Electromagnetism I – Problem sheet 6 – Solutions

Method 1. You can compute the electrostatic energy by computing the work that is needed to build up the cylinder step by step, in this case by bringing in cylindrical shells of charge with the same density.

1. Consider first the set up at an intermediate stage when you have already built a cylinder with a radius $r < a$, and charge density ρ.

   (a) What is the electric field E of this cylinder at radius $x > r$? Express the result as a function of x and ρ.
   The electric field at x , generated by the cylinder of radius r is:

$$E(x) = \frac{\rho \pi r^2}{2\pi\epsilon_0}\frac{1}{x} = \frac{\rho r^2}{2\epsilon_0}\frac{1}{x}$$

[1 mark]

   (b) What is the work, ΔW, done in bringing charge Δq from a radius $R > r$ down to the radius r?

$$\Delta W = -\int_R^r \Delta q E(x)\, dx = -\Delta q \int_R^r \frac{\rho r^2}{2\epsilon_0}\frac{dx}{x} = +\Delta q \frac{\rho r^2}{2\epsilon_0}\ln\frac{R}{r}$$

[2 marks]

2. As we build up the cylinder in steps, using cylindrical shells of infinitesimal width δr and length ℓ, write down an expression for Δq in terms of r, δr, ℓ and ρ.

$$\Delta q = \rho l 2\pi r \delta r$$

[1 mark]

3. Compute now the total work per unit length done in building up the full cylinder of radius a by integrating the work ΔW.

$$\Delta W = \Delta q \frac{\rho r^2}{2\epsilon_0}\ln\frac{R}{r} = \rho l 2\pi r \delta r \frac{\rho r^2}{2\epsilon_0}\ln\frac{R}{r}$$

$$\Delta W = \frac{\pi}{\epsilon_0}\rho^2 l r^3 \ln\frac{R}{r}\delta r$$

[1 mark]

   Hence infinitesimal work per unit length is

$$\delta w = \frac{\Delta W}{l} = \frac{\pi}{\epsilon_0}\rho^2 r^3 \ln\frac{R}{r}\delta r$$

[1 mark]

   Total work done:

$$w = \int_0^a dw = \frac{\pi}{\epsilon_0}\rho^2\left(\int_0^a r^3 \ln\frac{R}{r} dr\right) = \frac{\pi}{\epsilon_0}\rho^2\left(\frac{a^4}{16} + \frac{a^4}{4}\ln\frac{R}{a}\right)$$

[1 mark]

Method 2. You should obtain the same result if you compute the energy per unit length by applying the known result that the energy density of the electrostatic field is

$$\frac{dU}{dV} = \frac{1}{2}\epsilon_0 E^2 .$$

6. Compute the energy per unit length u_{ins} inside the cylinder.
   E-field inside the cylinder is $E_{in} = \frac{\rho r}{2\epsilon_0}$ and $dV = 2\pi rldr$ so energy inside cylinder is:

   $$U_{in} = \frac{1}{2}\epsilon_0 \int_0^a \left(\frac{\rho}{2\epsilon_0}r\right)^2 2\pi lrdr = \frac{\pi\rho^2 l}{4\epsilon_0}\int_0^a r^3 dr$$

   $$U_{in} = \frac{\pi\rho^2 l}{4\epsilon_0}\frac{a^4}{4} = \frac{\pi\rho^2 l}{\epsilon_0}\frac{a^4}{16}$$

   So energy per unit length inside is just:

   $$u_{in} = \frac{\pi\rho^2}{\epsilon_0}\frac{a^4}{16}$$

   [2 marks]

   Which is the first term from the result of the previous method.

7. Compute the energy per unit length u_{ext} external to the cylinder up to a radius $R > r$.
   The E-field outside the cylinder is $E_{out} = \frac{\rho a^2}{2\epsilon_0 r}$ so energy outside the cylinder is:

   $$Uout = \frac{1}{2}\epsilon_0 \int_a^R \left(\frac{\rho a^2}{2\epsilon_0 r}\right)^2 2\pi lrdr = \frac{\pi\rho^2 l}{4\epsilon_0}a^4 \ln\frac{R}{a}$$

   So energy per unit length outside is just:

   $$u_{out} = \frac{\pi\rho^2}{4\epsilon_0}a^4 \ln\frac{R}{a}$$

   [1 mark]