VGLA: Vectors Practice Questions

The following questions relate to Chapter 1, Vectors. Questions are ranked in difficulty from A (basic) to C (challenging).

(A) Question 1. For each of the following sets of points U, V and W calculate $\vec{UV}$, $\vec{UW}$ and hence determine whether U, V and W are co-linear:

(a) $U = (1, 3, -1)$, $V = (5, 1, -2)$ and $W = (3, 2, -3)$;

(b) $U = (2, 1, 4)$, $V = (1, 4, 2)$ and $W = (4, -5, 8)$.

Solution:

(a) $\vec{UV} = \mathbf{v} - \mathbf{u} = (5, 1, -2) - (1, 3, -1) = (4, -2, -1) = 4\mathbf{i} - 2\mathbf{j} - \mathbf{k}$.

$\vec{UW} = (2, -1, -2) = 2\mathbf{i} - \mathbf{j} - 2\mathbf{k}$.

Since $\vec{UV} \neq \alpha\vec{UW}$ for any $\alpha \in \mathbb{R}$, we conclude that U, V and W are not co-linear.

(b) $\vec{UV} = (-1, 3, -2) = -\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$.

$\vec{UW} = (2, -6, 4) = 2\mathbf{i} - 6\mathbf{j} + 4\mathbf{k}$.

Since $\vec{UW} = -2\vec{UV}$, we conclude that U, V and W are co-linear.

■

(A) Question 2. In each of the following cases, find numbers s and t, if they exist, such that $\mathbf{w} = s\mathbf{u}+t\mathbf{v}$:

(a) $\mathbf{u} = (3, -1, 1)$, $\mathbf{v} = (4, 3, -3)$ and $\mathbf{w} = (17, 3, -3)$;

(b) $\mathbf{u} = (2, -3, 5)$, $\mathbf{v} = (-1, 4, 6)$ and $\mathbf{w} = (8, -17, 2)$.

(If $\mathbf{w} = s\mathbf{u} + t\mathbf{v}$, $\mathbf{w}$ is said to be a *linear combination* of $\mathbf{u}$ and $\mathbf{v}$.)

Solution:

(a) Observe that

$$\begin{aligned}
\mathbf{w} = s\mathbf{u} + t\mathbf{v} &\iff (17, 3, -3) = s(3, -1, 1) + t(4, 3, -3) \\
&\iff (17, 3, -3) = (3s + 4t, -s + 3t, s - 3t) \\
&\iff \begin{cases} 3s + 4t &= 17 \\ -s + 3t &= 3 \\ s - 3t &= -3 \end{cases} \\
&\iff \begin{cases} 3s + 4t &= 17 \\ -s + 3t &= 3 \end{cases} \\
&\iff s = 3,\ t = 2.
\end{aligned}$$

Hence, $\mathbf{w} = 3\mathbf{u} + 2\mathbf{t}$. *Note that you can check this answer directly by substitution of s and t into $\boldsymbol{w} = s\boldsymbol{u} + t\boldsymbol{v}$.*

(b)

$$\begin{aligned}
\mathbf{w} = s\mathbf{u} + t\mathbf{v} &\iff (8, -17, 2) = s(2, -3, 5) + t(-1, 4, 6) \\
&\iff (8, -17, 2) = (2s - t, -3s + 4t, 5s + 6t)
\end{aligned}$$

$$\iff \begin{cases} 2s - t &= 8 \\ -3s + 4t &= -17 \\ 5s + 6t &= \ 2 \end{cases}$$

$$\iff \begin{cases} 2s - t &= 8 \\ 5s &= 15 \\ 5s + 6t &= \ 2 \end{cases}$$

$$\iff \begin{cases} s &= 3 \\ t &= -2 \\ 5(3) + 6(-2) = 3 &\neq \ 2 \end{cases}$$

and hence, there does not exist $s, t \in \mathbb{R}$ such that $\mathbf{w} = s\mathbf{u} + t\mathbf{v}$. *Note that the set of all linear combinations of 2 vectors* $\boldsymbol{u}$ *and* $\boldsymbol{v}$ *in* E^3 *(i.e.* $s\boldsymbol{u}+t\boldsymbol{v}$ *for* $s, t \in \mathbb{R}$*) defines a plane. If a point is not on that plane, then it cannot be represented by a linear combination of* $\boldsymbol{u}$ *and* $\boldsymbol{v}$*.*

■

(A) Question 3. Find the vector $\mathbf{u}$ which has magnitude 6 and has the same direction as the vector $\mathbf{v} = (1, -2, 1)$.

Solution: The vector $\mathbf{u}$ must satisfy $\mathbf{u} = \lambda\mathbf{v}$ and $|\mathbf{u}| = 6$ for some $\lambda \in \mathbb{R}_+$. Thus,

$$|\mathbf{u}| = \sqrt{(\lambda)^2 + (-2\lambda)^2 + (\lambda)^2} = \sqrt{6}\lambda \iff \lambda = \sqrt{6}.$$

We conclude that **the** vector $\mathbf{u}$ is given by

$$\mathbf{u} = \sqrt{6}\mathbf{v} = \left(\sqrt{6}, -2\sqrt{6}, \sqrt{6}\right).$$

■

(A) Question 4. In each of the following cases, find the unit vector with the same direction as $\mathbf{v}$ and the unit vector with the opposite direction to $\mathbf{v}$:

(a) $\mathbf{v} = (3, -2, -6)$;

(b) $\mathbf{v} = (0, -3, 5)$.

Solution:

(a) $|\mathbf{v}| = \sqrt{(3)^2 + (-2)^2 + (-6)^2} = \sqrt{49} = 7$ and hence the unit vector in the same direction as $\mathbf{v}$ is given by

$$\mathbf{e}_v = \frac{1}{7}\mathbf{v} = \left(\frac{3}{7}, -\frac{2}{7}, -\frac{6}{7}\right).$$

Moreover, the unit vector in the opposite direction to $\mathbf{v}$ is given by

$$\mathbf{e}_{-v} = -\frac{1}{7}\mathbf{v} = \left(-\frac{3}{7}, \frac{2}{7}, \frac{6}{7}\right).$$

(b) $|\mathbf{v}| = \sqrt{(0)^2 + (-3)^2 + (5)^2} = \sqrt{34}$ and hence the unit vector in the same direction as $\mathbf{v}$ is given by

$$\mathbf{e}_v = \frac{1}{\sqrt{34}}\mathbf{v} = \left(0, -\frac{3}{\sqrt{34}}, \frac{5}{\sqrt{34}}\right).$$

Moreover, the unit vector in the opposite direction to $\mathbf{v}$ is given by

$$\mathbf{e}_{-v} = -\frac{1}{\sqrt{34}}\mathbf{v} = \left(0, \frac{3}{\sqrt{34}}, -\frac{5}{\sqrt{34}}\right).$$

■

(A) Question 5. For each of the following pairs of vectors $\mathbf{u}$ and $\mathbf{v}$, determine whether they are perpendicular:

(a) $\mathbf{u} = (2, -3, 1)$ and $\mathbf{v} = (5, 4, 2)$;

(b) $\mathbf{u} = (\cos\theta, \sin\theta, 1)$ and $\mathbf{v} = (\sin\theta, -\cos\theta, 1)$;

(c) $\mathbf{u} = (\cos\theta, \sin\theta, 1)$ and $\mathbf{v} = (\cos\theta, \sin\theta, -1)$.

Solution:

(a) Since $\mathbf{u}$ and $\mathbf{v}$ are non-zero vectors and $\mathbf{u}.\mathbf{v} = (2,-3,1).(5,4,2) = 2.5+(-3).4+1.2 = 10-12+2 = 0$, it follows that $\mathbf{u} \perp \mathbf{v}$.

(b) Since $\mathbf{u}$ and $\mathbf{v}$ are non-zero vectors and

$$\mathbf{u}.\mathbf{v} = (\cos\theta, \sin\theta, 1).(\sin\theta, -\cos\theta, 1) = \cos\theta.\sin\theta - \sin\theta.\cos\theta + 1 = 1 \neq 0$$

for all $\theta \in [0, 2\pi)$, it follows that for all $\theta \in [0, 2\pi)$ we have $\mathbf{u} \not\perp \mathbf{v}$.

(c) Since $\mathbf{u}$ and $\mathbf{v}$ are non-zero vectors and

$$\mathbf{u}.\mathbf{v} = (\cos\theta, \sin\theta, 1).(\cos\theta, \sin\theta, -1) = \cos^2\theta + \sin^2\theta - 1 = 0$$

for all $\theta \in [0, 2\pi)$, it follows that for all $\theta \in [0, 2\pi)$, we have $\mathbf{u} \perp \mathbf{v}$.

■

(A) Question 6. Find all values of λ, if any, for which the vectors $\mathbf{u}$ and $\mathbf{v}$ are perpendicular:

(a) $\mathbf{u} = (3,-2,1)$ and $\mathbf{v} = (4,\lambda,-2)$;

(b) $\mathbf{u} = (\lambda,2,7)$ and $\mathbf{v} = (\lambda,-3,1)$;

(c) $\mathbf{u} = (1,\lambda,\lambda)$ and $\mathbf{v} = (-2,\lambda,1)$.

Solution: Since none of the vectors above are the zero vector, it follows that $\mathbf{u} \perp \mathbf{v} \iff \mathbf{u}.\mathbf{v} = 0$. Hence:

(a) $\mathbf{u} \perp \mathbf{v} \iff (3,-2,1).(4,\lambda,-2) = 0 \iff 12 - 2\lambda - 2 = 0 \iff \lambda = 5$. So $\lambda \in \{5\}$.

(b) $\mathbf{u} \perp \mathbf{v} \iff (\lambda,2,7).(\lambda,-3,1) \iff \lambda^2 - 6 + 7 = 0 \iff \lambda^2 + 1 = 0$. Since the quadratic equation has no real solutions, it follows that $\mathbf{u} \not\perp \mathbf{v}$ for all $\lambda \in \mathbb{R}$.

(c) $\mathbf{u} \perp \mathbf{v} \iff (1,\lambda,\lambda).(-2,\lambda,1) \iff -2+\lambda^2+\lambda = 0 \iff (\lambda+2)(\lambda-1) = 0 \iff \lambda = -2$ or $\lambda = 1$. So $\lambda \in \{1,-2\}$.

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(A) Question 7. If $\mathbf{u} = (2,3,-1)$, $\mathbf{v} = (-2,-1,2)$ and $\mathbf{w} = (1,2,1)$, calculate:

(a) $\mathbf{u} \times \mathbf{v}$;

(b) $(\mathbf{u} \times \mathbf{v}) \times \mathbf{w}$;

(c) $\mathbf{v} \times \mathbf{w}$;

(d) $\mathbf{u} \times (\mathbf{v} \times \mathbf{w})$;

(e) $\mathbf{u}.(\mathbf{v} \times \mathbf{w})$.

(f) Deduce from (a), the two unit vectors that are perpendicular to both $\mathbf{u}$ and $\mathbf{v}$.

Solution:

(a)

$$\begin{aligned}
\mathbf{u} \times \mathbf{v} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 3 & -1 \\ -2 & -1 & 2 \end{vmatrix} \\
&= ((3)(2) - (-1)(-1))\mathbf{i} - ((2)(2) - (-1)(-2))\mathbf{j} + ((2)(-1) - (3)(-2))\mathbf{k} \\
&= (5,-2,4).
\end{aligned}$$

An incomplete check that your calculations are correct involves computing $\boldsymbol{u}.(\boldsymbol{u} \times \boldsymbol{v})$ and $\boldsymbol{v}.(\boldsymbol{u} \times \boldsymbol{v})$, and checking that they equal 0. This does not check the right hand rule is satisfied though. For the remaining parts of the question, we just give the answer.

(b) $(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = (-10, -1, 12)$.

(c) $\mathbf{v} \times \mathbf{w} = (-5, 4, -3)$.

(d) $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (-5, 11, 23)$;

(e) $\mathbf{u}.(\mathbf{v} \times \mathbf{w}) = (2, 3, -1).(-5, 4, -3) = -10 + 12 + 3 = 5$.

(f) Since $(\mathbf{u} \times \mathbf{v}) \perp \mathbf{u}$ and $(\mathbf{u} \times \mathbf{v}) \perp \mathbf{v}$ it follows that the two unit vectors that are perpendicular to both $\mathbf{u}$ and $\mathbf{v}$ are given by

$$\frac{\pm 1}{|\mathbf{u} \times \mathbf{v}|}(\mathbf{u} \times \mathbf{v}) = \frac{\pm 1}{\sqrt{5^2 + (-2)^2 + 4^2}}(5, -2, 4) = \pm \left(\frac{5}{\sqrt{45}}, -\frac{2}{\sqrt{45}}, \frac{4}{\sqrt{45}} \right).$$

■

(A) Question 8. In each of the following cases, find an equation for the plane π:

(a) π is parallel to the yz-plane and intersects the point $(1, 2, 3)$;

(b) π is parallel to the zx-plane and intersects the point $(3, -1, 4)$;

(c) π intersects the point $(2, -1, -4)$ and has normal vector $(2, 1, 0)$;

(d) π intersects the points $U = (1, 1, 3)$, $V = (-1, 3, 2)$ and $W = (1, -2, 5)$.

Solution:

(a) $x = 1$.

(b) $y = -1$.

(c) If the plane intersects $(2, -1, -4)$ and has normal vector $\mathbf{n} = (2, 1, 0)$ then the equation of the plane is given by

$$((x, y, z) - (2, -1, -4)).(2, 1, 0) = 0 \iff 2x + y = 3.$$

(d) A normal vector to the plane Π can be found by $\vec{UV} \times \vec{UW}$ i.e.

$$\mathbf{n} = \vec{UV} \times \vec{UW} = (-2, 2, -1) \times (0, -3, 2) = (1, 4, 6).$$

Therefore, the equation of the plane Π is given by

$$((x, y, z) - (1, 1, 3)).(1, 4, 6) = 0 \iff x + 4y + 6z = 23.$$

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(A) Question 9. Obtain a set of parametric equations for the straight line L that intersects the points $P = (3, 1, 4)$ and $Q = (-1, -2, 8)$. Find the coordinates of the points of intersection of L and the plane π in part (A) Question 8, part (d).

Solution: The direction of the line L is $\vec{PQ} = (-4, -3, 4)$. Thus, the set of parametric equations for the line L is

$$\begin{cases} x &= 3 - 4\lambda \\ y &= 1 - 3\lambda \\ z &= 4 + 4\lambda. \end{cases}$$

The point(s) of intersection between the line L and the plane Π (if they exist) can be determined by substitution of the parametric equations for the line, into the Cartesian equation for the plane i.e.

$$(3 - 4\lambda) + 4(1 - 3\lambda) + 6(4 + 4\lambda) = 23 \iff 8\lambda = 23 - 31 \iff \lambda = -1.$$

Therefore the point of intersection has co-ordinates

$$\begin{cases} x &= 3 - 4(-1) = 7 \\ y &= 1 - 3(-1) = 4 \\ z &= 4 + 4(-1) = 0 \end{cases}$$

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(A) Question 10. In each of the following cases, find the volume of the parallelepiped with vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ as adjacent edges:

(a) $\mathbf{u} = (0, 2, 2)$, $\mathbf{v} = (3, 1, 1)$ and $\mathbf{w} = (3, -5, 1)$;

(b) $\mathbf{u} = (1, 0, 2)$, $\mathbf{v} = (1, 1, 0)$ and $\mathbf{w} = (0, 1, 1)$.

Solution: The volume V of the parallelepiped defined above is given by $|\mathbf{u}.(\mathbf{v} \times \mathbf{w})|$.

(a) Since $\mathbf{v} \times \mathbf{w} = (6, 0, -18)$, it follows that $V = |(0, 2, 2).(6, 0, -18)| = 36$.

(b) Since $\mathbf{v} \times \mathbf{w} = (1, -1, 1)$, it follows that $V = |(1, 0, 2).(1, -1, 1)| = 3$.

■

(A) Question 11. For the vectors $\mathbf{u} = (1, -1, 0)$, $\mathbf{v} = (0, 1, 1)$ and $\mathbf{w} = (2, 0, -1)$ calculate:

(a) $\mathbf{u} \times \mathbf{v}$;

(b) $(\mathbf{u} \times \mathbf{v}) \times \mathbf{w}$;

(c) $\mathbf{u}.\mathbf{w}$;

(d) $\mathbf{v}.\mathbf{w}$.

Verify directly for vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ that

$$(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = (\mathbf{u}.\mathbf{w})\mathbf{v} - (\mathbf{v}.\mathbf{w})\mathbf{u}.$$

Solution:

(a)

$$\begin{aligned}
\mathbf{u} \times \mathbf{v} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 0 \\ 0 & 1 & 1 \end{vmatrix} \\
&= ((-1)(1) - (0)(1))\mathbf{i} - ((1)(1) - (0)(0))\mathbf{j} + ((1)(1) - (-1)(0))\mathbf{k} \\
&= (-1, -1, 1).
\end{aligned}$$

(b) $(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = (-1, -1, 1) \times (2, 0, -1) = (1, 1, 2)$.

(c) $\mathbf{u}.\mathbf{w} = (1, -1, 0).(2, 0, -1) = 2$.

(d) $\mathbf{v}.\mathbf{w} = (0, 1, 1).(2, 0, -1) = -1$.

Finally, since $(\mathbf{u}.\mathbf{w})\mathbf{v} - (\mathbf{v}.\mathbf{w})\mathbf{u} = 2(0, 1, 1) - (-1)(1, -1, 0) = (1, 1, 2)$, it follows from (b) that for the vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ the following identity holds:

$$(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = (\mathbf{u}.\mathbf{w})\mathbf{v} - (\mathbf{v}.\mathbf{w})\mathbf{u}.$$

■

(A) Question 12. For the vectors $\mathbf{a} = (3, 0, 0)$ and $\mathbf{b} = (1, -2, 2)$, find the following:

(a) $|\mathbf{a}|$ and $|\mathbf{b}|$;

(b) $\mathbf{a} \cdot \mathbf{b}$;

(c) unit vectors parallel to $\mathbf{b}$ in the same/opposite direction;

(d) the projection of the vector $\mathbf{a}$ unto the direction of vector $\mathbf{b}$, i.e. $\text{proj}_{\mathbf{b}}(\mathbf{a})$;

(e) the vectors $\mathbf{u}$ and $\mathbf{w}$ such that $\mathbf{a} = \mathbf{u} + \mathbf{w}$ with $\mathbf{u} \parallel \mathbf{b}$ and $\mathbf{u} \perp \mathbf{w}$.

Solution:

(a) $|\mathbf{a}| = 3$ and $|\mathbf{b}| = \sqrt{1 + 4 + 4} = 3$.

(b) $\mathbf{a} \cdot \mathbf{b} = 3.1 + 0 + 0 = 3$.

(c) Since $|\mathbf{b}| = 3$ it follows that

$$\mathbf{e_b} = \left(\frac{1}{3}, -\frac{2}{3}, \frac{2}{3}\right) \text{ and } \mathbf{e_{\text{-}b}} = \left(-\frac{1}{3}, \frac{2}{3}, -\frac{2}{3}\right).$$

(d)

$$\text{proj}_{\mathbf{b}}(\mathbf{a}) = (\mathbf{a}.\mathbf{e_b})\,\mathbf{e_b} = \left((3,0,0).\left(\frac{1}{3}, -\frac{2}{3}, \frac{2}{3}\right)\right)\left(\frac{1}{3}, -\frac{2}{3}, \frac{2}{3}\right) = \left(\frac{1}{3}, -\frac{2}{3}, \frac{2}{3}\right).$$

(e) It follows that $\mathbf{u} = \text{proj}_{\mathbf{b}}(\mathbf{a})$ and $\mathbf{w} = \mathbf{a} - \mathbf{u}$ i.e.

$$\mathbf{u} = \left(\frac{1}{3}, -\frac{2}{3}, \frac{2}{3}\right) \text{ and } \mathbf{w} = (3,0,0) - \left(\frac{1}{3}, -\frac{2}{3}, \frac{2}{3}\right) = \left(\frac{8}{3}, \frac{2}{3}, -\frac{2}{3}\right).$$

■

(A) Question 13. Consider the vectors $\mathbf{a} = (3, -2, 0)$ and $\mathbf{b} = (1, -2, 2)$, and the points $P(1, -1, 2)$, $Q(1, 0, 0)$ and $R(-3, 0, 1)$.

(a) Determine $\mathbf{a} \times \mathbf{b}$;

(b) Find two distinct unit vectors perpendicular to the plane containing the vectors $\mathbf{a}$ and $\mathbf{b}$;

(c) Find an equation of the plane Π_1 that intersects the point P and is perpendicular to the vector $\mathbf{b}$;

(d) Find an equation of the plane Π_2 parallel to the plane Π_1 that intersects the origin;

(e) Find an equation of the plane Π_3 that intersects the points P, Q and R;

(f) Find an equation of the line L_1 that intersects the point A with position vector $\mathbf{a}$ and is parallel to the vector $\mathbf{b}$;

(g) Find an equation of the plane Π_4 which contains the line L_1 and the point P;

(h) Find the point of intersection of the line L_1 and the plane Π_5 with equation $2x - 3y + z = -3$.

Solution:

(a)

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -2 & 0 \\ 1 & -2 & 2 \end{vmatrix} = -4\mathbf{i} - 6\mathbf{j} - 4\mathbf{k}.$$

(b) Since $|\mathbf{a} \times \mathbf{b}| = \sqrt{(-4)^2 + (-6)^2 + (-4)^2} = \sqrt{68}$, it follows that the two unit vectors perpendicular to the plane are

$$\mathbf{e}_{\pm} = \pm\left(\frac{4}{\sqrt{68}}, \frac{6}{\sqrt{68}}, \frac{4}{\sqrt{68}}\right).$$

(c) Since $\mathbf{b}$ is a normal vector to Π_1, it follows that the plane Π_1 is given by the equation

$$(x, y, z).(1, -2, 2) = (1, -1, 2).(1, -2, 2) \iff x - 2y + 2z = 7.$$

(d) Via (c), the plane Π_2 has equation

$$x - 2y + 2z = 0.$$

(e) Since

$$\vec{PQ} \times \vec{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & -2 \\ -4 & 1 & -1 \end{vmatrix} = \mathbf{i} + 8\mathbf{j} + 4\mathbf{k},$$

and $Q \in \Pi_3$, it follows that the plane Π_3 is given by

$$(x, y, z).(1, 8, 4) = (1, 0, 0).(1, 8, 4) \iff x + 8y + 4z = 1.$$

(f) The line L_1 is given by $\mathbf{r}(t) = \mathbf{a} + t\mathbf{b} = (3+t, -2-2t, 2t)$ for $t \in \mathbb{R}$.

(g) Since

$$\vec{AP} \times \vec{OB} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 1 & 2 \\ 1 & -2 & 2 \end{vmatrix} = 6\mathbf{i} + 6\mathbf{j} + 3\mathbf{k},$$

and $A \in \Pi_4$, it follows that the plane Π_4 is given by

$$(x, y, z).(6, 6, 3) = (3, -2, 0).(6, 6, 3) \iff 6x + 6y + 3z = 6.$$

(h) Substituting the parametric equations for L_1 into Π_5 gives

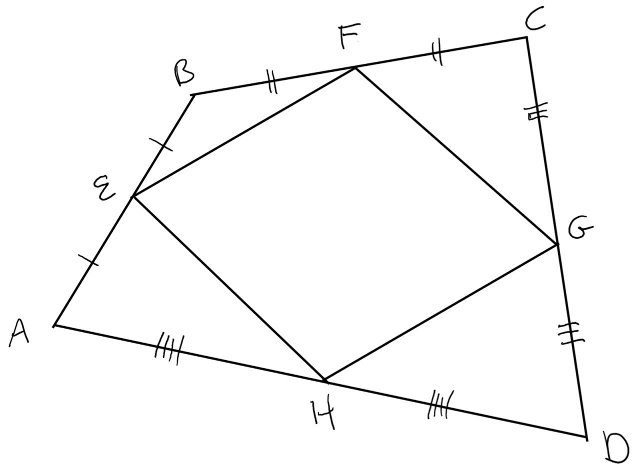
$$2(3+t) - 3(-2-2t) + (2t) = -3 \iff 10t = -15 \iff t = -\frac{3}{2},$$

and hence the point of intersection of L_1 and Π_5 is $(\frac{3}{2}, 1, -3)$.

■

(B) Question 14. The mid-points of the sides AB, BC, CD and DA of a quadrilateral $ABCD$ are E, F, G and H respectively. Show, **using vectors**, that $EFGH$ is a parallelogram. *Hint: Take position vectors relative to an origin O and use the result that a quadrilateral with a pair of opposite sides which are parallel and equal in length is a parallelogram.*

Solution:



A sketch of terms in the question.

It follows that

$$\vec{AB} + \vec{BC} + \vec{CD} + \vec{DA} = \mathbf{0}, \tag{1}$$

$$\vec{EF} + \vec{FG} + \vec{GH} + \vec{HE} = \mathbf{0}, \tag{2}$$

$$\vec{EF} = \frac{1}{2}(\vec{AB} + \vec{BC}),\ \vec{FG} = \frac{1}{2}(\vec{BC} + \vec{CD}),\ \vec{GH} = \frac{1}{2}(\vec{CD} + \vec{DA}) \text{ and } \vec{HE} = \frac{1}{2}(\vec{DA} + \vec{AB}). \tag{3}$$

Substituting expressions for $\vec{EF}$ and $\vec{GH}$ into (2) gives

$$\frac{1}{2}(\vec{AB} + \vec{BC}) + \vec{FG} + \frac{1}{2}(\vec{CD} + \vec{DA}) + \vec{HE} = \mathbf{0}$$

$$\frac{1}{2}(\vec{AB} + \vec{BC} + \vec{CD} + \vec{DA}) + \vec{FG} + \vec{HE} = \mathbf{0} \qquad \text{(+ is associative/commutative)}$$

$$\vec{FG} = \vec{EH} \qquad \text{(via (1))}. \tag{4}$$

Substituting (4) into (2) then gives

$$\vec{EF} + \vec{GH} + (\vec{FG} - \vec{EH}) = \mathbf{0} \Longrightarrow \vec{EF} = \vec{HG}. \tag{5}$$

It follow from (4) and (5) that EFGH is a parallelogram, as required. ■

(B) Question 15. Let $\mathbf{u}$ and $\mathbf{v}$ be the distinct position vectors relative to an origin O of the points U and V respectively. Additionally, let W be a point on either UV (outside the closed line segment [UV]) or VU (outside the closed line segment [VU]) such that $UW : WV = s : t$.

Establish that the position vector $\mathbf{w}$ of W relative to O is given by

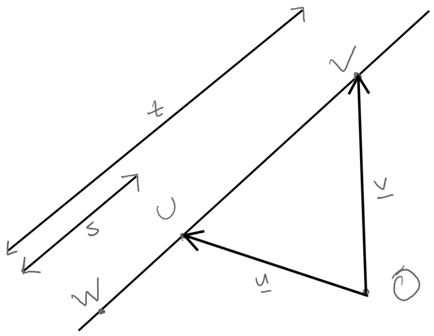
$$\mathbf{w} = \left(\frac{s}{s-t}\right)\mathbf{v} - \left(\frac{t}{s-t}\right)\mathbf{u}.$$

If $U = (2, -4, 3)$ and $V = (-3, 1, -2)$, find the coordinates of W when

(a) W is on VU and $UW : WV = 2 : 7$;

(b) W is on UV and $UW : WV = 7 : 2$.

Solution:



A sketch of terms in the question.

Let $\mathbf{w}$ be the position vector of W relative to O. Then $\mathbf{w} = \mathbf{u} + \lambda(\mathbf{v} - \mathbf{u})$ for some $\lambda \in \mathbb{R}$. Since,

$$t|\mathbf{w} - \mathbf{u}| = s|\mathbf{w} - \mathbf{v}|,$$

$\mathbf{w} - \mathbf{u}$ and $\mathbf{w} - \mathbf{v}$ are parallel, and W is not contained between U and V, it follows that

$$t(\mathbf{w} - \mathbf{u}) = s(\mathbf{w} - \mathbf{v}) \iff \mathbf{w} = \left(\frac{s}{s-t}\right)\mathbf{v} - \left(\frac{t}{s-t}\right)\mathbf{u}. \tag{6}$$

(a) For $s = 2$ and $t = 7$, it follows from (6) that

$$\mathbf{w} = -\frac{2}{5}(-3, 1, -2) + \frac{7}{5}(2, -4, 3) = (4, -6, 5).$$

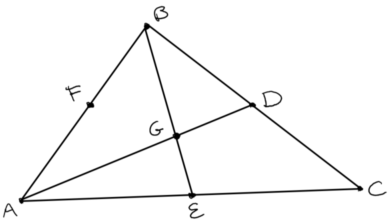
(b) For $s = 7$ and $t = 2$, it follows from (6) that

$$\mathbf{w} = \frac{7}{5}(-3, 1, 2) - \frac{2}{5}(2, -4, 3) = (-5, 3, -4).$$

Note that you can verify these answers are correct, first by observing that they are on the correct line segment, but also by calculation $|\boldsymbol{u} - \boldsymbol{w}|$ and $|\boldsymbol{v} - \boldsymbol{w}|$. Additionally ... what happens to the point $\boldsymbol{w}$ as $\frac{s}{t} \to 1$? ■

(B) Question 16. If D, E and F are the mid-points of the sides BC, CA and AB respectively of $\triangle ABC$, the line segments AD, BE and CF are known as the **medians** of $\triangle ABC$. If A, B and C have position vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ relative to an origin O, find the position vector of the point G on AD s.t. $AG : GD = 2 : 1$ and establish that G also lies on BE and CF. Hence prove that the medians of a triangle are concurrent (they all intersect).

Solution:



A sketch of terms in the question.

If G is on the line segment AD (between A and D, see Example 45) such that $AG : GD = 2 : 1$, then $\mathbf{g} = \frac{1}{3}\mathbf{a} + \frac{2}{3}\mathbf{d}$. Since $\mathbf{d} = \frac{1}{2}(\mathbf{b}+\mathbf{c})$ it follows that

$$\mathbf{g} = \frac{1}{3}\mathbf{a} + \frac{1}{3}\mathbf{b} + \frac{1}{3}\mathbf{c} = \frac{1}{3}(\mathbf{a}+\mathbf{b}+\mathbf{c}). \tag{7}$$

It follows that if G' is on the line segment BE (between B and E) such that $BG' : G'E = 2 : 1$, then $\mathbf{g}' = \frac{1}{3}\mathbf{b} + \frac{2}{3}\mathbf{e}$. Since $\mathbf{e} = \frac{1}{2}(\mathbf{a}+\mathbf{c})$, it follows from (7) that

$$\mathbf{g} = \frac{1}{3}(\mathbf{a}+\mathbf{b}+\mathbf{c}) = \frac{1}{3}\mathbf{b} + \frac{2}{3}\left(\frac{1}{2}(\mathbf{a}+\mathbf{c})\right) = \frac{1}{3}\mathbf{b} + \frac{2}{3}\mathbf{e} = \mathbf{g}'. \tag{8}$$

Similarly, if G'' is on the line segment CF (between C and F) such that $CG'' : G''F = 2 : 1$, then $\mathbf{g}'' = \frac{1}{3}\mathbf{c} + \frac{2}{3}\mathbf{f}$. Since $\mathbf{f} = \frac{1}{2}(\mathbf{a}+\mathbf{b})$, it follows again from (7) that

$$\mathbf{g} = \frac{1}{3}(\mathbf{a}+\mathbf{b}+\mathbf{c}) = \frac{1}{3}\mathbf{c} + \frac{2}{3}\left(\frac{1}{2}(\mathbf{a}+\mathbf{b})\right) = \frac{1}{3}\mathbf{c} + \frac{2}{3}\mathbf{f} = \mathbf{g}''. \tag{9}$$

Therefore, via (7), (8) and (9) the medians of a triangle are concurrent (at the same point). ■

(B) Question 17. The triangle OAB has vertices at O (the origin) and at points A and B with position vectors $\mathbf{a}$ and $\mathbf{b}$ respectively. The point P is the midpoint of OA and Q is the midpoint of PB. The line OQ (extended) intersects AB at R.

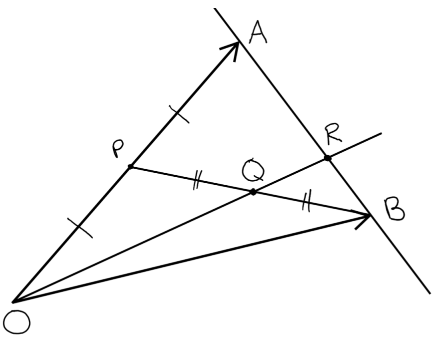
(a) Sketch a diagram of the information above.

(b) Write down, in terms of $\mathbf{a}$ and $\mathbf{b}$, the position vectors $\mathbf{p}$ and $\mathbf{q}$ of the points P and Q.

(c) Find the vector form of the lines AB and OQ and hence find the position vector $\mathbf{r}$ of R. In what ratio does R split AB?

(d) If A has coordinates $(4, -3)$ and B has coordinates $(1, 3)$ show that OR is perpendicular to AB and find the cosine of the angle $A\hat{O}R$.

Solution:



(a)

(b) $\mathbf{p} = \frac{1}{2}\mathbf{a}$ and $\mathbf{q} = \frac{1}{2}\mathbf{a} + \frac{1}{2}(\mathbf{b} - \frac{1}{2}\mathbf{a}) = \frac{1}{4}\mathbf{a} + \frac{1}{2}\mathbf{b}$.

(c) The line L_{AB} is given by $\mathbf{r}' = \mathbf{a} + \lambda(\mathbf{b}-\mathbf{a})$ for $\lambda \in \mathbb{R}$. The line L_{OQ} is given by $\mathbf{r}' = \mu\left(\frac{1}{4}\mathbf{a} + \frac{1}{2}\mathbf{b}\right)$ for $\mu \in \mathbb{R}$. To find the point of intersection of L_{AB} and L_{OQ} set the equations for L_{AB} and L_{OQ} equal, i.e.

$$\mathbf{a} + \lambda(\mathbf{b}-\mathbf{a}) = \mu\left(\frac{1}{4}\mathbf{a} + \frac{1}{2}\mathbf{b}\right) \iff \left(1 - \lambda - \frac{\mu}{4}\right)\mathbf{a} + \left(\lambda - \frac{\mu}{2}\right)\mathbf{b} = \mathbf{0}. \tag{10}$$

Since **a** and **b** are not co-linear, (10) holds if and only if

$$\left(1-\lambda-\frac{\mu}{4}\right)=0 \text{ and } \left(\lambda-\frac{\mu}{2}\right)=0 \iff \mu=\frac{4}{3} \text{ and } \lambda=\frac{2}{3},$$

i.e. the position vector of R is $\mathbf{r}=\frac{1}{3}\mathbf{a}+\frac{2}{3}\mathbf{b}$. We conclude that $AR:RB=2:1$.

(d) $\vec{OR}=\mathbf{r}=\frac{1}{3}(4,-3)+\frac{2}{3}(1,3)=(2,1)$ and $\vec{AB}=(\mathbf{b}-\mathbf{a})=(-3,6)$. Since $(2,1).(-3,6)=-6+6=0$ it follows that $\vec{OR}\perp\vec{AB}$. Additionally, since $\vec{OA}.\vec{OR}=|\vec{OA}||\vec{OR}|\cos(\theta_{AOR})$, it follows that

$$\cos(\theta_{AOR})=\frac{(4,-3).(2,1)}{\sqrt{25}\sqrt{5}}=\frac{1}{\sqrt{5}}.$$

■

(B) Question 18. Find the (non-reflex) angle between the vectors $\mathbf{u}=(1,1,0)$ and $\mathbf{v}=(0,-1,1)$.

Solution: Since $|\mathbf{u}|=\sqrt{2}$, $|\mathbf{v}|=\sqrt{2}$, it follows that if θ is the angle between **u** and **v**, then

$$\mathbf{u}.\mathbf{v}=-1=\sqrt{2}\sqrt{2}\cos\theta.$$

We conclude that $\cos\theta=-\frac{1}{2}$ i.e. that $\theta=\frac{2}{3}\pi$. ■

(B) Question 19. If $U=(6,-2,1)$, $V=(5,4,2)$ and $W=(6,-3,4)$ respectively, determine:

(i) the length of the sides of $\triangle UVW$;

(ii) if $\angle VUW$ is acute or obtuse.

Solution:

(i) Observe that

$$\begin{aligned}|\vec{UV}|&=\sqrt{(\vec{UV}.\vec{UV})}=\sqrt{(-1)^2+6^2+1^2}=\sqrt{38},\\ |\vec{VW}|&=\sqrt{(\vec{VW}.\vec{VW})}=\sqrt{(1)^2+(-7)^2+2^2}=\sqrt{54},\\ |\vec{WU}|&=\sqrt{(\vec{WU}.\vec{WU})}=\sqrt{(0)^2+(-1)^2+3^2}=\sqrt{10}.\end{aligned}$$

(ii) An angle θ is: *acute* if $0<\theta<\frac{\pi}{2}$; and *obtuse* if $\frac{\pi}{2}<\theta<\pi$. Let $\theta=\angle VUW$. Since

$$\vec{UV}.\vec{UW}=(-1,6,1).(0,-1,3)=-3<0,$$

it follows that

$$\cos\theta=-\frac{3}{|\vec{UV}||\vec{UW}|}<0.$$

Also, since $0\le\theta<\pi$ (non-reflex), it follows that $\frac{\pi}{2}<\theta$, i.e. θ is obtuse.

■

(B) Question 20. Find the components of the two **unit** vectors $\mathbf{w}=(w_1,w_2,w_3)$ which make an angle of $\pi/3$ rad. with the vector $\mathbf{u}=(1,0,-1)$ and an angle of $\pi/4$ rad with the vector $\mathbf{v}=(1,-2,-2)$. Show that these two unit vectors are perpendicular.

Solution: Since $\mathbf{w}=(w_1,w_2,w_3)$ is a unit vector, i.e.

$$w_1^2+w_2^2+w_3^2=1, \tag{11}$$

and makes an angle of $\pi/3$ rad with the vector $\mathbf{u}=(1,0,-1)$ and an angle of $\pi/4$ rad with the vector $\mathbf{v}=(1,-2,-2)$, it follow that

$$\mathbf{w}.\mathbf{u}=w_1-w_3=|\mathbf{w}||\mathbf{u}|\cos(\pi/3)=\frac{1}{\sqrt{2}}, \tag{12}$$

$$\mathbf{w}.\mathbf{v} = w_1 - 2w_2 - 2w_3 = |\mathbf{w}||\mathbf{v}|\cos(pi/4) = \frac{3}{\sqrt{2}}. \tag{13}$$

Since $\mathbf{w}$ must satisfy (12) and (13), it follows from subtracting (12) from (13), we have the system:

$$\begin{cases} w_1 - w_3 &= \frac{1}{\sqrt{2}} \\ -2w_2 - w_3 &= \sqrt{2}. \end{cases} \tag{14}$$

Therefore, the solution set to (12) and (13) is the solution to the (14) given by

$$\left\{(w_1, w_2, w_3) = \left(\frac{1}{\sqrt{2}} + t, -\left(\frac{1}{\sqrt{2}} + \frac{t}{2}\right), t\right) : t \in \mathbb{R}\right\}. \tag{15}$$

The elements in the set in (15) that also satisfy (11) must satisfy

$$\left(\frac{1}{\sqrt{2}} + t\right)^2 + \left(-\frac{1}{\sqrt{2}} - \frac{t}{2}\right)^2 + (t)^2 = 1 \iff t\left(\frac{9}{4}t + \frac{3\sqrt{2}}{2}\right) = 0 \iff t \in \left\{0, -\frac{2\sqrt{2}}{3}\right\},$$

and hence,

$$\mathbf{w}^{(1)} = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right) \text{ and } \mathbf{w}^{(2)} = \left(-\frac{\sqrt{2}}{6}, -\frac{\sqrt{2}}{6}, -\frac{2\sqrt{2}}{3}\right).$$

Since, for the unit vectors $\mathbf{w}_1$ and $\mathbf{w}_2$, we have $\mathbf{w}^{(1)}.\mathbf{w}^{(2)} = -\frac{1}{6} + \frac{1}{6} + 0 = 0$, it follows that $\mathbf{w}^{(1)} \perp \mathbf{w}^{(2)}$, as required. ■

(B) Question 21. If $\mathbf{u}$ is a non-zero vector and $\mathbf{u}\times\mathbf{v} = \mathbf{u}\times\mathbf{w}$, is it necessarily true that $\mathbf{v} = \mathbf{w}$? Justify your answer. When $\mathbf{u} = (2, 3, -4)$ and $\mathbf{v} = (-3, -1, 2)$, find a vector $\mathbf{w}$ ($\neq \mathbf{v}$), if one exists, such that $\mathbf{u} \times \mathbf{v} = \mathbf{u} \times \mathbf{w}$.

Solution: No. Via the distributive property, $\mathbf{u} \times \mathbf{v} = \mathbf{u} \times \mathbf{w} \iff \mathbf{u} \times (\mathbf{v} - \mathbf{w}) = \mathbf{0}$. So provided that $\mathbf{v} - \mathbf{w} \parallel \mathbf{u}$, then $\mathbf{u} \times \mathbf{v} = \mathbf{u} \times \mathbf{w}$. For instance, for $\mathbf{u} = (2, 3, -4)$ and $\mathbf{v} = (-3, -1, 2)$, if $\mathbf{w} = (-1, 2, -2)$ (which is not equal to $\mathbf{v}$) then

$$\mathbf{u} \times (\mathbf{w} - \mathbf{v}) = \mathbf{u} \times \mathbf{u} = \mathbf{0} \iff \mathbf{u} \times \mathbf{w} = \mathbf{u} \times \mathbf{v}.$$

■

(B) Question 22. Determine all values of α, if any, for which $\mathbf{u} \times \mathbf{v}$ is perpendicular to $\mathbf{u} \times \mathbf{w}$ when

(a) $\mathbf{u} = (1, 2, -1)$, $\mathbf{v} = (1, \alpha, 1)$ and $\mathbf{w} = (2, -1, \alpha)$;

(b) $\mathbf{u} = (1, -1, 1)$, $\mathbf{v} = (1, 3, \alpha)$ and $\mathbf{w} = (1, \alpha, -1)$.

Solution:

(a) Since

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & -1 \\ 1 & \alpha & 1 \end{vmatrix} = (2 + \alpha, -2, \alpha - 2) \text{ and } \mathbf{u} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & -1 \\ 2 & -1 & \alpha \end{vmatrix} = (2\alpha - 1, -(\alpha + 2), -5),$$

it follows that since $\mathbf{u}$ and $\mathbf{w}$ are non-zero,

$$\begin{aligned} \mathbf{u} \times \mathbf{v} \perp \mathbf{u} \times \mathbf{w} &\iff (\mathbf{u} \times \mathbf{v}).(\mathbf{u} \times \mathbf{w}) = 0 \\ &\iff (2 + \alpha).(2\alpha - 1) + 2(\alpha + 2) - 5(\alpha - 2) \\ &\iff \alpha^2 + 6 = 0, \end{aligned}$$

and since $\alpha^2 + 6 = 0$ has no real solutions we conclude that $\mathbf{u} \times \mathbf{v} \not\perp \mathbf{u} \times \mathbf{w}$ for all $\alpha \in \mathbb{R}$.

(b) Since

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 1 \\ 1 & 3 & \alpha \end{vmatrix} = (-(\alpha + 3), 1 - \alpha, 4) \text{ and } \mathbf{u} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 1 \\ 1 & \alpha & -1 \end{vmatrix} = (1 - \alpha, 2, \alpha + 1),$$

it follows that since $\mathbf{u}$ and $\mathbf{w}$ are non-zero,

$$\begin{aligned}\mathbf{u}\times\mathbf{v}\perp\mathbf{u}\times\mathbf{w} &\iff (\mathbf{u}\times\mathbf{v}).(\mathbf{u}\times\mathbf{w})=0\\ &\iff -(\alpha+3)(1-\alpha)+2(1-\alpha)+4(1+\alpha)\\ &\iff \alpha^2+4\alpha+3=0\\ &\iff (\alpha+3)(\alpha+1)=0,\end{aligned}$$

and hence $\mathbf{u}\times\mathbf{v}\perp\mathbf{u}\times\mathbf{w}$ for $\alpha\in\mathbb{R}$ if and only if, $\alpha=-3$ or $\alpha=-1$.

■

(B) Question 23. Using the definition given in lectures for $\mathbf{u}\times\mathbf{v}$ in the case where $\mathbf{u}$ and $\mathbf{v}$ are two non-zero vectors which are not parallel, establish that if $\mathbf{u}$ is a non-zero vector and $\mathbf{v}$ is a vector such that

$$\mathbf{u}=\mathbf{v}\times(\mathbf{u}\times\mathbf{v}),$$

then $\mathbf{v}$ is a unit vector which is perpendicular to $\mathbf{u}$.

Solution: First observe that

$$\mathbf{u}=\mathbf{v}\times(\mathbf{u}\times\mathbf{v}) \implies \mathbf{v}.\mathbf{u}=\mathbf{v}.(\mathbf{v}\times(\mathbf{u}\times\mathbf{v}))=0,$$

since $\mathbf{v}\perp(\mathbf{v}\times(\mathbf{u}\times\mathbf{v}))$. We conclude that $\mathbf{v}\perp\mathbf{u}$. Consequently, we have

$$|\mathbf{u}|=|\mathbf{v}\times(\mathbf{u}\times\mathbf{v})|=|\mathbf{v}||\mathbf{u}\times\mathbf{v}|\sin\left(\tfrac{\pi}{2}\right)=|\mathbf{v}|^2|\mathbf{u}|\left(\sin\left(\tfrac{\pi}{2}\right)\right)^2 \iff |\mathbf{v}|^2=1 \iff |\mathbf{v}|=1.$$

Therefore, $\mathbf{v}$ is a unit vector perpendicular to $\mathbf{u}$, as required. ■

(B) Question 24. Obtain the components of a vector which is perpendicular to the vectors represented by $\vec{AB}$ and $\vec{CD}$ where

$$A=(1,-2,-3),\quad B=(4,1,1),\quad C=(0,1,-1)\ \text{ and }\ D=(-6,1,3).$$

Hence obtain scalar equations for two parallel planes π and π' where π contains A and B, and π' contains C and D. Calculate the distance between Π and Π'.

Solution: Since $\vec{AB}=(3,3,4)$ and $\vec{CD}=(-6,0,4)$, it follows that a vector perpendicular to $\vec{AB}$ and $\vec{CD}$ is

$$\vec{AB}\times\vec{CD}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\3&3&4\\-6&0&4\end{vmatrix}=(12,-36,18).$$

Since $\vec{AB}\times\vec{CD}$ is perpendicular to both planes, Π and Π', it follows that Π and Π' are given respectively by:

$$(12,-36,18).(x,y,z)=(12,-36,18).(1,-2,-3) \iff 12x-36y+18z=30 \tag{16}$$

$$(12,-36,18).(x,y,z)=(12,-36,18).(0,1,-1) \iff 12x-36y+18z=-54. \tag{17}$$

Therefore, the distance between the two planes is given by

$$\frac{|d-d'|}{|\mathbf{n}|}=\frac{|30-(-54)|}{\sqrt{6^2(2^2+6^2+3^2)}}=\frac{84}{42}=2.$$

Note that to apply the formula to calculate the distance between the two planes above, we need to keep the LHS of (16) *and* (17) *equal (not a (non-1) scalar multiple of the other).* ■

(B) Question 25. Find an equation of the plane Π which contains the point $U=(5,0,2)$ and the line L given by the parametric equations $x=1+3t$, $y=4-2t$, $z=-3+t$.

Solution: Since the direction of the line L is $(3,-2,1)$, and for $t=0$, $(1,4,-3)$ is on L with $(1,4,-3)\not\parallel(3,-2,1)$, it follows that a normal vector to the plane Π is given by

$$((5,0,2)-(1,4,-3))\times(3,-2,1)=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\4&-4&5\\3&-2&1\end{vmatrix}=(6,11,4).$$

Therefore, the plane Π is given by $6x+11y+4z=38$. ■

(B) Question 26. Prove that if $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are distinct non-zero vectors such that

$$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = \mathbf{0} \qquad \text{and} \qquad (\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = \mathbf{0},$$

then either $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are parallel or $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are mutually perpendicular.

Solution: Since $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = \mathbf{0}$ it follows that we have two cases to consider:

(i) If $\mathbf{v} \times \mathbf{w} = \mathbf{0}$ then since $\mathbf{v}$ and $\mathbf{w}$ are non-zero vectors, it follows that

$$\mathbf{v} \parallel \mathbf{w} \Longrightarrow (\mathbf{u} \times \mathbf{v}) \parallel (\mathbf{u} \times \mathbf{w}) \text{ or } (\mathbf{u} \mid\mid \mathbf{v} \mid\mid \mathbf{w}). \tag{18}$$

Now, suppose that $\mathbf{u} \not\parallel \mathbf{v}$. Then since $\mathbf{u}$ and $\mathbf{v}$ are non-zero it follows that $\mathbf{u} \times \mathbf{v} \neq \mathbf{0}$ and moreover, from the second equation above that

$$(\mathbf{u} \times \mathbf{v}) \parallel \mathbf{w} \tag{19}$$

Thus, via (18) and (19) it follows that $(\mathbf{u} \times \mathbf{w}) \parallel (\mathbf{u} \times \mathbf{v}) \parallel \mathbf{w}$ which is a contradiction. Therefore, $\mathbf{u} \parallel \mathbf{v} \parallel \mathbf{w}$, as required.

(ii) If $\mathbf{v} \times \mathbf{w} \neq \mathbf{0}$, then it follows from the the hypothesis that $\mathbf{u} \parallel (\mathbf{v} \times \mathbf{w})$ which implies that $\mathbf{u} \perp \mathbf{v}$ and $\mathbf{u} \perp \mathbf{w}$. Consequently, the hypothesis implies that $\mathbf{w} \parallel (\mathbf{u} \times \mathbf{v})$ which further implies that $\mathbf{v} \perp \mathbf{w}$. Therefore, $\mathbf{u} \perp \mathbf{v} \perp \mathbf{w} \perp \mathbf{u}$, as required.

$\blacksquare$