Relativistic time dilation

Another manifestation of Lorentz transformation is the fact that measuring time intervals in different inertial frames will lead to different results. This can be seen from the following considerations.

Imagine that one is able to place a clock in every point x of the frame K in which the observer is at rest. We can synchronise those clocks so all of them show the same time. Now consider a frame K' moving with respect to K in positive x-direction. The observer in this frame will have its clock changing constantly its position x, so accordingly to Lorentz transformation the time in his frame

$$t' = \gamma(v)(t - vx/c^2)$$

will be different and all the clocks he sees will not be synchronised.

Therefore it is again important to define *proper time* interval τ_0 as the duration of some physical process, like period of a swinging pendulum, in the frame, where this clock is *at rest*. In other words, τ_0 is the time which can be measured by the *same clock*. In terms of relativistic events, we can take $E_1 = (x'_1, t'_1)$ and $E_2 = (x'_2, t'_2) = (x'_1, t'_1 + \tau_0)$. Applying Lorentz transformation to the time coordinates in the frame K we get

$$\tau = t_2 - t_1 = \gamma(v)\left(t'_2 - t'_1 + v\left(x'_2 - x'_1\right)/c^2\right) = \gamma(v)(\tau_0 + 0) = \frac{\tau_0}{\sqrt{1 - v^2/c^2}} .$$

We see that this time interval is larger or equal to the interval measured in the rest frame of the clock, $\tau \geq \tau_0$. One can reformulate is as "moving clocks run slower". This effect is called *"time dilation"* and Fig. 1 shows the corresponding Minkowski diagram [1].

Muon decay

The time dilation was observed in experiments with muons – elementary particles resulting from cosmic rays hitting upper atmosphere. These are unstable particles moving with relativistic speeds $v \simeq c$ and decaying after a typical life-time of $\tau_0 \simeq 1\mu\text{s} = 10^{-6}\text{s}$. The typical length they should travel before decaying is thus $l \simeq c \times \tau_0 \simeq 3 \times 10^8\text{m/s} \times 10^{-6}\text{s} = 300\text{m}$. However the muons were observed much closer to the earth, having travelled for $l' = 3000\text{m}$. This is due to relativistic time dilation, $l' = c \times \tau$, where

$$\tau = \frac{\tau_0}{\sqrt{1 - v^2/c^2}} .$$

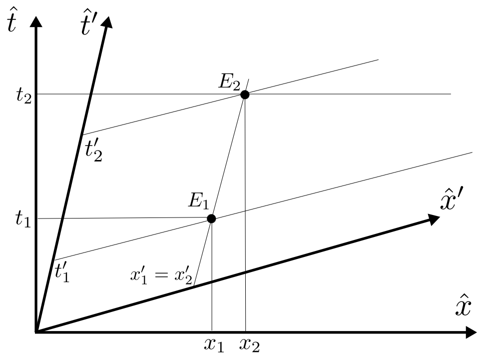

Figure 1: Measuring time by a moving clock.

Given $l'/l = \tau/\tau_0 = 10$ one deduces that $v/c = \sqrt{1 - (1/10)^2} \simeq 0.994$.

[1] It might appear from this diagram that the length of the interval $\tau = t_2 - t_1$ is smaller than $\tau_0 = t'_2 - t'_1$. This is however not true as the distances and time intervals along tilted axes are different from those on straight ones, so one must be careful when comparing them graphically. Only results based on Lorentz transformation can be trusted.

Twin paradox and its Minkowski diagram

The relativistic time dilation could affect future space travel and lead to the following paradox: one twin remains on the earth while another twin travels on a spaceship with a relativistic speed and comes back after many years. Thinking about the travelling twin in the same way as a relativistic muon, we understand that time dilation should affect him and that his biological ageing goes slower that his brother's, so that he must appear younger. This is what usually shown in sci-fi movies.

On the other hand, in the travelling reference frame it is the earth which is moving away with relativistic speed, so the earth sibling should be younger. Below we show that this is not correct by using Minkowski diagram and analysing carefully the corresponding relativistic events.

To do so we change the story slightly. Instead of using the same space ship for flying away and coming back, the travelling twin is using some kind of regular space bus service: he flies away on one space bus going in positive x direction with speed v, then, after time $T/2$ according to the earth clock he jumps onto a bus going in the opposite direction with the same speed v and thus comes back after another $T/2$ time interval. The total time of travel according to the earth clock is $T/2 + T/2 = T$. The corresponding trajectory is shown in Fig. 2 in the earth reference frame K. For convenience the origin $(x,t) = (0,0)$ is take to be the event of changing ships, so the earth is remaining at $x = -vT/2$ for all times in frame K. One can write the departure and arrival events as $(-vT/2, -T/2)$ and $(-vT/2, T/2)$ correspondingly.

Once in the departing spaceship the travelling twin is at rest in the moving reference frame K'. In this frame the traveller moves along $\hat{t}'$ axis so its coordinate is $x' = 0$ for times $-T\sqrt{1 - v^2/c^2}/2 < t' < 0$. Indeed, the Lorentz transformation of the departure event $(-vT/2, -T/2)$ gives

$$\begin{aligned} x' &= \gamma(x - vt) = \gamma(-vT/2 - v(-T/2)) = 0 \\ t' &= \gamma(t - vx/c^2) = \gamma(-T/2 - (v/c^2)(-vT/2)) = -\gamma(1 - v^2/c^2)T/2 = -\sqrt{1 - v^2/c^2}T/2\,. \end{aligned}$$

Similarly, after changing ships, the travelling twin has coordinate $x'' = 0$ in the frame K'' associated with the returning ship and $0 < t'' < T\sqrt{1 - v^2/c^2}/2$.

Before changing ships the travelling twin wonders what is the earth time t_- according to his sibling's clock (*i.e.* in the earth frame K). Thus he uses the Lorentz transformation from K' to K of the event $(x', t') = (-vT/2\gamma, 0)$ and obtains the result

$$t_- = \gamma\left(t' + vx'/c^2\right) = \gamma\left(0 + (v/c^2)(-vT/2\gamma)\right) = -\frac{v^2}{c^2}\frac{T}{2}\,.$$

Once on the earth bound ship the travelling twin recalculates the time experienced by the earth twin. This time he has to use transformation of the event $(x'', t'') = (-vT/2\gamma, 0)$ in K'' frame of the ship to the earth frame K. He obtains a different result

$$t_+ = \gamma\left(t'' - vx''/c^2\right) = \gamma\left(0 - (v/c^2)(-vT/2\gamma)\right) = +\frac{v^2}{c^2}\frac{T}{2}\,.$$

He understands that by changing ships he missed the whole time interval $[t_-, t_+]$ passed on the earth[2].

[2]if he was making a softer U-turn (taking a finite time) instead of changing the direction of motion instantaneously he would notice time on the earth running very fast, giving eventually the same results as our simplified model.

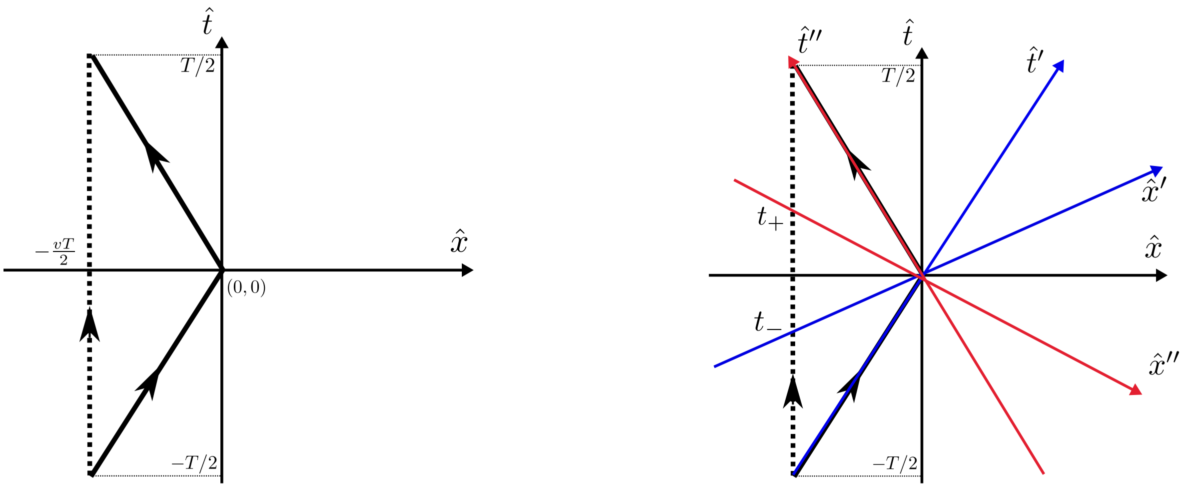

Figure 2: Left: space-time trajectory of the travelling twin (solid) and the earth twin (dashed) in the earth reference frame K. Right: transformation to inertial frames K' (blue axes) and K'' (red axes) associated with the outgoing and returning ship respectively.

The total time experienced by the earth sibling according to the travelling twin's calculation is thus

$$\frac{1}{\gamma}\left(\frac{T}{2\gamma}+\frac{T}{2\gamma}\right)+t_+-t_-=\frac{T}{\gamma^2}+\frac{v^2}{c^2}T=T$$

as it should be. There is no paradox.