Optics and Waves (week 4)

At the boundary between two strings of density μ_1 and μ_2, an incident wave $y_1 = A\cos(k_1 x - \omega t)$ is partially transmitted as partially reflected. The transmitted and reflected waves are $y_t = B\cos(k_2 x - \omega t)$ and $y_r = C\cos(k_1 x + \omega t)$, respectively.

We know that $B = \frac{2k_1}{k_1 + k_2} A, \quad C = \frac{k_1 - k_2}{k_1 + k_2} A$.

Show that the sum of the energy carried by the transmitted wave y_t and that by the reflected wave y_r equals the energy from the initial wave y_1.

The average energy related to a sinusoidal travelling wave over a whole wavelength is:

$$E_{aver} = \frac{1}{2}\mu A^2 \omega^2 \lambda$$

The average energy for y_t and y_r are:

$$E_t = \frac{1}{2}\mu_2 B^2 \omega^2 \lambda_2,$$

$$E_r = \frac{1}{2}\mu_1 C^2 \omega^2 \lambda_1,$$

$$E_t + E_r = \frac{1}{2}\mu_2 B^2 \omega^2 \lambda_2 + \frac{1}{2}\mu_1 C^2 \omega^2 \lambda_1 = \frac{1}{2}\omega^2 [\frac{2k_1}{k_1 + k_2} A]^2 \mu_2 \lambda_2 + \frac{1}{2}\omega^2 [\frac{k_1 - k_2}{k_1 + k_2} A]^2 \mu_1 \lambda_1$$

$$= \frac{1}{2}\omega^2 A^2 \mu_1 \lambda_1 \left[[\frac{2k_1}{k_1 + k_2}]^2 \frac{\mu_2 \lambda_2}{\mu_1 \lambda_1} + [\frac{k_1 - k_2}{k_1 + k_2}]^2 \right]$$

$$= \frac{1}{2}\omega^2 A^2 \mu_1 \lambda_1 \left[[\frac{2k_1}{k_1 + k_2}]^2 \frac{\mu_2}{\mu_1} \sqrt{\frac{\mu_1}{\mu_2}} + [\frac{k_1 - k_2}{k_1 + k_2}]^2 \right]$$

$$= \frac{1}{2}\omega^2 A^2 \mu_1 \lambda_1 \left[[\frac{2k_1}{k_1 + k_2}]^2 \sqrt{\frac{\mu_2}{\mu_1}} + [\frac{k_1 - k_2}{k_1 + k_2}]^2 \right]$$

Substitute the following into the above equation.

$$k_1 = \frac{\omega}{\sqrt{\frac{T}{\mu_1}}} = \omega\sqrt{\frac{\mu_1}{T}},$$

$$k_2 = \omega\sqrt{\frac{\mu_2}{T}},$$

It can be shown that the terms inside the brackets equal 1,

and

$$\frac{1}{2}\omega^2 A^2 \mu_1 \lambda_1$$

is exactly the energy per wavelength for the incident wave.