Electromagnetism

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Lecture 20
Physics beyond the Stand Module!
Maxwell's Equations
Week 10
(Week 11 of Semester 2)

Last Lecture

- Paramagnetic materials
- Magnetic Susceptibility
- Relative permeability
- Ferromagnetic materials
- Diamagnetic materials
- Displacement Current Density

This Lecture

- Ampere-Maxwell Law
- Maxwell’s Equations in integral form
- Vector calculus
- Divergence and Stokes’ theorems
- Maxwell’s Equations in differential form
- The End

Extra Material – Towards Maxwell's Equations

- **Physics beyond the Stand Module!**
- This lecture is not in the syllabus so you can relax.
- However, we've got this far so may as well get to Maxwell's Equations.

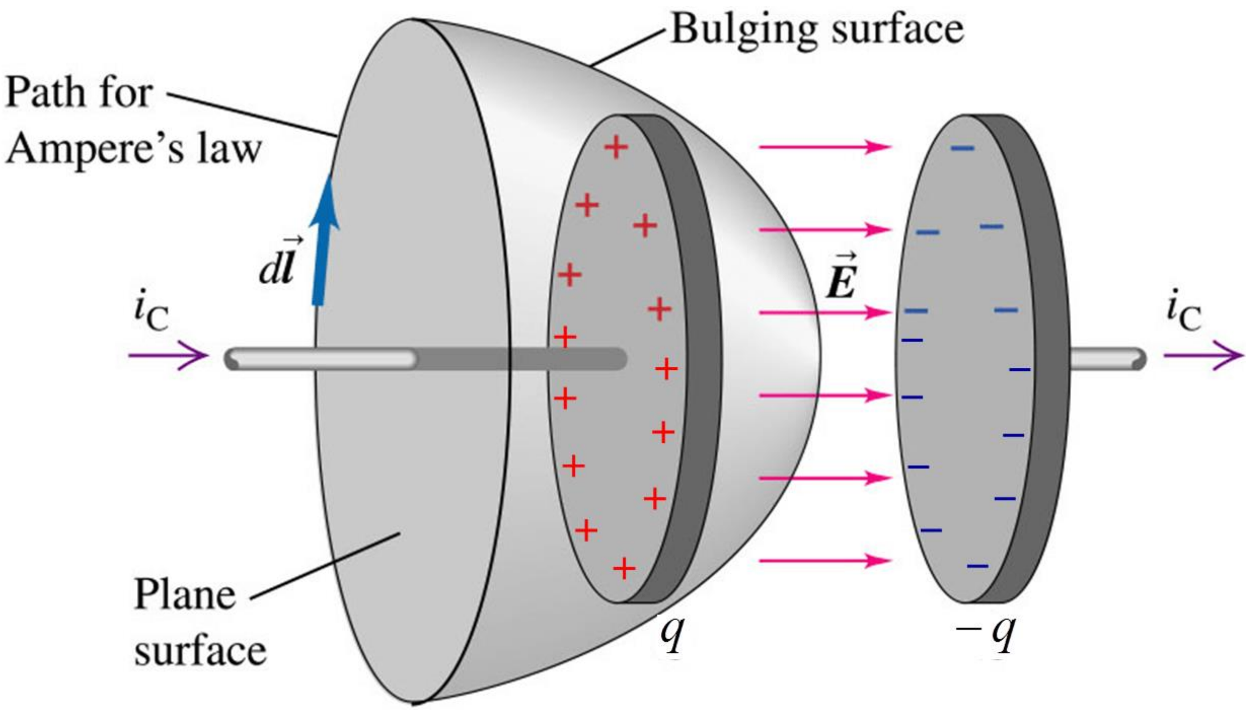
Ampere-Maxwell Law

- Changing B-fields induce a changing E-Field.
- $\oint \underline{\boldsymbol{E}} \cdot d\underline{\boldsymbol{l}} = -\frac{d\Phi_m}{dt}$
- And $\phi_m = \int_S \underline{\boldsymbol{B}} \cdot d\underline{\boldsymbol{S}}$
- So: $\oint \underline{\boldsymbol{E}} \cdot d\underline{\boldsymbol{l}} = -\int_S \frac{\partial \underline{\boldsymbol{B}}}{\partial t} \cdot d\underline{\boldsymbol{S}}$ (Maxwell's third eq^n)
- **So, can a changing E-field induce a changing B-field?**

YES

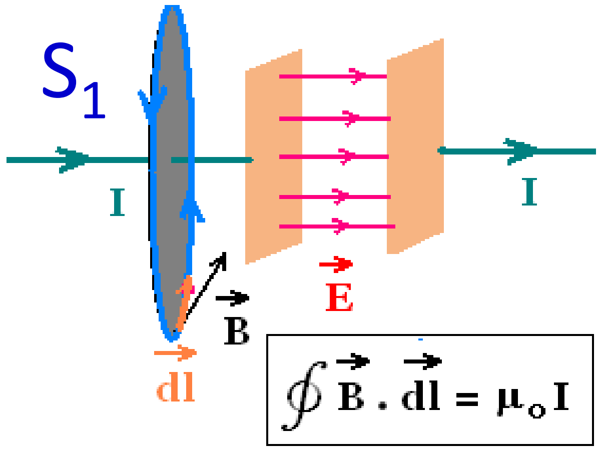
Ampere-Maxwell Law

- Consider an electrical circuit with a capacitor in.
- For an AC voltage, there will be an AC current in the circuit.



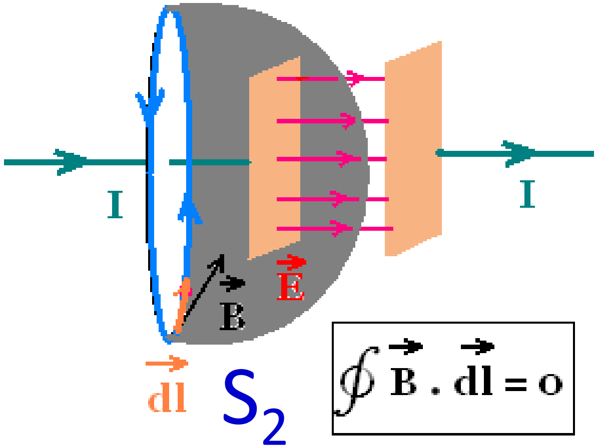
Apply Ampere's Law around the wire, in front of a capacitor plate.

Ampere-Maxwell Law



Consider plane surface, S_1.

$$\oint \underline{\boldsymbol{B}} \cdot d\underline{\boldsymbol{l}} = \int_{S_1} \underline{\boldsymbol{J}} \cdot d\underline{\boldsymbol{S}} = \mu_0 I$$



Now consider the surface, S_2.

$$\oint \underline{\boldsymbol{B}} \cdot d\underline{\boldsymbol{l}} = \int_{S_2} \underline{\boldsymbol{J}} \cdot d\underline{\boldsymbol{S}} = 0$$

We seem to have a problem!

Displacement Current Density

- There is an AC current in the wires but not between the capacitor plates.
- However, there must be something and that is an alternative electric field.
- The plates and discharging and charging as the voltage alternates.
- Current is rate of change of charge $I = \frac{dQ}{dt}$
- And Current density, $J = \frac{d\sigma}{dt}$

Displacement Current Density

- Current density, $J_D = \frac{d\sigma}{dt}$
- But for capacitor plates, $E = \frac{\sigma}{\varepsilon_0}$
- So $\boxed{J_D = \varepsilon_0 \frac{\partial E}{\partial t}}$ This is known as the **Displacement Current Density**
- It's not a current, it's a changing electric field but has the same effect as a current i.e. produces B-field.

Ampere-Maxwell Law

- Ampere's law needs to be modified to take the displacement current into account:
- **Ampere-Maxwell Law**

$$\oint \underline{\boldsymbol{B}} \cdot d\underline{\boldsymbol{l}} = \mu_0 \int_S \left(\underline{\boldsymbol{J}} + \varepsilon_0 \frac{\partial \underline{\boldsymbol{E}}}{\partial t}\right) \cdot d\underline{\boldsymbol{s}}$$

Maxwell's Equations

In free space, in integral form:

- $\int_S \underline{\boldsymbol{E}} \cdot d\underline{\boldsymbol{S}} = \int_V \frac{\rho}{\varepsilon_0} dV$ M1
- $\int_S \underline{\boldsymbol{B}} \cdot d\underline{\boldsymbol{S}} = 0$ M2
- $\oint \underline{\boldsymbol{E}} \cdot d\underline{\boldsymbol{l}} = -\int_S \frac{\partial \underline{\boldsymbol{B}}}{\partial t} \cdot d\underline{\boldsymbol{S}}$ M3
- $\oint \underline{\boldsymbol{B}} \cdot d\underline{\boldsymbol{l}} = \mu_0 \int_S \left(\underline{\boldsymbol{J}} + \varepsilon_0 \frac{\partial \underline{\boldsymbol{E}}}{\partial t}\right) \cdot d\underline{\boldsymbol{S}}$ M4

James Clerk Maxwell

13 June 1831 – 5 November 1879

- British physicist
- formulate the classical theory of electromagnetic radiation (unifications of electric and magnetic forces)
- Proposed light is EM radiation
- helped develop the Maxwell–Boltzmann distribution, to describe kinetic theory of gases.

Considered the 3rd greatest physicist of all time (after Newton and Einstein)

Vector Calculus
Year 2 mathematics

- $\nabla = \frac{\partial}{\partial x}\underline{\hat{\boldsymbol{\imath}}} + \frac{\partial}{\partial y}\underline{\hat{\boldsymbol{\jmath}}} + \frac{\partial}{\partial z}\underline{\hat{\boldsymbol{k}}}$
- grad(a) → $\nabla a = \frac{\partial a}{\partial x}\underline{\hat{\boldsymbol{\imath}}} + \frac{\partial a}{\partial y}\underline{\hat{\boldsymbol{\jmath}}} + \frac{\partial a}{\partial z}\underline{\hat{\boldsymbol{k}}}$
  - (a is any scalar)
- div $\underline{\boldsymbol{A}}$ → $\nabla \cdot \underline{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$
- curl $\underline{\boldsymbol{A}}$ → $\nabla \wedge \underline{\boldsymbol{A}} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\underline{\hat{\boldsymbol{\imath}}} - \left(\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z}\right)\underline{\hat{\boldsymbol{\jmath}}} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\underline{\hat{\boldsymbol{k}}}$ (describes the rotation of a vector field)

Two Vector Theorems (year 2 maths)

- *Divergence Theorem*
- $\int_S \underline{\boldsymbol{A}} \cdot d\underline{\boldsymbol{S}} = \int_V \nabla \cdot \underline{\boldsymbol{A}}\, dV$

where S is the enclosed surface around a volume V.

- *Stokes' Theorem*
- $\oint \underline{\boldsymbol{A}} \cdot d\underline{\boldsymbol{l}} = \int_S (\nabla \wedge \underline{\boldsymbol{A}}) \cdot d\underline{\boldsymbol{S}}$

where l is length of the boundary around the area S.

- Both true for any vector $\underline{\boldsymbol{A}}$.

Maxwell's Equation M1

- $\int_S \underline{\boldsymbol{E}} \cdot d\underline{\boldsymbol{S}} = \int_V \frac{\rho}{\varepsilon_0} dV$
- Applying the divergence theorem to the LHS
- $\int_S \underline{\boldsymbol{E}} \cdot d\underline{\boldsymbol{S}} = \int_V \nabla \cdot \underline{\boldsymbol{E}}\, dV$
- Hence $\int_V \nabla \cdot \underline{\boldsymbol{E}}\, dV = \int_V \frac{\rho}{\varepsilon_0} dV$
- If this is true for <u>*any*</u> volume, then:

$$\nabla \cdot \underline{\boldsymbol{E}} = \frac{\rho}{\varepsilon_0}$$

Maxwell's Equation M2

- $\int_S \underline{\boldsymbol{B}} \cdot d\underline{\boldsymbol{S}} = 0$
- Applying the divergence theorem to the LHS
- $\int_S \underline{\boldsymbol{B}} \cdot d\underline{\boldsymbol{S}} = \int_V \nabla \cdot \underline{\boldsymbol{B}}\, dV$
- Hence $\int_V \nabla \cdot \underline{\boldsymbol{B}}\, dV = 0$
- If this is true for <u>*any*</u> volume, then:

$$\nabla \cdot \underline{\boldsymbol{B}} = 0$$

Maxwell's Equation M3

- $\oint \underline{\boldsymbol{E}} \cdot d\underline{\boldsymbol{l}} = -\int_S \frac{\partial \underline{\boldsymbol{B}}}{\partial t} \cdot d\underline{\boldsymbol{S}}$
- Applying Stokes' theorem to the LHS
- $\oint \underline{\boldsymbol{E}} \cdot d\underline{\boldsymbol{l}} = \int_S \left(\nabla \wedge \underline{\boldsymbol{E}}\right) \cdot d\underline{\boldsymbol{S}}$
- Hence $\int_S \left(\nabla \wedge \underline{\boldsymbol{E}}\right) \cdot d\underline{\boldsymbol{S}} = -\int_S \frac{\partial \underline{\boldsymbol{B}}}{\partial t} \cdot d\underline{\boldsymbol{S}}$
- If this is true for *any* surface, then:

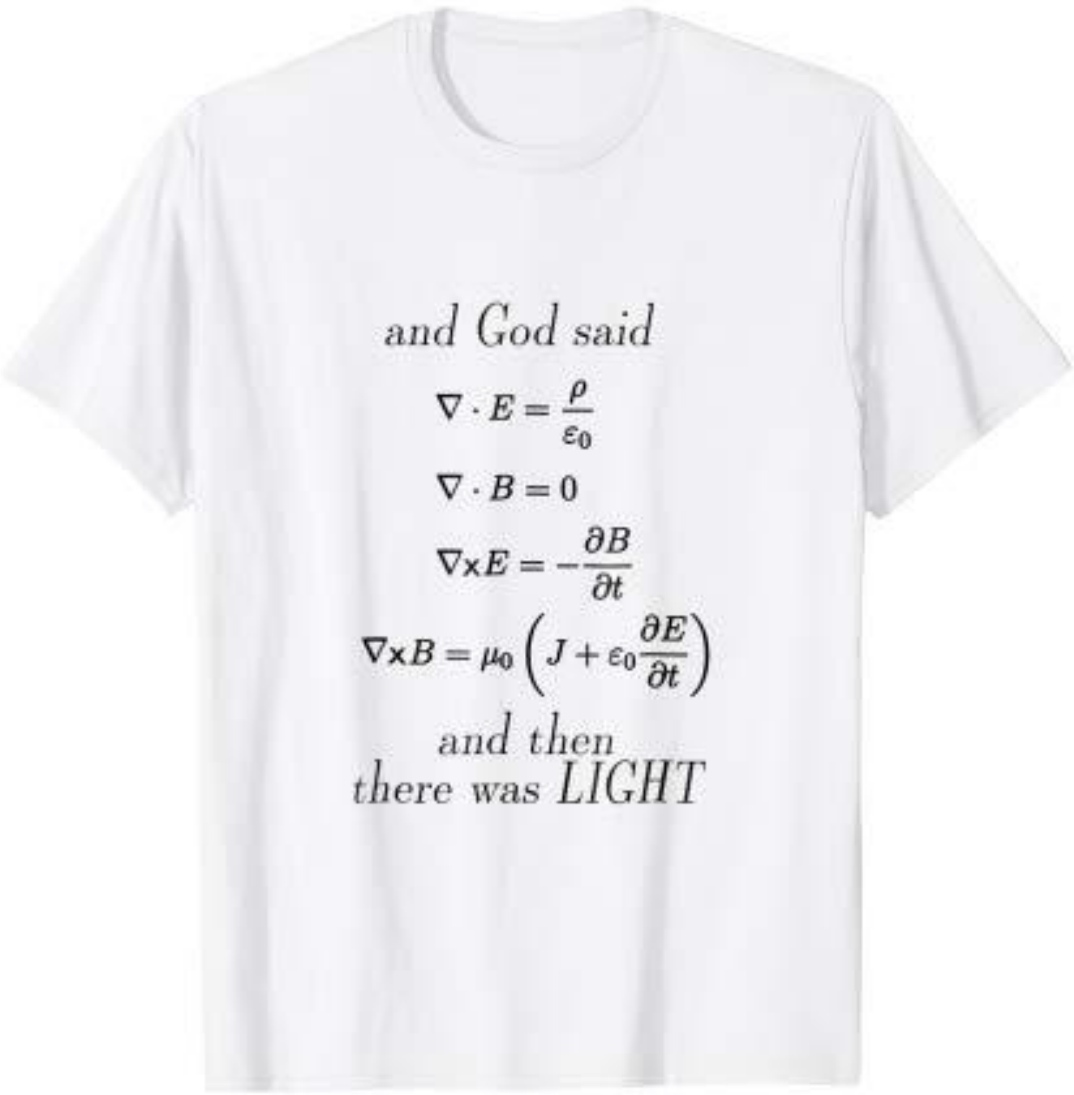
$$\nabla \wedge \underline{\boldsymbol{E}} = -\frac{\partial \underline{\boldsymbol{B}}}{\partial t}$$

Maxwell's Equation M4

- $\oint \underline{\boldsymbol{B}} \cdot d\underline{\boldsymbol{l}} = \mu_0 \int_S \left(\underline{\boldsymbol{J}} + \varepsilon_0 \frac{\partial \underline{\boldsymbol{E}}}{\partial t}\right) \cdot d\underline{\boldsymbol{S}}$
- Applying Stokes' theorem to the LHS
- $\oint \underline{\boldsymbol{B}} \cdot d\underline{\boldsymbol{l}} = \int_S \left(\nabla \wedge \underline{\boldsymbol{B}}\right) \cdot d\underline{\boldsymbol{S}}$
- Hence $\int_S \left(\nabla \wedge \underline{\boldsymbol{B}}\right) \cdot d\underline{\boldsymbol{S}} = \mu_0 \int_S \left(\underline{\boldsymbol{J}} + \varepsilon_0 \frac{\partial \underline{\boldsymbol{E}}}{\partial t}\right) \cdot d\underline{\boldsymbol{S}}$
- If this is true for <u>*any*</u> surface, then:

$$\nabla \wedge \underline{\boldsymbol{B}} = \mu_0 \left(\underline{\boldsymbol{J}} + \varepsilon_0 \frac{\partial \underline{\boldsymbol{E}}}{\partial t}\right)$$

Maxwell's Equations



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