A34927

No Calculator

UNIVERSITY OF BIRMINGHAM

School of Mathematics

Programmes in the School of Mathematics — First Examination

Programmes involving Mathematics — First Examination

1SAS 06 34047 Level C

LC Sequences and Series

May/June Examinations 2023-24

One Hour and Thirty Minutes

Full marks will be obtained with complete answers to BOTH questions. Each question carries equal weight. You are advised to initially spend no more than 45 minutes on each question and then to return to any incomplete questions if you have time at the end. An indication of the number of marks allocated to parts of questions is shown in square brackets.

No calculator is permitted in this examination.

Section A

1. (a) (i) Define what it means for a sequence (a_n) of real numbers to *tend to infinity*.

(ii) Using the definition show that the sequence (a_n) given by $a_n = n^3$ tends to infinity.

[5]

(b) (i) Define what it means for a sequence (a_n) of real numbers to *converge to a real number ℓ*.

(ii) Using the definition show that the sequence (a_n) given by

$$a_n = \frac{n^2+3n+1}{2n^2-n+1}$$

converges to $\frac{1}{2}$.

[6]

(c) Which of the following series converge? Justify your answers.

(i)

$$\sum_{n=1}^{\infty} \frac{1}{n^{3/2}+2n+1}.$$

(ii)

$$\sum_{n=1}^{\infty} \frac{2^n(n!)^2}{(2n)!}.$$

(iii)

$$\sum_{n=1}^{\infty} \left(\frac{n+3}{2n+1}\right)^n.$$

(iv)

$$\sum_{n=1}^{\infty} \frac{\cos(n)}{n^2}.$$

(v)

$$\sum_{n=1}^{\infty} (-1)^n.$$

[In this question you may appeal to standard limit theorems for sequences and convergence tests for series.] [14]

Section B

2. (a) A sequence (a_n) of positive real numbers is given recursively by the formula

$$a_{n+1} = \frac{1}{5}(1+a_n)^2,$$

where $a_1 = 0$.

(i) Prove that (a_n) is increasing.

(ii) Prove that $a_n \leq 2$ for all n.

(iii) State a theorem that allows you to conclude that (a_n) converges.

(iv) Find the limit of (a_n), carefully justifying any assertions that you make.

[10]

(b) (i) Define what it means for a series

$$\sum_{n=1}^{\infty} a_n$$

of real numbers to *converge*.

(ii) Prove that if a series

$$\sum_{n=1}^{\infty} a_n$$

converges, then

$$\sum_{n=N+1}^{2N} a_n \to 0 \quad \text{as} \quad N \to \infty.$$

[7]

(c) Find the radius of convergence of the power series

$$\sum_{n=0}^{\infty} \frac{n}{n^2+1} x^n.$$

For which real numbers x does this series converge? Justify any assertions that you make. [8]

A34927

LC Sequences and Series

Do not complete the attendance slip, fill in the front of the answer book or turn over the question paper until you are told to do so.

Important Reminders

- Coats and outer-wear should be placed in the designated area.
- Unauthorised materials (e.g. notes or Tippex) **MUST** be placed in the designated area.
- Check that you **DO NOT** have any unauthorised materials with you (e.g. in your pockets, pencil case).
- Mobile phones and smart watches **MUST** be switched off and placed in the designated area or under your desk. They must not be left on your person or in your pockets.
- You are **NOT** permitted to use a mobile phone as a clock. If you have difficulty in seeing a clock, please alert an Invigilator.
- You are **NOT** permitted to have writing on your hand, arm or other body part.
- Check that you do not have writing on your hand, arm or other body part – if you do, you must inform an Invigilator immediately.
- Alert an Invigilator immediately if you find any unauthorised item upon you during the examination.

Any students found with non-permitted items upon their person during the examination, or who fail to comply with Examination rules may be subject to the Student Conduct procedures.