Mechanics: Newton's laws in one dimension

Rosemary Dyson

1 Introduction

Having set up the framework of *kinematics* last week, i.e. ideas about movement, this week we will move on to the main topic of this module, *mechanics*, where we include the effects of the forces which cause (or prevent!) motion.

2 Newton's laws of motion

Newton's laws relate how a force acting on a particle relates to the motion of that particle, and originate from the 1600s. Forces are vectors with both magnitude and direction; the magnitude gives a measure of the strength of the force and is typically measured in the SI unit of Newtons (1N=1kgms^{-2}). For example a stone dropped from a height experiences a gravitational force pulling it downwards of strength mg where m is the mass of the stone and g is the acceleration due to gravity (9.81ms^{-2} on the surface of the Earth). A particle will respond to the net force acting on it, that is the sum (resultant) of all the forces acting on the particle.

Newton's first law: In the absence of any resultant forces, a particle moves with constant velocity $\mathbf{v}$.

Newton's second law: If a net force $\mathbf{F}$ acts on a particle of constant mass m, then the acceleration $\mathbf{a}$ of the particle is related to $\mathbf{F}$ and m by

$$\mathbf{F} = m\mathbf{a}.$$

We can also define the linear momentum of a particle as mass times velocity

$$\mathbf{p} = m\mathbf{v},$$

and hence this also gives

$$\mathbf{F} = \frac{d\mathbf{p}}{dt}.$$

The rate of change of linear momentum is equal to the sum of the forces acting on it. This also works when the particle mass is not constant.

Newton's third law: For every force there is an equal and opposite reaction force.

Definition: Inertial frame An inertial frame is one in which Newton's laws hold. (We'll come back to this later!)

Intuitively.... Imagine throwing a ball on a train. When the train is moving at a constant speed in a straight line this is no different from throwing a ball in the garden - the same trajectory will be observed. If, however, the train is going round a corner (i.e. it is accelerating as the velocity is not constant), then we will see a very different motion - the frame is not an inertial frame. [See video on Canvas, taken from `https://twitter.com/EvoBiomech/status/1351589830698921986` with permission, thanks to the Evolutionary Biomechanics Lab at Imperial.]

3 Standard procedure

The basic procedure in mechanics is as follows:

- Form the governing equations using Newton's laws (often II), and the initial conditions from the initial location/velocity of the particle (as appropriate).
- Solve the resulting system.
- Look at what this tells us - for example
  - what is the shape of the particle path?
  - what is the speed of motion?
  - how far does the particle go?
  - when/where does a given event (for example a crash) occur?

We start by looking at motion in one direction, before moving on to two and three dimensions next week.

4 One dimensional motion

Example 1: Mass moving with a time-dependent force

A particle of mass m moves in a straight line subject to a time dependent force of the form $F = F_0 \sin(\omega t)$. At time $t = 0$ the particle is located at $x = 0$ and has velocity v_0. What is the motion?

Solution. Take x to measure distance in a straight line from the starting point. The equation of motion is found from Newton's second law:

$$\text{Force} = \text{mass x acceleration}$$

and hence we find

$$F_0 \sin(\omega t) \quad = \quad m\ddot{x}.$$

This can be integrated once to find

$$\begin{aligned} \dot{x} &= \frac{F_0}{m} \int \sin(\omega t) \ \mathrm{d}t + c_1, \\ &= -\frac{F_0}{m\omega} \cos(\omega t) + c_1, \end{aligned}$$

and again to find

$$x \quad = \quad -\frac{F_0}{m\omega^2} \sin(\omega t) + c_1 t + c_2,$$

where c_1, c_2 are constants of integration.

To find the constants of integration we use the initial conditions that at $t = 0$, $x = 0$ and $\dot{x} = v_0$ (velocity). Using these we find:

$$x = 0 \qquad \Longrightarrow \qquad c_2 = 0,$$

and

$$\begin{aligned} \dot{x} &= v_0, \\ \Longrightarrow \ -\frac{F_0}{m\omega} + c_1 &= v_0, \\ \Longrightarrow c_1 &= v_0 + \frac{F_0}{m\omega}. \end{aligned}$$

Thus we have found the particle position for all values of $t > 0$, given by

$$x \quad = \quad -\frac{F_0}{m\omega^2} \sin(\omega t) + \left(v_0 + \frac{F_0}{m\omega}\right) t.$$

The particle therefore oscillates (the sin term) with some drift (the t term) either to left or right. When:

- $v_0 > -F_0/m\omega$, then $x \to \infty$ as $t \to \infty$ (tends to the right/positive x direction).

- $v_0 < -F_0/m\omega$, then $x \to -\infty$ as $t \to \infty$ (tends to the left/negative x direction).
- $v_0 = -F_0/m\omega$, then $x = -\dfrac{F_0}{m\omega^2}\sin(\omega t)$ and particle oscillates with no drift about $x = 0$.

See https://www.geogebra.org/m/guxameb2. ◀

Activity: *You should now be able to tackle question 4 on this week's problem sheet.*

Aside: Governing equations formulated using force = mass × acceleration tend to be of the form

$$m\ddot{x} \;=\; g(x, \dot{x}, t).$$

When this can be written as

$$a\ddot{x} + b\dot{x} + cx \;=\; f(t),$$

where a, b, c are constants this is a linear, second-order differential equation with constant coefficients; being able to solve such equations is key to this module! You will learn the formal details later in 1RA, but we will cover the method now (see Section **??** of the crib sheet).

Example 2: Mass on a spring, neglecting gravity

If a particle of mass m is attached to a spring (natural length l, spring constant k), which is fixed at the opposite end, which starts at its natural length at time $t = 0$ moving with speed u, what is the motion?

Solution. We first set up an appropriate coordinate system for the problem. We choose $x = 0$ to be when the spring is unstretched (i.e. it is at natural length), with x pointing away from the fixed end of the spring. x therefore gives the extension in the spring. See Fig 1.

Hooke's law then says that force is proportional to extension, so we have $F = -kx$, where k is the spring constant, and the minus sign is because the force works to pull the particle back to the natural length. Hence Newton's second law gives

$$\begin{aligned} m\ddot{x} &= -kx, \\ \implies \ddot{x} + \frac{k}{m}x &= 0. \end{aligned}$$

Since k and m are both positive, we can write $k/m = \omega^2$ which forces this to be true for real ω. Thus the model becomes

$$\ddot{x} + \omega^2 x \;=\; 0.$$

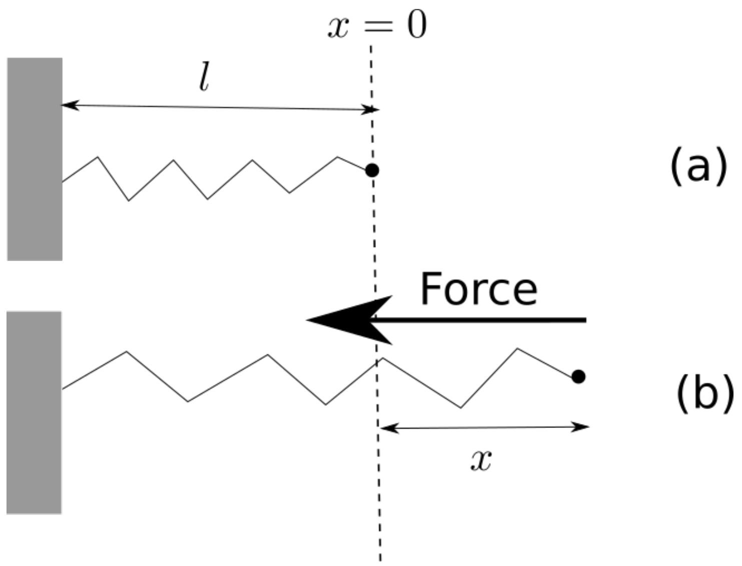

Figure 1: Sketch of the coordinate system showing (a) the unstretched mass/spring and (b) the stretched mass/spring.

[This is the equation of *simple harmonic motion.*] We solve this by first forming the characteristic equation:

$$\lambda^2 + \omega^2 = 0,$$

giving $\lambda = \pm i\omega$ and hence

$$x = A\cos(\omega t) + B\sin(\omega t).$$

[This is the other reason I defined ω like that!]. We now use the initial conditions to find A and B. Since the particle starts at natural length with speed u, this gives the initial conditions $x = 0$, $\dot{x} = u$ at $t = 0$. Hence we find

- at $t = 0$, $x = 0$, gives $A = 0$.
- $\dot{x} = \omega B\cos\omega t$ and hence $\omega B = u$ at $t = 0$, giving $B = u/\omega$.

Thus the solution is

$$x = \frac{u}{\omega}\sin\omega t.$$

In the absence of damping/other forcing, the mass will oscillate forever. ◀

Example 3: Trainee parachute jump

Consider a trainee practicing parachute jumps by jumping from a tall building of height H from the ground. They open their parachute immediately and fall in a straight line to the ground. How long does it take them to hit the ground, and what speed are they going when they make contact?

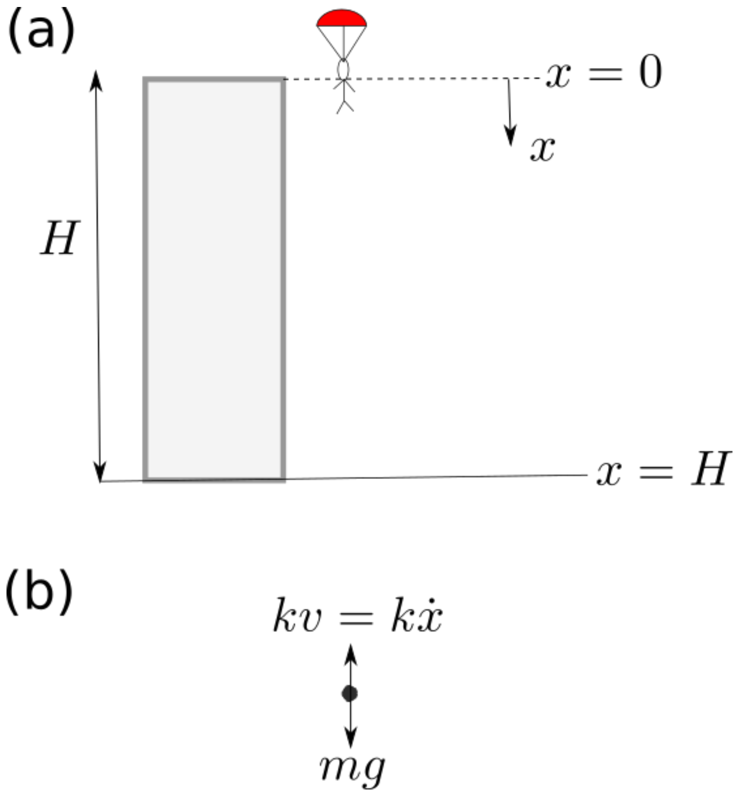

Figure 2: Sketch of the problem set up: (a) the geometry of the problem, indicating the jumper, the building and the ground, (b) the force balance on the particle between gravity pulling downwards and air resistance pulling upwards.

Solution. We first have to make a series of assumptions which turn the physical situation into a mathematical model. We assume:

- the person can be approximated as a particle.
- the motion is one dimensional.
- distance is measured from the top of the tower downwards; $x = 0$ gives the top and $x = H$ is the ground (see Fig. 2).
- $x(t)$ gives the location of the person, with $\dot{x}(t)$ giving the speed.

To use Newton's second law we need to calculate the forces acting on the person. These are:

- the weight of the person, acting downwards (i.e. in the positive x direction). This is given by mg where m is the mass of the person and g is the acceleration due to gravity (9.81ms^{-1}).

- the force of the parachute pulling upwards. This is due to air resistance and requires a model assumption. Air resistance gets stronger the faster you're moving, so assume it is proportional to the speed of the person acting upwards $-kv$ (negative x direction). This could also be e.g. speed squared.

Hence Newton's second law gives

$$\begin{aligned} m\ddot{x} &= mg - k\dot{x}, \\ \implies \ddot{x} + \frac{k}{m}\dot{x} &= g. \end{aligned} \tag{1}$$

The initial conditions are that the trainee starts from rest at the top of the tower, hence $x = 0$, $\dot{x} = 0$ at $t = 0$.

We now solve equation (1). We first consider the homogeneous problem

$$\ddot{x}_c + \frac{k}{m}\dot{x}_c = 0,$$

to find the complementary function. This has characteristic equation

$$\lambda^2 + \frac{k}{m}\lambda = 0,$$

and hence $\lambda = 0$, or $\lambda = -k/m$, giving

$$x_c = C_1 + C_2 e^{-kt/m}.$$

The particular integral will be of the form

$$x_p = A_1 t;$$

note that we need the linear in t term as constant is part of the complementary function. Hence $\dot{x}_p = A_1$, $\ddot{x}_p = 0$ and so

$$\begin{aligned} \ddot{x}_p + \frac{k}{m}\dot{x}_p &= g, \\ \implies \frac{k}{m}A_1 &= g, \\ \implies A_1 &= \frac{gm}{k}. \end{aligned}$$

Thus the full solution is

$$x = C_1 + C_2 e^{-kt/m} + \frac{gm}{k}t.$$

We now use the initial conditions to find the constants of integration.

- $x(0) = 0$ give $x(0) = C_1 + C_2 = 0$.
- $\dot{x}(0)$ gives $\dot{x}(0) = -\frac{kC_2}{m} + \frac{gm}{k} = 0$

and hence we find

$$\begin{aligned} C_2 &= \frac{gm^2}{k^2}, \\ C_1 &= -\frac{gm^2}{k^2}, \end{aligned}$$

giving the full solution

$$x = -\frac{gm^2}{k^2} + \frac{gm^2}{k^2}e^{-kt/m} + \frac{gm}{k}t.$$

The trainee hits the ground when $t = T$ such that $x(T) = H$ - i.e. the value of time at which they've fallen a distance H. This is given by

$$H = -\frac{gm^2}{k^2} + \frac{gm^2}{k^2}e^{-kT/m} + \frac{gm}{k}T,$$

which could be solved numerically to find T. They therefore are moving with speed

$$\dot{x}(T) = -\frac{gm}{k}e^{-kT/m} + \frac{gm}{k},$$

upon impact. The general expression for speed is given by

$$\dot{x} = \frac{gm}{k}\left(1 - e^{-kt/m}\right),$$

so as time tends to infinity, $\dot{x} \to \frac{gm}{k}$ which gives a constant terminal velocity. ◀

This gives some examples in one dimension where all the motion is in a straight line. This clearly doesn't cover most cases! Next week we will move on to consider examples where the motion is in more than one dimension.

Activity: *You should now be able to tackle question 5 on this week's problem sheet.*