University of Birmingham
School of Mathematics

1RA Real Analysis Summer 2024

Final Exam
Generic Feedback

The performance of the students in the final exam (summer exam 2024, Questions 1 and 3), in particular their work on Question 1 on the final exam, indicates a solid understanding of some fundamental concepts in calculus, such as the $\varepsilon - \delta$ definition of convergence and derivative, and was able to use them in simple calculations. However, the difficulties they faced with L'Hospital's rule in non-standard scenarios suggest a need for more targeted practice in recognizing and handling indeterminate forms, such as $0 \cdot \infty$ and 1^∞ forms. Question 3 (b) was especially hard for students, only some students observed the relationship between zero derivatives and constant functions. Question 3 (a) (iii) was another difficult question. Many students can apply the logarithm trick to reduce the 1^∞ to a standard $0/0$ form, but most fail to complete the computation. Many students also failed to provide a complete and rigorous proof of Q 3 (d).

Q1. The majority of students answered this question very well. There are some minor concerns on how to write mathematical proofs by using the $\varepsilon - \delta$ definition in a logical manner, and failed to provide enough intermediate work.

(a). This is bookwork and was done well. Some students didn't exclude $x = \alpha$ when defining the convergence. It should be $|f(x) - \ell| < \varepsilon$ provided $0 < |x - \alpha| < \delta$. Many students failed to produce a logical and structured proof when applying the $\varepsilon - \delta$ definition.

(b). This is also standard bookwork, and was done well. Some students lost points because they did carry the limit sign $\lim\limits_{x \to 2}$ all the way in their computation. Some students used a wrong formula when expanding $(x + h)^3$.

(c). This question is, in general, done well. Students lost marks due to incomplete justification of their computation. Please note the question asked for "provided you clearly state what you are using".

(d). This question is, in general, done well. Many students just gave the final answer and failed to provide any intermediate work, which may result in losing step marks.

Q2. Overall, parts (a) and (c) were not answered particularly well, whilst part (d) was answered well.

(a). This function is not integrable. There is a similar example where this is proved in the Lecture Notes.

(b). Marks were lost for not using the correct logical order and quantifiers in the statement of Riemann's Integrability Criterion.

(c). The key here is to recognise that because h is *continuous*, the First Fundamental Theorem of Calculus guarantees that the function $H(x) := \int_a^x h$ for all $x \in [a, b]$ is differentiable with $H'(x) = h(x)$. The identity $H'(x) = h(x)$ is all that is required for H to be an antiderivative of h (recall the definition of an antiderivative from the Lecture Notes).

(d). The substitution $u = \cos(x)$ reduces this to the integral of a polynomial.

(e). The substitution $x = 3\tan(\theta)$ reduces this to a trigonometric integral, which can then be computed using the substitution $u = \sin(\theta)$ to reduce to the integral of a polynomial. It was important to use trigonometry to avoid inverse trigonometric functions in the final answer. Many students had difficulty with these computations.

(f). This part required computing the integration factor $I(x) = x^9$. Marks were lost for not using the initial data to determine the constant of integration.

(g). This is a second-order differential equation with constant coefficients, so the Characteristic Polynomial method can be used to determine the general solution y_c of the homogeneous equation. A particular solution can be found by using the trial function $y_p = Ax + B$, since the right-hand side of the equation is a first-order polynomial. The general solution is then given by $y = y_c + y_p$.

Q3. There were only a few students can finish all questions in this section. Most students struggled with Q3 (a) (ii), (iii), Q3 (b), and Q (d).

(a.i) The solution to this question is a standard application of L'Hospital's rule to the 0/0 indeterminate form. One may apply another LH rule to the limit

$$\lim_{x\to 0} \frac{\sin x}{x},$$

or refer it to the notable limit from Lecture notes.

(a.ii) This question is a $0 \cdot \infty$ indeterminate form. To apply LH rule, we need to first rewrite it as

$$\lim_{x\to 0+} x^{0.001} \ln x = \lim_{x\to 0+} \frac{\ln x}{x^{-0.001}},$$

which is a standard $\frac{\infty}{\infty}$ form.

(a.iii) This question is a 1^∞ indeterminate form. To apply LH rule, we need to first rewrite it by using the log tricks

$$\ln\left[\left(\frac{4^{1/x} + 9^{1/x}}{2}\right)^x\right] = x \ln\left(\frac{4^{1/x} + 9^{1/x}}{2}\right),$$

which is a $\infty \cdot 0$ form. We need to rewrite it further as

$$x \ln\left(\frac{4^{1/x} + 9^{1/x}}{2}\right) = \frac{\ln\left(\frac{4^{1/x}+9^{1/x}}{2}\right)}{\frac{1}{x}}.$$

This is a $\frac{0}{0}$ form, and we may apply the LH rule. Some students failed to get the correct formula for the derivative of $\ln\left(\frac{4^{1/x}+9^{1/x}}{2}\right)$.

(b) This is a typical unseen question. However, many students didn't get it correctly. We need to rewrite the condition of this question in a quotient form as

$$\left|\frac{f(x) - f(y)}{x - y}\right| \leq 100|x - y|^{0.001}$$

for $x \neq y$. The key observation is that the above inequality implies zero derivatives over $[1, 3]$, i.e.

$$f'(y) = \lim_{x \to y} \frac{f(x) - f(y)}{x - y} = \lim_{x \to y} \left| \frac{f(x) - f(y)}{x - y} \right| = 0$$

by using the squeeze theorem. The zero derivative further implies constant functions.

(c) Students answered this question well. However, there were some cases where the second derivative of the function f was not computed correctly.

(d) This question is an unseen question, but it is not challenging. However, most students didn't get full marks. The most direct proof should be contradictory, i.e., assume there is more than one solution on the interval and then disprove this.

Q4. Overall, good progress was made on parts (a) and (b), whilst part (c) was the most challenging.

(a). This requires using the method of partial fractions. The denominator involves an irreducible quadratic factor, so a correct partial fraction expansion is

$$\frac{A}{x-1} + \frac{Bx + C}{x^2 + 4}.$$

There is a similar example in the Lecture Notes.

(b). This requires two applications of integration-by-parts and introducing limits to demonstrate that the improper integral is being understood as a limit of proper integrals. Marks were lost for not appropriately justifying limit computations such as $\lim_{R \to \infty} R^2 e^{-3R} = 0$, which can be done using L'Hôpital's Rule.

(c). The key here was to recall the Partition Lemma and combine it with the technique, encountered in several proofs in Lectures, of defining the finer partition $R_n := P_n \cup Q_n$ by taking the union of the two given partitions P_n and Q_n. Most marks were lost for not defining a partition R_n. It was then necessary to introduce $\epsilon > 0$, choose $n \in \mathbb{N}$ so that the condition in (ii) implies that $U(f, P_n) - L(f, P_n) < \epsilon$, and then use Riemann's Criterion to deduce integrability. Many students included these key ingredients of the proof.

(d). This is an application of the First Fundamental Theorem of Calculus. The function underneath the integral, i.e. $t \mapsto \sin(t^2)$, is continuous because it is the composition of two continuous functions (the Sine function and a polynomial). Marks were lost for not observing and also not justifying this continuity, as it is essential for the application of the First Fundamental Theorem of Calculus. It was then necessary to correctly define an auxiliary function, e.g. $G(x) := \int_0^x \sin(t^2)\, \mathrm{d}t$, on which the First Fundamental Theorem of Calculus could be applied to deduce that $G'(x) = \sin(x^2)$. The Chain Rule could then be used to conclude that g is differentiable, and to compute $g'(x)$, since $g(x) = G(2x\cos^2(3x))$ and so $g'(x) = G'(2x\cos^2(3x))(2x\cos^2(3x))'$. Marks were lost here for not defining an auxiliary function and not correctly applying the Chain Rule.