Mechanics week 7: Bounded orbits and angular momentum

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1 Introduction

Last week we used the framework for Central Forces problems to consider some examples. This week we will consider some more examples, along with some more general theory.

2 Final central forces example

Example 3: Comet Consider a comet approaching the sun from far away, with constant velocity V. If we neglected the effect of the sun on the comet, the velocity would remain as V, and it would come closest to the Sun at a distance P. What is the path of the comet, and the angle through which it is deflected?

Solution. The force acting on the comet will be

$$F(r) = \frac{GMm}{r^2},$$

where m is the mass of the comet, M is the mass of the sun, G is the gravitational constant and r is the distance between the sun and the comet such that $\mathbf{r} = r\mathbf{e}_r$ is the position vector of the comet with respect to the sun.

We let $u = 1/r$, then u will satisfy

$$\begin{aligned}\frac{d^2u}{d\theta^2} + u &= -\frac{F(1/u)}{mh^2u^2}, \\ &= -\frac{-GMmu^2}{mh^2u^2}, \\ &= \frac{GM}{h^2}.\end{aligned}$$

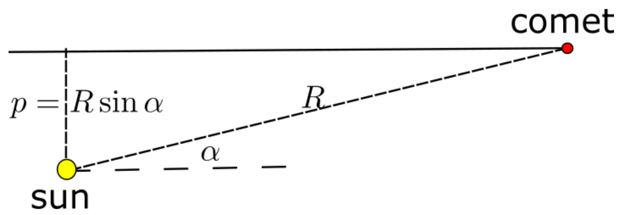

Figure 1: Initial set up, showing comet and sun

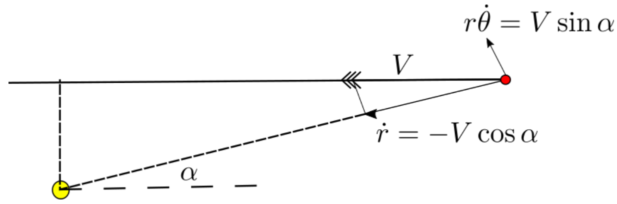

Figure 2: Initial velocity in component form

This is the same as the previous example, so will have general solution

$$u = A\sin\theta + B\cos\theta + \frac{GM}{h^2}.$$

We now consider the initial conditions and set up. Let the initial distance between the comet and the sun be $R \gg 1$, at angle $\alpha \ll 1$. Then at $t = 0$ we have $r = R$, $\dot{r} = -V\cos\alpha$, $r\dot{\theta} = V\sin\alpha$. We also know $p = R\sin\alpha$. Hence the constant h is given by

$$\begin{aligned} h &= r^2\dot{\theta}, \\ &= r \cdot r\dot{\theta}, \\ &= R \cdot V\sin\alpha, \\ &= R\sin\alpha \cdot V, \\ &= pV. \end{aligned}$$

Since $\alpha \ll 1$, $\cos\alpha \approx 1$ giving $\dot{r} = -V$ at time $t = 0$.

Now, we can **choose** where $\theta = 0$ (since it's essentially rotating the axes to whereever we like), so we take the comet to be at $\theta = 0$ at $t = 0$. Hence we can rewrite our initial conditions ($t = 0$) in terms of $\theta = 0$ to find

$$\begin{aligned} u &= \frac{1}{R} \approx 0, \\ \frac{du}{d\theta} &= -\frac{\dot{r}}{h}, \\ &= \frac{V}{pV} = \frac{1}{p}, \end{aligned}$$

at $\theta = 0$.

Hence

$$\begin{aligned} u &= A\sin\theta + B\cos\theta + \frac{GM}{h^2}, \\ &= A\sin\theta + B\cos\theta + \frac{GM}{p^2V^2}. \end{aligned}$$

Then

$$\begin{aligned} u(0) &= B + \frac{GM}{p^2V^2}, \\ &= 0, \\ \Longrightarrow B &= \frac{-GM}{p^2V^2}, \end{aligned}$$

and

$$\begin{aligned} \frac{du}{d\theta} &= A\cos\theta - B\sin\theta, \\ &= A, \\ &= 1/p, \end{aligned}$$

at $\theta = 0$. Hence the complete solution is

$$u = \frac{1}{p}\sin\theta + \frac{GM}{p^2V^2}\left(1 - \cos\theta\right).$$

What does this tell us?

The comet goes off into space as $r \to \infty$, i.e. when $u \to 0$. This happens when

$$\frac{1}{p}\sin\theta + \frac{GM}{p^2V^2}\left(1 - \cos\theta\right) = 0,$$

which holds when $\theta = 0$ (so $\sin\theta = 0$, $\cos\theta = 1$), or when $\theta = \Theta$ nonzero satisfies

$$\begin{aligned} \frac{1}{p}\sin\Theta + \frac{GM}{p^2V^2}\left(1 - \cos\Theta\right) &= 0, \\ \Longrightarrow \frac{\cos\Theta - 1}{\sin\Theta} &= \frac{pV^2}{GM}. \end{aligned}$$

Now we expect the Sun to deflect the comet from its undisturbed trajectory. If we take 2δ to be the angle between the asymptotes of the trajectory (see Figure 3), we will have

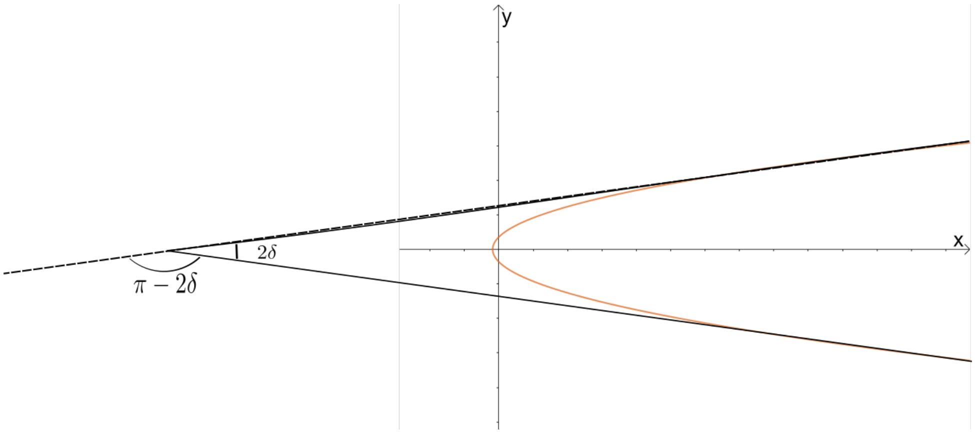

Figure 3: The deflection of the comet by the sun, showing 2δ as the angle at the "point" where the asymptotes meet, and $\pi - 2\delta$ as the deflection of the comet.

$\Theta = 2\pi - 2\delta$. Then

$$\begin{aligned} \cos(2\pi - 2\delta) &= \cos(2\pi)\cos(-2\delta) - \sin(2\pi)\sin(-2\delta), \\ &= \cos(2\delta), \\ &= 2\cos^2\delta - 1, \\ \sin(2\pi - 2\delta) &= \sin(2\pi)\cos(-2\delta) + \cos(2\pi)\sin(-2\delta), \\ &= -\sin(2\delta), \\ &= -2\sin\delta\cos\delta, \end{aligned}$$

and hence

$$\begin{aligned} \frac{pV^2}{GM} &= \frac{2(\cos^2\delta - 1)}{-2\sin\delta\cos\delta}, \\ &= \frac{\sin^2\delta}{\sin\delta\cos\delta}, \\ &= \tan\delta. \end{aligned}$$

This gives a deflection of $\pi - 2\delta$, since the path would otherwise go from $\theta = 0$ to $\theta = \pi$.

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3 Circular orbits and stability

Circular orbits which are *stable* are very important, particularly when considering satellites in orbit about a planet (e.g. for GPS). We need satellites to remain the same distance away from the Earth even if there are slight disturbances. This leads us to the mathematical concept known as stability - if we have a small *perturbation* to an otherwise steady solution, does that perturbation decay, so we return to the steady solution, or grow, so we move away? You will learn more about this in later modules, but we shall show an example of this here.

We first determine what force $F(r)$ leads to a circular orbit. This will be such that $r = a$, for some constant value a. Since we know that

$$m\left(\ddot{r} - h^2/r^3\right) = F(r), \tag{1}$$

must be satisfied (from week 04), for $r = a$ (and hence $\ddot{r} = 0$) to be a solution we must have

$$-mh^2/a^3 = F(a).$$

Therefore if the force F satisfies $F(a) = -mh^2/a^3$ for the particular setup (i.e. the value of h), then $r = a$ could be a solution to the model.

We also need this to be stable, so that once the particle reaches $r = a$ it remains in this orbit. We test this by putting

$$r = a + \epsilon P(t),$$

where $\epsilon P(t)$ is a small perturbation (i.e. $\epsilon \ll 1$). If $P(t)$ grows (i.e. $P \to \infty$ as $t \to \infty$) in time we move further and further away from the constant radius case. Substituting this into the governing equation (1) and using $\ddot{r} = \epsilon\ddot{P}$ gives

$$m\left(\epsilon\ddot{P} - \frac{h^2}{(a+\epsilon P)^3}\right) = F(a+\epsilon P),$$

for fixed h.

Since ϵ is small we can expand the second term on the left hand side, and the right hand side:

$$\begin{aligned}\frac{h^2}{(a+\epsilon P)^3} &= \frac{h^2}{a^3}\left(1+\epsilon P/a\right)^{-3}, \\ &= \frac{h^2}{a^3}\left(1 - 3\epsilon P/a + \ldots\right),\end{aligned}$$

using a Binomial expansion and

$$F(a+\epsilon P) = F(a) + \epsilon P F'(a) + \ldots,$$

using a Taylor expansion, where $'$ denotes d/dr. This gives

$$m\left(\epsilon\ddot{P} - \frac{h^2}{a^3}\left(1 - 3\epsilon P/a + \ldots\right)\right) = F(a) + \epsilon P F'(a) + \ldots.$$

Since ϵ is small, we can approximate this equation at different orders. At leading order (i.e. when we neglect any term which has an ϵ in it), we require

$$\frac{h^2}{a^3} = F(a).$$

This is precisely the requirement we already had to have a circular orbit in the first place. We then look at next order, by equating all the terms which are linear in ϵ, and neglecting anything which is ϵ^2 or smaller (small thing times small thing is a really small thing):

$$\begin{aligned} m\left(\ddot{P} + \frac{3h^2}{a^4}P\right) &= PF'(a), \\ \implies m\ddot{P} + \left(\frac{3mh^2}{a^4} - F'(a)\right)P &= 0. \end{aligned}$$

This is a linear second order ODE with constant coefficients, so which we can solve using the characteristic equation

$$m\lambda^2 + \left(\frac{3mh^2}{a^4} - F'(a)\right) = 0.$$

The solution will be in the form of exponentials if $\dfrac{3mh^2}{a^4} - F'(a) < 0$ or sines and cosines if $\dfrac{3mh^2}{a^4} - F'(a) > 0$. Exponential solutions will grow and move away from the constant radius solution, whilst sines and cosines will oscillate, remaining in the orbit, so we require $\dfrac{3mh^2}{a^4} - F'(a) > 0$ for a stable orbit.

Activity: *You should now be able to tackle question 3 on this week's problem sheet.*

4 Angular momentum (or moment of momentum)

Having seen the principles in action, we now return to some more general theory about angular momentum (or moment of momentum). The moment of a general force $\mathbf{F}$ about an origin O is

$$\mathbf{r} \times \mathbf{F}$$

where $\mathbf{r}$ gives the location of the point where the force is applied relative to O.

For example, a force F applied vertically, a horizontal distance d from the pivot, gives a moment $\mathbf{r} \times \mathbf{F}$ about the pivot where $\mathbf{r} = d\mathbf{i}$, $\mathbf{F} = -F\mathbf{k}$ and hence

$$\begin{aligned} \mathbf{r} \times \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ d & 0 & 0 \\ 0 & 0 & -F \end{vmatrix}, \\ &= Fd\mathbf{j}, \end{aligned}$$

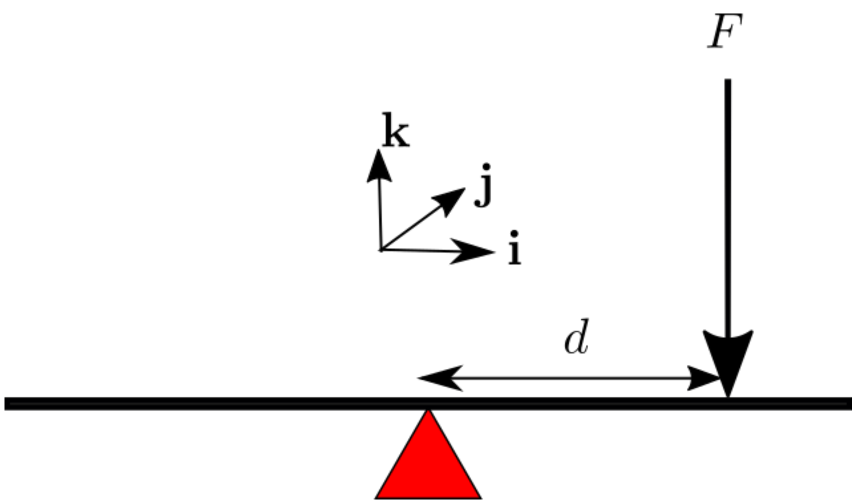

Figure 4: A seesaw showing a force applied vertically, at a horizontal distance d to the pivot.

as you probably recognised from high school (but now with added direction!).

We now recall the definition of momentum from earlier: $\mathbf{p} = m\dot{\mathbf{r}}$. Therefore the moment of momentum about an origin O is given by

$$\mathbf{r} \times (m\dot{\mathbf{r}}),$$

this is the angular momentum. [Note that we're assuming the pivot and the coordinate system origin at the same.]

Then Newton's second law gives

$$\mathbf{F} = m\ddot{\mathbf{r}},$$

and taking the cross product of both sides with $\mathbf{r}$ gives

$$\begin{aligned} \mathbf{r} \times \mathbf{F} &= \mathbf{r} \times (m\ddot{\mathbf{r}}), \\ &= m\mathbf{r} \times \ddot{\mathbf{r}}. \end{aligned}$$

Now the cross product of any vector with itself is zero, as they are parallel to each other, so $\dot{\mathbf{r}} \times \dot{\mathbf{r}} = 0$. So we can write

$$\begin{aligned} \mathbf{r} \times \mathbf{F} &= m(\mathbf{r} \times \ddot{\mathbf{r}} + \dot{\mathbf{r}} \times \dot{\mathbf{r}}), \\ &= m\frac{d}{dt}(\mathbf{r} \times \dot{\mathbf{r}}), \\ &= \frac{d}{dt}(\mathbf{r} \times (m\dot{\mathbf{r}})). \end{aligned}$$

Here the left hand side is the moment about O of the force $\mathbf{F}$ and the right hand side is the derivative of the moment about O of momentum, or the angular momentum. [Expand out the derivative to check the first and second lines are the same!]

Special case: Central force

If $\mathbf{F} = F(r)\mathbf{e}_r$ then $\mathbf{r} \times \mathbf{F} = 0$ since $\mathbf{r}$ and $\mathbf{F}$ are parallel vectors. Hence

$$\frac{d}{dt}\left(\mathbf{r} \times \dot{\mathbf{r}}\right) = 0,$$

and so

$$\begin{aligned} \mathbf{r} \times \dot{\mathbf{r}} &= \text{constant}, \\ &= \mathbf{h}. \end{aligned}$$

If we now dot with $\mathbf{r}$ we find

$$\mathbf{r} \cdot (\mathbf{r} \times \dot{\mathbf{r}}) = \mathbf{r} \cdot \mathbf{h}.$$

Now $\mathbf{r} \times \dot{\mathbf{r}}$ must be perpendicular to $\mathbf{r}$ (as it's in the cross product), so

$$\mathbf{r} \cdot \mathbf{h} = 0.$$

This gives the equation of a plane through the origin, so the motion takes place entirely in a plane (as we assumed earlier).

Finally, if

$$\begin{aligned} \mathbf{h} &= \mathbf{r} \times \dot{\mathbf{r}}, \\ &= r\mathbf{e}_r \times \left(\dot{r}\mathbf{e}_r + r\dot{\theta}\mathbf{e}_\theta\right), \\ &= (r\mathbf{e}_r \times \dot{r}\mathbf{e}_r) + \left(r\mathbf{e}_r \times r\dot{\theta}\mathbf{e}_\theta\right), \\ &= r^2\dot{\theta}\left(\mathbf{e}_r \times \mathbf{e}_\theta\right), \\ &= r^2\dot{\theta}\mathbf{k}, \end{aligned}$$

where $\mathbf{k}$ is a vector coming out of the plane. [Using the fact that $\mathbf{e}_r \times \mathbf{e}_r = 0$.] Hence $r^2\dot{\theta}$ is conserved as before.

Activity: *You should now be able to tackle question 4 on this week's problem sheet.*

This concludes the Chapter on Central Forces, next week we shall start on Energy.