Consequences of Lorentz transformation.

Once we have established Lorentz transformations we use them to study relativistic phenomena. The most important are length contraction and time dilation. In each case in order to get correct results it is necessary to define the physical question and reformulate it as a question about space-time points (events) in appropriate inertial frame. Applying Lorentz transformation to these events and collecting information from the transformed coordinates and times will lead to the answer in terms of observed physical quantities.

Relativistic length contraction

Let us take an elongated object moving along x-axis with velocity v in some inertial frame K. *In the frame* K' *where the object is at rest* (co-moving frame) its length is l_0. It is very important to specify the frame in which the length of the object was measured, because as we will see the results of the measurement depend on the frame. In particular, we ask the question what is the length l of the object in the frame K in which it moves with velocity v.

We need to reformulate the question in terms of some space-time events (Fig. 1). To measure length in the co-moving frame K' we can consider two events, $E_1 = (x'_1, t'_1)$ and $E_2 = (x'_2, t'_2)$ associated with the positions of the rear and the front of the object respectively, so that $x'_2 = x'_1 + l_0$. In principle, as the object is at rest in K', the values of t'_1 and t'_2 can be arbitrary, but it is convenient to use simultaneous (in K' !) events $t'_1 = t'_2$.

From the point of view of an observer in K in order to measure the length l of the moving object one has to find the space-time coordinates (x_1, t_1), (x_2, t_2) of the events E_1, E_2 and deduce

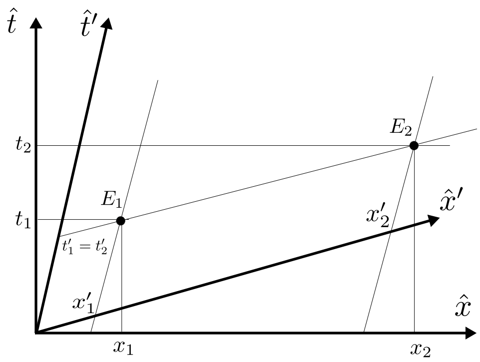

Figure 1: Measuring length of a moving object.

$$l = x_2 - x_1 - v(t_2 - t_1)\,.$$

The second term accounts for the distance the object travels during time interval $t_2 - t_1$ as the events are not simultaneous in K:

$$t_2 - t_1 = \gamma(v)\left(t'_2 - t'_1 + v\left(x'_2 - x'_1\right)/c^2\right) = \gamma(v)(0 + l_0 v/c^2) = \frac{v}{c^2}\frac{l_0}{\sqrt{1 - v^2/c^2}}\,.$$

Here we used the fact that Lorentz transformation is linear, so it can be applied to the differences $t_2 - t_1$, $t'_2 - t'_1$, $x_2 - x_1$ in one go. Using the transformation for coordinates we obtain

$$\begin{aligned} l = x_2 - x_1 - v(t_2 - t_1) &= \frac{x'_2 - x'_1 + v\left(t'_2 - t'_1\right)}{\sqrt{1 - v^2/c^2}} - \frac{v}{c^2}\frac{l_0}{\sqrt{1 - v^2/c^2}} = \frac{1 - v^2/c^2}{\sqrt{1 - v^2/c^2}} l_0 \\ &= l_0\sqrt{1 - v^2/c^2}\,. \end{aligned}$$

We see that for any velocity v the length l of a moving object is smaller than its *proper* length l_0, *i.e.* the length measured in its co-moving frame.

Problem: A low cost airline does allow on board any luggage longer than 45cm. A passenger has only a suitcase 50cm long. Find the velocity v with which the passenger must move through the gate at the airport in order to take the luggage with him on board and avoid paying extra fees.
Answer:

$$\sqrt{1 - v^2/c^2} = \frac{45}{50} = 0.9 \quad \Rightarrow \quad v/c = \sqrt{1 - 0.9^2} = \sqrt{0.19} \simeq 0.44\,.$$

The passenger must move at $v = 0.44c \simeq 132\,000$km/s.

Relativistic length contraction can be used to explain the outcome (or rather the absence whereof) of Michelson-Morley experiment. Take the naïve calculation of the time it takes a light signal to go along the interferometer arm directed parallel to the direction of motion

$$t_1 = \frac{2l_0}{c}\frac{1}{1 - v^2/c^2}\,.$$

Here l_0 is the length of the interferometer arm measured at rest, *i.e.* its proper length. By using the length contraction it must be replaced by $l = l_0\sqrt{1 - v^2/c^2}$ leading to the result

$$t_1 = \frac{2l_0}{c}\frac{1}{\sqrt{1 - v^2/c^2}} = t_2$$

identical to the time it takes the light to go along the perpendicular arm to the motion (of course, as lengths are not contracted in the perpendicular direction on can take $l = l_0$ in the expression for t_2).

Ladder Paradox:

A ladder of length ℓ_0 (measured in its own rest frame) moves with relativistic velocity v past a garage of length ℓ, Fig. 2. To fit inside the garage between its front F and rear R doors it is enough to built a garage of length

$$\ell = \sqrt{1 - v^2/c^2}\,\ell_0 < \ell_0 \tag{1}$$

as predicted by Special Relativity. However, from the point of view of the person moving together with the ladder it is the *garage that is moving* so it is too short for the ladder to fit in.

Resolution:

When saying "to fit" we mean that the front and the rear end of the ladder should coincide with the front and the rear doors of the garage *at the same moment of time.* While these two events, $E_f = (t, \ell)$

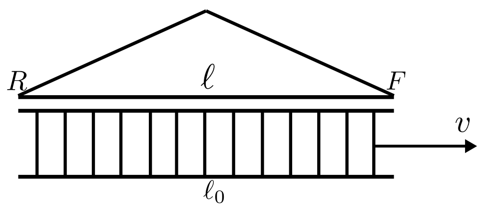

Figure 2: Ladder paradox.

and $E_r = (t, 0)$ are simultaneous in the garage's frame of reference, they are not simultaneous in the rest frame of the ladder! Indeed, the Lorentz transformation gives in the moving frame

$$E_f = (t'_f, x'_f) = (\gamma(t - v\ell/c^2), \gamma(\ell - vt)) \tag{2}$$

$$E_r = (t'_r, x'_r) = (\gamma t, -\gamma vt) \tag{3}$$

$$\tag{4}$$

So there is a time delay $\Delta t' = t'_r - t'_f = \gamma v\ell/c^2$ between these events. During this time the garage moves by $\Delta x' = v\Delta t' = \gamma\ell(v^2/c^2) = \ell_0(v^2/c^2)$. Adding this length to the length of the garage $\ell\sqrt{1 - v^2/c^2} = \ell_0(1 - v^2/c^2)$ (as observed from the ladder and therefore Lorentz contracted) results in the total length ℓ_0 and there is no paradox.