UNIVERSITY OF BIRMINGHAM

School of Physics and Astronomy

DEGREE OF B.Sc. & M.Sci. WITH HONOURS

FIRST YEAR EXAMINATION

03 19749

LC SPECIAL RELATIVITY/PROBABILITY AND RANDOM PROCESSES

SEMESTER 1 EXAMINATIONS 2021/2022

Time Allowed: 1 hour 30 minutes

Answer Section 1 and two questions from Section 2.

Section 1 consists of four questions and carries 40% of the marks.
Answer ***all four*** questions from this section.

Section 2 consists of three questions and carries 60% of the marks.
Answer ***two*** questions from this Section. If you answer more than two questions, credit will only be given for the best two answers.

The approximate allocation of marks to each part of a question is shown in brackets **[ ]**.

All symbols have their usual meanings.

Calculators may be used in this examination but must not be used to store text. Calculators with the ability to store text should have their memories deleted prior to the start of the examination.

A table of physical constants and units that may be required will be found at the end of this question paper.

SECTION 1

Answer **all four** *questions from this section.*

1. In an inertial frame Σ two events occur at distance 10^9 m away from each other and are separated by a time interval of 5 s.

   Calculate the space-time interval Δs^2 between the events. Using the value of the space-time interval you found answer the following questions:

   - Is there an inertial frame in which these events happen simultaneously?
   - Is there an inertial frame where the events happen at the same point in space?

   In the case where your answer is yes, finds the frame's velocity relative to Σ. **[10]**

2. Spaceship B appears to have velocity components u_x, u_y as observed in the reference frame Σ of spaceship A. Spaceship C moves in the positive x direction relative to spaceship A with velocity v. In the reference frame Σ' of spaceship C, the spaceship B moves vertically (along the y direction). Find v and the velocity u'_x of B in the frame Σ. **[10]**

3. Lying to the public increases the chances of a politician being elected by a factor of 1.5. There are typically four out of five honest politicians. Find the probability that an elected politician is a liar. **[10]**

4. Consider the generating function

$$G(z) = A \cosh z = \sum_{n=0}^{\infty} P(n) z^n$$

   of a discrete probability distribution $P(n)$.

   (a) Find the value of the constant A and calculate the probabilities for $n = 0$ and for $n = 1$. **[5]**

   (b) Write down the definitions for the mean μ and the variance σ^2 of the distribution $P(n)$ and calculate them using the generating function above. **[5]**

SECTION 2

Answer **two** *questions from this Section. If you answer more than two questions, credit will only be given for the best two answers.*

5. Two lovers, R and J, have an argument and fly away from Earth in opposite directions with velocities of equal magnitude V. J is moving in the positive x direction. According to her clock she travels for time $T_J/2$ before receiving a radio signal from R, in which he apologises and says that he misses J and will not proceed with his journey until he hears from J again. J accepts the apologies and sends a radio signal back to R saying that she is returning immediately (with the same speed V). Once R receives the message from J he starts the engines and heads back towards Earth with speed V.

   (a) Draw the Minkowski space-time diagram of the journeys of R and J. Use the frame in which the Earth is at rest and the starting event $E_s = (t_s, x_s) = (0, 0)$. Indicate

   i. the event E_1 when R sends his message and stops,
   ii. the event E_J when J receives and replies to R's message,
   iii. the event E_2 when R hears back from J and starts his return journey,
   iv. the event E_f when R and J meet again.

   Give an argument why R and J's reunion happens on Earth. **[8]**

   (b) Give the definition of *proper time*.

   Using this definition calculate the total time of J and R's journey, *i.e.*the time elapsed from the moment they separated until the moment they meet again, as measured by a clock in the Earth's reference frame. **[7]**

   (c) Calculate the time t_1 in the Earth reference frame when R sent the first signal to J. Show that he was waiting for time

$$t_w = \frac{2V/c}{1 + V/c} \frac{T_J}{\sqrt{1 - V^2/c^2}}$$

   before hearing back from J. **[8]**

   (d) R was 20 and J was 3 years older than him when they separated. Given that J was travelling for $T_J = 16$ years with speed $V = 0.6c$, deduce the age of R at the (happy) end of his journey. **[7]**

6. Two photons are moving towards each other. The energy of each photon is $E_{\mathrm{ph}} = h\nu$, where ν is the frequency of each photon and h is Planck's constant. The photons produce an electron-positron pair, with each particle having the rest mass m_{e}.

   (a) Calculate the minimum frequency ν_{min} of a photon for which the pair-production process can take place. **[5]**

   (b) Assuming that $\nu > \nu_{\mathrm{min}}$, find the energy and momentum of the electron. **[5]**

   (c) Find the magnitude v of the electron's velocity as a function of the frequency ν. **[3]**

   (d) Given the frequency ν has a probability density $p(\nu) = Ae^{-\nu/\nu_{\mathrm{min}}}$, $\nu > 0$, calculate the probability p_0 that the pair production process takes place, *i.e.* that $\nu > \nu_{\mathrm{min}}$. Your answer should contain only Euler's constant e. **[5]**

   (e) Calculate the mean, $\langle \nu \rangle$, and the variance, $\sigma_\nu^2 = \langle \nu^2 \rangle - \langle \nu \rangle^2$, of the photon's frequency. **[6]**

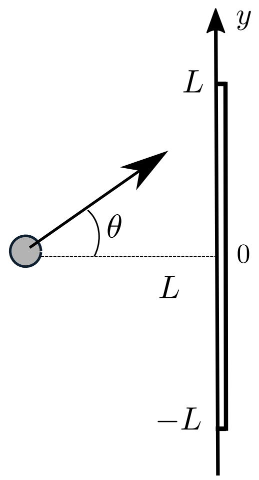
   (f) Explain why the probability distribution of the electron's speed is given by

$$P(v) = \frac{1}{p_0} p(\nu(v)) \frac{\mathrm{d}\nu}{\mathrm{d}v} \,,$$

   where p_0 is the probability calculated in part (d). Write down an explicit expression for $P(v)$. **[6]**

7. A footballer kicks a ball at an angle θ to the centre-line of the goal, as shown, where $-\pi/2 < \theta < \pi/2$. The angle θ is randomly distributed with probability density

$$p(\theta) = C\cos^2\theta\,.$$



(a) Find the value of the constant C. **[5]**

(b) Calculate the standard deviation $\sigma = \sqrt{\langle\theta^2\rangle - \langle\theta\rangle^2}$.
You may use the fact that

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \theta^2 \cos 2\theta \mathrm{d}\theta = -\frac{\pi}{2}$$ **[6]**

(c) The ball hits the y axis, situated at a distance L from the footballer, at the point y. Show that the probability density $P(y)$ is given by

$$P(y) = \frac{2}{\pi L}\frac{1}{[1+(y/L)^2]^2}.$$ **[8]**

(d) p_G is the probability that the ball hits the goal which is defined by the interval $-L < y < L$. Using either $p(\theta)$ or $P(y)$ show that

$$p_G = 1/2 + 1/\pi \simeq 0.8.$$ **[6]**

(e) The game is over once the ball hits the goal. Calculate the average number of kicks in this game. You may use the fact that

$$\sum_{k=1}^{\infty} k a^k = \frac{a}{(1-a)^2}\,, \qquad |a| < 1\,.$$ **[5]**

Useful Formulae for Special Relativity & Probability

Lorentz Transformations in Standard Configuration

$$\begin{array}{lll} ct' = \gamma(ct - \beta x) & ct = \gamma(ct' + \beta x') & \beta = v/c \\ x' = \gamma(x - \beta ct) & x = \gamma(x' + \beta ct') & \gamma = 1/\sqrt{1-\beta^2} \\ y' = y & y = y' & \\ z' = z & z = z' & \end{array}$$

Velocity Transformation

$$u'_x = \frac{u_x - v}{\left(1 - \frac{u_x v}{c^2}\right)}, \qquad u'_y = \frac{u_y}{\gamma(v)\left(1 - \frac{u_x v}{c^2}\right)}, \qquad u'_z = \frac{u_z}{\gamma(v)\left(1 - \frac{u_x v}{c^2}\right)}.$$

Invariant Interval

$$\Delta s^2 = (c\Delta t)^2 - \Delta x^2 - \Delta y^2 - \Delta z^2$$

Energy and Momentum

$$E = \frac{mc^2}{\sqrt{1 - v^2/c^2}}, \qquad \mathbf{p} = \frac{m\mathbf{v}}{\sqrt{1 - v^2/c^2}}, \qquad E^2 = p^2c^2 + m^2c^4.$$

Conditional Probability

$$P(A|B)P(B) = P(A \cap B)$$

Binomial Distribution

$$P(n; N, p) = \binom{N}{n} p^n (1-p)^{N-n}$$

Poisson Distribution

$$P(n; \mu) = \frac{\mu^n}{n!} e^{-\mu}$$

Probability Generating Function

$$G(z) = \sum_{n=0}^{\infty} P(n) z^n$$

Normal Distribution

$$p(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[\frac{-(x-\mu)^2}{2\sigma^2}\right]$$

Change of Variable in Probability Density Function

$$p(y) = p(x)\left|\frac{dx}{dy}\right|$$

Physical Constants and Units

Acceleration due to gravity	g	9.81 $\mathrm{m\,s^{-2}}$
Gravitational constant	G	$6.674 \times 10^{-11}\,\mathrm{N\,m^2\,kg^{-2}}$
Ice point	T_{ice}	273.15 K
Avogadro constant	N_A	$6.022 \times 10^{23}\,\mathrm{mol^{-1}}$
		[*N.B.* 1 mole $\equiv 1$ *gram-molecule*]
Gas constant	R	$8.314\,\mathrm{J\,K^{-1}\,mol^{-1}}$
Boltzmann constant	k, k_B	$1.381 \times 10^{-23}\,\mathrm{J\,K^{-1}} \equiv 8.62 \times 10^{-5}\,\mathrm{eV\,K^{-1}}$
Stefan constant	σ	$5.670 \times 10^{-8}\,\mathrm{W\,m^{-2}\,K^{-4}}$
Rydberg constant	R_∞	$1.097 \times 10^{7}\,\mathrm{m^{-1}}$
	$R_\infty hc$	13.606 eV
Planck constant	h	$6.626 \times 10^{-34}\,\mathrm{J\,s} \equiv 4.136 \times 10^{-15}\,\mathrm{eV\,s}$
$h/2\pi$	$\hbar$	$1.055 \times 10^{-34}\,\mathrm{J\,s} \equiv 6.582 \times 10^{-16}\,\mathrm{eV\,s}$
Speed of light *in vacuo*	c	$2.998 \times 10^{8}\,\mathrm{m\,s^{-1}}$
	$\hbar c$	197.3 MeV fm
Charge of proton	e	$1.602 \times 10^{-19}\,\mathrm{C}$
Mass of electron	m_e	$9.109 \times 10^{-31}\,\mathrm{kg}$
Rest energy of electron		0.511 MeV
Mass of proton	m_p	$1.673 \times 10^{-27}\,\mathrm{kg}$
Rest energy of proton		938.3 MeV
One atomic mass unit	u	$1.66 \times 10^{-27}\,\mathrm{kg}$
Atomic mass unit energy equivalent		931.5 MeV
Electric constant	ϵ_0	$8.854 \times 10^{-12}\,\mathrm{F\,m^{-1}}$
Magnetic constant	μ_0	$4\pi \times 10^{-7}\,\mathrm{H\,m^{-1}}$
Bohr magneton	μ_B	$9.274 \times 10^{-24}\,\mathrm{A\,m^2}$ ($\mathrm{J\,T^{-1}}$)
Nuclear magneton	μ_N	$5.051 \times 10^{-27}\,\mathrm{A\,m^2}$ ($\mathrm{J\,T^{-1}}$)
Fine-structure constant	$\alpha = e^2/4\pi\epsilon_0\hbar c$	$7.297 \times 10^{-3} = 1/137.0$
Compton wavelength of electron	$\lambda_c = h/m_e c$	$2.426 \times 10^{-12}\,\mathrm{m}$
Bohr radius	a_0	$5.2918 \times 10^{-11}\,\mathrm{m}$
angstrom	Å	$10^{-10}\,\mathrm{m}$
barn	b	$10^{-28}\,\mathrm{m^2}$
torr (mm Hg at 0 °C)	torr	133.32 Pa ($\mathrm{N\,m^{-2}}$)

Do not complete the attendance slip, fill in the front of the answer book or turn over the question paper until you are told to do so

Important Reminders

- Coats/outwear should be placed in the designated area.
- Unauthorised materials (e.g. notes or Tippex) must be placed in the designated area.
- Check that you do not have any unauthorised materials with you (e.g. in your pockets, pencil case).
- Mobile phones and smart watches **must** be switched off and placed in the designated area or under your desk. They must not be left on your person or in your pockets.
- You are not permitted to use a mobile phone as a clock. If you have difficulty seeing a clock, please alert an Invigilator.
- You are not permitted to have writing on your hand, arm or other body part.
- Check that you do not have writing on your hand, arm or other body part – if you do, you must inform an Invigilator immediately
- Alert an Invigilator immediately if you find any unauthorised item upon you during the examination.

Any students found with non-permitted items upon their person during the examination, or who fail to comply with Examination rules may be subject to Student Conduct procedures.