Einstein postulates of special relativity and Lorentz transformations

To explain results of Michelson-Morley experiment in a consistent way one could either give up the relativity principle that all laws are the same in any inertial frame or to modify the Galilean transformation law. In the later case it was the principle that time is the same in all inertial frames that had to be abandoned.

Albert Einstein proposed to give up (or rather modify) Galilean transformation and its idea of universal time. He argued that in order to measure time in different locations we need to use a procedure of *clock synchronisation*. This is only possible by sending information about clocks in these locations. The universal time would only make sense if the information had travelled infinitely fast, *i.e.* instantaneously. This is not the case. The maximum speed of information propagation is speed of light c, and it has to be the same at every point in space.

The following two statements were proposed as postulates of special relativity:

1. All inertial frames are equivalent to all laws of physics.
2. The speed of light in vacuum is the same (invariant) in all inertial frames.

Thus a new transformation law of coordinates and time must replace Galilean Transformation law. This new transformation should lead to a new law of velocity composition preserving the speed of light.

Dutch scientist Hendrik Lorentz discovered such transformations even before Einstein formulated his postulates, by studying Maxwell equations. These transformations are known to us as Lorentz transformations, even though they were formulated in the final form by French mathematician Henri Poincaré. Below we will derive them by exploiting idea of *simaltaneity.*

Imagine three equidistant points, A, B, C which are at rest in some inertial frame K. We send a signal (light) from B towards A and C. The signal arrives to these points at the *same time*, *i.e.* simultaneously, as shown graphically in Fig. .

Now imagine that the system of three points A, B, C is moving in the positive x-direction with velocity v. The light starting in B at $t = 0$ arrives to point A earlier than to point C as shown in Fig. . Of course this is expected. What is not expected is that in the moving frame K' in which the points are stationary the Galilean transformation also predicts $t'_A < t'_C$.

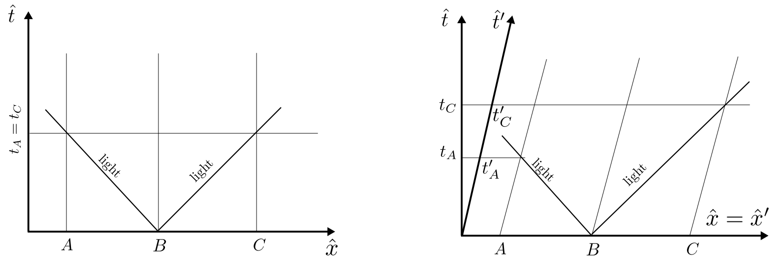

Figure 1: Left: light arrives to points A and C at the same time in stationary frame K. Right: for points A,B, C moving with constant velocity v the light from B arrives to points A and C at different times $t_A < t_C$. Galilean transformation to the moving frame K' in which all points are at rest predicts $t'_A < t'_C$ as well.

Indeed, from the point of view of an observer moving with B, the light should reach the points A and C *at the same time*! Thus something must be done to the way *time* is transformed so that the transformation predicts $t'_A = t'_C$. Graphically this means that the line $t' = t'_A = t'_C$ must be parallel to to the new $\hat{x}'$-axis, so the latter must be tilted as shown in a modified diagram Fig. 2

We see that simultaneity is relative notion and depends on the frame chosen. Of course it contradicts the universal time $t' = t$ of Galilean transformation. Therefore we are led to a general transformation which includes both space and time,

$$x = ax' + bt' \tag{1}$$

$$t = fx' + gt' \,, \tag{2}$$

where the coefficient a, b, f, g depend on velocity v.

To find these coefficients we note first, that the trajectory of the origin of K, $x = 0$ for any t, corresponds to $x' = -vt'$, *i.e.* viewed from the moving frame K' the origin moves to the left with velocity v. This gives the relation:

$$b = av\,.$$

Next we look at the trajectory of a light beam starting from the origin at time $t = t' = 0$. We have in both frames:

$$x = ct\,, \qquad x' = ct' \tag{3}$$

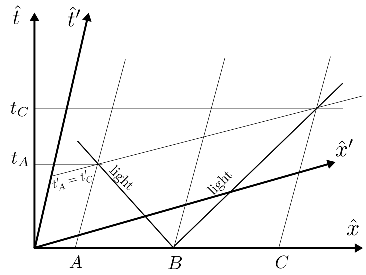

Figure 2: In order to make the times of arrival to points A and C equal in the moving frame, $t'_A = t'_C$, the $\hat{x}'$-axis must be tilted.

These equations can be rewritten using the transformation (1) as

$$ct = x = ax' + bt' = act' + avt' = a(c+v)t'$$
$$ct' = x' = a(c-v)t\,,$$

where the last equation follows from the fact that the inverse transformation (from K' to K) can be obtained simply by changing $(x,t) \to (x',t')$ and $v \to -v$. We have

$$ct = a(c+v)t' = a(c+v)(c-v)t/c \quad \Rightarrow \quad a^2 = \frac{c^2}{c^2 - v^2}\,.$$

It is clear that $a > 0$, so the first line of Lorentz transformation is

$$x = \frac{x' + vt'}{\sqrt{1 - v^2/c^2}} = \gamma(v)(x' + vt')\,, \tag{4}$$

where we renamed the relativistic factor $\gamma(v) = a = 1/\sqrt{1 - v^2/c^2}$. To find the second line of Lorentz transformation we use $x' = ax - bt$ so that

$$t = \frac{ax - x'}{b} = \frac{(a^2-1)x' + abt'}{b} = \frac{\sqrt{1 - v^2/c^2}}{v}\left(\frac{1}{1 - v^2/c^2} - 1\right)x' + \frac{t'}{\sqrt{1 - v^2/c^2}}$$
$$= \frac{vx'/c^2}{\sqrt{1 - v^2/c^2}} + \frac{t'}{\sqrt{1 - v^2/c^2}} = \frac{t' + vx'/c^2}{\sqrt{1 - v^2/c^2}} = \gamma(v)(t' + vx'/c^2)\,.$$

This provides the second line of Lorentz transformation. The coordinates perpendicular to the direction of motion are unaffected:

$$y = y' \,, \qquad z = z' \,.$$

By changing $(x, t) \to (x', t')$ and $v \to -v$ we get the inverse transformation[1]

$$\begin{aligned} x' &= \frac{x - vt}{\sqrt{1 - v^2/c^2}} = \gamma(v)(x - vt) \\ t' &= \frac{t - vx/c^2}{\sqrt{1 - v^2/c^2}} = \gamma(v)(t - vx/c^2) \,. \end{aligned}$$

We see that for $v/c \to 0$ Lorentz transformations reduce to Galilean transformations, in particular $t' = t$.

[1] We used the convention that the primed frame K' moves in positive x direction with respect to K. In this case a simple rule can be used to memorise the signs in Lorentz transformation: "either prime or minus" (V.I. Nikolaev, professor of physics at Moscow State University who taught me in my first year there).