Solution Sheet 0: Ordinary linear differential equations in time

These examples test both solving ordinary linear differential equations and integration. One can guess the particular integral and then use the exponential ansatz to find the complementary functions if desired. The solutions here will use the integrating factor technique:
Differentiation of a product provides

$$\frac{d}{dt}\left[e^{-at}x\right] = e^{-at}\left[\frac{dx}{dt} - ax\right]$$

and this works for any function $x(t)$ so we may write an identity

$$e^{at}\frac{d}{dt}e^{-at} \equiv \frac{d}{dt} - a$$

which generalises to

$$\frac{d^2x}{dt^2} - (a_1 + a_2)\frac{dx}{dt} + a_1a_2x = \left[\frac{d}{dt} - a_1\right]\left[\frac{d}{dt} - a_2\right]x = e^{a_1t}\frac{d}{dt}\left[e^{-a_1t}e^{a_2t}\frac{d}{dt}\left(e^{-a_2t}x\right)\right]$$

and all we need to do is find the roots of the relevant polynomial and we can turn all of these questions into pure integration.

1. Solve, for the general solution,

$$\frac{dx}{dt} + x = 1$$

$$\frac{dx}{dt} + x = e^{2t}$$

$$\frac{dx}{dt} + x = \sin t$$

$$\frac{dx}{dt} + x = e^{-t}$$

Answer 1. To solve the general problem

$$\frac{dx}{dt} + x = f(t)$$

we may rewrite

$$e^{-t}\frac{d}{dt}\left(e^{t}x\right) = \frac{dx}{dt} + x = f(t) \quad \Rightarrow \quad x(t) = e^{-t}\left[A + \int^t dse^s f(s)\right]$$

and then the four integrals are

$$e^{-t}\int^t dse^s = 1$$

$$e^{-t}\int^t dse^s e^{2s} = \frac{e^{2t}}{3}$$

$$e^{-t}\int^t dse^s \sin s = e^{-t}\frac{1}{2i}\int^t dse^s\left[e^{is}-e^{-is}\right] = e^{-t}\frac{1}{2i}\left[\frac{e^{t+it}}{1+i}-\frac{e^{t-it}}{1-i}\right]$$

$$= \frac{1}{2i}\left[\frac{1-i}{2}e^{it}-\frac{1+i}{2}e^{-it}\right] = \frac{1}{2}\sin t - \frac{1}{2}\cos t$$

$$e^{-t}\int^t dse^s e^{-s} = te^{-t}$$

2. Solve, for the general solution,

$$\frac{d^2x}{dt^2} - x = 1$$

$$\frac{d^2x}{dt^2} - x = e^{2t}$$

$$\frac{d^2x}{dt^2} - x = \sin t$$

$$\frac{d^2x}{dt^2} - x = e^{-t}$$

Answer 2. To solve the general problem

$$\frac{d^2x}{dt^2} - x = f(t)$$

we may rewrite

$$\frac{d^2x}{dt^2} - x = \left[\frac{d}{dt}+1\right]\left[\frac{d}{dt}-1\right]x = e^{-t}\frac{d}{dt}\left[e^te^t\frac{d}{dt}\left(e^{-t}x\right)\right] = f(t)$$

and then integrate once to get

$$e^te^t\frac{d}{dt}\left(e^{-t}x\right) = A + \int^t dse^s f(s)$$

and then integrate again to get

$$x(t) = e^t\left[B - \frac{A}{2}e^{-2t} + \int^t dpe^{-2p}\int^p dse^s f(s)\right]$$

and then the four double integrals are

$$e^t\int^t dpe^{-2p}\int^p dse^s = e^t\int^t dpe^{-2p}e^p = -e^te^{-t} = -1$$

$$e^t \int^t dp e^{-2p} \int^p ds e^s e^{2s} = e^t \int^t dp e^{-2p} \frac{e^{3p}}{3} = e^t \frac{e^t}{3} = \frac{e^{2t}}{3}$$

$$e^t \int^t dp e^{-2p} \int^p ds e^s \sin s = e^t \int^t dp e^{-2p} e^p \frac{1}{2} [\sin p - \cos p]$$

where we used the answer to question 1 to do the first integral

$$= e^t \int^t dp e^{-p} \frac{1}{2} \left[\frac{1-i}{2i} e^{ip} - \frac{1+i}{2i} e^{-ip} \right] = e^t \frac{1}{2} \left[-\frac{1}{2i} e^{-t+it} + \frac{1}{2i} e^{-t-it} \right] = -\frac{1}{2} \sin t$$

and clearly the particular integral idea would have been better here

$$e^t \int^t dp e^{-2p} \int^p ds e^s e^{-s} = e^t \int^t dp e^{-2p} p = e^t \left[-\frac{1}{2} e^{-2t} t + \frac{1}{2} \int^t dp e^{-2p} \right]$$

$$= -\frac{1}{2} e^{-t} - \frac{1}{4} t e^{-t}$$

where we used integration by parts. This calculation is easier using the integrating factor.

3. Solve, for the general solution,

$$\frac{d^2x}{dt^2} + \frac{dx}{dt} = 1$$

$$\frac{d^2x}{dt^2} + \frac{dx}{dt} = e^{2t}$$

$$\frac{d^2x}{dt^2} + \frac{dx}{dt} = \sin t$$

$$\frac{d^2x}{dt^2} + \frac{dx}{dt} = e^{-t}$$

Answer 3. Once again we may rewrite the general problem

$$\frac{d^2x}{dt^2} + \frac{dx}{dt} = f(t)$$

as

$$\frac{d^2x}{dt^2} + \frac{dx}{dt} = \frac{d}{dt} \left[e^{-t} \frac{d}{dt} \left(e^t x \right) \right] = f(t)$$

and then integrate

$$e^{-t} \frac{d}{dt} \left(e^t x \right) = A + \int^t ds f(s)$$

and integrate again

$$x(t) = e^{-t} \left[B + A e^t + \int^t dp e^p \int^p ds f(s) \right]$$

and then the four double integrals are

$$e^{-t}\int^t dpe^p\int^p ds = e^{-t}\int^t dpe^p p = e^{-t}\left[te^t - \int^t dpe^p\right] = t-1$$

$$e^{-t}\int^t dpe^p\int^p dse^{2s} = e^{-t}\int^t dpe^p\frac{1}{2}e^{2p} = \frac{1}{6}e^{2t}$$

$$e^{-t}\int^t dpe^p\int^p ds\sin s = -e^{-t}\int^t dpe^p\cos p = -e^{-t}\left[\int^t dpe^p\frac{1}{2}\left(e^{ip}+e^{-ip}\right)\right]$$

$$= -e^{-t}\frac{1}{2}\left[\frac{1}{1+i}e^{t+it}+\frac{1}{1-i}e^{t-it}\right] = -e^{-t}\frac{1}{2}\left[\frac{1-i}{2}e^{t+it}+\frac{1+i}{2}e^{t-it}\right]$$

$$= \frac{1}{2}\left[-\cos t - \sin t\right]$$

$$e^{-t}\int^t dpe^p\int^p dse^{-s} = -e^{-t}\int^t dpe^p e^{-p} = -e^{-t}t$$

4. Solve, for the general solution,

$$\frac{d^2x}{dt^2}+x = 1$$

$$\frac{d^2x}{dt^2}+x = e^{2t}$$

$$\frac{d^2x}{dt^2}+x = \sin t$$

$$\frac{d^2x}{dt^2}+x = e^{-t}$$

Answer 4. Once again we may rewrite the general problem

$$\frac{d^2x}{dt^2}+x = f(t)$$

as

$$\frac{d^2x}{dt^2}+x = \left[\frac{d}{dt}+i\right]\left[\frac{d}{dt}-i\right]x = e^{-it}\frac{d}{dt}\left[e^{it}e^{it}\frac{d}{dt}\left(e^{-it}x\right)\right] = f(t)$$

and then integrate

$$e^{it}e^{it}\frac{d}{dt}\left(e^{-it}x\right) = A + \int^t dse^{is}f(s)$$

and integrate again

$$x(t) = e^{it}\left[B - \frac{A}{2i}e^{-2it} + \int^t dpe^{-2ip}\int^p dse^{is}f(s)\right]$$

and then the four double integrals are

$$e^{it}\int^t dpe^{-2ip}\int^p dse^{is} = e^{it}\int^t dpe^{-2ip}\frac{1}{i}e^{ip} = e^{it}\frac{1}{-i}\frac{1}{i}e^{-it} = 1$$

$$e^{it}\int^t dpe^{-2ip}\int^p dse^{is}e^{2s} = e^{it}\int^t dpe^{-2ip}\frac{1}{2+i}e^{2p+ip} = e^{it}\frac{1}{2+i}\frac{1}{2-i}e^{2t-it} = \frac{1}{5}e^{2t}$$

$$e^{it}\int^t dpe^{-2ip}\int^p dse^{is}\sin s = e^{it}\int^t dpe^{-2ip}\int^p dse^{is}\frac{1}{2i}\left(e^{is}-e^{-is}\right)$$

$$= e^{it}\int^t dpe^{-2ip}\frac{1}{2i}\left(\frac{1}{2i}e^{2ip}-p\right) = -\frac{1}{4}e^{it}t + e^{it}\frac{1}{2i}\frac{1}{2i}te^{-2it} + e^{it}\frac{1}{4}\int^t dpe^{-2ip}$$

$$= -\frac{1}{2}t\cos t - \frac{1}{8i}e^{-it}$$

where we used integration by parts and the final term is a complementary function.

$$e^{it}\int^t dpe^{-2ip}\int^p dse^{is}e^{-s} = e^{it}\int^t dpe^{-2ip}\frac{1}{i-1}e^{-p+ip} = e^{it}\frac{1}{i-1}\frac{1}{-i-1}e^{t-it} = \frac{1}{2}e^{-t}$$

5. Solve, for the general solution,

$$\frac{d^2x}{dt^2} + 2\nu\frac{dx}{dt} + (\nu^2+1)x = 1$$

$$\frac{d^2x}{dt^2} + 2\nu\frac{dx}{dt} + (\nu^2+1)x = e^{2t}$$

$$\frac{d^2x}{dt^2} + 2\nu\frac{dx}{dt} + (\nu^2+1)x = \sin t$$

$$\frac{d^2x}{dt^2} + 2\nu\frac{dx}{dt} + (\nu^2+1)x = e^{-t}$$

Answer 5. This example is very germane to the course. We can apply the same technique to the general problem

$$\frac{d^2x}{dt^2} + 2\nu\frac{dx}{dt} + (\nu^2+1)x = f(t)$$

to obtain

$$\frac{d^2x}{dt^2} + 2\nu\frac{dx}{dt} + (\nu^2+1)x = \left[\frac{d}{dt}+\nu+i\right]\left[\frac{d}{dt}+\nu-i\right]x$$

$$= e^{-\nu t-it}\frac{d}{dt}\left[e^{\nu t+it}e^{-\nu t+it}\frac{d}{dt}\left(e^{\nu t-it}x\right)\right] = f(t)$$

and then integrate

$$e^{2it}\frac{d}{dt}\left(e^{\nu t-it}x\right) = A + \int^t dse^{\nu s+is}f(s)$$

and then integrate again

$$x(t) = e^{-\nu t+it}\left[B - \frac{A}{2i}e^{-2it} + \int^t dpe^{-2ip}\int^p dse^{\nu s+is}f(s)\right]$$

and all we have left is the four double integrals

$$e^{-\nu t+it}\int^t dpe^{-2ip}\int^p dse^{\nu s+is} = e^{-\nu t+it}\int^t dpe^{-2ip}\frac{1}{\nu+i}e^{\nu p+ip}$$

$$= e^{-\nu t+it}\frac{1}{(\nu+i)}\frac{1}{(\nu-i)}e^{\nu t-it} = \frac{1}{\nu^2+1}$$

$$e^{-\nu t+it}\int^t dpe^{-2ip}\int^p dse^{\nu s+is}e^{2s} = e^{-\nu t+it}\int^t dpe^{-2ip}\frac{1}{2+\nu+i}e^{2p+\nu p+ip}$$

$$= e^{-\nu t+it}\frac{1}{(2+\nu+i)}\frac{1}{(2+\nu-i)}e^{2t+\nu t-it} = \frac{1}{(2+\nu)^2+1}e^{2t}$$

$$e^{-\nu t+it}\int^t dpe^{-2ip}\int^p dse^{\nu s+is}\sin s = e^{-\nu t+it}\int^t dpe^{-2ip}\int^p dse^{\nu s+is}\frac{1}{2i}\left(e^{is}-e^{-is}\right)$$

$$= e^{-\nu t+it}\int^t dpe^{-2ip}\frac{1}{2i}\left(\frac{1}{\nu+2i}e^{\nu p+2ip} - \frac{1}{\nu}e^{\nu p}\right)$$

$$= \frac{1}{2i}\left(e^{it}\frac{1}{\nu+2i}\frac{1}{\nu} - e^{-it}\frac{1}{\nu}\frac{1}{\nu-2i}\right) = \frac{1}{\nu}\left[\frac{\nu\sin t - 2\cos t}{\nu^2+4}\right]$$

$$e^{-\nu t+it}\int^t dpe^{-2ip}\int^p dse^{\nu s+is}e^{-s} = e^{-\nu t+it}\int^t dpe^{-2ip}\frac{1}{\nu-1+i}e^{\nu p-p+ip}$$

$$= e^{-\nu t+it}\frac{1}{(\nu-1+i)}\frac{1}{(\nu-1-i)}e^{\nu t-t-it} = \frac{1}{(\nu-1)^2+1}e^{-t}$$

There are many ways to solve these equations, the technique employed here is technically good because it gives an answer even if the resulting integral may not be obtained in closed form. One may also feed in the boundary conditions after each integration which is also advantageous (sometimes). Feel free to use a simpler particular method attuned to the problem if you can think of one....