Mechanics 2024-25

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Chapter 1

Units, dimensions and kinematics

1.1 Introduction

Classical Newtonian mechanics is the foundation of applied mathematics and is an astonishingly powerful tool for explaining physical systems, from projectiles to planetary motion to the design of racing cars. It acts as a natural starting point for any serious discussion of mathematical modelling in broader areas. This module uses ideas such as forces, moments, Newton's Laws of Motion and energy to model practical situations. These models can then be analysed using a wide range of techniques from pure mathematics such as trigonometry, algebra, calculus and, in particular, vector methods. We will make use of some techniques you have already seen in other modules, and introduce further concepts which will be taught more formally/rigorously in later modules (e.g. in VGLA or in second year Differential Equations)- this serves as motivation as to why these methods are interesting and useful! See the crib sheet on Canvas or the appendix of the full notes if you need any refreshers!

This week we will start by considering units and dimensions, followed by *kinematics* i.e. how we represent the motion of a particle.

1.2 Notation

First a quick note about notation:

Vectors I will use the printed notation $\mathbf{a}$ for a vector, with handwritten notation $\underline{a}$.

Differentiation I will use "dot" to mean differentiation with respect to time, so that

$$\dot{x} = \frac{dx}{dt},$$

and

$$\ddot{x} = \frac{d^2x}{dt^2}.$$

1.3 Units and dimensions

We begin with some discussion about units and dimensions - this is a crucial tool for checking whether something is right or not, and really fundamental to producing accurate and useful results.

Throughout this course we shall use SI units, which were adopted internationally in 1960. The system has seven *basic units*, one for each of the so called basic quantities: mass, length, time, electric current, temperature, luminous intensity and amount of a substance. All other quantities are measured in terms of *derived units* which are obtained from the basic units by multiplication or division. For example, volume is measured in terms of the unit of length raised to the third power, while velocity is measured in terms of units of length per unit of time.

Basic Quantity	Symbol	Basic Unit	Symbol
mass	M	kilogram	kg
length	L	metre	m
time	T	second	s
electric current	i	ampere	A
temperature	Θ	Kelvin	K
luminous intensity	I	candela	cd
amount of a substance	N	mole	mol

The dimensions of a quantity show how it is related to the basic quantities. We use square brackets to indicate that we are specifying the dimensions of the quantity.

Example 1:

- Volume V has dimensions of length cubed which we write as

$$[V] = [L^3], \qquad \text{SI units m}^3.$$

- Density of a substance (often denoted ρ) is mass per unit volume so that

$$[\rho] = [ML^{-3}] \qquad \text{SI units kgm}^{-3}.$$

- A velocity v is distance per time, hence

$$[v] = [LT^{-1}] \qquad \text{SI units ms}^{-1}.$$

Every derived quantity has dimensions. Some derived SI units also have names e.g. forces are often denoted by Newtons (1 N= 1 kg m s^{-2}).

Both derivatives and integrals of physical quantities have dimensions. Suppose that V is a volume then the rate of change of volume with time t is $\frac{dV}{dt}$ which has dimensions

$$\left[\frac{dV}{dt}\right] = \frac{[V]}{[t]} = \frac{[L^3]}{[T]} = [L^3T^{-1}].$$

The second derivative can be thought of as

$$\frac{d^2V}{dt^2} = \frac{d}{dt}\left(\frac{dV}{dt}\right),$$

and hence

$$\left[\frac{d^2V}{dt^2}\right] = \frac{1}{[T]}\left[\frac{dV}{dt}\right] = \frac{[V]}{[t^2]} = \frac{[L^3]}{[T^2]} = [L^3T^{-2}].$$

Similarly an integral with respect to time t is essentially multiplying by a time, so has dimensions

$$\left[\int V dt\right] = [V][t] = [L^3][T] = [L^3T].$$

Note that the arguments of e.g. trig functions, exponentials and logarithms must all be nondimensional, so $\sin(\omega t)$ must have that ωt is nondimensional, and hence $[\omega t]$=[1], so $[\omega]$=[T^{-1}].

Note! Using a consistent set of units is essential! Particularly when working with other disciplines, who may have a tendency to use very odd and inconsistent unit systems, care is needed. There are various examples of things going very wrong when people make mistakes with inconsistent units (for example the Hubble Space Telescope and the Mars Climate Orbiter).

1.4 Principle of dimensional homogeneity

An equation derived from physical principles must be dimensionally homogeneous, that is

- The dimensions of the right-hand side must be the same as the dimensions of the left-hand side.
- The dimensions of each additive terms must be the same.

This is necessary (but not sufficient) for the equation to be valid. If you end up with a dimensionally inhomogeneous equation it is definitely wrong! Comparing dimensions can often help find the source of the error.

Example 2: Dimensions

Is the equation

$$\rho v = \frac{m}{At^2} + \frac{1}{V}\frac{dm}{dt}\int v dt$$

where

- ρ is density ($[ML^{-3}]$),
- v is velocity ($[LT^{-1}]$),
- m is mass ($[M]$),
- A is area ($[L^2]$),
- V is volume ($[L^3]$),
- t is time ($[T]$),

dimensionally consistent?

Solution. Each term has dimensions:

$$[\rho v] = [ML^{-3}][LT^{-1}] = [ML^{-2}T^{-1}].$$

$$\left[\frac{m}{At^2}\right] = [ML^{-2}T^{-2}].$$

$$\left[\frac{1}{V}\frac{dm}{dt}\int v dt\right] = [MT^{-1}][L]/[L^3] = [ML^{-2}T^{-1}].$$

Thus our equation is dimensionally inhomogeneous, so it must be wrong and we suspect there may be an error with the first term on the right hand side. ◀

1.4.1 Dimensions, units and writing equations

When you write a model based on real world principles you should denote everything which might change or which has dimensions as a symbol rather than using the numerical value. For example you should use "g" rather than "9.81m/s^2" for acceleration due to gravity, or "c" rather than "3 x 10^8 m/s" for the speed of light. This means that your equation is as general as possible (e.g. it will still work on the moon rather than on Earth) and it will hold in whichever set of (consistent!) units you choose. This then means that any actual numbers which appear will always be nondimensional.

Beware, however, that some people (particularly non-mathematicians!) don't always follow this rule, so you should approach any model that you've not written yourself with caution!

Activity: *You should now be able to tackle question 1 on this week's problem sheet.*

1.5 Newtonian mechanics

Much of classical mechanics is based on the fundamentals laid down by Newton in the 1600s, requiring the ideas of calculus and differential equations to be developed. We now know that Newtonian mechanics does not describe very small or very fast things well (e.g. atoms or galaxies) where we need quantum theory or relativity, but for many situations Newtonian mechanics provides sufficient accuracy to answer many fundamental questions. The techniques used also form the building blocks for much of applied mathematics.

1.6 Position, velocity and acceleration - kinematics

We first need a geometric framework in which to formulate models. We will consider three-dimensional space described using a position vector relative to some origin and a set of (probably) perpendicular axes e.g. Cartesian space with axes **i**, **j**, **k** in the x, y, z direction respectively (see Fig 1.1). This gives the *frame of reference*.

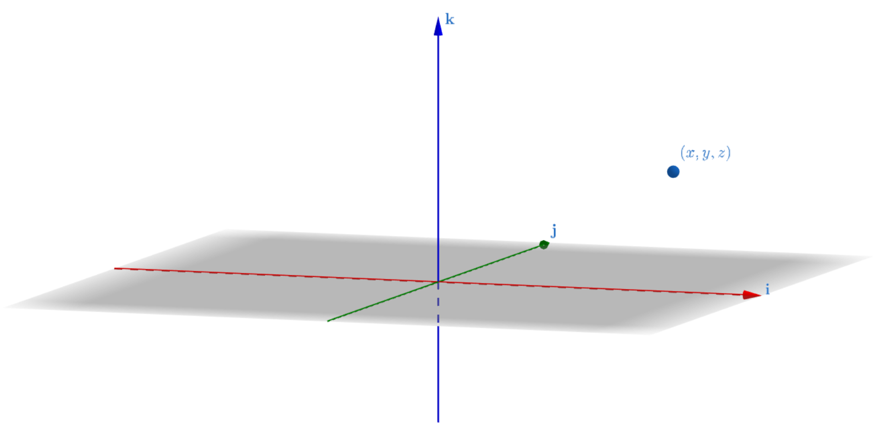

Figure 1.1: Cartesian axes showing perpendicular vectors **i**, **j**, **k** and a point with coordinates (x, y, z).

We will consider how a particle moves in time through space. A particle is a mathematical object which has mass, but does not have size or shape. This is a good approximation for an object which is heavy but small compared to the other lengths in the system. For example representing the Earth as a particle is a good approximation when considering planetary motion around the sun, but bad when considering the moon's orbit around the Earth.

We pick an origin O, then the location of a particle is given by its position vector $\mathbf{r}$ measured from the origin. If the particle is moving then

$$\mathbf{r} = \mathbf{r}(t)$$

gives the particle path as a three-dimensional curve, with the particle location depending on the current time. In Cartesians this can be written as

$$\mathbf{r} = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k}.$$

See Fig 1.2.

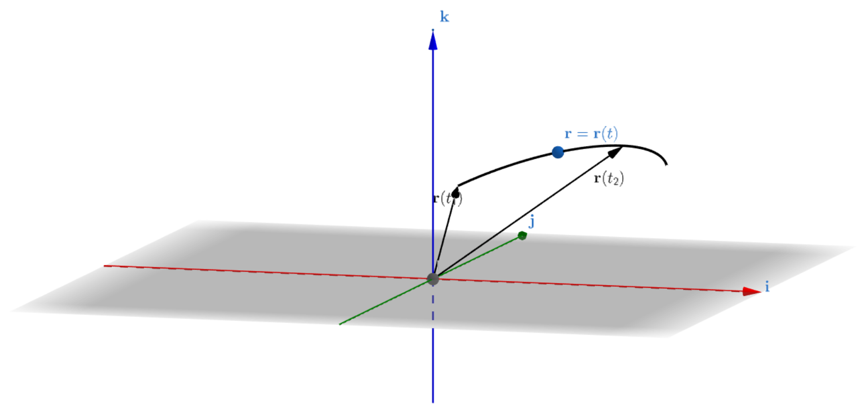

Figure 1.2: Cartesian axes showing perpendicular vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ and a line represented by a position vector $\mathbf{r}$. Points are marked for times t_1 and t_2.

Given the particle's position vector we can then calculate the velocity of the particle, which is given by the instantaneous rate of change of position with respect to time

$$\begin{aligned} \mathbf{v} &= \frac{d\mathbf{r}}{dt}, \\ &= \frac{dx(t)}{dt}\mathbf{i} + \frac{dy(t)}{dt}\mathbf{j} + \frac{dz(t)}{dt}\mathbf{k}, \end{aligned}$$

in Cartesians. We will use the notation $\dot{} \equiv d/dt$ so that

$$\mathbf{v} = \dot{\mathbf{r}}.$$

Acceleration of a particle $\mathbf{a}$ is defined as the instantaneous rate of change of velocity with time:

$$\begin{aligned}\mathbf{a} &= \frac{d\mathbf{v}}{dt} \\ &= \frac{d^2\mathbf{r}}{dt^2} \\ &= \frac{d^2x}{dt^2}\mathbf{i} + \frac{d^2y}{dt^2}\mathbf{j} + \frac{d^2z}{dt^2}\mathbf{k},\end{aligned}$$

in Cartesians.

Note that position, velocity and acceleration are all vectors so they have magnitude and direction. The magnitude of position is

$$r = |\mathbf{r}| = \sqrt{x^2 + y^2 + z^2},$$

the magnitude of velocity (also known as speed) is

$$v = |\mathbf{v}| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}$$

and the magnitude of acceleration is

$$a = |\mathbf{a}| = \sqrt{\left(\frac{d^2x}{dt^2}\right)^2 + \left(\frac{d^2y}{dt^2}\right)^2 + \left(\frac{d^2z}{dt^2}\right)^2}.$$

In general, position, velocity and acceleration will all be pointing in different directions, and will have different magnitude. In particular, it is possible to have varying velocity but constant speed if the direction of motion is changing but the magnitude is fixed. Given position, velocity or acceleration (and appropriate initial conditions if required) we can calculate the others by differentiating/integrating.

Example 3: Parametric position vector

Consider a particle with position vector

$$\mathbf{r} = \cos(\omega t)\mathbf{i} + \sin(\omega t)\mathbf{j}(+0\mathbf{k}).$$

What are the velocity and acceleration vectors? What shape does the particle path make?

Solution. We can calculate the particle's velocity by differentiating with respect to time to give

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{d}{dt}\left(\cos(\omega t)\right)\mathbf{i} + \frac{d}{dt}\left(\sin(\omega t)\right)\mathbf{j},$$
$$= -\omega\sin(\omega t)\mathbf{i} + \omega\cos(\omega t)\mathbf{j},$$

and acceleration by differentiating again to give

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d}{dt}\left(-\omega\sin(\omega t)\mathbf{i} + \omega\cos(\omega t)\mathbf{j}\right),$$
$$= -\omega^2\cos(\omega t)\mathbf{i} - \omega^2\sin(\omega t)\mathbf{j}.$$

Note that $\mathbf{a} = -\omega^2\mathbf{r}$ - in this case the acceleration acts in the opposite direction to the position vector.

We can find the shape of the particle path by eliminating t from the position vector. Since $x = \cos(\omega t)$, $y = \sin(\omega t)$, then

$$x^2 + y^2 = \cos^2(\omega t) + \sin^2(\omega t),$$
$$= 1.$$

i.e. the particle moves in a circle. The acceleration is therefore pointing towards the centre of the circle, with the velocity tangent to the circle (note that $\mathbf{v}\cdot\mathbf{r} = -\omega\sin(\omega t)\cos(\omega t) + \omega\cos(\omega t)\sin(\omega t) = 0$). See fig 1.3.

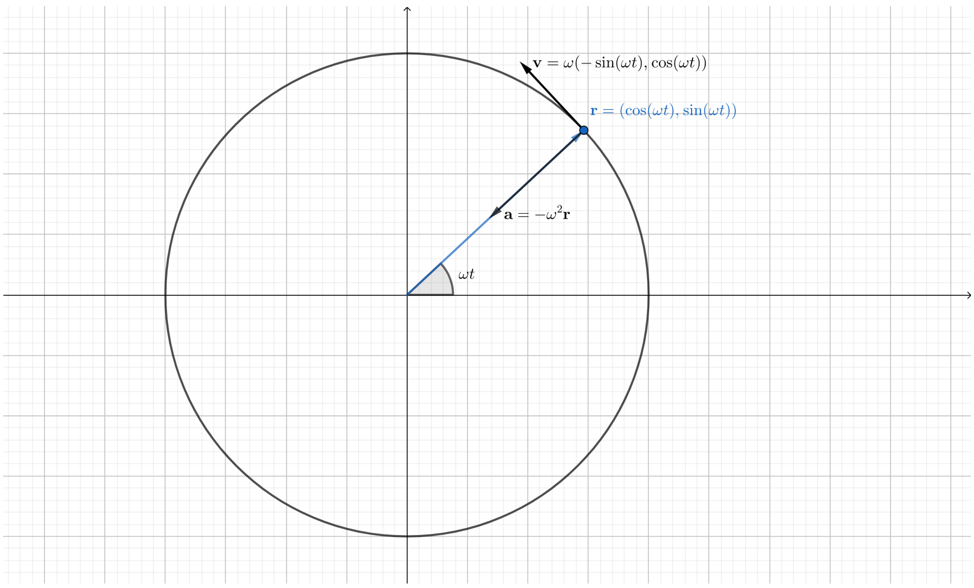

Figure 1.3: A plot showing the position, velocity and acceleration vectors.

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Example 4: Parametric acceleration vector

Similarly, if we had been given the acceleration we could have gone in the other direction by integrating to find the velocity and position vectors. In this case we would also need to know where the particle started, and how quickly it was moving at that point - i.e. the initial conditions. For example, if a particle starts at $(1, 0)$ with velocity $(0, \omega)$ subject to an acceleration of the form

$$\mathbf{a} = -\omega^2 \cos(\omega t)\mathbf{i} - \omega^2 \sin(\omega t)\mathbf{j},$$

what are the velocity and position vectors?

Solution. Then

$$\begin{aligned}\mathbf{v} &= \int \mathbf{a}\ \mathrm{d}t, \\ &= \left(\int -\omega^2 \cos(\omega t)\ \mathrm{d}t + c_1\right)\mathbf{i} + \left(\int -\omega^2 \sin(\omega t)\ \mathrm{d}t + c_2\right)\mathbf{j}, \\ &= \left(-\omega \sin(\omega t) + c_1\right)\mathbf{i} + \left(\omega \cos(\omega t) + c_2\right)\mathbf{j},\end{aligned}$$

where $\mathbf{c} = c_1\mathbf{i} + c_2\mathbf{j}$ gives the (now vector) constant of integration. This is found using the initial condition that $\mathbf{v} = (0, \omega) = \omega\mathbf{j}$. Hence

$$\omega\mathbf{j} = c_1\mathbf{i} + (\omega + c_2)\,\mathbf{j},$$

and so $c_1 = 0$ and $\omega + c_2 = \omega$, hence $c_2 = 0$ by equating coefficients of $\mathbf{i}$ and $\mathbf{j}$.

Then

$$\begin{aligned}\mathbf{r} &= \int \mathbf{v}\ \mathrm{d}t, \\ &= \int -\omega \sin(\omega t)\mathbf{i} + \omega \cos(\omega t)\mathbf{j}\ \mathrm{d}t, \\ &= \left(\cos(\omega t) + d_1\right)\mathbf{i} + \left(\sin(\omega t) + d_2\right)\mathbf{j},\end{aligned}$$

with $\mathbf{d} = (d_1, d_2)$ giving the constant of integration. Now, since $\mathbf{r} = (1, 0) = \mathbf{i}$ initially, this gives

$$\mathbf{i} = (1 + d_1)\mathbf{i} + d_2\mathbf{j},$$

and hence $d_1 = 0$, $d_2 = 0$ and

$$\mathbf{r} = \cos(\omega t)\,\mathbf{i} + \sin(\omega t)\,\mathbf{j},$$

as before. ◀

Expressions such as these allow us to describe the *kinematics* (i.e. motion) of the particle, but in general we want to work out what those kinematics will be in reponse to external forces. For this we need Newton's laws, which we will learn about next week.

Activity: *You should now be able to tackle questions 2 and 3 on this week's problem sheet.*

Chapter 2

Newton's laws in one dimension

2.1 Introduction

Having set up the framework of *kinematics* last week, i.e. ideas about movement, this week we will move on to the main topic of this module, *mechanics*, where we include the effects of the forces which cause (or prevent!) motion.

2.2 Newton's laws of motion

Newton's laws relate how a force acting on a particle relates to the motion of that particle, and originate from the 1600s. Forces are vectors with both magnitude and direction; the magnitude gives a measure of the strength of the force and is typically measured in the SI unit of Newtons (1N=1kgms^{-2}). For example a stone dropped from a height experiences a gravitational force pulling it downwards of strength mg where m is the mass of the stone and g is the acceleration due to gravity (9.81ms^{-2} on the surface of the Earth). A particle will respond to the net force acting on it, that is the sum (resultant) of all the forces acting on the particle.

Newton's first law: In the absence of any resultant forces, a particle moves with constant velocity $\mathbf{v}$.

Newton's second law: If a net force $\mathbf{F}$ acts on a particle of constant mass m, then the acceleration $\mathbf{a}$ of the particle is related to $\mathbf{F}$ and m by

$$\mathbf{F} = m\mathbf{a}.$$

We can also define the linear momentum of a particle as mass times velocity

$$\mathbf{p} = m\mathbf{v},$$

and hence this also gives

$$\mathbf{F} = \frac{d\mathbf{p}}{dt}.$$

The rate of change of linear momentum is equal to the sum of the forces acting on it. This also works when the particle mass is not constant.

Newton's third law: For every force there is an equal and opposite reaction force.

Definition: Inertial frame An inertial frame is one in which Newton's laws hold. (We'll come back to this later!)

Intuitively.... Imagine throwing a ball on a train. When the train is moving at a constant speed in a straight line this is no different from throwing a ball in the garden - the same trajectory will be observed. If, however, the train is going round a corner (i.e. it is accelerating as the velocity is not constant), then we will see a very different motion - the frame is not an inertial frame. [See video on Canvas, taken from https://twitter.com/EvoBiomech/status/1351589830698921986 with permission, thanks to the Evolutionary Biomechanics Lab at Imperial.]

2.3 Standard procedure

The basic procedure in mechanics is as follows:

- Form the governing equations using Newton's laws (often II), and the initial conditions from the initial location/velocity of the particle (as appropriate).
- Solve the resulting system.
- Look at what this tells us - for example
  - what is the shape of the particle path?
  - what is the speed of motion?
  - how far does the particle go?
  - when/where does a given event (for example a crash) occur?

We start by looking at motion in one direction, before moving on to two and three dimensions next week.

2.4 One dimensional motion

Example 5: Mass moving with a time-dependent force

A particle of mass m moves in a straight line subject to a time dependent force of the form $F = F_0 \sin(\omega t)$. At time $t = 0$ the particle is located at $x = 0$ and has velocity v_0. What is the motion?

Solution. Take x to measure distance in a straight line from the starting point. The equation of motion is found from Newton's second law:

$$\text{Force} = \text{mass x acceleration}$$

and hence we find

$$F_0 \sin(\omega t) \quad = \quad m\ddot{x}.$$

This can be integrated once to find

$$\begin{aligned} \dot{x} &= \frac{F_0}{m} \int \sin(\omega t)\ \mathrm{d}t + c_1, \\ &= -\frac{F_0}{m\omega} \cos(\omega t) + c_1, \end{aligned}$$

and again to find

$$x \quad = \quad -\frac{F_0}{m\omega^2} \sin(\omega t) + c_1 t + c_2,$$

where c_1, c_2 are constants of integration.

To find the constants of integration we use the initial conditions that at $t = 0$, $x = 0$ and $\dot{x} = v_0$ (velocity). Using these we find:

$$x = 0 \quad \Longrightarrow \quad c_2 = 0,$$

and

$$\begin{aligned} \dot{x} &= v_0, \\ \Longrightarrow \ -\frac{F_0}{m\omega} + c_1 &= v_0, \\ \Longrightarrow c_1 &= v_0 + \frac{F_0}{m\omega}. \end{aligned}$$

Thus we have found the particle position for all values of $t > 0$, given by

$$x \quad = \quad -\frac{F_0}{m\omega^2} \sin(\omega t) + \left(v_0 + \frac{F_0}{m\omega}\right) t.$$

The particle therefore oscillates (the sin term) with some drift (the t term) either to left or right. When:

- $v_0 > -F_0/m\omega$, then $x \to \infty$ as $t \to \infty$ (tends to the right/positive x direction).
- $v_0 < -F_0/m\omega$, then $x \to -\infty$ as $t \to \infty$ (tends to the left/negative x direction).
- $v_0 = -F_0/m\omega$, then $x = -\dfrac{F_0}{m\omega^2}\sin(\omega t)$ and particle oscillates with no drift about $x = 0$.

See https://www.geogebra.org/m/guxameb2. ◀

Activity: *You should now be able to tackle question 4 on this week's problem sheet.*

Aside: Governing equations formulated using force = mass × acceleration tend to be of the form

$$m\ddot{x} \;=\; g(x, \dot{x}, t).$$

When this can be written as

$$a\ddot{x} + b\dot{x} + cx \;=\; f(t),$$

where a, b, c are constants this is a linear, second-order differential equation with constant coefficients; being able to solve such equations is key to this module! You will learn the formal details later in 1RA, but we will cover the method now (see Section 12.3 of the crib sheet).

Example 6: Mass on a spring, neglecting gravity

If a particle of mass m is attached to a spring (natural length l, spring constant k), which is fixed at the opposite end, which starts at its natural length at time $t = 0$ moving with speed u, what is the motion?

Solution. We first set up an appropriate coordinate system for the problem. We choose $x = 0$ to be when the spring is unstretched (i.e. it is at natural length), with x pointing away from the fixed end of the spring. x therefore gives the extension in the spring. See Fig 2.1.

Hooke's law then says that force is proportional to extension, so we have $F = -kx$, where k is the spring constant, and the minus sign is because the force works to pull the particle back to the natural length. Hence Newton's second law gives

$$\begin{aligned} m\ddot{x} &= -kx, \\ \implies \ddot{x} + \frac{k}{m}x &= 0. \end{aligned}$$

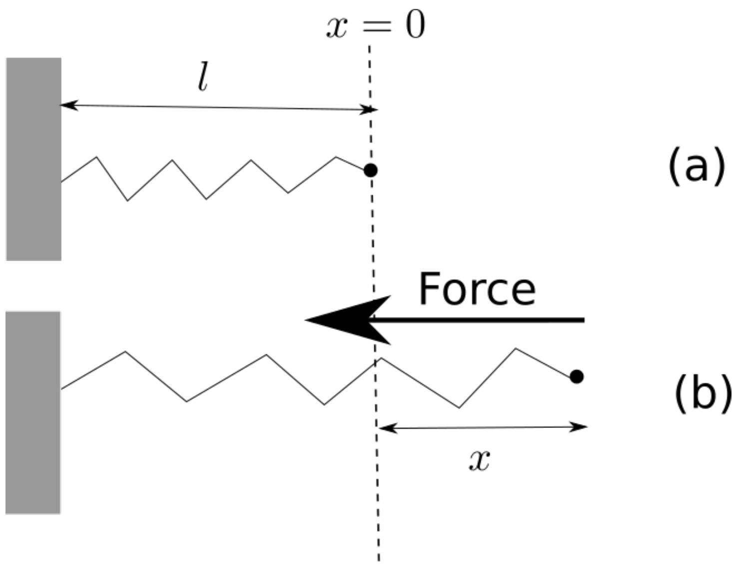

Figure 2.1: Sketch of the coordinate system showing (a) the unstretched mass/spring and (b) the stretched mass/spring.

Since k and m are both positive, we can write $k/m = \omega^2$ which forces this to be true for real ω. Thus the model becomes

$$\ddot{x} + \omega^2 x = 0.$$

[This is the equation of *simple harmonic motion.*] We solve this by first forming the characteristic equation:

$$\lambda^2 + \omega^2 = 0,$$

giving $\lambda = \pm i\omega$ and hence

$$x = A\cos(\omega t) + B\sin(\omega t).$$

[This is the other reason I defined ω like that!]. We now use the initial conditions to find A and B. Since the particle starts at natural length with speed u, this gives the initial conditions $x = 0$, $\dot{x} = u$ at $t = 0$. Hence we find

- at $t = 0$, $x = 0$, gives $A = 0$.
- $\dot{x} = \omega B \cos\omega t$ and hence $\omega B = u$ at $t = 0$, giving $B = u/\omega$.

Thus the solution is

$$x = \frac{u}{\omega}\sin\omega t.$$

In the absence of damping/other forcing, the mass will oscillate forever. ◀

Example 7: Trainee parachute jump

Consider a trainee practicing parachute jumps by jumping from a tall building of height H from the ground. They open their parachute immediately and fall in a straight line to the ground. How long does it take them to hit the ground, and what speed are they going when they make contact?

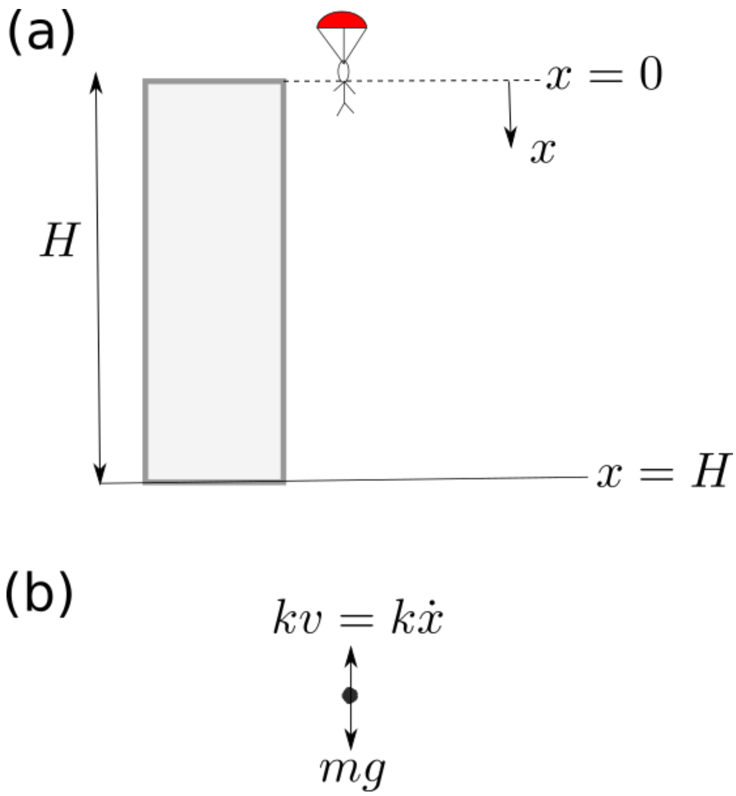

Figure 2.2: Sketch of the problem set up: (a) the geometry of the problem, indicating the jumper, the building and the ground, (b) the force balance on the particle between gravity pulling downwards and air resistance pulling upwards.

Solution. We first have to make a series of assumptions which turn the physical situation into a mathematical model. We assume:

- the person can be approximated as a particle.
- the motion is one dimensional.
- distance is measured from the top of the tower downwards; $x = 0$ gives the top and $x = H$ is the ground (see Fig. 2.2).
- $x(t)$ gives the location of the person, with $\dot{x}(t)$ giving the speed.

To use Newton's second law we need to calculate the forces acting on the person. These are:

- the weight of the person, acting downwards (i.e. in the positive x direction). This is given by mg where m is the mass of the person and g is the acceleration due to gravity ($9.81\mathrm{ms}^{-1}$).

- the force of the parachute pulling upwards. This is due to air resistance and requires a model assumption. Air resistance gets stronger the faster you're moving, so assume it is proportional to the speed of the person acting upwards $-kv$ (negative x direction). This could also be e.g. speed squared.

Hence Newton's second law gives

$$\begin{aligned} m\ddot{x} &= mg - k\dot{x}, \\ \implies \ddot{x} + \frac{k}{m}\dot{x} &= g. \end{aligned} \tag{2.1}$$

The initial conditions are that the trainee starts from rest at the top of the tower, hence $x = 0$, $\dot{x} = 0$ at $t = 0$.

We now solve equation (2.1). We first consider the homogeneous problem

$$\ddot{x}_c + \frac{k}{m}\dot{x}_c = 0,$$

to find the complementary function. This has characteristic equation

$$\lambda^2 + \frac{k}{m}\lambda = 0,$$

and hence $\lambda = 0$, or $\lambda = -k/m$, giving

$$x_c = C_1 + C_2 e^{-kt/m}.$$

The particular integral will be of the form

$$x_p = A_1 t;$$

note that we need the linear in t term as constant is part of the complementary function. Hence $\dot{x}_p = A_1$, $\ddot{x}_p = 0$ and so

$$\begin{aligned} \ddot{x}_p + \frac{k}{m}\dot{x}_p &= g, \\ \implies \frac{k}{m}A_1 &= g, \\ \implies A_1 &= \frac{gm}{k}. \end{aligned}$$

Thus the full solution is

$$x = C_1 + C_2 e^{-kt/m} + \frac{gm}{k}t.$$

We now use the initial conditions to find the constants of integration.

- $x(0) = 0$ give $x(0) = C_1 + C_2 = 0$.
- $\dot{x}\,(0)$ gives $\dot{x}\,(0) = -\frac{kC_2}{m} + \frac{gm}{k} = 0$

and hence we find

$$\begin{aligned} C_2 &= \frac{gm^2}{k^2}, \\ C_1 &= -\frac{gm^2}{k^2}, \end{aligned}$$

giving the full solution

$$x = -\frac{gm^2}{k^2} + \frac{gm^2}{k^2}e^{-kt/m} + \frac{gm}{k}t.$$

The trainee hits the ground when $t = T$ such that $x(T) = H$ - i.e. the value of time at which they've fallen a distance H. This is given by

$$H = -\frac{gm^2}{k^2} + \frac{gm^2}{k^2}e^{-kT/m} + \frac{gm}{k}T,$$

which could be solved numerically to find T. They therefore are moving with speed

$$\dot{x}\,(T) = -\frac{gm}{k}e^{-kT/m} + \frac{gm}{k},$$

upon impact. The general expression for speed is given by

$$\dot{x} = \frac{gm}{k}\left(1 - e^{-kt/m}\right),$$

so as time tends to infinity, $\dot{x} \to \frac{gm}{k}$ which gives a constant terminal velocity. ◀

This gives some examples in one dimension where all the motion is in a straight line. This clearly doesn't cover most cases! Next week we will move on to consider examples where the motion is in more than one dimension.

Activity: *You should now be able to tackle question 5 on this week's problem sheet.*

Chapter 3

Newton's laws in more dimensions

3.1 Introduction

Last week we considered a particle which moves only in a straight line. If, however, the particle moves in a plane (2D) or in all space (3D) we follow the same procedures as before, but now we use vectors as direction is also important. Here the path the particle follows is often critical to find. Choosing how to set up your coordinates and axes is often the key to being able to solve these problems!

3.2 Examples in 2 and 3 dimensions

Example 8: Throwing a stone with no air resistance Suppose a stone is thrown with velocity $\mathbf{V}$ at an angle α to the horizontal, from the top of a wall of height h. Ignoring air resistance, what is the path of the stone and its horizontal range?

Solution.

Develop model: If there is no air resistance everything will happen in a two dimensional (vertical) plane. Choose the origin O to be at ground level at the bottom of the wall, with unit vectors $\mathbf{i}$, $\mathbf{k}$ pointing in the horizontal and vertical directions respectively. Then the position of the stone over time is given by

$$\mathbf{r} = x(t)\mathbf{i} + z(t)\mathbf{k},$$

then $\dot{\mathbf{r}}$ gives the velocity and $\ddot{\mathbf{r}}$ gives the acceleration. The only force acting on the stone

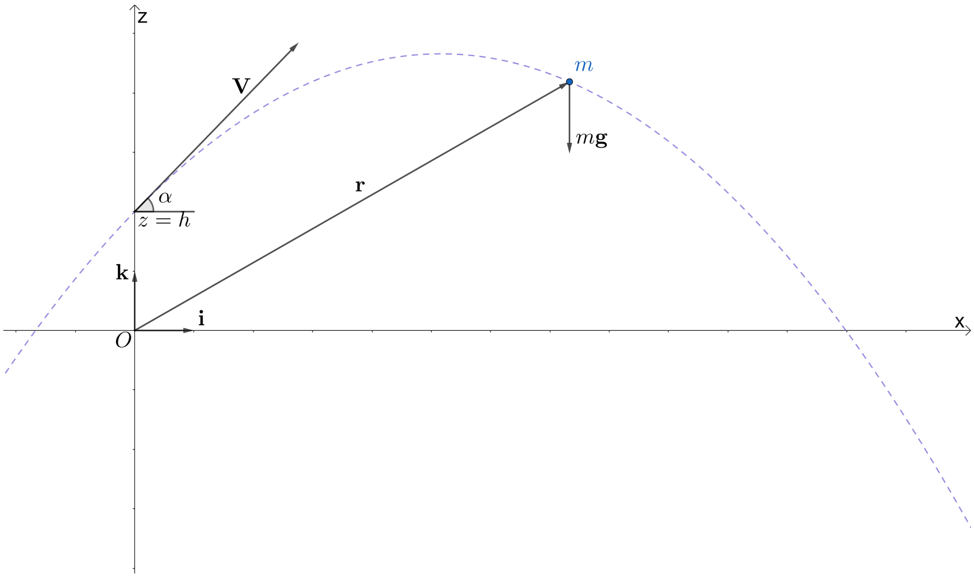

Figure 3.1: Sketch of the coordinate system.

is gravity $\mathbf{g}$, so

$$\begin{aligned} \mathbf{F} &= m\mathbf{g}, \\ &= -mg\mathbf{k}, \end{aligned}$$

as gravity acts downwards. Newton's second law then gives

$$m\ddot{\mathbf{r}} = m\mathbf{g}. \tag{3.1}$$

We also know two initial conditions: that the stone starts at the top of the wall,

$$\mathbf{r}(0) = h\mathbf{k},$$

and that it's moving with velocity $\mathbf{V}$,

$$\dot{\mathbf{r}}(0) = \mathbf{V}.$$

We are also told that the velocity makes an angle α with the horizontal. This means we can resolve the velocity such that $\mathbf{V} = V\cos\alpha\mathbf{i} + V\sin\alpha\mathbf{k}$ where $V = |\mathbf{V}|$. **Note that distinguishing between vectors and scalars is crucial!!**

Solve model: This gives a full defined system which we can now solve. We solve (3.1) in the same way as we would tackle a scalar equation - we can just integrate twice as $m\mathbf{g}$

are just constants. Hence

$$
\begin{aligned}
m\ddot{\mathbf{r}} &= m\mathbf{g}, \\
\Longrightarrow \ddot{\mathbf{r}} &= \mathbf{g}, \\
\Longrightarrow \dot{\mathbf{r}} &= \mathbf{g}t + \mathbf{c}_1, \\
\Longrightarrow \mathbf{r} &= \frac{1}{2}\mathbf{g}t^2 + \mathbf{c}_1 t + \mathbf{c}_2,
\end{aligned}
$$

where $\mathbf{c}_1$ and $\mathbf{c}_2$ are now vector constants which we find using the initial conditions.

We first use $\mathbf{r}(0) = h\mathbf{k}$ to find:

$$
\begin{aligned}
\mathbf{r}(0) &= \frac{0^2}{2}\mathbf{g} + 0\mathbf{c}_1 + \mathbf{c}_2, \\
\Longrightarrow h\mathbf{k} &= \mathbf{c}_2.
\end{aligned}
$$

Then $\dot{\mathbf{r}}(0) = \mathbf{V}$ gives

$$
\begin{aligned}
\dot{\mathbf{r}}(0) &= 0\mathbf{g} + \mathbf{c}_1, \\
\Longrightarrow \mathbf{V} &= \mathbf{c}_1.
\end{aligned}
$$

Hence the overall solution is given by

$$
\begin{aligned}
\mathbf{r} &= \frac{t^2}{2}\mathbf{g} + t\mathbf{V} + h\mathbf{k}, \\
&= \frac{-gt^2}{2}\mathbf{k} + Vt\cos\alpha\mathbf{i} + Vt\sin\alpha\mathbf{k} + h\mathbf{k}, \\
&= Vt\cos\alpha\mathbf{i} + \left(Vt\sin\alpha + h - \frac{gt^2}{2}\right)\mathbf{k},
\end{aligned}
$$

which gives the horizontal position as a function of time as

$$
x(t) = Vt\cos\alpha,
$$

and the vertical position as

$$
z(t) = Vt\sin\alpha + h - \frac{gt^2}{2}.
$$

What happens? The stone hits the ground when $z = 0$, or equivalently $\mathbf{r} \cdot \mathbf{k} = 0$, at time T which we want to find. Hence

$$\begin{aligned}
VT\sin\alpha + h - \frac{gT^2}{2} &= 0, \\
\implies T &= \frac{-V\sin\alpha \pm \sqrt{V^2\sin^2\alpha + 2gh}}{-g}, \\
&= \frac{V\sin\alpha \mp \sqrt{V^2\sin^2\alpha + 2gh}}{g},
\end{aligned}$$

using the quadratic formula. This gives two potential values (one for the plus sign and one for the minus sign) - we want the positive value, so we choose the plus sign to give

$$T = \frac{V\sin\alpha + \sqrt{V^2\sin^2\alpha + 2gh}}{g}.$$

The value of $x(T)$ gives the distance from the wall where the stone will hit the ground:

$$\begin{aligned}
x(T) &= VT\cos\alpha, \\
&= \frac{V\left(V\sin\alpha + \sqrt{V^2\sin^2\alpha + 2gh}\right)\cos\alpha}{g}.
\end{aligned}$$

We can also find the path of the stone by eliminating t from the expressions for x and z:

$$\begin{aligned}
x(t) &= Vt\cos\alpha, \\
\implies t &= \frac{x}{V\cos\alpha}, \\
\implies z &= Vt\sin\alpha + h - \frac{gt^2}{2}, \\
\implies z &= V\sin\alpha\frac{x}{V\cos\alpha} + h - \frac{g}{2}\left(\frac{x}{V\cos\alpha}\right)^2, \\
&= -\frac{g\sec^2\alpha}{2V^2}x^2 + \tan\alpha x + h,
\end{aligned}$$

which gives a parabola (see Figure 3.1 and the interactive version on Canvas). ◀

Activity: *You should now be able to tackle question 1 on this week's problem sheet.*

Example 9: Throwing a ball with air resistance A ball is thrown with velocity $\mathbf{V}$ in a steady crosswind of velocity $\mathbf{u}$. If air resistance is linearly proportional to the relative velocity between the ball and the air, what is the particle path?

Solution.

Develop model The ball will be subject to two forces - gravity ($m\mathbf{g}$) and air resistance.

The air resistance force will be given by

$$-k\left(\dot{\mathbf{r}}-\mathbf{u}\right),$$

i.e. proportional to the difference in the velocities between the ball and the air. Then Newton's second law gives

$$\begin{aligned} m\ddot{\mathbf{r}} &= m\mathbf{g}-k\left(\dot{\mathbf{r}}-\mathbf{u}\right), \\ \implies \ddot{\mathbf{r}}+\frac{k}{m}\dot{\mathbf{r}} &= \mathbf{g}+\frac{k}{m}\mathbf{u}. \end{aligned}$$

We also need initial conditions: assume that the ball starts at the origin with velocity $\mathbf{V}$, giving

$$\begin{aligned} \mathbf{r}=0 \quad &\text{at} \quad t=0, \\ \dot{\mathbf{r}}=\mathbf{V} \quad &\text{at} \quad t=0. \end{aligned}$$

Solve model We could go into components of the vector equation here, but then we'd have to solve two or three equations, so it's often easier to stay in terms of vectors as long as possible. The solution technique is as for a scalar equation - this is a linear, second order, inhomogeneous ODE with constant coefficients which we've seen before, just with vector coefficients.

We start with the homogeneous equation:

$$\ddot{\mathbf{r}}_c+\frac{k}{m}\dot{\mathbf{r}}_c = 0,$$

and guess the solution $\mathbf{r}_c=\mathbf{B}e^{\lambda t}$, where we need to find $\mathbf{B}$ (from the initial conditions) and (two values of) λ from the governing equation. Upon substituting we find

$$\begin{aligned} \lambda^2\mathbf{B}e^{\lambda t}+\frac{k}{m}\lambda\mathbf{B}e^{\lambda t} &= 0, \\ \implies \lambda^2+\frac{k}{m}\lambda &= 0, \end{aligned}$$

which gives the characteristic equation. This has solutions $\lambda=0$, or $\lambda=-k/m$, and hence we find the complementary solution

$$\mathbf{r}_c = \mathbf{B}_1+\mathbf{B}_2e^{-kt/m},$$

for two constant (vectors) $\mathbf{B}_1$, $\mathbf{B}_2$.

We now look for a particular solution. The right hand side is constant, and we have a constant in our complementary solution, so guess the particular solution is of the form $\mathbf{r}_p=\mathbf{A}t$, where $\mathbf{A}$ is a constant. This gives $\dot{\mathbf{r}}_p=\mathbf{A}$, $\ddot{\mathbf{r}}_p=0$. When we substitute this into the original equation this gives

$$\begin{aligned}
\ddot{\mathbf{r}}_p + \frac{k}{m}\dot{\mathbf{r}}_p &= \mathbf{g} + \frac{k}{m}\mathbf{u}, \\
\frac{k}{m}\mathbf{A} &= \mathbf{g} + \frac{k}{m}\mathbf{u}, \\
\implies \mathbf{A} &= \frac{m}{k}\mathbf{g} + \mathbf{u}, \\
\implies \mathbf{r}_p &= \left(\frac{m}{k}\mathbf{g} + \mathbf{u}\right) t.
\end{aligned}$$

This gives the complete solution

$$\begin{aligned}
\mathbf{r} &= \mathbf{r}_c + \mathbf{r}_p, \\
&= \mathbf{B}_1 + \mathbf{B}_2 e^{-kt/m} + \left(\frac{m}{k}\mathbf{g} + \mathbf{u}\right) t.
\end{aligned}$$

We now use the initial conditions to find the constants $\mathbf{B}_1$, $\mathbf{B}_2$. The first condition is $\mathbf{r} = 0$ at $t = 0$ which gives

$$\mathbf{0} = \mathbf{B}_1 + \mathbf{B}_2.$$

The second condition is $\dot{\mathbf{r}} = \mathbf{V}$ at $t = 0$. Since

$$\dot{\mathbf{r}} = -\frac{k}{m}\mathbf{B}_2 e^{-kt/m} + \left(\frac{m}{k}\mathbf{g} + \mathbf{u}\right),$$

we have

$$\mathbf{V} = -\frac{k}{m}\mathbf{B}_2 + \frac{m}{k}\mathbf{g} + \mathbf{u}.$$

This gives

$$\mathbf{B}_2 = -\frac{m}{k}\mathbf{V} + \frac{m^2}{k^2}\mathbf{g} + \frac{m}{k}\mathbf{u},$$

and hence

$$\begin{aligned}
\mathbf{B}_1 &= -\mathbf{B}_2, \\
&= \frac{m}{k}\mathbf{V} - \frac{m^2}{k^2}\mathbf{g} - \frac{m}{k}\mathbf{u},
\end{aligned}$$

giving the complete solution

$$\mathbf{r} = \frac{m}{k}\left(\mathbf{V} - \frac{m}{k}\mathbf{g} - \mathbf{u}\right)\left(1 - e^{-kt/m}\right) + \left(\frac{m}{k}\mathbf{g} + \mathbf{u}\right) t.$$

What does this tell us?

This gives the position of the ball at all points in time - but what does it tell us? We could write this as

$$\mathbf{r} \;=\; \frac{m}{k}\left(1-e^{-kt/m}\right)\mathbf{V}+\left(t-\frac{m}{k}\left(1-e^{-kt/m}\right)\right)\left(\frac{m}{k}\mathbf{g}+\mathbf{u}\right),$$

which gives the motion in a plane spanned by the two vectors $\mathbf{V}$ and $\frac{m}{k}\mathbf{g}+\mathbf{u}$. This is not necessarily a vertical plane because of the cross wind!

If, say,

$$\begin{aligned}\mathbf{V} &= V\cos\alpha\mathbf{i}+V\sin\alpha\mathbf{k},\\ \mathbf{g} &= -g\mathbf{k},\\ \mathbf{u} &= u\mathbf{j},\end{aligned}$$

where $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are cartesian vectors such that $\mathbf{k}$ points vertically upwards, $V=|\mathbf{V}|$, $u=|\mathbf{u}|$, and $g=|\mathbf{g}|$ is acceleration due to gravity, then we have a horizontal cross wind which is perpendicular to the ball's initial motion. In this case

$$\mathbf{r} \;=\; \frac{m}{k}\left(1-e^{-kt/m}\right)\left(V\cos\alpha\mathbf{i}+\left(V\sin\alpha+\frac{m}{k}g\right)\mathbf{k}-u\mathbf{j}\right)+t\left(-\frac{mg}{k}\mathbf{k}+u\mathbf{j}\right),$$

so the ball is moving in three dimensions in Cartesian space, but in a 2D plane defined by the two vectors $\mathbf{V}$ and $\frac{m}{k}\mathbf{g}+\mathbf{u}=-mg\mathbf{k}/k+u\mathbf{j}$.

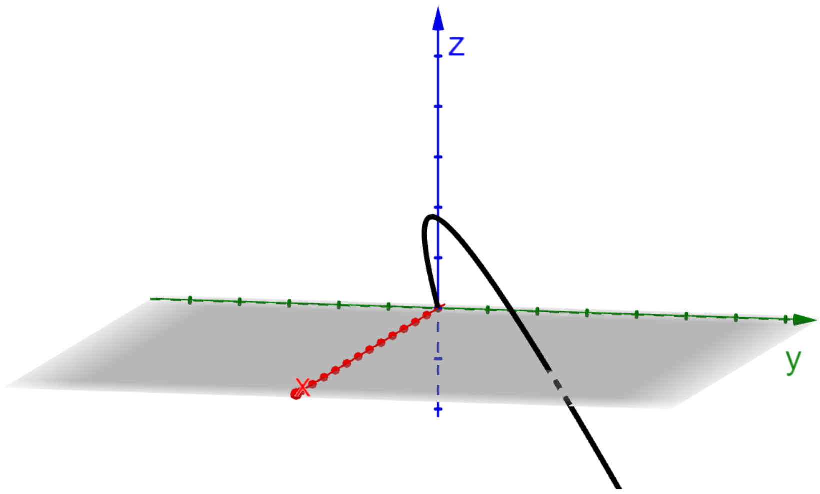

Figure 3.2: The particle path traced out in 3D, showing the particle moving in a non-vertical plane. See the link to the interactive version on Canvas!

◀

Activity: *You should now be able to tackle question 2 on this week's problem sheet.*

3.3 Moving frames of reference

In the above examples we have assumed that we can choose a frame of reference, locate the origin where we like and that Newton's laws will hold. In particular, we have assumed that the Earth's surface gives an inertial frame, despite the fact that the Earth is rotating both about its own axis and the sun. For the examples we considered, over the time and distance of the particle being airborne, the velocity of the frame of reference can be taken as uniform relative to the centre of the Earth. But if, for example, you were throwing things whilst spinning on a roundabout that would not be the case.

If we have an inertial frame S with origin O and axes $Oxyz$ then a change of axes $Oxyz$ to $Ox'y'z'$ and/or a change of origin by a fixed vector $\mathbf{a}$, $(\mathbf{O} = \mathbf{a} + \mathbf{O}')$ both give another inertial frame S'.

Proposition Suppose that S is an inertial frame with origin O. Then a frame of reference S' whose origin O' is moving with speed $\mathbf{u}$ with respect to S is also an inertial frame.

Proof Suppose that initially the origins O and O' are separated by a vector $\mathbf{a}$ at time $t = 0$. Then, since O' moves with velocity $\mathbf{u}$ with respect to S, so the vector from O to O' is given by $\mathbf{a} + \mathbf{u}t$. If $\mathbf{r}$ and $\mathbf{r}'$ are the position vectors of the same particle with respect to frames S and S' respectively then

$$\mathbf{r}' = \mathbf{r} - (\mathbf{u}t + \mathbf{a}) , \tag{3.2}$$

and hence $\ddot{\mathbf{r}}' = \ddot{\mathbf{r}}$, implying that Newton's second law also applies in the new frame S', as it applies in S. Hence S' is also inertial.

Chapter 4

Central forces (bookwork)

This week we will begin learning about problems with *Central Forces*. This week we will focus on deriving the framework (what's generally known as the "bookwork") for the problem, and we will look at examples next week. It is important to fully understand the bookwork, but in general you can use the findings without derivation, as long as you clearly state them.

4.1 Introduction

Newtonian mechanics was originally inspired by the motion of the planets around the sun. The attractive force between two gravitating bodies of mass m and M (e.g. Earth and sun) is

$$\frac{GMm}{r^2}, \tag{4.1}$$

along the line joining the two bodies. Here G is the universal gravitational constant and r is the distance between the bodies.

In particular, we might want to show/understand Kepler's laws, which he determined through observation in about 1609. These are:

Kepler's First Law: The planets move about the sun in an elliptical orbit with the sun at one focus.

Kepler's Second Law: The straight line joining a planet and the sun sweeps out equal areas in equal time.

Kepler's Third Law: The square of the period of the orbit is equal to the cube of the semi-major axis of the orbit.

Kepler's laws are phenomenological: his laws agreed with his physical observations, but he was not able to develop a mathematical theory to explain why they are true - this needed Newton's *Principia* in 1687, which includes a statement of Newton's law of gravity (as above).

This is an example of a wider class of forces known as *central forces*.

Definiton

A central force $\mathbf{F}$ acting on a particle P depends on the distance of that particle from some fixed central origin O in an inertial frame, and is directed along the line joining the particle to O. If $\mathbf{r}$ is the position vector of the particle from O then

$$\mathbf{F} = F(r)\mathbf{e}_r,$$

where $\mathbf{e}_r$ is the unit vector in the same direction as $\mathbf{r}$.

4.2 Derivation of central forces framework

4.2.1 Geometry

We first set up the geometrical framework we're going to work in. We will assume that all motion will take place in a 2D plane (we'll show this is true later). Since a central force only depends on the distance from the origin, this suggests using *polar coordinates* r (radius, i.e. the distance to the particle) and θ (angle with the x axis) to describe the motion.

We will use non-constant unit vectors $\mathbf{e}_r$, $\mathbf{e}_\theta$ in the directions of increasing r and θ - these change depending on where the particle is! Then

$$\begin{aligned}\mathbf{e}_r &= \cos\theta\mathbf{i} + \sin\theta\mathbf{j}, \\ \mathbf{e}_\theta &= -\sin\theta\mathbf{i} + \cos\theta\mathbf{j}\end{aligned}$$

in terms of Cartesian vectors $\mathbf{i}$ and $\mathbf{j}$. Note that $\mathbf{e}_r \cdot \mathbf{e}_\theta = -\cos\theta\sin\theta + \cos\theta\sin\theta = 0$ and hence $\mathbf{e}_r$ and $\mathbf{e}_\theta$ are perpendicular. As the particle moves these vectors change direction

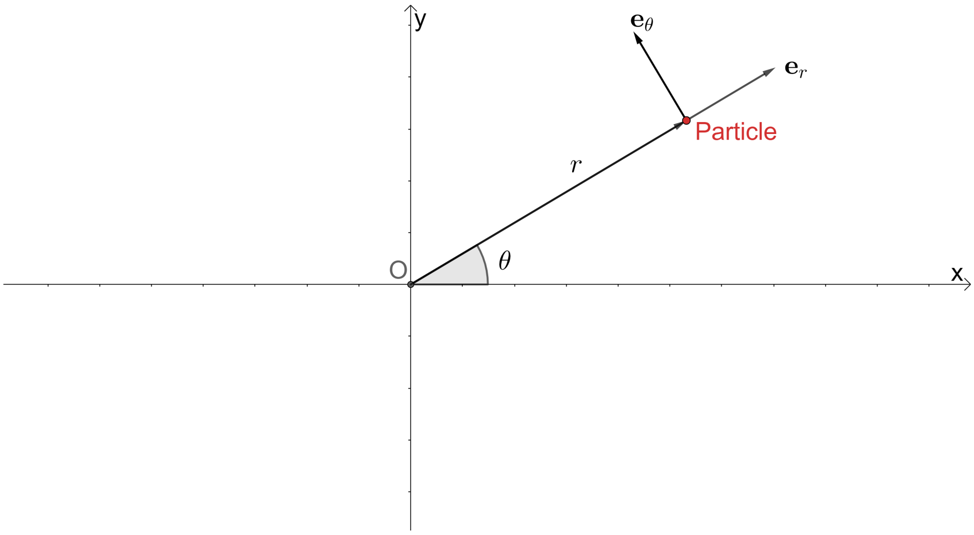

Figure 4.1: Geometry of a central forces problem. Shows the $\mathbf{e}_r$, $\mathbf{e}_\theta$ vectors, r the distance between the particle and the origin, and θ the angle the $\mathbf{e}_r$ vector makes with the x axis.

- therefore they are a function of time in the position vector. We will need

$$\begin{aligned}
\dot{\mathbf{e}}_r &= \frac{d}{dt}(\cos\theta)\,\mathbf{i} + \frac{d}{dt}(\sin\theta)\,\mathbf{j}, \\
&= -\dot{\theta}\sin\theta\mathbf{i} + \dot{\theta}\cos\theta\mathbf{j}, \\
&= \dot{\theta}\mathbf{e}_\theta, \\
\dot{\mathbf{e}}_\theta &= \frac{d}{dt}(-\sin\theta)\,\mathbf{i} + \frac{d}{dt}(\cos\theta)\,\mathbf{j}, \\
&= -\dot{\theta}\cos\theta\mathbf{i} - \dot{\theta}\sin\theta\mathbf{j}, \\
&= -\dot{\theta}\mathbf{e}_r.
\end{aligned}$$

Then the position vector of a particle is given by

$$\begin{aligned}
\mathbf{r} &= r\mathbf{e}_r, \\
&= r\cos\theta\mathbf{i} + r\sin\theta\mathbf{j}.
\end{aligned}$$

Hence

$$\begin{aligned}
\dot{\mathbf{r}} &= \frac{d}{dt}(r\mathbf{e}_r) = r\frac{d\mathbf{e}_r}{dt} + \frac{dr}{dt}\mathbf{e}_r, \\
&= r\dot{\theta}\mathbf{e}_\theta + \dot{r}\mathbf{e}_r.
\end{aligned}$$

This gives the radial and transverse components of velocity (see Fig 4.2).

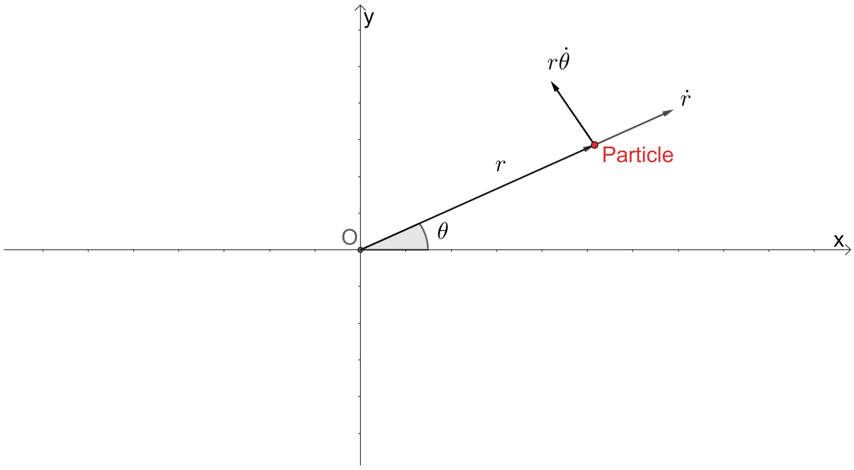

Figure 4.2: The direction of the velocity components.

Similarly acceleration is given by

$$\begin{aligned}
\ddot{\mathbf{r}} &= \frac{d}{dt}\left(\dot{\mathbf{r}}\right), \\
&= \frac{d}{dt}\left(r\dot{\theta}\mathbf{e}_\theta + \dot{r}\mathbf{e}_r\right), \\
&= \dot{r}\dot{\theta}\mathbf{e}_\theta + r\ddot{\theta}\mathbf{e}_\theta + r\dot{\theta}\dot{\mathbf{e}}_\theta + \ddot{r}\mathbf{e}_r + \dot{r}\dot{\mathbf{e}}_r, \\
&= \dot{r}\dot{\theta}\mathbf{e}_\theta + r\ddot{\theta}\mathbf{e}_\theta - r\dot{\theta}^2\mathbf{e}_r + \ddot{r}\mathbf{e}_r + \dot{r}\dot{\theta}\mathbf{e}_\theta, \\
&= \left(\ddot{r} - r\dot{\theta}^2\right)\mathbf{e}_r + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\mathbf{e}_\theta.
\end{aligned}$$

Note that

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} = \frac{1}{r}\frac{d}{dt}\left(r^2\dot{\theta}\right),$$

(expand it out to check) and hence

$$\ddot{\mathbf{r}} = \left(\ddot{r} - r\dot{\theta}^2\right)\mathbf{e}_r + \frac{1}{r}\frac{d}{dt}\left(r^2\dot{\theta}\right)\mathbf{e}_\theta.$$

This gives the radial and transverse components of acceleration (see Fig 4.3).

Activity: *You should now be able to tackle question 3(a) on this week's problem sheet.*

4.2.2 Newton's second law

Newton's second law therefore becomes

$$\begin{aligned}
m\ddot{\mathbf{r}} &= F(r)\mathbf{e}_r, \\
\implies m\left(\ddot{r} - r\dot{\theta}^2\right)\mathbf{e}_r + \frac{m}{r}\frac{d}{dt}\left(r^2\dot{\theta}\right)\mathbf{e}_\theta &= F(r)\mathbf{e}_r.
\end{aligned}$$

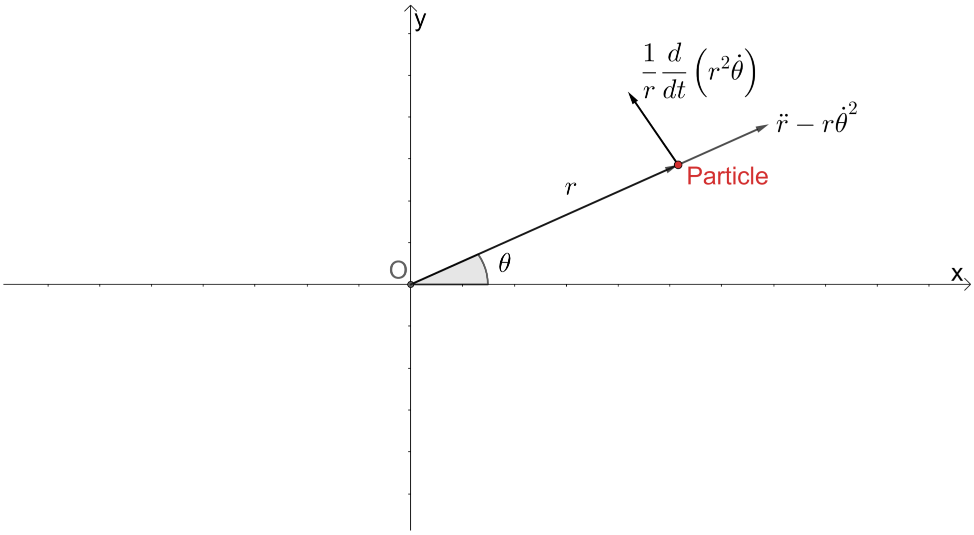

Figure 4.3: The direction of the acceleration components.

Equating coefficients of the vectors gives

$$m\left(\ddot{r}-r\dot{\theta}^2\right)=F(r),$$
$$\frac{m}{r}\frac{d}{dt}\left(r^2\dot{\theta}\right) = 0.$$

We immediately see that $r^2\dot{\theta}$ is a constant, typically denoted h. This is **always true for any central force**. [This also gives Kepler's second law that a straight line joining the sun and a planet sweeps out equal area in equal time since

$$\frac{dA}{dt} = \frac{1}{2}r^2\dot{\theta}, \quad = \frac{1}{2}h,$$

where $A = r^2\theta/2$ is the area of the sector.]

We can therefore write $\dot{\theta} = h/r^2$, and hence

$$m\left(\ddot{r}-r\left(h/r^2\right)^2\right) = F(r),$$
$$m\left(\ddot{r}-h^2/r^3\right) = F(r),$$

which gives a scalar equation for r.

4.2.3 Equation of a path

We could solve to find $r(t)$, $\theta(t)$, but often what we want to find is the equation of the path of the particle, i.e. $r(\theta)$ - so we'd like to find a single equation for this (rather than having to solve two equations in terms of t and then eliminate). To do this we first need to convert d/dt to $d/d\theta$, and then get a differential equation for r as a function of θ.

Now

$$\begin{aligned}\frac{dr}{dt} &= \frac{dr}{d\theta}\frac{d\theta}{dt}, \\ &= \frac{dr}{d\theta}\frac{h}{r^2},\end{aligned}$$

which we notice could be written as a total derivative as

$$= -h\frac{d}{d\theta}\left(\frac{1}{r}\right).$$

This gives us an idea - instead of finding an equation for $r(\theta)$, it may be easier to work in terms of $u = 1/r$ as a function of θ.

This gives

$$\dot{r} = -h\frac{du}{d\theta}, \tag{4.2}$$

and so we can find

$$\begin{aligned}\ddot{r} &= \frac{d\dot{r}}{dt}, \\ &= \frac{d\dot{r}}{d\theta}\frac{d\theta}{dt}, \\ &= \dot{\theta}\frac{d}{d\theta}\left(-h\frac{du}{d\theta}\right), \\ &= -h\dot{\theta}\frac{d^2u}{d\theta^2}, \\ &= -\frac{h^2}{r^2}\frac{d^2u}{d\theta^2}, \\ &= -h^2u^2\frac{d^2u}{d\theta^2}.\end{aligned}$$

Hence

$$\begin{aligned}m\left(\ddot{r} - h^2/r^3\right) &= F(r), \\ \implies m\left(-h^2u^2\frac{d^2u}{d\theta^2} - h^2u^3\right) &= F(1/u), \\ \implies \frac{d^2u}{d\theta^2} + u &= -\frac{F(1/u)}{mh^2u^2},\end{aligned}$$

which gives an equation for $u(\theta)$ for any central force F. **N.B.** be careful about signs! Some textbooks define F as the inward pointing force which flips the sign of the right hand side!

Activity: *You should now be able to tackle question 3(b) on this week's problem sheet.*

4.3 Initial conditions and finding "h"

The key to solving central forces problems is generally setting up the coordinate system in a sensible way, and finding "h" and the initial conditions correctly.

For any coordinate system you have (at least!) two choices to make: where you put the origin, and how you align the axes (i.e. the orientation), these are equivalent to translation and rotation. It makes sense to locate the origin of your coordinate system at the point where the force originates, e.g. the centre of the planet/sun/..., so that you do have a central force problem. You can then **choose** how to line up your $\mathbf{i}$, $\mathbf{j}$ axes or equivalently how to pick where $\theta = 0$. It's generally best to pick the axes in a such a way that the particle is initially located at $\theta = 0$ (for some value of r).

We then need to calculate the value of the constant h, and find the initial conditions for u in terms of θ. Typically we will know the initial location and the velocity of the particle in terms of $\mathbf{r}$ and $\dot{\mathbf{r}}$, specified at $t = 0$, so we need to convert from $\mathbf{r}(t)$ to $u(\theta)$.

Example 10: If a particle is initially located a distance a from the origin, moving with speed V purely in the transverse direction, find h and appropriate initial conditions for $u = 1/r$.

Solution. At $t = 0$ we know that $r = a$, $\dot{r} = 0$, $r\dot{\theta} = V$, and we **choose** our axes such that $\theta = 0$. Since $h = r^2\dot{\theta}$ is constant throughout the motion, its value will always be its initial value. Hence

$$h \quad = \quad r^2\dot{\theta} = r \cdot r\dot{\theta} = aV,$$

initially and hence throughout the motion.

We then convert the initial conditions from $r(t)$, $\theta(t)$ terms to $u(\theta)$. Firstly note that since $\theta = 0$ at $t = 0$, the u initial conditions will be applied at $\theta = 0$. Now, since $u = 1/r$, and $r = a$ at $t = 0$, this means $u = 1/a$ at $\theta = 0$. Similarly we need an expression for $\dfrac{du}{d\theta}$ at $\theta = 0$. From (5.1) we know that

$$\dot{r} \quad = \quad -h\frac{du}{d\theta},$$

giving

$$\frac{du}{d\theta} \quad = \quad -\frac{\dot{r}}{h}.$$

Hence at $\theta = 0$ (equivalently $t = 0$),

$$\begin{aligned} \frac{du}{d\theta} &= -\frac{\dot{r}}{h}, \\ &= -\frac{0}{aV}, \end{aligned}$$

using $\dot{r} = 0$ at $t = 0$ and $h = aV$ always. ◀

This sets up the framework to solve central forces problems, we will cover some examples on using it next week.

Activity: *You should now be able to tackle question 4 on this week's problem sheet.*

Chapter 5

Central forces (examples)

5.1 Introduction

Last week we set up the framework for Central Forces problems, this week we will consider some examples of it in action. We start with a summary of what we found last week.

5.2 Summary

The position vector of a particle is given by

$$\begin{aligned}\mathbf{r} &= r\mathbf{e}_r, \\ &= r\cos\theta\mathbf{i} + r\sin\theta\mathbf{j}.\end{aligned}$$

The velocity of the particle is therefore

$$\dot{\mathbf{r}} = r\dot{\theta}\mathbf{e}_\theta + \dot{r}\mathbf{e}_r.$$

The acceleration is thus

$$\ddot{\mathbf{r}} = \left(\ddot{r} - r\dot{\theta}^2\right)\mathbf{e}_r + \frac{1}{r}\frac{d}{dt}\left(r^2\dot{\theta}\right)\mathbf{e}_\theta.$$

For a **outward pointing** central force, $F(r)\mathbf{e}_r$, $u = 1/r$ satisfies

$$\frac{d^2u}{d\theta^2} + u = -\frac{F(1/u)}{mh^2u^2},$$

with $h = r^2\dot{\theta}$ constant throughout the motion. We also have

$$\dot{r} = -h\frac{du}{d\theta}. \tag{5.1}$$

5.3 Examples

Example 11: Motion under a $1/r^2$ central force Typical orbit problems often have an inverse square law central force. Let

$$F(r) = \frac{-GMm}{r^2},$$

so we have a gravitational force between two bodies of mass M and m (e.g. the sun and a planet), with initial conditions

$$\begin{array}{llcr} r = d, & \theta = 0, & \text{specifying location at} & t = 0, \\ \dot{r} = 0, & r\dot{\theta} = v & \text{specifying radial and transverse velocity components at} & t = 0. \end{array}$$

Note the negative sign as the force is directed inwards.

Solution. We know

$$\frac{d^2u}{d\theta^2} + u = -\frac{F(1/u)}{mh^2u^2},$$

where $u = 1/r$ always holds for a central force. Here $F(r)$ is of the form

$$F(r) = \frac{-GMm}{r^2},$$

and hence

$$\begin{aligned} F(1/u) &= \frac{-GMm}{(1/u)^2}, \\ &= -GMmu^2. \end{aligned}$$

This gives

$$\begin{aligned} \frac{d^2u}{d\theta^2} + u &= -\frac{-GMmu^2}{mh^2u^2}, \\ &= \frac{GM}{h^2}. \end{aligned}$$

We first find h from initial conditions (since it is a constant, it is the initial value for all time). It's normally a good idea to find the value of h as early as possible!

Now,

$$\begin{aligned} h &= r^2\dot{\theta}, \\ &= r \cdot r\dot{\theta}, \\ &= dv, \end{aligned}$$

at $t = 0$, and hence for all time. Thus we need to solve

$$\frac{d^2u}{d\theta^2} + u = \frac{GM}{d^2v^2}.$$

This is a linear second order ODE with constant coefficients, so we use the standard method to solve it. Start with the homogeneous equation:

$$\frac{d^2u_c}{d\theta^2} + u_c = 0,$$

which has characteristic equation

$$\begin{aligned} \lambda^2 + 1 &= 0, \\ \implies \lambda &= \pm i, \end{aligned}$$

and hence

$$\implies u_c = A\sin\theta + B\cos\theta,$$

where A and B are constants to be found from the initial conditions. The particular integral is given by

$$\begin{aligned} u_p &= C, \\ \implies C &= \frac{GM}{d^2v^2}, \end{aligned}$$

giving the full solution

$$u = A\sin\theta + B\cos\theta + \frac{GM}{d^2v^2}.$$

We now need to find A, B using the initial conditions. We first need to rewrite these in terms of $u(\theta)$instead of $r(t)$. Firstly $r = d$ gives $u = 1/r = 1/d$, and

$$\begin{aligned} \dot{r} &= -h\frac{du}{d\theta}, \\ &= 0, \\ \implies \frac{du}{d\theta} &= 0. \end{aligned}$$

Since $\theta = 0$ at $t = 0$, both of these hold for $\theta = 0$ so they become

$$r = d, \frac{du}{d\theta} = 0 \quad \text{at} \quad \theta = 0.$$

Hence

$$
\begin{aligned}
u(0) &= B + \frac{GM}{d^2v^2}, \\
&= \frac{1}{d}, \\
\implies B &= \frac{1}{d} - \frac{GM}{d^2v^2},
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{du}{d\theta} &= A\cos\theta - B\sin\theta, \\
&= A \text{ at } \theta = 0, \\
&= 0.
\end{aligned}
$$

This gives the complete solution

$$
u = \left(\frac{1}{d} - \frac{GM}{d^2v^2}\right)\cos\theta + \frac{GM}{d^2v^2}.
$$

What shape is this? We have

$$
\begin{aligned}
r &= \frac{1}{u}, \\
&= \frac{1}{\left(\frac{1}{d} - \frac{GM}{d^2v^2}\right)\cos\theta + \frac{GM}{d^2v^2}}, \\
&= \frac{d^2v^2/GM}{\left(\frac{dv^2}{GM} - 1\right)\cos\theta + 1}.
\end{aligned}
$$

Firstly note that if $\frac{dv^2}{GM} = 1$ this gives a circle (i.e. $r =$constant). A general conic with focus at $(0,0)$ has a polar representation

$$
r(\theta) = \frac{e\hat{d}}{1 \pm e\cos\theta},
$$

where $x = \hat{d}$ is the directrix ($\hat{d} > 0$) and e is the eccentricity such that

- $0 < e < 1$ gives an ellipse,
- $e = 1$ gives a parabola,
- $e > 1$ gives a hyperbola.

(See the crib sheet for more details.) Hence we have

$$
\pm e = \frac{dv^2}{GM} - 1,
$$

which implies we require

$$\left|\frac{dv^2}{GM}-1\right| \quad < \quad 1,$$

for the path to be an ellipse. Hence

$$\begin{aligned} -1 < \frac{dv^2}{GM} - 1 < 1, \\ \implies (0 <)\frac{dv^2}{GM} < 2, \\ \implies v^2 < \frac{2GM}{d}. \end{aligned}$$

So we have an elliptical orbit depending on the mass of the sun and the initial distance and velocity of the planet - this gives Kepler's first rule. See https://www.geogebra.org/m/hekwfhc2 for an interactive solution. ◀

Example 12: Not an inverse square law Suppose a particle of mass m is acted on by a force

$$m\left(\frac{\alpha}{r^2}+\frac{\beta}{r^3}\right),$$

acting towards the origin $r = 0$ of an inertial frame, such that $\beta = \alpha a/2$. The particle is initially a distance a from the origin and moving with velocity $\sqrt{\alpha/a}$ perpendicular to the radial vector.

What are the maximum and minimum distances between the particle and the origin, and what angle is turned through by the radius vector in travelling between them?

Solution. We have that $u = 1/r$ satisfies

$$\begin{aligned} \frac{d^2u}{d\theta^2}+u &= -\frac{F(1/u)}{mh^2u^2}, \\ &= \frac{\alpha u^2+\beta u^3}{h^2u^2}, \end{aligned}$$

where $h = r^2\dot{\theta}$ is a constant. Hence

$$\frac{d^2u}{d\theta^2}+\left(1-\frac{\beta}{h^2}\right)u \quad = \quad \frac{\alpha}{h^2},$$

will give the shape of the path.

We next consider the initial conditions. We **choose** $\theta = 0$ at $t = 0$, and from the problem statement we have

- $r = a$ at $t = 0$ from the initial location of the particle.
- $\dot{r} = 0$, $r\dot{\theta} = \sqrt{\alpha/a}$ at $t = 0$ from the radial and transverse velocity components.

We first calculate h such that

$$\begin{aligned} h &= r \cdot r\dot{\theta}, \\ &= a \cdot \sqrt{\alpha/a}, \\ &= \sqrt{\alpha a}. \end{aligned}$$

We next convert the initial conditions into terms of $u(\theta)$ to give

- $u = 1/r = 1/a$ at $\theta = 0$.
- $\frac{du}{d\theta} = -\dot{r}/h = 0$ at $\theta = 0$.

Hence

$$\begin{aligned} \frac{d^2u}{d\theta^2} + \left(1 - \frac{\beta}{h^2}\right) u &= \frac{\alpha}{h^2}, \\ \implies \frac{d^2u}{d\theta^2} + \left(1 - \frac{\beta}{\alpha a}\right) u &= \frac{\alpha}{\alpha a}, \\ \implies \frac{d^2u}{d\theta^2} + \left(1 - \frac{1}{2}\right) u &= \frac{1}{a}, \\ \implies \frac{d^2u}{d\theta^2} + \frac{u}{2} &= \frac{1}{a}, \end{aligned}$$

since $\beta = \alpha a/2$. This is a second order, linear ODE with constant coefficients, so we proceed as before, starting by solving the homogeneous system, which will have characteristic equation:

$$\lambda^2 + \frac{1}{2} = 0,$$

and hence $\lambda = \pm i/\sqrt{2}$ and thus

$$u_c = A \cos \frac{\theta}{\sqrt{2}} + B \sin \frac{\theta}{\sqrt{2}},$$

where A and B are constants. The particular integral is then of the form $u_p = C$ is constant, hence

$$\begin{aligned} \frac{1}{2} C &= \frac{1}{a}, \\ \implies u_p &= \frac{2}{a}. \end{aligned}$$

The complete solution is thus

$$u = A\cos\frac{\theta}{\sqrt{2}} + B\sin\frac{\theta}{\sqrt{2}} + \frac{2}{a}.$$

Finally, we use the initial conditions to find the values of A and B, such that

$$\begin{aligned} u(0) &= A + \frac{2}{a}, \\ &= \frac{1}{a}, \\ \implies A = -\frac{1}{a}, \end{aligned}$$

and

$$\frac{du}{d\theta} = -\frac{A}{\sqrt{2}}\sin\frac{\theta}{\sqrt{2}} + \frac{B}{\sqrt{2}}\cos\frac{\theta}{\sqrt{2}},$$

so evaluating this at $\theta = 0$ gives

$$\begin{aligned} \frac{du}{d\theta} &= \frac{B}{\sqrt{2}} = 0, \\ \implies B = 0. \end{aligned}$$

This gives the solution

$$u = \frac{2}{a} - \frac{1}{a}\cos\frac{\theta}{\sqrt{2}}.$$

We want to find the maximum and the minimum distances, so the maximum and minimum values of r. This is equivalent to minimum and maximum u.

- Minimum u (maximum r) is when $\cos\dfrac{\theta}{\sqrt{2}} = 1$, so $\theta = 0$. Hence $u = 1/a$, and $r = a$.
- Maximum u (minimum r) is when $\cos\dfrac{\theta}{\sqrt{2}} = -1$, so $\theta = \sqrt{2}\pi$. Hence $u = 3/a$, and $r = a/3$.

The angle turned through in the difference in these values of θ, i.e. $\sqrt{2}\pi$. See Figure 5.1, and the interactive version at https://www.geogebra.org/m/wsazcjre.

◀

Activity: *You should now be able to tackle questions 1 and 2 on this week's problem sheet.*

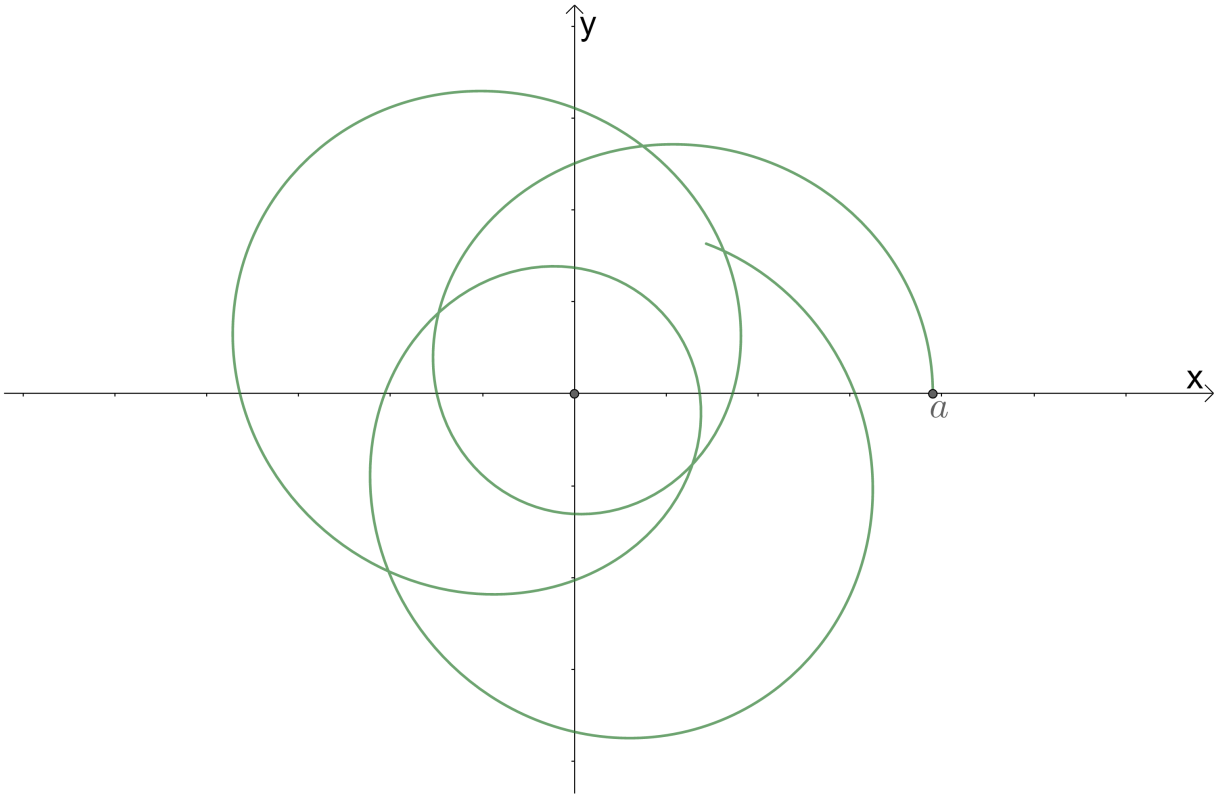

Figure 5.1: The path of the particle from example 2, spiralling from its initial location around the origin

Chapter 6

Consolidation week

No new lecture material in week 6.

Chapter 7

Bounded orbits and angular momentum

7.1 Introduction

Last week we used the framework for Central Forces problems to consider some examples. This week we will consider some more examples, along with some more general theory.

7.2 Final central forces example

Example 13: Comet Consider a comet approaching the sun from far away, with constant velocity V. If we neglected the effect of the sun on the comet, the velocity would remain as V, and it would come closest to the Sun at a distance P. What is the path of the comet, and the angle through which it is deflected?

Solution. The force acting on the comet will be

$$F(r) = \frac{GMm}{r^2},$$

where m is the mass of the comet, M is the mass of the sun, G is the gravitational constant and r is the distance between the sun and the comet such that $\mathbf{r} = r\mathbf{e}_r$ is the position vector of the comet with respect to the sun.

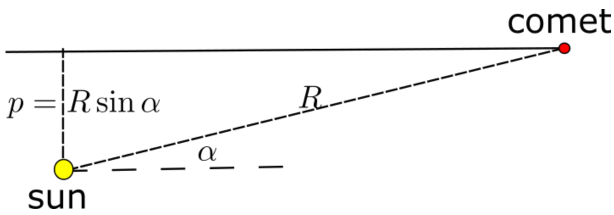

Figure 7.1: Initial set up, showing comet and sun

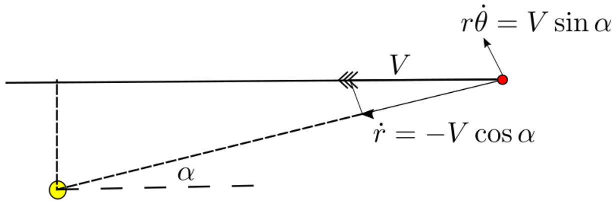

Figure 7.2: Initial velocity in component form

We let $u = 1/r$, then u will satisfy

$$\begin{aligned}\frac{d^2u}{d\theta^2} + u &= -\frac{F(1/u)}{mh^2u^2},\\ &= -\frac{-GMmu^2}{mh^2u^2},\\ &= \frac{GM}{h^2}.\end{aligned}$$

This is the same as the previous example, so will have general solution

$$u = A\sin\theta + B\cos\theta + \frac{GM}{h^2}.$$

We now consider the initial conditions and set up. Let the initial distance between the comet and the sun be $R \gg 1$, at angle $\alpha \ll 1$. Then at $t = 0$ we have $r = R$, $\dot{r} = -V\cos\alpha$, $r\dot{\theta} = V\sin\alpha$. We also know $p = R\sin\alpha$. Hence the constant h is given by

$$\begin{aligned}h &= r^2\dot{\theta},\\ &= r\cdot r\dot{\theta},\\ &= R\cdot V\sin\alpha,\\ &= R\sin\alpha\cdot V,\\ &= pV.\end{aligned}$$

Since $\alpha \ll 1$, $\cos\alpha \approx 1$ giving $\dot{r} = -V$ at time $t = 0$.

Now, we can **choose** where $\theta = 0$ (since it's essentially rotating the axes to whereever we like), so we take the comet to be at $\theta = 0$ at $t = 0$. Hence we can rewrite our initial conditions ($t = 0$) in terms of $\theta = 0$ to find

$$\begin{aligned} u &= \frac{1}{R} \approx 0, \\ \frac{du}{d\theta} &= -\frac{\dot{r}}{h}, \\ &= \frac{V}{pV} = \frac{1}{p}, \end{aligned}$$

at $\theta = 0$.

Hence

$$\begin{aligned} u &= A \sin\theta + B \cos\theta + \frac{GM}{h^2}, \\ &= A \sin\theta + B \cos\theta + \frac{GM}{p^2 V^2}. \end{aligned}$$

Then

$$\begin{aligned} u(0) &= B + \frac{GM}{p^2 V^2}, \\ &= 0, \\ \implies B &= \frac{-GM}{p^2 V^2}, \end{aligned}$$

and

$$\begin{aligned} \frac{du}{d\theta} &= A \cos\theta - B \sin\theta, \\ &= A, \\ &= 1/p, \end{aligned}$$

at $\theta = 0$. Hence the complete solution is

$$u = \frac{1}{p} \sin\theta + \frac{GM}{p^2 V^2} \left(1 - \cos\theta\right).$$

What does this tell us?

The comet goes off into space as $r \to \infty$, i.e. when $u \to 0$. This happens when

$$\frac{1}{p} \sin\theta + \frac{GM}{p^2 V^2} \left(1 - \cos\theta\right) = 0,$$

which holds when $\theta = 0$ (so $\sin\theta = 0$, $\cos\theta = 1$), or when $\theta = \Theta$ nonzero satisfies

$$\begin{aligned} \frac{1}{p} \sin\Theta + \frac{GM}{p^2 V^2} \left(1 - \cos\Theta\right) &= 0, \\ \implies \frac{\cos\Theta - 1}{\sin\Theta} &= \frac{pV^2}{GM}. \end{aligned}$$

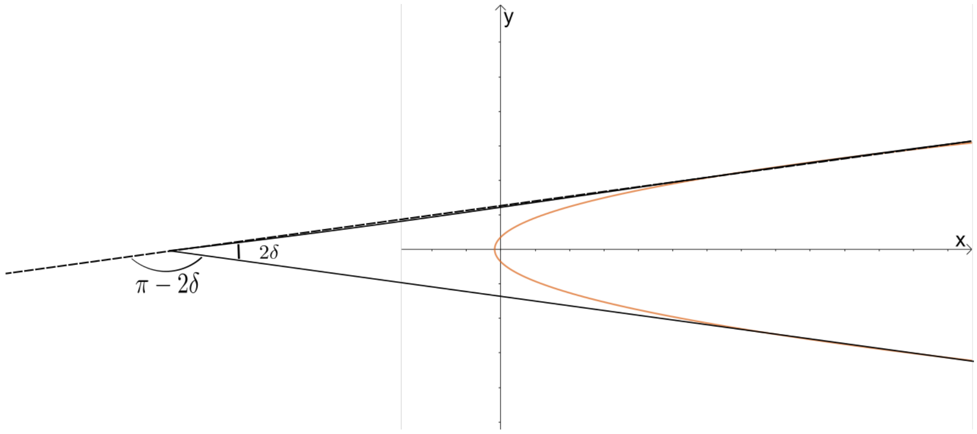

Figure 7.3: The deflection of the comet by the sun, showing 2δ as the angle at the "point" where the asymptotes meet, and $\pi - 2\delta$ as the deflection of the comet.

Now we expect the Sun to deflect the comet from its undisturbed trajectory. If we take 2δ to be the angle between the asymptotes of the trajectory (see Figure 7.3), we will have $\Theta = 2\pi - 2\delta$. Then

$$\begin{aligned}
\cos(2\pi - 2\delta) &= \cos(2\pi)\cos(-2\delta) - \sin(2\pi)\sin(-2\delta),\\
&= \cos(2\delta),\\
&= 2\cos^2\delta - 1,\\
\sin(2\pi - 2\delta) &= \sin(2\pi)\cos(-2\delta) + \cos(2\pi)\sin(-2\delta),\\
&= -\sin(2\delta),\\
&= -2\sin\delta\cos\delta,
\end{aligned}$$

and hence

$$\begin{aligned}
\frac{pV^2}{GM} &= \frac{2(\cos^2\delta - 1)}{-2\sin\delta\cos\delta},\\
&= \frac{\sin^2\delta}{\sin\delta\cos\delta},\\
&= \tan\delta.
\end{aligned}$$

This gives a deflection of $\pi - 2\delta$, since the path would otherwise go from $\theta = 0$ to $\theta = \pi$.

◀

7.3 Circular orbits and stability

Circular orbits which are *stable* are very important, particularly when considering satellites in orbit about a planet (e.g. for GPS). We need satellites to remain the same distance away from the Earth even if there are slight disturbances. This leads us to the mathematical concept known as stability - if we have a small *perturbation* to an otherwise steady solution, does that perturbation decay, so we return to the steady solution, or grow, so we move away? You will learn more about this in later modules, but we shall show an example of this here.

We first determine what force $F(r)$ leads to a circular orbit. This will be such that $r = a$, for some constant value a. Since we know that

$$m\left(\ddot{r} - h^2/r^3\right) = F(r), \tag{7.1}$$

must be satisfied (from week 04), for $r = a$ (and hence $\ddot{r} = 0$) to be a solution we must have

$$-mh^2/a^3 = F(a).$$

Therefore if the force F satisfies $F(a) = -mh^2/a^3$ for the particular setup (i.e. the value of h), then $r = a$ could be a solution to the model.

We also need this to be stable, so that once the particle reaches $r = a$ it remains in this orbit. We test this by putting

$$r = a + \epsilon P(t),$$

where $\epsilon P(t)$ is a small perturbation (i.e. $\epsilon \ll 1$). If $P(t)$ grows (i.e. $P \to \infty$ as $t \to \infty$) in time we move further and further away from the constant radius case. Substituting this into the governing equation (7.1) and using $\ddot{r} = \epsilon \ddot{P}$ gives

$$m\left(\epsilon\ddot{P} - \frac{h^2}{(a+\epsilon P)^3}\right) = F(a + \epsilon P),$$

for fixed h.

Since ϵ is small we can expand the second term on the left hand side, and the right hand side:

$$\begin{aligned}\frac{h^2}{(a+\epsilon P)^3} &= \frac{h^2}{a^3}\left(1 + \epsilon P/a\right)^{-3}, \\ &= \frac{h^2}{a^3}\left(1 - 3\epsilon P/a + \ldots\right),\end{aligned}$$

using a Binomial expansion and

$$F(a + \epsilon P) = F(a) + \epsilon P F'(a) + \ldots,$$

using a Taylor expansion, where $'$ denotes d/dr. This gives

$$m\left(\epsilon\ddot{P} - \frac{h^2}{a^3}\left(1 - 3\epsilon P/a + \ldots\right)\right) = F(a) + \epsilon P F'(a) + \ldots.$$

Since ϵ is small, we can approximate this equation at different orders. At leading order (i.e. when we neglect any term which has an ϵ in it), we require

$$\frac{h^2}{a^3} = F(a).$$

This is precisely the requirement we already had to have a circular orbit in the first place. We then look at next order, by equating all the terms which are linear in ϵ, and neglecting anything which is ϵ^2 or smaller (small thing times small thing is a really small thing):

$$\begin{aligned} m\left(\ddot{P} + \frac{3h^2}{a^4}P\right) &= PF'(a), \\ \implies m\ddot{P} + \left(\frac{3mh^2}{a^4} - F'(a)\right)P &= 0. \end{aligned}$$

This is a linear second order ODE with constant coefficients, so which we can solve using the characteristic equation

$$m\lambda^2 + \left(\frac{3mh^2}{a^4} - F'(a)\right) = 0.$$

The solution will be in the form of exponentials if $\dfrac{3mh^2}{a^4} - F'(a) < 0$ or sines and cosines if $\dfrac{3mh^2}{a^4} - F'(a) > 0$. Exponential solutions will grow and move away from the constant radius solution, whilst sines and cosines will oscillate, remaining in the orbit, so we require $\dfrac{3mh^2}{a^4} - F'(a) > 0$ for a stable orbit.

Activity: *You should now be able to tackle question 3 on this week's problem sheet.*

7.4 Angular momentum (or moment of momentum)

Having seen the principles in action, we now return to some more general theory about angular momentum (or moment of momentum). The moment of a general force $\mathbf{F}$ about an origin O is

$$\mathbf{r} \times \mathbf{F}$$

where $\mathbf{r}$ gives the location of the point where the force is applied relative to O.

For example, a force F applied vertically, a horizontal distance d from the pivot, gives a

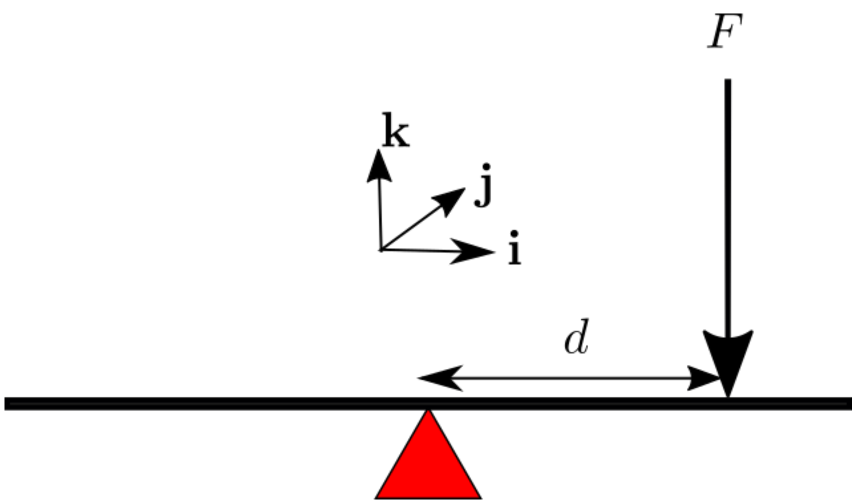

Figure 7.4: A seesaw showing a force applied vertically, at a horizontal distance d to the pivot.

moment $\mathbf{r} \times \mathbf{F}$ about the pivot where $\mathbf{r} = d\mathbf{i}$, $\mathbf{F} = -F\mathbf{k}$ and hence

$$\begin{aligned} \mathbf{r} \times \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ d & 0 & 0 \\ 0 & 0 & -F \end{vmatrix}, \\ &= Fd\mathbf{j}, \end{aligned}$$

as you probably recognised from high school (but now with added direction!).

We now recall the definition of momentum from earlier: $\mathbf{p} = m\dot{\mathbf{r}}$. Therefore the moment of momentum about an origin O is given by

$$\mathbf{r} \times (m\dot{\mathbf{r}}),$$

this is the angular momentum. [Note that we're assuming the pivot and the coordinate system origin at the same.]

Then Newton's second law gives

$$\mathbf{F} = m\ddot{\mathbf{r}},$$

and taking the cross product of both sides with $\mathbf{r}$ gives

$$\begin{aligned} \mathbf{r} \times \mathbf{F} &= \mathbf{r} \times (m\ddot{\mathbf{r}}), \\ &= m\mathbf{r} \times \ddot{\mathbf{r}}. \end{aligned}$$

Now the cross product of any vector with itself is zero, as they are parallel to each other, so $\dot{\mathbf{r}} \times \dot{\mathbf{r}} = 0$. So we can write

$$\begin{aligned}
\mathbf{r} \times \mathbf{F} &= m\left(\mathbf{r} \times \ddot{\mathbf{r}} + \dot{\mathbf{r}} \times \dot{\mathbf{r}}\right), \\
&= m\frac{d}{dt}\left(\mathbf{r} \times \dot{\mathbf{r}}\right), \\
&= \frac{d}{dt}\left(\mathbf{r} \times (m\dot{\mathbf{r}})\right).
\end{aligned}$$

Here the left hand side is the moment about O of the force $\mathbf{F}$ and the right hand side is the derivative of the moment about O of momentum, or the angular momentum. [Expand out the derivative to check the first and second lines are the same!]

Special case: Central force

If $\mathbf{F} = F(r)\mathbf{e}_r$ then $\mathbf{r} \times \mathbf{F} = 0$ since $\mathbf{r}$ and $\mathbf{F}$ are parallel vectors. Hence

$$\frac{d}{dt}\left(\mathbf{r} \times \dot{\mathbf{r}}\right) = 0,$$

and so

$$\begin{aligned}
\mathbf{r} \times \dot{\mathbf{r}} &= \text{constant}, \\
&= \mathbf{h}.
\end{aligned}$$

If we now dot with $\mathbf{r}$ we find

$$\mathbf{r} \cdot (\mathbf{r} \times \dot{\mathbf{r}}) = \mathbf{r} \cdot \mathbf{h}.$$

Now $\mathbf{r} \times \dot{\mathbf{r}}$ must be perpendicular to $\mathbf{r}$ (as it's in the cross product), so

$$\mathbf{r} \cdot \mathbf{h} = 0.$$

This gives the equation of a plane through the origin, so the motion takes place entirely in a plane (as we assumed earlier).

Finally, if

$$\begin{aligned}
\mathbf{h} &= \mathbf{r} \times \dot{\mathbf{r}}, \\
&= r\mathbf{e}_r \times \left(\dot{r}\mathbf{e}_r + r\dot{\theta}\mathbf{e}_\theta\right), \\
&= \left(r\mathbf{e}_r \times \dot{r}\mathbf{e}_r\right) + \left(r\mathbf{e}_r \times r\dot{\theta}\mathbf{e}_\theta\right), \\
&= r^2\dot{\theta}\left(\mathbf{e}_r \times \mathbf{e}_\theta\right), \\
&= r^2\dot{\theta}\mathbf{k},
\end{aligned}$$

where $\mathbf{k}$ is a vector coming out of the plane. [Using the fact that $\mathbf{e}_r \times \mathbf{e}_r = 0$.] Hence $r^2\dot{\theta}$ is conserved as before.

Activity: *You should now be able to tackle question 4 on this week's problem sheet.*

This concludes the Chapter on Central Forces, next week we shall start on Energy.

Chapter 8

Energy in a straight line (one dimension)

8.1 Introduction

We will now move on to using ideas of energy in mechanics, we will focus on this topic for the next three weeks.

When you throw a ball in the air, the ball starts with a lot of velocity, gains height whilst slowing down, reaches a maximum height when it becomes stationary before falling and gaining speed again. In the absence of other forces, as you gain height you lose speed and vice versa. This is because energy is being conserved. When energy is conserved, this can be a quick and easy way of determining important information, for example the maximum distance or speed a particle can attain, but doesn't tend to be the best method to get the full particle path. This week we will focus on examples where the motion happens in a one dimensional straight line, before moving on to more complex examples next week.

8.2 Energy in one dimension

If you consider a particle of mass m moving in a straight line, which is being acted on by a force F **which depends only on the position of the particle** x, then Newton's second law gives

$$m\ddot{x} \quad = \quad F(x),$$

and the particle has two types of energy:

- the **kinetic energy** $\frac{1}{2}m\dot{x}^2$ because the particle is moving and
- the **potential energy** due to the force acting on it.

The potential energy is essentially the energy that is "stored" to be converted into kinetic energy at a later time. There are a number of different types of potential energy, for example gravitational energy (if something is high) or elastic potential energy (if something is stretched or compressed). We can also have e.g. electrical, chemical or nuclear potential energy.

The potential energy $V(x)$ is defined to be

$$V(x) \quad = \quad -\int F(x)dx.$$

The potential energy is therefore defined only up to an additive constant, so we can **choose** where the potential energy equals zero for example. Equivalently

$$F(x) = -\frac{dV}{dx}.$$

If the force acting on a particle can be derived from a potential energy then the total energy is conserved.

8.2.1 Conservation of energy

Newton's second law gives

$$F(x) \quad = \quad m\ddot{x}.$$

We multiply both sides by $\dot{x}$ to find

$$\begin{aligned} F(x)\dot{x} &= m\ddot{x}\dot{x}, \\ &= \frac{d}{dt}\left(\frac{m}{2}\dot{x}^2\right). \end{aligned}$$

This can then be integrated with respect to time:

$$\begin{aligned} \int F(x)\frac{dx}{dt}\,\mathrm{d}t &= \int \frac{d}{dt}\left(\frac{m}{2}\dot{x}^2\right)\,\mathrm{d}t, \\ \int F(x)dx + \text{const} &= \frac{1}{2}m\dot{x}^2, \\ -V(x) + \text{const} &= \frac{1}{2}m\dot{x}^2. \end{aligned}$$

Hence we find conservation of energy:

$$\underbrace{\frac{1}{2}m\dot{x}^2}_{\text{kinetic energy}} \quad + \quad \underbrace{V(x)}_{\text{potential energy}} \quad = \quad E,$$

where E the constant total energy can be found from the initial conditions, since the value initially must be the value for all time if it is constant. Hence a loss in kinetic energy leads to an increase in potential energy and vice versa - as the ball gets higher it gets slower.

8.2.2 Gravitational potential energy

If the particle is moving vertically under the action of gravity, with x measured upwards, the force is given by $F = -mg$ and hence

$$\begin{aligned} V(x) &= -\int F \, \mathrm{d}x, \\ &= \int mg \, \mathrm{d}x, \\ &= mgx + \text{constant}. \end{aligned}$$

Since potential is only defined up to an additive constant (as it's an integral) we can **choose** where we set it to be zero to make the maths as easy as possible - the constant is absorbed into the energy E. For this example we pick $V = 0$ at $x = 0$, so the constant is also zero - this means we're measuring potential energy from $x = 0$. Hence

$$\frac{1}{2}m\dot{x}^2 + mgx = E$$

gives the energy conservation equation for a particle moving under gravity in a straight line.

Ideas of energy conservation lead to first order (nonlinear) ODEs as opposed to second order ODEs from Newton's second law directly, which may (or may not!) be easier to solve. If there is no analytical solution, energy conservation can give useful information (e.g. bounds on the motion) which Newton's second law can't give.

Example 14: A ball falling under gravity If we throw a ball of mass m vertically upwards with velocity v from the ground, how high will it go?

Solution. Let $x(t)$ be the height of the ball at time t. If we **choose** to take the potential energy to be zero at $x = 0$ (i.e. ground level), conservation of energy gives

$$\frac{1}{2}m\dot{x}^2 + mgx = E.$$

Taking $\dot{x} = v$, $x = 0$ at $t = 0$, we first find the value of the constant E to be

$$E = \frac{1}{2}mv^2,$$

giving

$$\frac{1}{2}m\dot{x}^2 + mgx = \frac{1}{2}mv^2.$$

Now, the highest point of the ball's flight will be the point at which its velocity is zero (so it's changing direction to come back down again). Hence the highest point is $x = h$ such that

$$\begin{aligned} mgh &= \frac{1}{2}mv^2, \\ \implies h = &= \frac{v^2}{2g}. \end{aligned}$$

◀

Activity: *You should now be able to tackle question 1 on this week's problem sheet.*

8.3 Elastic potential energy

Example 15: Mass on a spring neglecting gravity

Consider a particle of mass m attached to a spring (constant k and natural length a) which is fixed at the opposite end. Neglecting gravity, how does the particle move?

Solution. We will measure x, the location of the particle, from the fixed end of the spring. Then Hooke's law will give the tension in the spring, such that the magnitude of the force is proportional to the extension from natural length, $F = -k(x-a)$, with the minus sign as it acts in the opposite direction to that that x is pointing in. Hence

$$\begin{aligned} V(x) &= -\int F(x)\,\mathrm{d}x, \\ &= k\int (x-a)\ \mathrm{d}x, \\ &= k\frac{(x-a)^2}{2} + \text{const}, \end{aligned}$$

using the chain rule (integrate it directly and then factorise to check it works!). We choose the constant so that the potential energy is zero at the natural length of the spring (i.e. $V(a) = 0$) and hence

$$V(x) = \frac{k(x-a)^2}{2},$$

(this is why I wrote V in that format in the first place!), and hence conservation of energy gives

$$\frac{1}{2}m\dot{x}^2 + \frac{k\left(x-a\right)^2}{2} \quad = \quad E$$

where E is constant. Here the first term is the kinetic energy, and the second is the *elastic* potential energy stored in the spring. Since both $\frac{1}{2}m\dot{x}^2$ and $\dfrac{k\left(x-a\right)^2}{2}$ are necessarily positive, this gives maximum values on the motion. The maximum value of K.E. is when the P.E. is zero, so

$$\begin{aligned} \frac{1}{2}m\dot{x}^2 &\leq E, \\ \dot{x}^2 &\leq \frac{2E}{m}, \end{aligned}$$

which limits how quickly the particle can move. Similarly

$$\begin{aligned} \frac{k\left(x-a\right)^2}{2} &\leq E, \\ \left(x-a\right)^2 &\leq \frac{2E}{k}, \end{aligned}$$

so this gives oscillatory motion between

$$a - \sqrt{\frac{2E}{k}} \leq x \leq a + \sqrt{\frac{2E}{k}}.$$

◀

8.4 Gravitational and elastic energy

Example 16: Mass on a spring with gravity

Consider a particle of mass m attached to a spring (constant k and natural length a) which is fixed at the opposite end, with gravity acting downwards. If we release the particle from rest at the natural length of the spring, what is the motion of the particle?

Solution. Let $x = 0$ be the fixed end of the spring with $x(t)$ giving the location of the particle (see figure 8.1). As before the tension in the spring is given by $k\left(x-a\right)$. Hence the total force on the particle (acting downwards) is

$$\begin{aligned} F(x) &= mg - k\left(x-a\right), \\ \text{and} \qquad V(x) &= -\int F(x)\,\mathrm{d}x. \end{aligned}$$

Now, integration is additive, so we can consider each part separately, and **define a zero**

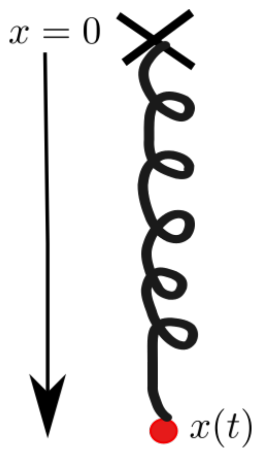

Figure 8.1: Sketch for example 16 showing the particle a distance x below the fixed point of the spring which is located at $x = 0$.

value location for each. Hence we have

- P.E. of gravity giving $-mgx$ where we choose it to be zero at $x = 0$.
- P.E. of the spring giving $k\left(x-a\right)^2/2$, where we choose it to be zero at $x = a$, the spring's natural length.

Hence the total potential energy is the sum of these such that

$$V(x) = -mgx + \frac{k\left(x-a\right)^2}{2},$$

and hence

$$\frac{1}{2}m\dot{x}^2 - mgx + \frac{k\left(x-a\right)^2}{2} = E$$

where E is constant, gives conservation of energy. The potential energy consists of two components; the gravitational potential energy $-mgx$ and the elastic potential energy $\frac{k\left(x-a\right)^2}{2}$.

We now use the initial conditions to find E; at $t = 0$ we have $\dot{x} = 0$ and $x = a$. Thus

$$\begin{aligned} E &= -mga, \\ \implies \frac{1}{2}m\dot{x}^2 - mgx + \frac{k\left(x-a\right)^2}{2} &= -mga, \\ \implies \frac{1}{2}m\dot{x}^2 &= mgx - \frac{k\left(x-a\right)^2}{2} - mga, \\ \implies \dot{x}^2 &= 2gx - \frac{k}{m}\left(x-a\right)^2 - 2ga, \\ &= 2g\left(x-a\right) - \frac{k}{m}\left(x-a\right)^2, \\ &= \left(x-a\right)\left(2g - \frac{k}{m}\left(x-a\right)\right), \\ &= \left(x-a\right)\left(2g + \frac{ak}{m} - \frac{kx}{m}\right) \end{aligned}$$

Now, $\dot{x}^2$ must be positive (since it's something squared), which means that the right hand side must also be greater than or equal to zero. Therefore the particle will always lie between $x = a$ and $x = \dfrac{2mg}{k} + a$. ◀

Example 17: Bungee jumper Consider a bungee jumper who steps from a high platform (distance H above the ground) with a stretchy elastic cord (unstretched length a and elastic constant k) tied around their ankles, with the other end attached to the platform. Initially they fall under gravity until they reach the natural length of the cord, at which point the cord starts stretching and exerting an additional upward force. Natural questions could be how fast is the jumper falling when the cord starts stretching, and how close to the ground do they get?

Solution. Let $x(t)$ be the location of the person (approximated as a particle of mass m), measured downwards from the platform. The cord is best approximated as a "string" rather than a "spring", that is it still obeys Hooke's law in tension (when it's stretched), but there is no response when it is compressed. There are therefore two regimes for the jumper; the first where the cord is not yet stretched, so isn't contributing mechanically, and G.P.E is converted into K.E, and the second where the cord begins to stretch, also storing up elastic potential energy. We consider each of these two cases separately initially.

Before the jumper reaches a distance a below the platform, at which point the cord begins to stretch, conservation of energy gives

$$\frac{1}{2}m\dot{x}^2 - mgx = E,$$

where we have set the zero of the G.P.E to be at the platform and recalling that x increases downwards (hence the minus sign). At $t = 0$ the jumper has no initial velocity $\dot{x} = 0$, and is at location $x = 0$. Thus we find the total energy as $E = 0$, giving

$$\frac{1}{2}m\dot{x}^2 - mgx = 0.$$

This equation will hold for $0 \leq x \leq a$, when the distance between the two ends of the cord is less than its unstretched length.

The cord will begin to stretch when $x = a$, so that the person is a distance a from the platform and the cord begins to be in tension, storing elastic potential energy. The extension in the cord as the person continues to fall is given by $x - a$, and therefore conservation of energy gives

$$\begin{aligned} \frac{1}{2}m\dot{x}^2 - mgx + \frac{k(x-a)^2}{2} &= E, \\ &= 0, \end{aligned}$$

since the total energy must still be the same as initially. This equation holds for $a \leq x \leq H$. Note that the two expressions are equivalent when $x = a$.

We can now quickly calculate the speed of the jumper at the point $x = a$, which satisfies

$$\begin{aligned} \frac{1}{2}m\dot{x}^2 - mga &= 0, \\ \implies \dot{x} &= \sqrt{2ga}, \end{aligned}$$

and the lowest point of the jumper ($x = L$ say), which is when $\dot{x} = 0$, so

$$\begin{aligned} -mgL + \frac{k(L-a)^2}{2} &= 0, \\ \implies (L-a)^2 - \frac{2mg}{k}L &= 0, \\ \implies L^2 - \left(\frac{2mg}{k} + 2a\right)L + a^2 &= 0, \end{aligned}$$

giving a quadratic for L which we solve to find

$$\begin{aligned} L &= \frac{2\left(a + \frac{mg}{k}\right) \pm \sqrt{4\left(a + \frac{mg}{k}\right)^2 - 4a^2}}{2}, \\ &= \left(a + \frac{mg}{k}\right) \pm \sqrt{\frac{2mga}{k} + \frac{m^2g^2}{k^2}}. \end{aligned}$$

◀

This gives the basic concepts of using conservation of energy. Next week we will move on to move complex behaviour, with motions which aren't necessarily in a straight line in one dimension.

Activity: *You should now be able to tackle question 2 on this week's problem sheet.*

Chapter 9

Energy in more than one dimension part 1

9.1 Introduction

We now consider a similar principle to last week, but in more than one dimension. This will work in a similar fashion to energy in one dimension. This week we will consider motion constrained to lie on a 1D (not necessarily straight) line or a flat plane. Next week we will look at motion on surfaces of revolution.

9.2 More than one dimension

Consider a particle of mass m moving in 3D under the action of a force $\mathbf{F}$. Then Newton's second law gives

$$\mathbf{F} = m\ddot{\mathbf{r}},$$

where $\mathbf{r}$ gives the position vector of the particle. If we take the dot product with $\dot{\mathbf{r}}$ we find

$$\begin{aligned} \mathbf{F}\cdot\dot{\mathbf{r}} &= m\ddot{\mathbf{r}}\cdot\dot{\mathbf{r}}, \\ &= \frac{d}{dt}\left(\frac{1}{2}m\left|\dot{\mathbf{r}}\right|^2\right). \end{aligned}$$

The left hand side here represents the rate at which the force does work on the particle (recall that work = force x distance), whilst the right hand side is the time derivative of the familiar kinetic energy, this time defined in 3D: $\frac{1}{2}m\left|\dot{\mathbf{r}}\right|^2$- "half mass times velocity squared". Note that $\left|\dot{\mathbf{r}}\right|^2 = \dot{\mathbf{r}}\cdot\dot{\mathbf{r}}$ is a scalar and so we can safely neglect the modulus signs without confusion.

For a **conservative** force, this can be integrated in time to give energy conservation. This involves some subtle definitions which we aren't going to cover here, in particular involving line integrals (which you'll see next year). Essentially everything works the same as in 1D provided the force doesn't dissipate energy, so no friction or air resistance type forces. We will only consider examples where conservation of energy holds, but please be aware that it doesn't always!

9.3 Motion constrained to 1D

We first consider an example where the motion occurs in a plane, but the particle is constrained to move along a one dimensional path, for example if it is attached to a pendulum, or moving along a wire. This motion is one dimensional (i.e. so it depends only on one coordinate), but not necessarily in a straight line.

Example 18: Motion of a simple pendulum

Consider a particle of mass m attached to a light rod (a rigid line with no mass, which constrains the particle to move in a circle) of length l. The rod is fixed in space at the top. What initial velocity is required for the pendulum to overturn?

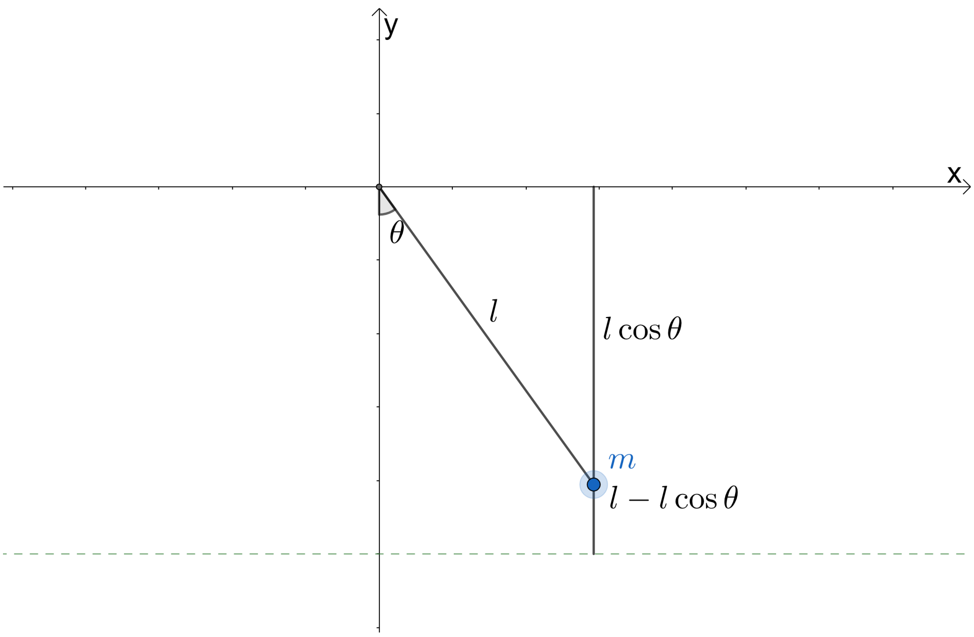

Figure 9.1: Mass moving on a pendulum on length l showing the relative height of the mass above the "bottom" of the pendulum where we set the GPE to be zero.

Solution. Let θ be the angle of deflection of the rod. (See Figure 9.1). Then the position vector of the particle, relative to an origin at the fixed top of the rod, is

$$\mathbf{r} \;=\; l\mathbf{e}_r,$$

and hence the velocity of the mass is

$$\dot{\mathbf{r}} \;=\; l\dot{\theta}\mathbf{e}_\theta,$$

(since $d\mathbf{e}_r/dt = \dot{\theta}\mathbf{e}_\theta$, similar to the Central Forces examples). Hence the kinetic energy is

$$\frac{1}{2}m\left|\dot{\mathbf{r}}\right|^2 \;=\; \frac{1}{2}m\left(l\dot{\theta}\right)^2.$$

We now need to find the gravitational potential energy. We will define the potential to be zero when $\theta = 0$. Then the height of the pendulum above this will be $l - l\cos\theta$ (see Figure 9.1). Hence the gravitational potential energy will be $mgl\left(1-\cos\theta\right)$.

This gives the total energy as

$$\frac{1}{2}m\left(l\dot{\theta}\right)^2 + mgl\left(1-\cos\theta\right) \;=\; E,$$

where E is the constant energy. We find E by using the initial conditions.

We set $\theta = 0$, $\dot{\theta} = \Omega$ at $t = 0$, so that the pendulum starts hanging straight down, with a nonzero initial velocity. Then

$$\begin{aligned}
\frac{1}{2}m\left(l\dot{\theta}\right)^2 + mgl\left(1-\cos\theta\right) &= \frac{1}{2}ml^2\Omega^2,\\
\implies \frac{1}{2}\dot{\theta}^2 + \frac{g}{l}\left(1-\cos\theta\right) &= \frac{1}{2}\Omega^2,\\
\implies \dot{\theta}^2 &= \Omega^2 - \frac{2g}{l}\left(1-\cos\theta\right).
\end{aligned}$$

Now $\dot{\theta}^2$ is necessarily positive so this means that

$$\Omega^2 - \frac{2g}{l}\left(1-\cos\theta\right) \;\geq\; 0,$$

throughout the motion. Hence we can calculate a maximum value of $|\theta|$ that the pendulum can achieve for a given initial motion. In particular, for the pendulum to overturn we need $\theta = \pi$ (i.e. it reaches the top of the circle), so that $\cos\theta = -1$ at some time t. For this to be possible the above equation must still hold, and so we require

$$\begin{aligned}
\Omega^2 &> \frac{2g}{l}\left(1+1\right),\\
&= \frac{4g}{l},\\
\text{i.e.}\qquad \Omega &> 2\sqrt{\frac{g}{l}}.
\end{aligned}$$

This gives the minimum angular velocity needed initially for the pendulum to overturn. ◄

Activity: *You should now be able to tackle question 3 on this week's problem sheet.*

9.4 Motion constrained to a flat plane

9.4.1 Special case - central force problems

Consider the special case where the particle moves under the action of a central force, such that $\mathbf{F} = F(r)\mathbf{e}_r$ in polar coordinates. As before all the motion will take place in a flat plane, and we have $\dot{\mathbf{r}} = \dot{r}\mathbf{e}_r + r\dot{\theta}\mathbf{e}_\theta$ as in the previous chapter. Thus

$$\mathbf{F} \cdot \dot{\mathbf{r}} = F(r)\dot{r},$$

since $\mathbf{e}_r \cdot \mathbf{e}_\theta = 0$. Hence

$$\begin{aligned}
\frac{d}{dt}\left(\frac{1}{2}m\,|\dot{\mathbf{r}}|^2\right) &= F(r)\frac{dr}{dt}, \\
\implies \frac{1}{2}m\,|\dot{\mathbf{r}}|^2 &= \int F(r)\frac{dr}{dt}dt, \\
&= \int F(r)dr + \text{const},
\end{aligned}$$

by integrating with respect to time. Hence

$$\underbrace{\frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right)}_{\text{K.E.}} \underbrace{- \int F(r)\,\mathrm{d}r}_{\text{P.E.}} = E,$$

where E is a constant, gives conservation of energy. In particular, the gravitational potential energy due to an inverse square law force of the form

$$F(r) = -\frac{GMm}{r^2}$$

where M, m are the masses of two bodies, G is the gravitational constant and r is the distance between them, is given by

$$\begin{aligned}
-\int F(r)\,\mathrm{d}r &= \int \frac{GMm}{r^2}\,\mathrm{d}r, \\
&= -\frac{GMm}{r}.
\end{aligned}$$

Hence conservation of energy gives

$$\frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - \frac{GMm}{r} = \text{const}.$$

NB: The argument for conservation of angular momentum still holds, so we will also have $h = r^2\dot{\theta}$ constant for a central forces problem as before.

Example 19: Rocket

Suppose that a rocket of mass m is launched from the surface of the Earth with velocity $\mathbf{V}$ relative to an inertial frame at the centre of the Earth. Under what condition will the rocket escape from the Earth (neglecting the effect of the atmosphere)?

Solution. Energy will be conserved, so the system will satisfy

$$\begin{aligned} \frac{1}{2}m\dot{\mathbf{r}}^2 - \frac{GMm}{r} &= \text{constant}, \\ &= \frac{1}{2}mV^2 - \frac{GMm}{R}, \end{aligned}$$

where $V = |\mathbf{V}|$ and R is the radius of the Earth, and we have used the initial conditions $\dot{\mathbf{r}} = \mathbf{V}$, $r = R$ at $t = 0$.

For the rocket to escape we need $r \to \infty$, so that the rocket gets infinitely far away from the Earth. We would then have

$$\frac{1}{2}m\dot{\mathbf{r}}^2 = \frac{1}{2}mV^2 - \frac{GMm}{R}.$$

Since $\dot{\mathbf{r}}^2$ is always positive, this means we require

$$\frac{1}{2}mV^2 - \frac{GMm}{R} \geq 0,$$

and so the minimum escape velocity is

$$V_e = \sqrt{\frac{2GM}{R}}.$$

Provided we launch with a velocity above this (in any direction which doesn't hit the ground!) the rocket should escape. ◀

Example 20: Satellite Suppose we launch a satellite with velocity V relative to the centre of the Earth along a circular orbit of a space ship orbiting at a distance D from the centre. If we know that the satellite does not escape the Earth's pull, what is the maximum distance of the satellite from the centre of the Earth? See Figure 9.2.

Solution. Initially, we have $r\dot{\theta} = V$, $\dot{r} = 0$, $r = D$ at $t = 0$. Hence conservation of energy

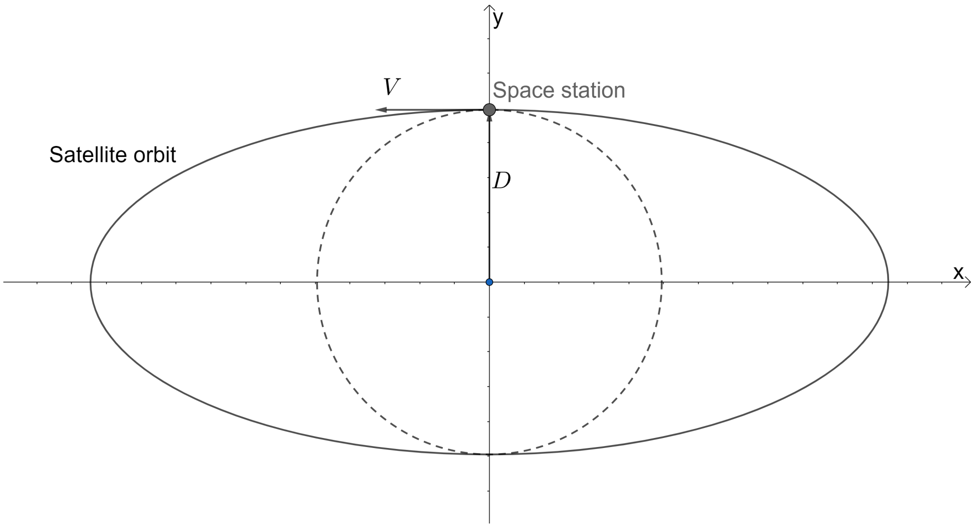

Figure 9.2: Space station orbiting the Earth with a circular orbit of radius D. Launching a satellite with tangential initial velocity V.

gives

$$\begin{aligned}\frac{1}{2}m\dot{\mathbf{r}}^2 - \frac{GMm}{r} &= E, \\ &= \frac{1}{2}mV^2 - \frac{GMm}{D},\end{aligned}$$

so if the satellite doesn't escape we require $V^2 < 2GM/D$ as in the previous example. As this is a central force, we also have conservation of angular momentum, so

$$\begin{aligned}r^2\dot{\theta} &= \text{const}, \\ &= r \cdot r\dot{\theta}, \\ &= DV.\end{aligned}$$

Hence we have

$$\begin{aligned}r^2\dot{\theta}^2 &= \frac{\left(r^2\dot{\theta}\right)^2}{r^2}, \\ &= \frac{h^2}{r^2}, \\ &= \frac{D^2V^2}{r^2},\end{aligned}$$

and so

$$\begin{aligned}\frac{1}{2}m\dot{\mathbf{r}}^2 - \frac{GMm}{r} &= \frac{1}{2}mV^2 - \frac{GMm}{D}, \\ \implies \dot{r}^2 + \frac{D^2V^2}{r^2} - \frac{2GM}{r} &= V^2 - \frac{2GM}{D}.\end{aligned}$$

The maximum value of $r = r_m$ occurs when $\dot{r} = 0$ as the particle has to change the direction of motion. Hence we have

$$\begin{aligned} \frac{D^2V^2}{r_m^2} - \frac{2GM}{r_m} &= V^2 - \frac{2GM}{D}, \\ \Longrightarrow \left(V^2 - \frac{2GM}{D}\right) r_m^2 + 2GMr_m - D^2V^2 &= 0. \end{aligned}$$

This may then be solved by noticing that since $r = D$ and $\dot{r} = 0$ initially, one root must be $r_m = D$ and hence

$$\begin{aligned} 0 &= \left(V^2 - \frac{2GM}{D}\right) r_m^2 + 2GMr_m - D^2V^2, \\ &= (r_m - D)\left(\left(V^2 - \frac{2GM}{D}\right) r_m + DV^2\right), \end{aligned}$$

immediately, or by using the quadratic formula (where the algebra gets a little bit gnarly!):

$$\begin{aligned} r_m &= \frac{-2GM \pm \sqrt{4G^2M^2 - 4D^2V^2\left(V^2 - 2GM/D\right)}}{2\left(V^2 - 2GM/D\right)}, \\ &= \frac{-GM \pm \sqrt{G^2M^2 - D^2V^2\left(V^2 - 2GM/D\right)}}{\left(V^2 - 2GM/D\right)}, \\ &= \frac{-GM \pm \sqrt{\left(GM - DV^2\right)^2}}{\left(V^2 - 2GM/D\right)}, \\ &= \frac{-GM \pm \left(GM - DV^2\right)}{\left(V^2 - 2GM/D\right)}, \end{aligned}$$

noting that $V^2 < 2GM/D$ so we've taken the positive square root. This gives us two values for r_m (one with the plus sign and one with the minus sign); one will give the maximum distance and one the minimum. The positive sign gives

$$r_m = -\frac{DV^2}{V^2 - 2GM/D},$$

and the negative sign

$$\begin{aligned} r_m &= \frac{-2GM + DV^2}{V^2 - 2GM/D}, \\ &= \frac{D\left(V^2 - 2GM/D\right)}{V^2 - 2GM/D}, \\ &= D, \end{aligned}$$

i.e. where the satellite started. Therefore

$$r_m = \frac{D^2V^2}{2GM - DV^2},$$

gives the maximum distance of the satellite from the planet. ◀

Activity: *You should now be able to tackle question 4 on this week's problem sheet.*

Next week we will conclude this "chapter" by considering the motion of a particle on a surface of revolution.

Chapter 10

Energy (motion of a particle on a surface under gravity)

10.1 Introduction

This week we will complete our study of conservation of energy, considering a particle moving in 3D, but constrained to a 2D curved surface, for example if it's sliding on the inside of a bowl or on the end of a spherical pendulum. We will consider the special case where the surface is axisymmetric, in particular where it is a surface of revolution, that is it can be constructed by rotating a curve in (e.g.) the $x-z$ plane about the z axis. At first glance this can seem quite complicated, but it's actually quite formulaic - focus on what you're trying to achieve at each step! This week's problem sheet questions are slightly longer, so the lecture videos are slightly lighter.

10.2 Motion of a particle on a surface under gravity

If a particle moves under the effect of gravity, on a surface in such a way that the reaction[1] between the particle and the surface is normal to the surface (and hence the motion which occurs on the surface), then energy is conserved since:

Newton's second law gives

$$m\ddot{\mathbf{r}} = m\mathbf{g} + \mathbf{R},$$

where $\mathbf{g}$ gives the acceleration due to gravity and $\mathbf{R}$ is the reaction force. We then take the dot product with $\dot{\mathbf{r}}$ to find

$$m\ddot{\mathbf{r}} \cdot \dot{\mathbf{r}} = m\mathbf{g} \cdot \dot{\mathbf{r}} + \mathbf{R} \cdot \dot{\mathbf{r}}.$$

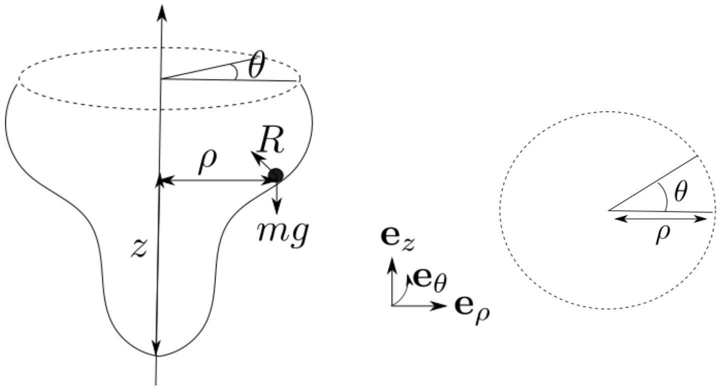

Figure 10.1: A particle on a surface of revolution from the "side" and the "top", showing cylindrical polar coordinates, and the reaction force and gravity acting on the particle.

Now, $\mathbf{R} \cdot \dot{\mathbf{r}} = 0$ since these are perpendicular vectors. We now integrate with respect to time to give

$$\begin{aligned} \frac{1}{2} m \dot{\mathbf{r}}^2 &= m\mathbf{g} \cdot \mathbf{r} + \text{const}, \\ \implies \frac{1}{2} m \dot{\mathbf{r}}^2 - m\mathbf{g} \cdot \mathbf{r} &= E, \end{aligned}$$

where E is the constant conserved energy.

Aside: Cylindrical polar coordinates For these types of problems we need to use cylindrical polar coordinates since the system is axisymmetric about the z axis (say). The surface can be considered to be made up of stacks of circles in the xy plane, with varying radii, piled up in the z direction, and hence we take cylindrical polar coordinates such that $x = \rho \cos\theta$, $y = \rho \sin\theta$, $z = z$ where ρ is the (varying) radius of the circle at a given z height, and θ measures the angle around circle; see Figure 10.1. Note that we've used ρ for the plane polar coordinate (i.e. in the xy plane) to avoid confusion with $\mathbf{r}$, and $r = |\mathbf{r}|$ the position vector.

Hence the position of the particle is given by

$$\mathbf{r} = \rho \mathbf{e}_\rho + z \mathbf{e}_z,$$

where $\mathbf{e}_\rho$, $\mathbf{e}_\theta$ and $\mathbf{e}_z$ are unit vectors in the ρ, θ and z direction respectively, and $|\mathbf{r}| = \sqrt{\rho^2 + z^2}$. The velocity is then given by

$$\dot{\mathbf{r}} = \dot{\rho} \mathbf{e}_\rho + \rho \dot{\theta} \mathbf{e}_\theta + \dot{z} \mathbf{e}_z,$$

with the acceleration

$$\ddot{\mathbf{r}} = \left(\ddot{\rho} - \rho \dot{\theta}^2 \right) \mathbf{e}_\rho + \frac{1}{\rho} \frac{d}{dt} \left(\rho^2 \dot{\theta} \right) \mathbf{e}_\theta + \ddot{z} \mathbf{e}_z,$$

i.e. the same as for plane polar coordinates but with an extra term in the z direction.

10.2.1 Surface of revolution

We are now going to consider motion on a surface of revolution i.e. we take a curve in ρz space and rotate is through 2π in θ to give an axisymmetric surface, with gravity pointing in the z direction. If we rotate a straight line going through the origin we'll get a cone; a semi circle will give a sphere etc (see e.g. https://www.geogebra.org/m/adb4bfbp, https://www.geogebra.org/m/PuBVTMep, or the picture at https://mathworld.wolfram.com/SurfaceofRevolution.html, but don't worry about the maths!).

If we have a particle on a surface of revolution it is subject to gravity and a reaction force $\mathbf{R}$. (See Figure 10.1). If the surface is smooth (i.e. no friction), then the (unknown) reaction is normal to the surface. Then:

- There is no component of $\mathbf{R}$ in the $\mathbf{e}_\theta$ direction, so the $\mathbf{e}_\theta$ component of $\mathbf{F} = m\ddot{\mathbf{r}}$ still gives

$$\begin{aligned} \frac{d}{dt}\left(\rho^2\dot{\theta}\right) &= 0, \\ \implies \qquad \rho^2\dot{\theta} &= \text{const.} \end{aligned}$$

  This is conservation of angular momentum as before.

- The velocity $\dot{\mathbf{r}}$ is parallel to the surface (since the motion is **along** the surface), so $\mathbf{R}\cdot\dot{\mathbf{r}} = 0$ is also zero and hence energy is conserved. This gives

$$\frac{1}{2}m\dot{\mathbf{r}}^2 + mgz = \text{const},$$

  and hence

$$\frac{1}{2}m\left(\dot{\rho}^2 + \left(\rho\dot{\theta}\right)^2 + \dot{z}^2\right) + mgz = E.$$

We will focus on one extended example this week; in particular you should think about what we're trying to do at each stage.

Example 21: Particle motion on a cone Suppose a particle lies on a cone with equation $z = \rho$, with z pointing vertically upwards. Initially the particle is at height $z = a$ with initial velocity V horizontally. How does the particle move and what are the limits on the motion?

Solution. To find the motion we consider the following steps.

Model: We start with the equations of conservation of energy and angular momentum, that is:

$$\frac{1}{2}m\left(\dot{\rho}^2 + \left(\rho\dot{\theta}\right)^2 + \dot{z}^2\right) + mgz = E = \text{constant}$$

and

$$\rho^2\dot{\theta} = h = \text{constant},$$

(as explained above).

We first find the **initial conditions**. We know that $z = a$, $\dot{z} = 0$, $\rho\dot{\theta} = V$ at $t = 0$. We then need to calculate the initial values of ρ and $\dot{\rho}$ since we know the particle lies on the cone $z = \rho$. Note that

$$\dot{\rho} = \dot{z};$$

in this case this is very simple - however, you may have to differentiate implicitly and substitute to find this relationship, remembering that you know how z and ρ are related since the particle lies on the cone (see questions on the problem sheet). Thus

$$\begin{aligned} z = a &\implies \rho = a, \\ \dot{z} = 0 &\implies \dot{\rho} = 0, \end{aligned}$$

at $t = 0$.

Find the constants: We use these initial conditions to find the constants h, the angular momentum, and E, the energy. Firstly

$$\begin{aligned} h = \rho^2\dot{\theta} &= \rho \cdot \rho\dot{\theta}, \\ &= aV. \end{aligned}$$

Secondly

$$\begin{aligned} E = \frac{1}{2}m\left(\dot{\rho}^2 + \left(\rho\dot{\theta}\right)^2 + \dot{z}^2\right) + mgz &= \frac{1}{2}m\left(0 + V^2 + 0\right) + mga, \\ &= \frac{1}{2}mV^2 + mga. \end{aligned}$$

Eliminate variables: Equation (10.1) is one equation for the particle motion in terms of ρ, z and θ, whilst $\rho^2\dot{\theta} = h$ and the equation of the surface ($\rho = z$ in this case) give relationships between ρ and θ, and ρ and z. To be able to solve this we need to rewrite it in terms of one variable only; we do this by eliminating θ and one of ρ or z using conservation of angular momentum and the equation of the surface. It depends on what you're trying to find out, and what your equation looks like as to whether you want to write it in terms of ρ or z. In this case we will eliminate ρ and find an equation for the height of the particle z.

We know $\dot{\rho} = \dot{z}$ from the equation of the surface, and we can find

$$\begin{aligned}
\rho^2\dot{\theta}^2 &= \left(\rho^2\dot{\theta}\right)^2/\rho^2, && \text{[rearrange]} \\
&= h^2/\rho^2, && \text{[cons. of angular momentum]} \\
&= h^2/z^2, && \text{[use equation of surface]} \\
&= a^2V^2/z^2. && \text{[sub in h]}
\end{aligned}$$

Hence we find

$$\begin{aligned}
\frac{1}{2}mV^2 + mga &= \frac{1}{2}m\left(\dot{\rho}^2 + \left(\rho\dot{\theta}\right)^2 + \dot{z}^2\right) + mgz, \\
\implies \frac{1}{2}V^2 + ga &= \frac{1}{2}\left(\dot{z}^2 + \frac{a^2V^2}{z^2} + \dot{z}^2\right) + gz,
\end{aligned}$$

i.e. an equation just in terms of z.

What does this tell us? We now investigate what the model can tell us, this will depend on the question we're trying to answer. First we rearrange to give

$$\begin{aligned}
\dot{z}^2 &= \frac{1}{2}V^2 + ga - \frac{1}{2}\frac{a^2V^2}{z^2} - gz, \\
&= \frac{1}{2}V^2\left(1 - \frac{a^2}{z^2}\right) + g\left(a - z\right),
\end{aligned}$$

this gives the $\dot{z}^2$ term on the left hand side, and we know this is necessarily positive, and also it will be zero at the points where the particle changes the direction of motion.

The question asks for the limits on the motion, so we suspect we want to find the heights between which the particle moves (since this seems the most likely thing to happen!). Since $\dot{z}^2 > 0$, we want to try and factorise the right hand side if possible - this will easily tell us when $\dot{z}$ is zero and hence when the particle changes direction. Hence

$$\begin{aligned}
\dot{z}^2 &= \frac{1}{2}\frac{V^2}{z^2}\left(z^2 - a^2\right) + g\left(a - z\right), \\
&= \frac{1}{2}\frac{V^2}{z^2}\left(z - a\right)\left(z + a\right) + g\left(a - z\right), \\
&= \frac{1}{2}\frac{V^2}{z^2}\left(z - a\right)\left(z + a\right) - g\left(z - a\right), \\
&= \frac{\left(z - a\right)g}{z^2}\left(\frac{V^2}{2g}\left(z + a\right) - z^2\right), \\
&= -\frac{\left(z - a\right)g}{z^2}\left(z^2 - \frac{V^2}{2g}z - \frac{V^2a}{2g}\right).
\end{aligned}$$

Since we knew that $\dot{z} = 0$ at $z = a$ by the initial conditions we were expecting this to be one of the roots. This then leaves us with a quadratic equation $z^2 - \frac{V^2}{2g}z - \frac{V^2a}{2g}$ to factorise. We can write this as

$$z^2 - \frac{V^2}{2g}z - \frac{V^2 a}{2g} = (z - z_1)(z - z_2),$$

where we need to find z_1, z_2 which are the roots of the quadratic equation

$$z^2 - \frac{V^2}{2g}z - \frac{V^2 a}{2g} = 0.$$

Therefore we use the quadratic formula to find:

$$\begin{aligned} z_{1,2} &= \frac{\frac{V^2}{2g} \pm \sqrt{\left(\frac{V^2}{2g}\right)^2 + 4\frac{V^2 a}{2g}}}{2}, \\ &= \frac{V^2}{4g} \pm \sqrt{\frac{V^4}{16g^2} + \frac{V^2 a}{2g}}, \end{aligned}$$

so

$$\begin{aligned} z_1 &= \frac{V^2}{4g} + \sqrt{\frac{V^4}{16g^2} + \frac{V^2 a}{2g}}, \\ z_2 &= \frac{V^2}{4g} - \sqrt{\frac{V^4}{16g^2} + \frac{V^2 a}{2g}}. \end{aligned}$$

We now want to think about whether these roots are physically realistic as the points at which the particle turns around. Note that z_2 must be negative, since the first term is smaller than the square root as

$$\frac{V^4}{16g^2} + \frac{V^2 a}{2g} > \frac{V^4}{16g^2},$$

as $\dfrac{V^2 a}{2g}$ is positive, so

$$\begin{aligned} \sqrt{\frac{V^4}{16g^2} + \frac{V^2 a}{2g}} &> \sqrt{\frac{V^4}{16g^2}}, \\ &= \frac{V^2}{4g}. \end{aligned}$$

But z can't be negative due to the geometry of the cone, so $z - z_2 > 0$.

Then since $\dot{z}^2 \geq 0$ we have

$$\underbrace{\dot{z}^2}_{\geq 0} = \underbrace{-\frac{(z-a)\,g}{z^2}(z - z_1)}_{\geq 0}\underbrace{(z - z_2)}_{\geq 0},$$

and hence

$$(z-a)(z-z_1) \leq 0.$$

This means the particle must lie between $z=a$ and $z=z_1$ (as a positive quadratic it'll be below the x axis (and hence negative) only between those point), since either

$$\begin{aligned} z-a \leq 0 \quad &\text{and} \quad z-z_1 \geq 0, \\ \Longrightarrow \quad & \quad z_1 \leq z \leq a, \end{aligned}$$

or

$$\begin{aligned} z-a \geq 0 \quad &\text{and} \quad z-z_1 \leq 0, \\ \Longrightarrow \quad & \quad a \leq z \leq z_1, \end{aligned}$$

with $\dot{z}=0$ at the maximum/minimum values. The sign of $\ddot{z}$ will give whether the particle is rising or falling. ◀

This concludes the section on conservation of energy.

Activity: *You should now be able to tackle questions 1 and 2 on this week's problem sheet. You may find question 2 easier to tackle than question 1!*

Chapter 11

Variable mass and conservation of momentum

11.1 Introduction

Everything we've studied so far has assumed that the particle has constant mass. This is clearly not realistic in many cases. This week we will relax this assumption, and reconsider some simple examples, in particular using Conservation of Momentum.

11.2 Conservation of Momentum and Newton's Second Law

We are used to Newton's Second Law in the form "Force equals mass times acceleration". This, however, only works when mass is constant. For examples with variable mass we need the more general law of "Conservation of Momentum", where force is equal to the rate of change of (linear) momentum. Recalling that momentum $\mathbf{p}$ is defined via $\mathbf{p} = m\mathbf{v}$ where $\mathbf{v}$ is velocity (and $\mathbf{v} = \dot{\mathbf{r}}$ for a particle with position vector $\mathbf{r}$), we therefore have

$$\mathbf{F} = \frac{d\mathbf{p}}{dt}.$$

Note that when mass is constant this reduces to

$$\begin{aligned} \mathbf{F} &= \frac{d\mathbf{p}}{dt}, \\ &= \frac{d(m\mathbf{v})}{dt}, \\ &= m\frac{d\mathbf{v}}{dt}, \\ &= m\mathbf{a}, \end{aligned}$$

where $\mathbf{a}$ gives the acceleration, as before.

This concept (along with a similar equation representing conservation of mass, saying that the amount of "stuff" you have remains the same) forms the basis for most mechanical models used in a wide variety of applications, from understanding how planes fly, to how cancer tumours grow, to how a chocolate fountain works, and will be used extensively in year 3 modules such as Continuum Mechanics and Advanced Mathematical Modelling.

11.3 No net force

Even if there is no net force acting on the particle, allowing for variable mass can allow changes in motion. For example consider standing on ice (or some other approximation of a zero friction surface e.g. standing on a skateboard or floating in space) and throwing a heavy item away from you. Since momentum over the whole system will be conserved, you will start to move in the opposite direction, despite the fact that no external force has acted on you. Rocket ships use this principle to generate thrust by burning fuel and expelling exhaust gas at high velocity to push the rocket forwards.

11.3.1 Instantaneous mass ejection

When all the mass that is going to ejected is thrown out instantaneously, conservation of momentum can be easily applied to work out the final velocity.

Example 22: Movement on ice

If a person of mass m_p is standing on smooth ice (so there is no friction) holding a heavy object of mass m_o which they push away from them with velocity v_o *relative to their new motion*, what is the resulting motion of the person?

Solution. We assume all motion happens in 1D, with the object thrown in the positive x direction. Since there are no net forces acting on the person, and the person (and hence also the object) is initially at rest, conservation of momentum across the whole system gives

$$m_p v_p + m_o (v_o + v_p) = 0,$$

where v_p is the velocity of the person, by summing the momentum of both the object and the person. Hence rearranging gives the velocity of the person as

$$v_p = -\frac{m_o}{m_p + m_o} v_o,$$

where the minus sign shows they are moving in the opposite direction to that which they threw the object. ◀

11.3.2 Continuous mass ejection

In contrast, when the mass is being ejected continuously (for example a rocket burning fuel), more care is needed as the ejected materal will have a different velocity depending on the time at which it was expelled. In such cases we consider what's happening over a small time δt, and then take the limit as δt tends to zero to find the governing equation. We will consider examples where the motion occurs only in one dimension, but things generalise to more than one dimension in the obvious way. It is also important to be very clear about what frame of reference velocities are being measured in.

Derivation We first derive the **Rocket Equation** for a rocket generating forward thrust by expelling exhaust gases. This equation has wider applicability, but it's easiest to be more specific in the derivation.

At time t let the rocket ship have mass $m(t)$ and be moving with velocity $v(t)$ relative to a fixed inertial frame (i.e. as a fixed observed would see it). The total momentum of the rocket at time t is then given by $P(t) = mv$.

We consider what happens over a short time interval δt, during which the rocket expels a small amount of exhaust gas backwards at a velocity v_{ex} relative to the rocket, or at a velocity $v + \delta v - v_{\text{ex}}$ in the fixed inertial reference frame, where δv gives the change in the velocity of the rocket. This leads to a change in the mass of the rocket which we denote δm (which we know will be negative), so the mass at time $t + \delta t$ will be $m + \delta m$ (see footnote for an alternative method [2]). This will also lead to a change in the velocity of the rocket $v + \delta v$, and hence the momentum of the rocket at time $t + \delta t$ is $P_{rocket}(t + \delta t) = (m + \delta m)(v + \delta v)$.

The expelled fuel also has momentum which is contributing to the overall momentum of the system. The expelled fuel has mass $-\delta m$ and velocity $v + \delta v - v_{\text{ex}}$ so that the fuel momentum at time $t + \delta t$ is given by $P_{fuel}(t + \delta t) = -\delta m(v + \delta v - v_{\text{ex}})$.

Thus the total momentum at time $t + \delta t$ is

$$\begin{aligned} P(t + \delta t) &= P_{rocket}(t + \delta t) + P_{fuel}(t + \delta t), \\ &= (m + \delta m)(v + \delta v) - \delta m(v + \delta v - v_{\text{ex}}). \end{aligned}$$

Since no forces are acting on the rocket, there is no change in momentum over the time δt, and hence

$$\begin{aligned} 0 &= P(t + \delta t) - P(t), \\ &= (m + \delta m)(v + \delta v) - \delta m(v + \delta v - v_{\text{ex}}) - mv. \end{aligned}$$

Now, since δt is small, we also expect δm and δv to be small. Therefore $\delta m \delta v$ is very small, and these terms can be neglected (in fact they cancel in this case anyway). Hence

$$0 = m\delta v + v_{\text{ex}}\delta m$$

since the leading order "mv" terms cancel.

Dividing through by δt and taking the limit as $\delta t \to 0$ we have

$$0 = m\frac{dv}{dt} + v_{\text{ex}}\frac{dm}{dt}.$$

The quantity $-v_{\text{ex}}\frac{dm}{dt}$ is often referred to as the thrust of the rocket.

Example 23: Rocket with no external force

A rocket of mass M_r neglecting fuel blasts off from rest and expels fuel at a constant rate k and velocity v_{ex} relative to the ship. Ignoring the effect of gravity, if the total mass of fuel initially is M_f, what will be the final velocity of the rocket?

Solution. The Rocket equation

$$0 = m\frac{dv}{dt} + v_{\text{ex}}\frac{dm}{dt},$$

will hold as derived above. If the fuel is burnt at a constant rate k, the total mass of fuel m_f will evolve according to

$$\frac{dm_f}{dt} = -k,$$

which we integrate to give

$$m_f = M_f - kt,$$

where M_f gives the initial mass of fuel. The total mass of the rocket is therefore

$$\begin{aligned} m(t) &= M_r + m_f(t), \\ &= M_r + M_f - kt, \end{aligned}$$

by summing the mass of the rocket and the mass of the fuel. This will hold until all the fuel has been burnt such that $m_f = 0$ at $t = M_f/k$.

Thus the rocket equation gives

$$\begin{aligned} 0 &= m\frac{dv}{dt} + v_{\text{ex}}\frac{dm}{dt}, \\ (M_r + M_f - kt)\frac{dv}{dt} &= kv_{\text{ex}}, \end{aligned}$$

since $dm/dt = -k$. We rearrange and integrate with respect to t to solve:

$$\begin{aligned}
\int \frac{dv}{dt}dt &= \int \frac{kv_{\text{ex}}}{M_r + M_f - kt}dt, \\
\implies \int dv &= -v_{\text{ex}} \ln \left(M_r + M_f - kt\right) + \text{constant}, \\
\implies v &= -v_{\text{ex}} \ln \left(M_r + M_f - kt\right) + v_{\text{ex}} \ln \left(M_r + M_f\right), \\
&= v_{\text{ex}} \ln \left(\frac{M_r + M_f}{M_r + M_f - kt}\right),
\end{aligned}$$

using the initial condition that $v = 0$ at $t = 0$.

Now, we run out of fuel when $t = M_f/k$ at which point the velocity is

$$v = v_{\text{ex}} \ln \left(\frac{M_r + M_f}{M_r}\right),$$

giving us the final velocity of the rocket. We see this is controlled by varying either the expulsion speed v_{ex}, or the rocket-to-fuel mass ratio M_f/M_r. The final velocity will be much more sensitive to the expulsion speed than the mass ratio due to the logarithm, so this should be targeted if you want to increase the rocket's final speed. ◀

11.4 Including a net force

If the rocket (for example) is also subject to a net force, most of the derivation above continues to hold. However the change in momentum of the system $P(t + \delta t) - P(t)$ no longer equals zero, but rather changes in momentum will be related to the net force applied $F(t)$ and the length of time over which it is applied δt. Hence instead we have

$$\begin{aligned}
F\delta t &= P(t + \delta t) - P(t), \\
&= m\delta v + v_{\text{ex}}\delta m,
\end{aligned}$$

again neglecting higher order terms of the form $\delta m \delta v$. Dividing by δt we find

$$\begin{aligned}
F &= \frac{P(t + \delta t) - P(t)}{\delta t}, \\
&= m\frac{\delta v}{\delta t} + v_{\text{ex}}\frac{\delta m}{\delta t},
\end{aligned}$$

and letting δt tend to zero gives

$$\begin{aligned}
F &= \frac{dP}{dt}, \\
&= m\frac{dv}{dt} + v_{\text{ex}}\frac{dm}{dt}.
\end{aligned}$$

This is very similar to before, but where the force F can effect changes in the overall momentum of the system.

Example 24: Rocket acted upon by gravity

In the early stages of a rocket flight gravity cannot be neglected. If we take a coordinate system in which gravity points downwards, where a rocket of mass M_r neglecting fuel blasts off from rest and expels fuel at a constant rate k and velocity v_{ex} relative to the ship, the governing equation becomes

$$-mg = m\frac{dv}{dt} + v_{ex}\frac{dm}{dt}.$$

As before $m(t) = M_r + M_f - kt$, and hence

$$-(M_r + M_f - kt)g = (M_r + M_f - kt)\frac{dv}{dt} - kv_{ex}.$$

Rearranging this we get

$$\frac{dv}{dt} = -g + \frac{kv_{ex}}{M_r + M_f - kt}.$$

Integrating this equation and using $v = 0$ at $t = 0$ we find

$$v = -gt + v_{ex}\ln\left(\frac{M_r + M_f}{M_r + M_f - kt}\right).$$

The final velocity is again when $kt = M_f$ and all the fuel has been burnt. This gives

$$v = v_{ex}\ln\left(\frac{M_r + M_f}{M_r}\right) - \frac{gM_f}{k}.$$

This shows that the final velocity in the presence of gravity is always less than the velocity in the absence of gravity. Also, for fixed values of M_r, M_f and v_{ex}, the influence of gravity can be reduced by increasing the value of k (i.e. burning the rocket fuel at a faster rate).

Example 25: Raindrop accumulating mass as it falls through a cloud

Suppose that a raindrop falls through an (assumed stationary) cloud, and accumulates mass at a rate kmv for some constant k, where m is the mass of the raindrop and v is its velocity. If the raindrop starts from rest with initial mass m_0, find its mass and velocity at time t.

Solution. Since the mass being added to the raindrop has zero initial velocity it also has zero initial momentum (note that this is not the case if the added mass is moving with an initial velocity when more care is needed!). Taking gravity to act in the positive direction conservation of momentum gives

$$\begin{aligned} mg &= \frac{d}{dt}(mv), \\ &= m\frac{dv}{dt} + v\frac{dm}{dt}. \end{aligned}$$

Now we know that the mass evolves according to

$$\frac{dm}{dt} = kmv,$$

giving

$$mg = m\frac{dv}{dt} + kmv^2.$$

To solve this we cancel through by m and rearrange to find

$$\frac{dv}{dt} = g - kv^2.$$

This is a separable equation and hence

$$\begin{aligned} \frac{1}{g-kv^2}\frac{dv}{dt} &= 1, \\ \implies \int \frac{1}{g-kv^2}\frac{dv}{dt}dt &= \int dt. \end{aligned}$$

We solve this using partial fractions, first noticing this will be easier is we define $\gamma^2 = g/k(>0)$. Then

$$\begin{aligned} \frac{1}{g-kv^2} &= \frac{1}{k}\frac{1}{g/k - v^2}, \\ &= \frac{1}{k}\frac{1}{\gamma^2 - v^2}, \\ &= \frac{1}{k}\frac{1}{(\gamma - v)(\gamma + v)}, \\ &= \frac{1}{2k\gamma}\left(\frac{1}{(\gamma - v)} + \frac{1}{(\gamma + v)}\right). \end{aligned}$$

Hence

$$\begin{aligned} \int \frac{1}{g-kv^2}\frac{dv}{dt}dt &= \int dt, \\ \implies \int \frac{1}{2k\gamma}\left(\frac{1}{(\gamma - v)} + \frac{1}{(\gamma + v)}\right)dv &= \int dt, \\ \implies \frac{1}{2k\gamma}\left(-\ln(\gamma - v) + \ln(\gamma + v)\right) &= t + \text{constant}. \end{aligned}$$

Using $v = 0$ at $t = 0$ we find

$$t = \frac{1}{2k\gamma} \ln\left(\frac{\gamma + v}{\gamma - v}\right).$$

This is rearranged to give

$$\begin{aligned}
\frac{\gamma + v}{\gamma - v} &= e^{2k\gamma t}, \\
\Longrightarrow \gamma + v &= (\gamma - v)\, e^{2k\gamma t}, \\
\Longrightarrow v\left(1 + e^{2k\gamma t}\right) &= \gamma\left(e^{2k\gamma t} - 1\right), \\
\Longrightarrow v &= \gamma \frac{e^{2k\gamma t} - 1}{1 + e^{2k\gamma t}}, \\
&= \gamma \tanh(\gamma k t), \\
&= \sqrt{\frac{g}{k}} \tanh\left(\sqrt{kg}t\right).
\end{aligned}$$

This gives the velocity of the raindrop.

We can now find the mass, since

$$\begin{aligned}
\frac{dm}{dt} &= kmv, \\
&= km\sqrt{\frac{g}{k}} \tanh\left(\sqrt{kg}t\right), \\
&= m\sqrt{kg} \tanh\left(\sqrt{kg}t\right).
\end{aligned}$$

This is again separable, so we find

$$\begin{aligned}
\int \frac{1}{m}\frac{dm}{dt} dt &= \sqrt{kg} \int \tanh\left(\sqrt{kg}t\right) dt, \\
\Longrightarrow \ln m &= \ln\cosh\left(\sqrt{kg}t\right) + \text{constant}, \\
\Longrightarrow \ln m &= \ln\cosh\left(\sqrt{kg}t\right) + \ln m_0,
\end{aligned}$$

where $m = m_0$ at $t = 0$ is the initial mass. Hence

$$m = m_0 \cosh\left(\sqrt{kg}t\right),$$

gives the evolving mass of the raindrop.

◀

This concludes the section on variable mass, and hence the content of the module!

Activity: *You should now be able to tackle questions 3 and 4 on this week's problem sheet.*

11.5 Summary

In this module we have covered some of the fundamental building blocks and concepts of applied mathematics. We've focussed on Newtonian mechanics, but the techniques and concepts of forming and solving mathematical models are more broadly applicable to a wide range of topics.

In particular, we have covered:

- Units and dimensions.
- Kinematics (position, velocity and acceleration).
- Newton's laws of motion.
- Solving second order linear ODEs with constant coefficients.
- Formulating and solving problems using Newton's second law in one or more dimensions.
- Talked briefly about moving frames of reference.
- Central forces.
- Kepler's laws.
- Formulating problems using polar coordinates, including calculating and using the components of velocity and acceleration in radial and transverse directions.
- Finding the equation of the particle path in terms of u as a function of θ.
- Showing that the angular momentum $h = r^2\dot{\theta}$ is constant.
- Formulating and solving central force problems, in particular using initial conditions to find the constant angular momentum h.
- Bounded orbits and precession (briefly touched on).
- Angular momentum/moment of momentum.
- Conservation of energy in one or more dimensions including:
  - kinetic energy.
  - potential energy for different forces.
- Formulating conservation of energy equations (including using initial conditions and conservation of angular momentum) and using them to find bounds on the motion of a particle.

- Using conservation of energy to describe the motion of a particle on surface under gravity, including on a surface of revolution.
- Using conservation of momentum to consider variable mass problems, including deriving evolution equations by considering the dynamics over a small time frame.

I hope you've enjoyed this module - relevant applied mathematics techniques will be taught in further modules including 2MVA and 2DE, with more modelling content in third year modules such as Advanced Mathematical Modelling and Continuum Mechanics.

Notes

[1]the reaction force is from Newton's third law - the surface pushes back on the particle

[2]This could also be written as $m\left(t+\delta t\right)$ etc. The derivation can then be replicated by Taylor expanding all functions for small δt.

Chapter 12

Appendix: Crib sheet

In this document we will define some general notation and recap/preview some of the fundamental mathematics we will use in this module. **This is not necessarily intended as something for you to work through now, but more as a reference for when needed.** This module will rely heavily on being able to solve second order linear ODEs, so this is something to focus on!

12.1 Notation

Vectors I will use the printed notation $\mathbf{a}$ for a vector, with handwritten notation $\underline{a}$.

Differentiation I will use "dot" to mean differentiation with respect to time, so that

$$\dot{x} = \frac{dx}{dt},$$

and

$$\ddot{x} = \frac{d^2x}{dt^2}.$$

12.2 The quadratic formula

The roots of the quadratic equation

$$ax^2 + bx + c = 0$$

are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

12.3 Solving differential equations

Many problems in mechanics ultimately come down to solving a (system of) differential equation(s) - calculus was originally invented in order to solve questions in mechanics. The details of techniques to solve ODEs and why they work will be covered later in 1RA, but we will frequently need to be able to solve such equations in this module before you have done it in 1RA, so I summarise the key points here:

12.3.1 Separable first order differential equations

Consider the first order differential equation given by

$$\dot{x} = \frac{dx}{dt} = f(x,t).$$

If $f(x,t)$ can be factored as $f(x,t) = g(x)h(t)$ then

$$\frac{dx}{dt} = g(x)h(t).$$

This equation can be solved by bringing all of the x's onto one side and all of the t's onto the other and then integrating

$$\int \frac{1}{g(x)} dx = \int h(t) dt.$$

The method of partial fractions may be useful. Note that it is not always possible to obtain a closed form solution to a separable differential equation (i.e. it might not be possible to calculate the x integral or the t integral analytically). Also, it not always possible to calculate x explicitly as a function of t.

12.3.2 Second-order linear ordinary differential equations with constant coefficients

A linear, second-order differential equation with constant coefficients is an equation of the form

$$a\ddot{x} + b\dot{x} + cx = f(t),$$

where a, b, c are constants, and $x(t)$ is a function of time t. This type of equation very often crops up in mechanics examples, so being able to solve them is essential!

Homogeneous equations

We start by considering the homogeneous case, this is where $f(t) = 0$. Hence we solve

$$a\ddot{x} + b\dot{x} + cx = 0.$$

This is a linear equation, so the solution will be the sum of all possible solutions. We "guess" these will be exponentials such that

$$x = C_1 e^{\lambda t},$$

for some constant C_1 (found using the initial conditions) and some value(s) of λ. We substitute this guess into the equation to find the value(s) of λ that work to solve the equation. As this is a second order equation we expect two values of λ. Hence

$$\begin{aligned} \dot{x} &= \lambda e^{\lambda t}, \\ \ddot{x} &= \lambda^2 e^{\lambda t}, \end{aligned}$$

which gives

$$\begin{aligned} a\lambda^2 e^{\lambda t} + b\lambda e^{\lambda t} + c e^{\lambda t} &= 0, \\ \implies a\lambda^2 + b\lambda + c &= 0. \end{aligned}$$

This is called the characteristic or auxillary equation of the ODE. This can be solved using the quadratic formula to give two solutions for λ such that

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Depending on the values of λ we find, the solutions are of different forms:

1. If we have **two real distinct** roots λ_1, λ_2 ($b^2 - 4ac > 0$) the the solution will be of the form

   $$x = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}.$$

   [Note that $\lambda_i = 0$ gives a constant term.]

2. If we have **one repeated root** λ ($b^2 - 4ac = 0$), then the solution will be of the form

   $$x = C_1 e^{\lambda t} + C_2 t e^{\lambda t}.$$

3. If we have **two complex conjugate** roots of the form $\lambda = \alpha \pm i\beta$, ($b^2 - 4ac < 0$) the the solution will be of the form

   $$x = e^{\alpha_1 t} \left(C_1 \cos(\beta t) + C_2 \sin(\beta t) \right).$$

Once we have found λ_1, λ_2, α, β and the appropriate form of the solution, we use the initial conditions to find C_1, C_2.

Example 26: Homogeneous ODE

Solve $\ddot{x} + \dot{x} - 2x = 0$ subject to $x(0) = 4$, $\dot{x}(0) = -5$.

Solution. The characteristic equation is given by

$$\begin{aligned} \lambda^2 + \lambda - 2 &= 0, \\ \implies (\lambda - 1)(\lambda + 2) &= 0, \end{aligned}$$

to give the roots $\lambda_1 = 1$, $\lambda_2 = -2$ [or use the quadratic formula]. This is two distinct real roots, so hence the solution is of the form

$$x = C_1 e^t + C_2 e^{-2t}.$$

We now use the initial conditions to find $C_{1,2}$. If

- $x(0) = 4$, then
$$C_1 + C_2 = 4. \tag{12.1}$$
- $\dot{x}(0) = -5$ then, since $\dot{x} = C_1 e^t - 2C_2 e^{-2t}$, we have
$$\dot{x}(0) = C_1 - 2C_2 = -5. \tag{12.2}$$

This gives two simultaneous equations for C_1, C_2. Equation (12.1)-(12.2) gives $3C_2 = 9$ and hence $C_2 = 3$. Substituting this into the (12.1) gives

$$\begin{aligned} C_1 + 3 &= 4, \\ \implies C_1 &= 1. \end{aligned}$$

The full solution is hence

$$x = e^t + 3e^{-2t}.$$

◀

Inhomogeneous equations

What about when $f(t) \neq 0$? This gives an inhomogeneous equation.

$$a\ddot{x} + b\dot{x} + cx = f(t).$$

The solution is now split into two parts:

- the solution to the homogeneous equation (as found above, called the "complementary function" x_c),
- a "particular solution" or "particular integral" which satisfies the whole equation, denoted x_p,

so that $x = x_c + x_p$. How to find x_p depends on the form of $f(t)$ - essentially we "guess" a x_p and then find values of the constants when it works. There are rules about what form to try:

	$f(t)$	Trial solution of the form x_p
polynomial of degree n:	$\alpha_0 + \alpha_1 t + \alpha_2 t^2$ $+ \ldots + \alpha_n t^n$	$A_0 + A_1 t + A_2 t^2 + \ldots + A_n t^n$
exponential:	$e^{\alpha t}$	$Ae^{\alpha t}$
trig functions:	$\cos \alpha t$ **or** $\sin \alpha t$	$A \cos \alpha t + B \sin \alpha t$ (**both!**)

then we substitute the trial solution into the equation and find the constants A_0, A_1, A, B etc by equating coefficients of the relevant terms in t.

There are two cases where this must be modified:

1. If $f(t)$ is several of the above functions multiplied together, then use a trial solution which is the relevant trial solutions multiplied together. For example, if $f(t) = t\cos(2t)$, then x_p is of the form

$$x_p = (A_0 + A_1 t)\,(B_1 \cos(2t) + B_2 \sin(2t))\,.$$

2. If your trial solution x_p is already contained in the complementary function x_c, multiply it by t. For example, if $f(t) = e^{2t}$ with $x_c = e^{2t}$, then $x_p = Ate^{2t}$.

Example 27: Inhomogeneous ODE

Solve the equation $\ddot{x} + 4x = 8t^2$.

Solution. The solution is $x = x_c + x_p$ where x_c is the complementary function (the solution to the homogeneous problem) and x_p is the particular integral.

We first solve the homogeneous function $\ddot{x}_c + 4x_c = 0$ by forming the characteristic equation

$$\lambda^2 + 4 = 0,$$

which has solutions $\lambda = \pm 2i$ and hence

$$y_c = C_1 \sin(2t) + C_2 \cos(2t),$$

where C_1, C_2 are constants.

We now look for a particular integral. Since $f(t) = 8t^2$ is a polynomial of degree 2, we guess a function of the form

$$x_p = A_0 + A_1 t + A_2 t^2, \tag{12.3}$$

where we now have to find the right A_i values to make it work. To substitute in we first calculate

$$\begin{aligned} \dot{x}_p &= A_1 + 2A_2 t, \\ \ddot{x}_p &= 2A_2. \end{aligned}$$

Hence we find

$$\begin{aligned} \ddot{x}_p + 4x_p &= 8t^2, \\ \Longrightarrow 2A_2 + 4\left(A_0 + A_1 t + A_2 t^2\right) &= 8t^2. \end{aligned}$$

We now equate coefficients of the t terms to find the correct values of A_i to make the test solution actually solve the equation.

- constant terms: $2A_2 + 4A_0 = 0$
- linear t terms: $4A_1 = 0$
- quadratic t^2 terms: $4A_2 = 8$.

Hence $A_2 = 2$, $A_0 = -1$ and thus $x_p = -1 + 2t^2$ and the full solution is

$$\begin{aligned} x &= x_c + x_p, \\ &= C_1 \sin(2t) + C_2 \cos(2t) - 1 + 2t^2. \end{aligned}$$

We would then use initial conditions to find C_1 and C_2. ◀

12.4 Vectors

Another crucial tool for mechanics are concepts using vectors.

Vectors have both length/magnitude and direction. They are typically identified in written mathematics with a bold font in typeset work (e.g. **a**) or underlined in handwritten work (e.g. $\underline{a}$). Distinguishing between vectors and scalars is crucial - r and $\mathbf{r}$ are not the same thing! Equations must be consistent, i.e. in a given equation every term is a scalar, or every term is a vector.

We will use $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ as the standard Cartesian vectors, i.e. unit vectors in the x, y, z directions respectively. The magnitude (i.e. length) of a vector $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} = (a_1, a_2, a_3)$ is denoted by $|\mathbf{a}|$ and given by $\sqrt{a_1^2 + a_2^2 + a_3^2}$.

The **scalar or dot product** between two Cartesian vectors $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} = (a_1, a_2, a_3)$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k} = (b_1, b_2, b_3)$ is given by

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3. \tag{12.4}$$

Hence $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$. We can also write this as

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos\theta,$$

where θ is the angle between the vectors $\mathbf{a}$ and $\mathbf{b}$. If $\mathbf{a}$ and $\mathbf{b}$ are perpendicular then $\mathbf{a} \cdot \mathbf{b} = \mathbf{0}$.

The **vector or cross product** is calculated (in Cartesians) as

$$\begin{aligned} \mathbf{a} \times \mathbf{b} &= \mathbf{i}\,(a_2 b_3 - b_2 a_3) - \mathbf{j}\,(a_1 b_3 - b_1 a_3) + \mathbf{k}\,(a_1 b_2 - b_1 a_2) && (12.5) \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}. && (12.6) \end{aligned}$$

The vector $\mathbf{a} \times \mathbf{b}$ is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$, with $\mathbf{a} \times \mathbf{b} = 0$ when $\mathbf{a}$ and $\mathbf{b}$ are parallel.

12.5 Geometry

Again, some of this may be unfamiliar until it is covered in VGLA later this term. We may use some of these ideas before they have been fully covered, but I'll give you enough detail for what you need at the time!

12.5.1 Equation of a plane

The equation of a plane consisting of points with position vector $\mathbf{r}$, with normal $\mathbf{n}$ is given by $\mathbf{r} \cdot \mathbf{n} = d$ for some constant d.

12.5.2 Equation of a circle

In (x, y) Cartesian coordinates, a circle, centre (a, b) of radius R is given by

$$(x-a)^2 + (y-b)^2 = R^2.$$

12.5.3 Parabola

In (x, y) Cartesian coordinates, an equation of the form

$$y = (x-a)^2 + b,$$

gives a parabola with a minimum at (a, b). Flip the sign to change from a positive parabola with a minimum to a negative parabola with a maximum.

12.5.4 Curve sketching

To sketch a curve you should think about (for example, as appropriate):

- The domain that is appropriate.
- Where the function intercepts the axes.
- What happens for large values?
- Does it have any maxima or minima or is it monotonic?
- Does the function increase or decrease?
- ...

You can also use curve sketching software such as GeoGebra, Desmos or Maple (plenty of others are available!).

12.5.5 Conics

A conic section is the shape you get when you take the intersection of a plane with a cone. Depending on the angle of your plane this could be an ellipse, a parabola or a hyperbola. A nice demonstration of these is available at `https://www.geogebra.org/m/GmTngth7#material/T8TV2JqG`. You will learn about these in VGLA, but not until the end of this semester!

Conics are constructed with a focus (which is a point), a directrix (a line) and have a property called eccentricity (a positive number). A general conic with focus at $(0,0)$ has a polar (i.e. radius r as a function of angle θ) representation

$$r(\theta) = \frac{e\hat{d}}{1 \pm e\cos\theta},$$

with a vertical directrix $x = \pm\hat{d}$ ($\hat{d} > 0$) or

$$r(\theta) = \frac{e\hat{d}}{1 \pm e\sin\theta},$$

with a horizontal directrix $y = \pm\hat{d}$ ($\hat{d} > 0$). The plus/minus sign is chosen to ensure that $\hat{d}$ is positive. The eccentricity e determines which shape the conic is, such that

- $0 < e < 1$ gives an ellipse,
- $e = 1$ gives a parabola,
- $e > 1$ gives a hyperbola.

When $e = 0$, we have the special case of a circle. In this case the polar representation is simply $r = R$ where R is a constant (i.e. the radius is constant).

You will learn about conic sections in VGLA, but at the moment you just need to be able to recognise the shape from the polar representation.

12.5.6 Taylor Series

The Taylor series of the function $y = f(x)$ about the point $x = c$ is

$$y = f(c) + f'(c)(x-c) + \frac{1}{2!}f''(c)(x-c)^2 + \frac{1}{3!}f'''(c)(x-c)^3 + \frac{1}{4!}f''''(c)(x-c)^4 + \dots$$

where $'$ denotes d/dx.

12.6 Binomial approximation

The binomial approximation is given by

$$(1+x)^n = 1 + nx + \frac{1}{2}n(n-1)x^2 + \dots,$$

for small x. This can be shown by applying a Taylor series expansion about $x = 0$.

12.7 GCSE/High school science concepts

The following concepts from GCSE science will come up:

Hooke's law The tension in a spring is proportional to its extension, i.e. the difference between its current and natural length. The constant of proportionality is called the spring constant.

The moment of a force A moment gives the turning effect of a force and is taken about a point (typically a pivot - think about a seesaw). The moment is given by Fd where F is the force, and d is the perpendicular distance from the pivot to the line of action.