UNIVERSITY OF BIRMINGHAM

School of Physics and Astronomy

DEGREE OF B.Sc. & M.Sci. WITH HONOURS

FIRST-YEAR EXAMINATION

03 19748 - CM
03 19718 - QM, O&W

LC CLASSICAL MECHANICS & RELATIVITY
QUANTUM MECHANICS / OPTICS & WAVES

SEMESTER 1 EXAMINATIONS 2021/22

Time Allowed: 3 hours

Answer five questions from Section 1 and three questions from Section 2.

Section 1 counts for 25% of the marks for the examination. Answer ***all five*** questions in this Section.
Section 2 consists of three questions and carries 75% of the marks.
Answer ***all three*** questions in this Section. Note that each question has two parts, of which only ***one part*** should be answered. If you answer both parts, credit will only be given for the best answer.

The approximate allocation of marks to each part of a question is shown in brackets **[ ]**.

PLEASE USE A SEPARATE ANSWER BOOK FOR SECTION 1 AND SECTION 2 QUESTIONS.

Calculators may be used in this examination but must not be used to store text. Calculators with the ability to store text should have their memories deleted prior to the start of the examination.

A formula sheet and a table of physical constants and units that may be required will be found at the end of this question paper.

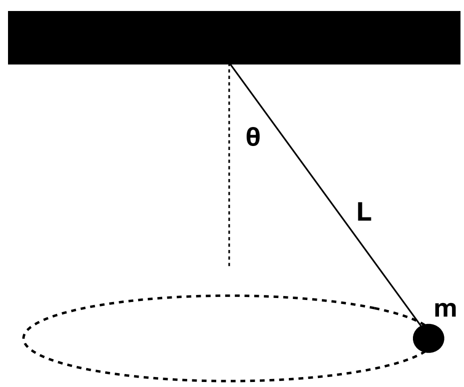
SECTION 1

*Answer **all five** questions in this Section.*

1. A pulse of muons travels at a speed of $0.995\,c$ in a straight line. The muons reach a particle detector, which detects the presence of 800 muons (no muon is absorbed in the process). A second particle detector is located $5\,\mathrm{km}$ away from the first one along the particles' trajectory. How many muons reach this second detector?

   In its rest-frame, the muon lifetime is $\tau = 2.2\times10^{-6}\,\mathrm{s}$. Particles decay according to $N(t) = N_0 e^{-t/\tau}$, where N_0 and $N(t)$ are the number of particles, initially and at proper time t, respectively. Assume both the detectors are 100% efficient. **[5]**

2. In a conical pendulum, a bob of mass m is attached at the end of a thin, inextensible wire of length L (see figure below). The bob moves in a horizontal circle with constant speed, with the wire making a fixed angle, θ, with the vertical. Find the period of the pendulum, *i.e.* the time for one revolution of the bob as a function of m, L and θ. **[5]**



3. A string of 0.01 kg m^{-1} is held fixed at one end and passes over a pulley at the other. The end of the string which passes over the pulley has a weight of 10 kg attached. The length of the string that is free to vibrate between the fixed end and the pulley is 30 cm. The string is plucked and oscillates in the fundamental mode with a maximum transverse displacement of 1 cm.

   (a) What is the frequency of the first harmonic?

   (b) The standing wave can be expressed as the superposition of two travelling waves. Write down expressions for the two travelling waves.

   (c) The assembly is placed at the rear end of a carriage of a train travelling at 100 km/h. What frequency would a passenger in the front end of the same carriage observe?

[5]

4. Unpolarised light, travelling through air (n_{air}=1), is incident on the top surface of a thin sheet of glass with refractive index $n_{glass} = 1.4$. The glass sheet has a thickness of 0.1 μm and the bottom surface of the glass is also in contact with air. The incident angle of the light beam is adjusted such the polarisation of the reflected ray from the top surface (ray 1) is maximised.

   (a) What is the incident angle of the light beam measured with respect to the normal to the glass surface?

   (b) Noting that the ray is refracted as it passes into the glass and then reflected from the second surface (ray 2), find the wavelength of light for which constructive interference of the two rays, 1 and 2, is maximised?

[5]

5. The line in the hydrogen emission spectrum corresponding to a transition from $n = 4$ to $n = 2$ has wavelength 486 nm. Use this information to estimate the value of the ionisation energy of hydrogen from its ground state. Express your answer in units of eV.

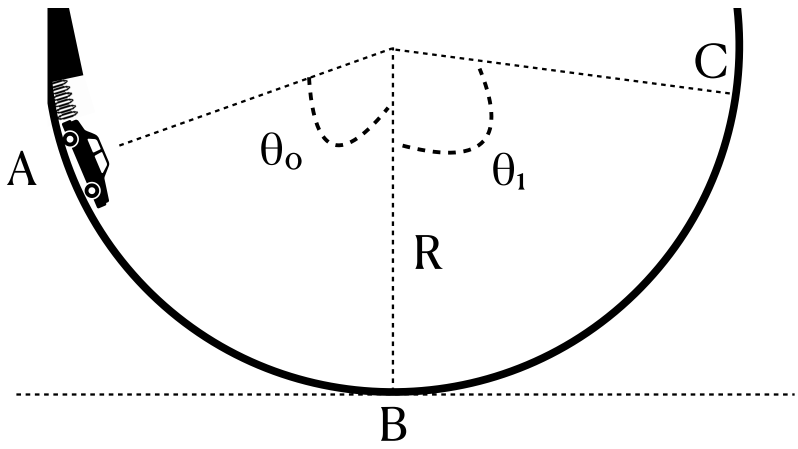
[5]

SECTION 2

*Answer **all three** questions in this Section. Note that each question has two parts, of which only **one part** should be answered. If you answer both parts, credit will only be given for the best answer.*

6. **EITHER (Part A)**

A toy-car of mass $50\,\mathrm{g}$ is on a vertical and frictionless circular track of radius $R = 2\,\mathrm{m}$, as shown in the diagremme below. The car is initially pushed along the track to compress a spring, with spring constant $900\,\mathrm{N\,m^{-1}}$, by $2\,\mathrm{cm}$. The car is then released from rest at point A, at an angle $\theta_0 = 35°$ to the vertical.



(a) What is the total energy of the car in A just before it is released? Assume the zero of the gravitational potential energy to be the bottom of the track (B). **[3]**

(b) What is the kinetic energy and speed of the car when it goes through B at the bottom of the track? **[3]**

(c) What is the total work from A to B done by the normal force? **[2]**

(d) What is the total work from A to B done by the gravitational force? **[2]**

(e) The car passes point B and climbs up the other side of the track as far as point C, which is at an angle θ_1 with respect to the vertical. Find θ_1. **[5]**

(f) Just before the toy-car starts to slide back from C to A, a child spills honey on the track. The coefficient of kinetic friction between the car and the track is now $\mu = 0.25$.

   i. Calculate the work done by friction from C to B. **[4]**

   ii. Calculate the speed of the car in B. **[6]**

OR (Part B)

A loading ramp of length $L = 9\,\mathrm{m}$ is tilted upwards at an angle $\theta = 37^\circ$ with respect to the horizontal.

A crate of mass $M = 80\,\mathrm{kg}$ is pulled from the bottom to the top of the the ramp along its sloping surface by a rope (of negligible mass) under constant tension $T = 900\,\mathrm{N}$ that is parallel to the ramp. The coefficient of friction between the crate and the ramp is $\mu = 0.25$, and the crate is initially at rest at the bottom of the ramp.

(a) What work is done on the crate by the tension T? **[2]**

(b) What work is done on the crate by the force of friction? **[2]**

(c) What is the increase in potential energy when the crate is at the top of the ramp? **[2]**

(d) Using the results above, what is the increase in kinetic energy and the speed of the crate at the top of the ramp? **[5]**

(e) By considering all forces acting on the crate, calculate its acceleration. Use this acceleration to calculate the crate's speed at the top of the ramp. **[4]**

(f) At the top of the ramp the crate is stopped. Unfortunately, the rope suddenly breaks, and the crate then slides back down the ramp. Half-way down, it hits another crate of mass $m = 20\,\mathrm{kg}$, which is initially at rest.

i. If the two crates stick together, how fast do they move just after the collision? **[7]**

ii. How much energy is dissipated during the collision? **[3]**

7. **EITHER (Part A)**

(a) The equation for light refracted from a glass surface, of radius of curvature r, when incident from air is

$$\frac{1}{s} + \frac{n_g}{s'} = \frac{n_g - 1}{r},$$

where n_g is the refractive index of the glass, s and s' are the object and image distance from the glass surface and it is assumed the refractive index of air, n_a = 1. Show that lens formula for a thin lens is given by

$$\frac{1}{s} + \frac{1}{s'} = \frac{1}{f}.$$

[8]

(b) What is the radius of curvature of a *biconcave* diverging lens (both surfaces are concave), which has equal magnitude radii of curvature for both faces, with a focal length of -20 cm? The refractive index of the glass from which the lens is constructed is n_g = 1.4. **[4]**

(c) An object is placed 10 cm to the left of the lens in part (b). Where is the image formed and what is the magnification? Sketch a ray diagram. **[6]**

(d) A converging lens, of focal length 20 cm, is placed to the right of the diverging lens in part (b). Where must this second lens be placed, with respect to the first lens, to create a real image 40 cm away from the second lens? What is the overall magnification? Draw the ray diagram for the two lens system. **[7]**

OR (Part B)

(a) By considering the transverse displacement y of a string of mass per unit length μ, held under tension T along the x-axis, derive the wave-equation

$$\frac{\partial^2 y}{\partial x^2} = \frac{\mu}{T}\frac{\partial^2 y}{\partial t^2}.$$

You may assume that the displacement y is small and small-angle approximations apply. **[8]**

(b) A string is held under tension T and has a length L. The mass per unit length has a form

$$\mu(x) = \mu_0 \left[(2L - x)/L\right].$$

Derive an expression for the length of time an impulse starting at $x = 0$ takes to reach the point $x = L$. **[9]**

(c) Two strings of length L, but different masses per unit length, are joined together and held under tension T. The mass per unit length of string 1, μ_1, is 4 times the mass per unit length of that of string 2, μ_2. An impulse is started at the left hand end of string 1 and travels across the join to the right hand end of string 2, whose end is held fixed. The reflected impulse then returns along the two-string system. If the amplitude of the initial impulse is A, what is the amplitude of the *first two* pulses to be reflected to the left end of string 1? **[8]**

8. **EITHER (Part A)**

(a) i. A photon of energy E scatters off a free electron of mass m_e, which is initially at rest. Show that the maximum kinetic energy of the electron after scattering is given by

$$T_{max} = \frac{E^2}{E + \frac{m_e c^2}{2}}.$$

You may use without proof the Compton equation for the shift in the wavelength, $\delta\lambda$, of a photon scattered through an angle θ from a free electron:

$$\delta\lambda = \frac{h}{m_e c}\left(1 - \cos\theta\right).$$

[8]

ii. When $E \gg m_e c^2$ show that the *minimum* energy of the scattered photon tends to a constant value. Express that value in units of keV. **[5]**

iii. If visible light were scattered from free electrons, explain whether the shift in wavelength would be observable to the human eye. **[2]**

(b) i. When $30\ \mathrm{eV}$ electrons are incident upon a crystalline silicon sample, peaks in scattered intensity are observed as the crystal is rotated. If the largest angle between the incident and scattered beams at which a peak is observed is $138\ \mathrm{degrees}$, calculate the lattice spacing and find the angles at which all further peaks in the scattered intensity would be observed. **[8]**

ii. What photon energy would give rise to peaks in the scattered intensity at the same angles? **[2]**

OR (Part B)

(a) The time-independent Schrödinger equation (TISE) in one dimension is

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + V(x)\psi(x) = E\psi(x).$$

Using the TISE as an example, explain the terms *eigenvalue equation*, *eigenfunction* and *eigenvalue* and identify the *operators* in the equation. **[5]**

(b) A particle of mass m is bound in an infinite one-dimensional potential well, which has potential $V(x) = 0$ in the region $0 \leq x \leq L$ and $V(x) = \infty$ elsewhere.

i. Show that the spatial wave function $\psi(x) = A\sin(kx)$ is a solution of the time independent Schrödinger equation inside the well. Hence find an expression for the total energy of the particle E in terms of the wave number k. **[3]**

ii. State and explain the boundary condition that the wave function must satisfy at the boundaries of the well. Hence show that there are an infinite number of bound states in the well and find an expression for the allowed values of k in terms of the width of the well. **[6]**

iii. Find the *expectation value* of the momentum of the particle inside the well and hence show that it is the same for any state. Explain the physical significance of the value you obtain. **[5]**

(c) Now consider a particle of mass m that is bound in a finite one-dimensional potential well, which has potential $V(x) = 0$ in the region $0 \leq x \leq L$ and $V(x) = V_0$ elsewhere, where V_0 is a positive constant.

i. Sketch the probability density distribution of the ground state (lowest) energy solution, indicating the expected functional form in different regions of x. **[4]**

ii. What can you say about the energy of the ground state in the finite well compared to an infinite well of the same width? (Hint: consider how the ground state wave function changes as $V_0 \rightarrow \infty$.) **[2]**

Formula Sheet

Useful Formulae for Classical Mechanics and Relativity 1

Work
$W = \int_1^2 \vec{F} \cdot d\vec{s}$

Kinetic energy
$E_k = \frac{1}{2}mv^2$

Work-energy theorem
$W = E_{k_2} - E_{k_1}$

Potential energy
$\vec{F} = -\nabla U$

Energy conservation
$W_{\text{ex}} + U_1 + E_{k_1} = U_2 + E_{k_2}$

Power
$P = \vec{F} \cdot \vec{v}$

Linear momentum
$\vec{p} = m\vec{v}$

Impulse
$\vec{J} = \int_1^2 \vec{F} dt = \vec{p}_2 - \vec{p}_1$

Torque
$\vec{\tau} = \vec{r} \times \vec{F}$

Angular momentum
$\vec{L} = \vec{r} \times \vec{p}$

Newton's law of gravitation
$\vec{F} = -G\frac{m_1 m_2}{r^2}\hat{r}$

Gravitational potential energy
$U = -G\frac{m_1 m_2}{r}$

Acceleration in uniform circular motion
$a = \frac{v^2}{R} = \frac{4\pi^2 R}{T^2}$

Restoring force of a spring:
$\vec{F} = -k\vec{x}$

Elastic potential energy
$U = \frac{1}{2}kx^2$

Angular frequency (simple harmonic motion)
$\omega^2 = \frac{k}{m}$

Period of simple pendulum
$T = 2\pi\sqrt{\frac{L}{g}}$

Angular frequency of damped oscillations
$\omega' = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}$

Amplitude of driven oscillator
$A = \frac{F_{\max}}{\sqrt{(k-m\omega_d^2)^2 + b^2\omega_d^2}}$

Lorentz factor
$\gamma = \frac{1}{\sqrt{1-u^2/c^2}}$

Time dilation
$\Delta t = \gamma \Delta t_0$

Length contraction
$\Delta l = \Delta l_0/\gamma$

Lorentz transformations:
$x' = \gamma\left(x - ut\right)$
$t' = \gamma\left(t - ux/c^2\right)$
$v_x' = \frac{v_x - u}{1 - uv_x/c^2}$

Doppler effect
$f = \sqrt{\frac{1+u/c}{1-u/c}} f_0$

Useful Formulae for Optics and Waves

Equation of a travelling wave:

$$y(x,t) = A\sin(kx - \omega t)$$

Equation of a standing wave:

$$y(x,t) = 2A\cos(\omega t)\sin(kx)$$

Wave equation:

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}$$

Velocity of a wave on a string:

$$v = \sqrt{\frac{T}{\mu}}$$

Velocity of a wave in a gas:

$$v = \sqrt{\frac{\gamma RT}{M}}$$

Kinetic energy of waves on string:

$$KE = \frac{1}{2}\mu\left(\frac{\partial y}{\partial t}\right)^2$$

Potential energy of waves on string:

$$PE = \frac{1}{2}T\left(\frac{\partial y}{\partial x}\right)^2$$

Reflection and transmission amplitudes:

$$B = \frac{2k_1}{k_1 + k_2}A, \;\; C = \frac{k_1 - k_2}{k_1 + k_2}A$$

Phase velocity:

$$v_p = \frac{\omega}{k}$$

Group velocity:

$$v_g = \frac{\Delta\omega}{\Delta k}$$

Doppler shift for sound:

$$f_r = \frac{v \pm u_r}{v \mp u_s}f_s$$

Doppler shift for light (for $v << c$):

$$\frac{\Delta f}{f} = \frac{v}{c}$$

Decibel scale:

$$\beta = 10\log\left(\frac{I}{I_0}\right)$$

Snell's law:

$$n_1\sin(\theta_1) = n_2\sin(\theta_2)$$

Mirror equation:

$$\frac{1}{s} + \frac{1}{s'} = \frac{2}{r}$$

Focus for curved surface:

$$\frac{n_1}{s} + \frac{n_2}{s'} = \frac{n_2 - n_1}{r}$$

Thin lens formula 1:

$$\frac{n_m}{s} + \frac{n_l}{s'} = (n_l - n_m)\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$$

Thin lens formula 2:

$$\frac{1}{s} + \frac{1}{s'} = \frac{1}{F}$$

Malus's law:

$$I = I_0\cos^2(\theta)$$

Diffraction from N slits:

$$I = I_0\frac{\sin^2(N\delta/2)}{\sin^2(\delta/2)}$$

Diffraction from single slit:

$$I = I_0\frac{\sin^2(\phi/2)}{(\phi/2)^2}$$

Physical Constants and Units

Acceleration due to gravity	g	9.81 m s^{-2}
Gravitational constant	G	6.674×10^{-11} N m^2 kg^{-2}
Ice point	T_{ice}	273.15 K
Avogadro constant	N_A	6.022×10^{23} mol^{-1}
		[*N.B.* 1 mole $\equiv 1$ *gram-molecule*]
Gas constant	R	8.314 J K^{-1} mol^{-1}
Boltzmann constant	k, k_B	1.381×10^{-23} J K^{-1} $\equiv 8.62 \times 10^{-5}$ eV K^{-1}
Stefan constant	σ	5.670×10^{-8} W m^{-2} K^{-4}
Rydberg constant	R_∞	1.097×10^{7} m^{-1}
	$R_\infty hc$	13.606 eV
Planck constant	h	6.626×10^{-34} J s $\equiv 4.136 \times 10^{-15}$ eV s
$h/2\pi$	$\hbar$	1.055×10^{-34} J s $\equiv 6.582 \times 10^{-16}$ eV s
Speed of light *in vacuo*	c	2.998×10^{8} m s^{-1}
	$\hbar c$	197.3 MeV fm
Charge of proton	e	1.602×10^{-19} C
Mass of electron	m_e	9.109×10^{-31} kg
Rest energy of electron		0.511 MeV
Mass of proton	m_p	1.673×10^{-27} kg
Rest energy of proton		938.3 MeV
One atomic mass unit	u	1.66×10^{-27} kg
Atomic mass unit energy equivalent		931.5 MeV
Electric constant	ϵ_0	8.854×10^{-12} F m^{-1}
Magnetic constant	μ_0	$4\pi \times 10^{-7}$ H m^{-1}
Bohr magneton	μ_B	9.274×10^{-24} A m^2 (J T^{-1})
Nuclear magneton	μ_N	5.051×10^{-27} A m^2 (J T^{-1})
Fine-structure constant	$\alpha = e^2/4\pi\epsilon_0\hbar c$	7.297×10^{-3}= 1/137.0
Compton wavelength of electron	$\lambda_c = h/m_e c$	2.426×10^{-12} m
Bohr radius	a_0	5.2918×10^{-11} m
angstrom	Å	10^{-10} m
barn	b	10^{-28} m^2
torr (mm Hg at 0 °C)	torr	133.32 Pa (N m^{-2})