UNIVERSITY OF BIRMINGHAM

School of Physics and Astronomy

DEGREE OF BSc & MSci WITH HONOURS

FIRST YEAR EXAMINATION

03 17483/03 20835

LC Chaos and Non-Linear Systems A/B

The total time allowed is 1 hour

SUMMER EXAMINATIONS 2018

*Students should attempt **two** questions.*

*If you attempt more than two questions credit will be given for the **best two** answers.*

*The **approximate** allocation of marks to each part of a question is shown in brackets **[ ]**.*

Calculators may be used in this examination but must not be used to store text. Calculators with the ability to store text should have their memories deleted prior to the start of the examination.

1. (a) Define what is meant by the terms *stable fixed point*, *unstable fixed point*, *stable 2-cycle* and *unstable 2-cycle* of an iterated map,

$$x_{n+1} = f(x_n).$$

State the conditions for stability and instability of a fixed point. Deduce from these the corresponding conditions for stability and instability of a 2-cycle. **[4]**

Consider the iterated map defined by

$$x_{n+1} = f(x) = rx_n - x_n^3,$$

where $r \geq 0$ is a variable parameter.

(b) Find the three fixed points of this map, and analyse their existence and stability as the parameter r is varied. **[3]**

(c) Given that the equation $f(f(x)) = x$ can be rewritten in the form

$$x(x^2 - r + 1)(x^2 - r - 1)(x^4 - rx^2 + 1) = 0,$$

deduce all the 2-cycles of the map. For what values of r are they stable? **[3]**
[Hint: You may find useful the identities

$$\left[\frac{\sqrt{r+2} \pm \sqrt{r-2}}{2}\right]^2 = \frac{r \pm \sqrt{r^2-4}}{2}$$
$$\left[\frac{\sqrt{r+2} + \sqrt{r-2}}{2}\right]\left[\frac{\sqrt{r+2} - \sqrt{r-2}}{2}\right] = 1.]$$

2. A competition model is given by the differential equations

$$\frac{dR}{dt} = f(R,S) = r_1 R(1-R) - r_1 a_1 RS$$
$$\frac{dS}{dt} = g(R,S) = r_2 S(1-S) - r_2 a_2 RS,$$

where $R(t)$ and $S(t)$ are the (suitably scaled) rabbit and sheep populations at time t, and r_1, r_2, a_1, a_2 are positive constants.

(a) Explain the biological meaning of the terms $r_1 R$, $-r_1 R^2$, $-r_1 a_1 RS$, $r_2 S$, $-r_2 S^2$, and $-r_2 a_2 RS$, which occur on the right hand side of the differential equations. What is the nature of the scaling which has been applied to reduce the equations to the above form? **[2]**

(b) Explain what is meant by a fixed point, (R^*, S^*), and deduce the four fixed points of the above system, and the values of parameters a_1 and a_2 for which they are biologically sensible. **[3]**

A linear stability analysis about a fixed point of the above dynamical system leads to solutions of the form

$$R - R^* = he^{\lambda t}, \quad S - S^* = ke^{\lambda t},$$

where the λ are solutions of the quadratic equation

$$\lambda^2 - (A+D)\lambda + AD - BC = 0,$$

and A, B, C, D, are given by

$$A = \frac{\partial f}{\partial R}(R^*, S^*), \quad B = \frac{\partial f}{\partial S}(R^*, S^*), \quad C = \frac{\partial g}{\partial R}(R^*, S^*), \quad D = \frac{\partial g}{\partial S}(R^*, S^*).$$

(c) What conditions must the two solutions, $\lambda_{1,2}$, of the above quadratic equation satisfy for the fixed point to be: (i) a stable node; (ii) an unstable node; (iii) a saddle point? **[2]**

(d) Deduce the nature of the four fixed points as a function of the parameters a_1 and a_2. [Hint: If either $B = 0$ or $C = 0$, the two solutions of the quadratic equation are $\lambda = A$ and $\lambda = D$.] **[3]**

3. (a) Define what is meant by *self-similarity*, and give an example of where it may be seen in a system that also exhibits chaos. **[2]**

   (b) Show that a square has fractal dimension 2, using a method which may also be used for non-integer dimensions. Carefully explain the steps in your calculation. **[2]**

   (c) The *Cantor set* can be constructed by taking the line $0 < x < 1$, and removing the middle third. This is repeated at each iteration. Draw the process, and find the associated fractal dimension. **[2]**

   (d) By considering their base 3 representation, show that the elements of the Cantor set may be put into one-to-one correspondence with the elements of the original line $0 < x < 1$. **[2]**

   (e) Show that $4/13$ is a member of the Cantor set, and deduce the element of the original line, $0 < x < 1$, which it may be put into one-to-one correspondence with. **[2]**

Do not complete the attendance slip, fill in the front of the answer book or turn over the question paper until you are told to do so

Important Reminders

- Coats/outwear should be placed in the designated area.
- Unauthorised materials (e.g. notes or Tippex) must be placed in the designated area.
- Check that you do not have any unauthorised materials with you (e.g. in your pockets, pencil case).
- Mobile phones and smart watches **must** be switched off and placed in the designated area or under your desk. They must not be left on your person or in your pockets.
- You are not permitted to use a mobile phone as a clock. If you have difficulty seeing a clock, please alert an Invigilator.
- You are not permitted to have writing on your hand, arm or other body part.
- Check that you do not have writing on your hand, arm or other body part – if you do, you must inform an Invigilator immediately
- Alert an Invigilator immediately if you find any unauthorised item upon you during the examination.

Any students found with non-permitted items upon their person during the examination, or who fail to comply with Examination rules may be subject to Student Conduct procedures.