UNIVERSITY OF BIRMINGHAM

School of Mathematics

Programmes in the School of Mathematics	First Examination
Programmes in the School of Mathematics	Second Examination
Programmes involving Mathematics	First Examination
Programmes involving Mathematics	Second Examination

1Mech 06 25661 Level C
LC Mechanics

1Mech2 06 27345 Level I
LI Mechanics

May/June Examinations 2023-24

One Hour and Thirty Minutes

Full marks will be obtained with complete answers to BOTH questions. Each question carries equal weight. You are advised to initially spend no more than 45 minutes on each question and then to return to any incomplete questions if you have time at the end. An indication of the number of marks allocated to parts of questions is shown in square brackets.

No calculator is permitted in this examination.

Section A

1. (a) Consider the equation

$$\int u \, \mathrm{d}t \quad = \quad mg + \ddot{x},$$

where u is velocity, t is time, m is mass, g is acceleration due to gravity, x is distance and double dot denotes the second derivative with respect to time. Is this equation dimensionally homogeneous? Justify your answer. [2]

(b) If a particle of mass m is subject to a force of the form $F = F_0 \cos(\omega t)$, where F_0 and ω are constants and t is time, find the acceleration, velocity and position of the particle as a function of time, given the particle starts at $x = 0$ with constant velocity u. [6]

(c) Starting from Newton's Second Law, show that

$$\frac{d}{dt}(\mathbf{r} \times \dot{\mathbf{r}}) \quad = \quad 0$$

for a particle moving under the action of a central force, where $\mathbf{r}$ is position, m is mass and t is time. Hence, prove that the motion is constrained to lie in a plane. [5]

(d) Formulate a complete model system for a situation where a particle of mass m is acted on by an outward pointing central force of the form

$$F(r) \quad = \quad -\frac{\alpha}{r^2} + \frac{\beta}{r^3},$$

where α and β are constants. The particle is initially located a distance a from the centre of the force, moving with radial velocity v and transverse velocity w. You may assume that $\dot{\mathbf{r}} = \dot{r}\mathbf{e}_r + r\dot{\theta}\mathbf{e}_\theta$ for a position vector $\mathbf{r}$ in polar coordinates, $\dot{r} = -h\dfrac{du}{d\theta}$ and

$$\frac{d^2u}{d\theta^2} + u \quad = \quad -\frac{F(1/u)}{mh^2u^2},$$

where $u = 1/r$ and you should carefully define h. Note that you are not asked to solve the resulting model. [12]

Section B

2. (a) First consider a particle of mass m moving in a straight line with position x. Starting from Newton's second law, prove the principle of Conservation of Energy for a force that depends only on x. [5]

Please turn over for the rest of the question.

(b) Now consider a particle of mass m at position $P(\theta,t)$ moving on a smooth wire shaped in a circle of radius a lying in a vertical plane, where θ is the polar angle that the particle makes with the centre of the circle O measured from the upwards vertical and t is time. The particle is attached to one end of a spring (spring constant k, natural length $a/2$) with the other end of the spring attached to the top of the wire $T(\theta = 0)$. See Figure 1.

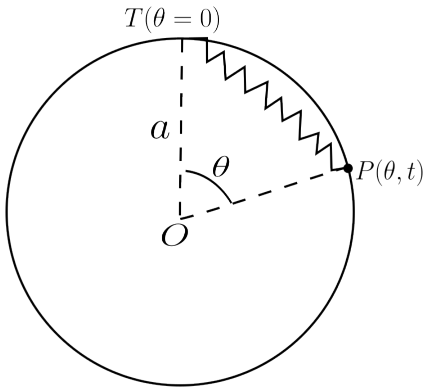

Figure 1: Sketch of the wire, mass and spring system for part (b), showing the top of the circle $T(\theta = 0)$, the particle at point $P(\theta,t)$ and the origin O at the centre of the circle.

(i) Show that the kinetic energy of the particle is given by

$$\frac{1}{2}ma^2\dot{\theta}^2,$$

where dot denotes the first derivative with respect to time.

(ii) Show that the extension in the spring is given by

$$2a\sin\left(\frac{\theta}{2}\right) - \frac{a}{2}.$$

(iii) Find the height of the particle above the centre of the circle.

(iv) Hence, show that the total energy of the system is given by

$$\frac{1}{2}ma^2\dot{\theta}^2 + mga\cos\theta + \frac{ka^2}{2}\left(2\sin\left(\frac{\theta}{2}\right) - \frac{1}{2}\right)^2 \quad = \quad E.$$

(v) If the particle starts from rest at $\theta = \pi/3$, find the value of θ when the particle will stop again. You may leave your answer in terms of a trigonometric function for θ. [20]

[Hint: Recall that $\sin\frac{\pi}{6} = \frac{1}{2}$, $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$ *and* $\cos 2\theta = 1 - 2\sin^2\theta$.*]*

This page is intentionally blank.

A38707

LC/LI Mechanics

Do not complete the attendance slip, fill in the front of the answer book or turn over the question paper until you are told to do so.

Important Reminders

- Coats and outer-wear should be placed in the designated area.
- Unauthorised materials (e.g. notes or Tippex) **MUST** be placed in the designated area.
- Check that you **DO NOT** have any unauthorised materials with you (e.g. in your pockets, pencil case).
- Mobile phones and smart watches **MUST** be switched off and placed in the designated area or under your desk. They must not be left on your person or in your pockets.
- You are **NOT** permitted to use a mobile phone as a clock. If you have difficulty in seeing a clock, please alert an Invigilator.
- You are **NOT** permitted to have writing on your hand, arm or other body part.
- Check that you do not have writing on your hand, arm or other body part – if you do, you must inform an Invigilator immediately.
- Alert an Invigilator immediately if you find any unauthorised item upon you during the examination.

Any students found with non-permitted items upon their person during the examination, or who fail to comply with Examination rules may be subject to the Student Conduct procedures.