UNIVERSITY OF BIRMINGHAM

School of Physics and Astronomy

DEGREE OF B.Sc. & M.Sci. WITH HONOURS

FIRST YEAR EXAMINATION

03 19749

LC SPECIAL RELATIVITY/PROBABILITY AND RANDOM PROCESSES

SUMMER EXAMINATIONS 2019

Time Allowed: 1 hour 30 minutes

Answer Section 1 and two questions from Section 2.

Section 1 counts for 40% of the marks for the examination.
Full marks for this Section can be obtained by correctly answering ***four*** questions.
You may attempt more questions, but marks in excess of 40% will be disregarded.

Section 2 consists of three questions and carries 60% of the marks.
Answer ***two*** questions from this Section. If you answer more than two questions, credit will only be given for the best two answers.

The approximate allocation of marks to each part of a question is shown in brackets **[ ]**.

All symbols have their usual meanings.

Calculators may be used in this examination but must not be used to store text. Calculators with the ability to store text should have their memories deleted prior to the start of the examination.

A table of physical constants and units that may be required will be found at the end of this question paper.

SECTION 1

*Full marks for this section can be obtained by correctly answering **four** questions. You may attempt as many questions as you wish, but any marks in excess of 40% will be disregarded.*

1. In an inertial frame Σ two events occur at distance 10×10^8 m away from each other and are separated by a time interval of 3 s.

   (a) Calculate the space-time interval Δs^2 between the events. Is there an inertial frame in which these events happen at the same space point? Justify your answer by using the value of the space-time interval you found. **[5]**

   (b) In some inertial frame Σ' the events happen simultaneously. Find the velocity of the frame Σ' relative to Σ **[5]**

2. Spaceship B appears to have velocity u in the positive x direction as observed from the spaceship A. Find the velocity V of the spaceship C relative to the spaceship A so that the spaceships A and B have opposite velocities of equal magnitude. **[10]**

3. Two particles each having rest mass m are approaching each other with velocity v (in some inertial frame).

   (a) Write down the total energy E of the particles and their total momentum p **[1]**

   (b) In the reference frame Σ' the rightmost particle is at rest. Assuming that the leftmost particle has velocity v' write down the total energy E' of the particles and their total momentum p' in the new frame of reference. **[2]**

   (c) Using invariance of the combination $E^2/c^2 - p^2$ find the velocity v' and show that it obeys the relativistic law of velocity composition. **[7]**

4. Obtaining a university degree doubles the chances of a successful career. Only one quarter of young people go to university. What is the probability that a person with a successful career does not have a university degree? **[10]**

5. Consider the generating function

$$g(z) = A \sinh z = \sum_{n=0}^{\infty} p_n z^n$$

of a discrete probability distribution p_n.

(a) Find the value of the constant A and calculate the probabilities for $n = 0$ and for $n = 1$. **[5]**

(b) Write down the definitions for the mean μ and the variance σ^2 of the distribution p_n and calculate them using the generating function above. **[5]**

6. A one-dimensional quantum mechanical *wavefunction* is

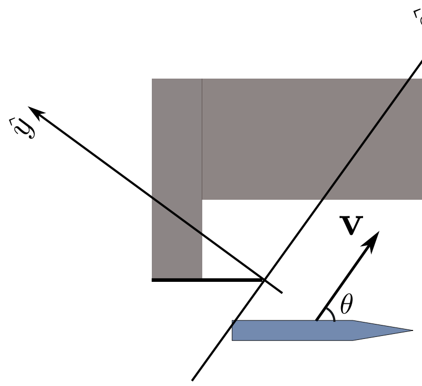
$$\psi(x) = Cxe^{-x}\,,$$

where the particle is restricted to the region $x \in [0, \infty]$ and C is the (real) normalisation constant. Find the normalisation constant C and calculate the uncertainty in position $\Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}$ of the particle. [*Hint: You can use the result* $\int_0^\infty s^k e^{-s} \mathrm{d}s = k!$ *for integer* k.] **[10]**

SECTION 2

Answer **two** *questions from this Section. If you answer more than two questions, credit will only be given for the best two answers.*

7. A space ship approaches a distant space station as shown. The space ship crew plans to enter inside the gate of length L using the shown trajectory keeping the space ship *parallel to the gate*. The velocity $\mathbf{v}$ (in the station's reference frame) makes the angle θ with the gate's plane. Choose a coordinate system with $\hat{x}$ axis parallel to the space ship velocity, $\mathbf{v} = v\hat{x}$, the $\hat{y}$ axis as shown and the origin coinciding with the left end of the opening.



(a) Give a definition of *proper length*. **[2]**

(b) Using the time measured by the space station's clocks, the event of the rear of the space ship entering the station is $E_r = (t_r, x_r, y_r, z_r) = (0, 0, 0, 0)$ and the event of the front of the space ship entering the station is $E_f = (t_f, x_f, y_f, z_f) = (0, X, Y, 0)$. Express X and Y in terms of the space ship length l measured in the station's reference frame and state the condition on the length l for the space ship to be able to enter the station gate. Neglect the width of the space ship. **[8]**

(c) The condition on the maximal length of the spaceship is communicated to the space ship crew and captain. The crew measures the length of their ship and find that it is actually twice longer than the communicated length limit. Nevertheless the captain decides the passage is still possible. Upon which relativistic phenomenon does he base his decision? **[5]**

(d) The angle $\theta = 30^\circ$ is communicated to the space ship crew from the station. Perform a Lorentz transformation to the reference frame moving with the space ship, and find the corresponding coordinates of the events E_r and E_f. By using these results or otherwise find the minimum velocity v which will allow it to enter the space station gate. **[15]**

8. (a) Define the *probability density* $p(x)$ of a random variable $a < x < b$ and explain how to normalise it. **[3]**

(b) If the random variable x is mapped onto y using the monotonically increasing function $y(x)$ show the following relation between their probability densities,

$$P(y) = p(x)\frac{dx}{dy}\,, \qquad \text{for} \qquad y(a) < y < y(b)\,.$$

[5]

A thermal source with temperature T produces relativistic particles of (rest) mass m. The particles move in the xy plane, so that the the momentum magnitude $p = \sqrt{p_x^2 + p_y^2}$ is distributed according to probability density

$$f(p) = A\,p\,e^{-E(p)/k_{\mathrm{B}}T}\,,$$

where A is a normalisation constant.

(c) State the relativistic relation $E(p)$ between the energy and momentum. **[3]**

(d) Show that the probability density $p(E)$ of the energy of the emitted particles is

$$p(E) = \frac{AE}{c^2}e^{-E/k_{\mathrm{B}}T}\,.$$

State the interval of energy E for which $p(E)$ is defined. **[4]**

(e) Find the constant A. **[5]**

(f) Calculate the mean energy $\bar{E}$ of the emitted particles. Write down the result in the limit $k_BT \ll mc^2$ and explain the physical meaning of the expression. **[10]**

9. An atom is found in a cubic container with size length L at random position with coordinates (x, y, z).

   (a) Find the probability $p(x)\mathrm{d}x$ for the coordinate x of the atom to be found in the interval $[x, x + \mathrm{d}x]$. By considering similar intervals for coordinates y and z show that the joint probability density $p(x, y, z)$ which allows to find probability to find the atom in a infinitesimal volume $\mathrm{d}x\mathrm{d}y\mathrm{d}z$ situated at (x, y, z) is

$$p(x, y, z) = 1/L^3$$

[5]

   (b) Using the probability density from the previous part, find the probability for an atom inside the cube to be outside the volume v. **[3]**

   (c) Suppose now that the cube contains N atoms with positions distributed randomly according to the probability distribution $p(x, y, z)$. Find the probability p_0 that the volume v is empty. Assume that atoms are independent and do not influence each other. **[5]**

   (d) Using the mean density $n = N/L^3$ of the atom show that in the limit $v/L^3 \to 0$ and for fixed n the probability p_0 becomes

$$p_0 = e^{-nv} \, .$$

. **[7]**

   (e) Find the probability distribution $f(v)$ of the size v of an empty region.

   [*Hint: Use the probabilistic definition of its cumulative distribution function* $F(S) = \int_0^S f(s)\mathrm{d}s$.] Calculate the mean distance between the points in the cube using this distribution.

[10]

Physical Constants and Units

Acceleration due to gravity	g	$9.81\,\mathrm{m\,s^{-2}}$
Gravitational constant	G	$6.674 \times 10^{-11}\,\mathrm{N\,m^2\,kg^{-2}}$
Ice point	T_{ice}	$273.15\,\mathrm{K}$
Avogadro constant	N_A	$6.022 \times 10^{23}\,\mathrm{mol^{-1}}$
		[*N.B.* 1 mole $\equiv 1$ *gram-molecule*]
Gas constant	R	$8.314\,\mathrm{J\,K^{-1}\,mol^{-1}}$
Boltzmann constant	k, k_B	$1.381 \times 10^{-23}\,\mathrm{J\,K^{-1}} \equiv 8.62 \times 10^{-5}\,\mathrm{eV\,K^{-1}}$
Stefan constant	σ	$5.670 \times 10^{-8}\,\mathrm{W\,m^{-2}\,K^{-4}}$
Rydberg constant	R_∞	$1.097 \times 10^{7}\,\mathrm{m^{-1}}$
	$R_\infty hc$	$13.606\,\mathrm{eV}$
Planck constant	h	$6.626 \times 10^{-34}\,\mathrm{J\,s} \equiv 4.136 \times 10^{-15}\,\mathrm{eV\,s}$
$h/2\pi$	$\hbar$	$1.055 \times 10^{-34}\,\mathrm{J\,s} \equiv 6.582 \times 10^{-16}\,\mathrm{eV\,s}$
Speed of light *in vacuo*	c	$2.998 \times 10^{8}\,\mathrm{m\,s^{-1}}$
	$\hbar c$	$197.3\,\mathrm{MeV\,fm}$
Charge of proton	e	$1.602 \times 10^{-19}\,\mathrm{C}$
Mass of electron	m_e	$9.109 \times 10^{-31}\,\mathrm{kg}$
Rest energy of electron		$0.511\,\mathrm{MeV}$
Mass of proton	m_p	$1.673 \times 10^{-27}\,\mathrm{kg}$
Rest energy of proton		$938.3\,\mathrm{MeV}$
One atomic mass unit	u	$1.66 \times 10^{-27}\,\mathrm{kg}$
Atomic mass unit energy equivalent		$931.5\,\mathrm{MeV}$
Electric constant	ϵ_0	$8.854 \times 10^{-12}\,\mathrm{F\,m^{-1}}$
Magnetic constant	μ_0	$4\pi \times 10^{-7}\,\mathrm{H\,m^{-1}}$
Bohr magneton	μ_B	$9.274 \times 10^{-24}\,\mathrm{A\,m^2}$ ($\mathrm{J\,T^{-1}}$)
Nuclear magneton	μ_N	$5.051 \times 10^{-27}\,\mathrm{A\,m^2}$ ($\mathrm{J\,T^{-1}}$)
Fine-structure constant	$\alpha = e^2/4\pi\epsilon_0\hbar c$	7.297×10^{-3}= 1/137.0
Compton wavelength of electron	$\lambda_c = h/m_e c$	$2.426 \times 10^{-12}\,\mathrm{m}$
Bohr radius	a_0	$5.2918 \times 10^{-11}\,\mathrm{m}$
angstrom	Å	$10^{-10}\,\mathrm{m}$
barn	b	$10^{-28}\,\mathrm{m^2}$
torr (mm Hg at 0 °C)	torr	$133.32\,\mathrm{Pa}$ ($\mathrm{N\,m^{-2}}$)

Do not complete the attendance slip, fill in the front of the answer book or turn over the question paper until you are told to do so

Important Reminders

- Coats/outwear should be placed in the designated area.
- Unauthorised materials (e.g. notes or Tippex) must be placed in the designated area.
- Check that you do not have any unauthorised materials with you (e.g. in your pockets, pencil case).
- Mobile phones and smart watches **must** be switched off and placed in the designated area or under your desk. They must not be left on your person or in your pockets.
- You are not permitted to use a mobile phone as a clock. If you have difficulty seeing a clock, please alert an Invigilator.
- You are not permitted to have writing on your hand, arm or other body part.
- Check that you do not have writing on your hand, arm or other body part – if you do, you must inform an Invigilator immediately
- Alert an Invigilator immediately if you find any unauthorised item upon you during the examination.

Any students found with non-permitted items upon their person during the examination, or who fail to comply with Examination rules may be subject to Student Conduct procedures.