Chaos

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Prior knowledge

You will need to know about, or be able to
(1) Differentiate
(2) Integrate
(3) Expand using low order Taylor's expansion
(4) Solve linear ordinary differential equations using
  (i) The exponential ansatz
  (ii) Particular integrals and complementary functions
  (iii) Integrating factors
(5) Employ the chain rule

$$\frac{d}{dx}f[g[x]] = \frac{df}{dg}[g[x]]\frac{dg}{dx}[x]$$

(6) The identities

$$e^{i\theta} = \cos\theta + i\sin\theta \qquad\qquad e^{-i\theta} = \cos\theta - i\sin\theta$$

$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} \qquad\qquad \sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$$

$$e^{i\theta}e^{-i\theta} = 1 = (\cos\theta)^2 + (\sin\theta)^2$$

These skills will be practiced in Example Sheet 0 and the first support class.

Chaos

By its very nature chaos is difficult to describe and even more difficult to control. It is important to realise, however, that chaos is not particular to a few systems but is believed to be generic behaviour for almost all time dependent systems of a fairly general class. We need to start out by looking at this general class and understand how to put systems into the general form that we want to study, the *fundamental equation*

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}[\mathbf{x}, t]$$

where $\mathbf{x}$ is a vector that lives in *phase space* and $\mathbf{f}[\mathbf{x}, t]$ are smooth functions of both the *dynamical variables*, $\mathbf{x}$, and of time. The vector $\mathbf{x}$ is not in general restricted to three dimensions but lives in a space of arbitrary dimensions, often a high dimensional space.

Higher-order differential equations

The main source of chaos has been from classical mechanics. One envisages a mechanical system, such as a pendulum, and then the classical equations of motion provide us with a dynamical system. The first problem that we encounter is that Newton's laws of motion relate forces to acceleration and acceleration is the second derivative of position with respect to time; providing a higher-order differential equation when we want a first order equation which relates the analogue of a velocity to an analogue of a force.

The first major challenge is to rewrite higher-order differential equations into our generic form, as a first order differential equation of an appropriate vector.

Let us examine one of the simplest systems; the one-dimensional harmonic oscillator

$$m\frac{d^2x}{dt^2} = -kx$$

where you should recognise Newton's law of mass times acceleration is equal to the force, which in this case is proportional to the position. To convert this equation into a linear equation we need to use velocity as a dynamical variable. One choice is

$$v = \frac{dx}{dt} \qquad \Rightarrow \qquad \frac{dv}{dt} = \frac{d^2x}{dt^2} = -\frac{k}{m}x$$

and we have an equivalent two-dimensional problem which involves only first derivatives

$$\frac{d}{dt}\begin{bmatrix} x \\ v \end{bmatrix} = \begin{bmatrix} v \\ -\frac{k}{m}x \end{bmatrix}$$

where $\mathbf{x} = \begin{bmatrix} x \\ v \end{bmatrix}$ and relabeling $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ we have $f_1 = x_2$ and $f_2 = -\frac{k}{m}x_1$. This choice is not unique. When you come to the second-year course on Lagrangian and Hamiltonian systems, it will be explained that the momentum, $p = m\frac{dx}{dt}$, is a more natural physical variable than the velocity. In terms of this choice we would find

$$\frac{d}{dt}\begin{bmatrix} x \\ p \end{bmatrix} = \begin{bmatrix} \frac{p}{m} \\ -kx \end{bmatrix}$$

an equivalent representation. In terms of natural scaling we might consider $y = \left[\frac{m}{k}\right]^{\frac{1}{2}}\frac{dx}{dt}$ and then

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \left[\frac{k}{m}\right]^{\frac{1}{2}}\begin{bmatrix} y \\ -x \end{bmatrix}$$

and the final rescaling $T = \left[\frac{k}{m}\right]^{\frac{1}{2}} t$ would eliminate all the parameters leaving a pure dimensionless problem. When we tackle problems numerically we almost always scale out such energy scales and work with a dimensionless problem.

A second problem is the pendulum, which involves a mass hanging on an inextensible string. Applying Newton's law in the direction of motion, to avoid the force along the string, we find

$$m\frac{d^2}{dt^2}(a\theta) = -mg\sin\theta \qquad \Rightarrow \qquad \frac{d^2\theta}{dt^2} = -\frac{g}{a}\sin\theta$$

which is again a second-order differential equation. We can turn this into a first-order differential equation using the angular velocity

$$\omega = \frac{d\theta}{dt} \qquad \Rightarrow \qquad \frac{d\omega}{dt} = -\frac{g}{a}\sin\theta$$

which provides

$$\frac{d}{dt}\begin{bmatrix}\theta\\ \omega\end{bmatrix} = \begin{bmatrix}\omega\\ -\frac{g}{a}\sin\theta\end{bmatrix}$$

In a parallel manner we could use the angular momentum

$$p_\theta = ma^2\frac{d\theta}{dt} \qquad \Rightarrow \qquad \frac{dp_\theta}{dt} = -mag\sin\theta$$

which provides

$$\frac{d}{dt}\begin{bmatrix}\theta\\ p_\theta\end{bmatrix} = \begin{bmatrix}\frac{p_\theta}{ma^2}\\ -mag\sin\theta\end{bmatrix}$$

The natural rescaling variable does not exist because $\sin\theta$ has a natural periodicity of 2π which already provides a natural angular scale. Rescaling time with $T{=}\left[\frac{g}{a}\right]^{\frac{1}{2}} t$ provides

$$\frac{d^2\theta}{dT^2} = -\sin\theta$$

and then using $\omega = \frac{d\theta}{dT}$ provides

$$\frac{d}{dT}\begin{bmatrix}\theta\\ \omega\end{bmatrix} = \begin{bmatrix}\omega\\ -\sin\theta\end{bmatrix}$$

which is natural for numerical calculations.

Another problem that one might tackle is celestial motion, the Earth moving around the sun for example. If we assume that the sun is immobile then from the inverse square law we would have that

$$m\frac{d^2x}{dt^2} = -GmM\frac{x}{[x^2+y^2+z^2]^{\frac{3}{2}}}$$

$$m\frac{d^2y}{dt^2} = -GmM\frac{y}{[x^2+y^2+z^2]^{\frac{3}{2}}}$$

$$m\frac{d^2z}{dt^2} = -GmM\frac{z}{[x^2+y^2+z^2]^{\frac{3}{2}}}$$

from Newton's laws. Now we would have a six dimensional problem if we use the velocities as natural parameters

$$v_x = \frac{dx}{dt} \qquad \Rightarrow \qquad \frac{dv_x}{dt} = -\frac{GMx}{r^3}$$

$$v_y = \frac{dy}{dt} \qquad \Rightarrow \qquad \frac{dv_y}{dt} = -\frac{GMy}{r^3}$$

$$v_z = \frac{dz}{dt} \qquad \Rightarrow \qquad \frac{dv_z}{dt} = -\frac{GMz}{r^3}$$

in terms of $r = \left[x^2 + y^2 + z^2\right]^{\frac{1}{2}}$, the separation between the two objects. This is

$$\frac{d}{dt}\begin{bmatrix} x \\ y \\ z \\ v_x \\ v_y \\ v_z \end{bmatrix} = \begin{bmatrix} v_x \\ v_y \\ v_z \\ -\frac{GMx}{r^3} \\ -\frac{GMy}{r^3} \\ -\frac{GMz}{r^3} \end{bmatrix}$$

in our generic form. If we allow the sun to move as well, then we will end up with a twelve dimensional problem, the derivation of which is left as a challenge.

Energy conservation

There are two types of chaos; Hamiltonian chaos and dissipative chaos. The key to Hamiltonian chaos is the energy, which is conserved. For the harmonic oscillator

$$E = \frac{p^2}{2m} + \frac{kx^2}{2} \qquad \Rightarrow \qquad \frac{dE}{dt} = \frac{p}{m}\frac{dp}{dt} + kx\frac{dx}{dt}$$

and then, using the equations of motion,

$$\frac{dx}{dt} = \frac{p}{m} \qquad \frac{dp}{dt} = -kx \qquad \Rightarrow \qquad \frac{dE}{dt} = 0$$

and the energy does not change as a function of time. For the pendulum

$$E = \frac{p_\theta^2}{2ma^2} + mag\left(1 - \cos\theta\right) \qquad \Rightarrow \qquad \frac{dE}{dt} = \frac{p_\theta}{ma^2}\frac{dp_\theta}{dt} + mag\sin\theta\frac{d\theta}{dt}$$

and, once again using the equations of motion,

$$\frac{d\theta}{dt} = \frac{p_\theta}{ma^2} \qquad \frac{dp_\theta}{dt} = -mag\sin\theta \qquad \Rightarrow \qquad \frac{dE}{dt} = 0$$

and the energy also does not change with time. For the star and planet motion

$$E = \frac{1}{2}m\left[v_x^2 + v_y^2 + v_z^2\right] - \frac{GmM}{\left[x^2 + y^2 + z^2\right]^{\frac{1}{2}}} \qquad \Rightarrow$$

$$\frac{dE}{dt} = m\left[v_x\frac{dv_x}{dt} + v_y\frac{dv_y}{dt} + v_z\frac{dv_z}{dt}\right] + \frac{GmM}{\left[x^2 + y^2 + z^2\right]^{\frac{3}{2}}}\left[x\frac{dx}{dt} + y\frac{dy}{dt} + z\frac{dz}{dt}\right]$$

and then using the equations of motion

$$v_x = \frac{dx}{dt} \quad v_y = \frac{dy}{dt} \quad v_z = \frac{dz}{dt} \quad \frac{dv_x}{dt} = -\frac{GM}{r^3}x \quad \frac{dv_y}{dt} = -\frac{GM}{r^3}y \quad \frac{dv_z}{dt} = -\frac{GM}{r^3}z$$

$$\Rightarrow \qquad \frac{dE}{dt} = 0$$

In all these cases this energy is conserved and for a general Hamiltonian system the energy is conserved.

Trajectories in phase space

The problem that we have elected to study

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}[\mathbf{x}, t]$$

involves the motion of a point in phase space. This fundamental equation provides the local velocity of this point at any point in phase space. One can imagine a fluid in phase space which is moving around under the action of the fundamental equation which gives the local direction of flow. The next concept is a *trajectory:* This is the path that a point will take as it moves around in phase space. The trajectory is locally parallel to $\mathbf{f}[\mathbf{x}, t]$ but the speed of progress is hidden in the trajectory. A picture of phase space, with the trajectories drawn on, is called a *phase space portrait.* One of the most important mathematical challenges is to find such a portrait and you will be tested....

There are two styles of systems that we investigate; firstly time independent systems

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}[\mathbf{x}]$$

where the local equation does not change as a function of time and secondly the initial problem when the local equation does depend on time. The first case is best described by a phase space portrait, includes Hamiltonian systems, and involves trajectories which do not cross. The second case can be described by a phase space portrait but because the problem changes as a function of time it is best to restrict to one trajectory for a phase space portrait. For the general time dependent problem one must include time as a separate dimension from phase space and the trajectories then wind around each other in phase space but always travel forwards in time.

Dissipation

Quantum mechanics conserves energy and is reversible, providing a Hamiltonian system. When we consider macroscopic systems we often greatly simplify the issues and replace complicated problems with much more elementary analogues. The previous case of a pendulum is a good example. We replace a complicated object on a complicated wire by an elementary point mass with no internal properties and an inextensible wire of fixed length. Obviously the wire is not inextensible and there will be complications caused by the stretching of the wire and the rippling along its length, we are simply ignoring these effects as too small to worry about. These approximations are not too disturbing but there is another approximation which is fundamentally more crucial; isolation and independence.

When a real pendulum moves it disturbs the surrounding air and this causes air resistance; more generally *dissipation.* Although energy is conserved, the energy of the motion in the pendulum gets transformed into the motion of the surrounding fluid, the air. Since we are not describing the air, for the subsystem that we are describing energy is not conserved and the pendulum slows down and stops. If we want to include the effects of air resistance we can modify the problem to mock up their effect.

We include a term in the equation of motion to include the dissipation

$$ma\frac{d^2\theta}{dt^2} + \gamma\frac{d\theta}{dt} + mg\sin\theta = 0$$

where the second term provides the dissipation. Note that using the previous energy

$$E = \frac{1}{2}ma^2\left(\frac{d\theta}{dt}\right)^2 + mag(1-\cos\theta) \quad \Rightarrow \quad \frac{dE}{dt} = ma^2\frac{d\theta}{dt}\frac{d^2\theta}{dt^2} + mag\sin\theta\frac{d\theta}{dt} = -a\gamma\left(\frac{d\theta}{dt}\right)^2$$

and the energy always decreases, as expected. The new modified fundamental equation is

$$\frac{d\theta}{dt} = \frac{p_\theta}{ma^2}$$

$$\frac{dp_\theta}{dt} = -mag\sin\theta - \frac{\gamma}{ma}p_\theta$$

Phase space portraits

The previous systems that we examined provide examples of phase space portraits that yield intuition. For the harmonic oscillator we have

$$E = \frac{p^2}{2m} + \frac{kx^2}{2} \quad \Rightarrow \quad \frac{x^2}{a^2} + \frac{p^2}{b^2} = 1$$

where $a^2 = \frac{2E}{k}$ and $b^2 = 2mE$ depend on the energy. For each energy the fact that this energy is conserved provides an ellipse. As the energy is varied so this ellipse is rescaled, leading to a series of concentric ellipses as the phase space portrait. It is important to understand how the underlying motion is described in this phase space portrait. The trajectory winds around the ellipse, with each complete oscillation corresponding to a single circuit of the ellipse. Obviously the x-axis provides the position of the oscillator but the p-axis provides the simultaneous momentum of the oscillator. If we start at $(x,p)=(a,0)$ then the oscillator is at maximum extension and is not moving. We follow the trajectory in a clockwise direction and so the momentum is negative and the position reduces towards the origin. The speed increases until it achieves maximum at $(0,-b)$ and the oscillator is at the origin. The speed then reduces again as the oscillator sweeps into the negative x region and it stops again at position $(-a,0)$. The upper half of the trajectory corresponds to to the forward motion of the oscillator with positive momentum until it reaches $(a,0)$ again and the motion repeats.

For this problem we can completely solve for the motion and we find

$$x = a\cos\left[\omega(t-t_0)\right] = \left(\frac{2E}{k}\right)^{\frac{1}{2}}\cos\left[\omega(t-t_0)\right]$$

$$p = -b\sin\left[\omega(t-t_0)\right] = -\left(2mE\right)^{\frac{1}{2}}\sin\left[\omega(t-t_0)\right]$$

where $\omega = \left(\frac{k}{m}\right)^{\frac{1}{2}}$ and t_0 is one of the times where the oscillator has maximum extension in the x-direction. The identity

$$(\sin\theta)^2 + (\cos\theta)^2 = 1$$

provides the ellipse and when considered as a mathematical problem we should think of t_0 and E as variables which must be chosen to fit the initial conditions.

We may add dissipation into the harmonic oscillator to provide the equations of motion

$$m\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + kx = 0$$

and then the fundamental description becomes

$$\frac{dx}{dt} = \frac{p}{m}$$

$$\frac{dp}{dt} = -kx - \frac{\gamma}{m}p$$

using the same definition of momentum. How is the phase space portrait affected? To investigate this we need to solve this problem mathematically.

The damped harmonic oscillator

The equation of motion is

$$m\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + kx = 0$$

and the best current method to mathematically solve this problem is the exponential ansatz; assume that $x{=}Ae^{Dt}$ and find the possible values of D. If we pursue this then

$$A\left[mD^2 + \gamma D + k\right]e^{Dt} = 0$$

and we have to solve a quadratic equation for D

$$\left(D + \frac{\gamma}{2m}\right)^2 = D^2 + \frac{\gamma}{m}D + \frac{\gamma^2}{4m^2} = \frac{\gamma^2}{4m^2} - \frac{k}{m} \equiv -\Omega^2$$

and we have a solution

$$x = Ae^{-\frac{\gamma}{2m}t}\cos\left[\Omega(t - t_0)\right]$$

where t_0 and A are arbitrary constants and we have chosen a representation to make the solution look similar to that of the undamped oscillator. Now

$$\frac{dx}{dt} = -Ae^{-\frac{\gamma}{2m}t}\left(\frac{\gamma}{2m}\cos\left[\Omega(t - t_0)\right] + \Omega\sin\left[\Omega(t - t_0)\right]\right)$$

and so the momentum is a bit of a mess. We can observe that

$$P = m\frac{dx}{dt} + \frac{\gamma}{2}x = -Am\Omega e^{-\frac{\gamma}{2m}t}\sin\left[\Omega(t - t_0)\right]$$

is very similar to the undamped oscillator, could we use this variable instead? If we define

$$\frac{dx}{dt} = \frac{P}{m} - \frac{\gamma}{2m}x$$

then

$$\frac{dP}{dt} = m\frac{d^2x}{dt^2} + \frac{\gamma}{2}\frac{dx}{dt} = -kx - \frac{\gamma}{2}\frac{dx}{dt} = -kx - \frac{\gamma}{2m}P + \frac{\gamma^2}{4m}x = -m\Omega^2 x - \frac{\gamma}{2m}P$$

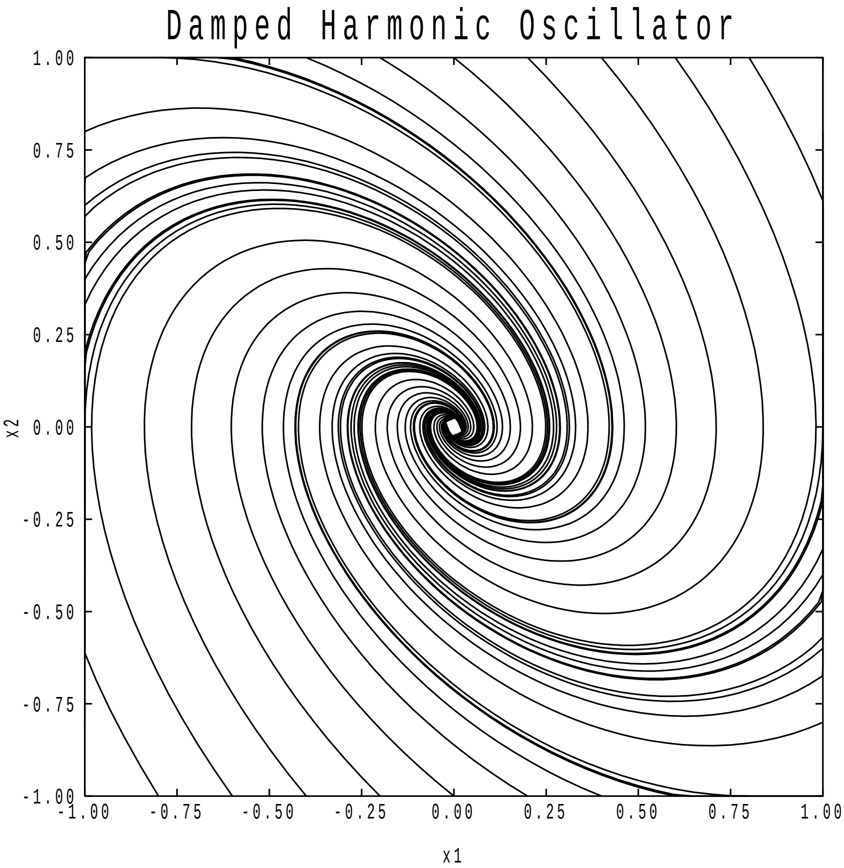
and we have another choice for our phase space variables, an equivalent but more balanced fundamental equation and a more natural solution. When we turn a second order differential equation into a first order vector equation in two phase space dimensions, there are many possible choices of variables and some are better than others. If we choose the new, more natural variables then we have that

$$x^2 + \left(\frac{P}{m\Omega}\right)^2 = A^2 e^{-\frac{\gamma}{m}(t-t_0)} \quad \Rightarrow \quad \left(\frac{x}{a(t)}\right)^2 + \left(\frac{P}{b(t)}\right)^2 = 1$$

with

$$a(t) = Ae^{-\frac{\gamma}{2m}(t-t_0)} \qquad b(t) = m\Omega A e^{-\frac{\gamma}{2m}(t-t_0)}$$

which is like a time dependent ellipse. We can imagine the trajectory circling around the ellipse, with $\omega \mapsto \Omega$, in comparison to the undamped case, and so the oscillations are occurring slower with a longer period, $\tau = \frac{2\pi}{\Omega}$. At the same time, on a separate time-scale, $T = \frac{2m}{\gamma}$, the ellipse is smoothly shrinking and so the trajectories are spiraling down towards the origin. In this new dissipative case the trajectories do not close up but go on for ever, spiraling down towards the origin



This behaviour leads to a new concept; the *attractor*. If we wait for a very long time what does the trajectory converge to? The attractor is the trajectory that we might converge to if we wait for ever, ie the limit for time tending to infinity. For the current problem the attractor is the single point at the origin, a *fixed point,* and wherever we start we always end up at the origin.

The current picture is for weak dissipation but we may also consider strong dissipation. There is a clear change of behaviour when $\gamma^2 = 4km$, at which point the oscillations stop and the system develops two rates of decay. We may define $\Lambda^2 = \frac{\gamma^2}{4m^2} - \frac{k}{m} = -\Omega^2$ and then

$$x(t) = e^{-\frac{\gamma}{2m}t}\left[Ae^{\Lambda t} + Be^{-\Lambda t}\right]$$

provides the solution, with constants A and B, which then yields

$$P(t) = m\frac{dx}{dt} + \frac{\gamma}{2}x = e^{-\frac{\gamma}{2m}t}\left[m\Lambda Ae^{\Lambda t} - m\Lambda Be^{-\Lambda t}\right]$$

and combining

$$x(t) + \frac{P(t)}{m\Lambda} = 2Ae^{-\frac{\gamma}{2m}t+\Lambda t}$$
$$x(t) - \frac{P(t)}{m\Lambda} = 2Be^{-\frac{\gamma}{2m}t-\Lambda t}$$

where the first equation has the longer time scale and corresponds to the slower behaviour while the second equation has the shorter time scale and corresponds to faster behaviour. There are two natural trajectories the first of which corresponds to A=0 and $B \neq$0 and forms the line $P = -m\Lambda x$ while the second of which corresponds to $A \neq$0 and B=0 and forms the line P=$m\Lambda x$. The first trajectory involves the faster behaviour and the second trajectory involves the slower behaviour. A general trajectory moves faster towards the line P=$m\Lambda x$ and drifts more slowly parallel to this line towards the origin. This provides a second style of fixed point. The first involves trajectories spiraling down towards the fixed point and now we find two lines with the trajectories collapsing faster towards one of the lines, parallel to the other, and slower along the line which is collapsed to. This type of behaviour is generic to many systems.

The pendulum

We now consider the phase space portrait of the pendulum, which shows yet further fundamental behaviour. This is a Hamiltonian system with an energy

$$E = \frac{p_\theta^2}{2ma^2} + mag\left[1 - \cos\theta\right]$$

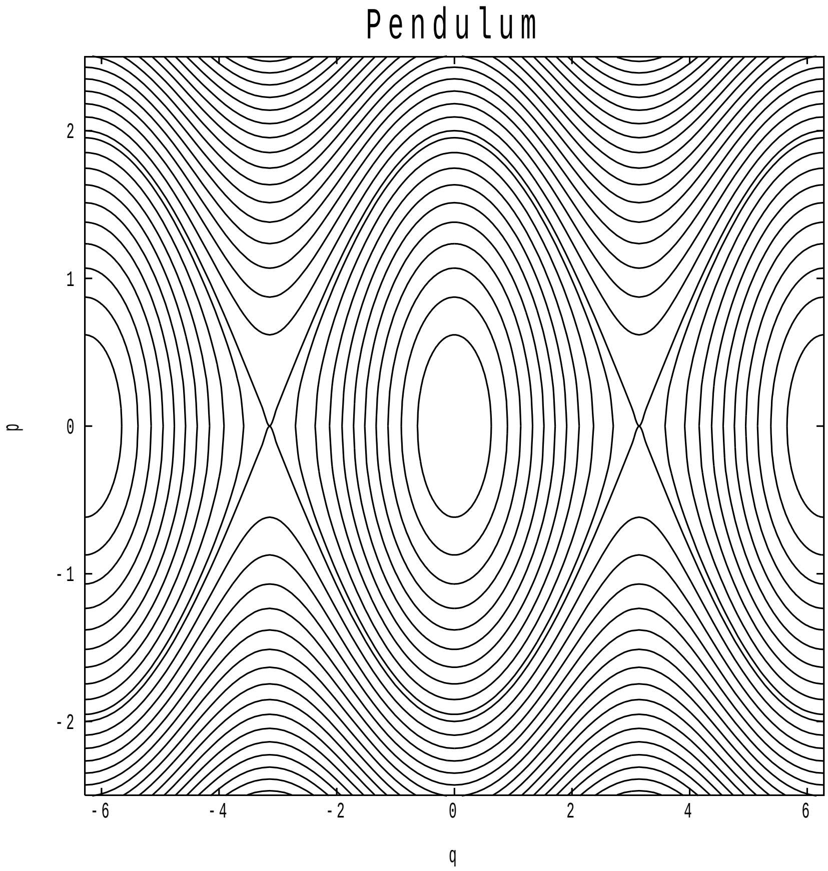
and then in phase space, (θ, p_θ), the different energies provide the trajectories which correspond to the motion of the pendulum. When the energy is small the pendulum only swings through a small angle and we may approximate $\cos\theta \mapsto 1 - \frac{\theta^2}{2} + ...$, where we ignore the quartic terms and beyond. This approximation leads to

$$E \mapsto \frac{p_\theta^2}{2ma^2} + \frac{mag\theta^2}{2} + ...$$

which is equivalent to the harmonic oscillator leading to a sequence of concentric ellipses as before. The larger the energy the worse the approximation and the more distorted the ellipses become. There is a major change of behaviour that occurs when $E{=}2mag$, at which point

$$\frac{p_\theta^2}{2ma^2} = mag\,[1+\cos\theta] = 2mag\left[\cos\frac{\theta}{2}\right]^2 \quad \Rightarrow \quad p_\theta = \pm 2ma\,(ag)^{\frac{1}{2}}\cos\frac{\theta}{2}$$

and this is a very special trajectory called the *separatrix.* The cosine curve in the full phase space portrait



This behaviour looks as if we still have an oscillating solution, but this is not true. The solution corresponds to

$$p_\theta = ma^2\frac{d\theta}{dt} = \pm 2ma\,(ag)^{\frac{1}{2}}\cos\frac{\theta}{2} \quad \Rightarrow \quad \frac{d\theta}{2\cos\frac{\theta}{2}} = \pm\left(\frac{g}{a}\right)^{\frac{1}{2}} dt$$

This equation can be solved using the observation that

$$\frac{d}{d\theta}\left[\sec\frac{\theta}{2}+\tan\frac{\theta}{2}\right] = \frac{1}{2}\sec\frac{\theta}{2}\left[\tan\frac{\theta}{2}+\sec\frac{\theta}{2}\right]$$

which leads to

$$\ln\left[\tan\frac{\theta}{2}+\sec\frac{\theta}{2}\right] = \pm\left(\frac{g}{a}\right)^{\frac{1}{2}}(t-t_0)$$

A sequence of mathematical identities

$$\sin\left(\frac{\pi}{2}-\theta\right)=\cos\theta$$

$$\cos\left(\frac{\pi}{2}-\theta\right)=\sin\theta$$

$$1+\cos\theta=2\left(\cos\frac{\theta}{2}\right)^2$$

$$\sin\theta=2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$$

allows us to rewrite

$$\sec\frac{\theta}{2}+\tan\frac{\theta}{2}=\frac{1+\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}=\frac{1+\cos\frac{\pi-\theta}{2}}{\sin\frac{\pi-\theta}{2}}=\frac{2\left[\cos\frac{\pi-\theta}{4}\right]^2}{2\sin\frac{\pi-\theta}{4}\cos\frac{\pi-\theta}{4}}=\frac{1}{\tan\frac{\pi-\theta}{4}}$$

and finally

$$\tan\frac{\pi-\theta}{4}=e^{\mp\left(\frac{g}{a}\right)^{\frac{1}{2}}(t-t_0)}$$

The top phase corresponds to increasing θ and has $p_\theta\geq 0$ whereas the bottom phase corresponds to decreasing θ and has $p_\theta\leq 0$. As time ranges from minus infinity to plus infinity, for the first case θ ranges from -π to π and for the second case θ ranges from π to -π. The motion is still clockwise on the phase space portrait but as time tends to infinity these trajectories stop at $(\pi,0)$ and $(-\pi,0)$ respectively, which gives us new types of fixed points for this problem.

The pendulum is a periodic system and if the angle is increased through 2π the pendulum is in the same position as it started out and is physically indistinguishable from its original configuration. One might then want to conclude that the only important variable is θ (mod 2π), but this is not altogether true. The actual angle also has the information of how many times the pendulum has turned over in the motion. Note that the new type of fixed point involves the pendulum being vertical where it is clearly unstable. In practice we tend to use periodic boundary conditions for phase space, plot only for $\theta\in(-\pi,\pi]$ and when θ increases past π we allow it to reappear at -π and continue on. This leads to a description of only the physical configurations but hides the knowledge of how many times the pendulum turns over.

The final class of trajectories occurs when $E>2mag$ and for this case there are no solutions to p_θ=0 and the angle endlessly either increases or decreases, for positive and negative values of p_θ respectively. Physically they correspond to the pendulum turning over and over repeatedly.

The damped pendulum

The equation of motion for the damped pendulum is

$$ma\frac{d^2\theta}{dt^2}+\gamma\frac{d\theta}{dt}+mg\sin\theta=0$$

and we are no longer able to mathematically control the solutions nor are we able to employ a conserved energy to help us. We are left with approximations and numerical analysis to resolve any issues. The fixed points at the origin and its periodic analogues remain and so do the unstable fixed points where the pendulum is vertical. It is natural to use

$$P_\theta = ma^2\frac{d\theta}{dt} + \frac{\gamma a}{2}\theta$$

and then

$$\frac{d\theta}{dt} = \frac{P_\theta}{ma^2} - \frac{\gamma}{2ma}\theta$$

$$\frac{dP_\theta}{dt} = -mag\sin\theta - \frac{\gamma a}{2}\frac{d\theta}{dt} = \frac{\gamma^2}{4m}\theta - mag\sin\theta - \frac{\gamma}{2ma}P_\theta$$

which makes a connection to the damped harmonic oscillator at small angles. The analogue of the separatrix still exists and occurs when θ=π and P_θ=$a\gamma\frac{\pi}{2}$ or when θ=-π and P_θ=-$a\gamma\frac{\pi}{2}$. We have disrupted the periodicity with this choice and now $\theta \mapsto \theta + 2\pi$ and $P_\theta \mapsto P_\theta + \gamma a\pi$ provides the analogue. The motion is now more subtle with a weird trajectory where the pendulum ends up vertical but almost all trajectories end up with the pendulum at rest pointing down. The generic trajectory involves the pendulum turning over and over but slowing down until it fails to reach the top and then it oscillates backwards and forwards, still slowing down, until it finally stops.

Forcing the system

The main problem with finding chaos in dissipative systems is that the energy leaks out of the system and the motion slows down and stops. The lowest energy solution is usually dull and so the eventual behaviour is usually trivial. To compensate for this problem we have to feed energy into the system continuously in order to achieve an interesting steady state. This occurs through forcing terms and provides the explicit time dependence in the fundamental equation.

The simplest example is the forced damped harmonic oscillator

$$m\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + kx = R\cos(2\pi ft)$$

where f, the frequency of the forcing term, is usually held constant and R, the strength of the forcing term, is used as a control parameter.

This problem is linear and hence solvable and we find an additional solution, the particular integral,

$$x = A\cos(2\pi ft) + B\sin(2\pi ft)$$

which yields

$$-m(2\pi f)^2\left[A\cos(2\pi ft) + B\sin(2\pi ft)\right] + \gamma(2\pi f)\left[-A\sin(2\pi ft) + B\cos(2\pi ft)\right]$$

$$+k\left[A\cos(2\pi ft) + B\sin(2\pi ft)\right] = R\cos(2\pi ft)$$

Equating the coefficients of the sin and cosine terms gives us two equations

$$B\left[k - m(2\pi f)^2\right] = A\gamma(2\pi f) \qquad A\left[k - m(2\pi f)^2\right] + B\gamma(2\pi f) = R$$

which solve to provide

$$B = \frac{\gamma(2\pi f)R}{\left[k - m(2\pi f)^2\right]^2 + \left[\gamma(2\pi f)\right]^2} \qquad A = \frac{\left[k - m(2\pi f)^2\right] R}{\left[k - m(2\pi f)^2\right]^2 + \left[\gamma(2\pi f)\right]^2}$$

which always provides a non-trivial solution. Note that

$$\frac{dx}{dt} = (2\pi f)\left[-A\sin(2\pi f t) + B\cos(2\pi f t)\right]$$

and so

$$x^2 + \left[\frac{1}{2\pi f}\frac{dx}{dt}\right]^2 = A^2 + B^2 = \frac{R^2}{\left[k - m(2\pi f)^2\right]^2 + \left[\gamma(2\pi f)\right]^2}$$

and we have a relationship in phase space for this particular integral. Using the velocity as the other variable provides an ellipse and this solution corresponds to a residual forced oscillation for the system; the oscillator is still oscillating which gives scope for more interesting behaviour. The complementary function solution is analogous to the previous unforced problem and decays down to zero if we wait long enough. This particular integral is consequently the attractor for this system and involves a residual oscillation for the oscillator at the forcing frequency. If you start the oscillator off inside this special ellipse then it tends to spiral outwards towards the attractor and if you start the oscillator off outside this special ellipse then it tends to spiral inwards towards the attractor.

An interesting aside is a pathology in this analysis. As $\gamma \mapsto 0$ we find a divergence in A when $k{=}m(2\pi f)^2$. This is known as *resonance* and if you force the oscillator at its natural frequency then it goes wild and gives a divergent response which is only cut off by the dissipation. This is another ingredient in the physical causes of chaos.

Chaos

We have developed the concepts and ideas so far but we have found no chaos. Systems have to be above a certain level of complexity before they can exhibit chaos but that level is not very high. Indeed, Hamiltonian systems with four degrees of freedom are usually chaotic and forced damped systems with two degrees of freedom are usually chaotic. We will take two relatively simple systems and examine them for chaos.

Our Hamiltonian system is the sprung pendulum, with natural length l,

$$E = \frac{m}{2}\left[\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2\right] - mgr\cos\theta + \frac{1}{2}k(r-l)^2$$

and if the radius, r, is held constant we can recognise a pendulum, whereas if θ is held constant we can recognise a harmonic oscillator, centred at $r_0 = l + \frac{mg}{k}\cos\theta$. We now have two equations of motion

$$m\frac{d^2r}{dt^2} = mr\left(\frac{d\theta}{dt}\right)^2 - k(r-l) + mg\cos\theta$$

$$m\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) = -mgr\sin\theta$$

which are much more difficult to derive from Newton's laws; the details are in an appendix. If we use

$$p_r = m\frac{dr}{dt} \qquad p_\theta = mr^2\frac{d\theta}{dt}$$

then we have

$$\frac{dr}{dt} = \frac{p_r}{m}$$

$$\frac{d\theta}{dt} = \frac{p_\theta}{mr^2}$$

$$\frac{dp_r}{dt} = \frac{p_\theta^2}{mr^3} - k(r-l) + mg\cos\theta$$

$$\frac{dp_\theta}{dt} = -mgr\sin\theta$$

as our fundamental equation.

Our dissipative system is the forced damped pendulum with

$$ma\frac{d^2\theta}{dt^2} + \gamma a\frac{d\theta}{dt} + mg\sin\theta = R\cos(2\pi ft)$$

and then

$$\frac{d\theta}{dt} = \omega$$

$$\frac{d\omega}{dt} = -\frac{\gamma}{m}\omega - \frac{g}{a}\sin\theta + \frac{R}{ma}\cos(2\pi ft)$$

provides our fundamental equation.

For our Hamiltonian system the energy is

$$E = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} - mgr\cos\theta + \frac{k}{2}(r-l)^2$$

and for a fixed energy this provides a three dimensional surface in four dimensional phase space that the trajectories are restricted to. How do we investigate the behaviour of these trajectories and decide whether or not they are chaotic?

For our dissipative system we can create a trajectory in two-dimensional phase space and then in principle follow the trajectory for a long time until it converges to the attractor and then we can analyse the attractor. How do we investigate the behaviour of the attractor and whether or not it is chaotic?

Poincare sections

So far we have dealt with some essentially trivial systems, for which we have either solved them completely or determined the form of the phase space portrait or found the attractor. Next we have introduced some much more complicated systems that we are going to claim have chaotic behaviour. We now run into a critical problem; chaotic systems are much too sophisticated to exactly solve and chaotic systems have unimaginably complicated phase space portraits and chaotic systems have such complicated attractors that the term 'strange attractor' is used to describe them. How are we supposed to uncover their behaviour?

There are two basic approaches that we will follow; firstly, one can employ the computer and numerically solve the system to machine accuracy and secondly, one can simplify the system until it is possible to solve it. The numerical approach is very instructive but suffers from the 'butterfly effect'; if an infinitesimal change is made to the initial conditions of the system, it can grow exponentially in time until it dominates and the behaviour is fundamentally different to the undistorted system. The intrinsic accuracy of the computer provides such small changes in boundary conditions which means that one cannot trust the long time veracity of the results. The simplification route suffers the problem that the original system may exhibit quite different behaviour because the approximations have wiped out the expected behaviour. Neither issue is thought to be critical, although the arguments for this will appear at the end of the course and they are quite sophisticated. We will develop the approximations and we will employ the numerical calculations without attempting to control the intrinsic errors.

The first major difficulty that we encounter is the number of dimensions involved in the description; for the sprung pendulum we have a four dimensional phase space and picturing a trajectory in phase space is consequently challenging. For this system we also have a natural constraint; the energy is conserved, leading to a three-dimensional subspace of phase space that the trajectory is restricted to and winds around in. Even picturing this subspace is very challenging. The way that we deal with these conceptual difficulties is to change the focus from the whole trajectory to some subset of the trajectory that is easier to picture; a *Poincare section.* For the current problem we would naturally choose a three dimensional subspace of phase space that the trajectory cannot avoid and then focus on when the trajectory passes through this subspace, ignoring the trajectory when the trajectory wanders around in the rest of phase space. Clearly we are throwing away almost all of the behaviour in order to focus on a tiny part, but if we can still recognise chaos in this tiny part then we can still answer the main question: Is this system chaotic?

Naively one would expect that by fixing one variable and looking at a subspace that is one dimension less than phase space we would require a (d-1)-dimensional picture, but this is not true. For a Hamiltonian system the energy is conserved and we can use this to specify another of the dynamical variables leaving behind a (d-2)-dimensional picture. It is best to appreciate these ideas through an example; the sprung pendulum. The pendulum swings backwards and forwards and so it is very difficult for the system to avoid θ=0 and so this is the natural choice for the Poincare section. For this particular choice we have

$$E = \frac{m}{2}\left[\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2\right] + \frac{k}{2}(r-l)^2 - mgr$$

which relates the other three dynamical variables. We can imagine solving this equation for $\frac{d\theta}{dt}$ and then, up to a sign, if we know r, $\frac{dr}{dt}$ and E then we can determine $\frac{d\theta}{dt}$ which may be presumed known. This then leaves two dynamical variables, r and $\frac{dr}{dt}$, which specify the trajectory as it passes across the surface characterised by θ=0. We depict the Poincare section using a two dimensional plot which involves r and $\frac{dr}{dt}$ as the x and y axes respectively. The fact that $\frac{dr}{dt}$ only appears as a square suggests that one might expect some symmetry under $\frac{dr}{dt} \mapsto -\frac{dr}{dt}$ so this dynamical variable makes a natural y-axis with the r behaviour providing the active x-axis. The energy being conserved still has an effect on this Poincare section since

$$\frac{1}{2}m\left(\frac{dr}{dt}\right)^2 - mgr + \frac{k}{2}(r-l)^2 = E - \frac{1}{2}mr^2\left(\frac{d\theta}{dt}\right)^2$$

and since the right-hand side is bounded by E, at fixed E the points that can occur in Poincare space are bounded by an ellipse constructed by setting $\frac{d\theta}{dt}$=0. The energy restriction means that the trajectory may only collide with the Poincare section within this ellipse, which is consequently the intersection of the energy surface with the Poincare surface. When the trajectory strikes the Poincare section we have a single point in this projected space

$$\left(r, \frac{dr}{dt}\right)_n \mapsto (x_n, y_n)$$

where we are using an integer variable, n, to denote the sequence, in time, for each collision. This provides us with a new type of fundamental problem; a *map* with

$$x_{n+1} = M_x[x_n, y_n]$$

$$y_{n+1} = M_y[x_n, y_n]$$

or more generally, in terms of vectors in Poincare space

$$\mathbf{x}_{n+1} = \mathbf{M}[\mathbf{x}_n]$$

where $\mathbf{M}[\mathbf{x}]$, analogous to $\mathbf{f}[\mathbf{x}]$, is a smooth function of the variables, the components of $\mathbf{x}$. To find this map we need to integrate the equations of motion between the collisions with the Poincare section and this is analytically intractable in general but numerically accessible. We will provide some examples of Poincare sections obtained with the computer.

To simplify these extremely complicated systems we will approximate the map, $\mathbf{M}[\mathbf{x}]$, by a very elementary analogue; the logistic map.

When we deal with dissipative systems there are similarities and differences. If we have damping only, then the system slows down and stops so we need forcing. Forcing leads to a smoothly changing system of equations and a lack of reproducibity from time to time. We tend to focus on the attractor, which does not depend on the time dependent changes in the equations, but we also study the irrelevant passage to the attractor in our modeling. For the forced damped pendulum we can depict the full trajectory in two dimensions but we can still create a Poincare section. For forced problems we tend to use time and the forcing term to set up the analogue of a Poincare section. We employ the whole of phase space but take a snapshot of the trajectory at a sequence of times which are separated by the period of the forcing term. This produces a map, as in the Hamiltonian case, and the map converges to a fixed point is the natural behaviour; as the trajectory converges to a single loop as happened in our exactly solvable forced damped harmonic oscillator example.

The forced damped pendulum: Part one

We need to develop intuition by studying an example and we choose to use the forced damped pendulum to provide this intuition. We rescale time to eliminate any unnecessary parameters and study

$$\frac{d^2\theta}{dt^2} + \nu\frac{d\theta}{dt} + \sin\theta = T\sin(2\pi F t)$$

and there are three essential parameters remaining; the dissipation, ν, the strength of the forcing, T, and the frequency of the forcing, F. One must fully understand the physical issues before investigating the system. If we set T=0 then the damped pendulum remains and this system is itself quite complicated, having multiple time-scales as was previously found. Close to complete decay we have the harmonic oscillator

$$\frac{d^2\theta}{dt^2} + \nu\frac{d\theta}{dt} + \theta = 0 \quad \Rightarrow \quad \left(D + \frac{\nu}{2}\right)^2 = \frac{\nu^2}{4} - 1 \equiv -\Omega^2$$

with solution

$$\theta(t) = Ae^{-\frac{\nu}{2}t}\cos\left[\Omega(t - t_0)\right]$$

There is a decay time, $t_d = \frac{2}{\nu}$, which controls the rate of approach to the origin. We enter our overdamped regime when $\nu > 2$ and we will always keep $\nu \ll 2$ in order to maintain the oscillations. There is an oscillation time, $t_o = \frac{2\pi}{\Omega}$, which is the natural time to complete one full oscillation. These considerations are valid when we have small oscillations but we also studied the case ν=0 and we know that there is another change of behaviour when the pendulum starts to turn over. The separatrix marks a trajectory that lasts forever, limiting towards the unstable fixed point with θ=π. For larger oscillations the oscillations take longer to complete, culminating in this separatrix where the time scale is infinite. Once the pendulum is turning over there is a second type of oscillation which is the time for a single turn and if we increase the energy this too gets shorter and shorter as we go further from the separatrix. The forcing term also has a time scale, $t_f = \frac{1}{F}$, which is the repeat time for the forcing. This forcing time sets the scale for the Poincare section. The final parameter, T, is crucial and controls the competition between all the previous effects; it is known as the *control parameter.*

When the control parameter is small the induced oscillations are small. This statement must be tempered by the possibility of *resonance;* if we force the system at its natural frequency we generate huge oscillations. In the absence of the dissipation the response can be divergent but with dissipation the response is always finite. We will use weak dissipation so resonance could be relevant. We will start out with the oscillation time being shorter than the forcing time. The forcing time is fixed but as the control parameter is increased the oscillation time is also increased and so we are expecting to pass through resonance at some level of forcing. At very high levels of forcing the pendulum does turn over and over very quickly, but on the forcing time scale the direction of rotation reverses. Solving the problem

$$\frac{d^2\theta}{dt^2} + \nu\frac{d\theta}{dt} = T\sin[2\pi F t]$$

verifies this, since $|\sin\theta| \leq 1$ provides a tiny perturbation when T is very large. We may integrate

$$\frac{d\theta}{dt} + \nu\theta = A - \frac{T}{2\pi F}\cos[2\pi F t]$$

employ an integrating factor

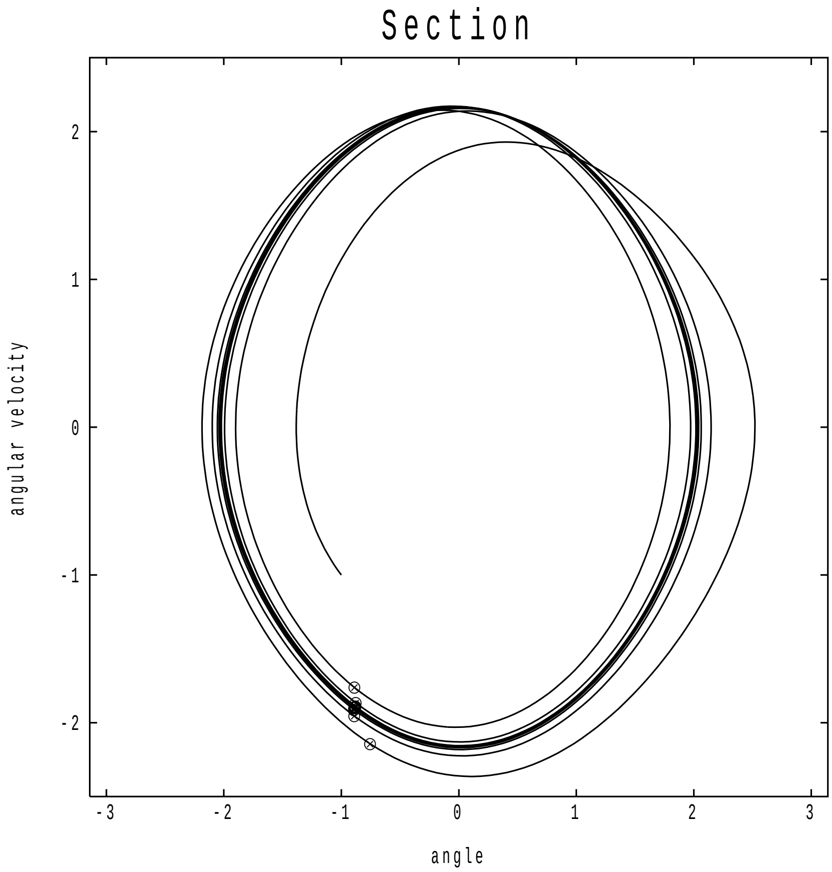
$$\frac{d}{dt}\left[e^{\nu t}\theta\right] = Ae^{\nu t} - \frac{T}{2\pi F}e^{\nu t}\cos[2\pi F t]$$

and then integrate again to find

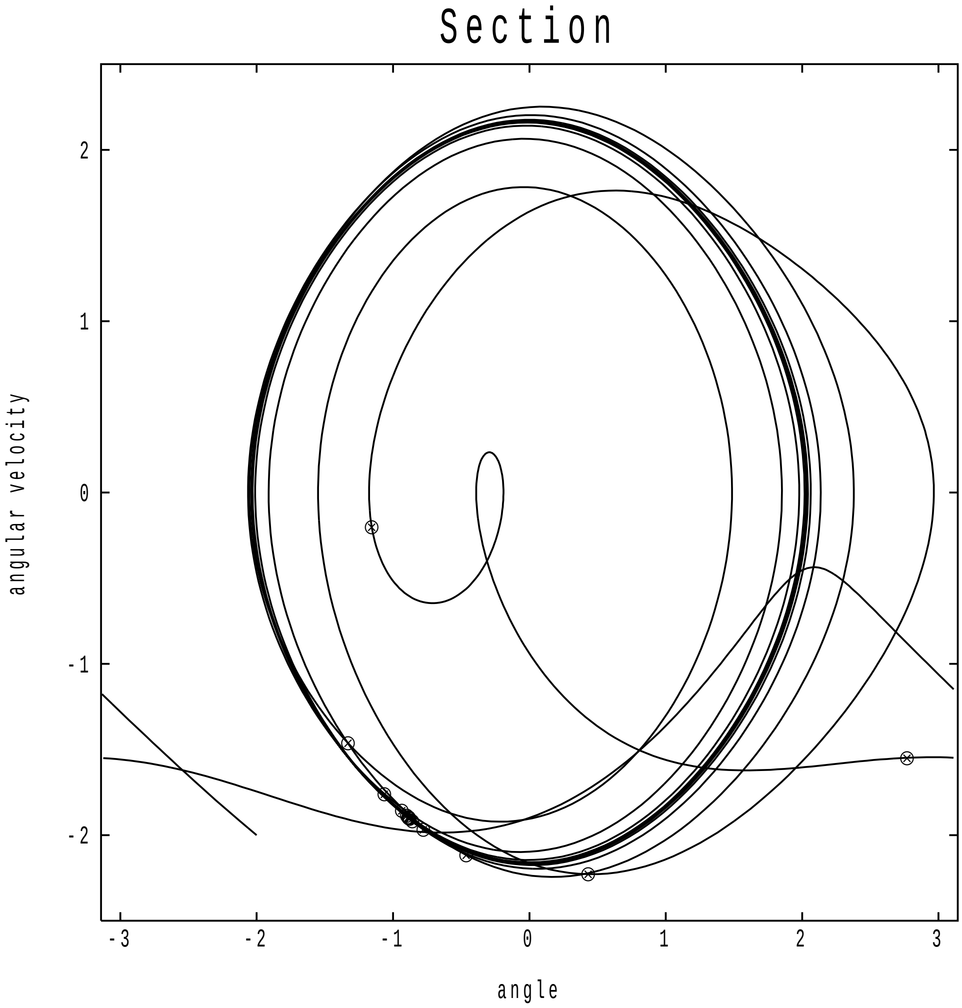
$$\theta(t) = \frac{A}{\nu} + Be^{-\nu t} - \frac{T}{(2\pi F)^2}\frac{1}{1+\left(\frac{\nu}{2\pi F}\right)^2}\left[\sin[2\pi F t] + \frac{\nu}{2\pi F}\cos[2\pi F t]\right]$$

which tells us that the scale of these oscillations is of order T which can be much larger than 2π which verifies the fact that the pendulum can turn over repeatedly during a single forcing cycle.

All this thought provides us with a very complicated picture for the forced damped pendulum, but reality is much worse. This is the next target for the course. We start out with a few examples to provide pictures of trajectories and associated Poincare sections. We use the same values of the parameters but choose different boundary conditions. The first example starts out quite close to the attractor



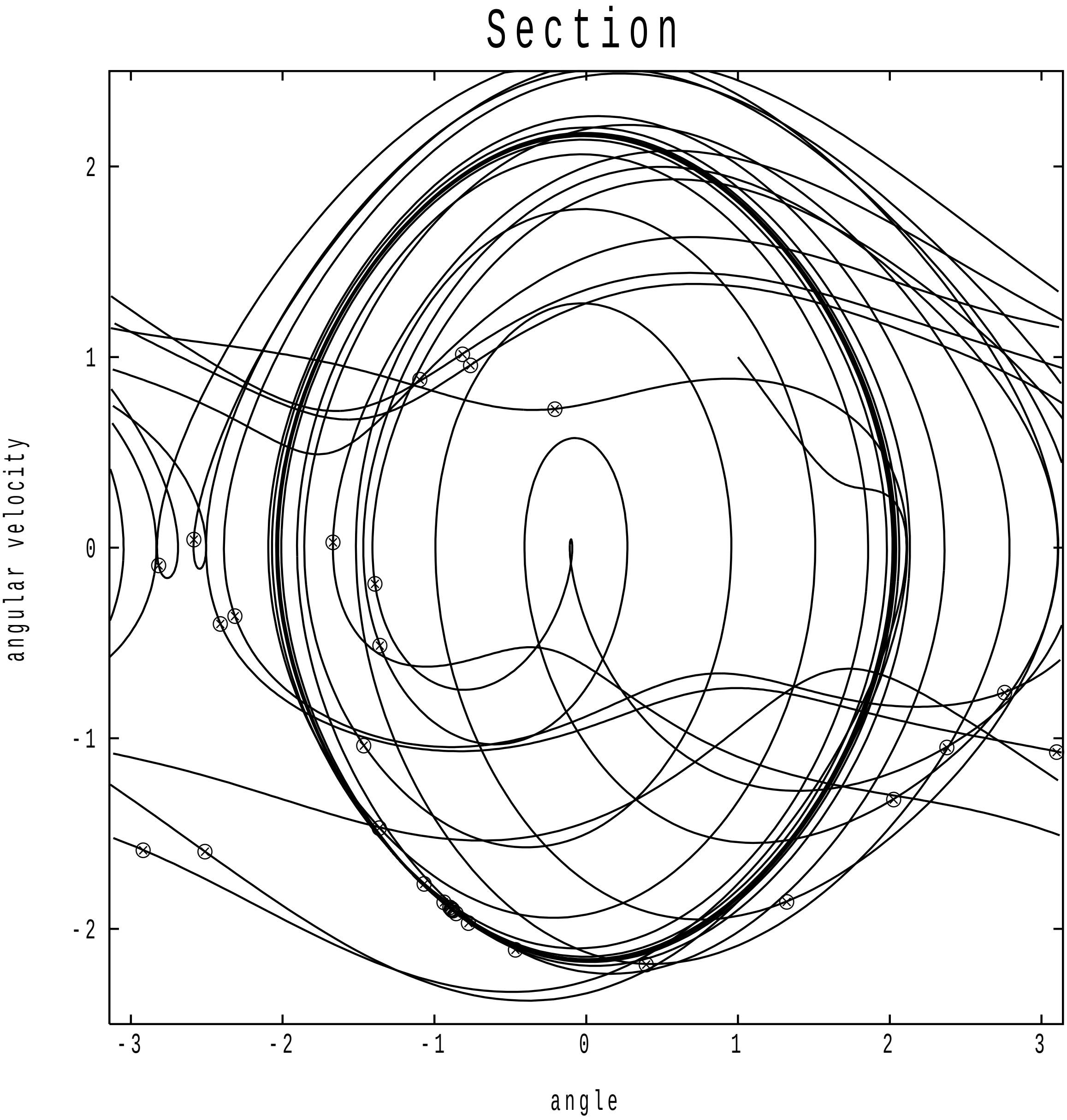
and we can see the trajectory wind around and around displaying multiple oscillations as it converges to the attractor which is a closed loop. We have also added the Poincare section as a superposition of symbols and one can observe a sequence of points converging to the fixed point that lies on the attractor. The second example involves the pendulum turning over twice before



it settles down to its oscillations and the eventual attractor. Note that we are using periodic boundary conditions but by following the trajectory we can count the number of times it turns over before it settles down to the final oscillations within a single window of $\theta \in (-\pi, \pi)$. The trajectory spends almost an entire cycle not doing very much early on before the forcing term gets it oscillating coherently.

There is a second style of Poincare section for this system which is more physical. If one focuses on $\frac{d\theta}{dt}=0$ then the trajectory crosses the line vertically and marks either a local maximum or a local minimum in the value of θ. The sequence of values of θ for which $\frac{d\theta}{dt}=0$ makes a one-dimensional Poincare section. If we employ the full range of θ then these points oscillate between maxima and minima and can be thought of as the end points of an infinite sequence of oscillations. It makes some more sense to focus further on alternating points, which provide either a maximum or a minimum and naturally converge to a single fixed point. The small loop in the middle of the previous trajectory corresponds to a brief period where the pendulum makes a tiny oscillation away from the origin but then picks up speed again. We would usually employ the time oriented Poincare section because it picks up the case where the pendulum turns over and over, whereas this more physical Poincare section would ignore this.

The third example is the bizarre

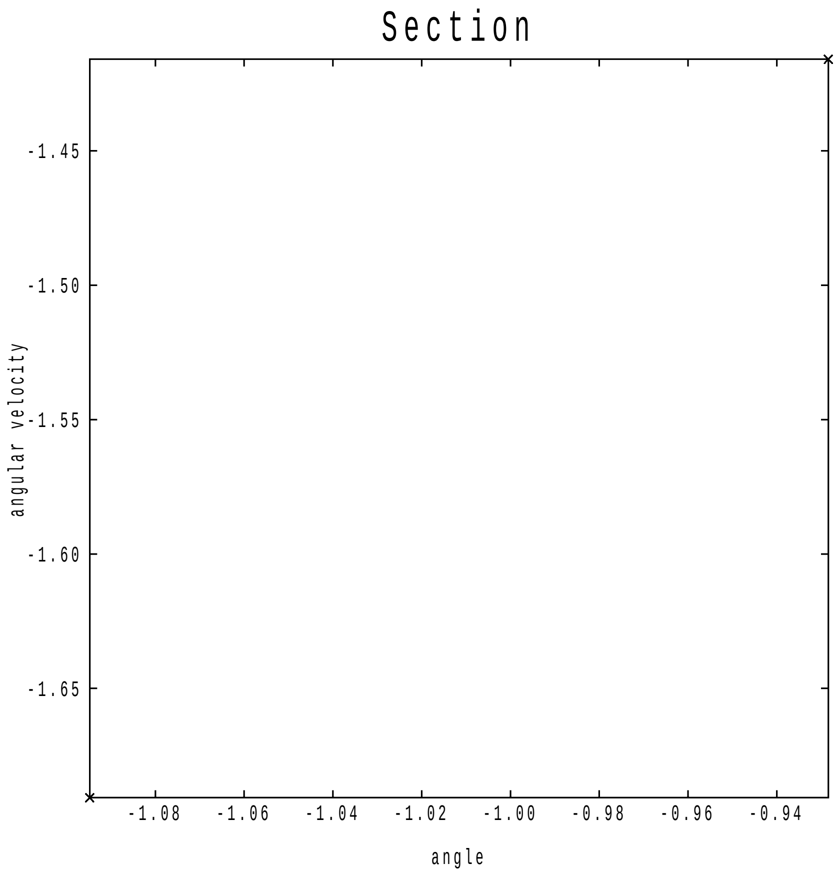


which we included to show the complexity of the motion in general. There are wild fluctuations, which are known as *transients,* but eventually the system settles down to a regular behaviour. For all trajectories we end up with a closed loop; the *attractor.* For the Poincare section we observe that the sequence of points in two dimensions converges to a fixed point; the *attractor.* When we investigate maps we need to keep in mind the behaviour of the underlying system from which the map is generated.

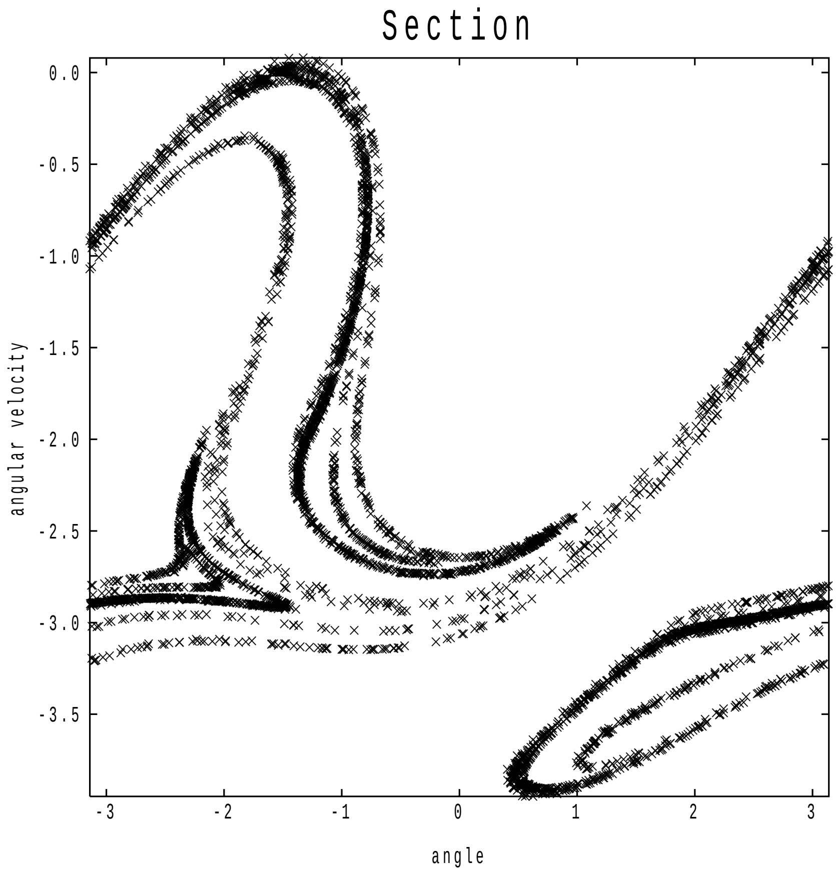
When considering the second style of Poincare section one can observe three tiny loops corresponding to tiny oscillations. The two together are seen to involve the change of behaviour where the pendulum makes its final two full rotations before it settles down to bounded oscillations centred on its final periodic motion. The second Poincare section provides physical insight into the motion but also corresponds directly to the approximate map that we will use to investigate dissipative chaos; the *logistic map.*

The numerical investigation is surprisingly sensitive to the chosen parameters. When $\nu \ll 1$ the trajectories take a very long time to converge, as one might expect, but more tricky is the sensitivity to the control parameter. The changes in behaviour occur over impressivly small intervals of control parameter. Although we would like to use small values for the parameter ν, we start out with ν=0.2 but eventually choose to investigate ν=0.4 in order to stabilise the behaviour and obtain quick and easy results. Finding the exotic behaviour is incredibly difficult for $\nu \ll 1$. This is an important motivation for studying an elementary, almost trivial, analogue; the *logistic map.*

The next step is to observe the problem. We chose to fix F=1 and ν=0.2 and then $T \ll$2.1637 gave us a fixed point. Once we reach T=2.1637, however, the attractor has subtly changed. To study the attractor we ran the simulation until it converges. The mathematical attractor is *never* achieved, no matter how long the equations are propagated. For a numerical calculation we only need to achieve machine accuracy and we have the attractor as accurately as a computer can calculate it. The resulting attractor is

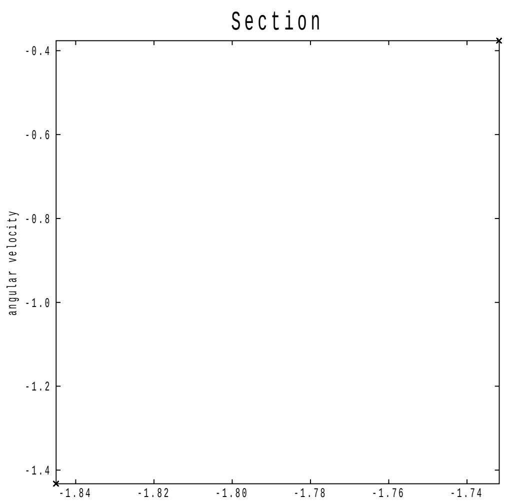


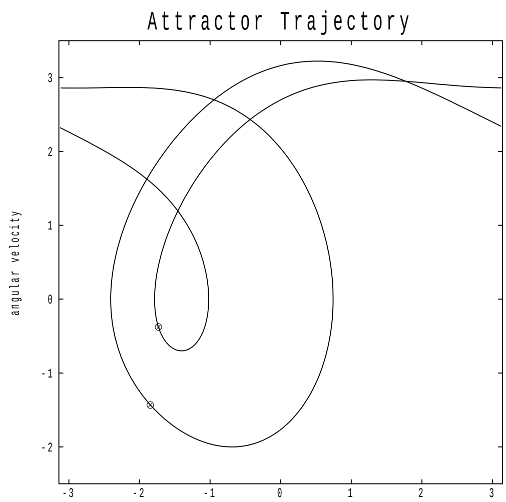
and it is clear that the system has settled down and the transients are gone. We find that the attractor is not a single fixed point for this system but involves two points. This type of behaviour is known as a 2-cycle and the map oscillates between the two points. The real interest in this problem is exhibited by a second case where T=2.2



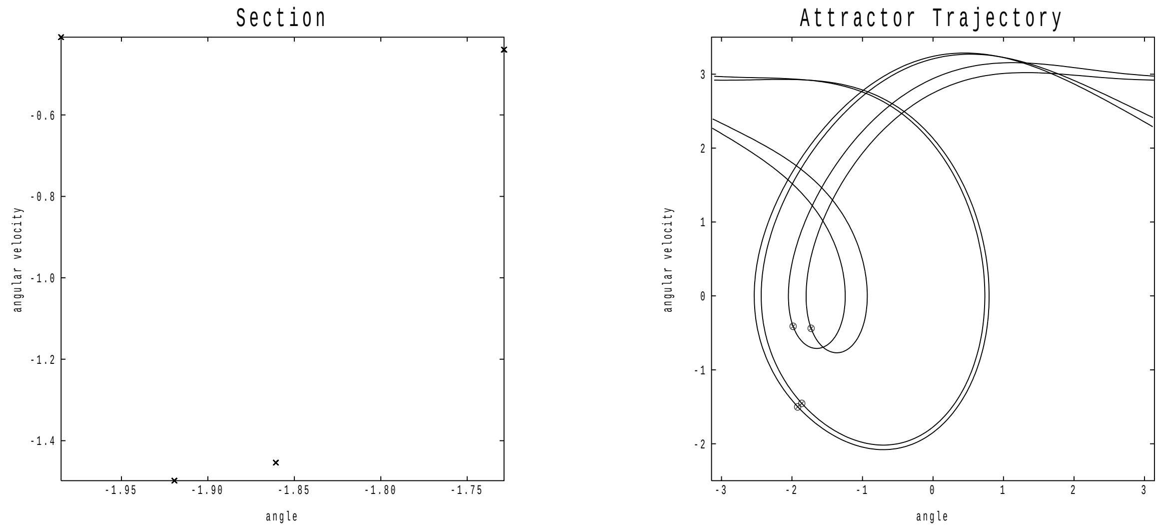
Once again, the simulation was propagated until numerical convergence, although this time it is not possible to control this convergence. The resulting attractor is clearly not a fixed point and is clearly very complicated but what type of set is it? In general such a set is known as a *strange attractor.*

To target the strange attractor immediately is too much. We will look at the first sign of the disease initially. We will fix F=1 and ν=0.4 and then T=2.4 gives us the previous 2-cycle





Here we have also provided the trajectory and one can see that the pendulum is turning over continuously, but has settled down into a motion that has two different cycles. When we increase the control parameter to T=2.45 we get a significant change of behaviour where



and now the motion appears to involve four cycles before it repeats. For this system we can converge the computer calculation to the attractor but now the attractor involves either two or four complete oscillations before it repeats. Is this behaviour natural? How does this behaviour develop into full blown chaos?

The current problem is too difficult to analyse and we need to simplify the problem through approximation. The simplest non-linear map is the *logistic map* and we will now spend several lectures on this topic....before we return to the forced damped pendulum again.

The logistic map

One of the most heavily studied maps is the logistic map

$$x_{n+1} = rx_n(1 - x_n)$$

The parameter r is a control parameter and each value of r provides a distinct map. For each map one starts with an x_0 and then the map provides an infinite sequence of values; $\{x_0, x_1, x_2, x_3, ...\}$. The greatest interest in this map is the eventual behaviour as $n \mapsto \infty$ and whether the map converges to a regular repeating limit or goes crazy and is hence chaotic.

Rescaling the logistic map

A simple rescaling of the variable demonstrates that $r \leq 1$ may be mapped onto $r \geq 1$ and so the characteristic behaviour of the map can be studied by restricting to $r \geq 1$.

We can employ a general change of origin and scale

$$x_n = a + bX_n$$

to obtain

$$a + bX_{n+1} = r(a + bX_n)(1 - a - bX_n)$$

so

$$X_{n+1} = \frac{1}{b}[ra(1-a) - a] + r(1-2a)X_n - rbX_n^2$$

and then the choice

$$ra(1-a) - a = 0 \qquad r(1-2a) \equiv R \qquad rb \equiv R$$

also provides the logistic map. Obviously a=0 and b=1 is a solution and corresponds to the original description. r=0 is pathological but there is an additional solution with

$$a = 1 - \frac{1}{r} \qquad b = 1 - 2a = \frac{2}{r} - 1 \qquad R = 2 - r$$

This provides a new copy of the logistic map

$$x_n = \frac{1}{r}[r - 1 + (2-r)X_n] \quad \Rightarrow \quad X_{n+1} = RX_n(1 - X_n)$$

with $R = 2 - r$. You may like to show that

$$X_n = \frac{1}{R}[R - 1 + (2 - R)x_n]$$

and the balanced

$$r + R = 2 \qquad r(1 - 2x_n) = R(1 - 2X_n)$$

The exact solution at r=2

There are two solvable values of r and we will set the scene with the case r=2 which is particularly simple. We may rewrite

$$x_{n+1} = 2x_n(1 - x_n) \qquad \Rightarrow \qquad 1 - 2x_{n+1} = (1 - 2x_n)^2$$

for this case and then using $z_n \equiv 1 - 2x_n$ we have that

$$z_{n+1} = z_n^2 \qquad \Rightarrow \qquad z_n = z_0^{2^n}$$

and so

$$x_n = \frac{1 - (1 - 2x_0)^{2^n}}{2}$$

is the general solution. There are essentially five styles of behaviour. If $x_0 \in \left(0, \frac{1}{2}\right)$ then $z_0 \in (0, 1)$ and the repeated squaring of a number which is less than unity leads to $z_n \mapsto 0$ as $n \mapsto \infty$ and so $x_n \mapsto \frac{1}{2}$. If $x_0 \in \left(\frac{1}{2}, 1\right)$ then $z_0 \in (-1, 0)$ and so the initial z_0 is negative. The first square ensures that z_1 is positive and less than unity and then the subsequent behaviour is like the first case and so $x_n \mapsto \frac{1}{2}$ as $n \mapsto \infty$. Note that after the first application of the map, $x_0 = \pm a$ provides an identical sequence. This map is not reversible because two starting points lead to the same endpoint. The case $x_0 = \frac{1}{2}$

gives z_0=0 and then the entire sequence remains at $x_n = \frac{1}{2}$. This is known as a fixed point and is one of the possible outcomes for the map as $n \mapsto \infty$. Indeed, we can see that all three cases so far tend towards this fixed point. The case of $x_0 = 0$ gives $z_0 = 1$ and we find a second fixed point with $x_n = 0$ for all values of n. The final case of $x_0 = 1$ gives $z_0 = -1$ and then $z_1 = 1$ and we are back on the second fixed point. The limiting behaviour of this particular map is very simple. If $x_0 \in \{0, 1\}$ we end up at the fixed point $x_n = 0$ and if $x_0 \in (0, 1)$ then we end up at the fixed point $x_n = \frac{1}{2}$. The second fixed point is said to be stable, because if we start out close to this fixed point we end up at this fixed point. The first fixed point is said to be unstable because if we start out close to this fixed point we end up at the other fixed point.

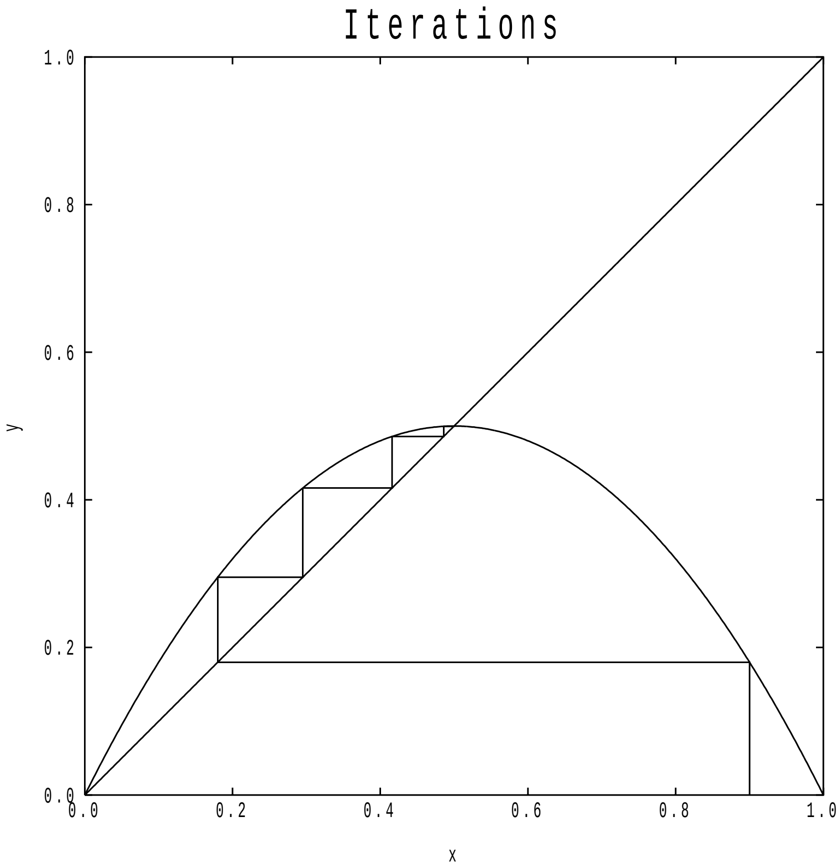
The behaviour of this map may be represented using a 'spider plot'. We think of the function

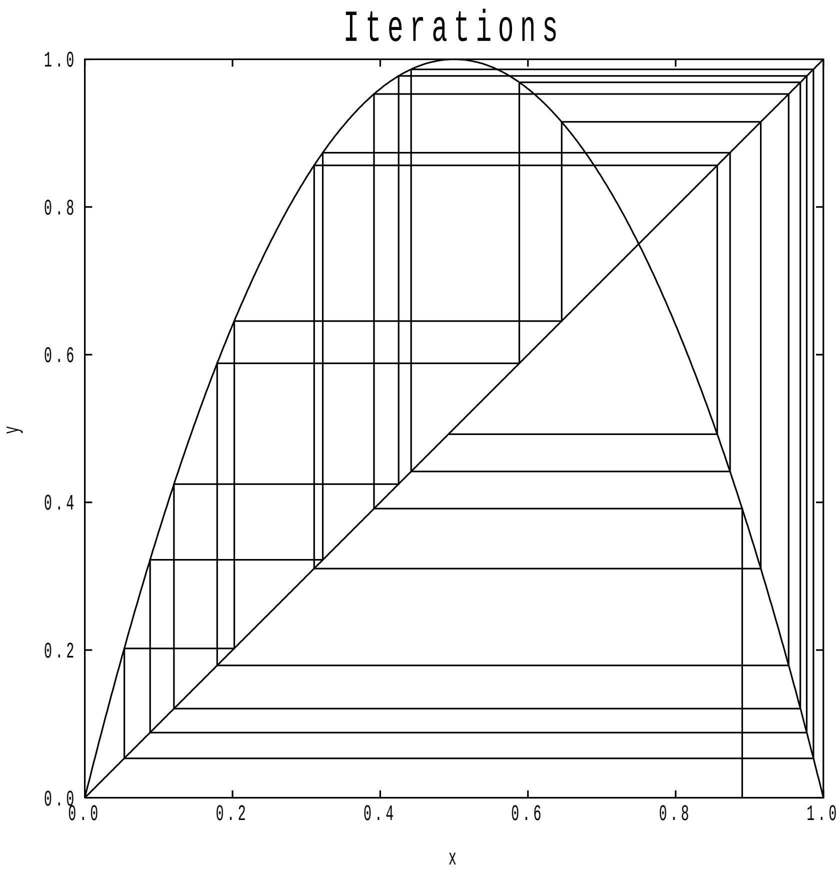
$$y(x) = rx(1 - x)$$

and then the map can be accomplished using

$$y_n = y(x_n) \quad and \quad x_{n+1} = y_n$$

We can view this as moving vertically to the function y(x) and then moving laterally to the line $y = x$ and then repeating. Pictorially this is





for the cases $r = 2$ and r=4 where each subsequent line takes us to the next point in the sequence $\{x_0, y_0, x_1, y_1, x_2, ...\}$ and clearly there is a duplication unnecessary to the desired sequence. The case of $r = 4$ is clearly all over the place, as depicted

From r=2 to r=4: analytic

The next step is to assess some of the possible outcomes for the map as $n \mapsto \infty$. We have observed fixed points, but there is the possibility of n-cycles. A 2-cycle is a situation where the map oscillates between two distinct points consecutively. An n-cycle is where a map cycles through n distinct points consecutively and then repeats. The exact solution to the logistic map at $r = 2$ demonstrates that there are only fixed points or 1-cycles for this case.

How many fixed points are there for a general value of r and where are they? A fixed point must satisfy

$$x^* = rx^*(1 - x^*) \quad \Rightarrow \quad x^* \in \big\{0, 1 - \frac{1}{r}\big\}$$

and so in general there are two fixed points, one at the origin and one that moves around as r is varied. Note that the special case r=1 finds only one fixed point at the origin but it is clearly a sort of 'double fixed point'. Note also that the previous mapping of the logistic map onto itself maps the two fixed points onto each other. This means that stability is transferred between the two fixed points in the mapping.

Now let us hunt down some 2-cycles for the logistic map. The first application of the map to the first point should give the second point and then the second application of the map should take us back to the first point. If we call the two points x and y, then we need $x \neq y$ and

$$y = rx(1-x) \qquad x = ry(1-y)$$

and the solution oscillates between x and y. If we subtract the two equations we get

$$x - y = r(x^2 - y^2 - x + y) = (x-y)r(x+y-1)$$

so either we get a fixed point, where y=x, or

$$x + y = 1 + \frac{1}{r}$$

Now we add the two equations to get

$$x + y = r(x + y - x^2 - y^2) = r(x + y - [x+y]^2 + 2xy)$$

and then substituting in for $x + y$

$$xy = \frac{1}{r}\left[1 + \frac{1}{r}\right]$$

We can construct a quadratic equation that the two roots satisfy, since

$$(z - z_1)(z - z_2) = z^2 - (z_1 + z_2)z + z_1 z_2$$

and the coefficient of z is related to the sum of the two roots while the constant term is the product of the two roots. We then have

$$(z-x)(z-y) = z^2 - (x+y)z + xy = z^2 - \left(1 + \frac{1}{r}\right)z + \frac{1}{r} + \frac{1}{r^2}$$

and then we can solve this quadratic to find x and y

$$\left[z - \frac{1}{2}\left(1 + \frac{1}{r}\right)\right]^2 = \frac{1}{4}\left(1 + \frac{1}{r}\right)^2 - \frac{1}{r} - \frac{1}{r^2} = \frac{1}{4} - \frac{1}{2r} - \frac{3}{4r^2} = \frac{1}{4r^2}(r^2 - 2r - 3)$$

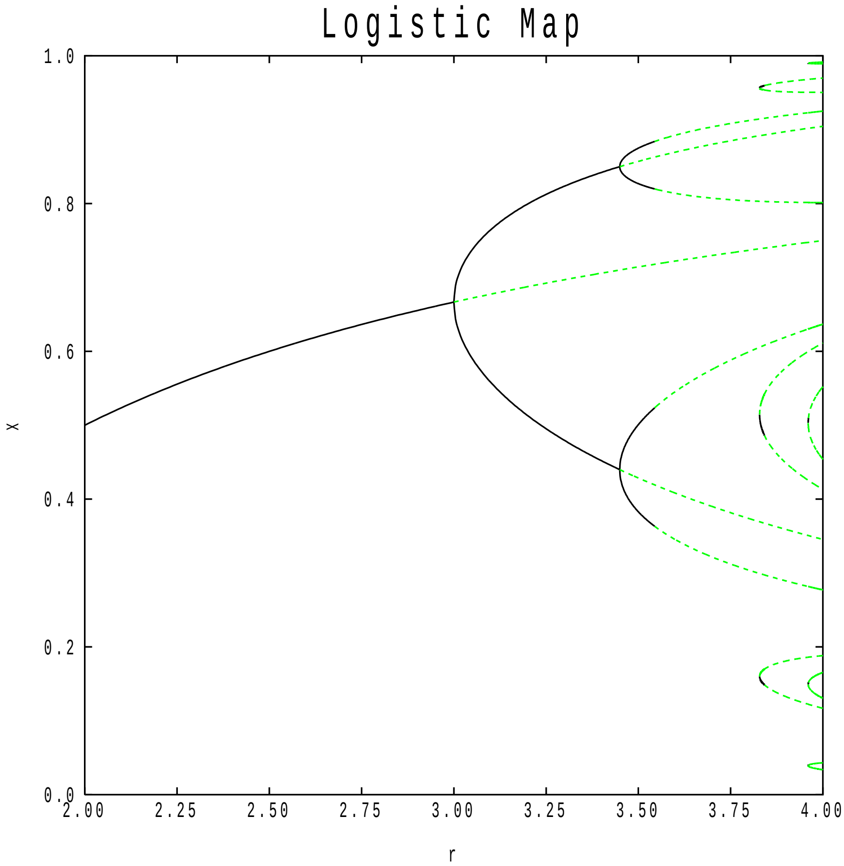
$$= \frac{1}{4r^2}([r-1]^2 - 4) = \frac{1}{4r^2}(r-3)(r+1)$$

so

$$x, y = \frac{1}{2}\left(1 + \frac{1}{r}\right) \pm \frac{1}{2r}\left[(r-1)^2 - 4\right]^{\frac{1}{2}}$$

Clearly the first time that a 2-cycle is allowed, for the real map, is at $r = 3$ and then this solution exists for $r \geq 3$. We could look for 3-cycles and 4-cycles using similar ideas but this is not appropriate for a first year course. You might like to try the algebra.... If you are interested in solving polynomial equations there is an appendix which shows how to deal with low order equations in general and solve for all cycles up to fourth order for the logistic map in particular.

In the investigation of complex mathematical problems, pure mathematics techniques allow access to a limited number of true statements. Usually the arguments are subtle and difficult to uncover but the unadulterated truth of the predictions is crucial to developing sensible intuition and more advanced guesswork. The analysis outlined here and fully explained in the appendix allows the following picture for the attractor; the stable long-time limits for the map

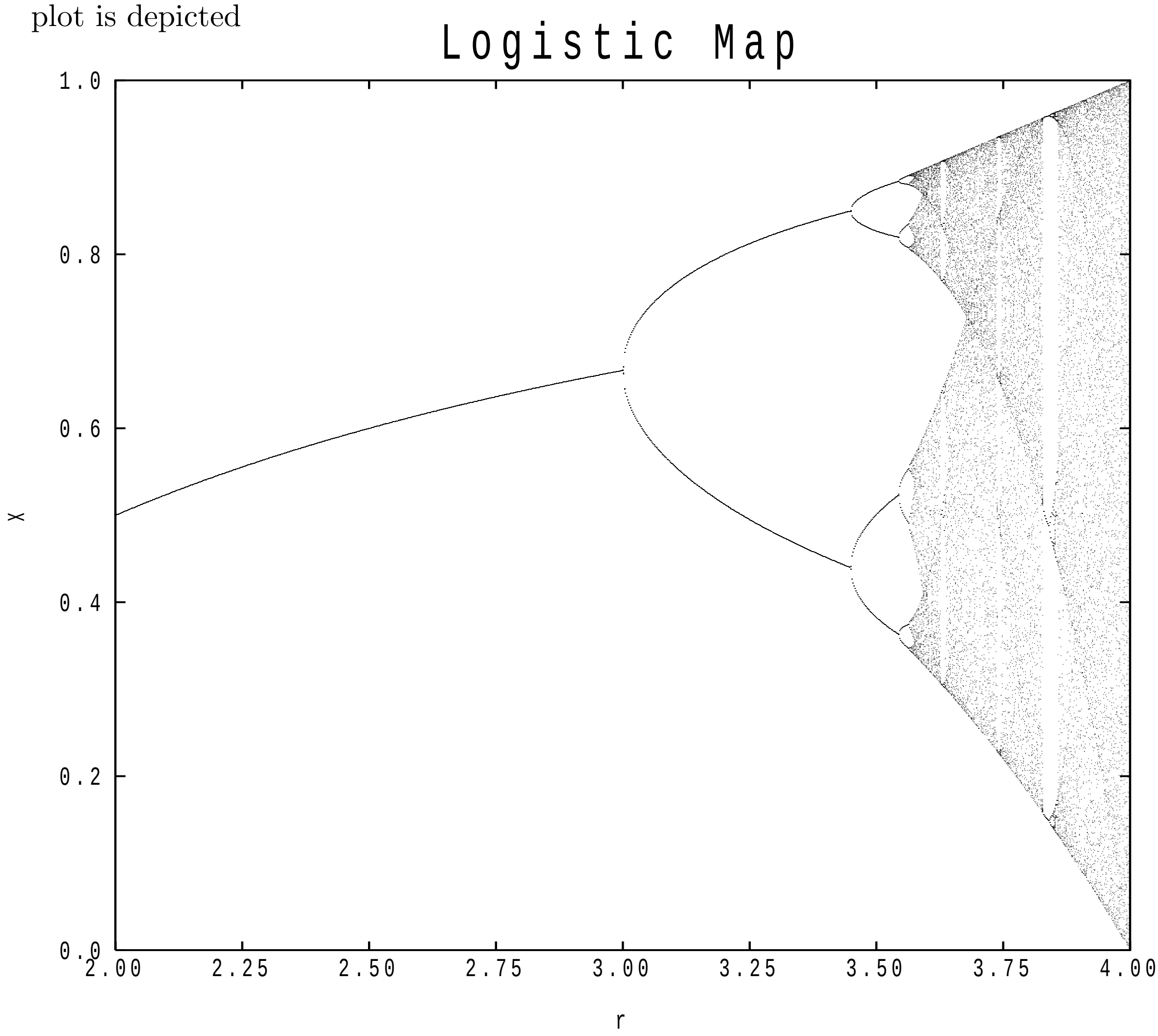


all of the curves amount to possible n-cycles, with $n \in \{1, 2, 3, 4\}$, and indeed, due to the exact nature of the calculations, all such possible n-cycles. We have used solid lines to denote the stable solutions and dotted lines to denote unstable solutions. The topic of the next section is stability.

The 1-cycle that is stable for the r=2 solution is seen to remain the attractor until the point r=3 after which the solution remains but the stability is transferred to the 2-cycle. The 2-cycle remains the attractor up until r=$1+\sqrt{6}$ at which point the 4-cycle becomes the attractor. The 2-cycle still exists but is no longer stable after this point. We can trace the 4-cycle on until it too loses its stability but we have not pursued what comes next. At r=$1+\sqrt{8}$ the pair of 3-cycles appear and one of them is briefly the attractor. At r=$1+2\left[1+\frac{3}{2^{\frac{4}{3}}}\right]^{\frac{1}{2}}$ a pair of 4-cycles appear and one of them is briefly the attractor. The behaviour, even at this low level, is becoming quite complicated.

The big picture is the 1-cycle passes stability to the 2-cycle which passes stability to the 4-cycle. It is natural to conjecture that the 4-cycle transfers stability to an 8-cycle which transfers stability to a 16-cycle and so on, but how can we establish whether this is true or not?

Numerical calculations provide an alternative complementary approach to exact mathematics. Numerical calculations tend to be much easier to perform than analytic but they tend also to have errors and often involve approximations; sometimes uncontrolled. To find the attractor numerically, one can run the map for a long time and assume that eventual convergence has been reached. Then one can slowly increase the control parameter and assume that the attractor will be mapped out. The resulting plot is depicted



but the problem is chaos. If the attractor is chaotic then convergence is not possible to establish in a finite time and, due to the finite time for each attempt at convergence, any cycle which takes longer than this time to converge is not accessible. The sequence of bifurcations is clearly visible and appears to cluster at a finite value of r; r_∞ say. A second numerical approach is to try to do the previous analytic calculations numerically. One can hunt transitions where n-cycles first start or merge by looking for double roots in multiple maps. This has only numerical accuracy and stability as fundamental issues.

Stability

As well as finding the possible n-cycles, we also want to assess whether or not they are stable and hence whether or not a wide range of initial conditions will converge to them. To do this we need to look very close to the cycle and analyse whether the map moves towards or away from the n-cycle. The mathematical technique we employ is Taylor's theorem. Let us start with a 1-cycle and employ a general map defined by a function $f(x)$. A 1-cycle satisfies

$$x = f(x)$$

for any and every real solution to this equation. We then start our sequence out very close to a solution using

$$x_n = x + \delta x_n$$

where we assume, at the outset, that all the δx_n are very small. We then look at the mapping using Taylor's expansion

$$x_{n+1} = f(x_n) \quad \Rightarrow \quad x+\delta x_{n+1} = f(x+\delta x_n) = f(x)+\delta x_n \frac{df}{dx}(x)+\frac{1}{2}(\delta x_n)^2 \frac{d^2 f}{dx^2}(x)+...$$

and we assume that the higher order terms are negligible. This then provides the approximate mapping that

$$\delta x_{n+1} = \frac{df}{dx}(x)\delta x_n$$

which immediately solves to provide

$$\delta x_n = \left[\frac{df}{dx}(x)\right]^n \delta x_0$$

If $-1 < \frac{df}{dx}(x) < 1$ then the solution exponentially vanishes as $n \mapsto \infty$ and the Taylor's expansion becomes better and better and the solution is stable. If $\frac{df}{dx}(x) > 1$ then the separation from the fixed point grows and the assumptions will break down and the sequence will move away from the fixed point. We cannot predict from this theory where it will go. If $\frac{df}{dx}(x) < -1$ then the separation will oscillate about the fixed point as it moves away and we also have an unstable situation. The special cases of $\frac{df}{dx}(x) = \pm 1$ mark the transition from stable to unstable and can be used to predict changes of behaviour.

For the logistic map we have

$$f(x) = rx(1-x) \qquad \frac{df}{dx}(x) = r(1-2x)$$

and we have the two previous fixed points

$$x = rx(1-x) \quad \Rightarrow \quad x \in \{0, 1-\frac{1}{r}\}$$

For the origin, x=0, we find

$$\frac{df}{dx}(0) = r$$

and so this fixed point is stable when $r \in (-1, 1)$. For the case $x = 1 - \frac{1}{r}$ we find

$$\frac{df}{dx}\left(1-\frac{1}{r}\right) = r\left[\frac{2}{r}-1\right] = 2-r$$

and so this fixed point is stable when $r \in (1,3)$. Note that at r=3 we have $\frac{df}{dx}\left(1-\frac{1}{r}\right) = -1$ and so the separation oscillates as it moves away from the fixed point, converging to the 2-cycle. We also found that r=3 was the point at which the 2-cycle first appeared and the current analysis is an independent way of finding this change of behaviour.

We now turn our attention to generic 2-cycles. The 2-cycle satisfies

$$y = f(x) \qquad x = f(y)$$

and once again we wish to start close to this 2-cycle. We now have a natural oscillation so we need to use the idea that even values of the n are close to x but odd values of n are close to y

$$x_{2m} = x + \delta x_{2m} \qquad x_{2m+1} = y + \delta x_{2m+1}$$

and then assuming that all the distortions are small we find

$$y + \delta x_{2m+1} = f(x + \delta x_{2m}) = f(x) + \delta x_{2m}\frac{df}{dx}(x) + \frac{1}{2}(\delta x_{2m})^2\frac{d^2 f}{dx^2}(x) + ...$$

$$x + \delta x_{2m+2} = f(y + \delta x_{2m+1}) = f(y) + \delta x_{2m+1}\frac{df}{dx}(y) + \frac{1}{2}(\delta x_{2m+1})^2\frac{d^2 f}{dx^2}(y) + ...$$

and assuming that the higher order terms vanish we find that

$$\delta x_{2m+1} = \delta x_{2m}\frac{df}{dx}(x) \qquad \delta x_{2m+2} = \delta x_{2m+1}\frac{df}{dx}(y)$$

and then combining the two equations we find that

$$\delta x_{2m+2} = \frac{df}{dx}(y)\frac{df}{dx}(x)\delta x_{2m}$$

which solves to provide

$$\delta x_{2m} = \left[\frac{df}{dx}(y)\frac{df}{dx}(x)\right]^m \delta x_0$$

and now stability is achieved by

$$-1 < \frac{df}{dx}(y)\frac{df}{dx}(x) < 1$$

Once again we have the important critical case when $\frac{df}{dx}(y)\frac{df}{dx}(x) = \pm 1$ which we can use to determine when the stability breaks down. We also have $\frac{df}{dx}(y)\frac{df}{dx}(x) > 1$ which corresponds to the 2-cycle growing and $\frac{df}{dx}(y)\frac{df}{dx}(x) < -1$ which corresponds to a 4-cycle growing.

For the logistic map we have the previous analysis of the possible 2-cycles

$$y = rx(1-x) \quad x = ry(1-y)$$

$$\Rightarrow \quad y - x = (x-y)r(1-x-y) \quad x + y = r\left[x + y - (x+y)^2 + 2xy\right]$$

$$\Rightarrow \quad x+y = 1+\frac{1}{r} \quad xy = \frac{1}{2}\left[\frac{1}{r}+\frac{1}{r^2}+\left(1+\frac{1}{r}\right)^2-\left(1+\frac{1}{r}\right)\right] = \frac{1}{r}+\frac{1}{r^2}$$

Stability is controlled by

$$\frac{df}{dx}(x) = r(1-2x) \quad \Rightarrow \quad \frac{df}{dx}(y)\frac{df}{dx}(x) = r^2(1-2x)(1-2y) = r^2\left[1-2(x+y)+4xy\right]$$

$$= r^2\left[1-2\left(1+\frac{1}{r}\right)+4\left(\frac{1}{r}+\frac{1}{r^2}\right)\right] = 4+2r-r^2 = 5-(r-1)^2$$

Note that the stability does not depend on which point is which and so the quadratic equation need not be solved when we determine the stability. The 2-cycle first goes unstable when $5-(r-1)^2 = 1 \quad \Rightarrow \quad r = -1, 3$ and the 2-cycle becomes unstable to a 4-cycle when $5-(r-1)^2 = -1$ and $r = 1-\sqrt{6},\ 1+\sqrt{6}$; in complete agreement with the previous analysis in both the main text and the appendix. Note that in the first case the 2-cycle ceases to exist whereas in the second case it still exists but becomes unstable. Some additional analysis is required to uncover the complete picture.

Obviously we can extend these ideas to higher-order cycles. For 3-cycles we have

$$y = f(x) \qquad z = f(y) \qquad x = f(z)$$

and the assumptions

$$x_{3m} = x+\delta x_{3m} \qquad x_{3m+1} = y+\delta x_{3m+1} \qquad x_{3m+2} = z+\delta x_{3m+2}$$

leads to

$$y+\delta x_{3m+1} = f(x)+\delta x_{3m}\frac{df}{dx}(x)+... \qquad z+\delta x_{3m+2} = f(y)+\delta x_{3m+1}\frac{df}{dx}(y)+...$$

$$x+\delta x_{3m+3} = f(z)+\delta x_{3m+2}\frac{df}{dx}(z)+...$$

which gives

$$\delta x_{3m+3} = \frac{df}{dx}(z)\frac{df}{dx}(y)\frac{df}{dx}(x)\delta x_{3m}$$

and you should be able to see that in general we need to find the n-cycle and then find the product of the $\frac{df}{dx}(x)$ evaluated at the points which make up the n-cycle in order to determine the stability. We analyse this stability for the logistic map at the end of the appendix.

The exact solution at r=4

The second exactly solvable point is at $r = 4$ but the solution for this case is more subtle. For this case we make the transformation

$$x_n = \left[\sin\left(\frac{\pi y_n}{2}\right)\right]^2$$

which maps $x_n \in [0,1]$ monotonically onto $y_n \in [0,1]$ for a particular choice of branch for y_n. The logistic equation becomes

$$x_{n+1} = 4x_n(1-x_n) = \left[\sin\left(\frac{\pi y_{n+1}}{2}\right)\right]^2 = 4\left[\sin\left(\frac{\pi y_n}{2}\right)\right]^2\left(1-\left[\sin\left(\frac{\pi y_n}{2}\right)\right]^2\right)$$

$$= \left[2\sin\left(\frac{\pi y_n}{2}\right)\cos\left(\frac{\pi y_n}{2}\right)\right]^2 = \left[\sin\left(\pi y_n\right)\right]^2$$

which reduces to

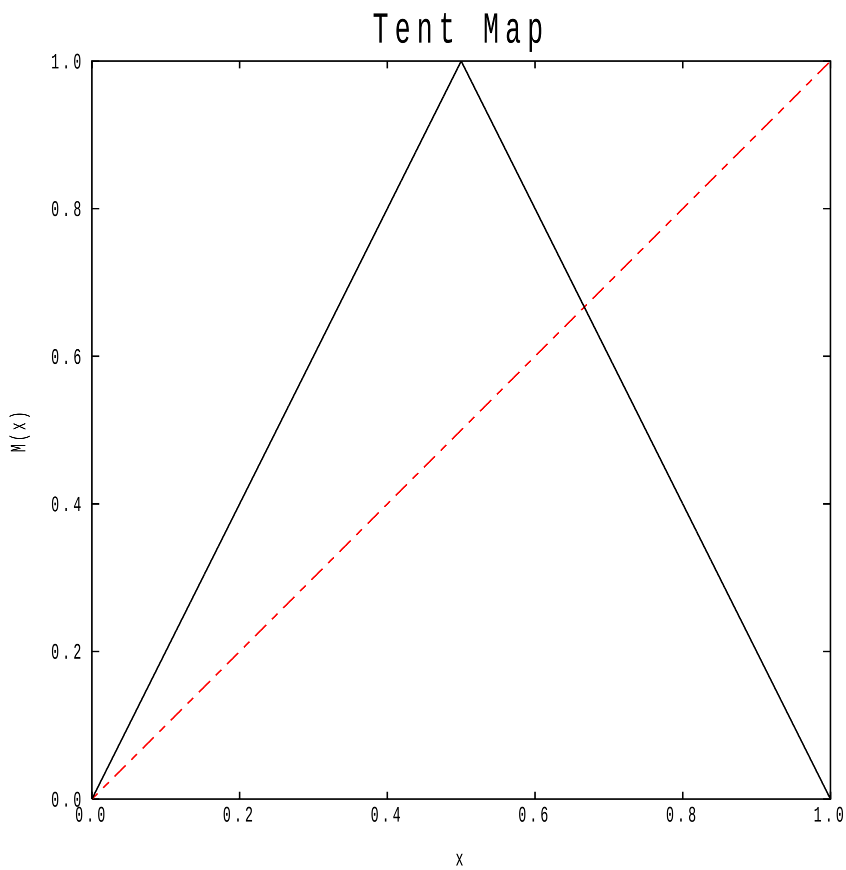
$$\sin\left(\frac{\pi y_{n+1}}{2}\right) = \sin\left(\pi y_n\right) = \sin\left(\pi\left[1-y_n\right]\right)$$

and then to

$$y_{n+1} = 2y_n \qquad y_n \in \left[0, \tfrac{1}{2}\right]$$

$$y_{n+1} = 2(1-y_n) \qquad y_n \in \left[\tfrac{1}{2}, 1\right]$$

where we choose the trigonometric solution to ensure that $y_n \in [0,1]$. This map depicts as



where we have included the identity curve to investigate 1-cycles. It is clear that y_n=0 provides a 1-cycle and the other 1-cycle satisfies

$$y = 2(1-y) \quad \Rightarrow \quad y_n = \frac{2}{3}$$

Of course in the original logistic map we need the transformation $x = \left[\sin\frac{\pi y}{2}\right]^2$ to relate these 1-cycles and although the origin is mapped onto the origin, $y = \frac{2}{3}$ maps onto

$$x = \left[\sin\frac{\pi}{3}\right]^2 = \frac{3}{4} = 1 - \frac{1}{4} = 1 - \frac{1}{r}$$

which agrees with the previous general analysis.

We would now like to answer the question of how many n-cycles there are for all possible integer values of n. At first sight this issue appears quite challenging, because the function is no longer smooth and changes behaviour at $y = \frac{1}{2}$. We could proceed on regardless, 'bull at a gate' tactics, and then we would need solutions to

$$y = f(x) \qquad\qquad x = f(y)$$

for 2-cycles and we could run through all the possible choices

$$y = 2x \qquad\qquad x = 2y$$

$$y = 2x \qquad\qquad x = 2(1-y)$$

$$y = 2(1-x) \qquad\qquad x = 2y$$

$$y = 2(1-x) \qquad\qquad x = 2(1-y)$$

The first case gives x=0, so one of the 1-cycles. The final case gives

$$x = 2(1-y) = 2(1-2(1-x)) = 4x-2 \quad x = \frac{2}{3}$$

the other 1-cycle while the second and third case give

$$x = 2(1-2x) \quad \Rightarrow \quad x = \frac{2}{5} \quad y = 2x = \frac{4}{5}$$

Obviously this technique works, but the number of different cases soon becomes unmanageable and a better idea is required; *multiple maps.*

Multiple maps

Another way to examine n-cycles is to consider multiple applications of the same map. We can introduce the notation

$$f^{(2)}(x) \equiv f(f(x)) \qquad f^{(3)}(x) \equiv f(f(f(x))) = f(f^{(2)}(x)) = f^{(2)}(f(x))$$

and so on. A 3-cycle for $f(x)$ is also a 1-cycle for the map $f^{(3)}(x)$. This idea allows us to analyse the stability of an n-cycle using an independent but equivalent route to the previous method. For a 2-cycle we also have a 1-cycle for $f^{(2)}(x)$. Stability for a 1-cycle is controlled by the derivative of the function but from the chain rule

$$\frac{df^{(2)}}{dx}(x) = \frac{d}{dx}f(f(x)) = \frac{df}{dx}(f(x))\frac{df}{dx}(x)$$

and since $f(x) = y$ is the other point of the 2-cycle we have

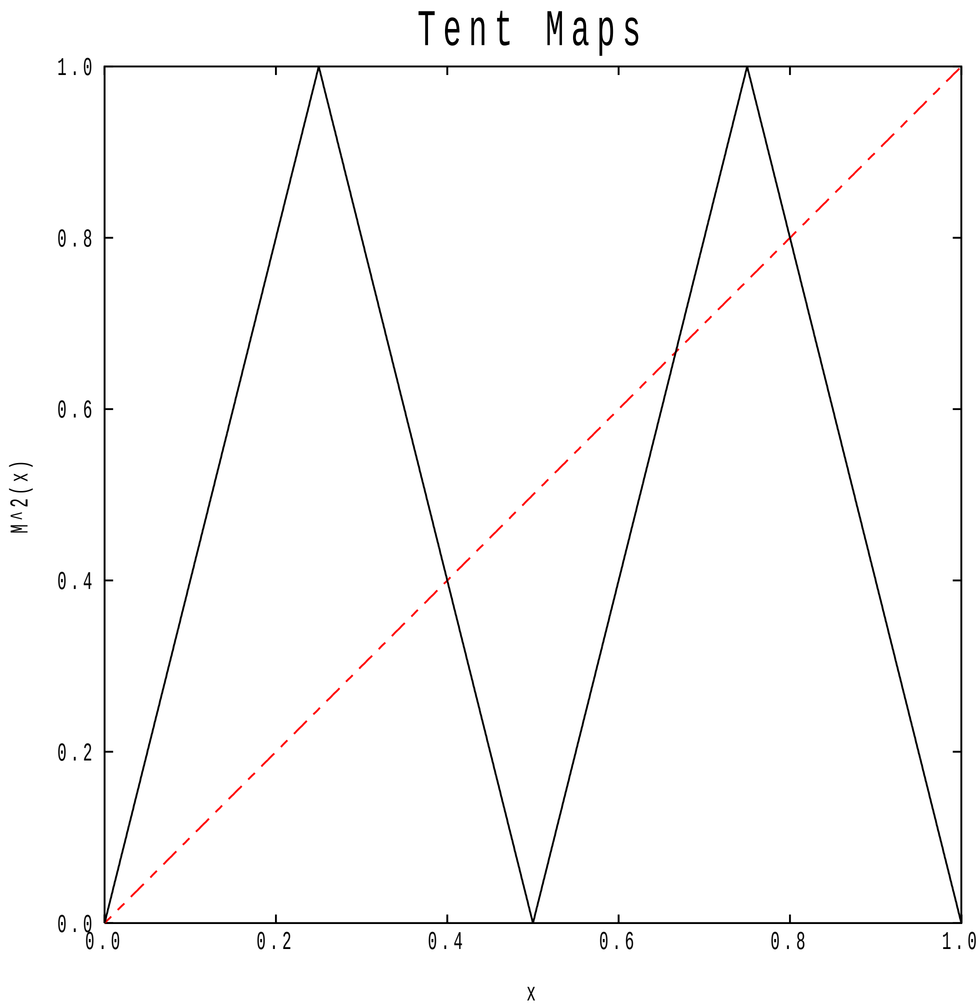
$$\frac{df^{(2)}}{dx}(x) = \frac{df}{dx}(y)\frac{df}{dx}(x)$$

identical to the previous result. For 3-cycles we have

$$\frac{df^{(3)}}{dx}(x) = \frac{d}{dx}f(f(f(x))) = \frac{df}{dx}(f(f(x)))\frac{d}{dx}f(f(x)) = \frac{df}{dx}(f(f(x)))\frac{df}{dx}(f(x))\frac{df}{dx}(x)$$

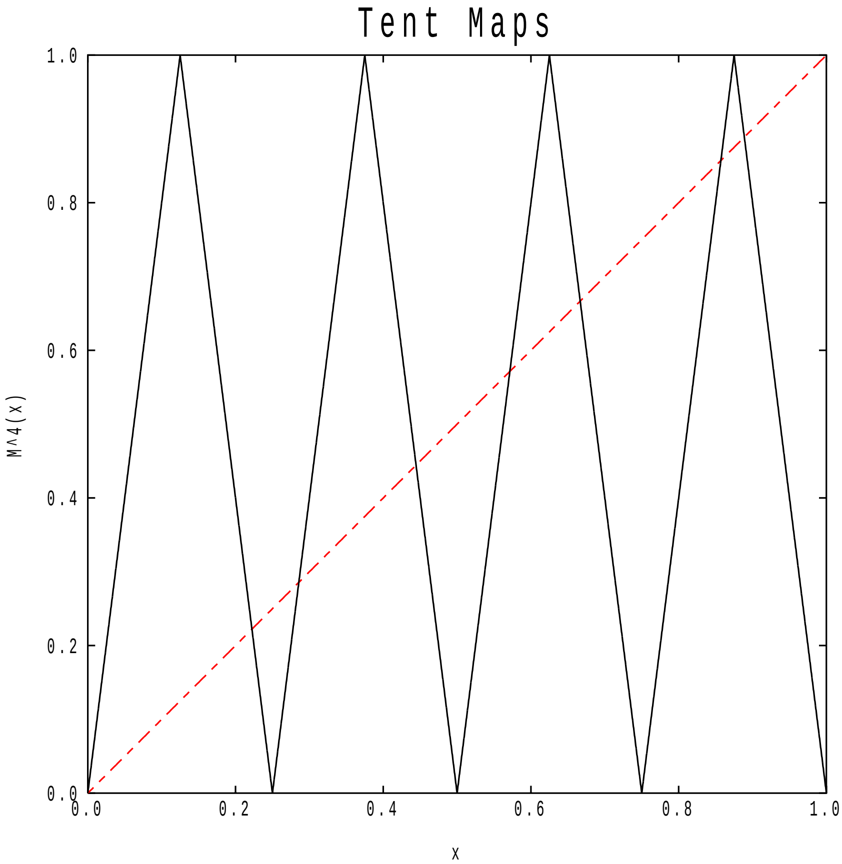
and the product of derivatives as we cycle around the points in the 3-cycle. Obviously this idea generalises to n-cycles.

We may investigate the n-cycles of the tent map, $T(x)$, using multiple maps. To find all the 2-cycles we may investigate all the 1-cycles of the map $T^{(2)}(x)$, since 1-cycles of this map are either 1-cycles of the original tent map or are 2-cycles. We can construct $T^{(2)}(x)$ by appreciating that $T(x)$ for $x \in \left[0, \frac{1}{2}\right]$ maps onto the entire region $x \in [0, 1]$ linearly, which then provides a copy of the tent map. For $x \in \left[\frac{1}{2}, 1\right]$ the tent map maps to the entire region $x \in [0, 1]$ but backwards. Since the tent map is symmetric $T(1 - x) = T(x)$ we again get a copy of the tent map so



and to find the 1-cycles we may set $y = x = T^{(2)}(x)$ and we can observe four solutions;

the two 1-cycles and a pair of points which create the unique 2-cycle. To construct $T^{(3)}(x)$ we can use $T^{(2)}(T(x))$ and we clearly get four copies of the tent map



To find the 1-cycles of this map we may put $y = x = T^{(3)}(x)$ and we clearly have eight solutions; the two 1-cycles and a pair of triples of points which create the two 3-cycles. We obtain clear agreement with the previous general analysis of the logistic map where we found these two solutions in general.

The analysis of 4-cycles is more subtle because both 1-cycles and 2-cycles will show up as a 1-cycle for $T^{(4)}(x)$. Indeed it should be clear that for a general map $T^{(n)}(x)$ and factors of $n{=}n_1n_2$ will also provide 1-cycles of the map since

$$T^{(n)}(x) = T^{(n_2)}[T^{(n_2)}[...T^{(n_2)}[T^{(n_2)}(x)]...]]$$

with n_1 applications and

$$T^{(n)}(x) = T^{(n_1)}[T^{(n_1)}[...T^{(n_1)}[T^{(n_1)}(x)]...]]$$

with n_2 applications.

For $T^{(4)}(x)$ we have sixteen solutions to $x = T^{(4)}(x)$ and this includes the two 1-cycles and the one 2-cycle leaving twelve points which make up the three 4-cycles that we found as general solutions to the logistic map.

Now that we have reproduced the results obtained for the general theory on the logistic map we can now extend the results for the exact solution at $r{=}4$ beyond 4-cycles to try to understand larger n-cycles. When we consider 5-cycles we need to examine $T^{(5)}(x)$ which involves sixteen copies of the tent map leading to thirty-two solutions to $x = T^{(5)}(x)$. Since five is prime, there are only two 1-cycles and then six distinct 5-cycles which lead to the remaining thirty solutions. When we consider 6-cycles we need to examine solutions to the equation $x = T^{(6)}(x)$ of which there are sixty-four. This includes the two 1-cycles, the one 2-cycle, the two 3-cycles and this leaves nine 6-cycles which form the remaining fifty-four points. For the general n-cycles we need to analyse $x = T^{(n)}(x)$ which has 2^n solutions in general. If n is prime then there are the two 1-cycles and the rest of the points make distinct n-cycles, but if n has factors then all the cycles for these factors are also contained within these solutions. For 7-cycles we have $128{=}2\times 1+18\times 7$ whereas for 8-cycles we have $256{=}2\times 1+1\times 2+3\times 4+30\times 8$ and for 9-cycles we have $512{=}2\times 1+2\times 3+56\times 9$ and for 10-cycles we have $1024{=}2\times 1+1\times 2+6\times 5+99\times 10$ and so on.

The larger the value of n, the larger the number of solutions to $x = T^{(n)}(x)$ but the growth is exponential. There are approximately $\frac{2^n}{n}$ distinct n-cycles and this also grows exponentially with n. The physical picture for the logistic map is that at r=2 there is only a single stable 1-cycle and a second unstable 1-cycle, whereas for r=4 there are a multitude of cycles for all orders with more and more cycles for higher and higher orders. All of these cycles must appear as r is increased and so the behaviour must be supremely complicated.

So far we have not discussed stability but since

$$\frac{dT}{dx}(x) = 2 \qquad x \in \left[0, \frac{1}{2}\right)$$

$$\frac{dT}{dx}(x) = -2 \qquad x \in \left(\frac{1}{2}, 0\right]$$

and does not exist when $x = \frac{1}{2}$, any product of these derivatives will be larger than unity so we are expecting *all* n-cycles to be unstable. The next section gives a representation in which stability is easy to understand and provides a physical picture for the behaviour that is useful; an intuitive picture of a chaotic trajectory.

Base 2

You should all be familiar with base 10, known as denary, where we represent integer numbers as

$$a_0 + a_1 \times 10 + a_2 \times 10^2 + a_3 \times 10^3 + ... \quad \mapsto \quad ...a_3 a_2 a_1 a_0$$

where $a_m \in \{0,1,2,3,4,5,6,7,8,9\}$. Base 2 is analogous to base 10 but with

$$a_0 + a_1 \times 2 + a_2 \times 2^2 + a_3 \times 2^3 + ... \quad \mapsto \quad ...a_3 a_2 a_1 a_0$$

where $a_m \in \{0,1\}$. It is also known as binary. We are more interested in fractions where

$$0.a_{\bar{1}} a_{\bar{2}} a_{\bar{3}} ... \equiv a_{\bar{1}} \times \frac{1}{2} + a_{\bar{2}} \times \frac{1}{2^2} + a_{\bar{3}} \times \frac{1}{2^3} + ...$$

Note that $a_{\bar{1}}$=0 means that $x \in \left[0, \frac{1}{2}\right)$ whereas $a_{\bar{1}}$=1 means $x \in \left[\frac{1}{2}, 1\right]$. This is extremely useful in controlling $T(x)$, since $a_{\bar{1}}$ controls which of the two branches is active. The action of $T(x)$ using binary representation is very instructive.

To apply $T(x)$ we need to be able to do two things: (1) multiply by 2 and (2) calculate 1-x and both of these things require some discussion. Since

$$...2 \times 2 = 2^2 \quad 2 \times 1 = 2 \quad 2 \times \frac{1}{2} = 1 \quad 2 \times \frac{1}{2^2} = \frac{1}{2}...$$

and so on we have that

$$2[0.a_{\bar{1}}a_{\bar{2}}a_{\bar{3}}...] = 2\left[a_{\bar{1}} \times \frac{1}{2} + a_{\bar{2}} \times \frac{1}{2^2} + a_{\bar{3}} \times \frac{1}{2^3} + ...\right] = a_{\bar{1}}.a_{\bar{2}}a_{\bar{3}}...$$

and, just like multiplying by ten in base 10, all the digits are moved up by one in the sequence. For the first half of the tent map we are only applying the multiply by two to numbers $x \in \left[0, \frac{1}{2}\right]$ and for these numbers the binary expansion is $0.0a_{\bar{2}}a_{\bar{3}}...$ and so

$$2 \times 0.0a_{\bar{2}}a_{\bar{3}}... = 0.a_{\bar{2}}a_{\bar{3}}...$$

and the initial 0 is eliminated and the digits move one place forward in the expansion. The calculation of 1-x is more subtle. For this we require the identity that

$$1 = 0.11111....$$

This may be deduced from a geometric series

$$0.1111.... = 1 \times \frac{1}{2} + 1 \times \frac{1}{2^2} + 1 \times \frac{1}{2^3} + ... = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = \frac{\frac{1}{2}}{1 - \frac{1}{2}} = 1$$

and so there is some ambiguity in the description as the same number may be represented in two distinct ways. We can now calculate

$$1 - 0.a_{\bar{1}}a_{\bar{2}}a_{\bar{3}}... = 0.111... - 0.a_{\bar{1}}a_{\bar{2}}a_{\bar{3}}... = 0.(1 - a_{\bar{1}})(1 - a_{\bar{2}})(1 - a_{\bar{3}})...$$

and to calculate 1-x we can replace all the digits by their complements

$$0.01101001... \mapsto 0.10010110...$$

For the tent map when $x \in \left[\frac{1}{2}, 1\right]$ we need to calculate 1-x and then multiply by two. These numbers start out as $0.1a_{\bar{2}}a_{\bar{3}}...$ and so when we take the complement we end up with $0.0(1 - a_{\bar{2}})(1 - a_{\bar{3}})...$ and then when we multiply by two we again eliminate the zero and move each digit one place forward in the representation.

We can now apply the tent map to an arbitrary number in binary representation and consider how to calculate n-cycles directly. The basic idea is to apply the tent map n times and arrive back at the original number. n applications of the tent map will eliminate the first n digits in the binary representation and transfer the remaining digits forward by n places to fill the gap. The actual digits will either be the same as the originals or will be the complements. The key to whether the n digits repeat or alternate with the complementary set of n digits is the final n'th digit. If the n'th digit is zero then we have repetition and if the n'th digit is unity then we have alternation. After n-1 digits have been eliminated the final digit is either the same as it was at the start or it is opposite; if it started out as zero then if it is still zero the final elimination leaves it as zero, whereas if it has changed to unity then the final elimination will take it back to zero. If it started out as unity and is still unity before the final application then the final application will flip it, whereas if it has flipped to zero before the final elimination then it will remain flipped.

For 1-cycles we have

$$0.0000... \mapsto 0.0000... \qquad 1-cycle$$

$$0.1010... \mapsto 0.1010... \qquad 1-cycle$$

For 2-cycles we have

$$0.0000... \mapsto 0.0000... \qquad 1-cycle$$

$$0.1010... \mapsto 0.1010... \qquad 1-cycle$$

$$0.0110... \mapsto 0.1100... \qquad 2-cycle$$

$$0.1100... \mapsto 0.0110... \qquad 2-cycle$$

For 3-cycles we have

$$0.000000... \mapsto 0.000000... \qquad 1-cycle$$

$$0.010010... \mapsto 0.100100... \qquad 3-cycle$$

$$0.100100... \mapsto 0.110110... \qquad 3-cycle$$

$$0.110110... \mapsto 0.010010... \qquad 3-cycle$$

$$0.001110... \mapsto 0.011100... \qquad 3-cycle$$

$$0.011100... \mapsto 0.111000... \qquad 3-cycle$$

$$0.111000... \mapsto 0.001110... \qquad 3-cycle$$

$$0.101010... \mapsto 0.101010... \qquad 1-cycle$$

We can find the actual fractions which correspond to our solutions in two ways. We can trace out the points and assess which branch of $T(x)$ we are on to decide whether we use $2x$ or 2(1-x). For the 2-cycle we have

$$y = 2x \quad x = 2(1-y) \quad \Rightarrow \quad x = \frac{2}{5} \quad y = \frac{4}{5}$$

For the first 3-cycle we have

$$y = 2x \quad z = 2(1-y) \quad x = 2(1-z) \quad \Rightarrow \quad x = 2(1-2(1-y)) = 4y-2 = 8x-2 \quad \Rightarrow$$

$$x = \frac{2}{7} \quad y = \frac{4}{7} \quad z = \frac{6}{7}$$

and for the second 3-cycle we have

$$y = 2x \quad z = 2y \quad x = 2(1-z) \quad \Rightarrow \quad x = 2(1-2y) = 2-4y = 2-8x \quad \Rightarrow$$

$$x = \frac{2}{9} \quad y = \frac{4}{9} \quad z = \frac{8}{9}$$

Alternatively we can take any repeating number and multiply by 2^n, where n is the repeat distance, to deduce the number

$$x = 0.111000111000... \quad \Rightarrow \quad 64x = 111000 + x \mapsto 63x = 8+16+32 = 56 \quad \Rightarrow \quad x = \frac{8}{9}$$

Note the mixture of denary and binary representations....

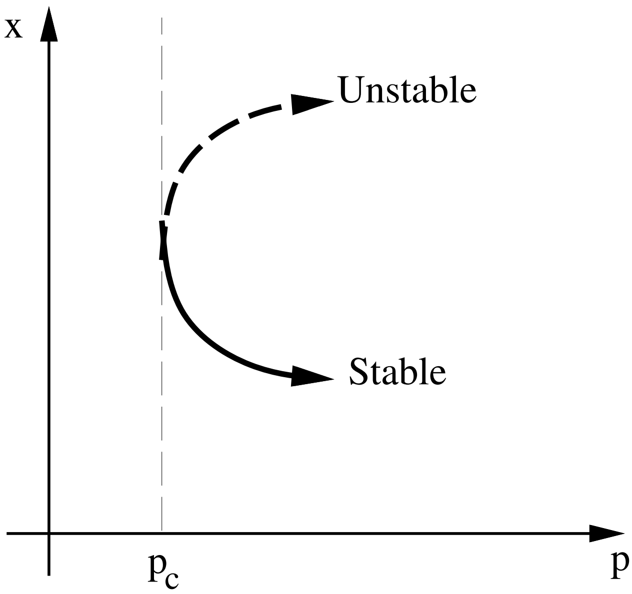
The best way to use this binary representation is in the understanding of stability. To test stability we need to start out close to an n-cycle and determine whether we get closer to the n-cycle or further from the n-cycle as we repeatedly apply the map. The initial issue is how to choose a number close to the n-cycle. Binary representation involves increasing accuracy as we progress along the digits. If two numbers agree for the first N digits then they are within $\frac{1}{2^N}$ of each-other. For two numbers to be close they must have the same initial string of digits up to some length and where they first disagree sets the scale for how far apart they are. A number which is both close to another and otherwise random agrees with the other up to some length and is then a random string of zeroes and ones after that. When we apply the tent map we lose the information in the first digit and transfer all the digits up by one along the number, sometimes replacing all the digits by their complements. If we start from two numbers which are close, then the first N digits agree but after that they become uncorrelated. The action of the tent map agrees for the first N application, in the sense that we are on the same branch each time, but then the resulting behaviour is uncorrelated. Another way to understand this behaviour is to interpret the difference between the two numbers as an error and then the error grows by a factor of two with each application of the map until it dominates and becomes the whole number.

This is an example of the ‘butterfly effect’. A very small distortion in the initial conditions eventually causes completely different behaviour. Here the regular behaviour is an n-cycle and adding an error in the initial conditions of order $\frac{1}{2^N}$ leads to chaotic behaviour after N applications of the map. An exponentially small distortion of the boundary conditions leads to chaos in a linear time. In general the chaotic behaviour is generated exponentially as $\delta\epsilon e^{\lambda t}$, where $\delta\epsilon$ is the initial error, t is time and λ is known as a Lyapunov exponent. For us, time is discrete and each application of the map is a unit of time and so $\delta\epsilon 2^N$ should be interpreted as t=N and then e^λ=2 corresponds to the Lyapunov exponent. Any positive Lyapunov exponent is interpreted as the system being chaotic.

The bifurcation route to chaos

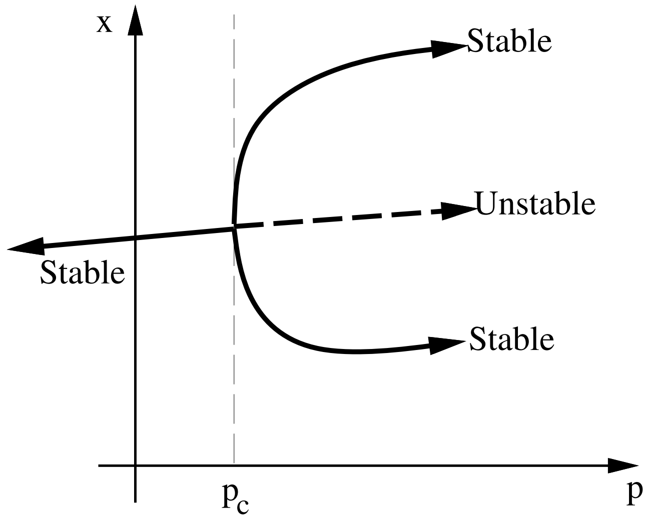
In our analysis of the logistic map we observed the stable 1-cycle which lost stability to a stable 2-cycle. We then observed the stable 2-cycle lose its stability to a stable 4-cycle. This 4-cycle also loses its stability and we presume that it is replaced by a stable 8-cycle. How does this sequence end? We also found a stable 3-cycle which loses its stability to, we presume, a stable 6-cycle and we found a second stable 4-cycle which loses its stability to, we presume, a stable 8-cycle. We investigated the point r=4 and uncovered a *huge* number of unstable n-cycles of all orders but with an exponentially increasing number of such cycles as n increases. When do these cycles appear, are any of them stable and what happens to them? The numerical investigation indicates that when n-cycles appear then they appear in pairs with one of the two cycles stable or they appear from a cycle with half the order through a bifurcation of the smaller cycle. The smaller cycle gives up its stability to the larger cycle.

There is a further mathematical calculation that one can perform to assess the appearance of n-cycles for a general map and the result of this calculation is an understanding of the only 'smooth' ways that the attractor can change its behaviour. The first possibility is that, as the control parameter is varied, a pair of 1-cycles can appear, with one being stable and the other being unstable; *pair production*



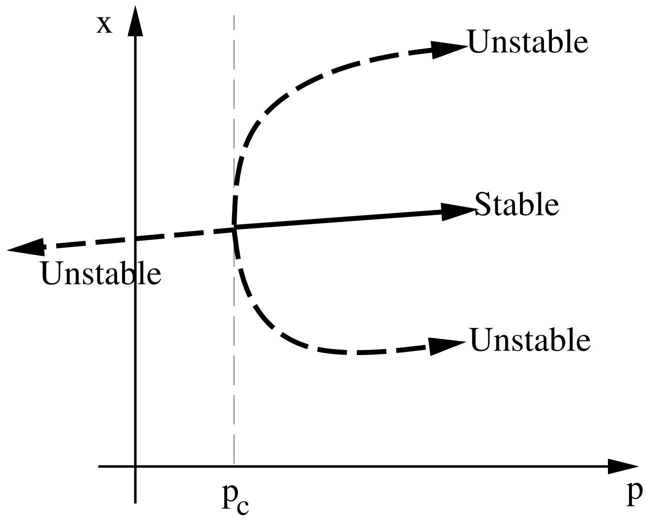
In our investigation of the logistic map we see this behaviour many times. All odd cycles appear in pairs and even cycles can appear as pairs in a similar fashion. In our exact analysis the pair of 3-cycles and the pair of 4-cycles both appear using this source.

The second possibility is that an existing stable 1-cycle can give up its stability to a stable 2-cycle without giving up its identity; *pitchfork bifurcation*



In our investigation of the logistic map we see this behaviour many times. Each stable n-cycle of all orders goes through the bifurcation route to chaos with a countable number of pitchfork bifurcations. In our exact analysis the 1-cycle turns into a 2-cycle and the 2-cycle turns into a 4-cycle using this source.

The third possibility is that a stable 1-cycle can appear from an unstable 1-cycle together with the appearance of an unstable 2-cycle



We do not see this possibility occur in the logistic map, but it is mathematically permitted. Considering multiple maps then allows n-cycles to appear as pairs and n-cycles can bifurcate into $2n$-cycles using the same ideas.

The bifurcation route to chaos: numerical

In our analytic attack on the problem we controlled the 1-cycle, the 2-cycle, the 4-cycle and achieved the point where the 8-cycle first appeared. This, however , is very far removed from the actual appearance of chaos for the first time and so we deal with how to use the computer to address much larger 2^n-cycles and how we approach the question: Even if we accept that there are a sequence of bifurcations, why should we believe that an infinite number of these bifurcations occur in a finite range of the control parameter?

The way to attack this question numerically is to use *finite-size scaling.* This is a procedure where we employ a sequence of numerically exact calculations on finite systems that we try to extrapolate to an infinite system. For a general map

$$x_{n+1} = f(x_n; r)$$

we can investigate N-cycles by generating the multiple maps

$$x_{n+N} = f^{(N)}(x_n; r)$$

and then hunting for solutions to

$$x = f^{(N)}(x; r)$$

This enables us to trace out the trajectory of any N-cycle by solving an algebraic equation, as a function of the control parameter, r. If we focus on the attractor then we want to ensure stability and so we also need to study

$$\frac{\partial}{\partial x} f^{(N)}(x; r)$$

which predicts stability when it ranges in the set $(-1, 1)$. The special point

$$\frac{\partial}{\partial x} f^{(N)}(x; r) = 1$$

provides the first appearance of the N-cycle and the special point

$$\frac{\partial}{\partial x} f^{(N)}(x; r) = -1$$

provides the point at which the stability is lost. It turns out that

$$\frac{\partial}{\partial x} f^{(N)}(x; r) = 0$$

is also interesting as between each appearance of an N-cycle and its loss of stability such a point exists and it is both physically interesting and also numerically more easy to handle. This special case, where the derivative vanishes, corresponds to the physical case where the convergence to the attractor is the fastest. The exact solution at r=2 corresponds to this case and we get convergence to the single stable fixed point much faster than an exponential. Indeed, at other points we get exponential convergence, except at the two cases where the cycle first appears or loses stability at which convergence is almost non-existent. This special case is numerically valuable because

$$\frac{\partial f^{(N)}}{\partial x}(x; r) = \frac{\partial f}{\partial x}(f^{(N-1)}(x; r); r) \frac{\partial f^{(N-1)}}{\partial x}(x; r)$$

$$= \frac{\partial f}{\partial x}(f^{(N-1)}(x;r);r)\frac{\partial f}{\partial x}(f^{(N-2)}(x;r);r)....\frac{\partial f}{\partial x}(f^{(2)}(x;r);r)\frac{\partial f}{\partial x}(f(x;r);r)\frac{\partial f}{\partial x}(x;r)$$

and so this quantity vanishes only if one of the factors vanishes and this is a much simpler problem to solve. For the logistic map

$$\frac{\partial f}{\partial x}(x;r) = r(1-2x)$$

and so this special case occurs when $x{=}\frac{1}{2}$ is one of the points in the N-cycle. For this case we may investigate

$$\frac{1}{2} = f^{(N)}\left(\frac{1}{2};r\right)$$

as an equation for r to find the position of this special case. For the appearance of the N-cycle or the loss of stability we need to solve the two equations

$$x = f^{(N)}(x;r) \qquad \frac{\partial}{\partial x}f^{(N)}(x;r) = \pm 1$$

for two unknowns, x and r, which is clearly numerically more difficult.

To make things more concrete, let us examine the first few supercycles. We need for the first case

$$\frac{1}{2} = f\left(\frac{1}{2}\right) = r\frac{1}{2}\left(1-\frac{1}{2}\right) = \frac{r}{4} \qquad \Rightarrow \qquad r = 2$$

and we find our exact solution. For the second case we need

$$\frac{1}{2} = f\left(f\left(\frac{1}{2}\right)\right) = f\left(\frac{r}{4}\right) = r\frac{r}{4}\left(1-\frac{r}{4}\right) \qquad \Rightarrow \qquad r^3 - 4r^2 + 8 = 0$$

$$\Rightarrow (r-2)(r^2-2r-4) = 0 \qquad \Rightarrow \qquad r = 2,\ 1 \pm \sqrt{5}$$

and the previous result is present as a 1-cycle together with the desired 2-cycle. At third order we find

$$\frac{1}{2} = f\left(f\left(f\left(\frac{1}{2}\right)\right)\right) = f\left(f\left(\frac{r}{4}\right)\right) = r\frac{r^2}{4}\left(1-\frac{r}{4}\right)\left[1-\frac{r^2}{4}\left(1-\frac{r}{4}\right)\right]$$

$$= \frac{r^3}{4} - \frac{r^4}{16} - \frac{r^5}{16}\left(1-\frac{r}{2}+\frac{r^2}{16}\right) \qquad \Rightarrow \qquad (r-2)(r^6-6r^5+4r^4+24r^3-16r^2-32r-64) = 0$$

and the complexity of the equation increases dramatically order by order.

The changes in stability are controlled by two equations. At the lowest order

$$x = f(x) = rx(1-x) \qquad \frac{df}{dx}(x) = r(1-2x) \equiv S$$

and then we need to eliminate x to find r

$$x = 1 - \frac{1}{r} \qquad x = \frac{1}{2}\left(1-\frac{S}{r}\right) \qquad \Rightarrow \qquad \frac{1}{2} = \frac{1}{r} - \frac{S}{2r} \qquad \Rightarrow \qquad r = 2 - S$$

and when S=1 we find r=1 and the 1-cycle first becomes stable and when S=-1 we find r=3 and the 1-cycle becomes unstable to the 2-cycle. At the next order we find

$$x = f(f(x)) = r\, rx(1-x)\,[1 - rx(1-x)] \qquad S = r(1-2x)r\,[1 - 2rx(1-x)]$$

and the general procedure of eliminating x is now quite daunting. The equation for the 2-cycles factorises into

$$x\left(x - 1 + \frac{1}{r}\right)\left[x^2 - x\left(1+\frac{1}{r}\right) + \frac{1}{r}\left(1+\frac{1}{r}\right)\right] = 0$$

and the stability is controlled by

$$4x^3 - 6x^2 + 2x\left(1+\frac{1}{r}\right) - \frac{1}{r} + \frac{S}{r^3} = 0$$

Substituting in for $4x^3$ and $6x^2$ gives

$$4x\left(x-\frac{1}{r}\right)\left(1+\frac{1}{r}\right) - 6\left(x-\frac{1}{r}\right)\left(1+\frac{1}{r}\right) + 2x\left(1+\frac{1}{r}\right) - \frac{1}{r} + \frac{S}{r^3}$$

$$= 4\left(1+\frac{1}{r}\right)\left[x^2 - x - \frac{x}{r}\right] + \frac{6}{r}\left(1+\frac{1}{r}\right) - \frac{1}{r} + \frac{S}{r^3} = 0$$

and then one final substitution gives

$$-4\left(1+\frac{1}{r}\right)\frac{1}{r}\left(1+\frac{1}{r}\right) + \frac{6}{r}\left(1+\frac{1}{r}\right) - \frac{1}{r} + \frac{S}{r^3}$$

$$= \frac{1}{r^3}\left[r^2 - 2r - 4 + S\right] = \frac{1}{r^3}\left[(r-1)^2 - 5 + S\right] = 0$$

providing us with the previous results. Clearly the supercycles calculation is much easier.

The numerical details of solving algebraic equations also involves maps. For a simple algebraic equation

$$g(x) = 0$$

we assume that we are quite close to a root and Taylor expand. If x is the current position and x^* is the root then $x = x^* + \delta x$ gives

$$0 = g(x^*) = g(x - \delta x) = g(x) - \delta x \frac{dg}{dx}(x) + \frac{1}{2}\delta x^2 \frac{d^2 g}{dx^2}(x) + ...$$

f we assume that the higher order terms are small then we can neglect them and get an *approximation* for δx

$$\delta x \sim \frac{g(x)}{\frac{dg}{dx}(x)}$$

we can then create a sequence of points by using this argument on the previous point, x_n, to create a next point, x_{n+1},

$$x_n - x_{n+1} = \frac{g(x_n)}{\frac{dg}{dx}(x_n)} \quad \Rightarrow \quad x_{n+1} = x_n - \frac{g(x_n)}{\frac{dg}{dx}(x_n)}$$

This technique is called Newton-Raphson and converges to a root much faster than exponential convergence.

If we generalise to two equations

$$f(x,r) = 0 \qquad\qquad g(x,r) = 0$$

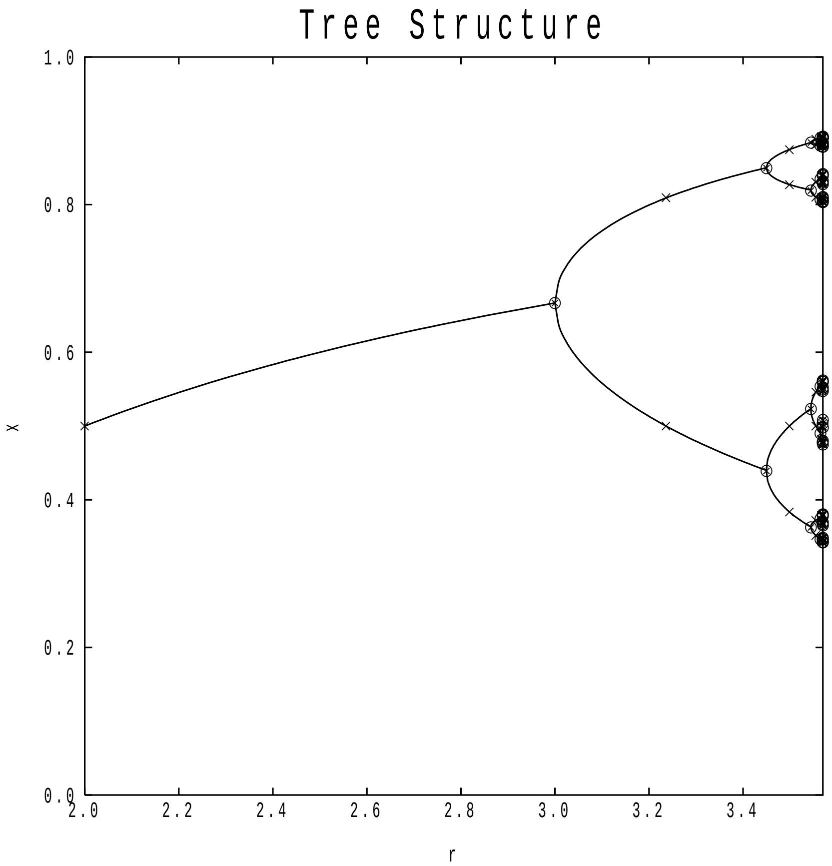
then an analogous argument provides the iterative equations

$$x_{n+1} = x_n - \frac{f\frac{\partial g}{\partial r} - \frac{\partial f}{\partial r}g}{\frac{\partial f}{\partial x}\frac{\partial g}{\partial r} - \frac{\partial f}{\partial r}\frac{\partial g}{\partial x}}(x_n, r_n)$$

$$r_{n+1} = r_n - \frac{\frac{\partial f}{\partial x}g - f\frac{\partial g}{\partial x}}{\frac{\partial f}{\partial x}\frac{\partial g}{\partial r} - \frac{\partial f}{\partial r}\frac{\partial g}{\partial x}}(x_n, r_n)$$

We can use these to quickly solve the pair of equations which create the endpoints for the regions.

To investigate the first appearance of chaos we investigate the stability windows of the 2^n-cycles for n=0,1,2,3... by solving the previous equations to machine accuracy. The results for the first few values of n are depicted below, up to sixth order,



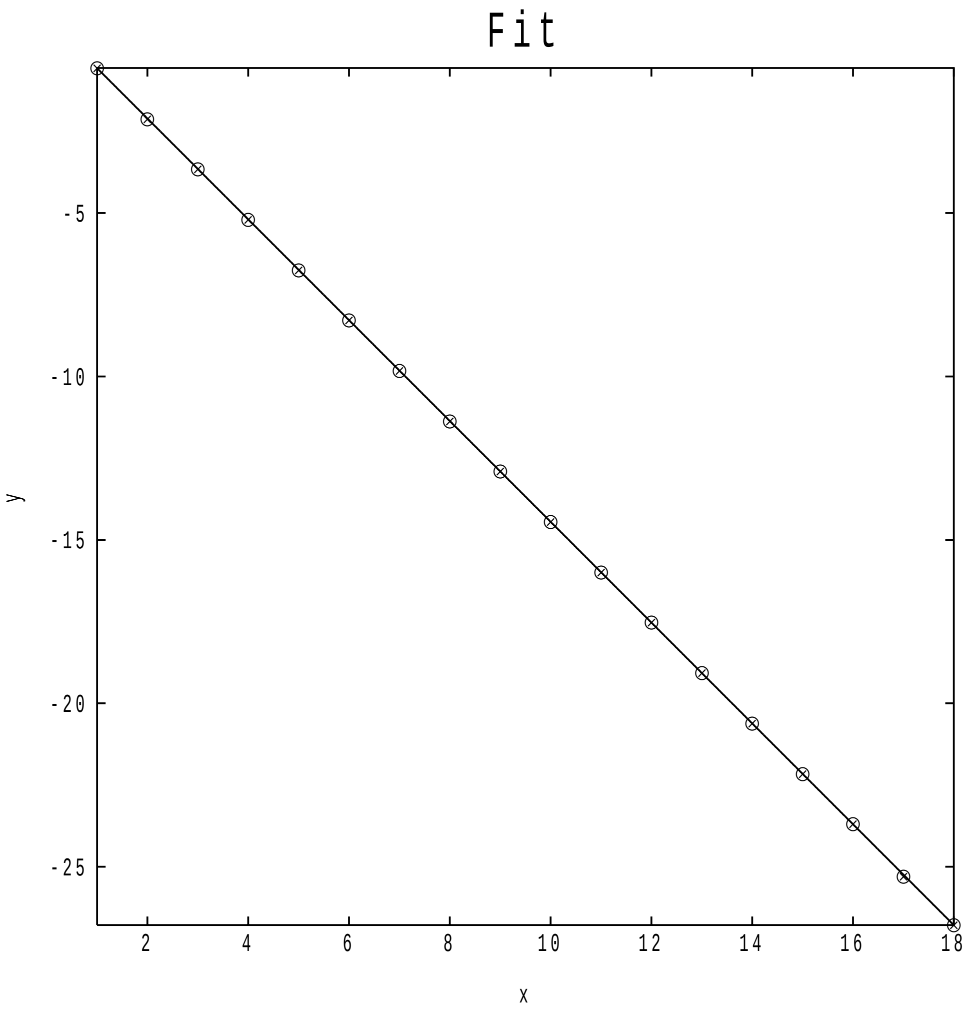
where we also found the trajectories of the 2^n-cycles when they were stable by solving

$$x = f^{(2^n)}(x; r)$$

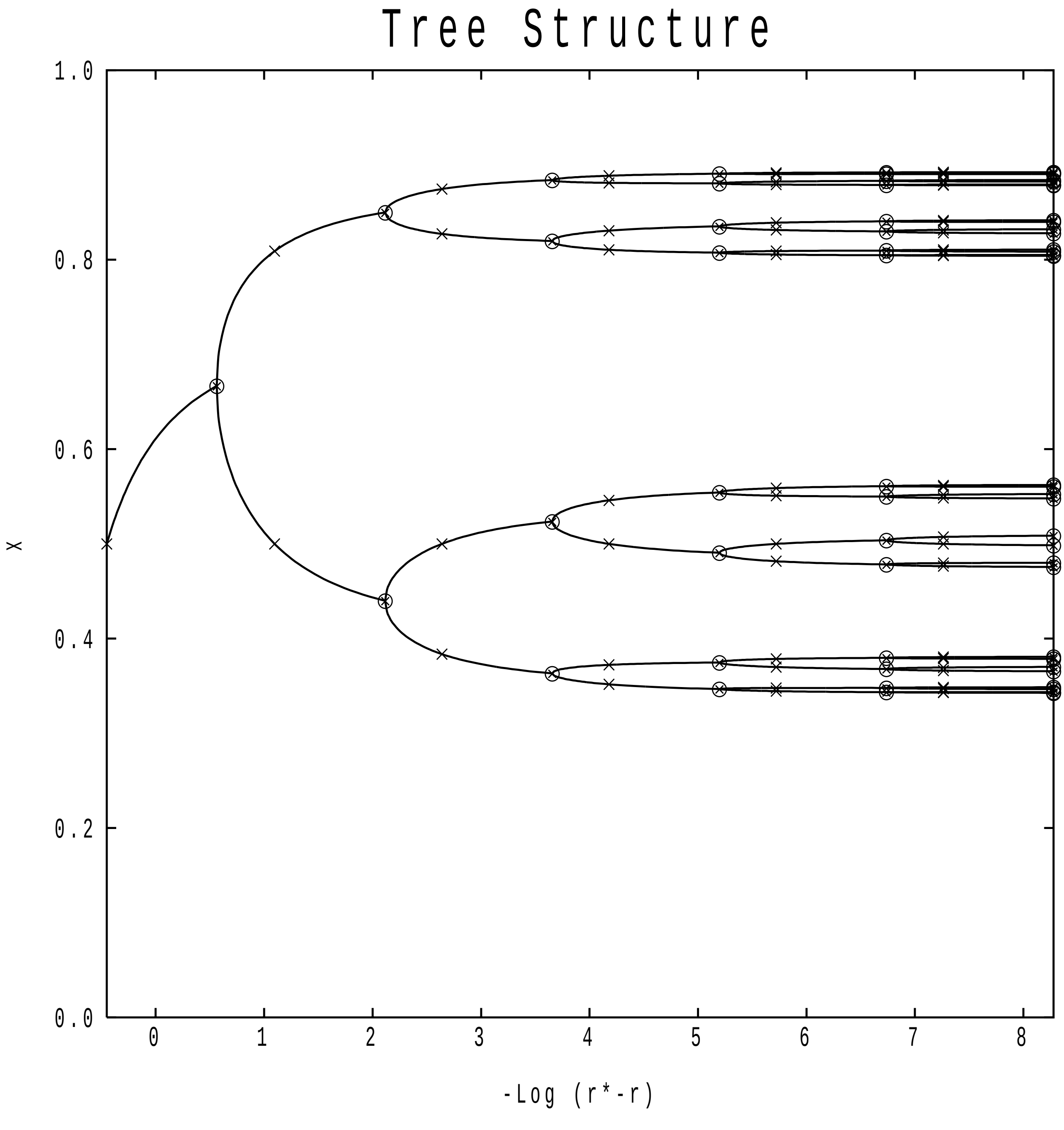
as r is smoothly varied between the appearance of the cycle and the loss of its stability. The clustering of the points as r is increased is an indication that the number of bifurcations might diverge in a finite range of r, but now we need to finite-size scale to try to verify this possibility. To apply this idea we need a guess for the behaviour that we can then try to fit and we guess that the behaviour is exponential

$$r_n = r_\infty - ab^n \quad \Rightarrow \quad \ln[r_\infty - r_n] = \ln a + n \ln b$$

and then a non-linear fitting procedure will provide a best estimate for r_∞ and the linear fit for the sequence of transitions can be examined. We worked up to 131072-cycles



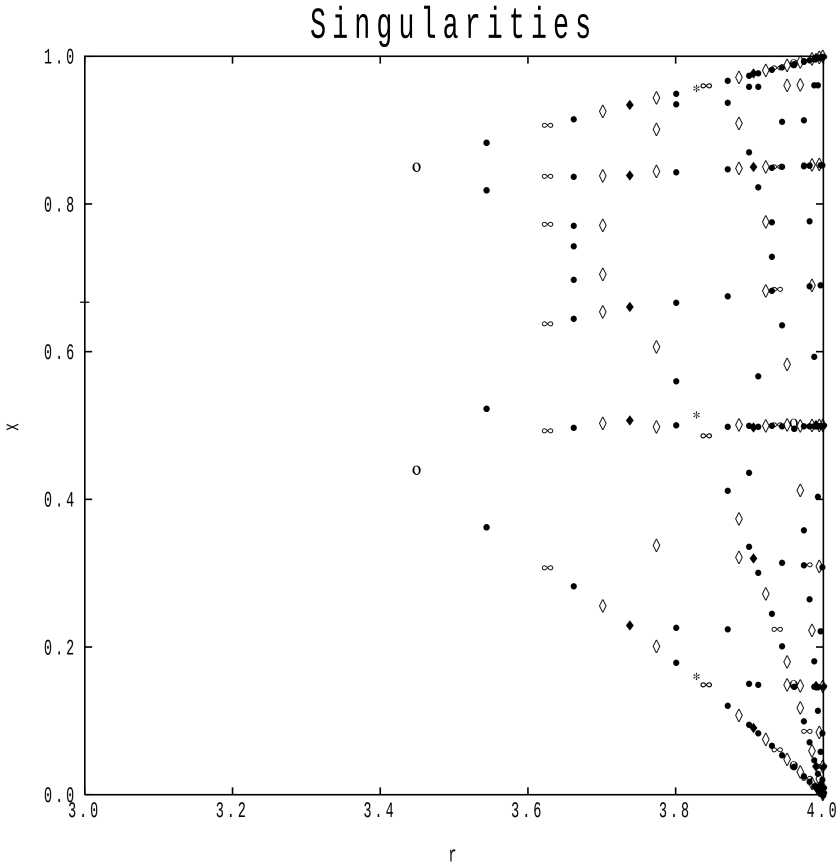
and clearly we see that r_∞=3.569945671870... yields an impressively linear fit giving excellent evidence that the route to chaos occurs in a finite range of control parameter. The bifurcations may then be redrawn on a logarithmic scale showing the bifurcations on an equal footing



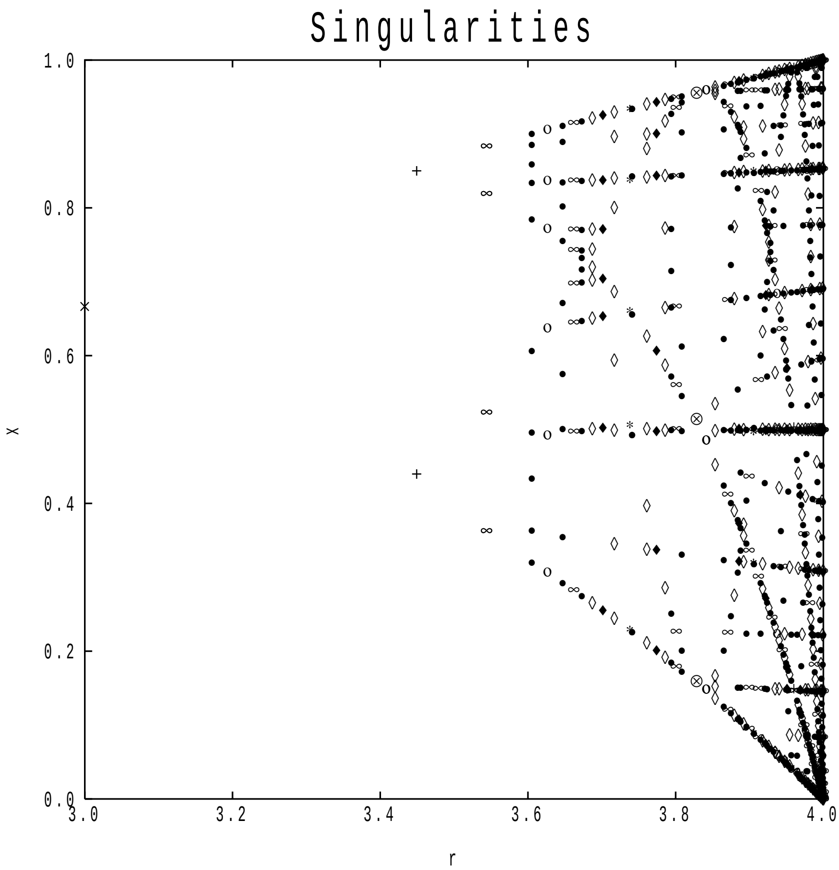
We will return to this exponential convergence later on in the course.

The windows of periodic n-cycles: numerical

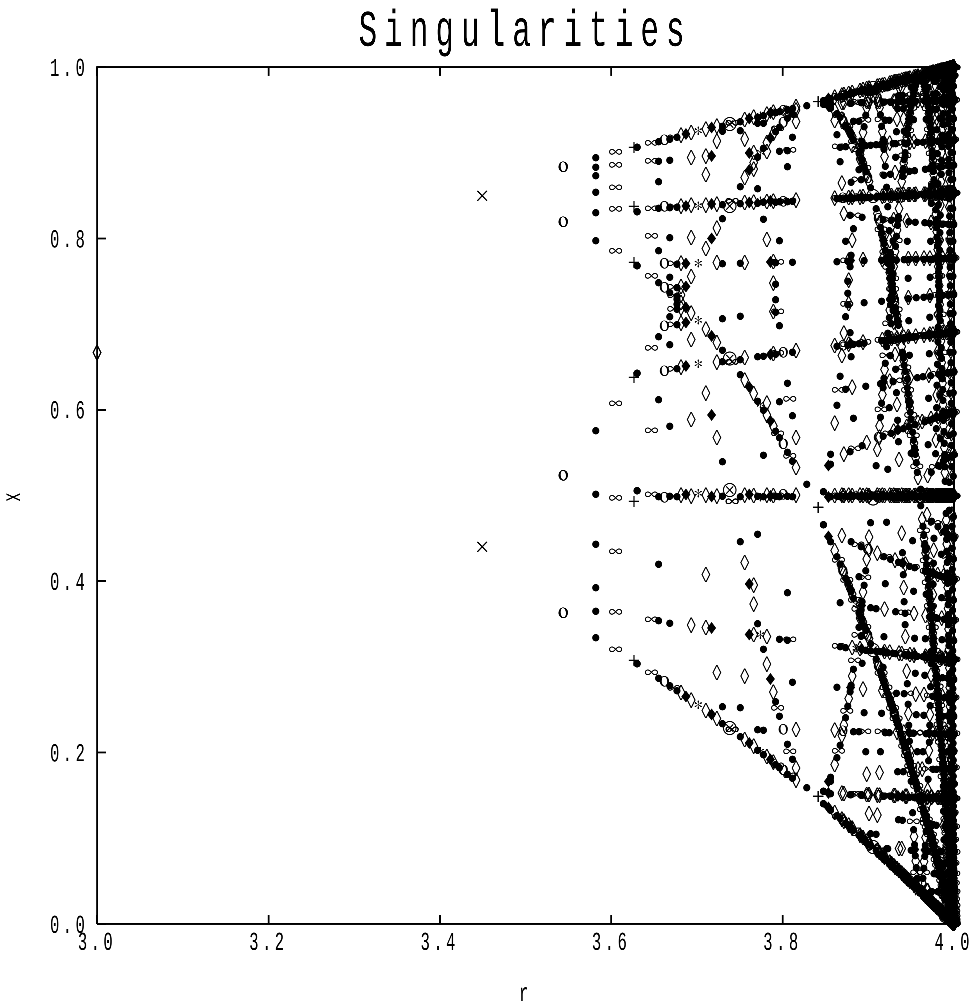
As well as the initial bifurcation route to chaos, we may also look at the higher lying N-cycles using the computer, providing the positions of their appearance accurately. We use the same ideas as for the 2^n-cycles but for general N-cycles and calculate the examples: for up to 8-cycles



for up to 10-cycles



and for all cycles up to 12-cycles. There is clearly structure and patterns in



these plots that we will not investigate.

A countably infinite sequence of bifurcations which occur for a finite interval of the control parameter which ends with chaotic behaviour is known as the 'bifurcation route to chaos'. The natural picture is then that n-cycles appear from the chaos and then disappear again from this infinite sequence of bifurcations. This picture is certainly consistent with the numerical calculations, but the inability to separate very high order n-cycles from chaos leads to some obvious questions. Due to the very large number of n-cycles, how do we know that there is any chaos at all? It appears plausible that the end of one bifurcation sequence tree could be precisely the start of the appearance of a new high order n-cycle and that we simply have a contiguous sequence of n-cycles of all orders appearing sequentially. To investigate this issue we need to appreciate how this question might be asked about mathematically and this requires a little bit of pure mathematics; counting and measure.

Countable and uncountable

Counting is straightforward when you have a finite set of objects to count but becomes more subtle when there are an infinite number of objects which need counting.

There is more than one type of infinity of which the first two are associated firstly with countable, ie possible to be listed in an infinite list, and secondly the number of real numbers between zero and unity, which interestingly, cannot be listed! These issues are surprisingly important in the understanding of chaos.

Clearly the positive integers are countable since listing them in order provides the required list. One might question whether the set of all the integers, both positive and negative, is countable because when listing them in order they go off to infinity in both directions, but if we organise them in the list

$$0 \;\; 1 \;\; -1 \;\; 2 \;\; -2 \;\; 3 \;\; -3....$$

we can see that we miss out none of them and so the set of all the integers is countable. Even though the positive integers are a subset of all the integers, we would say that there are the same number for both sets; a bit weird but the best mathematical choice.

There is an important theorem in counting; 'a countable set of countable sets is countable'. To prove this we can annotate all the elements with

$$\begin{array}{ccccc} a_1 & b_1 & c_1 & d_1 & ... \\ a_2 & b_2 & c_2 & d_2 & ... \\ a_3 & b_3 & c_3 & d_3 & ... \\ . & . & . & . & ... \end{array}$$

where $\{a_n\}$ is the first countable set, $\{b_n\}$ is the second countable set and we are listing all the sets in a horizontal list. We can then list all of the elements of all of the sets in a single list

$$a_1, b_1, a_2, c_1, b_2, a_3, d_1, c_2, b_3, a_4, e_1, d_2, c_3, b_4, a_5, ...$$

and so on. Clearly we miss out none of the objects so this set is countable. This, in turn, allows us to count the rational numbers, which can be annotated (with some double counting) using n/m

n/m	1	2	3	4	5	...
1	1/1	2/1	3/1	4/1	5/1	...
2	1/2	2/2	3/2	4/2	5/2	...
3	1/3	2/3	3/3	4/3	5/3	...
4	1/4	2/4	3/4	4/4	5/4	...
5	1/5	2/5	3/5	4/5	5/5	...
.	.	.	.	.	.	...

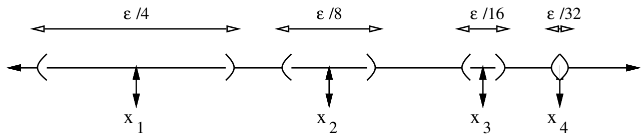
which is a countable set of countable sets and is hence countable. Interestingly, it is not possible to count the real numbers between zero and unity. To demonstrate this we will use measure.

Measure: Length in one-dimension

Length is an everyday concept that requires a third year undergraduate course in mathematics to properly appreciate. Here we will use some of the obvious properties of length in order to make some obvious deductions. The first deduction that we will make is that all countable sets have no finite length. To do this we employ a geometric series

$$1 = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + ... = \sum_{n=1}^{\infty} \frac{1}{2^n}$$

that we previously used for base 2. We may find a set of arbitrarily small length which contains the countable set of real numbers. We take our arbitrarily small length, ϵ say and multiply the geometric series by this number. This breaks the length up into a countable number of smaller lengths, $\frac{\epsilon}{2^n}$, for $n = 1, 2, 3....$ For any countable set of numbers, we may create a set of length less than or equal to ϵ which contains the countable set by simply taking each smaller length, $\frac{\epsilon}{2^n}$, in sequence, and putting that length down on the real line centred on the point in our countable set. Pictorially



and the set below has less than or equal length ϵ and clearly contains the countable set. We get a set which may be of length less than ϵ because some of the sets could overlap. If the length of the countable set exists, then it must be zero because we can always find a set with a length that is smaller than any proposed length which still contains the countable set. The length of any countable set vanishes.

The first major use of this result is to appreciate the conceptual horror of the relationship between real numbers and rational numbers. Rational numbers are both countable and *dense* in the real numbers. Dense is a term that means that for every real number, however close we get to that real number there are always rational numbers which are closer. Using our base 2 representation, order by order, we get a sequence of rational numbers which get arbitrarily close to the real number in question; the rational numbers are dense in the real numbers.

When we apply the cover of the rational numbers with a set of length ϵ we end up with a disturbing but crucial picture: The real line ends up speckled with tiny sets. If we pick any point these sets cluster around this point and there are no clear gaps without speckles. All pairs of points have a countably infinite number of speckles between them, so there is no smoothness. The most disturbing aspect, however, is that almost none of the real numbers are covered by this set, because the length of the real line is sizable but the length of the speckles in total is arbitrarily small. This picture appears very unnatural to us but is mathematically crucial for understanding chaos; known as ‘reals in rationals’.

The logistic map: ‘reals in rationals’

We stopped looking at the solutions to the logistic map because we ran into conceptual difficulties. We now have the mathematical tools to understand the answer. The speckles of the previous section are real finite sets when we think about the logistic map. The two styles of behaviour are, firstly, regular motion with pure n-cycles as the attractor and, secondly, chaotic motion that we still do not have a picture for. We can find the set of numbers for which the 1-cycle is stable; this makes a first regular set. For $r \in [2,3]$ the 1-cycle is stable, which makes up the piece of $r \in [2,4]$ which has a stable 1-cycle. There is a single region where the 2-cycle is stable, for $r \in (3, 1+\sqrt{6}]$, which makes a second regular set, connected to the first. Although there are two 3-cycles, only one is stable and has a brief period of stability starting at $r=1+\sqrt{8}$, the third regular set disconnected from the first two. For 4-cycles there are three, one which is connected to the 2-cycle and two which appear as a pair, one of which is briefly stable, two new regular sets, one connected to the previous cluster and one disconnected from the previous sets. Clearly we can investigate each n-cycle in turn and find new regular sets. For 5-cycles we find six which arrive as three pairs with one of each pair being stable for a brief period; three new disconnected regular sets. As we go to higher and higher order we get more and more regular sets. For even values of n we get sets which connect to previous sets, extending them, and other pairs of cycles, one of which is stable and provides a new regular set.

The set of regular sets is countable: For each value of n there are a finite number of n-cycles and so we can count them in order. Although we cannot find the length of these sets analytically in general, we can find the length numerically and then we can add up all the lengths of these sets and estimate numerically whether or not it converges to a limit (which of course it does) and then find out if there is any length left over unaccounted for. When we pursue this line of argument, we find that once the first chaotic point occurs, then only a very small fraction of the remaining region of r corresponds to regular behaviour. In this remaining region the speckles of regular n-cycles are dense and so we pretty much have chaos and a ‘reals in rationals’ picture for how the regular behaviour is interspersed with the chaotic behaviour.

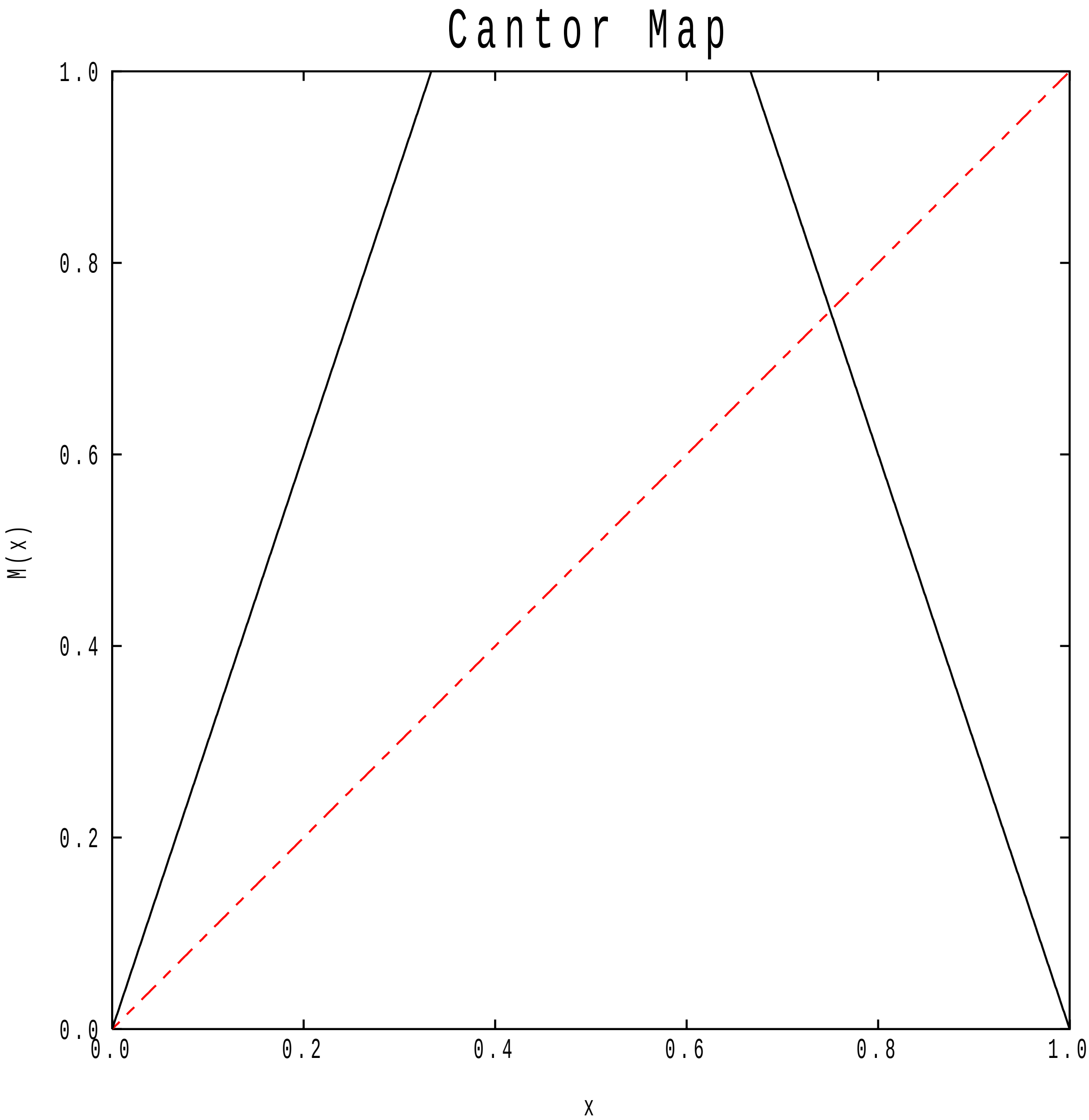
This is one of the disturbing aspects to chaos; the lack of continuity. If we pick a value for the control parameter at random, $r \in (3.6, 4)$, then only rarely does it provide a regular n-cycle. If this happens then there is a local region surrounding that point where the n-cycle is stable; continuity. Usually, however, the behaviour is chaotic and never repeats. When this happens there is no continuity with both regular and chaotic behaviour interspersed however close we get to the chaotic point.

Fractals and strange attractors

Our last task here is to picture the chaotic behaviour and for this we need to appreciate what a fractal is; because the attractor at a chaotic point tends to be a fractal; a ‘strange attractor’. We will introduce these ideas through base 3 and the Cantor set.

The Cantor map

The logistic map has chaos which is too subtle to control, so we need a simpler problem to commence our intuition and understanding. The Cantor map, $C(x)$, provides a simple intuitive picture



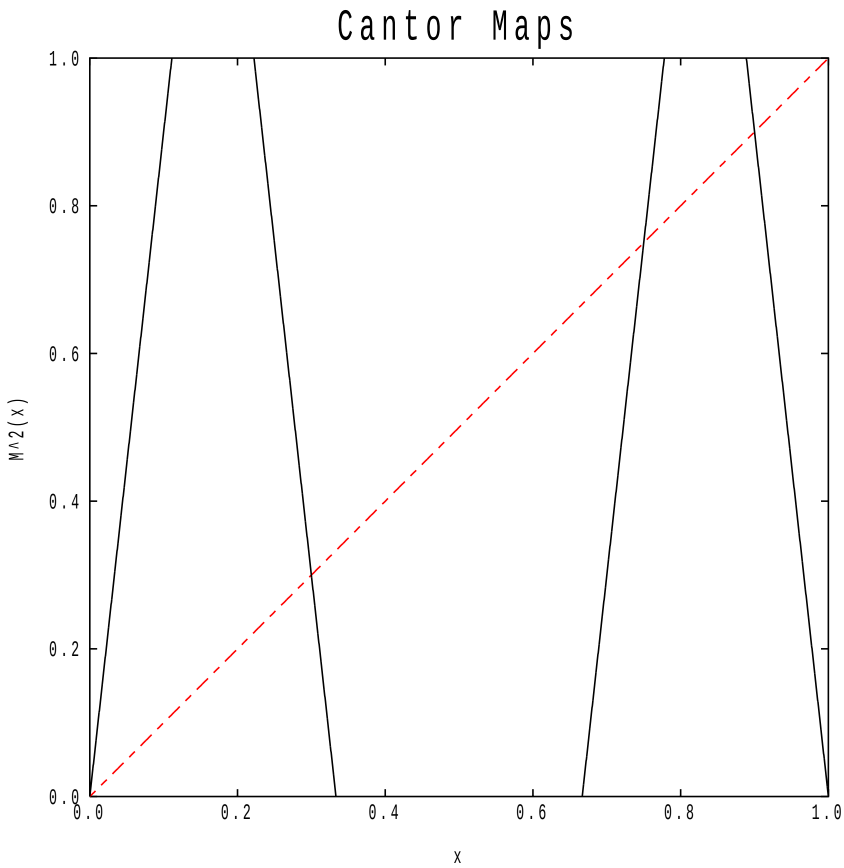
This map is similar to the tent map. For the first region and last region we have a linear rescaling but for the middle region all points map to unity

$$C(x) = 3x \qquad x \in \left[0, \frac{1}{3}\right]$$

$$C(x) = 1 \qquad x \in \left[\frac{1}{3}, \frac{2}{3}\right]$$

$$C(x) = 3(1-x) \qquad x \in \left[\frac{2}{3}, 1\right]$$

As with the tent map we are most interested in the behaviour after many applications of the map and what the attractor is. We now know how to investigate n-cycles; multiple maps are instructive. We can investigate $x = C(x)$ to find 1-cycles. We can investigate $x = C^{(2)}(x)$ to find 2-cycles and we can investigate $x = C^{(n)}(x)$ to investigate general n-cycles. The form of $C^{(2)}(x)$



and higher order maps tells us that the number of solutions is completely equivalent to what happened in the tent map, so the number of n-cycles of all orders is the same. Unlike the tent map, for this map most of the points get mapped onto zero eventually, which is one of the two 1-cycles. How much of the original length gets mapped onto zero eventually? This problem is all about 'middle thirds'. The middle third of the original set $x \in [0, 1]$ gets mapped onto zero after two applications of the map. The two middle thirds of the two end thirds, which are actually ninths, get mapped onto zero after three applications of the map. This leaves us with four active regions, the first, third seventh and ninth, ninths. The four middle thirds, which are actually twenty-sevenths, get mapped onto zero after four applications of the map. This process proceeds, order by order, and so the total length that is mapped onto zero is

$$\frac{1}{3} + 2\left(\frac{1}{9}\right) + 4\left(\frac{1}{27}\right) + 8\left(\frac{1}{81}\right) + ... = \frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{2}{3}\right)^n = \frac{\frac{1}{3}}{1-\frac{2}{3}} = 1$$

and length-wise this is the whole line. Are there any points left which are not mapped onto zero? To answer this we use base 3, just like we used base 2 to control the tent map.

In base 3, sometimes called ternary, we represent fractions as

$$a_{\bar{1}} \times \frac{1}{3} + a_{\bar{2}} \times \frac{1}{3^2} + a_{\bar{3}} \times \frac{1}{3^3} + ... \equiv 0.a_{\bar{1}}a_{\bar{2}}a_{\bar{3}}...$$

where now $a_n \in \{0,1,2\}$. Multiplying by three translates all the digits forward by one, analogous to binary and denary. We still have some ambiguity, because

$$0.2222... = 2 \times \frac{1}{3} + 2 \times \frac{1}{3^2} + 2 \times \frac{1}{3^3} + 2 \times \frac{1}{3^4} + ... = 2\sum_{n=1}^{\infty} \frac{1}{3^n} = \frac{\frac{2}{3}}{1-\frac{1}{3}} = 1$$

and we have two representations for the same number. We can now deduce that

$$1 - 0.a_{\bar{1}}a_{\bar{2}}a_{\bar{3}}... = 0.222... - 0.a_{\bar{1}}a_{\bar{2}}a_{\bar{3}}... = 0.(2-a_{\bar{1}})(2-a_{\bar{2}})(2-a_{\bar{3}})...$$

For the first third, $x \in \left[0, \frac{1}{3}\right]$, the ternary numbers take the form $0.0a_{\bar{2}}a_{\bar{3}}a_{\bar{4}}...$ and $C(x)$=$3x$. For the second third, $x \in \left[\frac{1}{3}, \frac{2}{3}\right]$, the ternary numbers take the form $0.1a_{\bar{2}}a_{\bar{3}}a_{\bar{4}}...$ and $C(x)$=1. For the final third, $x \in \left[\frac{2}{3}, 1\right]$, the ternary numbers take the form $0.2a_{\bar{2}}a_{\bar{3}}a_{\bar{4}}...$ and $C(x)$=3(1-x). In ternary we have

$$C[0.0a_{\bar{2}}a_{\bar{3}}a_{\bar{4}}...] = 0.a_{\bar{2}}a_{\bar{3}}a_{\bar{4}}...$$

$$C[0.1a_{\bar{2}}a_{\bar{3}}a_{\bar{4}}...] = 1 = 0.2222...$$

$$C[0.2a_{\bar{2}}a_{\bar{3}}a_{\bar{4}}...] = 0.(2-a_{\bar{2}})(2-a_{\bar{3}})(2-a_{\bar{4}})...$$

and when we apply $C(x)$ we lose the first digit and then either we are mapped onto unity or the digits move forward by one and in one case the digits are conserved and in the other the digits are mapped onto their complements; $d \mapsto 2-d$. This complement conserves digits which are unity and turns zeroes into twos and twos into zeroes.

In ternary we can easily find the set of numbers that eventually map to zero, these are the numbers for which at least one of the digits is unity. Every application of $C(x)$ moves the digits forward by one, conserving the units, and when the first time a unit reaches the head of the number the number is mapped onto unity followed by being mapped onto zero. The numbers that do not map onto zero are then the numbers that are represented as strings of zeroes and twos. The theory of finding the n-cycles is then equivalent. For 1-cycles

$$0.0000... \mapsto 0.0000... \qquad 1-cycle$$

$$0.2020... \mapsto 0.2020... \qquad 1-cycle$$

for 2-cycles

$$0.0000... \mapsto 0.0000... \qquad 1-cycle$$

$$0.2020... \mapsto 0.2020... \qquad 1-cycle$$

$$0.0220... \mapsto 0.2200... \qquad 2-cycle$$

$$0.2200... \mapsto 0.0220... \qquad 2-cycle$$

for 3-cycles

$$0.000000... \mapsto 0.000000... \qquad 1-cycle$$

$$0.020020... \mapsto 0.200200... \qquad 3-cycle$$

$$0.200200... \mapsto 0.220220... \qquad 3-cycle$$

$$0.220220... \mapsto 0.020020... \qquad 3-cycle$$

$$0.002220... \mapsto 0.022200... \qquad 3-cycle$$

$$0.022200... \mapsto 0.222000... \qquad 3-cycle$$

$$0.222000... \mapsto 0.002220... \qquad 3-cycle$$

$$0.202020... \mapsto 0.202020... \qquad 1-cycle$$

which is equivalent to the tent map calculation. We can find the actual points as fractions using multiplication by powers of three

$$9 \times 0.2020... = 20 + 0.2020... = 6 + 0.2020... \quad \Rightarrow \quad 0.2020... = \frac{6}{8} = \frac{3}{4}$$

$$81 \times 0.02200220... = 220 + 0.02200220... = 24 + 0.02200220... \quad \Rightarrow$$
$$0.02200220... = \frac{24}{80} = \frac{3}{10}$$
$$81 \times 0.22002200... = 2200 + 0.22002200... = 72 + 0.22002200... \quad \Rightarrow$$
$$0.22002200... = \frac{72}{80} = \frac{9}{10}$$

and so on. When dealing with numbers that have alternating strings of digits and their complements, we can also use the idea that 1-x provides us with the complements to simplify the conversion to a fraction. For example, in binary

$$x = 0.10001\dot{1}10001\dot{1}$$

can be analysed either by multiplying by 2^6 or by 2^3

$$2^6x = 100011 + x \quad 2^3x = 100 + 1 - x \quad \Rightarrow \quad 63x = 32+2+1 \quad 9x = 4+1 \quad \Rightarrow \quad x = \frac{5}{9}$$

For example in ternary

$$x = 0.01221\dot{0}01221\dot{0}$$

can be analysed by either multiplying by 3^6 or by 3^3

$$3^6x = 12210 + x \quad 3^3x = 12 + 1 - x \quad \Rightarrow \quad 728x = 81 + 2\times 27 + 2 \times 9 + 3 \quad 28x = 3+2+1$$

$$\Rightarrow \quad x = \frac{3}{14}$$

Clearly the use of 1-x is numerically easier. We could also use

$$x = 3(1-x) \quad \Rightarrow \quad x = \frac{3}{4}$$

and

$$y = 3x \qquad x = 3(1-y) \quad \Rightarrow \quad x = 3(1-3x) \quad \Rightarrow \quad x = \frac{3}{10} \quad y = \frac{9}{10}$$

This is completely analogous to the tent map. For the tent map the n-cycles are countable and so almost all of the behaviour is chaotic and irregular. For the Cantor map almost all of the points map to zero but there are the same number of n-cycles of all orders as there are for the tent map, but how about the chaotic behaviour, is there any such behaviour for the Cantor map?

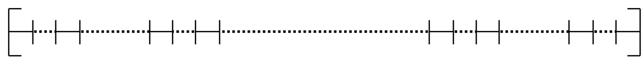
The points which do not map to zero have a ternary representation which involves only zeroes and twos. We can map these points one to one onto the real numbers between zero and unity. We simply map the numbers composed of zeroes and twos onto the binary representation of real numbers

$$0.22020002022... \mapsto 0.11010001011...$$

and we have the same number of chaotic points in both cases; the chaotic points are dominant over the periodic solutions but neither have any measure for the Cantor map. These chaotic solutions form a fractal.

Fractals

Fractals are self-similar stuctures with a dimension which is less than the dimension of the space in which they live but is usually between this dimension and the previous dimension and is greater than the dimension of the creature forming the object. The Cantor map provides such a self-similar fractal. If we picture the set of active points, by colouring in all the points which map onto zero, then we get the picture (only up to third order)



where the middle thirds are recurrently extracted for ever. The self-similar observation is that if we look at the first third, then it is a scaled version of the whole object, scaled down by a factor of one third. If we assume that, in the regions where there is mass, $M \propto L^d$, and that there is some dimension, d, then as $L \mapsto \frac{L}{3}$ then $M \mapsto \frac{M}{2}$ and we see that

$$M = \lambda L^d \quad \Rightarrow \quad \frac{M}{2} \mapsto \lambda \left(\frac{L}{3}\right)^d$$

and then taking the ratio

$$3^d = 2 \quad \Rightarrow \quad d = \frac{\ln 2}{\ln 3}$$

and we have a fractal dimension which is less than unity. In fact, in general chaotic systems, fractal dimension is a subtle concept; in most cases there is more than one fractal dimension that can be defined. For a standard dissipative chaotic system the attractor tends to be a fractal and the label 'strange attractor' has been coined to describe them. There is one final concept which is worthy of note and that is *ergodicity.*

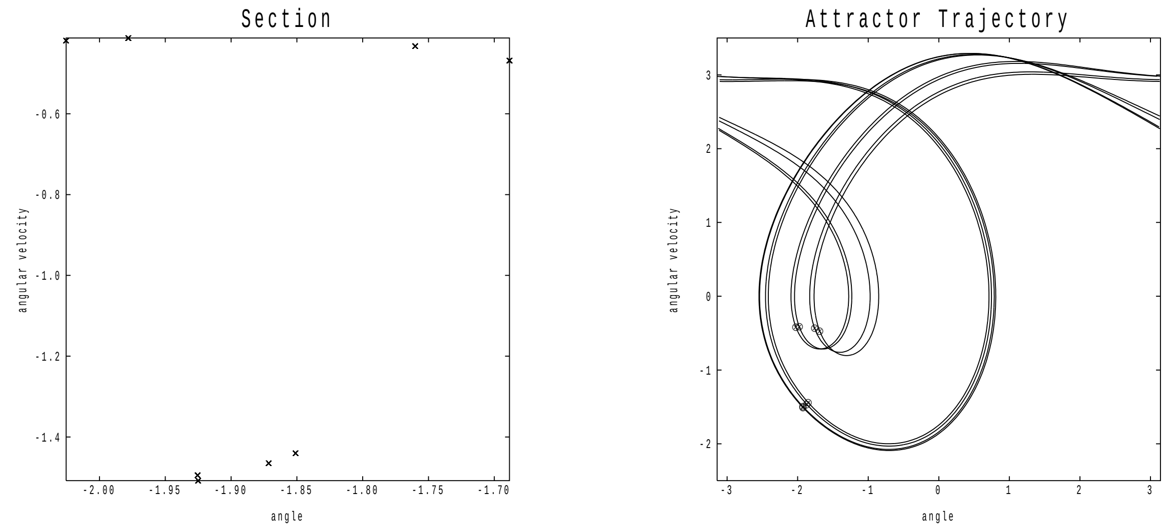
Ergodicity

A trajectory is said to be ergodic if it approaches arbitrarily close to all the points in phase space. If you wait long enough a trajectory will do every possible behaviour that is accessible to it with arbitrary precision.

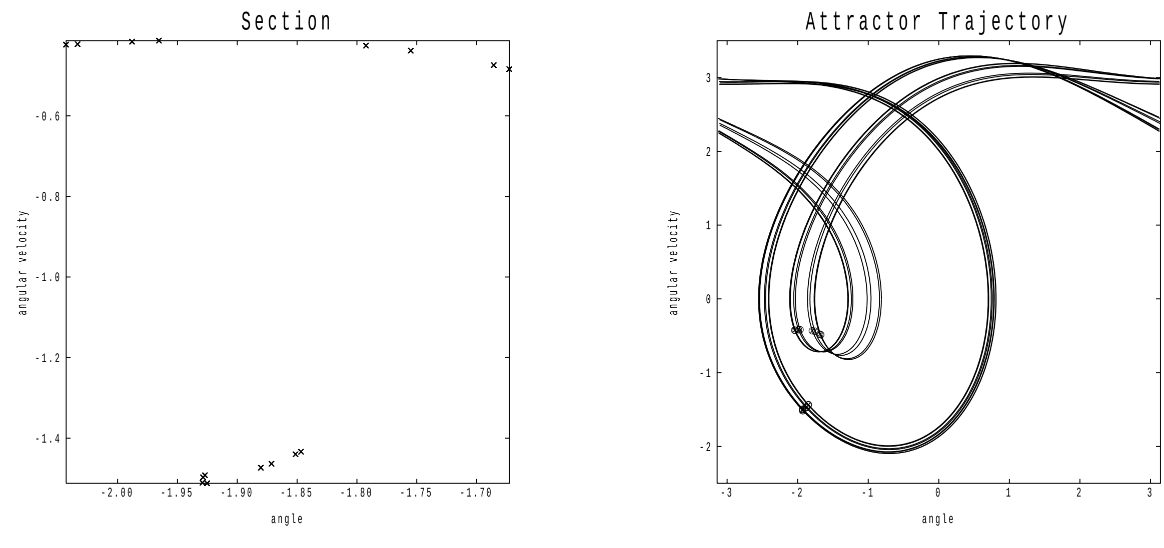
In fact our tent map gives an excellent example of this behaviour. If we select a random number then the digits in the binary representation are random. If we further select a target number then we can find a number arbitrarily close to our chosen number by demanding that the digits agree up to some order. When we hunt along the digits of our random number there is a non-zero probability of finding zero followed by our target approximation or alternatively one followed by the complement of our target approximation. When we apply the tent map sequentially to our random number, every time the zero reaches the front of the number or the one reaches the front of the number, the next application of the tent map maps the number to as close to out targeted number as the targeted accuracy required. As our random number is mapped, it becomes arbitrarily close to any and all real numbers if we apply the map enough times; this is ergodic behaviour and a good intuitive picture of chaotic motion.

The forced damped pendulum: Part two

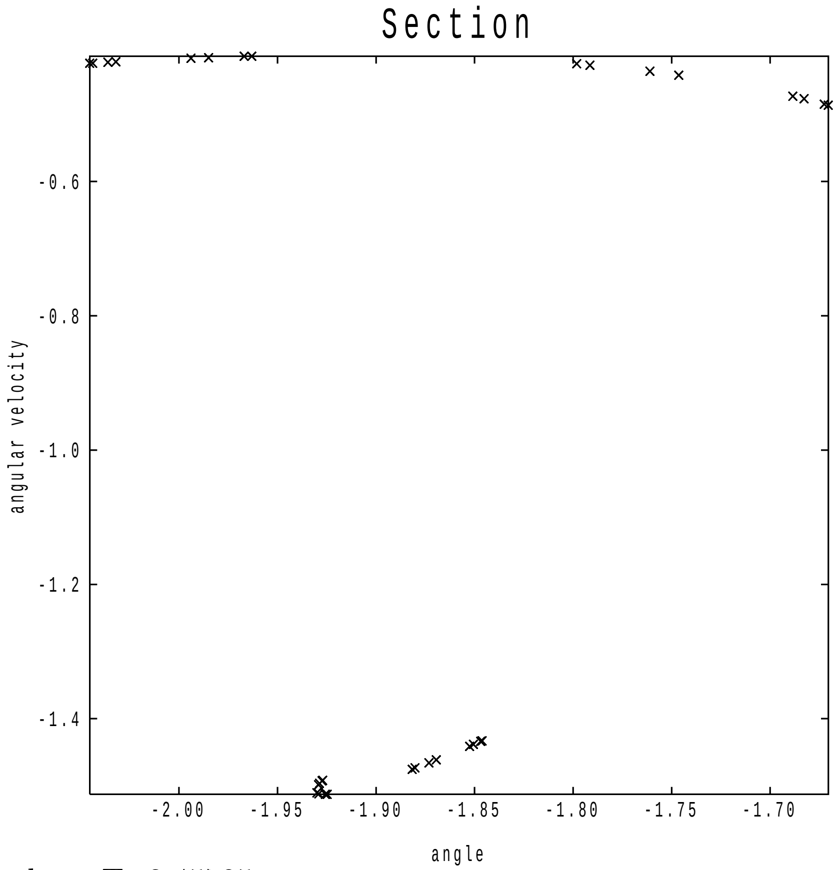
The logistic map provided us with the bifurcation route to chaos and the previous analysis provided us with a 2-cycle and a 4-cycle. Does this system follow the bifurcation route to chaos? When we continue increasing the amplitude of the forcing term, T, we can create an 8-cycle with T=2.455



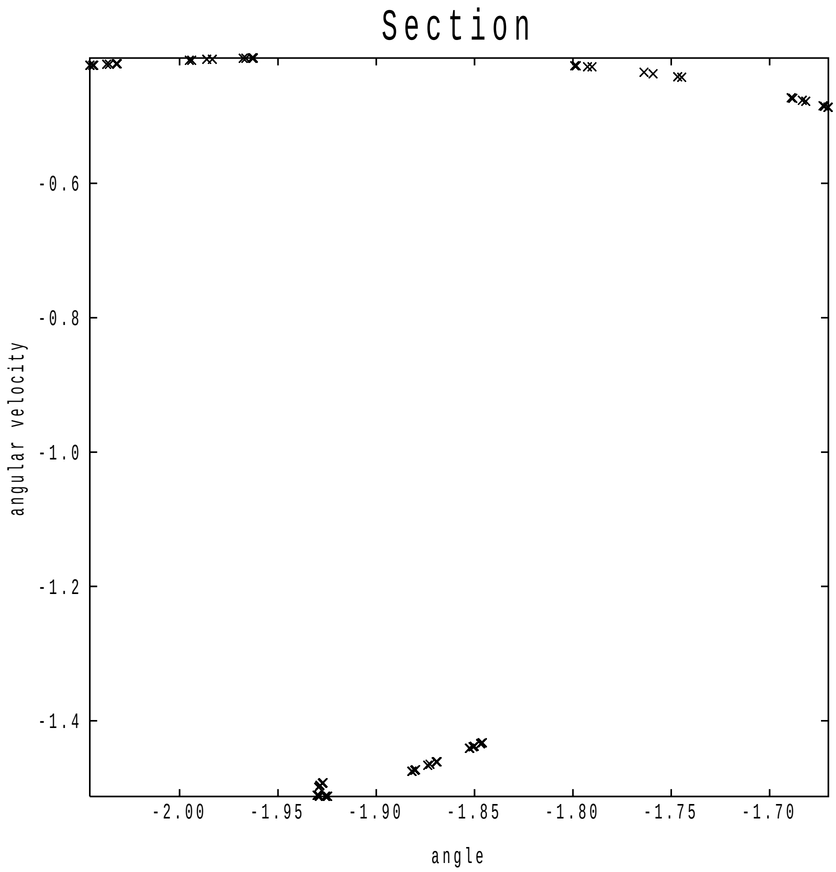
a 16-cycle with T=2.4575



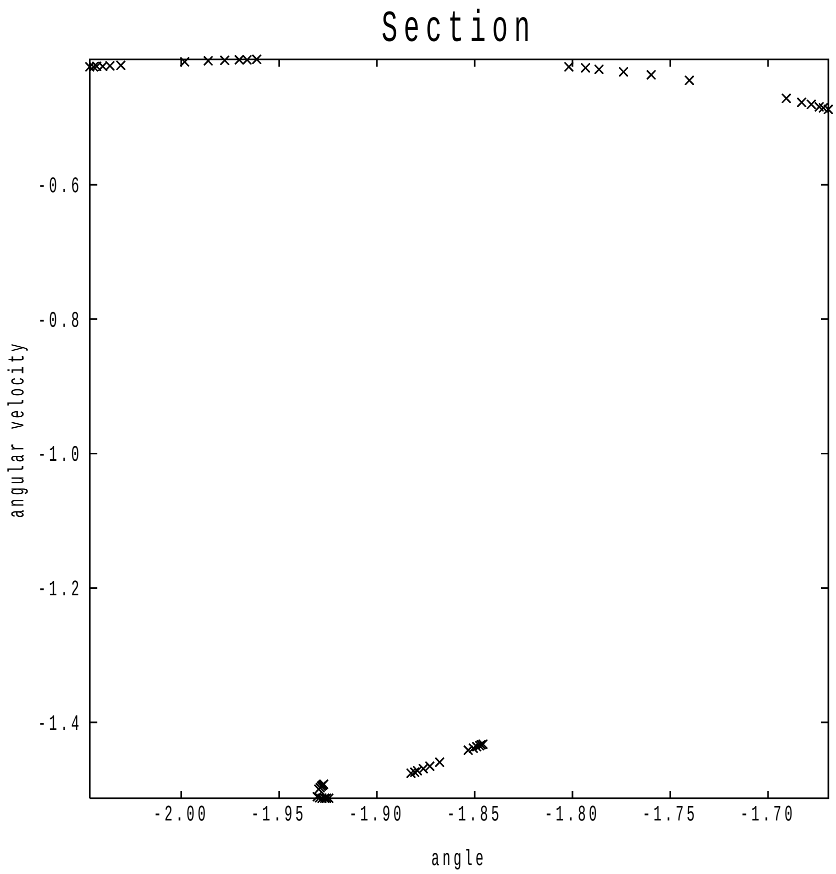
and even a 32-cycle at T=2.4578



and a 64-cycle at T=2.45785



It is much more difficult to analyse this model than the logistic map but the behaviour appears analogous and it is believed that these two systems are in a particular sense equivalent; *renormalisation.* The case of T=2.458 provides a 48-cycle



so the picture of brief stability of n-cycles in a sea of chaos looks an excellent description for the behaviour of this system.

Hamiltonian chaos

In dissipative chaos we focus on the attractor. Although regular behaviour can be observed, such as fixed points and n-cycles, there is also the possibility of chaos. Chaos manifests as fractal sets for the attractor; *strange attractors.* Hamiltonian systems are different, since all points in phase space are available at all times; the long-time limit is not special for Hamiltonian systems. All we know at the moment, is that energy is conserved and so each trajectory winds around in a (d-1)-dimensional subspace of d-dimensional phase space. It is the trajectory itself which exhibits the chaos.

To appreciate Hamiltonian chaos we need to understand ordinary non-chaotic systems; *integrable systems.* A simple example is the d-dimensional harmonic oscillator with $2d$ equations of motion

$$\frac{dx_n}{dt} = \frac{p_n}{m} \qquad \frac{dp_n}{dt} = -k_n x_n$$

where x_n are the components of position, p_n are the components of momentum, m is the mass but we apply a different linear force in each of the dimensions, with a spring constant k_n. The energy is

$$E = \frac{1}{2m}\left(p_1^2 + p_2^2 + ... + p_d^2\right) + \frac{1}{2}\left(k_1 x_1^2 + k_2 x_2^2 + ... + k_d x_d^2\right)$$

which forms a (2d-1)-dimensional phase space. This system has more conservation laws, however, since each energy for each individual component is conserved

$$E_n = \frac{p_n^2}{2m} + \frac{k_n x_n^2}{2} \qquad \Rightarrow \qquad \frac{dE_n}{dt} = \frac{p_n}{m}\frac{dp_n}{dt} + k_n \frac{dx_n}{dt} = 0$$

and so there are d conservation laws and each trajectory is restricted to a d-dimensional subspace of phase space; a high dimensional torus. A $2d$-dimensional system is called an integrable system if it has d independent conservation laws. The trajectories of such a system are very simple and correspond to winding around the d independent loops which create a d-dimensional torus in phase space. Clearly an integrable system does not explore the whole of the energy subspace. Chaotic systems explore subsets which are only one dimension less than the whole of phase space. The physical idea is that trajectories do almost everything that is energetically possible for them to do with arbitrary precision; if you wait long enough every possible thing that can happen does happen.

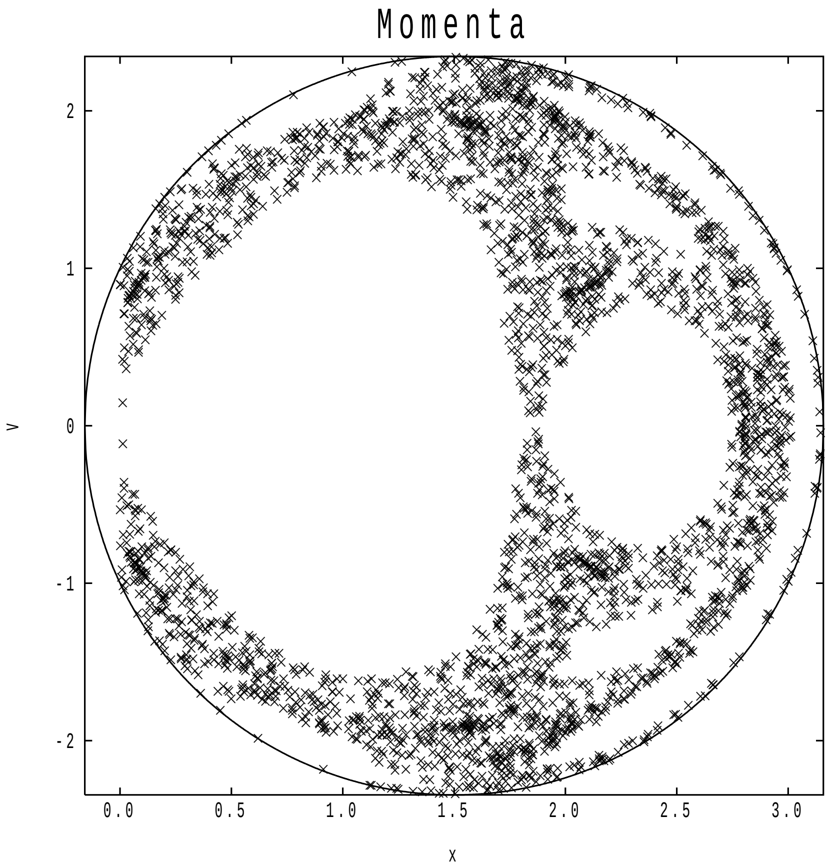
Ergodicity is the technical term for a trajectory that comes arbitrarily close to all possible behaviours available, subject to conservation of energy. Very few systems are fully ergodic but chaotic systems exhibit a weaker form of behaviour; mixed systems, where some of the trajectories are ergodic on a subspace of the energy surface but with the same dimension.

The sprung pendulum

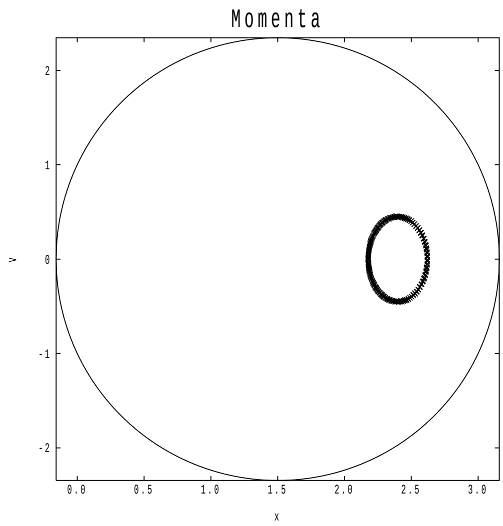
We can rescale the energy of the sprung pendulum, using time into

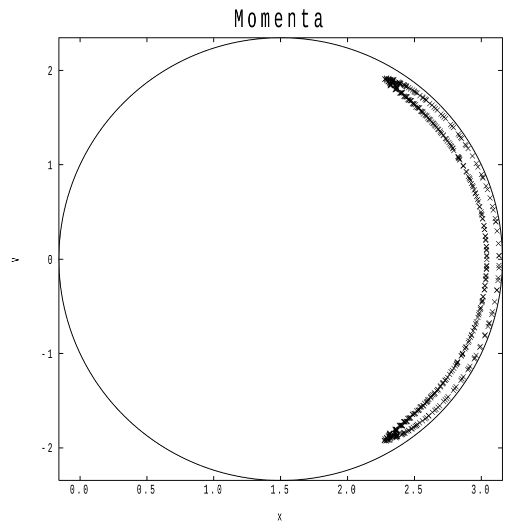
$$E = \frac{1}{2}\left[\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2\right] + \frac{1}{2}r(r - 2r_0\cos\theta - 2r_1)$$

and then our chosen Poincare section is θ=0, and the collision points are bounded by a circle. For an integrable system the trajectory is restricted to a two-dimensional subspace, a torus, and the intersection with the Poincare section is expected to be a loop. For an ergodic system we would expect the entire two dimensional subspace to be approached arbitrarily closely by the trajectory. Note that a trajectory is one dimensional and a section through a trajectory is a collection of points and possesses zero measure. To find a Poincare section we choose a point on the section and then use the energy to create the initial point in phase space. We then integrate the equations of motion using numerical algorithms and employ conservation of energy to assess numerical errors. Every time the trajectory collides with the section we add a symbol to create a scatter plot of where the trajectory intersects the section. A particular example is depicted

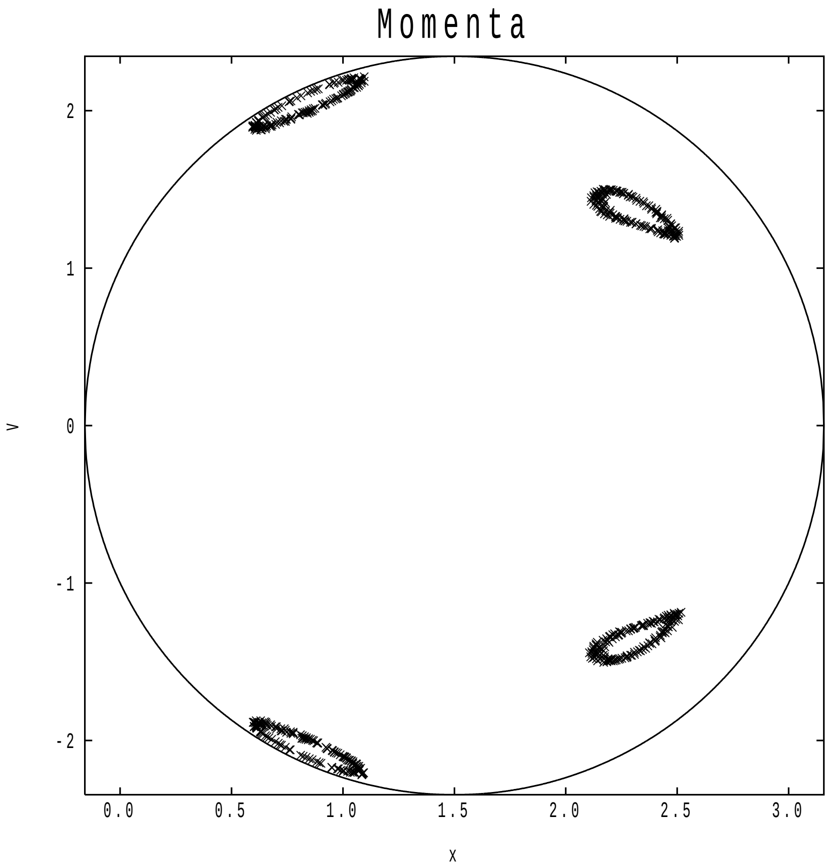


This example exhibits a space filling region together with some holes that the trajectory does not penetrate. This trajectory is chaotic. We can commence separate trajectories within these holes and we obtain new sections

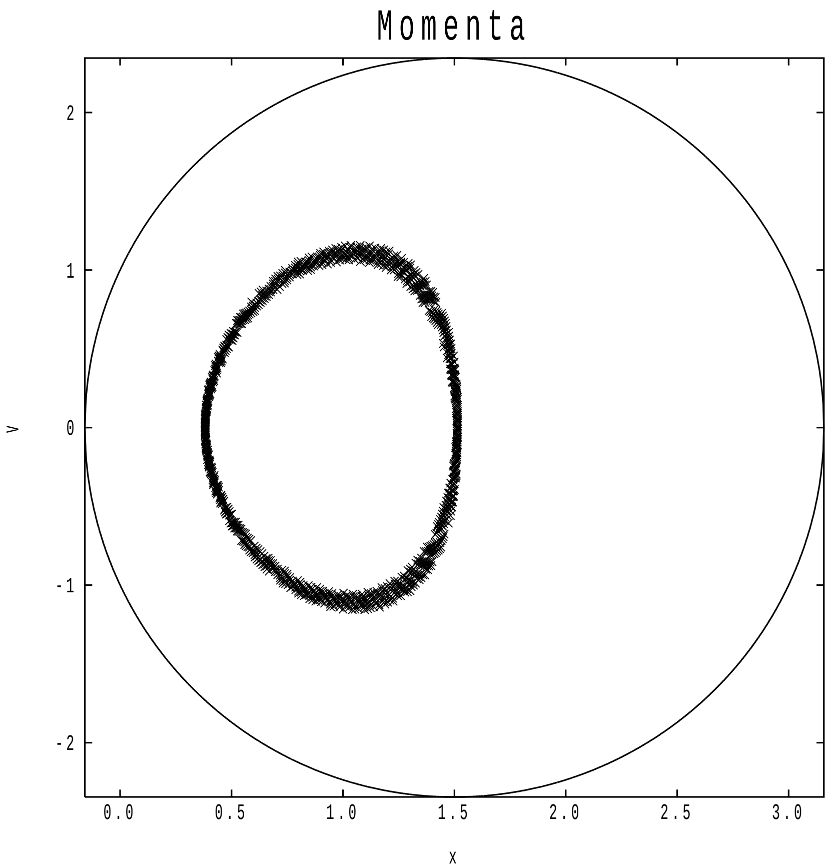




which provides a single loop and are clearly integrable trajectories. We also find a trajectory which provides several loops. We still view this as an integrable trajectory but the torus is now topologically quite complicated and winds around phase space in a more complicated analogue to a ‘figure of eight’ in two dimensions.



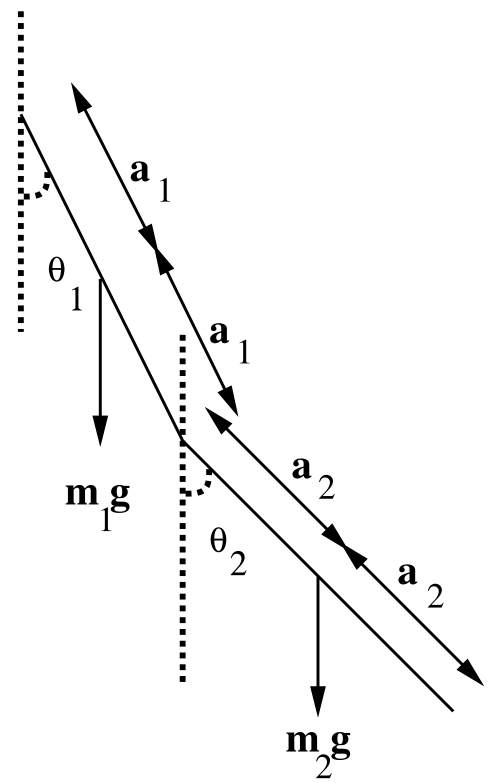
The final trajectory



looks like a 'fluffy loop' and we would still interpret this as an integrable trajectory but the topology of the torus is a bit mysterious for this trajectory.

The double pendulum

One of the most widely cited chaotic systems is the double pendulum



and this system is also susceptible to a Poincare analysis. The potential energy is simple

$$V = -m_1 g a_1 \cos\theta_1 - m_2 g\left(2a_1\cos\theta_1 + a_2\cos\theta_2\right)$$

but the kinetic energy is much more complicated

$$T = \frac{1}{2}\frac{1}{3}m_1 a_1^2\left(\frac{d\theta_1}{dt}\right)^2 + \frac{1}{2}\frac{1}{3}m_2 a_2^2\left(\frac{d\theta_2}{dt}\right)^2 + \frac{1}{2}m_1 a_1^2\left(\frac{d\theta_1}{dt}\right)^2$$
$$+\frac{1}{2}m_2\left[4a_1^2\left(\frac{d\theta_1}{dt}\right)^2 + a_2^2\left(\frac{d\theta_2}{dt}\right)^2 + 4a_1a_2\left(\frac{d\theta_1}{dt}\right)\left(\frac{d\theta_2}{dt}\right)\cos(\theta_1-\theta_2)\right]$$

where the first two terms are the rotational energy of the two rods about their centres of mass and the final two terms are the kinetic energies of the two centres of mass. The details are relegated to an appendix but the equations of motion are

$$\frac{d}{dt}\left[a\frac{d\theta_1}{dt} + c\frac{d\theta_2}{dt}\cos(\theta_1-\theta_2)\right] = -d\sin\theta_1 - c\frac{d\theta_1}{dt}\frac{d\theta_2}{dt}\sin(\theta_1-\theta_2)$$

$$\frac{d}{dt}\left[b\frac{d\theta_2}{dt} + c\frac{d\theta_1}{dt}\cos(\theta_1-\theta_2)\right] = -e\sin\theta_2 - c\frac{d\theta_1}{dt}\frac{d\theta_2}{dt}\sin(\theta_2-\theta_1)$$

where

$$a = \frac{1}{3}m_1 a_1^2 + m_1 a_1^2 + 4m_2 a_1^2 \qquad b = \frac{1}{3}m_2 a_2^2 + m_2 a_2^2$$

$$c = 2m_2 a_1 a_2 \qquad d = (m_1 + 2m_2)a_1 g \qquad e = m_2 a_2 g$$

and the energy is

$$E = \frac{1}{2}a\left(\frac{d\theta_1}{dt}\right)^2 + \frac{1}{2}b\left(\frac{d\theta_2}{dt}\right)^2 + c\frac{d\theta_1}{dt}\frac{d\theta_2}{dt}\cos(\theta_1 - \theta_2) - d\cos\theta_1 - e\cos\theta_2$$

and if we set θ_1=0 and $\frac{d\theta_1}{dt}$=0 we reduce to

$$E = \frac{1}{2}b\left(\frac{d\theta_2}{dt}\right)^2 - e\cos\theta_2$$

and a pendulum. If we set θ_1=θ_2 we get $\frac{d\theta_1}{dt}$=$\frac{d\theta_2}{dt}$ and

$$E = \frac{1}{2}(a + b + 2c)\left(\frac{d\theta_1}{dt}\right)^2 - (d + e)\cos\theta_1$$

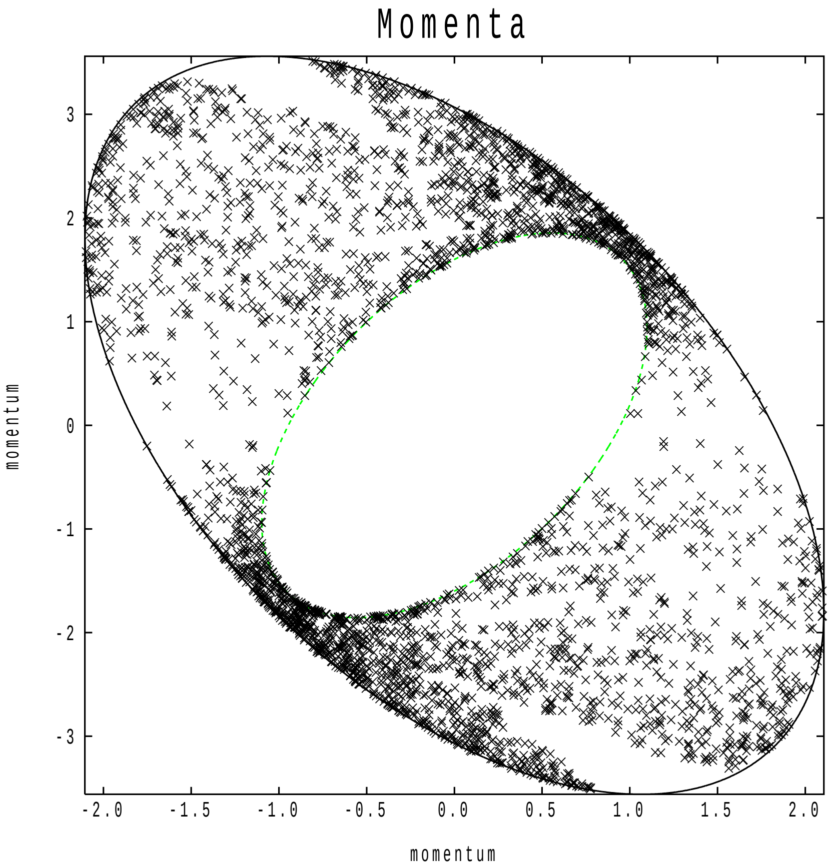
and another pendulum; as one would anticipate.

The natural Poincare section is θ_2=0, which corresponds to the lower pendulum pointing vertically down and then we can use the energy to determine a value for θ_1. This leaves a Poincare section in the two dimensional space of $\left(\frac{d\theta_1}{dt}, \frac{d\theta_2}{dt}\right)$. Once again the energy provides a restriction and the active region is bounded by the curves when $\cos\theta_1$=1 and $\cos\theta_1$=-1

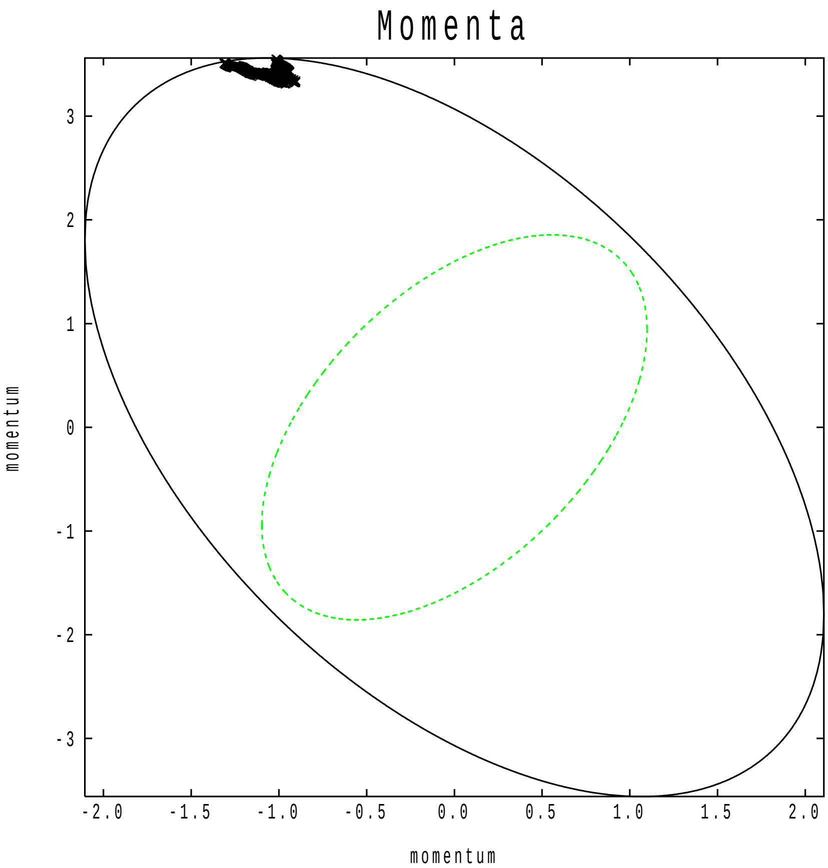
$$\frac{1}{2}a\left(\frac{d\theta_1}{dt}\right)^2 + \frac{1}{2}b\left(\frac{d\theta_2}{dt}\right)^2 + c\frac{d\theta_1}{dt}\frac{d\theta_2}{dt} = E + d + e$$

$$\frac{1}{2}a\left(\frac{d\theta_1}{dt}\right)^2 + \frac{1}{2}b\left(\frac{d\theta_2}{dt}\right)^2 - c\frac{d\theta_1}{dt}\frac{d\theta_2}{dt} = E - d + e$$

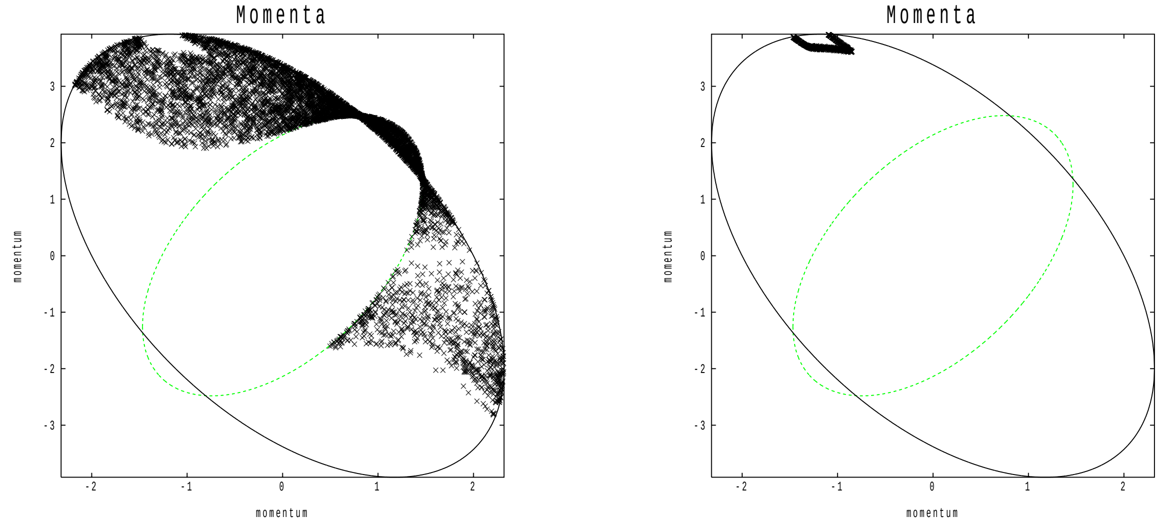
which are two ellipses (details in an example sheet). We can integrate the equations of motion to find the Poincare section and again we find that the system is a mixed system. Typical trajectories are



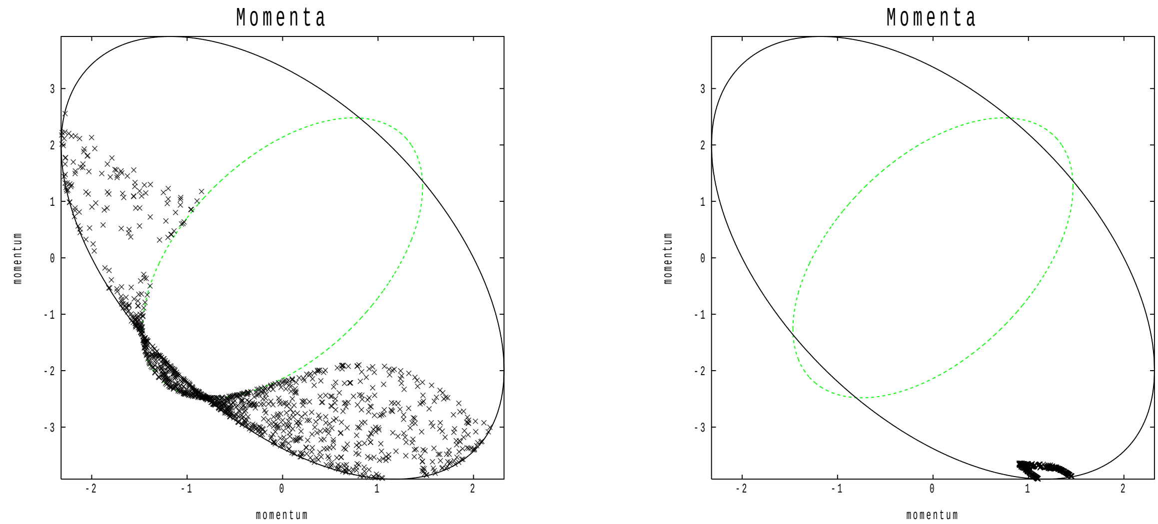
for the space filling chaotic trajectory and



for the integrable motion. Other choices of parameters provide



and



and we find a weird splitting of the space filling trajectory into two disconnected pieces.

Perhaps not as pretty as the sprung pendulum but analogous.

Renormalisation: Intuitive

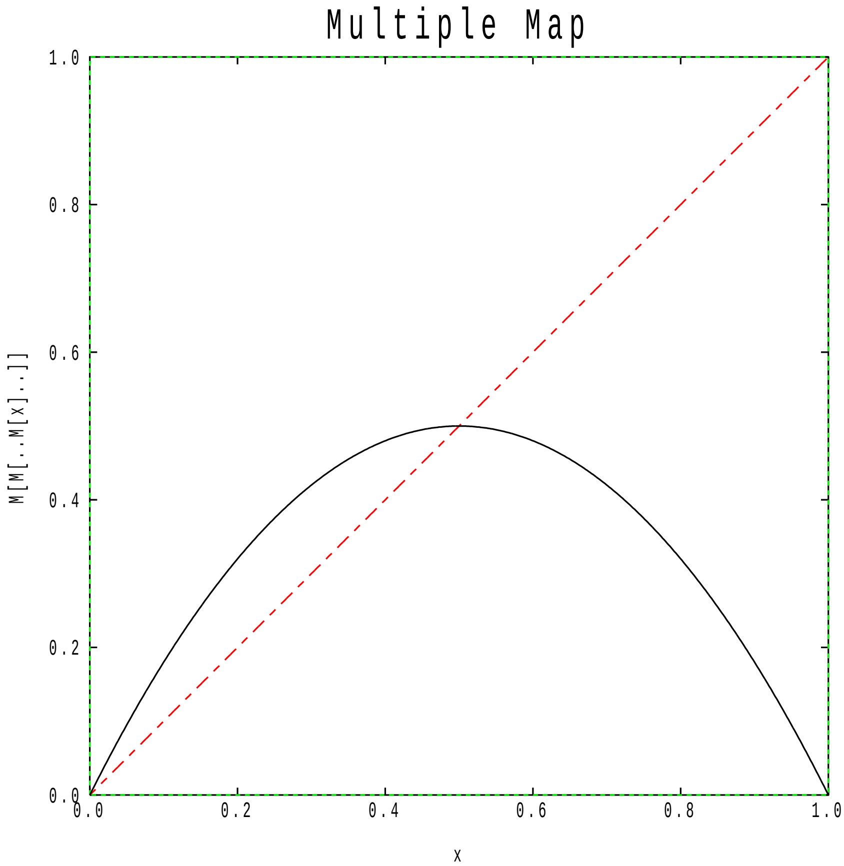
In this section we take another look at the bifurcation route to chaos but try to get an intuitive picture for what happens. We will use the logistic map as our example. For 1-cycles we have

$$x = f(x) \quad \Rightarrow \quad x = rx(1-x) \quad \Rightarrow \quad x = 0 \qquad x = 1 - \frac{1}{r}$$

and so we have two 1-cycles and

$$\frac{df}{dx}(x) = r(1-2x) \quad \Rightarrow \quad \frac{df}{dx}(0) = r \qquad \frac{df}{dx}\left(1-\frac{1}{r}\right) = 2-r$$

The key here is to focus on the region of parameter space $r \in [1,3]$ and to look at the picture coming from $y{=}x$ and $y{=}f(x)$. Now $f(1-x){=}f(x)$, and is symmetric, and consequently the maximum always occurs at $x=\frac{1}{2}$. We naturally use $x \in [0,1]$ in order to analyse the symmetric region including both 1-cycles. At $r=1$ the two 1-cycles are on top of each other and as r increases they move apart until at $r=3$ the stable 1-cycle is at $x=\frac{2}{3}$ and loses stability. In this process the slope of the curve $f(x)$ at the stable fixed point smoothly changes from unity, when the two fixed points are on top of each other, to minus unity when the stable fixed point loses its stability. On the way we pass through the supercycle, at $r=2$, when the fixed point is at a half and the snapshot picture is



Now let us focus on the 2-cycle which becomes stable. We have

$$y = f(x) \quad x = f(y) \quad \Rightarrow \quad y = rx(1-x) \quad x = ry(1-y)$$

$$\Rightarrow \quad x + y = 1 + \frac{1}{r} \quad xy = \frac{1}{r} + \frac{1}{r^2}$$

and we have the points of interest

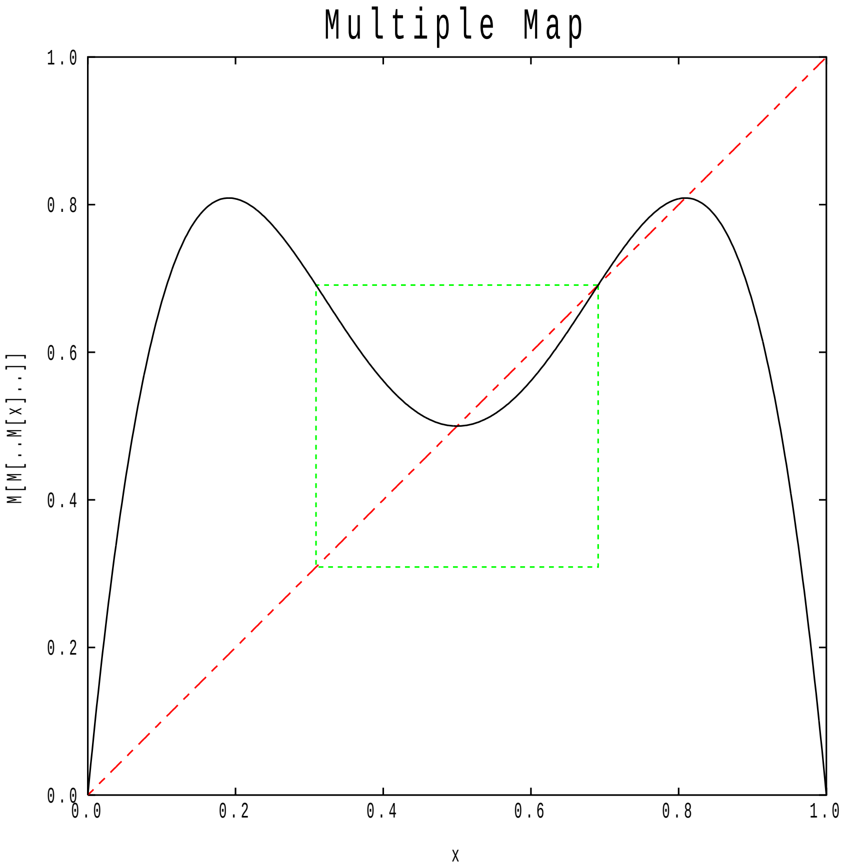
$$z = 1 - \frac{1}{r} \qquad x = \frac{1}{2r}\left(1 + r - [(r-1)(r-3)]^{\frac{1}{2}}\right) \quad y = \frac{1}{2r}\left(1 + r + [(r-1)(r-3)]^{\frac{1}{2}}\right)$$

the unstable 1-cycle and the stable 2-cycle respectively. The stability is controlled by

$$\frac{df^{(2)}}{dx}(x) = \frac{df}{dx}(y)\frac{df}{dx}(x) = r^2(1-2x)(1-2y) = r^2\left[1 - 2(x+y) + 4xy\right]$$

$$= r^2 \left[1 - 2\left(1 + \frac{1}{r}\right) + 4\left(\frac{1}{r} + \frac{1}{r^2}\right)\right] = 4 + 2r - r^2 = 5 - (r-1)^2$$

At r=3 this derivative is unity and at r=1+$\sqrt{6}$ this derivative is minus unity and the 2-cycle loses its stability. The derivative smoothly goes between these two values and on the way, at r=1+$\sqrt{5}$, we have another supercycle and we obtain the plot



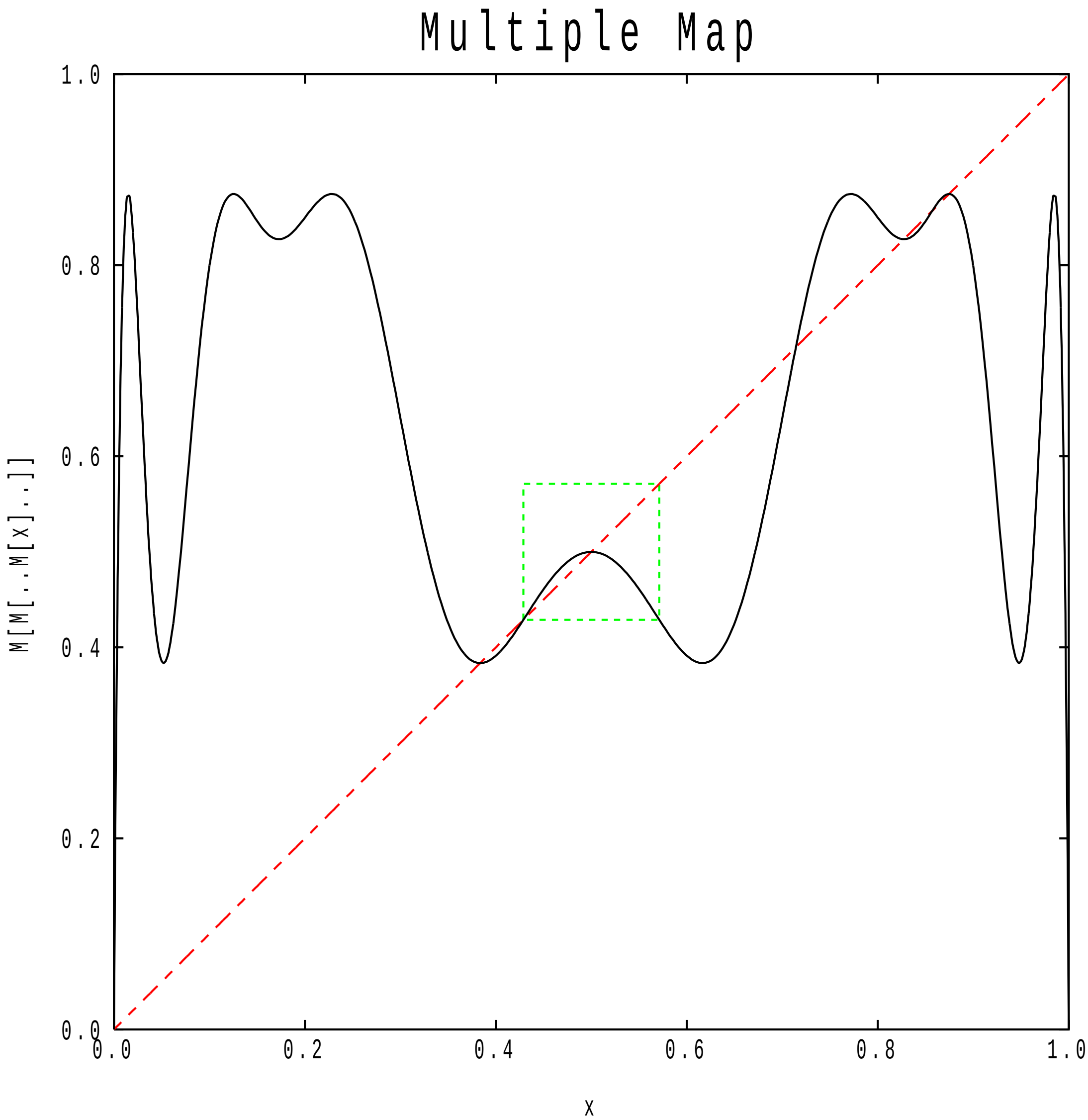
and the three points of interest are clearly visible where the function y=$f^{(2)}(x)$ crosses the line y=x. Now

$$\frac{df^{(2)}}{dx}(x) = \frac{df}{dx}(f(x))\frac{df}{dx}(x) = r\left[1 - 2rx(1-x)\right] r(1-2x)$$

and so the minimum remains at x=$\frac{1}{2}$ for the entire time and the curve is still symmetric $f^{(2)}(1-x)$=$f^{(2)}(x)$. What we need to do now is *renormalise.* When we looked at the life and death of the 1-cycle we focused on the symmetric region bounded by the unstable 1-cycle. We can do the same for the 2-cycle if we focus on the symmetric region bounded by the point z=1-$\frac{1}{r}$, the green box on the previous figure. As r ranges from 3 to 1+$\sqrt{6}$ the point z moves, but we can imagine tracking the green box with this motion and we would obtain a very similar set of pictures for y=x and y=$f^{(2)}(x)$ in this green region as we did for y=x and y=$f(x)$ for the whole region in x; except that it is upside down.

We can now move on to the 4-cycle and imagine how to generalise this picture. At r=1+$\sqrt{6}$ we would look at the plot of y=$f^{(4)}(x)$ and y=x and the 2-cycle and the 4-cycle would be superimposed. As we increase r we would now use the unstable 2-cycle to control the new analogue of the green box; the previous point

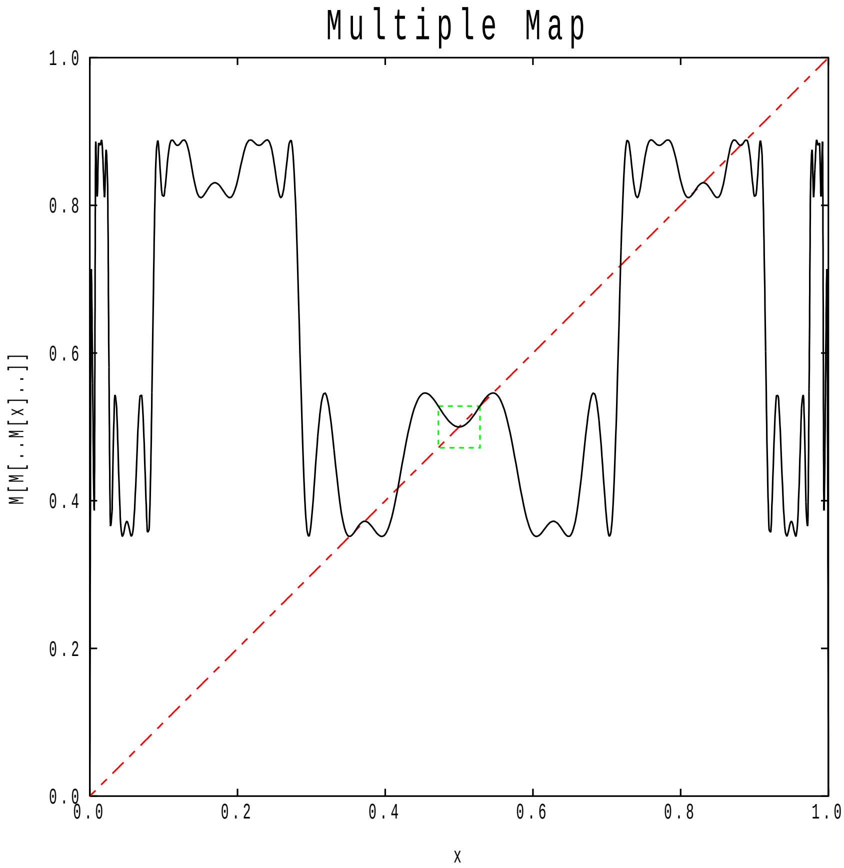
$x = \frac{1}{2r}\left(1 + r - [(r-1)(r-3)]^{\frac{1}{2}}\right)$. The stable 4-cycle would now migrate upwards, pass through the point x=$\frac{1}{2}$ with the next supercycle



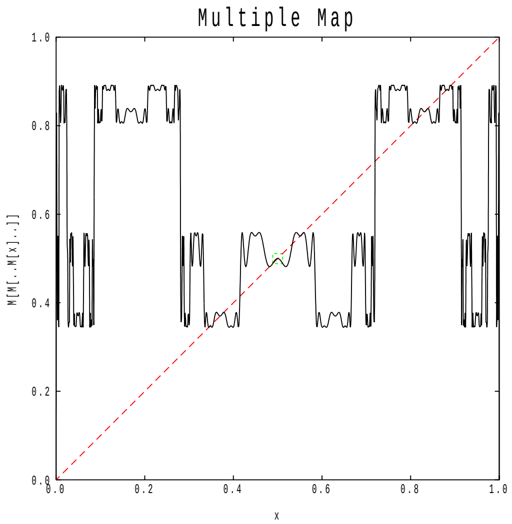
until this 4-cycle loses stability. Clearly the point x still moves around and we have to track the green box with it, but we generate another similar progression of behaviour close to x=$\frac{1}{2}$.

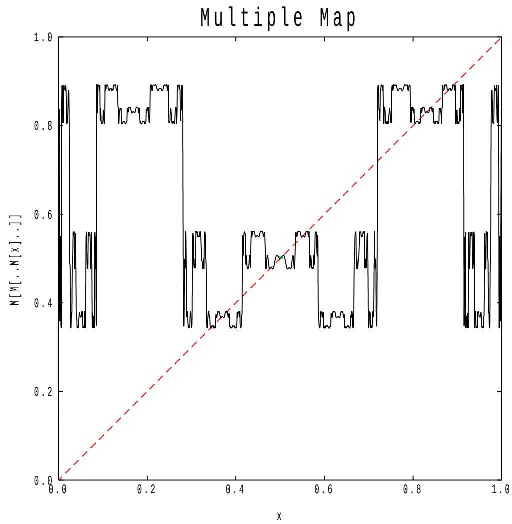
If we look at the whole picture then we have eight points which satisfy $x = f^{(4)}(x)$; the two unstable 1-cycles, the unstable 2-cycle and the currently stable 4-cycle. As well as the green box, we could imagine a box with z=1-$\frac{1}{r}$, the 1-cycle, as the corner. This would provide an upside down, scaled analogue of the previous map $x = f^{(2)}(x)$. There is a hierarchy of all the previous maps contained in the next map, scaled down and alternately flipped over.

The same analysis may be performed for the resulting 8-cycle and an analogous picture for the supercycle is



once again with a new green box added. The behaviour is now very complicated, because if we look at the big picture we have eight solutions to $x = f^{(8)}(x)$ which form the stable 8-cycle. We have four which create the unstable 4-cycle, two more which create the unstable 2-cycle and a further two to provide the two unstable 1-cycles; sixteen in all. In renormalisation we focus in on a smaller and smaller region to try to pick up analogous behaviour to the previous behaviour and lose the big picture. The supercycles for 16-cycles and 32-cycles are

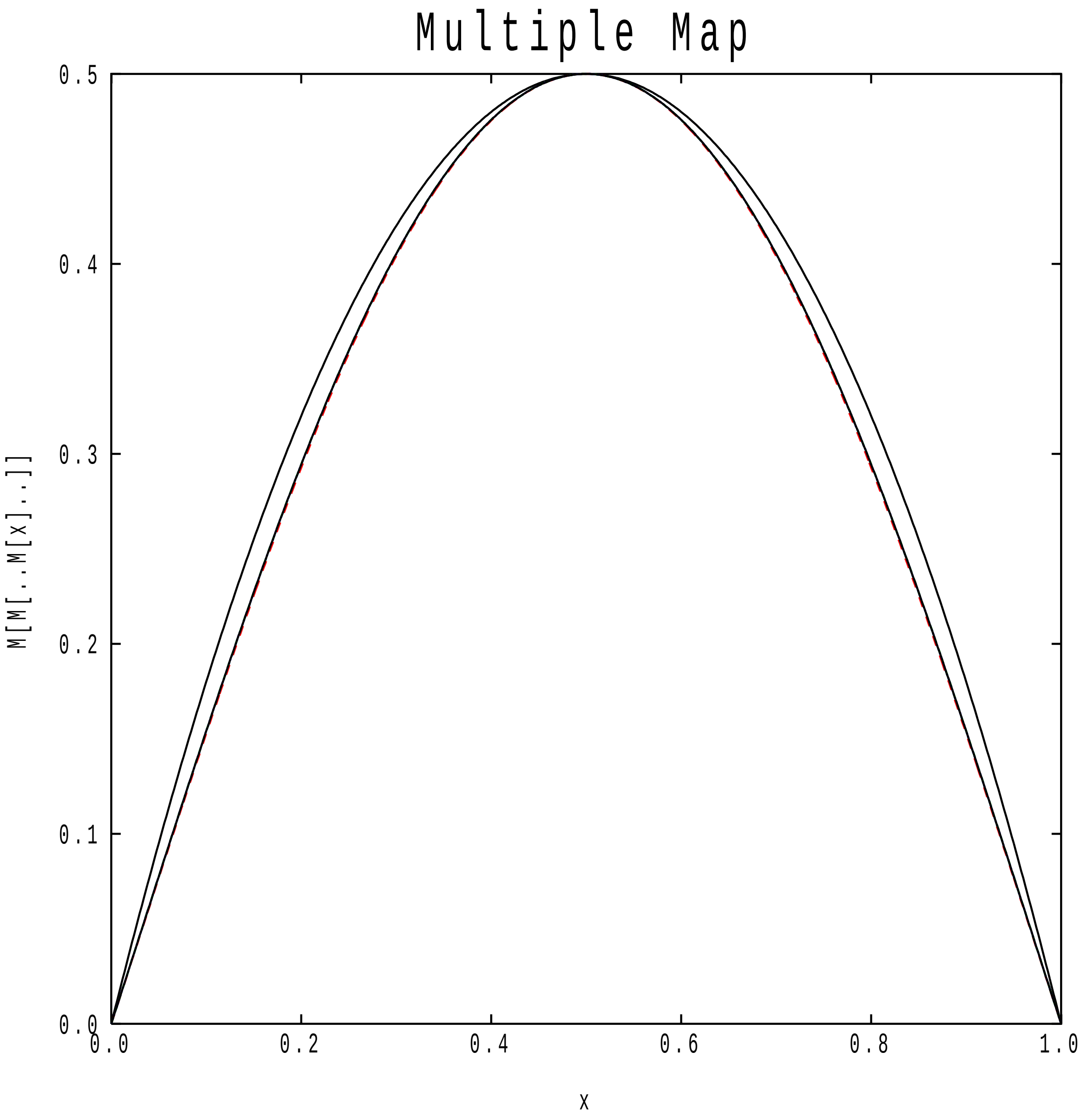




As we go to higher and higher order 2^n-cycles how do these analogous intuitive pictures change; do they converge?

Renormalisation and universality: Numerical

What we do here is superimpose all the green boxes and look for numerical convergence

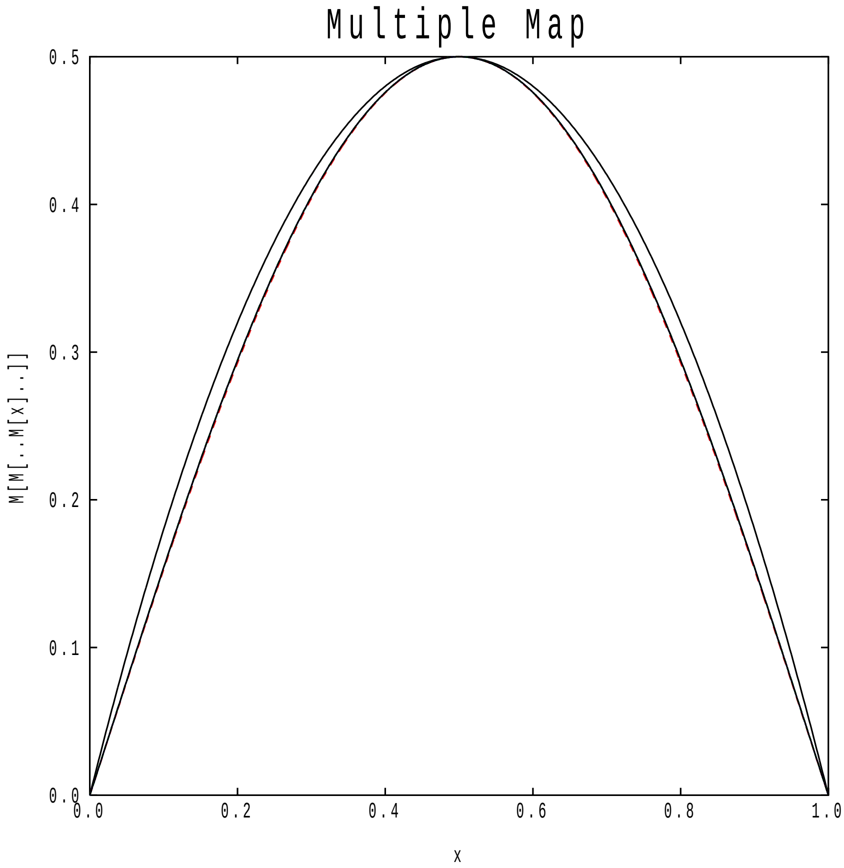


It is interesting to note that there is an impressively quick (exponential) convergence to a single map which, in the limit, controls all the high order bifurcations.

What do we get when we change the map? We looked at

$$f(x) = r \sin(\pi x)$$

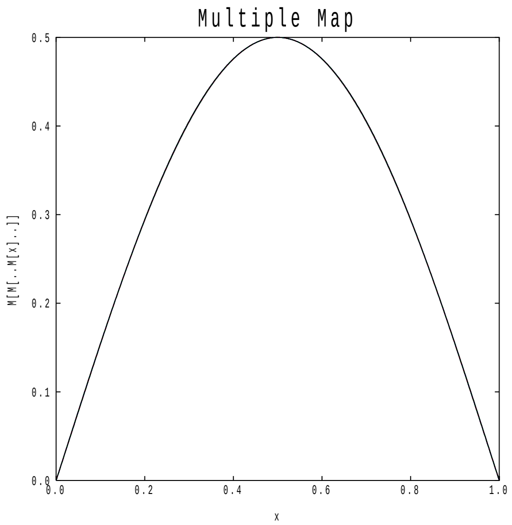
found the sequence of supercycles and overlaid them to find that this map also converges to its own limit.

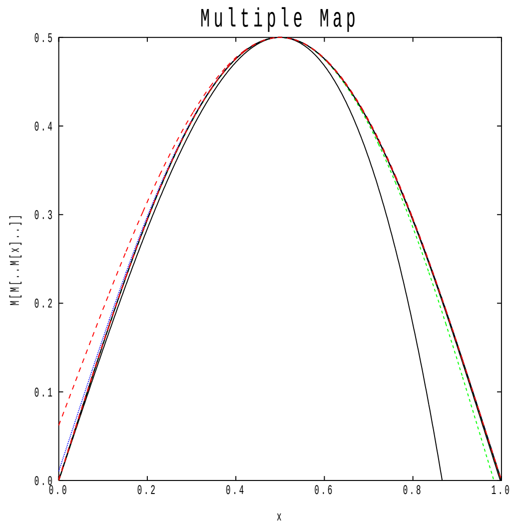


We then followed up with the two maps

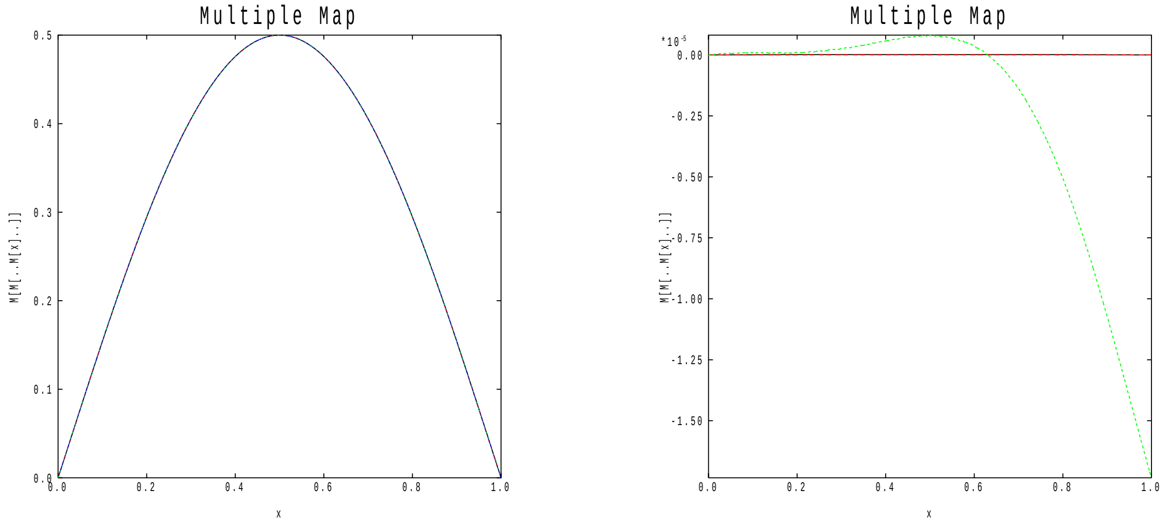
$$f(x) = r - x^2 \qquad f(x) = rx(1 - x^2)$$

and these maps also converge to their own natural limits





Finally we overlay all the limiting maps on the same plot and then plot out the difference between the first plot and the other three in the second plot



Almost to machine accuracy all of these converged maps are identical; this is *universality.*

We focused on the original map, $f(x)$, for our renormalisation, the green box, but instead we could have focused on $f(f(x)) \equiv f^{(2)}(x)$. This would employ the current stable 2^n-cycle, the previous now unstable 2^{n-1}-cycle and the 2^{n-2}-cycle which provides the corner of the larger box. This extended map converges to an extended universal map. Indeed, we could take any finite N and examine $f^{(2^N)}(x)$ and attempt the renormalisation procedure which would converge to a larger, more complicated, piece of the universal map. This universal map should be thought of as extending over all scales and going on for ever to make up the big picture.

It does not matter which original map you have, when you are close to chaos there is a unique map, that all maps converge to, which controls the phenomenon. If we obtained our map from a complicated dynamical system through a Poincare section, or we just made it up...the bifurcation route to chaos should be the same!

Phase space portraits

Prior knowledge

You need to know about vectors and matrices and how to multiply them. In two dimensions

$$\mathbf{v} = \begin{bmatrix} x \\ y \end{bmatrix} \qquad \mathbf{v}^T = [\,x \quad y\,] \qquad M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

are the three types of object and multiplying is

$$M\mathbf{v} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix} \qquad \mathbf{v}^T M = [\,xa + yc \quad xb + yd\,]$$

In three dimensions

$$\mathbf{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad \mathbf{v}^T = [\,x \quad y \quad z\,] \qquad M = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & k \end{bmatrix}$$

are the three types of object and multiplying is

$$M\mathbf{v} = \begin{bmatrix} ax+by+cz \\ dx+ey+fz \\ gx+hy+kz \end{bmatrix} \qquad \mathbf{v}^T M = [\, xa+yd+zg \quad xb+ye+zh \quad xc+yf+zk \,]$$

You also need the identity matrix, denoted I, which satisfies $IM = MI = M$ and which is explicitly

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

in two and three dimensions respectively. These ideas can be generalised into any number of dimensions but you will not need these extensions for this course.

Phase space portraits

The regular behaviour found in phase space portraits can be analysed using a local argument. We take the time independent fundamental equation

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}[\mathbf{x}]$$

and look at the local solution. To do this we Taylor expand the vector $\mathbf{f}[\mathbf{x}]$ in the vicinity of the target point and solve the resulting approximate problem. To zeroth order we can approximate at the point $\mathbf{x}_0$, using $\mathbf{x} = \mathbf{x}_0 + \boldsymbol{\delta}\mathbf{x}$

$$\mathbf{f}[\mathbf{x}_0 + \boldsymbol{\delta}\mathbf{x}] = \mathbf{f}[\mathbf{x}_0] + O[\mid \boldsymbol{\delta}\mathbf{x} \mid]$$

and then solve the approximate problem

$$\frac{d\mathbf{x}}{dt} = \frac{d\boldsymbol{\delta}\mathbf{x}}{dt} = \mathbf{f}[\mathbf{x}_0] \quad \Rightarrow \quad \mathbf{x}(t) = \mathbf{x}_0 + \boldsymbol{\delta}\mathbf{x}_0 + \mathbf{f}[\mathbf{x}_0](t - t_0)$$

where $\boldsymbol{\delta}\mathbf{x}$ is small. This solution provides a picture of trajectories which are locally straight lines which are parallel to $\mathbf{f}[\mathbf{x}_0]$. This behaviour is observed at almost all points but this argument can fail at special points, known as *fixed points,* where $\mathbf{f}[\mathbf{x}_0]=\mathbf{0}$. Special points provide entire trajectories because $\mathbf{x}(t)=\mathbf{x}_0$ is a solution to the fundamental equation. What do nearby trajectories look like?

To investigate this we need to expand to the next order about the fixed point, $\mathbf{x} = \mathbf{x}_0 + \boldsymbol{\delta}\mathbf{x}$

$$\frac{d\mathbf{x}}{dt} = \frac{d\boldsymbol{\delta}\mathbf{x}}{dt} = \mathbf{f}[\mathbf{x}_0 + \boldsymbol{\delta}\mathbf{x}] = \mathbf{f}[\mathbf{x}_0] + \boldsymbol{\delta}\mathbf{x}.\nabla\mathbf{f}[\mathbf{x}_0] + O[\mid \boldsymbol{\delta}\mathbf{x} \mid^2] \mapsto \boldsymbol{\delta}\mathbf{x}.\nabla\mathbf{f}[\mathbf{x}_0]$$

but this equation requires some mathematical analysis to understand.

Stability matrices

The vectors may be described by their components and then the fundamental equation becomes, in d dimensions

$$\frac{dx_1}{dt} = f_1[x_1, ..., x_d]$$
$$\frac{dx_2}{dt} = f_2[x_1, ..., x_d]$$
$$...$$
$$\frac{dx_d}{dt} = f_d[x_1, ..., x_d]$$

the vicinity of $\mathbf{x}_0$, defined by $\mathbf{x} = \mathbf{x}_0 + \boldsymbol{\delta}\mathbf{x}$, is then controlled by the components of $\boldsymbol{\delta}\mathbf{x}$ and we find

$$\frac{d\delta x_1}{dt} = f_1[x_1^0 + \delta x_1, x_2^0 + \delta x_2, ..., x_d^0 + \delta x_d]$$
$$...$$
$$\frac{d\delta x_d}{dt} = f_d[x_1^0 + \delta x_1, x_2^0 + \delta x_2, ..., x_d^0 + \delta x_d]$$

and the Taylor's expansion provides

$$\frac{d\delta x_n}{dt} = f_n[x_1^0, x_2^0, ..., x_d^0] + \delta x_1 \frac{\partial f_n}{\partial x_1}[x_1^0, x_2^0, ..., x_d^0] + \delta x_2 \frac{\partial f_n}{\partial x_2}[x_1^0, x_2^0, ..., x_d^0] + ...$$

$$... + \delta x_d \frac{\partial f_n}{\partial x_d}[x_1^0, x_2^0, ..., x_d^0] + O[\delta x_m^2]$$

where there is an error for all values of m and there is an equation for each coefficient labeled by n. For a fixed point $f_n[x_1^0, x_2^0, ..., x_d^0]$=0 and we tend to describe these complicated equations using *matrices,* arrays of numbers, and for the current problem the controlling matrix is

$$M = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & ... & \frac{\partial f_1}{\partial x_d} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & ... & \frac{\partial f_2}{\partial x_d} \\ . & . & ... & . \\ \frac{\partial f_d}{\partial x_1} & \frac{\partial f_d}{\partial x_2} & ... & \frac{\partial f_d}{\partial x_d} \end{bmatrix}$$

where all derivatives are taken at the point $\mathbf{x}_0$. This matrix is known as the *stability matrix.* We may then rewrite the equations as

$$\begin{bmatrix} \frac{d\delta x_1}{dt} \\ \frac{d\delta x_2}{dt} \\ ... \\ \frac{d\delta x_d}{dt} \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & ... & \frac{\partial f_1}{\partial x_d} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & ... & \frac{\partial f_2}{\partial x_d} \\ . & . & ... & . \\ \frac{\partial f_d}{\partial x_1} & \frac{\partial f_d}{\partial x_2} & ... & \frac{\partial f_d}{\partial x_d} \end{bmatrix} \begin{bmatrix} \delta x_1 \\ \delta x_2 \\ ... \\ \delta x_d \end{bmatrix}$$

and in short-hand

$$\frac{d\boldsymbol{\delta}\mathbf{x}}{dt} = M\boldsymbol{\delta}\mathbf{x} \qquad\qquad M_{ij} = \frac{\partial f_i}{\partial x_j}$$

where $M \equiv \nabla \mathbf{f}$ is the depicted matrix.

Previous example: We previously studied the harmonic oscillator which can be rescaled into

$$\frac{dx_1}{dt} = x_2 \qquad \Rightarrow \qquad f_1[x_1, x_2] = x_2$$

$$\frac{dx_2}{dt} = -x_1 \qquad \Rightarrow \qquad f_2[x_1, x_2] = -x_1$$

To have a fixed point we need that $f_1[x_1, x_2]$=0 and that $f_2[x_1, x_2]$=0 and this only occurs at the origin. For this example we do not need $\mathbf{x}_0$ so we have

$$\begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Previous example: We previously studied the pendulum which can be rescaled to

$$\frac{dx_1}{dt} = x_2 \qquad \Rightarrow \qquad f_1[x_1, x_2] = x_2$$

$$\frac{dx_2}{dt} = -\sin x_1 \qquad \Rightarrow \qquad f_2[x_1, x_2] = -\sin x_1$$

and we may limit $x_1 \in (-\pi, \pi]$ due to the periodicity of the angle. To have a fixed point we need that $f_1[x_1, x_2]$=0 and that $f_2[x_1, x_2]$=0. Although we have the unique x_2=0, there are two solutions for x_1; we have x_1=0 and we also have x_1=π. We can now construct the stability matrix

$$M = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\cos x_1 & 0 \end{bmatrix}$$

For the first fixed point we do not need $\mathbf{x}_0$ so we have

$$\begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

and this local approximation provides the previous harmonic oscillator. For the second fixed point we have $x_1 = \pi + \delta x_1$ and $x_2 = \delta x_2$ so

$$\begin{bmatrix} \frac{d\delta x_1}{dt} \\ \frac{d\delta x_2}{dt} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \delta x_1 \\ \delta x_2 \end{bmatrix}$$

The general approach to this local argument involves solving $\mathbf{f}[\mathbf{x}]$=$\mathbf{0}$ in order to find the possible fixed points. Then, for each fixed point, $\mathbf{x}_0$, we employ $\mathbf{x}$=$\mathbf{x}_0$+$\boldsymbol{\delta}\mathbf{x}$ and construct the stability matrix M. This gives us a linear differential equation to solve

$$\frac{d\boldsymbol{\delta}\mathbf{x}}{dt} = M\boldsymbol{\delta}\mathbf{x}$$

but how do we solve this equation?

There is a general technique, called *matrix diagonalisation,* but first we will solve a particular example to provide some insight.

Example: A two-dimensional dynamical system has a fundamental equation

$$\frac{dx_1}{dt} = x_1(x_2 - 1) \equiv f_1 \qquad \frac{dx_2}{dt} = x_2(x_1 - 1) \equiv f_2$$

find the fixed points and establish the behaviour of the trajectories in their vicinities.

We start out with our general procedure and try to find the fixed points

$$x_1(x_2 - 1) = 0 \qquad x_2(x_1 - 1) = 0$$

and for the first equation we have two choices; firstly

$$x_1 = 0 \quad \Rightarrow \quad x_2(0 - 1) = 0 \quad \Rightarrow \quad x_1 = 0,\ x_2 = 0$$

and secondly

$$x_2 = 1 \quad \Rightarrow \quad 1(x_1 - 1) = 0 \quad \Rightarrow \quad x_1 = 1,\ x_2 = 1$$

We can construct the stability matrix

$$M = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} x_2 - 1 & x_1 \\ x_2 & x_1 - 1 \end{bmatrix}$$

Close to the origin we have

$$\begin{bmatrix} \frac{d\delta x_1}{dt} \\ \frac{d\delta x_2}{dt} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \delta x_1 \\ \delta x_2 \end{bmatrix} \quad \Rightarrow \quad \frac{d\delta x_1}{dt} = -\delta x_1 \quad \frac{d\delta x_2}{dt} = -\delta x_2$$

where finally we have ignored the matrix description. We can then integrate the two equations to get

$$\delta x_1 = Ae^{-t} \qquad \delta x_2 = Be^{-t} \quad \Rightarrow \quad \delta x_2 = \frac{B}{A}\delta x_1$$

and the trajectories are straight lines which pass through the fixed point.

Close to the point $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ we have

$$\begin{bmatrix} \frac{d\delta x_1}{dt} \\ \frac{d\delta x_2}{dt} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \delta x_1 \\ \delta x_2 \end{bmatrix} \quad \Rightarrow \quad \frac{d\delta x_1}{dt} = \delta x_2 \quad \frac{d\delta x_2}{dt} = \delta x_1$$

and again we have ignored the matrix description in the final representation. One way to proceed is to eliminate δx_2 to obtain

$$\frac{d^2\delta x_1}{dt^2} = \delta x_1 \quad \Rightarrow \quad \delta x_1 = Ae^t + Be^{-t}$$

using the exponential ansatz. We can then deduce that

$$\delta x_2 = \frac{d\delta x_1}{dt} = Ae^t - Be^{-t}$$

There are two arbitrary constants and

$$\delta x_1 + \delta x_2 = 2Ae^t \qquad \delta x_1 - \delta x_2 = 2Be^{-t}$$

which provides

$$(\delta x_1 + \delta x_2)(\delta x_1 - \delta x_2) = \delta x_1^2 - \delta x_2^2 = 4AB$$

and the local trajectories are hyperbolae. These linear relationships are a general phenomenon in these systems and can be mathematically targeted. The mathematical process is called *matrix diagonalisation.*

Matrix diagonalisation

There is a specific mathematical problem that controls this issue. The controlling equations are

$$M\mathbf{v} = \lambda\mathbf{v} \qquad \tilde{\mathbf{v}}^T M = \lambda\tilde{\mathbf{v}}^T$$

where all of λ, $\tilde{\mathbf{v}}$ and $\mathbf{v}$ are variables in this equation that need to be determined; $\mathbf{v}$ is known as a right eigenvector, $\tilde{\mathbf{v}}^T$ is known as a left eigenvector and λ is known as an eigenvalue. If we can find such a solution then it leads to one of these linear relationships

$$\frac{d\boldsymbol{\delta}\mathbf{x}}{dt} = M\boldsymbol{\delta}\mathbf{x} \quad \Rightarrow \quad \tilde{\mathbf{v}}^T\frac{d\boldsymbol{\delta}\mathbf{x}}{dt} = \tilde{\mathbf{v}}^T M\boldsymbol{\delta}\mathbf{x} = \lambda\tilde{\mathbf{v}}^T\boldsymbol{\delta}\mathbf{x}$$

and so

$$\frac{d}{dt}\left(\tilde{\mathbf{v}}^T\boldsymbol{\delta}\mathbf{x}\right) = \lambda\left(\tilde{\mathbf{v}}^T\boldsymbol{\delta}\mathbf{x}\right) \quad \Rightarrow \quad \tilde{\mathbf{v}}^T\boldsymbol{\delta}\mathbf{x} = Ae^{\lambda t}$$

which provides a linear relationship.

For the previous example we had, close to the origin

$$M = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = -1\begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = -1\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -1\begin{bmatrix} 1 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -1\begin{bmatrix} 0 & 1 \end{bmatrix}$$

and this gives

$$\begin{bmatrix} 1 & 0 \end{bmatrix}\boldsymbol{\delta}\mathbf{x} = \delta x_1 = Ae^{-t} \qquad \begin{bmatrix} 0 & 1 \end{bmatrix}\boldsymbol{\delta}\mathbf{x} = \delta x_2 = Be^{-t}$$

as we previously found. Close to the point $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ we found

$$M = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 1\begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = -1\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$[\,1 \quad 1\,]\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} = 1\,[\,1 \quad 1\,] \qquad [\,1 \quad -1\,]\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} = -1\,[\,1 \quad -1\,]$$

and this gives

$$[\,1 \quad 1\,]\,\boldsymbol{\delta}\mathbf{x} = \delta x_1 + \delta x_2 = Ae^{t} \qquad [\,1 \quad -1\,]\,\boldsymbol{\delta}\mathbf{x} = \delta x_1 - \delta x_2 = Be^{-t}$$

as we previously found.

The $\mathbf{v}$ and $\tilde{\mathbf{v}}$ are intimately related. As the examples show, there are usually more than one of these solutions and so we may label them with an index, n say

$$M\mathbf{v}_n = \lambda_n \mathbf{v}_n \qquad \tilde{\mathbf{v}}_n^T M = \lambda_n \tilde{\mathbf{v}}_n^T$$

We may then consider a pair of solutions, $n \neq n'$, and

$$\tilde{\mathbf{v}}_n^T M \mathbf{v}_{n'} = \tilde{\mathbf{v}}_n^T \lambda_{n'} \mathbf{v}_{n'} = \lambda_n \tilde{\mathbf{v}}_n^T \mathbf{v}_{n'} \quad \Rightarrow \quad (\lambda_n - \lambda_{n'})\tilde{\mathbf{v}}_n^T \mathbf{v}_{n'} = 0$$

and if the two solutions have different values of the eigenvalues, λ, $\lambda_n \neq \lambda_{n'}$, then $\tilde{\mathbf{v}}_n^T \mathbf{v}_{n'} = 0$. This allows us to completely solve the problem when all the eigenvalues are distinct. If we assume that

$$\boldsymbol{\delta}\mathbf{x} = \tilde{A}_1 \mathbf{v}_1 + \tilde{A}_2 \mathbf{v}_2 + \ldots + \tilde{A}_d \mathbf{v}_d$$

then

$$\tilde{\mathbf{v}}_n^T \boldsymbol{\delta}\mathbf{x} = \tilde{A}_n \tilde{\mathbf{v}}_n^T \mathbf{v}_n = A_n e^{\lambda_n t}$$

because all the other terms vanish. We can then see that

$$\boldsymbol{\delta}\mathbf{x} = \frac{A_1}{\tilde{\mathbf{v}}_1^T \mathbf{v}_1} e^{\lambda_1 t} \mathbf{v}_1 + \frac{A_2}{\tilde{\mathbf{v}}_2^T \mathbf{v}_2} e^{\lambda_2 t} \mathbf{v}_2 + \ldots + \frac{A_d}{\tilde{\mathbf{v}}_d^T \mathbf{v}_d} e^{\lambda_d t} \mathbf{v}_d$$

Since the A_n are arbitrary constants, we can absorb the factors of $\tilde{\mathbf{v}}_n^T \mathbf{v}_n$ into these constants to provide

$$\boldsymbol{\delta}\mathbf{x} = A_1 e^{\lambda_1 t} \mathbf{v}_1 + A_2 e^{\lambda_2 t} \mathbf{v}_2 + \ldots + A_d e^{\lambda_d t} \mathbf{v}_d$$

which is the solution to the behaviour of a trajectory in the neighbourhood of all fixed points that have non-degenerate eigenvalues.

The eigenvectors, $\mathbf{v}_n$, set up natural directions in phase space and the coefficients, $A_n e^{\lambda_n t}$, provide the appropriate motion parallel to each direction. Since the A_n are arbitrary constants, for the case where the vectors are real, we can choose all bar one to vanish and this leads to special trajectories which are restricted to the natural directions. We usually start by depicting these special trajectories.

To determine the general solution to our previous example, close to the point $\mathbf{x}_0 = \begin{bmatrix}1\\ 1\end{bmatrix}$ we need the vectors $\mathbf{v}_n$ and

$$\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}1\\ 1\end{bmatrix} = 1\begin{bmatrix}1\\ 1\end{bmatrix} \qquad \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}1\\ -1\end{bmatrix} = -1\begin{bmatrix}1\\ -1\end{bmatrix}$$

and hence

$$\begin{bmatrix}\delta x_1\\ \delta x_2\end{bmatrix} = Ae^t\begin{bmatrix}1\\1\end{bmatrix} + Be^{-t}\begin{bmatrix}1\\-1\end{bmatrix}$$

which, (up to an irrelevant factor of two) provides the previous solutions.

We now need to examine the technical details of how to find the eigenvalues and eigenvectors of a general matrix.

Diagonalising a 2X2 matrix

The mathematics of a 2X2 matrix may be rewritten

$$M = \begin{bmatrix}a & b\\ c & d\end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix}a & b\\ c & d\end{bmatrix}\begin{bmatrix}\alpha\\ \beta\end{bmatrix} = \lambda\begin{bmatrix}\alpha\\ \beta\end{bmatrix} = \begin{bmatrix}\lambda & 0\\ 0 & \lambda\end{bmatrix}\begin{bmatrix}\alpha\\ \beta\end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix}a-\lambda & b\\ c & d-\lambda\end{bmatrix}\begin{bmatrix}\alpha\\ \beta\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

where α, β and λ are the variables to be found. There is a quantity known as the *adjoint* of M

$$adjM = \begin{bmatrix}d & -b\\ -c & a\end{bmatrix}$$

which satisfies

$$adjMM = \begin{bmatrix}d & -b\\ -c & a\end{bmatrix}\begin{bmatrix}a & b\\ c & d\end{bmatrix} = \begin{bmatrix}ad-bc & 0\\ 0 & ad-bc\end{bmatrix} = (ad-bc)\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

$$\equiv detM\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} = detM\ I$$

and a new quantity, the determinant, $detM = ad - bc$, appears.

We can apply these ideas to our diagonalisation problem

$$adj\begin{bmatrix}a-\lambda & b\\ c & d-\lambda\end{bmatrix} = \begin{bmatrix}d-\lambda & -b\\ -c & a-\lambda\end{bmatrix} \quad \Rightarrow \quad adj\begin{bmatrix}a-\lambda & b\\ c & d-\lambda\end{bmatrix}\begin{bmatrix}a-\lambda & b\\ c & d-\lambda\end{bmatrix}\begin{bmatrix}\alpha\\ \beta\end{bmatrix}$$

$$= \begin{bmatrix}d-\lambda & -b\\ -c & a-\lambda\end{bmatrix}\begin{bmatrix}a-\lambda & b\\ c & d-\lambda\end{bmatrix}\begin{bmatrix}\alpha\\ \beta\end{bmatrix} = [(a-\lambda)(d-\lambda)-bc]\begin{bmatrix}\alpha\\ \beta\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

There are two choices; either the pre-factor does not vanish, in which case α=0 and β=0 and we have no solution, or the prefactor vanishes and we have a non-trivial solution.

We have a quadratic equation

$$(a-\lambda)(d-\lambda) - bc = \lambda^2 - \lambda(a+d) + ad - bc = \left(\lambda - \frac{a+d}{2}\right)^2 - \left(\frac{a-d}{2}\right)^2 - bc$$

$$\Rightarrow \quad \lambda_{\pm} = \frac{a+d}{2} \pm \left[\left(\frac{a-d}{2}\right)^2 + bc\right]^{\frac{1}{2}} \equiv \frac{a+d}{2} \pm D$$

and then we can determine the ratio of α to β from

$$\begin{bmatrix} a-\lambda_\pm & b \\ c & d-\lambda_\pm \end{bmatrix} \begin{bmatrix} \alpha_\pm \\ \beta_\pm \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad \left[\frac{a-d}{2} \mp D\right] \alpha_\pm + b\beta_\pm = 0$$

$$c\alpha_\pm + \left[\frac{d-a}{2} \mp D\right] \beta_\pm = 0$$

Although there appear to be two equations here, they are actually proportional to each other. Taking the ratios of the coefficients we find

$$\frac{\left[\frac{a-d}{2} \mp D\right]}{c} = \frac{b}{\left[\frac{d-a}{2} \mp D\right]} \quad \Rightarrow \quad D^2 - \left(\frac{a-d}{2}\right)^2 = bc$$

which is true from the definition of D.

If we investigate our example, then near $\mathbf{x}=\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ we have

$$M = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad M - \lambda I = \begin{bmatrix} -\lambda & 1 \\ 1 & -\lambda \end{bmatrix} \quad \Rightarrow \quad det[M - \lambda I] = \lambda^2 - 1$$

and we have two solutions; λ=1 which gives

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \beta_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad \alpha_1 = \beta_1 \quad \Rightarrow \quad \mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

and λ=-1 which gives

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \alpha_{\bar{1}} \\ \beta_{\bar{1}} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad \alpha_{\bar{1}} + \beta_{\bar{1}} = 0 \quad \Rightarrow \quad \mathbf{v}_{\bar{1}} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

and we find the previous solutions.

Finding the vector $\tilde{\mathbf{v}}^T$ can be tackled in a similar way

$$[\,\tilde{\alpha} \quad \tilde{\beta}\,] \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \lambda\,[\,\tilde{\alpha} \quad \tilde{\beta}\,] \quad \Rightarrow \quad [\,\tilde{\alpha} \quad \tilde{\beta}\,] \begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix} = [\,0 \quad 0\,]$$

we now need

$$MadjM = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = (ad-bc) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and so $adjMM = MadjM$ and multiplying on the right by $adjM$ gives

$$[\,\tilde{\alpha} \quad \tilde{\beta}\,]\,[(a-\lambda)(d-\lambda) - bc] = [\,0 \quad 0\,]$$

and when a non-trivial solution to $\begin{bmatrix} \alpha \\ \beta \end{bmatrix}$ exists, so does a non-trivial solution to $[\,\tilde{\alpha} \quad \tilde{\beta}\,]$. The same eigenvalues then provide

$$[\,\tilde{\alpha}_\pm \quad \tilde{\beta}_\pm\,] \begin{bmatrix} a-\lambda_\pm & b \\ c & d-\lambda_\pm \end{bmatrix} = [\,0 \quad 0\,] \quad \Rightarrow \quad \tilde{\alpha}_\pm \left[\frac{a-d}{2} \mp D\right] + c\tilde{\beta}_\pm = 0$$

$$b\tilde{\alpha}_\pm + \tilde{\beta}_\pm \left[\frac{d-a}{2} \mp D\right] = 0$$

These equations are almost the same as for $\alpha_\pm$ and $\beta_\pm$ except that the roles of b and c are interchanged.

For our example, since b=c we can immediately write down that

$$\tilde{\mathbf{v}}_1^T = [\,\tilde{\alpha}_1 \quad \tilde{\beta}_1\,] = [\,1 \quad 1\,] \qquad\qquad \tilde{\mathbf{v}}_{\bar{1}}^T = [\,\tilde{\alpha}_{\bar{1}} \quad \tilde{\beta}_{\bar{1}}\,] = [\,1 \quad -1\,]$$

and we find the previous solutions.

We also have the previous example of the pendulum which can be analysed using these ideas. The point $(\pi, 0)$ is identical to the previous example close to $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ with

$$M = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \Rightarrow \quad \delta x_1 + \delta x_2 = Ae^t \qquad \delta x_1 - \delta x_2 = Be^{-t}$$

The origin, which corresponds to the harmonic oscillator, provides

$$M = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad \Rightarrow \quad [\,\tilde{\alpha} \quad \tilde{\beta}\,] \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \lambda\,[\,\tilde{\alpha} \quad \tilde{\beta}\,] = [\,\tilde{\alpha} \quad \tilde{\beta}\,] \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

$$\Rightarrow \quad [\,\tilde{\alpha} \quad \tilde{\beta}\,] \begin{bmatrix} -\lambda & 1 \\ -1 & -\lambda \end{bmatrix} = [\,0 \quad 0\,] \quad \Rightarrow \quad (\lambda^2 + 1)\,[\,\tilde{\alpha} \quad \tilde{\beta}\,] = [\,0 \quad 0\,]$$

and now we have two complex solutions

$$[\,\tilde{\alpha}_\pm \quad \tilde{\beta}_\pm\,] \begin{bmatrix} \mp i & 1 \\ -1 & \mp i \end{bmatrix} = [\,0 \quad 0\,] \quad \Rightarrow \quad \mp i\tilde{\alpha}_\pm - \tilde{\beta}_\pm = 0 \;\; \tilde{\alpha}_\pm \mp i\tilde{\beta}_\pm = 0 \quad \Rightarrow \quad [\,1 \quad \mp i\,]$$

and the resulting linear relationships are

$$\delta x_1 \mp i\delta x_2 = A_\pm e^{\pm it}$$

The fact that the dynamical variables are real now has an impact. Taking the complex conjugate

$$\delta x_1 - i\delta x_2 = A_+ e^{it} \quad \Rightarrow \quad \delta x_1 + i\delta x_2 = \bar{A}_+ e^{-it}$$

and so $A_- = \bar{A}_+$ and then

$$(\delta x_1 + i\delta x_2)(\delta x_1 - i\delta x_2) = \delta x_1^2 + \delta x_2^2 = A_+\bar{A}_+ = \mid A_+ \mid^2$$

and the trajectories are the circles that we previously encountered.

When we look for the general solution we now have $b \neq c$ and so the vectors $\tilde{\mathbf{v}}^T$ and $\mathbf{v}$ are no longer trivially related.

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \lambda \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} -\lambda & 1 \\ -1 & -\lambda \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad (\lambda^2+1)\begin{bmatrix}\alpha\\ \beta\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$

and the eigenvalues are $\lambda_\pm = \pm i$. The eigenvector is determined from

$$\begin{bmatrix}\mp i & 1\\ -1 & \mp i\end{bmatrix}\begin{bmatrix}\alpha_\pm\\ \beta_\pm\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix} \quad \Rightarrow \quad \mp i\alpha_\pm + \beta_\pm = 0 \quad -\alpha_\pm \mp i\beta_\pm = 0 \quad \Rightarrow \quad \begin{bmatrix}1\\ \pm i\end{bmatrix}$$

and so the general trajectory is

$$\begin{bmatrix}\delta x_1\\ \delta x_2\end{bmatrix} = A_+ e^{it}\begin{bmatrix}1\\ i\end{bmatrix} + A_- e^{-it}\begin{bmatrix}1\\ -i\end{bmatrix}$$

which, up to a factor of two, is consistent with the previous solution. Note that to maintain reality we need $A_- = \bar{A}_+$ and if we use $A_+ =\mid A \mid e^{-it_0}$ we end up with

$$\begin{bmatrix}\delta x_1\\ \delta x_2\end{bmatrix} =\mid A \mid e^{i(t-t_0)}\begin{bmatrix}1\\ i\end{bmatrix} + \mid A \mid e^{i(t_0-t)}\begin{bmatrix}1\\ -i\end{bmatrix} = \begin{bmatrix}2 \mid A \mid \cos(t-t_0)\\ -2 \mid A \mid \sin(t-t_0)\end{bmatrix}$$

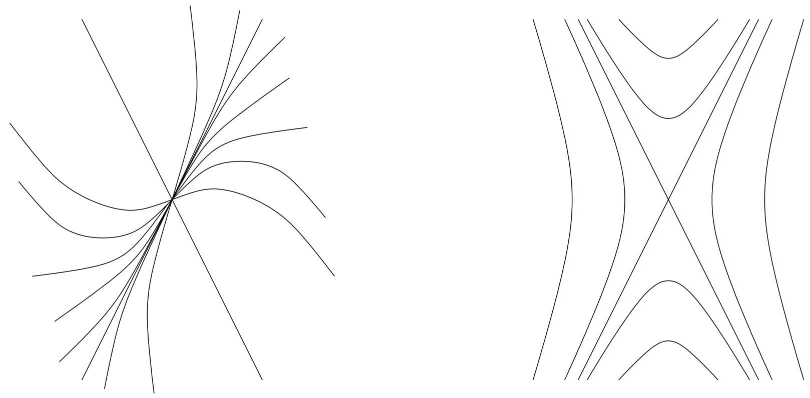
essentially the same representation as at the start of the course.

Depicting the trajectories near the fixed point

When we wish to picture the phase space portrait we need to be able to turn the mathematical answer into trajectories. Although we have previously found straight lines, hyperbolae and circles as examples, it is useful to have a general procedure. It is the general solution that leads to the desired picture when the eigenvalues are real. There are two special trajectories that are straight lines

$$\boldsymbol{\delta}\mathbf{x} = A_1 e^{\lambda_1 t}\mathbf{v}_1 + A_2 e^{\lambda_2 t}\mathbf{v}_2$$

one occurs when A_2=0 and provides a trajectory in the direction of $\mathbf{v}_1$ and the other occurs when A_1=0 and provides a trajectory in the direction of $\mathbf{v}_2$. As a function of time, the exponential, $e^{\lambda_n t}$, provides both the rate of progress and the direction of progress. If $\lambda_n > 0$ then the trajectory is moving away from the fixed point and if $\lambda_n < 0$ then the trajectory is moving towards the fixed point; $\tau_n=\frac{1}{\lambda_n}$ provides a time scale. If τ_n is small the point moves along the trajectory quickly and if τ_n is large the point moves along the trajectory slowly. For a general point both time scales are relevant. One can consider the coefficients of the two vectors as components of a vector description and the time dependence controls how these coefficients vary. If both λ_n are positive then the trajectory moves away from the fixed point in both directions but ends up being parallel to the direction with the fastest motion, the largest value of λ_n. If both λ_n are negative then the trajectory moves towards the fixed point and the fixed point is said to be *stable* and can be the *attractor*. The fastest motion, largest value of λ_n, collapses the component quickly and leaves the system following the trajectory along the direction associated with the slow motion, smallest value of λ_n. If one of the λ_n is positive while the other is negative then the negative component marks the loss of the associated coefficient leaving the coefficient associated with the positive value of λ_n which grows exponentially carrying the trajectory away from the fixed point.



where the first plot is for $\lambda_1 > \lambda_2 > 0$ or $\lambda_1 < \lambda_2 < 0$ and the second is for $\lambda_1 > 0 > \lambda_2$ or $\lambda_1 < 0 < \lambda_2$

Example:

$$\frac{dx_1}{dt} = x_1(1-x_1)(1-x_2) \equiv f_1[x_1,x_2] \qquad \frac{dx_2}{dt} = x_2(2-x_1) \equiv f_2[x_1,x_2]$$

First we need to find the fixed points. Start with the easiest equation first

$$f_2 = 0 \quad \Rightarrow \quad x_2 = 0, f_1 = x_1(1-x_1) = 0 \quad \mathit{or} \quad x_1 = 2, f_1 = 2(x_2-1)$$

which leads to three fixed points $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$. We can calculate the stability matrix

$$\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} (1-2x_1)(1-x_2) & -x_1(1-x_1) \\ -x_2 & 2-x_1 \end{bmatrix}$$

and then investigate the fixed points one at a time. For the origin, $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$, we have

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \lambda \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \quad \Rightarrow \quad \lambda = 1 \quad \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \lambda = 2 \quad \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

and so we have an unstable fixed point with

$$\begin{bmatrix} \delta x_1 \\ \delta x_2 \end{bmatrix} = Ae^t \begin{bmatrix} 1 \\ 0 \end{bmatrix} + Be^{2t} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \Rightarrow \quad \left[\frac{\delta x_1}{A}\right]^2 = \frac{\delta x_2}{B}$$

and we have a collection of parabolae with the fast dependence carrying us parallel to δx_2 eventually. For the point $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ we have

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \lambda \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \quad \Rightarrow \quad \lambda = -1 \quad \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \lambda = 1 \quad \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

and so
$$\begin{bmatrix}\delta x_1\\ \delta x_2\end{bmatrix} = Ae^{-t}\begin{bmatrix}1\\0\end{bmatrix} + Be^{t}\begin{bmatrix}0\\1\end{bmatrix} \quad\Rightarrow\quad \frac{\delta x_1}{A}\frac{\delta x_2}{B} = 1$$

and we have a collection of hyperbolae. The final point, $\begin{bmatrix}2\\1\end{bmatrix}$, is the most interesting

$$\begin{bmatrix}0 & 2\\ -1 & 0\end{bmatrix}\begin{bmatrix}\alpha\\ \beta\end{bmatrix} = \lambda\begin{bmatrix}\alpha\\ \beta\end{bmatrix} \quad\Rightarrow\quad \lambda = i\sqrt{2}\ \begin{bmatrix}\sqrt{2}\\ i\end{bmatrix} \qquad \lambda = -i\sqrt{2}\ \begin{bmatrix}\sqrt{2}\\ -i\end{bmatrix}$$

and then
$$\begin{bmatrix}\delta x_1\\ \delta x_2\end{bmatrix} = Ae^{i\sqrt{2}t}\begin{bmatrix}\sqrt{2}\\ i\end{bmatrix} + \bar{A}e^{-i\sqrt{2}t}\begin{bmatrix}\sqrt{2}\\ -i\end{bmatrix}$$

and so

$$\delta x_1 = \mid A\mid 2\sqrt{2}\cos\sqrt{2}(t-t_0) \quad \delta x_2 = -\mid A\mid 2\sin\sqrt{2}(t-t_0) \quad\Rightarrow\quad \delta x_1^2 + 2\delta x_2^2 = 8\mid A\mid^2$$

and a collection of concentric ellipses.

It is interesting that this example has a conservation law; an analogue of an energy. Taking the ratio of the two equations

$$\frac{dx_1}{dt} = x_1(1-x_1)(1-x_2) \quad \frac{dx_2}{dt} = x_2(2-x_1) \quad\Rightarrow\quad \frac{dx_1}{dx_2} = \frac{x_1(1-x_1)(1-x_2)}{x_2(2-x_1)}$$

leads to a separation of variables

$$\frac{dx_1(2-x_1)}{x_1(1-x_1)} = \frac{dx_2(1-x_2)}{x_2} = dt(2-x_1)(1-x_2)$$

which can be integrated

$$2\frac{dx_1}{x_1} + \frac{dx_1}{1-x_1} = \frac{dx_2}{x_2} - dx_2 \quad\Rightarrow\quad 2\ln\mid x_1\mid - \ln\mid 1-x_1\mid - \ln\mid x_2\mid - x_2 + A$$

and then
$$\frac{x_1^2}{1-x_1} = 2Ex_2e^{-x_2}$$

where x_1=1 is a trajectory, x_2=0 is a trajectory and x_1=0 is also a trajectory so the modulus signs may be absorbed into the analogue of the energy, E. We can even solve for x_1 in terms of x_2 by completing the square

$$\left[x_1 + Ex_2e^{-x_2}\right]^2 = \left[Ex_2e^{-x_2}\right]^2 + 2Ex_2e^{-x_2}$$

We can analyse the previous fixed points using this relationship. We find the value of E which corresponds to the fixed point and then choose a small value for the change in E and Taylor expand. Close to the origin

$$\frac{x_1^2}{1-x_1} = 2Ex_2e^{-x_2} \mapsto x_1^2 = 2Ex_2$$

and we find our parabolae. Close to the point $\begin{bmatrix}1\\0\end{bmatrix}$ we use $x_1 = 1 + \delta x_1$ and we have

$$\frac{x_1^2}{1-x_1} = 2Ex_2e^{-x_2} \mapsto \frac{-1}{\delta x_1} = 2Ex_2$$

and we find our hyperbolae when E is very large, $\frac{1}{E} \mapsto \delta$. Close to the point $\begin{bmatrix}2\\1\end{bmatrix}$ we need to do more analysis, $x_1 = 2 + \delta x_1$ and $x_2 = 1 + \delta x_2$

$$\frac{x_1^2}{1-x_1} = 2Ex_2e^{-x_2} = \frac{4+4\delta x_1 + \delta x_1^2}{-1-\delta x_1} = 2E(1+\delta x_2)e^{-1}e^{-\delta x_2}$$

$$\mapsto -4 - \delta x_1^2 = 2\frac{E}{e}(1+\delta x_2)\left[1 - \delta x_2 + \frac{1}{2}\delta x_2^2\right] = 2\frac{E}{e}\left[1 - \frac{1}{2}\delta x_2^2 ...\right]$$

and so $2\frac{E}{e} \mapsto -4 - \delta$ and

$$\delta x_1^2 + 2\delta x_2^2 = \delta$$

and we recognise the final concentric ellipses.

Join the dots

Once the local trajectories in the vicinity of all the fixed points have been determined, the phase space portrait may be completed using the idea that the flow of trajectories is smooth everywhere else....join the dots.

Example:

$$\frac{dx_1}{dt} = x_2(2-x_2) \equiv f_1[x_1, x_2] \qquad \frac{dx_2}{dt} = x_1(1-x_1) \equiv f_2[x_1, x_2]$$

Firstly we need to find the fixed points. The two equations are independent so we find either x_2=0 or x_2=2, together with either x_1=0 or x_1=1. This gives four possible fixed points

$$\begin{bmatrix}0\\0\end{bmatrix} \quad \begin{bmatrix}1\\0\end{bmatrix} \quad \begin{bmatrix}0\\2\end{bmatrix} \quad \begin{bmatrix}1\\2\end{bmatrix}$$

We can calculate the stability matrix in general

$$\begin{bmatrix}\frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2}\\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2}\end{bmatrix} = \begin{bmatrix}0 & 2-2x_2\\ 1-2x_1 & 0\end{bmatrix}$$

and then we can look at each fixed point in turn. The origin, $\begin{bmatrix}0\\0\end{bmatrix}$, provides

$$\begin{bmatrix}0 & 2\\ 1 & 0\end{bmatrix}\begin{bmatrix}\sqrt{2}\\1\end{bmatrix} = \sqrt{2}\begin{bmatrix}\sqrt{2}\\1\end{bmatrix} \qquad \begin{bmatrix}0 & 2\\ 1 & 0\end{bmatrix}\begin{bmatrix}\sqrt{2}\\-1\end{bmatrix} = -\sqrt{2}\begin{bmatrix}\sqrt{2}\\-1\end{bmatrix}$$

and the solution

$$\begin{bmatrix}\delta x_1\\ \delta x_2\end{bmatrix} = Ae^{\sqrt{2}t}\begin{bmatrix}\sqrt{2}\\1\end{bmatrix} + Be^{-\sqrt{2}t}\begin{bmatrix}\sqrt{2}\\-1\end{bmatrix}$$

which is an unstable fixed point with collapse in one direction and expansion in the other. The two special trajectories are the lines $x_1{=}\pm\sqrt{2}x_2$. The point $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ provides

$$\begin{bmatrix} 0 & 2 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} \sqrt{2} \\ i \end{bmatrix} = \sqrt{2}i\begin{bmatrix} \sqrt{2} \\ i \end{bmatrix} \qquad \begin{bmatrix} 0 & 2 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} \sqrt{2} \\ -i \end{bmatrix} = -\sqrt{2}i\begin{bmatrix} \sqrt{2} \\ -i \end{bmatrix}$$

and the solution

$$\begin{bmatrix} \delta x_1 \\ \delta x_2 \end{bmatrix} = Ae^{i\sqrt{2}t}\begin{bmatrix} \sqrt{2} \\ i \end{bmatrix} + Be^{-i\sqrt{2}t}\begin{bmatrix} \sqrt{2} \\ -i \end{bmatrix}$$

and then the parameterisation $B = \bar{A} =\mid A \mid e^{i\sqrt{2}t_0}$ leads to

$$\begin{bmatrix} \delta x_1 \\ \delta x_2 \end{bmatrix} = 2 \mid A \mid \begin{bmatrix} \sqrt{2}\cos\sqrt{2}(t-t_0) \\ -\sin\sqrt{2}(t-t_0) \end{bmatrix}$$

and the trajectories loop clockwise around concentric ellipses. The other two points can be solved by realising that

$$\begin{bmatrix} 0 & -2 \\ -1 & 0 \end{bmatrix} = -\begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & -2 \\ 1 & 0 \end{bmatrix} = -\begin{bmatrix} 0 & 2 \\ -1 & 0 \end{bmatrix}$$

and so the two new points have the same eigenvectors as the previous two but the eigenvalues have the opposite sign. This is equivalent to the trajectories having time reversed. For the point $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ we find

$$\begin{bmatrix} \delta x_1 \\ \delta x_2 \end{bmatrix} = Ae^{-\sqrt{2}t}\begin{bmatrix} \sqrt{2} \\ 1 \end{bmatrix} + Be^{\sqrt{2}t}\begin{bmatrix} \sqrt{2} \\ -1 \end{bmatrix}$$

and for the point $\begin{bmatrix} 0 \\ 2 \end{bmatrix}$ we find

$$\begin{bmatrix} \delta x_1 \\ \delta x_2 \end{bmatrix} = 2 \mid A \mid \begin{bmatrix} \sqrt{2}\cos\sqrt{2}(t-t_0) \\ \sin\sqrt{2}(t-t_0) \end{bmatrix}$$

The direction of the trajectories is a great help in joining the dots. We find a special pair of trajectories that start and finish at the unstable fixed points but take a circuit around their nearest stable fixed point with the ellipses. The other trajectories either wriggle around between these two regions or avoid this distortion entirely.

Interestingly this example also has a conservation law. Taking the ratio of the two equations gives

$$\frac{dx_1}{dt} = x_2(2-x_2) \qquad \frac{dx_2}{dt} = x_1(1-x_1) \quad \Rightarrow \quad \frac{dx_1}{dx_2} = \frac{x_2(2-x_2)}{x_1(1-x_1)}$$

which separates

$$dx_1 x_1(1-x_1) = dx_2 x_2(2-x_2) = dt x_1(1-x_1)x_2(2-x_2)$$

which integrates to give

$$\frac{x_1^2}{2} - \frac{x_1^3}{3} = x_2^2 - \frac{x_2^3}{3} + E$$

and the special trajectories are provided by E=0 at the origin and E=-$\frac{7}{6}$ for the point $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

Diagonalising a 3X3 matrix

A general 3X3 matrix provides the problem

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & k \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \lambda \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} a-\lambda & b & c \\ d & e-\lambda & f \\ g & h & k-\lambda \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Once again a general matrix has an adjoint matrix

$$M = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & k \end{bmatrix} \qquad adjM = \begin{bmatrix} ek-fh & ch-bk & bf-ce \\ fg-dk & ak-cg & cd-af \\ dh-eg & bg-ah & ae-bd \end{bmatrix}$$

and careful multiplication leads to

$$M\ adjM = adjM\ M = detM\ I$$

where

$$detM = aek - afh + bfg - bdk + cdh - ceg$$

The diagonalisation problem now involves a cubic equation

$$(a-\lambda)(e-\lambda)(k-\lambda) - (a-\lambda)fh + bfg - bd(k-\lambda) + cdh - c(e-\lambda)g = 0$$

The algebra is much worse but the basic ideas remain.

Example: The nuclear decay of an isotope through an unstable intermediary nucleus to a stable nucleus is described by

$$\begin{aligned} \frac{dN_1}{dt} &= -\lambda_1 N_1 \\ \frac{dN_2}{dt} &= \lambda_1 N_1 - \lambda_2 N_2 \\ \frac{dN_3}{dt} &= \lambda_2 N_2 \end{aligned}$$

and the controlling matrix equation is

$$\begin{bmatrix} -\lambda_1 & 0 & 0 \\ \lambda_1 & -\lambda_2 & 0 \\ 0 & \lambda_2 & 0 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \lambda \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}$$

and the solutions are easy to find

$$\lambda = 0 \quad \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad \lambda = -\lambda_2 \quad \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \qquad \lambda = -\lambda_1 \quad \begin{bmatrix} \lambda_1 - \lambda_2 \\ -\lambda_1 \\ \lambda_2 \end{bmatrix}$$

and lead to the final answer

$$\begin{bmatrix} N_1 \\ N_2 \\ N_3 \end{bmatrix} = A \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + Be^{-\lambda_2 t} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + Ce^{-\lambda_1 t} \begin{bmatrix} \lambda_1 - \lambda_2 \\ -\lambda_1 \\ \lambda_2 \end{bmatrix}$$

Alternatively we can look to the linear relationships through

$$[\,\tilde{\alpha} \quad \tilde{\beta} \quad \tilde{\gamma}\,] \begin{bmatrix} -\lambda_1 & 0 & 0 \\ \lambda_1 & -\lambda_2 & 0 \\ 0 & \lambda_2 & 0 \end{bmatrix} = \lambda\,[\,\tilde{\alpha} \quad \tilde{\beta} \quad \tilde{\gamma}\,]$$

which solves with

$$\lambda = 0 \;\; [1 \quad 1 \quad 1] \qquad \lambda = -\lambda_2 \;\; [\lambda_1 \quad \lambda_1 - \lambda_2 \quad 0] \qquad \lambda = -\lambda_1 \;\; [1 \quad 0 \quad 0]$$

and provides

$$N_1 + N_2 + N_3 = A \qquad \lambda_1 N_1 + (\lambda_1 - \lambda_2) N_2 = Be^{-\lambda_2 t} \qquad N_1 = Ce^{-\lambda_1 t}$$

completely consistent with the previous solution but giving the important conservation law that the total number of nuclei is conserved.