University of Birmingham
School of Mathematics

Real Analysis – Integration – Spring 2025

Problem Sheet 9
Model Solutions

Instructions: You are strongly encouraged to attempt all of the Questions (Q) below, and as many of the Extra Questions (EQ) as you can, to help prepare for the final exam. Model solutions will only be released for the Questions (Q1–Q4).

Questions

Q1. (a) Suppose that $f : [0,4] \to [0,1]$ is given by

$$f(x) := \begin{cases} x, & \text{if } 0 \le x \le 1; \\ 1, & \text{if } 1 < x \le 4. \end{cases}$$

Find a solution $y : [0,4) \to \mathbb{R}$ of the initial value problem

$$y' = f(x), \quad y(0) = 1.$$

You must prove that your solution is indeed differentiable on $(0,4)$.

(b) Find a solution $y : [0,\infty) \to \mathbb{R}$ of the initial value problem

$$yy' = \log(x), \quad y(0) = 2.$$

You must justify all limit computations.

Solution. (a) Here we can apply Theorem 9.1.3 from the Lecture Notes, or proceed with Direct Integration, to obtain solutions to the ordinary differential equation on the *open* intervals $(0,1)$ and $(0,4)$ separately: If $x \in (0,1)$, then $y'(x) = f(x) = x$ so $y(x) = \frac{1}{2}x^2 + C$ for any $C \in \mathbb{R}$; If $x \in (1,4)$, then $y'(x) = f(x) = 1$ so $y(x) = x + D$ for any $D \in \mathbb{R}$.

The solution of the initial value problem must be continuous on $[0,4)$ with $y(0) = 1$. To ensure the continuity at $x = 0$ we must have $0 + C = 1$, hence $C = 1$. To ensure the continuity at $x = 1$, we must have

$$\tfrac{3}{2} = \lim_{x\to 1^-} (\tfrac{1}{2}x^2 + 1) = \lim_{x\to 1^+} (x + D) = 1 + D,$$

hence $D = \frac{1}{2}$.

Now define the $y : [0,4) \to \mathbb{R}$ by

$$y(x) := \begin{cases} \frac{1}{2}x^2 + 1, & \text{if } 0 \le x \le 1; \\ x + \frac{1}{2}, & \text{if } 1 < x < 4. \end{cases}$$

This function is continuous on $[0,4)$ by design, it is differentiable and satisfies the differential equation on $(0,1) \cup (1,4)$, and also $y(0) = 1$. It only remains to check that y is differentiable and satisfies the differential equation when $x = 1$. To this end, observe that

$$\lim_{h\to 0^+} \frac{y(1+h) - y(1)}{h} = \lim_{h\to 0^+} \frac{(\frac{1}{2}(1+h)^2 + 1) - \frac{3}{2}}{h} = \lim_{h\to 0^+} (1 + \tfrac{h}{2}) = 1 = f(1)$$

whilst

$$\lim_{h\to 0^+} \frac{y(1) - y(1-h)}{h} = \lim_{h\to 0^+} \frac{\frac{3}{2} - (1 - h + \frac{1}{2})}{h} = 1 = f(1),$$

hence y is differentiable when $x = 1$ with $y'(1) = f(1)$, so y is indeed a solution of the initial value problem.

(b) We apply the method of Separation of Variables. Assume that y is a solution on $(0,\infty)$ such that $y(x) \neq 0$ for all $x \in (0,\infty)$, so a formal application of the Substitution Formula gives

$$\begin{aligned} yy' = \log(x) &\implies \int y \,\mathrm{d}y = \int \log(x)\,\mathrm{d}x \\ &\implies \tfrac{1}{2}y^2 = x\log(x) - x + C \\ &\implies y(x) = \sqrt{2x\log(x) - 2x + C} \end{aligned}$$

for some $C \in \mathbb{R}$. The solution must be continuous on $[0,\infty)$, so the initial condition $y(0) = 2$ requires that

$$\begin{aligned} \lim_{x\to 0^+} y(x) = 2 &\implies \lim_{x\to 0^+} \sqrt{2x\log(x) - 2x + C} = 2 \\ &\implies \lim_{x\to 0^+} \sqrt{C} = 2 \\ &\implies C = 4, \end{aligned}$$

where we used the Algebra of Limits and L'Hôpital's Rule, observing that

$$\lim_{x\to 0^+} x\log(x) = \lim_{x\to 0^+} \frac{(\log(x))'}{(x^{-1})'} = \lim_{x\to 0^+} \frac{x^{-1}}{-x^{-2}} = 0.$$

These computations muse be done using limits, as $\log(x)$ is not defined at $x = 0$. Altogether, this shows that the function $y : [0,\infty) \to \mathbb{R}$ given by

$$y(x) := \begin{cases} 2, & \text{if } x = 0; \\ \sqrt{2x\log(x) - 2x + 4}, & \text{if } x > 0 \end{cases}$$

is a solution of the initial value problem.

Q2. A swimming pool by the sea has a capacity of 5 000 000 L and the concentration of salt in the seawater is 0.045 kg L^{-1}. The pool is initially filled with pure water. The concentration of salt in the pool is then increased by pumping in seawater at a rate of 3 000 L min^{-1} whilst the pool is drained at the same rate. Assume that the mixture in the pool is instantly and uniformly mixed:

(a) Let $y(t)$ denote the mass (in kilograms) of salt in the pool at time t (in minutes) after mixing begins. Formulate an initial value problem to model the flow $y'(t)$.

(b) Find a solution to your initial value problem and determine how long it will take for the salt concentration in the swimming pool to reach 0.0035 kg L^{-1}?

(c) Suppose instead that the pump operates at 1 000 L min^{-1} whilst the pool is drained at 3 000 L min^{-1}. Formulate and solve an initial value problem to determine the mass of salt in the swimming pool after t minutes of mixing for all $t \in [0,+\infty)$.

Solution. (a) The flow equation $y'(t) =$ rate in $-$ rate out (in kg min^{-1}) becomes

$$y'(t) = \left(0.045\frac{\text{kg}}{\text{L}}\right)\left(3\times 10^3\frac{\text{L}}{\text{min}}\right) - \left(\frac{y(t)}{5\times 10^6}\frac{\text{kg}}{\text{L}}\right)\left(3\times 10^3\frac{\text{L}}{\text{min}}\right).$$

The flow is thus modelled by solutions $y : [0,+\infty) \to \mathbb{R}$ to the initial value problem

$$y'(t) = 135 - \frac{3y}{5000} = \frac{675000 - 3y}{5000}, \quad y(0) = 0.$$

(b) This is a separable first-order differential equation. Observe that $675000 - 3y = 0$ when $y = 225000$, but we ignore the constant solution $y(t) = 225000$ because it does not satisfy the initial condition $y(0) = 0$. Next, assume that y is a solution on $(0, +\infty)$ such that $y(t) \neq 225000$ for all $t \in (0, +\infty)$, so proceeding formally we obtain

$$\begin{aligned}
y'(t) = \frac{675000 - 3y}{5000} &\implies \frac{1}{675000 - 3y} \frac{\mathrm{d}y}{\mathrm{d}t} = \frac{1}{5000} \\
&\implies \int \frac{1}{675000 - 3y} \, \mathrm{d}y = \int \frac{1}{5000} \, \mathrm{d}t \\
&\implies -\tfrac{1}{3} \log |675000 - 3y| = \tfrac{t}{5000} + C \\
&\implies |675000 - 3y| = Ce^{-\frac{3t}{5000}} \\
&\implies 675000 - 3y = Ce^{-\frac{3t}{5000}} \\
&\implies y(t) = 225000 + Ce^{-\frac{3t}{5000}}
\end{aligned}$$

for some $C \in \mathbb{R}$. The initial condition $y(0) = 0$ then requires that $C = -225000$.

The function $y : [0, +\infty) \to \mathbb{R}$ given by $y(t) := 225000(1 - e^{-\frac{3t}{5000}})$ is differentiable on $(0, \infty)$ and continuous on $[0, \infty)$, since it is the composition of such functions. It also satisfies the differential equation and initial condition by construction, hence it is a solution of the initial value problem.

The salt concentration in the swimming pool will reach 0.0035 kg L^{-1} when

$$\begin{aligned}
\frac{y(t)}{5000000} = 0.0035 &\implies 225000(1 - e^{-\frac{3t}{5000}}) = 17500 \\
&\implies e^{-\frac{3t}{5000}} = 1 - \frac{175}{2250} \\
&\implies -\tfrac{3t}{5000} = \log\left(1 - \frac{175}{2250}\right) \\
&\implies t = \tfrac{5000}{3} \log\left(\frac{2250}{2075}\right) \approx 135,
\end{aligned}$$

that is, after about 135 minutes.

(c) The flow equation $y'(t) =$ rate in $-$ rate out (in kg min^{-1}) now becomes

$$y'(t) = \left(0.045 \frac{\mathrm{kg}}{\mathrm{L}}\right) \left(1000 \frac{\mathrm{L}}{\mathrm{min}}\right) - \left(\frac{y(t)}{5000000 - 2000t} \frac{\mathrm{kg}}{\mathrm{L}}\right) \left(3000 \frac{\mathrm{L}}{\mathrm{min}}\right).$$

In particular, since the output flow is now 2 000 L min^{-1} greater than the input flow, the volume in the swimming pool decreases from its 5 000 000 L capacity to $5000000 - 2000t$ at time $t \in (0, 2500)$, until the swimming pool is completely drained at $t = 2500$. The flow is thus modelled by solutions $y : [0, 2500) \to \mathbb{R}$ to the initial value problem

$$y'(t) = 45 - \frac{3}{5000 - 2t} y, \quad y(0) = 0.$$

This is a linear first-order differential equation, which in standard form becomes

$$y'(t) + \frac{3}{5000 - 2t} y = 45, \quad y(0) = 0.$$

We introduce the integration factor

$$I(t) := e^{\int \frac{3}{5000-2t} \, \mathrm{d}x} = e^{-(3/2) \log |5000 - 2t|} = |5000 - 2t|^{-\frac{3}{2}} = (5000 - 2t)^{-\frac{3}{2}},$$

where the final equality is justified because we are only concerned with $t \in [0, 2500)$. Next, we multiply the differential equation by I and proceed formally to obtain

$$\begin{aligned} I(t)\left[y'(t) + \tfrac{3}{5000-2t}y(t)\right] &= 45(5000-2t)^{-\frac{3}{2}} \\ \implies (Iy)'(t) &= 45(5000-2t)^{-\frac{3}{2}} \\ \implies (Iy)(t) &= 45\textstyle\int (5000-2t)^{-\frac{3}{2}}\,\mathrm{d}t \\ \implies (5000-2t)^{-\frac{3}{2}}y(t) &= 45(5000-2t)^{-\frac{1}{2}} + C \\ \implies y(t) &= 45(5000-2t) + C(5000-2t)^{\frac{3}{2}}, \end{aligned}$$

where $C \in \mathbb{R}$. The initial condition $y(0) = 0$ requires that $C = -\frac{45}{\sqrt{5000}}$. The function $y : [0, 2500) \to \mathbb{R}$ given by $y(t) := 45(5000-2t) - \frac{45}{\sqrt{5000}}(5000-2t)^{\frac{3}{2}}$ for all $t \in [0, 2500)$ is differentiable on $(0, 2500)$ and continuous on $[0, 2500)$, since it is a composition of such functions. It also satisfies the differential equation and initial condition by construction, hence it is a solution of the initial value problem. More generally, since the swimming pool will be empty when $t \geq 2500$, the function $\widetilde{y} : [0, +\infty) \to \mathbb{R}$ given by

$$\widetilde{y}(t) := \begin{cases} 45(5000-2t) - \frac{45}{\sqrt{5000}}(5000-2t)^{\frac{3}{2}}, & t \in [0, 2500) \\ 0, & t \geq 2500 \end{cases}$$

models the mass of salt in the swimming pool after t minutes of mixing for all $t \in [0, +\infty)$.

Q3. Find the general solution of the following homogeneous equations on $\mathbb{R}$, and where specified, find a solution of the initial value problem or boundary value problem.

(a) $y'' - 7y' + 12y = 0$.
(b) $y'' = -64y, \quad y(0) = 0, \quad y'(0) = 3, \quad y : [0, \infty) \to \mathbb{R}$.
(c) $y'' - 2y' + y = 0, \quad y(0) = 1, \quad y(1) = 2, \quad y : [0, 1] \to \mathbb{R}$.

Solution. (a) The characteristic equation $\lambda^2 - 7\lambda + 12 = (\lambda - 3)(\lambda - 4) = 0$ has the two real roots $\lambda = 3$ and $\lambda = 4$. Using Theorem 10.1.4 in the Lecture Notes, the general solution is thus $y(x) := C_1e^{3x} + C_2e^{4x}$ for all $x \in \mathbb{R}$, where $C_1, C_2 \in \mathbb{R}$.

(b) The characteristic equation $\lambda^2 + 64 = 0$ has the complex roots $\lambda = \pm 8i$, so by Theorem 11.3.4 the general solution is $y(x) := C_1 \cos(8x) + C_2 \sin(8x)$ for all $x \in \mathbb{R}$, where $C_1, C_2 \in \mathbb{R}$. The requirement that $y(0) = 0$ implies that $C_1 = 0$, whilst $y'(0) = 3$ requires that $8C_2 = 3$ so $C_2 = \frac{3}{8}$. The differentiable function $y : [0, \infty) \to \mathbb{R}$ given by $y(x) := \frac{3}{8}\sin(8x)$ for all $x \in [0, \infty)$ is thus a solution of the initial value problem.

(c) The characteristic equation $\lambda^2 - 2\lambda + 1 = (\lambda - 1)^2 = 0$ has only the single real root $\lambda = 1$, so by Theorem 11.3.4 the general solution is $y(x) := C_1e^x + C_2xe^x$ for all $x \in \mathbb{R}$, where $C_1, C_2 \in \mathbb{R}$. The requirement that $y(0) = 1$ implies that $C_1 = 1$, whilst $y(1) = 2$ requires that $e + C_2e = 2$ so $C_2 = (2 - e)/e$. The continuous function $y : [0, 1] \to \mathbb{R}$ given by $y(x) := e^x + (\frac{2}{e} - 1)xe^x$ for all $x \in [0, 1]$ is thus a solution of the boundary value problem.

Q4. Find the general solution of the following inhomogeneous equations on $\mathbb{R}$, and where specified, find a solution of the initial value problem or boundary value problem.

(a) $y'' - 2y' + 10y = e^x$.
(b) $y'' + 5y' + 4y = 3 - 2x, \quad y(0) = 0, \quad y'(0) = 0, \quad y : [0, \infty) \to \mathbb{R}$.
(c) $y'' + 9y = x \cos x, \quad y(0) = 1, \quad y(\frac{\pi}{2}) = \frac{1}{32}, \quad y : [0, \frac{\pi}{2}] \to \mathbb{R}$.

Solution. (a) We know from Theorem 10.2.1 in the Lecture Notes that the general solution is given by $y = y_c + y_p$:

- To find the general homogeneous solution y_c, we solve the characteristic equation $\lambda^2 - 2\lambda + 10 = (\lambda-1)^2 + 9 = 0$ to obtain the complex roots $\lambda = 1 \pm 3i$, thus $y_c(x) := C_1 e^x \cos(3x) + C_2 e^x \sin(3x)$ for all $x \in \mathbb{R}$, where $C_1, C_2 \in \mathbb{R}$.
- To find a particular solution y_p of the inhomogeneous equation, we substitute $y_p(x) := Ae^x$ into the differential equation to obtain

$$Ae^x - 2Ae^x + 10Ae^x = e^x.$$

  We equate the coefficients of e^x to find $9A = 1$, so $A = \frac{1}{9}$ and $y_p(x) = \frac{1}{9}e^x$ for all $x \in \mathbb{R}$.

The general solution is thus $y(x) = C_1 e^x \cos(3x) + C_2 e^x \sin(3x) + \frac{1}{9}e^x$ for all $x \in \mathbb{R}$, where $C_1, C_2 \in \mathbb{R}$.

(b) We know from Theorem 10.2.1 in the Lecture Notes that the general solution is given by $y = y_c + y_p$:

- To find the general homogeneous solution y_c, we solve the characteristic equation $\lambda^2 + 5\lambda + 4 = (\lambda+4)(\lambda+1) = 0$ to obtain the roots $\lambda = -4$ and $\lambda = -1$, thus $y_c(x) := C_1 e^{-4x} + C_2 e^{-x}$ for all $x \in \mathbb{R}$, where $C_1, C_2 \in \mathbb{R}$.
- To find a particular solution y_p of the inhomogeneous equation, we substitute $y_p(x) := Ax + B$, $y_p'(x) = A$ and $y_p''(x) = 0$ into the differential equation to obtain

$$0 + 5A + 4(Ax + B) = 3 - 2x.$$

  We equate the coefficients of x and x^0 to find $5A + 4B = 3$ and $4A = -2$, which has the solution $A = -\frac{1}{2}$ and $B = \frac{11}{8}$, so $y_p(x) = -\frac{1}{2}x + \frac{11}{8}$ for all $x \in \mathbb{R}$.

The general solution is thus $y(x) = C_1 e^{-4x} + C_2 e^{-x} - \frac{1}{2}x + \frac{11}{8}$ for all $x \in \mathbb{R}$, where $C_1, C_2 \in \mathbb{R}$. The requirements that $y(0) = 0$ and $y'(0) = 0$ imply that

$$\begin{aligned} C_1 + C_2 + \tfrac{11}{8} &= 0 \\ -4C_1 - C_2 - \tfrac{1}{2} &= 0, \end{aligned}$$

from which it follows that $C_1 = \frac{7}{24}$ and $C_2 = -\frac{5}{3}$. Altogether, the differentiable function $y : [0, \infty) \to \mathbb{R}$ given by

$$y(x) := \tfrac{7}{24}e^{-4x} - \tfrac{5}{3}e^{-x} - \tfrac{1}{2}x + \tfrac{11}{8}$$

for all $x \in [0, \infty)$ is thus a solution of the boundary value problem.

(c) We know from Theorem 10.2.1 in the Lecture Notes that the general solution is given by $y = y_c + y_p$:

- To find the general homogeneous solution y_c, we solve the characteristic equation $\lambda^2 + 9 = 0$ to obtain the complex roots $\lambda = \pm 3i$, thus

$$y_c(x) := C_1 \cos(3x) + C_2 \sin(3x)$$

  for all $x \in \mathbb{R}$, where $C_1, C_2 \in \mathbb{R}$.
- To find a particular solution y_p of the inhomogeneous equation, we substitute $y_p(x) := (Ax + B)\cos(x) + (Cx + D)\sin(x)$, as well as

$$\begin{aligned} y_p'(x) &= (Cx + D + A)\cos(x) + (-Ax - B + C)\sin(x), \\ y_p''(x) &= (-Ax - B + 2C)\cos(x) + (-Cx - D - 2A)\sin(x), \end{aligned}$$

  into the differential equation to obtain

$$\begin{aligned} &[(-Ax - B + 2C) + 9(Ax + B)]\cos(x) \\ &+ [(-Cx - D - 2A) + 9(Cx + D)]\sin(x) = x\cos x. \end{aligned}$$

We equate the coefficients of $\cos(x)$, $x\cos(x)$, $\sin(x)$ and $x\sin(x)$ to find

$$\begin{aligned} 2C+8B&=0\\ 8A&=1\\ -2A+8D&=0\\ 8C&=0, \end{aligned}$$

which has the solution $A=\frac{1}{8}$, $B=0$, $C=0$ and $D=\frac{1}{32}$, so

$$y_p(x)=\tfrac{1}{8}x\cos(x)+\tfrac{1}{32}\sin(x)$$

for all $x\in\mathbb{R}$.

The general solution is thus

$$y(x)=C_1\cos(3x)+C_2\sin(3x)+\tfrac{1}{8}x\cos(x)+\tfrac{1}{32}\sin(x)$$

for all $x\in\mathbb{R}$, where $C_1,C_2\in\mathbb{R}$. The requirements that $y(0)=1$ and $y(\frac{\pi}{2})=\frac{1}{32}$ imply that

$$\begin{aligned} C_1&=1\\ -C_2+\tfrac{1}{32}&=\tfrac{1}{32}, \end{aligned}$$

from which it follows that $C_1=1$ and $C_2=0$. Altogether, the continuous function $y:[0,\frac{\pi}{2}]\to\mathbb{R}$ given by

$$y(x):=\cos(3x)+\tfrac{1}{8}x\cos(x)+\tfrac{1}{32}\sin(x)$$

for all $x\in[0,\frac{\pi}{2}]$ is thus a solution of the boundary value problem.

Extra Questions

EQ1. (a) Find three solutions of the differential equation $y'=\cos\left(3x+\frac{\pi}{3}\right)$ on $\mathbb{R}$.

(b) Find a solution $y:[0,\infty)\to\mathbb{R}$ of the initial value problem

$$x^2y'=y^3,\quad y(0)=0.$$

(c) Find solutions $y:[0,R)\to\mathbb{R}$, for some $R\in[0,\infty]$, of the initial value problem

$$y'=y^2+y-12,\quad y(0)=y_0$$

for each $y_0\in\{2,3,5\}$.

EQ2. Find solutions $y:[0,\infty)\to\mathbb{R}$ of the following initial value problems:

(a) $y'+y=\cos(e^x),\quad y(0)=9.$

(b) $y'+2xy=4x,\quad y(0)=y_0\in\mathbb{R}.$

(c) $xy'=y+x^3+3x^2-2x,\quad y(0)=0.$

EQ3. Suppose that a, b, $c\in\mathbb{R}$ with $a\neq 0$. Let $y_1:[0,1]\to\mathbb{R}$ and $y_2:[0,1]\to\mathbb{R}$ denote solutions of the respective boundary value problems below:

$$\begin{aligned} ay_1''+by_1'+cy_1=0,\quad y_1(0)=1,\quad y_1(1)=3;\\ ay_2''+by_2'+cy_2=0,\quad y_2(0)=2,\quad y_2(1)=-5. \end{aligned}$$

(a) Prove that $y_3:=2y_1+3y_2$ is a solution of the boundary value problem

$$ay''+by'+cy=0,\quad y(0)=8,\quad y(1)=-9.$$

(b) Let α, $\beta\in\mathbb{R}$. Find a solution $y_4:[0,1]\to\mathbb{R}$ of the boundary value problem

$$ay''+by'+cy=0,\quad y(0)=\alpha,\quad y(1)=\beta$$

in terms of y_1 and y_2.

EQ4. Hooke's Law states that a spring with spring constant $k > 0$ exerts a force $F = -ky$ (in Newtons) when stretched a distance y (in metres) from its equilibrium position. Suppose that an object with mass $m = 5$ kg is attached to the end of a spring with spring constant $k = 100$. Combine Hooke's Law with Newton's Law $F = my''$ to formulate initial value problems or boundary value problems, as appropriate, to model each of the following scenarios. In each case, determine the distance $y(t)$ of the object from its equilibrium position at time t (in seconds) after it is released:

(a) The object is released at rest at a distance of 1 m from its equilibrium position.

(b) The object is released at a distance of 1 m from its equilibrium position so that after 1 second it has travelled 0.7 m.

(c) The object is released at rest at a distance of 1 m from its equilibrium position and it is subject to an additional force of $5\sin(2\sqrt{5}t)$ N at time t.