University of Birmingham
School of Mathematics

1SAS Sequences and Series Autumn 2024

Problem Sheet 2
(Issued Week 3)

Q1. Use standard limit theorems, such as the Algebra of Limits, or the Sandwich Theorem, to show that

(i)
$$\frac{1 + n - 2024n^2 + n^3}{3 + 49n^2 + 4n^3} \to \frac{1}{4}$$

(ii)
$$\frac{7n^4 - 3n^2 \sin(2n) + n}{5n^4 + 7n - 4} \to \frac{7}{5}$$

(iii)
$$\frac{n-1}{n^2+1} + \frac{n-1}{n^2+2} + \cdots + \frac{n-1}{n^2+n} \to 1$$

Q2. Which of the following statements are true? If a statement is true, provide a proof. If it is false, provide a counterexample.

(i) If $a_n \to \infty$ then $\frac{1}{a_n} \to 0$.
(ii) If $a_n \to 0$ then $\frac{1}{a_n} \to \infty$.
(iii) Suppose $a_n \to \ell$ and $b_n \to m$. If $a_n < b_n$ for all $n \in \mathbb{N}$ then $\ell < m$.
(iv) If the sequences $(a_n + b_n)$ and $(a_n - b_n)$ both converge, then (a_n) and (b_n) both converge.
(v) If the sequences $(a_n + b_n)$ and $(a_n b_n)$ both converge, then (a_n) and (b_n) both converge.

Q3. A sequence of positive real numbers (a_n) is defined recursively by the formula

$$a_{n+1} = \frac{1}{5}(a_n^2 + 2a_n + 2),$$

where $a_1 = 0$.

(i) Use induction to prove that $a_n \leq 1$ for all n.
(ii) Prove that

$$a_{n+2} - a_{n+1} = \frac{1}{5}(a_{n+1} + a_n + 2)(a_{n+1} - a_n)$$

for all $n \in \mathbb{N}$.
(iii) Use induction to prove that (a_n) is increasing.
(iv) Deduce that (a_n) converges, and find its limit, justifying any assertions that you make.

Q4. Consider again the sequence defined in Question 3 above. Using Parts (i) and (ii) of that question, or otherwise, show that

$$a_{n+1} - a_n \leq \frac{1}{2}\left(\frac{4}{5}\right)^n$$

for all $n \in \mathbb{N}$.

$\boxed{\text{SUM}}$ **Q5**. A sequence of nonnegative real numbers (a_n) is defined recursively by the formula

$$a_{n+1} = \frac{a_n + 1}{a_n + 2},$$

where $a_1 = 0$.

(i) Show that (a_n) is bounded above.

(ii) Prove that

$$a_{n+2} - a_{n+1} = \frac{a_{n+1} - a_n}{(a_n + 2)(a_{n+1} + 2)}$$

for all $n \in \mathbb{N}$.

(iii) Use induction to prove that (a_n) is increasing.

(iv) Use the Monotone Convergence Theorem to show that (a_n) converges. Find the limit of (a_n), justifying any assertions that you make.

(v) Show further that

$$a_{n+2} - a_{n+1} \le \frac{1}{4}(a_{n+1} - a_n)$$

for all $n \in \mathbb{N}$.

Q6. Returning to Question 5 above, show further that

$$0 \le a_{n+1} - a_n \le c4^{-n}$$

for all $n \in \mathbb{N}$, where c is some positive constant that you should identify.

Extra Questions

EQ1. Suppose that (a_n) is a bounded sequence and $b_n \to 0$. Prove that $a_n b_n \to 0$.

EQ2. Prove that if a sequence (a_n) converges then

$$a_{n+1} - a_n \to 0.$$

Is the converse of this statement true; i.e. is it true that if $a_{n+1} - a_n \to 0$ then (a_n) converges? If it is true, provide a proof. If it is false, provide a counterexample.

EQ3. Recall the following theorem:

Theorem (Monotone Convergence Theorem)**.**
An increasing sequence of real numbers which is bounded above converges.

Propose a similar statement involving an increasing sequence which is *not* bounded above. Can you prove your assertion?

EQ4. A sequence (a_n) is defined recursively by the formula

$$a_{n+1} = 4 + 2a_n^{1/3},$$

where $a_1 = 1$.

(i) Prove by induction that $1 \le a_n \le 8$ for all $n \in \mathbb{N}$.

(ii) Prove that $\frac{a_{n+1}}{a_n} \ge 1$ for all $n \in \mathbb{N}$.

(iii) State a theorem which allows you to conclude that (a_n) converges.

(iv) Find the limit of the sequence (a_n), justifying any assertions that you make.

EQ5. A sequence (a_n) is defined recursively by the formula

$$a_{n+1} = \frac{1}{5}(a_n^2 + 6),$$

where $a_1 = 5/2$.

(i) Prove that $2 < a_n < 3$ for all $n \in \mathbb{N}$.
(ii) Prove that (a_n) is decreasing.
(iii) Deduce that (a_n) converges, and find its limit. Justify any assertions that you make.

EQ6. Define what it means for a sequence of real numbers (a_n) to *tend to infinity*. Let (a_n) be the sequence of real numbers defined recursively by

$$a_{n+1} = a_n + \frac{1}{\sqrt{n+1}},$$

where $a_1 = 1$.

(i) Use induction to prove that

$$a_n = \sum_{k=1}^{n} \frac{1}{\sqrt{k}}$$

for all $n \in \mathbb{N}$.
(ii) Prove that $a_n \geq \sqrt{n}$ for all $n \in \mathbb{N}$.
(iii) Deduce that $a_n \to \infty$.