Video Material Week 06

Inductors

1. Introduction

An inductor is another circuit element that gives rise to time-dependent voltages and currents in an electric circuit. Once we have reviewed how inductors work, I will show that we can apply the same shorthand approach to analyse circuits containing inductors as we did for capacitors. Consequently, there is some element of revision in this video material.

Before we get started, it may be useful to keep the following idea in mind. Capacitors and inductors are devices that are able to store electrical energy. In the case of the capacitor, energy is stored in the electric field that develops across a capacitor when a battery is connected. In the case of the inductor, energy is stored in a magnetic field. Under certain circumstances, this enables capacitors and inductors to behave a bit like batteries or current sources, albeit rather limited ones.

The physics of inductors, as the name suggests, requires an understanding of the process of electromagnetic induction. This is covered in your course on electromagnetism. I will only review the essential facts that are needed to understand inductors as circuit elements. If you haven't encountered inductors before, the recommended reading given at the end of this video may help fill in some of the blanks.

2. Inductors

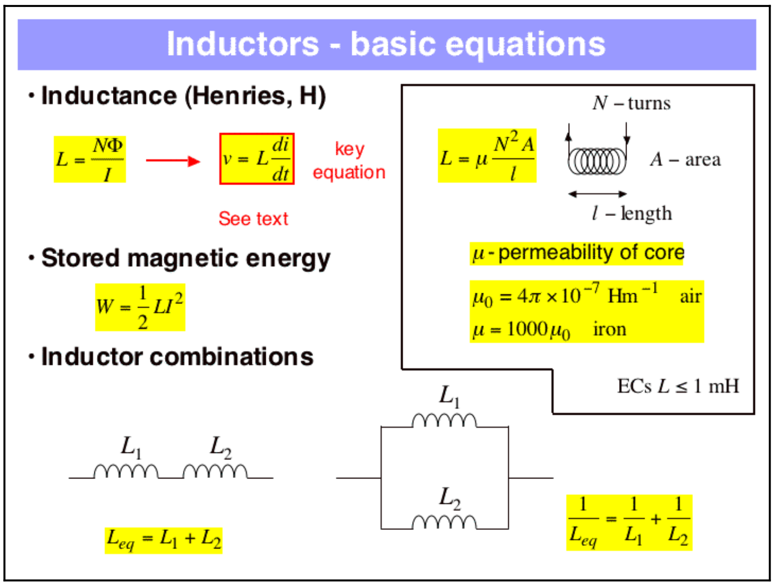

Figure 6.1: Basic equations involving inductors.

You can think of an inductor as nothing more than a coil of wire, like a solenoid, as shown in Figure 6.1. If you pass a current through a wire that forms a solenoid it produces a magnetic field inside. It turns out that the total magnetic flux (see definition below) is directly proportional to the current in the coil. The constant of proportionality is the self-inductance of the coil, L. For a coil with N turns, the inductance is given by

$$L = \frac{N \Phi_m}{I} \tag{1}$$

Magnetic flux is defined as the magnetic field strength, measured in Teslas, multiplied by the area within one loop of the coil, measured in metres squared. It is a measure of the number of field lines contained within each loop.

Since L is a constant, any change in the current must be accompanied by a corresponding change in the magnetic flux. You can also induce a current in a coil by changing the flux, by inserting a permanent magnet into a solenoid for example. Faraday's law says that such a change induces an EMF in the circuit, which is proportional to the rate of change of the flux.

$$E = -N \frac{d \Phi_m}{dt} = -L \frac{di}{dt} \tag{2}$$

EMF stands for Electro Motive Force. You sometimes find batteries referred to as sources of EMF. Tipler uses this terminology. It is an unfortunate term, since an EMF is not a force at all. It has units of joules per coulomb (or volts), so Faraday's law says that a change of flux in the coil will induce a potential difference across its ends. It also follows from equation (2) that a change in the current, also induces a potential difference across the ends of a coil.

Lenz's law tells us that the induced EMF (or potential difference) opposes the change. That is, if we increase the current in the solenoid, the solenoid temporarily works like a current source acting in the opposite direction. If we decrease the current, the solenoid temporarily works like a current source trying to maintain the current at its present level. Because of this **opposition** to changes in the current, the voltage across an inductor is actually given by

$$V = -E = +L \frac{di}{dt} \tag{3}$$

Notice that if the current is not changing there is no potential difference across the inductor. That is, the inductor looks like a short-circuit to a constant current.

This is the basic equation that allows us to treat inductors as circuit elements (see

Figure 6.1) since it defines the relationship between the voltage across the inductor and the current flowing through it. The plus sign in equation (3) is important. This point will be highlighted in the following discussion.

3. Inductors as circuit elements

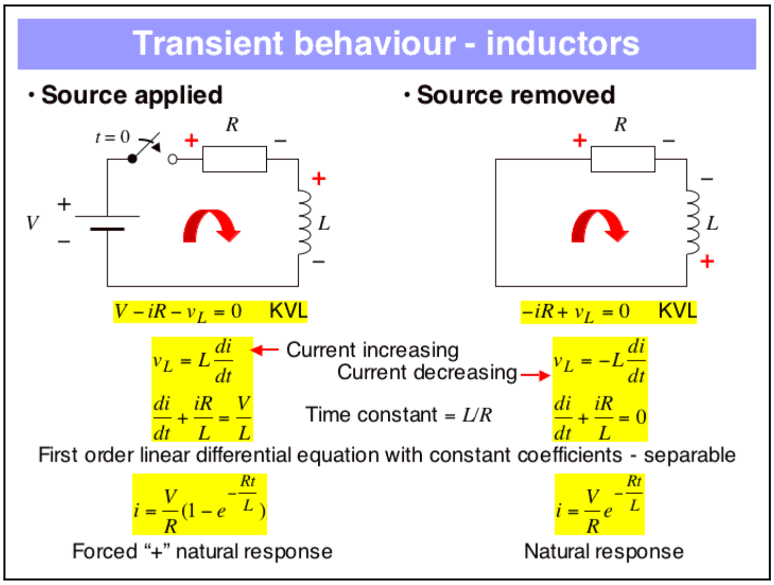

Figure 6.2: Transient behaviour in an inductor.

Let's consider what happens in a simple RL circuit when a source is first applied and then removed. In Figure 6.2 I've indicated the direction in which positive charge would flow (conventional current) by the red arrow. I've also indicated which end of each component is at the higher electrical potential, using the knowledge that resistors dissipate energy and inductors act to oppose the change occurring in the circuit. We can now consider the two cases individually. As before, this will involve a first order linear differential equation. Since this equation will have the same form as we found for RC circuits, I will only repeat the essential steps here.

3.1. Source applied (current increasing)

When the switch is closed a current may flow around the circuit. As current begins to flow some of the source voltage is dropped across the resistor. The remainder falls across the inductor, which tries to oppose the increase. By opposing the current, the inductor behaves like a source that acts against the battery. This tells us how to assign a polarity to the inductor. We can now use Kirchhoff's voltage law to sum up the potential rise and drops around the loop.

$$V - iR - v_L = 0 \tag{4}$$

This equation has two unknowns, the current and voltage across the inductor. We can eliminate one of these by using equation (3) to express the voltage across the inductor in terms of the rate of change of current in the circuit. Given that we know the inductor gives rise to a potential drop, it should now be apparent that the expression for the voltage must be $+Ldi/dt$ and not $-Ldi/dt$, as the latter would change the sign of the v_L in equation (4).

Making this substitution and arranging the equation in standard form, we find

$$\frac{di}{dt}+\frac{Ri}{L}=\frac{V}{L} \tag{5}$$

Since the first term has dimensions of current over time, it follows from the second term that the ratio L/R must have dimensions of time (because each term in any equation must have the same dimensions). As we shall see, this defines the time constant of the circuit. Solving this differential equation, it can be shown that

$$i=\frac{V}{R}\left(1-e^{-\frac{Rt}{L}}\right) \tag{6}$$

Check how we did this for capacitors in the previous lecture and see if you can verify that equation (6) is indeed a solution of equation (5) for yourself.

As you might expect from the material in the previous week, there are two terms in this equation. A constant term, known as the ***forced response***, and a time dependent term, known as the ***transient response***. The forced response is due to the battery, which determines the final current. The transient part is the exponential term that dies away with time and determines how quickly the current reaches that value. Notice that the initial current (when t = 0) is zero. A long time after the switch is closed (when $t \rightarrow \infty$) the current is $i = V/R$. What does this imply about the initial and final behaviour of the inductor after the switch is closed? (See section 4 below.)

3.2. Source removed (current decreasing)

A similar procedure can be followed to find the current when the battery is removed. Once again, the inductor opposes the change and this time acts to maintain the current. This determines the polarities that are shown in figure 6.2. The crucial point here is that the current decreases with time. In order to convey this, we must insert a minus sign in front of the differential term. This indicates a negative time-rate-of-change, or a decreasing trend.

$$v_L = -L\frac{di}{dt} \tag{7}$$

This time, the differential equation we obtain does not have a constant (or forcing) term because we no longer have a battery.

$$\frac{di}{dt} + \frac{Ri}{L} = 0 \tag{8}$$

To solve this equation we must specify the initial current. If the battery was connected for a long time before it being removed, the starting current would be given by $i = V/R$. The solution to equation (8) is then found to be

$$i = \frac{V}{R} e^{-\frac{Rt}{L}} \tag{9}$$

In the absence of a forcing term (no battery) we are left with only a transient term that decays away with time. Here, the magnetic energy stored in the inductor is given back to the circuit and dissipated in the resistor.

4. Initial and final response of an inductor

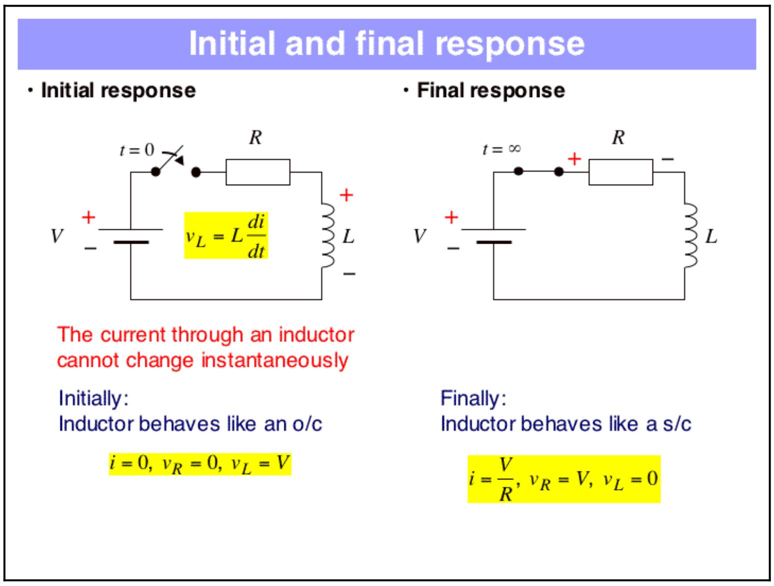

Figure 6.3: The initial and final response of an inductor.

When the switch is closed the current cannot suddenly jump from zero to some finite value. This is because the voltage across the inductor is given by $v_L = L\,di/dt$. A sudden jump in the current would represent an infinite rate of change in the current and induce an infinite potential difference across the inductor. This is just not possible.

What actually happens is that the EMF generated in the inductor initially opposes the current. That is, the inductor looks like an open-circuit (o/c) to the battery at the **instant** the switch is closed ($i = 0$ at $t = 0$). Therefore, all the applied voltage first appears across the inductor. However, the rate of increase in the current, di/dt, is not zero, so the current begins to increase. As the current begins to increase, more and more voltage is dropped across the resistor ($V = IR$). The increase finally stops when all the applied voltage is dropped across the resistor. When this happens, the inductor looks like a short-circuit (s/c) to the battery.

When the current is increasing, there is a potential drop across the inductor. Electrical potential energy is therefore given up to the inductor. *Where does this energy go?* It goes into building up the magnetic field inside the inductor. This energy is stored, rather than lost from the circuit. It is given back to the circuit if the current should start to decrease (for example if the source is removed) as the inductor will try and maintain the current at its present level. In this way, the inductor acts a bit like a current source when the battery is removed. The important point to remember is that the current in the inductor cannot change instantaneously. The initial value of the current in an inductor is determined by its value just before the change is made.

5. General solution for RL circuits

Using the preceding observations, we can deduce a general solution for RL circuits in exactly the same way as we did for RC circuits in the last exercise. We observe that the response of the RL circuit (see equations (6) and (9)) depends only on the initial and final values of the current. The transient part is an exponential characterised by the time-constant of the circuit. The general solution is summarised in Figure 6.4.

General solution for RC/RL circuits

- **Shorthand method for finding a solution**

  The response of the circuit may be deduced using:

  $$\text{Total response} = \text{Final Value} + [\text{Initial value} - \text{Final value}]e^{-\frac{t}{\tau}}$$

  [Voltage or current] [Forced response] [Natural response]

- **Comments**

  1. The transient part of the solution is given by $Ae^{-t/\tau}$ where

     $\tau = RC$ for an RC circuit

     $\tau = L/R$ for an RL circuit

  2. Works in circuits containing ...

     a single constant energy source, and a capacitor or inductor

     one or more resistors

     a switch that opens or closes at a known time (usually $t = 0$)

  3. The circuit must only contain linear components

Figure 6

6.4: The general solution for RL circuits.

Let's check this general method against the solutions we obtained when applying and removing a source to a simple RL circuit as shown in Figure 6.2.

Case 1. Current increasing.

The initial value of $i_L = 0$, since the current cannot change instantaneously.

The final value of $v_L = V/R$, since the inductor behaves like a short-circuit.

The time constant is L/R.

Inserting these values into the shorthand equation we obtain

$$i_L = \frac{V}{R} + \left(0 - \frac{V}{R}\right) e^{-\frac{Rt}{L}} \Rightarrow i_L = \frac{V}{R}\left(1 - e^{-\frac{Rt}{L}}\right)$$

Case 2: Current decreasing.

In the initial value of $i_L = V/R$, equal to current just before the source was removed.

The final value of $i_L = 0$, since there are no other sources present.

The time constant is L/R, as before.

Inserting these values into the shorthand equation we obtain

$$i_L = 0 + \left(\frac{V}{R} - 0\right) e^{-\frac{Rt}{L}} \Rightarrow i_L = \frac{V}{R} e^{-\frac{Rt}{L}}$$