii) By Parts (i) and (ii), the sequence (a_n) converges by the Monotone Convergence Theorem. Let us denote its limit by ℓ. Now, taking limits in the recursive formula, applying the Algebra of Limits, we have

$$\ell = \frac{1}{2}\left(\ell^2 + \frac{1}{2}\right).$$

This is a quadratic equation that we can solve explicitly to obtain the solutions $\ell = 1 \pm \frac{\sqrt{2}}{2}$.
Finally, since $\ell \leq 1$ (by Part (a) and Corollary 6.2), we conclude that $\ell = 1 - \frac{\sqrt{2}}{2}$.

□

Remark. In Part (i) we could have established a different upper bound. We only went for 1 as it was convenient (while being true!). You might like to try establishing a smaller upper bound here.

Remark. Sometimes when proving that a sequence (a_n) is increasing it is convenient to work with the difference $a_{n+1} - a_n$, and show that this is nonnegative (we could have done that in the example above). Other times we might work with the quotient $\frac{a_{n+1}}{a_n}$, and show that this is greater than or equal to 1. These are little "tricks" that sometimes help, and are good to bear in mind.

Please see Problem Sheet 2 for further examples of recursively defined sequences.

10. Another application of the MCT: Euler's sequence

The Monotone Convergence Theorem tells us that if we wish to prove that a given sequence converges, the it is enough to prove that it is both increasing and bounded above (or decreasing and bounded below). This often turns out to be a fruitful strategy, as we illustrate again now.

Proposition 10.1. *(Euler's sequence converging to $e = 2.71828...$). The sequence (a_n) given by*

$$a_n = \left(1 + \frac{1}{n}\right)^n \tag{1}$$

converges.

Proof. We begin by rewriting a_n using the binomial theorem

$$a_n := \left(1 + \frac{1}{n}\right)^n = \sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array} \right) \left(\frac{1}{n}\right)^k .$$

Here

$$\left(\begin{array}{c} n \\ k \end{array} \right) = \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!}$$

are the binomial coefficients. Using this we have, for each term in this sum that

$$\begin{aligned}
\left(\begin{array}{c} n \\ k \end{array} \right) \left(\frac{1}{n}\right)^k &= \frac{n(n-1)\cdots(n-k+1)}{k!}\frac{1}{n^k} \\
&= \frac{1}{k!}\frac{n}{n}\frac{(n-1)}{n}\frac{(n-2)}{n}\cdots\frac{(n-k+1)}{n} \\
&= \frac{1}{k!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\cdots\left(1 - \frac{k-1}{n}\right),
\end{aligned}$$

so that

$$a_n = \sum_{k=0}^{n} \frac{1}{k!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\cdots\left(1 - \frac{k-1}{n}\right),$$

and thus

$$\begin{aligned}
a_{n+1} &= \sum_{k=0}^{n+1} \frac{1}{k!}\left(1 - \frac{1}{n+1}\right)\left(1 - \frac{2}{n+1}\right)\cdots\left(1 - \frac{k-1}{n+1}\right) \\
&= \sum_{k=0}^{n} \frac{1}{k!}\left(1 - \frac{1}{n+1}\right)\left(1 - \frac{2}{n+1}\right)\cdots\left(1 - \frac{k-1}{n+1}\right) \\
&\quad + \frac{1}{(n+1)!}\left(1 - \frac{1}{n+1}\right)\left(1 - \frac{2}{n+1}\right)\cdots\left(1 - \frac{n}{n+1}\right).
\end{aligned}$$

We now compare the expressions for a_n and a_{n+1}. In the expression for a_{n+1} we see that all terms in the final summation above are greater than or equal to the corresponding terms in the summation for a_n. Moreover, in the expression for a_{n+1} there is an extra nonnegative term. Therefore $a_{n+1} \geq a_n$ for all n. That is to say the sequence (a_n) is *increasing.*

Returning to the binomial expansion we see that

$$a_n = \sum_{k=0}^{n} \frac{1}{k!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\cdots\left(1 - \frac{k-1}{n}\right) \leq \sum_{k=0}^{n} \frac{1}{k!},$$

since each of the factors in the summands is less than one. Further, if $k \geq 1$ then

$$k! = k(k-1)\cdots 3 \times 2 \times 1 \geq 2^{k-1},$$

so that for such k we have

$$\frac{1}{k!} \leq \frac{1}{2^{k-1}}.$$

Using this estimate we have

$$a_n \leq 1 + \sum_{k=1}^{n} \left(\frac{1}{2}\right)^{k-1} < 3.$$

In the last inequality we have used the well-known formula for the sum of a geometric progression (more on this later). This proves that the sequence (a_n) is *bounded above* by 3. Thus (a_n) is an increasing sequence which is bounded above. Hence by the Monotone Convergence Theorem we conclude that (a_n) converges. □

Remark. The limit of this sequence is, in fact, a special number. It is usually denoted by the letter e so that

$$e := \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n.$$

This constant was first considered by Leonhard Euler (pronounce Euler as "oiler").

Notice that our proof above establishes that $e \le 3$ by Corollary 6.2. How might you go about getting a better (smaller) upper bound on e?

11. Subsequences

Given a sequence

$$a_1, a_2, a_3, a_4, a_5, \ldots,$$

we can form a new sequence by selecting only some of the terms (or, equivalently, deleting some of the terms). For example, we may wish to consider the new sequence

$$a_1, a_3, a_5, a_7, a_9, \ldots,$$

where we have taken alternate terms (starting from the first) only. Or, we may wish to form a new sequence by simply deleting the 4th term; i.e.

$$a_1, a_2, a_3, a_5, a_6, \ldots$$

More generally, given any strictly increasing sequence of natural numbers

$$1 \le n_1 < n_2 < n_3 < n_4 < n_5 \cdots,$$

we may form a new sequence

$$a_{n_1}, a_{n_2}, a_{n_3}, a_{n_4}, a_{n_5}, \ldots,$$

which we may denote by $(a_{n_k})_{k=1}^{\infty}$ or simply (a_{n_k}).

Such new sequences are called *subsequences* of the original sequence.

Definition 11.1 (Subsequence)**.** Let $(n_k)_{k=1}^{\infty}$ be a strictly increasing sequence of natural numbers. Then the sequence $(a_{n_k})_{k=1}^{\infty}$ is called a *subsequence* of $(a_n)_{n=1}^{\infty}$.

Examples.

Consider the familiar sequence (a_n) given by $a_n = \frac{(-1)^n}{n}$; that is, the sequence

$$-1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, -\frac{1}{5}, \frac{1}{6}, -\frac{1}{7}, \ldots.$$

There are of course many subsequences of this sequence. [18] For example, if $n_k = 2k-1$, then the subsequence $(a_{n_k})_{k=1}^{\infty} = (a_{2k-1})_{k=1}^{\infty}$ is just the subsequence of (a_n) obtained by taking alternate terms, beginning with the first; that is, the sequence

$$-1, -\frac{1}{3}, -\frac{1}{5}, -\frac{1}{7}, -\frac{1}{9}, \ldots.$$

This we may observe from the formula for a_n since

$$a_{2k-1} = \frac{(-1)^{2k-1}}{2k-1} = -\frac{1}{2k-1} \quad \text{for all} \quad k \in \mathbb{N}.$$

[18] There are as many as there are choices of strictly increasing sequences of natural numbers $(n_k)_{k=1}^{\infty}$.

Here we have used the fact that $2k-1$ is always odd, so that $(-1)^{2k-1} = -1$ for all $k \in \mathbb{N}$.

Remark. In the above discussion the letters n and k are so-called dummy variables. In particular, the expressions $(a_{2k-1})_{k=1}^{\infty}$ and $(a_{2j-1})_{j=1}^{\infty}$ describe the same sequence. We could also have written $(a_{2n-1})_{n=1}^{\infty}$ for this particular subsequence of $(a_n)_{n=1}^{\infty}$.

Exercise. Describe the subsequence (a_{2n}) for this example.

The most basic theorem concerning subsequences is the following.

Theorem 11.2. *If (a_n) is a sequence of real numbers converging to ℓ, then every subsequence of (a_n) converges to ℓ.*

Proof. Suppose (a_n) is a sequence converging to ℓ, and let (a_{n_k}) be a subsequence. We are required to prove that

$$a_{n_k} \to \ell \quad \text{as} \quad k \to \infty.$$

To this end let $\epsilon > 0$ be arbitrary.

Since (a_n) converges to ℓ there exists $N \in \mathbb{N}$ such that

$$(\dagger) \qquad |a_n - \ell| < \epsilon \quad \text{for all} \quad n > N.$$

Since

$$1 \leq n_1 < n_2 < n_3 < \cdots < n_k$$

for every $k \in \mathbb{N}$, we have that $n_k \geq k$ for every $k \in \mathbb{N}$.[19] Hence, if $k > N$ then $n_k > N$. Combining this observation with $(\dagger)$ we obtain

$$|a_{n_k} - \ell| < \epsilon \quad \text{for all} \quad k > N.$$

□

The above theorem has the following immediate corollary, which is extremely useful for proving that certain sequences *do not* converge.

Corollary 11.3. *If a sequence (a_n) possesses two subsequences which converge to distinct limits, then (a_n) does not converge.*

Proof. Suppose for a contradiction that (a_n) is a convergent sequence which has two subsequences converging to distinct limits; i.e. suppose that (a_{n_k}) and $(a_{n'_k})$ are subsequences converging to α and β respectively, with $\alpha \neq \beta$.

Since (a_n) converges (to ℓ say), by the above theorem, the subsequences (a_{n_k}) and $(a_{n'_k})$ both converge to ℓ. Hence $\alpha = \ell$ and $\beta = \ell$, and so $\alpha = \beta$. This is a contradiction. □

Example. (i) Prove that the sequence (a_n) given by

$$a_n = (-1)^n$$

does not converge.[20]

[19] You may wish to provide a proof of this claim by induction.

[20] We have of course been working on the assumption that this is the case in our discussions so far. However, it is only now that we actually *prove* this.

Proof. Consider the subsequences (a_{2n}) and (a_{2n-1}). Since $a_{2n} = 1$ for all $n \in \mathbb{N}$, we have that

$$a_{2n} \to 1 \quad \text{as} \quad n \to \infty.$$

Since $a_{2n-1} = -1$ for all $n \in \mathbb{N}$, we have that

$$a_{2n-1} \to -1 \quad \text{as} \quad n \to \infty.$$

Since $1 \neq -1$, we conclude that (a_n) does *not* converge. $\square$

(ii) Prove that the sequence (a_n) given by

$$a_n = \sin(n\pi/8)$$

does not converge.

(iii) Prove that the sequence (a_n) given by

$$a_n = \sin n$$

does not converge. (This one requires rather more thought.)

We leave these as exercises.

12. The Bolzano–Weierstrass Theorem

In this lecture we will state and prove a particularly beautiful theorem concerning the convergence of subsequences. This is the *Bolzano–Weierstrass Theorem*, and is one of the highlights of our treatment of sequences of real numbers.

So far, all of the sequences that we have discussed in this lecture have at least one convergent subsequence. For example, if $a_n := (-1)^n$ then (a_{2n}) is convergent, as is (a_{2n-1}).

Question. Do all sequences possess convergent subsequences?

With a little thought we quickly conclude that the answer is NO. Take for example the sequence (a_n) given by

$$a_n = n \quad \text{for all} \quad n \in \mathbb{N}.$$

This has *no* convergent subsequence. This example tells us that, in general, *unbounded* sequences do not possess convergent subsequences.

With this in mind, a more reasonable question might be: Do all *bounded* sequences possess convergent subsequences?

The answer to this question is a particularly celebrated and useful theorem.

Theorem 12.1 (Bolzano–Weierstrass)**.** *Every bounded sequence of real numbers has a convergent subsequence.*

Proof. By the Monotone Convergence Theorem it will be enough to prove that every bounded sequence of real numbers has a monotone subsequence.

Let (a_n) be a bounded sequence of real numbers.

Consider the set

$$S := \{n \in \mathbb{N} : a_m \leq a_n \quad \text{for all} \quad m > n\}.$$

Since we don't know anything about the sequence (a_n), other that it is a bounded sequence, we know nothing about the set S, other than it is a subset of $\mathbb{N}$. In particular, we don't know whether it is a finite set or an infinite set. Let us consider these two possibilities in turn.

Possibility 1: S is finite.

Let n_1 be any natural number that is larger than all of the elements of S (such a number exists since S is finite). Then we have that $n_1 \notin S$. Hence, by the definition of S, there exists $n_2 > n_1$ such that $a_{n_2} > a_{n_1}$. Similarly, since $n_2 \notin S$ there exists $n_3 > n_2$ such that $a_{n_3} > a_{n_2}$. Continuing in this way we obtain an increasing subsequence $(a_{n_k})_{k=1}^{\infty}$, as desired.

Possibility 2: S is infinite.

Since S is infinite, we may write

$$S = \{n_k : k \in \mathbb{N}\},$$

where $(n_k)_{k=1}^{\infty}$ is a strictly increasing sequence of natural numbers. We claim that the subsequence (a_{n_k}) is decreasing. To prove this, let $k \in \mathbb{N}$ be arbitrary and observe that since $n_k \in S$ we have that $a_m \le a_{n_k}$ for all $m > n_k$. Since $n_{k+1} > n_k$, in particular we have that $a_{n_{k+1}} \le a_{n_k}$. Thus (a_{n_k}) is decreasing.

Since we are able to extract a monotone subsequence in either case, the proof is complete. $\square$

13. Cauchy's Convergence Criterion

You may have noticed that in order to use the definition of convergence to prove that a given sequence converges, you have to be able to come up with a candidate for the limit in advance (i.e. you have to "guess" the limit, and then go on to use the definition to prove that your guess is in fact correct). This begs a simple question.

Question. Is there a convenient convergence criterion which makes no reference to a limit ℓ?

Rather than a criterion which asks that the terms in the sequence "get closer and closer to a fixed number" (ℓ, say), perhaps we can find a criterion which asks that the terms in the sequence "get closer and closer to each other"...

A first guess: Could it be true that a sequence (a_n) converges if and only if the difference between consecutive terms tends to zero; i.e. iff $a_{n+1} - a_n \to 0$ and $n \to \infty$?

Certainly if (a_n) converges, then $a_{n+1} - a_n \to 0$ by the algebra of limits. However, the converse statement (that is, "if $a_{n+1} - a_n \to 0$ then (a_n) converges") fails terribly: consider the example $a_n = \sqrt{n}$; it is true that $\sqrt{n+1} - \sqrt{n} \to 0$, but $(\sqrt{n})$ does not converge.

It turns out that a slightly more sophisticated version of this first guess does work...

Definition 13.1 (Cauchy Sequence)**.** A sequence (a_n) is a *Cauchy Sequence* if given any $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$|a_n - a_m| < \epsilon \quad \text{for all} \quad m, n > N.$$

Remark. Since $|a_n - a_m| = |a_m - a_n|$, we see that the part "for all $m, n > N$" may be replaced by "for all $n > m > N$". Such symmetry considerations are often helpful in mathematics.

Theorem 13.2 (Cauchy's Convergence Criterion)**.** *A sequence of real numbers converges if and only if it is a Cauchy sequence.*

Remarks. Notice that the definition of a Cauchy sequence makes no reference to any limit ℓ. This is the main point. Therefore one may, in principle at least, set about proving that a given sequence converges, without having any idea about what the limit might be. We will consider an example after the proof of the theorem.

Proof. Since the above theorem is an "if and only if" statement, we have two things to prove:

(i) *if* (a_n) *converges then it is a Cauchy sequence*, and

(ii) *if* (a_n) *is a Cauchy sequence then it converges.*

Proof of (i). Suppose that (a_n) converges to some $\ell \in \mathbb{R}$.

Let $\epsilon > 0$ be arbitrary.

Since (a_n) converges to ℓ, there exists $N \in \mathbb{N}$ such that

$$|a_n - \ell| < \frac{\epsilon}{2} \quad \text{for all} \quad n > N.$$

Now, if $m, n > N$, then by the triangle inequality we get

$$|a_n - a_m| = |(a_n - \ell) - (a_m - \ell)| \leq |a_n - \ell| + |a_m - \ell| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Since $\epsilon > 0$ was arbitrary, we conclude that (a_n) is a Cauchy sequence.

Proof of (ii). Let us begin by showing that a Cauchy sequence is a bounded sequence. To do this we mimic the approach to proving that a convergent sequence is a bounded sequence.

Since (a_n) is a Cauchy sequence there exists $N \in \mathbb{N}$ such that

$$|a_n - a_m| < 1 \quad \text{for all} \quad m, n > N.$$

Hence, in particular,

$$|a_n - a_{N+1}| < 1 \quad \text{for all} \quad n > N.$$

Now, by the triangle inequality, we get that

$$|a_n| = |(a_n - a_{N+1}) + a_{N+1}| \leq |a_n - a_{N+1}| + |a_{N+1}| < 1 + |a_{N+1}| \quad \text{for all} \quad n > N.$$

Thus

$$|a_n| \leq M \quad \text{for all} \quad n \in \mathbb{N},$$

where $M := \max\{|a_1|, |a_2|, \ldots, |a_N|, |a_{N+1}| + 1\}$; i.e. (a_n) is a bounded sequence.

Now, since (a_n) is a bounded sequence, by the Bolzano–Weierstrass Theorem it must have a convergent subsequence (a_{n_k}).

Let ℓ be the limit of this subsequence. We shall prove that the full sequence (a_n) actually converges to ℓ.

To this end let $\epsilon > 0$ be arbitrary.

Since (a_n) is a Cauchy sequence there exists $N_1 \in \mathbb{N}$ such that

$$|a_n - a_m| < \frac{\epsilon}{2} \quad \text{for all} \quad m, n > N_1.$$

Hence, in particular,

$$(\dagger) \qquad |a_n - a_{n_k}| < \frac{\epsilon}{2} \quad \text{for all} \quad n, k > N_1.$$

Here we have used the fact that $n_k \geq k$ for all k.

Since (a_{n_k}) converges to ℓ there exists $N_2 \in \mathbb{N}$ such that

$$(\dagger\dagger) \qquad |a_{n_k} - \ell| < \frac{\epsilon}{2} \quad \text{for all} \quad k > N_2.$$

Now let $N = \max\{N_1, N_2\}$, and fix k to be any natural number larger than N (such as $N+1$). By ($\dagger$),($\dagger\dagger$) and the triangle inequality we conclude that

$$|a_n - \ell| = |(a_n - a_{n_k}) + (a_{n_k} - \ell)| \leq |a_n - a_{n_k}| + |a_{n_k} - \ell| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

for all $n > N$.

Since $\epsilon > 0$ was chosen arbitrarily, we conclude that (a_n) converges. $\square$

Example. Consider the sequence (s_n) given by

$$s_n = \sum_{k=1}^{n} \frac{(-1)^k}{2^{k^2}} = -\frac{1}{2} + \frac{1}{16} - \frac{1}{512} + \cdots + \frac{(-1)^n}{2^{n^2}}.$$

Prove that (s_n) converges.

First observe that (s_n) is *not* monotone since $s_{n+1} - s_n = \frac{(-1)^{n+1}}{2^{(n+1)^2}}$, which alternates in sign. So, we cannot use the Monotone Convergence Theorem (directly, at least) to show that (s_n) converges.

Notice also that there is no obvious candidate for a limit, so it would be very difficult to use the definition of convergence to prove that (s_n) converges ("what would we take ℓ to be?").

However, we can use Cauchy's Convergence Criterion relatively easily, as follows.

Let us begin by inspecting the quantity $|s_n - s_m|$. By symmetry we may suppose that $n > m$. Now, by the triangle inequality, we obtain

$$|s_n - s_m| = \left| \sum_{k=m+1}^{n} \frac{(-1)^k}{2^{k^2}} \right| \leq \sum_{k=m+1}^{n} \frac{1}{2^{k^2}} \leq \sum_{k=m+1}^{n} \frac{1}{2^k} \leq \sum_{k=m+1}^{\infty} \frac{1}{2^k} = \frac{1}{2^m} \leq \frac{1}{m}$$

for all $m \in \mathbb{N}$. Here we have also used the facts that $k^2 \geq k$ and $2^k \geq k$ for all $k \in \mathbb{N}$, along with the formula for the sum of a geometric series.

Now let $\epsilon > 0$.

Choose N to be any natural number larger than $1/\epsilon$.

Now, if $n > m > N$, then

$$|s_n - s_m| \leq \frac{1}{m} < \frac{1}{N} < \epsilon.$$

Since $\epsilon > 0$ was chosen arbitrarily, (s_n) is a Cauchy sequence. Hence (s_n) converges by Cauchy's convergence criterion.

Remark. As you will appreciate upon reflection, what we have done in this example is prove that the *series*

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{2^{n^2}}$$

converges. In the next part of this course ("Series") we will develop more systematic ways of doing such things.

Next week we will begin our detailed treatment of *series* of real numbers.

For next lecture: Think about what it should mean for a general series of the form

$$\sum_{n=1}^{\infty} a_n$$

to converge.

CHAPTER II: SERIES

14. What is a Series?

An infinite sum, such as

$$a_1 + a_2 + a_3 + a_4 + \cdots$$

is known as an *infinite series* (or simply a *series*). In this section we would like to examine when and how we can make sense of such an infinite sum. The rest of this course provides techniques for dealing with them. In particular we don't know if this is a finite amount, or if a limit exists, whatever that might mean. Let us start with an example. We want to try to sum the following:

$$\frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \cdots$$

where the general term is $\frac{1}{n(n+1)}$. That is to say, we are trying to find

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}.$$

Obviously we can't do this in one go. So a sensible approach might be to look at some initial segment – a *partial sum*. That is to say, we choose some N and calculate the finite partial sum

$$s_N := \frac{1}{2} + \frac{1}{6} + \cdots + \frac{1}{n(n+1)} + \cdots + \frac{1}{N(N+1)} = \sum_{n=1}^{N} \frac{1}{n(n+1)}.$$

We can use partial fractions on each of the terms in this sum and write

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$$

so that

$$s_N = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right) + \cdots + \left(\frac{1}{N} - \frac{1}{N+1}\right).$$

By cancelling adjacent terms this *telescoping sequence* simplifies to

$$S_N = 1 - \frac{1}{N+1}.$$

We can easily prove, using the ideas of the previous chapter, that the sequence (S_N) converges to the limit 1. Therefore, it seems reasonable to conclude that

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \cdots = 1. \tag{2}$$

Why bother with this "partial sums" business? Why not simply pair up terms in the infinite sum? The next example illustrates the problem of simply pairing things in an infinite setting.

Take the infinite series

$$\sum_{n=1}^{\infty} a_n \quad \text{where} \quad a_n := (-1)^n. \tag{3}$$

That is to say we sum

$$-1 + 1 - 1 + 1 - 1 + \cdots .$$

If we pair these up as

$$(-1 + 1) + (-1 + 1) + \cdots$$

this obviously equals zero. If we pair these up as

$$-1 + (1 - 1) + (1 - 1) + \cdots$$

then the answer obviously equals -1. So we have the apparent paradox

$$-1 = \sum_{n=1}^{\infty} (-1)^n = 0 \quad !?$$

Worse, let us write $1 = \frac{1}{2} + \frac{1}{2}$. Then we try to sum

$$\begin{aligned} &-1 + 1 - 1 + 1 - 1 + \cdots \\ &\qquad -\left(\frac{1}{2} + \frac{1}{2}\right) + \left(\frac{1}{2} + \frac{1}{2}\right) - \left(\frac{1}{2} + \frac{1}{2}\right) + \left(\frac{1}{2} + \frac{1}{2}\right) - \left(\frac{1}{2} + \frac{1}{2}\right) + \cdots \\ &\qquad\quad -\frac{1}{2} + \left(-\frac{1}{2} + \frac{1}{2}\right) + \left(+\frac{1}{2} - \frac{1}{2}\right) + \left(-\frac{1}{2} + \frac{1}{2}\right) + \left(+\frac{1}{2} - \frac{1}{2}\right) + \cdots = -\frac{1}{2}. \end{aligned}$$

Even worse we may write $1 = \epsilon + (1 - \epsilon)$ and repeat this to give the sum $-\epsilon$, for any ϵ!

To avoid such problems we only allow ourselves to pair off terms and cancel in a finite sum. Inspired by these examples we introduce the following definition of convergence of an infinite series.

Definition 14.1 (Convergence of series)**.**
The infinite series $\sum_{n=1}^{\infty} a_n$ is said to *converge* to a real number s if the sequence (s_N) of partial sums

$$s_N := \sum_{n=1}^{N} a_n$$

converges to s.

Language.

(i) If $\sum_{n=1}^{\infty} a_n$ converges to some s, then we say that the series *converges*, or is *convergent*.

(ii) We often use the shorthand

$$\sum_{n=1}^{\infty} a_n = s$$

to mean "the series $\sum_{n=1}^{\infty} a_n$ converges to s". We often refer to s as the *sum of the series*.

(iii) Any series that does not converge is said to *diverge*.

(iv) In the expression

$$\sum_{n=1}^{\infty} a_n,$$

n is called the *index of summation* (or *variable of summation*). This is a dummy variable, and so we are just as happy writing

$$\sum_{k=1}^{\infty} a_k.$$

To return to our previous "misbehaving" example we calculate the partial sums. Let $N \in \mathbb{N}$. Then, a short calculation shows that

$$s_{2N} = 0 \quad \text{(even terms)}, \quad s_{2N-1} = -1 \quad \text{(odd terms)}.$$

So the sequence of partial sums is

$$-1, 0, -1, 0, \cdots .$$

The subsequence of terms s_{2N-1} is constant at -1, the subsequence s_{2N} is constant and 0. The sequence (s_N) has two different convergent subsequences which converge to different limits. Thus, the sequence of partial sums, (s_N) cannot converge. Thus the series (3) does not converge: by our definition it diverges.

Let us consider the process of re-bracketing terms in a series more closely. Take the series $\sum_{n=1}^{\infty} a_n$ generated by the sequence (a_n). For the series to converge, the sequence of partial sums (s_N) must converge (this is the definition). Thus, by Theorem 11.2, any *subsequence* will converge, to the same limit. If we combine adjacent terms in the sequence (a_n), we effectively re-bracket. For example

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + \cdots = a_1 + (a_2 + a_3) + a_4 + (a_5 + a_6) + \cdots .$$

When this happens, the sequence of partial sums for the re-bracketed series is a *subsequence* of (s_N). Thus, by Theorem 11.2, re-bracketing of a series that converges does not change the limit of the series. However, you need to know that the series converges *before* you re-bracket. You can't re-bracket to prove a series converges because the converse of Theorem 11.2 is false.

Warning. We may re-bracket terms in a convergent series. But, we may *not* change the order in which they are taken.

For example, if the series $\sum_{n=1}^{\infty} a_n$ converges then the series

$$a_2 + a_1 + a_4 + a_3 + a_6 + a_5 + \cdots ,$$

is a *different* series. It may converge, it may not. We can't tell, even when $\sum_{n=1}^{\infty} a_n$ converges. Even if it does converge, the limit may be different! See the discussion in Lecture 1 for an explicit example of a convergent series which behaves "badly" under rearrangement. This delicate matter will be dealt with later.

15. Geometric series

Perhaps the most important[21] example of a series is the so-called *geometric series*. This is one in which the ratio of consecutive terms is constant (that is, independent of the index of summation).

Let $r \in \mathbb{R}$ and consider the geometric series

$$1 + r + r^2 + r^3 + \cdots = \sum_{n=0}^{\infty} r^n. \tag{4}$$

The partial sums of this are

$$s_N = 1 + r + r^2 + \cdots + r^N.$$

If $r = 1$ then $s_N = N + 1$, else if $r \neq 1$ then multiplying by $(1 - r)$ we have

$$\begin{aligned}(1-r)s_N &= (1-r)(1 + r + r^2 + \cdots + r^N)\\ &= (1-r) + (r - r^2) + (r^2 - r^3) + \cdots + (r^N - r^{N+1}) = 1 - r^{N+1},\end{aligned}$$

[21]Important in the sense that they crop up extremely often in applications, and in the study of other (trickier) series. We will see this happen many times over the next few weeks.

so that

$$s_N = \frac{1 - r^{N+1}}{1 - r}.$$

Since the sequence $(r^N)_{N=1}^{\infty}$ converges to zero if and only if $|r| < 1$, we conclude that the series converges if and only if $|r| < 1$, in which case

$$\sum_{n=0}^{\infty} r^n = \lim_{N\to\infty} \frac{1 - r^{N+1}}{1 - r} = \frac{1}{1 - r}. \tag{5}$$

You should memorise this formula for the sum of a geometric series as you will need to use it many *times.*

16. An important family of examples

In this lecture we will establish the convergence/divergence of series of the form

$$\sum_{n=1}^{\infty} \frac{1}{n^\alpha},$$

for various values of $\alpha \in \mathbb{R}$. Series of this form will crop up many times over the coming weeks.

Example 1. Use the definition to prove that the series

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

converges.

Proof. Since

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

is a series of nonnegative terms, the sequence of partial sums (s_N), given by

$$s_N = \sum_{n=1}^{N} \frac{1}{n^2}$$

is increasing. Hence by the Monotone Convergence Theorem it suffices to prove that the sequence (s_N) is bounded.

Observe that since $2^N \geq N$ for all $N \in \mathbb{N}$, we have that

$$s_N = \sum_{n=1}^{N} \frac{1}{n^2} \leq \sum_{n=1}^{2^N} \frac{1}{n^2} = s_{2^N}$$

for all $N \in \mathbb{N}$. Hence it remains to show that the subsequence (s_{2^N}) is bounded.

To see this we begin by observing that for each $j \in \mathbb{N}$,

$$\sum_{n=2^{j-1}+1}^{2^j} \frac{1}{n^2} \leq (2^j - 2^{j-1}) \times \frac{1}{(2^{j-1}+1)^2},$$

since, for each j the above sum consists of $2^j - 2^{j-1}$ terms of which the largest is $\frac{1}{(2^{j-1}+1)^2}$. Since $2^{j-1}+1 \geq 2^{j-1}$,

$$\sum_{n=2^{j-1}+1}^{2^j} \frac{1}{n^2} \leq (2^j - 2^{j-1}) \times \frac{1}{(2^{j-1})^2} \leq \frac{2^{j-1}}{(2^{j-1})^2} = \left(\frac{1}{2}\right)^{j-1}. \tag{6}$$

Now,

$$s_{2^N} = 1 + \sum_{j=1}^{N} \left(\sum_{n=2^{j-1}+1}^{2^j} \frac{1}{n^2} \right),$$

and so by (6)

$$s_{2^N} \leq 1 + \sum_{j=1}^{N} \left(\frac{1}{2}\right)^{j-1}.$$

Hence

$$s_{2^N} \leq 1 + \sum_{j=1}^{\infty} \left(\frac{1}{2}\right)^{j-1} = 1 + 1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \cdots = 1 + \frac{1}{1-\frac{1}{2}} = 3,$$

for all $N \in \mathbb{N}$, by the formula for the sum of a geometric series. We conclude that the sequence (s_{2^N}) is bounded, as required. $\square$

Notice also that this argument gives that $s_N \leq 3$ for all $N \in \mathbb{N}$. Hence, by Proposition 6.1,

$$\lim_{N\to\infty} s_N \leq 3;$$

in other words

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \leq 3.$$

(With a bit more care we can improve this bound of 3 considerably, although we do not pursue this.)

This series is known as Euler's series, and, as you will discover if you take the Year 3 module 3FFAb "Fourier Analysis", actually converges to $\pi^2/6$. Similarly,

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{\pi^6}{945}.$$

(There are similar identities when the power of n is even – the interested reader is referred to texts on *Fourier series*.)

Example 2. Use the definition to prove that the series

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$$

diverges.

Proof. For this series the sequence of partial sums (s_N) is given by

$$s_N = \sum_{n=1}^{N} \frac{1}{\sqrt{n}}.$$

Now, for each N, this is a sum of N positive terms, and the smallest of these is $1/\sqrt{N}$. Hence

$$s_N \geq N \times \frac{1}{\sqrt{N}} = \sqrt{N}.$$

Since $\sqrt{N} \to \infty$, we conclude that the sequence (s_N) is unbounded, and thus does *not* converge; i.e. the series

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$$

diverges. □

Example 3. Use the definition to prove that the series

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

diverges.

Proof. It suffices to show that the sequence of partial sums (s_N) has a subsequence which does not converge. Let us consider the subsequence

$$s_{2^N} = \sum_{n=1}^{2^N} \frac{1}{n}.$$

To see that (s_{2^N}) does not converge, we begin by observing that for each $j \in \mathbb{N}$,

$$\sum_{n=2^{j-1}+1}^{2^j} \frac{1}{n} \geq (2^j - 2^{j-1}) \times \frac{1}{2^j} = \frac{1}{2}, \tag{7}$$

since, for each j the above sum consists of $2^j - 2^{j-1}$ terms of which the smallest is $\frac{1}{2^j}$. Now,

$$s_{2^N} = 1 + \sum_{j=1}^{N} \left(\sum_{n=2^{j-1}+1}^{2^j} \frac{1}{n} \right),$$

and so by (7),

$$s_{2^N} \geq 1 + \sum_{j=1}^{N} \frac{1}{2} = 1 + \frac{N}{2}.$$

Since $1 + \frac{N}{2} \to \infty$ as $N \to \infty$, we conclude that $s_{2^N} \to \infty$. In particular, (s_{2^N}) does not converge. □

The series

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

is called the *harmonic series*, and its divergence is responsible for a number of famous mathematical paradoxes.

Exercise. Adapt the proofs in the examples above to show that

$$\sum_{n=1}^{\infty} \frac{1}{n^{\alpha}}$$

converges if and only if $\alpha > 1$. This exercise forms part of the problem sheet this week.

Remark. The function $\zeta : (1, \infty) \to \mathbb{R}$ given by

$$\zeta(x) = \sum_{n=1}^{\infty} \frac{1}{n^x}$$

is called the *Zeta Function*, and is intimately related to the distribution of the prime numbers. There is a famous, and longstanding, conjecture (unsolved problem) concerning this function, for which there is a million dollar prize...

17. The Linearity of Summation

Recall the definition of convergence of a series $\sum_{n=1}^{\infty} a_n$: A series

$$\sum_{n=1}^{\infty} a_n$$

converges if the sequence of partial sums (s_N), given by

$$s_N = \sum_{n=1}^{N} a_n,$$

converges.

Thus the convergence of a series is a statement about a corresponding *sequence*. As a result, the theorems about sequences that we have established so far may be reinterpreted as theorems about series. An important example is the following:

Theorem 17.1 (Linearity of summation)**.** *If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are convergent series and $\lambda, \mu \in \mathbb{R}$, then $\sum_{n=1}^{\infty}(\lambda a_n + \mu b_n)$ is convergent and*

$$\sum_{n=1}^{\infty}(\lambda a_n + \mu b_n) = \lambda \sum_{n=1}^{\infty} a_n + \mu \sum_{n=1}^{\infty} b_n.$$

Proof. Let (s_N) be the sequence of partial sums of the series $\sum_{n=1}^{\infty} a_n$, and let (t_N) be the sequence of partial sums of the series $\sum_{n=1}^{\infty} b_n$. By the definition of convergence we are required to show that the Nth partial sum

$$\sum_{n=1}^{N}(\lambda a_n + \mu b_n)$$

converges to $\lambda \sum_{n=1}^{\infty} a_n + \mu \sum_{n=1}^{\infty} b_n$ as $N \to \infty$. However,

$$\sum_{n=1}^{N}(\lambda a_n + \mu b_n) = \lambda \sum_{n=1}^{N} a_n + \mu \sum_{n=1}^{N} b_n = \lambda s_N + \mu t_N,$$

since the sums involved here are all finite sums. Since $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge, then the sequences (s_N) and (t_N) converge, and we denote their limits by $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ respectively. Hence by the algebra of limits

$$\lambda s_N + \mu t_N \to \lambda \sum_{n=1}^{\infty} a_n + \mu \sum_{n=1}^{\infty} b_n$$

as $N \to \infty$, as required. $\square$

18. Series Convergence Tests

Our goal for the next few weeks is to establish a number of easy-to-use tests for determining whether a given series converges or not. The simplest test is the following.

Theorem 18.1. *If a series $\sum_{n=1}^{\infty} a_n$ converges then $a_n \to 0$ as $n \to \infty$.*

We often refer to this theorem as the "*null sequence test*".

Proof. Since $\sum_{n=1}^{\infty} a_n$ converges, the sequence of partial sums (s_N), given by

$$s_N = \sum_{n=1}^{N} a_n$$

converges to some real number s. Now, since

$$a_n = s_n - s_{n-1},$$

we have that

$$a_n \to s - s = 0,$$

by the algebra of limits (for sequences). □

Warning. The converse to Theorem 18.1 is *not* true; i.e. if $a_n \to 0$ as $n \to \infty$, then the series

$$\sum_{n=1}^{\infty} a_n$$

may or may not converge. You should convince yourself of this by looking over the examples we have discussed so far in this section.

Example. Prove that the series

$$\sum_{n=1}^{\infty} \frac{n^2+n}{n^2+1}$$

diverges (does not converge).

Proof. Observe that the nth term in the series

$$a_n := \frac{n^2+n}{n^2+1} = \frac{1+\frac{1}{n}}{1+\frac{1}{n^2}} \to \frac{1}{1} = 1$$

as $n \to \infty$, by the algebra of limits. Since $a_n \to 1 \neq 0$, the series diverges by the null sequence test. □

Series of non-negative terms. You may have noticed that series of *nonnegative* terms are particularly nice since their sequence of partial sums forms an *increasing* sequence (and is thus primed for a potential application of the Monotone Convergence Theorem). As a result, a number of the series convergence tests that we will derive will be for series of positive terms only.

18.1. **The Comparison Test.**

Theorem 18.2 (The comparison test)**.**
Suppose that $0 \leq a_n \leq b_n$ *for all* $n \in \mathbb{N}$*.*

(1) *If* $\sum_{n=1}^{\infty} b_n$ *converges then* $\sum_{n=1}^{\infty} a_n$ *converges.*
(2) *If* $\sum_{n=1}^{\infty} a_n$ *diverges then* $\sum_{n=1}^{\infty} b_n$ *diverges.*

Proof. Let (s_N) be the sequence of partial sums for $\sum_{n=1}^{\infty} a_n$ and (t_N) be the sequence of partial sums for $\sum_{n=1}^{\infty} b_n$. Since all terms are non-negative both (s_N) and (t_N) are increasing sequences.

(1) Since $a_n \leq b_n$ for all $n \in \mathbb{N}$ then $s_N \leq t_N$ for all $N \in \mathbb{N}$. Since (t_N) converges it is bounded above. Hence (s_N) is also bounded above, and so converges by the Monotone Convergence Theorem.

(2) Since $\sum_{n=1}^{\infty} a_n$ is divergent, (s_N) is unbounded. Thus (t_N) must be is unbounded, and so does not converge. $\square$

In order to use the comparison test it is of course necessary that we have a number of standard series with which to compare. Often we end up comparing with geometric series, and often we end up comparing with series of the form

$$\sum_{n=1}^{\infty} \frac{1}{n^{\alpha}},$$

for $\alpha \in \mathbb{R}$. Let us see some examples...

Example 1. Use the comparison test to show that

$$\sum_{n=1}^{\infty} \left(\frac{n+1}{n^2+2} \right)^3 \tag{8}$$

converges.

Proof. This is a series of non-negative terms, so we may apply the comparison test.
First note that for all $n \in \mathbb{N}$ we have

$$\begin{aligned} \frac{n+1}{n^2+2} &\leq \frac{2n}{n^2+2} \quad \text{since } 1 \leq n, \\ &< \frac{2n}{n^2} \quad \text{since } n^2 < n^2+2, \\ &= \frac{2}{n}. \end{aligned}$$

So that for all $n \in \mathbb{N}$,

$$a_n := \left(\frac{n+1}{n^2+2} \right)^3 < \left(\frac{2}{n} \right)^3 = \frac{8}{n^3} := b_n.$$

So that for all $n \in \mathbb{N}$, $0 \leq a_n \leq b_n$.

We know $\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges, as it is of the form $\sum_{n=1}^{\infty} \frac{1}{n^{\alpha}}$ for some $\alpha > 1$.

By the algebra of limits for series so does $\sum_{n=1}^{\infty} \frac{8}{n^3}$.

Therefore, by the comparison test, (8) converges also. $\square$

Example 2. Use the comparison test to prove that the series

$$\sum_{n=1}^{\infty} \frac{n2^{-n}}{3n+1}$$

converges.

Proof. This is a series of non-negative terms, so we may apply the comparison test. First note that for all $n \in \mathbb{N}$ we have

$$a_n := \frac{n2^{-n}}{3n+1} \leq \frac{n2^{-n}}{3n} = \frac{1}{3}\left(\frac{1}{2}\right)^n$$

for all $n \in \mathbb{N}$. Hence if we set

$$b_n = \frac{1}{3}\left(\frac{1}{2}\right)^n,$$

then $0 < a_n \leq b_n$ for all $n \in \mathbb{N}$.

We know that $\sum_{n=1}^{\infty} \frac{1}{3}\left(\frac{1}{2}\right)^n$ converges, as it is a geometric series with common ratio $1/2$.

Therefore $\sum_{n=1}^{\infty} \frac{n2^{-n}}{3n+1}$ converges by the comparison test. $\square$

18.2. **D'Alembert's Ratio Test.**

Theorem 18.3 (D'Alembert's ratio test)**.**
Let (a_n) be a sequence of non-negative real numbers such that

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = r,$$

then the series $\sum_{n=1}^{\infty} a_n$

(1) *converges if $r < 1$, and*
(2) *diverges if $r > 1$.*

In general, if $r = 1$ the ratio test tells us nothing.

Proof. (1) Assume that $r < 1$, then choose some real number β such that $r < \beta < 1$ and set $\epsilon := \beta - r$. By the definition of convergence there exists some $N \in \mathbb{N}$ such that

$$\left|\frac{a_{n+1}}{a_n} - r\right| < \epsilon = \beta - r$$

for all $n \geq N$. So that using the triangle inequality

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{a_{n+1}}{a_n} - r + r\right| \leq \left|\frac{a_{n+1}}{a_n} - r\right| + r < \beta.$$

for all $n \geq N$. Hence

$$\left|\frac{a_{n+1}}{a_n}\right| < \beta \quad \text{for all} \quad n \geq N.$$

Since all the terms are positive

$$\frac{a_{n+1}}{a_n} = \left|\frac{a_{n+1}}{a_n}\right| < \beta.$$

for all $n \geq N$.

Now, if $n > N$ then

$$a_n = \frac{a_n}{a_{n-1}} \frac{a_{n-1}}{a_{n-2}} \frac{a_{n-2}}{a_{n-3}} \cdots \frac{a_{N+1}}{a_N} a_N < \beta^{n-N} a_N.$$

Define $\alpha_n := a_{n+N}$, then for all $n \in \mathbb{N}$, $0 \leq \alpha_n < \beta^n a_N$. Since $\beta < 1$ and a_N is constant, the series $\sum_{n=1}^{\infty} \beta^n a_N$ converges. Hence by the comparison test $\sum_{n=1}^{\infty} \alpha_n$ also converges. Lastly we note that

$$\begin{aligned} \sum_{n=1}^{\infty} a_n &= \sum_{n=1}^{N} a_n + \sum_{n=N+1}^{\infty} a_n \\ &= \sum_{n=1}^{N} a_n + \sum_{n=1}^{\infty} \alpha_n, \end{aligned}$$

proving that $\sum_{n=1}^{\infty} a_n$ converges.

(2) Assume that $r > 1$, then choose $\beta \in \mathbb{R}$ so that $1 < \beta < r$. By the definition of convergence there exists some $N \in \mathbb{N}$ such that for all $n \geq N$

$$\left| \frac{a_{n+1}}{a_n} - r \right| < r - \beta.$$

Thus

$$\beta - r < \frac{a_{n+1}}{a_n} - r < r - \beta$$

and so using the first of these

$$1 < \beta < \frac{a_{n+1}}{a_n}$$

This shows that $a_n < a_{n+1}$ and so (a_n) is an increasing sequence of positive terms – this cannot converge to zero. Hence by the null sequence test, the series diverges. $\square$

Example 1. Use the ratio test to prove that the series $\sum_{n=1}^{\infty} \frac{1}{n!}$ converges.

Proof. We begin by observing that we are working with a series of nonnegative terms (a requirement of the ratio test).

We take the ratio of successive terms in the series to obtain

$$\frac{a_{n+1}}{a_n} = \frac{n!}{(n+1)!} = \frac{1}{n+1}.$$

We know that

$$r = \lim_{n \to \infty} \frac{1}{n+1} = 0.$$

Since $r = 0 < 1$, the series converges by the ratio test. $\square$

Example 2. Prove that the series

$$\sum_{n=1}^{\infty} \frac{n}{3^n}$$

converges.

Proof. We begin by observing that we are working with a series of nonnegative terms (a requirement of the ratio test).

If $a_n = n/3^n$ then

$$\frac{a_{n+1}}{a_n} = \frac{n+1}{3^{n+1}} \times \frac{3^n}{n} = \frac{1}{3}\left(\frac{n+1}{n}\right) = \frac{1}{3}\left(1 + \frac{1}{n}\right)$$

for all $n \in \mathbb{N}$. Hence

$$\frac{a_{n+1}}{a_n} \to \frac{1}{3}(1+0) = \frac{1}{3}$$

as $n \to \infty$, by the algebra of limits. Since $1/3 < 1$, the series in question converges by the ratio test. $\square$

Example 3. Use the ratio test to determine the values of $\alpha \geq 0$ for which $\sum_{n=1}^{\infty} n^2\alpha^n$ converges.

We begin by observing that we are working with a series of nonnegative terms (a requirement of the ratio test).

We take the ratio of successive terms in the series

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^2\alpha^{n+1}}{n^2\alpha^n} = \left(\frac{n+1}{n}\right)^2 \alpha = \left(1+\frac{1}{n}\right)^2 \alpha.$$

We know that

$$r = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^2 \alpha = \alpha,$$

by the algebra of limits, so that if $0 \leq \alpha < 1$ this series converges by the ratio test. Furthermore, if $\alpha > 1$ then it diverges by the ratio test. If $\alpha = 1$ the ratio test is inconclusive, so must treat this case separately.

For $\alpha = 1$ the series is simply

$$\sum_{n=1}^{\infty} n^2.$$

This series diverges by the null sequence test as $n^2 \not\to 0$ as $n \to \infty$.

Example 4. Use the ratio test to determine the convergence or divergence of

$$\sum_{n=1}^{\infty} \frac{n^{10}}{n!}. \tag{9}$$

We begin by observing that we are working with a series of nonnegative terms (a requirement of the ratio test).

Define a_n as

$$a_n := \frac{n^{10}}{n!}$$

and compute the ratio $\frac{a_{n+1}}{a_n}$;

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^{10}}{(n+1)!}\frac{n!}{n^{10}} = \left(\frac{n+1}{n}\right)^{10} \frac{1}{n+1}.$$

Since $\lim_{n\to\infty} \frac{n+1}{n} = 1$ it follows by the algebra of limits that

$$\lim_{n\to\infty} \left(\frac{n+1}{n}\right)^{10} = 1.$$

Thus, using the algebra of limits again, we have that

$$r = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^{10} \frac{1}{n+1} = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^{10} \times \lim_{n\to\infty} \frac{1}{n+1} = 0.$$

Since $r = 0 < 1$ the ratio test confirms that this series converges.

Warning. As mentioned, if

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1,$$

we can in general conclude nothing. For example, applying the ratio test to terms in the harmonic series gives

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{n}{n+1} = 1.$$

The harmonic series diverges.

Applying the ratio test to terms in Euler's series gives

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{n^2}{(n+1)^2} = 1,$$

but Euler's series converges!

Do not use the ratio test if $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$. To prove convergence or otherwise you need to use an alternative test.

18.3. **The Root Test.** This test is very similar to the ratio test. The only difference is that the ratio a_{n+1}/a_n is replaced by $a_n^{1/n}$, that is, the nth root of the nth term.

Theorem 18.4 (The root test)**.** *Let (a_n) be a sequence of nonnegative real numbers such that*

$$\lim_{n\to\infty} a_n^{1/n} = r,$$

then the series $\sum_{n=1}^{\infty} a_n$

(1) *converges if $r < 1$, and*
(2) *diverges if $r > 1$.*

In general, if $r = 1$ the root test tells us nothing.

Due to time constraints we omit the proof of this theorem.

Example. Use the root test to prove that the series

$$\sum_{n=1}^{\infty} \left(\frac{n+5}{4n-1}\right)^n$$

converges.

Proof. We begin by observing that we are working with a series of nonnegative terms (a requirement of the root test).

Let $a_n = \left(\frac{n+5}{4n-1}\right)^n$, and observe that

$$a_n^{1/n} = \frac{n+5}{4n-1} = \frac{1+\frac{5}{n}}{4-\frac{1}{n}} \to \frac{1}{4}$$

as $n \to \infty$, by the algebra of limits. Since $1/4 < 1$, the series converges by the root test. $\square$

18.4. **The integral test.** Our next test capitalises on the fact that sums may often be estimated effectively by integrals.

Theorem 18.5 (Integral test)**.**
Let $f : \mathbb{R} \to \mathbb{R}$ be continuous, decreasing, and positive on $[1, \infty)$. Then

$$\sum_{n=1}^{\infty} f(n)$$

converges if and only if the sequence of integrals

$$\int_1^n f(x)dx$$

converges as $n \to \infty$.

The proof of this theorem is left as an exercise in applying the Comparison Test.

Example. Let $\alpha \in \mathbb{R}$. Define $f : [1, \infty) \to (0, \infty)$ by

$$f(x) = \frac{1}{x^\alpha}.$$

Then

$$\int_1^n f(x)dx = \int_1^n \frac{1}{x^\alpha}dx = \left[\frac{x^{1-\alpha}}{1-\alpha}\right]_1^n = \frac{n^{1-\alpha}-1}{1-\alpha}.$$

If $\alpha > 1$ then

$$\lim_{n\to\infty} n^{1-\alpha} = 0.$$

Thus

$$\lim_{n\to\infty} \frac{n^{1-\alpha}-1}{1-\alpha} = \frac{1}{\alpha-1}.$$

Hence, the integral test shows us that

$$\sum_{n=1}^{\infty} \frac{1}{n^\alpha}$$

converges as $n \to \infty$. (Of course we have already proved this result by a different method.)

Similar considerations reveal that

$$\sum_{n=1}^{\infty} \frac{1}{n^\alpha}$$

diverges if $\alpha \leq 1$ (again, something that we already knew). We leave this as an exercise.

Exercise. Use the integral test to show that the series

$$\sum_{n=2}^{\infty} \frac{1}{n\log(n)}$$

diverges. [You may wish to first observe that $\frac{d}{dx}\log(\log(x)) = \frac{1}{x\log(x)}$.]

Other series convergence tests (that don't require nonnegativity)

So far, with the mere exception of the null sequence test, our series convergence test have only been relevant for series with non-negative terms. The following test applies to series whose terms are alternately positive and negative.

18.5. **The alternating series test.** The following, also sometimes called Leibniz's Theorem, is one of the few general results which deals with series with both an infinite number of positive, and an infinite number of negative terms. Although the theorem is stated in terms of a non-negative sequence, this is test for convergence of a special kind of series with alternating positive and negative terms.

Theorem 18.6 (Alternating series test)**.**
Suppose the non-negative sequence (a_n)

(i) *is decreasing,*
(ii) *and converges to zero.*

Then

$$a_1 - a_2 + a_3 - a_4 + \cdots = \sum_{n=1}^{\infty} (-1)^{n+1} a_n$$

converges.

In order to prove this theorem we need a simple lemma about sequences.

Lemma 18.7 (Covering lemma)**.** *Let (a_n) be a sequence of real numbers and $a \in \mathbb{R}$. If both of the subsequences (a_{2n}) and (a_{2n+1}) converge to a, then the full sequence (a_n) converges to a.*

Proof. Let $\epsilon > 0$ be arbitrary.

Since $a_{2n} \to a$, there exists $N_1 \in \mathbb{N}$ such that

$$(\dagger) \qquad |a_{2n} - a| < \epsilon \quad \text{for all} \quad n > N_1.$$

Similarly, since $a_{2n+1} \to a$, there exists $N_2 \in \mathbb{N}$ such that

$$(\dagger\dagger) \qquad |a_{2n+1} - a| < \epsilon \quad \text{for all} \quad n > N_2.$$

Now let $N = \max\{2N_1, 2N_2 + 1\}$, and suppose that $n > N$. There are two possibilities to consider:-

Possibility 1: n is even.

If n is even then since $n > 2N_1$ we may write $n = 2k$ for some $k > N_1$. Hence by $(\dagger)$,

$$|a_n - a| = |a_{2k} - a| < \epsilon.$$

Possibility 2: n is odd.

If n is odd then since $n > 2N_2 + 1$ we may write $n = 2k + 1$ for some $k > N_2$. Hence by $(\dagger\dagger)$,

$$|a_n - a| = |a_{2k+1} - a| < \epsilon.$$

In either case we have that

$$|a_n - a| < \epsilon.$$

Hence

$$|a_n - a| < \epsilon \quad \text{for all} \quad n > N,$$

as required. □

We now turn to the proof of Theorem 18.6.

Proof. Let S_N be the Nth partial sum of this series. We begin by considering the subsequence of even partial sums (S_{2N}). Observe that

$$S_{2N+2} = S_{2N} + (-1)^{2N+2}a_{2N+1} + (-1)^{2N+3}a_{2N+2} = S_{2N} + a_{2N+1} - a_{2N+2}.$$

Thus

$$S_{2N+2} - S_{2N} = a_{2N+1} - a_{2N+2} \geq 0,$$

and so the subsequence of even partial sums (S_{2N}) is increasing. Furthermore

$$\begin{aligned} S_{2N} &= a_1 - a_2 + a_3 - a_4 + a_5 - a_6 + \cdots - a_{2N} \\ &= a_1 - (a_2 - a_3) - (a_4 - a_5) - \cdots - a_{2N} \leq a_1 \end{aligned}$$

for all $N \in \mathbb{N}$. Since the sequence (S_{2N}) is an increasing sequence that is bounded above, it must converge, to s say, by the Monotone Convergence Theorem.

We now consider the subsequence (S_{2N+1}), which satisfies

$$S_{2N+1} = S_{2N} + a_{2N+1}.$$

Since (S_{2N}) converges to s, and (a_{2N+1}) converges to zero, it follows, by the algebra of limits that the sequence (S_{2N+1}) converges to s also.

Hence the full sequence of partial sums converges to s by the covering lemma. □

Example 1. Use the alternating series test to show that the series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$

converges.

Proof. If $a_n := \frac{1}{n}$, then (a_n) is a decreasing sequence of non-negative terms that converges to zero. Thus

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \sum_{n=1}^{\infty} (-1)^{n+1} a_n$$

converges by the alternating series test. □

In fact, although we don't prove it, the sum of the above alternating series is $\log(2)$.

It is important to note that *both* the conditions of the theorem must hold. If either fails, the theorem fails.

(i) If the sequence (a_n) fails to be decreasing then we can build an example that fails. For example define the odd and even terms by

$$a_{2n-1} := \frac{1}{n}, \quad a_{2n} := \frac{1}{n^2}.$$

Examining the first few terms we have

$$a_1 = 1,\ a_2 = 1,\ a_3 = \frac{1}{2},\ a_4 = \frac{1}{2^2} = \frac{1}{4},\ a_5 = \frac{1}{3},\ a_6 = \frac{1}{3^2},\ \cdots$$

so $a_4 < a_5$. Thus (a_n) is not decreasing. If we calculate the partial sums then

$$S_{2N} := \sum_{n=1}^{2N}(-1)^{n+1}a_n = \sum_{n=1}^{N}(-1)^{2n}a_{2n-1} + \sum_{n=1}^{N}(-1)^{2n+1}a_{2n}$$
$$= \sum_{n=1}^{N}\frac{1}{n} - \sum_{n=1}^{N}\frac{1}{n^2} \geq \sum_{n=1}^{N}\frac{1}{n} - \frac{\pi^2}{6}.$$

But the first term is the harmonic series which we know does not converge. So our series does not converge.

(ii) If (a_n) does not converge to zero, then the *sequence* generated by $(-1)^{n+1}a_n$ doesn't converge to zero either. Hence the series

$$\sum_{n=1}^{\infty}(-1)^{n+1}a_n$$

does not converge by the null sequence test.

19. Absolute convergence

As we saw in the last section on the alternating series test, the series

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}$$

converges. This is not the first time we have had cause to discuss this particular series – in the introductory lecture we claimed that this series behaves rather badly under rearrangement; in particular we claimed that

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} + \frac{1}{11} - \cdots = \log 2,$$

which is about 0.69, but the rearranged sum

$$1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \frac{1}{11} - \frac{1}{6} + \cdots = \frac{3}{2}\log 2 \neq \log 2.$$

This appears to be somewhat of a contradiction. However, there is no contradiction. This example illustrates the fact that in general, if one changes the order in which one sums the terms of (even a convergent) series, one changes the sum of the series.

As we will see below, the concept of *absolute convergence* will help us decide when we may rearrange terms in a series without affecting the sum.

Definition 19.1 (Absolute convergence)**.**
A series $\sum_{n=1}^{\infty} a_n$ is said to be *absolutely convergent* if the series $\sum_{n=1}^{\infty} |a_n|$ is convergent.

A word about the terminology: The phrase "absolute convergence" is chosen to be evocative. However it is a definition. Do not be lulled into the mistake of assuming an absolutely convergent series converges. This is something we need to *prove*.

Theorem 19.2. *If $\sum_{n=1}^{\infty} a_n$ is absolutely convergent then $\sum_{n=1}^{\infty} a_n$ is convergent in the usual sense.*

Proof. Suppose that

$$\sum_{n=1}^{\infty} a_n$$

is absolutely convergent. We are required to show that the series

$$\sum_{n=1}^{\infty} a_n$$

converges. By the definition of a convergent series, we are required to show that the sequence (s_n) of partial sums given by

$$s_n = \sum_{k=1}^{n} a_k$$

converges.

We shall do this by showing that (s_n) is a Cauchy sequence. This will be enough thanks to Cauchy's Convergence Criterion.

To this end let $\epsilon > 0$ be arbitrary.

Since the series

$$\sum_{n=1}^{\infty} a_n$$

is absolutely convergent, the series

$$\sum_{n=1}^{\infty} |a_n|$$

is convergent (this is the definition of absolute convergence). By the definition of a convergent series, this means that the sequence of partial sums (t_n) given by

$$t_n = \sum_{k=1}^{n} |a_k|$$

converges. Now, by Cauchy's convergence criterion, the sequence (t_n) is a Cauchy sequence, and so there exists $N \in \mathbb{N}$ such that

$$|t_m - t_n| < \epsilon$$

for all $m > n > N$; i.e.

$$\sum_{k=n+1}^{m} |a_k| < \epsilon$$

for all $m > n > N$. Hence by the triangle inequality,

$$|s_m - s_n| = \left| \sum_{k=n+1}^{m} a_k \right| \leq \sum_{k=n+1}^{m} |a_k| < \epsilon$$

for all $m > n > N$.

Since $\epsilon > 0$ was chosen arbitrarily, we conclude that (s_n) is a Cauchy sequence, as required. $\square$

The above definition tells us what we mean by absolute convergence. Of course, if for all $n \in \mathbb{N}$, $a_n \geq 0$ and $\sum_{n=1}^{\infty} a_n$ converges then since $a_n = |a_n|$ for all $n \in \mathbb{N}$ it follows that $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} |a_n|$, and so the series will be absolutely convergent. An example of an absolutely convergent series with an infinite number of positive, and an infinite number of negative terms is

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}. \tag{10}$$

By the alternating series test this converges. Since

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}}{n^2} \right| = \sum_{n=1}^{\infty} \frac{1}{n^2},$$

this converges (it is Euler's series). So the original series (10) is absolutely convergent. In fact, the limit of this series is $\frac{\pi^2}{12}$.

However, as we will see, not all convergent series are absolutely convergent.

Warning. The converse of Theorem 19.2, viz

> "If a series is convergent then it is absolutely convergent",

is false. For example, by the alternating series test the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \tag{11}$$

converges. However,

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n}$$

diverges (it is the so-called harmonic series). Thus (11) does not converge absolutely.

Definition 19.3 (Conditional convergence)**.**
If $\sum_{n=1}^{\infty} a_n$ converges, but does *not* converge absolutely, then it is said to be *conditionally convergent.*

The next section explains the importance of absolutely convergent series.

19.1. **Good properties of absolutely convergent series.** The two theorems in this section essentially say that

- You may rearrange the terms in an absolutely convergent series without affecting the limit.
- You may rearrange the terms in a conditionally convergent series and build a convergent series that converges to any limit you choose.

Furthermore

- You can make sense of the "product" of two absolutely convergent series.

A rearrangement of a series is defined as follows. Let $\sigma : \mathbb{N} \to \mathbb{N}$ be any permutation of the natural numbers. That is σ is a bijection of $\mathbb{N}$ to itself. If $b_n := a_{\sigma(n)}$ then (b_n) is said to be a *rearrangement* of (a_n).

These results are more formally stated as follows:

Theorem 19.4 (Dirichlet's Theorem: rearrangements of series)**.**
Let $\sum_{n=1}^{\infty} a_n$ *be an absolutely convergent series of real numbers. If* (b_n) *is any rearrangement of* (a_n) *then*

(1) $\sum_{n=1}^{\infty} b_n$ *is an absolutely convergent series.*
(2) $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} a_n$.

Proof. [See Spivak, Calculus, pg 402]
[See Hart, Theorem 3.4.4]

Theorem 19.5 (Conditional convergence)**.**
If the series $\sum_{n=1}^{\infty} a_n$ *converges but does* not *converge absolutely (i.e.* $\sum_{n=1}^{\infty} a_n$ *is conditionally convergent) and* $\gamma \in \mathbb{R}$ *is* any real number*, then there exists a rearrangement* (b_n) *of the sequence* (a_n) *so that*

$$\sum_{n=1}^{\infty} b_n = \gamma.$$

Proof. [See Spivak, Calculus, pg 401]

20. Products of series

Looking back at the section on the linearity of summation we see that, while we understand how to sum two convergent series, so far we have no rule for calculating the **product** of two series, such as

$$\left(\sum_{n=1}^{\infty} a_n\right) \times \left(\sum_{n=1}^{\infty} b_n\right)$$

How would we do this?

It is natural to expect that the normal rules of algebra apply to products of infinite sums

$$(a_1 + a_2 + a_3 + \cdots) \times (b_1 + b_2 + b_3 + \cdots),$$

allowing us to express this as a sum of terms of the form

$$a_n b_m$$

where n and m take all possible values in $\mathbb{N}$. Using the "$\cdots$" notation, these terms are

$$\begin{array}{ccccc} a_1b_1 & a_1b_2 & a_1b_3 & a_1b_4 & \cdots \\ a_2b_1 & a_2b_2 & a_2b_3 & a_2b_4 & \cdots \\ a_3b_1 & a_3b_2 & a_3b_3 & a_3b_4 & \cdots \\ a_4b_1 & a_4b_2 & a_4b_3 & a_4b_4 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{array}$$

When we form the sum (or "double series")

$$\sum_{n,m=1}^{\infty} a_n b_m$$

from these terms, *in what order should we take them?*

We need to ensure that the order is unimportant. This is precisely where the concept of absolute convergence comes to our rescue.

One way to arrange them is to ensure that $n + m = k$ for each $k \geq 2$,

e.g. $k = 2$

$$a_1 b_1$$

e.g. $k = 3$

$$a_1 b_2 + a_2 b_1$$

e.g. $k = 4$

$$a_1 b_3 + a_2 b_2 + a_3 b_1$$

In general

$$\sum_{r=1}^{k-1} a_r b_{k-r}, \quad k \geq 2,$$

which corresponds to taking diagonals in the above array. This is known as the *Cauchy product*

$$\sum_{k=2}^{\infty} \sum_{r=1}^{k-1} a_r b_{k-r},$$

and equals

$$a_1 b_1 + (a_1 b_2 + a_2 b_1) + (a_1 b_3 + a_2 b_2 + a_3 b_1) + \cdots .$$

Theorem 20.1. *Assume $\sum_{n=1}^{\infty} a_n$, $\sum_{n=1}^{\infty} b_n$ are absolutely convergent series of real numbers. Then*

$$\sum_{n,m=1}^{\infty} a_n b_m$$

is absolutely convergent[22] *and*

$$\sum_{n,m=1}^{\infty} a_n b_m = \left(\sum_{n=1}^{\infty} a_n \right) \times \left(\sum_{n=1}^{\infty} b_n \right).$$

See Hart, pg 98. □

Our final topic of discussion before the vacation will be *Power Series*. A Power series is a particular type of infinite series of the form

$$\sum_{k=0}^{\infty} a_k x^k = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots ,$$

where the real number x is viewed as a variable. This allows us to define a function $f(x)$ as

$$f(x) := \sum_{k=0}^{\infty} a_k x^k.$$

This of course ties in well with the lectures of Dr. Wang.

[22] As a consequence of this the double series

$$\sum_{n,m=1}^{\infty} a_n b_m$$

is rearrangement-invariant, meaning that it doesn't matter in which order we do the summation. A popular choice would be the Cauchy product, described above.

21. Power Series

For us a power series is a series of the form

$$\sum_{n=0}^{\infty} a_n x^n$$

where (a_n) is a sequence of real numbers and $x \in \mathbb{R}$.

We consider (a_n) as fixed for this series, and x as a variable.

Clearly the series might converge for some values of x and not for others. Given a power series, we would of course like to be able to establish which values of x lead to convergence, and which lead to divergence.

Define the subset of real numbers

$$S := \left\{ x \in \mathbb{R} : \sum_{n=0}^{\infty} a_n x^n \ \text{ converges} \right\}.$$

Note that $S \neq \emptyset$, because $0 \in S$.

We may consider the power series $\sum_{n=0}^{\infty} a_n x^n$ as a *function*

$$f : S \longrightarrow \mathbb{R},$$

defined by

$$f(x) := \sum_{n=0}^{\infty} a_n x^n, \quad x \in S.$$

Example. Consider the geometric series

$$f(x) = \sum_{n=0}^{\infty} x^n.$$

We already know that for the geometric series, $S = (-1, 1)$. Moreover, in this particular case we have a "closed form" expression for f, namely

$$f(x) = \frac{1}{1-x}, \quad \forall x \in S = (-1, 1).$$

Question. Can the set of convergence S be *any* subset of $\mathbb{R}$, or does it have to take some special form? For example, can we find a power series such that the corresponding S is the union of two disjoint intervals?

The answer to this question is beautifully simple.

Theorem 21.1. *Given a power series $\sum_{n=0}^{\infty} a_n x^n$, either it converges absolutely for all $x \in \mathbb{R}$, or there exists $R \in [0, \infty)$ such that*

(1) *it converges absolutely when $|x| < R$*
(2) *it diverges when $|x| > R$.*

We refer to R as the *Radius of Convergence* of the power series. If the power series is absolutely convergent for all $x \in \mathbb{R}$ then we say that it has infinite radius of convergence, and write $R = \infty$.

Remark. We can restate the theorem as

$$(-R, R) \subseteq S \subseteq [-R, R]$$

and the power series converges absolutely in $(-R, R)$. In particular we see that S is always an *interval*.

Remark. Sometimes $(-R, R) \subsetneq S \subsetneq [-R, R]$; that is, it might be the case that $S = [-R, R)$ or $S = (-R, R]$. Indeed, consider for example the power series

$$\sum_{n=1}^{\infty} \frac{x^n}{n}.$$

When $|x| > 1$ the general term $x^n/n \not\to 0$, and so the series diverges for $|x| > 1$ by the null sequence test.

When $|x| < 1$ the series converges (absolutely) by the comparison test because $|x^n/n| \leq |x|^n$ for all $n \in \mathbb{N}$.

It follows that $R = 1$.

For $x = 1$ we have the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ which diverges.

For $x = -1$ we have the alternating series $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$ which converges by the alternating series test.

It follows that $S = [-1, 1)$ for this particular power series.

Fortunately there are some very easy-to-use theorems that allow us to find the radius of convergence of a given power series with ease (most of the time). These simply follow from applying the ratio and root tests to a general power series.

Theorem 21.2 (Ratio test for power series)**.** *Consider the power series*

$$\sum_{n=0}^{\infty} a_n x^n.$$

Suppose that

$$\frac{|a_{n+1}|}{|a_n|} \to \ell, \quad \text{as} \quad n \to \infty.$$

Then

$$R = \begin{cases} 0 & \text{if } \ell = \infty, \\ \frac{1}{\ell} & \text{if } \ell \in \mathbb{R} \backslash \{0\}, \\ \infty & \text{if } \ell = 0. \end{cases}$$

Proof. Suppose that $\ell \in \mathbb{R}$. By the ratio test, the series

$$\sum_{n=0}^{\infty} a_n x^n$$

converges absolutely if

$$\lim_{n\to\infty} \frac{|a_{n+1}x^{n+1}|}{|a_n x^n|} = |x| \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = |x|\ell < 1,$$

and diverges if

$$\lim_{n\to\infty} \frac{|a_{n+1}x^{n+1}|}{|a_n x^n|} = |x| \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = |x|\ell > 1.$$

Hence if $\ell \neq 0$, then $R = \frac{1}{\ell}$. If $\ell = 0$ then $|x|\ell < 1$ for all $x \in \mathbb{R}$, and so $R = \infty$.

If $\ell = \infty$, i.e if

$$\frac{|a_{n+1}|}{|a_n|} \to \infty$$

as $n \to \infty$, then the power series diverges for all $x \neq 0$ by the null sequence test. Hence $R = 0$. □

Theorem 21.3 (Root test for power series)**.** *Consider the power series*

$$\sum_{n=0}^{\infty} a_n x^n.$$

Suppose that

$$|a_n|^{\frac{1}{n}} \to \ell, \quad \text{as} \quad n \to \infty.$$

Then

$$R = \begin{cases} 0 & \text{if } \ell = \infty, \\ \frac{1}{\ell} & \text{if } \ell \in \mathbb{R}\backslash\{0\}, \\ \infty & \text{if } \ell = 0. \end{cases}$$

The proof is very similar to the proof of the ratio test for power series, and so we leave it as an exercise.

Example 1. Find the radius of convergence of the power series

$$\sum_{n=0}^{\infty} (-1)^n n^2 x^n.$$

What is the convergence set S for this power series?

We shall use the ratio test for power series. Let $a_n = (-1)^n n^2$ and observe that

$$\frac{|a_{n+1}|}{|a_n|} = \frac{(n+1)^2}{n^2} = \left(1 + \frac{1}{n}\right)^2 \to 1$$

as $n \to \infty$, by the algebra of limits. Hence the radius of convergence

$$R = \frac{1}{1} = 1.$$

Thus $(-1, 1) \subseteq S \subseteq [-1, 1]$, and so it remains to determine whether the power series converges for $x = \pm 1$. If $x = 1$, then the power series becomes the series

$$\sum_{n=1}^{\infty} (-1)^n n^2,$$

which diverges by the null sequence test as $(-1)^n n^2 \not\to 0$. Thus $1 \notin S$. Similarly $-1 \notin S$, and so

$$S = (-1, 1).$$

Example 2. Find the radius of convergence of the power series

$$\sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

What is the convergence set S for this power series? We shall use the ratio test for power series. Let $a_n = 1/n!$ and observe that

$$\frac{|a_{n+1}|}{|a_n|} = \frac{n!}{(n+1)!} = \frac{1}{n+1} \to 0$$

as $n \to \infty$. Hence $R = \infty$ by the ratio test for power series, and so the power series converges (absolutely) for all $x \in \mathbb{R}$; i.e. $S = \mathbb{R}$.

21.1. **Differentiability of power series.** In this brief section we discuss the calculus of power series. Due to time constraints we omit the proofs of the statements we make.

Suppose that

$$\sum_{n=0}^{\infty} a_n x^n$$

is a power series with convergence set S (we know already that $(-R, R) \subseteq S \subseteq [-R, R]$, where R is the radius of convergence). Define the function $f : S \to \mathbb{R}$ by

$$f(x) = \sum_{n=0}^{\infty} a_n x^n.$$

Question. Is f differentiable? If so, is it true that

$$f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} \qquad ?$$

This would certainly be true if f were a *finite* sum of powers of x, as f would simply be a polynomial, and we know that polynomials are differentiable. The function f is an *infinite* sum of powers of x, and it is not immediately apparent what the answer to this question should be.

Theorem 21.4 (Differentiability of power series)**.** *Suppose*

$$\sum_{n=0}^{\infty} a_n x^n$$

is a power series with radius of convergence R. Then the function $f : (-R, R) \to \mathbb{R}$ given by

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

is differentiable, and

$$f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}.$$

Remark. Implicit in the statement of the above theorem is that if a power series

$$\sum_{n=0}^{\infty} a_n x^n$$

has radius of convergence R then the power series

$$\sum_{n=1}^{\infty} n a_n x^{n-1}$$

also has radius of convergence R.

Many of the special functions that we work with in mathematics, such as the exponential and trigonometric functions, are often most conveniently defined as power series.

21.2. **The exponential function.** The exponential function $\exp : \mathbb{R} \to \mathbb{R}$ is defined by

$$\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

By the ratio test, the radius of convergence of the power series is ∞, and so exp is indeed defined on the whole of $\mathbb{R}$. Moreover, by the previous theorem, exp is differentiable on $\mathbb{R}$ and

$$(\dagger) \qquad \frac{\mathrm{d}}{\mathrm{d}x} \exp(x) = \sum_{n=1}^{\infty} \frac{nx^{n-1}}{n!} = \sum_{n=0}^{\infty} \frac{x^n}{n!} = \exp(x) \quad \forall x \in \mathbb{R}.$$

Note also that by definition,

$$(\dagger\dagger) \qquad \exp(0) = 1.$$

Remark. It can be proved that there exists only one differentiable function on $\mathbb{R}$ satisfying both the differential equation ($\dagger$) and the condition ($\dagger\dagger$).

Theorem 21.5 (Properties of exp)**.** *The exponential function has the following properties:*

1. $\exp(0) = 1$.
2. *For any* $x, y \in \mathbb{R}$, $\exp(x + y) = \exp(x)\exp(y)$.
3. *For any* $x \in \mathbb{R}$, $\exp(x) > 0$.
4. *For any* $x \in \mathbb{R}$, $\exp(-x) = \frac{1}{\exp(x)}$.
5. $\exp(1) = e$, *where* e *denotes Euler's constant.*

Remark. In particular, the above theorem tells us that exp is a homomorphism from the group $(\mathbb{R}, +)$ into the group $(\mathbb{R}_+, \cdot)$, where $\mathbb{R}_+ := (0, \infty)$. (Actually, exp is an isomorphism of groups!)

Notation. We often write e^x, rather than $\exp(x)$. In practice, the exp notation is useful if you wish to raise e to a complicated expression. For example, rather than writing

$$e^{\int_1^\alpha \sin\theta d\theta + \alpha^3},$$

it is clearer (and more elegant) to write

$$\exp\left(\int_1^\alpha \sin\theta d\theta + \alpha^3\right).$$

Proof. Point (1) is a consequence of the definition of exp.

We now prove (2). By definition,

$$\exp(x + y) = \sum_{n=0}^{\infty} \frac{(x+y)^n}{n!}.$$

Recall the binomial formula

$$(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} x^k y^{n-k}.$$

Thus, we have that

$$\exp(x+y) = \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{x^k}{k!}\frac{y^{n-k}}{(n-k)!}.$$

The right hand side of the above formula is the Cauchy product of the two absolutely convergent series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ and $\sum_{n=0}^{\infty} \frac{y^n}{n!}$. Thus

$$\exp(x+y) = \left(\sum_{n=0}^{\infty} \frac{x^n}{n!}\right) \times \left(\sum_{n=0}^{\infty} \frac{y^n}{n!}\right) = \exp(x)\exp(y).$$

Points (3) and (4) follow from (1) and (2). Indeed, by definition of exp, we have that

$$\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} \geq 1 > 0 \quad \forall x \geq 0.$$

Moreover, by (1) and (2)

$$1 = \exp(0) = \exp(x-x) = \exp(x)\exp(-x) \quad \forall x \in \mathbb{R}.$$

It follows that $\exp(x) > 0$ for all $x \in \mathbb{R}$, and

$$\exp(-x) = \frac{1}{\exp(x)}.$$

In order to prove that $e = \exp(1)$, we are required to show that

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = \sum_{k=0}^{\infty}\frac{1}{k!}$$

By using the binomial theorem,

$$\left(1+\frac{1}{n}\right)^n = \sum_{k=0}^{n}\frac{1}{k!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{k-1}{n}\right) \leq \sum_{k=0}^{n}\frac{1}{k!}.$$

Thus

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n \leq \sum_{k=0}^{\infty}\frac{1}{k!} = \exp(1).$$

It remains to prove that

$$e \geq \exp(1).$$

The first step is to show that for all m and n

$$\left(1+\frac{1}{n}\right)^{n+m} \geq \sum_{k=0}^{m}\frac{1}{k!}.$$

By the Binomial theorem,

$$\left(1+\frac{1}{n}\right)^{n+m} = \sum_{k=0}^{n+m}\frac{1}{k!}\frac{(n+m)!}{(n+m-k)!}\frac{1}{n^k} \quad \geq \sum_{k=0}^{m}\frac{1}{k!}\frac{(n+m)!}{(n+m-k)!}\frac{1}{n^k},$$

where we have thrown away the last n terms of the sum. So

$$\begin{aligned}\left(1+\frac{1}{n}\right)^{n+m} &\geq \sum_{k=0}^{m}\frac{1}{k!}\frac{(n+m)(n+m-1)\cdots(n+m-k+1)}{n^k}\\ &\geq \sum_{k=0}^{m}\frac{1}{k!}.\end{aligned}$$

Now fix m and let $n \to \infty$. Then,

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^m \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{m+n} \geq \sum_{k=0}^{m}\frac{1}{k!}$$

We have proved that

$$e \geq \sum_{k=0}^{m}\frac{1}{k!} \quad \forall m \in \mathbb{N}.$$

Then by taking the limit as $m \to \infty$, we get

$$e \geq \sum_{k=0}^{\infty}\frac{1}{k!} = \exp(1),$$

as required to finish the proof. □

21.3. **Maclaurin and Taylor Series.** As we have seen above, power series provide a deep connection between the theory of series (developed in these notes) and the theory of functions (developed by Dr. Wang in parallel). In particular, and as we shall see, the theory of the *calculus* provides us with a way of expressing a variety of "well-behaved" functions as power series. The series constructed in this way are referred to as Maclaurin or Taylor series. Due to time constraints out treatment of Maclaurin/Taylor series will be mainly methodological (i.e. without thorough proofs).

Often it is convenient to express a power series in terms of powers of $(x-c)$ for some real number c, rather than simply powers of x. Such a series, that is, a series of the form

$$\sum_{n=0}^{\infty} a_n(x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + \cdots,$$

is called a *power series about the point c.*

For a power series in this form, Theorem 21.1 becomes:

Theorem 21.6. *Given a power series*

$$\sum_{n=0}^{\infty} a_n(x-c)^n,$$

we have

(1) $\sum_{n=0}^{\infty} a_n x^n$ *converges absolutely when* $|x-c| < R$
(2) $\sum_{n=0}^{\infty} a_n x^n$ *diverges when* $|x-c| > R$.

Here R is the radius of convergence of the power series

$$\sum_{n=0}^{\infty} a_n x^n,$$

as defined at the beginning of Section 3.

Remarks. Just to be clear:

- If $R = 0$ the power series converges at $x = c$ only.
- If $R \in \mathbb{R}\backslash\{0\}$ then the power series
  - converges (absolutely) for every x such that $|x-c| < R$,

  - diverges for every x such that $|x-c|>R$, and
  - may or may not converge when $|x-c|=R$, i.e. when $x=c-R$ or $x=c+R$.

  In this case the power series defines a function from $(c-R,c+R)$ to $\mathbb{R}$.
- If $R=\infty$ the power series converges (absolutely) for all x, and defines a function on the whole of $\mathbb{R}$.

Terminology. Suppose that f is a function and that the power series

$$\sum_{n=0}^{\infty} a_n(x-c)^n$$

converges to $f(x)$ for each $x\in(c-R,c+R)$. Then the power series is said to be a *power series representation* of f on $(c-R,c+R)$.

Example. The function $f(x)=\frac{1}{1-x}$ has power series representation $\sum x^n$ for $x\in(-1,1)$. This representation does not hold for $|x|\geq 1$ as we have already discussed. Notice here that the power series is about 0 - i.e. $c=0$.

Theorem 21.7. *Suppose that the power series*

$$\sum_{n=0}^{\infty} a_n(x-c)^n$$

converges to a function $f(x)$ for all $c-R<x<c+R$ where $0<R\leq\infty$. Then

$$a_n=\frac{f^{(n)}(c)}{n!}$$

for each $n\in\mathbb{N}$.

Proof. If

$$f(x)=\sum_{n=0}^{\infty} a_n(x-c)^n,$$

then differentiating m times (as we may by our theorem on the differentiability of power series) we get

$$f^{(m)}(x)=\sum_{n=m}^{\infty} n(n-1)\cdots(n-m+1)a_n(x-c)^{n-m},$$

so that

$$f^{(m)}(x)=m!a_m+\sum_{n=m+1}^{\infty} n(n-1)\cdots(n-m+1)a_n(x-c)^{n-m}$$

whenever $|x-c|<R$. Setting $x=c$ in this last expression we see that the terms in the sum for $n\geq m+1$ vanish, leaving

$$f^{(m)}(c)=m!a_m;$$

i.e. $a_m=\frac{f^{(m)}(c)}{m!}$, as claimed. □

Definition 21.8. If the function f has a power series representation on the interval $(c-R,c+R)$, then the power series

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n \\ &= \frac{f(c)}{0!}+\frac{f'(c)(x-c)}{1!}+\frac{f''(c)(x-c)^2}{2!}+\frac{f'''(c)(x-c)^3}{3!}+\cdots \end{aligned}$$

is called the *Taylor Series of the function f about c.*

In the particular case that $c = 0$, then Taylor series of f is usually called the *Maclaurin series* of f:

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \\ &= \frac{f(0)}{0!} + \frac{f'(0)x}{1!} + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} + \cdots \end{aligned}$$

Tips for finding Taylor and Maclaurin series.
In applications it can be quite painstaking to calculate the series directly by differentiating the function repeatedly. It is often quicker to manipulate known series. Because power series representations of functions are unique, *any valid method of calculating it will suffice.*

21.4. **Taylor and Maclaurin Series of Common Functions. Some common expansions:**

1. For any $x \in \mathbb{R}$

$$\begin{aligned} e^x &= 1 + x + \frac{x^2}{2} + \cdots + \frac{x^n}{n!} + \cdots = \sum_{n=0}^{\infty} x^n/n! \\ e^{-x} &= 1 - x + \frac{x^2}{2} - \frac{x^3}{3!} \cdots + (-1)^n \frac{x^n}{n!} + \cdots \end{aligned}$$

2. For any $x \in \mathbb{R}$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \cdots = \sum_{n=0}^{\infty} (-1)^n x^{2n+1}/(2n+1)!$$

3. For any $x \in \mathbb{R}$

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!} + \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots = \sum_{n=0}^{\infty} (-1)^n x^{2n}/(2n)!$$

4. The Binomial Theorem: for any $\alpha \in \mathbb{R}$ and x such that $|x| < 1$

$$\begin{aligned} (1+x)^\alpha &= 1 + \alpha x + \frac{\alpha(\alpha-1)}{2} x^2 + \cdots + \frac{\alpha(\alpha-1)\ldots(\alpha-n+1)}{n!} x^n + \cdots \\ &= \sum_{n\geq 0} \frac{\alpha(\alpha-1)\ldots(\alpha-n+1)}{n!} x^n \end{aligned}$$

5. From 4. we have, for any x such that $|x| < 1$,

$$\begin{aligned} \frac{1}{1-x} &= 1 + x + x^2 + x^3 + \cdots + x^n + \cdots = \sum_{n=0}^{\infty} x^n \\ \frac{1}{1+x} &= 1 - x + x^2 - x^3 + \cdots + (-1)^n x^n + \cdots = \sum_{n=0}^{\infty} (-x)^n \end{aligned}$$

6. For any x such that $|x| < 1$

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + (-1)^{n+1} \frac{x^n}{n} + \cdots = \sum_{n\geq 1} (-1)^{n+1} \frac{x^n}{n}$$

21.5. **Manipulation of Power Series.**

Calculating Taylor and Maclaurin series directly:

Examples. (a) Find the Taylor series for $f(x) = e^x$ about $x = 1$ as far as the term in $(x-1)^4$.

(b) Find the Maclaurin of $f(x) = (1+x)^p$, where $0 \neq p \in \mathbb{R}$.

(a) By definition we want to find the Taylor series when $c = 1$:

$$\begin{aligned} e^x &= \sum_{n=0}^{\infty} \frac{f^{(n)}(1)}{n!}(x-1)^n \\ &= \frac{f(1)}{0!} + \frac{f'(1)(x-1)}{1!} + \frac{f''(1)(x-1)^2}{2!} + \frac{f'''(1)(x-1)^3}{3!} + \frac{f''''(1)(x-1)^4}{4!} + \cdots . \end{aligned}$$

We must calculate the coefficients of the Taylor series. Notice that $f^{(n)}(x) = e^x$ for all n, so that

$$f(1) = f'(1) = f''(1) = f'''(1) = f''''(1) = e.$$

Hence

$$e^x = e + e(x-1) + \frac{e}{2}(x-1)^2 + \frac{e}{6}(x-1)^3 + \frac{e}{24}(x-1)^4 + \cdots .$$

The expansion is valid for all $x \in \mathbb{R}$

(b) By definition we want to find

$$(1+x)^p = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

Now it is easy to see that

$$\left(\frac{d}{dx}\right)^n (1+x)^p = p(p-1)\cdots\big(p-(n-1)\big)(1+x)^{p-n}$$

so that $f^{(n)}(0) = p(p-1)\cdots(p-n+1)$. Hence

$$(1+x)^p = \sum_{n=0}^{\infty} \frac{p(p-1)\cdots(p-n+1)}{n!} x^n$$

Note that if $p = m$ for some positive integer m, then $p - m = 0$ so all the coefficient of all terms involving x^n for $n > m$ is 0. In this case the series is finite.

Simple use of known examples

Examples. (a) Find the Taylor series for $f(x) = e^x$ about $x = 1$.

(b) Calculate the Maclaurin series of $f(x) = e^{-3x}$.

(c) Calculate the Maclaurin series of $f(x) = \cos(2x)$.

(d) Calculate the Taylor series of $f(x) = \dfrac{1}{3x+2}$ about $x = -2$.

(a) We know that

$$e^t = 1 + t + \frac{t^2}{2} + \cdots + \frac{t^n}{n!} + \cdots$$

for any t. We want an expansion in terms of $(x-1)$. Putting $t = (x-1)$ in the above, we get

$$\begin{aligned} e^x = e^{(x-1)+1} = ee^{(x-1)} &= e\left(1 + (x-1) + \frac{(x-1)^2}{2} + \cdots + \frac{(x-1)^n}{n!} + \cdots\right) \\ &= e + e(x-1) + \cdots + \frac{e}{n!}(x-1)^n + \cdots \end{aligned}$$

(b) We know that

$$e^t = 1 + t + \frac{t^2}{2} + \cdots + \frac{t^n}{n!} + \cdots$$

Putting $t = -3x$, we get

$$e^{-3x} = 1 - 3x + \frac{9}{2}x^2 - \frac{9}{2}x^3 + \cdots + \frac{(-3)^n}{n!}x^n + \cdots$$

The expansion is valid for all x.

(c) We know that

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!} + \cdots + (-1)^n\frac{x^{2n}}{(2n)!} + \cdots$$

Replacing x with $2x$, we get

$$\begin{aligned} \cos 2x &= 1 - \frac{(2x)^2}{2} + \frac{(2x)^4}{4!} + \cdots + (-1)^n\frac{(2x)^{2n}}{(2n)!} + \cdots \\ &= 1 - \frac{4x^2}{2} + \frac{16x^4}{4!} + \cdots + (-1)^n\frac{2^{2n}x^{2n}}{(2n)!} + \cdots \end{aligned}$$

The expansion is valid for all x.

(d) We know that for any x such that $|x| < 1$,

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots = \sum_{n=0}^{\infty} x^n$$

Now we want an expansion involving powers of $(x - -2) = (x+2)$. Since

$$3x + 2 = 3(x+2) - 6 + 2 = 3(x+2) - 4 = -4\big(1 - 3(x+2)/4\big)$$

we have, replacing x by $3(x+2)/4$ in the expansion of $1/(1-x)$

$$\begin{aligned} \frac{1}{3x+2} &= -\frac{1}{4}\frac{1}{1-3(x+2)/4} \\ &= -\frac{1}{4}\left(1 + \frac{3(x+2)}{4} + \frac{9(x+2)^2}{16} + \cdots + \frac{3^n(x+2)^n}{4^n} + \cdots\right) \\ &= -\frac{1}{4} - \frac{3}{16}(x+2) - \cdots - \frac{3^n}{4^{n+1}}(x+2)^n + \cdots \end{aligned}$$

The expansion is valid for $|3(x+2)/4| < 1$, i.e. for $|x+2| < 4/3$.

Term by term addition of power series: Let $\sum_{n=0}^{\infty} a_n(x-c)^n$ have radius of convergence $R_1 > 0$ and $\sum_{n=0}^{\infty} b_n(x-c)^n$ have radius of convergence $R_2 > 0$. By the linearity of summation (or the "Algebra of Limits for Sums") the sum of the two series is

$$\sum_{n=0}^{\infty}(a_n + b_n)(x-c)^n,$$

and has radius of convergence $R \geq \min\{R_1, R_2\}$.

Example. Calculate the Maclaurin series of $f(x) = e^{-3x} - \cos(2x)$ as far as terms involving x^4.

From above we have

$$\begin{aligned} e^{-3x} &= 1 - 3x + \frac{9}{2}x^2 - \frac{9}{2}x^3 + \frac{(-3)^4}{4!}x^4 + \cdots \\ \cos 2x &= 1 - \frac{4x^2}{2} + \frac{16x^4}{4!} + \cdots \end{aligned}$$

so that

$$\begin{aligned} f(x) &= e^{-3x} - \cos(2x) \\ &= \left(1 - 3x + \frac{9}{2}x^2 - \frac{9}{2}x^3 + \frac{(-3)^4}{4!}x^4 + \cdots\right) \\ &- \left(1 - \frac{4x^2}{2} + \frac{16x^4}{4!} + \cdots\right) \\ &= -3x + \frac{13}{2}x^2 - \frac{9}{2}x^3 + \frac{65}{24}x^4 + \cdots \end{aligned}$$

Term by term multiplication of power series: Let $\sum_{n=0}^{\infty} a_n(x-c)^n$ have radius of convergence $R_1 > 0$ and $\sum_{n=0}^{\infty} b_n(x-c)^n$ have radius of convergence $R_2 > 0$. By Theorem 20.1 the product of the two series is

$$\left(\sum_{n=0}^{\infty} a_n x^n\right)\left(\sum_{n=0}^{\infty} b_n x^n\right) = \sum_{n=0}^{\infty} c_n x^n$$

where

$$c_n = a_0 b_n + a_1 b_{n-1} + \cdots + a_{n-1} b_1 + a_n b_0,$$

and has radius of convergence $R \geq \min\{R_1, R_2\}$.

Example. Calculate the Maclaurin series of $f(x) = e^{-3x}\cos(2x)$ as far as terms involving x^4.

It is often easier to work directly, rather than use the above (Cauchy product) formula. We know

$$\begin{aligned} e^{-3x} &= 1 - 3x + \frac{9}{2}x^2 - \frac{9}{2}x^3 + \frac{(-3)^4}{24}x^4 + \cdots \\ \cos 2x &= 1 - \frac{4x^2}{2} + \frac{16x^4}{24} + \cdots \end{aligned}$$

so that

$$
\begin{aligned}
f(x) &= e^{-3x}\cos(2x) \\
&= \left(1 - 3x + \frac{9}{2}x^2 - \frac{9}{2}x^3 + \frac{(-3)^4}{24}x^4 + \cdots\right)\left(1 - \frac{4x^2}{2} + \frac{16x^4}{24} + \cdots\right) \\
&= 1 - 3x + \left(\frac{9}{2} - \frac{4}{2}\right)x^2 + \left(\frac{12}{2} - \frac{9}{2}\right)x^3 + \left(-\frac{36}{4} + \frac{16}{24} + \frac{(-3)^4}{24}\right)x^4 + \cdots \\
&= 1 - 3x + \frac{5}{2}x^2 + \frac{3}{2}x^3 - \frac{119}{24}x^4 + \cdots
\end{aligned}
$$

Term by term differentiation of power series: By our theorem on the differentiability of power series, if the power series

$$\sum_{n=1}^{\infty} a_n(x-c)^n$$

converges to $f(x)$ for $|x - c| < R$, where $R > 0$ then f is differentiable on $(c - R, c + R)$ and

$$f'(x) = \sum_{n=1}^{\infty} na_n(x-c)^{n-1} = a_1 + 2a_2(x-c) + 3a_3(x-c)^2 + \cdots$$

Example. Find the Maclaurin series for $f(x) = \dfrac{1}{(1-x)^2}$, when $|x| < 1$.

We know that, for $|x| < 1$,

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots \quad \text{and that} \quad \frac{d}{dx}\frac{1}{(1-x)} = \frac{1}{(1-x)^2}.$$

Differentiating the series term by term we have

$$
\begin{aligned}
\frac{1}{(1-x)^2} = \frac{d}{dx}\frac{1}{(1-x)} &= \frac{d}{dx}\left(1 + x + x^2 + x^3 + \cdots\right) \\
&= 1 + 2x + 3x^3 + \cdots + nx^{n-1} + \cdots
\end{aligned}
$$