Mechanics week 9: Energy in more than one dimension part 1

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1 Introduction

We now consider a similar principle to last week, but in more than one dimension. This will work in a similar fashion to energy in one dimension. This week we will consider motion constrained to lie on a 1D (not necessarily straight) line or a flat plane. Next week we will look at motion on surfaces of revolution.

2 More than one dimension

Consider a particle of mass m moving in 3D under the action of a force $\mathbf{F}$. Then Newton's second law gives

$$\mathbf{F} = m\ddot{\mathbf{r}},$$

where $\mathbf{r}$ gives the position vector of the particle. If we take the dot product with $\dot{\mathbf{r}}$ we find

$$\begin{aligned}\mathbf{F}\cdot\dot{\mathbf{r}} &= m\ddot{\mathbf{r}}\cdot\dot{\mathbf{r}},\\ &= \frac{d}{dt}\left(\frac{1}{2}m\left|\dot{\mathbf{r}}\right|^2\right).\end{aligned}$$

The left hand side here represents the rate at which the force does work on the particle (recall that work = force x distance), whilst the right hand side is the time derivative of the familiar kinetic energy, this time defined in 3D: $\frac{1}{2}m\left|\dot{\mathbf{r}}\right|^2$- "half mass times velocity squared". Note that $\left|\dot{\mathbf{r}}\right|^2 = \dot{\mathbf{r}}\cdot\dot{\mathbf{r}}$ is a scalar and so we can safely neglect the modulus signs without confusion.

For a **conservative** force, this can be integrated in time to give energy conservation. This involves some subtle definitions which we aren't going to cover here, in particular involving line integrals (which you'll see next year). Essentially everything works the same as in 1D provided the force doesn't dissipate energy, so no friction or air resistance type forces. We will only consider examples where conservation of energy holds, but please be aware that it doesn't always!

3 Motion constrained to 1D

We first consider an example where the motion occurs in a plane, but the particle is constrained to move along a one dimensional path, for example if it is attached to a pendulum, or moving along a wire. This motion is one dimensional (i.e. so it depends only on one coordinate), but not necessarily in a straight line.

Example 1: Motion of a simple pendulum

Consider a particle of mass m attached to a light rod (a rigid line with no mass, which constrains the particle to move in a circle) of length l. The rod is fixed in space at the top. What initial velocity is required for the pendulum to overturn?

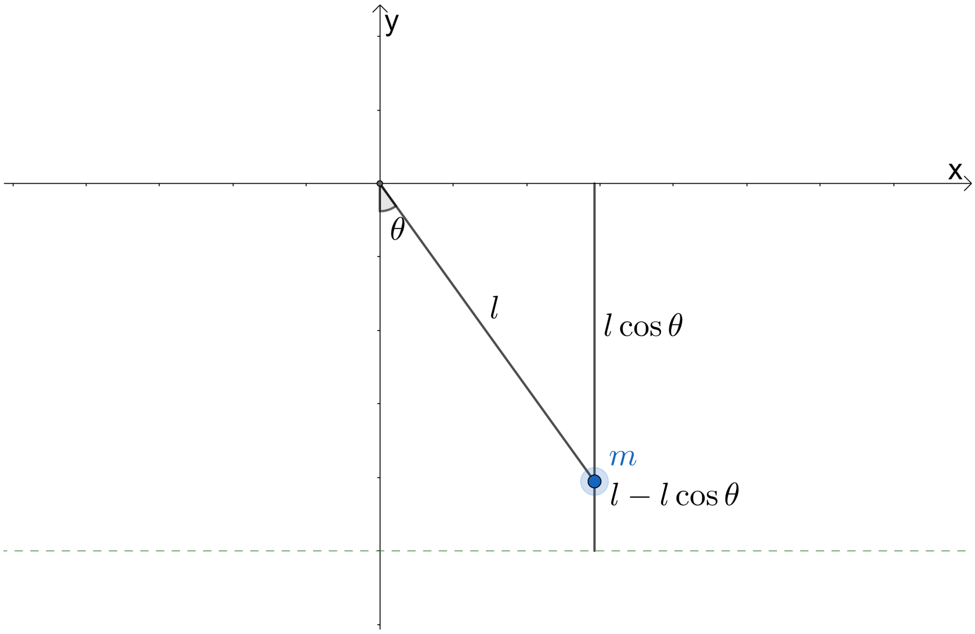

Figure 1: Mass moving on a pendulum on length l showing the relative height of the mass above the "bottom" of the pendulum where we set the GPE to be zero.

Solution. Let θ be the angle of deflection of the rod. (See Figure 1). Then the position vector of the particle, relative to an origin at the fixed top of the rod, is

$$\mathbf{r} = l\mathbf{e}_r,$$

and hence the velocity of the mass is

$$\dot{\mathbf{r}} = l\dot{\theta}\mathbf{e}_\theta,$$

(since $d\mathbf{e}_r/dt = \dot{\theta}\mathbf{e}_\theta$, similar to the Central Forces examples). Hence the kinetic energy is

$$\frac{1}{2}m\left|\dot{\mathbf{r}}\right|^2 = \frac{1}{2}m\left(l\dot{\theta}\right)^2.$$

We now need to find the gravitational potential energy. We will define the potential to be zero when $\theta = 0$. Then the height of the pendulum above this will be $l - l\cos\theta$ (see Figure 1). Hence the gravitational potential energy will be $mgl\left(1 - \cos\theta\right)$.

This gives the total energy as

$$\frac{1}{2}m\left(l\dot{\theta}\right)^2 + mgl\left(1 - \cos\theta\right) = E,$$

where E is the constant energy. We find E by using the initial conditions.

We set $\theta = 0$, $\dot{\theta} = \Omega$ at $t = 0$, so that the pendulum starts hanging straight down, with a nonzero initial velocity. Then

$$\begin{aligned} \frac{1}{2}m\left(l\dot{\theta}\right)^2 + mgl\left(1 - \cos\theta\right) &= \frac{1}{2}ml^2\Omega^2, \\ \implies \frac{1}{2}\dot{\theta}^2 + \frac{g}{l}\left(1 - \cos\theta\right) &= \frac{1}{2}\Omega^2, \\ \implies \dot{\theta}^2 &= \Omega^2 - \frac{2g}{l}\left(1 - \cos\theta\right). \end{aligned}$$

Now $\dot{\theta}^2$ is necessarily positive so this means that

$$\Omega^2 - \frac{2g}{l}\left(1 - \cos\theta\right) \geq 0,$$

throughout the motion. Hence we can calculate a maximum value of $|\theta|$ that the pendulum can achieve for a given initial motion. In particular, for the pendulum to overturn we need $\theta = \pi$ (i.e. it reaches the top of the circle), so that $\cos\theta = -1$ at some time t. For this to be possible the above equation must still hold, and so we require

$$\begin{aligned} \Omega^2 &> \frac{2g}{l}\left(1 + 1\right), \\ &= \frac{4g}{l}, \\ \text{i.e.} \qquad \Omega &> 2\sqrt{\frac{g}{l}}. \end{aligned}$$

This gives the minimum angular velocity needed initially for the pendulum to overturn.

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Activity: *You should now be able to tackle question 3 on this week's problem sheet.*

4 Motion constrained to a flat plane

4.1 Special case - central force problems

Consider the special case where the particle moves under the action of a central force, such that $\mathbf{F} = F(r)\mathbf{e}_r$ in polar coordinates. As before all the motion will take place in a flat plane, and we have $\dot{\mathbf{r}} = \dot{r}\mathbf{e}_r + r\dot{\theta}\mathbf{e}_\theta$ as in the previous chapter. Thus

$$\mathbf{F} \cdot \dot{\mathbf{r}} = F(r)\dot{r},$$

since $\mathbf{e}_r \cdot \mathbf{e}_\theta = 0$. Hence

$$\begin{aligned} \frac{d}{dt}\left(\frac{1}{2}m\,|\dot{\mathbf{r}}|^2\right) &= F(r)\frac{dr}{dt}, \\ \implies \frac{1}{2}m\,|\dot{\mathbf{r}}|^2 &= \int F(r)\frac{dr}{dt}dt, \\ &= \int F(r)dr + \text{const}, \end{aligned}$$

by integrating with respect to time. Hence

$$\underbrace{\frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right)}_{\text{K.E.}} \underbrace{- \int F(r)\;\mathrm{d}r}_{\text{P.E.}} = E,$$

where E is a constant, gives conservation of energy. In particular, the gravitational potential energy due to an inverse square law force of the form

$$F(r) = -\frac{GMm}{r^2}$$

where M, m are the masses of two bodies, G is the gravitational constant and r is the distance between them, is given by

$$\begin{aligned} -\int F(r)\;\mathrm{d}r &= \int \frac{GMm}{r^2}\;\mathrm{d}r, \\ &= -\frac{GMm}{r}. \end{aligned}$$

Hence conservation of energy gives

$$\frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - \frac{GMm}{r} = \text{const.}$$

NB: The argument for conservation of angular momentum still holds, so we will also have $h = r^2\dot{\theta}$ constant for a central forces problem as before.

Example 2: Rocket

Suppose that a rocket of mass m is launched from the surface of the Earth with velocity $\mathbf{V}$ relative to an inertial frame at the centre of the Earth. Under what condition will the rocket escape from the Earth (neglecting the effect of the atmosphere)?

Solution. Energy will be conserved, so the system will satisfy

$$\begin{aligned} \frac{1}{2}m\dot{\mathbf{r}}^2 - \frac{GMm}{r} &= \text{constant}, \\ &= \frac{1}{2}mV^2 - \frac{GMm}{R}, \end{aligned}$$

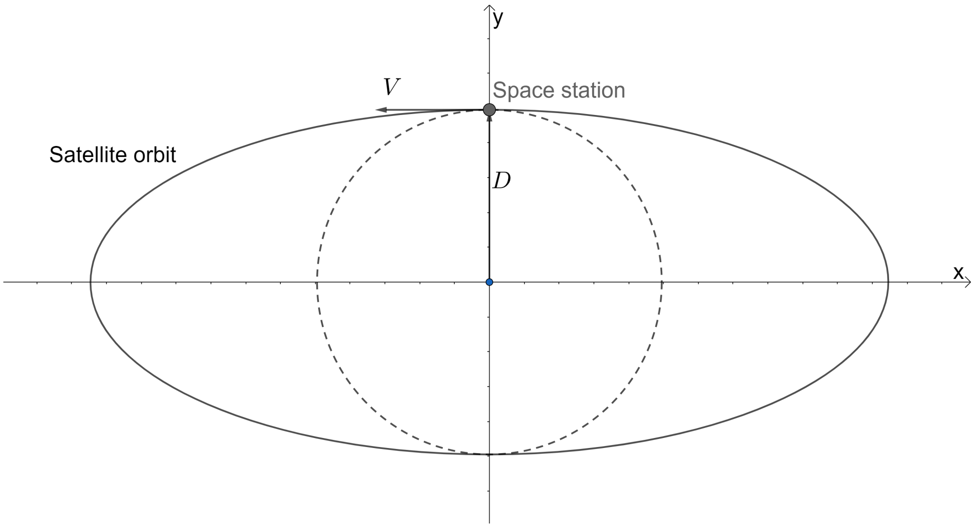

Figure 2: Space station orbiting the Earth with a circular orbit of radius D. Launching a satellite with tangential initial velocity V.

where $V = |\mathbf{V}|$ and R is the radius of the Earth, and we have used the initial conditions $\dot{\mathbf{r}} = \mathbf{V}$, $r = R$ at $t = 0$.

For the rocket to escape we need $r \to \infty$, so that the rocket gets infinitely far away from the Earth. We would then have

$$\frac{1}{2}m\dot{\mathbf{r}}^2 \quad = \quad \frac{1}{2}mV^2 - \frac{GMm}{R}.$$

Since $\dot{\mathbf{r}}^2$ is always positive, this means we require

$$\frac{1}{2}mV^2 - \frac{GMm}{R} \geq 0,$$

and so the minimum escape velocity is

$$V_e \quad = \quad \sqrt{\frac{2GM}{R}}.$$

Provided we launch with a velocity above this (in any direction which doesn't hit the ground!) the rocket should escape. ◀

Example 3: Satellite Suppose we launch a satellite with velocity V relative to the centre of the Earth along a circular orbit of a space ship orbiting at a distance D from the centre. If we know that the satellite does not escape the Earth's pull, what is the maximum distance of the satellite from the centre of the Earth? See Figure 2.

Solution. Initially, we have $r\dot{\theta} = V$, $\dot{r} = 0$, $r = D$ at $t = 0$. Hence conservation of energy gives

$$\begin{aligned}\frac{1}{2}m\dot{\mathbf{r}}^2-\frac{GMm}{r} &= E,\\ &= \frac{1}{2}mV^2-\frac{GMm}{D},\end{aligned}$$

so if the satellite doesn't escape we require $V^2 < 2GM/D$ as in the previous example. As this is a central force, we also have conservation of angular momentum, so

$$\begin{aligned}r^2\dot{\theta} &= \text{const},\\ &= r\cdot r\dot{\theta},\\ &= DV.\end{aligned}$$

Hence we have

$$\begin{aligned}r^2\dot{\theta}^2 &= \frac{\left(r^2\dot{\theta}\right)^2}{r^2},\\ &= \frac{h^2}{r^2},\\ &= \frac{D^2V^2}{r^2},\end{aligned}$$

and so

$$\begin{aligned}\frac{1}{2}m\dot{\mathbf{r}}^2-\frac{GMm}{r} &= \frac{1}{2}mV^2-\frac{GMm}{D},\\ \implies \dot{r}^2+\frac{D^2V^2}{r^2}-\frac{2GM}{r} &= V^2-\frac{2GM}{D}.\end{aligned}$$

The maximum value of $r = r_m$ occurs when $\dot{r} = 0$ as the particle has to change the direction of motion. Hence we have

$$\begin{aligned}\frac{D^2V^2}{r_m^2}-\frac{2GM}{r_m} &= V^2-\frac{2GM}{D},\\ \implies \left(V^2-\frac{2GM}{D}\right)r_m^2+2GMr_m-D^2V^2 &= 0.\end{aligned}$$

This may then be solved by noticing that since $r = D$ and $\dot{r} = 0$ initially, one root must be $r_m = D$ and hence

$$\begin{aligned}0 &= \left(V^2-\frac{2GM}{D}\right)r_m^2+2GMr_m-D^2V^2,\\ &= (r_m-D)\left(\left(V^2-\frac{2GM}{D}\right)r_m+DV^2\right),\end{aligned}$$

immediately, or by using the quadratic formula (where the algebra gets a little bit gnarly!):

$$\begin{aligned} r_m &= \frac{-2GM \pm \sqrt{4G^2M^2 - 4D^2V^2\left(V^2 - 2GM/D\right)}}{2\left(V^2 - 2GM/D\right)}, \\ &= \frac{-GM \pm \sqrt{G^2M^2 - D^2V^2\left(V^2 - 2GM/D\right)}}{\left(V^2 - 2GM/D\right)}, \\ &= \frac{-GM \pm \sqrt{\left(GM - DV^2\right)^2}}{\left(V^2 - 2GM/D\right)}, \\ &= \frac{-GM \pm \left(GM - DV^2\right)}{\left(V^2 - 2GM/D\right)}, \end{aligned}$$

noting that $V^2 < 2GM/D$ so we've taken the positive square root. This gives us two values for r_m (one with the plus sign and one with the minus sign); one will give the maximum distance and one the minimum. The positive sign gives

$$r_m = -\frac{DV^2}{V^2 - 2GM/D},$$

and the negative sign

$$\begin{aligned} r_m &= \frac{-2GM + DV^2}{V^2 - 2GM/D}, \\ &= \frac{D\left(V^2 - 2GM/D\right)}{V^2 - 2GM/D}, \\ &= D, \end{aligned}$$

i.e. where the satellite started. Therefore

$$r_m = \frac{D^2V^2}{2GM - DV^2},$$

gives the maximum distance of the satellite from the planet. ◀

Activity: *You should now be able to tackle question 4 on this week's problem sheet.*

Next week we will conclude this "chapter" by considering the motion of a particle on a surface of revolution.