1VGLA: DETERMINANTS PRACTICE QUESTIONS

The following questions relate to Chapter 7, Determinants. Questions are ranked in difficulty from A (basic) to C (challenging). For questions with multiple cases a fully justified solution will be given for at least 1 case.

(A) Question 1. Find the number of inversions in the following permutations:

(a) $\sigma = [5, 4, 3, 2, 1]$, (b) $\sigma = [1, 2, 3, 4, 5]$,
(c) $\sigma = [5, 3, 2, 4, 6, 1]$, (d) $\sigma = [2, 1, 3, 4]$.

Solution:

(a) $N([5, 4, 3, 2, 1]) = 10$,
(b) $N([1, 2, 3, 4, 5]) = 0$,
(c) $N([5, 3, 2, 4, 6, 1]) = 9$,
(d) $N([2, 1, 3, 4]) = 1$.

■

(A) Question 2. A matrix $\mathbf{A} = [a_{ij}]$ in $\mathcal{M}_{22}(\mathbb{R})$ is such that $a_{ij} = 2i - 5j$ for $i, j = 1, 2$. Find $\det(\mathbf{A})$.

Solution: Since

$$\mathbf{A} = \begin{pmatrix} 2(1) - 5(1) & 2(1) - 5(2) \\ 2(2) - 5(1) & 2(2) - 5(2) \end{pmatrix} = \begin{pmatrix} -3 & -8 \\ -1 & -6 \end{pmatrix}$$

it follows that $|\mathbf{A}| = (-3)(-6) - (-8)(-1) = 18 - 8 = 10$. ■

(A) Question 3. A matrix $\mathbf{A} = [a_{ij}]$ in $\mathcal{M}_{22}(\mathbb{R})$ is such that $a_{ii} = 1$ for $i = 1, 2$ and $a_{ij} = -3$ if $i \neq j$ and $i, j = 1, 2$. Find $\det(\mathbf{A})$.

Solution: Since

$$\mathbf{A} = \begin{pmatrix} 1 & -3 \\ -3 & 1 \end{pmatrix}$$

it follows that $|\mathbf{A}| = (1)(1) - (-3)(-3) = -8$. ■

(A) Question 4. Justify if you can/cannot compute the determinant of $\mathbf{A} \in \mathcal{M}_{23}(\mathbb{R})$.

Solution: We cannot since the determinant is only defined for $n \times n$ matrices. ■

(A) Question 5. Evaluate $|\mathbf{A}|$ for each of the following matrices $\mathbf{A}$:

(a) $\mathbf{A} = \begin{pmatrix} 5 & 7 & 9 \\ 1 & 3 & 2 \\ 3 & 9 & 6 \end{pmatrix}$, (b) $\mathbf{A} = \begin{pmatrix} 5 & 3 & 5 \\ 2 & -7 & 3 \\ 5 & 3 & 5 \end{pmatrix}$,

(c) $\mathbf{A} = \begin{pmatrix} 9 & 5 & 0 \\ 4 & -3 & -7 \\ 19 & 11 & 0 \end{pmatrix}$, (d) $\mathbf{A} = \begin{pmatrix} 3 & 0 & 7 \\ 5 & -3 & 2 \\ -2 & 0 & 3 \end{pmatrix}$,

(e) $\mathbf{A} = \begin{pmatrix} 3 & 5 & -7 & 11 \\ -4 & -3 & 4 & -13 \\ 0 & -2 & 0 & 0 \\ 0 & -5 & 3 & 0 \end{pmatrix}$, (f) $\mathbf{A} = \begin{pmatrix} 2 & -1 & 0 & 3 \\ 5 & 0 & 7 & 0 \\ 0 & 0 & 1 & 0 \\ -2 & -1 & -11 & 5 \end{pmatrix}$,

Solution: r_i and c_i refer to row i and column i in what follows.

(a) Observe that

$$|\mathbf{D}| = \sum_{\sigma\in S_n}(-1)^{N(\sigma}\begin{vmatrix} 5 & 7 & 9 \\ 1 & 3 & 2 \\ 3 & 9 & 6 \end{vmatrix} = 3\begin{vmatrix} 5 & 7 & 9 \\ 1 & 3 & 2 \\ 1 & 3 & 2 \end{vmatrix} (r_3 \to \tfrac{r_3}{3})$$
$$= 0 \qquad \text{(since } r_2 = r_3\text{)}$$

(b) The determinant is 0 since $r_1 = r_3$.

(c) Observe that

$$\begin{vmatrix} 9 & 5 & 0 \\ 4 & -3 & -7 \\ 19 & 11 & 0 \end{vmatrix} = (-7)(-1)^{2+3}\begin{vmatrix} 9 & 5 \\ 19 & 11 \end{vmatrix} \qquad \text{(expand along } c_3\text{)}$$
$$= 7(99-95)$$
$$= 28$$

(d) Observe that

$$\begin{vmatrix} 3 & 0 & 7 \\ 5 & -3 & 2 \\ -2 & 0 & 3 \end{vmatrix} = (-3)(-1)^{2+2}\begin{vmatrix} 3 & 7 \\ -2 & 3 \end{vmatrix} \qquad \text{(expand along } c_2\text{)}$$
$$= (-3)(9+14)$$
$$= -69$$

(e) Observe that

$$\begin{vmatrix} 3 & 5 & -7 & 11 \\ -4 & -3 & 4 & -13 \\ 0 & -2 & 0 & 0 \\ 0 & -5 & 3 & 0 \end{vmatrix} = (-2)(-1)^{3+2}\begin{vmatrix} 3 & -7 & 11 \\ -4 & 4 & -13 \\ 0 & 3 & 0 \end{vmatrix} \qquad \text{(expand along } r_3\text{)}$$
$$= 2.3.(-1)^{3+2}\begin{vmatrix} 3 & 11 \\ -4 & -13 \end{vmatrix} \qquad \text{(expand along } r_3\text{)}$$
$$= (-6)(-39+44)$$
$$= -30$$

(f) Observe that

$$\begin{vmatrix} 2 & -1 & 0 & 3 \\ 5 & 0 & 7 & 0 \\ 0 & 0 & 1 & 0 \\ -2 & -1 & -11 & 5 \end{vmatrix} = 1(-1)^{3+3}\begin{vmatrix} 2 & -1 & 3 \\ 5 & 0 & 0 \\ -2 & -1 & 5 \end{vmatrix} \qquad \text{(expand along } r_3\text{)}$$
$$= 5.(-1)^{2+1}\begin{vmatrix} -1 & 3 \\ -1 & 5 \end{vmatrix} \qquad \text{(expand along } r_2\text{)}$$
$$= (-5)(-5+3)$$
$$= 10$$

■

(A) Question 6. Calculate the determinant, $|\mathbf{A}|$, of the matrices $\mathbf{A}$ given below and explain why the calculation is true. *Refer to the property of determinants you are using (no need to reference the exact number of the result in the notes).*

(a) $\mathbf{A} = \begin{pmatrix} 2 & -3 & 5 \\ 0 & 1 & -7 \\ 0 & 0 & -4 \end{pmatrix}$, (b) $\mathbf{A} = \begin{pmatrix} 3 & -5 & 3 \\ 4 & -7 & 4 \\ 3 & 8 & 3 \end{pmatrix}$.

Solution:

(a) $|\mathbf{A}| = 2.1.(-4) = -8$ since $\mathbf{A}$ is upper triangular (i.e. $|\mathbf{A}|$ is the product of the terms on the diagonal).
(b) $|\mathbf{A}| = 0$ since columns 1 and 3 in $\mathbf{A}$ are equal.

■

(A) Question 7. Calculate the determinant, $|\mathbf{A}|$ of the matrix $\mathbf{A}$ given below by an expansion along a row or column of your choice, giving details of your calculation.

$$\mathbf{A} = \begin{pmatrix} -2 & -3 & 0 \\ 3 & -5 & -4 \\ -3 & -2 & 1 \end{pmatrix}.$$

Solution: One of many ways to calculate the determinant is as follows:

$$\begin{aligned} \begin{vmatrix} -2 & -3 & 0 \\ 3 & -5 & -4 \\ -3 & -2 & 1 \end{vmatrix} &= (-2)(-1)^{1+1}\begin{vmatrix} -5 & -4 \\ -2 & 1 \end{vmatrix} + (-3)(-1)^{1+2}\begin{vmatrix} 3 & -4 \\ -3 & 1 \end{vmatrix} \quad \text{(expand along } r_1\text{)} \\ &= (-2)(-5-8) + 3(3-12) \\ &= 26 - 27 \\ &= -1. \end{aligned}$$

■

(A) Question 8. A diagonal matrix $\mathbf{D} = [d_{ij}]$ in $\mathcal{M}_{44}(\mathbb{R})$ has integer entries and $|\mathbf{D}| = 12$. Find all possible values of d_{22} and d_{44} if $d_{11} = 3$ and $d_{33} = 1$.

Solution: Every diagonal matrix $\mathbf{D} \in \mathcal{M}_{44}(\mathbb{R})$ satisfies $|\mathbf{D}| = d_{11}d_{22}d_{33}d_{44}$. Since for these diagonal matrix in question,

$$12 = d_{11}d_{22}d_{33}d_{44} = 3d_{22}d_{44} \iff 4 = d_{22}d_{44}.$$

Also, since $d_{22}, d_{44} \in \mathbb{Z}$ it follows that

$$(d_{22}, d_{44}) \in \{(1,4),\ (-1,-4),\ (4,1), (-4,-1)\ (2,2), (-2,-2)\} \subset \mathbb{Z}^2.$$

■

(A) Question 9. Let $\mathbf{A}, \mathbf{B} \in \mathcal{M}_{nn}(\mathbb{R})$. Prove, using determinants, that $\mathbf{A}\cdot\mathbf{B}$ is invertible iff $\mathbf{A}$ and $\mathbf{B}$ are invertible.

Solution: Recall that for $\mathbf{A} \in \mathcal{M}_{nn}(\mathbb{R})$ we have $\mathbf{A}^{-1}$ exists iff $|\mathbf{A}| \neq 0$, and, $|\mathbf{A} \cdot \mathbf{B}| = |\mathbf{A}||\mathbf{B}|$. Therefore,

$$\begin{aligned} (\mathbf{A}\cdot\mathbf{B})^{-1} \text{ exists} &\iff |\mathbf{A}\cdot\mathbf{B}| \neq 0 \\ &\iff |\mathbf{A}||\mathbf{B}| \neq 0 \\ &\iff |\mathbf{A}| \neq 0 \text{ and } |\mathbf{B}| \neq 0. \\ &\iff \mathbf{A}^{-1} \text{ exists and } \mathbf{B}^{-1} \text{ exists,} \end{aligned}$$

as required. ■

(A) Question 10. Let $\mathbf{A}$ and $\mathbf{B}$ be invertible matrices in $\mathcal{M}_{nn}(\mathbb{R})$. Determine the value of

$$\det(\mathbf{A} \cdot \mathbf{B} \cdot \mathbf{A}^{-1} \cdot \mathbf{B}^{-1}).$$

Recall, matrix multiplication in $\mathcal{M}_{nn}(\mathbb{R})$ is not commutative.

Solution: Recall that $|\mathbf{A}^{-1}| = |\mathbf{A}|^{-1}$. Then, we have

$$|\mathbf{A} \cdot \mathbf{B} \cdot \mathbf{A}^{-1} \cdot \mathbf{B}^{-1}| = |\mathbf{A}||\mathbf{B}||\mathbf{A}^{-1}||\mathbf{B}^{-1}| = |\mathbf{A}||\mathbf{B}|\frac{1}{|\mathbf{A}|}\frac{1}{|\mathbf{B}|} = 1.$$

Note that we use associative and commutative properties of multiplication of real numbers in the last equality. ■

(A) Question 11. $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are matrices in $\mathcal{M}_{nn}(\mathbb{R})$ such that $\det(\mathbf{A}) = 2$, $\det(\mathbf{B}) = -\frac{1}{2}$ and $\det(\mathbf{C}) = \frac{1}{4}$. Determine the value of

(a) $\det(\mathbf{C}^3 \cdot \mathbf{B}^{-1} \cdot \mathbf{A}^2 \cdot (\mathbf{B}^T)^2 \cdot \mathbf{C}^T \cdot (\mathbf{A}^{-1})^3)$,
(b) $\det((\mathbf{A}^{-1})^2 \cdot (\mathbf{B}^{-1})^2 \cdot \mathbf{C}^T \cdot (\mathbf{A}^T)^2 \cdot \mathbf{B}^T \cdot (\mathbf{C}^{-1})^2)$.

Solution:

(a)

$$\begin{aligned}
\det(\mathbf{C}^3 \cdot \mathbf{B}^{-1} \cdot \mathbf{A}^2 \cdot (\mathbf{B}^T)^2 \cdot \mathbf{C}^T \cdot (\mathbf{A}^{-1})^3) &= |\mathbf{C}^3||\mathbf{B}^{-1}||\mathbf{A}^2||(\mathbf{B}^T)^2||\mathbf{C}^T||(\mathbf{A}^{-1})^3| \\
&= |\mathbf{C}|^3\frac{1}{|\mathbf{B}|}|\mathbf{A}|^2|\mathbf{B}|^2|\mathbf{C}|\frac{1}{|\mathbf{A}|^3} \\
&= |\mathbf{C}|^4|\mathbf{B}|\frac{1}{|\mathbf{A}|} \\
&= \left(\frac{1}{4}\right)^4\left(-\frac{1}{2}\right)\left(\frac{1}{2}\right) \\
&= -\frac{1}{1024}.
\end{aligned}$$

(b) $\det((\mathbf{A}^{-1})^2 \cdot (\mathbf{B}^{-1})^2 \cdot \mathbf{C}^T \cdot (\mathbf{A}^T)^2 \cdot \mathbf{B}^T \cdot (\mathbf{C}^{-1})^2) = -8.$

■

(A) Question 12. One expression for a certain invertible matrix $\mathbf{A}$ in $\mathcal{M}_{33}(\mathbb{R})$ as a product of elementary matrices is given below. Find the value of $|\mathbf{A}|$ without calculating the elements of $\mathbf{A}$.

(a) $\mathbf{A} = \mathbf{E}_{[2,-4]} \cdot \mathbf{E}_{(2,3,-3)} \cdot \mathbf{E}_{(1,3,2)} \cdot \mathbf{E}_{[3,\frac{1}{2}]} \cdot \mathbf{E}_{(2,3)} \cdot \mathbf{E}_{(2,1,-4)} \cdot \mathbf{E}_{[1,-1]} \cdot \mathbf{E}_{(1,3)}$,
(b) $\mathbf{A} = \mathbf{E}_{(2,1,\sqrt{5})} \cdot \mathbf{E}_{(2,3)} \cdot \mathbf{E}_{[3,\sqrt{3}]} \cdot \mathbf{E}_{(2,3,\frac{5}{11})} \cdot \mathbf{E}_{[2,\frac{7}{\sqrt{3}}]} \cdot \mathbf{E}_{(2,11,-\sqrt{11})} \cdot \mathbf{E}_{[3,-2]} \cdot \mathbf{E}_{(1,3,11)}$.

Recall that

- $\boldsymbol{E}_{[i,\lambda]}$ *is the elementary matrix which multiplies row i by λ;*
- $\boldsymbol{E}_{(i,j)}$ *is the elementary matrix which swaps rows i and j; and*
- $\boldsymbol{E}_{(i,j,\lambda)}$ *is the elementary matrix which adds λ times row j to row i.*

Solution: Now, we have,

(a)

$$\begin{aligned}
|\mathbf{A}| &= \left|\mathbf{E}_{[2,-4]} \cdot \mathbf{E}_{(2,3,-3)} \cdot \mathbf{E}_{(1,3,2)} \cdot \mathbf{E}_{[3,\frac{1}{2}]} \cdot \mathbf{E}_{(2,3)} \cdot \mathbf{E}_{(2,1,-4)} \cdot \mathbf{E}_{[1,-1]} \cdot \mathbf{E}_{(1,3)}\right| \\
&= |\mathbf{E}_{[2,-4]}||\mathbf{E}_{(2,3,-3)}||\mathbf{E}_{(1,3,2)}||\mathbf{E}_{[3,\frac{1}{2}]}||\mathbf{E}_{(2,3)}||\mathbf{E}_{(2,1,-4)}||\mathbf{E}_{[1,-1]}||\mathbf{E}_{(1,3)}| \\
&= (-4)(1)(1)\left(\tfrac{1}{2}\right)(-1)(1)(-1)(-1) \\
&= 2
\end{aligned}$$

(b) $|\mathbf{A}| = |\mathbf{E}_{(2,1,\sqrt{5})} \cdot \mathbf{E}_{(2,3)} \cdot \mathbf{E}_{[3,\sqrt{3}]} \cdot \mathbf{E}_{(2,3,\frac{5}{11})} \cdot \mathbf{E}_{[2,\frac{7}{\sqrt{3}}]} \cdot \mathbf{E}_{(2,11,-\sqrt{11})} \cdot \mathbf{E}_{[3,-2]} \cdot \mathbf{E}_{(1,3,11)}| = 14.$

■

(A) Question 13. A matrix $\mathbf{A} \in \mathcal{M}_{nn}(\mathbb{R})$ is called an *orthogonal* matrix if

$$\mathbf{A} \cdot \mathbf{A}^T = \mathbf{A}^T \cdot \mathbf{A} = \mathbf{I}_n.$$

List all elements of the set

$$\mathcal{S} = \{x \in \mathbb{R} \ : \ n \in \mathbb{N}, \ \mathbf{A} \in \mathcal{M}_{nn}(\mathbb{R}) \text{ is an orthogonal matrix and } |\mathbf{A}| = x\}.$$

Solution: For any orthogonal matrix

$$\mathbf{A} \cdot \mathbf{A}^T = \mathbf{I}_n \implies |\mathbf{A}||\mathbf{A}^T| = |\mathbf{I}_n| \implies |\mathbf{A}|^2 = 1.$$

Therefore, $\mathcal{S} \subset \{-1, 1\}$. For $n \in \mathbb{N}$, the matrix $\mathbf{I}_n$ is an orthogonal matrix with $|\mathbf{I}_n| = 1$. Moreover, for odd $n \in \mathbb{N}$, the matrix $-\mathbf{I}_n$ is an orthogonal matrix with $|-\mathbf{I}_n| = -1$. Hence, $\mathcal{S} = \{-1, 1\}$. ■

(A) Question 14. For each of the following invertible matrices $\mathbf{A} \in \mathcal{M}_{33}(\mathbb{R})$, find

(i) the cofactor matrix $\mathbf{C}(\mathbf{A})$ of $\mathbf{A}$,
(ii) the adjoint matrix $\operatorname{adj}(\mathbf{A})$ of $\mathbf{A}$,
(iii) $\det(\mathbf{A})$,
(iv) the inverse $\mathbf{A}^{-1}$ of $\mathbf{A}$,
(v) the solution of the system of linear equations $\mathbf{A} \cdot \mathbf{x} = \mathbf{b}$, for $\mathbf{x} = (x_1 \ x_2 \ x_3)^T$.

(a) $\mathbf{A} = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 4 \\ 4 \\ 7 \end{pmatrix}$,

(b) $\mathbf{A} = \begin{pmatrix} 1 & -2 & -1 \\ 1 & -5 & -6 \\ 5 & -4 & 6 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 3 \\ 7 \\ 8 \end{pmatrix}$,

(c) $\mathbf{A} = \begin{pmatrix} -9 & -1 & -9 \\ -10 & -8 & -4 \\ 3 & 4 & 1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$.

Solution:

(a) (i)

$$\mathbf{C}(\mathbf{A}) = \begin{pmatrix} (0)(8) - (3)(-3) & (-1)((1)(8) - (3)(4)) & (1)(-3) - (0)(4) \\ (-1)((1)(8) - (2)(-3)) & (0)(8) - (2)(4) & -((0)(-3) - (1)(4)) \\ (1)(3) - (2)(0) & -((0)(3) - (2)(1)) & (0)(0) - (1)(1) \end{pmatrix}$$
$$= \begin{pmatrix} 9 & 4 & -3 \\ -14 & -8 & 4 \\ 3 & 2 & -1 \end{pmatrix}$$

(ii)

$$\operatorname{adj}(\mathbf{A}) = \mathbf{C}(\mathbf{A})^T = \begin{pmatrix} 9 & -14 & 3 \\ 4 & -8 & 2 \\ -3 & 4 & -1 \end{pmatrix}$$

(iii)

$$|\mathbf{A}| = -((1)(8) - (3)(4)) + 2((1)(-3) - (0)(4)) = 4 - 6 = -2.$$

(iv)

$$\mathbf{A}^{-1} = \frac{1}{|\mathbf{A}|}\text{adj}(\mathbf{A}) = \begin{pmatrix} -\frac{9}{2} & 7 & -\frac{3}{2} \\ -2 & 4 & -1 \\ \frac{3}{2} & -2 & \frac{1}{2} \end{pmatrix}$$

(v)

$$\mathbf{x} = \mathbf{A}^{-1} \cdot \mathbf{b} = \begin{pmatrix} -\frac{9}{2} & 7 & -\frac{3}{2} \\ -2 & 4 & -1 \\ \frac{3}{2} & -2 & \frac{1}{2} \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 4 \\ 7 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \\ 1 \\ \frac{3}{2} \end{pmatrix}$$

(b) (i) $\mathbf{C}(\mathbf{A}) = \begin{pmatrix} -54 & -36 & 21 \\ 16 & 11 & -6 \\ 7 & 5 & -3 \end{pmatrix}$

(ii) $\text{adj}(\mathbf{A}) = \begin{pmatrix} -54 & 16 & 7 \\ -36 & 11 & 5 \\ 21 & -6 & -3 \end{pmatrix}$

(iii) $|\mathbf{A}| = -3$

(iv) $\mathbf{A}^{-1} = \begin{pmatrix} 18 & -\frac{16}{3} & -\frac{7}{3} \\ 12 & -\frac{11}{3} & -\frac{5}{3} \\ -7 & 2 & 1 \end{pmatrix}$

(v) $\mathbf{x} = \begin{pmatrix} -2 \\ -3 \\ 1 \end{pmatrix}$

(c) (i) $\mathbf{C}(\mathbf{A}) = \begin{pmatrix} 8 & -2 & -16 \\ -35 & 18 & 33 \\ -68 & 54 & 62 \end{pmatrix}$

(ii) $\text{adj}(\mathbf{A}) = \begin{pmatrix} 8 & -35 & -68 \\ -2 & 18 & 54 \\ -16 & 33 & 62 \end{pmatrix}$

(iii) $|\mathbf{A}| = 74$

(iv) $\mathbf{A}^{-1} = \begin{pmatrix} \frac{4}{37} & -\frac{35}{74} & -\frac{34}{37} \\ -\frac{1}{37} & \frac{9}{37} & \frac{27}{37} \\ -\frac{8}{37} & \frac{33}{74} & \frac{31}{37} \end{pmatrix}$

(v) $\mathbf{x} = \begin{pmatrix} \frac{53}{74} \\ -\frac{2}{37} \\ -\frac{69}{74} \end{pmatrix}$

■

(A) Question 15. Use Cramer's rule to solve the systems of simultaneous linear equations

(a) $\left\{ \begin{array}{rcrcr} 11x_1 & + & 61x_2 & = & 2 \\ 19x_1 & - & 31x_2 & = & -3, \end{array} \right.$ (b) $\left\{ \begin{array}{rcrcr} 4x_1 & - & 6x_2 & = & 2 \\ -5x_1 & + & 9x_2 & = & 5, \end{array} \right.$

(c) $\left\{ \begin{array}{rcrcr} -x_1 & - & x_2 & = & -2 \\ -2x_1 & & & = & 3, \end{array} \right.$ (d) $\left\{ \begin{array}{rcrcr} -5x_1 & - & x_2 & = & -5 \\ 5x_1 & - & 3x_2 & = & 0. \end{array} \right.$

Solution:

(a) Since

$$|\mathbf{A}| = \begin{vmatrix} 11 & 61 \\ 19 & -31 \end{vmatrix} = 11.(-31) - 61.19 = -341 - 1159 = -1500$$

$$|\mathbf{A}_1| = \begin{vmatrix} 2 & 61 \\ -3 & -31 \end{vmatrix} = -62 + 183 = 121$$

$$|\mathbf{A}_2| = \begin{vmatrix} 11 & 2 \\ 19 & -3 \end{vmatrix} = -33 - 38 = -71,$$

it follows from Cramer's rule that

$$x = \frac{|\mathbf{A}_1|}{|\mathbf{A}|} = -\frac{121}{1500} \text{ and } y = \frac{|\mathbf{A}_2|}{|\mathbf{A}|} = \frac{71}{1500}.$$

(b) Since $|\mathbf{A}| = 6$, $|\mathbf{A}_1| = 48$ and $|\mathbf{A}_2| = 30$ it follows from Cramer's rule that $x = \frac{48}{6} = 8$ and $y = \frac{30}{6} = 5$.

(c) Since $|\mathbf{A}| = -2$, $|\mathbf{A}_1| = 3$ and $|\mathbf{A}_2| = -7$ it follows from Cramer's rule that $x = \frac{3}{-2} = -\frac{3}{2}$ and $y = \frac{-7}{-2} = \frac{7}{2}$.

(d) Since $|\mathbf{A}| = 20$, $|\mathbf{A}_1| = 15$ and $|\mathbf{A}_2| = 25$ it follows from Cramer's rule that $x = \frac{15}{20} = \frac{3}{4}$ and $y = \frac{25}{20} = \frac{5}{4}$.

■

(A) Question 16. Use Cramer's rule to find the coordinates of the point of intersection of the planes with equations

$$\text{(a)} \left\{ \begin{array}{rcrcrcr} -2x & & & + & 2z & = & 2 \\ -2x & - & y & + & 3z & = & 1 \\ & & & - & 4z & = & -1, \end{array} \right. \qquad \text{(b)} \left\{ \begin{array}{rcrcrcr} 4x & - & 2y & + & 2z & = & 1 \\ 3x & - & 3y & - & 5z & = & -1 \\ 3x & - & 2y & + & z & = & 0. \end{array} \right.$$

Solution:

(a) Since $|\mathbf{A}| = -8$, $|\mathbf{A}_1| = 6$, $|\mathbf{A}_2| = -10$ and $|\mathbf{A}_3| = -2$ it follows from Cramer's rule that $x = \frac{6}{-8} = -\frac{3}{4}$, $y = \frac{-10}{-8} = \frac{5}{4}$ and $z = \frac{-2}{-8} = \frac{1}{4}$.

(b) Since $|\mathbf{A}| = -10$, $|\mathbf{A}_1| = -11$, $|\mathbf{A}_2| = -16$ and $|\mathbf{A}_3| = 1$ it follows from Cramer's rule that $x = \frac{-11}{-10} = \frac{11}{10}$, $y = \frac{-16}{-10} = \frac{8}{5}$ and $z = \frac{1}{-10} = -\frac{1}{10}$.

■

(A) Question 17. Determine whether or not the following homogeneous systems have non-trivial solutions. *You do not need to find these solutions if they exist!*

$$\text{(a)} \left\{ \begin{array}{rcrcrcr} -3x & + & 3y & - & 4z & = & 0 \\ & - & 3y & & & = & 0 \\ -3x & + & 2y & - & 4z & = & 0, \end{array} \right. \qquad \text{(b)} \left\{ \begin{array}{rcrcrcr} 5x & - & 4y & - & z & = & 0 \\ -5x & + & 6y & - & z & = & 0 \\ -4x & + & 9y & - & 5z & = & 0. \end{array} \right.$$

Solution: A homogeneous system of n linear equations in n unknowns, say, $\mathbf{A} \cdot \mathbf{x} = \mathbf{0}$ will have non-trivial solutions if and only if $|\mathbf{A}| = 0$.

(a) Since

$$\begin{vmatrix} -3 & 3 & -4 \\ 0 & -3 & 0 \\ -3 & 2 & -4 \end{vmatrix} = (-3)((-3)(-4) - (-4)(-3)) = 0,$$

it follows that the system of linear equations has non-trivial solutions.

(b) Since

$$\begin{vmatrix} 5 & -4 & -1 \\ -5 & 6 & -1 \\ -4 & 9 & -5 \end{vmatrix} = \begin{vmatrix} 5 & -4 & -1 \\ -10 & 10 & 0 \\ -29 & 29 & 0 \end{vmatrix} = 0,$$

it follows that the system of linear equations has non-trivial solutions.

■

(A) Question 18. Is it true of false that a homogeneous system of n linear equations, say, $\mathbf{A} \cdot \mathbf{x} = \mathbf{0}$, has non-trivial solutions if and only if $|\mathbf{A}| = 0$?

Solution: It is false since not all linear systems of n equations are in n unknowns i.e. $\mathbf{A}$ does not necessarily have a determinant. To clarify, consider $\mathbf{A} \in \mathcal{M}_{23}(\mathbb{R})$ given by

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

for which the induced linear system has non-trivial solutions but $\mathbf{A}$ does not have a determinant. ■

(B) Question 19. For $\sigma \in S_n$ with $N(\sigma) > 0$, establish that there exists $i \in \Sigma_n$ such that $\sigma(i) > \sigma(i+1)$.

Solution: Suppose that there does not exist $i \in \Sigma_n$ such that $\sigma(i) > \sigma(i+1)$. Then

$$1 \leq \sigma(1) \leq \sigma(2) \leq \sigma(3) \leq \ldots \leq \sigma(n-1) \leq \sigma(n) \leq n.$$

Since σ is a bijection from Σ_n to Σ_n it follows that $\sigma(i) = i$ for $i = 1, \ldots, n$ and hence $N(\sigma) = 0$, which is a contradiction. Therefore, there exists $i \in \Sigma_n$ such that $\sigma(i) > \sigma(i+1)$. ■

(B) Question 20. Evaluate $\det(\mathbf{A})$ for each of the following matrices $\mathbf{A}$:

$$\text{(a)}\quad \mathbf{A} = \begin{pmatrix} 1 & 2 & -1 & 5 \\ -2 & -2 & 5 & -3 \\ 3 & 0 & -11 & -2 \\ -1 & 4 & 3 & 2 \end{pmatrix}, \quad \text{(b)}\quad \mathbf{A} = \begin{pmatrix} 3 & 1 & 0 & 0 \\ 1 & 3 & 1 & 0 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 1 & 3 \end{pmatrix}.$$

Solution:

(a)

$$\begin{aligned}
|\mathbf{A}| &= \begin{vmatrix} 1 & 2 & -1 & 5 \\ -2 & -2 & 5 & -3 \\ 3 & 0 & -11 & -2 \\ -1 & 4 & 3 & 2 \end{vmatrix} \\
&= \begin{vmatrix} 1 & 2 & -1 & 5 \\ 0 & 2 & 3 & 7 \\ 0 & -6 & -8 & -17 \\ 0 & 6 & 2 & 7 \end{vmatrix} && \begin{pmatrix} r_2 \to r_2 + 2r_1 \\ r_3 \to r_3 - 3r_1 \\ r_4 \to r_4 + r_1 \end{pmatrix} \\
&= 1(-1)^{1+1} \begin{vmatrix} 2 & 3 & 7 \\ -6 & -8 & -17 \\ 6 & 2 & 7 \end{vmatrix} \\
&= \begin{vmatrix} 2 & 3 & 7 \\ 0 & 1 & 4 \\ 0 & -7 & -14 \end{vmatrix} && \begin{pmatrix} r_2 \to r_2 + 3r_1 \\ r_3 \to r_3 - 3r_1 \end{pmatrix} \\
&= 2(-1)^{1+1} \begin{vmatrix} 1 & 4 \\ -7 & -14 \end{vmatrix} \\
&= 2(-14 + 28) \\
&= 28
\end{aligned}$$

(b)

$$|\mathbf{A}| = \begin{vmatrix} 3 & 1 & 0 & 0 \\ 1 & 3 & 1 & 0 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 1 & 3 \end{vmatrix}$$

$$
\begin{aligned}
&= 3(-1)^{1+1}\begin{vmatrix} 3 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 3 \end{vmatrix} + 1(-1)^{1+2}\begin{vmatrix} 1 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 1 & 3 \end{vmatrix} \\
&= 3.3(-1)^{1+1}\begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix} + 3.1(-1)^{1+2}\begin{vmatrix} 1 & 1 \\ 0 & 3 \end{vmatrix} - 1.(-1)^{1+1}\begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix} \\
&= 9(9-1) - 3(3-0) - (9-1) \\
&= 55
\end{aligned}
$$

■

(B) Question 21. For each of the following matrices $\mathbf{A} \in \mathcal{M}_{22}(\mathbb{R})$, find all real values of x, if any, for which $|\mathbf{A}| = 0$:

(a) $\mathbf{A} = \begin{pmatrix} 3x+4 & -3 \\ 1 & 2x-1 \end{pmatrix}$, (b) $\mathbf{A} = \begin{pmatrix} x & x-1 \\ 1 & x^2+1 \end{pmatrix}$,

Solution:

(a)

$$
\begin{aligned}
\begin{vmatrix} 3x+4 & -3 \\ 1 & 2x-1 \end{vmatrix} &= (3x+4)(2x-1) + 3 \\
&= 6x^2 + 8x - 3x - 4 + 3 \\
&= 6x^2 + 5x - 1 \\
&= (6x-1)(x+1).
\end{aligned}
$$

Therefore, $|\mathbf{A}| = 0 \iff x \in \{-1, \frac{1}{6}\}$.

(b)

$$\begin{vmatrix} x & x-1 \\ 1 & x^2+1 \end{vmatrix} = x^3 + x - x + 1 = x^3 + 1 = (x+1)(x^2 - x + 1).$$

The quadratic on the RHS of the equation above is irreducible so $|\mathbf{A}| = 0 \iff x \in \{-1\}$.

■

(B) Question 22. If a, b, c are distinct real numbers, show that

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = (a-b)(b-c)(c-a).$$

This matrix is called a Vandermonde matrix and can be used to prove results related to polynomial interpolation.

Solution:

$$
\begin{aligned}
\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} &= \begin{vmatrix} 1 & 0 & 0 \\ a & b-a & c-a \\ a^2 & b^2-a^2 & c^2-a^2 \end{vmatrix} \qquad \begin{pmatrix} c_2 \to c_2 - c_1 \\ c_3 \to c_3 - c_1 \end{pmatrix} \\
&= 1.(-1)^{1+1}\begin{vmatrix} b-a & c-a \\ b^2-a^2 & c^2-a^2 \end{vmatrix} \\
&= (b-a)(c^2-a^2) - (c-a)(b^2-a^2) \\
&= (b-a)(c-a)(c+a) - (c-a)(b-a)(b+a) \\
&= (b-a)(c-a)(c+a-b-a) \\
&= (a-b)(b-c)(c-a)
\end{aligned}
$$

$\blacksquare$

(B) Question 23. For each of the following matrices $\mathbf{A} \in \mathcal{M}_{33}(\mathbb{R})$, find all real numbers $\lambda \in \mathbb{C}$ such that $|\mathbf{A} - \lambda \mathbf{I}_3| = 0$.

(a) $\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 6 & -6 \\ 1 & 2 & -1 \end{pmatrix}$, (b) $\mathbf{A} = \begin{pmatrix} 1 & 2 & -3 \\ 0 & -4 & 0 \\ 7 & 8 & 9 \end{pmatrix}$,

Solution:

(a)

$$\begin{aligned} |\mathbf{A} - \lambda \mathbf{I}_3| &= \begin{vmatrix} 1-\lambda & 0 & 0 \\ 2 & 6-\lambda & -6 \\ 1 & 2 & -1-\lambda \end{vmatrix} \\ &= (1-\lambda)(-1)^{1+1} \begin{vmatrix} 6-\lambda & -6 \\ 2 & -1-\lambda \end{vmatrix} \\ &= (1-\lambda)(-(6-\lambda)(1+\lambda)+12) \\ &= (1-\lambda)(\lambda^2 - 5\lambda + 6) \\ &= (1-\lambda)(\lambda-2)(\lambda-3). \end{aligned}$$

Therefore $|\mathbf{A} - \lambda \mathbf{I}_3| = 0 \iff \lambda \in \{1, 2, 3\}$.

(b)

$$\begin{aligned} |\mathbf{A} - \lambda \mathbf{I}_3| &= \begin{vmatrix} 1-\lambda & 2 & -3 \\ 0 & -4-\lambda & 0 \\ 7 & 8 & 9-\lambda \end{vmatrix} \\ &= -(\lambda+4)(-1)^{2+2} \begin{vmatrix} 1-\lambda & -3 \\ 7 & 9-\lambda \end{vmatrix} \\ &= -(\lambda+4)((1-\lambda)(9-\lambda)+21) \\ &= -(\lambda+4)(\lambda^2 - 10\lambda + 30). \end{aligned}$$

Since the roots of $(\lambda^2 - 10\lambda + 30)$ are given by

$$\lambda_{\pm} = \frac{10 \pm (100-120)^{1/2}}{2} = 5 \pm \sqrt{5}i,$$

it follows that $|\mathbf{A} - \lambda \mathbf{I}_3| = 0 \iff \lambda \in \{-4,\ 5+\sqrt{5}i, 5-\sqrt{5}i\}$.

$\blacksquare$

(B) Question 24. Let $\mathbf{A}, \mathbf{P} \in \mathcal{M}_{nn}(\mathbb{R})$ with $\mathbf{P}$ invertible. Prove that

(a) $\det(\mathbf{P}^{-1} \cdot \mathbf{A} \cdot \mathbf{P}) = \det(\mathbf{A})$,

(b) $\det(\mathbf{P}^{-1} \cdot \mathbf{A} \cdot \mathbf{P} - \lambda \mathbf{I}_n) = \det(\mathbf{A} - \lambda \mathbf{I}_n)$ for any $\lambda (\neq 0) \in \mathbb{R}$. *It may be helpful to write $\lambda \boldsymbol{I}_n$ as $\boldsymbol{P}^{-1} \cdot (\lambda \boldsymbol{I}_n) \cdot \boldsymbol{P}$.*

(c) If

$$\mathbf{A} = \begin{pmatrix} b+c & c-a & b-a \\ c-b & c+a & a-b \\ b-c & a-c & a+b \end{pmatrix} \text{ and } \mathbf{P} = \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}$$

find $\mathbf{P}^{-1}$ and then directly calculate $\mathbf{P}^{-1} \cdot \mathbf{A} \cdot \mathbf{P}$. Hence find $|\mathbf{A}|$.

Solution:

(a) The result follows from

$$|\mathbf{P}^{-1} \cdot \mathbf{A} \cdot \mathbf{P}| = |\mathbf{P}^{-1}||\mathbf{A}||\mathbf{P}| = \frac{1}{|\mathbf{P}|}|\mathbf{A}||\mathbf{P}| = |\mathbf{A}|.$$

(b) Since $\lambda \mathbf{I}_n = \mathbf{P}^{-1} \cdot (\lambda \mathbf{I}_n) \cdot \mathbf{P}$, it follows (using the distributive property) that

$$\begin{aligned} |\mathbf{P}^{-1} \cdot \mathbf{A} \cdot \mathbf{P} - \lambda \mathbf{I}_n| &= |\mathbf{P}^{-1} \cdot \mathbf{A} \cdot \mathbf{P} - \mathbf{P}^{-1} \cdot (\lambda \mathbf{I}_n) \cdot \mathbf{P}| \\ &= |\mathbf{P}^{-1} \cdot (\mathbf{A} - \lambda \mathbf{I}_n) \cdot \mathbf{P}| \\ &= |\mathbf{P}^{-1}||\mathbf{A} - \lambda \mathbf{I}_n||\mathbf{P}| \\ &= \frac{1}{|\mathbf{P}|}|\mathbf{P}||\mathbf{A} - \lambda \mathbf{I}_n| \\ &= |\mathbf{A} - \lambda \mathbf{I}_n|. \end{aligned}$$

This calculation establishes that similar matrices have the same eigenvalues (see Chapter 10).

(c) It follows that

$$\mathbf{P}^{-1} = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix},$$

and

$$\mathbf{P}^{-1} \cdot \mathbf{A} \cdot \mathbf{P} = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix} \cdot \begin{pmatrix} b+c & c-a & b-a \\ c-b & c+a & a-b \\ b-c & a-c & a+b \end{pmatrix} \cdot \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix} = \begin{pmatrix} 2a & 0 & 0 \\ 0 & 2b & 0 \\ 0 & 0 & 2c \end{pmatrix}.$$

Therefore, $|\mathbf{A}| = |\mathbf{P}^{-1} \cdot \mathbf{A} \cdot \mathbf{P}| = 8abc$.

■

(B) Question 25. (a) If $\mathbf{A} \in \mathcal{M}_{nn}(\mathbb{R})$, find a set $\mathcal{S}$ of cardinality less than 4 which contains the possible real numbers $\det(\mathbf{A})$ in each of the following cases?

(i) $\mathbf{A}^4 = \mathbf{I}_n$,
(ii) $\mathbf{A}^3 = \mathbf{A}$,
(iii) $\mathbf{A}^2 = -\mathbf{A}$ with n odd.
(iv) $\mathbf{A}^2 = -\mathbf{A}$ with n even.

(b) Does there exist a matrix $\mathbf{A} \in \mathcal{M}_{33}(\mathbb{R})$ such that $\mathbf{A}^2 = -\mathbf{I}_3$? Justify your answer.

Solution:

(a) (i) $\mathbf{A}^4 = \mathbf{I}_n \implies |\mathbf{A}|^4 - 1 = 0 \implies |\mathbf{A}| \in \{-1, 1\}$. Therefore, let $\mathcal{S} = \{-1, 1\}$.
(ii) $\mathbf{A}^3 = \mathbf{A} \implies |\mathbf{A}|^3 - |\mathbf{A}| = 0 \iff |\mathbf{A}|(|\mathbf{A}|^2 - 1) = 0 \iff |\mathbf{A}| \in \{-1, 0, 1\}$. Therefore, let $\mathcal{S} = \{-1, 0, 1\}$.
(iii) If n is odd, then $\mathbf{A}^2 = -\mathbf{A} \implies |\mathbf{A}|^2 + |\mathbf{A}| = 0 \iff |\mathbf{A}|(|\mathbf{A}| + 1) = 0 \iff |\mathbf{A}| \in \{-1, 0\}$. Therefore, let $\mathcal{S} = \{-1, 0\}$.
(iv) If n is even, then $\mathbf{A}^2 = -\mathbf{A} \implies |\mathbf{A}|^2 - |\mathbf{A}| = 0 \iff |\mathbf{A}|(1 - |\mathbf{A}|) = 0 \iff |\mathbf{A}| \in \{0, 1\}$. Therefore, let $\mathcal{S} = \{0, 1\}$.

(b) No, since such a matrix would satisfy $|\mathbf{A}|^2 = |-\mathbf{I}_3| = -1$ which has no real solutions.

■

(B) Question 26. Let $\mathbf{A}, \mathbf{B} \in \mathcal{M}_{nn}(\mathbb{R})$. Prove that

(a) $\det(\mathbf{A} \cdot \mathbf{B}) = \det(\mathbf{B} \cdot \mathbf{A})$,
(b) $\det(\mathbf{A}^T + \mathbf{B}) = \det(\mathbf{A} + \mathbf{B}^T)$.

Recall that matrix multiplication in $\mathcal{M}_{nn}(\mathbb{R})$ is not commutative and $(\boldsymbol{P} + \boldsymbol{Q})^T = \boldsymbol{P}^T + \boldsymbol{Q}^T$.

Solution:

(a) Since multiplication of real numbers is a commutative operation, it follows that

$$|\mathbf{A} \cdot \mathbf{B}| = |\mathbf{A}||\mathbf{B}| = |\mathbf{B}||\mathbf{A}| = |\mathbf{B} \cdot \mathbf{A}|.$$

(b) From the hint, it follow that

$$|\mathbf{A}^T + \mathbf{B}| = |(\mathbf{A}^T + \mathbf{B})^T| = |(\mathbf{A}^T)^T + \mathbf{B}^T| = |\mathbf{A} + \mathbf{B}^T|.$$

$\blacksquare$

(B) Question 27. Is the following argument acceptable as an answer to the following question which was (apparently) set on an examination paper?

Question: "Prove that if $\mathbf{A}$ is a skew-symmetric matrix in $\mathcal{M}_{nn}(\mathbb{R})$ with n an *odd* positive integer, then $\det(\mathbf{A}) = 0$."

(Recall that $\mathbf{A}$ is skew-symmetric if $\mathbf{A}^T = -\mathbf{A}$.)

Answer: "Let

$$\mathbf{A} = \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix}$$

where $a, b, c \in \mathbb{R}$. Then by expanding elements of row 1 with cofactors,

$$\begin{aligned} \det(\mathbf{A}) = \begin{vmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{vmatrix} &= (-a)\begin{vmatrix} -a & c \\ -b & 0 \end{vmatrix} + b\begin{vmatrix} -a & 0 \\ -b & -c \end{vmatrix} \\ &= -abc + abc \\ &= 0.'' \end{aligned}$$

If you decide that the argument above is not acceptable as an answer, state why not and then give an argument which proves that the statement in the question is true.

Solution: This proof is not valid since it only establishes the result when $n = 3$. To prove the result, let $\mathbf{A} \in \mathcal{M}_{nn}(\mathbb{R})$ be a skew-symmetric matrix for odd $n \in \mathbb{N}$. Then,

$$\mathbf{A}^T = -\mathbf{A} \Longrightarrow |\mathbf{A}^T| = |-\mathbf{A}| \iff |\mathbf{A}| = -|\mathbf{A}| \iff |\mathbf{A}| = 0.$$

Note that if n was even then $|-\boldsymbol{A}| = |\boldsymbol{A}|$. $\blacksquare$

(B) Question 28. Using Cramer's rule, find

(a) x such that $\left\{\begin{array}{rcrcrcrcr} w & + & x & + & y & + & z & = & 3 \\ 7w & + & 3x & - & y & + & z & = & 1 \\ 2w & - & 2x & - & 3y & + & 3z & = & 4 \\ w & + & x & + & y & + & 8z & = & 7 \end{array}\right.$

(b) z such that $\left\{\begin{array}{rcrcrcrcr} -w & - & 2x & + & y & + & z & = & 3 \\ w & - & x & + & 2y & - & 2z & = & -1 \\ -2w & - & x & + & & + & z & = & 1 \\ 2w & + & 2x & + & & - & 3z & = & 0 \end{array}\right.$

Solution:

(a) Observe that

$$\begin{aligned} |\mathbf{A}| = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 7 & 3 & -1 & 1 \\ 2 & -2 & -3 & 3 \\ 1 & 1 & 1 & 8 \end{vmatrix} &= \begin{vmatrix} 1 & 1 & 1 & 1 \\ 7 & 3 & -1 & 1 \\ 2 & -2 & -3 & 3 \\ 0 & 0 & 0 & 7 \end{vmatrix} \\ &= 7(-1)^{4+4}\begin{vmatrix} 1 & 1 & 1 \\ 7 & 3 & -1 \\ 2 & -2 & -3 \end{vmatrix} \\ &= 7((-9-2) - (-21+2) + (-14-6)) \\ &= 7(-11+19-20) \end{aligned}$$

$$= -84$$

and

$$|\mathbf{A}_2| = \begin{vmatrix} 1 & 3 & 1 & 1 \\ 7 & 1 & -1 & 1 \\ 2 & 4 & -3 & 3 \\ 1 & 7 & 1 & 8 \end{vmatrix} = \begin{vmatrix} 1 & 3 & 1 & 1 \\ 0 & -20 & -8 & -6 \\ 0 & -2 & -5 & 1 \\ 0 & 4 & 0 & 7 \end{vmatrix}$$

$$= 1(-1)^{1+1} \begin{vmatrix} -20 & -8 & -6 \\ -2 & -5 & 1 \\ 4 & 0 & 7 \end{vmatrix}$$

$$= 4(-1)^{1+3}(-8-30) + 7(-1)^{3+3}(100-16)$$

$$= -152 + 588$$

$$= 436.$$

Therefore, by Cramer's rule

$$x = \frac{|\mathbf{A}_2|}{|\mathbf{A}|} = \frac{436}{-84} = -\frac{109}{21}.$$

(b) Here $|\mathbf{A}| = -1$ and $|\mathbf{A}_4| = 14$. Therefore, by Cramer's rule $z = \frac{14}{-1} = -14$.

Note that exact steps one uses to calculate a determinant is not limited to the approach above. There are most likely more efficient ways to perform these calculations which depend on each individual matrix. ■

(B) Question 29. Read the question below and then study the given 'suggested' answer with a view to deciding if this answer is correct. If you conclude that the 'suggested' answer is wrong, then produce a correct answer.

Question: Decide whether the following statement is true or false. "If $\mathbf{A}$ and $\mathbf{B}$ are any two matrices in $\mathcal{M}_{22}(\mathbb{R})$, then

$$\det(\mathbf{A}+\mathbf{B}) = \det(\mathbf{A}) + \det(\mathbf{B})."$$

Give a proof if you conclude that the statement is true or a counterexample if you think it is false.

Answer: The statement is true. Let

$$\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \text{ and } \mathbf{B} = \begin{pmatrix} 1 & 3 \\ 3 & 4 \end{pmatrix}.$$

Then $\det(\mathbf{A}) = 2-1 = 1$, $\det(\mathbf{B}) = 4-9 = -5$ and hence, $\det(\mathbf{A}) + \det(\mathbf{B}) = -4$. Also

$$\mathbf{A}+\mathbf{B} = \begin{pmatrix} 2 & 4 \\ 4 & 6 \end{pmatrix} \implies \det(\mathbf{A}+\mathbf{B}) = 12 - 16 = -4.$$

Hence $\det(\mathbf{A}+\mathbf{B}) = \det(\mathbf{A}) + \det(\mathbf{B})$ for all $\mathbf{A}, \mathbf{B} \in \mathcal{M}_{22}(\mathbb{R})$.

Solution: The given answer is incorrect since it only establishes that the result holds for the 2 matrices considered. The statement is false. A sufficient counter-example is given by $\mathbf{A} = \mathbf{I}_2$ and $\mathbf{B} = -\mathbf{I}_2$, since, $|\mathbf{A}| = |\mathbf{B}| = 1$ and

$$|\mathbf{A}+\mathbf{B}| = |\mathbf{0}| = 0 \neq 2 = |\mathbf{A}| + |\mathbf{B}|.$$

■

(C) Question 30. A matrix $\mathbf{A} = [a_{ij}] \in \mathcal{M}_{nn}(\mathbb{R})$ with $n \geq 2$ is such that $a_{ij} = 0$ if $i + j < n + 1$. Establish that $|\mathbf{A}| = (-1)^k a_{1,n}a_{2,n-1}\dots a_{n-1,2}a_{n,1}$ and $k = \frac{n(n-1)}{2}$.

Solution: It follows from definition of a_{ij} that

$$|\mathbf{A}| = \sum_{\sigma \in S_n} (-1)^{N(\sigma)} \prod_{i=1}^{n} a_{\sigma(i)i} = (-1)^{N(\sigma^*)} \prod_{i=1}^{n} a_{\sigma^*(i)i}$$

where $\sigma^*(i) = n+1-i$ for $i = 1, \ldots, n$ (since the upper left triangle in the matrix, above the diagonal, consists of only zeros). Since σ^* has $\sum_{i=1}^{n-1} i$ inversions, it follows that

$$|\mathbf{A}| = (-1)^k \prod_{i=1}^{n} a_{n+1-i,i} = (-1)^k a_{n,1}a_{n-1,2}a_{n-2,3}\ldots a_{2,n-1}a_{1,n}.$$

You can alternatively obtain this result by swapping rows to convert the matrix into diagonal form. ■

(C) Question 31. Establish that if $\mathbf{A} \in \mathcal{M}_{33}(\mathbb{R})$ is the matrix

$$\mathbf{A} = \begin{pmatrix} b+c & a^2 & a \\ c+a & b^2 & b \\ a+b & c^2 & c \end{pmatrix}$$

then $|\mathbf{A}| = -(a-b)(b-c)(c-a)(a+b+c)$. Deduce the conditions on a, b and c under which the system

$$\begin{array}{ccccccc} (b+c)x & + & a^2y & + & az & = & a^3, \\ (c+a)x & + & b^2y & + & bz & = & b^3, \\ (a+b)x & + & c^2y & + & cz & = & c^3 \end{array}$$

of three linear equations in three unknowns possesses a unique solution.

Solution: The determinant can be computed directly. The system of linear equations has a unique solution precisely when $|\mathbf{A}| \neq 0$ i.e. when $a \neq b \neq c \neq a$ and $(a+b+c) \neq 0$. ■

(C) Question 32. For some fixed $k \in \{1, \ldots, n\}$, let $\mathbf{A} = [a_{ij}], \mathbf{B} = [b_{ij}], \mathbf{C} = [c_{ij}] \in \mathcal{M}_{nn}(\mathbb{R})$ be given by

$$b_{ij} = \begin{cases} a_{ij}, & j \neq k \\ b_{ik}, & j = k, \end{cases} \qquad c_{ij} = \begin{cases} a_{ij} & j \neq k \\ c_{ik} & j = k, \end{cases}$$

and $a_{ik} = b_{ik} + c_{ik}$, for $i = 1, \ldots, n$. Prove that $|\mathbf{A}| = |\mathbf{B}| + |\mathbf{C}|$.
Solution: Observe that the 3 matrices have the same entries, except in column k. Therefore,

$$\begin{aligned}
|\mathbf{A}| &= \sum_{\sigma \in S_n} (-1)^{N(\sigma)} \prod_{i=1}^{n} a_{\sigma(i)i} \\
&= \sum_{\sigma \in S_n} (-1)^{N(\sigma)} \left(\prod_{\substack{i=1 \\ i \neq k}}^{n} a_{\sigma(i)i} \right) (b_{\sigma(k)k} + c_{\sigma(k)k}) \\
&= \left(\sum_{\sigma \in S_n} (-1)^{N(\sigma)} \left(\prod_{\substack{i=1 \\ i \neq k}}^{n} a_{\sigma(i)i} \right) (b_{\sigma(k)k}) \right) + \left(\sum_{\sigma \in S_n} (-1)^{N(\sigma)} \left(\prod_{\substack{i=1 \\ i \neq k}}^{n} a_{\sigma(i)i} \right) (c_{\sigma(k)k}) \right) \\
&= \left(\sum_{\sigma \in S_n} (-1)^{N(\sigma)} \prod_{i=1}^{n} b_{\sigma(i)i} \right) + \left(\sum_{\sigma \in S_n} (-1)^{N(\sigma)} \prod_{i=1}^{n} c_{\sigma(i)i} \right) \\
&= |\mathbf{B}| + |\mathbf{C}|.
\end{aligned}$$

■