Mechanics: Newton's laws in more dimensions

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1 Introduction

Last week we considered a particle which moves only in a straight line. If, however, the particle moves in a plane (2D) or in all space (3D) we follow the same procedures as before, but now we use vectors as direction is also important. Here the path the particle follows is often critical to find. Choosing how to set up your coordinates and axes is often the key to being able to solve these problems!

2 Examples in 2 and 3 dimensions

Example 1: Throwing a stone with no air resistance Suppose a stone is thrown with velocity $\mathbf{V}$ at an angle α to the horizontal, from the top of a wall of height h. Ignoring air resistance, what is the path of the stone and its horizontal range?

Solution.

Develop model: If there is no air resistance everything will happen in a two dimensional (vertical) plane. Choose the origin O to be at ground level at the bottom of the wall, with unit vectors $\mathbf{i}$, $\mathbf{k}$ pointing in the horizontal and vertical directions respectively. Then the position of the stone over time is given by

$$\mathbf{r} = x(t)\mathbf{i} + z(t)\mathbf{k},$$

then $\dot{\mathbf{r}}$ gives the velocity and $\ddot{\mathbf{r}}$ gives the acceleration. The only force acting on the stone is gravity $\mathbf{g}$, so

$$\begin{aligned} \mathbf{F} &= m\mathbf{g}, \\ &= -mg\mathbf{k}, \end{aligned}$$

as gravity acts downwards. Newton's second law then gives

$$m\ddot{\mathbf{r}} = m\mathbf{g}. \tag{1}$$

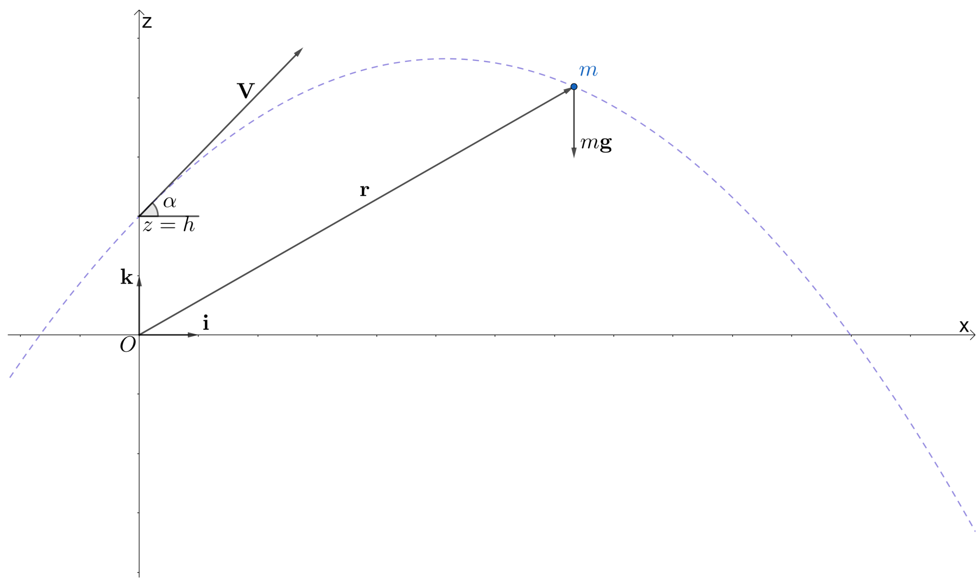

Figure 1: Sketch of the coordinate system.

We also know two initial conditions: that the stone starts at the top of the wall,

$$\mathbf{r}(0) = h\mathbf{k},$$

and that it's moving with velocity $\mathbf{V}$,

$$\dot{\mathbf{r}}(0) = \mathbf{V}.$$

We are also told that the velocity makes an angle α with the horizontal. This means we can resolve the velocity such that $\mathbf{V} = V\cos\alpha\mathbf{i} + V\sin\alpha\mathbf{k}$ where $V = |\mathbf{V}|$. **Note that distinguishing between vectors and scalars is crucial!!**

Solve model: This gives a full defined system which we can now solve. We solve (1) in the same way as we would tackle a scalar equation - we can just integrate twice as $m\mathbf{g}$ are just constants. Hence

$$\begin{aligned} m\ddot{\mathbf{r}} &= m\mathbf{g}, \\ \implies \ddot{\mathbf{r}} &= \mathbf{g}, \\ \implies \dot{\mathbf{r}} &= \mathbf{g}t + \mathbf{c}_1, \\ \implies \mathbf{r} &= \frac{1}{2}\mathbf{g}t^2 + \mathbf{c}_1 t + \mathbf{c}_2, \end{aligned}$$

where $\mathbf{c}_1$ and $\mathbf{c}_2$ are now vector constants which we find using the initial conditions.

We first use $\mathbf{r}(0) = h\mathbf{k}$ to find:

$$\begin{aligned} \mathbf{r}(0) &= \frac{0^2}{2}\mathbf{g} + 0\mathbf{c}_1 + \mathbf{c}_2, \\ \implies h\mathbf{k} &= \mathbf{c}_2. \end{aligned}$$

Then $\dot{\mathbf{r}}(0) = \mathbf{V}$ gives

$$\begin{aligned} \dot{\mathbf{r}}(0) &= 0\mathbf{g} + \mathbf{c}_1, \\ \implies \mathbf{V} &= \mathbf{c}_1. \end{aligned}$$

Hence the overall solution is given by

$$\begin{aligned} \mathbf{r} &= \frac{t^2}{2}\mathbf{g} + t\mathbf{V} + h\mathbf{k}, \\ &= \frac{-gt^2}{2}\mathbf{k} + Vt\cos\alpha\mathbf{i} + Vt\sin\alpha\mathbf{k} + h\mathbf{k}, \\ &= Vt\cos\alpha\mathbf{i} + \left(Vt\sin\alpha + h - \frac{gt^2}{2}\right)\mathbf{k}, \end{aligned}$$

which gives the horizontal position as a function of time as

$$x(t) = Vt\cos\alpha,$$

and the vertical position as

$$z(t) = Vt\sin\alpha + h - \frac{gt^2}{2}.$$

What happens? The stone hits the ground when $z = 0$, or equivalently $\mathbf{r}\cdot\mathbf{k} = 0$, at time T which we want to find. Hence

$$\begin{aligned} VT\sin\alpha + h - \frac{gT^2}{2} &= 0, \\ \implies T &= \frac{-V\sin\alpha \pm \sqrt{V^2\sin^2\alpha + 2gh}}{-g}, \\ &= \frac{V\sin\alpha \mp \sqrt{V^2\sin^2\alpha + 2gh}}{g}, \end{aligned}$$

using the quadratic formula. This gives two potential values (one for the plus sign and one for the minus sign) - we want the positive value, so we choose the plus sign to give

$$T = \frac{V\sin\alpha + \sqrt{V^2\sin^2\alpha + 2gh}}{g}.$$

The value of $x(T)$ gives the distance from the wall where the stone will hit the ground:

$$\begin{aligned} x(T) &= VT\cos\alpha, \\ &= \frac{V\left(V\sin\alpha + \sqrt{V^2\sin^2\alpha + 2gh}\right)\cos\alpha}{g}. \end{aligned}$$

We can also find the path of the stone by eliminating t from the expressions for x and z:

$$\begin{aligned}
x(t) &= Vt\cos\alpha, \\
\implies t &= \frac{x}{V\cos\alpha}, \\
\implies z &= Vt\sin\alpha + h - \frac{gt^2}{2}, \\
\implies z &= V\sin\alpha\frac{x}{V\cos\alpha} + h - \frac{g}{2}\left(\frac{x}{V\cos\alpha}\right)^2, \\
&= -\frac{g\sec^2\alpha}{2V^2}x^2 + \tan\alpha x + h,
\end{aligned}$$

which gives a parabola (see Figure 1 and the interactive version on Canvas). ◀

Activity: *You should now be able to tackle question 1 on this week's problem sheet.*

Example 2: Throwing a ball with air resistance A ball is thrown with velocity $\mathbf{V}$ in a steady crosswind of velocity $\mathbf{u}$. If air resistance is linearly proportional to the relative velocity between the ball and the air, what is the particle path?

Solution.

Develop model The ball will be subject to two forces - gravity ($m\mathbf{g}$) and air resistance. The air resistance force will be given by

$$-k(\dot{\mathbf{r}} - \mathbf{u}),$$

i.e. proportional to the difference in the velocities between the ball and the air. Then Newton's second law gives

$$\begin{aligned}
m\ddot{\mathbf{r}} &= m\mathbf{g} - k(\dot{\mathbf{r}} - \mathbf{u}), \\
\implies \ddot{\mathbf{r}} + \frac{k}{m}\dot{\mathbf{r}} &= \mathbf{g} + \frac{k}{m}\mathbf{u}.
\end{aligned}$$

We also need initial conditions: assume that the ball starts at the origin with velocity $\mathbf{V}$, giving

$$\begin{aligned}
\mathbf{r} = 0 \quad &\text{at} \quad t = 0, \\
\dot{\mathbf{r}} = \mathbf{V} \quad &\text{at} \quad t = 0.
\end{aligned}$$

Solve model We could go into components of the vector equation here, but then we'd have to solve two or three equations, so it's often easier to stay in terms of vectors as long as possible. The solution technique is as for a scalar equation - this is a linear, second order, inhomogeneous ODE with constant coefficients which we've seen before, just with vector coefficients.

We start with the homogeneous equation:

$$\ddot{\mathbf{r}}_c + \frac{k}{m}\dot{\mathbf{r}}_c = 0,$$

and guess the solution $\mathbf{r}_c = \mathbf{B}e^{\lambda t}$, where we need to find $\mathbf{B}$ (from the initial conditions) and (two values of) λ from the governing equation. Upon substituting we find

$$\begin{aligned} \lambda^2\mathbf{B}e^{\lambda t} + \frac{k}{m}\lambda\mathbf{B}e^{\lambda t} &= 0, \\ \Longrightarrow \lambda^2 + \frac{k}{m}\lambda &= 0, \end{aligned}$$

which gives the characteristic equation. This has solutions $\lambda = 0$, or $\lambda = -k/m$, and hence we find the complementary solution

$$\mathbf{r}_c = \mathbf{B}_1 + \mathbf{B}_2 e^{-kt/m},$$

for two constant (vectors) $\mathbf{B}_1$, $\mathbf{B}_2$.

We now look for a particular solution. The right hand side is constant, and we have a constant in our complementary solution, so guess the particular solution is of the form $\mathbf{r}_p = \mathbf{A}t$, where $\mathbf{A}$ is a constant. This gives $\dot{\mathbf{r}}_p = \mathbf{A}$, $\ddot{\mathbf{r}}_p = 0$. When we substitute this into the original equation this gives

$$\begin{aligned} \ddot{\mathbf{r}}_p + \frac{k}{m}\dot{\mathbf{r}}_p &= \mathbf{g} + \frac{k}{m}\mathbf{u}, \\ \frac{k}{m}\mathbf{A} &= \mathbf{g} + \frac{k}{m}\mathbf{u}, \\ \Longrightarrow \mathbf{A} &= \frac{m}{k}\mathbf{g} + \mathbf{u}, \\ \Longrightarrow \mathbf{r}_p &= \left(\frac{m}{k}\mathbf{g} + \mathbf{u}\right)t. \end{aligned}$$

This gives the complete solution

$$\begin{aligned} \mathbf{r} &= \mathbf{r}_c + \mathbf{r}_p, \\ &= \mathbf{B}_1 + \mathbf{B}_2 e^{-kt/m} + \left(\frac{m}{k}\mathbf{g} + \mathbf{u}\right)t. \end{aligned}$$

We now use the initial conditions to find the constants $\mathbf{B}_1$, $\mathbf{B}_2$. The first condition is $\mathbf{r} = 0$ at $t = 0$ which gives

$$\mathbf{0} = \mathbf{B}_1 + \mathbf{B}_2.$$

The second condition is $\dot{\mathbf{r}} = \mathbf{V}$ at $t = 0$. Since

$$\dot{\mathbf{r}} = -\frac{k}{m}\mathbf{B}_2 e^{-kt/m} + \left(\frac{m}{k}\mathbf{g} + \mathbf{u}\right),$$

we have

$$\mathbf{V} = -\frac{k}{m}\mathbf{B}_2 + \frac{m}{k}\mathbf{g} + \mathbf{u}.$$

This gives

$$\mathbf{B}_2 = -\frac{m}{k}\mathbf{V} + \frac{m^2}{k^2}\mathbf{g} + \frac{m}{k}\mathbf{u},$$

and hence

$$\begin{aligned} \mathbf{B}_1 &= -\mathbf{B}_2, \\ &= \frac{m}{k}\mathbf{V} - \frac{m^2}{k^2}\mathbf{g} - \frac{m}{k}\mathbf{u}, \end{aligned}$$

giving the complete solution

$$\mathbf{r} = \frac{m}{k}\left(\mathbf{V} - \frac{m}{k}\mathbf{g} - \mathbf{u}\right)\left(1 - e^{-kt/m}\right) + \left(\frac{m}{k}\mathbf{g} + \mathbf{u}\right)t.$$

What does this tell us?

This gives the position of the ball at all points in time - but what does it tell us? We could write this as

$$\mathbf{r} = \frac{m}{k}\left(1 - e^{-kt/m}\right)\mathbf{V} + \left(t - \frac{m}{k}\left(1 - e^{-kt/m}\right)\right)\left(\frac{m}{k}\mathbf{g} + \mathbf{u}\right),$$

which gives the motion in a plane spanned by the two vectors $\mathbf{V}$ and $\frac{m}{k}\mathbf{g} + \mathbf{u}$. This is not necessarily a vertical plane because of the cross wind!

If, say,

$$\begin{aligned} \mathbf{V} &= V\cos\alpha\mathbf{i} + V\sin\alpha\mathbf{k}, \\ \mathbf{g} &= -g\mathbf{k}, \\ \mathbf{u} &= u\mathbf{j}, \end{aligned}$$

where $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are cartesian vectors such that $\mathbf{k}$ points vertically upwards, $V = |\mathbf{V}|$, $u = |\mathbf{u}|$, and $g = |\mathbf{g}|$ is acceleration due to gravity, then we have a horizontal cross wind which is perpendicular to the ball's initial motion. In this case

$$\mathbf{r} = \frac{m}{k}\left(1 - e^{-kt/m}\right)\left(V\cos\alpha\mathbf{i} + \left(V\sin\alpha + \frac{m}{k}g\right)\mathbf{k} - u\mathbf{j}\right) + t\left(-\frac{mg}{k}\mathbf{k} + u\mathbf{j}\right),$$

so the ball is moving in three dimensions in Cartesian space, but in a 2D plane defined by the two vectors $\mathbf{V}$ and $\frac{m}{k}\mathbf{g} + \mathbf{u} = -mg\mathbf{k}/k + u\mathbf{j}$.

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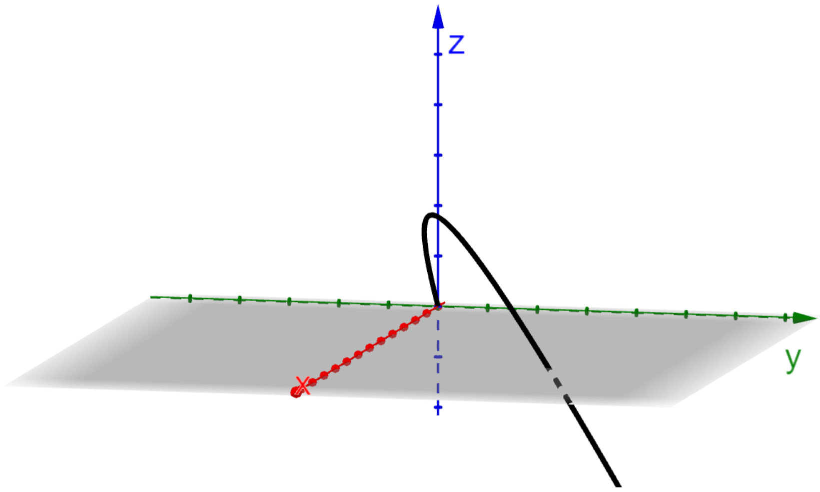

Figure 2: The particle path traced out in 3D, showing the particle moving in a non-vertical plane. See the link to the interactive version on Canvas!

Activity: *You should now be able to tackle question 2 on this week's problem sheet.*

3 Moving frames of reference

In the above examples we have assumed that we can choose a frame of reference, locate the origin where we like and that Newton's laws will hold. In particular, we have assumed that the Earth's surface gives an inertial frame, despite the fact that the Earth is rotating both about its own axis and the sun. For the examples we considered, over the time and distance of the particle being airborne, the velocity of the frame of reference can be taken as uniform relative to the centre of the Earth. But if, for example, you were throwing things whilst spinning on a roundabout that would not be the case.

If we have an inertial frame S with origin O and axes $Oxyz$ then a change of axes $Oxyz$ to $Ox'y'z'$ and/or a change of origin by a fixed vector $\mathbf{a}$, ($\mathbf{O} = \mathbf{a} + \mathbf{O}'$) both give another inertial frame S'.

Proposition Suppose that S is an inertial frame with origin O. Then a frame of reference S' whose origin O' is moving with speed $\mathbf{u}$ with respect to S is also an inertial frame.

Proof Suppose that initially the origins O and O' are separated by a vector $\mathbf{a}$ at time $t = 0$. Then, since O' moves with velocity $\mathbf{u}$ with respect to S, so the vector from O to O' is given by $\mathbf{a} + \mathbf{u}t$. If $\mathbf{r}$ and $\mathbf{r}'$ are the position vectors of the same particle with respect to frames S and S' respectively then

$$\mathbf{r}' = \mathbf{r} - (\mathbf{u}t + \mathbf{a}) , \tag{2}$$

and hence $\ddot{\mathbf{r}}' = \ddot{\mathbf{r}}$, implying that Newton's second law also applies in the new frame S', as it applies in S. Hence S' is also inertial.