Week 11: Filter circuits

- **Aims and Objectives**

  In this lecture we will use complex impedance to study the application of RC and RL circuits as filters.

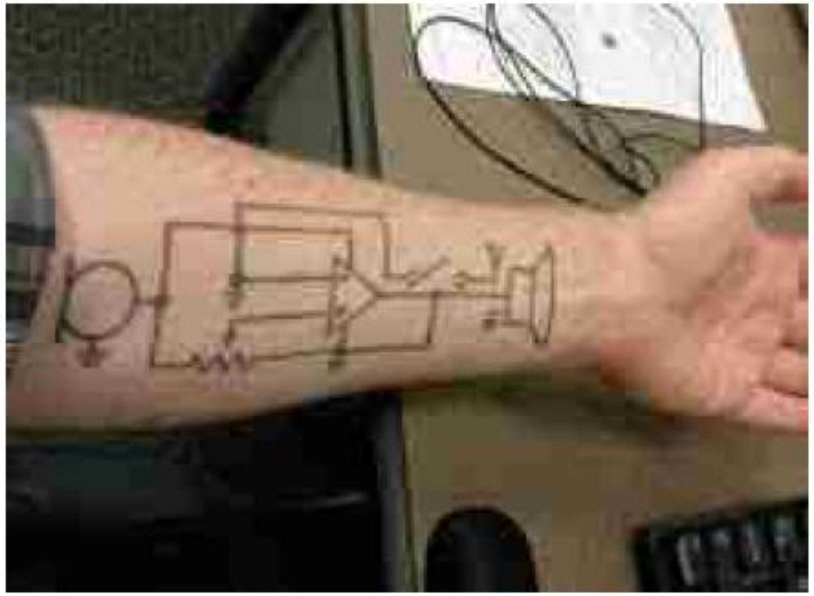
- **Material to be covered**

  The decibel scale

  RC low pass filter

  RC high pass filter

  Bode plots



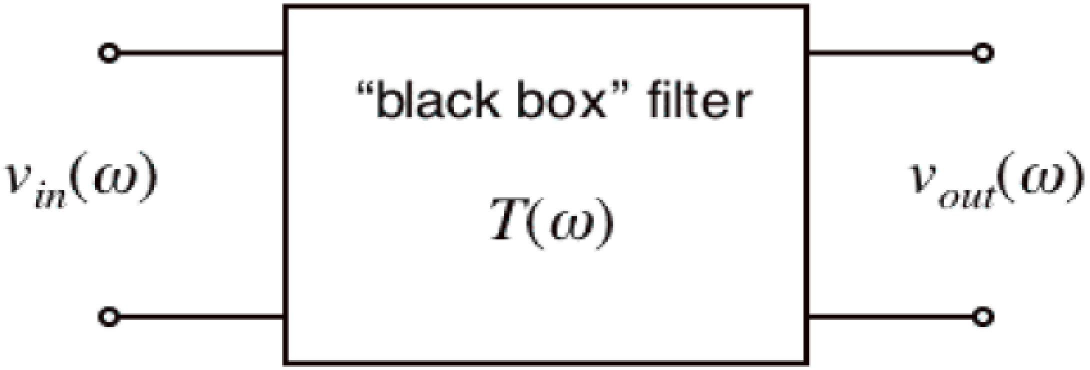
Filter circuits

What is a filter circuit?

- A circuit that transmits signals (voltages) at frequencies of interest, while blocking other frequencies.

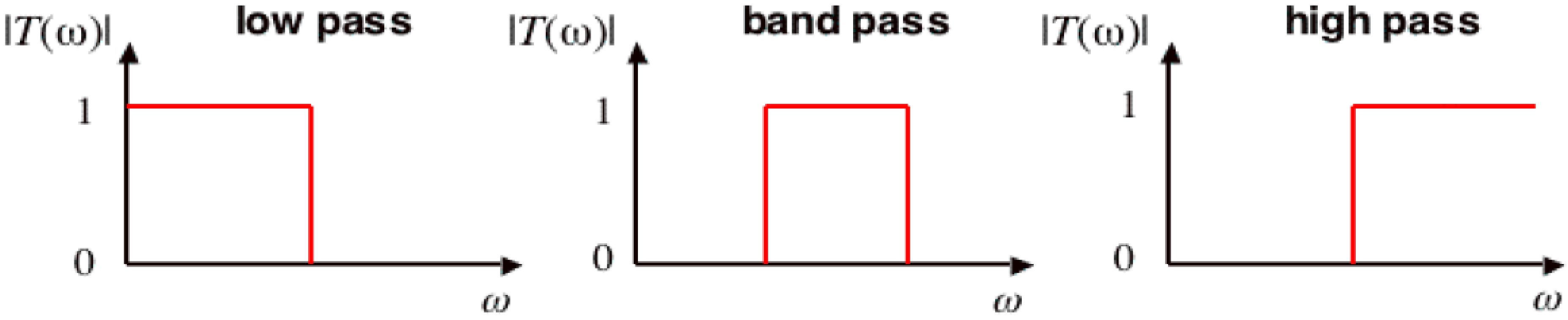
Filter circuits

- **A general filter circuit**



- **The filter is defined by its transfer function,** $T(\omega)$

$$v_{out} = T(\omega)v_{in} \qquad T(\omega) = \frac{v_{out}}{v_{in}}$$



Decibel scale

- **The decibel scale is defined in terms of power ratios**

The gain (or loss) is defined as

$$\text{gain or loss in dB} = 10\log_{10}\left(\frac{P_1}{P_2}\right)$$

Since $P = V^2/R$ for a resistor

$$\text{gain or loss in dB} = 20\log_{10}\left(\frac{V_1}{V_2}\right)$$

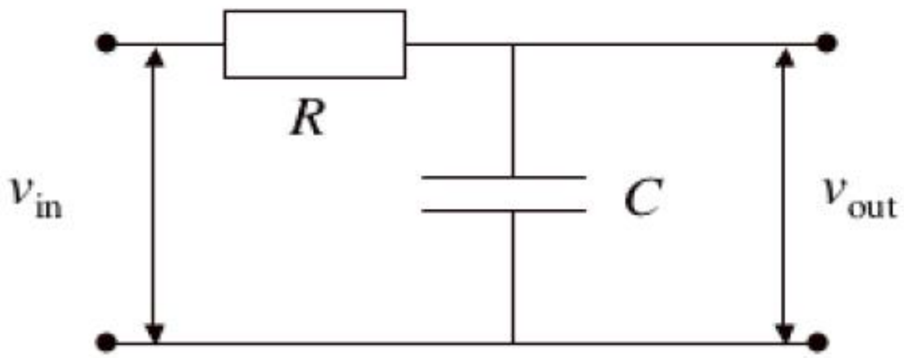
A power gain or loss of ×2 corresponds to ±3 dB

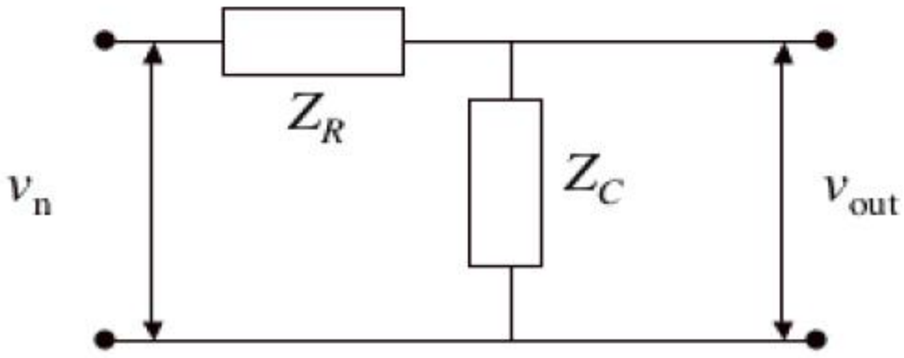
X10 corresponds to ±10 dB

$\frac{V_2}{V_1}$	dB
1000	60
100	40
10	20
$\sqrt{2}$	3.01
1	0
$1/\sqrt{2}$	–3.01
0.1	–20
0.01	–40
0.001	–60

RC low pass filter

- **A generalised a.c. potential divider**





$$\frac{v_{out}}{v_{in}} = \frac{Z_c}{Z_c + Z_R} = \frac{1/j\omega C}{1/j\omega C + R} = \frac{1}{1 + j\omega CR}$$

Find the modulus and argument

$$\left|\frac{v_{out}}{v_{in}}\right| = \frac{1}{\sqrt{1+(\omega CR)^2}} = \frac{1}{\sqrt{1+\left(\omega/\omega_0\right)^2}}$$

$$\omega_0 = 1/CR = 1/\tau$$

$$\phi = -\tan^{-1}(\omega CR)$$

As $\omega \to 0$ ($\omega \ll \omega_0$)

$$\left|\frac{v_{out}}{v_{in}}\right| \to 1, \quad \phi = 0, \quad \text{gain} = 0\,\text{dB}$$

When $\omega = \omega_0$

$$\left|\frac{v_{out}}{v_{in}}\right| = \frac{1}{\sqrt{2}}, \quad \phi = -\frac{\pi}{4}, \quad \text{gain} = -3\,\text{dB}$$

As $\omega \to \infty$ ($\omega \gg \omega_0$)

$$\left|\frac{v_{out}}{v_{in}}\right| \propto \frac{1}{\omega}, \quad \phi = -\frac{\pi}{2}, \quad \text{gain} = -20\,\text{dB/decade}$$

Frequency dependence

- At low frequency ($\omega \to 0$), the magnitude of the **transfer function** is $|T(\omega)| \approx 1$, so that $|v_{out}| = |v_{in}|$. The phase angle is zero, $\phi \approx 0$, so v_{out} is in phase with v_{in}. Thus we have demonstrated that low frequency signals are passed unattenuated and with no change of phase.

- At high frequency ($\omega \to \infty$), the magnitude of the transfer function tends to zero, and v_{out} **lags** v_{in} by 90 degrees. At frequencies much larger than ω_0, the transfer function is proportional to the reciprocal of the angular frequency.

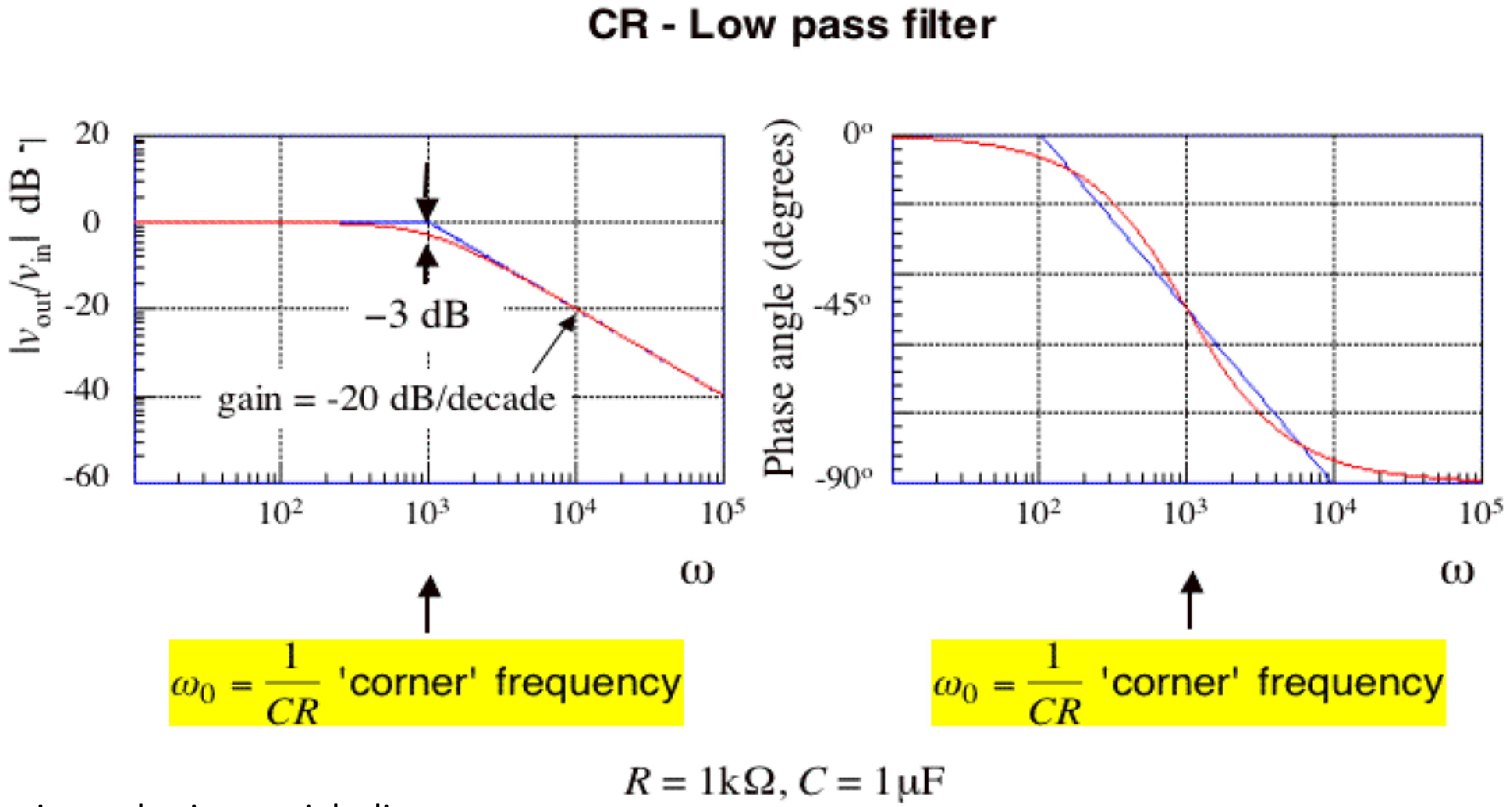
$$\left|T\left(w \gg w_0\right)\right| \to \frac{w_0}{w}$$

  This means that for every factor of 10 (or decade) increase in frequency, the ratio $|V_{out}|/|V_{in}|$ drops by another factor of 10 (or ~20 dB). In short, we say that the transfer function falls off at ~20 dB

- When $\omega = \omega_0$, $|v_{out}|/|v_{in}| = 1/\sqrt{2}$ (or –3 dB) and $\phi = \tan^{-1}(1) = 45$ degrees

RC low pass filter response

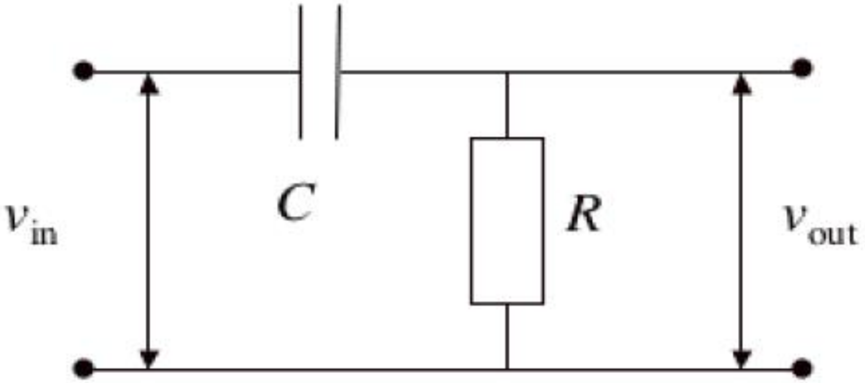
- **Bode plot**

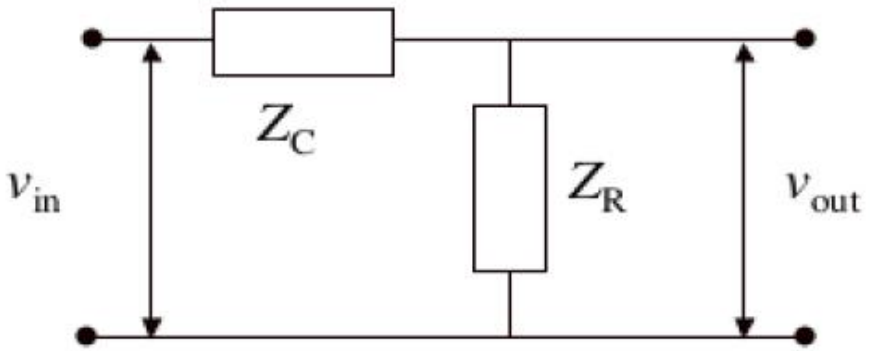


Also approximated using straight line segments.

High pass filter

- **A generalised a.c. potential divider**





$$\frac{v_{out}}{v_{in}} = \frac{Z_R}{Z_R + Z_C} = \frac{R}{R + 1/\mathrm{j}\omega C} = \frac{\mathrm{j}\omega CR}{1 + \mathrm{j}\omega CR}$$

Find the modulus and argument.

$$\left|\frac{v_{out}}{v_{in}}\right| = \frac{\omega CR}{\sqrt{1 + (\omega CR)^2}} = \frac{\omega/\omega_0}{\sqrt{1 + (\omega/\omega_0)^2}}$$

$$\omega_0 = 1/CR$$

$$\phi = \frac{\pi}{2} - \tan^{-1}(\omega CR)$$

As $\omega \to 0$ $(\omega \ll \omega_0)$

$$\left|\frac{v_{out}}{v_{in}}\right| \propto \omega, \quad \phi = \frac{\pi}{2}, \quad \text{gain} = +20\,\text{dB/decade}$$

When $\omega = \omega_0$

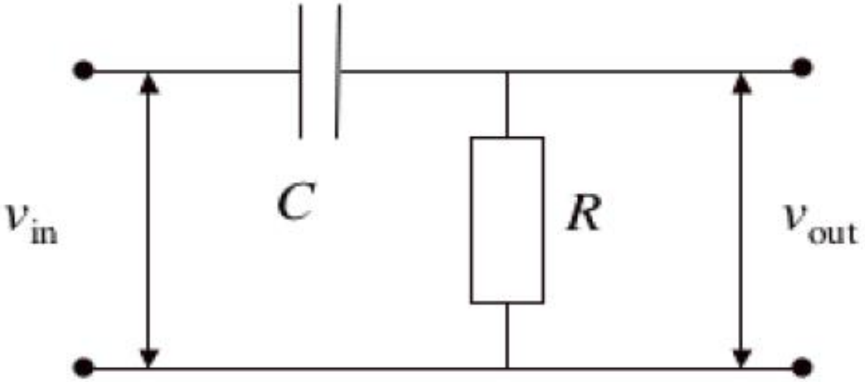
$$\left|\frac{v_{out}}{v_{in}}\right| = \frac{1}{\sqrt{2}}, \quad \phi = \frac{\pi}{4}, \quad \text{gain} = -3\,\text{dB}$$

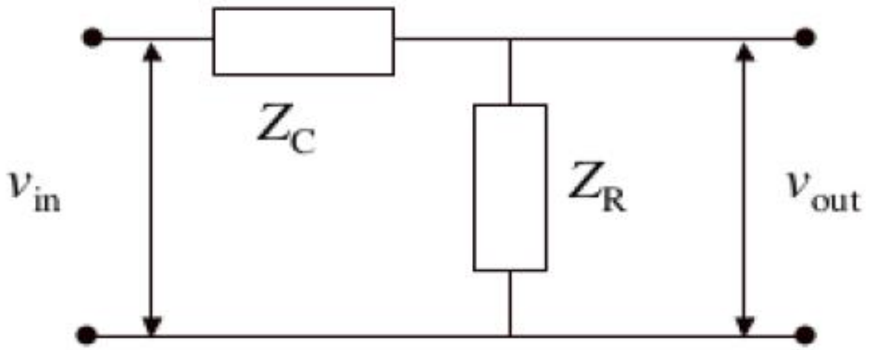
As $\omega \to \infty$ $(\omega \gg \omega_0)$

$$\left|\frac{v_{out}}{v_{in}}\right| \to 1, \quad \phi = 0, \quad \text{gain} = 0\,\text{dB}$$

High pass filter

- A generalised a.c. potential divider





$$\frac{v_{out}}{v_{in}} = \frac{Z_R}{Z_R + Z_C} = \frac{R}{R + 1/\mathrm{j}\omega C} = \frac{\mathrm{j}\omega CR}{1 + \mathrm{j}\omega CR}$$

Find the modulus and argument.

$$\left|\frac{v_{out}}{v_{in}}\right| = \frac{\omega CR}{\sqrt{1+(\omega CR)^2}} = \frac{\omega/\omega_0}{\sqrt{1+\left(\omega/\omega_0\right)^2}}$$

$$\tan^{-1}(1/x) = \frac{\pi}{2} - \tan^{-1} x$$

$$\omega_0 = 1/CR$$

$$\phi = \frac{\pi}{2} - \tan^{-1}(\omega CR)$$

As $\omega \to 0$ $(\omega \ll \omega_0)$

$$\left|\frac{v_{out}}{v_{in}}\right| \propto \omega, \quad \phi = \frac{\pi}{2}, \quad \text{gain} = +20\,\text{dB/decade}$$

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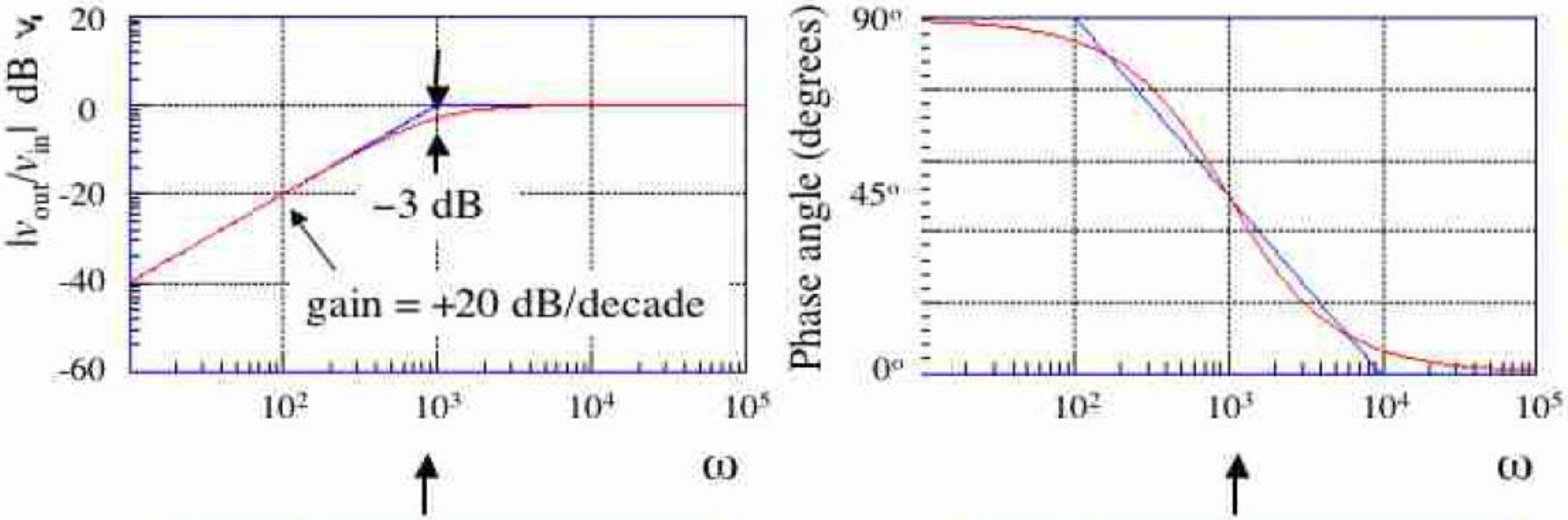
High pass filter response

- **Bode plot**

RC - High pass filter



$\omega_0 = \dfrac{1}{CR}$ 'corner' frequency

$\omega_0 = \dfrac{1}{CR}$ 'corner' frequency

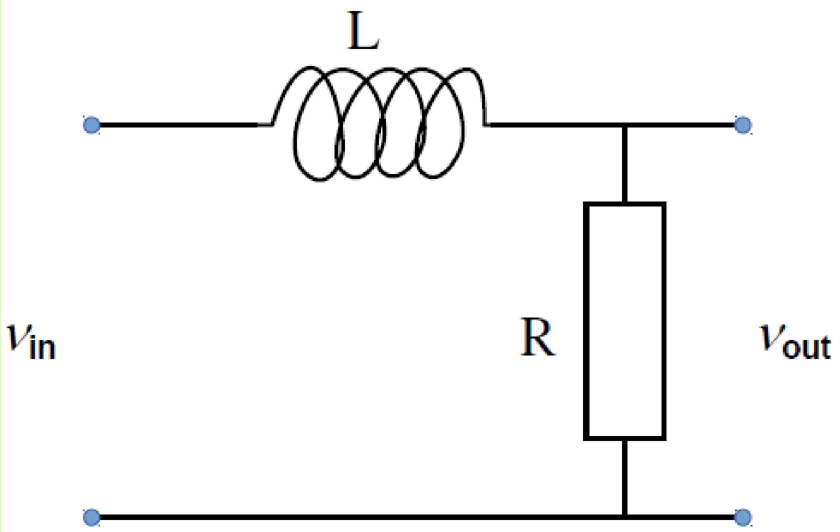
$R = 1\text{k}\Omega,\ C = 1\,\mu\text{F}$

Try it yourself: RL filters

- In order to help you get to grips with this week's material, and get used to working with complex numbers, I would like you to repeat this exercise replacing the capacitor in Slides 6 and 9 with an inductor. You should be able to derive the expressions summarised on the next slide. Notice that when you make the substitution $\boxed{\omega_0 = R/L}$ you are left with exactly the same expression for the low and high pass RL filter as we found for RC filters.
- Once again, ω_0 is just the reciprocal of the time constant of the circuit. You can therefore build an equivalent filter using either an RC circuit or an RL circuit. However, the filters will have different input impedances in the limit of low and high frequency. The choice of circuit is usually determined by which configuration best matches the input, based upon a consideration of the maximum power transfer that we encountered earlier in this course.

RL filters

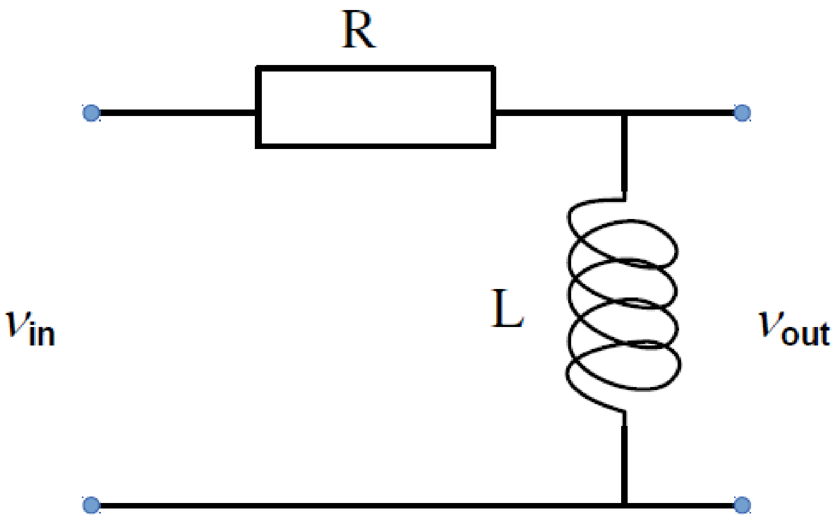
Low pass filter



$$|T(\omega)| = \frac{R}{\sqrt{R^2 + (\omega L)^2}} = \frac{1}{\sqrt{1 + (\omega/\omega_0)^2}}$$

$$\phi = \tan^{-1}(-\omega\, L/R) = \tan^{-1}(-\,\omega/\omega_0)$$

High pass filter



$$|T(\omega)| = \frac{\omega L}{\sqrt{R^2 + (\omega L)^2}} = \frac{\omega/\omega_0}{\sqrt{1 + (\omega/\omega_0)^2}}$$

$$\phi = \frac{\pi}{2} - \tan^{-1}(\omega\, L/R) = \frac{\pi}{2} - \tan^{-1}(\omega/\omega_0)$$