University of Birmingham
School of Mathematics

1SAS — Sequences and Series — Autumn 2024

Problem Sheet 3

(Issued Week 6)

In this problem sheet you may appeal to any results covered in the lectures (such as the Algebra of Limits, the Monotone Convergence Theorem, results about subsequences etc.) provided you make it clear when you are doing so.

Q1. Which of the following sequences (a_n) converge? If a sequence converges find its limit. Justify any assertions that you make.

(i)
$$a_n = \left(1 + \frac{1}{n}\right)^{100}$$

(ii)
$$a_n = \left(1 + \frac{1}{n+100}\right)^{n}$$

(iii)
$$a_n = \left(2 + \frac{1}{n}\right)^{n}$$

(iv)
$$a_n = \left(1 + \frac{1}{3n-1}\right)^{3n-1}$$

(v)
$$a_n = \left(1 + \frac{1}{n}\right)^{100n}$$

[Recall that the sequence $\left(1 + \frac{1}{n}\right)^n$ converges to Euler's constant e. You may refer to this fact.]

Q2. A sequence (a_n) is defined recursively by

$$a_{n+1} = \frac{1}{2}\left(1 + \frac{1}{a_n}\right), \quad a_1 = 1/2.$$

Observe that (a_n) is not a monotone sequence.

(a) (i) Show that $a_n \geq \frac{1}{2}$ for all $n \in \mathbb{N}$, and deduce further that $a_n \leq \frac{3}{2}$ for all $n \in \mathbb{N}$. [Hint: It is best to avoid induction this time.]

(ii) State a theorem that guarantees that (a_n) has a convergent subsequence.

(b) Let (b_n) be the subsequence of (a_n) given by $b_n = a_{2n}$ for each $n \in \mathbb{N}$.

(i) Prove that

$$b_{n+1} = \frac{1}{2}\left(1 + \frac{2b_n}{b_n + 1}\right)$$

for all $n \in \mathbb{N}$.

(ii) Prove further that

$$b_{n+2} - b_{n+1} = \frac{b_{n+1} - b_n}{(b_n + 1)(b_{n+1} + 1)}$$

for all $n \in \mathbb{N}$, and deduce that (b_n) is decreasing.

(iii) State a theorem that allows you to conclude that (b_n) converges and find its limit.

(iv) Show further that the subsequence (c_n) of (a_n) given by $c_n = a_{2n+1}$ also converges, and to the same limit as that of (b_n). [Hint: write down a formula relating the subsequences (b_n) and (c_n).]

Q3. For each $n \in \mathbb{N}$ let

$$a_n = \sum_{k=1}^{n} \frac{1}{n+k}.$$

(i) Prove that $\frac{1}{2} \leq a_n \leq 1$ for all n.

(ii) Prove that (a_n) is monotone.

(iii) Deduce that (a_n) converges to a limit $\ell \in \mathbb{R}$ where $\frac{1}{2} \leq \ell \leq 1$. [You should refer explicitly to any results that you apply from the module.]

Q4. (i) Show that the sequence (a_n) given by

$$a_n = \cos\left(\frac{n^2\pi}{2}\right),$$

does not converge.

(ii) For $n \in \mathbb{N}$ let p_n denote the nth prime number, and let

$$a_n = \frac{1}{p_{n+1} - p_n}.$$

Show that (a_n) possesses a convergent subsequence.

(iii) Suppose that (a_n) is a sequence of real numbers, and (a_{n_k}) is a subsequence. By appealing to the appropriate definitions, prove that if (a_n) tends to infinity then so does (a_{n_k}).

Extra Questions

EQ1. Study the proof (from recording 1SAS 1.14 or the lecture notes) of the convergence of the sequence (a_n), given by

$$a_n = \left(1 + \frac{1}{n}\right)^n .$$

(i) Why does this proof break down when applied to the sequence (b_n), given by

$$b_n = \left(1 - \frac{1}{n}\right)^n ?$$

(ii) Show that

$$a_{n-1} b_n = \left(1 - \frac{1}{n}\right) \quad \text{for all} \quad n \geq 2.$$

What does this allow us to conclude about the sequence (b_n)? Justify your assertions.

(iii) Consider the sequence (c_n), given by

$$c_n = \left(1 + \frac{(-1)^n}{n}\right)^n .$$

Does this sequence converge? Justify your answer.

EQ2. Suppose (a_n) is a sequence which is *not* bounded above.

(i) Prove that there exists a strictly increasing sequence of natural numbers (n_k) such that

$$a_{n_k} > k$$

for each k.

(ii) Deduce that (a_n) has a subsequence that tends to infinity.

EQ3. Can you find two sequences (a_n) and (b_n), which are not the same sequence, but such that each is a subsequence of the other?