1 Atoms, matter - solutions

The problems are roughly in order of difficulty. The ones with ♣ are the hardest ones, which might only occur as a "sting in the tail" at the end of a long examination question.

Problem 1.1 **Aluminium atomic spacing**
Aluminium has an Atomic Weight of 27 and a density of 2700 kg m^{-3}. Estimate the typical spacing of the Aluminium atoms.

$$\begin{aligned}\text{Volume of 1 mol of Al} &= \frac{\text{Mass of one mol}}{\text{Density}} \\ &= \frac{27\times10^{-3}\text{kg}}{2700\ \text{kg m}^{-3}} \\ &= 10^{-5}\text{m}^3\end{aligned}$$

The volume is occupied by $N_\text{A} = 6\times10^{23}$ atoms, so

$$\text{Volume occupied by one atom} = \frac{10^{-5}}{6\times10^{23}} = 1.7\times10^{-29}\text{m}^3\ .$$

Now treat the volume as a cube, and interpret the length of one side as being the inter-atomic spacing. So

$$\begin{aligned}\text{Typical inter-atomic spacing} &= \sqrt[3]{1.7\times10^{-29}}\text{m} \\ &= 2.5\times10^{-10}\text{m}\ .\end{aligned}$$

Problem 1.2 **Lennard-Jones forces**
See the figure below with superimposed graphs of the Lennard-Jones potential and the corresponding force.

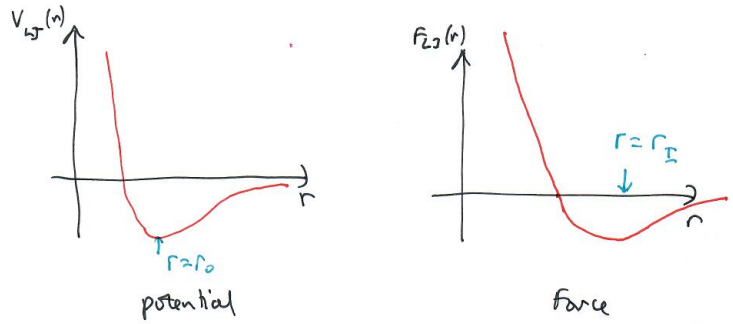

Figure 1.1: The Lennard Jones potential and the force derived from it. Notice the maximum force (in magnitude) is at the point of inflection of the potential, and the zero of the force is at the equilibrium point.

Problem 1.3 **Derivation of size of atom by dimensional analysis**
We may calculate d by dimensional analysis as well, but it gives little physical insight. Let $[X]$ denote the dimensions of the quantity X, then (avoiding confusion between latent heat, $\mathcal{L}$, and L, the dimension of length)

$$\begin{aligned}[\ell] &= [\gamma]^\alpha [\mathcal{L}]^\beta [\rho]^\delta \\ &= (ML^2T^{-2}L^{-2})^\alpha (ML^2T^{-2}M^{-1})^\beta (ML^{-3})^\delta \\ \Rightarrow \quad L &= M^{\alpha+\delta} T^{-2(\alpha+\beta)} L^{2\beta-3\delta} \ .\end{aligned}$$

We now compare powers of M, T and L to get three linear equations:

$$\text{equation for } M \qquad 0 = \alpha + \delta \tag{1.1}$$
$$\text{equation for } T \qquad 0 = -2(\alpha + \beta) \tag{1.2}$$
$$\text{equation for } L \qquad 1 = 2\beta - 3\delta \tag{1.3}$$

Then we may deduce from Eqns. (1.1) and (1.2) that

$$2\delta - 2\beta = 0 \ .$$

So substituting $\delta = \beta$ into Eqn. (1.3) we see that $\delta = -1$, and then $\beta = -1$ and finally $\alpha = 1$. So

$$\ell = \frac{\gamma}{\mathcal{L}\rho} = \frac{d}{4} \ .$$

I.e. reproducing the physical calculation, but with no physical insight.

Problem 1.4 **Three Lennard-Jones atoms and fracture**
The total potential energy, $V^{\rm tot}(x_1, x_2, x_3)$, is:

$$V^{\rm tot}(x_1, x_2, x_3) = V_{\rm LJ}(x_2 - x_1) + V_{\rm LJ}(x_3 - x_2) \ .$$

Even spacing of the atoms yields $x_1 = -a$, $x_2 = 0$ and $x_3 = a$. To see if atom 2 is at a point of equilibrium, let us differentiate $V^{\rm tot}(x_1 = a, x_2, x_3 = a)$ with respect to x_2, note that since atoms 1 and 2 are fixed the derivative is an ordinary derivative:

$$\begin{aligned}\frac{{\rm d}V^{\rm tot}}{{\rm d}x_2} &= \frac{{\rm d}V_{\rm LJ}(x_2 + a)}{{\rm d}x_2} + \frac{{\rm d}V_{\rm LJ}(a - x_2)}{{\rm d}x_2} \\ \Rightarrow \quad \left.\frac{{\rm d}V^{\rm tot}}{{\rm d}x_2}\right|_{x_2=0} &= V'_{\rm LJ}(a) - V'_{\rm LJ}(a) = 0 \ ,\end{aligned}$$

where the minus sign in the second term comes from the chain rule. Thus $x_2 = 0$ is indeed a point of equilibrium.

But is it stable? Let us examine the second derivative:

$$\begin{aligned}\frac{{\rm d}^2V^{\rm tot}}{{\rm d}x_2^2} &= \frac{{\rm d}^2V_{\rm LJ}(x_2 + a)}{{\rm d}x_2^2} + \frac{{\rm d}^2V_{\rm LJ}(a - x_2)}{{\rm d}x_2^2} \\ \Rightarrow \quad \left.\frac{{\rm d}^2V^{\rm tot}}{{\rm d}x_2^2}\right|_{x_2=0} &= V''_{\rm LJ}(a) + (-1)^2 V''_{\rm LJ}(a) \\ &= 2V''_{\rm LJ}(a) \\ &= \begin{cases} 2|V''_{\rm LJ}(a)| & \text{if } a < r_{\rm I}, \\ -2|V''_{\rm LJ}(a)| & \text{if } a > r_{\rm I}. \end{cases}\end{aligned}$$

Thus if the distance between atoms 1 and 3 is less than twice the distance to the point of inflection in the LJ potential, ie $2a < 2r_{\rm I}$, the equilibrium at x_2 is *stable*. But if $2a > 2r_{\rm I}$, then the equilibrium is unstable. Upon any slight perturbation atom 2 will accelerate towards either 1 or 3. The "chain" of three atoms breaks and this is a simple model of fracture.

Exactly at $a = r_{\rm I}$, atom 2 is in neutral equilibrium.

Note this analysis does not rule out new equilibrium points occurring away from $x_2 = 0$ before that position becomes unstable.

♣*Problem 1.5* **Difference between sand and flour**
The key difference between a grain of sand and a particle of flour is their size. Typically a grain of sand might be 1 mm, whereas for flour it is 0.05mm. (A table of size nomenclature for geological entities (from boulders to silt) is here.)

Given the hint in the question, Hamaker's solution to the van der Waals interaction between two spheres, note in the limit where their separation, d, is small compared to their size, R, then $V^{\rm int} \simeq -U_0(R/d)$. To form a "clump" one must consider the gravitational potential energy of placing one grain on top of another. This is $\Delta V^{\rm grav} \simeq MgR$, where M is the mass of the grain.

But

$$Mg R = \frac{4\pi}{3}\rho R^3 {\rm g} R \ .$$

where ρ is the mass-density of the material. Thus the total energy, adding these two contributions is:

$$\begin{aligned}\mathcal{E}^{\rm tot} &= V^{\rm int} + \Delta V^{\rm grav} \\ &= -U_0\frac{R}{d} + \underbrace{\frac{4\pi}{3}{\rm g}d\rho d^3}_{\text{microscopic energy of lifting one atom}} \left(\frac{R}{d}\right)^4\end{aligned}$$

To compare the two terms let us take their ratio, say v:

$$v = \frac{\Delta V^{\rm grav}}{V^{\rm int}} = -\frac{4\pi}{3}\frac{{\rm g}d\rho d^3}{U_0}\left(\frac{R}{d}\right)^3 \ .$$

As the size of the grain, R, gets smaller this ratio gets smaller. The pre-factor is already very small, being the ratio of (for, say, atomic weight of 100 and $d = 1$Å) 1.5×10^{-34}J to 5×10^{-21}J (assuming the spacing of the atoms is around 1 Å.

The critical size for clumping should be when this ratio is around unity, i.e. $v \simeq 1$. This predicts:

$$\frac{R^{\rm crit}}{d} \simeq \sqrt[3]{\frac{5\times10^{-21}}{1.5\times10^{-34}}} \simeq 3.2\times10^4 \quad \Rightarrow \quad R^{\rm crit} \simeq 3.2\mu{\rm m} = 0.0032 \text{ mm} \ .$$

This is a bit small and shows that the flour grains do not approach *on average* to an atomic distance (presumably due to asperities on their surfaces).

♣*Problem 1.6* **Bodies held together by gravity - the Roche limit**
Firstly let us evaluate the gravitational force (*per unit mass*) due to the planet at the edges

of the asteroid, in the radial direction:

$$\begin{aligned}\mathbf{F}(r \pm a) &= -\frac{\mathrm{G}M}{(r \pm a)^2}\hat{\mathbf{r}} \\ &\simeq -\frac{\mathrm{G}M}{r^2}\left(1 \mp 2\frac{a}{r}\right)\end{aligned}$$

So if we now consider a model of the asteroid as being two point particles each of mass m, at $r \pm a$, then there is a "tidal" force, $\delta\mathbf{F}$, on each away from the centre of mass (away from the planet for the "outer" mass and towards the planet for the inner mass).

This should be contrasted with the mutual attraction of the two masses,

$$\mathbf{F}^{\mathrm{att}} = \pm\frac{\mathrm{G}m^2}{(2a)^2}\hat{\mathbf{r}} \ .$$

Here the plus sign is for the inner mass and the minus sign is for the outer mass.

Then the total force, $\mathbf{F}^{\mathrm{tot}}_{\pm}$, (relative to the centre of mass), where the plus sign is the force on the inner mass, is:

$$\mathbf{F}^{\mathrm{tot}}_{\pm} = \left(\mp\frac{2\mathrm{G}Mma}{(r)^3} \pm \frac{\mathrm{G}m^2}{(2a)^2}\right)\hat{\mathbf{r}} \ .$$

We find the two forces balance when $r = r_{\mathrm{R}}$, the Roche radius

$$\begin{aligned} 0 &= \frac{\mathrm{G}Mm}{r_{\mathrm{R}}^2}\left(-2\frac{a}{r_{\mathrm{R}}} + \frac{m}{M}\left(\frac{r_{\mathrm{R}}}{2a}\right)^2\right) \\ \Rightarrow \quad r_{\mathrm{R}}^3 &= (2a)^3\frac{M}{m} \\ \Rightarrow \quad r_{\mathrm{R}} &= 2a\ \sqrt[3]{\frac{M}{m}}\end{aligned}$$

is the Roche limit.

It is interesting to note that most of Saturn's rings are inside the Roche limit (using a as their width). That could be evidence of primordial satellites being torn apart by the tidal forces.