Lorentz transformation of energy and momentum. Four-vectors

In this lecture we reconsider Lorentz transformations by treating them by analogy to rotations in the plane which we mentioned in the beginning of this course. The main points we will concentrate on are:

1. A combination of two consecutive Lorentz transformation with velocities v_1 and v_2 is another Lorentz transformation with velocity v_3 calculated in accordance with relativistic velocity composition law.

2. Time and space intervals are transforming accordingly to Lorentz transformation when considered in different reference frames. Are there other quantities which transform in this way? We shall see that energy and momentum provide such an example.

To simplify the calculations we use units in which $c = 1$. This means velocities are measured in units of speed of light [1] and distances have the same units as time. Lorentz transformation to the frame moving with velocity v becomes

$$\begin{aligned} t' &= \frac{t}{\sqrt{1-v^2}} - \frac{vx}{\sqrt{1-v^2}} = at - bx \\ x' &= -\frac{vt}{\sqrt{1-v^2}} + \frac{x}{\sqrt{1-v^2}} = -bt + ax\,, \end{aligned}$$

where $a = \gamma(v) = 1/\sqrt{1-v^2}$, $b = v/\sqrt{1-v^2}$. It is easy to see that coefficients a and b are related by $a^2 - b^2 = 1$. This suggests that they can be written as $a = \cosh\theta$, $b = \sinh\theta$ using one single parameter θ such that

$$\tanh\theta = \frac{\sinh\theta}{\cosh\theta} = \frac{b}{a} = v\,.$$

Lorentz transformation in the form

$$\begin{aligned} t' &= \cosh\theta\, t - \sinh\theta\, x \\ x' &= -\sinh\theta\, t + \cosh\theta\, x \end{aligned}$$

should be compared with the transformation

$$\begin{aligned} x' &= \cos\alpha\, x + \sin\alpha\, y \\ y' &= -\sin\alpha\, x + \cos\alpha\, y\,. \end{aligned}$$

of xy plane rotated by angle α. The coefficients $\cos\alpha$, $\sin\alpha$ satisfy Pythagoras theorem $\cos^2\alpha + \sin^2\alpha = 1$ and the transformation preserves the circles $x^2 + y^2 = R^2 = x'^2 + y'^2$. Lorentz transformations, on the contrary, preserve hyperbolas $t^2 - x^2 = s^2 = t'^2 - x'^2$ (invariance of relativistic interval) and are sometimes described as "hyperbolic rotations".

We now want to combine two Lorentz transformation

$$\begin{aligned} t' &= \cosh\theta\, t - \sinh\theta\, x \\ x' &= -\sinh\theta\, t + \cosh\theta\, x \end{aligned}$$

[1] For example a 60 mph speed limit will look (approximately) like 10^{-7}.

and

$$t'' = \cosh\phi\, t' - \sinh\phi\, x'$$
$$x'' = -\sinh\phi\, t' + \cosh\phi\, x'$$

by substituting t', x' from the first one,

$$t'' = \cosh\phi\,(\cosh\theta\, t - \sinh\theta\, x) - \sinh\phi\,(-\sinh\theta\, t + \cosh\theta\, x)$$
$$x'' = -\sinh\phi\,(\cosh\theta\, t - \sinh\theta\, x) + \cosh\phi\,(-\sinh\theta\, t + \cosh\theta\, x)$$

Using hyperbolic function properties $\sinh(\theta+\phi) = \sinh\theta\cosh\phi + \cosh\theta\sinh\phi$, and $\cosh(\theta + \phi) = \cosh\theta\cosh\phi + \sinh\theta\sinh\phi$ one arrives at

$$t'' = \cosh(\theta+\phi)t - \sinh(\theta+\phi)x$$
$$x'' = -\sinh(\theta+\phi)t + \cosh(\theta+\phi)x\,,$$

which shows that a combination of Lorentz transformations with parameters θ and ϕ is againa Lorentz transformation with parameter $\theta + \phi$. The velocity corresponding to this transformation is obtained from

$$u = \tanh(\theta+\phi) = \frac{\tanh\theta + \tanh\phi}{1 + \tanh\theta\tanh\phi}$$

Using $\tanh\theta = v$ and $\tanh\phi = w$ we recognise the relativistic velocity composition,

$$u = \frac{v+w}{1+vw}\,,$$

or, reintroducing the speed of light $v \to u/c$, $w \to w/c$, $u \to u/c$,

$$u = \frac{v+w}{1+vw/c^2}\,,$$

Using the parametrisation $v = \tanh\theta$ we can write the relativistic momentum and energy of a particle with mass m as

$$p' = \frac{mv}{\sqrt{1-v^2}} = m\sinh\theta$$
$$E' = \frac{m}{\sqrt{1-v^2}} = m\cosh\theta$$

(The primed notations are used as we consider this frame moving) Now we consider the same particle in the inertial frame which moves with velocity $-w = -\tanh\phi$ and which we consider stationary. In this frame the particle has velocity $u = (v+w)/(1+vw) = \tanh(\theta+\phi)$. The momentum p and energy E in this frame are

$$p = \frac{mu}{\sqrt{1-u^2}} = m\sinh(\theta+\phi) = m\sinh\theta\cosh\phi + m\cosh\theta\sinh\phi$$
$$E = \frac{u}{\sqrt{1-u^2}} = m\cosh(\theta+\phi) = m\cosh\theta\cosh\phi + m\sinh\theta\sinh\phi$$

This is nothing but a Lorentz transformation

$$
\begin{aligned}
p &= \cosh\phi\, p' + \sinh\phi\, E' = \frac{p' + (w/c^2)E'}{\sqrt{1 - w^2/c^2}} \\
E &= \cosh\phi\, E' + \sinh\phi\, p' = \frac{E' + w'p}{\sqrt{1 - w^2/c^2}}\,.
\end{aligned}
$$

In the last equation we have reintroduced the speed of light for clarity.

Let us recall that in the full four-dimensional time-space with $p = p_x$, the transverse components p_y, p_z are not affected by Lorentz transformation to a frame moving in x-direction. We see that four quantities $(E/c, p_x, p_y, p_z)$ transform exactly in the same way as (ct, x, y, z). The quantities which are transformed this way are called *four-vectors* and are denoted as (a^0, a^1, a^2, a^3) using their four components a^μ with Greek index $\mu = 0, 1, 2, 3$.