Electromagnetism

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Lecture 9

Capacitance

Week 5

Last Week - Dipoles

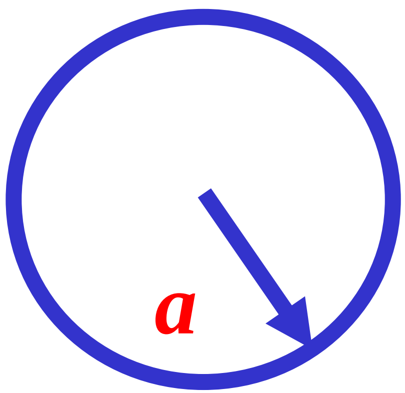
- Define dipole moment as $\underline{\boldsymbol{p}} = q\underline{\boldsymbol{a}}$
- $V_p \approx \dfrac{p\cos\theta}{4\pi\varepsilon_0 r^2}$ (for r >> a)
- $\underline{\boldsymbol{E}} \approx \dfrac{2\,p\cos\theta}{4\pi\varepsilon_0 r^3}\underline{\hat{\boldsymbol{r}}} + \dfrac{p\sin\theta}{4\pi\varepsilon_0 r^3}\underline{\widehat{\boldsymbol{\theta}}}$ (for r >> a)
- For Dipole in external uniform E-field
- $\underline{\boldsymbol{\tau}} = q\underline{\boldsymbol{a}} \wedge \underline{\boldsymbol{E}} = \underline{\boldsymbol{p}} \wedge \underline{\boldsymbol{E}}$
- $\boldsymbol{U} = -\underline{\boldsymbol{p}} \cdot \underline{\boldsymbol{E}}$

This Lecture - Capacitance

- Earthing / Grounding
- To introduce the concept of capacitance, C
  - Definition of capacitance
- Energy stored in a capacitor
- To calculate C of ideal capacitors
  - Parallel plates
  - Co-axial cables
  - Spherical capacitors

Earthing

Consider two conducting spheres



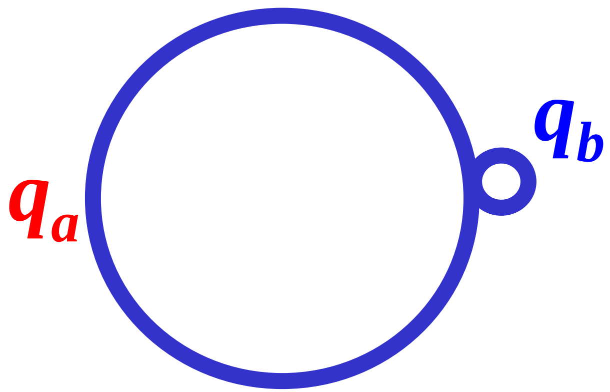
No charge

Radius *b*

Charge q

Earthing

Put spheres together



$$V_a = \left(\frac{1}{4\pi\varepsilon_0}\right)\frac{q_a}{a}$$

$$V_b = \left(\frac{1}{4\pi\varepsilon_0}\right)\frac{q_b}{b}$$

The two spheres acquire the same potential

Earthing

- Same potential implies: $\frac{q_a}{a} = \frac{q_b}{b}$
- Total charge is constant: $q = q_a + q_b$
- $\frac{q_a}{a} = \frac{q_b}{b} = \frac{q-q_a}{b} \rightarrow \ (a+b)q_a = aq$
- $q_a = \frac{aq}{a+b} \approx q$ if $a \gg b$
- Hence $q_b \approx 0$ (if $a \gg b$)

Earthing: Conclusion

- Earthed isolated charged bodies share their charge with the Earth - effectively lose that charge. The bodies and the Earth acquire a common potential - *called zero of potential*

Earth

- The potential of the Earth does not change

$$V_a = \left(\frac{1}{4\pi\varepsilon_0}\right)\frac{Q_a}{a}$$

- because a is very large
- Similar to the use of sea level as a reference for altitude

Earthing

- Earthing or grounding

Capacitance for Charge

- Q placed on a conductor changes the conductor's potential by $V \propto Q$
- We define the capacitance C of the conductor by the equation:

$$\boxed{C = \frac{Q}{V}}$$

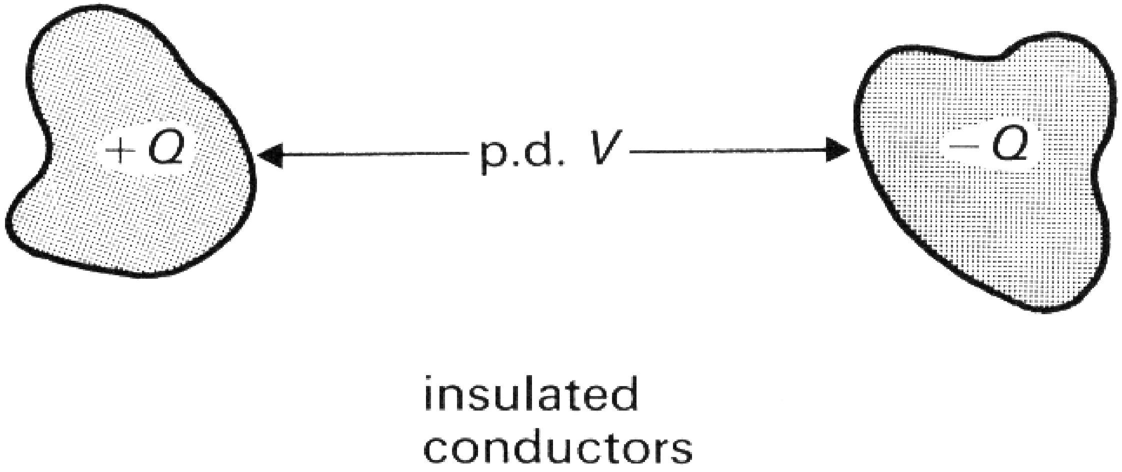
- Unit of C - Coulomb Volt^{-1} - Farad (F)
- Named after Faraday

Capacitance

- So anything that can hold a charge has a capacitance (not just parallel plates).
- Example: an isolated conducting sphere of radius R and carrying a charge Q:
- $V = \frac{Q}{4\pi\varepsilon_0 R} \rightarrow C = \frac{Q}{V} = 4\pi\varepsilon_0 R$
- A sphere of radius 9×10^9 m (more than $10^3 \times$ that of the Earth) would have a capacitance of about 1 farad.
- *Common capacitors in use: pF-μF*

Capacitors

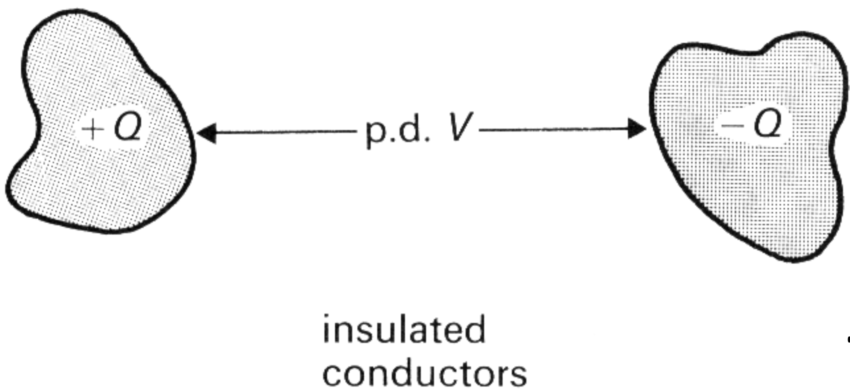
- It is a system designed for the storage of separated positive and negative charges



- Capacitors store charge and hence energy.

Energy Stored in Capacitor

- Transfer δq between the two conductors whilst at a potential difference of V_q



Work done
$\delta W = \delta q \; V_q$

Total work = $\int_0^Q V_q dq$

- $q = CV_d \rightarrow dq = C\, dV_q$
- Work done = $W = C \int_0^V V_q \, dV_q = \frac{1}{2} CV^2 = U$

Energy Stored in Capacitor

- Capacitance defined as:

$$C = \frac{Q}{V}$$

- Energy stored in the electric field between the "plates" of a capacitor

$$\boxed{U = \frac{1}{2}CV^2 = \frac{1}{2}QV = \frac{1}{2}\frac{Q^2}{C}}$$

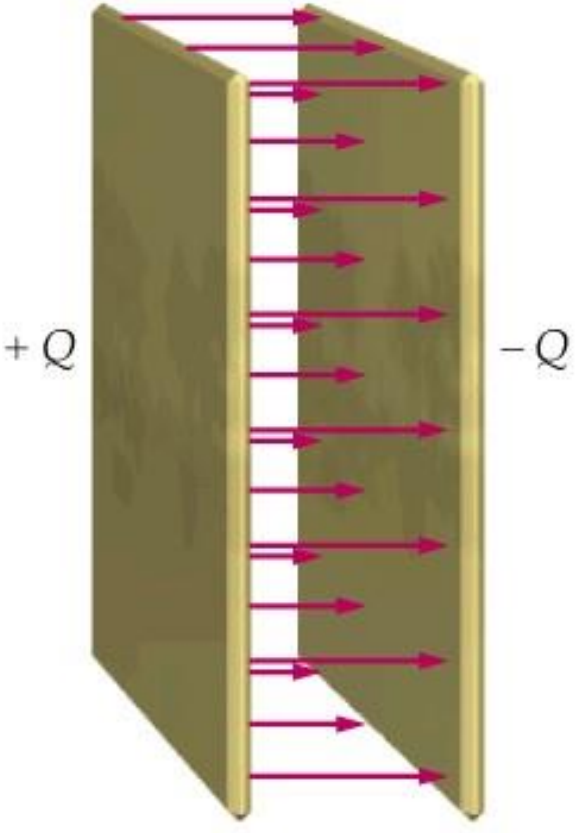
Calculation of Capacitance

- *Procedure:*

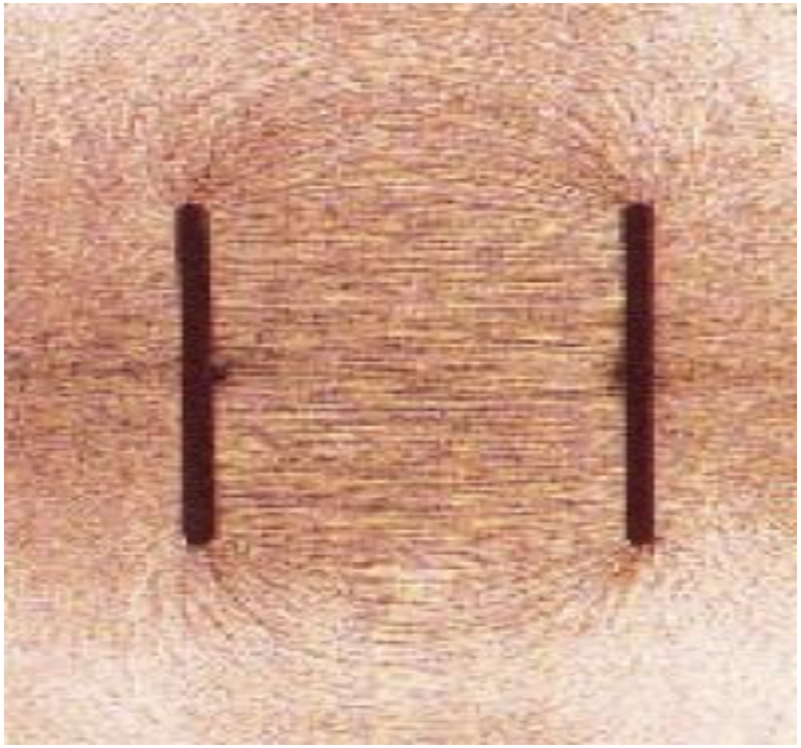
1. Determine E as a function of Q
2. Calculate the change of V
3. Apply $C = Q / V$

- We will consider 3 geometries - planar, cylindrical, and spherical

(1) The Parallel Plate Capacitor

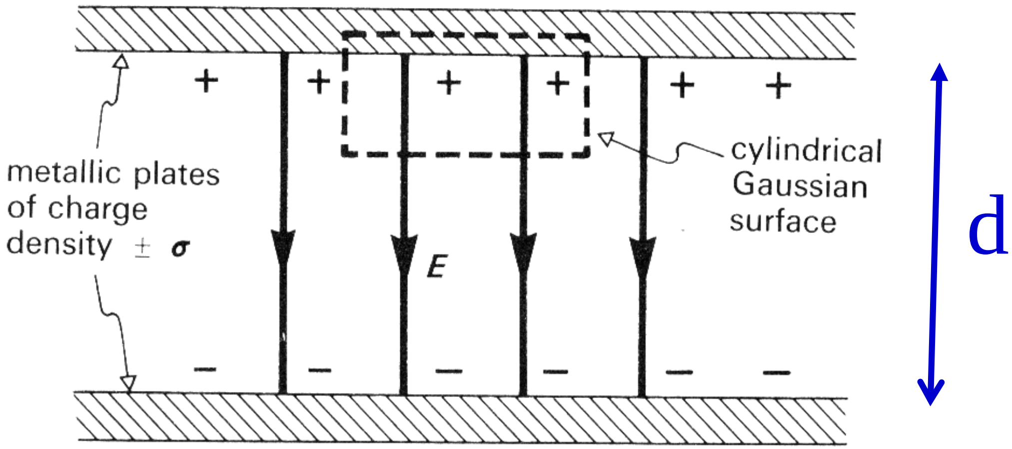


(a)



We will ignore edge (fringing) effects – treat like infinite plates

Capacitance of parallel Plates



Consider parallel plates of area, A

From Gauss's Law: $E = \frac{\sigma}{\varepsilon_0} = \frac{Q}{\varepsilon_0 A}$

As E is constant between plates, potential difference,

$$V = -\int_d^0 \underline{E} \cdot d\underline{x} = Ed$$

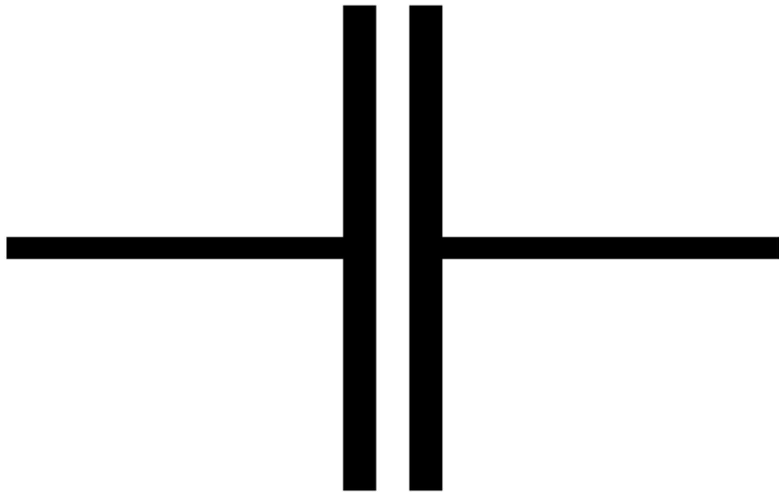
Capacitance of parallel Plates

From Gauss's Law: $E = \frac{\sigma}{\varepsilon_0} = \frac{Q}{\varepsilon_0 A}$

$$V = V_+ - V_- = Ed = \frac{Qd}{\varepsilon_0 A}$$

$$\boxed{C = \frac{Q}{V} = \frac{\varepsilon_0 A}{d}}$$

Symbol for Capacitor

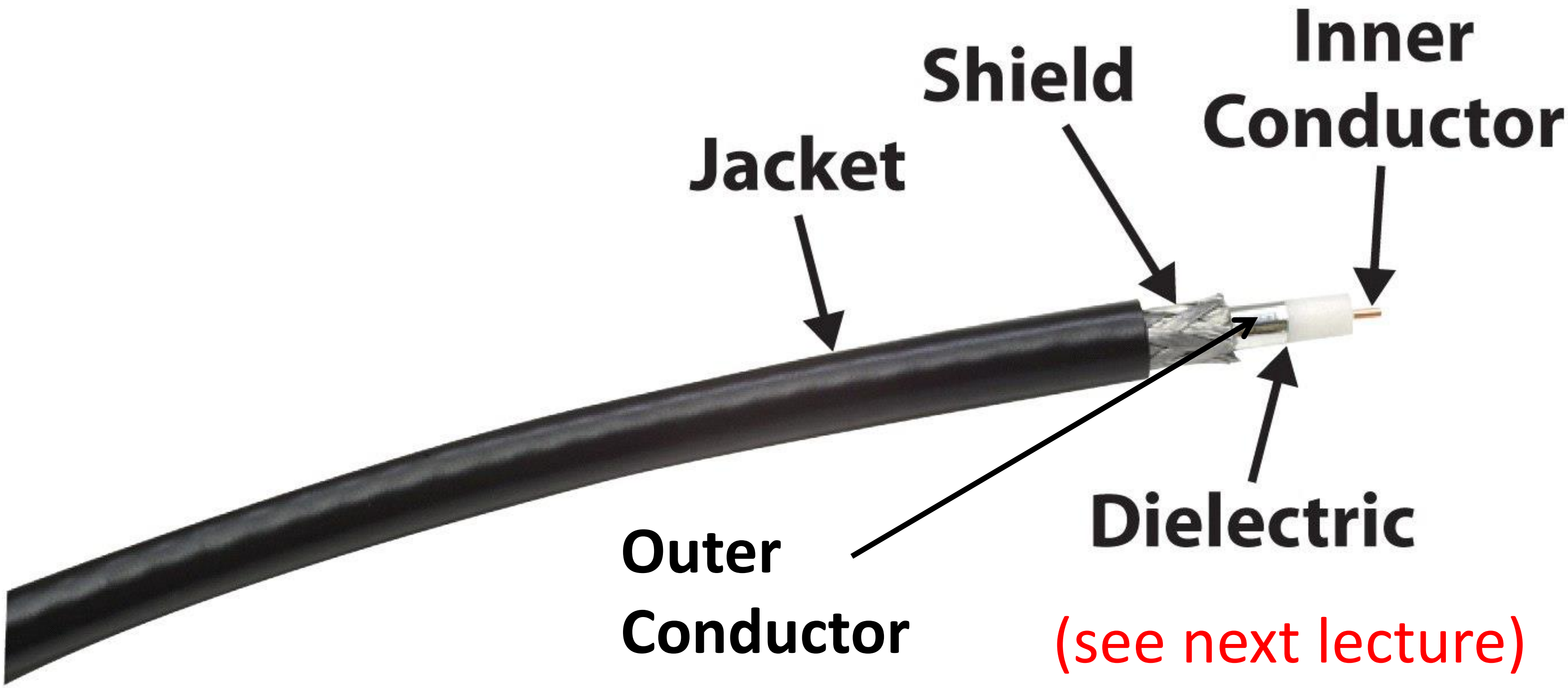


- This is the symbol for a capacitor in an electric circuit.

For the same amount of charge, the capacitor with a small voltage is said to have a large capacity (in storing electric charges).

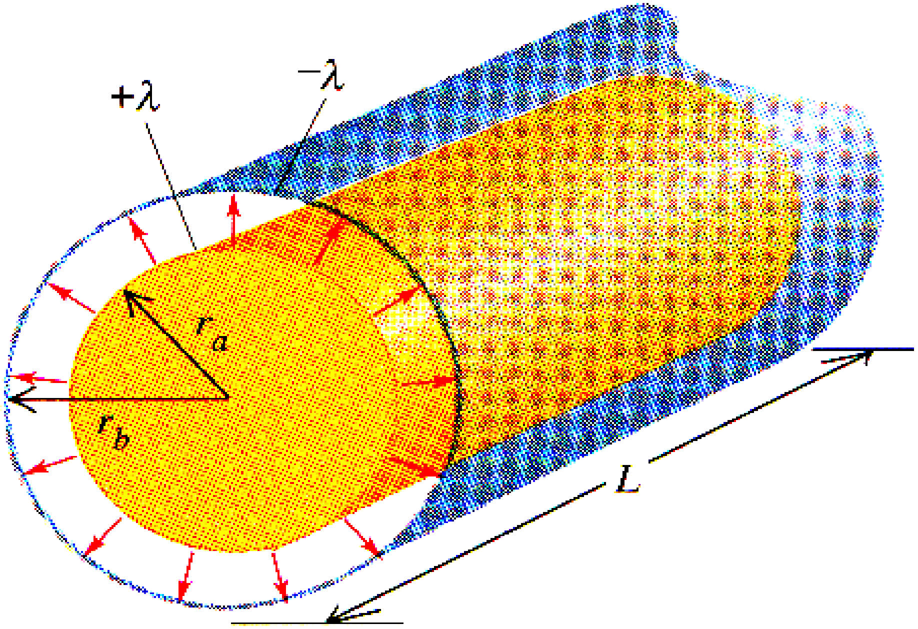
(2) A Long Cylindrical Capacitor

- A Long Cylindrical Capacitor (co-axial cable)



(see next lecture)

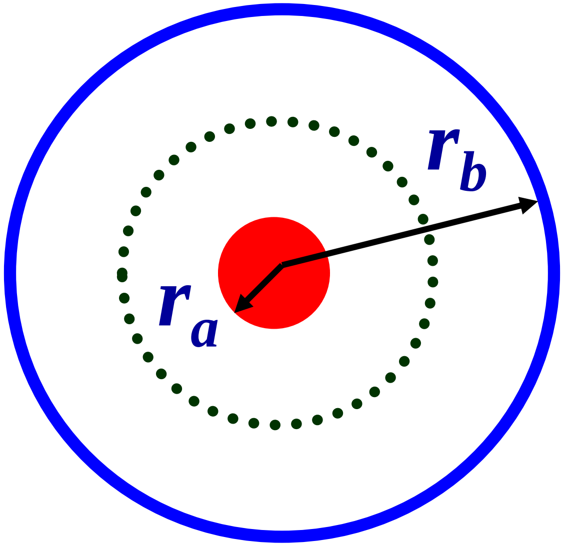
(2) A Long Cylindrical Capacitor



- What is the capacitance per unit length?
- Important in determining the transmission characteristics of the cable.

(2) A Long Cylindrical Capacitor

- Treat as infinite co-axial cable (i.e. ignore edge effects). Charge per unit length, λ.
- Choose a cylindrical Gaussian surface length, l



$$\int \underline{E} \cdot d\underline{S} = \frac{Q_{enc}}{\varepsilon_0} = \frac{\lambda l}{\varepsilon_0}$$

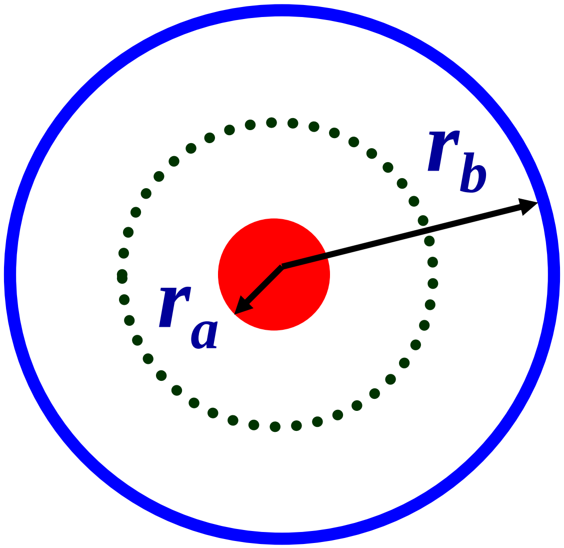
By symmetry LHS = $E\ 2\pi r l$

$$\underline{\boldsymbol{E}} = \frac{\lambda}{2\pi\varepsilon_0 r}\,\underline{\hat{\boldsymbol{r}}}$$

(2) A Long Cylindrical Capacitor

- Now find V (let's do it on the visualizer)

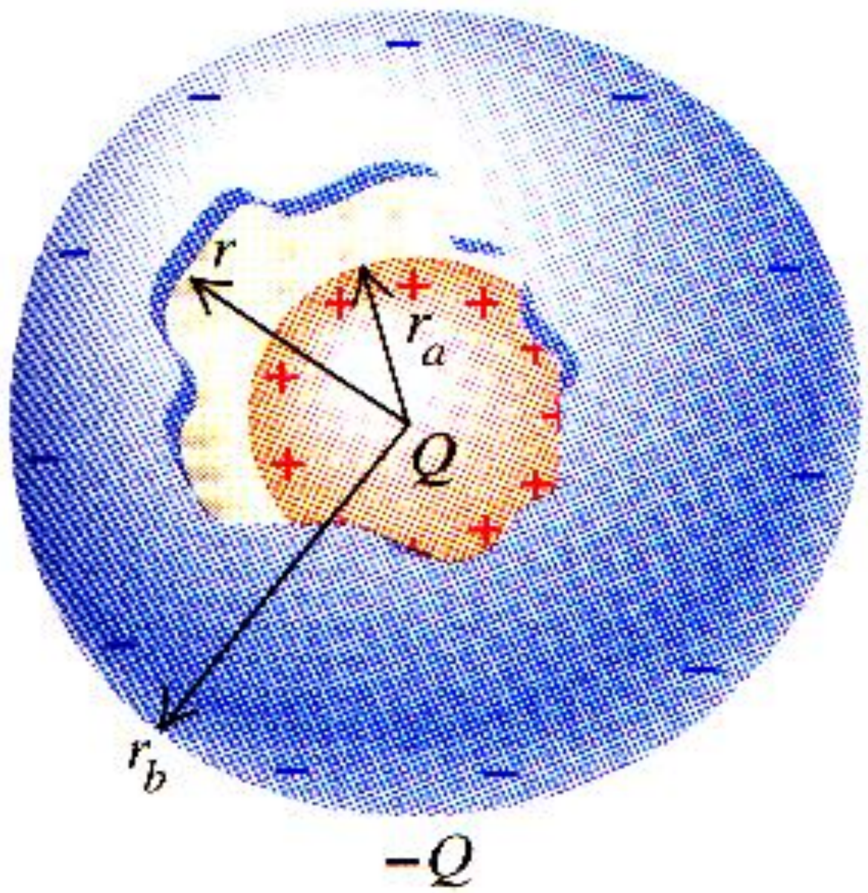
$$V = V_a - V_b = -\int_{r_b}^{r_a} \underline{E} \cdot d\underline{r} = \frac{\lambda}{2\pi\varepsilon_0} ln\left(\frac{r_b}{r_a}\right)$$



Capacitance per unit length = $\frac{\lambda}{V}$

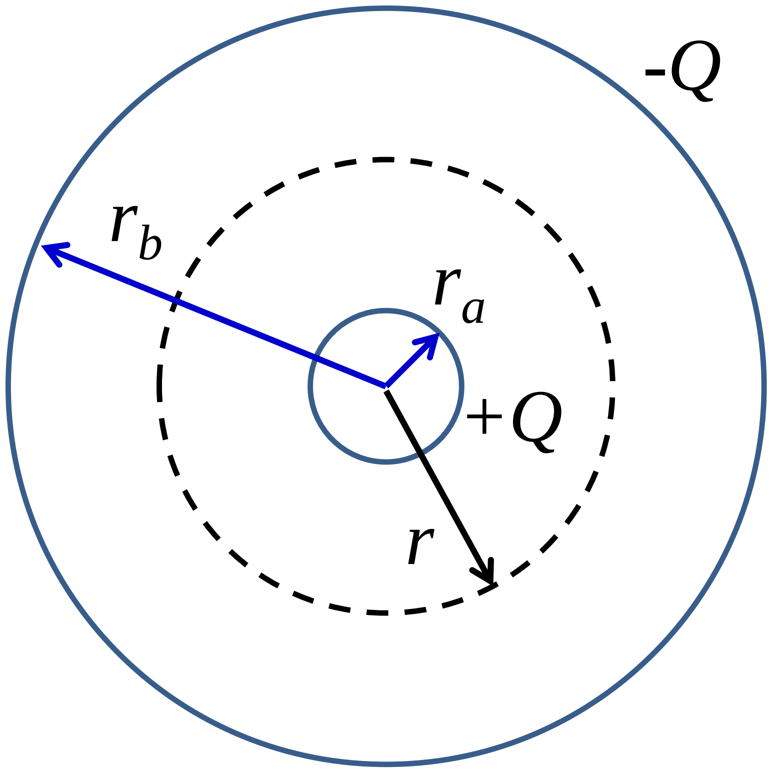
$$C = \frac{\lambda}{V} = \frac{2\pi\varepsilon_0}{ln\left(\frac{r_b}{r_a}\right)}$$

(3) Spherical Capacitor



Consider two concentric charged spheres as shown

(3) Spherical Capacitor



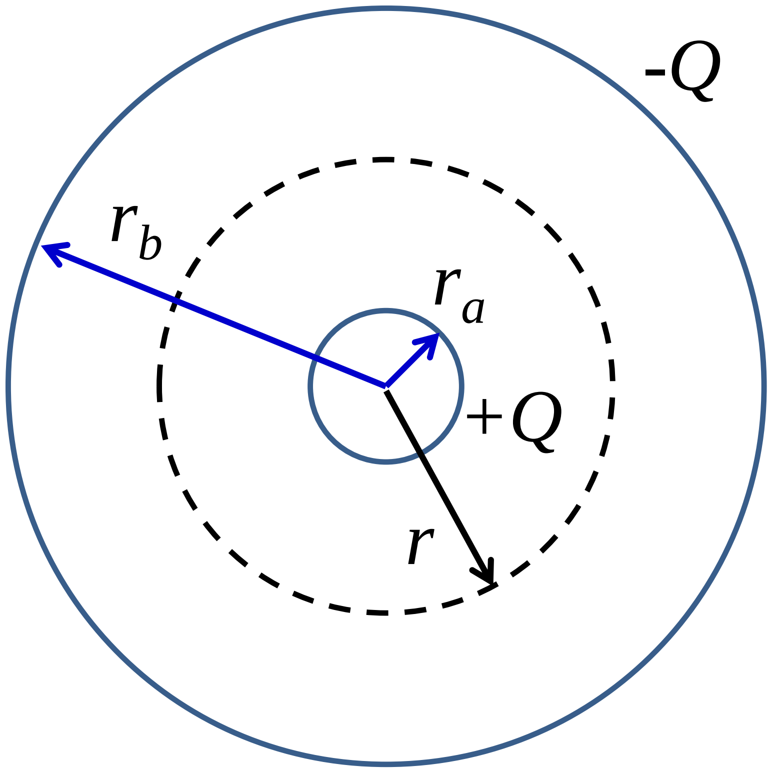
The appropriate Gaussian surface is a sphere concentric with, and between, the conducting spheres

$$\int \underline{E} \cdot d\underline{S} = \frac{Q_{enc}}{\varepsilon_0} = \frac{Q}{\varepsilon_0}$$

By symmetry LHS = $E\ 4\pi r^2$

$$\underline{\boldsymbol{E}} = \frac{Q}{4\pi\varepsilon_0 r^2}\ \underline{\hat{\boldsymbol{r}}}$$

(3) Spherical Capacitor



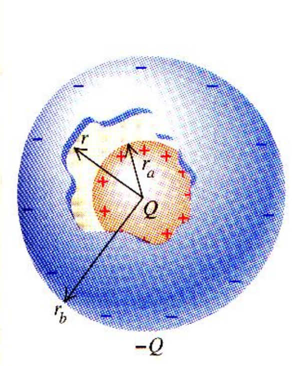
$$\underline{\boldsymbol{E}} = \frac{Q}{4\pi\varepsilon_0 r^2}\,\underline{\hat{\boldsymbol{r}}}$$

$$V = -\int_{r_b}^{r_a} \underline{E} \cdot d\underline{r}$$

Let’s do it on the visualizer

(3) Spherical Capacitor

- $C = \frac{Q}{V} = \frac{4\pi\varepsilon_0 r_a r_b}{r_b - r_a}$

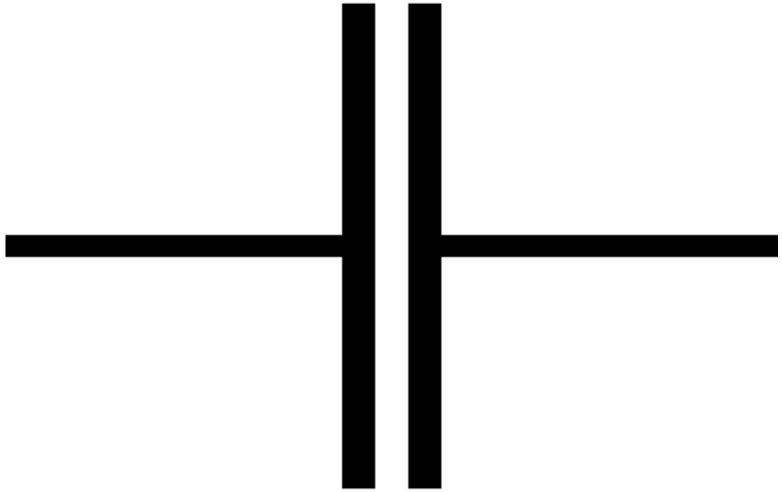


- What if $r_b \to \infty$?

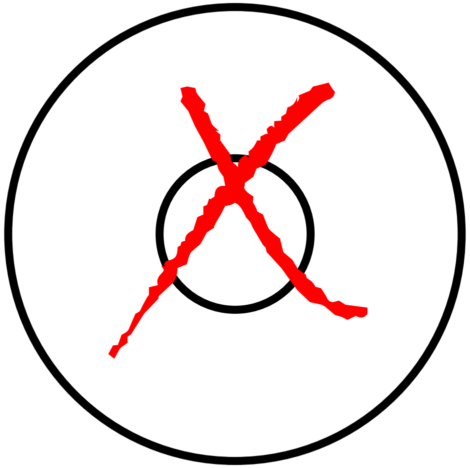
- $C = \frac{4\pi\varepsilon_0 r_a r_b}{r_b - r_a} \approx \frac{4\pi\varepsilon_0 r_a r_b}{r_b} = 4\pi\varepsilon_0 r_a$

- This is the capacitance of an isolated sphere.

Symbol for Spherical Capacitor



Not:



Summary

- **Calculation of Capacitance**
- *Procedure:*

1. Determine E (e.g. using Gauss's Law)
2. Use $V = -\int \underline{E} \cdot d\underline{l}$
3. Apply $C = Q / V$