University of Birmingham
School of Mathematics

1RA - Real Analysis: Differentiation Autumn 2024

Summative Question Sheet 1
issued Week 2
Questions

(SUM) **Q1**. Determine the following limits and prove that your answer is correct by directly appealing to the definition of the limit.

(a) $\lim_{x\to 10} 2x+3$.

(b) $\lim_{x\to -7} -6x-2$.

Make sure to check out Question 2 in Problem Sheet 1 for similar questions and a guide on how to write proofs.

(SUM) **Q2**. In this question, you may use the Algebra of Limits. You should not need to use L'Hôpital's rule.

(a) Evaluate the limit $\displaystyle\lim_{h\to 0} \frac{2(-3+h)^2-18}{h}$.

(b) Evaluate the limit $\displaystyle\lim_{t\to 4} \frac{t-\sqrt{3t+4}}{4-t}$.

(c) Evaluate $\displaystyle\lim_{x\to 2} \frac{\sqrt{3x-2}-\sqrt{5x-6}}{\sqrt{2x-1}-\sqrt{x+1}}$.

Make sure to check out Questions 8 and EQ2 in Practice Problem Sheet 1 for similar questions and a guide on how to write proofs.

(SUM) **Q3**. Let $a,b,c\in\mathbb{R}$. Let $f:\mathbb{R}\to\mathbb{R}$ be the polynomial

$$f(x) = x^3+ax^2+bx+c.$$

(i) Prove that

$$\lim_{x\to-\infty} f(x) = -\infty \qquad \text{and} \qquad \lim_{x\to\infty} f(x) = \infty.$$

(ii) Prove that there exist $x_1, x_2\in\mathbb{R}$ such that

$$f(x_1)<0 \qquad \text{and} \qquad f(x_2)>0.$$

[Hint: definition of limit. Also, check out Question 6 in Practice Problem Sheet 1 for a similar argument.]

(iii) Prove that there exists $x_0\in\mathbb{R}$ such that

$$f(x_0)=0,$$

i.e. f has at least one zero x_0 in $\mathbb{R}$.

[Note: This shows that every polynomial of degree 3 has a zero in $\mathbb{R}$. A similar argument proves that every polynomial of odd degree has a zero in $\mathbb{R}$; in turn, this fact can be used as a starting point for a proof of the Fundamental Theorem of Algebra.]