Quantum Mechanics 1 – Solution 3

a) This question requires the use of the photoelectric equation:

$$KE_{max} = hf - \phi,$$

where KE_{max} is the maximum kinetic energy of the photoelectrons, h is Planck's constant, f is the frequency of the incident light and ϕ is the work function of the material.

First, use $f = c/\lambda$ to find the frequencies corresponding to the two wavelengths:

$$\lambda = 512 \text{ nm} \implies f = 5.86 \times 10^{14} \text{ s}^{-1}$$
$$\lambda = 365 \text{ nm} \implies f = 8.22 \times 10^{14} \text{ s}^{-1}$$

[2 marks]

Substituting the values of electron kinetic energy and frequency into the photoelectric equation leads to a pair of simultaneous equations:

$$0.44 \text{ eV} = h \times 5.86 \times 10^{14} - \phi \quad (1)$$
$$1.41 \text{ eV} = h \times 8.22 \times 10^{14} - \phi \quad (2)$$

[1 mark]

To solve for Planck's constant, subtract equation (1) from equation (2):

$$1.41 - 0.44 \text{ eV} = (8.22 - 5.86) \times 10^{14} \times h\, s^{-1}$$
$$0.97 \text{ eV} = 2.36 \times 10^{14} \times h$$
$$h = 4.11 \times 10^{-15} \text{ eVs}$$
$$h = 6.58 \times 10^{-34} \text{ Js}$$

[1 mark]

Substituting the value of h back into either (1) or (2) gives:

$$\phi = 1.97 \text{ eV } (3.15 \times 10^{-19} \text{ J})$$

[1 mark]

The question did not specify in which units to give the answer. Either eV or J is acceptable, although eV would be the more natural choice as this was used in the question.

c) The question suggests using energy and momentum conservation to solve the problem. First, let us define some variables. Let $E_i = hc/\lambda_i$ be the initial (or incoming) photon energy, $E_f = hc/\lambda_f$ be the final (or outgoing) photon energy and T_e be the final electron kinetic energy. Energy conservation demands that:

$$E_i + m_e c^2 = E_f + T_e + m_e c^2$$

Note that the total energy of the electron includes its rest mass energy, $E_e = T_e +$

$m_e c^2$, where $T_e = 0$ before the collision. The rest mass energy cancels to give

$$E_i = E_f + T_e. \quad (1)$$

Using $E^2 = p^2c^2$, the momentum of the incoming photon is $p_i = h/\lambda_i$ and the momentum of the outgoing photon is $p_f = h/\lambda_f$. The initial momentum of the electron Is zero, since the electron is stationary to begin with, and the final non-relativistic momentum of the electron is $p_e = \sqrt{2m_eT_e}$, where m_e is the mass of the electron. Momentum conservation demands that:

$$p_i = p_e - p_f. \quad (2)$$

Note that momentum is a vector quantity. Writing it in this way, we are defining the direction of the incoming photon as being positive and, since the photon is scattered through 180 degrees, the momentum of the outgoing photon as being negative. After scattering, the electron will be travelling in the same direction as the initial photon.

[1 mark]

Now, substituting for E_i and E_f in equation (1) we get

$$\frac{hc}{\lambda_i} = \frac{hc}{\lambda_f} + T_e. \quad (3)$$

And substituting for p_i and p_f in equation (2) we get

$$\frac{h}{\lambda_i} = p_e - \frac{h}{\lambda_f}. \quad (4)$$

[1 mark]

We want to find the wavelength of the initial (or incoming) photon, λ_i . Rearranging equation (4) we find

$$\frac{h}{\lambda_f} = p_e - \frac{h}{\lambda_i},$$

and substituting for h/λ_f in equation (3) we find

$$\frac{hc}{\lambda_i} = \left(p_e - \frac{h}{\lambda_i}\right)c + T_e.$$

Rearranging to make λ_i the subject of the equation we end up with

$$\lambda_i = \frac{2hc}{p_e c + T_e},$$

[2 marks]

where $T_e = 35\ \mathrm{keV} = 5.6 \times 10^{-15}\ \mathrm{J}$ and
$p_e = \sqrt{2m_e T_e} = 1.01 \times 10^{-22}\ \mathrm{kg\ m\ s^{-1}}$.

Finally, solving for λ_i,

$$\lambda_i = \frac{2hc}{p_e c + T_e} = \frac{2 \times 6.626 \times 10^{-34} \times 3 \times 10^8}{1.01 \times 10^{-22} \times 3 \times 10^8 + 5.6 \times 10^{-15}} \frac{\mathrm{Jm}}{\mathrm{J}},$$

$$\lambda_i = 1.11 \times 10^{-11}\mathrm{m}.$$

[1 mark]

Note that we can calculate the relativistic momentum of the electron using
$E_e^2 = p_e^2 c^2 + m_e^2 c^4$, where $E_e = T_e + m_e c^2$.

Substituting for E_e, we can rearrange to find p_e:

$$p_e = \sqrt{\frac{T_e^2}{c^2} + 2T_e m_e}.$$

Plugging in values for T_e, m_e, and c in SI units, we find:
$p_e = 1.0278 \times 10^{-22}\ \mathrm{kg\ m\ s^{-1}}$ and $\lambda_i = 1.091 \times 10^{-11}\mathrm{m}$.

The value of the initial wavelength obtained using the non-relativistic calculation of the electron momentum is only 1.4% different from the value obtained using the relativistic energy equation. The relativistic equation is (always) more accurate but the error in this case in quite small. This raises the question when is it necessary to use a relativistic treatment of energy (momentum)?

As a rough rule of thumb, we should use a relativistic treatment of energy and momentum when the velocity of particle is more than 10% the speed of light. In this case, the electron is actually travelling at 35% the speed of light. Another rule of thumb is that relativistic effects become important when the kinetic energy is similar in magnitude the rest energy of the particle (due to its mass). In this case, the kinetic energy is about 7% of the rest energy of an electron. Ultimately, the decision depends on the accuracy you want or need. If you want or need the wavelength to better than 1% accuracy, then a relativistic treatment of energy and momentum for the electron is unavoidable. Of course, the photon is always treated relativistically.