Solution Sheet 2: Maps

1. Find all the 1-cycles for the map

$$x_{n+1} = a - x_n^2$$

Answer 1. The 1-cycles satisfy

$$x = a - x^2 \quad \Rightarrow \quad \left(x + \frac{1}{2}\right)^2 = \frac{1}{4} + a \quad \Rightarrow \quad x = -\frac{1}{2} \pm \frac{1}{2}\left[1 + 4a\right]^{\frac{1}{2}}$$

The 1-cycles only exist when $a > -\frac{1}{4}$.

2. Find the stability region of a for all the 1-cycles of the map

$$x_{n+1} = a - x_n^2$$

Answer 2. To find stability in general we need

$$-1 < S \equiv \frac{df}{dx}(x) < 1$$

and for this particular map

$$-1 < S = -2x < 1 \quad \Rightarrow \quad -1 < 1 \mp \left[1 + 4a\right]^{\frac{1}{2}} < 1 \quad \Rightarrow \quad -2 < \mp\left[1 + 4a\right]^{\frac{1}{2}} < 0$$

and so only the top sign is possible and gives

$$0 < 1 + 4a < 4 \quad \Rightarrow \quad -\frac{1}{4} < a < \frac{3}{4}$$

The stable limit cycle has $x = -\frac{1}{2} + \frac{1}{2}\left[1 + 4a\right]^{\frac{1}{2}}$ and the lower stability limit is the merging of the two 1-cycles.

3. Find all the 2-cycles for the map

$$x_{n+1} = a - x_n^2$$

Answer 3. We need to solve the equations

$$y = a - x^2 \qquad\qquad x = a - y^2$$

Subtracting the two equations gives

$$y - x = y^2 - x^2 = (y - x)(y + x)$$

and so if we avoid 1-cycles, $x \neq y$, we have $x + y$=1. Adding the two equations gives

$$y + x = 2a - x^2 - y^2 = 2a - (x + y)^2 + 2xy \quad \Rightarrow \quad xy = \frac{1}{2}\left[(x + y)^2 + (x + y)\right] - a = 1 - a$$

and then we can construct a quadratic equation with these points as solutions

$$(z-x)(z-y) = z^2 - z(x+y) + xy = z^2 - z + 1 - a = \left(z - \frac{1}{2}\right)^2 - \frac{1}{4} + 1 - a$$

$$= \left(z - \frac{1}{2}\right)^2 - \frac{1}{4}[4a-3] = 0 \quad \Rightarrow \quad x, y = \frac{1}{2} \pm \frac{1}{2}[4a-3]^{\frac{1}{2}}$$

The 2-cycles only exist when $a > \frac{3}{4}$.

4. Find the stability region of a for all the 2-cycles of the map

$$x_{n+1} = a - x_n^2$$

Answer 4. To find the stability of a 2-cycle we need

$$S = \frac{df}{dx}(x)\frac{df}{dx}(y) = 4xy$$

for the two points which make up the 2-cycle. This gives

$$-1 < S = 4(1-a) < 1 \quad \Rightarrow \quad \frac{3}{4} < a < \frac{5}{4}$$

The lower limit is the merge into the stable 1-cycle.

5*. Find all the 3-cycles for the map

$$x_{n+1} = a - x_n^2$$

Answer 5. To find the 3-cycles we need to solve the equations

$$y = a - x^2 \qquad z = a - y^2 \qquad x = a - z^2$$

Obviously we need to target the coefficients of the cubic equation

$$(w-x)(w-y)(w-z) = w^3 - (x+y+z)w^2 + (yz+zx+xy)w - xyz \equiv w^3 - b_1 w^2 + b_2 w - b_3$$

We can sum the equations

$$x + y + z = 3a - x^2 - y^2 - z^2 \quad \Rightarrow \quad b_1 = 3a - b_1^2 + 2b_2$$

We can multiply by the sum of the other two roots and sum

$$(y+z)y + (z+x)z + (x+y)x = a(y+z+z+x+x+y) - (y+z)x^2 - (z+x)y^2 - (x+y)z^2$$

$$\Rightarrow \quad b_1^2 - b_2 = 2ab_1 - b_1 b_2 + 3b_3$$

The third equation can be obtained by taking the difference between neighbouring equations

$$z - y = x^2 - y^2 \qquad x - z = y^2 - z^2 \qquad y - x = z^2 - x^2$$

producting them together and assuming that there is no 1-cycle

$$(x + y)(y + z)(z + x) = -1 \quad \Rightarrow \quad b_1 b_2 - b_3 = -1$$

We can then eliminate b_3 using the third equation, eliminate b_2 using the first equation to get a cubic equation for b_1

$$\left(b_1 + \frac{3}{2}\right)\left(b_1^2 - b_1 - a + 2\right) = 0$$

and the relevant solutions are

$$b_1 = \frac{1}{2} \pm \frac{1}{2}\left[4a - 7\right]^{\frac{1}{2}}$$

The 3-cycles only exist when $a > \frac{7}{4}$.

6*. Find the stability region of a for all the 3-cycles of the map

$$x_{n+1} = a - x_n^2$$

Answer 6. To determine stability we need

$$S = \frac{df}{dx}(x)\frac{df}{dx}(y)\frac{df}{dx}(z) = -8xyz = -8b_3 = -8 - 4b_1(b_1^2 + b_1 - 3a) = 8 - 4a \pm 4a\left[4a - 7\right]^{\frac{1}{2}}$$

where a lot of algebra is required to do this calculation. We have the quadratic equation for b_1

$$b_1^2 = b_1 + a - 2$$

which we can substitute in

$$S = -8 - 4b_1\left[2b_1 - 2a - 2\right] = -8 + 8ab_1 - 8\left[b_1^2 - b_1\right] = -8 + 8ab_1 - 8\left[a - 2\right]$$

$$= 8 - 8a + 8ab_1 = 8 - 8a + 4a\left(1 \pm \left[4a - 7\right]^{\frac{1}{2}}\right) = 8 - 4a \pm 4a\left[4a - 7\right]^{\frac{1}{2}}$$

where we substituted the quadratic a second time. Clearly the bottom sign leads to the stable solution. Clearly, when S=1, $a = \frac{7}{4}$ provides the start of stability and when $S = -1$ we find a nasty cubic equation to solve which requires a computer. When $4a = 7 + \frac{1}{9}$ we find

$$S \mapsto \frac{8}{9} - \left[7 + \frac{1}{9}\right]\frac{1}{3} = -\frac{40}{27}$$

and when $4a = 7 + \frac{1}{16}$ we find

$$S \mapsto \frac{15}{16} - \left[7 + \frac{1}{16}\right]\frac{1}{4} = -\frac{27}{64}$$

which bracket the solution.

7*. Find all the 4-cycles for the map

$$x_{n+1} = a - x_n^2$$

Answer 7. To find the 4-cycles we need to solve the equations

$$y = a - x^2 \qquad z = a - y^2 \qquad w = a - z^2 \qquad x = a - w^2$$

Taking the difference of the first and third equations and then the difference of the second and fourth equations give

$$y - w = z^2 - x^2 \qquad z - x = w^2 - y^2$$

and then the product of these two and the assumption that there are no 2-cycles gives

$$(z + x)(y + w) = -1$$

The sum of the first and third equations and then the sum the second and fourth equations gives

$$(y+w) = 2a-x^2-z^2 = 2a-(x+z)^2+2xz \qquad x+z = 2a-y^2-w^2 = 2a-(y+w)^2+2yw$$

which should be considered as equations for the products

$$2xz = (x + z)^2 + (y + w) - 2a \qquad 2yw = (y + w)^2 + (x + z) - 2a$$

The final equation comes from taking the first difference equation times $(z - x)$ and then adding the second difference equation times $(w - y)$ to give

$$(z-x)^2(z+x)+(w-y)^2(w+y) = 0 = \left[(z + x)^2 - 4xz\right](z+x)+\left[(w + y)^2 - 4wy\right](w+y)$$

and then eliminating the products

$$\left[(z + x)^2 - 2\left((x + z)^2 + (y + w) - 2a\right)\right](z + x)$$

$$+\left[(y + w)^2 - 2\left((y + w)^2 + (x + z) - 2a\right)\right](y + w) = 0$$

which simplifies to

$$(x + z)^3 + (y + w)^3 = 4a(x + z + y + w) - 4(x + z)(y + w) = 4a(x + z + y + w) + 4$$

We can also deduce that

$$(x + z + y + w)^3 = (x + z)^3 + (y + w)^3 + 3(x + z)(y + w)(x + z + y + w)$$

$$= (x + z)^3 + (y + w)^3 - 3(x + z + y + w)$$

and then we find an equation for $(x+z+y+w) \equiv b_1$

$$b_1^3 - (4a-3)b_1 - 4 = 0$$

We solve this for b_1 then we solve

$$(q-x-z)(q-y-w) = q^2 - b_1 q - 1 = 0$$

to provide $x+z$ and $y+w$ and finally

$$(p-x)(p-z) = p^2 - (x+z)p + xz = 0 \qquad (p-y)(p-w) = p^2 - (y+w)p + yw = 0$$

to find all the points. Note that when $a = \frac{5}{4}$ then b_1=2 is a solution, providing the appearance of a 4-cycle.

8*. Find the stability region of a for all the 4-cycles of the map

$$x_{n+1} = a - x_n^2$$

Answer 8. The stability is controlled by

$$S = \frac{df}{dx}(x)\frac{df}{dx}(y)\frac{df}{dx}(z)\frac{df}{dx}(w) = 16xyzw$$

$$= 4\left[(x+z)^2 + (y+w) - 2a\right]\left[(y+w)^2 + (x+z) - 2a\right] = 4\Big[(x+z)^2(y+w)^2 + (x+z)^3$$

$$+(y+w)^3 + (x+z)(y+w) - 2a(x+z)^2 - 2a(y+w)^2 - 2a(x+z+y+w) + 4a^2\Big]$$

and then in terms of b_1

$$S = 4(b_1^3 + 3b_1) - 8a(b_1^2 + 2) - 8ab_1 + 16a^2 = 4b_1^3 - 8ab_1^2 + (12-8a)b_1 + 16a^2 - 16a$$

which together with the governing equation

$$b_1^3 - (4a-3)b_1 - 4 = 0$$

can be used to provide a in terms of S. The algebra is unpleasant. The stability is controlled by

$$b_1^3 - 2ab_1^2 + (3-2a)b_1 + 4a^2 - 4a - \frac{S}{4} = 0$$

and subtraction provides

$$2ab_1^2 - 2ab_1 - 4a^2 + 4a - 4 + \frac{S}{4} = 0 \quad \Rightarrow \quad b_1^2 - b_1 - 2a + 2 - \frac{2}{a} + \frac{S}{8a} = 0$$

Multiplying this by b_1 and subtracting gives

$$b_1^2 - \left(2a - 1 - \frac{2}{a} + \frac{S}{8a}\right)b_1 - 4 = 0$$

and then subtracting the last two gives

$$\left(2a - 2 - \frac{2}{a} + \frac{S}{8a}\right) b_1 = 2a - 6 + \frac{2}{a} - \frac{S}{8a}$$

and substituting this in gives

$$\left(2a - 6 + \frac{2}{a} - \frac{S}{8a}\right)^2 - \left(2a - 6 + \frac{2}{a} - \frac{S}{8a}\right)\left(2a - 2 - \frac{2}{a} + \frac{S}{8a}\right)$$
$$- \left(2a - 2 + \frac{2}{a} - \frac{S}{8a}\right)\left(2a - 2 - \frac{2}{a} + \frac{S}{8a}\right)^2 = 0$$

and with $\delta \equiv 1 - \frac{S}{16}$

$$[a(a-3) + \delta]\,[(a(a-3) + \delta) - (a(a-1) - \delta)]\,a = 2\,[a(a-1) + \delta]\,[a(a-1) - \delta]^2$$

$$[a(a-3) + \delta]\,(\delta - a)a = \left[a^2(a-1)^2 - \delta^2\right][a(a-1) - \delta]$$

Note that with $a = \frac{5}{4}$ and $S = 1 \Rightarrow \delta = \frac{15}{16}$ we find

$$a(a-1) - \delta = -\frac{5}{8} \qquad a(a-1) + \delta = \frac{5}{4} \qquad a(a-3) + \delta = -\frac{5}{4}$$

which provides a solution, but in general we have a sextic equation.

9. Verify that the mapping

$$r\left[x_n - \frac{1}{2}\right] = R\left[X_n - \frac{1}{2}\right] \equiv z_n$$

with r+R=2 maps the logistic map, $x_{n+1} = rx_n(1 - x_n)$, onto itself. Show that z_n satisfies

$$z_{n+1} = a - z_n^2$$

and find the relationship between a and r. What interval of a, for the new map, corresponds to the original logistic map? Use the sepcial cases, r=2 and r=4, to find exact solutons to the map

$$z_{n+1} = a - z_n^2$$

Answer 9. Solving for x_n gives

$$x_n = \frac{1}{2} + \frac{R}{r}\left[X_n - \frac{1}{2}\right] = \frac{r - R}{2r} + \frac{R}{r}X_n = \frac{1 - R}{2 - R} + \frac{RX_n}{2 - R}$$

so

$$\frac{1 - R + RX_{n+1}}{2 - R} = (2 - R)\frac{[1 - R + RX_n]}{2 - R}\frac{[1 - RX_n]}{2 - R} \qquad \Rightarrow$$

$$1 - R + RX_{n+1} = 1 - R + RX_n - (1 - R)RX_n - R^2X_n^2 \quad \Rightarrow \quad X_{n+1} = RX_n(1 - X_n)$$

Solving for x_n gives

$$x_n = \frac{1}{2} + \frac{z_n}{r}$$

which yields

$$\frac{1}{2} + \frac{z_{n+1}}{r} = r\left[\frac{1}{2} + \frac{z_n}{r}\right]\left[\frac{1}{2} - \frac{z_n}{r}\right] = \frac{r}{4} - \frac{z_n^2}{r} \qquad \Rightarrow \qquad z_{n+1} = \frac{r^2}{4} - \frac{r}{2} - z_n^2 \equiv a - z_n^2$$

so

$$a = \frac{r^2}{4} - \frac{r}{2} = \frac{(r-1)^2}{4} - \frac{1}{4} \qquad \Rightarrow \qquad a + \frac{1}{4} \geq 0$$

When r=2 we find a=0 and so

$$z_{n+1} = -z_n^2 \qquad \Rightarrow \qquad z_n = (-1)z_0^{2^n}$$

for $n > 0$. When r=4 we find a=2 and so

$$z_{n+1} = 2 - z_n^2$$

The transformation $z_n = -2\cos(\pi y_n)$ then provides

$$\cos(\pi y_{n+1}) = 2\left[\cos(\pi y_n)\right]^2 - 1 = \cos(2\pi y_n) = \cos(2\pi[1 - y_n])$$

and we recognise the tent map.

10. Find all the 1-cycles for the map

$$x_{n+1} = r x_n (1 - x_n)(1 - 2x_n)$$

Answer 10. The 1-cycles satisfy

$$x = rx(1-2x)(1-x) \qquad \Rightarrow \qquad x = 0, \quad x^2 - \frac{3}{2}x + \frac{1}{2} - \frac{1}{2r} = 0$$

$$\left(x - \frac{3}{4}\right)^2 = \frac{9}{16} - \frac{1}{2} + \frac{1}{2r} = \left(\frac{1}{4}\right)^2\left[1 + \frac{8}{r}\right] \equiv \left(\frac{\Delta}{4}\right)^2$$

and so there are three 1-cycles

$$x = 0, \qquad x = \frac{3 \pm \Delta}{4}$$

where the final two only exist when $r > 0$ or $r < -8$.

11. Find the stability region of r for all the 1-cycles of the map

$$x_{n+1} = r x_n (1 - x_n)(1 - 2x_n)$$

Answer 11. The stability is controlled by

$$\frac{df}{dx}(x) = r\left[1 - 6x + 6x^2\right]$$

for the origin

$$-1 < \frac{df}{dx}(0) = r < 1$$

but for the other two 1-cycles

$$\frac{df}{dx}(x) = r\left[1 - 6x + 6x^2\right] = r\left[6\left(x - \frac{1}{2}\right)^2 - \frac{1}{2}\right]$$

and so

$$\frac{df}{dx}\left(\frac{3 \pm \Delta}{4}\right) = r\left[6\left(\frac{1 \pm \Delta}{4}\right)^2 - \frac{1}{2}\right] = r\left[\frac{3}{8}\left(1 \pm 2\Delta + \Delta^2\right) - \frac{1}{2}\right]$$

$$= r\left[\frac{3}{4} + \frac{3}{r} \pm \frac{3}{4}\Delta - \frac{1}{2}\right] = 3 + \frac{r}{4} \pm \frac{3}{4}r\Delta$$

Employing S to describe this quantity we have

$$\pm 3r\Delta = 4S - 12 - r \quad \Rightarrow \quad 9r^2 + 72r = (r + 12 - 4S)^2 \mapsto (r+8)^2 \quad \mathit{or} \quad (r+16)^2$$

for the two special cases where stability first appears and is lost respectively. For the first case

$$8r^2 + 56r - 64 = 8(r^2 + 7r - 8) = 8(r+8)(r-1) = 0$$

so r=1 and Δ=3 or r=-8 and Δ=0. For the second case

$$8r^2 + 40r - 256 = 8(r^2 + 5r - 32) = 8\left(\left[r + \frac{5}{2}\right]^2 - \frac{25}{4} - \frac{128}{4}\right) = 8\left(\left[r + \frac{5}{2}\right]^2 - \frac{9}{4}17\right)$$

and so $r = -\frac{5}{2} \pm \frac{3}{2}\sqrt{17}$. Clearly only one sign works for each case and so only one of these two 1-cycles is stable at one time.

12. Consider the map

$$x_{n+1} = r(2x_n^2 - 1)$$

Find all the 1-cycles and establish the regions of control parameter for which they are stable. Find all the 2-cycles and establish the regions of control parameter for which they are stable. Employ $x_n = \cos \pi y_n$ to establish an exact solution which is controlled by the tent map and find the value of r for which the exact solution exists.

Answer 12. The 1-cycles are controlled by

$$x = r(2x^2 - 1) \quad \Rightarrow \quad \left(x - \frac{1}{4r}\right)^2 = \frac{1}{16r^2} + \frac{1}{2} = \frac{[1 + 8r^2]}{(4r)^2} \equiv \left[\frac{\Delta}{4r}\right]^2$$

so

$$x = \frac{1 \pm \Delta}{4r} \qquad \Delta^2 = 1 + 8r^2$$

To determine stability we need

$$\frac{df}{dx}(x) = 4rx = 1 \pm \Delta \equiv S \qquad \Rightarrow \qquad \Delta^2 = 1 + 8r^2 = (S-1)^2$$

One of the 1-cycles loses stability to a 2-cycle when $r^2 = \frac{3}{8}$. For the 2-cycles we need to solve

$$y = r(2x^2 - 1) \qquad\qquad x = r(2y^2 - 1)$$

Subtracting gives

$$(y - x)\left[1 + 2r(x+y)\right] = 0 \qquad \Rightarrow \qquad x + y = -\frac{1}{2r}$$

and adding gives

$$y + x = 2r\left[(x+y)^2 - 2xy - 1\right] \qquad \Rightarrow \qquad xy = \frac{1}{4}\left[2(x+y)^2 - 2 - \frac{x+y}{r}\right]$$

$$= \frac{1}{8r^2} - \frac{1}{2} + \frac{1}{8r^2} = \frac{1}{4r^2} - \frac{1}{2} = \frac{1-2r^2}{4r^2}$$

and so the points for the 2-cycle satisfy

$$(z-x)(z-y) = z^2 - (x+y)z + xy = 0 \qquad \Rightarrow \qquad \left(z + \frac{1}{4r}\right)^2 = \frac{1}{16r^2} + \frac{1}{2} - \frac{1}{4r^2} \equiv \left[\frac{\Gamma}{4r}\right]^2$$

$$\Gamma^2 = 8r^2 - 3$$

so we have

$$x,\ y = \frac{-1 \pm \Gamma}{4r} \qquad\qquad \Gamma^2 = 8r^2 - 3$$

The stability is controlled by

$$\frac{df}{dx}(x)\frac{df}{dx}(y) = 4rx4ry = 16r^2xy = 4 - 8r^2 \equiv S$$

The 2-cycle first appears when S=1 and hence when $r^2 = \frac{3}{8}$. The 2-cycle loses stability when S=-1 and hence when $r^2 = \frac{5}{8}$. Substituting in the transformation gives

$$\cos \pi y_{n+1} = r\left[2\left(\cos \pi y_n\right)^2 - 1\right] = r \cos 2\pi y_n$$

and when r=1 we have

$$\cos \pi y_{n+1} = \cos 2\pi y_n = \cos 2\pi(1 - y_n)$$

Restricting $y_n \in [0,1]$ then gives $y_{n+1} = M[y_n]$ with

$$\begin{aligned} M[y] =& 2y & \qquad y \in \left[0, \frac{1}{2}\right] \\ =& 2(1-y) & \qquad y \in \left[\frac{1}{2}, 1\right] \end{aligned}$$

the tent map.

13. Consider the map

$$x_{n+1} = rx_n(3 - 4x_n^2)$$

Find all the 1-cycles and establish the regions of control parameter for which they are stable. Find all the 2-cycles and establish the regions of control parameter for which they are stable. Employ $x_n = \sin\frac{\pi y_n}{2}$ to establish an exact solution. Find the value of r for which the exact solution exists and find the map that controls it.

Answer 13. The 1-cycles are controlled by

$$x = rx(3 - 4x^2) \quad \Rightarrow \quad x = 0,\ \pm\frac{1}{2}\left(3 - \frac{1}{r}\right)^{\frac{1}{2}}$$

Stability is controlled by

$$\frac{df}{dx}(x) = 3r(1 - 4x^2) \equiv S$$

and so the origin is stable when $-\frac{1}{3} < r < \frac{1}{3}$. The other two 1-cycles are controlled by

$$S = 3r\left(1 - 3 + \frac{1}{r}\right) = 3 - 6r \quad \Rightarrow \quad \frac{1}{3} < r < \frac{2}{3}$$

The 2-cycles are controlled by

$$y = rx(3 - 4x^2) \qquad x = ry(3 - 4y^2)$$

subtracting gives

$$(y - x)\left[\frac{1}{r} + 3 - 4(y^2 + yx + x^2)\right] = 0$$

and adding gives

$$(y + x)\left[\frac{1}{r} - 3 + 4(y^2 - xy + x^2)\right] = 0$$

We may abolish 1-cycles but we do have

$$y = -x \quad \Rightarrow \quad x^2 = \frac{1}{4}\left(3 + \frac{1}{r}\right)$$

The other 2-cycles satisfy

$$y^2 + x^2 = \frac{3}{4} \qquad xy = \frac{1}{4r}$$

and so

$$(x + y)^2 = y^2 + x^2 + 2xy = \frac{3}{4} + \frac{1}{2r} = \left(\frac{\Delta}{2}\right)^2 \qquad \Delta^2 = 3 + \frac{2}{r}$$

The points which make up the 2-cycle satisfy

$$(z - x)(z - y) = z^2 - (x + y)z + xy = 0 \quad \Rightarrow \quad z^2 \pm \frac{\Delta}{2}z + \frac{1}{4r} = 0 \quad \Rightarrow$$

$$\left(z \pm \frac{\Delta}{4}\right)^2 = \left(\frac{\Delta}{4}\right)^2 - \frac{1}{4r} = \frac{1}{16}\left[3 + \frac{2}{r} - \frac{4}{r}\right] = \frac{1}{16}\left[3 - \frac{2}{r}\right]$$

Stability is controlled by

$$\frac{df}{dx}(x)\frac{df}{dx}(y) = 9r^2(1-4x^2)(1-4y^2) \equiv S$$

The case y=-x gives

$$S = 9r^2\left(1 - 3 - \frac{1}{r}\right)^2 = (3+6r)^2$$

and the only possibility is S=1 which occurs when

$$3 + 6r = \pm 1 \qquad \Rightarrow \qquad r = -\frac{1}{3},\ -\frac{2}{3}$$

the first case marks the appearence and the second marks Δ=0 and they disappear again. The other 2-cycles give

$$S = 9r^2\left[1 - 4(y^2+x^2) + 16(xy)^2\right] = 9r^2\left[1 - 3 + \frac{1}{r^2}\right] = 9(1-2r^2)$$

$$\Rightarrow \qquad r^2 = \frac{1}{2}\left(1 - \frac{S}{9}\right)$$

and we find the 2-cycle first appears when $r^2 = \frac{4}{9}$ and then loses stability when $r^2 = \frac{5}{9}$. Substituting in the transformation gives

$$\sin\frac{\pi y_{n+1}}{2} = r\sin\frac{\pi y_n}{2}\left[3 - 4\left(\sin\frac{\pi y_n}{2}\right)^2\right] = r\sin\frac{3\pi y_n}{2}$$

We need de moivre's theorem

$$\cos 3x + i\sin 3x = e^{3ix} = \left(e^{ix}\right)^3 = (\cos x + i\sin x)^3 = \cos^3 x - 3\cos x\sin^2 x$$

$$+ i(3\cos^2 x\sin x - \sin^3 x) = 4\cos^3 x - 3\cos x + i(3\sin x - 4\sin^3 x)$$

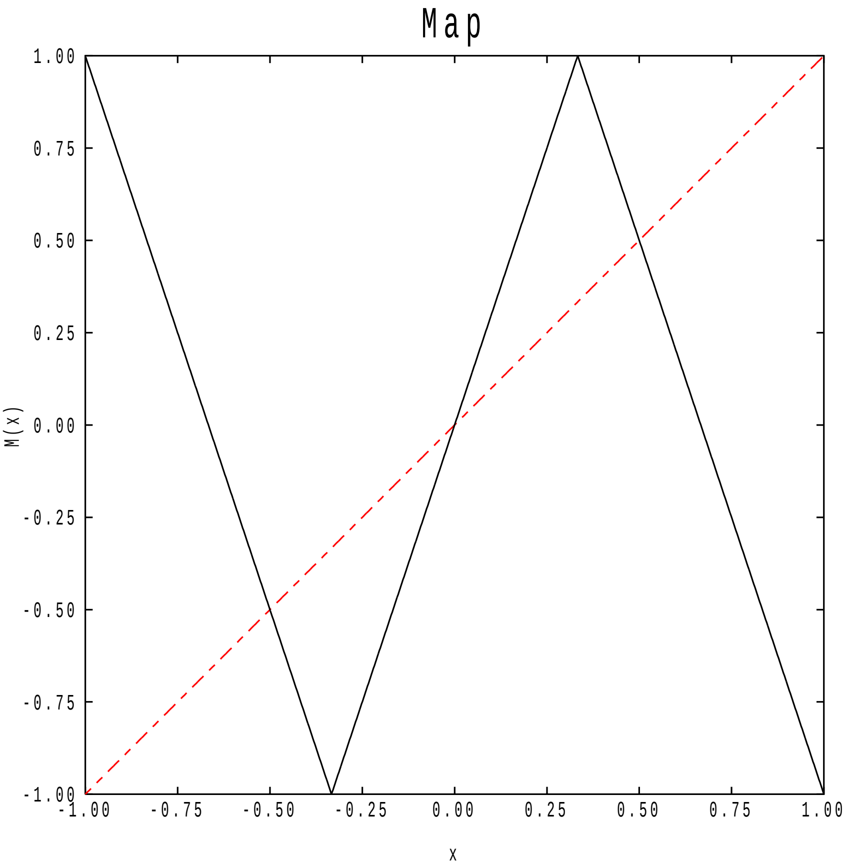
and then when r=1 we find

$$\sin\frac{\pi y_{n+1}}{2} = \sin\frac{3\pi y_n}{2} = \sin\frac{2\pi - 3\pi y_n}{2} = \sin\frac{-2\pi - 3\pi y_n}{2}$$

Restricting $y_n \in [-1, 1]$ then gives $y_{n+1} = M[y_n]$ with

$$\begin{aligned} M[y] =& -2 - 3y & y \in \left[-1, -\frac{1}{3}\right] \\ =& 3y & y \in \left[-\frac{1}{3}, \frac{1}{3}\right] \\ =& 2 - 3y & y \in \left[\frac{1}{3}, 1\right] \end{aligned}$$

which depicts as



14. Consider the mapping

$$x_{n+1} = r x_n (x_n^2 - 1)$$

where r is a control parameter. Employ the transformation

$$x_n = \frac{2}{\sqrt{3}} \cos \pi y_n$$

to provide an exact solution for a value of r that you should determine. Using $y_{n+1} = M[y_n]$ for this solution find the map $M[y]$ and depict it. Find the 1-cycles. Find the 2-cycles.

Answer 14. We simply substitute the transformation into the equation

$$\frac{2}{\sqrt{3}} \cos \pi y_{n+1} = r \frac{2}{\sqrt{3}} \cos \pi y_n \left[\frac{4}{3} (\cos \pi y_n)^2 - 1 \right]$$

which leads to

$$\cos \pi y_{n+1} = \frac{r}{3} \left[4 (\cos \pi y_n)^3 - 3 (\cos \pi y_n) \right] = \frac{r}{3} \cos 3\pi y_n$$

Where we used an identity stemming from de moivre's theorem

$$\cos 3x + i \sin 3x = e^{3ix} = (e^{ix})^3 = (\cos x + i \sin x)^3$$

$$= \cos^3 x + 3i \cos^2 x \sin x - 3 \cos x \sin^2 x - i \sin^3 x$$

from which

$$\cos 3x = \cos^3 x - 3 \cos x \sin^2 x = 4 \cos^3 x - 3 \cos x$$

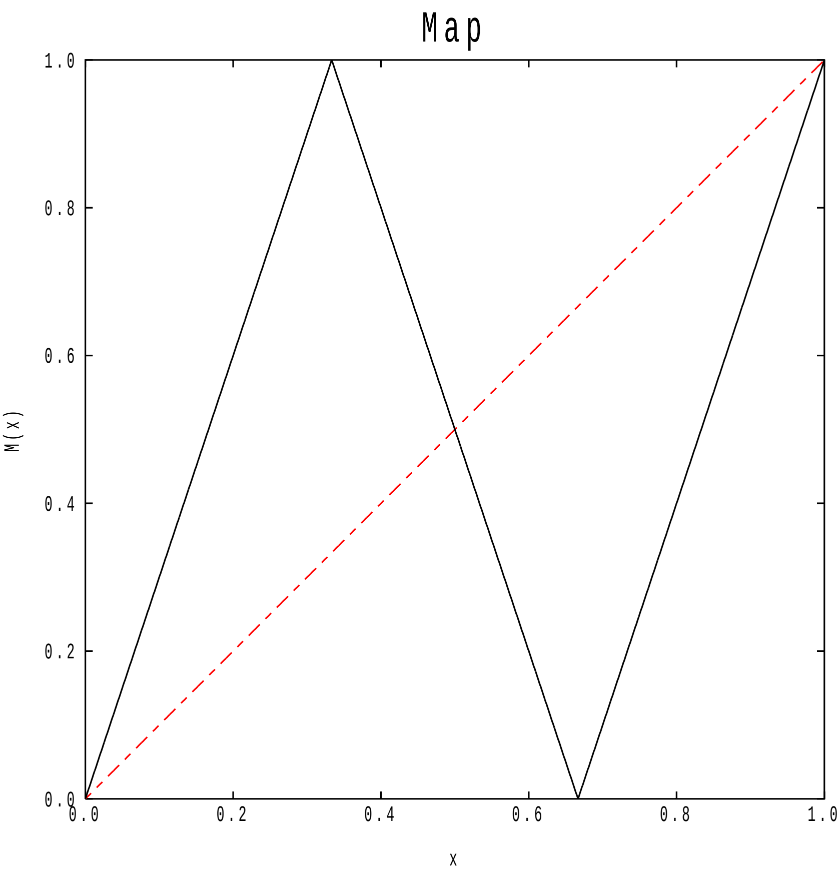
The special value of control parameter is r=3 and with the help of

$$\cos \pi y_{n+1} = \cos 3\pi y_n = \cos(2\pi - 3\pi y_n) = \cos(3\pi y_n - 2\pi)$$

We can deduce that

$$\begin{aligned} M[y] =& 3y & y \in \left[0, \frac{1}{3}\right] \\ =& 2 - 3y & y \in \left[\frac{1}{3}, \frac{2}{3}\right] \\ =& 3y - 2 & y \in \left[\frac{2}{3}, 1\right] \end{aligned}$$

which depicts as



For the 1-cycles we have

$$y = 3y \qquad y = 2 - 3y \qquad y = 3y - 2$$

which provides us with

$$y = 0,\ \frac{1}{2},\ 1 \quad \Rightarrow \quad x = \frac{2}{\sqrt{3}},\ 0,\ -\frac{2}{\sqrt{3}}$$

For the 2-cycles we have

$$z = 3y \qquad y = 3z \qquad \Rightarrow \qquad y = z = 0$$

$$z = 2 - 3y \qquad y = 2 - 3z \qquad \Rightarrow \qquad y = z = \frac{1}{2}$$

$$z = 3y - 2 \qquad y = 3z - 2 \qquad \Rightarrow \qquad y = z = 1$$

$$z = 3y \qquad y = 2 - 3z \qquad \Rightarrow \qquad y = \frac{1}{5} \qquad z = \frac{3}{5}$$

$$z = 3y \qquad y = 3z - 2 \qquad \Rightarrow \qquad y = \frac{1}{4} \qquad z = \frac{3}{4}$$

$$z = 2 - 3y \quad y \quad = 3z - 2 \qquad \Rightarrow \qquad y = \frac{2}{5} \qquad z = \frac{4}{5}$$

15. Consider the mapping

$$x_{n+1} = x_n \left[f(x_n)\right]^2$$

and employ the transformation $x_n = z_n^2$ to provide a second representation. Use this theory to deduce the 1-cycles and 2-cycles for the map

$$x_{n+1} = r^2 x_n (1 - x_n)^2$$

from the solution to the map

$$z_{n+1} = r z_n (1 - z_n^2)$$

Answer 15. Direct substitution provides

$$z_{n+1}^2 = z_n^2 \left[f(z_n^2)\right]^2 \qquad \Rightarrow \qquad z_{n+1} = \pm z_n f(z_n^2)$$

With the choice $f(x) = r(1 - x)$ we find

$$x_{n+1} = r^2 x_n (1 - x_n)^2 \qquad \Rightarrow \qquad z_{n+1} = r z_n (1 - z_n^2)$$

where r can have either sign in the second representation. For 1-cycles we need to solve

$$z = rz(1 - z^2) \qquad \Rightarrow \qquad z = 0, \;\; z^2 = 1 - \frac{1}{r}$$

$$x = r^2 x (1 - x)^2 \quad x = 0, \;\; x = 1 \pm \frac{1}{r}$$

For 2-cycles we need to solve

$$y = rx(1 - x^2) \qquad x = ry(1 - y^2)$$

Subtracting gives

$$(y - x)\left[\frac{1}{r} + 1 - x^2 - xy - y^2\right] = 0$$

Adding gives

$$(y+x)\left[\frac{1}{r}-1+x^2-xy+y^2\right]=0$$

We get the origin, 1-cycles $x^2=1-\frac{1}{r}$,

$$y=-x \qquad x^2=1+\frac{1}{r}$$

and

$$x^2+y^2=1 \quad xy=\frac{1}{r} \quad \Rightarrow \quad (x+y)^2=1+\frac{2}{r} \quad (x-y)^2=1-\frac{2}{r}$$

which gives

$$x,y=\frac{1}{2}\left[\left(1+\frac{2}{r}\right)^{\frac{1}{2}}\pm\left(1-\frac{2}{r}\right)^{\frac{1}{2}}\right] \quad or \quad \frac{1}{2}\left[-\left(1+\frac{2}{r}\right)^{\frac{1}{2}}\pm\left(1-\frac{2}{r}\right)^{\frac{1}{2}}\right]$$

which provides

$$x^2=\frac{1}{2}\pm\frac{1}{2}\left(1-\frac{4}{r}\right)^{\frac{1}{2}}$$

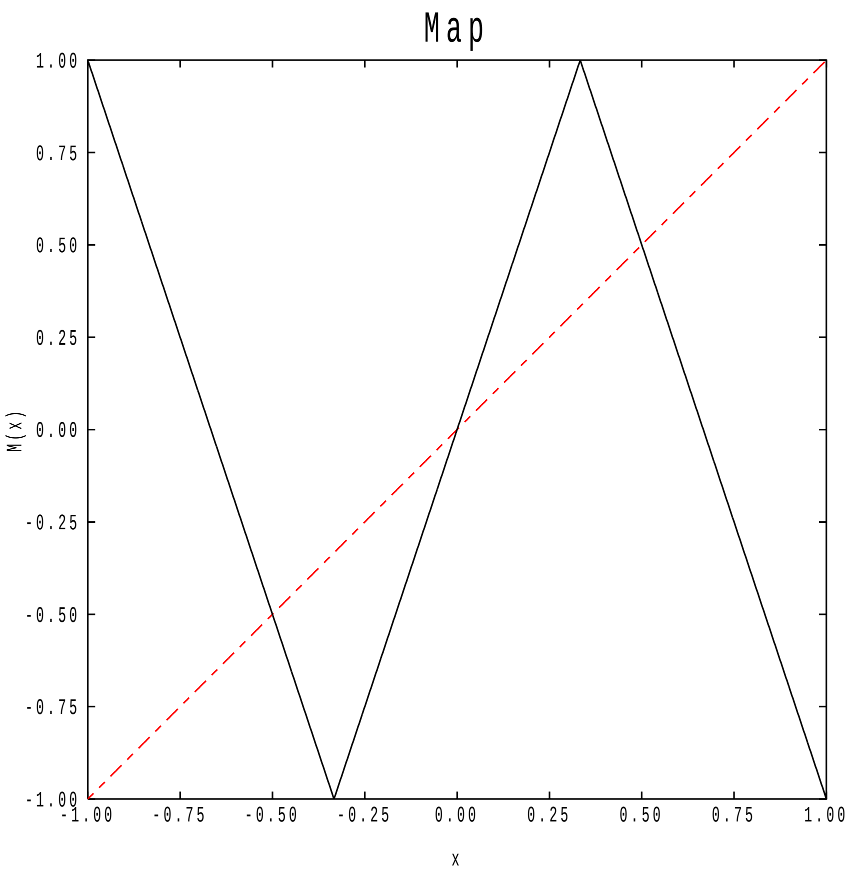
for the other system.

16. Consider the map

$$\begin{aligned} M[x]=&-2-3x \qquad && x\in\left[-1,-\frac{1}{3}\right] \\ =&3x && x\in\left[-\frac{1}{3},\frac{1}{3}\right] \\ =&2-3x && x\in\left[\frac{1}{3},1\right] \end{aligned}$$

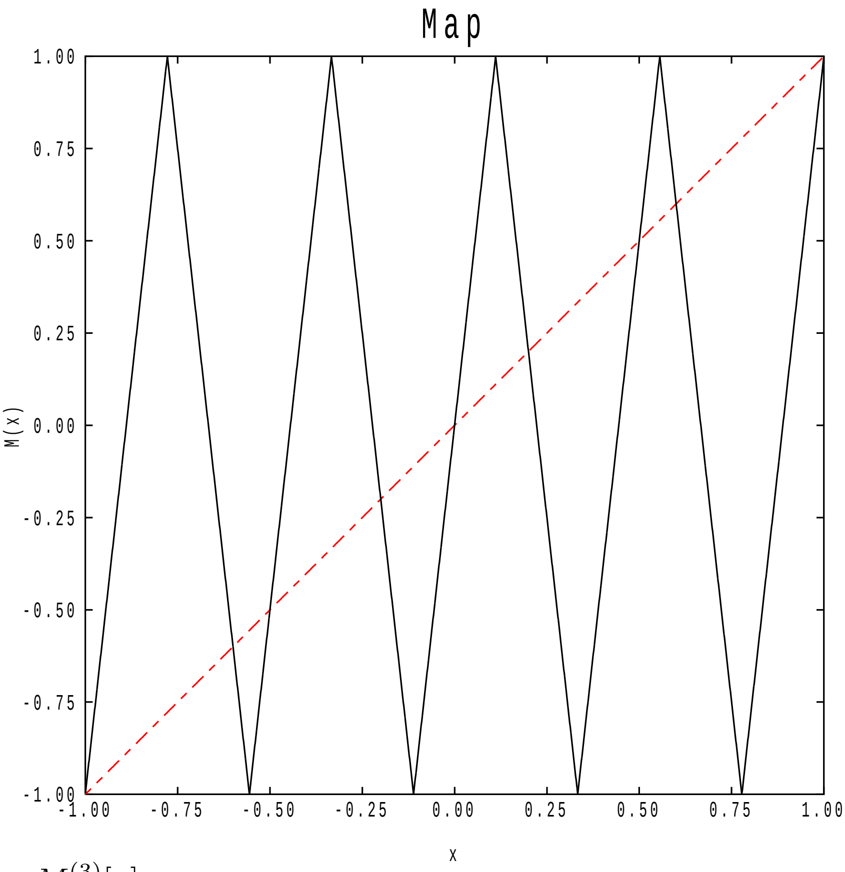
Depict $M[x]$, $M^{(2)}[x]$ and $M^{(3)}[x]$. Determine the number of solutions to $x=M^{(N)}[x]$ and use this result to calculate the number of n-cycles for $n\leq 10$.

Answer 16. We can depict the map $M[x]$

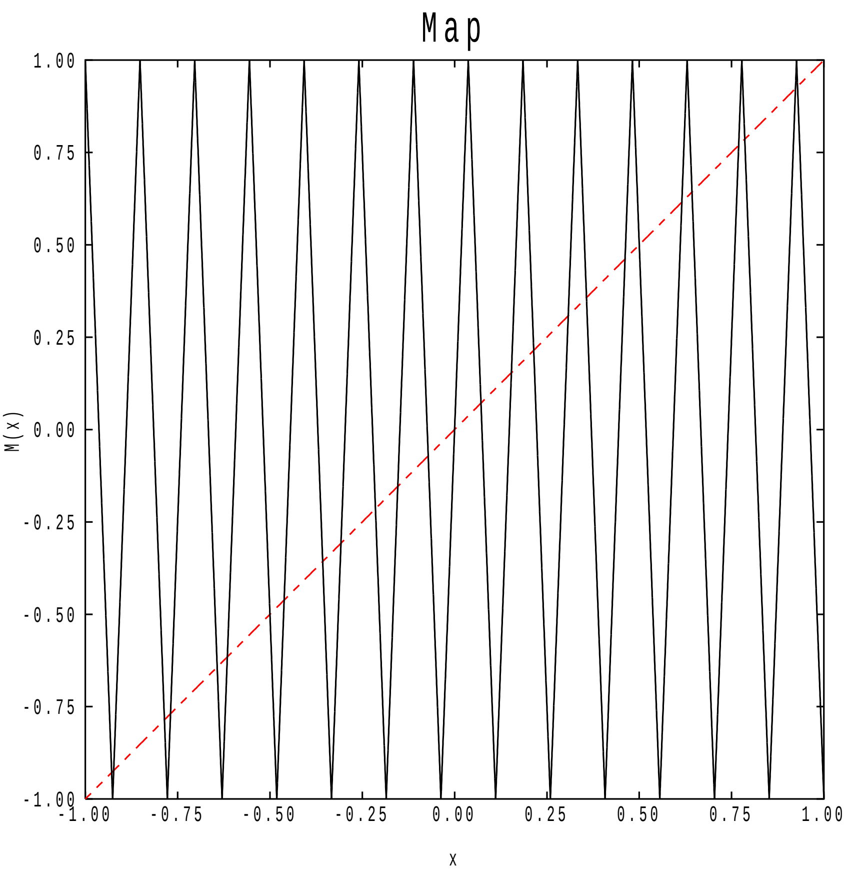


and then since each third maps onto the whole region we get three copies of the original map. Since this map is anti-symmetric, when we map from one down to minus one, we get the map reversed. This gives us two copies of the reversed map at the edges and one copy in the middle. Similar arguments work for all multiple maps. The map $M^{(2)}[x]$

depicts as



and the map $M^{(3)}[x]$ as



The number of solutions to $x = M^{(N)}[x]$ is clearly 3^N. We can readily count the n-cycles using the idea that all previous cycles with factors that divide n are also encountered

n=1	$\Rightarrow$	3×1-cycles			
n=2	$\Rightarrow$	3×1-cycles	3×2-cycles		
n=3	$\Rightarrow$	3×1-cycles	8×3-cycles		
n=4	$\Rightarrow$	3×1-cycles	3×2-cycles	18×4-cycles	
n=5	$\Rightarrow$	3×1-cycles	48×5-cycles		
n=6	$\Rightarrow$	3×1-cycles	3×2-cycles	8×3-cycles	116×6-cycles
n=7	$\Rightarrow$	3×1-cycles	312×7-cycles		
n=8	$\Rightarrow$	3×1-cycles	3×2-cycles	18×4-cycles	810×8-cycles
n=9	$\Rightarrow$	3×1-cycles	8×3-cycles	2184×9-cycles	
n=10	$\Rightarrow$	3×1-cycles	3×2-cycles	48×5-cycles	5880×10-cycles

17. Consider the map

$$x_{n+1} = rx_n \left[1 - x_n^2 + \frac{x_n^4}{5}\right]$$

Find all the 1-cycles and establish the region of r for which each is stable. Employ the transformation

$$x_n = 2 \sin \frac{\pi y_n}{2}$$

to find an exact solution and determine the particular value of r associated with this solution. Using $y_{n+1} = M[y_n]$ determine the map $M[y]$.

Answer 17. To find the 1-cycles we solve

$$x = rx\left(1 - x^2 + \frac{x^4}{5}\right) \quad \Rightarrow \quad x = 0 \quad \left(x^2 - \frac{5}{2}\right)^2 = \frac{25}{4} - 5 + \frac{5}{r} = \frac{1}{4}\left[5 + \frac{20}{r}\right] \equiv \left[\frac{\Delta}{2}\right]^2$$

$$\Rightarrow \quad x^2 = \frac{1}{2}(5 \pm \Delta)$$

To establish stability we need

$$\frac{df}{dx}(x) = r\left[1 - 3x^2 + x^4\right] \equiv S$$

and for the origin we find $-1 < r < 1$ whereas for the other solutions we find

$$S = r\left(1 - \frac{3}{2}[5 \pm \Delta] + \frac{1}{4}[5 \pm \Delta]^2\right) = r + 5 \pm r\Delta$$

Squaring out the quantity Δ then gives

$$r^2\Delta^2 = 5r^2 + 20r = (r - S + 5)^2 = r^2 + 2(5 - S)r + (5 - S)^2$$

$$\Rightarrow \quad 4r^2 + 2(5 + S)r - (5 - S)^2 = 0$$

and when S=1 we find

$$4r^2 + 12r - 16 = 4(r^2 + 3r - 4) = 0 \quad \Rightarrow \quad r = 1,\ -4$$

whilst when S=-1 we find

$$4r^2 + 8r - 36 = 4(r^2 + 2r - 9) = 0 \quad \Rightarrow \quad r = -1 \pm \sqrt{10}$$

Substituting in the transformation gives

$$2 \sin \frac{\pi y_{n+1}}{2} = 2r \sin \frac{\pi y_n}{2} \left[1 - 4 \sin^2 \frac{\pi y_n}{2} + \frac{16}{5} \sin^4 \frac{\pi y_n}{2} \right]$$

which leads to

$$\sin \frac{\pi y_{n+1}}{2} = \frac{r}{5} \left[5 \sin \frac{\pi y_n}{2} - 20 \sin^3 \frac{\pi y_n}{2} + 16 \sin^5 \frac{\pi y_n}{2} \right]$$

We now need an identity stemming from de moivre's theorem

$$\cos 5x + i \sin 5x = e^{5ix} = (e^{ix})^5 = (\cos x + i \sin x)^5$$

$$= \cos^5 x + 5i \cos^4 x \sin x - 10 \cos^3 x \sin^2 x - 10i \cos^2 x \sin^3 x + 5 \cos x \sin^4 x + i \sin^5 x$$

from which we can deduce that

$$\sin 5x = 5 \cos^4 x \sin x - 10 \cos^2 x \sin^3 x + \sin^5 x = 5 \sin x - 20 \sin^3 x + 16 \sin^5 x$$

The special choice of r=5 then gives

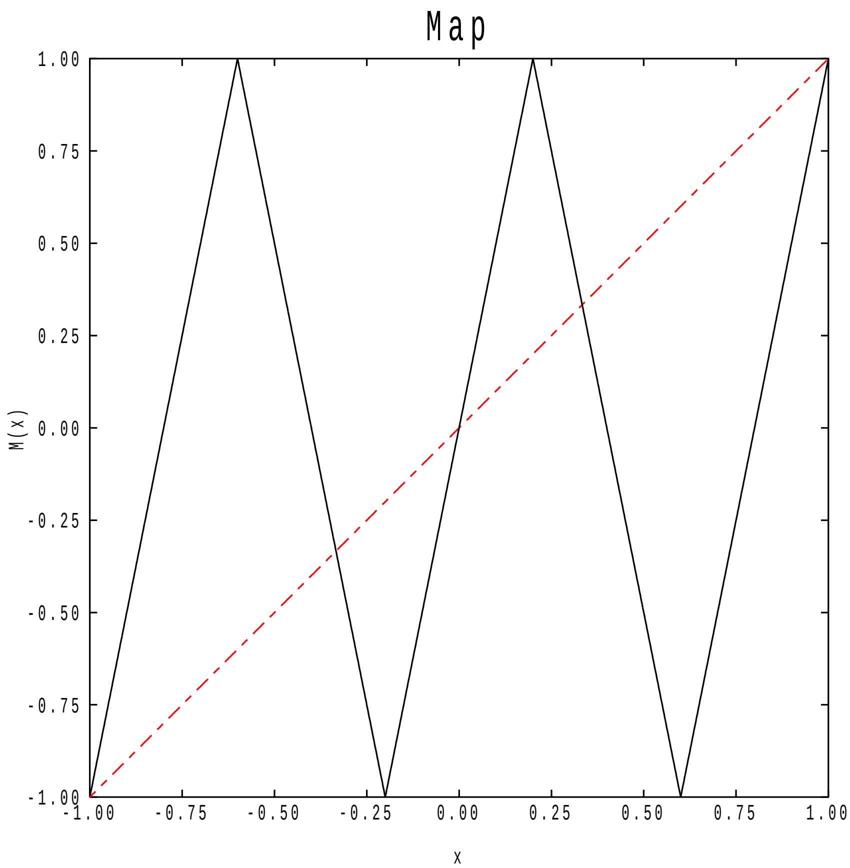
$$\sin \frac{\pi y_{n+1}}{2} = \sin \frac{5\pi y_n}{2} = \sin \left(\pi - \frac{5\pi y_n}{2} \right) = \sin \left(\frac{5\pi y_n}{2} - 2\pi \right)$$

$$= \sin \left(-\pi - \frac{5\pi y_n}{2} \right) = \sin \left(\frac{5\pi y_n}{2} + 2\pi \right)$$

The choice of $y_{n+1} = M[y_n]$ then provides the map

$$\begin{aligned} M[y] =& 4 + 5y & y \in \left[-1, -\frac{3}{5} \right] \\ =& -2 - 5y & y \in \left[-\frac{3}{5}, -\frac{1}{5} \right] \\ =& 5y & y \in \left[-\frac{1}{5}, \frac{1}{5} \right] \\ =& 2 - 5y & y \in \left[\frac{1}{5}, \frac{3}{5} \right] \\ =& 5y - 4 & y \in \left[\frac{3}{5}, 1 \right] \end{aligned}$$

which depicts as



18*. Consider the map

$$x_{n+1} = a - x_n^3$$

Show that there is only one 1-cycle and find the region of stability in a. Show that there is always at least one 2-cycle and find the region of a for which there are three 2-cycles. Show that these 2-cycles are never unstable to 4-cycles and find the region of stability in a. Describe the attractor as a function of a.

Answer 18. Consider the function

$$g(x) = x^3 + x - a \qquad \Rightarrow \qquad \frac{dg}{dx} = 3x^2 + 1$$

Since $g(x)$ has a positive derivative it must be monatonic which means that there is only one root. Now

$$x = a - x^3 \equiv f(x) \qquad \Rightarrow \qquad \frac{df}{dx}(x) = -3x^2$$

and so the 1-cycle can only be unstable to a 2-cycle and this occurs when

$$3x^2 = 1 \qquad x^3 + x = a \qquad \Rightarrow \qquad x = \pm\frac{1}{\sqrt{3}} \quad a = \pm\frac{4}{3\sqrt{3}}$$

so the 1-cycle is stable when $-\frac{4}{3\sqrt{3}} < a < \frac{4}{3\sqrt{3}}$. To look at 2-cycles we need to solve

$$y = a - x^3 \qquad x = a - y^3$$

and subtracting

$$y - x = y^3 - x^3 = (y-x)(y^2 + yx + x^2) = (y-x)\left[(y+x)^2 - yx\right]$$

Avoiding 1-cycles we find

$$xy = (x+y)^2 - 1$$

Adding gives

$$y + x = 2a - y^3 - x^3 = 2a - (y+x)(y^2 - yx + x^2) = 2a - (y+x)\left[(y+x)^2 - 3xy\right]$$

and substituting for xy

$$y + x = 2a - (y+x)\left[(y+x)^2 - 3\left[(y+x)^2 - 1\right]\right] \quad \Rightarrow \quad (y+x)^3 - 2(y+x) + a = 0$$

We change from one root to three roots when there is a double root so (using $z \equiv x+y$)

$$z^3 - 2z + a = 0 \quad 3z^2 - 2 = 0 \quad \Rightarrow \quad z = \pm\frac{\sqrt{2}}{\sqrt{3}} \quad a = \pm\frac{4\sqrt{2}}{3\sqrt{3}}$$

and we have three 2-cycles when $-\frac{4\sqrt{2}}{3\sqrt{3}} < a < \frac{4\sqrt{2}}{3\sqrt{3}}$. The stability of the 2-cycles is controlled by

$$S \equiv \frac{df}{dx}(x)\frac{df}{dx}(y) = (-3x^2)(-3y^2) = (3xy)^2 = \left[3(z^2-1)\right]^2$$

and we cannot have a 2-cycle turn into a 4-cycle. The appearence of 2-cycles is controlled by S=1 and

$$3(z^2 - 1) = \pm 1 \quad z^3 - z - a = 0 \quad \Rightarrow \quad z = \pm\frac{\sqrt{2}}{\sqrt{3}},\ \pm\frac{2}{\sqrt{3}}$$

The first roots, $z = \pm\frac{\sqrt{2}}{\sqrt{3}}$ correspond to the merging of the 2-cycles as they disappear and the second roots, $z = \pm\frac{2}{\sqrt{3}}$ correspond to x=y=$\pm\frac{1}{\sqrt{3}}$ and the appearence of the 2-cycle from the 1-cycle. The attractor is the 1-cycle for $-\frac{4}{3\sqrt{3}} < a < \frac{4}{3\sqrt{3}}$, is a 2-cycle for $-\frac{4\sqrt{2}}{3\sqrt{3}} < a < -\frac{4}{3\sqrt{3}}$ and $\frac{4}{3\sqrt{3}} < a < \frac{4\sqrt{2}}{3\sqrt{3}}$ and the trajectories diverge outside of this region.

19. Demonstrate explicitely that the transformation

$$X_n = \frac{1}{R}\left[R - 1 + (2-R)x_n\right] \qquad r + R = 2$$

maps the 2-cycle of the logistic map for $r \geq 1$ onto itself for $R \leq 1$.

Answer 19. The 2-cycle is described by

$$x + y = 1 + \frac{1}{r} = \frac{3 - R}{2 - R} \qquad xy = \frac{1}{r}\left(1 + \frac{1}{r}\right) = \frac{3 - R}{(2 - R)^2}$$

and so

$$X + Y = \frac{1}{R}\left[2R - 2 + (2 - R)(x + y)\right] = \frac{1}{R}\left[2R - 2 + 3 - R\right] = 1 + \frac{1}{R}$$

$$\begin{aligned} XY &= \frac{1}{R^2}\left[(R-1)^2 + (R-1)(2-R)(x+y) + (2-R)^2 xy\right] \\ &= \frac{1}{R^2}\left[(R-1)^2 + (R-1)(3-R) + (3-R)\right] = \frac{1+R}{R^2} \end{aligned}$$

20.* Show that the map

$$x_{n+1} = r^3 x_n (1 - x_n)^3$$

can be transformed into the map

$$z_{n+1} = r z_n (1 - z_n^3)$$

using the transformation $x_n = z_n^3$ and use this idea to find the 1-cycles and 2-cycles for the original map.

Answer 20. The mapping is trivial and a real cube root only has a single solution, so we can have any sign for r and we have a one to one mapping. To solve for 1-cycles we need to solve

$$z = rz(1 - z^3) \quad \Rightarrow \quad z = 0, \quad z^3 = 1 - \frac{1}{r}$$

which leads to

$$x = 0, \quad x = 1 - \frac{1}{r}$$

which can be verified directly. To find the 2-cycles we need to solve

$$w = rz(1 - z^3) \qquad z = rw(1 - w^3)$$

Subtract to find

$$(w - z)\left[\frac{1}{r} + 1 - (w + z)(w^2 + z^2)\right] = 0$$

where we used

$$w^4 - z^4 = (w^2 - z^2)(w^2 + z^2) = (w - z)(w + z)(w^2 + z^2)$$

Add to find

$$(w + z)\left[1 - \frac{1}{r}\right] = w^4 + z^4 = (w^2 + z^2)^2 - 2w^2z^2 = \left[(w + z)^2 - 2wz\right]^2 - 2w^2z^2$$

$$= (w + z)^4 - 4wz(w + z)^2 + 2w^2z^2$$

so we have the two equations

$$(w+z)^3 - 2wz(w+z) - 1 - \frac{1}{r} = 0$$

$$(w+z)^4 - 4wz(w+z)^2 + 2w^2z^2 - \left(1 - \frac{1}{r}\right)(w+z) = 0$$

Multiply the first by $2(w+z)$ and subtract

$$(w+z)^4 - 2w^2z^2 - \left(1 + \frac{3}{r}\right)(w+z) = 0$$

Multiply by $2(w+z)^2$ and substitute

$$2(w+z)^6 - [2wz(w+z)]^2 - 2\left(1 + \frac{3}{r}\right)(w+z)^3 = 0 \quad \Rightarrow$$

$$2(w+z)^6 - \left[(w+z)^3 - 1 - \frac{1}{r}\right]^2 - 2\left(1 + \frac{3}{r}\right)(w+z)^3 = 0$$

and so a final quadratic equation

$$(w+z)^6 - \frac{4}{r}(w+z)^3 - \left(1 + \frac{1}{r}\right)^2 = 0$$

which solves to provide

$$(w+z)^3 = \frac{2}{r} \pm \left[\frac{4}{r^2} + \left(1 + \frac{1}{r}\right)^2\right]^{\frac{1}{2}} \equiv \frac{2}{r} \pm \Delta$$

$$2wz(w+z) = \frac{1}{r} - 1 \pm \left[\frac{4}{r^2} + \left(1 + \frac{1}{r}\right)^2\right]^{\frac{1}{2}} \equiv \frac{1}{r} - 1 \pm \Delta$$

and so

$$w^3 + z^3 = \frac{2}{r} \pm \Delta - \frac{3}{2}\left[\frac{1}{r} - 1 \pm \Delta\right] = \frac{3}{2} + \frac{1}{2r} \mp \frac{\Delta}{2}$$

Finally, producting the two original equations gives

$$(1-w^3)(1-z^3) = \frac{1}{r^2} = w^3z^3 + 1 - w^3 - z^3 \quad \Rightarrow \quad w^3z^3 = \frac{1}{2}\left[1 + \frac{1}{r} + \frac{2}{r^2} \pm \Delta\right]$$

and we can find

$$(w^3 - z^3)^2 = (1 - z^3 - 1 + w^3)^2 = (1 - w^3 + 1 - z^3)^2 - 4(1-w^3)(1-z^3)$$

$$= \frac{1}{4}\left[1 - \frac{1}{r} \pm \Delta\right]^2 - \frac{4}{r^2} = \frac{1}{4}\left[\left(1 - \frac{1}{r}\right)^2 + \left(1 + \frac{1}{r}\right)^2 + \frac{4}{r^2}\right] \pm \frac{1}{2}\left(1 - \frac{1}{r}\right)\Delta - \frac{4}{r^2}$$

$$= \frac{1}{2} - \frac{5}{2r^2} \pm \frac{1}{2}\left(1 - \frac{1}{r}\right)\Delta$$

and we can then find w^3 and z^3.