Average KE (Maxwell - Boltzmann)

$$\langle KE \rangle = \frac{1}{2} m \langle v^2 \rangle \qquad P_r(v) = 4\pi \left(\frac{m}{2\pi k_B T}\right)^{\frac{3}{2}} v^2 e^{-\left(\frac{mv^2}{2k_B T}\right)}$$

$$\langle v^2 \rangle = \int_0^\infty P_r(v) v^2 \, dv$$

Remember Feynman's trick: $\displaystyle\int_0^\infty e^{-ax^2} x^{2n} \, dx = \frac{1}{2}\left(-\frac{\partial}{\partial a}\right)^n \sqrt{\frac{\pi}{a}}$

with $a = \frac{m}{2k_B T}$ and $x = v$

$$\frac{1}{2} m \langle v^2 \rangle = \frac{1}{2} m \int_0^\infty P_r(v) v^2 \, dv$$

I_2

$$= \frac{1}{2} m \, 4\pi \left(\frac{m}{2k_B \pi T}\right)^{\frac{3}{2}} \int_0^\infty e^{-\frac{mv^2}{2k_B T}} v^4 \, dv$$

$$= 2\pi m \left(\frac{m}{2k_B T \pi}\right)^{\frac{3}{2}} \times \frac{1}{2}\left(-\frac{\partial}{\partial a}\right)^2 \sqrt{\frac{\pi}{a}}$$

$$= 2\pi^{\frac{3}{2}} m \left(\frac{m}{2k_B T \pi}\right)^{\frac{3}{2}} \times -\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right) \frac{1}{a^{\frac{5}{2}}}$$

$$= \frac{3}{4} \pi^{\frac{3}{2}} m \left(\frac{m}{2\pi k_B T}\right)^{\frac{3}{2}} \left(\frac{2k_B T}{m}\right)^{\frac{5}{2}}$$

$$= \boxed{\frac{3}{2} k_B T} \quad \text{after cancelling}$$

$$\sqrt{\langle v^2 \rangle} = \sqrt{\frac{3k_B T}{m}} = \sqrt{\frac{3}{2}} \, V_{\text{Most probable}}$$

root mean square speed

(rms)