1VGLA-1RA Problem Sheet Weeks 1 & 2

These questions relate to the introductory lectures in the first two weeks of semester. We will not have covered all of the material until the end of week two. The work will be marked but is formative, so does not count towards your module marks. Try as many as you can and hand in your solutions. Some of your work will be marked.

Question 1. (a) Prove that for each $0 < n \in \mathbb{N}$, the sum of the first n positive odd integers is n^2.

(b) Red wooden blocks are 4 cm high, blue wooden blocks are 5 cm high. You can have as many blocks of either colour as you need. Prove that there is a number N such that for any $n \geqslant N$, you can make a tower of blocks exactly n cm tall. What is N?

(c) Comment on the following argument. There are many breeds of dog but for any n, in any pack of n dogs all of the dogs are the same breed. To prove this, as a base step note that if there is one dog in a pack, then all of the dogs in that pack are the same breed. For the inductive step, suppose that it is true that in any pack of k dogs, all of the dogs are the same breed. Suppose now that you have a pack of $k+1$ dogs. Take one dog, Sookie say, for a walk. You are left with a pack of k dogs, so by hypothesis, they are all the same breed. Bring Sookie back and take Archie for a walk. Again you are left with a pack of k dogs, which are all the same breed. So Sookie and Archie must be the same breed as well and all dogs in the pack are the same breed. The result follows by induction.

Question 2. Solve the following inequalities.

(a)

$$|x+2| > x^2 - 1.$$

(b)

$$\frac{x^2-1}{x^3+x^2-6x} \leqslant 0$$

(c)

$$\big||x|-3\big| > 1.$$

(d)

$$\frac{|x|-1}{|x|-2} \geqslant 0.$$

Question 3. Let A, B and C be sets. Prove the following.

(a) $(A \cap B) \cap C = A \cap (B \cap C)$.

(b) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

(c) $A - (B \cap C) = (A - B) \cup (A - C)$.

Question 4 (Challenge question)**.** (a) Prove that the following defines an ordered pair, that is to say $(x, y) = (w, z)$ if and only if $x = w$ and $y = z$:

$$(x, y) := \{\{x\}, \{x, y\}\}\,.$$

(Remember that two sets are equal if and only if they have exactly the same elements.)

(b) Let X be a set and let $A = \{x \in X : x \notin x\}$. Is A a subset of X? Is A an element of X? If it is, is it an element of itself or not? Deduce that there is no set of all sets (and hence that the naïve approach to set theory does not work).

(c) Let A, B and C be sets. Show that $A \cup B \cup C = \bigcup\{A, B, C\}$.

(d) Let $\mathcal{C}$ be the collection of all open intervals $(-r, r)$, for $0 < r \in \mathbb{R}$. What are $\bigcap \mathcal{C}$ and $\bigcup \mathcal{C}$.

(e) Let $\mathbb{P}$ denote the set of prime numbers. Find a collection of subsets, $\mathcal{D}$, of the natural numbers $\mathbb{N}$ such that $\bigcup \mathcal{D} = \mathbb{N}$ and $\bigcap \mathcal{D} = \mathbb{P}$.

Question 5. (a) Let g be the rule that assigns to each real number x the digit in the third decimal place when x is expressed as a decimal. Is g a well-defined rule?

(b) The function $f : \mathbb{R} \to \mathbb{R}$ is defined by the rule $f : x \mapsto 9^x$, for all $x \in \mathbb{R}$. Find the outputs of f corresponding to the inputs: (i) -3, (ii) $-5/2$, (iii) $-1/2$, (iv) $1/2$, (v) $3/2$. (vi) What input has 81 as its output? (vii) Are 0, 1 outputs of f? (viii) What is the image of f?

(c) Give examples of functions such that: i) the domain is not equal to the codomain; ii) the domain is not equal to the image; iii) the codomain is not equal to the image.

(d) Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ that is not onto (surjective). Is your example injective?

(e) Give an example of a function $g : \mathbb{R} \to \mathbb{R}$ that is not one-to-one (injective). Is your example surjective?

(f) Let $f : X \to Y$. Prove that the following are equivalent:

(i) f is injective;

(ii) $x = x'$ whenever $f(x) = f(x')$;

(iii) $f(x) \neq f(x')$ whenever $x \neq x'$.

Question 6. (a) Suppose that $f : X \to Y$ and $g : Y \to Z$ are functions. Show that the composition $g \circ f : X \to Z$ defined by the rule $g \circ f(x) = g\big(f(x)\big)$ is, indeed, a function from X to Z.

(b) Find examples of two functions $f, g : \mathbb{R} \to \mathbb{R}$ such that $f \circ g$ and $g \circ f$ are not equal.

(c) Is it possible for $g \circ f$ to be a bijection even though neither f nor g are bijections. If so, then what can you say about f and g?