University of Birmingham
School of Mathematics

Real Analysis – Integration – Spring 2025

Problem Sheet 5

Model Solutions

Instructions: You are strongly encouraged to attempt all of the questions (Q) below, and as many of the extra questions (EQ) as you can, to help prepare for the final exam. Model solutions will only be released for questions Q1-Q4.

Q1. (a) Suppose that $X \subseteq \mathbb{R}$ is not empty and that $N \in \mathbb{R}$ is an upper bound for X. Prove that if for each $\epsilon > 0$, there exists x in X such that $N - \epsilon < x$, then $N = \sup X$.

(b) Find, with proof, the infimum of the set

$$A := \{x : 3 < |x - 2| < 5\}$$

(c) Suppose that $P = \{0, 4, 12, 17, 20\}$ and $g : [0, 20] \to \mathbb{R}$ is given by

$$g(x) := 3x + 7.$$

(i) Calculate the Riemann–Darboux sums $L(g, P)$ and $U(g, P)$.

(ii) For each $n \in \mathbb{N}$, let P_n denote the partition of $[0, 20]$ into n subintervals of equal width. Find a natural number N such that $U(g, P_N) - L(g, P_N) < \frac{1}{1{,}000}$.

The Riemann–Darboux sums calculator (Spring Week 1 Materials on Canvas) may be helpful in part (c).

Solution. (a)

Proof. We know that N is an upper bound for X, we only need to prove that N is the *least* upper bound for X (recall (3)(b) in Definition 1.1.1 in the Lecture Notes). To that end, suppose that M is an upper bound for X. Assume for a contradiction that $N > M$, so then $\epsilon := N - M > 0$, and by the assumption in the question there exists x in X such that $N - \epsilon < x$. Now observe that $N - \epsilon = M$, but $M < x \in X$ is in contradiction with the fact M is an upper bound for X. This shows that $N \le M$ whenever M is an upper bound for X, hence $N = \sup X$, as required. □

(b) Observe that

$$\begin{aligned} A &= \{x : 3 < (x - 2) < 5\} \cup \{x : 3 < -(x - 2) < 5\} \\ &= \{x : 5 < x < 7\} \cup \{x : -3 > (x - 2) > -5\} \\ &= \{x : 5 < x < 7\} \cup \{x : -1 > x > -3\} \\ &= (-3, -1) \cup (5, 7) \end{aligned}$$

We claim that $\inf A = -3$. To prove this, observe that $x \ge -3$ for all $x \in A$, so -3 is a lower bound for A. Next, if $\epsilon \in (0, 1)$, then $-3 + \epsilon \ge -3 + \frac{\epsilon}{2} \in A$, so $-3 + \epsilon$ is not a lower bound for A. Finally, if $\epsilon \ge 1$, then $-3 + \epsilon \ge -3 + 1 = -2 \in A$, so $-3 + \epsilon$ is not a lower bound for A. This show that -3 is the greatest lower bound for A, hence $\inf A = -3$.

(c)(i) Using the notation $P = \{0, 4, 12, 17, 20\} =: \{x_0, x_1, x_2, x_3, x_4\}$, we calculate

$$\begin{aligned} L(g, P) &= \sum_{i=1}^{4} m_i(x_i - x_{i-1}) \\ &= g(0)(4 - -0) + g(4)(12 - 4) + g(12)(17 - 12) + g(17)(20 - 17) \\ &= 7(4) + 19(8) + 43(5) + 58(3) \\ &= 28 + 152 + 215 + 174 = 569 \end{aligned}$$

and

$$\begin{aligned} U(g,P) &= \sum_{i=1}^{4} M_i(x_i - x_{i-1}) \\ &= g(4)(4 - -0) + g(12)(12-4) + g(17)(17-12) + g(20)(20-17) \\ &= 19(4) + 43(8) + 58(5) + 67(3) \\ &= 76 + 344 + 290 + 201 = 911. \end{aligned}$$

Alternatively, these calculations can be done using the Riemann–Darboux Sums Calculator on Canvas, but a screenshot as in Figure 1 below must be provided.

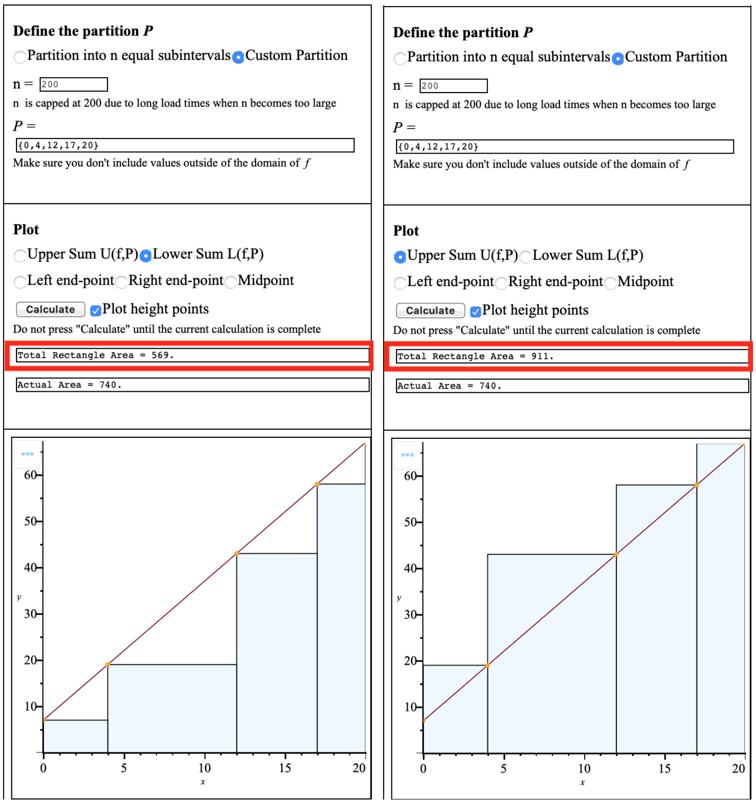

FIGURE 1. Two graphs of $g(x) := 3x + 7$ showing the areas covered by $L(g,Q) = 569$ and $U(g,Q) = 911$ for $Q = \{0, 4, 12, 17, 20\}$.

(c)(ii) The Riemann–Darboux Sums Calculator does not provide the required accuracy. Instead, the key point is to recognise that the function g is increasing. Suppose that $n \in \mathbb{N}$ and consider the equal-width partition $P_n := \{x_0, x_1, \dots, x_n\}$ of $[0, 20]$ into n sub-intervals of width $20/n$. The majority of the terms in $U(g, P_n) - L(g, P_n)$ cancel, as we find that

$$\begin{aligned} U(g,P_n) - L(g,P_n) &= \sum_{i=1}^{n} M_i(x_i - x_{i-1}) - \sum_{i=1}^{n} m_i(x_i - x_{i-1}) \\ &= \sum_{i=1}^{n} g(x_i)\frac{20}{n} - \sum_{i=1}^{n} g(x_{i-1})\frac{20}{n} \\ &= \frac{20}{n}[(g(x_1) + \cdots + g(x_n)) - (g(x_0) + \cdots + g(x_{n-1}))] \\ &= \frac{20}{n}[g(x_n) - g(x_0)] \\ &= \frac{20}{n}[g(20) - g(0)] \\ &= \frac{1,200}{n}. \end{aligned}$$

Therefore, we need a natural number n such that $\frac{1{,}200}{n} < \frac{1}{1{,}000}$, which is the same as requiring that $n > 1,200,000$. Now choose $N := 1,200,001$. The preceding computation shows that $U(g, P_N) - L(g, P_N) = \frac{1{,}200}{N} = \frac{1{,}200}{1{,}200{,}001} < \frac{1}{1{,}000}$, as required. Of course, there is no need to be so precise with the choice of N, as any natural number larger than 1,200,000 is an equally valid choice to answer this question, provided the justification above is given.

Alternatively, replacing x with $3x + 7$ and setting $b = 20$ in Exercise 1.2.7 of the Lecture Notes, we can compute

$$L(g,P_n) = \sum_{i=1}^{n} m_i(x_i - x_{i-1}) = \sum_{i=1}^{n}[3\tfrac{(i-1)b}{n} + 7]\tfrac{b}{n} = 3\tfrac{n-1}{n}\tfrac{b^2}{2} + \tfrac{7b}{n}\sum_{i=1}^{n} 1 = 3\tfrac{n-1}{n}\tfrac{b^2}{2} + 7b$$

and

$$U(g,P_n) = \sum_{i=1}^{n} M_i(x_i - x_{i-1}) = \sum_{i=1}^{n}[3\tfrac{ib}{n} + 7]\tfrac{b}{n} = 3\tfrac{n+1}{n}\tfrac{b^2}{2} + \tfrac{7b}{n}\sum_{i=1}^{n} 1 = 3\tfrac{n+1}{n}\tfrac{b^2}{2} + 7b,$$

so that

$$U(g,P_n) - L(g,P_n) = \tfrac{3b^2}{2}\left(\tfrac{n+1}{n} - \tfrac{n-1}{n}\right) = \tfrac{3b^2}{n} = \tfrac{1{,}200}{n}$$

and we can then proceed as above. □

Q2. (a) Calculate the Riemann–Darboux sums $L(f, P)$ and $U(f, P)$ for each of the following functions f and partitions P:

(i) $f : [0, 10] \to \mathbb{R},\ f(x) = x^2,\ P = \{0, 2, 7, 10\}$

(ii) $f : [0, 10] \to \mathbb{R},\ f(x) = e^{-x},\ P = \{0, 1, 5, 8, 9, 10\}$

(iii) $f : [0, \pi] \to \mathbb{R},\ f(x) = \sin(x),\ P = \{0, \frac{\pi}{2}, \frac{3\pi}{4}, \pi\}$

(b) Use the Riemann–Darboux sums calculator (Spring Week 1 Materials on Canvas) to visualize and check your answers above (note that $\frac{3\pi}{4}$ is entered as 3*Pi/4).

(c) For each of the functions f above, use the Riemann–Darboux sums calculator to calculate $L(f, P_n)$ and $U(f, P_n)$ when $n = 5$, $n = 10$ and $n = 100$, where P_n is the partition of the domain of f into n subintervals of equal width.

(d) For each $\delta \in (0, 1/10)$, find an expression for the sums $L(f, P_\delta)$ and $U(f, P_\delta)$ when $f : [-2, 2] \to \mathbb{R},\ f(x) = \begin{cases} 5, & x \in \mathbb{N} \\ 3, & x \notin \mathbb{N} \end{cases}$ and $P_\delta = \{-2, 1 - \delta, 1 + \delta, 2 - \delta, 2\}$.

Solution. (a)(i) For $f:[0,10]\to\mathbb{R},\ f(x)=x^2,\ P=\{0,2,7,10\}$, we calculate

$$\begin{aligned} L(f,P) &= \textstyle\sum_{i=1}^{3} m_i(x_i-x_{i-1}) \\ &= 0(2-0)+4(7-2)+49(10-7) \\ &= 167 \end{aligned}$$

and

$$\begin{aligned} U(f,P) &= \textstyle\sum_{i=1}^{3} M_i(x_i-x_{i-1}) \\ &= 4(2-0)+49(7-2)+100(10-7) \\ &= 553. \end{aligned}$$

(a)(ii) For $f:[0,10]\to\mathbb{R},\ f(x)=e^{-x},\ P=\{0,1,5,8,9,10\}$, we calculate

$$\begin{aligned} L(f,P) &= \textstyle\sum_{i=1}^{5} m_i(x_i-x_{i-1}) \\ &= e^{-1}(1-0)+e^{-5}(5-1)+e^{-8}(8-5) \\ &\qquad +e^{-9}(9-8)+e^{-10}(10-9) \\ &= e^{-1}+4e^{-5}+3e^{-8}+e^{-9}+e^{-10} \end{aligned}$$

and

$$\begin{aligned} U(f,P) &= \textstyle\sum_{i=1}^{5} M_i(x_i-x_{i-1}) \\ &= e^{-0}(1-0)+e^{-1}(5-1)+e^{-5}(8-5) \\ &\qquad +e^{-8}(9-8)+e^{-9}(10-9) \\ &= 1+4e^{-1}+3e^{-5}+e^{-8}+e^{-9}. \end{aligned}$$

(a)(iii) For $f:[0,\pi]\to\mathbb{R},\ f(x)=\sin(x),\ P=\{0,\frac{\pi}{2},\frac{3\pi}{4},\pi\}$, we calculate

$$\begin{aligned} L(f,P) &= \textstyle\sum_{i=1}^{3} m_i(x_i-x_{i-1}) \\ &= \sin(0)(\tfrac{\pi}{2}-0)+\sin(\tfrac{3\pi}{4})(\tfrac{3\pi}{4}-\tfrac{\pi}{2})+\sin(\pi)(\pi-\tfrac{3\pi}{4}) \\ &= \tfrac{\sqrt{2}}{2}\tfrac{\pi}{4} \end{aligned}$$

and

$$\begin{aligned} U(f,P) &= \textstyle\sum_{i=1}^{3} M_i(x_i-x_{i-1}) \\ &= \sin(\tfrac{\pi}{2})(\tfrac{\pi}{2}-0)+\sin(\tfrac{\pi}{2})(\tfrac{3\pi}{4}-\tfrac{\pi}{2})+\sin(\tfrac{3\pi}{4})(\pi-\tfrac{3\pi}{4}) \\ &= \tfrac{\pi}{2}+\tfrac{\pi}{4}+\tfrac{\sqrt{2}}{2}\tfrac{\pi}{4} \\ &= (3+\tfrac{\sqrt{2}}{2})\tfrac{\pi}{4}. \end{aligned}$$

(b)(i) For $f:[0,10]\to\mathbb{R},\ f(x)=x^2,\ P=\{0,2,7,10\}$, the Riemann–Darboux Sums Calculator depicts the areas covered by $L(f,P)$ and $U(f,P)$ in Figure 2 below.

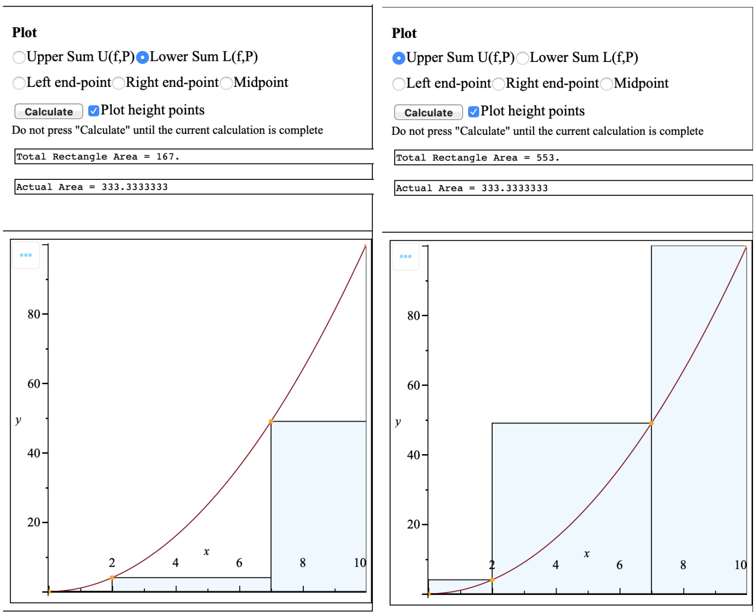

FIGURE 2. Two graphs of $f : [0, 10] \to \mathbb{R},\ f(x) = x^2$, showing the areas covered by $L(f, P)$ and $U(f, P)$ for $P = \{0, 2, 7, 10\}$.

(b)(ii) For $f : [0, 10] \to \mathbb{R},\ f(x) = e^{-x},\ P = \{0, 1, 5, 8, 9, 10\}$, the Riemann–Darboux Sums Calculator depicts the areas covered by $L(f, P)$ and $U(f, P)$ in Figure 3 below.

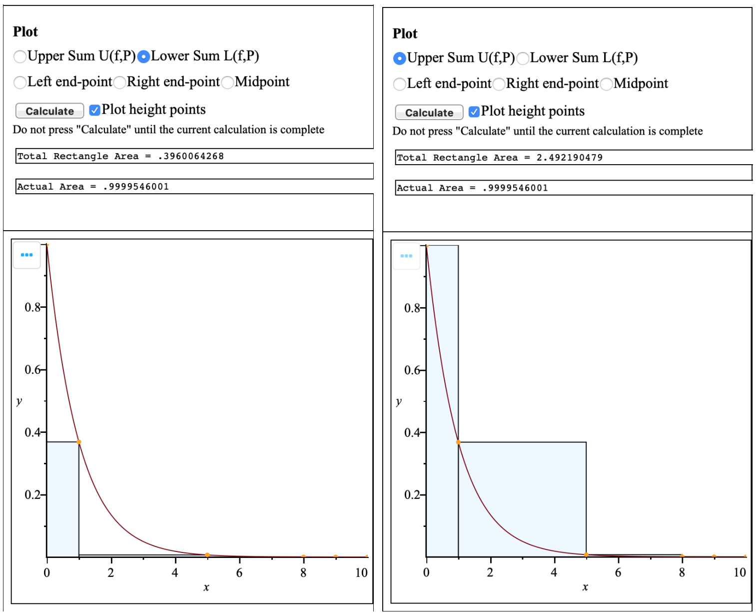

FIGURE 3. Two graphs of $f : [0, 10] \to \mathbb{R},\ f(x) = e^{-x}$, showing the areas covered by $L(f, P)$ and $U(f, P)$ for $P = \{0, 1, 5, 8, 9, 10\}$.

(b)(iii) For $f : [0, \pi] \to \mathbb{R},\ f(x) = \sin(x),\ P = \{0, \frac{\pi}{2}, \frac{3\pi}{4}, \pi\}$, the Riemann–Darboux Sums Calculator depicts the areas covered by $L(f, P)$ and $U(f, P)$ in Figure 4 below.

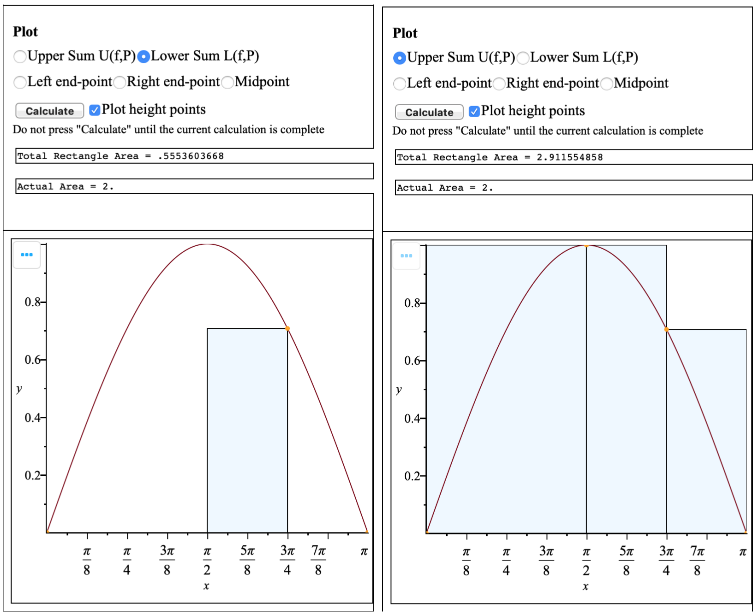

FIGURE 4. Two graphs of $f : [0, \pi] \to \mathbb{R},\ f(x) = \sin(x)$, showing the areas covered by $L(f, P)$ and $U(f, P)$ for $P = \{0, \frac{\pi}{2}, \frac{3\pi}{4}, \pi\}$.

(c) Table 1 shows the values (correct to three decimal places) for $L(f, P_n)$ and $U(f, P_n)$ when $n = 5$, $n = 10$ and $n = 100$ for each function f from part (a). These values were found using the Riemann–Darboux Sums Calculator:

	$L(f, P_5)$	$L(f, P_{10})$	$L(f, P_{100})$	Area	$U(f, P_{100})$	$U(f, P_{10})$	$U(f, P_5)$
(i)	240	285	328.35	333.333	338.35	385	440
(ii)	0.313	0.582	0.951	1.000	1.051	1.582	2.313
(iii)	1.336	1.669	1.968	2	2.030	2.298	2.193

TABLE 1. The values of $L(f, P_n)$ and $U(f, P_n)$ when $n = 5$, $n = 10$ and $n = 100$ for each function f from part (a).

(d) For $f : [-2, 2] \to \mathbb{R},\ f(x) = \begin{cases} 5, & x \in \mathbb{N} \\ 3, & x \notin \mathbb{N} \end{cases}$ and $P_\delta = \{-2, 1-\delta, 1+\delta, 2-\delta, 2\}$, we calculate

$$\begin{aligned} L(f, P_\delta) &= \textstyle\sum_{i=1}^{4} m_i(x_i - x_{i-1}) \\ &= 3(3-\delta) + 3(2\delta) + 3(1-2\delta) + 3(\delta) \\ &= 12 \end{aligned}$$

and

$$\begin{aligned} U(f, P_\delta) &= \textstyle\sum_{i=1}^{4} M_i(x_i - x_{i-1}) \\ &= 3(3-\delta) + 5(2\delta) + 3(1-2\delta) + 5(\delta) \\ &= 12 + 6\delta \end{aligned}$$

for each $\delta \in (0, 1/10)$. □

Q3. It is proved in Lecture 2.1 of the Integration Lecture Notes that if $f : [a, b] \to [0, \infty)$ is a bounded function, where $-\infty < a < b < \infty$, and P, Q are partitions of $[a, b]$ such that $P \subseteq Q$, then $L(f, P) \leq L(f, Q)$. List the changes that are needed to prove that $U(f, P) \geq U(f, Q)$, and in particular, explain how the inequalities $m_1 \leq m_1'$ and $m_1 \leq m_1''$ need to be modified.

Solution. We follow the same procedure as in the Lecture Notes, except instead of considering the quantities m_1, m_1', and m_1'', we introduce

$$\begin{aligned} M_1 &:= \sup\{f(x) : x \in [x_0, x_1]\}, \\ M_1' &:= \sup\{f(x) : x \in [x_0, y]\}, \\ M_1'' &:= \sup\{f(x) : x \in [y, x_1]\}. \end{aligned}$$

The inequalities $m_1 \leq m_1'$ and $m_1 \leq m_1''$ are then reversed in that $M_1 \geq M_1'$ and $M_1 \geq M_1''$ to obtain

$$M_1(x_1 - x_0) \geq M_1'(y - x_0) + M_1''(x_1 - y)$$

and thus ultimately $U(f, P) \geq U(f, Q)$. $\square$

Q4. Let $f : [0, b] \to [0, \infty)$ be defined by $f(x) = x^2$, where $b \in (0, \infty)$:

(a) For each $n \in \mathbb{N}$, let $P_n = \{x_i : i = 0, 1, \ldots, n\}$ denote the partition of $[0, b]$ into n subintervals of equal width. Express x_i in terms of i and b.

(b) Use the formula $\sum_{j=1}^{k} j^2 = \frac{1}{6}k(k + 1)(2k + 1)$ to prove that

$$L(f, P_n) = \frac{b^3}{6n^3}(n - 1)n(2n - 1) \quad \text{and} \quad U(f, P_n) = \frac{b^3}{6n^3}n(n + 1)(2n + 1).$$

(c) Find, with proof, the values $\sup\{L(f, P_n) : n \in \mathbb{N}\}$ and $\inf\{U(f, P_n) : n \in \mathbb{N}\}$.

(d) Find, with proof, the lower integral $\underline{\int_0^b} f$ and the upper integral $\overline{\int_0^b} f$.

(e) Prove that f is bounded and integrable, then find $\int_0^b f$ (without using calculus).

Solution. (a) For each $n \in \mathbb{N}$, we have

$$P_n = \{0, \tfrac{b}{n}, \tfrac{2b}{n}, \ldots, \tfrac{(n-1)b}{n}, 1\} = \{\tfrac{ib}{n} : i = 0, 1, \ldots, n\},$$

so $x_i = \frac{ib}{n}$ for each $i \in \{0, 1, \ldots, n\}$.

(b) For each $i \in \{0, 1, \ldots, n\}$, we have

$$\begin{aligned} m_i &= \inf\{x^2 : x \in [\tfrac{(i-1)b}{n}, \tfrac{ib}{n}]\} = \left(\tfrac{(i-1)b}{n}\right)^2, \\ M_i &= \sup\{x^2 : x \in [\tfrac{(i-1)b}{n}, \tfrac{ib}{n}]\} = \left(\tfrac{ib}{n}\right)^2. \end{aligned}$$

We use these to obtain

$$\begin{aligned} L(f, P_n) &= \textstyle\sum_{i=1}^{n} m_i(x_i - x_{i-1}) \\ &= \textstyle\sum_{i=1}^{n} \left(\tfrac{(i-1)b}{n}\right)^2 \tfrac{b}{n} \\ &= \tfrac{b^3}{n^3} \textstyle\sum_{j=0}^{n-1} j^2 \\ &= \tfrac{b^3}{n^3} \textstyle\sum_{j=1}^{n-1} j^2 \\ &= \tfrac{b^3}{6n^3}(n - 1)n(2n - 1), \end{aligned}$$

where we used the given sum with $k = n - 1$ in the penultimate equality, and

$$\begin{aligned} U(f, P_n) &= \textstyle\sum_{i=1}^n M_i(x_i - x_{i-1}) \\ &= \textstyle\sum_{i=1}^n \left(\frac{ib}{n}\right)^2 \frac{b}{n} \\ &= \frac{b^3}{n^3} \textstyle\sum_{i=1}^n i^2 \\ &= \frac{b^3}{6n^3}(n+1)n(2n+1), \end{aligned}$$

where we used the given sum with $k = n$ in the penultimate equality.

(c) We can now calculate

$$\sup\{L(f, P_n) : n \in \mathbb{N}\} = \sup\{\tfrac{b^3}{6}(2 - \tfrac{3}{n} + \tfrac{1}{n^2}) : n \in \mathbb{N}\} = \tfrac{b^3}{3}$$

and

$$\inf\{U(f, P_n) : n \in \mathbb{N}\} = \inf\{\tfrac{b^3}{6}(2 + \tfrac{3}{n} + \tfrac{1}{n^2}) : n \in \mathbb{N}\} = \tfrac{b^3}{3}.$$

To prove these facts, it suffices to observe that $a_n := \frac{b^3}{6}(2 - \frac{3}{n} + \frac{1}{n^2})$ is an increasing sequence with $\lim_{n\to\infty} a_n = \frac{b^3}{3}$, whilst $b_n := \frac{b^3}{6}(2 + \frac{3}{n} + \frac{1}{n^2})$ is a decreasing sequence with limit $\lim_{n\to\infty} b_n = \frac{b^3}{3}$.

(d) Now, as

$$\{L(f, P_n) : n \in \mathbb{N}\} \subseteq \{L(f, P) : P \text{ is a partition of } [0, b]\},$$

we have

$$\underline{\int_0^b} f := \sup_P L(f, P) \geq \sup_{n\in\mathbb{N}} L(f, P_n) = \tfrac{b^3}{3}.$$

Moreover, as

$$\{U(f, P_n) : n \in \mathbb{N}\} \subseteq \{U(f, P) : P \text{ is a partition of } [0, b]\},$$

we have

$$\overline{\int_0^b} f := \inf_P U(f, P) \leq \inf_{n\in\mathbb{N}} U(f, P_n) = \tfrac{b^3}{3}.$$

We also know from Proposition 2.2.2 in the Lecture Notes that the lower integrals is always less than or equal to the upper integral, hence

$$\tfrac{b^3}{3} \leq \underline{\int_0^b} f \leq \overline{\int_0^b} f \leq \tfrac{b^3}{3},$$

which implies that $\frac{b^3}{3} = \underline{\int_0^b} f = \overline{\int_0^b} f = \frac{b^3}{3}$.

(e) The preceding equality proves that f is integrable with $\int_0^b f = \frac{b^3}{6}$. $\square$

Extra Questions

EQ1. (a) Find, with proof, the infimum and supremum of each of the following sets:

(i) $A = (0, 1] \cup (2, 3]$

(ii) $B = \{x^2 - 4x + 5 : x \in (1, 3]\}$

(iii) $C = \{(2n + 3)/n : n \in \mathbb{N}\}$

(iv) $D = \{n^2 - 6n + 10 : n \in \mathbb{N}\}$

(b) Let X denote a nonempty bounded subset of $\mathbb{R}$. Prove that for each $\epsilon > 0$, there exists x in X such that $\inf X \leq x < \inf X + \epsilon$.

EQ2. For each $n \in \mathbb{N}$, let P_n denote the partition of $[0, 1]$ into n subintervals of equal width. Find an expression, possibly involving summation notation, for the sums $L(f, P_n)$ and $U(f, P_n)$ for $f : [0, 1] \to \mathbb{R}$ in each of the following cases:

(a) $f(x) = x^2 + x$

(b) $f(x) = \cos(x)$

(c) $f(x) = \begin{cases} 5, & x \in \mathbb{Q} \\ 3, & x \notin \mathbb{Q} \end{cases}$

(d) $f(x) = \begin{cases} 1, & x \in \{0,1\}, \\ 0, & x \notin \{0,1\}. \end{cases}$

EQ3. Let P_1 denote a partition of $[a,b]$ and $P_2 = P_1 \cup \{c\}$, where $-\infty < a < c < b < \infty$. Use results from lectures to prove that $U(f,P_2) - L(f,P_2) \le U(f,P_1) - L(f,P_1)$.

EQ4. Let $f : [a,b] \to \mathbb{R}$ be defined by $f(x) = x^3$, where $0 \le a < b < \infty$. Use the procedure outlined in **Q4** and the formula $\sum_{j=1}^{k} j^3 = \frac{1}{4}k^4 + \frac{1}{2}k^3 + \frac{1}{4}k^2$ to prove that f is integrable with $\int_a^b f = \frac{1}{4}b^4 - \frac{1}{4}a^4$.