Electromagnetism

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Lecture 3
Gauss's Law
Week 2

Last Lecture

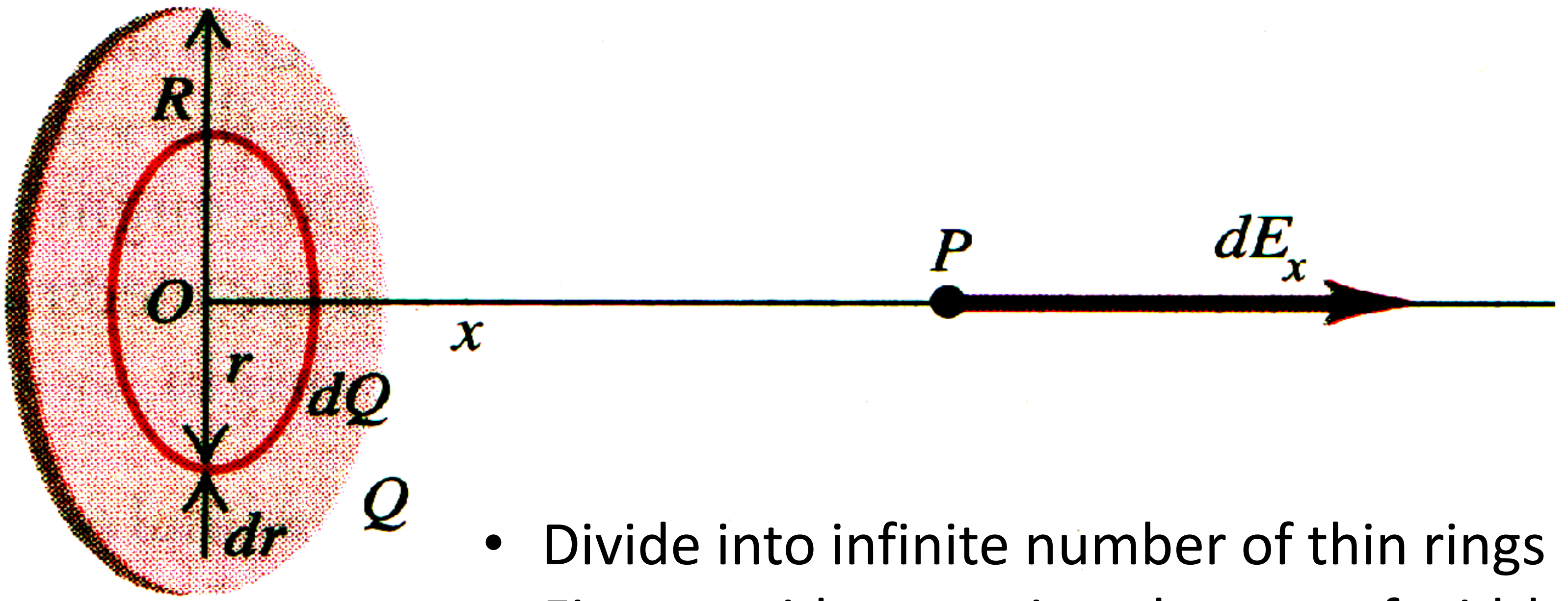
- Visualisation of E-fields
  - Electric field lines
- Continuous Charge Distributions
  - Using Coulomb's Law and integration
- **Some examples: E-field from .....**
  - **Line of continuous charge**
  - **Uniform charged thin ring**

Lecture 3 Content

- Some more examples of continuous charge distributions
  - Uniform charged circular plane
  - Infinite plane
  - Inside charged hollow sphere
- Electric flux
- Gauss's Law
  - Examples using Gauss's Law

Example 3.1 – Large Circular Plane

- Find the E-field due to large circular plane of uniform surface charge density, σ.



- Divide into infinite number of thin rings
- First consider one ring element of width δr.

Example 3.1

Do example on the visualizer

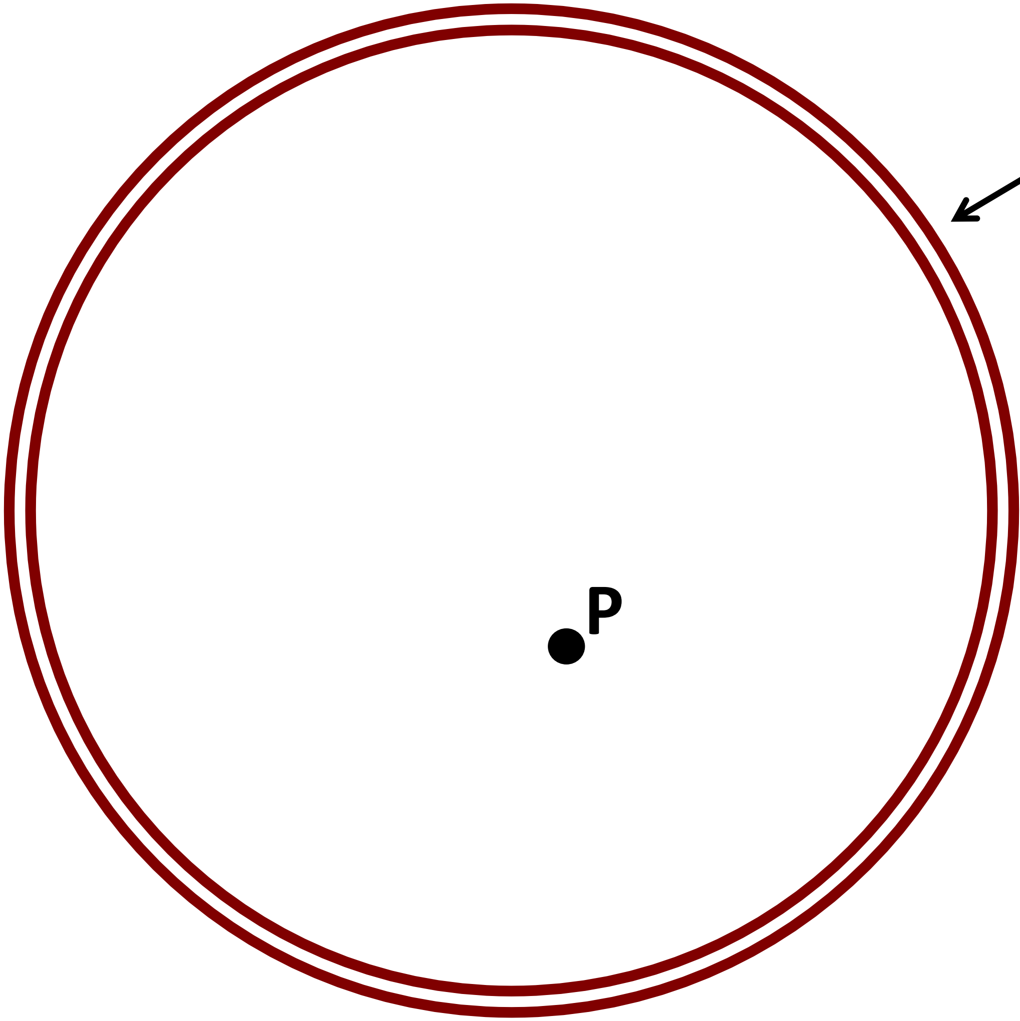
For an infinite plane sheet of charge the <u>E</u>-field produced is independent of the distance from the sheet

$$\boxed{E = \frac{s}{2e_0}}$$

This result is true <u>*close*</u> to the surface of any charge distribution

E-field Inside Charged Hollow Sphere

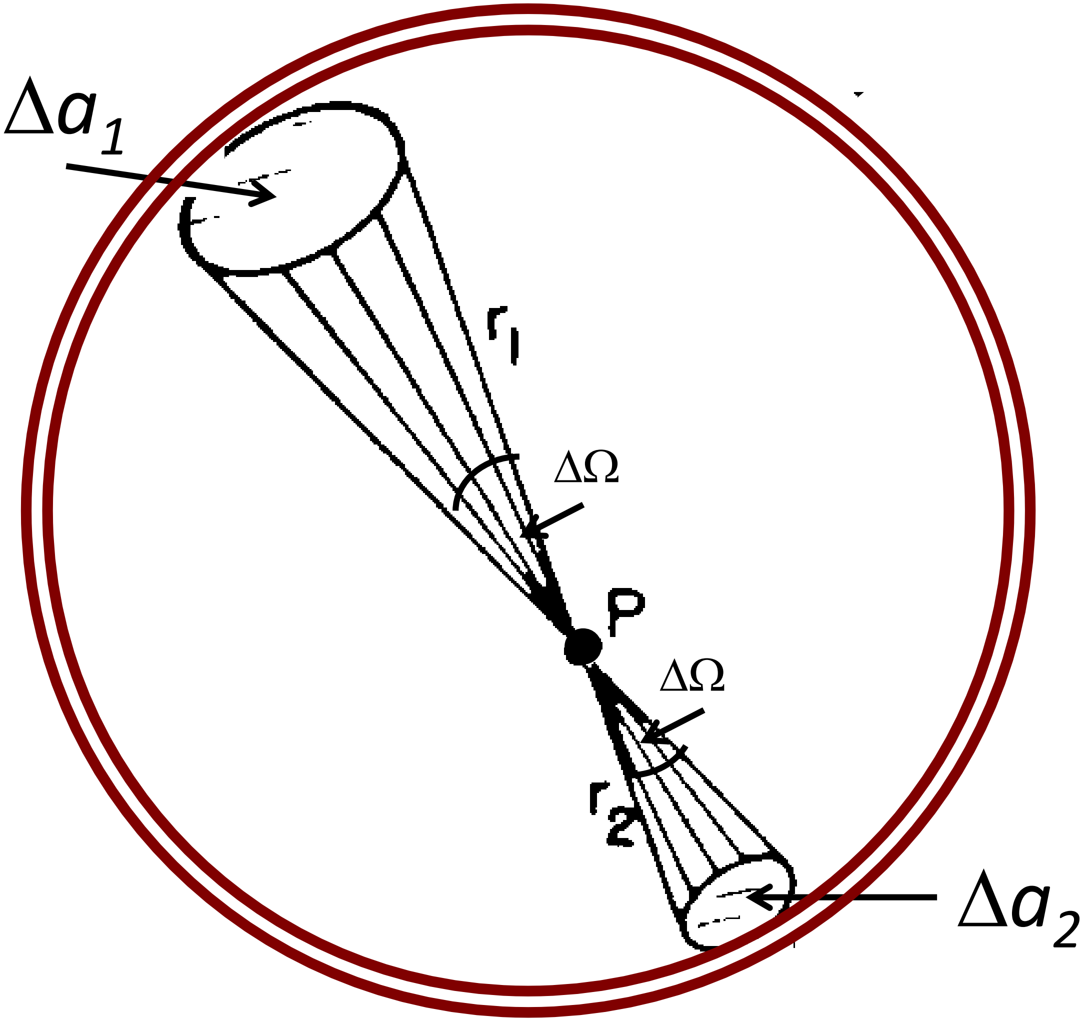
Consider a point, P inside the hollow sphere



Uniform charged shell Surface charge density, σ

E-field Inside Charged Hollow Sphere

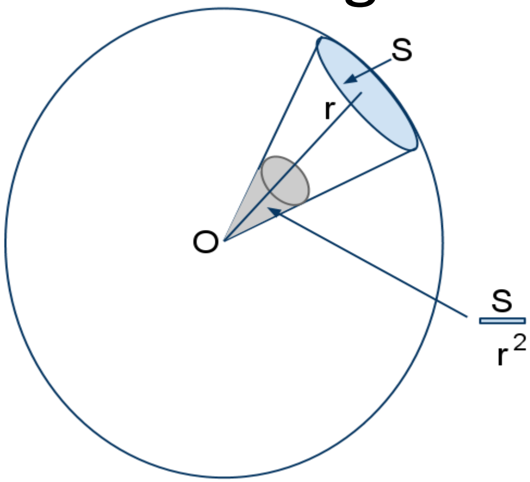
Consider back-to-back solid angle, $\Delta\Omega$, from point P to the surface of the sphere



$\Delta\Omega = \Delta a_1/r_1^2$
$= \Delta a_2/r_2^2$

Aside – Solid Angle

- 3D version of angle
  - Angle, θ = length of arc / radius (radians)
- Solid angle = area of spherical surface / radius squared
  - $\Omega = A/r^2$ (dimensionless unit: steradians (Sr))
- Note: surface area of sphere is $4\pi r^2$ so total solid angle (i.e. looking in all directions) is 4π
- In differential form: $\Delta\Omega = \sin(\theta)\Delta\theta\Delta\phi$
  - i.e. $\Omega = \int d\varphi \int \sin\theta \; d\theta$
  - In spherical polar coordinates

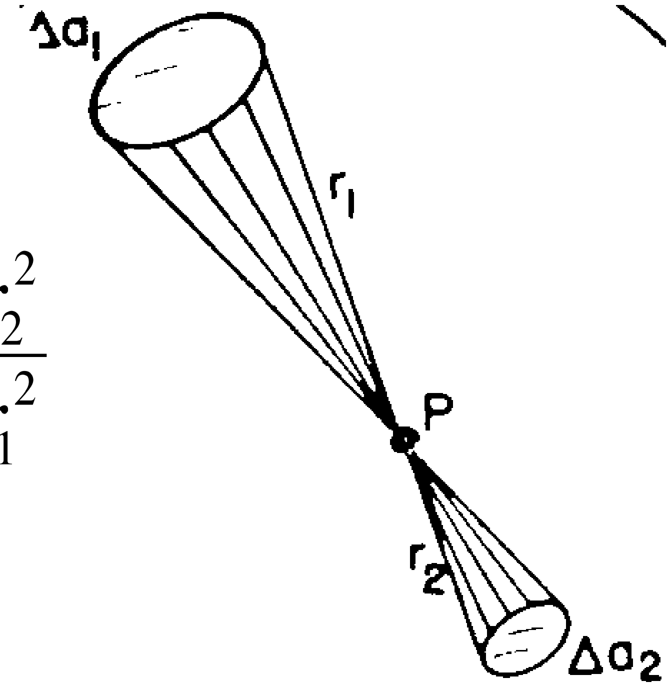


E-field in charged Sphere

All parts of the surface can be paired off

$$\frac{Da_2}{Da_1} = \frac{r_2^2}{r_1^2} \quad \text{so} \quad \frac{\Delta q_2}{\Delta q_1} = \frac{\sigma \Delta a_2}{\sigma \Delta a_1} = \frac{r_2^2}{r_1^2}$$

$$\frac{E_2}{E_1} = \frac{\Delta q_2 / r_2^2}{\Delta q_1 / r_1^2} = \frac{\Delta q_2}{\Delta q_1}\frac{r_1^2}{r_2^2} = 1$$

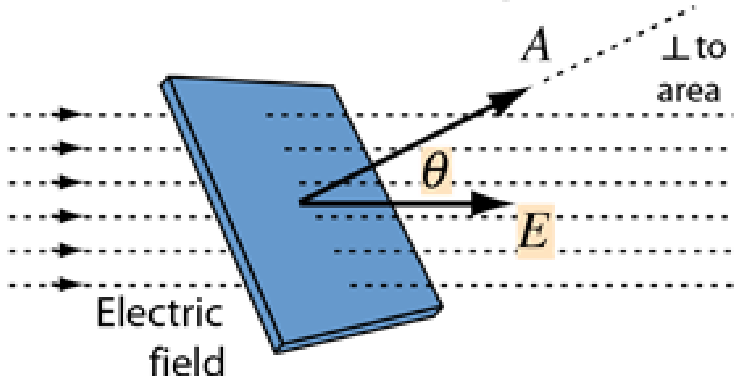


Total *E*-Field at P is *ZERO*

Electric Flux

- Electric flux is the amount of E-field passing across a surface
  - Defined in terms of a normal unit vector, $\underline{\hat{\boldsymbol{n}}}$ perpendicular to the surface

$$\text{flux} = \Phi = EA\cos\theta$$



- For a constant E-field and flat surface:

$$\Phi_E = A\underline{\boldsymbol{E}} \cdot \underline{\hat{\boldsymbol{n}}} = EA\cos\theta$$

- In general:

$$\Phi_E = \int \underline{\boldsymbol{E}} \cdot \underline{\hat{\boldsymbol{n}}}\, dS = \int \underline{\boldsymbol{E}} \cdot d\underline{\boldsymbol{S}}$$

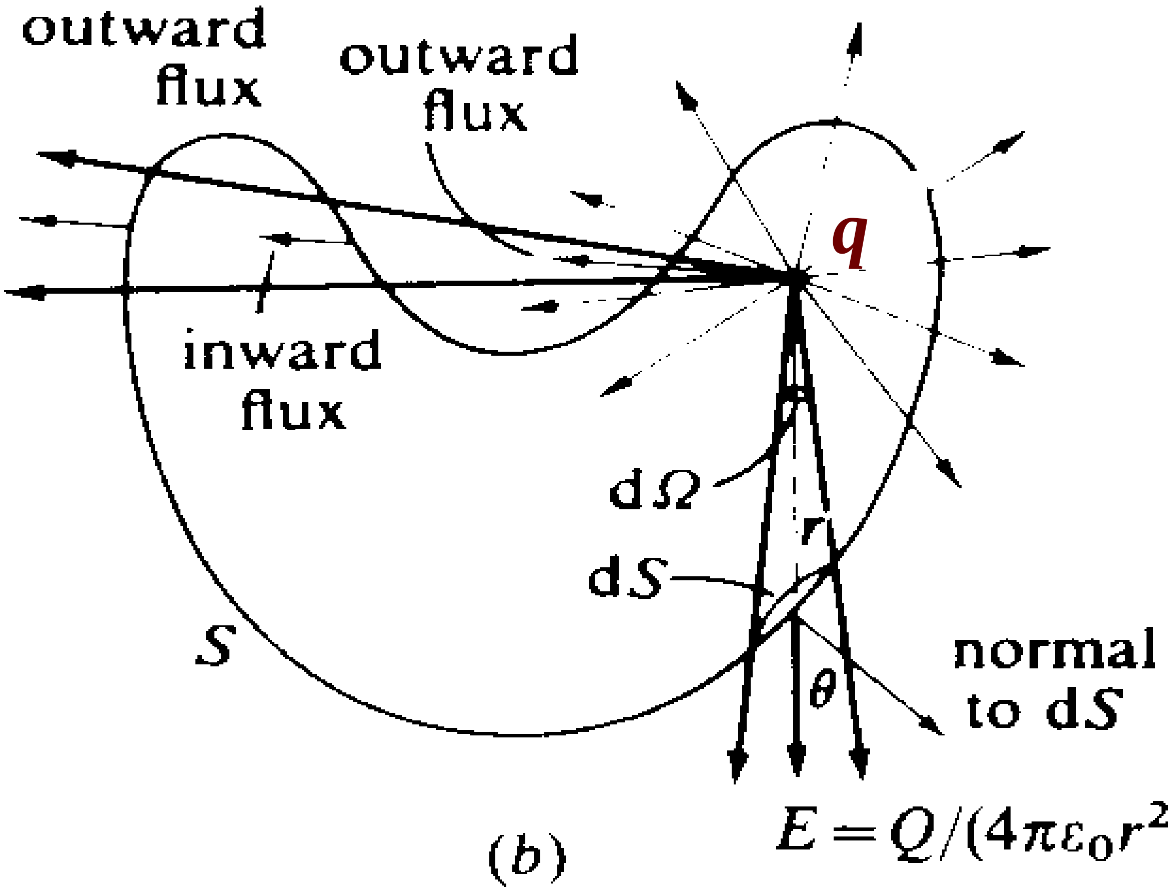
(I use dS as element of area but use dA if you like)

Gauss's Law

Consider an imaginary closed surface around a charge q.

The E-field at an element of surface, a distance r from the charge is:

$$\underline{\boldsymbol{E}} = \frac{q}{4\pi\varepsilon_0 r^2}\underline{\hat{\boldsymbol{r}}}$$

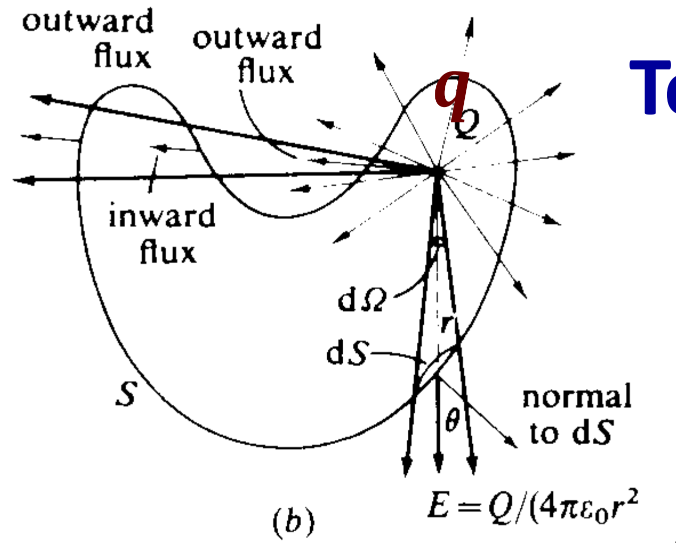


Gauss's Law

- So flux of **E** out of element of area d**S** is:

$$d\Phi = \underline{E} \cdot d\underline{S} = \underline{E} \cdot \underline{\hat{n}}\, dS = E \cos\theta\, dS$$

$$d\Phi = \frac{q}{4\pi\varepsilon_o r^2} \cos\theta\; dS = \frac{q}{4\pi\varepsilon_o} \frac{\cos\theta\, dS}{r^2} = \frac{q}{4\pi\varepsilon_o}\; d\Omega$$



Total flux out of imaginary surface:

$$\Phi_E = \frac{q}{4\pi\varepsilon_o} \int_S d\Omega = \frac{q}{4\pi\varepsilon_o} 4\pi = \frac{q}{\varepsilon_o}$$

Gauss's Law

Gauss's Law: the net electric flux of $\underline{E}$ out of any closed surface, enclosing a total charge $Q_{enclosed}$ situated in a vacuum (air for practical purposes) is

$$\Phi_E = \int_S \underline{E} \cdot d\underline{S} = \frac{Q_{encl}}{\varepsilon_0}$$

$Q_{encl} = \sum_i q_i$ or $Q_{encl} = \int_V \rho \, dV$

Gauss's Law

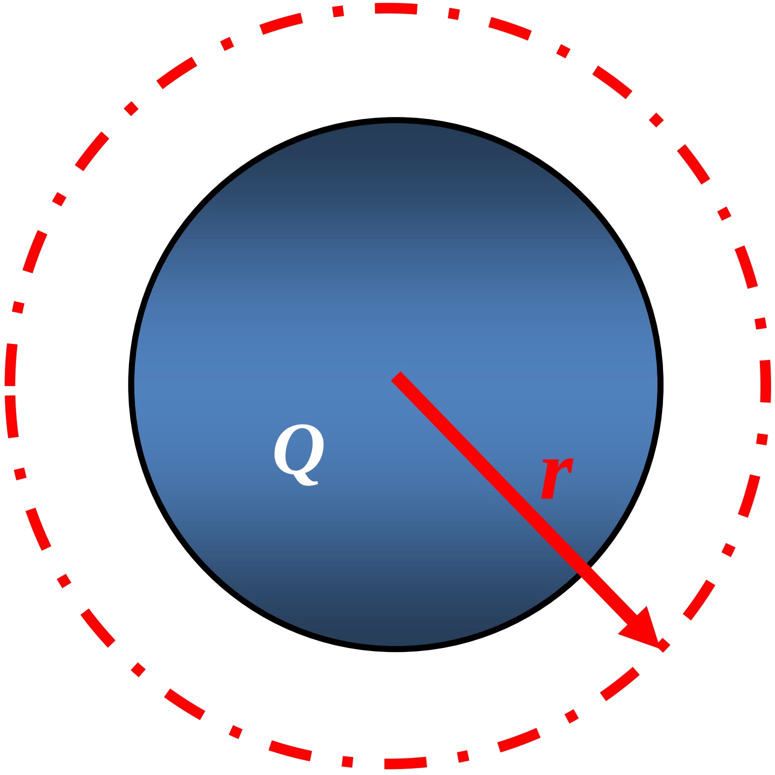
- This is Gauss's Law
  - You need to know this and know how to use it

$$\boxed{\int_S \underline{E} \cdot d\underline{S} = \frac{Q_{encl}}{\varepsilon_0}}$$

- Very useful for solving problems where there's symmetry (see examples).

Solid Sphere with Uniform Charge

Example: Sphere of radius *R* uniformly charged throughout its volume. Total charge *Q*



(a) $\underline{E}$-field for $r > R$

(b) $\underline{E}$-field for $r < R$

Method: set up an (imaginary) *Gaussian Surface* and use *symmetry*

Let's solve this using the visualizer