Introduction to Probability

Lecture 3

Today

We will arrive at the formula

$$P(A \cap B) = P(A) + P(B) - P(A \cup B)$$

$$\text{Probability}(\text{A } \mathbf{and} \text{ B}) = \text{Probability}(\text{A}) + \text{Probability}(\text{B}) - \text{Probability}(\text{A } \mathbf{or} \text{ B})$$

Together with how (or why and when) to add events

$$P(e_1 \cup e_2 \cup \cdots \cup e_N) = P(e_1) + P(e_2) + \cdots P(e_N)$$

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Summary

	With Replacement	**Without Replacement**
Keep Order	$\lvert\Omega\rvert = N^k$	$\lvert\Omega\rvert = \frac{N!}{(N-k)!}$
Ignore Order	$\lvert\Omega\rvert = ?$	$\lvert\Omega\rvert = \frac{N!}{k!\,(N-k)!}$

Uniform Probability

$$P(A) = \frac{|A|}{|\Omega|} = \frac{\text{Number of events in } A}{\text{Number of events in } \Omega}$$

Remember: $P(\Omega) = 1$

Example (from last time)

A bag contains 10 red and 6 orange balls. What is the probability of drawing two red and two orange balls?

We pick 4 balls out of 16: $|\Omega| = \binom{16}{4}$

There are $\binom{10}{2}$ ways to get red

There are $\binom{6}{2}$ ways to get orange

$$P = \frac{\binom{10}{2}\binom{6}{2}}{\binom{16}{4}} = \frac{10}{16}\frac{9}{15}\frac{6}{14}\frac{5}{13} \times \binom{\mathbf{4}}{\mathbf{2}}$$

Set Theory

Set Theory

Previously we introduced **sets**.

A set is a collection of things (elements)

Elements in sets are **unordered** and **unique**.

Now we introduce two operations on sets.

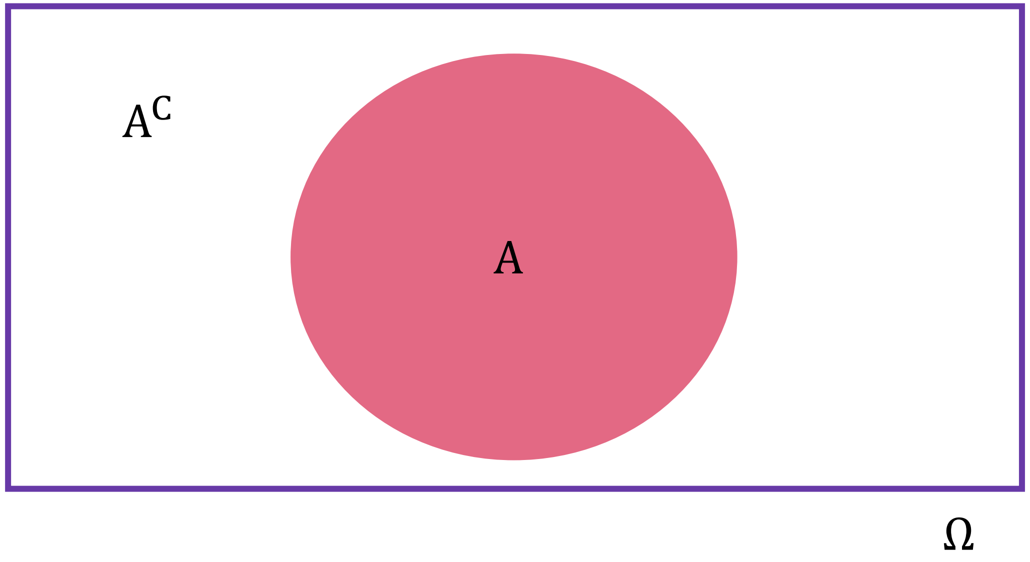
First, we need **Venn Diagrams**.

Venn Diagrams

We have a space Ω.

We then have a **subset** of Ω labelled A.

We also have the part of Ω that is **not** A called the **complement** written as A^{C}.



Also can define A as:

$$A = \{x | x \in A\}$$
$$A^C = \{x | x \notin A\}$$

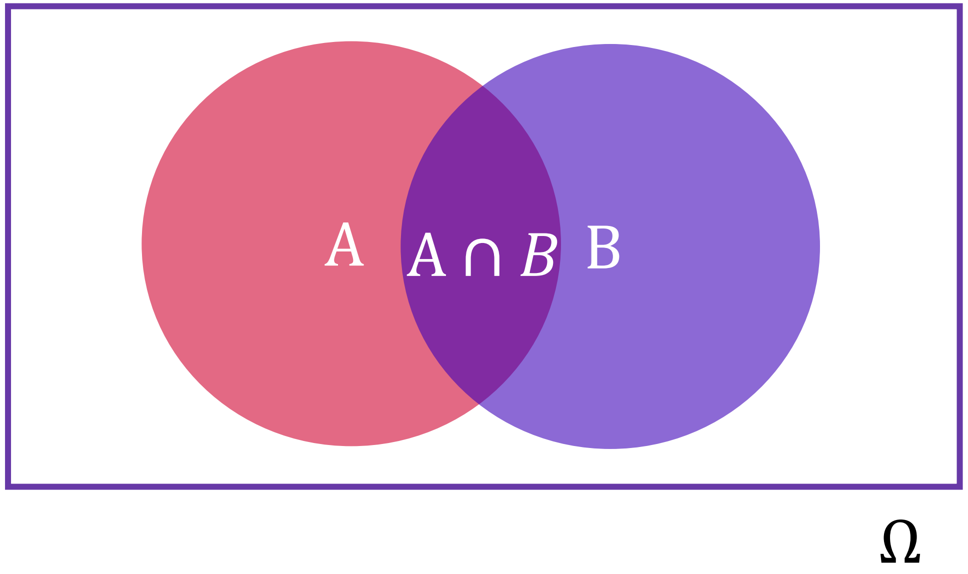
Venn Diagrams (2)

We have a space Ω.

We have two **subsets** of Ω labelled A and B.

The two sets overlap and the part in the middle is called the **intersection** written:

$$\mathrm{A} \cap B$$



It is the elements in A **and** B.

$$A \cap B = \{x | x \in A \textbf{ and } x \in B\}$$

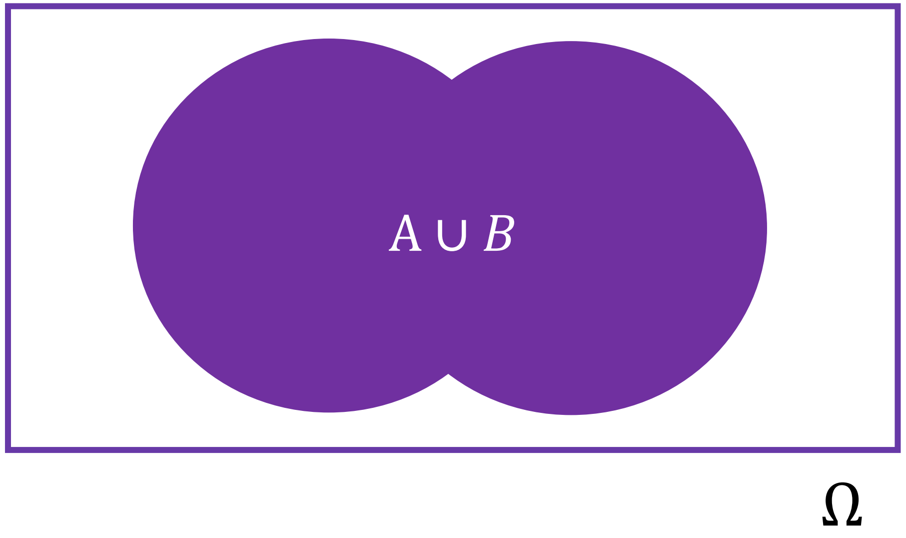
Venn Diagrams (3)

We have a space Ω.

We have two **subsets** of Ω labelled A and B.

Everything that is in A **or** B is called the **union** written as

$$A \cup B$$



$$A \cup B = \{x | x \in A \textbf{ or } x \in B\}$$

Empty Set

A special case is the empty set, written as

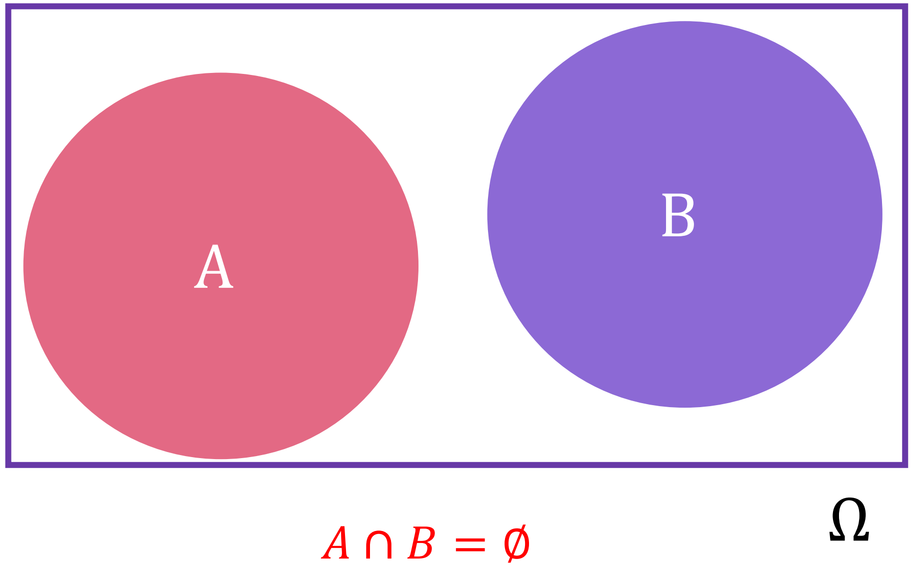
$$\emptyset$$

If two sets have no common elements, then:

- The intersection is $\emptyset$
- The sets are **pairwise disjoint (mutually exclusive)**

The empty set is the **complement** of Ω

$$\Omega = \emptyset^C$$

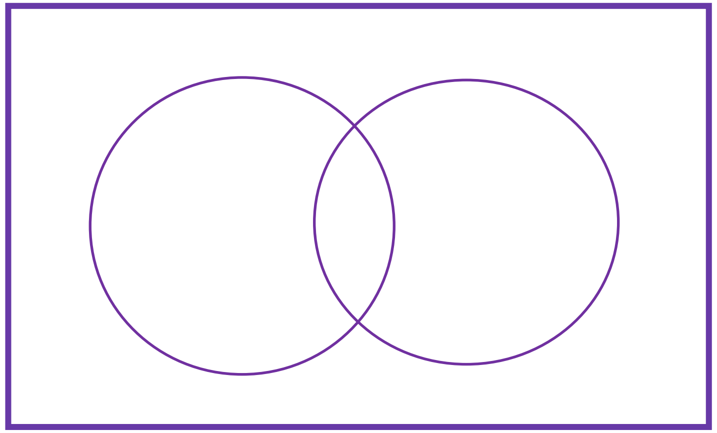


$A \cap B = \emptyset$

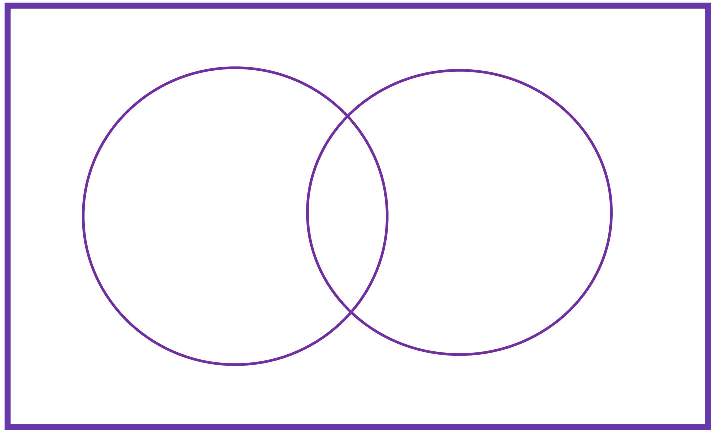
Note

$$A \cap A^C = \emptyset$$
$$A \cup A^C = \Omega$$

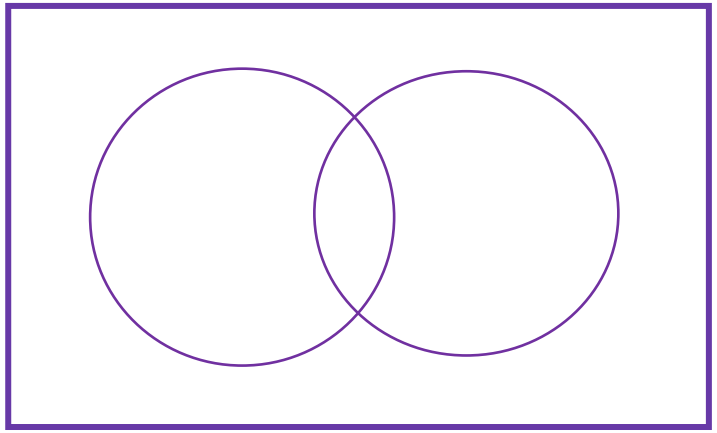
De Morgan’s Laws



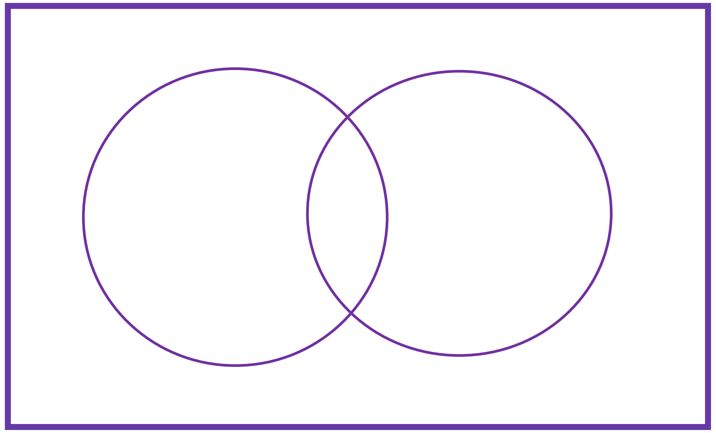
$(A \cup B)^C$



$A^C \cap B^C$



$(A \cap B)^C$



$A^C \cup B^C$

Examples

Example

If:

$\Omega = \{1,2,3,4,5,6\}$
$\mathrm{A} = \{1,3,5\}$ i.e. odd numbers
$\mathrm{B} = \{2,4,6\}$ i.e. even numbers
$\mathrm{C} = \{1,2,3\}$ i.e. numbers less than 4

Then what is:

1. $A \cup B$
2. $B \cap C$
3. $\Omega \cap B$
4. $A \cap B$

1. $A \cup B = \{1,2,3,4,5,6\} = \Omega$
2. $B \cap C = \{2\}$
3. $\Omega \cap B = \{2,4,6\} = \mathrm{B}$
4. $A \cap B = \emptyset$

Class Examples

If Ω is a deck of cards, what is:

1. $\text{Red Cards} \cup \text{Black Cards}$
2. $\text{Hearts} \cup \text{Spades} \cup \text{Diamonds} \cup \text{Clubs}$
3. $\text{Cards with Numbers} \cap \text{Diamonds}$
4. $\text{Spades} \cap \text{Jacks}$

1. Every card is either black or red, so

$$\text{Red Cards} \cup \text{Black Cards} = \Omega$$

2. Every card has a suite

$$\text{Hearts} \cup \text{Spades} \cup \text{Diamonds} \cup \text{Clubs} = \Omega$$

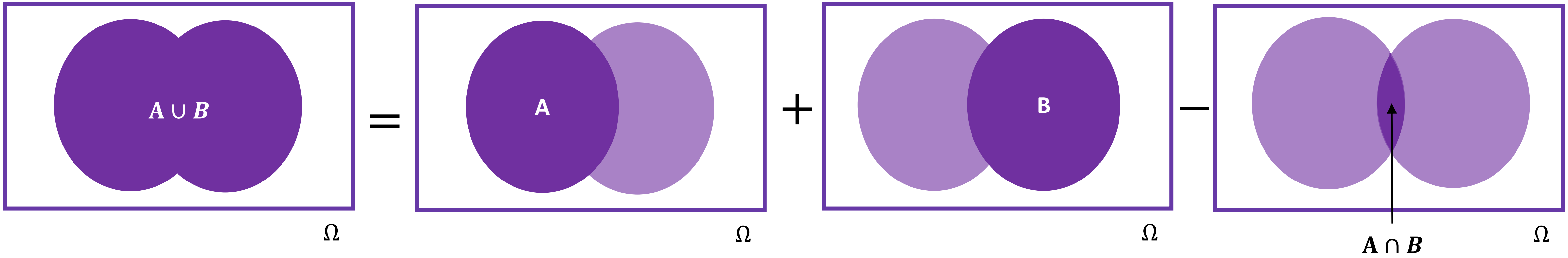
3. $\text{Cards with Numbers} \cap \text{Diamonds} = \{2 \diamond \cdots 9 \diamond, 10 \diamond\}$

4. $\text{Spades} \cap \text{Jacks} = \{\text{Jack of Spades}\}$

Inclusion-Exclusion Principle

Probability and Sets

The number of elements in the sets A and B satisfy



$$|A \cup B| = |A| + |B| - |A \cap B|$$

Probability Functions

The same is true of probability functions

$$P(A) + P(B) = P(A \cup B) + P(A \cap B)$$

In words:

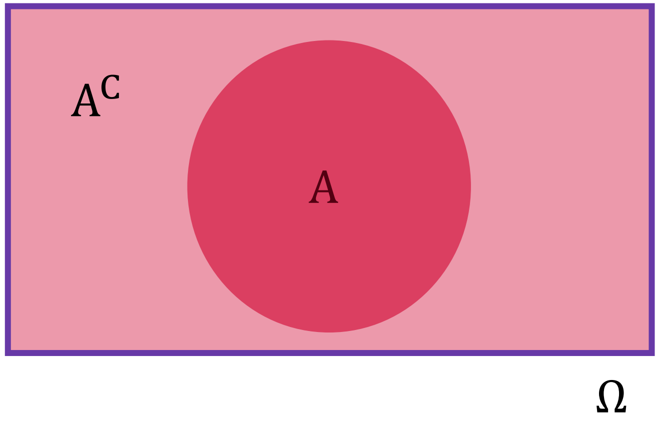
$$\text{Probability of A} + \text{Probability of B}$$
$$= \text{Probability of A } \mathbf{or} \text{ B} + \text{Probability of A } \mathbf{and} \text{ B}$$

$A \cup B$ and $A \cap B$ are events just as much as A or B were. They are just **sets**.

Consequences

$$P(\emptyset) = 0$$

$$P(A) = p$$
$$\rightarrow P(A^C) = 1 - p$$



$$P(A) = \frac{|A|}{|\Omega|}$$
$$|\emptyset| = 0 \rightarrow P(\emptyset) = 0$$

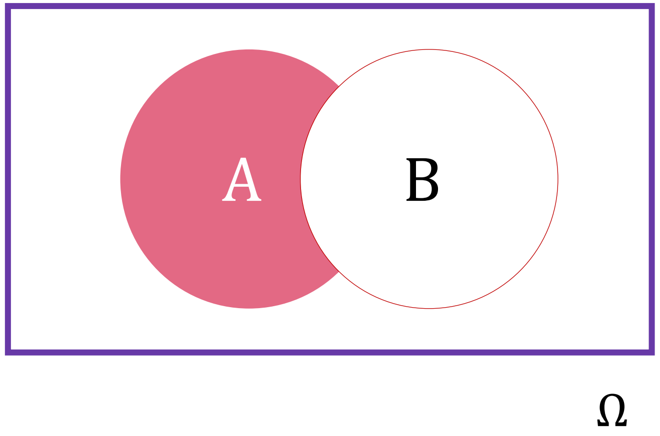
$$\Omega = A \cup A^C$$
$$P(\Omega) = P(A) + P(A^C) - P(A \cap A^C)$$
$$A \cap A^C = \emptyset$$

so

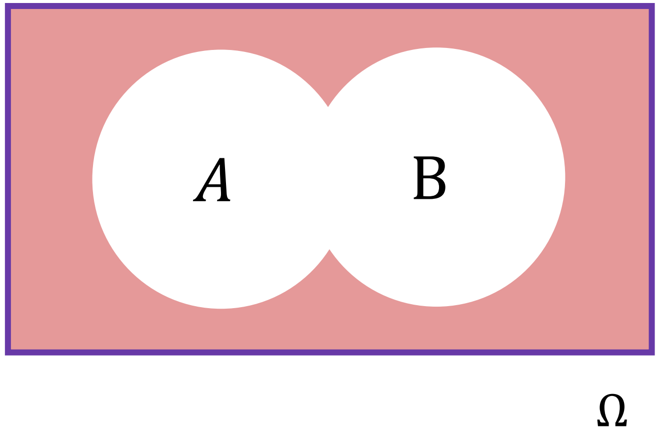
$$1 = P(A) + P(A^C)$$

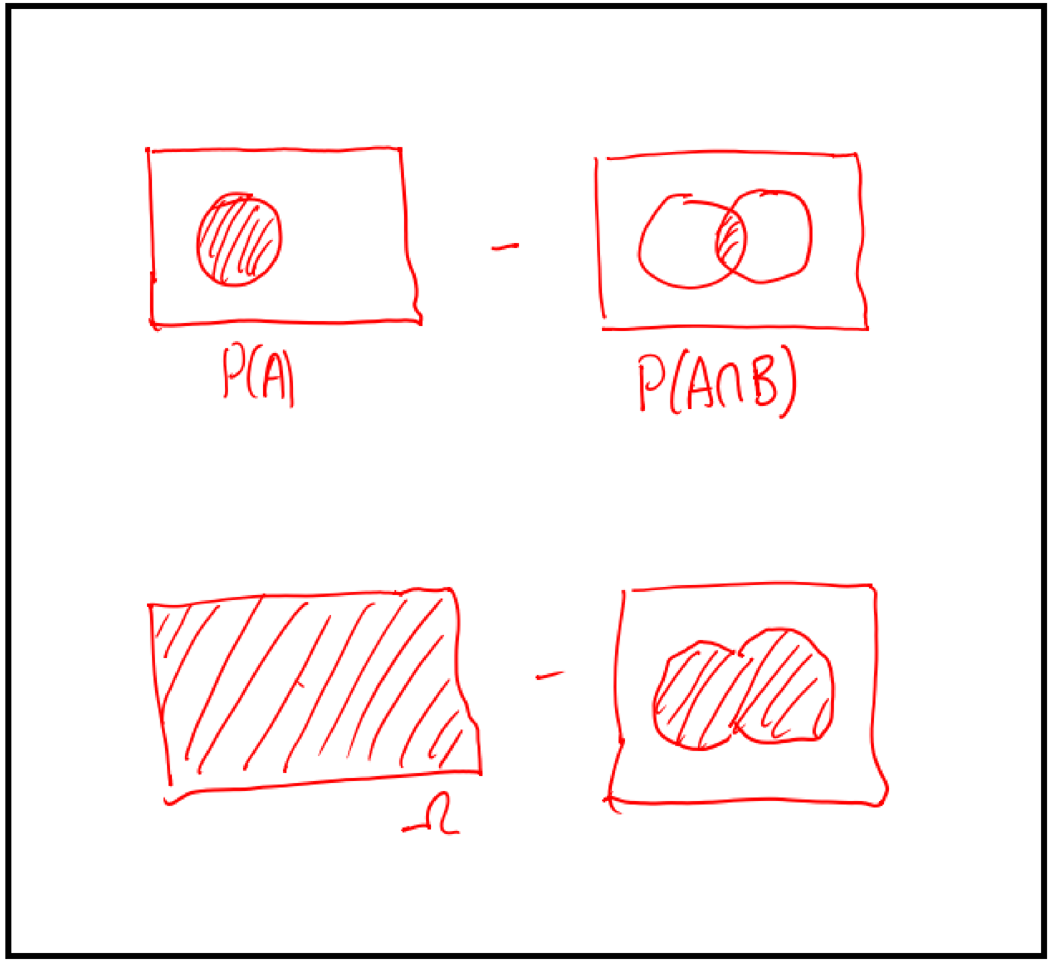
Consequences (2)

$$P(A \cap B^C) = P(A) - P(A \cap B)$$



$$P(A^C \cap B^C) = 1 - P(A \cup B)$$





Example

$P(A \cup B) = 0.4$, $P(A) = 0.1$ and $P(B) = 0.3$.

What is $P(A \cap B)$?

Using the formula

$$P(A \cap B) = P(A) + P(B) - P(A \cup B)$$

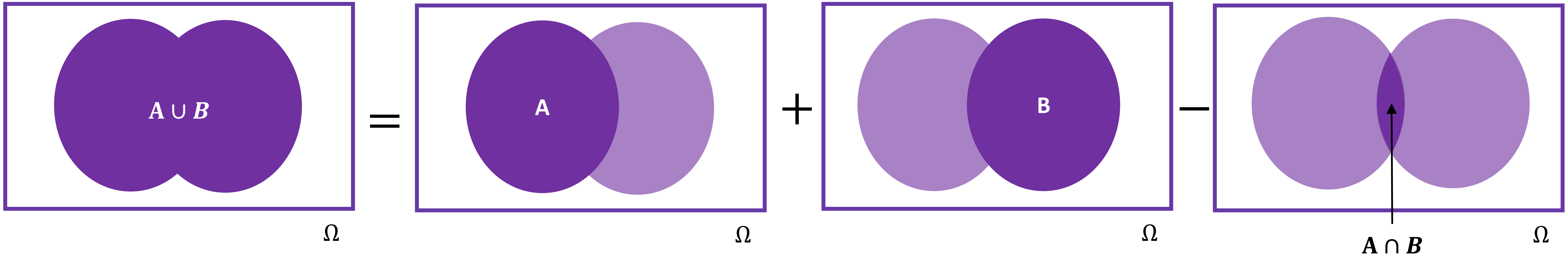
So $P(A \cap B) = 0.1 + 0.3 - 0.4 = 0$

This means the events A and B are pairwise disjoint.

Multiple Events

Multiple Events

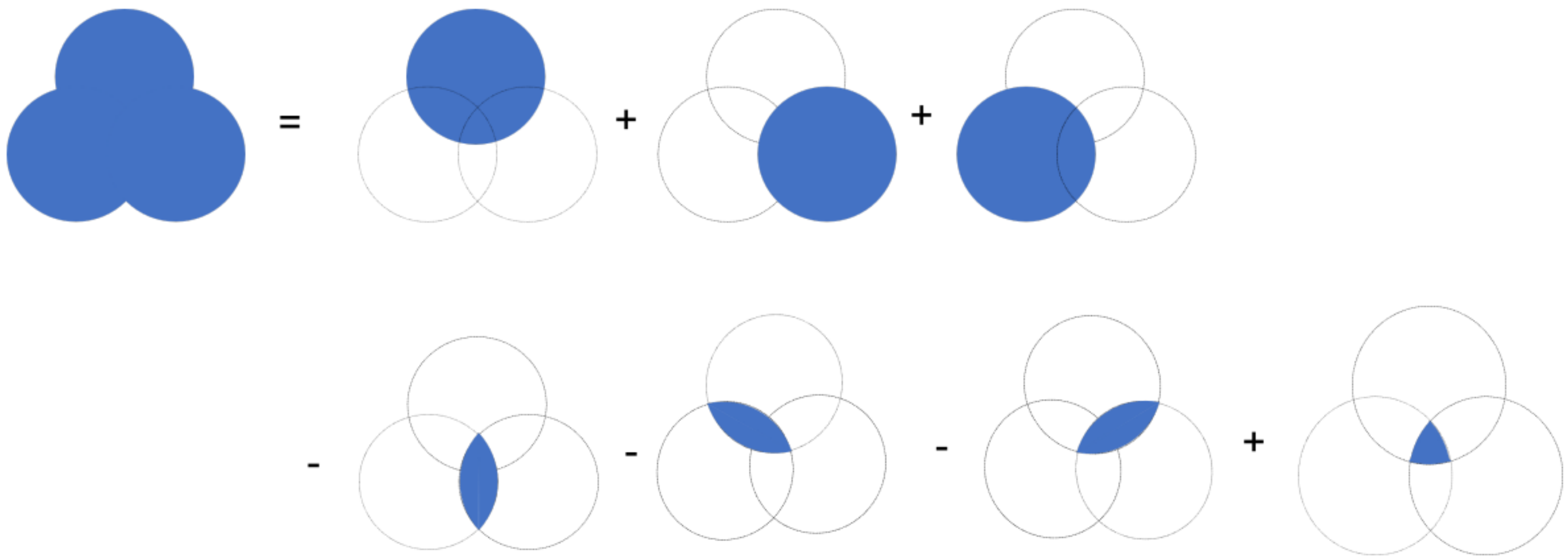
$$P(e_1 \cup e_2) = P(e_1) + P(e_2) - P(e_1 \cap e_2)$$



If $e_1 \cap e_2 = \emptyset$

$$P(e_1 \cup e_2) = P(e_1) + P(e_2)$$

Multiple Events (2)



$$P(e_1 \cup e_2 \cup e_3) = P(e_1) + P(e_2) + P(e_3) - P(e_1 \cap e_2) - P(e_1 \cap e_3) - P(e_2 \cap e_3) + P(e_1 \cap e_2 \cap e_3)$$

If $e_i \cap e_j = \emptyset$

$$P(e_1 \cup e_2 \cup e_3) = P(e_1) + P(e_2) + P(e_3)$$

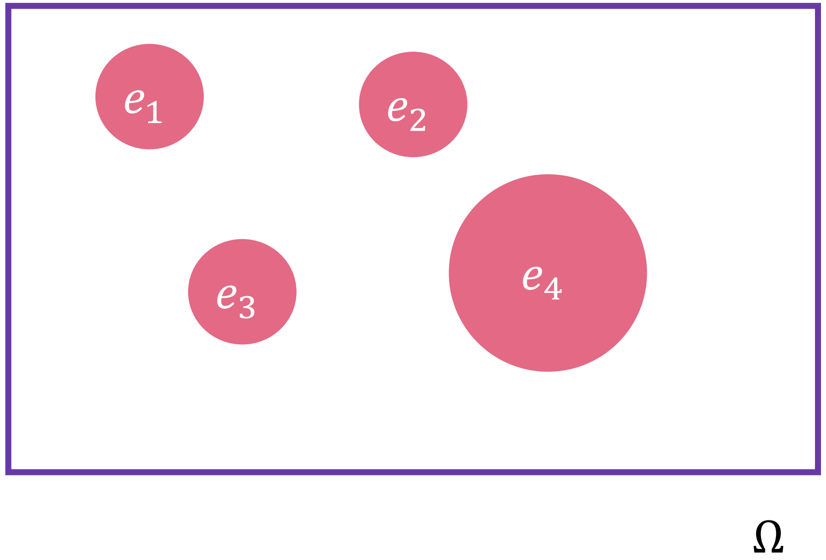
Multiple Events (3)

If events $e_1, e_2 \ldots e_N$ are **all** mutually exclusive then

$$P(e_1 \cup e_2 \cup \cdots \cup e_N) = P(e_1) + P(e_2) + \cdots P(e_N)$$

Or

$$P\left(\bigcup_{n=1}^{N} e_n\right) = \sum_{n=1}^{N} P(e_n)$$



Normalisation

If e_1, e_2 … e_N are mutually exclusive and

$$\Omega = e_1 \cup e_2 \ldots e_N$$

Then

$$\mathrm{P}(\Omega) = \mathrm{P}(\mathrm{e}_1) + \mathrm{P}(\mathrm{e}_2) + \cdots P(e_N) = 1$$

Example

$$\Omega = e_1 \cup e_2 \cup e_3$$

All disjoint and $P(e_1) = P(e_2) = P(e_3)$.
What is $P(e_1)$?

We have that

$$1 = P(e_1) + P(e_2) + P(e_3)$$

$$1 = 3P(e_1)$$

$$P(e_1) = \frac{1}{3}$$

Example

If I toss a fair 6 sided die:

1. What is the probability that I see an even number?

2. What is the probability that I see an even number or 1,2 or 3?

The even events are 2, 4 and 6 and these are **mutually exclusive**, so

$$P(2 \cup 4 \cup 6) = P(2) + P(4) + P(6) = \frac{3}{6} = \frac{1}{2}$$

Let $A = \{2,4,6\}$ and $B = \{1,2,3\}$ and these are not **mutually exclusive**, so

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$
$$= \frac{1}{2} + \frac{1}{2} - \frac{1}{6} = \frac{5}{6}$$

Summary

Concept	Formula
Union	$A \cup B$
Intersection	$A \cap B$
Empty Set (disjoint)	$\mathrm{A} \cap B = \emptyset$
Sample Space	$\mathrm{A} \cup A^C = \Omega$
Inclusion-Exclusion	$\|A \cup B\| = \|A\| + \|B\| - \|A \cap B\|$

$$P(A \cap B) = P(A) + P(B) - P(A \cup B)$$

$$P(e_1 \cup e_2 \cup \cdots \cup e_N) = P(e_1) + P(e_2) + \cdots P(e_N)$$

Examples

Example

Someone is interested in how often they eat ice cream, but unfortunately

they only have data relating to the weather as well. Historically, the following facts are true:

1. The probability of rain on any day is 1/4
2. It rains and the person eats ice with probability 1/10
3. The probability that it rains or the person eats ice cream is 1/3

What is the probability the person eats ice cream?

r: rain

i: ice cream

$$P(\mathrm{r}) = \frac{1}{4};\ P(\mathrm{r} \cap \mathrm{i}) = \frac{1}{10};\ P(\mathrm{r} \cup \mathrm{i}) = \frac{1}{3}$$

$$P(\mathrm{r} \cup \mathrm{i}) = P(r) + P(i) - P(r \cap i)$$
$$\rightarrow P(\mathrm{i}) = P(r \cup i) + P(r \cap i) - P(r)$$

$$P(\mathrm{i}) = \frac{1}{3} + \frac{1}{10} - \frac{1}{4}$$
$$= \frac{11}{60}$$

Class Examples

1. If $A = \{1,2,3, \ldots 16\}$ and B are all the positive numbers that perfectly divide by 3, what is $|A \cap B|$?

2. If $A = \{1,2,3, \ldots 16\}$, B are all the numbers that divide by 3, and C are the numbers that divide 5 what is $|A \cap B \cap C|$?

3. If I pick two numbers from $A = \{1,2,3, \ldots 16\}$, what is the probability that at least one divides 3?

Solutions

1. We need the set $B = \{3,6,9,\dots\}$. Then $A \cap B = \{3,6,9,12,15\}$

2. We also need the set $C = \{5,10,15\dots\}$. Then $A \cap B \cap C = \{15\}$.

3. Number of pairs in $\{1,2,3,\dots 16\}$ is $\binom{16}{2}$. The number of pairs with no divisors of three is $\binom{11}{2}$ so

$$P(\text{at least 1 divisor of three}) = 1 - \frac{\binom{11}{2}}{\binom{16}{2}} = \frac{13}{24}$$