1Mech — Mechanics

Mechanics exercises 3 (weeks 5 and 7)

This sheet is formative, but you should attempt all questions to discuss in your tutorial.

1. A particle of mass m is attracted towards the origin by a force of the form mc/r^2, where r, θ are polar coordinates and $c > 0$ is a constant. The particle is initially located a distance a from the origin and moving with speed $\sqrt{c/a}$ perpendicular to the radial direction.

   (a) Show that $u = 1/r$ satisfies

   $$\frac{d^2u}{d\theta^2} + u = \frac{c}{h^2},$$

   where $h = r^2\dot{\theta}$ is constant.

   (b) Find expressions for the initial conditions for the $u(\theta)$ equation, and the value of h.

   (c) Hence solve to find the particle path. What shape is it?

2. Suppose that a particle of mass m is subject to a central force acting **away** the origin of magnitude

   $$\frac{3ma^2\beta^2}{r^3},$$

   with β a positive constant, and r the distance between the particle and the origin. The particle is initially at $r = a$ with initial velocity $\dot{r} = -2\beta$, $r\dot{\theta} = \beta$.

   (a) Show that $u = 1/r$ satisfies

   $$\frac{d^2u}{d\theta^2} + 4u = 0,$$

   subject to $u = 1/a$, $du/d\theta = 2/a$ at $\theta = 0$.

   (b) Solve the system to find the particle path. What is the distance of closest approach to the origin? At what angle does the particle fly off to infinity?

3. For the following question you may find it advantageous to convert into $u(\theta)$ where $u = 1/r$. Note the question does not ask about stability of these paths.

   (a) If a particle under a central force $F(r)$ moves along the spiral $r = e^{-k\theta}$ where k is a constant, show that $F(r) = -C/r^3$ where C is a constant.

   (b) If a particle under a central force $F(r)$ moves along a circular arc terminating at $r = 0$, show that $r = a\cos\theta$ gives the particle path, where a gives the diameter of the circle, and hence show that $F(r) = -D/r^5$ where D is a constant. [Hint: Sketch the semi circular part of the particle path and draw a line joining $r = 0$ to any point on the particle path to find a right angled triangle.]

4. A comet moves under the inverse square law attraction of the Sun. The force is given by

$$\mathbf{F} = -\frac{GMm}{r^2}\mathbf{e}_r = -\frac{GMm}{r^3}\mathbf{r},$$

where $\mathbf{r}$ is the position vector of the comet relative to the Sun, r gives the distance between the Sun and the comet such that $\mathbf{r} = r\mathbf{e}_r$, G is the gravitational constant and M is the mass of the sun. Starting from Newton's second law show that

(a) the moment of momentum of the comet with respect to the Sun is constant.

(b) the orbit of the comet lies in a plane containing the Sun.

(c) $r^2\dot{\theta}$ is constant.