Solution Sheet 3: Bases

1. Convert the following binary numbers into fractions

$$0.010101\dot{0}\dot{1} \qquad 0.01100110\dot{0}11\dot{0} \qquad 0.101101\dot{1}0\dot{1}$$

Answer 1. We can tackle the first number by multiplying by 2^2 to get

$$2^2 x = 2^2 \times 0.010101\dot{0}\dot{1} = 1.010101\dot{0}\dot{1} = 1 + x \quad \Rightarrow \quad x = \frac{1}{3}$$

We can tackle the second number by multiplying by 2^4

$$2^4 x = 2^4 \times 0.01100110\dot{0}11\dot{0} = 110.01100110\dot{0}11\dot{0} = 2 + 4 + x \quad \Rightarrow \quad x = \frac{6}{15} = \frac{2}{5}$$

alternatively we could multiply by 2^2

$$2^2 x = 2^2 \times 0.01100110\dot{0}11\dot{0} = 1.10011001100\dot{0}11\dot{0} = 1 + 1 - x \quad \Rightarrow \quad x = \frac{2}{5}$$

We can tackle the third number by multiplying by 2^3

$$2^3 x = 2^3 \times 0.101101\dot{1}0\dot{1} = 101.101101\dot{1}0\dot{1} = 1 + 4 + x \quad \Rightarrow \quad x = \frac{5}{7}$$

2. Convert the following fractions into binary numbers

$$\frac{3}{7} \qquad \frac{4}{9} \qquad \frac{1}{15} \qquad \frac{1}{17} \qquad \frac{3}{4} \qquad \frac{1}{11}$$

Answer 2. This is a more intriguing question. We can use the previous ideas in reverse. Since 7=8-1 we have

$$7x = 3 \quad \Rightarrow \quad 8x = 3 + x$$

and since in binary three is 011 we can deduce that

$$\frac{3}{7} = 0.011011\dot{0}1\dot{1}$$

Since 9=8+1 we have

$$9x = 4 \quad \Rightarrow \quad 8x = 3 + 1 - x$$

and since in binary three is 011 we can deduce that

$$\frac{4}{9} = 0.011100011100\dot{0}1110\dot{0}$$

Since 15=16-1 we have

$$15x = 1 \quad \Rightarrow \quad 16x = 1 + x \quad \Rightarrow \quad x = 0.00010001\dot{0}00\dot{1}$$

Since 17=16+1 we have

$$17x = 1 \quad \Rightarrow \quad 16x = 0 + 1 - x \quad \Rightarrow \quad x = 0.0000111100001111\dot{0}000111\dot{1}$$

$$\frac{3}{4} = 0.11$$

For a general number we need to find an appropriate representation

$$\frac{1}{11} = \frac{3}{33}$$

since 33=32+1

$$33x = 3 \quad \Rightarrow \quad 32x = 2 + 1 - x \quad \Rightarrow \quad x = 0.0001011101\dot{0}00101110\dot{1}$$

3. Convert the following ternary numbers into fractions

$$0.012012\dot{0}1\dot{2} \qquad 0.01100110\dot{0}11\dot{0} \qquad 0.101101\dot{1}0\dot{1} \qquad 0.001221\dot{0}0122\dot{1}$$

Answer 3. We can tackle the first number by multiplying by 3^3 to get

$$3^3x = 3^3 \times 0.012012\dot{0}1\dot{2} = 12.012012\dot{0}1\dot{2} = 2 + 3 + x \quad \Rightarrow \quad x = \frac{5}{26}$$

We can tackle the second number by multiplying by 3^4 to get

$$3^4x = 3^4 \times 0.01100110\dot{0}11\dot{0} = 110.01100110\dot{0}11\dot{0} = 3 + 9 + x \quad \Rightarrow \quad x = \frac{12}{80} = \frac{3}{20}$$

We can tackle the third number by multiplying by 3^3 to get

$$3^3x = 3^3 \times 0.101101\dot{1}0\dot{1} = 101.101101\dot{1}0\dot{1} = 1 + 9 + x \quad \Rightarrow \quad x = \frac{10}{26} = \frac{5}{13}$$

We can tackle the final number by multiplying by 3^3 and we get

$$3^3x = 3^3 \times 0.001221\dot{0}0122\dot{1} = 1.221\dot{0}0122\dot{1} = 1 + 1 - x \quad \Rightarrow \quad x = \frac{2}{28} = \frac{1}{14}$$

4. Convert the following fractions into ternary numbers

$$\frac{1}{2} \qquad \frac{1}{8} \qquad \frac{1}{9} \qquad \frac{1}{10}$$

Answer 4. Since 2=3-1 we can use

$$2x = 1 \quad \Rightarrow \quad 3x = 1 + x \quad \Rightarrow \quad x = 0.11\dot{1}$$

Since 8=9-1 we can use

$$8x = 1 \quad \Rightarrow \quad 9x = 1 + x \quad \Rightarrow \quad x = 0.0101\dot{0}\dot{1}$$

Obviously $\frac{1}{9}$=0.01 and since 10=9+1 we can use

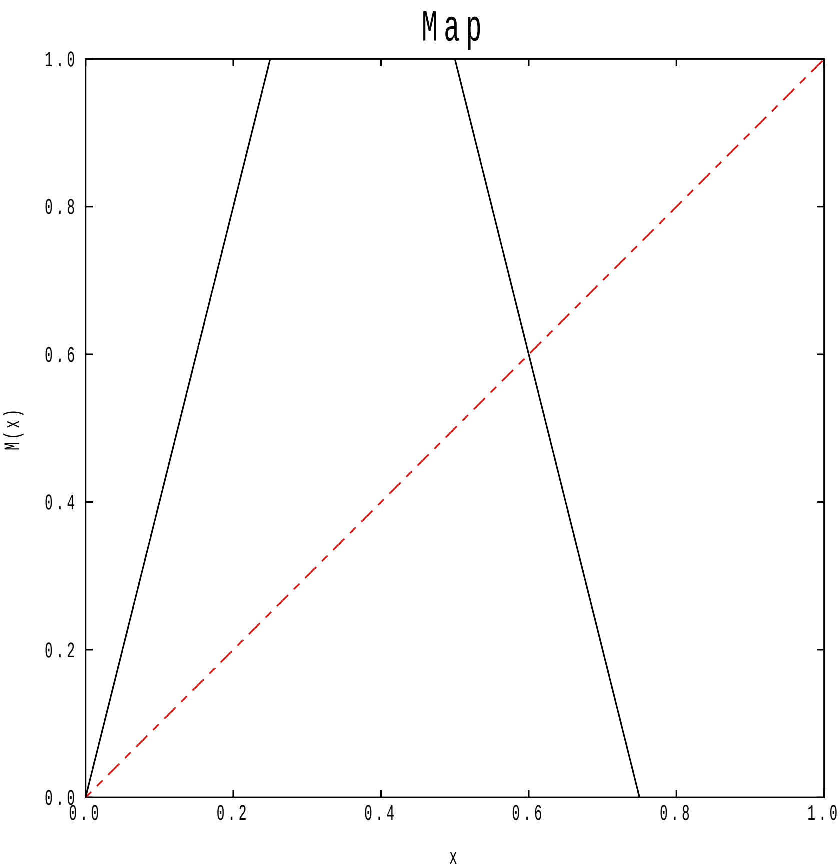
$$10x = 1 \quad \Rightarrow \quad 9x = 1 - x \quad \Rightarrow \quad x = 0.00220022\dot{0}02\dot{2}$$

5. A function is defined by

$$\begin{aligned} f(x) &= 4x & x &\in \left[0, \frac{1}{4}\right] \\ &= 1 & x &\in \left[\frac{1}{4}, \frac{1}{2}\right] \\ &= 4\left[\frac{3}{4} - x\right] & x &\in \left[\frac{1}{2}, \frac{3}{4}\right] \\ &= 0 & x &\in \left[\frac{3}{4}, 1\right] \end{aligned}$$

Depict this function and show how to apply it to a number which is represented in base 4. Find a representation for the numbers which do not end up at zero eventually.

Answer 5. The function depicts as



and clearly base 4 is sensible to describe this map. For the first quarter we have

$$M[0.0a_{\bar{2}}a_{\bar{3}}a_{\bar{4}}...] = 0.a_{\bar{2}}a_{\bar{3}}a_{\bar{4}}...$$

and the digits advance by one along the number. For the second quarter

$$M[0.1a_{\bar{2}}a_{\bar{3}}a_{\bar{4}}...] = 0.3333333....$$

For the third quarter $x = \frac{1}{2} + y$ leads to

$$M[x] = M\left[\frac{1}{2} + y\right] = 4\left[\frac{1}{4} - y\right] = 1 - 4y$$

and so

$$M[0.2a_{\bar{2}}a_{\bar{3}}a_{\bar{4}}...] = 0.3333.... - 0.a_{\bar{2}}a_{\bar{3}}a_{\bar{4}}...] = 0.(3 - a_{\bar{2}})(3 - a_{\bar{3}})(3 - a_{\bar{4}}).....$$

and for the final quarter

$$M[0.3a_{\bar{2}}a_{\bar{3}}a_{\bar{4}}...] = 0.0000000....$$

We can now investigate multiple maps. If the digit at the head of the number is odd then the number will be mapped onto zero after either one step, digit is three, or two steps, digit is one. If the digit is zero at the head of the number then the digits are preserved and move up by one but if the digit is two then the digits are mapped onto the complements and move up by one. This maps odd digits onto even digits and even digits onto odd digits. A surviving number must start with an arbitrary string of zero digits followed by a two. After this two we have an arbitrary string of three digits followed by a one. This pattern then repeats. A regular example is

$$0.0002333100023331....$$

and a more 'random' example is

$$0.2102310023310002333 1.... \mapsto 0.231023310023331000 2... \mapsto 0.023100233100023331...$$

$$\mapsto 0.23100233100023331... \mapsto 0.0233100233310002... \mapsto 0.233100233310002...$$

$$\mapsto 0.00233100023331... \mapsto 0.0233100023331... \mapsto 0.233100023331...$$

$$\mapsto 0.00233310002... \mapsto 0.0233310002... \mapsto 0.233310002...$$

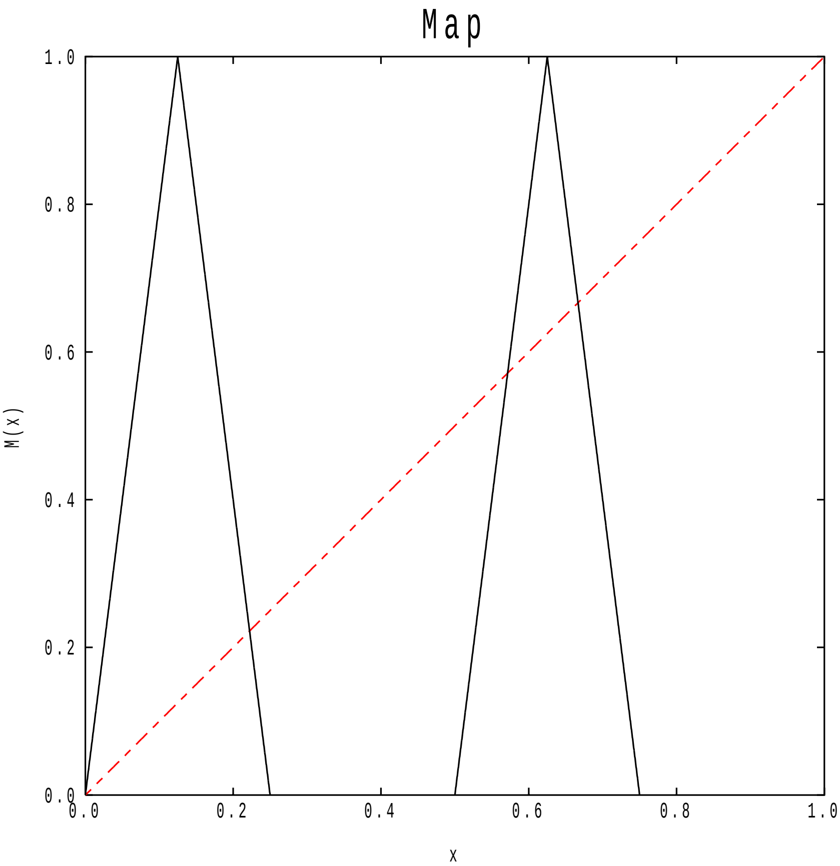
$$\mapsto 0.00023331... \mapsto 0.0023331...$$

6. A function is defined by

$$\begin{aligned} f(x) &= 8x & x &\in \left[0, \frac{1}{8}\right] \\ &= 8\left[\frac{1}{4} - x\right] & x &\in \left[\frac{1}{8}, \frac{1}{4}\right] \\ &= 0 & x &\in \left[\frac{1}{4}, \frac{1}{2}\right] \\ &= 8\left[x - \frac{1}{2}\right] & x &\in \left[\frac{1}{2}, \frac{5}{8}\right] \\ &= 8\left[\frac{3}{4} - x\right] & x &\in \left[\frac{5}{8}, \frac{3}{4}\right] \\ &= 0 & x &\in \left[\frac{3}{4}, 1\right] \end{aligned}$$

Depict this function and show how to apply it to a number which is represented in base 8. Find a representation for the numbers which do not end up at zero eventually.

Answer 6. The function depicts as



and clearly base 8 is suitable to represent the map. For the first eighth we have

$$M[0.0a_2a_3a_4...] = 0.a_2a_3a_4...$$

and for the second eighth we can use $x = \frac{1}{8} + y$ to get

$$M[x] = M\left[\frac{1}{8} + y\right] = 1 - 8y$$

and so

$$M[0.1a_2a_3a_4...] = 0.7777... - 0.a_2a_3a_4... = 0.(7-a_2)(7-a_3)(7-a_4)...$$

for the third and fourth eighths we have

$$M[0.2a_2a_3a_4...] = 0.0000...$$

$$M[0.3a_2a_3a_4...] = 0.0000...$$

for the fifth eighth we can use $x = \frac{1}{2} + y$ to get

$$M[x] = M\left[\frac{1}{2} + y\right] = 8y$$

and so

$$M[0.4a_{\bar{2}}a_{\bar{3}}a_{\bar{4}}...] = 0.a_{\bar{2}}a_{\bar{3}}a_{\bar{4}}...$$

for the sixth eighth we can use $x = \frac{5}{8} + y$ to get

$$M[x] = M\left[\frac{5}{8} + y\right] = 1 - 8y$$

and so

$$M[0.5a_{\bar{2}}a_{\bar{3}}a_{\bar{4}}...] = 0.7777... - 0.a_{\bar{2}}a_{\bar{3}}a_{\bar{4}}... = 0.(7 - a_{\bar{2}})(7 - a_{\bar{3}})(7 - a_{\bar{4}})...$$

for the seventh and final eighths we have

$$M[0.6a_{\bar{2}}a_{\bar{3}}a_{\bar{4}}...] = 0.0000...$$

$$M[0.7a_{\bar{2}}a_{\bar{3}}a_{\bar{4}}...] = 0.0000...$$

We can now investigate multiple maps. If the first digit is zero or four the rest of the digits will be preserved as the digits move up by one. If the first digit is one or five then the digits are mapped onto their complements as the digits are moved up by one. zero and four get mapped onto seven and three, whereas one and five get mapped onto six and two. The numbers that continue to cycle indefinitely start from an arbitrary sequence of zero and four and then we must have a one or a five. We then have an arbitrary sequence of seven and three followed by either a six or a two. This sequence then repeats. An example is

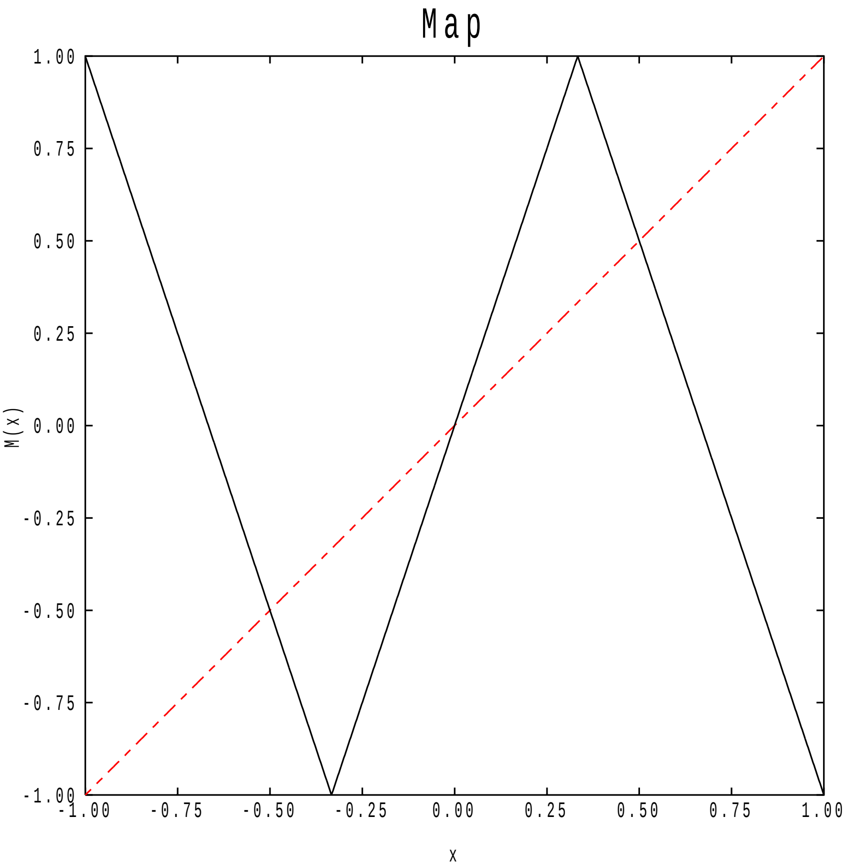
$$0.004453773601200445377360 12...$$

7. Consider the map

$$\begin{aligned} M[x] =& -2 - 3x & x \in \left[-1, -\frac{1}{3}\right] \\ =& 3x & x \in \left[-\frac{1}{3}, \frac{1}{3}\right] \\ =& 2 - 3x & x \in \left[\frac{1}{3}, 1\right] \end{aligned}$$

Depict this map. Find a representation for this map in base 3. Find all the 1-cycles, 2-cycles and 3-cycles using base 3. Convert these cycles into fractions.

Answer 7. The depiction is



To convert into base 3 we employ $z \equiv 0.0a_{\bar{2}}a_{\bar{3}}...$ to hive off the first digit

$$M[-0.2a_{\bar{2}}a_{\bar{3}}...] = M\left[-\frac{2}{3} - z\right] = 3z = 0.a_{\bar{2}}a_{\bar{3}}...$$

$$M[-0.1a_{\bar{2}}a_{\bar{3}}...] = M\left[-\frac{1}{3} - z\right] = -1+3z = -0.222...+0.a_{\bar{2}}a_{\bar{3}}... = -0.(2-a_{\bar{2}})(2-a_{\bar{3}})...$$

$$M[-0.0a_{\bar{2}}a_{\bar{3}}...] = M\left[-z\right] = -3z = -0.a_{\bar{2}}a_{\bar{3}}...$$

$$M[0.0a_{\bar{2}}a_{\bar{3}}...] = M\left[z\right] = 3z = 0.a_{\bar{2}}a_{\bar{3}}...$$

$$M[0.1a_{\bar{2}}a_{\bar{3}}...] = M\left[\frac{1}{3} + z\right] = 1-3z = 0.222... - 0.a_{\bar{2}}a_{\bar{3}}... = 0.(2-a_{\bar{2}})(2-a_{\bar{3}})...$$

$$M[0.2a_{\bar{2}}a_{\bar{3}}...] = M\left[\frac{2}{3} + z\right] = -3z = -0.a_{\bar{2}}a_{\bar{3}}...$$

The key to this map is that the first digit is lost and all the digits move up by one. When there is a unit at the head of the number the digits map onto their complements, $d \mapsto 2-d$, and when there is a two at the head of the number there is a change of sign. We can determine all cycles using numbers composed of finite strings which either repeat or alternate with their complements. If there are an odd number of units in the string we have to use the complements and if there are an odd number of two-digits, once they have been made complements by any units, in the string then the sign flips and we end up with a 2d-cycle instead of a d-cycle, if there are d digits. Using a single digit we have

$$0.0\dot{0} \mapsto 0.0\dot{0} \qquad x = 0$$

$$0.1\dot{1} \mapsto 0.1\dot{1} \qquad 3x = 1 + x \quad \Rightarrow \quad x = \frac{1}{2}$$

$$-0.1\dot{1} \mapsto -0.1\dot{1} \qquad 3x = -1 + x \quad \Rightarrow \quad x = -\frac{1}{2}$$

$$0.2\dot{2} \mapsto -0.2\dot{2} \qquad 3x = 2 + x \quad \Rightarrow \quad x = 1$$

$$-0.2\dot{2} \mapsto 0.2\dot{2} \qquad 3x = -2 + x \quad \Rightarrow \quad x = -1$$

and we find the three 1-cycles and a 2-cycle. Using two digits we find

$$0.00\dot{0}\dot{0} \mapsto 0.00\dot{0}\dot{0} \qquad x = 0$$

$$0.11\dot{1}\dot{1} \mapsto 0.11\dot{1}\dot{1} \qquad 9x = 11 + x \quad \Rightarrow \quad x = \frac{4}{8} = \frac{1}{2}$$

$$-0.11\dot{1}\dot{1} \mapsto -0.11\dot{1}\dot{1} \qquad 9x = -11 + x \quad \Rightarrow \quad x = -\frac{4}{8} = -\frac{1}{2}$$

$$0.22\dot{2}\dot{2} \mapsto -0.22\dot{2}\dot{2} \qquad 9x = 22 + x \quad \Rightarrow \quad x = \frac{8}{8} = 1$$

$$-0.22\dot{2}\dot{2} \mapsto 0.22\dot{2}\dot{2} \qquad 9x = -22 + x \quad \Rightarrow \quad x = -\frac{8}{8} = -1$$

$$0.1210\dot{1}21\dot{0} \mapsto 0.0121\dot{0}12\dot{1} \qquad 9x = 12 + 1 - x \quad \Rightarrow \quad x = \frac{6}{10} = \frac{3}{5}$$

$$0.0121\dot{0}121\dot{0} \mapsto 0.1210\dot{1}21\dot{0} \qquad 9x = 1 + 1 - x \quad \Rightarrow \quad x = \frac{2}{10} = \frac{1}{5}$$

$$-0.1210\dot{1}21\dot{0} \mapsto -0.0121\dot{0}12\dot{1} \qquad 9x = -12 - 1 - x \quad \Rightarrow \quad x = -\frac{6}{10} = -\frac{3}{5}$$

$$-0.0121\dot{0}121\dot{0} \mapsto -0.1210\dot{1}21\dot{0} \qquad 9x = -1 - 1 - x \quad \Rightarrow \quad x = -\frac{2}{10} = -\frac{1}{5}$$

$$0.02\dot{0}\dot{2} \mapsto 0.20\dot{2}\dot{0} \qquad 9x = 2 + x \quad \Rightarrow \quad x = \frac{2}{8} = \frac{1}{4}$$

$$0.20\dot{2}\dot{0} \mapsto -0.02\dot{0}\dot{2} \qquad 9x = 20 + x \quad \Rightarrow \quad x = \frac{6}{8} = \frac{3}{4}$$

$$-0.02\dot{0}\dot{2} \mapsto -0.20\dot{2}\dot{0} \qquad 9x = -2 + x \quad \Rightarrow \quad x = -\frac{2}{8} = -\frac{1}{4}$$

$$-0.20\dot{2}\dot{0} \mapsto 0.02\dot{0}\dot{2} \qquad 9x = -20 + x \quad \Rightarrow \quad x = -\frac{6}{8} = -\frac{3}{4}$$

$$0.1012\dot{1}01\dot{2} \mapsto 0.2101\dot{2}10\dot{1} \qquad 9x = 10 + 1 - x \quad \Rightarrow \quad x = \frac{4}{10} = \frac{2}{5}$$

$$0.2101\dot{2}10\dot{1} \mapsto -0.1210\dot{1}21\dot{0} \qquad 9x = 21 + 1 - x \quad \Rightarrow \quad x = \frac{8}{10} = \frac{4}{5}$$

$$-0.1012\dot{1}012\dot{2} \mapsto -0.2101\dot{2}101\dot{1} \qquad 9x = -10 - 1 - x \quad \Rightarrow \quad x = -\frac{4}{10} = -\frac{2}{5}$$

$$-0.2101\dot{2}101\dot{1} \mapsto 0.1210\dot{1}210\dot{0} \qquad 9x = -21 - 1 - x \quad \Rightarrow \quad x = -\frac{8}{10} = -\frac{4}{5}$$

and the three 1-cycles, three 2-cycles and two 4-cycles. At d=3 there are fifty-three numbers to process....

$$0.000\dot{0}0\dot{0} \mapsto 0.000\dot{0}0\dot{0} \qquad x = 0$$

$$0.111\dot{1}1\dot{1} \mapsto 0.111\dot{1}1\dot{1} \qquad 27x = 111 + x \quad \Rightarrow \quad x = \frac{13}{26} = \frac{1}{2}$$

$$-0.111\dot{1}1\dot{1} \mapsto -0.111\dot{1}1\dot{1} \qquad 27x = -111 + x \quad \Rightarrow \quad x = -\frac{13}{26} = -\frac{1}{2}$$

$$0.222\dot{2}2\dot{2} \mapsto -0.222\dot{2}2\dot{2} \qquad 27x = 222 + x \quad \Rightarrow \quad x = \frac{26}{26} = 1$$

$$-0.222\dot{2}2\dot{2} \mapsto 0.222\dot{2}2\dot{2} \qquad 27x = -222 + x \quad \Rightarrow \quad x = -\frac{26}{26} = -1$$

$$0.022\dot{0}2\dot{2} \mapsto 0.220\dot{2}2\dot{0} \qquad 27x = 22 + x \quad \Rightarrow \quad x = \frac{8}{26} = \frac{4}{13}$$

$$0.220\dot{2}2\dot{0} \mapsto -0.202\dot{2}0\dot{2} \qquad 27x = 220 + x \quad \Rightarrow \quad x = \frac{24}{26} = \frac{12}{13}$$

$$-0.202\dot{2}0\dot{2} \mapsto 0.022\dot{0}2\dot{2} \qquad 27x = -202 + x \quad \Rightarrow \quad x = -\frac{20}{26} = -\frac{10}{13}$$

$$-0.022\dot{0}2\dot{2} \mapsto -0.220\dot{2}2\dot{0} \qquad 27x = -22 + x \quad \Rightarrow \quad x = -\frac{8}{26} = -\frac{4}{13}$$

$$-0.220\dot{2}2\dot{0} \mapsto 0.202\dot{2}0\dot{2} \qquad 27x = -220 + x \quad \Rightarrow \quad x = -\frac{24}{26} = -\frac{12}{13}$$

$$0.202\dot{2}0\dot{2} \mapsto -0.022\dot{0}2\dot{2} \qquad 27x = 202 + x \quad \Rightarrow \quad x = \frac{20}{26} = \frac{10}{13}$$

$$0.110\dot{1}1\dot{0} \mapsto 0.121\dot{1}2\dot{1} \qquad 27x = 110 + x \quad \Rightarrow \quad x = \frac{12}{26} = \frac{6}{13}$$

$$0.121\dot{1}2\dot{1} \mapsto 0.011\dot{0}1\dot{1} \qquad 27x = 121 + x \quad \Rightarrow \quad x = \frac{16}{26} = \frac{8}{13}$$

$$0.011\dot{0}1\dot{1} \mapsto 0.110\dot{1}1\dot{0} \qquad 27x = 11 + x \quad \Rightarrow \quad x = \frac{4}{26} = \frac{2}{13}$$

$$-0.110\dot{1}1\dot{0} \mapsto -0.121\dot{1}2\dot{1} \qquad 27x = -110 + x \quad \Rightarrow \quad x = -\frac{12}{26} = -\frac{6}{13}$$

$$-0.121\dot{1}2\dot{1} \mapsto -0.011\dot{0}1\dot{1} \qquad 27x = -121 + x \quad \Rightarrow \quad x = -\frac{16}{26} = -\frac{8}{13}$$

$$-0.011\dot{0}1\dot{1} \mapsto -0.110\dot{1}1\dot{0} \qquad 27x = -11 + x \quad \Rightarrow \quad x = -\frac{4}{26} = -\frac{2}{13}$$

$$0.100122\dot{1}0012\dot{2} \mapsto 0.221001\dot{2}2100\dot{1} \qquad 27x = 100 + 1 - x \quad \Rightarrow \quad x = \frac{10}{28} = \frac{5}{14}$$

$$0.221001\dot{2}2100\dot{1} \mapsto -0.210012\dot{2}1001\dot{2} \qquad 27x = 221 + 1 - x \quad \Rightarrow \quad x = \frac{26}{28} = \frac{13}{14}$$

$$-0.210012\dot{2}1001\dot{2} \mapsto 0.100122\dot{1}0012\dot{2} \qquad 27x = -210 - 1 - x \quad \Rightarrow \quad x = -\frac{22}{28} = -\frac{11}{14}$$

$$-0.100122\dot{1}0012\dot{2} \mapsto -0.221001\dot{2}2100\dot{1} \qquad 27x = -100 - 1 - x \quad \Rightarrow \quad x = -\frac{10}{28} = -\frac{5}{14}$$

$$-0.221001\dot{2}2100\dot{1} \mapsto 0.210012\dot{2}1001\dot{2} \qquad 27x = -221 - 1 - x \quad \Rightarrow \quad x = -\frac{26}{28} = -\frac{13}{14}$$

$$0.210012\dot{2}1001\dot{2} \mapsto -0.100122\dot{1}0012\dot{2} \qquad 27x = 210 + 1 - x \quad \Rightarrow \quad x = \frac{22}{28} = \frac{11}{14}$$

$$0.122100\dot{1}2210\dot{0} \mapsto 0.001221\dot{0}0122\dot{1} \qquad 27x = 122 + 1 - x \quad \Rightarrow \quad x = \frac{18}{28} = \frac{9}{14}$$

$$0.001221\dot{0}0122\dot{1} \mapsto 0.012210\dot{0}1221\dot{0} \qquad 27x = 1 + 1 - x \quad \Rightarrow \quad x = \frac{2}{28} = \frac{1}{14}$$

$$0.012210\dot{0}1221\dot{0} \mapsto 0.122100\dot{1}2210\dot{0} \qquad 27x = 12 + 1 - x \quad \Rightarrow \quad x = \frac{6}{28} = \frac{3}{14}$$

$$-0.122100\dot{1}2210\dot{0} \mapsto -0.001221\dot{0}0122\dot{1} \qquad 27x = -122 - 1 - x \quad \Rightarrow \quad x = -\frac{18}{28} = -\frac{9}{14}$$

$$-0.001221\dot{0}0122\dot{1} \mapsto -0.012210\dot{0}1221\dot{0} \qquad 27x = -1 - 1 - x \quad \Rightarrow \quad x = -\frac{2}{28} = -\frac{1}{14}$$

$$-0.012210\dot{0}1221\dot{0} \mapsto -0.122100\dot{1}2210\dot{0} \qquad 27x = -12 - 1 - x \quad \Rightarrow \quad x = -\frac{6}{28} = -\frac{3}{14}$$

$$0.002\dot{0}0\dot{2} \mapsto 0.020\dot{0}2\dot{0} \qquad 27x = 2 + x \quad \Rightarrow \quad x = \frac{2}{26} = \frac{1}{13}$$

$$0.020\dot{0}2\dot{0} \mapsto 0.200\dot{2}0\dot{0} \qquad 27x = 20 + x \quad \Rightarrow \quad x = \frac{6}{26} = \frac{3}{13}$$

$$0.200\dot{2}0\dot{0} \mapsto -0.002\dot{0}0\dot{2} \qquad 27x = 200 + x \quad \Rightarrow \quad x = \frac{18}{26} = \frac{9}{13}$$

$$-0.002\dot{0}0\dot{2} \mapsto -0.020\dot{0}2\dot{0} \qquad 27x = -2 + x \quad \Rightarrow \quad x = -\frac{2}{26} = -\frac{1}{13}$$

$$-0.020\dot{0}2\dot{0} \mapsto -0.200\dot{2}0\dot{0} \qquad 27x = -20 + x \quad \Rightarrow \quad x = -\frac{6}{26} = -\frac{3}{13}$$

$$-0.200\dot{2}0\dot{0} \mapsto 0.002\dot{0}0\dot{2} \qquad 27x = -200 + x \quad \Rightarrow \quad x = -\frac{18}{26} = -\frac{9}{13}$$

$$0.112\dot{1}1\dot{2} \mapsto 0.101\dot{1}0\dot{1} \qquad 27x = 112 + x \quad \Rightarrow \quad x = \frac{14}{26} = \frac{7}{13}$$

$$0.101\dot{1}0\dot{1} \mapsto 0.211\dot{2}1\dot{1} \qquad 27x = 101 + x \quad \Rightarrow \quad x = \frac{10}{26} = \frac{5}{13}$$

$$0.211\dot{2}1\dot{1} \mapsto -0.112\dot{1}1\dot{2} \qquad 27x = 211 + x \quad \Rightarrow \quad x = \frac{22}{26} = \frac{11}{13}$$

$$-0.112\dot{1}1\dot{2} \mapsto -0.101\dot{1}0\dot{1} \qquad 27x = -112 + x \quad \Rightarrow \quad x = -\frac{14}{26} = -\frac{7}{13}$$

$$-0.101\dot{1}0\dot{1} \mapsto -0.211\dot{2}1\dot{1} \qquad 27x = -101 + x \quad \Rightarrow \quad x = -\frac{10}{26} = -\frac{5}{13}$$

$$-0.211\dot{2}1\dot{1} \mapsto 0.112\dot{1}1\dot{2} \qquad 27x = -211 + x \quad \Rightarrow \quad x = -\frac{22}{26} = -\frac{11}{13}$$

$$0.102120\dot{1}02120\dot{0} \mapsto 0.201021\dot{2}01021\dot{1} \qquad 27x = 102 + 1 - x \quad \Rightarrow \quad x = \frac{12}{28} = \frac{3}{7}$$

$$0.201021\dot{2}01021\dot{1} \mapsto -0.010212\dot{0}10212\dot{2} \qquad 27x = 201 + 1 - x \quad \Rightarrow \quad x = \frac{20}{28} = \frac{5}{7}$$

$$-0.010212\dot{0}10212\dot{2} \mapsto -0.102120\dot{1}02120\dot{0} \qquad 27x = -10 - 1 - x \quad \Rightarrow \quad x = -\frac{4}{28} = -\frac{1}{7}$$

$$-0.102120\dot{1}02120\dot{0} \mapsto -0.201021\dot{2}01021\dot{1} \qquad 27x = -102 - 1 - x \quad \Rightarrow \quad x = -\frac{12}{28} = -\frac{3}{7}$$

$$-0.201021\dot{2}01021\dot{1} \mapsto 0.010212\dot{0}10212\dot{2} \qquad 27x = -201 - 1 - x \quad \Rightarrow \quad x = -\frac{20}{28} = -\frac{5}{7}$$

$$0.010212\dot{0}10212\dot{2} \mapsto 0.102120\dot{1}02120\dot{0} \qquad 27x = 10 + 1 - x \quad \Rightarrow \quad x = \frac{4}{28} = \frac{1}{7}$$

$$0.120102\dot{1}20102\dot{2} \mapsto 0.021201\dot{0}21201\dot{1} \qquad 27x = 120 + 1 - x \quad \Rightarrow \quad x = \frac{16}{28} = \frac{4}{7}$$

$$0.021201\dot{0}21201\dot{1} \mapsto 0.212010\dot{2}12010\dot{0} \qquad 27x = 21 + 1 - x \quad \Rightarrow \quad x = \frac{8}{28} = \frac{2}{7}$$

$$0.212010\dot{2}12010\dot{0} \mapsto -0.120102\dot{1}20102\dot{2} \qquad 27x = 212 + 1 - x \quad \Rightarrow \quad x = \frac{24}{28} = \frac{6}{7}$$

$$-0.120102\dot{1}20102\dot{2} \mapsto -0.021201\dot{0}21201\dot{1} \qquad 27x = -120 - 1 - x \quad \Rightarrow \quad x = -\frac{16}{28} = -\frac{4}{7}$$

$$-0.021201\dot{0}21201\dot{1} \mapsto -0.212010\dot{2}12010\dot{0} \qquad 27x = -21 - 1 - x \quad \Rightarrow \quad x = -\frac{8}{28} = -\frac{2}{7}$$

$$-0.212010\dot{2}12010\dot{0} \mapsto 0.120102\dot{1}20102\dot{2} \qquad 27x = -212 - 1 - x \quad \Rightarrow \quad x = -\frac{24}{28} = -\frac{6}{7}$$

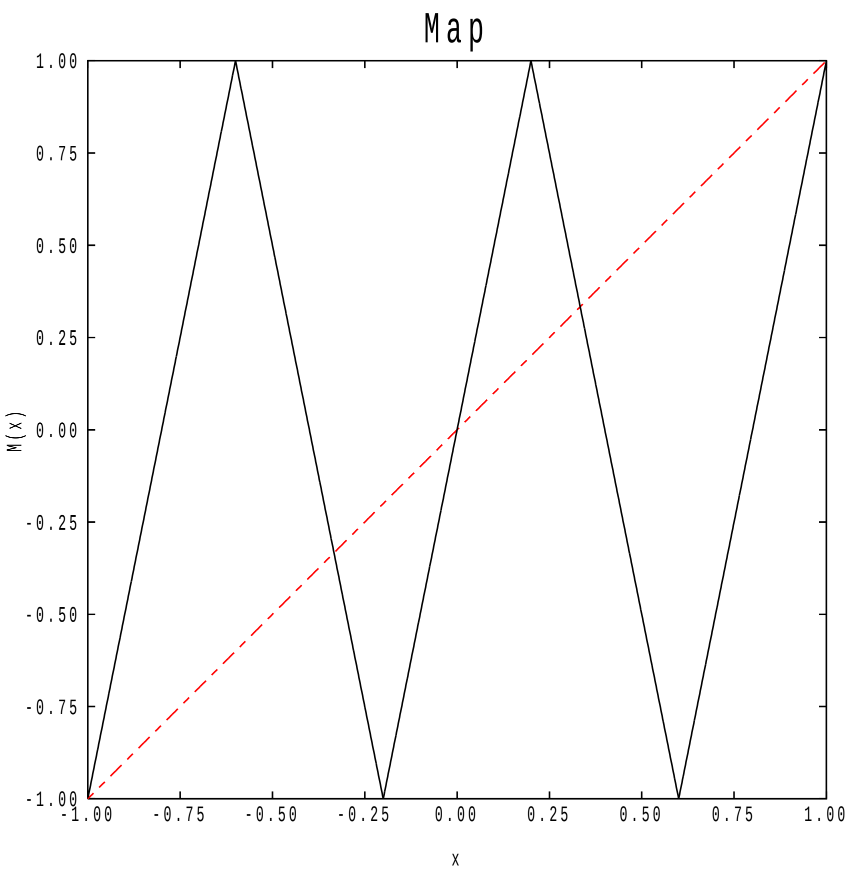
and we find three 1-cycles, one 2-cycle, eight 3-cycles and four 6-cycles.

8. Consider the map

$$\begin{aligned} M[x] =& 4 + 5x && x \in \left[-1, -\frac{3}{5}\right] \\ =& -2 - 5x && x \in \left[-\frac{3}{5}, -\frac{1}{5}\right] \\ =& 5x && x \in \left[-\frac{1}{5}, \frac{1}{5}\right] \\ =& 2 - 5x && x \in \left[\frac{1}{5}, \frac{3}{5}\right] \\ =& 5x - 4 && x \in \left[\frac{3}{5}, 1\right] \end{aligned}$$

Depict this map. Find a base 5 representation for this map and use it to find all the 1-cycles.

Answer 8. The map depicts as



and we may find the 1-cycles directly by solving

$$x = 4 + 5x \qquad x = -2 - 5x \qquad x = 5x \qquad x = 2 - 5x \qquad x = 5x - 4$$

which gives $x \in \left\{ -1, -\frac{1}{3}, 0, \frac{1}{3}, 1 \right\}$. To use base 5 we represent using $z \equiv 0.0a_{\bar{2}}a_{\bar{3}}...$ and then

$$M[-0.4a_{\bar{2}}a_{\bar{3}}...] = M\left[-\frac{4}{5} - z\right] = -5z = -0.a_{\bar{2}}a_{\bar{3}}...$$

$$M[-0.3a_{\bar{2}}a_{\bar{3}}...] = M\left[-\frac{3}{5} - z\right] = 1 - 5z = 0.444... - 0.a_{\bar{2}}a_{\bar{3}}... = 0.(4 - a_{\bar{2}})(4 - a_{\bar{3}})...$$

$$M[-0.2a_{\bar{2}}a_{\bar{3}}...] = M\left[-\frac{2}{5} - z\right] = 5z = 0.a_{\bar{2}}a_{\bar{3}}...$$

$$M[-0.1a_{\bar{2}}a_{\bar{3}}...] = M\left[-\frac{1}{5} - z\right] = 5z - 1 = -0.444... + 0.a_{\bar{2}}a_{\bar{3}}... = -0.(4 - a_{\bar{2}})(4 - a_{\bar{3}})...$$

$$M[-0.0a_{\bar{2}}a_{\bar{3}}...] = M[-z] = -5z = -0.a_{\bar{2}}a_{\bar{3}}...$$

$$M[0.0a_{\bar{2}}a_{\bar{3}}...] = M[z] = 5z = 0.a_{\bar{2}}a_{\bar{3}}...$$

$$M[0.1a_{\bar{2}}a_{\bar{3}}...] = M\left[\frac{1}{5}+z\right] = 1-5z = 0.444...-0.a_{\bar{2}}a_{\bar{3}}... = 0.(4-a_{\bar{2}})(4-a_{\bar{3}})...$$

$$M[0.2a_{\bar{2}}a_{\bar{3}}...] = M\left[\frac{2}{5}+z\right] = -5z = -0.a_{\bar{2}}a_{\bar{3}}...$$

$$M[0.3a_{\bar{2}}a_{\bar{3}}...] = M\left[\frac{3}{5}+z\right] = 5z-1 = -0.444...+0.a_{\bar{2}}a_{\bar{3}}... = -0.(4-a_{\bar{2}})(4-a_{\bar{3}})...$$

$$M[0.4a_{\bar{2}}a_{\bar{3}}...] = M\left[\frac{4}{5}+z\right] = 5z = 0.a_{\bar{2}}a_{\bar{3}}...$$

The crucial point is that the first digit is lost and all the remaining digits move up by one. When the first digit is odd we map onto the complements $d \mapsto 4-d$. When the digit is two or three we flip sign. Using repeating numbers or those that repeat with complements

$$0.0\dot{0} \mapsto 0.0\dot{0} \qquad x=0$$

$$0.4\dot{4} \mapsto 0.4\dot{4} \qquad 5x=4+x \quad x=1$$

$$-0.4\dot{4} \mapsto -0.4\dot{4} \qquad 5x=-4+x \quad x=-1$$

$$0.2\dot{2} \mapsto -0.2\dot{2} \qquad 5x=2+x \quad x=\frac{1}{2}$$

$$-0.2\dot{2} \mapsto 0.2\dot{2} \qquad 5x=-2+x \quad x=-\frac{1}{2}$$

$$0.13\dot{1}\dot{3} \mapsto 0.13\dot{1}\dot{3} \qquad 5x=1+1-x \qquad x=\frac{1}{3}$$

$$-0.13\dot{1}\dot{3} \mapsto -0.13\dot{1}\dot{3} \qquad 5x=-1-1-x \qquad x=-\frac{1}{3}$$

$$0.31\dot{3}\dot{1} \mapsto -0.31\dot{3}\dot{1} \qquad 5x=3+1-x \qquad x=\frac{2}{3}$$

$$-0.31\dot{3}\dot{1} \mapsto 0.31\dot{3}\dot{1} \qquad 5x=-3-1-x \qquad x=-\frac{2}{3}$$

and we have the previous five 1-cycles and a couple of 2-cycles.