W7120 Specified Calculator

THE UNIVERSITY OF BIRMINGHAM

First year single honours programmes in mathematics
Second year joint honours programmes including mathematics

06 11240 / 06 02422 / 06 15005

MSM1Cb: Computational and Applied Mathematics
MSM1A01b: Dynamical Systems
MSM2A01b: Computational and Applied Mathematics

Summer Examinations 2007 2 hours

Full marks may be obtained with complete answers to ALL questions in Section A (worth a total of 60 marks) and TWO (out of THREE) questions from Section B (worth 20 marks each). Only the best TWO answers from Section B will be credited. Calculator may be used as specified by the School as follows: only calculators with no equals key may be used

SECTION A

1. In this question we assume we are in a standard two-dimensional (x, y) Cartesian coordinate system with corresponding unit vectors $(\mathbf{i}, \mathbf{j})$.)

(a) A particle of mass m has a position vector given by

$$\mathbf{r}(t) = (2t^4 + 6)\mathbf{i} + \sin(2t)\mathbf{j}.$$

Determine the acceleration of the particle, and hence determine the force acting on the particle. [5]

(b) A particle of mass m moves in a force-field $\mathbf{F} = m((y^3 - 4)\mathbf{i} - \mathbf{j})$. If the particle starts from rest at the origin, determine the path of the particle as a function of time. [5]

2. A baseball is dropped from rest over the edge of a tall cliff and is subject to quadratic air resistance. What is the terminal fall speed v_{ter} of the baseball? The same baseball is thrown vertically upwards from $y = 0$ with initial speed v_o and is again subject to quadratic air resistance. Write down an equation for the upward motion of the ball and show that the maximum rise height is given by

$$y_{max} = \frac{v_{ter}^2}{2g} \ln\left(\frac{v_{ter}^2 + v_o^2}{v_{ter}^2}\right).$$

(Note that if $\frac{dy}{dt} = v$ then $\frac{d^2y}{dt^2} = v\frac{dv}{dy}$.) [10]

SECTION B

3. A particle of mass m is attached to a spring with a Hooke's constant of k. Suppose that the particle is on a horizontal frictionless surface and x measures the distance of the particle away from the equilibrium length of the spring.

(a) Use Hooke's law to determine the force acting on the particle due to the spring. [2]

(b) Write down Newton's second law for the particle and hence determine the location of the particle as a function of time, if at time $t = 0$, $x = x_o$ and $\frac{dx}{dt} = 0$. [5]

(c) Show that the sum of the kinetic energy and potential energy for the particle described in (b) is constant. [5]

(d) If we now assume that the particle is subject to a frictional resistance force equal to $-b\frac{dx}{dt}$, determine the location of the particle as a function of time, where once again at time $t = 0$, $x = x_o$ and $\frac{dx}{dt} = 0$. The constants k and b are related by $k = \frac{3b^2}{16m}$. In this case, what is $\lim_{t\to\infty} x$? [8]

4. In polar coordinates Newton's second law for a particle of mass m can be written as

$$\mathbf{F} = F_r \mathbf{e}_r + F_\theta \mathbf{e}_\theta = m \left(\left[\frac{d^2 r}{dt^2} - r \left(\frac{d\theta}{dt} \right)^2 \right] \mathbf{e}_r + \left[2 \frac{dr}{dt} \frac{d\theta}{dt} + r \frac{d^2\theta}{dt^2} \right] \mathbf{e}_\theta \right).$$

Assume that the particle is subject to a central force with $F_r = -g(r)$ (for some function g) and $F_\theta = 0$.

(a) Show that the quantity

$$h = r^2 \frac{d\theta}{dt}$$

is constant and hence or otherwise show that the position vector of the particle sweeps out equal areas about the origin in equal times. [6]

(b) If $u = 1/r$ show that the radial component of Newton's second law becomes

$$\frac{d^2 u}{d\theta^2} + u = \frac{g\,(1/u)}{m h^2 u^2},$$

where h is the constant identified above. [6]

(c) Determine the general form of the orbit $r(\theta)$ of the particle if $g(r) = \gamma / r^2$ where γ is a dimensional constant. [6]

(d) Under what condition are the orbits calculated above bounded? [2]