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UNIVERSITY OF BIRMINGHAM

School of Mathematics

Programmes in the School of Mathematics — First Examination

Programmes involving Mathematics — First Examination

1RA 06 34051 Level C

LC Real Analysis

Alternative Assessment

January Examinations 2021-22

Three Hours

Full marks will be obtained with complete answers to all FOUR questions. Each question carries equal weight. You are advised to initially spend no more than 45 minutes on each question and then to return to any incomplete questions if you have time at the end. An indication of the number of marks allocated to parts of questions is shown in square brackets.

Section A

1. (a) Let $f : \mathbb{R} \to \mathbb{R}$. Define what it means that $\lim_{x\to-\infty} f(x) = \infty$. [2]

(b) By using the definition of limit from part (a), prove that $\lim_{x\to-\infty} \sqrt[5]{x^2} = \infty$. [6]

(c) Determine whether the following limits exist, and if so find their values. Justify your answers. You can use any of the results discussed in lectures, provided you clearly state what you are using.

(i) $\lim_{x\to 0} \dfrac{(e^{5x}-1)\tan x}{\cos(2x)-1}$.

(ii) $\lim_{x\to\infty} \dfrac{3x^3+x^2+\sin(e^x)}{5x-8x^3+\arctan(\log x)}$. [10]

(Here $\log y = \log_e y$ denotes the natural logarithm of y.)

(d) Compute the derivative of the function $g : (0,\infty) \to \mathbb{R}$ defined by

$$g(x) = x^{\sin(x^2)}$$

for all $x \in (0,\infty)$. Justify all the steps in your computations. You can use any of the results discussed in lectures, provided you clearly state what you are using. [7]

2. (a) Suppose that $P = \{0,1,2,3\}$ and that $f : [0,3] \to \mathbb{R}$ is given by

$$f(x) := \begin{cases} 4, & \text{if } x = 0; \\ x^2 + 1, & \text{if } x \in (0,3); \\ 12, & \text{if } x = 3. \end{cases}$$

Compute **both** the lower sum $L(f,P)$ and the upper sum $U(f,P)$. [4]

(b) Suppose that $h : (0,3) \cup (3,\pi) \to \mathbb{R}$ is given by

$$h(x) := \begin{cases} x\sqrt{9-x^2}, & \text{if } x \in (0,3); \\ \sec^6(x)\tan^4(x), & \text{if } x \in (3,\pi). \end{cases}$$

Find an antiderivative of h, showing all of your calculations. [8]

(c) Compute the improper integral $\displaystyle\int_0^4 \frac{x^3-2}{|x-1|}\,\mathrm{d}x$, showing all of your calculations, or prove that it is divergent. [5]

(d) Suppose that $a \in \mathbb{R} \setminus \{0\}$. Find a solution $y : [0,+\infty) \to \mathbb{R}$ of the initial value problem

$$y' - ay = x^2, \qquad y(0) = 5a.$$

[8]

Section B

3. (a) Let $f : A \to \mathbb{R}$, where $A \subseteq \mathbb{R}$, and let $c \in A$. Define what it means that f is differentiable at c. [2]

(b) Let $g : (-\pi/6, \pi/6) \to \mathbb{R}$ be defined by

$$g(x) = \begin{cases} \dfrac{\sin(x^3)}{1-\cos(3x)} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

By using the definition from part (a), prove that g is differentiable at 0, and determine the value of $g'(0)$. You can use any of the results discussed in lectures, provided you clearly state what you are using. [6]

(c) Let h be the real-valued function of a real variable defined by the formula

$$h(x) = \sqrt{\frac{x^3}{x^2-2}}.$$

(i) Determine the domain of h according to the Domain Convention.

(ii) Find $\inf h$ and $\sup h$, and determine whether h has a maximum and/or a minimum.

(iii) Determine all the local minimum points of h.

(iv) Determine whether h is surjective (here the codomain of h is assumed to be $\mathbb{R}$).

Justify your answers. You can use any of the results discussed in lectures, provided you clearly state what you are using. [17]

4. (a) Suppose that $f : [1,3] \to \mathbb{R}$ is given by

$$f(x) := \begin{cases} 4, & \text{if } x \in \mathbb{N}; \\ 7, & \text{if } x \notin \mathbb{N}. \end{cases}$$

(i) Use Riemann's Criterion to prove that f is integrable.

(ii) Use the definition of the integral to compute, with proof, the value of $\int_1^3 f$.

[11]

(b) Suppose that $f : \mathbb{R} \to \mathbb{R}$ is differentiable and that $F : [1,10] \to \mathbb{R}$ is given by

$$F(x) := \int_1^{3x^4} \sin(f(t)) \, \mathrm{d}t$$

for all $x \in [1,10]$. Prove that F is differentiable, and find an expression for the derivative $F'(x)$, for all $x \in [1,10]$, in terms of f but which does **not** contain the integral or antiderivative symbol $\int$. [7]

(c) Suppose that $b \in \mathbb{R} \setminus \{0\}$. Find a solution $y : [0,+\infty) \to \mathbb{R}$ of the initial value problem

$$y'' - 4by' + 4b^2 y = x^2, \qquad y(0) = 0, \qquad y'(0) = \frac{1}{2b^3}.$$

[7]