2. Introduction to Data Analysis and Error Handling

Calculating with Errors

Often the physical quantity we are interested in is not what we measure directly, but some function of one or more different measurements made in an experiment. For example, if we wish to find the area of a circle by measuring its radius, we need to use the formula $A = \pi r^2$; or if we wish to measure the area of a rectangle, we must first measure its length and width, and then use the formula $A = \ell w$. The question then arises of how to work out the error in the final physical quantity given the errors in the original physical quantities we measure. To solve this problem, we need to remember that the error δx in a measurement x is the standard deviation of the measurement over a probability distribution $p(x)$, whilst the actual value x_0 is the mean of the measurement,

$$\delta x^2 \equiv \sigma(x)^2 = E\big[x^2\big] - E\big[x\big]^2, \qquad x_0 \equiv \mu(x) = E\big[x\big].$$

Some Simple Situations

We first consider some simple situations where we combine measurements of variables x, y, $z\ldots$ with errors δx, δy, $\delta z\ldots$ We will assume statistical independence between the variables which enter into our formulae.

Scaling: $y = ax$, where a is a constant. To evaluate $\mu(y)$ and $\sigma(y)$ we note that

$$\begin{aligned} E\big[y\big] &= E\big[ax\big] = aE\big[x\big], \\ E\big[y^2\big] &= E\big[a^2x^2\big] = a^2E\big[x^2\big], \end{aligned}$$

from which it follows that

$$\begin{aligned} \mu(y) &= E\big[y\big] = aE\big[x\big] = a\mu(x), \\ \delta^2(y) &= E\big[y^2\big] - E\big[y\big]^2 = a^2E\big[x^2\big] - a^2E\big[x\big]^2 = a^2\delta^2(x). \end{aligned}$$

Hence $\mu(y) = a\mu(x)$ and $\delta(y) = a\delta(x)$, or in less precise language, $y_0 = ax_0$ and $\delta y = a\delta x$.

Addition: $z = x + y$. To evaluate $\mu(z)$ and $\sigma(z)$ we note that

$$\begin{aligned} E\big[z\big] &= E\big[x+y\big] = E\big[x\big] + E\big[y\big], \\ E\big[z^2\big] &= E\big[x^2 + 2xy + y^2\big] = E\big[x^2\big] + 2E\big[x\big]E\big[y\big] + E\big[y^2\big], \end{aligned}$$

from which it follows that

$$\begin{aligned} \mu(z) &= E\big[z\big] = E\big[x\big] + E\big[y\big] = \mu(x) + \mu(y), \\ \sigma^2(z) &= E\big[z^2\big] - E\big[z\big]^2 = E\big[x^2\big] - E\big[x\big]^2 + E\big[y^2\big] - E\big[y\big]^2 = \sigma^2(x) + \sigma^2(y). \end{aligned}$$

Hence $\mu(z) = \mu(x)+\mu(y)$ and $\sigma^2(z) = \sigma^2(x)+\sigma^2(y)$, or in less precise language, $z_0 = x_0+y_0$ and $\delta z^2 = \delta x^2 + \delta y^2$.

Linear Combinations: $z = ax + by$, where a and b are constants. Combining the results of the previous two sections gives $\mu(z) = a\mu(x) + b\mu(y)$ and $\delta^2(z) = a^2\delta^2(x) + b^2\delta^2(y)$. or in less precise language

$$z_0 = ax_0 + by_0 \qquad \text{and} \qquad \delta z^2 = a^2\delta x^2 + b^2\delta y^2.$$

Multiplication: $z = xy$. To evaluate $\mu(z)$ and $\sigma(z)$ we note that

$$\begin{aligned} E\big[z\big] &= E\big[xy\big] = E\big[x\big]E\big[y\big], \\ E\big[z^2\big] &= E\big[x^2y^2\big] = E\big[x^2\big]E\big[y^2\big], \end{aligned}$$

from which it follows that

$$\begin{aligned} \mu(z) &= E\big[z\big] = E\big[x\big]E\big[y\big] = \mu(x)\mu(y), \\ \sigma^2(z) &= E\big[z^2\big] - E\big[z\big]^2 = E\big[x^2\big]E\big[y^2\big] - E\big[x\big]^2E\big[y\big]^2 \\ &= \Big[\mu^2(x) + \sigma^2(x)\Big]\Big[\mu^2(y) + \sigma^2(y)\Big] - \mu^2(x)\mu^2(y) \\ &= \mu^2(x)\sigma^2(y) + \mu^2(y)\sigma^2(x) + \sigma^2(x)\sigma^2(y). \end{aligned}$$

We may combine these two equations to give

$$\frac{\sigma^2(z)}{\mu^2(z)} = \frac{\sigma^2(x)}{\mu^2(x)} + \frac{\sigma^2(y)}{\mu^2(y)} + \frac{\sigma^2(x)}{\mu^2(x)}\frac{\sigma^2(y)}{\mu^2(y)} \approx \frac{\sigma^2(x)}{\mu^2(x)} + \frac{\sigma^2(y)}{\mu^2(y)},$$

if we ignore terms of the second order of smallness. These results may be summarised as

$$z_0 = x_0y_0 \qquad \text{and} \qquad \left(\frac{\delta z}{z_0}\right)^2 = \left(\frac{\delta x}{x_0}\right)^2 + \left(\frac{\delta y}{y_0}\right)^2.$$

The extension to a product of three or more independent variables is straightforward.

General Powers

To consider more general cases, we must use approximations and develop a suitable notation. We will write x_0 for the true value of a quantity, x, and define Δx by $x = x_0 + \Delta x$ i.e. Δx is the deviation of a single measurement from the true value. The error, which is the standard deviation of x, is then given by $\delta x^2 = E\big[\Delta x^2\big]$.

If we have $z = ax^\alpha$, then using the binomial theorem

$$z = z_0 + \Delta z = a(x_0 + \Delta x)^\alpha \approx ax_0^\alpha + a\alpha x_0^{\alpha-1}\Delta x,$$

from which it follows that

$$z_0 = ax_0^\alpha \qquad \text{and} \qquad \delta z^2 = a^2\alpha^2x_0^{2(\alpha-1)}\delta x^2.$$

In terms of the relative uncertainties, this gives the simple result

$$\left(\frac{\delta z}{z_0}\right)^2 = \alpha^2\left(\frac{\delta x}{x_0}\right)^2.$$

Exercise 1: *A loaded beam with dimensions ℓ, b and t is supported at the ends; the sag s at the midpoint is given by*

$$s = \frac{k\ell^3}{tb^3},$$

where k is a constant. If there is a 1% error in ℓ, a 2% error in b, and a 3% error in t, what is the percentage error in s?

Exercise 2: *Physically observable variables x, y and z are related by*

$$z = ax^{\alpha}y^{\beta},$$

where a, α and β are constants. Show that their mean values and standard deviations are related by

$$\mu(z) = a\mu(x)^{\alpha}\mu(y)^{\beta}, \qquad \left(\frac{\sigma(z)}{\mu(z)}\right)^2 = \left(\alpha\frac{\sigma(x)}{\mu(x)}\right)^2 + \left(\beta\frac{\sigma(y)}{\mu(y)}\right)^2.$$

Exercise 3: *Physically observable variables x and z are related by $z = f(x)$. Use Taylor's theorem to show that their mean values and standard deviations are related by*

$$\mu(z) = f(\mu(x)), \qquad \sigma(z) = |f'(\mu(x))|\,\sigma(x).$$

Exercise 4: *Physically observable variables x, y and z are related by $z = f(x, y)$. Use Taylor's theorem to show that their mean values and standard deviations are related by*

$$\begin{aligned}
&\mu(z) = f(\mu(x), \mu(y)),\\
&\sigma^2(z) = \left|\frac{\partial f}{\partial x}(\mu(x), \mu(y))\right|^2 \sigma^2(x) + \left|\frac{\partial f}{\partial y}(\mu(x), \mu(y))\right|^2 \sigma^2(y).
\end{aligned}$$