

University of Birmingham
School of Mathematics

1RA

Real Analysis

January 2023

Final Exam - Q1 and Q3
Generic Feedback

The performance of the students in the final exam (January exam 2023, Questions 1 and 3) was satisfactory overall. They demonstrated a good understanding of some fundamental concepts in calculus, such as function, convergence, and derivative, and were able to use them in simple calculations. However, they struggled with applying the concept of differentiation to more complex scenarios, such as optimization and L'Hospital's rule. Question 3 (c) was especially challenging, as it involved several techniques, such as limits at infinity, monotonicity and derivatives, continuity, and the intermediate value theorem. Many students failed to provide a complete and rigorous proof of this problem.

Q1.

- (a). This is bookwork, and was done well. However, some students lost points because they did not mention that “one element of Y ”.
- (b). This is also standard bookwork, and was done well. A good understanding of the definition of convergence is needed to apply it in the proof of (ii).
- (c). This is also standard bookwork, and was done well. Some students lost points because they did carry the limit sign $\lim_{x \rightarrow 5}$ all the way in their computation.
- (d). This question is in general done well. Please note that the increasing/decreasing intervals may include endpoints, i.e. they may be closed intervals.
- (d.i) To solve this problem, the first step is to calculate the derivative of f , which is denoted by f' . The sign of the derivative gives us the increasing/decreasing intervals. The next step is to use sign analysis to determine where f' is negative or positive. If f' is negative, then f is decreasing. If f' is positive, then f is increasing.
- (d.ii) This is just a consequence of (c.i). The points where f' changes sign are called critical points, and they are candidates for local minima or maxima of f . By comparing the values of f at the critical points and the the nearby domain, we can identify the local extrema of f .
- (d.iii) The second derivative of f tells us how the graph of the function curves. If the second derivative is positive, the function is convex. If the second derivative is negative, the function is concave. To find the convex and concave intervals of f , we need to compute the second derivative of f and find where it changes sign.

Q3.

- (a) This is a typical optimization question using calculus. To solve this problem, we need to identify the function that we want to optimize, which is the surface area of the can, and the constraint that limits the possible values of the variables, which comes from the fixed volume of the can. Then, we use techniques such as differentiation to find the optimal solution. Also note that the can has a bottom but no top, so the surface area of such a can is

$$A = 2\pi rh + \pi r^2$$

where r is the radius and h is the height of the cylinder. As the volume of the can is 27, which gives the constraint

$$r^2 h = 27 \text{ or } h = \frac{27}{r^2}.$$

The key step, that many got wrong, is plugging the constraint to the function of A to get

$$A(r) = \frac{54\pi}{r} + \pi r^2.$$

Thus the question reduces to minimize $A(r)$, which is a routine computation.

(b.1) To solve this problem, we need to apply the concept of continuity. Note that all the functions involved in the problem are continuous at the limit point, and that the denominator of the fraction does not equal zero at the point where we want to find the limit. Then we just need to plug $x = 0$ to the function and get its limit.

(b.2) To solve this question, we need to apply continuity several times. Recall that if f is continuous at the x_0 , then by definition we would have

$$\lim_{x \rightarrow x_0} f(x) = f(\lim_{x \rightarrow x_0} x) = f(x_0).$$

This rule allows us to simplify limits computations involving composition functions. The first step is to rewrite the given limit as a fraction, i.e. by continuity of e^x , we have

$$\lim_{x \rightarrow 0} \left(\frac{\pi}{2} - \arctan x \right)^{\frac{1}{\ln x}} = e^{\lim_{x \rightarrow 0} \frac{\ln \left(\frac{\pi}{2} - \arctan x \right)}{\ln x}}.$$

We note that

$$\frac{\ln \left(\frac{\pi}{2} - \arctan x \right)}{\ln x}$$

whose numerator has a finite limit $\ln(\pi/2)$ when $x \rightarrow 0$ (here we used continuity of $\ln x$ at $\pi/2$).

Please note if we change the limit $\lim_{x \rightarrow 0}$ to $\lim_{x \rightarrow \infty}$, then L'Hospital's rule would be needed.

(c) This is a challenging question, as it involved several techniques, such as limits at infinity, monotonicity and derivatives, continuity, and the intermediate value theorem. Many students failed to provide a complete and rigorous proof of this problem.

One first check the limit of $f(x)$ at ∞ and $-\infty$, which are ∞ and $-\infty$, respectively. Then one finds M and M' such that $f(M) > 0 > f(M')$. This, together with the continuity of f , enables us to apply the intermediate value theorem and find a point x_0 such that $f(x_0) = 0$.

To show that this is the only solution to $f(x) = 0$, we need to work further. Note the derivative of f is strictly positive on \mathbb{R} , which implies that f is strictly increasing on \mathbb{R} . Thus the solution must be unique.