Electromagnetism I – Problem sheet 6

A cylindrical volume of radius a is filled with charge of uniform volume density ρ. We want to know the *potential energy per unit length* of this cylinder of charge as measured from a radial distance $R > a$ from the cylinder. You are asked to compute this in two ways (and they should give the same answer!)

Method 1. You can compute the electrostatic energy by computing the work that is needed to build up the cylinder step by step, from a distance R, in this case by bringing in cylindrical shells of charge Δq with the same density.

1. Consider first the set up at an intermediate stage when you have already built a cylinder with a radius $r < a$, and charge density ρ.

   (a) What is the electric field E of this cylinder at radius $x > r$? Express the result as a function of x and ρ. **[1]**

   (b) What is the work, ΔW, done in bringing charge Δq from a radius $R > r$ down to the radius r? **[2]**

2. As we build up the cylinder in steps, using cylindrical shells of infinitesimal width δr and length ℓ, write down an expression for Δq in terms of r, δr, ℓ and ρ. **[1]**

3. Compute now the total work per unit length done in building up the full cylinder of radius a by integrating the work ΔW. **[3]**

   You may use the following integral result:

$$\int_0^A x^3 \ln \frac{B}{x} dx = \frac{A^4}{16} + \frac{A^4}{4} \ln \frac{B}{A}$$

Method 2. You should obtain the same result if you compute the energy per unit length by applying the known result that the energy density of the electrostatic field is

$$\frac{dU}{dV} = \frac{1}{2}\epsilon_0 \, E^2 \, .$$

6. Compute the energy per unit length $u_{\rm ins}$ inside the cylinder. **[2]**

7. Compute the energy per unit length $u_{\rm ext}$ external to the cylinder up to a radius $R > r$. **[1]**

You should find that $u_{\rm ins} + u_{\rm ext}$ is equal to the result of point (3). If it isn't, check your work.