Mechanics week 5: Central forces (examples)

Rosemary Dyson

1 Introduction

Last week we set up the framework for Central Forces problems, this week we will consider some examples of it in action. We start with a summary of what we found last week.

2 Summary

The position vector of a particle is given by

$$\begin{aligned}\mathbf{r} &= r\mathbf{e}_r, \\ &= r\cos\theta\mathbf{i} + r\sin\theta\mathbf{j}.\end{aligned}$$

The velocity of the particle is therefore

$$\dot{\mathbf{r}} = r\dot{\theta}\mathbf{e}_\theta + \dot{r}\mathbf{e}_r.$$

The acceleration is thus

$$\ddot{\mathbf{r}} = \left(\ddot{r} - r\dot{\theta}^2\right)\mathbf{e}_r + \frac{1}{r}\frac{d}{dt}\left(r^2\dot{\theta}\right)\mathbf{e}_\theta.$$

For a **outward pointing** central force, $F(r)\mathbf{e}_r$, $u = 1/r$ satisfies

$$\frac{d^2u}{d\theta^2} + u = -\frac{F(1/u)}{mh^2u^2},$$

with $h = r^2\dot{\theta}$ constant throughout the motion. We also have

$$\dot{r} = -h\frac{du}{d\theta}. \tag{1}$$

3 Examples

Example 1: Motion under a $1/r^2$ central force Typical orbit problems often have an inverse square law central force. Let

$$F(r) = \frac{-GMm}{r^2},$$

so we have a gravitational force between two bodies of mass M and m (e.g. the sun and a planet), with initial conditions

$$\begin{array}{llcr} r = d, & \theta = 0, & \text{specifying location at} & t = 0, \\ \dot{r} = 0, & r\dot{\theta} = v & \text{specifying radial and transverse velocity components at} & t = 0. \end{array}$$

Note the negative sign as the force is directed inwards.

Solution. We know

$$\frac{d^2u}{d\theta^2} + u = -\frac{F(1/u)}{mh^2u^2},$$

where $u = 1/r$ always holds for a central force. Here $F(r)$ is of the form

$$F(r) = \frac{-GMm}{r^2},$$

and hence

$$\begin{aligned} F(1/u) &= \frac{-GMm}{(1/u)^2}, \\ &= -GMmu^2. \end{aligned}$$

This gives

$$\begin{aligned} \frac{d^2u}{d\theta^2} + u &= -\frac{-GMmu^2}{mh^2u^2}, \\ &= \frac{GM}{h^2}. \end{aligned}$$

We first find h from initial conditions (since it is a constant, it is the initial value for all time). It's normally a good idea to find the value of h as early as possible!

Now,

$$\begin{aligned} h &= r^2\dot{\theta}, \\ &= r \cdot r\dot{\theta}, \\ &= dv, \end{aligned}$$

at $t = 0$, and hence for all time. Thus we need to solve

$$\frac{d^2u}{d\theta^2} + u = \frac{GM}{d^2v^2}.$$

This is a linear second order ODE with constant coefficients, so we use the standard method to solve it. Start with the homogeneous equation:

$$\frac{d^2u_c}{d\theta^2} + u_c = 0,$$

which has characteristic equation

$$\begin{aligned} \lambda^2 + 1 &= 0, \\ \implies \lambda &= \pm i, \end{aligned}$$

and hence

$$\implies u_c = A \sin\theta + B\cos\theta,$$

where A and B are constants to be found from the initial conditions. The particular integral is given by

$$\begin{aligned} u_p &= C, \\ \implies C &= \frac{GM}{d^2v^2}, \end{aligned}$$

giving the full solution

$$u = A\sin\theta + B\cos\theta + \frac{GM}{d^2v^2}.$$

We now need to find A, B using the initial conditions. We first need to rewrite these in terms of $u(\theta)$instead of $r(t)$. Firstly $r = d$ gives $u = 1/r = 1/d$, and

$$\begin{aligned} \dot{r} &= -h\frac{du}{d\theta}, \\ &= 0, \\ \implies \frac{du}{d\theta} &= 0. \end{aligned}$$

Since $\theta = 0$ at $t = 0$, both of these hold for $\theta = 0$ so they become

$$r = d, \frac{du}{d\theta} = 0 \quad \text{at} \quad \theta = 0.$$

Hence

$$\begin{aligned}
u(0) &= B + \frac{GM}{d^2v^2}, \\
&= \frac{1}{d}, \\
\implies B &= \frac{1}{d} - \frac{GM}{d^2v^2},
\end{aligned}$$

and

$$\begin{aligned}
\frac{du}{d\theta} &= A\cos\theta - B\sin\theta, \\
&= A \text{ at } \theta = 0, \\
&= 0.
\end{aligned}$$

This gives the complete solution

$$u = \left(\frac{1}{d} - \frac{GM}{d^2v^2}\right)\cos\theta + \frac{GM}{d^2v^2}.$$

What shape is this? We have

$$\begin{aligned}
r &= \frac{1}{u}, \\
&= \frac{1}{\left(\frac{1}{d} - \frac{GM}{d^2v^2}\right)\cos\theta + \frac{GM}{d^2v^2}}, \\
&= \frac{d^2v^2/GM}{\left(\frac{dv^2}{GM} - 1\right)\cos\theta + 1}.
\end{aligned}$$

Firstly note that if $\frac{dv^2}{GM} = 1$ this gives a circle (i.e. $r =$constant). A general conic with focus at $(0,0)$ has a polar representation

$$r(\theta) = \frac{e\hat{d}}{1 \pm e\cos\theta},$$

where $x = \hat{d}$ is the directrix ($\hat{d} > 0$) and e is the eccentricity such that

- $0 < e < 1$ gives an ellipse,
- $e = 1$ gives a parabola,
- $e > 1$ gives a hyperbola.

(See the crib sheet for more details.) Hence we have

$$\pm e = \frac{dv^2}{GM} - 1,$$

which implies we require

$$\left|\frac{dv^2}{GM} - 1\right| \quad < \quad 1,$$

for the path to be an ellipse. Hence

$$\begin{aligned}
-1 < \frac{dv^2}{GM} - 1 < 1, \\
\implies (0 <)\frac{dv^2}{GM} < 2, \\
\implies v^2 < \frac{2GM}{d}.
\end{aligned}$$

So we have an elliptical orbit depending on the mass of the sun and the initial distance and velocity of the planet - this gives Kepler's first rule. See https://www.geogebra.org/m/hekwfhc2 for an interactive solution. ◀

Example 2: Not an inverse square law Suppose a particle of mass m is acted on by a force

$$m\left(\frac{\alpha}{r^2} + \frac{\beta}{r^3}\right),$$

acting towards the origin $r = 0$ of an inertial frame, such that $\beta = \alpha a/2$. The particle is initially a distance a from the origin and moving with velocity $\sqrt{\alpha/a}$ perpendicular to the radial vector.

What are the maximum and minimum distances between the particle and the origin, and what angle is turned through by the radius vector in travelling between them?

Solution. We have that $u = 1/r$ satisfies

$$\begin{aligned}
\frac{d^2u}{d\theta^2} + u &= -\frac{F(1/u)}{mh^2u^2}, \\
&= \frac{\alpha u^2 + \beta u^3}{h^2u^2},
\end{aligned}$$

where $h = r^2\dot{\theta}$ is a constant. Hence

$$\frac{d^2u}{d\theta^2} + \left(1 - \frac{\beta}{h^2}\right)u \quad = \quad \frac{\alpha}{h^2},$$

will give the shape of the path.

We next consider the initial conditions. We **choose** $\theta = 0$ at $t = 0$, and from the problem statement we have

- $r = a$ at $t = 0$ from the initial location of the particle.
- $\dot{r} = 0$, $r\dot{\theta} = \sqrt{\alpha/a}$ at $t = 0$ from the radial and transverse velocity components.

We first calculate h such that

$$\begin{aligned} h &= r \cdot r\dot{\theta}, \\ &= a \cdot \sqrt{\alpha/a}, \\ &= \sqrt{\alpha a}. \end{aligned}$$

We next convert the initial conditions into terms of $u(\theta)$ to give

- $u = 1/r = 1/a$ at $\theta = 0$.
- $\frac{du}{d\theta} = -\dot{r}/h = 0$ at $\theta = 0$.

Hence

$$\begin{aligned} \frac{d^2u}{d\theta^2} + \left(1 - \frac{\beta}{h^2}\right) u &= \frac{\alpha}{h^2}, \\ \implies \frac{d^2u}{d\theta^2} + \left(1 - \frac{\beta}{\alpha a}\right) u &= \frac{\alpha}{\alpha a}, \\ \implies \frac{d^2u}{d\theta^2} + \left(1 - \frac{1}{2}\right) u &= \frac{1}{a}, \\ \implies \frac{d^2u}{d\theta^2} + \frac{u}{2} &= \frac{1}{a}, \end{aligned}$$

since $\beta = \alpha a/2$. This is a second order, linear ODE with constant coefficients, so we proceed as before, starting by solving the homogeneous system, which will have characteristic equation:

$$\lambda^2 + \frac{1}{2} = 0,$$

and hence $\lambda = \pm i/\sqrt{2}$ and thus

$$u_c = A\cos\frac{\theta}{\sqrt{2}} + B\sin\frac{\theta}{\sqrt{2}},$$

where A and B are constants. The particular integral is then of the form $u_p = C$ is constant, hence

$$\begin{aligned} \frac{1}{2}C &= \frac{1}{a}, \\ \implies u_p &= \frac{2}{a}. \end{aligned}$$

The complete solution is thus

$$u = A\cos\frac{\theta}{\sqrt{2}} + B\sin\frac{\theta}{\sqrt{2}} + \frac{2}{a}.$$

Finally, we use the initial conditions to find the values of A and B, such that

$$\begin{aligned} u(0) &= A + \frac{2}{a}, \\ &= \frac{1}{a}, \\ \implies A = -\frac{1}{a}, & \end{aligned}$$

and

$$\frac{du}{d\theta} = -\frac{A}{\sqrt{2}}\sin\frac{\theta}{\sqrt{2}} + \frac{B}{\sqrt{2}}\cos\frac{\theta}{\sqrt{2}},$$

so evaluating this at $\theta = 0$ gives

$$\begin{aligned} \frac{du}{d\theta} &= \frac{B}{\sqrt{2}} = 0, \\ \implies B = 0. & \end{aligned}$$

This gives the solution

$$u = \frac{2}{a} - \frac{1}{a}\cos\frac{\theta}{\sqrt{2}}.$$

We want to find the maximum and the minimum distances, so the maximum and minimum values of r. This is equivalent to minimum and maximum u.

- Minimum u (maximum r) is when $\cos\frac{\theta}{\sqrt{2}} = 1$, so $\theta = 0$. Hence $u = 1/a$, and $r = a$.
- Maximum u (minimum r) is when $\cos\frac{\theta}{\sqrt{2}} = -1$, so $\theta = \sqrt{2}\pi$. Hence $u = 3/a$, and $r = a/3$.

The angle turned through in the difference in these values of θ, i.e. $\sqrt{2}\pi$. See Figure 1, and the interactive version at https://www.geogebra.org/m/wsazcjre.

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Activity: *You should now be able to tackle questions 1 and 2 on this week's problem sheet.*

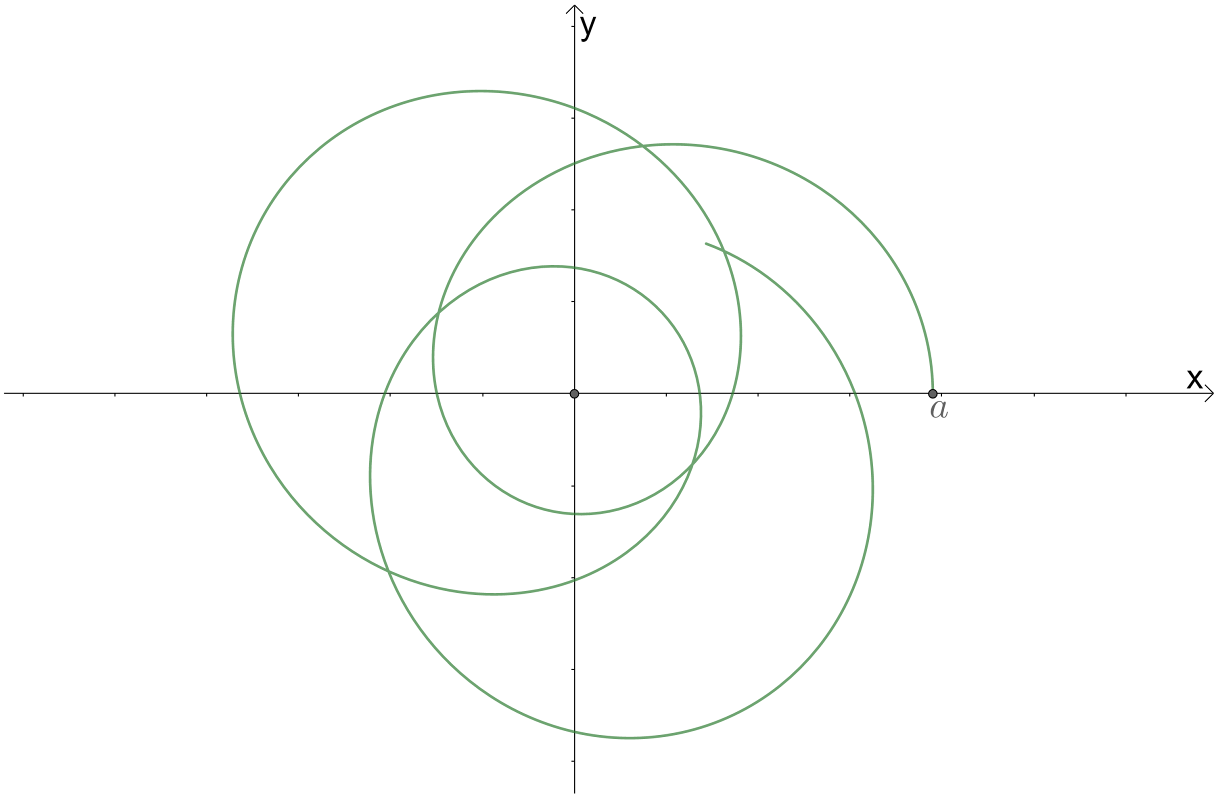

Figure 1: The path of the particle from example 2, spiralling from its initial location around the origin