Solutions to Mock Exam

1. A forced damped pendulum has an equation

$$\frac{d^2\theta}{dt^2} + 6\frac{d\theta}{dt} + 25\sin\theta = R\cos 4t \quad .$$

When $R \ll 1$, find the approximate solution for the attractor and transients. **[6]**
Using the two choices

$$p \equiv \frac{d\theta}{dt} \qquad P \equiv \frac{d\theta}{dt} + 3\theta$$

find the fundamental equations and propose which of these is more useful for the attractor and which is more useful for the transients; with an explanation for your choice. **[3]**
On what scale is the error? **[1]**

Answer 1. As $R \ll 1$ we can use the approximation $\sin\theta \mapsto \theta$ which gives the forced damped oscillator, a solvable problem. For the transients

$$\theta = Ae^{Dt} \qquad \Rightarrow \qquad D^2 + 6D + 25 = (D+3)^2 + 16 = 0$$

so

$$\theta = Ae^{-3t}\cos 4(t - t_0)$$
$$p = -Ae^{-3t}\left[3\cos 4(t-t_0) + 4\sin 4(t-t_0)\right]$$
$$P = -4Ae^{-3t}\sin 4(t-t_0)$$

and P is more natural

$$\theta^2 + \left[\frac{P}{4}\right]^2 = A^2 e^{-6t}$$

For the attractor

$$\theta = B\cos 4t + C\sin 4t$$

which, when substituted in, provides

$$-16\left[B\cos 4t + C\sin 4t\right] + 24\left[-B\sin 4t + C\cos 4t\right] + 25\left[B\cos 4t + C\sin 4t\right] = R\cos 4t$$

Equating coeficients gives

$$9C - 24B = 0 \quad 9B + 24C = R \quad \Rightarrow \quad 3C = 8B \quad 9B + 64B = R$$

$$\Rightarrow \quad B = \frac{R}{73} \qquad C = \frac{8}{3}\frac{R}{73}$$

Now we can find

$$p = -4B\sin 4t + 4C\cos 4t$$

$$P = -4B\sin 4t + 4C\cos 4t + 3B\cos 4t + 3C\sin 4t$$

and p is more natural

$$\theta^2 + \left[\frac{p}{4}\right]^2 = B^2 + C^2 = \frac{3^2 + 8^2}{(3\times 73)^2}R^2 = \frac{R^2}{9\times 73}$$

The error stems from

$$\sin\theta = \theta - \frac{\theta^3}{6}...$$

and so we expect an error of size

$$\frac{1}{6}\left[\frac{R}{3\sqrt{73}}\right]^3 [\cos(4t-\lambda)]^3$$

where $\tan\lambda = \frac{8}{3}$.

2. Consider the map

$$x_{n+1} = rx_n(1-x_n)^2$$

where $r > 0$ is a control parameter. Find all the possible 1-cycles and establish for which range of control parameter they are stable. **[4]**
Find all the possible 2-cycles and establish for which range of control parameter they are stable. **[5]**
How do you think this sequence of cycles continues? **[1]**

Answer 2. The 1-cycles satisfy the equation

$$x = rx(1-x)^2 \quad \Rightarrow \quad x = 0, \quad x = 1 \pm \frac{1}{\sqrt{r}}$$

Stability is controlled by

$$\frac{df}{dx}(x) = r\left[1 - 4x + 3x^2\right] = r(1-x)(1-3x)$$

For x=0 we have

$$-1 < \frac{df}{dx}(0) = r < 1 \quad \Rightarrow \quad -1 < r < 1$$

For x=1+$\frac{1}{\sqrt{r}}$ we have

$$-1 < \frac{df}{dx}\left(1+\frac{1}{\sqrt{r}}\right) = -r\frac{1}{\sqrt{r}}\left[-2-\frac{3}{\sqrt{r}}\right] = 2\sqrt{r}+3 < 1 \quad \Rightarrow \quad -2 < \sqrt{r} < -1$$

and so this 1-cycle is never stable. For x=1-$\frac{1}{\sqrt{r}}$ we have

$$-1 < \frac{df}{dx}\left(1-\frac{1}{\sqrt{r}}\right) = r\frac{1}{\sqrt{r}}\left[-2+\frac{3}{\sqrt{r}}\right] = -2\sqrt{r}+3 < 1 \quad \Rightarrow \quad 1 < \sqrt{r} < 2$$

$$\Rightarrow \quad 1 < r < 4$$

The 2-cycle calculation is much harder. The 2-cycles satisfy

$$y = rx(1-x)^2 \qquad x = ry(1-y)^2$$

and we can divide by r and subtract the two equations

$$(y-x)\left[\frac{1}{r} + 1 - 2(y+x) + y^2 + xy + x^2\right] = 0$$

using the two identities

$$y^2 - x^2 = (y-x)(y+x) \qquad y^3 - x^3 = (y-x)(y^2+xy+x^2)$$

We may also product the two equations to get

$$xy\left[(r(1-x)(1-y))^2 - 1\right] = 0$$

If we are hunting for 2-cycles we may assume $x \neq y$ and $xy \neq 0$ because x=0 is a 1-cycle. We can then observe that we are left with

$$(1-x)(1-y) = 1-(y+x)+xy = \frac{\sigma}{r} \qquad \frac{1}{r}+1-2(y+x)+(y+x)^2 - xy = 0$$

where $\sigma = \pm 1$ and we are trying to save effort by tackling both at once. Adding the two equations gives

$$(y+x)^2 - 3(y+x) + 2 + \frac{1-\sigma}{r} = 0 \quad \Rightarrow \quad \left[y+x-\frac{3}{2}\right]^2 = \frac{9}{4} - 2 - \frac{1-\sigma}{r} = \frac{1}{4} - \frac{1-\sigma}{r}$$

This gives us an easy solution, when σ=1

$$y+x = 1 \quad or \quad 2 \qquad xy = y+x-1+\frac{1}{r} = \frac{1}{r} \quad or \quad 1+\frac{1}{r}$$

We can build quadratic equations for the roots

$$(z-x)(z-y) = z^2 - z(y+x) + xy \quad \Rightarrow \quad z^2 - z + \frac{1}{r} = 0 \quad or \quad z^2 - 2z + 1 + \frac{1}{r} = 0$$

We may also determine the stability using

$$S \equiv \frac{df}{dx}(x)\frac{df}{dx}(y) = r^2(1-x)(1-y)(1-3x)(1-3y) = r^2(1-(y+x)+xy)(1-3(y+x)+9xy)$$

and then for the first case

$$S = r^2\frac{1}{r}\left[-2+\frac{9}{r}\right] = -2r+9 \quad \Rightarrow \quad -1 < -2r+9 < 1 \quad \Rightarrow \quad 4 < r < 5$$

which connects up to the stable 1-cycle. For the second case

$$S = r^2\frac{1}{r}\left[4+\frac{9}{r}\right] = 4r+9 \quad \Rightarrow \quad -1 < 4r+9 < 1 \quad \Rightarrow \quad -\frac{5}{2} < r < -2$$

which is never relevant. The other pair of solutions is more taxing, using $\Delta_\pm = \pm\left[1-\frac{8}{r}\right]^{\frac{1}{2}}$ we have

$$y+x = \frac{3}{2}+\frac{\Delta_\pm}{2} \qquad xy = \frac{1}{2}+\frac{\Delta_\pm}{2}-\frac{1}{r}$$

from which we can create the quadratic equations and stability is controlled by

$$S = -r^2 \frac{1}{r} \left[1 - 3 \left(\frac{3}{2} + \frac{\Delta_\pm}{2} \right) + 9 \left(\frac{1}{2} + \frac{\Delta_\pm}{2} - \frac{1}{r} \right) \right] = 9 - 3r\Delta_\pm - r$$

and so

$$9r^2\Delta_\pm^2 = 9(r^2 - 8r) = (r + S - 9)^2 \qquad \Rightarrow \qquad 8r^2 - (72 - 18 + 2S)r - (S - 9)^2 = 0$$

When S=1 we find

$$8r^2 - 56r - 64 = 8(r^2 - 7r - 8) = 8(r - 8)(r + 1) = 0$$

and for S=-1 we find

$$8r^2 - 52r - 100 = 8 \left(r^2 - \frac{13}{2} r - \frac{25}{2} \right) = 8 \left(\left[r - \frac{13}{4} \right]^2 - \frac{169}{16} - \frac{200}{16} \right) = 0$$

$$\Rightarrow \qquad r = \frac{13}{4} + \frac{3\sqrt{41}}{4}$$

which is a number slightly bigger than 8 which gives a brief window of stability for the pair of 2-cycles which appear at r=8. This is clearly the first few steps on the bifurcation route to chaos.

3. Consider the map

$$x_{n+1} = r x_n (1 - x_n)^2 \quad .$$

where r is a control parameter. Employ the transformation

$$x_n = \frac{4}{3} \left[\sin \frac{\pi y_n}{2} \right]^2$$

to rewrite the map as

$$\left[\sin \frac{\pi y_{n+1}}{2} \right]^2 = \left[\sin \frac{3\pi y_n}{2} \right]^2$$

at a particular value of r that you should determine. **[5]**
On the assumption that $y_n \in [0, 1]$, find the map that constitutes $y_{n+1} = M[y_n]$ and depict this map. **[4]**
Find all the possible 1-cycles of both maps and for which range of control parameter they are stable. **[1]**

Answer 3. We may substitute the transformation into the original equation to get

$$\frac{4}{3} \left[\sin \frac{\pi y_{n+1}}{2} \right]^2 = r \frac{4}{3} \left[\sin \frac{\pi y_n}{2} \right]^2 \left(1 - \frac{4}{3} \left[\sin \frac{\pi y_n}{2} \right]^2 \right)^2$$

and then rescaling

$$\left[\sin \frac{\pi y_{n+1}}{2} \right]^2 = \frac{r}{9} \left[\sin \frac{\pi y_n}{2} \right]^2 \left(3 - 4 \left[\sin \frac{\pi y_n}{2} \right]^2 \right)^2$$

We now need the identity

$$e^{3i\phi} = \cos 3\phi + i \sin 3\phi = (\cos\phi + i\sin\phi)^3$$

$$= \cos^3\phi - 3\cos\phi\sin^2\phi + i(3\cos^2\phi\sin\phi - \sin^3\phi)$$

which gives us that

$$\sin 3\phi = 3\sin\phi - 4\sin^3\phi$$

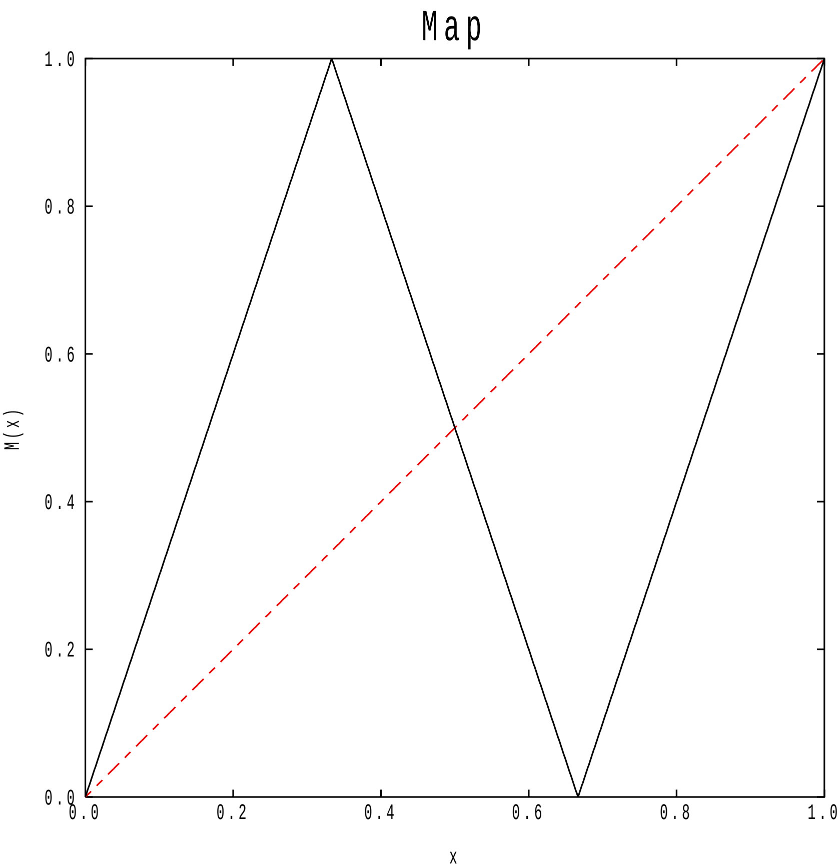
and when r=9 we find

$$\left[\sin\frac{\pi y_{n+1}}{2}\right]^2 = \left[\sin\frac{3\pi y_n}{2}\right]^2$$

Using y_{n+1}=$M[y_n]$ we can see that

$$\begin{aligned} M[y] =& 3y \qquad & y \in \left[0, \frac{1}{3}\right] \\ =& 2 - 3y \qquad & y \in \left[\frac{1}{3}, \frac{2}{3}\right] \\ =& 3y - 2 \qquad & y \in \left[\frac{2}{3}, 1\right] \end{aligned}$$

which depicts as



We needed

$$\sin(\pi - z) = \sin z = -\sin(z - \pi)$$

with $z=\frac{3\pi y}{2}$ in the derivation. The 1-cycles satisfy

$$y = 3y \qquad y = 2 - 3y \qquad y = 3y - 2 \quad \Rightarrow \quad y = 0,\ \frac{1}{2},\ 1$$

since $\frac{df}{dy}(y)=\pm 3$ these cycles are never stable.

4. Consider the map defined by

$$x_{n+1} = M[x_n]$$

where

$$\begin{aligned} M[x] =& 3x && x \in \left[0, \frac{1}{3}\right] \\ =& 2 - 3x && x \in \left[\frac{1}{3}, \frac{2}{3}\right] \\ =& 3x - 2 && x \in \left[\frac{2}{3}, 1\right] \end{aligned}$$

Employ Base 3 to determine a useful representation of this map. **[5]**
Find all the possible 1-cycles and 2-cycles of this map using Base 3. **[3]**
Check your answers directly. **[2]**

Answer 4. In the first third of the region the ternary representation is $0.0a_{\bar{2}}a_{\bar{3}}a_{\bar{4}}...$ and multiplying by three moves each of the digits forward by one so

$$M[0.0a_{\bar{2}}a_{\bar{3}}a_{\bar{4}}...] = 0.a_{\bar{2}}a_{\bar{3}}a_{\bar{4}}...$$

In the second third of the region the ternary representation is $0.1a_{\bar{2}}a_{\bar{3}}a_{\bar{4}}... \equiv \frac{1}{3}+z$ where z is in the first third, so

$$M[0.1a_{\bar{2}}a_{\bar{3}}a_{\bar{4}}...] = M\left[\frac{1}{3} + z\right] = 1 - 3z = 1 - 3 \times 0.0a_{\bar{2}}a_{\bar{3}}a_{\bar{4}}... = 1 - 0.a_{\bar{2}}a_{\bar{3}}a_{\bar{4}}...$$

$$= 0.222... - 0.a_{\bar{2}}a_{\bar{3}}a_{\bar{4}}... = 0.(2 - a_{\bar{2}})(2 - a_{\bar{3}})(2 - a_{\bar{4}})...$$

and the digits move forward by one but are also mapped onto their complements, $d \mapsto 2 - d$. For the final third of the region the ternary representation is $0.2a_{\bar{2}}a_{\bar{3}}a_{\bar{4}}... \equiv \frac{2}{3} + z$ where z is in the first third, so

$$M[0.2a_{\bar{2}}a_{\bar{3}}a_{\bar{4}}...] = M\left[\frac{2}{3} + z\right] = 3z = 0.a_{\bar{2}}a_{\bar{3}}a_{\bar{4}}...$$

and the digits move forward along the number by one with the first digit being lost.

To find 1-cycles using base 3 we need to use the idea that the number repeats or if we have a unit digit at the front it maps onto its complements. Since the complement of one is one we have

$$0.000.... \qquad 0.111.... \qquad 0.222....$$

and then multiplying by three

$$x = 0.111.... \quad \Rightarrow \quad 3x = 1.111.... = 1 + x \quad \Rightarrow \quad x = \frac{1}{2}$$

$$x = 0.222.... \quad \Rightarrow \quad 3x = 2.222.... = 2 + x \quad \Rightarrow \quad x = 1$$

For 2-cycles we can use numbers that repeat by two digits if there are an even number of ones but repeat with their complements if there is an odd number of ones

$$0.0000..., 0.0202..., 0.2020..., 0.2222..., 0.1111..., 0.0121..., 0.2101..., 0.1012..., 0.1210...$$

provide the nine possibilities. We can see the three 1-cycles and then applying the map we get the pairs

$$x = 0.0202... \quad y = 0.2020... \quad \Rightarrow \quad 3x = y \quad 3y = 2 + x = 9x \quad \Rightarrow \quad x = \frac{1}{4} \quad y = \frac{3}{4}$$

$$x = 0.0121... \quad y = 0.1210... \quad \Rightarrow \quad 3x = y \quad 27y = 1 + 6 + 9 + x$$
$$\Rightarrow \quad x = \frac{16}{80} = \frac{1}{5} \quad y = \frac{3}{5}$$
$$x = 0.1012... \quad y = 0.2101... \quad \Rightarrow \quad 3y = 2 + x \quad 27x = 1 + 9 + y = 81y - 54$$
$$\Rightarrow \quad y = \frac{64}{80} = \frac{4}{5} \quad x = \frac{2}{5}$$

If you substitute them in they do indeed work.

5. A mass feels a potential

$$V(\mathbf{x}) = \frac{1}{2}\left[k_1 x_1^2 + k_2 x_2^2\right] - mgx_2$$

Find the equations of motion and determine a fundamental equation for the system. **[4]**
Solve for the motion in general and show that the system is integrable. **[4]**
Employ a Poincare section with x_1=0 and find the effective map. When does this map have n-cycles as a solution and when is it ergodic on the Poincare surface? **[2]**

Answer 5. Using a Lagraangian, $L = T - V$, with

$$T = \frac{1}{2}m\left[\left(\frac{dx_1}{dt}\right)^2 + \left(\frac{dx_2}{dt}\right)^2\right] \qquad V = \frac{1}{2}\left[k_1 x_1^2 + k_2 x_2^2\right] - mgx_2$$

we get

$$\frac{d}{dt}\frac{\partial L}{\partial \frac{dx_1}{dt}} = m\frac{d^2 x_1}{dt^2} = \frac{\partial L}{\partial x_1} = -k_1 x_1$$
$$\frac{d}{dt}\frac{\partial L}{\partial \frac{dx_2}{dt}} = m\frac{d^2 x_2}{dt^2} = \frac{\partial L}{\partial x_2} = mg - k_2 x_2$$

The fundamental equation is

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} \frac{p_1}{m} \\ \frac{p_2}{m} \\ -k_1 x_1 \\ mg - k_2 x_2 \end{bmatrix}$$

The energies of the individual components are conserved so $E = E_1 + E_2$

$$E_1 = \frac{p_1^2}{2m} + \frac{k_1}{2}x_1^2 \qquad E_2 = \frac{p_2^2}{2m} + \frac{k_2}{2}x_2^2 - mgx_2$$

and

$$\frac{dE_1}{dt} = \frac{p_1}{m}\frac{dp_1}{dt} + k_1 x_1 \frac{dx_1}{dt} = 0$$
$$\frac{dE_2}{dt} = \frac{p_2}{m}\frac{dp_2}{dt} + (k_2 x_2 - mg)\frac{dx_2}{dt} = 0$$

are both independently conserved, so the system is integrable. We can completely solve

$$x_1 = A\cos\left[\frac{k_1}{m}\right]^{\frac{1}{2}}(t - t_1) \qquad p_1 = -\left[k_1 m\right]^{\frac{1}{2}} A \sin\left[\frac{k_1}{m}\right]^{\frac{1}{2}}(t - t_1)$$

$$x_2 = \frac{mg}{k_2} + B\cos\left[\frac{k_2}{m}\right]^{\frac{1}{2}}(t - t_2) \qquad p_2 = -\left[k_2 m\right]^{\frac{1}{2}} B \sin\left[\frac{k_2}{m}\right]^{\frac{1}{2}}(t - t_2)$$

The Poincare section involves the vanishing of x_1 which occurs when

$$\left[\frac{k_1}{m}\right]^{\frac{1}{2}}(T_n - t_1) = \frac{\pi}{2} + n\pi$$

We have a Poincare surface controlled by an ellipse

$$\left[x_2 - \frac{mg}{k_2}\right]^2 + \left[\frac{p_2}{\left[k_2 m\right]^{\frac{1}{2}}}\right]^2 = B^2$$

and then the angle

$$\theta_n = \left[\frac{k_2}{m}\right]^{\frac{1}{2}}(T_n - t_2) = \left[\frac{k_2}{m}\right]^{\frac{1}{2}}(t_1 - t_2) + \left[\frac{k_2}{k_1}\right]^{\frac{1}{2}}\left(\frac{\pi}{2} + n\pi\right)$$

provides the simple map

$$\theta_{n+1} = \theta_n + \left[\frac{k_2}{k_1}\right]^{\frac{1}{2}}\pi$$

We only get an N-cycle if

$$N\left[\frac{k_2}{k_1}\right]^{\frac{1}{2}}\pi = 2\pi M \quad \Rightarrow \quad k_2 = k_1\left(\frac{2M}{N}\right)^2$$

where M is an integer and so the spring constants have to be intimately related.

6. A dynamical system is described by a fundamental equation

$$\frac{dx_1}{dt} = -x_1 + x_2 \qquad \frac{dx_2}{dt} = -x_1 - x_2 + 4x_1x_2 + 2x_1^2 - 2x_1^3 - 2x_2^3$$

Find the fixed points and determine the local trajectories in the vicinity of these fixed points. **[7]**
Depict the phase space portrait. **[3]**

Answer 6. First we need to find the fixed points

$$f_1[x_1, x_2] \equiv -x_1 + x_2 \qquad f_2[x_1, x_2] \equiv -x_1 - x_2 + 4x_1x_2 + 2x_1^2 - 2x_1^3 - 2x_2^3$$

and so $f_1[x_1, x_2]$=0 and $f_2[x_1, x_2]$=0 lead to

$$x_1 = x_2 \qquad -2x_1 + 6x_1^2 - 4x_1^3 \quad \Rightarrow \quad x_1 = x_2 = 0,\ \frac{1}{2},\ 1$$

Next we need to determine the stability matrix

$$\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -1 + 4x_2 + 4x_1 - 6x_1^2 & -1 + 4x_1 - 6x_2^2 \end{bmatrix}$$

At the three fixed points, $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ respectively we find

$$\begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix} \qquad \begin{bmatrix} -1 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix} \qquad \begin{bmatrix} -1 & 1 \\ 1 & -3 \end{bmatrix}$$

Close to the origin

$$M = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix} \quad \Rightarrow \quad \lambda_\pm = -1 \pm i \qquad \begin{bmatrix} \mp i & 1 \\ -1 & \mp i \end{bmatrix} \begin{bmatrix} \alpha_\pm \\ \beta_\pm \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \mathbf{v}_\pm = \begin{bmatrix} 1 \\ \pm i \end{bmatrix}$$

and so

$$\begin{bmatrix} \delta x_1 \\ \delta x_2 \end{bmatrix} = A_+ e^{-t+it} \begin{bmatrix} 1 \\ i \end{bmatrix} + \bar{A}_+ e^{-t-it} \begin{bmatrix} 1 \\ -i \end{bmatrix} = 2 \mid A_+ \mid e^{-t} \begin{bmatrix} \cos(t - t_0) \\ -\sin(t - t_0) \end{bmatrix}$$

and a circular spiral in towards the origin. close to $\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$ we find

$$M = \begin{bmatrix} -1 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix} \quad \Rightarrow \quad (\lambda + 1)\left(\lambda + \frac{1}{2}\right) - \frac{3}{2} = \left(\lambda + \frac{3}{4}\right)^2 - \frac{9}{16} - 1$$

$$= \left(\lambda - \frac{1}{2}\right)(\lambda + 2) = 0 \qquad \begin{bmatrix} -\frac{3}{2} & 1 \\ \frac{3}{2} & -1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 \\ \frac{3}{2} & \frac{3}{2} \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

and so

$$\begin{bmatrix} \delta x_1 \\ \delta x_2 \end{bmatrix} = Ae^{\frac{t}{2}} \begin{bmatrix} 2 \\ 3 \end{bmatrix} + Be^{-2t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

and we have an unstable fixed point. The trajectories move towards the origin along the line x_2=-x_1 at the fastest rate and move more slowly away from the origin along the line x_2=$\frac{3}{2}x_1$. Close to $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ we find

$$M = \begin{bmatrix} -1 & 1 \\ 1 & -3 \end{bmatrix} \quad \Rightarrow \quad (\lambda + 1)(\lambda + 3) - 1 = (\lambda + 2)^2 - 4 + 3 - 1 = (\lambda + 2)^2 - 2 = 0$$

$$\begin{bmatrix} 1 \mp \sqrt{2} & 1 \\ 1 & -1 \mp \sqrt{2} \end{bmatrix} \begin{bmatrix} \alpha_\pm \\ \beta_\pm \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} \alpha_\pm \\ \beta_\pm \end{bmatrix} = \begin{bmatrix} 1 \\ \pm\sqrt{2} - 1 \end{bmatrix} \;\; or \;\; \begin{bmatrix} 1 \pm \sqrt{2} \\ 1 \end{bmatrix}$$

and so

$$\begin{bmatrix} \delta x_1 \\ \delta x_2 \end{bmatrix} = Ae^{-2t+\sqrt{2}t} \begin{bmatrix} 1 + \sqrt{2} \\ 1 \end{bmatrix} + Be^{-2t-\sqrt{2}t} \begin{bmatrix} 1 \\ -1 - \sqrt{2} \end{bmatrix}$$

and we have a stable fixed point. The trajectories move fast towards the origin along the line $x_2 = -(\sqrt{2} + 1)x_1$ and move more slowly towards the origin along the line $x_2 = (\sqrt{2} - 1)x_1$.