University of Birmingham
School of Mathematics

Real Analysis – Integration – Spring 2025

Problem Sheet 7
Issued Spring Week 5

Instructions: You are strongly encouraged to attempt all of the Questions (Q) below, and as many of the Extra Questions (EQ) as you can, to help prepare for the final exam. Model solutions will only be released for the Questions (Q1–Q4).

Questions

Q1. (a) State the First Fundamental Theorem of Calculus.

(b) Use the First Fundamental Theorem of Calculus to prove that each of the following functions is differentiable and find its derivative in terms of elementary functions:

(i) $F : [3,5] \to \mathbb{R},\ F(x) := \displaystyle\int_2^x 3t^{2t}\ \mathrm{d}t,\ x \in [3,5]$

(ii) $G : (\pi, 2\pi) \to \mathbb{R},\ G(x) := \displaystyle\int_1^{3\sin^2(x)+1} \frac{e^{-t}}{t}\ \mathrm{d}t,\ x \in (\pi, 2\pi)$

(iii) $H : [0, \frac{1}{10}] \to \mathbb{R},\ H(x) := \displaystyle\int_x^{2x} \frac{\sqrt{1-9t^2}}{\sqrt{1-t^2}}\ \mathrm{d}t,\ x \in [0, \frac{1}{10}]$

Q2. (a) State the Second Fundamental Theorem of Calculus.

(b) Suppose that $-\infty < a < b < \infty$. Use the Second Fundamental Theorem of Calculus to prove that

$$\int_a^b |x|\ \mathrm{d}x = \begin{cases} \frac{1}{2}(b^2 - a^2), & a \geq 0; \\ \frac{1}{2}(a^2 + b^2), & a < 0 \leq b; \\ \frac{1}{2}(a^2 - b^2), & b < 0. \end{cases}$$

You must verify all hypotheses required to apply the Second Fundamental Theorem of Calculus. In particular, if you use the fact that a certain function is an antiderivative of the absolute value function, then you must prove this fact (be careful proving differentiability at the origin).

(c) Let $f : \mathbb{R} \setminus [-3,-2] \to \mathbb{R}$ be defined by $f(x) := |x|$ for all $x \in \mathbb{R} \setminus [-3,-2]$. Find two antiderivatives F_1 and F_2 of f such that

$$F_1(x) - F_2(x) = \begin{cases} 9, & x > -2; \\ 3, & x < -3. \end{cases}$$

You must prove that your choices for F_1 and F_2 are antiderivatives of f.

Q3. Find the following antiderivatives and integrals (henceforth $\log(x) := \log_e(x)$):

(a) $\displaystyle\int_1^e x^2 \log(x)\ \mathrm{d}x$

(b) $\displaystyle\int (\log(x))^2\ \mathrm{d}x$

(c) $\displaystyle\int_0^\pi e^x \cos(x)\ \mathrm{d}x$

Q4. Find the following antiderivatives and integrals:

(a) $\displaystyle\int \frac{x-4}{x^2-5x+6}\ \mathrm{d}x$

(b) $\displaystyle\int_1^2 \frac{x^5+x-1}{x^3+1}\ \mathrm{d}x$

(c) $\displaystyle\int \frac{x^2+2x-1}{x^3-x}\ \mathrm{d}x$

Extra Questions

EQ1. Use the First Fundamental Theorem of Calculus and apply it to prove that each function below is differentiable and to find its derivative in terms of elementary functions:

(a) $F : [0,2] \to \mathbb{R},\ F(x) := \displaystyle\int_0^x \sin(t^2)\ \mathrm{d}t,\ x \in [0,2]$

(b) $G : [1,2] \to \mathbb{R},\ G(x) := \displaystyle\int_1^x \sin(t^2)\ \mathrm{d}t,\ x \in [1,2]$

(c) $H : [0,1] \to \mathbb{R},\ H(x) := \displaystyle\int_x^1 \sin(t^2)\ \mathrm{d}t,\ x \in [0,1]$

(d) $I : [0,1] \to \mathbb{R},\ I(x) := \displaystyle\int_0^{2x^3} \sin(t^2)\ \mathrm{d}t,\ x \in [0,1]$

(e) $J : [0,2] \to \mathbb{R},\ J(x) := \left(\displaystyle\int_0^x \sin(t^2)\ \mathrm{d}t\right)^2,\ x \in [0,2]$

The formula for the derivative $J'(x)$ may contain an integral expression.

EQ2. Suppose that $f : [1,3] \to \mathbb{R}$ is differentiable and that its derivative f' is continuous:

(a) If $f(1) = 10$ and $\int_1^3 f' = 16$, then calculate $f(3)$.

(b) Explain why the continuity of f' allowed for the application of the Fundamental Theorem of Calculus in part (a).

(c) State a weaker condition on f' that would suffice to apply the Fundamental Theorem of Calculus in part (a).

EQ3. Find the following antiderivatives and integrals:

(a) $\displaystyle\int x\sin(5x)\ \mathrm{d}x$

(b) $\displaystyle\int_1^2 \frac{(\log(x))^2}{x^3}\ \mathrm{d}x$

(c) $\displaystyle\int e^{2x}\sin(3x)\ \mathrm{d}x$

EQ4. Find the following antiderivatives and integrals:

(a) $\displaystyle\int \frac{x^2+1}{x+4}\ \mathrm{d}x$

(b) $\displaystyle\int \frac{10}{(x-1)(x^2+4)}\ \mathrm{d}x$

(c) $\displaystyle\int_3^4 \frac{x^2+1}{x^2-4x+4}\ \mathrm{d}x$