Electromagnetism I – Problem sheet 8

Consider a square loop of side length a. A current I (ignore resistance) circulates anticlockwise in the loop. Point P is directly above the middle of side (4) at distance $r \gg a$ from the centre of the loop, as seen in the figure below. The loop and P are in the (x, y) plane, with the z-axis of the coordinate system pointing out of the page.

1. Explain why the magnitudes of the magnetic field, $\underline{\mathbf{B}}$, at P generated by the current flowing through sides (1) and (3) are negligible compared to those from sides (2) and (4).

   The Biot-Savart Law can be used to calculate the magnetic field:

   $$\delta\underline{\mathbf{B}} = \frac{\mu_0}{4\pi}\frac{I}{r^2}\delta\underline{\mathbf{l}} \wedge \hat{\underline{\mathbf{r}}}$$

   [1 Mark]

   where $\delta\underline{\mathbf{l}}$ is in the direction of the current and $\hat{\underline{\mathbf{r}}}$ is from the element $\delta\underline{\mathbf{l}}$ to point P.
   Note to markers: 1 mark for using Biot-Savart Law anywhere in this problem.
   As $r \gg a$, sides (1) and (3) are almost parallel to $\hat{\underline{\mathbf{r}}}$ hence

   $$\delta\underline{\mathbf{l}} \wedge \hat{\underline{\mathbf{r}}} \approx 0 \implies \underline{\mathbf{B}} \approx 0$$

   [1 Mark]

   Likewise, the sides (2) and (4) are almost perpendicular to $\hat{\underline{\mathbf{r}}}$ and as $a \ll r$ we can approximate $\underline{\mathbf{a}} \approx \delta\underline{\mathbf{l}}$ hence

   $$|\delta\underline{\mathbf{l}} \wedge \hat{\underline{\mathbf{r}}}| \approx |\underline{\mathbf{a}} \wedge \hat{\underline{\mathbf{r}}}| \approx a \implies \underline{\mathbf{B}} \approx \frac{\mu_0}{4\pi}\frac{Ia}{r^2}$$

   [1 Mark]

   therefore magnetic field from each of (1) and (3) individually is much less than from each of (2) and (4).

2. Determine the **total** contribution from the current flowing through sides (2) **and** (4) to $\underline{\mathbf{B}}(P)$ to leading order in (a/r) (magnitude and direction).

   The magnetic fields from sides (2) and (4) are in the opposite direction. B_2 is out of the page and B_4 is into the page. As $a \ll r$:

   $$B_2 \approx \frac{\mu_0 I}{4\pi}\frac{a}{\left(r + \frac{a}{2}\right)^2} \approx \frac{\mu_0 I}{4\pi}\frac{a}{r^2}\left(1 - \frac{a}{r}\right)$$

   Likewise, for side 4:

   $$B_4 \approx \frac{\mu_0 I}{4\pi}\frac{a}{\left(r - \frac{a}{2}\right)^2} \approx \frac{\mu_0 I}{4\pi}\frac{a}{r^2}\left(1 + \frac{a}{r}\right)$$

   to order $\frac{a}{r}$.

   $$B_2 + B_4 = \frac{\mu_0 I}{4\pi}\frac{a}{r^2}\left[\left(1 - \frac{a}{r}\right) - \left(1 + \frac{a}{r}\right)\right]\hat{\underline{\mathbf{z}}}.$$

$$= \frac{\mu_0 I}{4\pi} \frac{a}{r^2} \left(-\frac{2a}{r} \right) \hat{\underline{\mathbf{z}}} = -\frac{\mu_0 I}{2\pi} \frac{a^2}{r^3} \hat{\underline{\mathbf{z}}}$$

[3 Marks]

Note to markers: 1 mark for direction ($-\hat{\underline{\mathbf{z}}}$) or stating "field into page"). 1 mark for the $\frac{a^2}{r^3}$ term, and 1 mark for the correct expression with all numerical factors.

3. Determine the **total** contribution from the current flowing through sides (1) **and** (3) to $\underline{\mathbf{B}}(P)$ to leading order in (a/r) (magnitude and direction).

Sides (1) and (3) also make a small contribution to the magnetic field at P. Consider side (3):

$$|\delta \underline{\mathbf{l}} \wedge \hat{\underline{\mathbf{r}}}| = \delta l \sin\theta \approx a \sin\theta \approx a \tan\theta = a\frac{a/2}{r} = \frac{a^2}{2r}$$

to leading order $\frac{a}{r}$. Therefore

$$\underline{\mathbf{B_3}} \approx \frac{\mu_0 I}{4\pi} \frac{1}{r^2} \frac{a^2}{2r} \hat{\underline{\mathbf{z}}}$$

The contribution is out of the page (positive $\hat{\underline{\mathbf{z}}}$)

The contribution from side (1) is also out of the page and is the same as side (3). Therefore $\underline{\mathbf{B}}_1 + \underline{\mathbf{B}}_3 = 2\underline{\mathbf{B}}_3$

$$= \frac{\mu_0 I}{4\pi} \frac{a^2}{r^3} \hat{\underline{\mathbf{z}}}$$

[3 Marks]

Note to markers: 1 mark for direction ($+\hat{\underline{\mathbf{z}}}$) or stating "field out of page"). 1 mark for the $\frac{a^2}{r^3}$ term, and 1 mark for the correct expression with all numerical factors.

4. Hence, determine the total magnetic field $\underline{\mathbf{B}}(P)$ from the whole loop to leading order in (a/r).

Hence total B-field at P is:

$$\frac{\mu_0 I}{4\pi} \frac{a^2}{r^3} \hat{\underline{\mathbf{z}}} - \frac{2\mu_0 I}{4\pi} \frac{a^2}{r^3} \hat{\underline{\mathbf{z}}} = -\frac{\mu_0 I}{4\pi} I \frac{a^2}{r^3} \hat{\underline{\mathbf{z}}}$$

[1 Mark]