Quantum Mechanics 1 – Solution 10

a) The kinetic energy operator is

$$\hat{T} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}.$$

Applying this to the wave function gives

$$\hat{T}\Psi = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}A\cos(kx)\,e^{-i\omega t} = \frac{\hbar^2 k^2}{2m}A\cos(kx)\,e^{-i\omega t} = \frac{\hbar^2 k^2}{2m}\Psi.$$

[1 mark]

This is an example of an eigenvalue equation:

operator x wave function = constant x <u>same</u> wave function.

Therefore, the wave function corresponds to an eigenfunction of kinetic energy.

[1 mark]

b) The momentum operator in the x-direction is

$$\hat{p}_x = -i\hbar\frac{\partial}{\partial x}.$$

Applying this to the wave function gives

$$\hat{p}_x\Psi = -i\hbar\frac{\partial}{\partial x}A\cos(kx)\,e^{-i\omega t} = i\hbar k A\sin(kx)\,e^{-i\omega t}$$

[1 mark]

This does not result in an eigenvalue equation as the operation does not return the original wavefunction multiplied by a constant. Therefore, the original wave function is <u>not</u> an eigenfunction of the momentum operator in the *x*-direction.

[1 mark]

c) The *expectation value* (or "average" value) of p_x is

$$\langle p_x\rangle = \int_{-\infty}^{+\infty}\Psi^*\,(\hat{p}_x\Psi)dx = i\hbar k A^2\int_{-L/2}^{+L/2}\cos(kx)\sin(kx)\,dx = 0.$$

[1 mark]

Note that the wavefunction is zero outside the well, so the integral is performed only between the boundaries of the well. To solve the integral:

EITHER: State that the integral involves an *odd* function integrated over *symmetric* limits, so the integral is zero by inspection.

OR: Solve the integral explicitly. The most straightforward approach is to use the double angle identity $\sin(2\theta) = 2\sin\theta\cos\theta$. Clearly, the integrand is still an odd function. Integrating gives a cosine, which is an even function. As the limits are symmetric about zero, the contributions to the integral at the upper and lower limits are the same (same magnitude and same sign), independent of the value of *k*, so the integral evaluates to zero. [1 mark]

d) The wave function is an eigenfunction of the kinetic energy operator, so it describes a particle with a ***well-defined value of kinetic energy.*** The particle is therefore in one of the energy eigenstates allowed within the infinite square well. [1 mark]

However, the wave function is not an eigenfunction of the momentum operator so it does ***does not have a well-defined value of momentum*** in the well. This is because the wavefunction of a particle in a 1-d potential well can be written as a superposition of ***two momentum eigenstates, simultaneously describing the particle moving in the positive x-direction with momentum $\boldsymbol{p_x = +\hbar k}$ and moving to in the negative x-direction with momentum $\boldsymbol{p_x = -\hbar k}$.*** [2 marks]

The *expectation value* of the x-component of momentum is equivalent to the average value of many measurements carried out on many particles prepared in the same quantum state, such that all are described by the same initial wave function. ***Under these circumstances, we would expect to find $\boldsymbol{p_x = +\hbar k}$, in half of the measurements and $\boldsymbol{p_x = -\hbar k}$ in the other half of the measurements. On <u>average</u> therefore, we would expect to measure zero momentum***. [1 mark]

There are two apparent discrepancies here: the fact that the kinetic energy is well defined in this state but not the momentum and the fact that the expectation value of the x-component of momentum is zero, when the particle cannot be in a zero-momentum state. These discrepancies can be explained by the fact that momentum is a vector quantity, having both a magnitude and a direction. Although the magnitude of the momentum of the particle is well defined in this state, its direction is not.