University of Birmingham
School of Mathematics

Real Analysis – Integration – Spring 2025

Problem Sheet 7
Model Solutions

Instructions: You are strongly encouraged to attempt all of the Questions (Q) below, and as many of the Extra Questions (EQ) as you can, to help prepare for the final exam. Model solutions will only be released for the Questions (Q1–Q4).

Questions

Q1. (a) State the First Fundamental Theorem of Calculus.
(b) Use the First Fundamental Theorem of Calculus to prove that each of the following functions is differentiable and find its derivative in terms of elementary functions:

(i) $F : [3,5] \to \mathbb{R},\ F(x) := \displaystyle\int_2^x 3t^{2t}\ \mathrm{d}t,\ x \in [3,5]$

(ii) $G : (\pi, 2\pi) \to \mathbb{R},\ G(x) := \displaystyle\int_1^{3\sin^2(x)+1} \frac{e^{-t}}{t}\ \mathrm{d}t,\ x \in (\pi, 2\pi)$

(iii) $H : [0, \frac{1}{10}] \to \mathbb{R},\ H(x) := \displaystyle\int_x^{2x} \frac{\sqrt{1-9t^2}}{\sqrt{1-t^2}}\ \mathrm{d}t,\ x \in [0, \frac{1}{10}]$

Solution. (a) The First Fundamental Theorem of Calculus is stated below. This is Theorem 5.1.1 in the Lecture Notes. It is good practice, although not essential, to use a theorem environment to express the result.

Theorem. Let $f : [a,b] \to \mathbb{R}$ denote a bounded integrable function and suppose that $F : [a,b] \to \mathbb{R}$ is given by $F(x) := \int_a^x f$ for all $x \in [a,b]$. If f is continuous at c, where $c \in [a,b]$, then F is differentiable at c with derivative $F'(c) = f(c)$.

(b)(i) Let $f : [2,4] \to \mathbb{R}$ be given by $f(x) := 3x^{2x}$ for all $x \in [2,5]$. This is a continuous function because the exponential function is continuous on $\mathbb{R}$. The First Fundamental Theorem of Calculus applies at each point c in the interval $[3,5]$, because f is continuous at each such point, so F is differentiable with $F'(x) = f(x) = 3x^{2x}$ for all $x \in [3,5]$.

(b)(ii) Let $f : [1,4] \to \mathbb{R}$ be given by $f(x) := e^{-x}/x$ for all $x \in [1,4]$. The First Fundamental Theorem of Calculus does not apply directly here because $3\sin^2(x)+1$ appears in upper endpoint of the integral. Instead, let $F : [1,4] \to \mathbb{R}$ be given by

$$F(x) := \int_1^x \frac{e^{-t}}{t}\ \mathrm{d}t = \int_1^x f(t)\ \mathrm{d}t \quad \text{for all} \quad x \in [1,4].$$

The function f is continuous, since it is the quotient of an exponential function and a nonzero polynomial on the interval $[1,4]$, so the First Fundamental Theorem of Calculus implies that F is differentiable with $F'(x) = f(x) = e^{-x}/x$ for all $x \in [1,4]$.

Next, let $g : (\pi, 2\pi) \to \mathbb{R}$ be given by $g(x) := 3\sin^2(x)+1$ for all $x \in (\pi, 2\pi)$. The function g is differentiable with derivative $g'(x) = 6\sin(x)\cos(x)$ for all $x \in (\pi, 2\pi)$. Moreover, we have $G = F \circ g$, where F is also differentiable, so the Chain Rule implies that G is differentiable with derivative

$$\begin{aligned} G'(x) &= (F \circ g)'(x) = F'(g(x))g'(x) \\ &= \frac{e^{-(g(x))}}{g(x)}(6\sin(x)\cos(x)) \\ &= \frac{e^{-(3\sin^2(x)+1)}}{3\sin^2(x)+1}(6\sin(x)\cos(x)) \end{aligned}$$

for all $x \in (\pi, 2\pi)$.

(b)(iii) Let $f : [0, \frac{1}{5}] \to \mathbb{R}$ be given by $f(x) := \sqrt{1-9x^2}/\sqrt{1-x^2}$ for all $x \in [0, \frac{1}{5}]$. The First Fundamental Theorem of Calculus does not apply directly here because the variable x appears in the lower endpoint of the integral. Instead, let $F : [0, \frac{1}{5}] \to \mathbb{R}$ be given by

$$F(x) := \int_0^x \frac{\sqrt{1-9t^2}}{\sqrt{1-t^2}}\ \mathrm{d}t = \int_0^x f(t)\ \mathrm{d}t \quad \text{for all} \quad x \in [0, \tfrac{1}{5}].$$

The function f is continuous, since the square-root function is continuous on $[0, \infty)$ and the polynomials in the definition of f are positive on $[0, \frac{1}{5}]$, so the First Fundamental Theorem of Calculus implies that F is differentiable with $F'(x) = f(x) = \sqrt{1-9x^2}/\sqrt{1-x^2}$ for all $x \in [0, \frac{1}{5}]$.

Next, we use the restriction and extension properties of the integral in Theorem 4.2.1 of the Lecture Notes to write

$$\begin{aligned} H(x) &= \int_x^{2x} \frac{\sqrt{1-9t^2}}{\sqrt{1-t^2}}\ \mathrm{d}t \\ &= \int_0^{2x} \frac{\sqrt{1-9t^2}}{\sqrt{1-t^2}}\ \mathrm{d}t - \int_0^{x} \frac{\sqrt{1-9t^2}}{\sqrt{1-t^2}}\ \mathrm{d}t \\ &= F(2x) - F(x) \end{aligned}$$

for all $x \in [0, \frac{1}{10}]$. The linearity of derivatives and the Chain Rule thus imply that H is differentiable with

$$\begin{aligned} H'(x) &= F'(2x)(2x)' - F'(x) \\ &= 2\frac{\sqrt{1-9(2x)^2}}{\sqrt{1-(2x)^2}} - \frac{\sqrt{1-9x^2}}{\sqrt{1-x^2}} \\ &= 2\frac{\sqrt{1-36x^2}}{\sqrt{1-4x^2}} - \frac{\sqrt{1-9x^2}}{\sqrt{1-x^2}} \end{aligned}$$

for all $x \in [0, \frac{1}{10}]$. □

Q2. (a) State the Second Fundamental Theorem of Calculus.

(b) Suppose that $-\infty < a < b < \infty$. Use the Second Fundamental Theorem of Calculus to prove that

$$\int_a^b |x|\ \mathrm{d}x = \begin{cases} \frac{1}{2}(b^2 - a^2), & a \geq 0; \\ \frac{1}{2}(a^2 + b^2), & a < 0 \leq b; \\ \frac{1}{2}(a^2 - b^2), & b < 0. \end{cases}$$

You must verify all hypotheses required to apply the Second Fundamental Theorem of Calculus. In particular, if you use the fact that a certain function is an antiderivative of the absolute value function, then you must prove this fact (be careful proving differentiability at the origin).

(c) Let $f : \mathbb{R} \setminus [-3,-2] \to \mathbb{R}$ be defined by $f(x) := |x|$ for all $x \in \mathbb{R} \setminus [-3,-2]$. Find two antiderivatives F_1 and F_2 of f such that

$$F_1(x) - F_2(x) = \begin{cases} 9, & x > -2; \\ 3, & x < -3. \end{cases}$$

You must prove that your choices for F_1 and F_2 are antiderivatives of f.

Solution. (a) The Second Fundamental Theorem of Calculus is stated below. It is good practice, although not essential, to use a theorem environment to express the result.

Theorem. If $f : [a,b] \to \mathbb{R}$ is a bounded integrable function, and there exists a differentiable function $g : [a,b] \to \mathbb{R}$ such that $f = g'$, then $\int_a^b f = g(b) - g(a)$.

(b) The function $f : [a,b] \to \mathbb{R}$ given by $f(x) := |x|$ for all $x \in [a,b]$ is continuous, hence it is integrable by Theorem 4.1.5 in the Lecture Notes. An antiderivative of f is any differentiable function g such that $f = g'$. We claim that $g : [a,b] \to \mathbb{R}$ given by

$$g(x) := \begin{cases} \frac{1}{2}x^2, & \text{if } x \geq 0; \\ -\frac{1}{2}x^2, & \text{if } x < 0, \end{cases}$$

is differentiable with $g'(x) = |x| = f(x)$ for all $x \in [a,b]$. To prove this, we consider the following three cases:

• If $a \geq 0$, then $g(x) = \frac{1}{2}x^2$ for all $x \in [a,b]$, so g is a polynomial around each point in its domain, and which is differentiable with derivative $g'(x) = x = |x|$ for all $x \in [a,b]$.

• If $b < 0$, then $g(x) = -\frac{1}{2}x^2$ for all $x \in [a,b]$, so g is a polynomial around each point in its domain, and which is differentiable with derivative $g'(x) = -x = |x|$ for all $x \in [a,b]$.

• If $a < 0 \leq b$, then g is differentiable on $[a,0)$ and on $(0,b]$ as in the above two cases. We must check if it is differentiable at the origin, however, because in this case g is *not* given by a *single* polynomial expression in any open interval containing the origin. To do this, we calculate

$$\lim_{h \to 0^+} \frac{g(0+h) - g(0)}{h} = \lim_{h \to 0^+} \frac{\frac{1}{2}h^2 - 0}{h} = \lim_{h \to 0^+} \tfrac{1}{2}h = 0$$

and

$$\lim_{h \to 0^-} \frac{g(0+h) - g(0)}{h} = \lim_{h \to 0^-} \frac{-\frac{1}{2}h^2 - 0}{h} = \lim_{h \to 0^-} -\tfrac{1}{2}h = 0.$$

The equality of these limits proves that g is differentiable at 0 with

$$g'(0) := \lim_{h \to 0} \frac{g(0+h) - g(0)}{h} = 0 = f(0).$$

It is important to recognise here that differentiability away from the origin is immediate from the differentiability of polynomials on $\mathbb{R}$. Around the origin, however, the function g is not given by a *single* polynomial, so we must instead argue directly using the definition of differentiability.

To conclude, since f is integrable, and we have found an antiderivative g (i.e. $f = g'$), we can apply the Second Fundamental Theorem of Calculus (Theorem 5.2.1 in the Lecture Notes) to obtain

$$\begin{aligned} \int_a^b |x| \, \mathrm{d}x &= \int_a^b f = \int_a^b g' \\ &= g(b) - g(a) = \begin{cases} \frac{1}{2}(b^2 - a^2), & a \geq 0; \\ \frac{1}{2}(a^2 + b^2), & a < 0 \leq b; \\ \frac{1}{2}(a^2 - b^2), & b < 0, \end{cases} \end{aligned}$$

as required.

(c) Now suppose that $f : \mathbb{R} \setminus [-3,-2] \to \mathbb{R}$ is defined by $f(x) := |x|$ for all $x \in \mathbb{R} \setminus [-3,-2]$. The domain of f is the set $\mathbb{R} \setminus [-3,-2] = (-\infty,-3) \cup (-2,\infty)$, which consists of two *disconnected* intervals $(-\infty,-3)$ and $(-2,\infty)$. It is this property which allows us to obtain antiderivatives, F_1 and F_2, which do *not* differ by an additive constant. For example, the functions $F_1 : \mathbb{R} \setminus [-3,-2] \to \mathbb{R}$ defined by

$$F_1(x) := \begin{cases} \frac{1}{2}x^2 + 9, & \text{if } x \in [0,\infty); \\ -\frac{1}{2}x^2 + 9, & \text{if } x \in (-2,0); \\ -\frac{1}{2}x^2 + 3, & \text{if } x \in (-\infty,-3), \end{cases}$$

and $F_2 : \mathbb{R} \setminus [-3,-2] \to \mathbb{R}$ defined by

$$F_2(x) := \begin{cases} \frac{1}{2}x^2, & \text{if } x \in [0,\infty); \\ -\frac{1}{2}x^2, & \text{if } x \in (-2,0); \\ -\frac{1}{2}x^2, & \text{if } x \in (-\infty,-3), \end{cases}$$

are both antiderivatives of f with

$$F_1(x) - F_2(x) = \begin{cases} 9, & x > -2; \\ 3, & x < -3, \end{cases}$$

as required. In particular, the functions F_1 and F_2 are both differentiable with $F_1' = F_2' = f$, as in part (b). It is enough to reference the analogous discussion and computations in part (b) for justification here, but the details are included below for completeness.

The functions F_1 and F_2 are each given by a single polynomial around each point in their domain $(-\infty,-3) \cup (-2,\infty)$, except at the origin. The differentiability of F_1 at the origin is proved by computing

$$\lim_{h\to 0^+} \frac{F_1(0+h) - F_1(0)}{h} = \lim_{h\to 0^+} \frac{(\frac{1}{2}h^2 + 9) - 9}{h} = \lim_{h\to 0^+} \tfrac{1}{2}h = 0$$

and

$$\lim_{h\to 0^-} \frac{F_1(0+h) - F_1(0)}{h} = \lim_{h\to 0^-} \frac{(-\frac{1}{2}h^2 + 9) - 9}{h} = \lim_{h\to 0^-} -\tfrac{1}{2}h = 0,$$

to deduce that $F_1'(0) = 0 = f(0)$. The differentiability of F_2 at the origin is proved likewise by computing

$$\lim_{h\to 0^+} \frac{F_2(0+h) - F_2(0)}{h} = \lim_{h\to 0^+} \frac{\frac{1}{2}h^2 - 0}{h} = \lim_{h\to 0^+} \tfrac{1}{2}h = 0$$

and

$$\lim_{h\to 0^-} \frac{F_1(0+h) - F_1(0)}{h} = \lim_{h\to 0^-} \frac{-\frac{1}{2}h^2 - 0}{h} = \lim_{h\to 0^-} -\tfrac{1}{2}h = 0,$$

to deduce that $F_2'(0) = 0 = f(0)$.

More generally, we could choose

$$F_1(x) := \begin{cases} \frac{1}{2}x^2 + c_1, & \text{if } x \in [0,\infty); \\ -\frac{1}{2}x^2 + c_1, & \text{if } x \in (-2,0); \\ -\frac{1}{2}x^2 + c_2, & \text{if } x \in (-\infty,-3), \end{cases}$$

and

$$F_2(x) := \begin{cases} \frac{1}{2}x^2 + c_3, & \text{if } x \in [0,\infty); \\ -\frac{1}{2}x^2 + c_3, & \text{if } x \in (-2,0); \\ -\frac{1}{2}x^2 + c_4, & \text{if } x \in (-\infty,-3), \end{cases}$$

for any real numbers c_1, c_2, c_3 and c_4, such that $c_1 - c_3 = 9$ and $c_2 - c_4 = 3$. □

Q3. Find the following antiderivatives and integrals (henceforth $\log(x) := \log_e(x)$):

(a) $\displaystyle\int_1^e x^2 \log(x)\, \mathrm{d}x$

(b) $\displaystyle\int (\log(x))^2 \, \mathrm{d}x$

(c) $\displaystyle\int_0^\pi e^x \cos(x) \, \mathrm{d}x$

Solution. (a) We apply Integration by Parts with

$$\begin{cases} u(x) = \log(x) \\ v(x) = \frac{1}{3}x^3 \end{cases} \qquad \text{and} \qquad \begin{cases} u'(x) = x^{-1} \\ v'(x) = x^2 \end{cases}$$

to obtain

$$\begin{aligned} \int_1^e x^2 \log(x) \, \mathrm{d}x &= \left[\tfrac{1}{3}x^3 \log(x)\right]_1^e - \int_1^e x^{-1}(\tfrac{1}{3}x^3) \, \mathrm{d}x \\ &= \tfrac{1}{3}e^3 - 0 - \left[\tfrac{1}{9}x^3\right]_1^e \\ &= \tfrac{1}{3}e^3 - \tfrac{1}{9}e^3 + \tfrac{1}{9} \\ &= \tfrac{2}{9}e^3 + \tfrac{1}{9}. \end{aligned}$$

(b) After recalling that $\int \log(x) \, \mathrm{d}x = x(\log(x) - 1)$ by Example 6.1.2 in the Lecture Notes, we apply Integration by Parts with

$$\begin{cases} u(x) = \log(x) \\ v(x) = x(\log(x) - 1) \end{cases} \qquad \text{and} \qquad \begin{cases} u'(x) = x^{-1} \\ v'(x) = \log(x) \end{cases}$$

to obtain

$$\begin{aligned} &\int (\log(x))^2 \, \mathrm{d}x \\ &\quad = (\log(x))x(\log(x) - 1) - \int x^{-1}x(\log(x) - 1) \\ &\quad = (\log(x))x(\log(x) - 1) - \int (\log(x) - 1) \\ &\quad = (\log(x))x(\log(x) - 1) - x(\log(x) - 1) + x \\ &\quad = (\log(x) - 1)x(\log(x) - 1) + x \\ &\quad = x(\log(x) - 1)^2 + x \\ &\quad = x(\log(x))^2 - 2x\log(x) + 2x. \end{aligned}$$

(c) We apply Integration by Parts twice. First with $u(x) = e^x$ and $v(x) = \sin(x)$, so $u'(x) = e^x$ and $v'(x) = \cos(x)$, to obtain

$$\int_0^\pi e^x \cos(x) \, \mathrm{d}x = [e^x \sin(x)]_0^\pi - \int_0^\pi e^x \sin(x) \, \mathrm{d}x.$$

Next, with $u(x) = e^x$ and $v(x) = -\cos(x)$, so $u'(x) = e^x$ and $v(x)' = \sin(x)$, to obtain

$$\begin{aligned} \int_0^\pi e^x \sin(x) \, \mathrm{d}x &= [-e^x \cos(x)]_0^\pi - \int_0^\pi e^x(-\cos(x)) \, \mathrm{d}x \\ &= [-e^x \cos(x)]_0^\pi + \int_0^\pi e^x \cos(x) \, \mathrm{d}x. \end{aligned}$$

We combine the previous two identities to obtain

$$\int_0^\pi e^x \cos(x) \, \mathrm{d}x = [e^x \sin(x)]_0^\pi - \left([-e^x \cos(x)]_0^\pi + \int_0^\pi e^x \cos(x) \, \mathrm{d}x\right),$$

hence

$$\int_0^\pi e^x \cos(x) \, \mathrm{d}x = \tfrac{1}{2}\left[e^x(\sin(x) + \cos(x))\right]_0^\pi = -\tfrac{1}{2}(e^\pi + 1).$$

Q4. Find the following antiderivatives and integrals:

(a) $\displaystyle\int \frac{x-4}{x^2-5x+6}\,\mathrm{d}x$

(b) $\displaystyle\int_1^2 \frac{x^5+x-1}{x^3+1}\,\mathrm{d}x$

(c) $\displaystyle\int \frac{x^2+2x-1}{x^3-x}\,\mathrm{d}x$

Solution. (a) The integrand is a rational function so we use the method of partial fractions. The denominator $x^2-5x+6=(x-3)(x-2)$ has two linear factors, so the appropriate form of the partial fraction expansion is

$$\frac{x-4}{x^2-5x+6}=\frac{A}{x-3}+\frac{B}{x-2}.$$

After finding a common denominator, we obtain

$$x-4=A(x-2)+B(x-3)=(A+B)x+(-2A-3B).$$

We equate the coefficients of both x and x^0 to obtain the simultaneous equations $A+B=1$ and $(-2A-3B)=-4$ with the solution $A=-1$ and $B=2$, hence

$$\begin{aligned}\int \frac{x-4}{x^2-5x+6}\,\mathrm{d}x &= \int \frac{-1}{x-3}\,\mathrm{d}x+\int \frac{2}{x-2}\,\mathrm{d}x\\ &= -\log|x-3|+2\log|x-2|.\end{aligned}$$

(b) The integrand is a rational function so we use the method of partial fractions. The degree of the numerator is larger than degree of the denominator, so we use polynomial long division to obtain

$$\frac{x^5+x-1}{x^3+1}=x^2+\frac{-x^2+x-1}{x^3+1}.$$

The denominator $x^3+1=(x+1)(x^2-x+1)$ has a linear factor and an irreducible quadratic factor, so the appropriate form of the partial fraction expansion is

$$\frac{-x^2+x-1}{(x+1)(x^2-x+1)}=\frac{A}{x+1}+\frac{Bx+C}{x^2-x+1}.$$

After finding a common denominator, we obtain

$$\begin{aligned}-x^2+x-1&=A(x^2-x+1)+(Bx+C)(x+1)\\ &=(A+B)x^2+(-A+B+C)x+(A+C).\end{aligned}$$

We equate the coefficients of x^2, x and x^0 to obtain the simultaneous equations

$$\begin{aligned}A+B&=-1\\ -A+B+C&=1\\ A+C&=-1\end{aligned}$$

with the solution $A=-1$, $B=0$ and $C=0$, hence

$$\begin{aligned}\int_1^2 \frac{x^5+x-1}{x^3+1}\,\mathrm{d}x &= \int_1^2 x^2\,\mathrm{d}x+\int_1^2 \frac{-1}{x+1}\,\mathrm{d}x\\ &= \left[\tfrac{1}{3}x^3-\log|x+1|\right]_1^2\\ &= \tfrac{7}{3}+\log(\tfrac{2}{3}).\end{aligned}$$

(c) The integrand is a rational function so we use the method of partial fractions. The denominator $x^3-x=x(x+1)(x-1)$ has three linear factors, so the correct form of the partial fraction decomposition is

$$\frac{x^2+2x-1}{x^3-x}=\frac{A}{x}+\frac{B}{x+1}+\frac{C}{x-1}.$$

After finding a common denominator, we obtain

$$\begin{aligned} x^2 + 2x - 1 &= A(x^2 - 1) + B(x^2 - x) + C(x^2 + x) \\ &= (A + B + C)x^2 + (-B + C)x + (-A). \end{aligned}$$

We equate the coefficients of x^2, x and x^0 to obtain the simultaneous equations

$$\begin{aligned} A + B + C &= 1 \\ -B + C &= 2 \\ -A &= -1 \end{aligned}$$

with the solution $A = 1$, $B = -1$ and $C = 1$, hence

$$\begin{aligned} \int \frac{x^2 + 2x - 1}{x^3 - x} \, \mathrm{d}x &= \int \frac{1}{x} \, \mathrm{d}x + \int \frac{-1}{x + 1} \, \mathrm{d}x + \int \frac{1}{x - 1} \, \mathrm{d}x \\ &= \log|x| - \log|x + 1| + \log|x - 1|. \end{aligned}$$

Extra Questions

EQ1. Use the First Fundamental Theorem of Calculus and apply it to prove that each function below is differentiable and to find its derivative in terms of elementary functions:

(a) $F : [0, 2] \to \mathbb{R}, \ F(x) := \displaystyle\int_0^x \sin(t^2) \, \mathrm{d}t, \ x \in [0, 2]$

(b) $G : [1, 2] \to \mathbb{R}, \ G(x) := \displaystyle\int_1^x \sin(t^2) \, \mathrm{d}t, \ x \in [1, 2]$

(c) $H : [0, 1] \to \mathbb{R}, \ H(x) := \displaystyle\int_x^1 \sin(t^2) \, \mathrm{d}t, \ x \in [0, 1]$

(d) $I : [0, 1] \to \mathbb{R}, \ I(x) := \displaystyle\int_0^{2x^3} \sin(t^2) \, \mathrm{d}t, \ x \in [0, 1]$

(e) $J : [0, 2] \to \mathbb{R}, \ J(x) := \left(\displaystyle\int_0^x \sin(t^2) \, \mathrm{d}t \right)^2, \ x \in [0, 2]$

The formula for the derivative $J'(x)$ may contain an integral expression.

EQ2. Suppose that $f : [1, 3] \to \mathbb{R}$ is differentiable and that its derivative f' is continuous:

(a) If $f(1) = 10$ and $\int_1^3 f' = 16$, then calculate $f(3)$.

(b) Explain why the continuity of f' allowed for the application of the Fundamental Theorem of Calculus in part (a).

(c) State a weaker condition on f' that would suffice to apply the Fundamental Theorem of Calculus in part (a).

EQ3. Find the following antiderivatives and integrals:

(a) $\displaystyle\int x \sin(5x) \, \mathrm{d}x$

(b) $\displaystyle\int_1^2 \frac{(\log(x))^2}{x^3} \, \mathrm{d}x$

(c) $\displaystyle\int e^{2x} \sin(3x) \, \mathrm{d}x$

EQ4. Find the following antiderivatives and integrals:

(a) $\displaystyle\int \frac{x^2+1}{x+4}\,\mathrm{d}x$

(b) $\displaystyle\int \frac{10}{(x-1)(x^2+4)}\,\mathrm{d}x$

(c) $\displaystyle\int_3^4 \frac{x^2+1}{x^2-4x+4}\,\mathrm{d}x$