4 Heat transfer – solutions

The problems are roughly in order of difficulty. The ones with ♣ are the hardest ones, which might only occur as a "sting in the tail" at the end of a long examination question.

Problem 4.1 **Simple conduction**
Use as the thermal conductivity for copper $\kappa_{\mathsf{Cu}} = 390$ Wm^{-1}K^{-1}.

1. the temperature gradient is:

$$\frac{T_2 - T_1}{L} = \frac{125}{0.25}\ \mathsf{Km}^{-1} = 500\ \mathsf{Km}^{-1}\ .$$

2. the rate of heat transfer

$$\begin{aligned}\dot{Q} &= \kappa A \frac{T_2 - T_1}{L} \\ &= 3.9 \times 10^2 \times 10^{-4} \times 500\ \mathsf{Js}^{-1} \\ &= 19.5\ \mathsf{Js}^{-1}\end{aligned}$$

3. The temperature, T_3, of the rod 10 cm from the high temperature end:

$$T_3 = \left(125 - \frac{10}{25}125\right)\ \mathsf{K} = 75\ \mathsf{K}\ .$$

Problem 4.2 **Melting ice**

1. The rate of heat transfer, $\dot{Q}$, is

$$\dot{Q} = 390 \times \left(4.8 \times 10^{-4}\right) \frac{100}{1.2}\ \mathsf{Js}^{-1} \simeq 15.6\ \mathsf{Js}^{-1}\ .$$

2. Find the rate at which ice melts at the cold end. Need to look up latent heat of melting of ice: $L = 3.34 \times 10^5$ J kg^{-1}. Then the rate of ice melting is

$$\frac{\dot{Q}}{L} = \frac{15.6}{3.34 \times 10^5}\ \mathsf{kg\,s}^{-1} = 4.8 \times 10^{-5}\ \mathsf{kg\,s}^{-1}\ .$$

Problem 4.3 **Dry and damp skiers**
We must realise at the start that to a good approximation the *shape* of the skier is irrelevant, as the clothes are 1 cm thick, but the dimensions of the limbs are around 10 cm, so we can treat the body as a plane of area $A = 1.8$ m^2.

1. Absorbing a minus sign into the definition of $\Delta T > 0$,

$$\dot{Q}_{\mathsf{dry}} = \kappa_{\mathsf{clothing}} A \frac{\Delta T}{L} = 4 \times 10^{-2} \times 1.8 \times \frac{38}{10^{-2}}\ \mathsf{Js}^{-1} = 274\ \mathsf{Js}^{-1}\ .$$

2. If the skier falls over so their clothes are now wet, we need the thermal conductivity of water, $\kappa_{\text{water}} = 0.6$ Wm^{-1}K^{-1} as the spaces which were full of air are now full of water. So the thermal flux is

$$\dot{Q}_{\text{wet}} = \frac{0.6}{0.04}\dot{Q}_{\text{dry}} = 15\dot{Q}_{\text{dry}} = 4100 \text{ Js}^{-1} \ .$$

This is why falling over is not a good idea.

Problem 4.4 **Cylinders and spheres**

For the cylindrical shell we see that (noting that temperature only depends on r) using $A = 2\pi r L$,

$$\begin{aligned}
\dot{Q} &= -\kappa(2\pi r L)\frac{\mathrm{d}T}{\mathrm{d}r} \\
\Rightarrow \quad \dot{Q}\frac{\mathrm{d}r}{\kappa(2\pi r L)} &= -\mathrm{d}T \\
\Rightarrow \quad \frac{\dot{Q}}{\kappa(2\pi L)}(\ln r - \ln r_1) &= -T(r) + T(r_1) \quad \text{(integrating from } r_1 \text{ to } r\text{)} \\
\Rightarrow \quad T(r_1) - \frac{\dot{Q}}{\kappa(2\pi L)}\ln\left(\frac{r}{r_1}\right) &= T(r)
\end{aligned}$$

For the spherical shell, $A = 4\pi r^2$, so:

$$\begin{aligned}
\dot{Q} &= -\kappa 4\pi r^2\frac{\mathrm{d}T}{\mathrm{d}r} \\
\Rightarrow \quad \dot{Q}\frac{\mathrm{d}r}{\kappa 4\pi r^2} &= -\mathrm{d}T \\
\Rightarrow \quad \frac{\dot{Q}}{\kappa 4\pi}\left(\frac{1}{r_1} - \frac{1}{r}\right) &= -T(r) + T(r_1) \\
\Rightarrow \quad T(r_1) - \frac{\dot{Q}}{\kappa 4\pi}\left(\frac{1}{r_1} - \frac{1}{r}\right) &= T(r)
\end{aligned}$$

Problem 4.5 **The heat equation**

1. The flux of heat through the two faces is (assuming that the area and thermal conductivity do no vary with x), noting the flow is *inwards* for x and *outwards* for $x + \mathrm{d}x$,

$$\begin{aligned}
\dot{Q}_{\text{net}} &= -\dot{Q}(x + \mathrm{d}x) + \dot{Q}(x) \\
&= \kappa A\left(\frac{\partial T}{\partial x}(x + \mathrm{d}x) - \frac{\partial T}{\partial x}(x)\right) \\
&\simeq \kappa A\left(\frac{\partial T}{\partial x}(x) + \mathrm{d}x\frac{\partial^2 T}{\partial x^2} + \cdots - \frac{\partial T}{\partial x}(x)\right) \\
&= \kappa A \mathrm{d}x\frac{\partial^2 T}{\partial x^2} \ . \qquad (4.1)
\end{aligned}$$

2. see first item.

3. The temperature, $T(t)$, of the section changes with time, because there is a net heat flux into it.

   If the specific heat per unit mass is c_V and so the specific heat *per unit volume* is $c_V\rho$ where ρ is the density. Then the specific heat of the section, which has volume $\mathrm{d}xA$, is
$$C = c_V\rho\mathrm{d}xA \ .$$
   Then we may use (from $\Delta Q = C\Delta T$)
$$\dot{Q}_{\text{net}} = C\frac{\partial T}{\partial t}(x,t) = c_V\rho\mathrm{d}xA\frac{\partial T}{\partial t}(x,t) \ .$$
   Equating this with the r.h.s of Eq. (4.1) we find:
$$\begin{aligned} c_V\rho\mathrm{d}xA\frac{\partial T}{\partial t}(x,t) &= \kappa A\mathrm{d}x\frac{\partial^2 T}{\partial x^2} \\ \Rightarrow \quad \frac{\partial T}{\partial t} &= \frac{\kappa}{c_V\rho}\frac{\partial^2 T}{\partial x^2} \\ \Rightarrow \quad \frac{\partial T}{\partial t} &= \alpha\frac{\partial^2 T}{\partial x^2} \ , \end{aligned}$$
   where the *thermal diffusivity*, α, is
$$\alpha = \frac{\kappa}{c_V\rho} \ .$$

This is the *heat equation*. We now examine some simple consequences of this.

4. If the temperature is *linear* in position i.e. $T(x) = \tilde{T}(t)x$, then $\partial^2 T/\partial x^2 = 0$ and so $\partial T/\partial t = 0$.
5. Thus taking $\tilde{T}(t) = \tau$, a constant, $\dot{Q}$ must be constant with x. This guarantees the net heat flux into each section is *zero* and so the temperature does not change with time.

Problem 4.6 **Series and parallel thermal conductors**

1. For there to be no build up at the boundaries the heat fluxes on both sides of the boundary must be equal. Thus
$$\begin{aligned} \dot{Q}_{\text{left}} &= \dot{Q}_{\text{right}} \\ \Rightarrow \quad -\kappa_1 A_1\frac{\partial T_{\text{left}}}{\partial x} &= -\kappa_2 A_2\frac{\partial T_{\text{right}}}{\partial x} \ . \end{aligned}$$
   Note if we wish no variation in temperature on each side of the junction, we need $\partial^2 T/\partial x^2 = 0$, so the temperature must be *linear* with x. Thus if the temperature at the interface is T_0, the temperature gradient on the left of the junction is
$$\frac{T_1 - T_0}{L_1} \quad \text{and on the right} \quad \frac{T_0 - T_2}{L_2} \ .$$

2. To deduce the temperature at the interface between the two materials, we require the heat flows on both sides of the junction to be equal:

$$
\begin{aligned}
\kappa_1 A_1 \frac{T_1 - T_0}{L_1} &= \kappa_2 A_2 \frac{T_0 - T_2}{L_2} \\
\Rightarrow \quad \frac{\kappa_1 A_1 T_1}{L_1} + \frac{\kappa_2 A_2 T_2}{L_2} &= \left(\frac{\kappa_1 A_1}{L_1} + \frac{\kappa_2 A_2}{L_2} \right) T_0 \\
\Rightarrow \quad \frac{\dfrac{\kappa_1 A_1 T_1}{L_1} + \dfrac{\kappa_2 A_2 T_2}{L_2}}{\dfrac{\kappa_1 A_1}{L_1} + \dfrac{\kappa_2 A_2}{L_2}} &= T_0 \\
\Rightarrow \quad \frac{K_1 T_1 + K_2 T_2}{K_1 + K_2} &= T_0 \ ,
\end{aligned}
$$

where the thermal conductances K_i are defined

$$K_i = \frac{\kappa_i A_i}{L_i} \ .$$

Returning to the heat flux:

$$
\begin{aligned}
\dot{Q} &= \kappa_1 A_1 \frac{T_1 - T_0}{L_1} \\
&= K_1 (T_1 - T_0) \\
&= K_1 \left(T_1 - \frac{K_1 T_1 + K_2 T_2}{K_1 + K_2} \right) \\
&= K_1 \left(\frac{T_1 (K_1 + K_2) - (K_1 T_1 + K_2 T_2)}{K_1 + K_2} \right) \\
&= K_1 \left(\frac{T_1 K_2 - K_2 T_2}{K_1 + K_2} \right) \\
&= \frac{K_1 K_2}{K_1 + K_2} (T_1 - T_2) \\
&= K_{\text{tot}}^{\text{series}} (T_1 - T_2) \ .
\end{aligned}
$$

I.e.

$$\frac{1}{K_{\text{tot}}^{\text{series}}} = \frac{1}{K_1} + \frac{1}{K_2} \ ,$$

like electrical *conductances*, not *resistances*.

If they are placed in *parallel* between the two heat baths (disregarding the inconvenient difference in lengths).

3. The heat flow through each conductor, $i = 1, 2$ is

$$\dot{Q}_i = \kappa_i A_i \frac{T_1 - T_2}{L_i} = K_i (T_1 - T_2) \ .$$

Thus the total heat flow through the two conductors is

$$
\begin{aligned}
\dot{Q}_{\text{tot}} &= \dot{Q}_1 + \dot{Q}_2 \\
&= (K_1 + K_2)(T_1 - T_2) \\
&= K_{\text{tot}}^{\text{parallel}} (T_1 - T_2) \ .
\end{aligned}
$$

So thermal conductances *add* when in parallel. (As do electrical *conductances* as against *resistances*.)

Problem 4.7 **Heat equation with internal heat generation**
This is really very straightforward. Recall the expression for the net heat flux into a section, but now modified by a term representing the creation of heat in the volume $\mathrm{d}xA$, $\dot{q}A\mathrm{d}x$:

$$\begin{aligned}
\dot{Q}_{\mathsf{net}} &= -\dot{Q}(x+\mathrm{d}x) + \dot{Q}(x) + \dot{q}A\mathrm{d}x \\
&= \kappa A\mathrm{d}x\frac{\partial^2 T}{\partial x^2} + \dot{q}A\mathrm{d}x \\
\Rightarrow \quad \rho c_{\mathrm{V}} A\mathrm{d}x\frac{\partial T}{\partial t} &= \kappa A\mathrm{d}x\frac{\partial^2 T}{\partial x^2} + \dot{q}A\mathrm{d}x \\
\Rightarrow \quad \frac{\partial T}{\partial t} &= \frac{\kappa}{\rho c_{\mathrm{V}}}\frac{\partial^2 T}{\partial x^2} + \frac{\dot{q}}{\rho c_{\mathrm{V}}} \\
\Rightarrow \quad \frac{\partial T}{\partial t} &= \alpha\frac{\partial^2 T}{\partial x^2} + \frac{\dot{q}}{\rho c_{\mathrm{V}}} \ .
\end{aligned}$$

Problem 4.8 **The Great Ocean Conveyor**
The water in the current at the ocean bottom moves a distance of the order of 10^4 km, at a speed of 1 cm s^{-1}, taking a time 10^9 s $\simeq 100$ years.

Noting that the bottom 2 km of the ocean is involved in this motion, we might consider a cuboid of 1 m^2 by 2 km, with a heat flux in at the sea bed of order 0.1 Wm^{-2}, so the total energy acquired by the cuboid is $\mathcal{E}_{\mathsf{cuboid}} = 10^8$ J.

The mass of the cuboid is $m_{\mathsf{cuboid}} = 2000 \times \rho_{\mathsf{water}} = 2 \times 10^6$ kg, and hence the heat capacity of the cuboid is $cm_{\mathsf{cuboid}} = \mathcal{C}_{\mathsf{cuboid}} = 2 \times 10^6 \times 4 \times 10^3$ J K^{-1} $= 8 \times 10^9$ J K^{-1}.

Thus the temperature rise is $\mathcal{E}_{\mathsf{cuboid}}/\mathcal{C}_{\mathsf{cuboid}} \simeq (10^8/10^{10})$ K $= 10^{-2}$K. This is very small compared to the change in temperature of the water due to interaction with solar radiation in the tropics and loss of heat to the atmosphere as the Arctic is approached which is of the order of 20 K.

So despite the long timescale of the interaction with the sea bed the time is too short to allow the water to warm up.

Problem 4.9 **Radiative transport in series with conduction**
Let the temperature at the end of the cylindrical conductor (Fig. (4.1)) be T_2.

Then in steady state the temperature gradient in the conductor must be constant, so the heat flux within the conductor is uniform. Then the flux in the conductor must be equal to that being lost radiatively. So

$$\sigma A T_2^4 = \kappa A\frac{T_1 - T_2}{L} \ . \tag{4.2}$$

There is only one unknown in this equation, T_2, but the equation is quartic in T_2, so the solution, in general, is complicated.

There is a natural temperature scale revealed by dividing through by σ, implying that the dimensions of the resulting product on the right hand side, $[\kappa/(\sigma L)] =$ (temperature)^3 $=$

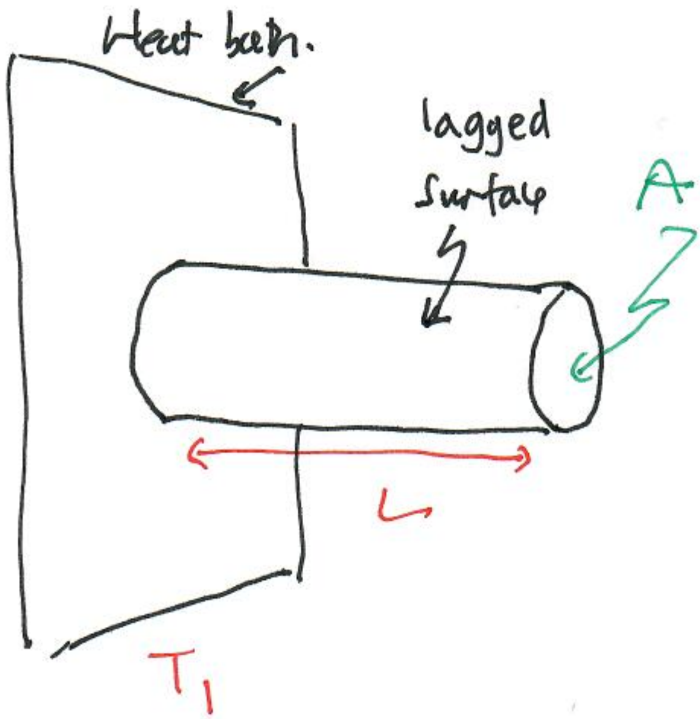

Figure 4.1: A copper rod connected to a heat bath at temperature T_1, cooling radiatively at the other end.

T_0^3. If we choose a copper rod, $\kappa_{\text{Cu}} \simeq 400$ Wm^{-1}K^{-1} and a length of 1 cm, we find that (using $\sigma \simeq 6 \times 10^{-8}$ Wm^{-2}K^{-4}),

$$T_0 \simeq 10^4 \text{ K} .$$

Thus we anticipate that the radiative cooling is only efficient when we are dealing with temperatures in that regime. Otherwise, there is almost no drop along the rod - since the very small gradient is enough to generate the weak thermal current to compensate for the very slow radiative loss.

♣ To be more accurate define the dimensionless temperature variables $\tau_i = T_i/T_0$. In terms of those variables we find that Eq. (4.2) becomes:

$$\tau_2^4 = \tau_1 - \tau_2 . \tag{4.3}$$

Let us examine the two limits.

When $\tau_1 \ll 1$, which imples $\tau_2 < \tau_1 \ll 1$, then the left hand side of Eq. (4.3) must be very small, so the dimensionless temperature difference along the rod must be very small. Motivated by that let us write

$$\tau_2 = \tau_1 - \delta \ , \text{ where } \delta \ll \tau_1 \ .$$

Upon substitution and only taking the leading term on the lhs of Eq. (4.3), we see:

$$\tau_1^4 \simeq \tau_1 - \tau_2 \Rightarrow \tau_2 \simeq \tau_1 - \tau_1^4 \ .$$

For example at $T_1 = 100$ K, the drop, $T_1 - T_2$, in temperature along the rod is around 10^{-4} K.

♣ Turning to the other limit, $\tau_i \gg 1$, then to balance the lefthand and righthand sides of Eq. (4.3) we see that

$$\tau_2^4 \simeq \tau_1 \text{ so } \tau_2 \simeq \tau_1^{1/4} \ll \tau_1 \ ,$$

consistently. Clearly the copper rod would be a pool of molten copper ... but ...We plot the result for small and large τ's interpolated by hand!

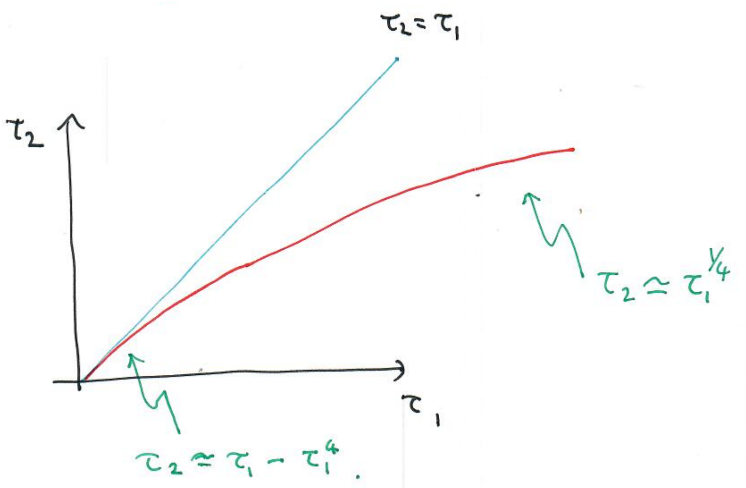

Figure 4.2: The reduced temperature τ_2 as a function of τ_1.

This disparity in radiative versus conductive heat transfer is used in the homely example of a vacuum flask, where there is vacuum between the outer casing and the container of the (hot or cold) liquid. Scientifically this is used in the storage of cold (cryogenic) liquids which would be gaseous at room temperature, eg N_2 or He. You can these in use daily in Physics East.

Problem 4.10 **Thermal capacitors and inductors?**

The variables in Kirchhoff's laws are voltage drops, ΔV, across components and currents, I. The counterparts are temperature drops, ΔT, and heat currents, $\dot{Q}$.

Kirchhoff's laws are: (i) voltage drops around a loop must equal zero (in the absence of time dependent magnetic fluxes threading the circuit!); (ii) at junctions of the circuit, the sum of the currents must be zero (conservation of electric charge and no charge build up at junctions).

The counterparts to these for thermal circuits are: (i) temperature drops round a thermal loop are zero; (ii) thermal currents at a node add to zero (again energy conserved and no temperature build up at junctions, at least in steady state).

A difference between the two types of circuits is that in the electrical case the components (resistors etc) are "lumped", connected by wires of negligible resistance, capacitance etc. Whereas in the thermal case, one cannot usually disregard the thermal resistance of the "wires", the components are "distributed".

For *ac effects*, it is hard to construct exact counterparts to capacitors and inductors. The usual type of electrical capacitor is an object with (at least) two plates which cannot pass a dc current. I cannot think of a counterpart. There is a different type of "thermal capacitor", which for both numerical and practical reasons[1] does not normally have a counterpart in electrical circuits: a component in a circuit with a large heat capacity. However, although this will damp out thermal transients, it is irrelevant for dc thermal currents.

For inductors again it is difficult to construct something that resists a change in thermal current. But there have been some recent (not very easy) results which indicate this may be possible.

It also seems that one can make thermal *rectifiers* .....

[1]Thanks to Mark Colclough for a discussion on that point.