University of Birmingham
School of Mathematics

1RA - Real Analysis: Differentiation Autumn 2024

Practice Problem Sheet 1
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Questions

Q1. Let $f : (-\infty, \alpha) \to \mathbb{R}$ for some $\alpha \in \mathbb{R}$. Define what is meant by $\lim_{x\to-\infty} f(x) = A$ for $A \in \mathbb{R}$. Prove $\lim_{x\to-\infty} \frac{1}{x} = 0$ by using this definition.

Q2. Determine the limit $\lim_{x\to-3} 3x$ and prove that your answer is correct by directly appealing to the definition of the limit.

Q3. Show that
$$\lim_{x\to 8} x^2 = 64,$$
by using the definition of limit.

Q4. Make minor adaptations to the proof of Theorem 2.6 to prove the following theorem.

Theorem 1 (Squeeze). *Suppose that f, g and h are real functions, and that for some $\alpha > 0$,*

$$f(x) \le h(x) \le g(x) \tag{1}$$

for all $x \in (\alpha, \infty)$, and that for some $A \in \mathbb{R}$,
$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = A.$$
Then $\lim_{x\to\infty} h(x) = A$.

Q5. Let $f, g : \mathbb{R} \to \mathbb{R}$. For each of the following statements, either prove it is true using the definition of the limit or give a counterexample to show that it is false.

(a) Suppose that $\lim_{x\to\infty} f(x) = a$ and $\lim_{x\to\infty} g(x) = b$. If $f(x) < g(x)$ for all $x \in \mathbb{R}$, then $a < b$.
(b) If $\lim_{x\to a} f(x) = \ell$ and $\lim_{x\to a} g(x) = \infty$, then $\lim_{x\to a} f(x)g(x) = \infty$.
(c) If $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = \ell$, then $f(a) = \ell$.
(d) If $\lim_{x\to b} f(x) = c$ and $\lim_{x\to a} g(x) = b$, then $\lim_{x\to a} f\big(g(x)\big) = c$.

Q6. Let $A \subseteq \mathbb{R}$. Let $f : A \to \mathbb{R}$ be continuous. Let $a \in A$. Prove that, if $f(a) > 0$, then there exists $\delta > 0$ such that $f(x) > 0$ for all $x \in A \cap (a - \delta, a + \delta)$.
[This is sometimes called the “sign-preserving property” of continuous functions: informally, if a continuous function is positive at a certain point, then it is also positive at nearby points.]

Q7. Demonstrate (referring to either definitions or theorems) that the following limits do not exist.

(a) $\lim_{x\to\infty} \cos x$.

(b) $\lim_{x\to 0} e^{-1/x}$.

Q8. Determine the value of the following limits. You can use any of the definitions and results discussed in lectures, provided you clearly state what you are using. **Only those materials that have been discussed in lectures can be used here. For instance, you can NOT use L'Hospital's rule here**.

(i) $\displaystyle\lim_{x\to 1} \frac{x^2-1}{2x^2-x-1}$.

(ii) $\displaystyle\lim_{x\to 0} \frac{(x-1)^3+(1-3x)}{x^2+2x^3}$.

(iii) $\displaystyle\lim_{x\to 1} \frac{x^n-1}{x^m-1}$, where $n, m \in \mathbb{N}$.

(iv) $\displaystyle\lim_{x\to 4} \frac{\sqrt{1+2x}-3}{\sqrt{x}-2}$.

(v) $\displaystyle\lim_{x\to 0} \frac{\tan x - \sin x}{x^3}$.

(vi) $\displaystyle\lim_{x\to\infty} x \sin\frac{1}{x}$.

Q9. Suppose $X \subset \mathbb{R}$ and let $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}$ be continuous functions. Define $p : X \to \mathbb{R}$ by $p(x) := \max\{f(x), g(x)\}$ and $q : X \to \mathbb{R}$ by $q(x) = \min\{f(x), g(x)\}$. Prove that p and q are continuous.

Q10. Find an example of a bounded function $f : [0,1] \to \mathbb{R}$ that has neither an absolute minimum nor an absolute maximum.

Q11. Let $f : (0,1) \to \mathbb{R}$ be a continuous function such that

$$\lim_{x\to 0} f(x) = \lim_{x\to 1} f(x) = 0.$$

Show that f achieves either an absolute minimum or an absolute maximum on $(0,1)$ (but perhaps not both).

Q12. Suppose for $f : [0,1] \to \mathbb{R}$ we have $|f(x)-f(y)| \le K|x-y|$ for all $x, y \in [0,1]$, and $f(0) = f(1) = 0$. Prove that $|f(x)| \le \frac{K}{2}$ for all $x \in [0,1]$.
Note: A function $f : X \to \mathbb{R}$ is called Lipschitz continuous if there exists a $K > 0$ such that

$$|f(x)-f(y)| \le K|x-y| \quad \forall\, x, y \in X.$$

Extra Questions

EQ1. For each of the following statements, either prove that it is true, or give a counterexample to show that it is false. You can use any of the definitions and results discussed in lectures, provided you clearly state what you are using.

(i) If $f : (0, 1) \to \mathbb{R}$ is continuous, then f is bounded.
(ii) If $g : (0, 1) \to \mathbb{R}$ is continuous, then g is differentiable.
(iii) If $k : [0, 1] \to \mathbb{R}$ is differentiable, then k is bounded.

EQ2. Determine the following limits and prove that your answer is correct by directly appealing to the definition of the limit.

(a) $\displaystyle\lim_{x\to\infty} \frac{3x^3 - 5x^2 - 13}{2x^3 + 1}$.
(b) $\displaystyle\lim_{x\to 1^+} 2x^2 - 3x + 5$.
(c) $\displaystyle\lim_{x\to 2^-} 1/(1 - x)$.
(d) $\displaystyle\lim_{x\to\infty} (1/x) \sin x$.

EQ3. Prove that the following limits exist and determine their value. You can use any of the definitions and results discussed in lectures, provided you clearly state what you are using.

(i) $\displaystyle\lim_{x\to -\infty} 2x^2 - 3x + \arctan x$.
(ii) $\displaystyle\lim_{x\to 2} \frac{1}{1 - x}$.
(iii) $\displaystyle\lim_{x\to 1/2} \frac{4x^2 - 1}{2x - 1}$.

EQ4. Suppose $g(x)$ is a monic polynomial of even degree d, that is,

$$g(x) = x^d + b_{d-1}x^{d-1} + \cdots + b_1 x + b_0 \quad \forall\, x \in \mathbb{R},$$

for $b_0, b_1, \ldots, b_{d-1} \in \mathbb{R}$. Show that g achieves an absolute minimum on $\mathbb{R}$. Use this to conclude that if $f(x)$ is a polynomial of degree d and $f(\mathbb{R}) = \mathbb{R}$, then d is odd.

EQ5. The number $x \in [0, 1]$ is called a fixed point of $f : [0, 1] \to [0, 1]$ if $x = f(x)$. If $f : [0, 1] \to [0, 1]$ is continuous, show that f has a fixed point.