Week 08

A.C. circuits

1. Introduction

So far we have seen how capacitors and inductors behave as temporary reservoirs of electrical energy. They also give rise to a time-changing current (or voltage) when a **constant supply** is either applied or removed from a circuit. We showed that the transient part of the response has an exponential form, characterised by the time constant of the circuit. This week we will study how resistors, capacitors and inductors behave in response to a **time-changing** supply. The most common time-dependent waveform encountered in electronics is the sine function. We will start with a brief review of the properties of sinusoidal waveforms, which is summarised in Figure 8.1.

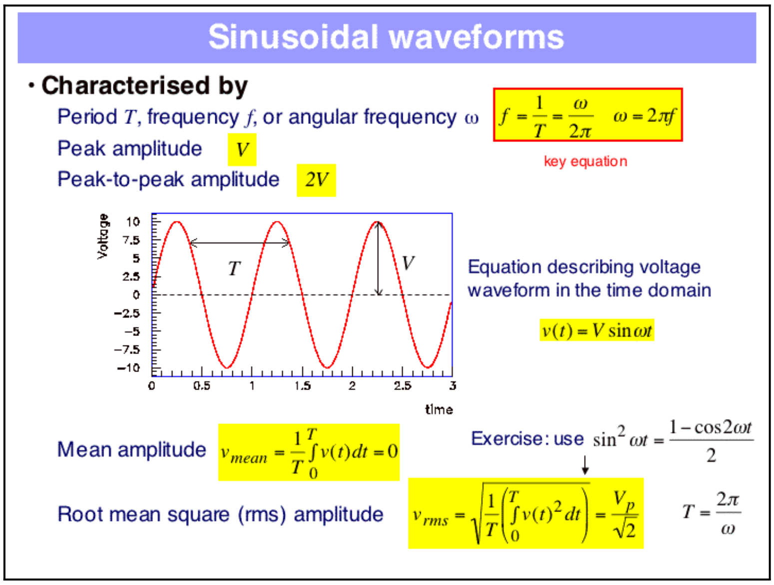

Figure 8.1: Sinusoidal waveforms – summary.

All periodic waveforms are characterised by their repetition frequency. If the waveform repeats itself every T seconds, the frequency is $f = 1/T$, where T is the period. Frequency is measured in cycles per second, or Hertz. A sine function repeats itself every 360 degrees, or 2π radians, so we can also define an angular frequency ω=2π/T radians per second. This tells us the time rate of change of the angle that is the argument of the sine function. Thus, we can write the instantaneous voltage (or current) as $v(t) = V \sin\omega t$.

Throughout this notes a lowercase v or i refers to the instantaneous (time-changing) voltage or current. An uppercase V or I refers to the voltage or current magnitude, or peak value.

The angular frequency is an important factor, as we shall see. However, waveforms are usually described by the frequency f, rather than the angular frequency ω. For example, the range of human hearing is said to be from 20 Hz to about 20 kHz; or we would say that the broadcast frequency of BBC Radio 4 is 92.7 MHz. We can translate these values into angular frequency by combining the expressions we have obtained for f and ω to give $\omega = 2\pi f$. You should try to memorise this relationship.

As shown in Figure 8.1, the average value of a sine wave over one period is zero. This is true of any alternating waveform. An alternative way to characterise an alternating waveform is to define its root-mean-square (rms) amplitude. For a sine wave this turns out to be the peak magnitude, divided by the square root of 2. I've left this integral as an exercise, but I only expect you to remember the result. We will see why the rms amplitude is useful a bit later.

Our next step is to derive the phase relationship between voltage and current in our three basic components: a resistor, capacitor and inductor. We will then combine these to find the phase relationship in circuits containing combinations of these components.

2. Phase relationships between voltage and current in R, C and L

It is quite straightforward to derive the phase relationships between voltage and current in a resistor, capacitor and inductor. We start by assuming a sinusoidal current (or voltage) and substitute this into the standard expressions that relate the voltage to the current (or vice versa) in each component. In each case, our aim is to find the voltage magnitude in relation to the current magnitude and the phase difference (if any) between the two.

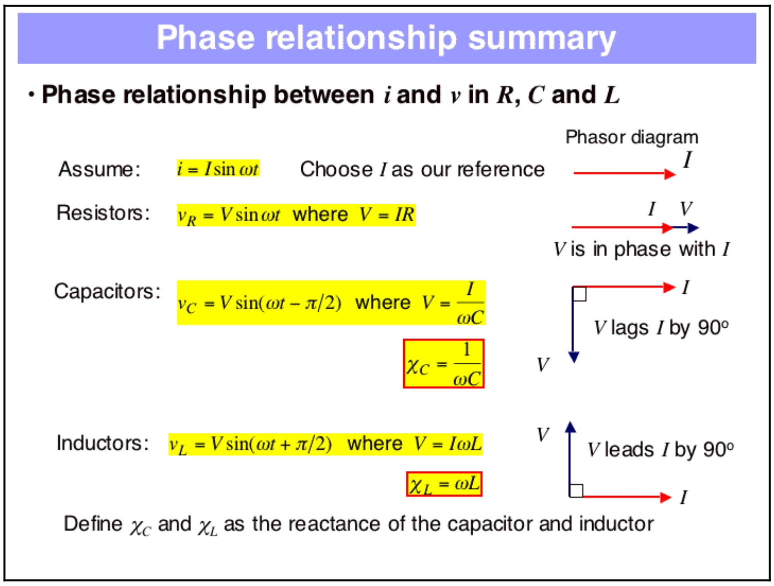

Figure 8.2: Phase relationship between the voltage and current in R, C and L.

We start with a sinusoidal current of the form $I = I \sin \omega t$. Capital "I" represents the current magnitude and this is our reference phasor. In other words, we will calculate all phase angles with respect to this phasor. We draw the reference phasor horizontally.

> It is a useful convention is to draw your first phasor horizontally. This makes it easier to calculate the phase angle. There is no loss of generality in doing this since, as I've already remarked, the phasors we are comparing are rotating at the same frequency. I am free to take a "snapshot" of the phasor diagram at any time I choose. It is simply easier to calculate an angle with respect to a horizontal line, than at any other angle. Note that Tipler doesn't follow this convention.

2.1 Resistor

This is the simplest case. In a resistor, Ohm's law relates the voltage to the current. Therefore, the voltage will also be of sinusoidal form, given by

$$v = iR = IR\sin wt = V\sin wt$$

The voltage across the resistor is represented by a new phasor of magnitude $V = IR$, which is in phase with the current. That is to say, the phase angle is zero.

2.2 Capacitor

The basic equation that relates the current and voltage in a capacitor is $i = C\,dv/dt$, which is obtained by differentiating $q = Cv$ with respect to time. Since we have postulated a sinusoidal current, we must integrate this expression to find the voltage waveform.

$$v = \frac{1}{C}\int i\,dt = \frac{I}{C}\int \sin\omega t\,dt = \frac{-I}{\omega C}\cos\omega t = V\sin(\omega t - \pi/2)$$

The voltage across the capacitor is represented by a new phasor of magnitude $V = I/\omega C$, which **lags** behind the current phasor by π/2 radians (or 90 degrees). You can show that $\sin(\omega t - \pi/2) = -\cos\omega t$, by using the compound angle formula. What is interesting is that we have found a new Ohm's law like relationship for the capacitor, which we can write as

$$V = I\,X_C\text{, where } X_C = \frac{1}{\omega C}$$

Here, χ_C is the *reactance* of the capacitor. It has units of ohms, since like resistance it is the ratio of a voltage and a current, but what distinguishes it from a resistance is that it depends on the angular frequency, ω.

2.3 Inductor

The basic equation that relates the current and voltage in an inductor is $v = Ldi/dt$. In order to find the voltage, we must differentiate the current.

$$v = L\frac{di}{dt} = LI\frac{d}{dt}(\sin\omega t) = \omega LI\cos\omega t = V\sin(\omega t + \pi/2)$$

The voltage across the inductor is represented by a new phasor of magnitude $V = I\omega L$, which **leads** the current phasor by $\pi/2$ radians (or 90 degrees). Once again we arrive at another Ohm's law like relationship for the inductor, which we can write as:

$$V = IX_L \text{, where } X_L = \omega L$$

Here, χ_L is the *reactance* of the inductor. It also has units of ohms and depends on the angular frequency, ω.

The idea that capacitors and inductors have a "resistance" which depends on frequency is extremely important. This is what makes a.c. circuits so versatile. We can now use these phase relationships to study circuits that contain combinations of resistors, capacitors and inductors.

3. A.C. circuits in complex notation

In the first two parts we have seen how a.c. circuits can be analysed using the phasor approach. This is a very convenient way of representing the magnitudes of voltages and currents in a circuit, allowing the phase relationships between those quantities to be determined. Although reasonably straightforward in the examples we have considered thus far, there is a more elegant way to analyse a.c. circuits using complex numbers. In this material I shall introduce the concept of complex impedance and show how this can be used to derive the phase relationship between the applied voltage and the current in simple circuits.

3.1 Complex number revision

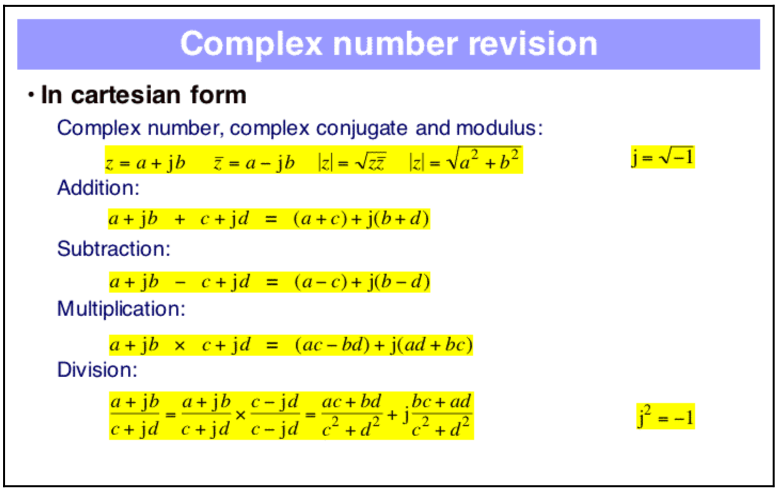

Figure 8.3: Complex arithmetic.

You will have met complex numbers in your maths course. Figure 8.3 summarises the basic mathematical operations involving complex numbers. In circuit analysis the imaginary part of a complex number is represented by the symbol j (as opposed to i in mathematics) to avoid confusion with the current.

Before continuing, take a moment to familiarise yourself with Figure 8.3. A complex number has a real part and an imaginary part. The modulus (or magnitude) of a complex number is given by the square root of the product of the complex number and its complex conjugate. That is, the magnitude is calculated by taking the square root of the real part squared, plus the imaginary part squared. Addition, subtraction and multiplication are pretty straightforward. When dividing two complex numbers, the numerator and the denominator must be multiplied by the complex conjugate of the denominator. This makes the denominator purely real and the result can then be separated into real and imaginary parts.

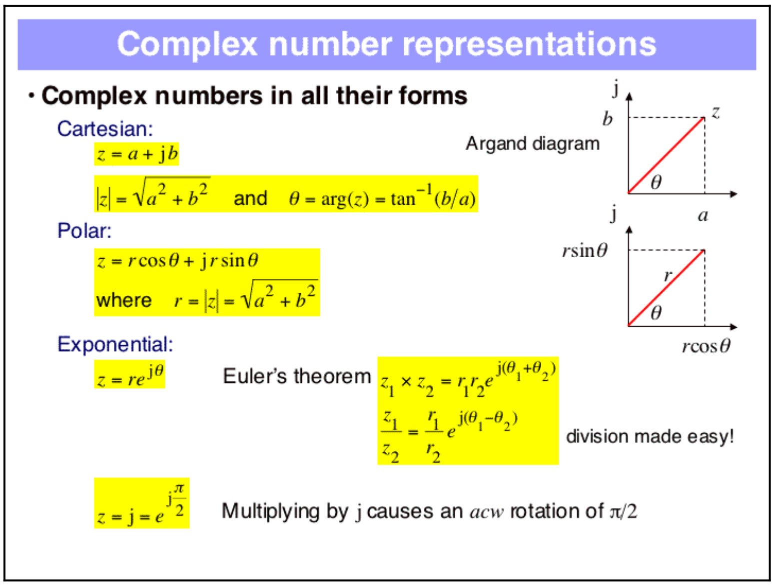

Figure 8.4: Complex number representations.

Complex numbers can be represented graphically on the **Argand diagram**, as shown in Figure 8.4. An Argand diagram consists of two orthogonal axes: a horizontal real axis and an imaginary vertical axis. Complex numbers can lie anywhere on the plane defined by these two axes (the complex plane). From the Argand diagram we can readily determine the magnitude and argument of a complex number. The argument is defined by the angle the complex vector makes with the horizontal (real) axis. The magnitude is found by applying Pythagoras' theorem.

$$z=a+jb|z|=\sqrt{a^2+b^2}q=\arg(z)=\tan^{-1}(b/a) \qquad (1)$$

From the Argand diagram it can be seen that a complex number can also be written in polar form. Here, the real and imaginary parts are expressed in terms of the magnitude and argument of the complex number.

$$z = r\cos\theta + \mathrm{j}r\sin\theta \qquad r = |z| \qquad (2)$$

Yet another way of representing a complex number is in exponential form. This follows from the fact that sine and cosine can be written as an infinite series.

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$$
$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots \qquad (3)$$

We can compare these series with that of an exponential below.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \qquad (4)$$

If we substitute $\mathrm{j}\theta$ for x in equation (4), then by comparison with equation (3), it can be shown that

$$e^{\mathrm{j}\theta} = \cos\theta + \mathrm{j}\sin\theta$$

This is Euler's theorem. Here, you need to remember that $\mathrm{j}^2 = -1$. Notice that the sine series contains only odd powers of x, hence there is a factor of j left in front of each term ($\mathrm{j}^3 = -\mathrm{j}$, $\mathrm{j}^5 = +\mathrm{j}$, $\mathrm{j}^7 = -\mathrm{j}$, and so on). In exponential form multiplying and dividing complex numbers is particularly easy, as shown in Figure 8.4. To multiply, you just take the product of the magnitudes and add the arguments. To divide, you take the ratio of the magnitudes and subtract the arguments. In electrical problems, voltage and currents are conveniently represented by complex exponentials.