Introduction to Probability

Lecture 9

Today

Continuous Probability

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Continuous Probability

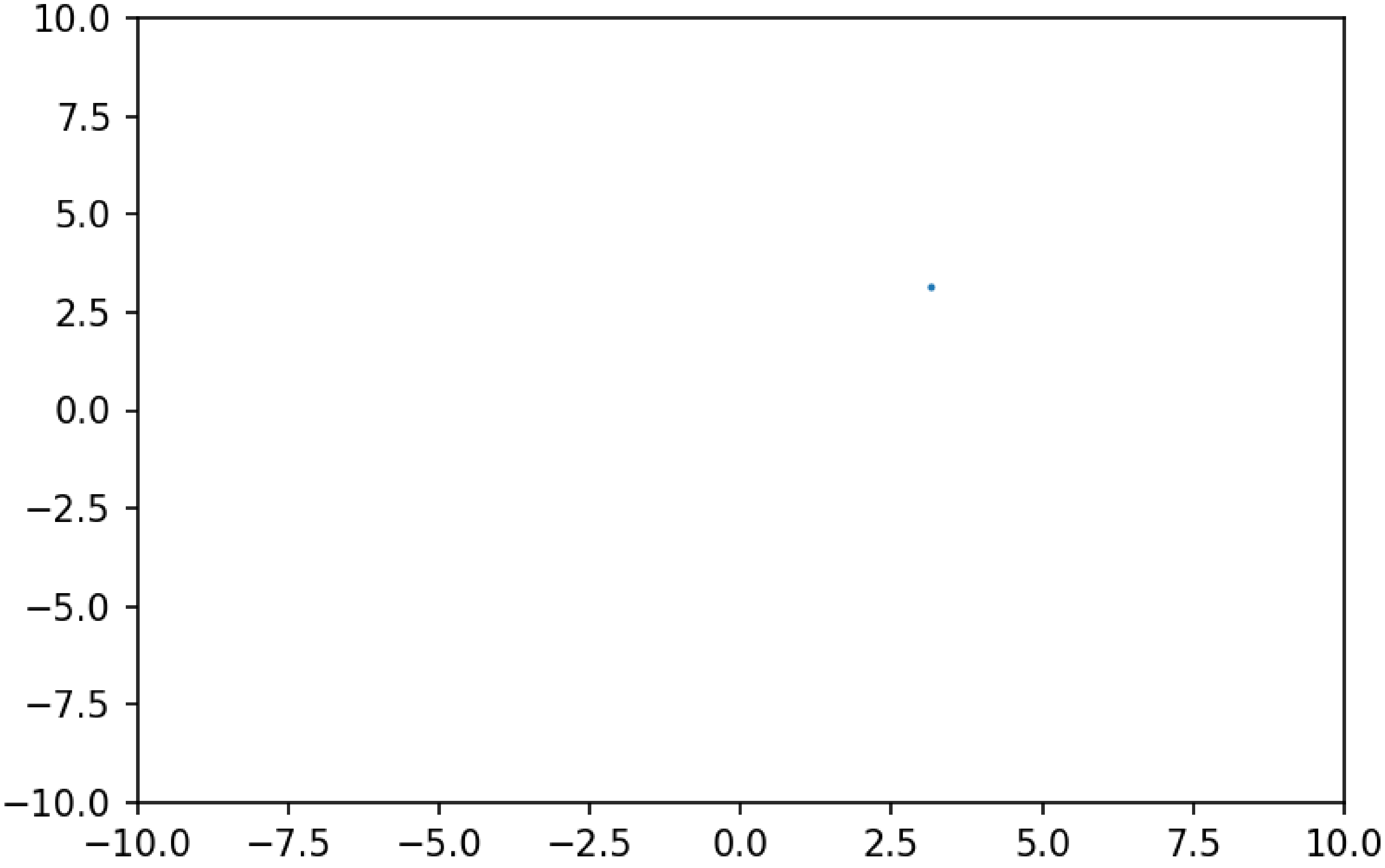
Discrete Probability

So far we have only discussed **discrete probability**

In this $P(x)$ represents the probability of x happening

Even if the state space Ω has an infinite number of outcomes, it still has the same meaning.

Continuous Random Walk



$$\begin{pmatrix} x_t \\ y_t \end{pmatrix} = \begin{pmatrix} x_{t-1} \\ y_{t-1} \end{pmatrix} + \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$

What is the probability it lands **exactly on** (π, π)?

What is the probability it lands **near** (π, π)?

Continuous Probability (1)

Continuous Probability allows the possible values to be **real numbers**.

Instead we have the real numbers (or a subset) for the sample space Ω.

We ask: "what is the probability that x lies in an **interval**?".

$$\mathrm{Prob}(\mathrm{a} \leq x \leq b) = \sum_{a \leq x \leq b} P(x) \rightarrow \int_a^b dx\, P(x)$$

Continuous Probability (2)

We can speak of the probability for an **interval** of values, but no particular value itself.

$$\text{Prob}(\underbrace{\mathrm{a} \le x \le a}_{x=a}) = \int_a^a dx\, P(x) = 0$$

But $P(a)$ itself may not be zero

We always need a window

$$P(x)dx$$

Probability Density Function

We call $P(x)$ the **Probability Density Function** (PDF).

$$P(x) \geq 0$$

$$1 = \int_{\Omega} P(x)\, dx$$

But $P(x)$ itself may be larger than 1.

Formulae

Replace Σ with $\int$

$$\langle x \rangle = \int_{\Omega} x\, P(x) dx$$

$$\langle f(x) \rangle = \int_{\Omega} f(x)\, P(x) dx$$

$$\mathrm{var}(x) = \int_{\Omega} (x - \langle x \rangle)^2 P(x) dx = \langle x^2 \rangle - \langle x \rangle^2$$

Just use integration instead!

All the old formulae hold!

Example

A PDF is given by

$$P(x) = \begin{cases} 3x^2 & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

What is $\langle x \rangle$?

$$\langle x \rangle = \int_\Omega x\, P(x)\, dx$$

$$= \int_0^1 x\, 3\, x^2\, dx = 3 \int_0^1 x^3\, dx$$

$$= \frac{3x^4}{4}\bigg|_0^1 = \frac{3}{4}$$

Example

A wavefunction is given by

$$\psi(x) = a(1 - i\,x^2)$$

For $0 \le x \le 1$.

In QM

$$P(x) = \psi^*(x)\psi(x)$$

Find the normalising constant, a and $\langle x \rangle$.

$$P(x) = a^2(1 + ix^2)(1 - ix^2) = a^2(1 + x^4)$$

Then

$$\frac{1}{a^2} = \int_0^1 dx\,(1 + x^4) = \left(x + \frac{x^5}{5}\right)\Big|_0^1 = 1 + \frac{1}{5} = \frac{6}{5}$$

So

$$a = \sqrt{\frac{5}{6}}$$

Then

$$\langle x \rangle = \frac{5}{6}\int_0^1 dx\,x(1 + x^4) = \frac{5}{6}\left(\frac{x^2}{2} + \frac{x^6}{6}\right)\Big|_0^1 = \frac{5}{6} \times \frac{4}{6} = \frac{5}{9}$$

Class Example

If $P(x) = 2x$ for $0 \leq x \leq 1$, what is $\mathrm{var}(x)$?

Note $\langle x \rangle = \frac{2}{3}$

Remember

$$\mathrm{var}(x) = \langle x^2 \rangle - \langle x \rangle^2$$

$$\langle x^2 \rangle \equiv \int x^2\, P(x) dx$$

$$\langle x^2 \rangle = \int_0^1 dx\, x^2 2x = 2\int_0^1 dx\, x^3$$

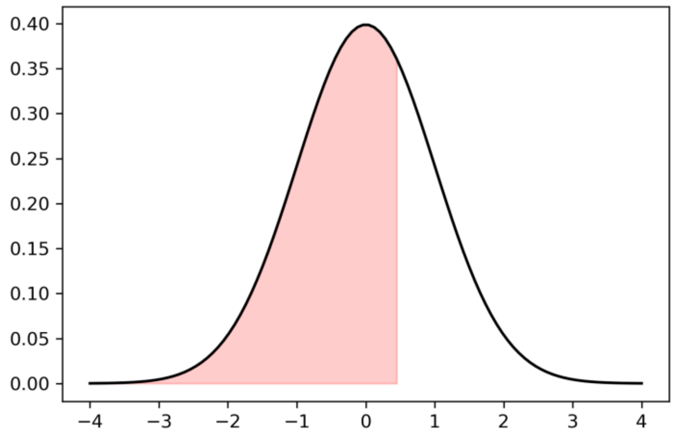
$$2 \times \frac{x^4}{4}\Big|_0^1 = \frac{1}{2}$$

$$\mathrm{var}(x) = \langle x^2 \rangle - \langle x \rangle^2 = \frac{1}{2} - \frac{4}{9} = \frac{1}{18}$$

Cumulative Distributions

Cumulative Distributions (1)

$$C(x) \equiv \text{Probability}(X \le x)$$



$$C(x) = \int_{-\infty}^{x} dx\, P(x)$$

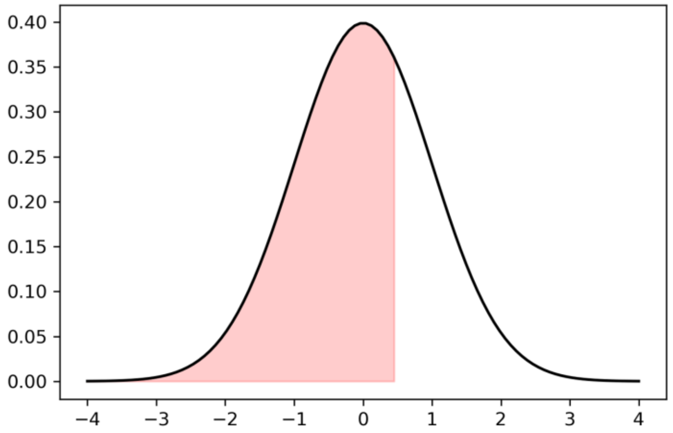
$$\frac{dC(x)}{dx} = P(x)$$

We also get

$$\text{Prob}(a \le x \le b) = \text{Prob}(x \le b) - \text{Prob}(x \le a)$$

$$= C(b) - C(a)$$

Cumulative Distributions (2)



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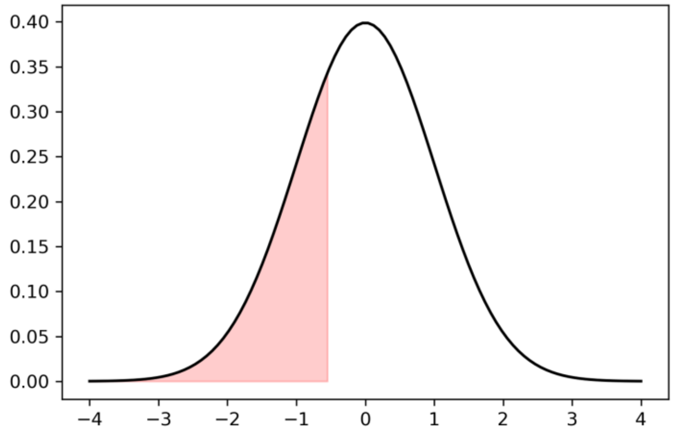

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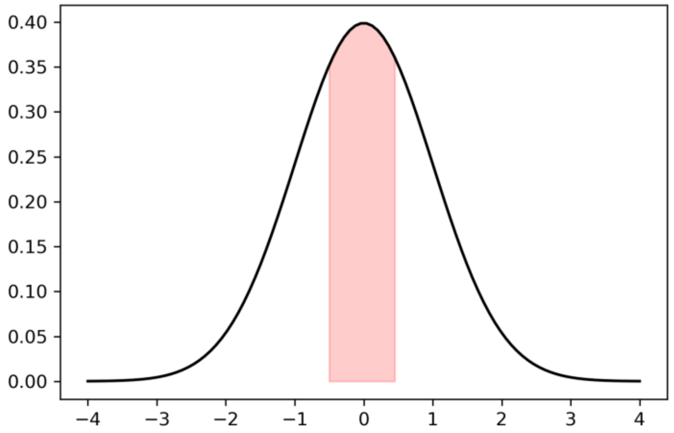


$$\text{Probability}\left(x \leq \frac{1}{2}\right)$$

$$\text{Probability}\left(x \leq -\frac{1}{2}\right)$$

$$\text{Probability}\left(-\frac{1}{2} \leq x \leq \frac{1}{2}\right)$$

Example

If a PDF is given by

$$P(x) = \frac{3}{2}(1 - x^2) \quad 0 \leq x \leq 1$$

Find

$$\text{Probability}\left(\frac{1}{4} \leq x \leq \frac{1}{2}\right)$$

Using the cumulative distribution

We need $C\left(\frac{1}{2}\right) - C\left(\frac{1}{4}\right)$

$$C(x) = \frac{3}{2}\int_0^x (1 - x^2)dx = \frac{3}{2}x - \frac{x^3}{2}$$

$$C\left(\frac{1}{2}\right) - C\left(\frac{1}{4}\right) = \frac{3}{2}\left(\frac{1}{2}\right) - \frac{1}{2}\left(\frac{1}{2}\right)^3 - \frac{3}{2}\left(\frac{1}{4}\right) + \frac{1}{2}\left(\frac{1}{4}\right)^3$$

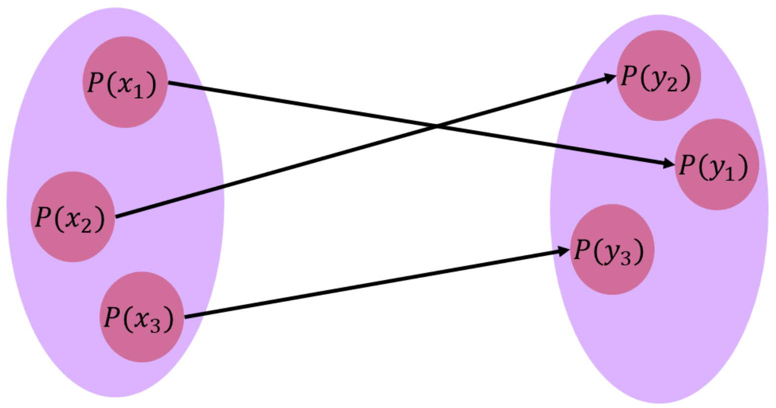
$$= \frac{41}{128}$$

Change of Variables

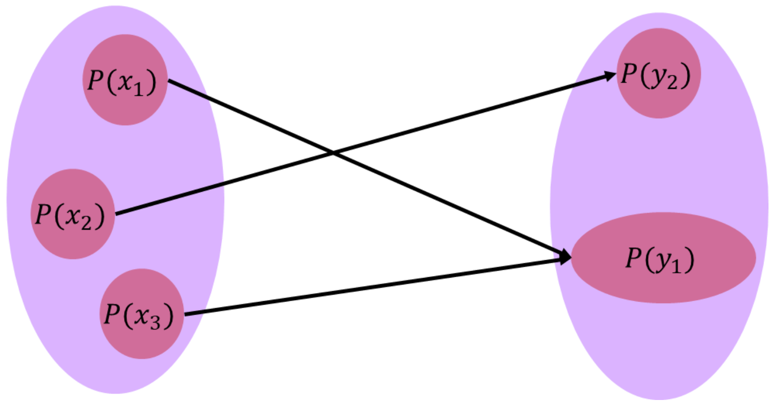
Change of Variables (1)

In discrete probability we considered if $x \sim P_x(x)$ and we set $y = f(x)$ what is $P_y(y)$?

$$P_y(y) = \sum_{x:f(x)=y} P_x(x)$$



We need the same thing for continuous probability.



Change of Variables (2)

$x \sim P_x(x)$ and we set $y = f(x)$ what is $P_y(y)$?

Assume $\Omega_x = [a, b]$

$$1 = \int_a^b dx\, P_x(x)$$

$$y = f(x) \to x = f^{-1}(y)$$

$$dx = \frac{df^{-1}}{dy} dy$$

$$1 = \int_{f(a)}^{f(b)} dy\, \frac{df^{-1}}{dy} P_x\left(f^{-1}(y)\right) = \int_{\Omega_y} P_y(y) dy$$

Change of Variables (3)

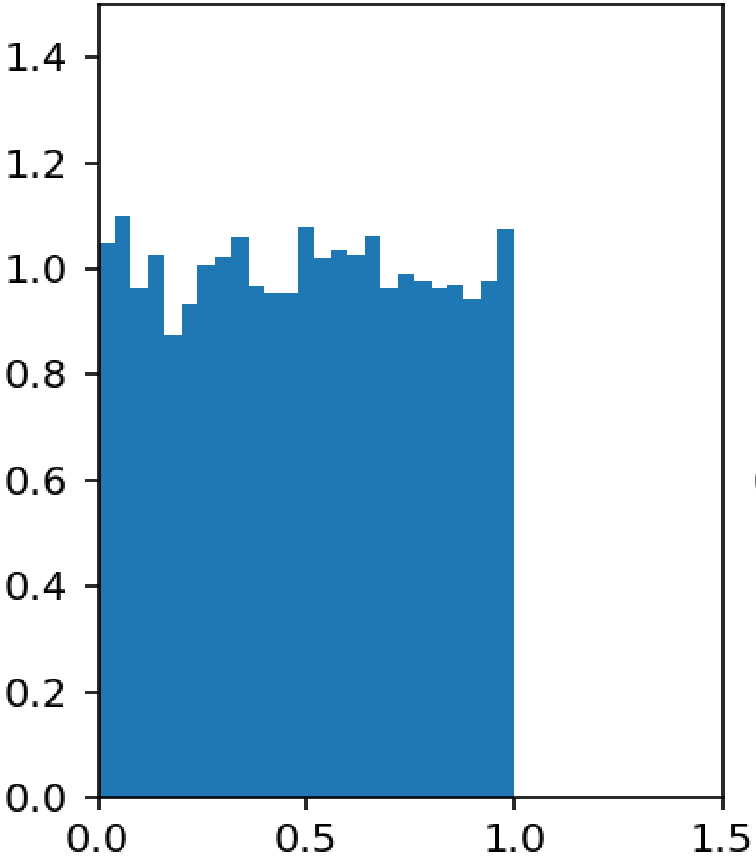
In general, if $y = f(x)$ is **monotonic** in Ω_x then

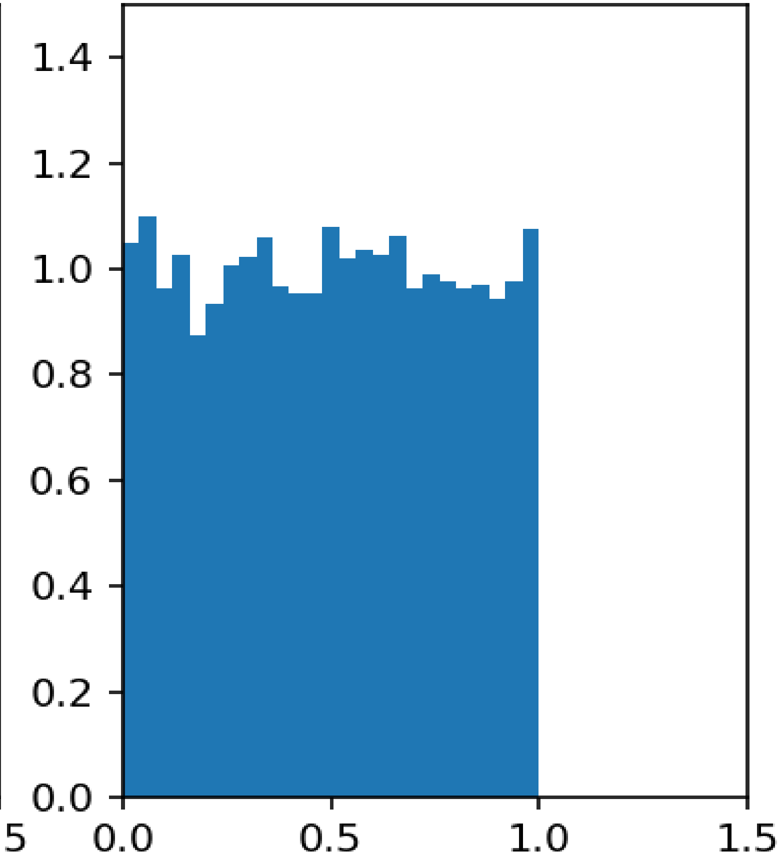
$$P_y(y) = \left|\frac{d}{dy} f^{-1}(y)\right| P_x(f^{-1}(y))$$

Simulation

$$f(x) = x$$

$x \sim P_x(x)$





$y \sim P_y(y)$

Example

If $P_x(x) = e^{-x}$ with $\Omega_x = [0, \infty)$ and $y = f(x) = x^2$ what is $P_y(y)$?

We note that x^2 maps $[0, \infty)$ onto itself so $\Omega_y = \Omega_x$

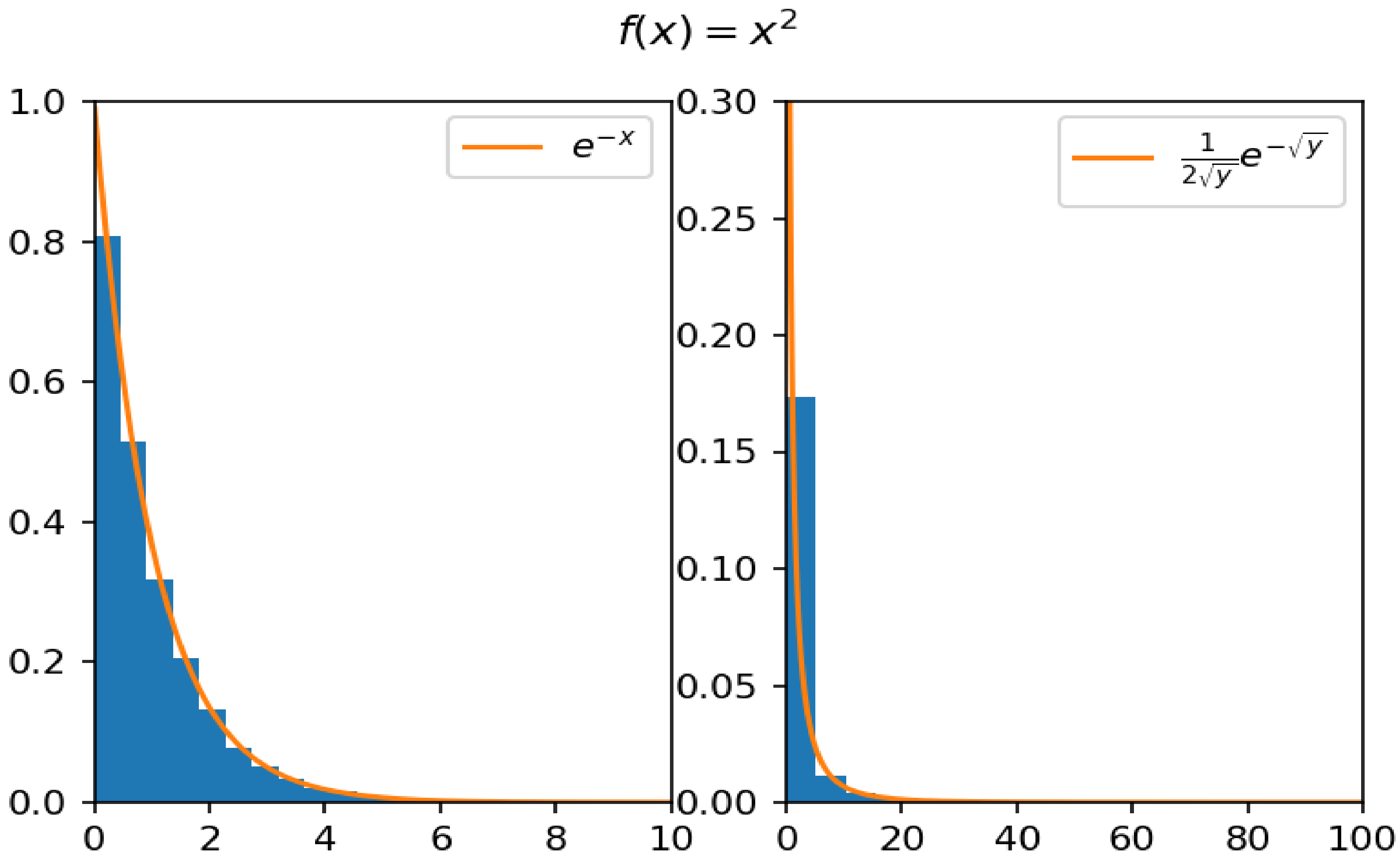
$$f(x) = x^2 \rightarrow f^{-1}(y) = \sqrt{y}$$

$$P_y(y) = \left| \frac{d}{dy} f^{-1}(y) \right| P_x(f^{-1}(y))$$

$$\frac{d}{dy} f^{-1}(y) = \frac{1}{2\sqrt{y}}$$

$$P_y(y) = \frac{1}{2\sqrt{y}} e^{-\sqrt{y}}$$

Simulation



$f(x) = x^2$
e^{-x}
$\frac{1}{2\sqrt{y}}e^{-\sqrt{y}}$
1.0
0.8
0.6
0.4
0.2
0.0
0 2 4 6 8 10
0.30
0.25
0.20
0.15
0.10
0.05
0.00
0 20 40 60 80 100

Class Example

If $P_x(x) = 1$ with $\Omega_x = [0,1]$ and $y = f(x) = x^4$ what is $P_y(y)$?

We note that x^4 maps $[0,1]$ onto itself so $\Omega_y = \Omega_x$

$$f(x) = x^2 \rightarrow f^{-1}(y) = y^{\frac{1}{4}}$$

$$P_y(y) = \left|\frac{d}{dy} f^{-1}(y)\right| P_x(f^{-1}(y))$$

$$\frac{d}{dy} f^{-1}(y) = \frac{1}{4y^{\frac{3}{4}}}$$

$$P_y(y) = \frac{1}{4y^{\frac{3}{4}}}$$

Summary

Essentially $\Sigma \to \int$

$$\frac{\mathrm{dC(x)}}{dx} = P(x)$$

$P(x)$ is the probability **density** and not the probability. The probability is given by (for some subset A)

$$\text{Probability}(a \le x \le b) = \int_a^b dx\, P(x)$$

$$P_y(y) = \left|\frac{d}{dy} f^{-1}(y)\right| P_x\left(f^{-1}(y)\right)$$

Examples

Example

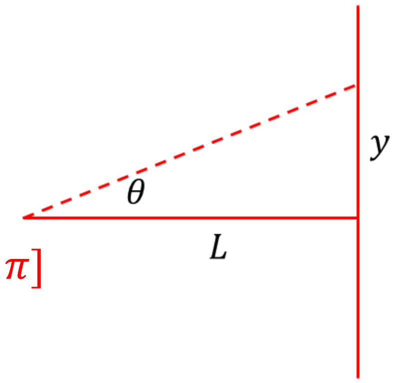
A particle emitter is placed at $(0,0)$. It emits angularly uniformly in $[\frac{\pi}{2},\frac{\pi}{2}]$.

If a vertical screen is L away, what is the probability distribution of particles on the screen?

You will need that

$$\frac{d}{dx}\tan^{-1}\left(\frac{x}{a}\right) = \frac{a}{a^2+x^2}$$

Using



$$P(\theta) = \frac{1}{\pi}; \quad \theta \in [-\pi,\pi]$$

$$y = \mathrm{L}\tan\theta$$

$$\rightarrow \theta = \tan^{-1}\frac{y}{L}$$

Then

$$P(y) = \left(\frac{d}{dy}\tan^{-1}\frac{y}{L}\right) \times \frac{1}{\pi}$$

$$P(y) = \frac{1}{\pi}\frac{L}{y^2+L^2}$$