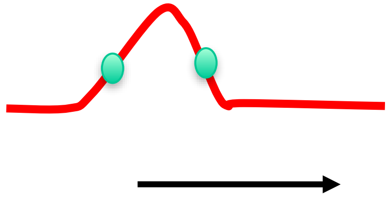
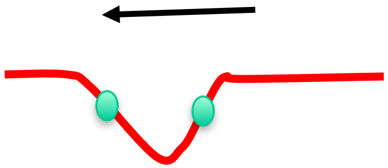
Optics and Waves

Lecture 3

Lecture 2: Transverse and longitudinal waves

General form of wave functions: $y(x, t) = f(x-vt)$





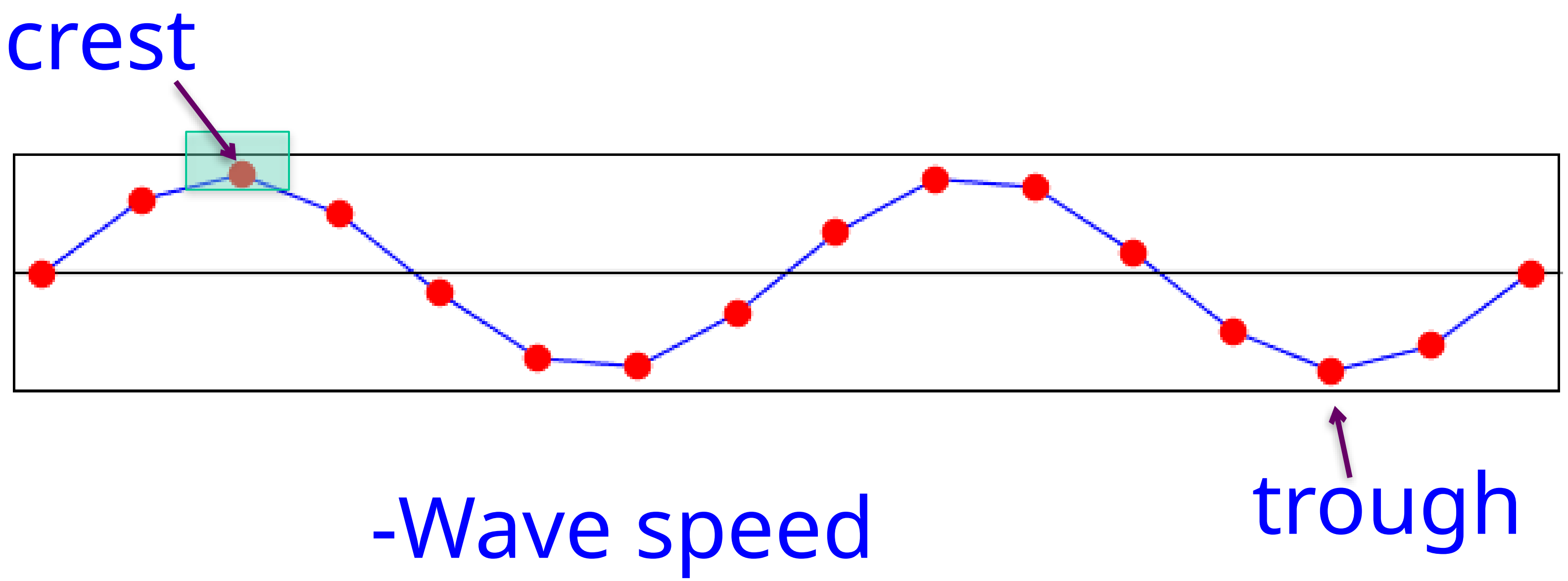
Lecture 3:
Properties of sinusoidal waves

-Wave speed

-Particle velocity

-Particle acceleration

-The wave equation($\neq$wave function)



The following four slides shows the derivation of the wavefunction for a sine wave. This is a differ approach from the one I used in Lecture 2.

The displacement of a particle at the left end of the string is given by:

$$y(x=0,t) = A\cos\omega t \qquad \text{Eq. 1}$$

What is $y(x, t)$ for x other than 0?

The same displacement is found at x at a later time. How much later?

x/v seconds later, because wave moves to the right with velocity v.

OR, The displacement at point x at time t is the same as the displacement at $x=0$ at an earlier time "$t-x/v$".

The displacement at $x=0$ at time $[t-x/v]$ is just

$$y(x=0, t-\frac{x}{v}) = A\cos\omega(t-\frac{x}{v}) \qquad \text{Eq. 2}$$

This is also the displacement at a point which has a distance *x* from the origin at time *t*.

In our analysis, we have chosen an arbitrary time and an arbitrary distance, thus, Eq. 2 applies to any distance and time. Therefore, the general expression of a sinusoidal wave:

$$y(x,t) = A\cos[\omega(t - \frac{x}{v})] = A\cos[\omega(\frac{x}{v} - t)]$$

$$y(x,t) = A\cos[2\pi f\frac{x}{v} - \omega t] = A\cos 2\pi[\frac{x}{\lambda} - \frac{t}{T}] = A\cos[2\pi\frac{x}{\lambda} - \omega t]$$

$k = \frac{2\pi}{\lambda}$ is called the wave number.

Hence, the wave function can be written as:

$$y(x,t) = A\cos(kx - \omega t)$$

Units: y in meters, A in meters, x in meters, k rad/m, ω rad/s

k and ω are the two key parameters for waves.

The above wave function describes a sinusoidal wave travelling in the $+x$ direction. For waves travelling in the $-x$ direction, we have:

$$y(x,t) = A\cos(kx + \omega t)$$

In both cases, the quantity $(kx \pm \omega t)$ is called the phase.

Adding a constant to the phase does not change the physical property of the wave

$$y(x,t) = A\cos(kx - \omega t + \varphi)$$

$$y(x,t) = A\sin(kx - \omega t) \qquad y(x,t) = A\cos(kx - \omega t)$$

Wave velocity.

Starting from the wave function:

$$y(x,t) = A\cos(kx - \omega t)$$

Rearrange:

$$y(x,t) = A\cos(kx - \omega t) = A\cos k(x - \frac{\omega}{k}t)$$

Compare with the general form of wave function $y = f(x - vt)$

We have

$$v = \frac{\omega}{k} \qquad v = \lambda f$$

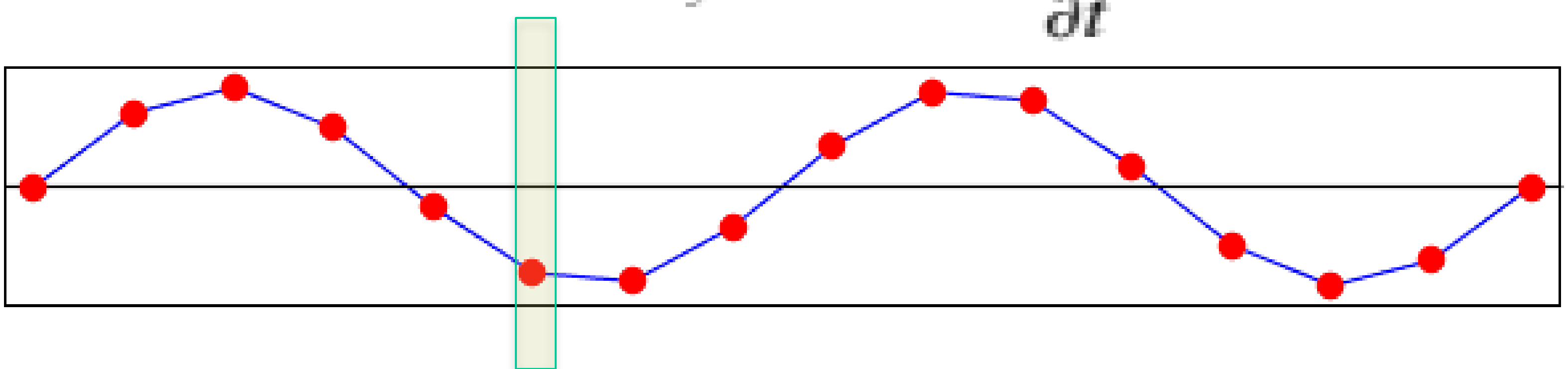
Particle velocity and acceleration in a sinusoidal wave

In a transverse wave, the particles oscillate in directions perpendicular to the direction that the wave is moving. Now, Let's find the transverse velocity of a particle, v_y.

v_y is not the wave velocity! We can find v_y using the wave function. We take the first derivative of the wave function with respect to t, keeping x constant.

$$y(x,t) = A\cos(kx - \omega t)$$

$$v_y(x,t) = \frac{\partial y(x,t)}{\partial t}$$



$$\frac{\partial y(x,t)}{\partial t}$$ is a partial derivative.

$$\frac{\partial y(x,t)}{\partial t} = \frac{dy(x,t)}{dt},$$ treating *x* as a constant.

$$v_y(x,t) = \frac{\partial y(x,t)}{\partial t} = \omega A \sin(kx - \omega t)$$

Since we have kept *x* fixed, the above equation hence shows the transverse velocity of a particle located at *x*. It also shows that the particle is a harmonic oscillator with a maximum speed ωA.

Is ωA greater or smaller than the wave velocity??

$$v = \frac{\omega}{k}$$

The acceleration of a particle is the second partial derivative of the wave function $y(x, t)$.

$$a_y(x,t) = \boxed{\frac{\partial^2 y(x,t)}{\partial t^2}} = \frac{\partial}{\partial t}\left(\frac{\partial y(x,t)}{\partial t}\right) = -\omega^2 A\cos(kx - \omega t) = \boxed{-\omega^2 y(x,t)}$$

Now let's compute the partial derivatives of $y(x, t)$ with respect to x, keeping t constant. Why?

$\dfrac{\partial y(x,t)}{\partial x}$ is the slope of the string at point x and time t,

$\dfrac{\partial^2 y(x,t)}{\partial x^2}$ is the curvature of the string.

$$\boxed{\frac{\partial^2 y(x,t)}{\partial x^2}} = -k^2 A\cos(kx - \omega t) = \boxed{-k^2 y(x,t)}$$

$$\left\{\begin{array}{ll}\dfrac{\partial^2 y(x,t)}{\partial t^2} = -\omega^2 y(x,t) & \text{Eq. i}\\[2ex] \dfrac{\partial^2 y(x,t)}{\partial x^2} = -k^2 y(x,t) & \text{Eq. ii}\end{array}\right.$$

Divide Eq. i by Eq ii:

$$\frac{\partial^2 y(x,t)/\partial t^2}{\partial^2 y(x,t)/\partial x^2} = \frac{\omega^2}{k^2} = v^2$$

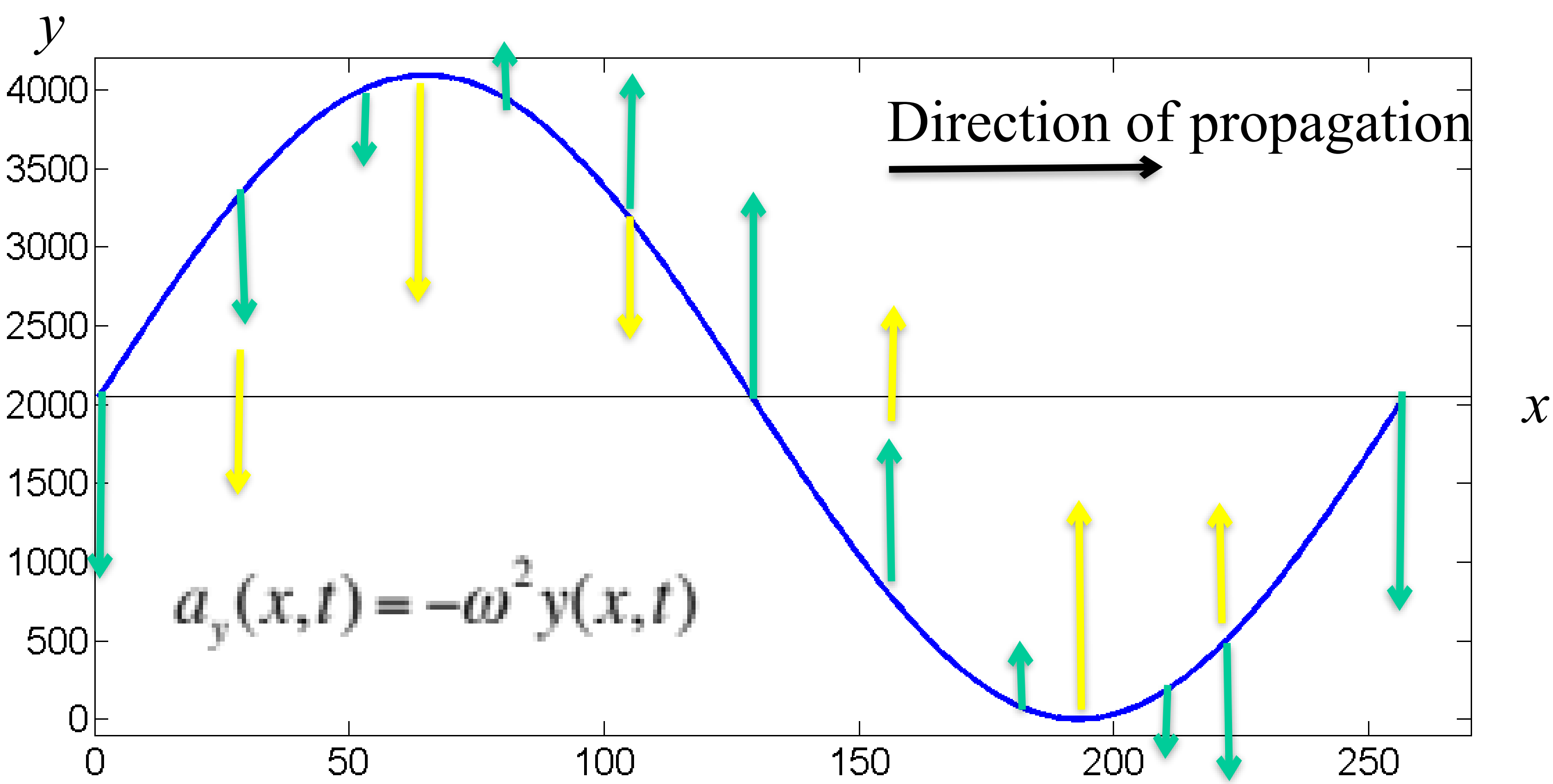
and

$$\boxed{\frac{\partial^2 y(x,t)}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y(x,t)}{\partial t^2}}$$

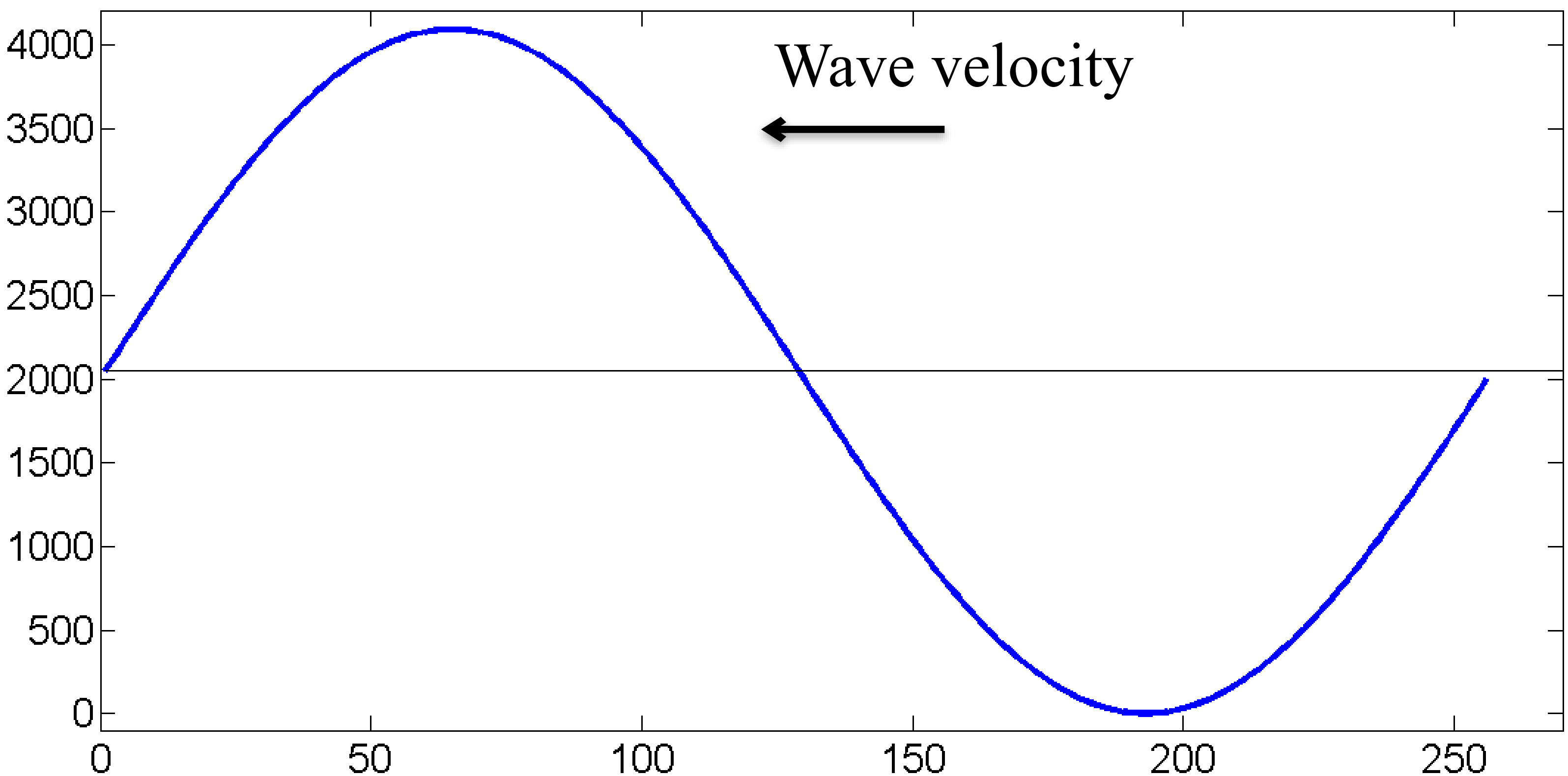
The wave equation

Any wave obeys this wave equation, no matter which direction the wave travels, or whether the wave is periodic or not! If *y(x,t)* does not satisfy this equation, then it is not a wave function.

Arrows in the figure below indicate the velocity/acceleration of the particles.



Acceleration and transverse velocity at each point on a string.



Wave velocity
4000
3500
3000
2500
2000
1500
1000
500
0
0
50
100
150
200
250