Examination Feedback, June 2023

The following topics were examined.
A = "Most students did well"; B = "Some did well"; C = "Few did well".

- Newton's Second Law: solving an equation of motion without applied force (Q1(a);A).
- Newton's Second Law: verifying the solution of an equation with applied force (Q1(a);B).
- Newton's Second Law: explaining particle behaviour without exact solution (Q1(a);C).
- Conservation of energy: particle motion restricted to a frictionless curve in 3D (Q1(b);B).
- The principle of dimensional homogeneity (Q2; A).
- Motion under a central force: deriving the particle path equation (Q2; B).
- Motion under a central force: verifying the solution of the particle path equation (Q2; B).
- Motion under a central force: interpreting the particle path (Q2; C).

1. (a) For part i, most students were able to solve the second-order linear ODE with constant coefficients and correctly apply the initial conditions.

   For part ii, some students misinterpreted the question and attempted to solve the equation $\ddot{x}+\omega^2 x = F_0\cos(\omega t)/m$ from scratch, when all they needed to do was check that the given solution satisfies the ODE. A number of students forgot to check that the initial condition $v(0) = 0$ is satisfied regardless of the value of C. The value of C must then be determined from the other initial condition, $x(0) = x_0$. Full marks could be gained from solving the equation directly, but it requires an unnecessary amount of work and increases the risk of making algebra mistakes.

   Part iii was difficult and there were very few good solutions. The key is to recognise that if $\gamma > 0$, then the complementary function must contain a factor of $e^{-(\gamma/2)t}$, which comes from $\lambda^2 + \gamma\lambda + \omega^2 = 0$ and hence $\lambda = \frac{1}{2}(-\gamma \pm \sqrt{\gamma^2 - 4\omega^2})$. Note that the expression inside the square root is less than γ^2, so it is impossible for λ to be positive. This exponential decay causes the complementary function to converge to 0 as $t \to \infty$. But the particular integral will now be simply a combination of $\cos$ and $\sin$, since the forcing term is no longer a part of the complementary function. As $t \to \infty$, the particular integral will oscillate periodically, so the general solution will do the same. Physically, a positive γ represents internal friction, causing any internally sustained motion (complementary function) to decay over time and leaving the particle with only the externally driven motion (particular integral).

   (b) Most students were able to derive the energy conservation equation by first finding the velocity vector and hence the kinetic energy. In part ii, most students were able to compute E given the initial conditions and hence derive an expression for $\dot{\theta}^2$. After that, however, many students lost sight of what the question is asking. The key is that "the particle never stops" means $\dot{\theta}^2 > 0$ for all time. If the expression for $\dot{\theta}^2$ has been correct, it should be immediately clear that $\dot{\theta}^2 > 0$ is equivalent to $\theta < 0$.

   Part iii was challenging. The key is that $\dot{\theta}^2$ has a maximum or minimum whenever $\frac{\mathrm{d}\dot{\theta}^2}{\mathrm{d}t} = 0$. Differentiating the expression for $\dot{\theta}^2$ is algebraically quite involved and obtaining the final equation relies on some clever manipulation. Even if one has been unable to derive the equation, one should still be able to attempt the final part of showing that the equation has infinitely many solutions. The key here is that the left-hand side oscillates with an increasing (to infinity) amplitude as θ decays towards $-\infty$, so its graph intersects any horizontal line at infinitely many values of θ. There were very few good interpretations of this result. The most relevant interpretation is that the particle's speed is maximised or minimised infinitely many times as it falls down the wire. Note that if the wire had been a *circular* helix, with $a = b$, then $\dot{\theta}^2$ is maximised or minimised if and only if $b^2 + c^2 = 0$, which is never. This is because a particle falling down a circular helix will do so with forever-increasing speed.

2. (a) This is bookwork, using the principle of dimensional homogeneity to explain or derive dimensions of physical quantities, and most students did well. For the dimensions of a, all that was required was stating that the exponent of an exponential must be dimensionless, so $[a]/[L] = 1$. Many students suffered from faulty or circular logic, for example using $[a] = [L]$ to derive $[\omega]$ and then using $[\omega]$ to "prove" $[a] = [L]$. Some students confused a with acceleration. Many students dropped a mark for careless notation.

   (b) Most students did well on this bookwork part, deriving the particle path equation from a given central force. A common mistake was not starting the derivation from Newton's Second Law, despite being explicitly asked to do so. Another common source of dropped marks was forgetting that the force should be negative. Some students then attempted to fix this error by magically vanishing a sign later in their answer, which cost them further marks.

   (c) Most students did well on this standard application of central force theory, deriving the initial conditions for the particle path equation. A common mistake was misapplying the initial velocity: many erroneously said $\dot{r} = a\omega$ instead of the correct $r\dot{\theta} = a\omega$, or found $\dot{r} = 0$ by differentiating $r = a$. Some students did not make clear the equivalence between $t = 0$ and $\theta = 0$.

   (d) High marks for this part were rare. Instead of *verifying* the given solution, the vast majority of students jumped into attempting to solve the ODE given in part (b); these attempts were universally unsuccessful. The remaining students fared better but often not by much: most could generally correctly verify the initial conditions of u but few could correctly differentiate u; fewer still could correctly differentiate u twice. Simple algebraic errors were often made, most commonly forgetting to square the h in the denominator or writing $e^{a+b} = e^a + e^b$, resulting in more complicated expressions that made life harder.

   (e) This part was very challenging and many students did not attempt it. It is useful to note that even if one did not manage to complete the previous parts, this part could still be attempted. Quite a few students believed that $\dot{\theta} > 0$ implied $\dot{\theta}$ was strictly increasing. The correct argument relies on noticing that $\dot{\theta} = hu^2 = a^2\omega u^2$. The $u(\theta)$ we are given is a strictly increasing function of θ, therefore so is $\dot{\theta}$. This means that the more θ grows, the more quickly it grows; and since θ is initially growing, it must grow indefinitely for all t at increasing rates, tending to ∞ as $t \to \infty$.

Here is some general advice based on this exam. Pay attention to instructions and action words such as "verify". Time pressure is high, so use the mark scheme as a guide for how much you need to write: for 4 marks, you most likely do not need 4 pages of algebra. Mind your notation: make clear the difference between a quantity, dimension and unit, or between a scalar and a vector. Do *not* attempt to fix errors by introducing another error, for example to cancel a minus sign that shoud not have been there: two errors will be penalized more heavily than one.