Electromagnetism 1 – Problem Sheet 3 – Solutions

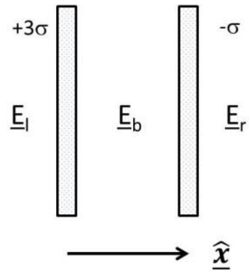
[Q1]

The electric field from a non-conducting sheet of surface charge density σ is

$\underline{E} = \frac{\sigma}{2\varepsilon_0}\underline{\hat{n}}$ where $\underline{\hat{n}}$ is the unit vector normal to the surface. **[1]**

(Students may quote this and are not required to derive it and don't need to include the unit vector. If they don't write this but still get (1) right they still get this mark)

Using the superposition principle: $\underline{E} \;= \underline{E}_{plate1} + \underline{E}_{plate2}$



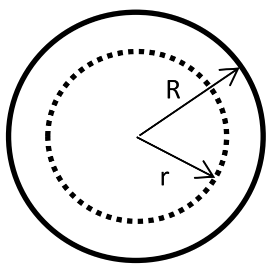
(1) to the left of the sheets: $\underline{E} = \left[-\frac{3\sigma}{2\varepsilon_0} + \frac{\sigma}{2\varepsilon_0}\right]\underline{\hat{x}} = \frac{-\sigma}{\varepsilon_0}\underline{\hat{x}}$ **[1 mark]**

(2) in between the sheets: $\underline{E} = \left[\frac{3\sigma}{2\varepsilon_0} + \frac{\sigma}{2\varepsilon_0}\right]\underline{\hat{x}} = \frac{2\sigma}{\varepsilon_0}\underline{\hat{x}}$ **[1 mark]**

(3) to the right of the sheets: $\underline{E} = \left[\frac{3\sigma}{2\varepsilon_0} - \frac{\sigma}{2\varepsilon_0}\right]\underline{\hat{x}} = \frac{\sigma}{\varepsilon_0}\underline{\hat{x}}$ **[1 mark]**

[Q2]

Using Gauss's Law: $\underline{E} = \int_S \underline{E}\cdot d\underline{S} = \frac{Q_{enc}}{\varepsilon_0}$, where Q_{enc} is the charged enclosed by the Gaussian surface S.



By symmetry $\int_S \underline{E}\cdot d\underline{S} = \int_S E\, dS = E\int_S dS = E\,4\pi r^2$

For r < R $\quad Q_{enc} = \int \rho(r)\, dV = \int \rho(r)\, 4\pi r^2 dr = \frac{4\pi\rho_0}{R^2}\int r^4 dr$

Hence $Q_{enc} = \frac{4\pi\rho_0}{R^2}\frac{r^5}{5}$

Putting the above into Gauss's Law: $E\,4\pi r^2 = \frac{4\pi\rho_0}{\varepsilon_0 R^2}\frac{r^5}{5} \rightarrow E = \frac{\rho_0 r^3}{5\varepsilon_0 R^2}$ (inside) **[2 marks]**

For r > R $\quad Q_{enc} = \frac{4\pi\rho_0}{R^2}\int_0^R r^4 dr = \frac{4\pi\rho_0}{R^2}\frac{R^5}{5} = \frac{4\pi\rho_0}{5}R^3$

Hence, putting the above into Gauss's Law: $E\,4\pi r^2 = \frac{4\pi\rho_0}{5\varepsilon_0} R^3 \;\rightarrow E = \frac{\rho_0 R^3}{5\varepsilon_0 r^2}$ (outside) **[1 mark]**

[Q3]

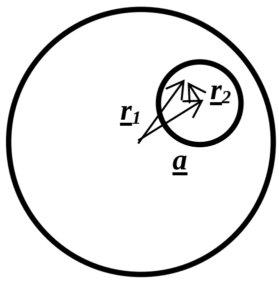
The simplest way to solve this problem is to see that the sphere with a cavity is equivalent to a solid sphere S_1 centred at the origin superimposed with a sphere S_2 with negative charge density -ρ, centred at $\underline{\boldsymbol{a}}$.

For sphere S_1 with no cavity, from Gauss's Law:

$$E\,4\pi r^2 = \frac{1}{\varepsilon_0}\rho\frac{4\pi}{3}r^3 \rightarrow \underline{E} = \frac{\rho}{3\varepsilon_0}\underline{r}$$ **[1 mark]**

The field is radial everywhere and outwards because the charge is positive.

Now, let's introduce a sphere S_2 with negative charge density.



The E-field due to the sphere S_1 with positive charge density is:

$\underline{E}_1 = \frac{\rho}{3\varepsilon_0}\underline{r}_1$ (from above) field due to S_1

The E-field due to the sphere S_2, with negative charge density, in terms of the vector $\underline{r}_2$ from the centre of S_2 is:

$\underline{E}_2 = \frac{-\rho}{3\varepsilon_0}\underline{r}_2$ field due to S_2

The total field inside the cavity is hence the vector sum of the $\underline{E}_1$ and $\underline{E}_2$ i.e.

$$\underline{E}_c = \frac{\rho}{3\varepsilon_0}(\underline{r}_1 - \underline{r}_2) = \frac{\rho}{3\varepsilon_0}\underline{a}$$ **[3 marks]**

[Note to markers: If the reasoning is correct, but there is an algebraic (not conceptual) error, in the calculation, subtract 1 mark.]