UNIVERSITY OF BIRMINGHAM

School of Mathematics

Programmes in the School of Mathematics	First Examination
Programmes in the School of Mathematics	Second Examination
Programmes involving Mathematics	First Examination
Programmes involving Mathematics	Second Examination

1Mech 06 25661 Level C

LC Mechanics

1Mech2 06 27345 Level I

LI Mechanics

May/June Examinations 2022-23

One Hour and Thirty Minutes

Full marks will be obtained with complete answers to BOTH questions. Each question carries equal weight. You are advised to initially spend no more than 45 minutes on each question and then to return to any incomplete questions if you have time at the end. An indication of the number of marks allocated to parts of questions is shown in square brackets.

No calculator is permitted in this examination.

Section A

1. (a) The displacement $x(t)$ of a mass m attached to a spring obeys the equation of motion,

$$\ddot{x} + \gamma\dot{x} + \omega^2 x = \frac{F(t)}{m},$$

where $\gamma \geq 0$ and $\omega > 0$ are constants and $F(t)$ is a driving force. Let the mass have initial displacement $x(0) = x_0 > 0$ and initial velocity $v(0) = 0$.

(i) Let $\gamma = 0$ and $F(t) = 0$. Find $x(t)$.

(ii) Let $\gamma = 0$ and $F(t) = F_0 \cos(\omega t)$, where $F_0 > 0$ is a constant. Verify that the function

$$x(t) = C\cos(\omega t) + \frac{F_0}{2m\omega} t \sin(\omega t),$$

where C is any constant, satisfies the equation of motion and the initial condition $v(0) = 0$. Determine C. Describe how $x(t)$ behaves in the limit $t \to \infty$.

(iii) Let $\gamma > 0$ and $F(t) = F_0 \cos(\omega t)$, where $F_0 > 0$ is a constant. Without finding $x(t)$, describe how $x(t)$ behaves in the limit $t \to \infty$ and justify your answer. [25]

(b) A particle of mass m moves under gravity along a coil of frictionless wire in the shape of an elliptical helix. The Cartesian coordinates of the particle are

$$x(t) = a\cos\theta(t), \quad y(t) = b\sin\theta(t), \quad z(t) = c\theta(t),$$

where $\theta(t)$ is some function of t, and a, b and c are positive constants with $a > b$.

(i) Find the particle's velocity vector, and hence show that

$$\frac{1}{2}m\dot{\theta}^2\left(a^2\sin^2\theta + b^2\cos^2\theta + c^2\right) + mgc\theta = E,$$

where E is a constant.

(ii) Let the particle be initially stationary at position $\theta(0) = 0$. Find an expression for $\dot{\theta}^2$. Assuming that the particle never stops after it is released, show that $\theta(t) < 0$ for all $t > 0$.

(iii) Show that $\dot{\theta}^2$ has a local maximum or local minimum whenever

$$(a^2 - b^2)\left(\sin^2\theta - \theta\sin(2\theta)\right) + b^2 + c^2 \;=\; 0. \qquad (\star)$$

Show that $(\star)$ holds for infinitely many values of $\theta < 0$, and interpret this result. [25]

Section B

2. In plane polar coordinates (r, θ), a particle of mass m is attracted towards the origin by a central force of magnitude

$$ma\omega^2 \left(\frac{a^3}{r^3} + \frac{a^2}{r^2} e^{-a/r} \right),$$

where a and ω are positive constants.

(a) Explain why a must have the dimensions of length. Derive the dimensions of ω. [4]

(b) Let $u = 1/r$. Starting from Newton's Second Law, show that the quantity $h = r^2\dot{\theta}$ is a constant, and that the particle's path satisfies

$$\frac{\mathrm{d}^2 u}{\mathrm{d}\theta^2} + u \;=\; \frac{a\omega^2}{h^2}(a^3 u + a^2 e^{-au}). \qquad (\dagger)$$

You may assume the following expressions for the particle's velocity $\dot{\mathbf{r}}$ and acceleration $\ddot{\mathbf{r}}$:

$$\begin{aligned} \dot{\mathbf{r}} &= \dot{r}\mathbf{e}_r + r\dot{\theta}\mathbf{e}_\theta, \\ \ddot{\mathbf{r}} &= \left(\ddot{r} - r\dot{\theta}^2\right)\mathbf{e}_r + \frac{1}{r}\left(\frac{\mathrm{d}}{\mathrm{d}t}\left(r^2\dot{\theta}\right)\right)\mathbf{e}_\theta. \end{aligned}$$

[16]

(c) Let the particle's initial position be given by $[r]_{t=0} = a$ and $[\theta]_{t=0} = 0$, and let its initial velocity be

$$[\dot{\mathbf{r}}]_{t=0} = a\omega\,\mathbf{e}_\theta.$$

Find the value of h, and the values of u and $\dfrac{\mathrm{d}u}{\mathrm{d}\theta}$ at $\theta = 0$. [10]

(d) Let $a = 1$ in some arbitrary units. Verify that the function

$$u(\theta) = 1 + \log\left(\cosh^2(c\theta)\right), \qquad (\ddagger)$$

where c is any constant, satisfies the initial conditions for u and $\dfrac{\mathrm{d}u}{\mathrm{d}\theta}$ at $\theta = 0$. Find the value of $c > 0$ such that ($\ddagger$) solves ($\dagger$). You may use the following without proof:

$$\frac{\mathrm{d}}{\mathrm{d}X}\sinh(X) = \cosh(X), \quad \frac{\mathrm{d}}{\mathrm{d}X}\cosh(X) = \sinh(X), \quad \cosh^2(X) - \sinh^2(X) = 1.$$

[12]

(e) Assume that the particle path is given by ($\ddagger$), with $c > 0$. Show that $\dot{\theta}$ is a strictly increasing function of θ, and hence explain why $\theta \to \infty$ as $t \to \infty$. What happens to r as $t \to \infty$? Justify your answer. Draw a sketch of the particle path in the (r, θ) plane. [8]

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LC/LI Mechanics

Do not complete the attendance slip, fill in the front of the answer book or turn over the question paper until you are told to do so.

Important Reminders

- Coats and outer-wear should be placed in the designated area.
- Unauthorised materials (e.g. notes or Tippex) **MUST** be placed in the designated area.
- Check that you **DO NOT** have any unauthorised materials with you (e.g. in your pockets, pencil case).
- Mobile phones and smart watches **MUST** be switched off and placed in the designated area or under your desk. They must not be left on your person or in your pockets.
- You are **NOT** permitted to use a mobile phone as a clock. If you have difficulty in seeing a clock, please alert an Invigilator.
- You are **NOT** permitted to have writing on your hand, arm or other body part.
- Check that you do not have writing on your hand, arm or other body part – if you do, you must inform an Invigilator immediately.
- Alert an Invigilator immediately if you find any unauthorised item upon you during the examination.

Any students found with non-permitted items upon their person during the examination, or who fail to comply with Examination rules may be subject to the Student Conduct procedures.