1Mech — Mechanics

Mechanics exercises 4 (weeks 8 and 9)

This sheet's assessed questions are question 2 and 3. If you are taking this module at level I (typically if you are a year 2 student) then you must also do the final part of question 3.

1. If we throw a particle of mass m upwards from the top of a wall of height h with velocity v, how high will it go and what will its velocity be when it hits the ground?

2. **Assessed** Let a particle of mass m be attached to two springs, both with spring constant k. The end of one spring (denoted α, with natural length a) is attached at a point A, with the end of the other spring (denoted β, with natural length $2a$) attached at a point B, a distance $4a$ directly above A. The mass is attached to the free end of both springs. The particle is at a location $x(t)$, where x is measured upwards such that $x = 0$ at point A.

   (a) Show that

      i. the extension in spring α is given by $x - a$.

      ii. the extension in spring β is given by $2a - x$.

   (b) Hence write down the equation for conservation of energy.

   (c) If the particle is initially at rest at $x = 2a$, find the value of the constant energy.

   (d) Find the height at which the particle will next come to rest.

3. **Assessed** A smooth (i.e. no friction) wire is in the shape of a helix so that $x = a\cos\theta\,(t)$, $y = a\sin\theta\,(t)$, $z = a\theta\,(t)/2$, with the central (z) axis pointing vertically upwards. A small bead of mass m moves along the wire, starting from height $z = 2\pi a$ with speed 0.

   (a) By writing down the position vector and differentiating, show that

   $$\dot{\mathbf{r}} \;=\; -a\dot{\theta}\sin\theta\mathbf{i} + a\dot{\theta}\cos\theta\mathbf{j} + \frac{a}{2}\dot{\theta}\mathbf{k},$$

   and hence that the kinetic energy of the bead is given by

   $$\frac{5ma^2}{8}\dot{\theta}^2.$$

   (b) Write down the potential energy of the bead in terms of θ, choosing the potential to be zero at $z = 0$.

   (c) Hence show that conservation of energy gives

   $$\frac{5ma^2}{8}\dot{\theta}^2 + \frac{mga}{2}\theta = 2mg\pi a.$$

   (d) By rearranging to find an equation for $\dot{\theta}^2$, find the maximum value that θ can attain. What does this mean physically?

   (e) **Extension for Level I only:** We will now calculate how long will it take for the bead to reach $z = 0$.

i. By square rooting, find an expression for $\dot{\theta}$. Ensure that you take the correct sign!

ii. By separating the variables in your expression and integrating both sides between $t = 0$ and $t = T$ (where T is the time at which the bead reaches $z = 0$), find how long it will take to reach $z = 0$.

4. A comet (mass m) which is travelling with speed V, approaches a stationary planet from a great distance. If the path of the comet was not affected by the planet, the distance of closest approach would be p. The comet experiences an attractive force GMm/r^2 towards the planet where r is the distance between them, G is the gravitational constant and M is the mass of the planet.

(a) **Briefly** explain why

$$\begin{aligned} r^2\dot{\theta} &= \text{constant}, \\ \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - \frac{GMm}{r} &= \text{constant}. \end{aligned}$$

(b) Using the initial conditions, find the values of the constants in part (a).

(c) By eliminating $\dot{\theta}$, show that

$$\dot{r}^2 = V^2 + \frac{2GM}{r} - \frac{p^2V^2}{r^2}.$$

(d) Hence calculate the actual distance of closest approach.