Week 11

Resonance

1. Resonance

We will now consider the behaviour of circuits that contain both a capacitor and an inductor connected in series. The inductor voltage leads the current by 90 degree and has a magnitude $V_L = I\omega L$. The capacitor voltage lags the current by 90 degrees and has a magnitude $V_C = I/\omega C$. The resistor voltage is in phase with the current and has a magnitude $V_R = IR$.

The inductor and capacitor voltages are 180 degrees out of phase. In other words, they oppose each other. When one is positive the other is negative. The net voltage across the combination of L and C depends on which is larger. In this example, I have chosen the inductor voltage to be about twice the capacitor voltage. This is the same as saying that the inductive reactance is twice the capacitive reactance, since the current is the same. Once the net voltage across L and C is determined, the applied voltage can be found by adding (vectorally) the resistor voltage. Applying Pythagoras' theorem we find

$$V^2 = V_R^2 + \left(V_L - V_C\right)^2 \tag{9}$$

Expressing the voltage magnitudes across each component in terms of the current we obtain an expression for the impedance of the circuit and the phase angle as a function of frequency.

$$Z = \frac{V}{I} = \sqrt{R^2 + \left(wL - \frac{1}{wC}\right)^2}$$

$$f = \tan^{-1}\left(\frac{V_L - V_C}{V_R}\right) = \tan^{-1}\left(\frac{wL}{R} - \frac{1}{wCR}\right) \tag{10}$$

You should recognise ωL as the reactance of the inductor and $1/\omega C$ as the reactance of the capacitor. We can investigate how the impedance changes with frequency by considering three special cases.

1. In the limit of very low frequency ($\omega \to 0$), reactance of the capacitor dominates and $Z \to \infty$.
2. In the limit of very high frequency ($\omega \to \infty$), the reactance of the inductor dominates and $Z \to \infty$.

3. When the frequency is such that the inductive reactance equals the capacitive reactance and $Z = R$. At this frequency the impedance is at a minimum. This occurs when

$$wL = \frac{1}{wC} \text{ or } w_0 = \frac{1}{\sqrt{LC}} \tag{11}$$

When this happens the circuit is said to be at resonance. Therefore ω_0 is known as the resonant frequency.

This situation is summarised in Figure 1 (a) When the inductive reactance is larger than the capacitive reactance, the applied voltage leads the current and the circuit overall appears to be inductive. (b) When the capacitive reactance exceeds the inductive reactance, the applied voltage lags the current and the circuit overall appears to be capacitive. (c) When the inductive reactance equals the capacitive reactance the applied voltage is in phase with the current and the circuit overall appears to be resistive.

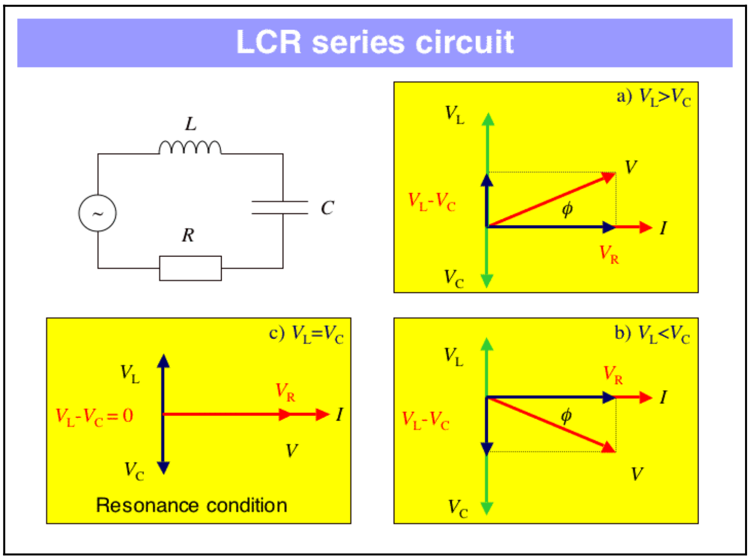

Figure 1: LCR series resonance.

Since the impedance is at a minimum at resonance, the current amplitude (I = V/Z) is at a maximum. The peak in the current magnitude can occur over a wide range of frequencies (a broad resonance) or over a very narrow range of frequencies (a sharp resonance) that depends on the value of the components. The sharpness of the resonance is characterised by the Q factor.

Notice that although the circuit behaves as if it is purely resistive at resonance (current and voltage are in phase) there is still a voltage across the capacitor and the inductor (equal in magnitude, but 180 degrees out of phase with each other). Energy is therefore being transferred back and forth between the capacitor and the inductor.

During one half-cycle the capacitor receives energy from the inductor, during the next half-cycle the inductor receives energy from the capacitor. As we shall see, the stored energy is a maximum when the circuit is at resonance.

1.1 Q factor

Figure 2 shows how the voltage magnitude across each of the components varies with the frequency. In this example, I have chosen L = 0.01 H and C = 1 μF so the resonant frequency from equation (11) is 10^4 radians per second (about 1.6 kHz). At this frequency the voltage magnitude across the resistor is equal to the applied voltage (100 V). What is striking about this plot is that the voltage across the capacitor and inductor is ten times larger than this. That is, 1 kV (!) appears across the capacitor and the inductor. This shows that a large amount of electrical energy is stored in the circuit when the circuit is at resonance. (Remember that the energy stored on a capacitor is given by $CV^2/2$.)

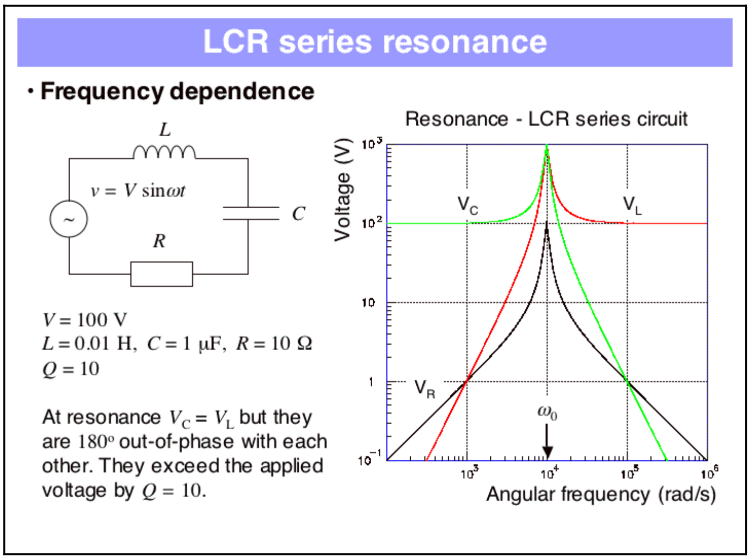

Figure 2: Exploring the quality, Q, of the resonance.

To visualise what is going on it may help to think of the mechanical analogy of a mass on the bottom of a vertical spring (See figure 14-23 on p. 450 in Chapter 14 of Tipler). This mechanical system will have a natural frequency such that small vertical displacements of the mass will cause the mass to oscillate up and down at this frequency. If you now imagine moving the top of the spring up and down, slowly at first but then at frequencies approaching the natural frequency, what you would see is that the vertical displacement of the mass would grow larger and larger. Correspondingly, the stored energy in the stretched spring would also grow larger. Above the natural frequency the oscillation would die down again, as the inertia of the mass makes it unable to respond to very rapid displacements at the top of the spring. (Compare the discussion in Tipler p. 450 and notice the similarity between Figure 14-24 and Figure 29-20 on p. 951.) In the electrical case it is the current that gives the large amplitude oscillations at resonance and the stored energy in the reactive components is at a maximum.

The sharpness of the resonance can be defined in a number of ways. The conventional way (see Figure 29-20 in Tipler and discussion on p. 952) is to define sharpness or Q factor as

$$Q = \frac{\omega_0}{\Delta\omega} \tag{12}$$

where ω_0 is the resonant frequency and $\Delta\omega$ the resonance width. The resonance width is taken to be the difference between the frequencies at which the power has dropped to half of its maximum (resonant) value. More conveniently, the Q factor can also be calculated from the ratio of the voltage magnitude across one of the reactive components to the applied voltage.

$$Q = \frac{V_L}{V} = \frac{I\omega_0 L}{IR} = \frac{\omega_0 L}{R} \tag{13}$$

Note that we can also calculate Q using the voltage across the capacitor, since at resonance the two voltages are the same. In the example shown in Figure 8.5, Q = 10. Resonant or "tuned" circuits have a number of useful applications. In a radio receiver, for example, the antenna is connected to a tuneable circuit where either the capacitance or the inductance can be varied. When tuned to a particular radio station frequency a large current will flow in the tuning circuit, and if the circuit has a sufficiently large Q, currents due to other stations operating at different frequencies will be negligible.

2. Try it for yourself

In the circuit above, find also (a) the Q factor and (b) the resonance width $\Delta\omega$. (c) What is the power factor when ω = 8000 rad/s?

(Answers: (a) 14.142, (b) 500 rad/s, (c) 0.275)

Appendix

Here are some useful trigonometric identities that you should know.

Aside - Useful identities

- Here are some useful identities for sinusoidal functions

$$\sin(\omega t+\phi)=\sin\omega t\cos\phi+\cos\omega t\sin\phi$$
$$\cos(\omega t+\phi)=\cos\omega t\cos\phi+\sin\omega t\sin\phi$$

Compound angle formulae

$$\sin 2\phi = 2\sin\phi\cos\phi$$
$$\cos 2\phi=\cos^2\phi-\sin^2\phi$$

Double angle formulae

- From the last one of these it follows:

$$\cos^2\phi=\frac{1+\cos 2\phi}{2}$$
$$\sin^2\phi=\frac{1-\cos 2\phi}{2}$$

$$\boxed{T=\frac{2\pi}{\omega}}$$

- Therefore ...

$$v_{rms}=\sqrt{\frac{1}{T}\int_0^T\left(V_p\sin\omega t\right)^2 dt}=V_p\sqrt{\frac{1}{T}\int_0^T\left(\frac{1-\cos 2\omega t}{2}\right)dt}=\frac{V_p}{\sqrt{2}}\sqrt{\frac{1}{T}\left[t-\frac{\sin 2\omega t}{2\omega}\right]_0^T}$$

Figure 3: Some useful trigonometric identities.