University of Birmingham
School of Mathematics

Real Analysis – Integration – Spring 2025

Problem Sheet 8 and Assignment 4
Model Solutions

Instructions: Your solution to the Assignment Question (**AQ1**) must be submitted via Assignments on the LC Real Analysis Canvas page before the following time:

$\boxed{\textbf{Due 17:00 Wednesday 2 April 2025}}$

You are strongly encouraged to attempt all of the remaining formative questions, and as many of the extra questions as you can, to prepare for the final exam, but only the Assignment Question should be submitted to Canvas. Model solutions will only be released for the Assignment Question AQ1 and Questions Q1-Q4.

Important: Late submissions will be penalised at a rate of 5% per day late up until exactly two days after the submission deadline, at which point the model solutions will be released and the Assignment will be closed to further submissions on Canvas. Your Assignment Question solutions must be submitted as a single PDF file. You may upload newer versions, BUT only the most recent upload will be viewed and graded. In particular, this means that subsequent uploads will need to contain ALL of your work, not just the parts which have changed. Moreover, if you upload a new version after the deadline, then your submission will be counted as late and the late penalty will be applied REGARDLESS of whether an older version was submitted before the deadline. In the interest of fairness to all students and staff, there will be no exceptions to these rules. All of this and more is explained in detail on the Submission of Continuous Assessment page in the Student Handbook on Canvas.

Assignment Question

AQ1. (a)
(b)
(c)

Questions

Q1. Find the following antiderivatives and integrals:

(a) $\displaystyle\int \sin^3(3x)\cos^5(3x)\ \mathrm{d}x$

(b) $\displaystyle\int \frac{\sin^7(x)}{\cos^4(x)}\ \mathrm{d}x$

(c) $\displaystyle\int_0^{\frac{\pi}{4}} (\tan(x)\sec(x))^8\ \mathrm{d}x$

Solution. (a) We substitute $u(x) = \sin(3x)$, noting that $u'(x) = 3\cos(3x)$, to obtain

$$\begin{aligned}\int \sin^3(3x)\cos^5(3x)\ \mathrm{d}x &= \int \sin^3(3x)\cos^4(3x)\cos(3x)\ \mathrm{d}x\\ &= \int \sin^3(3x)(1-\sin^2(3x))^2\cos(3x)\ \mathrm{d}x\\ &= \tfrac{1}{3}\int u^3(1-u^2)^2\ \mathrm{d}u\\ &= \tfrac{1}{3}\int (u^7 - 2u^5 + u^3)\ \mathrm{d}u\\ &= \tfrac{1}{24}\sin^8(3x) - \tfrac{1}{9}\sin^6(3x) + \tfrac{1}{12}\sin^4(3x).\end{aligned}$$

Alternatively, using the substitution $v(x) = \cos(3x)$, noting that $v'(x) = -3\sin(3x)$, we obtain the alternative antiderivative

$$\int \sin^3(3x)\cos^5(3x)\ \mathrm{d}x = \tfrac{1}{24}\cos^8(3x) - \tfrac{1}{18}\cos^6(3x).$$

(b) We substitute $u(x) = \cos(x)$, noting that $u'(x) = -\sin(x)$, to obtain

$$\begin{aligned}\int \frac{\sin^7(x)}{\cos^4(x)}\ \mathrm{d}x &= \int \sin^6(x)\cos^{-4}(x)\sin(x)\ \mathrm{d}x\\ &= \int (1-\cos^2(x))^3\cos^{-4}(x)\sin(x)\ \mathrm{d}x\\ &= -\int (1-u^2)^3u^{-4}\ \mathrm{d}u\\ &= \int (-u^{-4} + 3u^{-2} - 3 + u^2)\ \mathrm{d}u\\ &= \tfrac{1}{3}\sec^3(x) - 3\sec(x) - 3\cos(x) + \tfrac{1}{3}\cos^3(x).\end{aligned}$$

(c) We substitute $u(x) = \tan(x)$, noting that $u'(x) = \sec^2(x)$, to obtain

$$\begin{aligned}\int_0^{\frac{\pi}{4}} (\tan(x)\sec(x))^8\ \mathrm{d}x &= \int_0^1 \tan^8(x)\sec^6(x)\sec^2(x)\ \mathrm{d}x\\ &= \int_0^1 \tan^8(x)(\tan^2(x)+1)^3\sec^2(x)\ \mathrm{d}x\\ &= \int_0^1 u^8(u^6 + 3u^4 + 3u^2 + 1)\ \mathrm{d}u\\ &= \left[\tfrac{1}{15}u^{15} + \tfrac{3}{13}u^{13} + \tfrac{3}{11}u^{11} + \tfrac{1}{9}u^9\right]_0^1\\ &= \tfrac{1}{15} + \tfrac{3}{13} + \tfrac{3}{11} + \tfrac{1}{9}.\end{aligned}$$

Q2. (a) For each $n \in \mathbb{N} \cup \{0\}$, suppose that $I_n = \int (\log(5x))^n \, \mathrm{d}x$. Prove that

$$I_n(x) = x(\log(5x))^n - nI_{n-1}(x)$$

for all $x > 0$ and all $n \in \mathbb{N}$, and use this formula to find $\int (\log(5x))^3 \, \mathrm{d}x$.

(b) Find the following antiderivatives:

(i) $\displaystyle\int \frac{3x+2}{\sqrt{x-5}} \, \mathrm{d}x$

(ii) $\displaystyle\int \frac{\sqrt{x+1}}{x} \, \mathrm{d}x$

(iii) $\displaystyle\int \frac{1}{\sqrt{4x^2+9}} \, \mathrm{d}x$

Solution. (a) For each $n \in \mathbb{N}$, we apply Integration by Parts with

$$\begin{cases} u(x) = (\log(5x))^n \\ v(x) = x \end{cases} \quad \text{and} \quad \begin{cases} u'(x) = n(\log(5x))^{n-1}x^{-1} \\ v'(x) = 1 \end{cases}$$

to obtain

$$\begin{aligned} I_n(x) &= \int (\log(5x))^n \, \mathrm{d}x \\ &= x(\log(5x))^n - \int n(\log(5x))^{n-1}x^{-1}x \, \mathrm{d}x \\ &= x(\log(5x))^n - n\int (\log(5x))^{n-1} \, \mathrm{d}x \\ &= x(\log(5x))^n - nI_{n-1}(x) \end{aligned}$$

for all $x > 0$, as required. We apply this formula recursively to obtain

$$\begin{aligned} \int (\log(5x))^3 \, \mathrm{d}x &= I_3(x) \\ &= x\log^3(5x) - 3I_2(x) \\ &= x\log^3(5x) - 3[x(\log(5x))^2 - 2I_1(x)] \\ &= x\log^3(5x) - 3x\log^2(5x) + 6[x\log(5x) - I_0(x)] \\ &= x\log^3(5x) - 3x\log^2(5x) + 6x\log(5x) - 6x, \end{aligned}$$

where in the last equality we used $I_0(x) = \int 1 \, \mathrm{d}x = x$.

(b)(i) We substitute $u(x) = \sqrt{x-5}$ for all $x \in [5, \infty)$ by the Domain Convention, and noting that $u : [5, \infty) \to [0, \infty)$ is bijective with inverse $x(u) = u^2 + 5$ and $x'(u) = 2u$, we obtain

$$\begin{aligned} \int \frac{3x+2}{\sqrt{x-5}} \, \mathrm{d}x &= \int \frac{3(u^2+5)+2}{u} 2u \, \mathrm{d}u \\ &= \int (6u^2 + 34) \, \mathrm{d}u \\ &= 2(x-5)^{\frac{3}{2}} + 34(x-5)^{\frac{1}{2}} \end{aligned}$$

(b)(ii) We substitute $u(x) = \sqrt{x+1}$ for all $x \in [-1, \infty)$ by the Domain Convention, and noting that $u : [-1, \infty) \to [0, \infty)$ is bijective with inverse $x(u) = u^2 - 1$ and $x'(u) = 2u$, we obtain

$$
\begin{aligned}
\int \frac{\sqrt{x+1}}{x}\ \mathrm{d}x &= \int \frac{u}{u^2-1} 2u\ \mathrm{d}u \\
&= 2\int \frac{u^2-1+1}{u^2-1}\ \mathrm{d}u \\
&= 2\int 1\ \mathrm{d}u + 2\int \frac{1}{u^2-1}\ \mathrm{d}u \\
&= 2u + \int \frac{-1}{u+1}\mathrm{d}u + \int \frac{1}{u-1}\ \mathrm{d}u \\
&= 2u - \log|u+1| + \log|u-1| \\
&= 2\sqrt{x+1} - \log|\sqrt{x+1}+1| + \log|\sqrt{x+1}-1|,
\end{aligned}
$$

where we use the method of partial fractions to obtain the fourth equality.

(b)(iii) We substitute $x = \frac{3}{2}\tan(\theta)$ and note that $x'(\theta) = \frac{3}{2}\sec^2(\theta)$, to obtain

$$
\begin{aligned}
\int \frac{1}{\sqrt{4x^2+9}}\ \mathrm{d}x &= \int \frac{\frac{3}{2}\sec^2(\theta)}{\sqrt{9(\tan^2(\theta)+1)}}\ \mathrm{d}\theta \\
&= \int \frac{\sec^2(\theta)}{2\sec(\theta)}\ \mathrm{d}\theta \\
&= \tfrac{1}{2}\int \sec(\theta)\ \mathrm{d}\theta \\
&= \tfrac{1}{2}\log|\sec(\theta)+\tan(\theta)|,
\end{aligned}
$$

where we used Example 7.1.2 in the Lecture Notes to obtain the final equality.

To conclude, we using trigonometry to note that $\tan(\theta) = \frac{2x}{3} = \frac{\text{opp}}{\text{adj}}$, where we regard θ as the acute angle in a right-angled triangle with opposite side-length $2x$ and adjacent side-length 3, hence $\sec(\theta) = \frac{\text{hyp}}{\text{adj}} = \frac{\sqrt{4x^2+9}}{3}$. We thus have

$$
\begin{aligned}
\int \frac{1}{\sqrt{4x^2+9}}\ \mathrm{d}x &= \tfrac{1}{2}\log\left|\tfrac{1}{3}(\sqrt{4x^2+9}+2x)\right| \\
&= \tfrac{1}{2}\log\left|\sqrt{4x^2+9}+2x\right| - \tfrac{1}{2}\log(3),
\end{aligned}
$$

where we could also omit the $-\frac{1}{2}\log(3)$ term since it is just an additive constant.

Q3. Find, or prove divergence of, the following antiderivatives and integrals:

(a) $\displaystyle\int \frac{6x+1}{x^2+3x+5}\ \mathrm{d}x$

(b) $\displaystyle\int \sec^4(3x)\tan^4(3x)\ \mathrm{d}x$

(c) $\displaystyle\int_2^\infty \frac{1}{\sqrt{2x-3}}\ \mathrm{d}x$

(d) $\displaystyle\int_5^6 \frac{1}{\sqrt{x^2-25}}\ \mathrm{d}x$

Solution. (a) The denominator $x^2+3x+5 = (x+\frac{3}{2})^2 + \frac{11}{4}$ is an irreducible quadratic factor (since it has complex conjugate roots) so the method of partial fractions does not provide any simplification here. Instead, we aim to express the integrand as a linear combination of the antiderivatives

$$
\int \frac{1}{x^2+a^2}\ \mathrm{d}x = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) \quad\text{and}\quad \int \frac{f'(x)}{f(x)}\ \mathrm{d}x = \log|f(x)|
$$

with $f(x) := x^2+3x+5 = (x+\frac{3}{2})^2+(\frac{\sqrt{11}}{2})^2 \neq 0$ for all $x \in \mathbb{R}$ and $a = \frac{\sqrt{11}}{2}$. In particular, since $f'(x) = 2x+3$, our aim is to find two real numbers α and β such that

$$6x+1 = \alpha(2x+3)+\beta.$$

This holds provided $2\alpha = 6$ whilst $3\alpha+\beta = 1$, thus $\alpha = 3$ and $\beta = -8$. We combine this with the antiderivatives above (and the Substitution Formula) to obtain

$$\begin{aligned}\int \frac{6x+1}{x^2+3x+5}\ \mathrm{d}x &= 3\int \frac{2x+3}{x^2+3x+5}\ \mathrm{d}x - 8\int \frac{1}{(x+\frac{3}{2})^2+(\frac{\sqrt{11}}{2})^2}\ \mathrm{d}x \\ &= 3\log|x^2+3x+5| - 8\tfrac{2}{\sqrt{11}}\tan^{-1}\left(\tfrac{2}{\sqrt{11}}(x+\tfrac{3}{2})\right) \\ &= 3\log|x^2+3x+5| - \tfrac{16}{\sqrt{11}}\tan^{-1}\left(\tfrac{1}{\sqrt{11}}(2x+3)\right).\end{aligned}$$

(b) We substitute $u(x) = \tan(3x)$, noting that $u'(x) = 3\sec^2(3x)$, to obtain

$$\begin{aligned}\int \sec^4(3x)\tan^4(3x)\ \mathrm{d}x &= \int (\tan^2(3x)+1)\sec^2(3x)\tan^4(3x)\ \mathrm{d}x \\ &= \tfrac{1}{3}\int (u^2+1)u^4\ \mathrm{d}u \\ &= \tfrac{1}{3}\int (u^6+u^4)\ \mathrm{d}u \\ &= \tfrac{1}{3}(\tfrac{1}{7}u^7+\tfrac{1}{5}u^5) \\ &= \tfrac{1}{21}\tan^7(3x)+\tfrac{1}{15}\tan^5(3x).\end{aligned}$$

(c) This is an improper integral, so we compute

$$\begin{aligned}\int_2^\infty \frac{1}{\sqrt{2x-3}}\ \mathrm{d}x &= \lim_{t\to\infty}\left(\int_2^t (2x-3)^{-\frac{1}{2}}\ \mathrm{d}x\right) \\ &= \lim_{t\to\infty}\left[\tfrac{2}{2}(2x-3)^{\frac{1}{2}}\right]_2^t \\ &= \lim_{t\to\infty}\left[\sqrt{2t-3}-1\right]_2^t \\ &= \infty,\end{aligned}$$

which proves that the improper integral is divergent.

(d) We first find an antiderivative for the integrand, substituting $x = 5\sec(\theta)$ and noting that $x'(\theta) = 5\sec(\theta)\tan(\theta)$, to obtain

$$\begin{aligned}\int \frac{1}{\sqrt{x^2-25}}\ \mathrm{d}x &= \int \frac{1}{\sqrt{25(\sec^2(\theta)-1)}}5\sec(\theta)\tan(\theta)\ \mathrm{d}\theta \\ &= \int \sec(\theta)\ \mathrm{d}\theta \\ &= \log|\sec(\theta)+\tan(\theta)| \\ &= \log\left(\frac{x}{5}+\frac{\sqrt{x^2-25}}{5}\right) \\ &= \log\left(x+\sqrt{x^2-25}\right)-\log(5),\end{aligned}$$

where we used trigonometry to calculate $\tan(\theta) = \frac{\sqrt{x^2-25}}{5}$, since $\sec(\theta) = \frac{x}{5}$.

We then use the Second Fundamental Theorem of Calculus to compute

$$\begin{aligned}\int_5^6 \frac{1}{\sqrt{x^2-25}}\ \mathrm{d}x &= \lim_{t\to 0^+}\left(\int_{5+t}^6 \frac{1}{\sqrt{x^2-25}}\ \mathrm{d}x\right)\\ &= \lim_{t\to 0^+}\left[\log\left(x+\sqrt{x^2-25}\right)\right]_{5+t}^6\\ &= \lim_{t\to 0^+}\left(\log(6+\sqrt{11})-\log(5+t+\sqrt{10t+t^2})\right)\\ &= \log(6+\sqrt{11})-\log(5).\end{aligned}$$

Q4. Find the value of the following improper integrals or prove that they are divergent:

(a) $\displaystyle\int_0^1 x\log(x)\ \mathrm{d}x$

(b) $\displaystyle\int_0^{10} \frac{x}{x-5}\ \mathrm{d}x$

(c) $\displaystyle\int_{-\frac{\pi}{2}}^0 \sec(x)\ \mathrm{d}x$

Solution. (a) This is an improper integral because $x\log(x)$ is not defined at $x=0$. We apply Integration by Parts with

$$\begin{cases} u(x)=\log(x)\\ v(x)=\frac{1}{2}x^2\end{cases} \qquad \text{and} \qquad \begin{cases} u'(x)=x^{-1}\\ v'(x)=x\end{cases}$$

to obtain

$$\begin{aligned}\int_0^1 x\log(x)\ \mathrm{d}x &= \lim_{\delta\to 0^+}\left(\int_\delta^1 x\log(x)\ \mathrm{d}x\right)\\ &= \lim_{\delta\to 0^+}\left(\left[\tfrac{1}{2}x^2\log(x)\right]_\delta^1-\int_\delta^1 \tfrac{1}{2}x\ \mathrm{d}x\right)\\ &= \lim_{\delta\to 0^+}\left(\left[\tfrac{1}{2}x^2\log(x)-\tfrac{1}{4}x^2\right]_\delta^1\right)\\ &= \lim_{\delta\to 0^+}\left(-\tfrac{1}{4}-\tfrac{1}{2}\delta^2\log(\delta)+\tfrac{1}{4}\delta^2\right)\\ &= -\tfrac{1}{4}+\tfrac{1}{2}\lim_{\delta\to 0^+}\left(\frac{-\log(\delta)}{\delta^{-2}}\right)\\ &= -\tfrac{1}{4}+\tfrac{1}{2}\lim_{\delta\to 0^+}\left(\frac{-\delta^{-1}}{-2\delta^{-3}}\right)\\ &= -\tfrac{1}{4}+\tfrac{1}{4}\lim_{\delta\to 0^+}\left(\delta^2\right)\\ &= -\tfrac{1}{4},\end{aligned}$$

where we applied L'Hôpital's Rule to obtain the sixth equality, which is justified since $\lim_{\delta\to 0^+}(-\log(\delta))=\infty=\lim_{\delta\to 0^+}\delta^{-2}$.

(b) This is an improper integral because $\frac{x}{x-5}$ is not defined at $x = 5$. It is convergent when both $\lim_{\delta \to 0^+} \int_0^{5-\delta} \frac{x}{x-5} \, \mathrm{d}x$ and $\lim_{\delta \to 0^+} \int_{5+\delta}^{10} \frac{x}{x-5} \, \mathrm{d}x$ are finite. We find that

$$\begin{aligned}
\int_0^5 \frac{x}{x-5} \, \mathrm{d}x &= \lim_{\delta \to 0^+} \left(\int_0^{5-\delta} \frac{x-5+5}{x-5} \, \mathrm{d}x \right) \\
&= \lim_{\delta \to 0^+} \left(\int_0^{5-\delta} 1 \, \mathrm{d}x + \int_0^{5-\delta} \frac{5}{x-5} \, \mathrm{d}x \right) \\
&= \lim_{\delta \to 0^+} \left([x + 5 \log|x-5|]_0^{5-\delta} \right) \\
&= \lim_{\delta \to 0^+} (5 - \delta + 5\log(\delta) - 5\log(5)) \\
&= -\infty.
\end{aligned}$$

The improper integral is therefore divergent (there is no need to calculate the other integral in such case).

(c) This is an improper integral because $\sec(x)$ is not defined at $x = -\frac{\pi}{2}$. Using Example 7.1.2 from the Lecture Notes, we have

$$\begin{aligned}
\int_{-\frac{\pi}{2}}^0 \sec(x) \, \mathrm{d}x &= \lim_{\delta \to 0^+} \left(\int_{-\frac{\pi}{2}+\delta}^0 \sec(x) \, \mathrm{d}x \right) \\
&= \lim_{\delta \to 0^+} \left(\big[\log|\sec(x) + \tan(x)| \big]_{-\frac{\pi}{2}+\delta}^0 \right) \\
&= \lim_{\delta \to 0^+} \left(-\log|\sec(\delta - \tfrac{\pi}{2}) + \tan(\delta - \tfrac{\pi}{2})| \right) \\
&= -\lim_{\delta \to 0^+} \left(\log \left| \frac{1 + \sin(\delta - \frac{\pi}{2})}{\cos(\delta - \frac{\pi}{2})} \right| \right) \\
&= \infty,
\end{aligned}$$

where the final equality holds because

$$\begin{aligned}
\lim_{\delta \to 0^+} \left| \frac{1 + \sin(\delta - \frac{\pi}{2})}{\cos(\delta - \frac{\pi}{2})} \right| &= \lim_{\delta \to 0^+} \frac{1 + \sin(\delta - \frac{\pi}{2})}{\cos(\delta - \frac{\pi}{2})} \\
&= \lim_{\delta \to 0^+} \frac{\cos(\delta - \frac{\pi}{2})}{-\sin(\delta - \frac{\pi}{2})} \\
&= 0
\end{aligned}$$

and $\lim_{x \to 0^+} \log(x) = -\infty$. In particular, note that the application of L'Hôpital's Rule here is justified because $\lim_{\delta \to 0^+} (1 + \sin(\delta - \frac{\pi}{2})) = 0 = \lim_{\delta \to 0^+} \cos(\delta - \frac{\pi}{2})$. The improper integral is therefore divergent.

Extra Questions

EQ1. A cylindrical water tank is mounted sideways, so it has a circular vertical cross-section. The tank is 10 meters in diameter. Calculate the percentage of the tank's capacity that is filled when the height of the water in the tank is 6 meters.

(a) Calculate the percentage of the tank's capacity that is filled with water when the height of the water in the tank is 6 meters.

(b) Calculate the height of the water in the tank when it is at 33% capacity.

EQ2. (a) For each $n \in \mathbb{N} \cup \{0\}$, suppose that $T_n(x) = \int \tan^n(4x)\ \mathrm{d}x$:

(i) Prove that

$$T_n(x) = \frac{\tan^{n-1}(4x)}{4(n-1)} - T_{n-2}(x)$$

for all $x \in (-\frac{\pi}{8}, \frac{\pi}{8})$ and all integers $n > 2$.

(ii) Use the reduction formula above to find $\int \tan^3(4x)\ \mathrm{d}x$.

(b) Find the following antiderivatives or integrals:

(i) $\displaystyle\int \tan^5(x) \sec^6(x)\ \mathrm{d}x$

(ii) $\displaystyle\int_0^{\pi/4} \cos^2(6x) \sin^2(6x)\ \mathrm{d}x$

(iii) $\displaystyle\int \frac{x^2 + 5x - 4}{x^3 - x}\ \mathrm{d}x$

EQ3. A bounded function $f : \mathbb{R} \to \mathbb{R}$ is integrable if there exists $a \in \mathbb{R}$ such that both $\int_{-\infty}^a f$ and $\int_a^\infty f$ are convergent and finite, in which case $\int_{-\infty}^\infty f := \int_{-\infty}^a f + \int_a^\infty f$.

(a) Use the properties of integrals to prove that the value of $\int_{-\infty}^\infty f$ does not depend on the value of a in this definition. (In other words, you need to prove that $\int_{-\infty}^a f + \int_a^\infty f = \int_{-\infty}^b f + \int_b^\infty f$ for all $a, b \in \mathbb{R}$ whenever each side is defined.)

(b) Prove that $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) := (1 + x^2)^{-1}$ for all $x \in \mathbb{R}$ is integrable and calculate the value of $\int_{-\infty}^\infty f$.

(c) Prove that $g : [1, \infty) \to \mathbb{R}$ given by $g(x) := e^{-x}(4x^2 + 3x)^{-1}$ for all $x \in [1, \infty)$ is integrable.

EQ4. Find all real numbers p such that the following properties hold:

(a) The integral $\int_0^1 x^p\ \mathrm{d}x$ is an improper integral.

(b) The improper integral $\int_0^1 x^p\ \mathrm{d}x$ is convergent.