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No Calculator

UNIVERSITY OF BIRMINGHAM

School of Mathematics

Programmes in the School of Mathematics — First Examination

Programmes involving Mathematics — First Examination

1RA 06 34051 Level C

LC Real Analysis

January Examinations 2022-23

Three Hours

Full marks will be obtained with complete answers to all FOUR questions. Each question carries equal weight. You are advised to initially spend no more than 45 minutes on each question and then to return to any incomplete questions if you have time at the end. An indication of the number of marks allocated to parts of questions is shown in square brackets.

No calculator is permitted in this examination.

Section A

1. (a) Let X and Y be sets. Define what it means for f to be a function from X to Y. [3]

(b) Let $f:\mathbb{R}\to\mathbb{R}$ be a function and $\alpha,\ell\in\mathbb{R}$. [8]

(i) Define what it means that

$$\lim_{x\to\alpha} f(x)=\ell.$$

(ii) By directly using the definition of limit from (i), show that

$$\lim_{x\to 1} 2x+100=102.$$

(c) Let $f:\mathbb{R}\to\mathbb{R}$ be a differentiable function and $\alpha\in\mathbb{R}$. [5]

(i) Define $f'(\alpha)$, the derivative of f at α.

(ii) By directly using the definition of derivative from (i), find the derivative of the function $f(x)=x^2$ at the point $x=5$.

(d) Let $f:\mathbb{R}\to\mathbb{R}$ with $f(x)=2x^3-6x^2-18x+7$ for all $x\in\mathbb{R}$. [9]

(i) Find the increasing and decreasing intervals of f.

(ii) Find all local maximum and local minimum points of f.

(iii) Find the convex and concave intervals of f.

Justify all the assertions that you make. You can use any of the results discussed in lectures, provided you clearly state what you are using.

2. (a) Define what it means for a bounded function $f : [0,1] \to \mathbb{R}$ to be integrable. [1]

(b) State the First Fundamental Theorem of Calculus. [3]

(c) Suppose that $P = \{2,4,10\}$ and that $f : [2,10] \to \mathbb{R}$ is given by

$$f(x) := \begin{cases} x, & \text{if } x \notin \{4,10\}; \\ 1, & \text{if } x = 4; \\ 9, & \text{if } x = 10. \end{cases}$$

Compute both the lower sum $L(f,P)$ and the upper sum $U(f,P)$. [4]

(d) Find an antiderivative of the function $g : (-2\pi, 3\pi) \to \mathbb{R}$ given by $g(x) := \sin^3(x)\cos^4(x)$ for all $x \in (-2\pi, 3\pi)$. [3]

(e) Compute the improper integral $\int_1^2 \frac{x}{x^2 - 4x + 3}\, \mathrm{d}x$ or show that it diverges. [6]

(f) Find a solution $y : [0,\infty) \to \mathbb{R}$ of the initial value problem [5]

$$y' = 2x\sin(5x), \quad y(0) = 9.$$

(g) Find the general solution of the differential equation $4y'' - 20y' + 25y = 0$ on $\mathbb{R}$. [3]

Section B

3. (a) We want to construct a cylindrical can with a bottom but no top that will have a volume of 27π cm^3. Determine the dimensions (the radius and height) of the can that will minimize the amount of material needed to construct the can. Justify the assertion that you make. You can use any of the results discussed in lectures, provided you clearly state what you are using. [8]

(b) Determine whether the following limits exist, and if so find their values. Justify your answers. You can use any of the results discussed in lectures, provided you clearly state what you are using. [10]

(i) $\displaystyle\lim_{x\to 0} \frac{e^x \cos x + 5\sqrt{1-x}}{1 + \arctan x + \ln(1-x)}$.

(ii) $\displaystyle\lim_{x\to 0} \left(\frac{\pi}{2} - \arctan x\right)^{\frac{1}{\ln x}}$.

(Here $\ln x = \log_e x$ denotes the natural logarithm of x.)

(c) Let $a, b \in \mathbb{R}$ and $a > 0$. Let $f : \mathbb{R} \to \mathbb{R}$ be the polynomial

$$f(x) = x^3 + ax + b.$$

Prove that there exists exactly one number $x_0 \in \mathbb{R}$ such that $f(x_0) = 0$. [7]

4. (a) Suppose that $f : [-6,-5] \to \mathbb{R}$ is given by

$$\begin{cases} -x, & \text{if } x \in \mathbb{Z}; \\ 8, & \text{if } x \notin \mathbb{Z}. \end{cases}$$

Use Riemann's Criterion or otherwise to prove that f is integrable. [8]

(b) Suppose that $R = \{0,1,3,4\}$, $S = \{2,3\}$ and $g : [0,4] \to [5,10]$. Prove directly, without using the Partition Lemma, that $L(g,R) \leq L(g,R \cup S)$. [5]

(c) Suppose that $h : (0,1) \to \mathbb{R}$ is given by

$$h(x) := \int_0^{2x+1} \sqrt{100 - t^4}\, \mathrm{d}t$$

for all $x \in (0,1)$. Prove that h is differentiable and find an expression for the derivative $h'(x)$, for all $x \in (0,1)$, that does not contain the integral or antiderivative symbol $\int$. [6]

(d) A soft drink company has a $10{,}000$ litre (L) industrial mixing tank filled with pure water. The tank is drained at a rate of 12 litres per minute $(\mathrm{L\ min^{-1}})$ whilst a syrup solution containing 10 kilograms per litre $(\mathrm{kg\ L^{-1}})$ of sugar is pumped into the tank at a rate of $10\ \mathrm{L\ min^{-1}}$. Assuming that the mixture is instantly and uniformly mixed, formulate and solve an initial value problem to model the mass of sugar $y(t)$ in the tank after t minutes of mixing for all $t \in [0,5000]$. [6]

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Do not complete the attendance slip, fill in the front of the answer book or turn over the question paper until you are told to do so.

Important Reminders

- Coats and outer-wear should be placed in the designated area.
- Unauthorised materials (e.g. notes or Tippex) **MUST** be placed in the designated area.
- Check that you **DO NOT** have any unauthorised materials with you (e.g. in your pockets, pencil case).
- Mobile phones and smart watches **MUST** be switched off and placed in the designated area or under your desk. They must not be left on your person or in your pockets.
- You are **NOT** permitted to use a mobile phone as a clock. If you have difficulty in seeing a clock, please alert an Invigilator.
- You are **NOT** permitted to have writing on your hand, arm or other body part.
- Check that you do not have writing on your hand, arm or other body part – if you do, you must inform an Invigilator immediately.
- Alert an Invigilator immediately if you find any unauthorised item upon you during the examination.

Any students found with non-permitted items upon their person during the examination, or who fail to comply with Examination rules may be subject to the Student Conduct procedures.