Optics and Waves, Week 2

(a)

$y(x,t) = f(x - vt),$

At time t_1, we find a point at x_1 on the wave that has the maximum displacement y_m. At a later time t_2, the point with y_m is found at x_2. Since t_2 is greater than t_1, x_2 must be greater than x_1.

Although we selected the point with maximum displacement, the argument is true for all points. Therefore, all points appear to move in the + x direction. Keeping the displacement constantm we should have :

$$x_1 - vt_1 = x_2 - vt_2$$

$$\frac{x_2 - x_1}{t_2 - t_1} = v$$

Which describes a movement in the + x direction with velocity v.

(b)

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}$$

$$y(x,t) = f(x - vt),$$

$$y(x,t) = f(u), u = x - vt$$

$$\frac{\partial^2 y}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{df}{du}\frac{\partial u}{\partial x}\right) = \frac{d^2 f}{du^2}$$

$$y(x,t) = f(u), u = x - vt$$

$$\frac{\partial^2 y}{\partial t^2} = \frac{\partial}{\partial t}\left(\frac{\partial f}{\partial t}\right) = \frac{\partial}{\partial t}\left(\frac{df}{du}\frac{\partial u}{\partial t}\right) = \frac{\partial}{\partial t}\left(\frac{df}{du}(-v)\right) = \frac{d^2 f}{du^2}v^2$$

Hence

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}$$

(c)

$$\frac{\partial^2 y}{\partial t^2} = \frac{\partial^2}{\partial t^2}(De^{-(Bx-Ct)^2}) = \frac{\partial}{\partial t}\left[2DC(Bx - Ct)\ e^{-(Bx-Ct)^2}\right]$$

$$= 4DC^2(Bx - Ct)^2 e^{-(Bx-Ct)^2} - 2DC^2 e^{-(Bx-Ct)^2}$$

$$\frac{\partial^2 y}{\partial x^2} = \frac{\partial^2}{\partial x^2}(De^{-(Bx-Ct)^2}) = \frac{\partial}{\partial x}\left[2DB(Bx-Ct)\ e^{-(Bx-Ct)^2}\right]$$
$$= 4DB^2(Bx-Ct)^2 e^{-(Bx-Ct)^2} - 2DB^2 e^{-(Bx-Ct)^2}$$

So $\frac{\partial^2 y}{\partial x^2} = \frac{B^2}{C^2}\frac{\partial^2 y}{\partial t^2}$

Hence the velocity is C/B.