Mechanics week 1: Units, dimensions and kinematics

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1 Introduction

Classical Newtonian mechanics is the foundation of applied mathematics and is an astonishingly powerful tool for explaining physical systems, from projectiles to planetary motion to the design of racing cars. It acts as a natural starting point for any serious discussion of mathematical modelling in broader areas. This module uses ideas such as forces, moments, Newton's Laws of Motion and energy to model practical situations. These models can then be analysed using a wide range of techniques from pure mathematics such as trigonometry, algebra, calculus and, in particular, vector methods. We will make use of some techniques you have already seen in other modules, and introduce further concepts which will be taught more formally/rigorously in later modules (e.g. in VGLA or in second year Differential Equations)- this serves as motivation as to why these methods are interesting and useful! See the crib sheet on Canvas or the appendix of the full notes if you need any refreshers!

This week we will start by considering units and dimensions, followed by *kinematics* i.e. how we represent the motion of a particle.

2 Notation

First a quick note about notation:

Vectors I will use the printed notation $\mathbf{a}$ for a vector, with handwritten notation $\underline{a}$.

Differentiation I will use "dot" to mean differentiation with respect to time, so that

$$\dot{x} = \frac{dx}{dt},$$

and

$$\ddot{x} = \frac{d^2x}{dt^2}.$$

3 Units and dimensions

We begin with some discussion about units and dimensions - this is a crucial tool for checking whether something is right or not, and really fundamental to producing accurate and useful results.

Throughout this course we shall use SI units, which were adopted internationally in 1960. The system has seven *basic units*, one for each of the so called basic quantities: mass, length, time, electric current, temperature, luminous intensity and amount of a substance. All other quantities are measured in terms of *derived units* which are obtained from the basic units by multiplication or division. For example, volume is measured in terms of the unit of length raised to the third power, while velocity is measured in terms of units of length per unit of time.

Basic Quantity	Symbol	Basic Unit	Symbol
mass	M	kilogram	kg
length	L	metre	m
time	T	second	s
electric current	i	ampere	A
temperature	Θ	Kelvin	K
luminous intensity	I	candela	cd
amount of a substance	N	mole	mol

The dimensions of a quantity show how it is related to the basic quantities. We use square brackets to indicate that we are specifying the dimensions of the quantity.

Example 1:

- Volume V has dimensions of length cubed which we write as

$$[V] = [L^3], \qquad \text{SI units m}^3.$$

- Density of a substance (often denoted ρ) is mass per unit volume so that

$$[\rho] = [ML^{-3}] \qquad \text{SI units kgm}^{-3}.$$

- A velocity v is distance per time, hence

$$[v] = [LT^{-1}] \qquad \text{SI units ms}^{-1}.$$

Every derived quantity has dimensions. Some derived SI units also have names e.g. forces are often denoted by Newtons (1 N= 1 kg m s^{-2}).

Both derivatives and integrals of physical quantities have dimensions. Suppose that V is a volume then the rate of change of volume with time t is $\frac{dV}{dt}$ which has dimensions

$$\left[\frac{dV}{dt}\right] = \frac{[V]}{[t]} = \frac{[L^3]}{[T]} = [L^3T^{-1}].$$

The second derivative can be thought of as

$$\frac{d^2V}{dt^2} = \frac{d}{dt}\left(\frac{dV}{dt}\right),$$

and hence

$$\left[\frac{d^2V}{dt^2}\right] = \frac{1}{[T]}\left[\frac{dV}{dt}\right] = \frac{[V]}{[t^2]} = \frac{[L^3]}{[T^2]} = [L^3T^{-2}].$$

Similarly an integral with respect to time t is essentially multiplying by a time, so has dimensions

$$\left[\int V\,dt\right] = [V][t] = [L^3][T] = [L^3T].$$

Note that the arguments of e.g. trig functions, exponentials and logarithms must all be nondimensional, so $\sin(\omega t)$ must have that ωt is nondimensional, and hence $[\omega t]$=[1], so $[\omega]$=$[\mathrm{T}^{-1}]$.

Note! Using a consistent set of units is essential! Particularly when working with other disciplines, who may have a tendency to use very odd and inconsistent unit systems, care is needed. There are various examples of things going very wrong when people make mistakes with inconsistent units (for example the Hubble Space Telescope and the Mars Climate Orbiter).

4 Principle of dimensional homogeneity

An equation derived from physical principles must be dimensionally homogeneous, that is

- The dimensions of the right-hand side must be the same as the dimensions of the left-hand side.
- The dimensions of each additive terms must be the same.

This is necessary (but not sufficient) for the equation to be valid. If you end up with a dimensionally inhomogeneous equation it is definitely wrong! Comparing dimensions can often help find the source of the error.

Example 2: Dimensions

Is the equation

$$\rho v = \frac{m}{At^2} + \frac{1}{V}\frac{dm}{dt}\int v dt$$

where

- ρ is density ($[ML^{-3}]$),
- v is velocity ($[LT^{-1}]$),
- m is mass ($[M]$),
- A is area ($[L^2]$),
- V is volume ($[L^3]$),
- t is time ($[T]$),

dimensionally consistent?

Solution. Each term has dimensions:

$$[\rho v] = [ML^{-3}][LT^{-1}] = [ML^{-2}T^{-1}].$$

$$\left[\frac{m}{At^2}\right] = [ML^{-2}T^{-2}].$$

$$\left[\frac{1}{V}\frac{dm}{dt}\int v dt\right] = [MT^{-1}][L]/[L^3] = [ML^{-2}T^{-1}].$$

Thus our equation is dimensionally inhomogeneous, so it must be wrong and we suspect there may be an error with the first term on the right hand side. ◀

4.1 Dimensions, units and writing equations

When you write a model based on real world principles you should denote everything which might change or which has dimensions as a symbol rather than using the numerical value. For example you should use "g" rather than "9.81m/s^2" for acceleration due to gravity, or "c" rather than "3 x 10^8 m/s" for the speed of light. This means that your equation is as general as possible (e.g. it will still work on the moon rather than on Earth) and it will hold in whichever set of (consistent!) units you choose. This then means that any actual numbers which appear will always be nondimensional.

Beware, however, that some people (particularly non-mathematicians!) don't always follow this rule, so you should approach any model that you've not written yourself with caution!

Activity: *You should now be able to tackle question 1 on this week's problem sheet.*

5 Newtonian mechanics

Much of classical mechanics is based on the fundamentals laid down by Newton in the 1600s, requiring the ideas of calculus and differential equations to be developed. We now know that Newtonian mechanics does not describe very small or very fast things well (e.g. atoms or galaxies) where we need quantum theory or relativity, but for many situations Newtonian mechanics provides sufficient accuracy to answer many fundamental questions. The techniques used also form the building blocks for much of applied mathematics.

6 Position, velocity and acceleration - kinematics

We first need a geometric framework in which to formulate models. We will consider three-dimensional space described using a position vector relative to some origin and a set of (probably) perpendicular axes e.g. Cartesian space with axes $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ in the x, y, z direction respectively (see Fig 1). This gives the *frame of reference*.

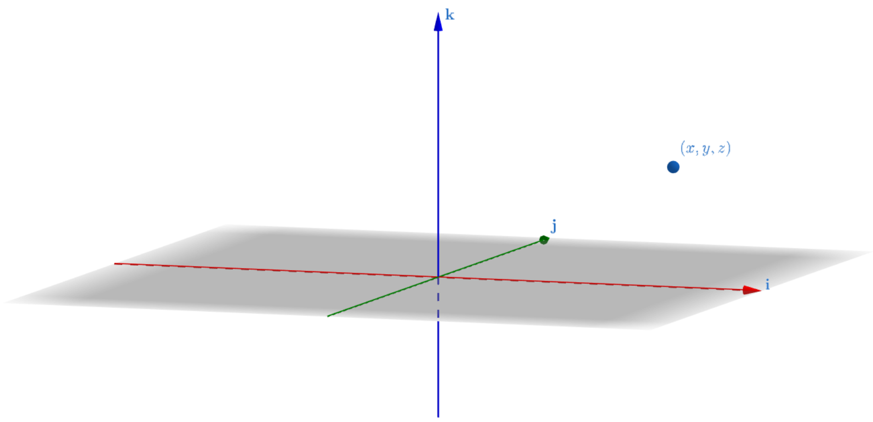

Figure 1: Cartesian axes showing perpendicular vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ and a point with coordinates (x, y, z).

We will consider how a particle moves in time through space. A particle is a mathematical object which has mass, but does not have size or shape. This is a good approximation for an object which is heavy but small compared to the other lengths in the system. For example representing the Earth as a particle is a good approximation when considering planetary motion around the sun, but bad when considering the moon's orbit around the Earth.

We pick an origin O, then the location of a particle is given by its position vector $\mathbf{r}$

measured from the origin. If the particle is moving then

$$\mathbf{r} \quad = \quad \mathbf{r}\,(t)$$

gives the particle path as a three-dimensional curve, with the particle location depending on the current time. In Cartesians this can be written as

$$\mathbf{r} \quad = \quad x\,(t)\,\mathbf{i} + y\,(t)\,\mathbf{j} + z\,(t)\,\mathbf{k}.$$

See Fig 2.

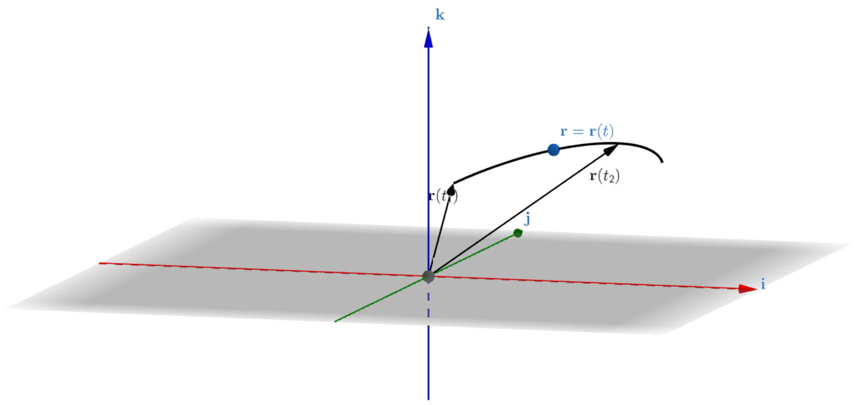

Figure 2: Cartesian axes showing perpendicular vectors **i**, **j**, **k** and a line represented by a position vector **r**. Points are marked for times t_1 and t_2.

Given the particle's position vector we can then calculate the velocity of the particle, which is given by the instantaneous rate of change of position with respect to time

$$\begin{aligned}\mathbf{v} \quad &= \quad \frac{d\mathbf{r}}{dt}, \\ &= \quad \frac{dx\,(t)}{dt}\mathbf{i} + \frac{dy\,(t)}{dt}\mathbf{j} + \frac{dz\,(t)}{dt}\mathbf{k},\end{aligned}$$

in Cartesians. We will use the notation $\dot{} \equiv d/dt$ so that

$$\mathbf{v} \quad = \quad \dot{\mathbf{r}}.$$

Acceleration of a particle **a** is defined as the instantaneous rate of change of velocity with time:

$$\begin{aligned}
\mathbf{a} &= \frac{d\mathbf{v}}{dt} \\
&= \frac{d^2\mathbf{r}}{dt^2} \\
&= \frac{d^2x}{dt^2}\mathbf{i} + \frac{d^2y}{dt^2}\mathbf{j} + \frac{d^2z}{dt^2}\mathbf{k},
\end{aligned}$$

in Cartesians.

Note that position, velocity and acceleration are all vectors so they have magnitude and direction. The magnitude of position is

$$r = |\mathbf{r}| = \sqrt{x^2 + y^2 + z^2},$$

the magnitude of velocity (also known as speed) is

$$v = |\mathbf{v}| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}$$

and the magnitude of acceleration is

$$a = |\mathbf{a}| = \sqrt{\left(\frac{d^2x}{dt^2}\right)^2 + \left(\frac{d^2y}{dt^2}\right)^2 + \left(\frac{d^2z}{dt^2}\right)^2}.$$

In general, position, velocity and acceleration will all be pointing in different directions, and will have different magnitude. In particular, it is possible to have varying velocity but constant speed if the direction of motion is changing but the magnitude is fixed. Given position, velocity or acceleration (and appropriate initial conditions if required) we can calculate the others by differentiating/integrating.

Example 3: Parametric position vector

Consider a particle with position vector

$$\mathbf{r} \;=\; \cos(\omega t)\mathbf{i} + \sin(\omega t)\mathbf{j}(+0\mathbf{k}).$$

What are the velocity and acceleration vectors? What shape does the particle path make?

Solution. We can calculate the particle's velocity by differentiating with respect to time to give

$$\begin{aligned}
\mathbf{v} = \frac{d\mathbf{r}}{dt} &= \frac{d}{dt}\left(\cos(\omega t)\right)\mathbf{i} + \frac{d}{dt}\left(\sin(\omega t)\right)\mathbf{j}, \\
&= -\omega \sin(\omega t)\mathbf{i} + \omega \cos(\omega t)\mathbf{j},
\end{aligned}$$

and acceleration by differentiating again to give

$$\begin{aligned}\mathbf{a} = \frac{d\mathbf{v}}{dt} &= \frac{d}{dt}\left(-\omega\sin(\omega t)\mathbf{i} + \omega\cos(\omega t)\mathbf{j}\right), \\ &= -\omega^2\cos(\omega t)\mathbf{i} - \omega^2\sin(\omega t)\mathbf{j}.\end{aligned}$$

Note that $\mathbf{a} = -\omega^2\mathbf{r}$ - in this case the acceleration acts in the opposite direction to the position vector.

We can find the shape of the particle path by eliminating t from the position vector. Since $x = \cos(\omega t)$, $y = \sin(\omega t)$, then

$$\begin{aligned}x^2 + y^2 &= \cos^2(\omega t) + \sin^2(\omega t), \\ &= 1.\end{aligned}$$

i.e. the particle moves in a circle. The acceleration is therefore pointing towards the centre of the circle, with the velocity tangent to the circle (note that $\mathbf{v}\cdot\mathbf{r} = -\omega\sin(\omega t)\cos(\omega t) + \omega\cos(\omega t)\sin(\omega t) = 0$). See fig 3.

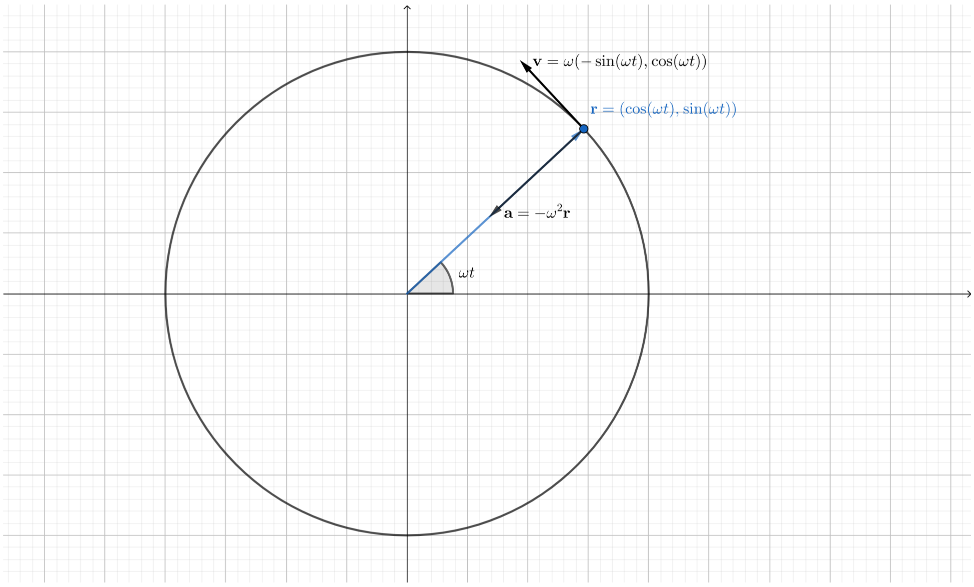

Figure 3: A plot showing the position, velocity and acceleration vectors.

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Example 4: Parametric acceleration vector

Similarly, if we had been given the acceleration we could have gone in the other direction by integrating to find the velocity and position vectors. In this case we would also need to know where the particle started, and how quickly it was moving at that point - i.e. the initial conditions. For example, if a particle starts at $(1,0)$ with velocity $(0,\omega)$ subject to an acceleration of the form

$$\mathbf{a} = -\omega^2\cos(\omega t)\mathbf{i} - \omega^2\sin(\omega t)\mathbf{j},$$

what are the velocity and position vectors?

Solution. Then

$$\begin{aligned}
\mathbf{v} &= \int \mathbf{a}\ \mathrm{d}t, \\
&= \left(\int -\omega^2\cos(\omega t)\ \mathrm{d}t + c_1\right)\mathbf{i} + \left(\int -\omega^2\sin(\omega t)\ \mathrm{d}t + c_2\right)\mathbf{j}, \\
&= \left(-\omega\sin(\omega t) + c_1\right)\mathbf{i} + \left(\omega\cos(\omega t) + c_2\right)\mathbf{j},
\end{aligned}$$

where $\mathbf{c} = c_1\mathbf{i} + c_2\mathbf{j}$ gives the (now vector) constant of integration. This is found using the initial condition that $\mathbf{v} = (0,\omega) = \omega\mathbf{j}$. Hence

$$\omega\mathbf{j} = c_1\mathbf{i} + (\omega + c_2)\mathbf{j},$$

and so $c_1 = 0$ and $\omega + c_2 = \omega$, hence $c_2 = 0$ by equating coefficients of $\mathbf{i}$ and $\mathbf{j}$.

Then

$$\begin{aligned}
\mathbf{r} &= \int \mathbf{v}\ \mathrm{d}t, \\
&= \int -\omega\sin(\omega t)\mathbf{i} + \omega\cos(\omega t)\mathbf{j}\ \mathrm{d}t, \\
&= \left(\cos(\omega t) + d_1\right)\mathbf{i} + \left(\sin(\omega t) + d_2\right)\mathbf{j},
\end{aligned}$$

with $\mathbf{d} = (d_1, d_2)$ giving the constant of integration. Now, since $\mathbf{r} = (1,0) = \mathbf{i}$ initially, this gives

$$\mathbf{i} = (1 + d_1)\mathbf{i} + d_2\mathbf{j},$$

and hence $d_1 = 0$, $d_2 = 0$ and

$$\mathbf{r} = \cos(\omega t)\,\mathbf{i} + \sin(\omega t)\,\mathbf{j},$$

as before. ◀

Expressions such as these allow us to describe the *kinematics* (i.e. motion) of the particle, but in general we want to work out what those kinematics will be in reponse to external forces. For this we need Newton's laws, which we will learn about next week.

Activity: *You should now be able to tackle questions 2 and 3 on this week's problem sheet.*