Quantum Mechanics 1 – Solution 9

a) To normalize the wave function find the integral

$$\int_{-\infty}^{+\infty} |\psi_n|^2\, dx = A^2 \int_0^L \sin^2\left(\frac{n\pi x}{L}\right) dx = 1,$$

where $n > 0$ is any positive integer. Note that the wave function is zero outside the well, so the integral is non-zero only over the limits of the well.

[1 mark]

Making the following substitution:

$$\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta),$$

the integral becomes

$$\frac{A^2}{2}\int_0^L 1 - \cos\left(\frac{2n\pi x}{L}\right) dx = 1\,.$$

[1 mark]

Integrating,

$$\frac{A^2}{2}\left[x - \frac{L}{2n\pi}\sin\left(\frac{2n\pi x}{L}\right)\right]_0^L = 1.$$

[1 mark]

Since n is any positive integer, the sine function is zero at the upper limit because $\sin(n2\pi) = 0$. The sine function is also zero at the lower limit because $\sin(0) = 0$. Thus, inserting the limits we find

$$\frac{A^2}{2}(L - 0 - 0 + 0) = 1.$$

Thus,

$$A = \sqrt{\frac{2}{L}}.$$

[1 mark]

Note that the constant A is independent of the quantum number n, therefore it is the same for each of the possible states within the well.

b) For the n = 1 state, the integral in the range $\frac{L}{4} \leq x \leq \frac{3L}{4}$ yields

$$\frac{1}{L}\left[x - \frac{L}{2\pi}\sin\left(\frac{2\pi x}{L}\right)\right]_{\frac{L}{4}}^{\frac{3L}{4}}.$$

[1 mark]

$$= \frac{1}{L}\left(\frac{3L}{4} - \frac{L}{2\pi}\sin\left(\frac{3\pi}{2}\right) - \frac{L}{4} + \frac{L}{2\pi}\sin\left(\frac{\pi}{2}\right)\right)$$

$$= \left(\frac{1}{2} + \frac{1}{2\pi}\left(\sin\left(\frac{\pi}{2}\right) - \sin\left(\frac{3\pi}{2}\right)\right)\right)$$

[1 mark]

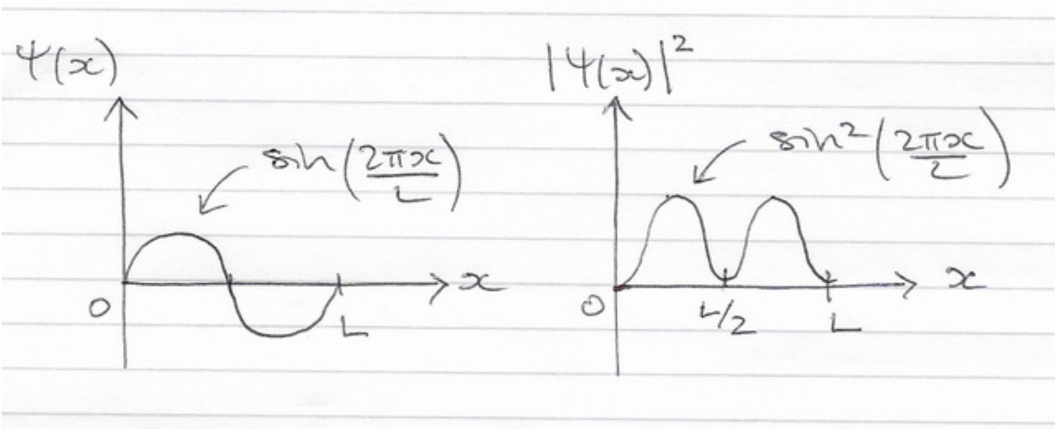
$$= \left(\frac{1}{2} + \frac{1}{\pi}\right) = 0.818 \text{ (or } 81.8\%\text{)}.$$

[1 mark]

The region $\frac{L}{4} \leq x \leq \frac{3L}{4}$ corresponds to half the width of the well. Classically, the particle travels at constant velocity across the well. Thus, the quantum mechanical probability of finding the particle in the central region of the well is significantly greater than the classically expected value of 0.5 (or 50%).

[1 mark]

c) The spatial wave function and probability density function for the n = 2 state are:



1 mark for each [2 marks]