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No Calculator

UNIVERSITY OF BIRMINGHAM

School of Mathematics

Programmes in the School of Mathematics — First Examination

Programmes involving Mathematics — First Examination

1SAS 06 34047 Level C

LC Sequences and Series

January Examinations 2022-23

One Hour and Thirty Minutes

Full marks will be obtained with complete answers to BOTH questions. Each question carries equal weight. You are advised to initially spend no more than 45 minutes on each question and then to return to any incomplete questions if you have time at the end. An indication of the number of marks allocated to parts of questions is shown in square brackets.

No calculator is permitted in this examination.

Section A

1. (a) (i) Define what it means for a sequence of real numbers (a_n) to *converge to a real number* ℓ.

(ii) Using the definition above, prove that the sequence (a_n) given by

$$a_n = \frac{n^2+2n+1}{2n^2+3n-1}$$

converges to $\frac{1}{2}$.

[6]

(b) (i) Define what it means for a sequence of real numbers (a_n) to *tend to infinity*.

(ii) Using the definition above, prove that the sequence (a_n) given by

$$a_n = \frac{2n^2+n+1}{3n+1}$$

tends to infinity.

[5]

(c) Establish which of the series below converge, justifying your assertions.

(i)

$$\sum_{n=1}^{\infty} \frac{2^n}{3^n}.$$

(ii)

$$\sum_{n=1}^{\infty} \frac{(2n)!}{3^n (n!)^2}.$$

(iii)

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}}.$$

(iv)

$$\sum_{n=1}^{\infty} \frac{2n+(-1)^n}{n^3+2}.$$

(v)

$$\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^n.$$

In this question you may appeal to standard results on the convergence of sequences and series, provided you make it clear that you are doing so. [14]

Section B

In this section you may appeal to standard results on the convergence of sequences and series, provided you make it clear when you do so.

2. (a) A sequence of nonnegative real numbers (a_n) is defined recursively by the formula

$$a_{n+1} = \frac{2+a_n}{3+a_n},$$

where $a_1 = 1$.

(i) Prove that

$$a_{n+2} - a_{n+1} = \frac{a_{n+1} - a_n}{(3+a_n)(3+a_{n+1})}$$

for all $n \in \mathbb{N}$.

(ii) Use induction to show that (a_n) is decreasing.

(iii) Show that (a_n) converges, and find its limit, carefully justifying any assertions that you make.

(iv) Show further that

$$a_{n+1} - a_n \leq \frac{1}{4}\left(\frac{1}{9}\right)^{n-1}$$

for all $n \in \mathbb{N}$. [12]

(b) (i) Define what it means for a series of real numbers

$$\sum_{n=1}^{\infty} a_n$$

to *converge*.

(ii) Use the definition to show that the series

$$\sum_{n=1}^{\infty} \frac{1}{n^2 - \frac{1}{4}}$$

converges, and find its sum. [6]

(c) Suppose (a_n) is a sequence of real numbers such that $|a_n|^{1/n} \to 1$. Prove that the series

$$\sum_{n=1}^{\infty} a_n x^n$$

converges whenever $|x| < 1$. [7]

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LC Sequences and Series

Do not complete the attendance slip, fill in the front of the answer book or turn over the question paper until you are told to do so.

Important Reminders

- Coats and outer-wear should be placed in the designated area.
- Unauthorised materials (e.g. notes or Tippex) **MUST** be placed in the designated area.
- Check that you **DO NOT** have any unauthorised materials with you (e.g. in your pockets, pencil case).
- Mobile phones and smart watches **MUST** be switched off and placed in the designated area or under your desk. They must not be left on your person or in your pockets.
- You are **NOT** permitted to use a mobile phone as a clock. If you have difficulty in seeing a clock, please alert an Invigilator.
- You are **NOT** permitted to have writing on your hand, arm or other body part.
- Check that you do not have writing on your hand, arm or other body part – if you do, you must inform an Invigilator immediately.
- Alert an Invigilator immediately if you find any unauthorised item upon you during the examination.

Any students found with non-permitted items upon their person during the examination, or who fail to comply with Examination rules may be subject to the Student Conduct procedures.