University of Birmingham
School of Mathematics

Real Analysis – Integration – Spring 2025

Problem Sheet 9
Issued Spring Week 10

Instructions: You are strongly encouraged to attempt all of the Questions (Q) below, and as many of the Extra Questions (EQ) as you can, to help prepare for the final exam. Model solutions will only be released for the Questions (Q1–Q4).

Questions

Q1. (a) Suppose that $f : [0,4] \to [0,1]$ is given by
$$f(x) := \begin{cases} x, & \text{if } 0 \le x \le 1; \\ 1, & \text{if } 1 < x \le 4. \end{cases}$$
Find a solution $y : [0,4) \to \mathbb{R}$ of the initial value problem
$$y' = f(x), \quad y(0) = 1.$$
You must prove that your solution is indeed differentiable on $(0,4)$.

(b) Find a solution $y : [0,\infty) \to \mathbb{R}$ of the initial value problem
$$yy' = \log(x), \quad y(0) = 2.$$
You must justify all limit computations.

Q2. A swimming pool by the sea has a capacity of 5 000 000 L and the concentration of salt in the seawater is 0.045 kg L^{-1}. The pool is initially filled with pure water. The concentration of salt in the pool is then increased by pumping in seawater at a rate of 3 000 L min^{-1} whilst the pool is drained at the same rate. Assume that the mixture in the pool is instantly and uniformly mixed:

(a) Let $y(t)$ denote the mass (in kilograms) of salt in the pool at time t (in minutes) after mixing begins. Formulate an initial value problem to model the flow $y'(t)$.

(b) Find a solution to your initial value problem and determine how long it will take for the salt concentration in the swimming pool to reach 0.0035 kg L^{-1}?

(c) Suppose instead that the pump operates at 1 000 L min^{-1} whilst the pool is drained at 3 000 L min^{-1}. Formulate and solve an initial value problem to determine the mass of salt in the swimming pool after t minutes of mixing for all $t \in [0, +\infty)$.

Q3. Find the general solution of the following homogeneous equations on $\mathbb{R}$, and where specified, find a solution of the initial value problem or boundary value problem.

(a) $y'' - 7y' + 12y = 0$.

(b) $y'' = -64y, \quad y(0) = 0, \quad y'(0) = 3, \quad y : [0,\infty) \to \mathbb{R}$.

(c) $y'' - 2y' + y = 0, \quad y(0) = 1, \quad y(1) = 2, \quad y : [0,1] \to \mathbb{R}$.

Q4. Find the general solution of the following inhomogeneous equations on $\mathbb{R}$, and where specified, find a solution of the initial value problem or boundary value problem.

(a) $y'' - 2y' + 10y = e^x$.
(b) $y'' + 5y' + 4y = 3 - 2x, \quad y(0) = 0, \quad y'(0) = 0, \quad y : [0, \infty) \to \mathbb{R}$.
(c) $y'' + 9y = x \cos x, \quad y(0) = 1, \quad y(\frac{\pi}{2}) = \frac{1}{32}, \quad y : [0, \frac{\pi}{2}] \to \mathbb{R}$.

Extra Questions

EQ1. (a) Find three solutions of the differential equation $y' = \cos\left(3x + \frac{\pi}{3}\right)$ on $\mathbb{R}$.
(b) Find a solution $y : [0, \infty) \to \mathbb{R}$ of the initial value problem

$$x^2 y' = y^3, \quad y(0) = 0.$$

(c) Find solutions $y : [0, R) \to \mathbb{R}$, for some $R \in [0, \infty]$, of the initial value problem

$$y' = y^2 + y - 12, \quad y(0) = y_0$$

for each $y_0 \in \{2, 3, 5\}$.

EQ2. Find solutions $y : [0, \infty) \to \mathbb{R}$ of the following initial value problems:

(a) $y' + y = \cos(e^x), \quad y(0) = 9$.
(b) $y' + 2xy = 4x, \quad y(0) = y_0 \in \mathbb{R}$.
(c) $xy' = y + x^3 + 3x^2 - 2x, \quad y(0) = 0$.

EQ3. Suppose that a, b, $c \in \mathbb{R}$ with $a \neq 0$. Let $y_1 : [0, 1] \to \mathbb{R}$ and $y_2 : [0, 1] \to \mathbb{R}$ denote solutions of the respective boundary value problems below:

$$ay_1'' + by_1' + cy_1 = 0, \quad y_1(0) = 1, \quad y_1(1) = 3;$$
$$ay_2'' + by_2' + cy_2 = 0, \quad y_2(0) = 2, \quad y_2(1) = -5.$$

(a) Prove that $y_3 := 2y_1 + 3y_2$ is a solution of the boundary value problem

$$ay'' + by' + cy = 0, \quad y(0) = 8, \quad y(1) = -9.$$

(b) Let α, $\beta \in \mathbb{R}$. Find a solution $y_4 : [0, 1] \to \mathbb{R}$ of the boundary value problem

$$ay'' + by' + cy = 0, \quad y(0) = \alpha, \quad y(1) = \beta$$

in terms of y_1 and y_2.

EQ4. Hooke's Law states that a spring with spring constant $k > 0$ exerts a force $F = -ky$ (in Newtons) when stretched a distance y (in metres) from its equilibrium position. Suppose that an object with mass $m = 5$ kg is attached to the end of a spring with spring constant $k = 100$. Combine Hooke's Law with Newton's Law $F = my''$ to formulate initial value problems or boundary value problems, as appropriate, to model each of the following scenarios. In each case, determine the distance $y(t)$ of the object from its equilibrium position at time t (in seconds) after it is released:

(a) The object is released at rest at a distance of 1 m from its equilibrium position.
(b) The object is released at a distance of 1 m from its equilibrium position so that after 1 second it has travelled 0.7 m.
(c) The object is released at rest at a distance of 1 m from its equilibrium position and it is subject to an additional force of $5\sin(2\sqrt{5}t)$ N at time t.