Example Sheet 4: Diagonalisation and phase space portraits

1. Diagonalise the matrix

$$\begin{bmatrix} -3 & 4 \\ 4 & 3 \end{bmatrix}$$

finding the eigenvalues and the right eigenvectors.

2. Diagonalise the matrix

$$\begin{bmatrix} 0 & 12 \\ 12 & 10 \end{bmatrix}$$

finding the eigenvalues and the right eigenvectors.

3. Diagonalise the matrix

$$\begin{bmatrix} -4 & 5 \\ -5 & 2 \end{bmatrix}$$

finding the eigenvalues and both types of eigenvector.

4. Two energy surfaces take the forms

$$E + d = \frac{1}{2}a\left(\frac{d\theta_1}{dt}\right)^2 + \frac{1}{2}b\left(\frac{d\theta_2}{dt}\right)^2 + c\frac{d\theta_1}{dt}\frac{d\theta_2}{dt}$$

and

$$E - d = \frac{1}{2}a\left(\frac{d\theta_1}{dt}\right)^2 + \frac{1}{2}b\left(\frac{d\theta_2}{dt}\right)^2 - c\frac{d\theta_1}{dt}\frac{d\theta_2}{dt}$$

Diagonalise the two matrices

$$M_+ \equiv \begin{bmatrix} a & c \\ c & b \end{bmatrix} \qquad M_- \equiv \begin{bmatrix} a & -c \\ -c & b \end{bmatrix}$$

where $a > b$ and $c > 0$ and normalise the eigenvectors with $\mathbf{v}^T\mathbf{v}$=1. Employ

$$\begin{bmatrix} \frac{d\theta_1}{dt} \\ \frac{d\theta_2}{dt} \end{bmatrix} \equiv x_1\mathbf{v}_1 + x_2\mathbf{v}_2$$

to recognise the energy surfaces.

5. Find the fixed points and stability matrix for the non-linear oscillator

$$\frac{d^2x}{dt^2} = x - x^3$$

where

$$x_1 = x \qquad x_2 = \frac{dx}{dt}$$

Diagonalise the stability matrix at the fixed points and depict the local trajectories in their vicinity. Can you find a conservation law? Depict the full phase space portrait.

6. Find the fixed points and stability matrix for the damped oscillator

$$\frac{d^2\theta}{dt^2} + 2\frac{d\theta}{dt} + 5\theta = 0$$

with the choice

$$x_1 = \theta \qquad\qquad x_2 = \frac{d\theta}{dt} + \theta$$

Diagonalise the stability matrix and find the general solution. Depict the phase space portrait.

7. Find the fixed points of the system

$$\frac{dx_1}{dt} = x_1(1-x_1)(2-x_2) \qquad\qquad \frac{dx_2}{dt} = x_2(1-x_2)(2-x_1)$$

and depict the local trajectories in the vicinity of each point. Can you find a conservation law? Depict the phase space portrait.

8. For a fairly general 2×2 matrix, $b > 0$ and $c > 0$,

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

show that the eigenvalues and eignvectors

$$M\mathbf{v} = \lambda\mathbf{v} \qquad\qquad \tilde{\mathbf{v}}^T M = \lambda\tilde{\mathbf{v}}^T$$

satisfy

$$\lambda_+ = \frac{a+d}{2} + \left[\left(\frac{a-d}{2}\right)^2 + bc\right]^{\frac{1}{2}} \equiv \frac{a+d}{2} + \Delta \quad \mathbf{v}_+ = \begin{bmatrix} \left[b\left(\Delta + \frac{a-d}{2}\right)\right]^{\frac{1}{2}} \\ \left[c\left(\Delta - \frac{a-d}{2}\right)\right]^{\frac{1}{2}} \end{bmatrix}$$

$$\tilde{\mathbf{v}}_+^T = \begin{bmatrix} \left[c\left(\Delta + \frac{a-d}{2}\right)\right]^{\frac{1}{2}} & \left[b\left(\Delta - \frac{a-d}{2}\right)\right]^{\frac{1}{2}} \end{bmatrix}$$

$$\lambda_- = \frac{a+d}{2} - \left[\left(\frac{a-d}{2}\right)^2 + bc\right]^{\frac{1}{2}} \equiv \frac{a+d}{2} - \Delta \quad \mathbf{v}_- = \begin{bmatrix} -\left[b\left(\Delta - \frac{a-d}{2}\right)\right]^{\frac{1}{2}} \\ \left[c\left(\Delta + \frac{a-d}{2}\right)\right]^{\frac{1}{2}} \end{bmatrix}$$

$$\tilde{\mathbf{v}}_-^T = \begin{bmatrix} -\left[c\left(\Delta - \frac{a-d}{2}\right)\right]^{\frac{1}{2}} & \left[b\left(\Delta + \frac{a-d}{2}\right)\right]^{\frac{1}{2}} \end{bmatrix}$$

and find $\tilde{\mathbf{v}}_+^T\mathbf{v}_+$, $\tilde{\mathbf{v}}_-^T\mathbf{v}_-$ and $\tilde{\mathbf{v}}_+^T\mathbf{v}_-$, $\tilde{\mathbf{v}}_-^T\mathbf{v}_+$.