University of Birmingham
School of Mathematics

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Problem Sheet 3 - self assessment
Model Solutions

Questions

Q1. Find the derivatives of following functions according to the definition, where they exist.
(a) $f(x) = x^2$.
(b) $f(x) = e^x$.

Solution. (a) We want to find $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$. Observe that for $h \neq 0$ we have

$$\begin{aligned}\frac{f(x+h)-f(x)}{h} &= \frac{1}{h}\left[((x+h)^2 - x^2)\right] \\ &= \frac{1}{h}\left[(x^2+2xh+h^2) - x^2)\right] \\ &= 2x + h.\end{aligned}$$

Hence

$$f'(x) = \lim_{h\to 0}(2x+h) = 2x.$$

(b) From the trigonometric identity $e^{x+h} = e^x e^h$, it follows that

$$\begin{aligned}\frac{e^{(x+h)} - e^x}{h} &= \frac{e^x e^h - e^x}{h} \\ &= \frac{(e^h - 1)}{h} e^x\end{aligned}$$

for all $x \in \mathbb{R}$ and $h \neq 0$. Therefore, we have that

$$\lim_{h\to 0} \frac{e^{x+h} - e^x}{h} = e^x \lim_{h\to 0} \frac{e^h - 1}{h} = e^x$$

where we used notable limit, for all $x \in \mathbb{R}$, as required.

□

Q2. Assume that $f(x)$ is an even function and differentiable at $x = 0$. Show that $f'(0) = 0$.

Solution. Since f is differentiable at $x = 0$, and $f(-x) = f(x)$ for all $x \in \mathbb{R}$, we have

$$f'(0) = \lim_{h\to 0} \frac{f(h) - f(0)}{h} = \lim_{h\to 0} \frac{f(-h) - f(0)}{h} = -\lim_{\tilde{h}\to 0} \frac{f(\tilde{h}) - f(0)}{\tilde{h}} = -f'(0).$$

Therefore, $f'(0) = 0$, as required. □

Q3. For each $n \in \{0, 1, 2\}$, define the function $f_n : \mathbb{R} \to \mathbb{R}$ by

$$f_n(x) = \begin{cases} x^n \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

These functions are all differentiable at all points x in $\mathbb{R} \setminus \{0\}$, and hence continuous there. Decide whether these functions are continuous or differentiable at 0.

Solution. We discuss these functions separately.

(1) f_0 is not continuous at 0, since $\lim_{x\to 0} f_0(x)$ does not exist. In particular, f_0 is not differentiable at $x = 0$ via Corollary 4.6, since f_0 is not continuous at $x = 0$.

(2) We claim that f_1 is continuous at 0, but is not differentiable at 0. We recall Example 2.17 which demonstrates that f_1 is continuous at $x = 0$. However, observe that

$$\frac{f_1(0+h) - f_1(0)}{h} = \frac{h\sin(\frac{1}{h}) - 0}{h} = \sin\frac{1}{h}.$$

Since $\lim_{h\to 0} \sin\frac{1}{h}$ does not exist, it follows that f_1 is not differentiable at 0.

(3) We claim that f_2 is differentiable at 0, and thus f_2 is also continuous at 0. To show the differentiability, observe that

$$\frac{f_2(0+h) - f_2(0)}{h} = \frac{f_2(h)}{h} = \frac{h^2\sin(\frac{1}{h})}{h} = h\sin\frac{1}{h}.$$

Again, recalling Example 2.17, it follows that $\lim_{h\to 0} h\sin\frac{1}{h} = 0$. Hence, $f_2'(0) = 0$ i.e. f_2 is differentiable at 0. Via Theorem 4.5, it follows that f is continuous at 0 too.

□

Q4. (a) If $f(x) = x/\sin x$, find the exact value of $f'(\pi/3)$

(b) If $y = \sqrt{1+\sqrt{x}}$, find $\frac{dy}{dx}$.

Note that $f'(\pi/3)$ means to find the $f'(x)$ first and substitute in $\pi/3$ for x. It is not the derivative of $f(\pi/3)$, which is 0.

Solution. (a) By the quotient rule, $f'(x) = (\sin x - x\cos x)/\sin^2 x$. Since $\sin(\pi/3) = \sqrt{3}/2$ and $\cos(\pi/3) = 1/2$, $f'(\pi/3) = (\sqrt{3}/2 - \pi/6)/(3/4) = 2/\sqrt{3} - 2\pi/9$.

(b) $y = \sqrt{1+\sqrt{x}}$, so by the chain rule, if $u = 1+\sqrt{x}$,

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \frac{d\sqrt{u}}{du}\frac{d(1+\sqrt{x})}{dx} = \frac{1}{2\sqrt{u}}.\frac{1}{2\sqrt{x}} = \frac{1}{4\sqrt{x}\sqrt{1+\sqrt{x}}}.$$

Although the curve y is defined for all $x \geq 0$, the derivative $\frac{dy}{dx}$ is only valid for $x > 0$.

□

Q5. (a) Find an expression (by implicit differentiation) for the derivative at the point (x, y) on the ellipse $x^2/3 + y^2/6 = 1$. Hence find the gradients of the tangent lines when $x = 1/4$.

(b) Differentiate $x^{\cos x}$ with respect to x.

Solution. (a) Differentiate both sides of $x^2/3 + y^2/6 = 1$ implicitly:

$$\begin{aligned} 0 = \frac{d}{dx}1 &= \frac{d}{dx}\left(\frac{x^2}{3} + \frac{y^2}{6}\right) \\ &= \frac{2}{3}x + \frac{1}{6}\frac{dy^2}{dx} \\ &= \frac{2}{3}x + \frac{1}{6}\frac{dy^2}{dy}\frac{dy}{dx} \\ &= \frac{2}{3}x + \frac{2}{6}y\frac{dy}{dx}. \end{aligned}$$

Therefore,

$$\frac{dy}{dx} = -\frac{2x}{y}.$$

Since $y^2/6 = 1 - x^2/3$ it follows that $y = \pm\sqrt{47/8}$ when $x = 1/4$. Hence the gradients of the tangent lines to the ellipse when $x = 1/4$ are given by

$$\frac{dy}{dx} = -2x/y = \pm\sqrt{2}/\sqrt{47}.$$

There are two answers because there are two points on the ellipse with x-value equal to $1/4$.

(b) You can use logarithmic differentiation or the chain rule here. Using logarithmic differentiation, let $y = x^{\cos x}$, so that $\ln y = \cos x \ln x$. Differentiating both sides with respect to x, implicitly on the left and using the product rule on the right, we get

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x}\cos x - \ln x \sin x,$$

so that

$$\frac{dy}{dx} = y\left(\frac{\cos x}{x} - \ln x \sin x\right) = x^{\cos x}\left(\frac{\cos x}{x} - \ln x \sin x\right).$$

Alternatively, by using the chain rule, $x = e^{\ln x}$ so $x^{\cos x} = \left(e^{\ln x}\right)^{\cos x} = e^{\ln x \cos x}$. Let $u = e^v$, so $u' = e^v$, and $v = \ln x \cos x$, so $v' = (\cos x)/x - \ln x \sin x$. By the chain Rule,

$$\begin{aligned} \frac{d}{dx}x^{\cos x} &= \frac{d}{dx}e^{\ln x \cos x} \\ &= \frac{du}{dv}.\frac{dv}{dx} \\ &= e^v\left(\frac{\cos x}{x} - \ln x \sin x\right) \\ &= x^{\cos x}\left(\frac{\cos x}{x} - \ln x \sin x\right). \end{aligned}$$

It is implicit that one should write the answer to this question in terms of x.

□

Q6. Prove that

$$(\sin x)^{(n)} = \sin\left(x + \frac{n\pi}{2}\right), \quad (\cos x)^{(n)} = \cos\left(x + \frac{n\pi}{2}\right)$$

Hint - use mathematical induction.

Solution. We prove the formula for $\sin x$. It is helpful to recall that $\sin(x + \pi/2) = \cos x$. Let $P(n)$ be the statement

$$(\sin x)^{(n)} = \sin\left(x + \frac{n\pi}{2}\right)$$

for each $n \in \mathbb{N}$. Then $P(1)$ is true since

$$(\sin(x))' = \cos x = \sin\left(x + \frac{\pi}{2}\right).$$

Suppose statement $P(k)$ is true for some $k \in \mathbb{N}$. Then,

$$(\sin x)^{(k+1)} = \frac{d}{dx}(\sin x)^{(k)} = \frac{d}{dx}\sin\left(x + \frac{k\pi}{2}\right) = \cos\left(x + \frac{k\pi}{2}\right) = \sin\left(x + \frac{(k+1)\pi}{2}\right).$$

Thus $P(k)$ is true implies $P(k+1)$ is true. Via mathematical induction, $P(n)$ is true for all $n \in \mathbb{N}$, as required. $\square$

Q7. Find the derivatives of the following functions.

(1) $y = \dfrac{x^2 + 4x + 3}{\sqrt{x}}$

(2) $g(x) = (x^2 + 1)^3 (x^2 + 2)^6$

(3) $B(u) = (u^3 + 1)(2u^2 - u - 6)$

(4) $g(t) = (t + 1)^{\frac{2}{3}} (2t^2 - 1)^3$

(5) $y = \dfrac{1}{t^3 - 2t^2 + 1}$

(6) $f(x) = \sqrt{\dfrac{1 + \sin x}{1 + \cos x}}$

(7) $y = \dfrac{x}{x + \frac{2}{x}}$

(8) $f(x) = \sqrt{x + \sqrt{x + \sqrt{x}}}$

(9) $y = \dfrac{t \sin t}{1 + t}$

(10) $y = [x + (x + \sin^2 x)^3]^4$

(11) $y = x \sin x \tan x$

(12) $y = \tan(\sec(\cos x))$

Solution. (Please note that you should provide full details for your assignments)

(1) $y = x^{3/2} + 4x^{1/2} + \dfrac{3}{x^{1/2}} \implies y' = \dfrac{3}{2}x^{1/2} + 2x^{-1/2} - \dfrac{3}{2}x^{-3/2}$.

(2) $g'(x) = 6x(x^2 + 1)^2(x^2 + 2)^6 + 12x(x^2 + 1)^3(x^2 + 2)^5$

(3) $B'(u) = 3u^2(2u^2 - u - 6) + (u^3 + 1)(4u - 1)$

(4) $g'(t) = \dfrac{2}{3}(t + 1)^{-1/3}(2t^2 - 1)^3 + 12t(t + 1)^{2/3}(2t^2 - 1)^2$

(5) $y' = -\dfrac{3t^2 - 4t}{(t^3 - 2t^2 + 1)^2}$

(6) $f'(x) = \dfrac{1}{2}\left(\dfrac{1 + \sin x}{1 + \cos x}\right)^{-1/2} \dfrac{\cos x(1 + \cos x) - (1 + \sin x)(-\sin x)}{(1 + \cos x)^2}$

(7) $y = \dfrac{x^2}{x^2 + 2} \implies y' = \dfrac{4x}{(x^2 + 2)^2}$

(8) $f'(x) = \dfrac{1}{2}\left(x + \sqrt{x + \sqrt{x}}\right)^{-1/2} \left(1 + \dfrac{1}{2}(x + \sqrt{x})^{-1/2}\right)\left(1 + \dfrac{1}{2}x^{-1/2}\right)$

(9) $y' = \dfrac{(\sin t + t\cos t)(1 + t) - t\sin t}{(1 + t)^2}$

(10) $y' = 4\left(x + (x + \sin^2 x)^3\right)^3 \left(1 + 3(x + \sin^2 x)^2(1 + 2\sin x \cos x)\right)$

(11) $y' = \sin x \tan x + x\cos x \tan x + x \sin x \sec^2 x$

(12) $y' = \sec^2(\sec(\cos x)) \sec(\cos x) \tan(\cos x)(-\sin x)$

$\square$

Q8. Use logarithmic differentiation to find the derivatives of the following curves $y = f(x)$:

$$(a)\ y = \sqrt{x}e^{x^2-x}(x+1)^{2/3}, \qquad (b)\ y = x^x,$$
$$(c)\ y = \sin(x^x), \qquad (d)\ y = x^{\sin x},$$
$$(e)\ y = (\sin x)^{\ln x}, \qquad (f)\ y = (\ln x)^{\sin x}.$$

Solution. (Please note that you should provide full details for your assignments)

$$(a)\ \ln y = \frac{1}{2}\ln x + (x^2 - x) + \frac{2}{3}\ln(x+1)$$
$$\implies y' = \sqrt{x}e^{x^2-x}(x+1)^{2/3}\left[\frac{1}{2x} + 2x - 1 + \frac{2}{3}\frac{1}{(x+1)}\right].$$
$$(b)\ \ln y = x\ln x \implies y' = x^x(\ln x + 1).$$
$$(c)\ y' = \cos(x^x)x^x(\ln x + 1).$$
$$(d)\ \ln y = \sin x \ln x \implies y' = x^{\sin x}\left(\cos x \ln x + \frac{\sin x}{x}\right).$$
$$(e)\ \ln y = \ln x \ln \sin x \implies y' = (\sin x)^{\ln x}\left(\frac{\ln \sin x}{x} + \ln x\frac{\cos x}{\sin x}\right).$$
$$(f)\ \ln y = \sin x \ln(\ln x)$$
$$\implies y' = (\ln x)^{\sin x}\left(\cos x \ln(\ln x) + \sin x \frac{1}{\ln x}\frac{1}{x}\right).$$

□

Q9. Find y' if $x^y = y^x$.

Solution. Using logarithmic differentitiation yields,

$$\begin{aligned} & y\ln x = x\ln y \\ & \implies y'\ln x + \frac{y}{x} = \ln y + \frac{x}{y}y' \\ & \implies y' = \frac{\ln y - \frac{y}{x}}{\ln x - \frac{x}{y}}. \end{aligned}$$

□

Q10. Suppose $f : \mathbb{R} \to \mathbb{R}$ is a function such that $|f(x) - f(y)| \leq |x - y|^\alpha$, for all $x, y \in \mathbb{R}$ with $\alpha > 1$. Show that $f(x) = C$ for some constant C. Hint: Show that f is differentiable at all points and compute the derivative.

Solution. If $f' = 0$ on the whole of $\mathbb{R}$ then f is a constant function. Observe that

$$0 \leq |f'(x)| = \lim_{h\to 0}\left|\frac{f(x+h) - f(x)}{h}\right| \leq \lim_{h\to 0}\frac{|h|^\alpha}{h} = 0 \quad \forall x \in \mathbb{R}.$$

Therefore, $f'(x) = 0$ for all $x \in \mathbb{R}$, and hence $f(x) = C$ for some $C \in \mathbb{R}$ for all $x \in \mathbb{R}$, as required. □

Q11. Let $f : (a, b) \to \mathbb{R}$ be an unbounded differentiable function. Show that $f' : (a, b) \to \mathbb{R}$ is unbounded.

Solution. Without loss of generality we assume that f is unbounded above. Since f is continuous, there exists a sequence $\{a_n\}$ such that $a_n \in (a,b)$ amd $f(a_n) = 2^n$ for each $n \in \mathbb{N}$ (via the IVT and definition of f being unbounded). Thus, via the mean value theorem, there exists a sequence $\{\xi_n\}$ with $\xi_n \in (a,b)$ for each $n \in \mathbb{N}$, such that

$$\frac{2^{n+1}-2^n}{b-a} \leq \frac{f(a_{n+1})-f(a_n)}{a_{n+1}-a_n} = f'(\xi_n) \quad \forall n \in \mathbb{N}.$$

Thus, f' is unbounded, as required. □

Q12. Using L'Hôpital's rule, or otherwise, prove that the function $f : (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$ given by

$$f(x) = \begin{cases} \frac{\tan x - x}{x^2}\,, & \text{if } x \neq 0 \\ 0\,, & \text{if } x = 0 \end{cases}$$

is differentiable at $x_0 = 0$ and state $f'(0)$. Is f continuous at $x_0 = 0$? Justify your answer.

Solution. Observe that

$$\lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0} \frac{\tan h - h}{h^3} = \lim_{h\to 0} \frac{\sec^2 h - 1}{3h^2} = \frac{2}{3} \lim_{h\to 0} \frac{\sec h - 1}{h^2} = \frac{1}{3}\,.$$

Thus $f'(0)$ exists and equals $\frac{1}{3}$. Since f is differentiable at $x = 0$, the necessarily f is continuous at $x = 0$ (via Theorem 4.5). □

Q13. Determine the following limits:

(a) $\displaystyle\lim_{x\to -1} \frac{x^2-1}{\sin(1+x)}$;

(b) $\displaystyle\lim_{x\to 0} \frac{k^x - 1}{x}$, where $k > 0$;

(c) $\displaystyle\lim_{x\to 0} \frac{x\sin x}{1-\cos x}$;

(d) $\displaystyle\lim_{x\to 0} \frac{e^x - 1 - x}{x^2}$.

Solution. (a) Let $f(x) = x^2 - 1$ and $g(x) = \sin(1+x)$. Note that $\lim_{x\to -1} f(x) = \lim_{x\to -1} g(x) = 0$. Moreover, observe that $f'(x) = 2x$ and $g'(x) = \cos(1+x)$. From L'Hôpital's rule, it follows that

$$\lim_{x\to -1} \frac{x^2-1}{\sin(1+x)} = \lim_{x\to -1} \frac{2x}{\cos(1+x)} = -2.$$

(b) Let $f(x) = k^x - 1$ and $g(x) = x$. Then $\lim_{x\to 0} f(x) = \lim_{x\to 0} g(x) = 0$. Note that $f'(x) = \ln(k)k^x$ and $g'(x) = 1$. From L'Hôpital's rule, it follows that

$$\lim_{x\to 0} \frac{k^x-1}{x} = \lim_{x\to 0} \frac{\ln(k)k^x}{1} = \ln k.$$

(c) Let $f(x) = x\sin x$ and $g(x) = 1 - \cos x$. Note that $\lim_{x\to 0} f(x) = \lim_{x\to 0} g(x) = 0$. Moreover, observe that $f'(x) = \sin x + x\cos x$ and $g'(x) = \sin x$. Thus $\lim_{x\to 0} f'(x) = \lim_{x\to 0} g'(x) = 0$. However, $f''(x) = \cos x + \cos x - x\sin x$ and $g''(x) = \cos x$. Thus, from L'Hôpital's rule applied twice, it follows that

$$\lim_{x\to 0} \frac{x\sin x}{1-\cos x} = \lim_{x\to 0} \frac{\sin x + x\cos x}{\sin x} = \lim_{x\to 0} \frac{\cos x + \cos x - x\sin x}{\cos x} = \frac{2}{1} = 2.$$

(d) Let $f(x) = e^x - 1 - x$ and $g(x) = x^2$. Note that $\lim_{x\to 0} f(x) = \lim_{x\to 0} g(x) = 0$. Moreover, observe that $f'(x) = e^x - 1$ and $g'(x) = 2x$. Thus $\lim_{x\to 0} f'(x) = \lim_{x\to 0} g'(x) = 0$. However $f''(x) = e^x$ and $g''(x) = 2$. Then, from L'Hôpital's rule applied twice

$$\lim_{x\to 0} \frac{e^x - 1 - x}{x^2} = \lim_{x\to 0} \frac{e^x - 1}{2x} = \lim_{x\to 0} \frac{e^x}{2} = \frac{1}{2}.$$

$\square$

Q14. Find $\frac{dy}{dx}$ by implicit differentiation.

(a) $x^2 - 4xy + y^2 = 4$.

(b) $\cos(xy) = x + \sin y$.

(c) $\tan\left(\frac{x}{y}\right) = x + y$.

Solution. (a) By implicit differentiation

$$2x - 4y - 4x\frac{dy}{dx} + 2y\frac{dy}{dx} = 0 \implies \frac{dy}{dx} = \frac{x - 2y}{2x - y}.$$

(b) By implicit differentiation

$$-\sin(xy)\left(y + x\frac{dy}{dx}\right) = 1 + \cos y\frac{dy}{dx} \implies \frac{dy}{dx} = -\frac{1 + y\sin(xy)}{\cos y + x\sin(xy)}.$$

(c) By implicit differentiation

$$\sec^2\left(\frac{x}{y}\right)\frac{y - x\frac{dy}{dx}}{y^2} = 1 + \frac{dy}{dx} \implies \frac{dy}{dx} = \frac{y\sec^2(\frac{x}{y}) - y^2}{y^2 + x\sec^2(\frac{x}{y})}.$$

$\square$

Q15. Show by implicit differentiation that the tangent to the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

at the point (x_0, y_0) is

$$\frac{x_0 x}{a^2} + \frac{y_0 y}{b^2} = 1\,.$$

Solution. Applying implicit differentiation we obtain

$$\frac{2x}{a^2} + \frac{2yy'}{b^2} = 0.$$

Rearranging this equation we obtain

$$y' = \frac{-b^2 x}{a^2 y}.$$

Therefore the tangent to the ellipse at (x_0, y_0) is given by

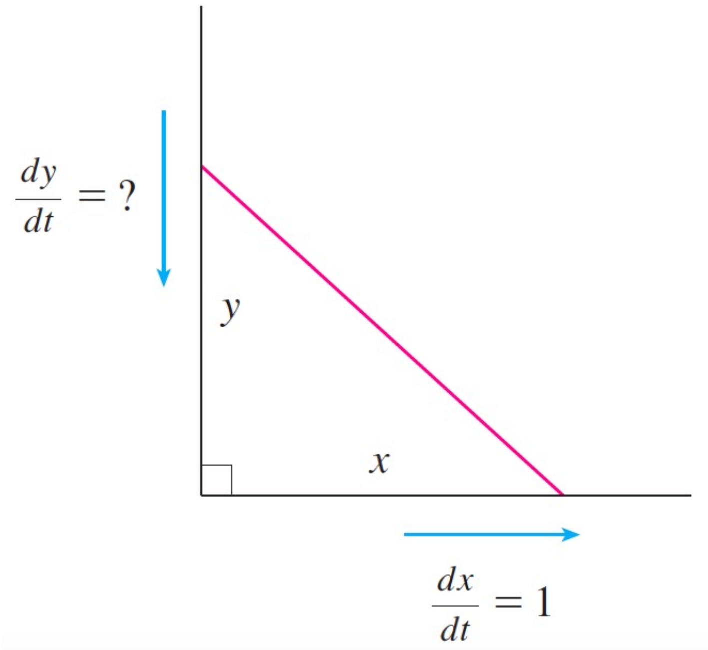
$$y - y_0 = \frac{-b^2 x_0}{a^2 y_0}(x - x_0).$$

Rearranging this equation and using $\frac{x_0^2}{a^2} + \frac{y_0^2}{b^2} = 1$ gives us

$$\frac{x_0 x}{a^2} + \frac{y_0 y}{b^2} = 1,$$

as required. $\square$

Q16. A ladder 10 m long rests against a vertical wall. If the bottom of the ladder slides away from the wall at a rate of 1 metre per second (m/s), how fast is the top of the ladder sliding down the wall when the bottom of the ladder is 6 m from the wall?



Solution. Let x m be the distance from the bottom of the ladder to the wall and y m the distance from the top of the ladder to the ground. Note that x and y are both functions of time t, measured in seconds s.

We are given that $\frac{dx}{dt} = 1$ m/s and we are asked to find $\frac{dy}{dt}$ when $x = 6$ m. The relationship between x and y is given by

$$x^2 + y^2 = 10^2 \tag{1}$$

i.e. both x and y are functions of t. Differentiating each side with respect to t using the chain rule, we have

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0.$$

Solving this equation for the desired rate, we obtain

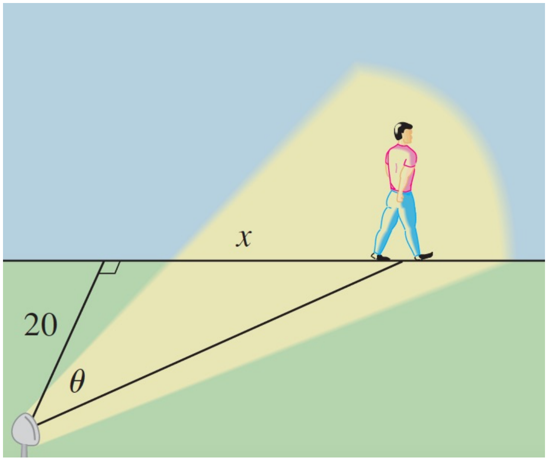
$$\frac{dy}{dt} = -\frac{x}{y}\frac{dx}{dt}. \tag{2}$$

When $x = 6$, then via (1), $y = 8$. Substituting $x = 6$, $y = 8$ and $\frac{dx}{dt} = 1$ into (2) gives

$$\frac{dy}{dt} = -\frac{3}{4} \text{ m/s}.$$

The fact that $\frac{dy}{dt}$ is negative means that the distance from the top of the ladder to the ground is decreasing at a rate of $\frac{3}{4}$ m/s. □

Q17. A man walks along a straight path at a speed of 4 m/s. A searchlight is located on the ground 20 m from the path and is kept focused on the man. At what rate is the searchlight rotating when the man is 15 m from the point on the path closest to the searchlight?



Solution. Let x be the distance from the man to the point on the path closest to the searchlight in m. Denote θ to be the angle in radians between the beam of the searchlight and the perpendicular to the path.

We are given that $\frac{dx}{dt} = 4$ m/s and are asked to find $\frac{d\theta}{dt}$ when $x = 15$. The equation that relates x and θ can be written as

$$x = 20\tan\theta. \tag{3}$$

By differentiating each side of (3) with respect to t, we get

$$\frac{dx}{dt} = 20\sec^2\theta\frac{d\theta}{dt}.$$

Since $\frac{dx}{dt} = 4$, it follows that

$$\frac{d\theta}{dt} = \frac{1}{20}\cos^2\theta\frac{dx}{dt} = \frac{1}{5}\cos^2\theta\,. \tag{4}$$

When $x = 15$ m, the length of the beam is 25 m. Hence $\cos\theta = \frac{4}{5}$. From (4) we conclude that

$$\frac{d\theta}{dt} = \frac{16}{125}\,\text{rads/s}.$$

□

Q18. A wire of length L is cut into two pieces. One piece is shaped into a circle and the other is shaped into a square. Let A_C be the area contained within the circle and A_S be the area contained within the square. What are the maximum and minimum values of $A_C + A_S$?

Solution. $0 \le x \le L$. Let x be the perimeter of the circle. Thus, the total area $A(x) = A_c(x) + A_s(x)$ is

$$A(x) = \pi\left(\frac{x}{2\pi}\right)^2 + \left(\frac{L-x}{4}\right)^2 = \left(\frac{1}{4\pi} + \frac{1}{16}\right)x^2 - \frac{L}{8}x + \frac{L^2}{16} \quad \forall x \in [0, L].$$

Thus,

$$A'(x) = \left(\frac{1}{2\pi} + \frac{1}{8}\right)x - \frac{L}{8} \quad \forall x \in [0, L].$$

It follows that

$$A'(x) = 0 \iff x = x_c = \frac{L}{\frac{4}{\pi} + 1}.$$

We note that $A(x_c) = \frac{L^2}{16+4\pi}$. We also note that $A(0) = \frac{L^2}{16}$ and that $A(L) = \frac{L^2}{4\pi}$. Therefore,

$$\min_{x\in[0,L]} A(x) = \frac{L^2}{16+4\pi} \le A(x) \le \frac{L^2}{4\pi} = \max_{x\in[0,L]} A(x) \quad \forall\, x \in [0,L],$$

as required. □

Q19. Suppose $f : [a,b] \to \mathbb{R}$ is differentiable and $c \in (a,b)$. Show there exists a sequence $\{x_n\}$ converging to c, with $x_n \neq c\ \forall n \in \mathbb{N}$, and such that

$$f'(c) = \lim_{n\to\infty} f'(x_n)\,.$$

Moreover, explain why this does not imply that f' is continuous.

Solution. Recall that

$$f'(c) = \lim_{h\to 0} \frac{f(c+h) - f(c)}{h}\,.$$

Let $h_n = c + 1/n$ for each $n \in \mathbb{N}$. Then via the mean value theorem, for each $n \in \mathbb{N}$ there exists $x_n \in (c, h_n)$ such that

$$\frac{f(c+h_n) - f(c)}{h_n} = f'(x_n)\,.$$

It follows that $x_n \to c$ and $f'(x_n) \to f'(c)$ as $n \to \infty$. This does not imply that f' is continuous at c, though. For continuity to hold for f' at c, we would require $f'(y_n) \to f'(c)$ to hold for any sequence $\{y_n\}$ for which $y_n \to c$ as $n \to \infty$. □

Q20. Assume that $f(x)$ is bounded on $[a,\infty)$ for some $a \in \mathbb{R}$, f is differentiable on (a,∞) and

$$\lim_{x\to\infty} f'(x) = b\,.$$

Prove that $b = 0$.

Solution. The mean value theorem implies that for any $x > a$, there exists $c \in (x, 2x)$, such that

$$\frac{f(2x) - f(x)}{x} = f'(c). \tag{5}$$

Since $\lim\limits_{x\to\infty} f'(x) = b$ exists, and $x \to \infty$ implies that $c \to \infty$ it follows that

$$\lim_{x\to\infty} f'(c) = b. \tag{6}$$

Alternatively, since $f(x)$ is bounded, we have

$$\lim_{x\to\infty} \frac{f(2x) - f(x)}{x} = 0\,. \tag{7}$$

Thus via (5), (6) and (7), we have $b = 0$, as required. □

Q21. Use L'Hôpital's rule to find the following limits, when it applies:

(a) $\lim\limits_{x\to\infty} \dfrac{\ln x}{\sqrt{x}}$.

(b) $\lim\limits_{x\to 0^+} \dfrac{\ln x}{x}$

(c) $\lim\limits_{x\to 1} \dfrac{x^8 - 1}{x^5 - 1}$

(d) $\lim\limits_{x\to 0} \left(\dfrac{1}{x} - \dfrac{1}{e^x - 1}\right)$

(e) $\lim_{x\to 0^+} (\tan 2x)^x$

(f) $\lim_{x\to\infty} \left(\frac{a^{1/x}+b^{1/x}}{2}\right)^x$ for $a, b > 0$

Solution.

(a) This is "$\frac{\infty}{\infty}$" form.

$$\lim_{x\to\infty} \frac{\ln x}{\sqrt{x}} = \lim_{x\to\infty} \frac{\frac{1}{x}}{\frac{1}{2}x^{\frac{1}{2}-1}} = 0.$$

(b) This is "$\frac{\infty}{0}$" form and L'Hôpital's rule does not apply.

$$\lim_{x\to 0^+} \frac{\ln x}{x} = -\infty\,.$$

(c) This is "$\frac{0}{0}$" form.

$$\lim_{x\to 1} \frac{x^8-1}{x^5-1} = \lim_{x\to 1} \frac{8x^7}{5x^4} = \frac{8}{5}.$$

(d) This is "$\infty-\infty$" form.

$$\begin{aligned}\lim_{x\to 0}\left(\frac{1}{x}-\frac{1}{e^x-1}\right) &= \lim_{x\to 0}\frac{e^x-1-x}{x(e^x-1)} \qquad \text{“}\frac{0}{0}\text{” form}\\ &= \lim_{x\to 0}\frac{e^x-1}{e^x-1+xe^x}\\ &= \lim_{x\to 0}\frac{e^x}{2e^x+xe^x}\\ &= \lim_{x\to 0}\frac{1}{2+x}\\ &= \frac{1}{2}.\end{aligned}$$

(e) This is "0^0" form.

$$\begin{aligned}\lim_{x\to 0^+}(\tan(2x))^x &= \lim_{x\to 0^+} e^{x\ln\tan(2x)}\\ &= \exp\left(\lim_{x\to 0^+}\frac{\ln(\tan(2x))}{\frac{1}{x}}\right) \qquad \text{“}\frac{\infty}{\infty}\text{” form}\\ &= \exp\left(\lim_{x\to 0^+}\frac{\frac{1}{\tan 2x}\sec^2(2x)2}{-\frac{1}{x^2}}\right)\\ &= \exp\left(\lim_{x\to 0^+}\frac{2x}{\sin 2x}\lim_{x\to 0^+}\frac{-2x}{\cos x}\right)\\ &= e^0\\ &= 1\,.\end{aligned}$$

(f) This is "1^∞" form.

$$\begin{aligned}
\lim_{x\to\infty}\left(\frac{a^{\frac{1}{x}}+b^{\frac{1}{x}}}{2}\right)^x &= \lim_{x\to\infty}\exp\left(\frac{\ln\left(\frac{a^{\frac{1}{x}}+b^{\frac{1}{x}}}{2}\right)}{\frac{1}{x}}\right) \qquad \text{“}\frac{0}{0}\text{” form}\\
&= \exp\left(\lim_{x\to\infty}\frac{\left(\frac{a^{1/x}+b^{1/x}}{2}\right)^{-1}\left(\frac{1}{2}a^{1/x}\ln a(-\frac{1}{x^2})+\frac{1}{2}b^{1/x}\ln b(-\frac{1}{x^2})\right)}{-\frac{1}{x^2}}\right)\\
&= \exp\left(\lim_{x\to\infty}\left(\frac{a^{1/x}+b^{1/x}}{2}\right)^{-1}\frac{1}{2}\left(a^{1/x}\ln a+b^{1/x}\ln b\right)\right)\\
&= e^{\frac{\ln a+\ln b}{2}}\\
&= \sqrt{ab}\,.
\end{aligned}$$

$\square$

Q22. Find the limit

$$\lim_{x\to\infty}\frac{(x+2)^{\frac{1}{x}}-x^{\frac{1}{x}}}{(x+3)^{\frac{1}{x}}-x^{\frac{1}{x}}}.$$

Solution. Observe that

$$\lim_{x\to\infty}\frac{(x+2)^{\frac{1}{x}}-x^{\frac{1}{x}}}{(x+3)^{\frac{1}{x}}-x^{\frac{1}{x}}}=\lim_{x\to\infty}\frac{\left(1+\frac{2}{x}\right)^{\frac{1}{x}}-1}{\left(1+\frac{3}{x}\right)^{\frac{1}{x}}-1}\tag{8}$$

which is a "$\frac{0}{0}$" form. Let

$$y=\left(1+\frac{a}{x}\right)^{1/x},$$

for constant $a>0$ and $x>0$. Then,

$$\ln y=\frac{1}{x}\ln\left(1+\frac{a}{x}\right)\implies y'=-\left(1+\frac{a}{x}\right)^{1/x}\left[\frac{a}{x^2}\left(\frac{1}{x+a}\right)+\frac{1}{x^2}\ln\left(1+\frac{a}{x}\right)\right].\tag{9}$$

Therefore, via L'Hôpital's rule, it follows from (8) and (9) that

$$\begin{aligned}
\lim_{x\to\infty}\frac{\left(1+\frac{2}{x}\right)^{\frac{1}{x}}-1}{\left(1+\frac{3}{x}\right)^{\frac{1}{x}}-1} &= \lim_{x\to\infty}\frac{\left(1+\frac{2}{x}\right)^{1/x}\left[\frac{2}{x^2}\frac{1}{x+2}+\frac{1}{x^2}\ln(1+\frac{2}{x})\right]}{\left(1+\frac{3}{x}\right)^{1/x}\left[\frac{3}{x^2}\frac{1}{x+3}+\frac{1}{x^2}\ln(1+\frac{3}{x})\right]}\\
&= \lim_{x\to\infty}\frac{\frac{2}{x+2}+\ln(1+\frac{2}{x})}{\frac{3}{x+3}+\ln(1+\frac{3}{x})}\\
&= \lim_{x\to\infty}\frac{-\frac{2}{(x+2)^2}-\frac{2}{x(x+2)}}{-\frac{3}{(x+3)^2}-\frac{3}{x(x+3)}}\\
&= \lim_{x\to\infty}\frac{(x+3)^2}{(x+2)^2}\lim_{x\to\infty}\frac{2(1+\frac{x+2}{x})}{3(1+\frac{x+3}{x})}\\
&= \frac{2}{3}\,.
\end{aligned}$$

$\square$

Q23. Prove that
$$\ln(1+x) < \frac{x}{\sqrt{1+x}}$$
for all $x > 0$.

Solution. Let $f : [0, \infty) \to \mathbb{R}$ be given by
$$f(x) = \ln(1+x) - \frac{x}{\sqrt{1+x}} \qquad \forall\, x \geq 0.$$
Thus $f(0) = 0$ and

$$f'(x) = \frac{1}{1+x} - \frac{1}{\sqrt{x+1}} + \frac{x}{2\sqrt{x+1}(x+1)} = \frac{2\sqrt{1+x}-2-x}{2\sqrt{1+x}(1+x)} \tag{10}$$

for all $x > 0$. Since
$$(2\sqrt{1+x}-2-x)_{x=0} = 0 \quad \text{ and } \quad (2\sqrt{1+x}-2-x)' = \frac{1}{\sqrt{1+x}} - 1 < 0 \quad \forall\, x > 0$$
it follows that $2\sqrt{1+x}-2-x < 0$ for all $x > 0$. Hence, via (10) it follows that $f'(x) < 0$ for all $x > 0$. This $f(x) < 0$ for all $x < 0$ which implies that
$$\ln(1+x) < \frac{x}{\sqrt{1+x}}$$
for all $x > 0$ as required. □

Q24. Use Taylor's Theorem to approximate the following functions $f : Dom(f) \to \mathbb{R}$ about points $x \in Dom(f)$. Moreover, state an appropriate error term for your approximation in each case.

(a) $f(x) = \tan x$ about $x = 0$, accurate to order 3 terms.
(b) $f(x) = e^x$ about $x = 0$, accurate to order 4 terms.
(c) $f(x) = \ln x$ about $x = 1$, accurate to order 4 terms.
(d) $f(x) = \cos x - 1$ about $x = 2\pi$, accurate to order 4 terms.

Solution. (a) Let $f : (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$ be given by $f(x) = \tan x$ for all $x \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Observe that
$$f(x) = \frac{\sin x}{\cos x},$$
$$f'(x) = \frac{1}{\cos^2 x},$$
$$f''(x) = \frac{2\sin x}{\cos^3 x},$$
$$f'''(x) = \frac{2}{\cos^4 x} + \frac{2\tan^2 x}{\cos^2 x},$$
for all $x \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Thus $f(0) = 0$, $f'(0) = 1$, $f''(0) = 0$ and $f'''(0) = 2$. Via Taylor's theorem it follows that
$$f(x) = \frac{0x^0}{0!} + \frac{1x}{1!} + \frac{0x^2}{2!} + \frac{2x^3}{3!} + R_3(x) = x + \frac{x^3}{3} + R_3(x),$$
with $R_3(x) = \frac{f^{(4)}(c)x^4}{4!}$ for some c between 0 and x.

(b) Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = e^x$ for all $x \in \mathbb{R}$. Observe that
$$f^{(n)}(x) = f(x) = e^x \quad \forall x \in \mathbb{R} \text{ and } n \in \mathbb{N} \cup \{0\}.$$
Thus $f^(n)(0) = 1$ for all $n \in \mathbb{N} \cup \{0\}$. Via Taylor's theorem it follows that

$$f(x) = \frac{1x^0}{0!} + \frac{1x}{1!} + \frac{1x^2}{2!} + \frac{1x^3}{3!} + \frac{1x^4}{4!} + R_4(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + R_4(x) \tag{11}$$

with $R_4(x) = \frac{e^c x^5}{5!}$ for some c between 0 and x.

(c) Let $f : (0, \infty) \to \mathbb{R}$ be given by $f(x) = \ln x$ for all $x \in (0, \infty)$. Observe that

$$\begin{aligned} f(x) &= \ln x, \\ f'(x) &= \frac{1}{x}, \\ f''(x) &= \frac{-1}{x^2}, \\ f'''(x) &= \frac{2}{x^3}, \\ f''''(x) &= \frac{-6}{x^4}, \end{aligned}$$

for all $x \in (0, \infty)$. Thus $f(1) = 0$, $f'(1) = 1$, $f''(1) = -1$, $f'''(1) = 2$ and $f''''(1) = -6$. Via Taylor's theorem it follows that

$$\begin{aligned} f(x) &= \frac{0(x-1)^0}{0!} + \frac{1(x-1)}{1!} + \frac{-1(x-1)^2}{2!} + \frac{2(x-1)^3}{3!} + \frac{-6(x-1)^4}{4!} + R_4(x) \\ &= (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + R_4(x) \end{aligned}$$

with $R_4(x) = \frac{f^{(5)}(c)(x-1)^5}{5!}$ for some c between 1 and x.

(d) Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = \cos x - 1$ for all $x \in \mathbb{R}$. Observe that

$$\begin{aligned} f(x) &= \cos x - 1, \\ f'(x) &= -\sin x, \\ f''(x) &= -\cos x, \\ f'''(x) &= \sin x, \\ f''''(x) &= \cos x, \end{aligned}$$

for all $x \in \mathbb{R}$. Thus $f(2\pi) = 0$, $f'(2\pi) = 0$, $f''(2\pi) = -1$, $f'''(2\pi) = 0$ and $f''''(2\pi) = 1$. Via Taylor's theorem it follows that

$$\begin{aligned} f(x) &= \frac{0(x-2\pi)^0}{0!} + \frac{0(x-2\pi)}{1!} + \frac{-1(x-2\pi)^2}{2!} + \frac{0(x-2\pi)^3}{3!} + \frac{1(x-2\pi)^4}{4!} + R_4(x) \\ &= -\frac{(x-2\pi)^2}{2!} + \frac{(x-2\pi)^4}{4!} + R_4(x) \end{aligned}$$

with $R_4(x) = \frac{-\sin(c)(x-1)^5}{5!}$ for some c between 2π and x.

$\square$

Q25. Determine the types of stationary points for $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = x^4 - 6x^2 + 8x + 1 \quad \forall x \in \mathbb{R}.$$

Solution. Observe that

$$\begin{aligned} f'(x) &= 4x^3 - 12x + 8, \\ f''(x) &= 12x^2 - 12, \end{aligned}$$

for all $x \in \mathbb{R}$. On sight $x = 1$ (and after factorisation) $x = -2$ are the only values of x at which $f'(x) = 0$. Since $f''(2) > 0$, f has a local minima at $x = 2$. Since $f''(1) = 0$, and $f''(1 + h) > 0$ with $f''(1 - h) < 0$ for all sufficiently small $h > 0$ then it follows that f has an inflection point at $x = 1$. $\square$

Q26. Sketch the curve

$$y = \frac{2x^2}{x^2 - 1}\,.$$

Solution. A The domain is $\{x \in \mathbb{R} \;:\; x^2 - 1 \neq 0\} = \{x \in \mathbb{R} \setminus \{-1, 1\}\}$.

B The x- and y- intercepts are both 0.

C The curve is symmetric about the y-axis. You can observe this since $\frac{2x^2}{x^2-1} = \frac{2(-x)^2}{(-x)^2-1}$.

D $\displaystyle\lim_{x\to\pm\infty} \frac{2x^2}{x^2-1} = 2$, therefore the line $y = 2$ is a horizontal asymptote. Since the denominator is 0 when $x = \pm 1$, we compute the following limits:

$$\lim_{x\to 1^-} \frac{2x^2}{x^2-1} = -\infty\,, \quad \lim_{x\to 1^+} \frac{2x^2}{x^2-1} = \infty$$

$$\lim_{x\to -1^+} \frac{2x^2}{x^2-1} = -\infty\,, \quad \lim_{x\to -1^+} \frac{2x^2}{x^2-1} = \infty\,.$$

Therefore the lines $x = 1$ and $x = -1$ are vertical asymptotes.

E Observe that

$$f'(x) = \frac{\left(x^2-1\right)(4x) - 2x^2 \cdot 2x}{\left(x^2-1\right)^2} = \frac{-4x}{\left(x^2-1\right)^2}$$

for all $x \in \mathbb{R} \setminus \{-1, 1\}$. Since $f'(x) > 0$ when $x < 0$ and $f'(x) < 0$ when $x > 0$, we infer that f is increasing on $(-\infty, 0) \setminus \{-1\}$ and decreasing on $(0, \infty) \setminus \{1\}$.

F $y' = 0 \iff x = 0$ so the only critical point is located at $x = 0$.

G Note that

$$f''(x) = \frac{\left(x^2-1\right)^2(-4) + 4x \cdot 2\left(x^2-1\right)2x}{\left(x^2-1\right)^4} = \frac{12x^2+4}{\left(x^2-1\right)^3}$$

for all $x \in \mathbb{R} \setminus \{-1, 1\}$. Since $f''(0) = -4 < 0$ if follows that there is a local maximum at $(0, 0)$. Moreover, $f''(x) > 0 \iff |x| > 1$. Thus the curve is concave upward on $\mathbb{R} \setminus [-1, 1]$ and concave downward on $(-1, 1)$. It has no points of inflection since 1 and -1 are not in the domain of f.

H For a sketch see Figure 1.

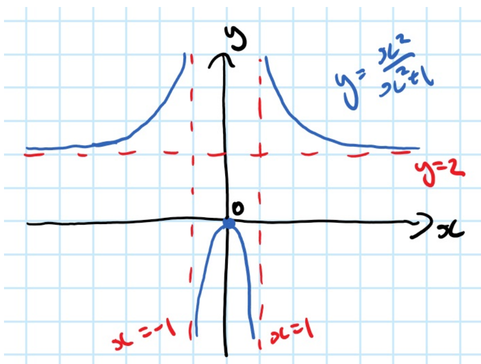


FIGURE 1. A sketch of the curve in Question 20

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Q27. Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by $f(x) = x^2 e^x$ for all $x \in \mathbb{R}$.

(i) Calculate f' and f''.
(ii) Find and determine the nature of the stationary points of f.
(iii) Find the points of inflection of f.
(iv) Determine the regions in which f is strictly increasing and decreasing.
(v) Determine the regions in which f is concave up and concave down.
(vi) Determine all the asymptotes of f.
(vii) Sketch the graph of f.

Solution. (i) Note that

$$f'(x) = x(x+2)e^x,$$
$$f''(x) = (x^2 + 4x + 2)e^x,$$

for all $x \in \mathbb{R}$.

(ii) Observe that $f'(x) = 0 \implies x = 0$ or $x = -2$. Moreover, $f''(0) > 0$ means there is a local minima at $x = 0$. Also $f''(-2) < 0$ means that there is a local maxima at $x = -2$.

(iii) Observe that $f''(x) = 0 \implies x = \frac{-4\pm\sqrt{8}}{2} = -2\pm\sqrt{2}$. It follows that f'' changes signs at these two points. So there are points of inflection at $x = -2 \pm \sqrt{2}$.

(iv) Observe that $f'(x) > 0 \iff x < -2$ or $x > 0$, where, f is strictly increasing. Also $f'(x) > 0 \iff -2 < x < 0$, where f is strictly decreasing.

(v) Observe that $f''(x) > 0 \iff x < -2-\sqrt{2}$ or $x > -2+\sqrt{2}$, where f is concave up. Also, $f''(x) < 0 \iff -2 - \sqrt{2} < x < -2 + \sqrt{2}$, where f is concave down.

(vi) There are no vertical asymptotes. Moreover, $f(x) \to 0$ as $x \to -\infty$ and $f(x) \to \infty$ as $x \to \infty$. Thus, there is a horizontal asymptote at $y = 0$.

(vii) See Figure 2.

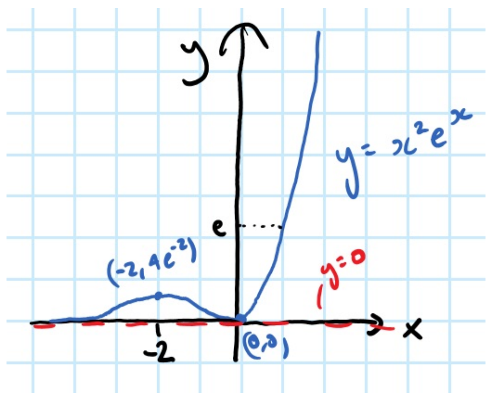

FIGURE 2. Sketch of $f(x) = x^2x^x$.

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