Lecture 21
Diffraction
Y&F Chapt 36

Fraunhofer and Fresnel Diffraction

Fraunhofer – far field
when incoming and outgoing waves
can be considered planar or parallel. Diffraction pattern
viewed at a long distance from the diffracting object.

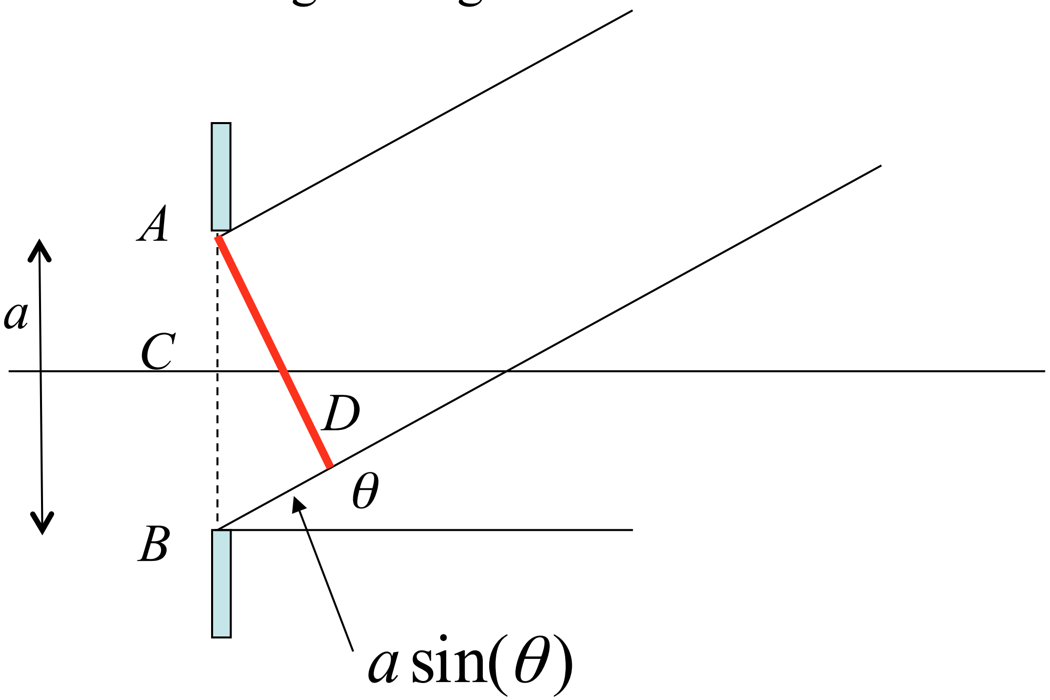
Fresnel – near field
when the slit and source are at finite
distances from each other

Fraunhofer if: $$d > \frac{a^2}{\lambda}$$

d is the smaller of source and screen distance from slit, and a is slit size

Here we consider only Fraunhofer .

Diffraction through a single slit

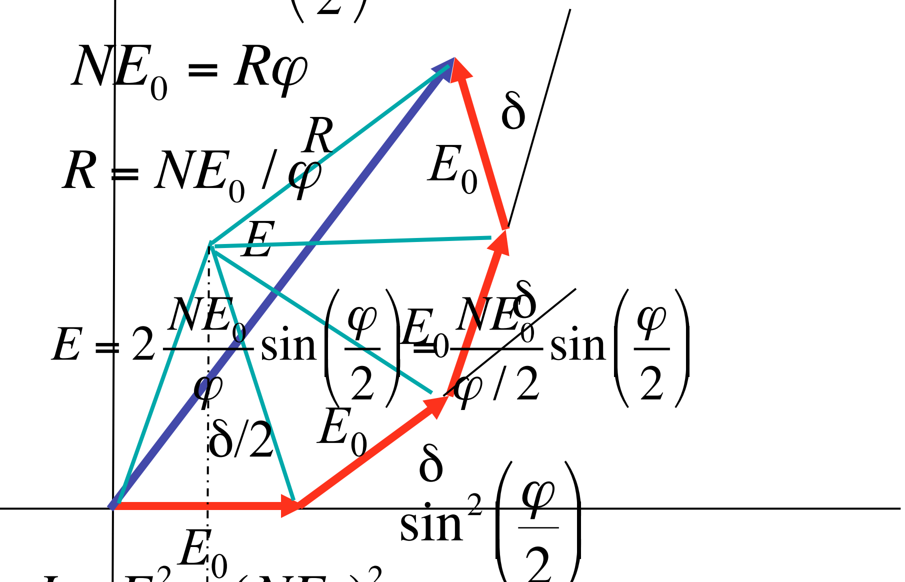


Consider wavefront to be composed of a large number of emitters A to B

Total phase difference between wavelets from A and B is:

$$\phi = 2\pi \frac{a\sin(\theta)}{\lambda}$$

Phase diff of each source is: $\dfrac{\phi}{N} = \dfrac{2\pi}{N}\dfrac{a\sin(\theta)}{\lambda}$



$$E = 2R\sin\left(\frac{\varphi}{2}\right)$$

$$NE_0 = R\varphi$$

$$R = NE_0/\varphi$$

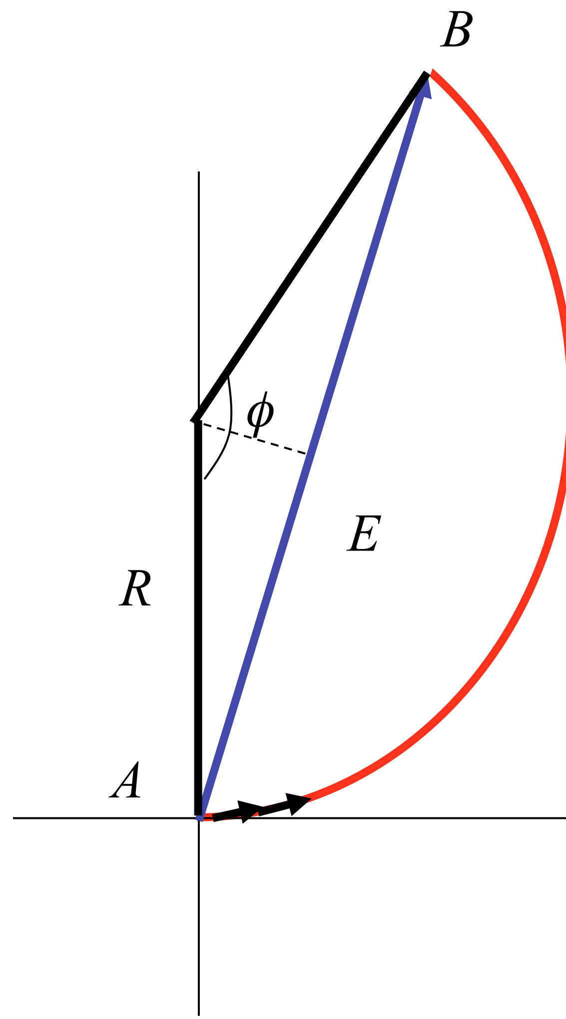
$$E = 2\frac{NE_0}{\varphi}\sin\left(\frac{\varphi}{2}\right) = \frac{NE_0}{\varphi/2}\sin\left(\frac{\varphi}{2}\right)$$

$$I \propto E^2 = (NE_0)^2\,\frac{\sin^2\left(\frac{\varphi}{2}\right)}{(\varphi/2)^2}$$

$\varphi = 0, E = NE_{0,}$

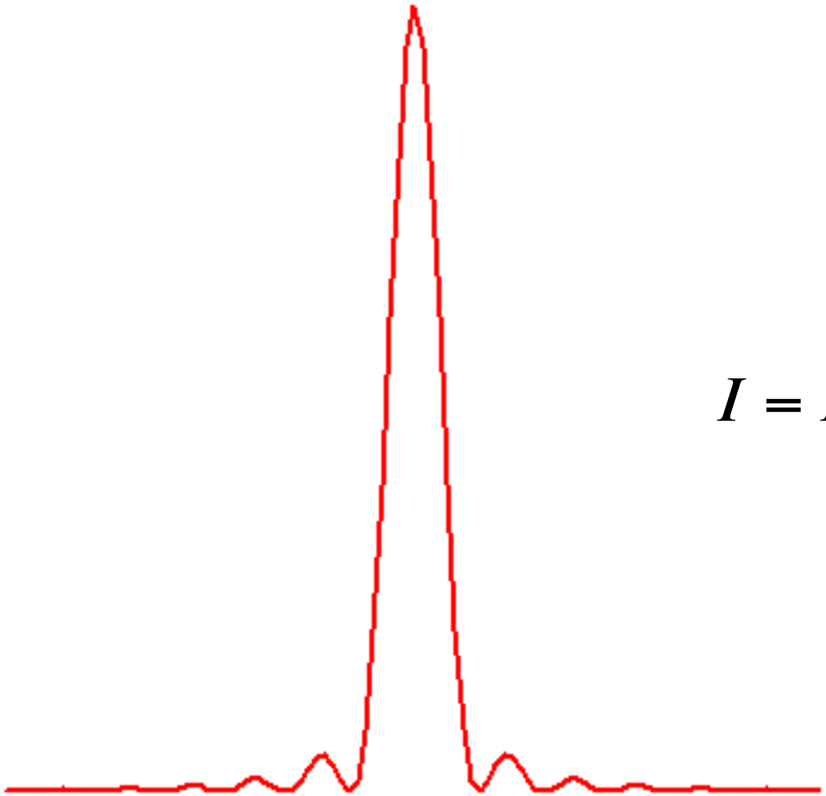
$I(0) = N^2E_0^2,$

this is the intensity for $\varphi = 0$, i.e. $\theta = 0$



$$\phi = 2\pi\frac{a\sin(\theta)}{\lambda}$$

$$\frac{I}{I(0)} = \frac{E^2}{N^2E_0^2} = \frac{\sin^2\left(\frac{\varphi}{2}\right)}{\left(\frac{\varphi}{2}\right)^2}$$



$$I = I_0 \frac{\sin^2\left(\frac{\varphi}{2}\right)}{\left(\frac{\varphi}{2}\right)^2}$$

$I_0 = I(0)$, the intensity at zero angle

Compute location of max and minima

I is a max/min when dI/dϕ=0

$$I = I_0 \frac{\sin^2\left(\frac{\varphi}{2}\right)}{\left(\frac{\varphi}{2}\right)^2} = uv, \quad \sin^2\left(\frac{\varphi}{2}\right) = u, \quad \frac{1}{\left(\frac{\varphi}{2}\right)^2} = v$$

$$\frac{dI}{d\varphi} = \frac{2\sin\left(\frac{\varphi}{2}\right)\cos\left(\frac{\varphi}{2}\right)\frac{1}{2}}{\left(\frac{\varphi}{2}\right)^2} + \frac{\sin^2\left(\frac{\varphi}{2}\right)(-2)\frac{1}{2}}{\left(\frac{\varphi}{2}\right)^3} = 0$$

$$\frac{\sin\left(\frac{\varphi}{2}\right)\cos\left(\frac{\varphi}{2}\right)}{\left(\frac{\varphi}{2}\right)^2} = \frac{\sin^2\left(\frac{\varphi}{2}\right)}{\left(\frac{\varphi}{2}\right)^3}$$

$$\sin\left(\frac{\varphi}{2}\right)\left(\frac{\varphi}{2}\cos\left(\frac{\varphi}{2}\right) - \sin\left(\frac{\varphi}{2}\right)\right) = 0$$

min. when

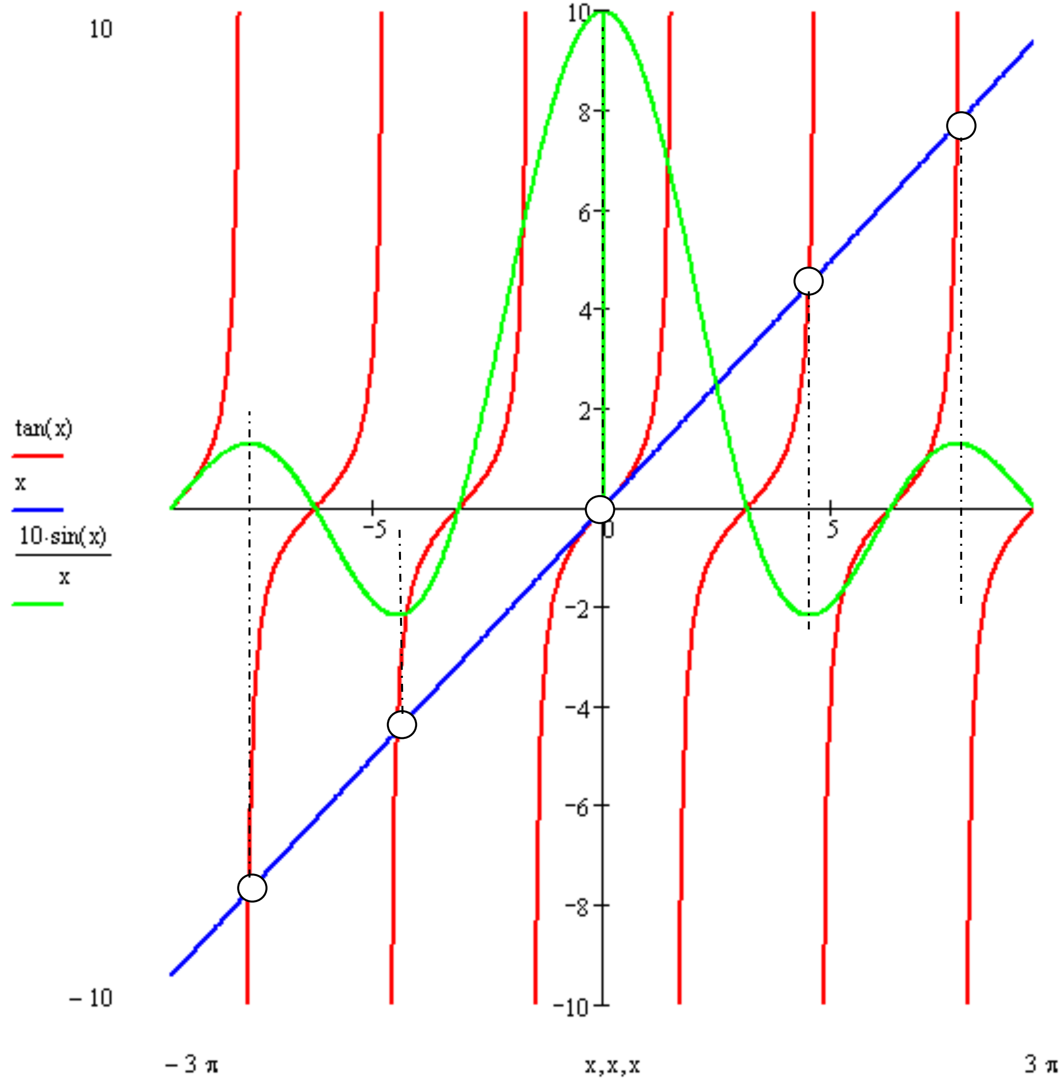
$$\sin\left(\frac{\varphi}{2}\right) = 0, \quad \text{excluding } \varphi = 0$$

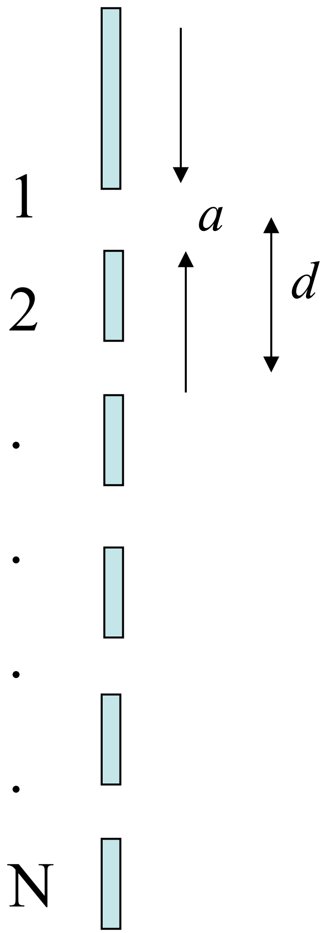
intensity is zero.

max

$$\left(\frac{\varphi}{2}\cos\left(\frac{\varphi}{2}\right) - \sin\left(\frac{\varphi}{2}\right)\right) = 0$$

$$\frac{\varphi}{2} = \tan\left(\frac{\varphi}{2}\right)$$





$$I_{\text{single}} = I_0 \frac{\sin^2\left(\frac{\varphi}{2}\right)}{\left(\frac{\varphi}{2}\right)^2} \quad \text{Single slit} \qquad \phi = 2\pi \frac{a\sin(\theta)}{\lambda}$$

$$I = I_{\text{single}} \frac{\sin^2\left(N\frac{\delta}{2}\right)}{\sin^2\left(\frac{\delta}{2}\right)} \quad \text{N-slits} \qquad \delta = \frac{2\pi}{\lambda} d\sin\theta$$

Combining both the response of the diffraction patterns

$$I = I_0 \frac{\sin^2\left(\frac{N\delta}{2}\right)}{\sin^2\left(\frac{\delta}{2}\right)^2} \frac{\sin^2\left(\frac{\phi}{2}\right)}{\left(\frac{\phi}{2}\right)^2}$$

Where are the maxima and the minima? (depends on δ and ϕ)

$$I = I_0 \frac{\sin^2\left(\frac{N\delta}{2}\right)}{\sin^2\left(\frac{\delta}{2}\right)} \frac{\sin^2\left(\frac{\phi}{2}\right)}{\left(\frac{\phi}{2}\right)^2}$$

single slit(blue)

For multi-slit interf. Max when:

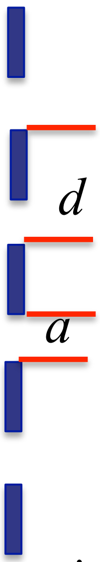
$$\delta = 2m\pi, \quad \frac{d\sin(\theta)}{\lambda} = m$$

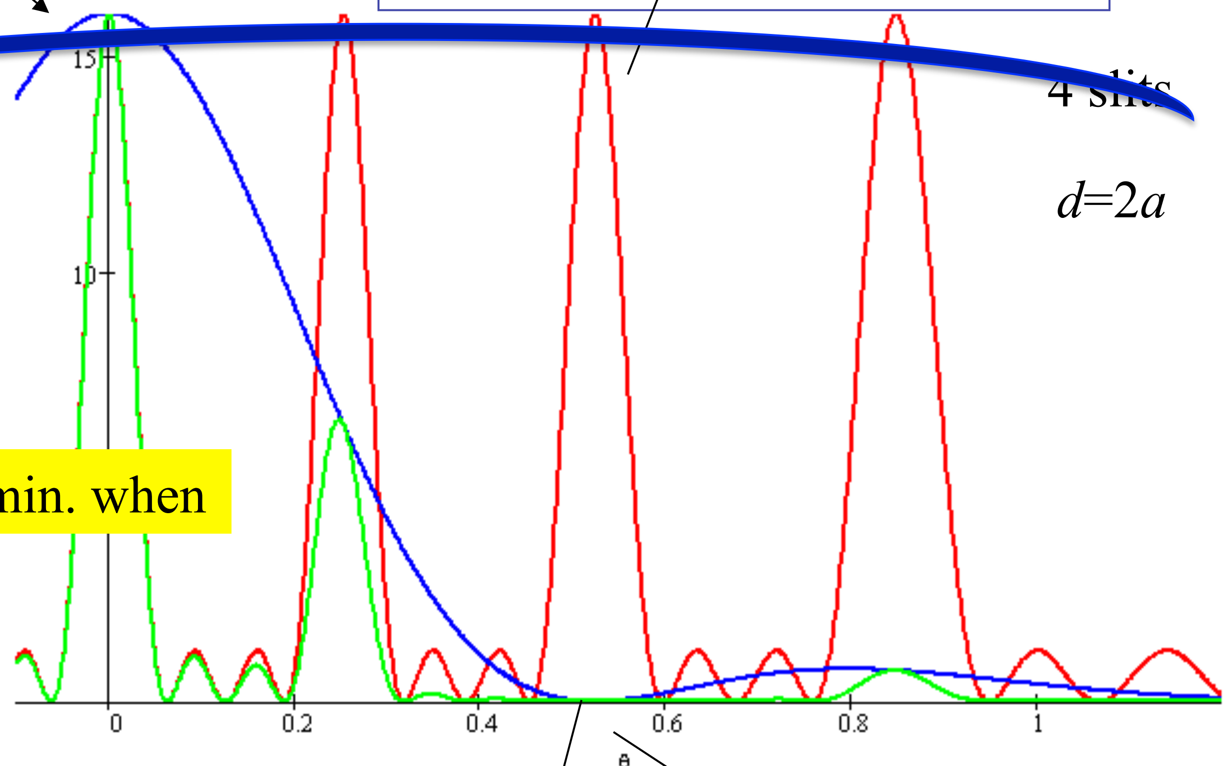
$$\sin(\theta) = m\lambda / d$$

$$\varphi = 2\pi \frac{a\sin(\theta)}{\lambda}$$

4 slits

d=2a





Diffraction has min. when

$$\sin\left(\frac{\varphi}{2}\right) = 0$$

$$i.e. \frac{\varphi}{2} = \pi, 2\pi, 3\pi, 4\pi, ..$$

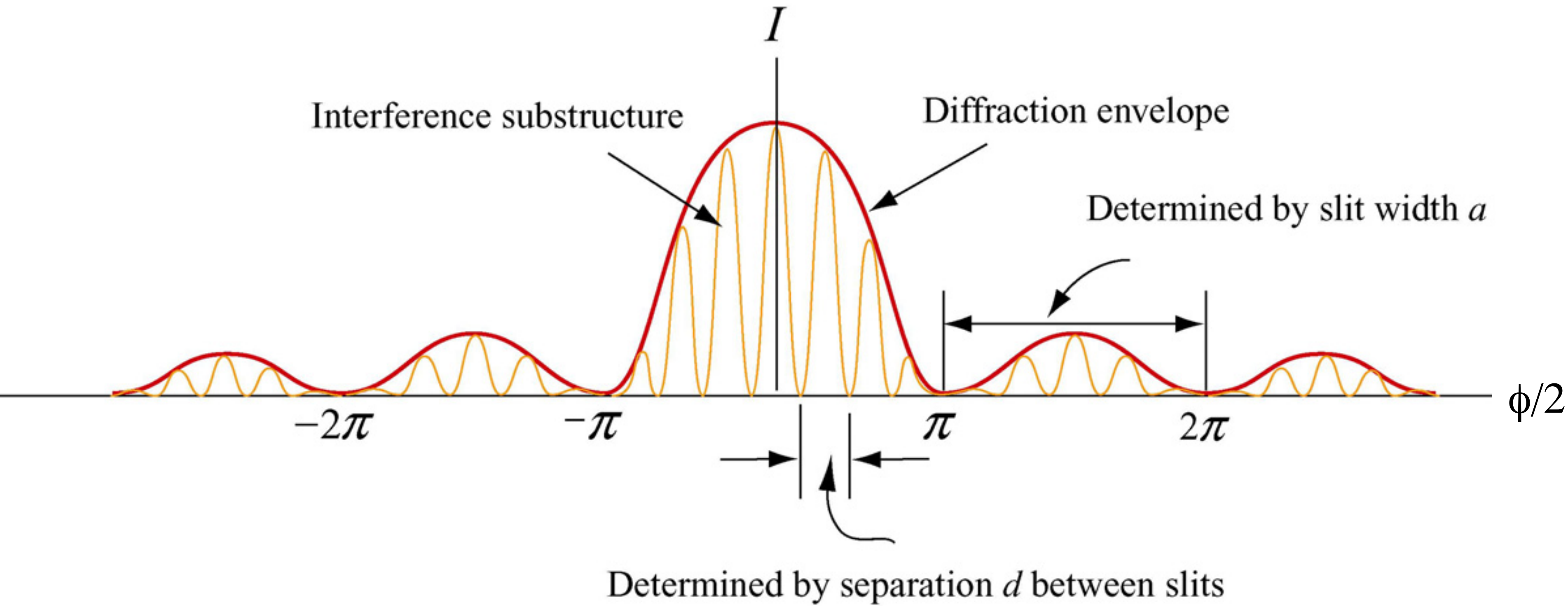
$$\frac{a\sin(\theta)}{\lambda} = 1, 2, 3, 4, ...$$

$$\sin(\theta) = \lambda / a,\ 2\lambda / a,\ 3\lambda / a, ...$$

$$\sin(\theta) = \lambda / a$$

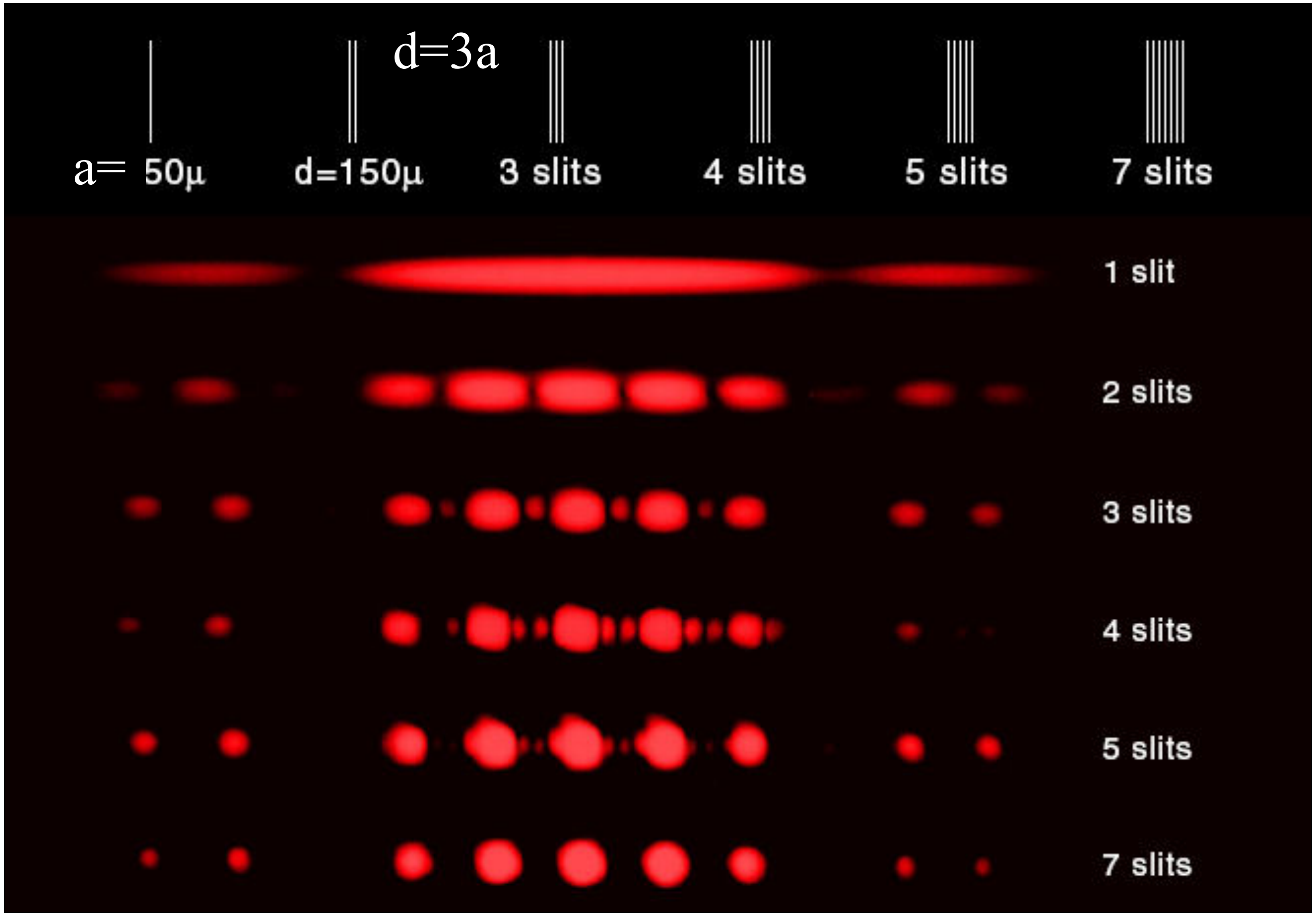
$$d\sin(\theta) = 2\lambda$$

Can get missing orders in the diffraction pattern

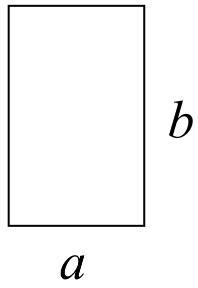


Double-slit interference with diffraction.

DiffractionDiff



Two dimensional diffraction (rectangular slit):

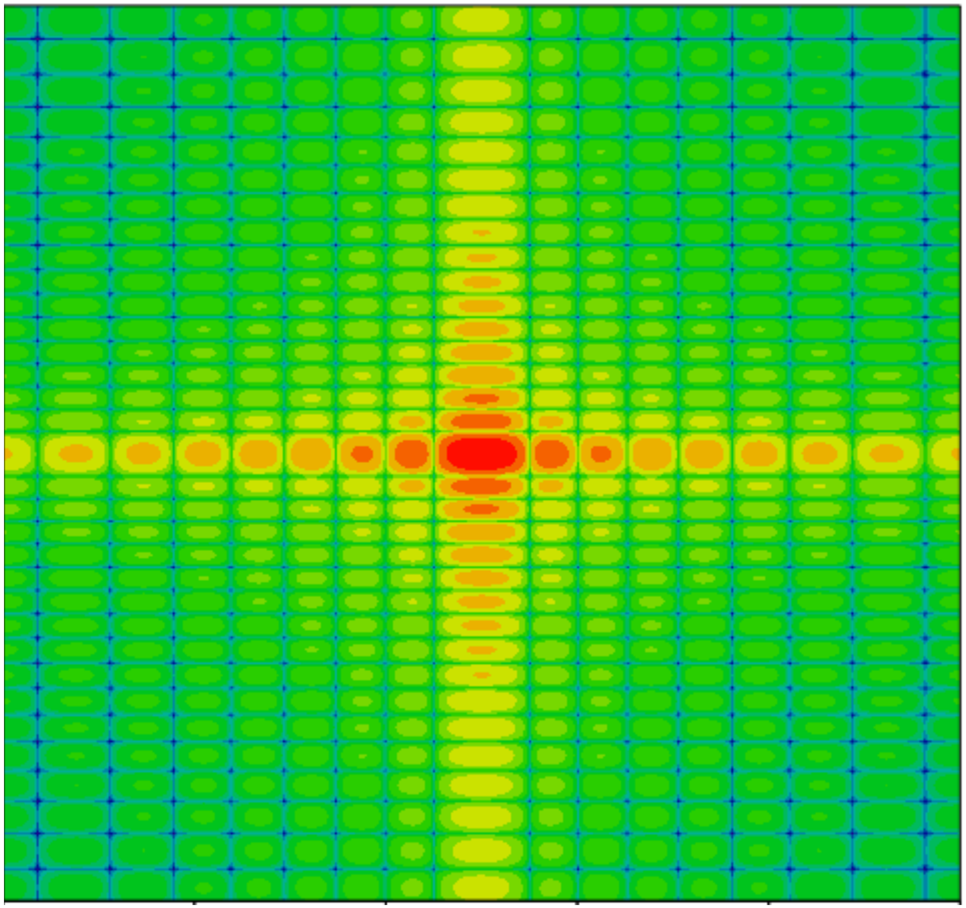


Rectangular slit

$$\alpha = 2\pi \frac{a \sin(\theta)}{\lambda}$$

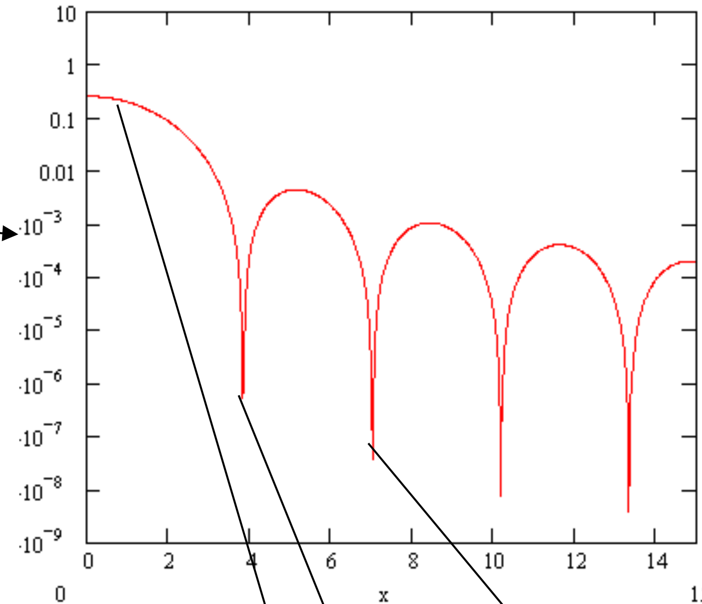
$$\beta = 2\pi \frac{b \sin(\theta')}{\lambda}$$

$$I = I_0 \frac{\sin^2\left(\frac{\alpha}{2}\right)}{\left(\frac{\alpha}{2}\right)^2} \frac{\sin^2\left(\frac{\beta}{2}\right)}{\left(\frac{\beta}{2}\right)^2}$$



The circular aperture

$$I = I_0 \left(\frac{2J_1\left(\frac{2\pi}{\lambda} a \sin\theta\right)}{\frac{2\pi}{\lambda} a \sin\theta} \right)^2$$



note:
log scale

J_1 is a first order Bessel function
$J_1(x)$=0 at x=0, 3.83, 7.018

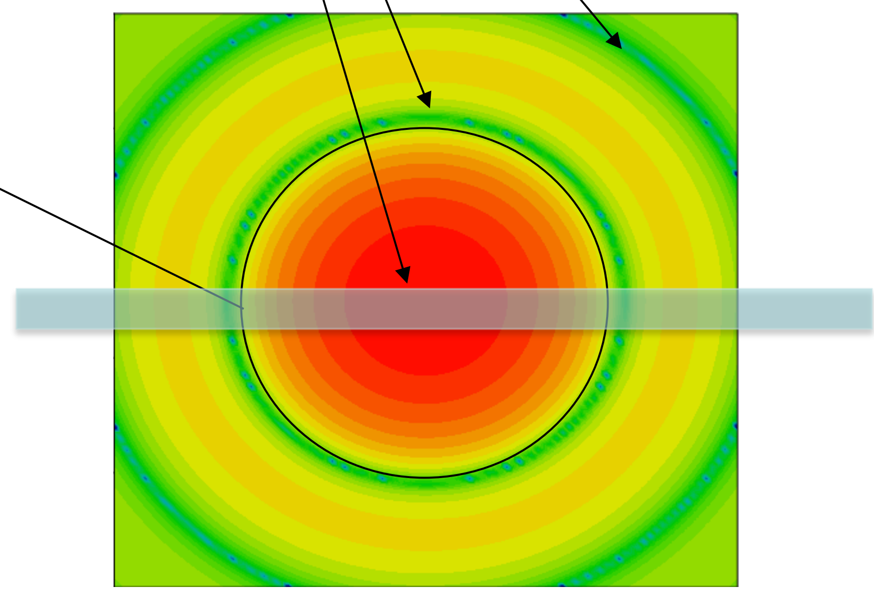
Airy disc

first dark ring

$$\frac{2\pi}{\lambda} a \sin\theta = 3.83$$

$$\sin\theta = \frac{3.83}{\pi} \frac{\lambda}{2a} = 1.22 \frac{\lambda}{D}$$

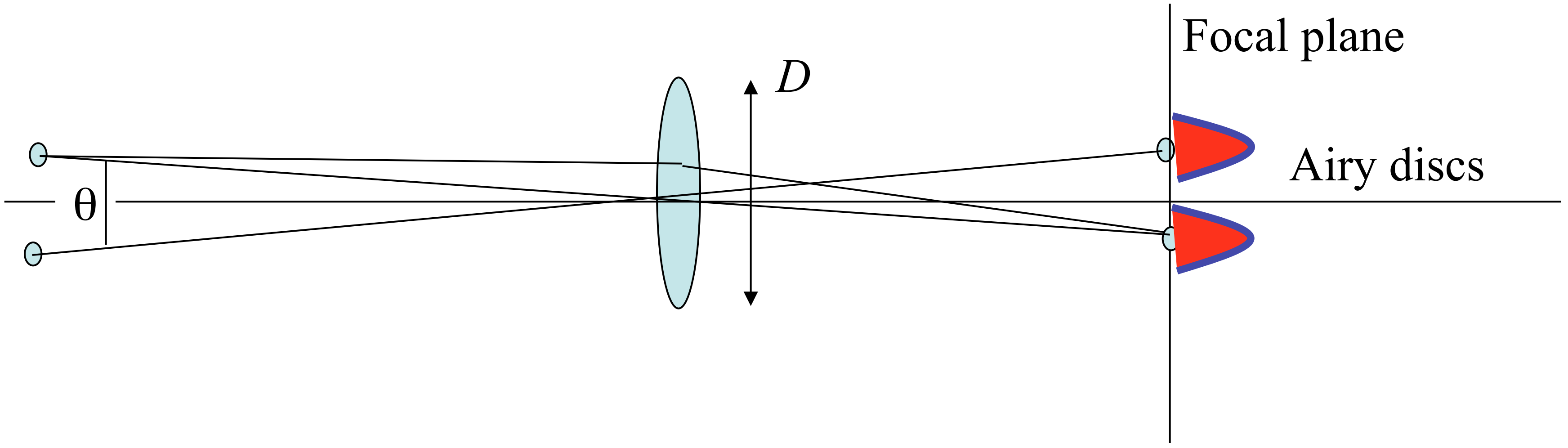
$D = 2a$ (diameter of aperture)



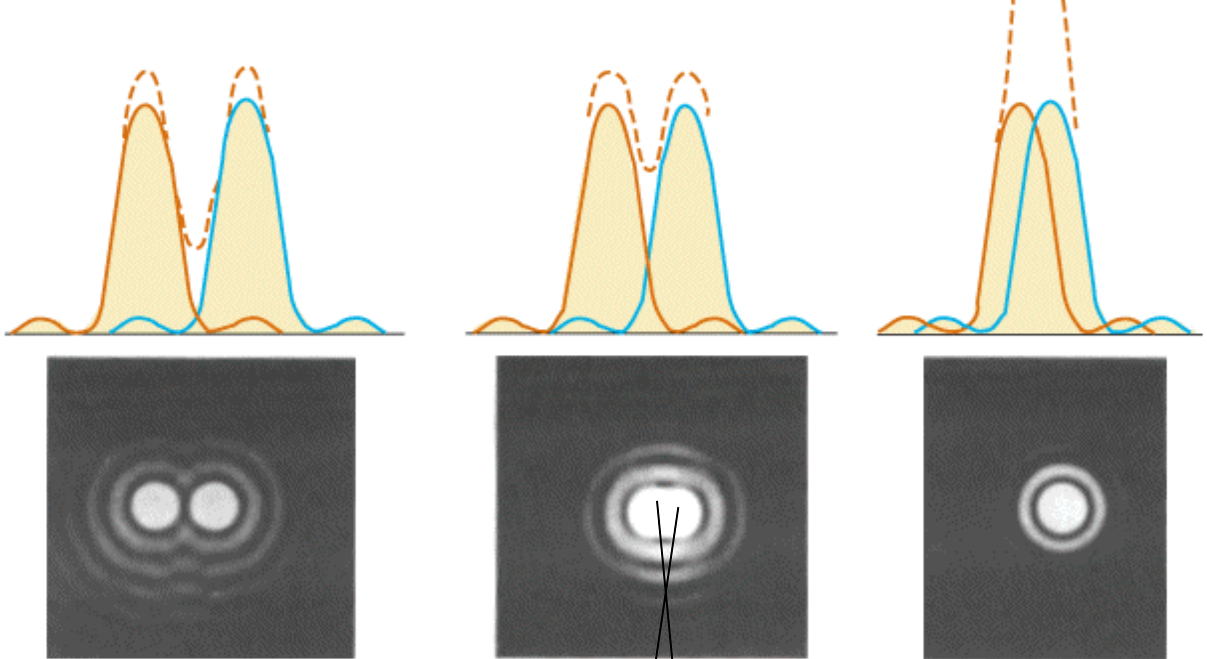
particle size

Resolution of Imaging Systems

Consider a lens system that forms an image of an object.
Lens forms an image which can be regarded as an Airy pattern (circular aperture).



Serway, Physics for Scientists and Engineers, 5/e
Figure 38.13 Harcourt, Inc.



(a) Clearly resolved

(b) just resolved (Rayleigh)

(c) Not resolved

To improve resolution:
Use light with small λ;
Use large aperture!

θ

The Rayleigh criterion: Centre of one disc occurs at the first minimum of the other.

$$\theta = 1.22\frac{\lambda}{D}$$

for small angles

Resolution of the human eye

$$\theta = 1.22\frac{\lambda}{D}$$

λ~ 500 nm, D=5 mm, (Size of pupil)

$$\theta = 1.22\frac{\lambda}{D} \sim 0.0001 \text{ rad} \sim 0.005 \text{ degrees}$$

Cannot resolve two objects separated by <1mm at 10 m distance.