University of Birmingham
School of Mathematics

1SAS Sequences and Series Autumn 2024

Problem Sheet 1
(Issued Week 1)

Q1. Recall the following definition from the lectures:

Definition. *A sequence of real numbers (a_n) tends to infinity if given any real number $A > 0$ there exists $N \in \mathbb{N}$ such that*

$$a_n > A \quad \text{for all } n > N.$$

(i) Using the definition prove that the sequence (a_n) given by

$$a_n = \sqrt{n}$$

tends to infinity.

(ii) Using the definition prove that the sequence (b_n) given by

$$b_n = \frac{n^4 + n^2 + 4}{3n^3 + 2n + 1}$$

tends to infinity.

Q2. Recall the following definition from the lectures:

Definition. *A sequence (a_n) of real numbers converges to a real number ℓ if given any $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that*

$$|a_n - \ell| < \varepsilon \quad \text{for all} \quad n > N.$$

A sequence (a_n) converges if it converges to ℓ for some real number ℓ.

(i) Using the definition prove that the sequence (a_n) given by

$$a_n = \frac{n+2}{n+1}$$

converges to 1.

(ii) Using the definition prove that the sequence (b_n) given by

$$b_n = \frac{n + (-1)^n}{2n + (-1)^n}$$

converges.

Q3. Prove (using the definition) that the sequence (a_n) tends to infinity in each of the following cases:
(i) $a_n = n^{1/4}$; (ii) $a_n = \frac{n^4+4n^3+1}{n^3+2}$; (iii) $a_n = (2 + (-1)^n)n$; (iv) $a_n = \left(n + \frac{1}{n}\right)^3 - n^3$.

Q4. For each of the following sequences (a_n) and values of ℓ, prove (using the definition) that $a_n \to \ell$:

(i) $a_n = \frac{1+(-1)^n}{n}$, $\quad \ell = 0$;

(ii) $a_n = \frac{n^2+4\sin(n)}{2n^2+3}$, $\quad \ell = \frac{1}{2}$.

Q5. (i) Prove that if $x, y > 0$ then

$$\sqrt{x} - \sqrt{y} = \frac{x-y}{\sqrt{x}+\sqrt{y}}.$$

(ii) Using the definition of convergence, prove that the sequence (a_n) given by

$$a_n = \sqrt{n+1} - \sqrt{n}$$

converges to 0.

(iii) Does the sequence (b_n) given by

$$b_n = \sqrt{n^2+n} - n$$

converge? If so, what does it converge to? Justify your assertions.

Extra Questions

These are some additional questions that you may find helpful, either now or at a later date.

EQ1. Consider the sequence

$$1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \frac{1}{5}, \cdots,$$

which has nth term $a_n = \frac{(-1)^{n+1}}{n}$.

(i) Find a natural number N for which $a_n \in (-\frac{1}{10}, \frac{1}{10})$ for all $n > N$.
[Here $(-\frac{1}{10}, \frac{1}{10})$ denotes the interval $\{x \in \mathbb{R} : -\frac{1}{10} < x < \frac{1}{10}\}$.]

(ii) Find a natural number N for which $a_n \in (-\frac{1}{1000}, \frac{1}{1000})$ for all $n > N$.

(iii) Let ε be *any* positive real number. Find a natural number N for which $a_n \in (-\varepsilon, \varepsilon)$ for all $n > N$. What does this prove?

EQ2. Let the sequence (a_n) be given by

$$a_n = \sum_{k=1}^{n} \frac{1}{\sqrt{n+k}}.$$

(i) Show that $a_n \geq \sqrt{\frac{n}{2}}$ for all $n \in \mathbb{N}$.

(ii) By appealing to the appropriate definition, deduce that $a_n \to \infty$.

EQ3. Suppose that $\ell > 0$ and that $a_n \to \ell$. Prove that there exists $N \in \mathbb{N}$ such that $a_n > 0$ for all $n > N$.

EQ4. Prove that if $a_n \to \infty$ and $b_n \to \infty$ then (a) $a_n + b_n \to \infty$, and (b) $a_n b_n \to \infty$.

EQ5. Suppose that $N_0 \in \mathbb{N}$ and that $a_n \geq b_n$ for all $n > N_0$. Prove that if $b_n \to \infty$ then $a_n \to \infty$.

EQ6. Give explicit examples of sequences (a_n) and (b_n), satisfying $a_n \to \infty$ and $b_n \to 0$, for which

(i) $a_n b_n \to 1$.
(ii) $a_n b_n \to 0$.
(iii) $a_n b_n \to \infty$.
(iv) $a_n b_n \to -\infty$.
(v) the sequence $(a_n b_n)$ neither converges nor tends to $\pm\infty$.
(vi) $b_n > 0$ for all $n \in \mathbb{N}$, and the sequence $(a_n b_n)$ neither converges nor tends to $\pm\infty$.

[Here $(a_n b_n)$ is the sequence whose nth term is the product $a_n b_n$.]

EQ7. Give explicit examples of sequences (a_n) and (b_n), satisfying $a_n \to \infty$ and $b_n \to \infty$, for which

(i) $a_n - b_n \to 0$,
(ii) $a_n - b_n \to 1$,
(iii) $a_n - b_n \to \infty$,
(iv) $a_n - b_n \to -\infty$,
(v) the sequence $(a_n - b_n)$ neither converges nor tends to $\pm\infty$.

[Here $(a_n - b_n)$ is the sequence whose nth term is the difference $a_n - b_n$.]

EQ8. Suppose that (a_n) is a sequence of positive real numbers converging to $\ell > 0$.

(i) Prove that
$$|\sqrt{a_n} - \sqrt{\ell}| \leq \frac{|a_n - \ell|}{\sqrt{\ell}}$$
for all $n \in \mathbb{N}$.
(ii) Using Part (i), or otherwise, prove that $\sqrt{a_n} \to \sqrt{\ell}$ as $n \to \infty$.
(iii) Deduce that
$$\frac{\sqrt{n^4 + 4n}}{n^2 + 1} \to 1,$$
justifying any assertions that you make.

EQ9.

(i) For which values of $n \in \mathbb{N}$ is it true that $2^n \geq n$? Prove your assertion.
Using your result, or otherwise, prove that
$$2^n \to \infty.$$
(ii) For which values of $n \in \mathbb{N}$ is it true that $2^n \geq n^2$? Prove your assertion.
Using your result, or otherwise, prove that
$$\frac{2^n}{n} \to \infty.$$
(iii) Outline a potential strategy for proving that
$$\frac{2^n}{n^k} \to \infty$$
for all $k = 0, 1, 2, 3, \ldots$.