UNIVERSITY OF BIRMINGHAM

School of Physics and Astronomy

DEGREE OF B.Sc. & M.Sci. WITH HONOURS

FIRST-YEAR EXAMINATION

03 19718 - QM, O&W

LC QUANTUM MECHANICS / OPTICS & WAVES

SEMESTER 1 EXAMINATIONS 2022/23

Time Allowed: 2 hours

Answer four questions from Section 1 and two questions from Section 2.

Section 1 consists of four questions and carries 28% of the marks for the examination. Answer ***all four*** questions from this Section.

Section 2 consists of two questions and carries 72% of the marks. Answer ***both*** questions in this Section. Note that each question has two parts, of which only ***one part*** should be answered. If you answer both parts, credit will only be given for the best answer.

The approximate allocation of marks to each part of a question is shown in brackets **[ ]**.

PLEASE USE A SEPARATE ANSWER BOOK FOR SECTION 1 AND SECTION 2 QUESTIONS.

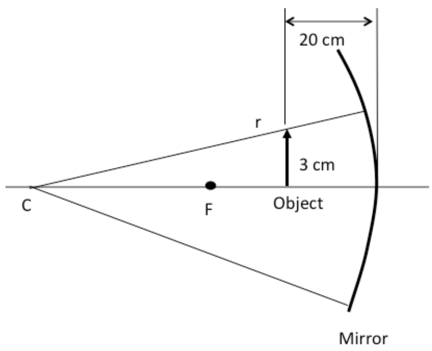
Calculators may be used in this examination but must not be used to store text. Calculators with the ability to store text should have their memories deleted prior to the start of the examination.

A formula sheet and a table of physical constants and units that may be required will be found at the end of this question paper.

SECTION 1

Answer **all four** *questions from this Section.*

1. A $3\,\mathrm{cm}$-high object is located $20\,\mathrm{cm}$ from a concave mirror. The mirror's radius of curvature is $80\,\mathrm{cm}$. Determine the position, orientation, and the height of the image. Sketch the special rays.



[5]

2. A violin string $15\,\mathrm{cm}$ long and fixed at both ends oscillates in its $n = 1$ mode. The speed of waves on the string is $250\,\mathrm{m\,s^{-1}}$, and the speed of sound in air is $348\,\mathrm{m\,s^{-1}}$. What are (i) the frequency and (ii) the wavelength of the emitted sound wave? What is the frequency of the sound perceived by a bird flying at $10\,\mathrm{m\,s^{-1}}$ toward the violin? **[5]**

3. A monoenergetic beam of neutrons scatters from a crystal with planes of atoms separated by $0.0910\ \mathrm{nm}$. A single intensity maximum is observed when the scattering angle between the incoming and outgoing beams of neutrons is $29°$. Given that the mass of the neutron is $1.675 \times 10^{-27}\ \mathrm{kg}$, find the kinetic energy of the neutrons in units of electron volts. **[5]**

4. A particle of mass m is confined in an infinite one-dimensional potential well of width L. Its spatial wave function is given by

$$\psi(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{3\pi x}{L}\right).$$

Show that the wave function is <u>not</u> an eigenfunction of the operator for the x-component of momentum.

Show that the wave function is an eigenfunction of the kinetic energy operator and determine its eigenvalue.

Explain the physical significance of these observations in this context. **[5]**

SECTION 2

*Answer **both** questions in this Section. Note that each question has two parts, of which only **one part** should be answered. If you answer both parts, credit will only be given for the best answer.*

5. **EITHER Part A**

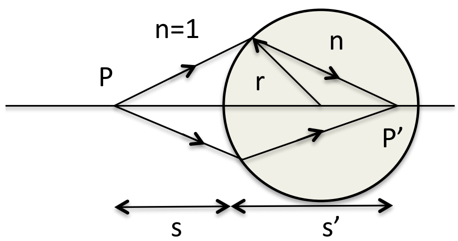
(a) State Fermat's principle for the time it takes for light to propagate from one point to another. Use it to derive Snell's law for light transmitting across the interface between two media

$$n_i \sin\theta_i = n_t \sin\theta_t \ ,$$

where the symbols have their usual meaning. **[5]**

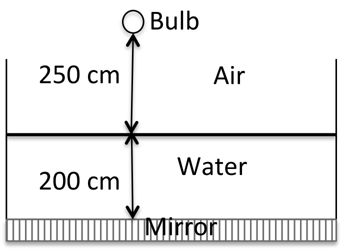
(b) The diagram below shows an object at P forms a real image at P' inside the glass sphere of radius r. Which direction will the image move under the following actions?

i. Increase r.
ii. Increase the index of refraction n.
iii. Increase the object distance.
iv. Decrease r
v. Decrease the index of refraction.



[5]

(c) A small light bulb is suspended $250\,\mathrm{cm}$ above the surface of the water ($n = 1.33$) in a swimming pool. The bottom of the pool is a large mirror.



i. When viewed under water, how far does the bulb appear from the surface of the water? How far is the image of the bulb behind the mirror? Use ray diagrams to back up your calculations/analysis. **[7]**

ii. When viewed from air, how far does the image of the bulb appear behind the mirror? Ignore the reflection at the surface of the water and hence ignore multiple reflections. [Hint: Consider the following processes: 1) refraction at the air/water interface; 2) reflection by the mirror; 3) The image behind the mirror subject to refraction at the water/air interface] **[8]**

You may find one of the following equations useful:

$$\frac{1}{s}+\frac{1}{s'}=\frac{1}{f},\ \frac{n_1}{s}+\frac{n_2}{s'}=\frac{n_2-n_1}{r},\ \frac{1}{f}=(n-1)\left(\frac{1}{r_1}-\frac{1}{r_2}\right),\ \frac{1}{s}+\frac{1}{s'}=\frac{2}{r}$$

OR Part B

(a) Use Huygen's principle to show that for single-slit diffraction, a dark fringe occurs when

$$\sin\theta = \frac{m\lambda}{a} ,$$

where $m = \pm 1, \pm 2, ...$, and a is the width of the slit. **[5]**

(b) In a double-slit interference experiment, the spacing between the two slits of negligible width is d and the distance to the viewing screen is L with $L \gg d$. Treating the two slits as two sources of identical in-phase waves, show with the help of a sketch that the mth bright fringe occurs at position:

$$y_m = \frac{m\lambda L}{d}, \quad m = 0, \pm 1, \pm 2, ...$$

[5]

(c) A beam of parallel coherent light of wavelength λ is incident normally on a set of N parallel slits. The slits have negligible width and each is seperated from its neighbour by a distance d.

Show that the resulting intensity of light emitted in direction θ is given by

$$I(\theta) = I_0 \frac{\sin^2 \frac{N\delta}{2}}{\sin^2 \frac{\delta}{2}} ,$$

where I_0 is the intensity from one slit alone and $\delta = \frac{2\pi d \sin\theta}{\lambda}$ in the usual notation. **[5]**

If the incident light consists of two wavelengths λ and $\lambda + \Delta\lambda$, where $\Delta\lambda \ll \lambda$, show that the angular separation of these two wavelengths in the mth order is

$$\Delta\theta = \frac{\Delta\lambda}{\sqrt{\left(\frac{d}{m}\right)^2 - \lambda^2}}.$$

[5]

The ability of the slits to seperate different wavelengths of light according to the above equation makes the set of slits as a spectrometer. Describe, qualitatively, how the property of such a spectrometer depends on the number of slits. **[5]**

6. **EITHER (Part A)**

(a) The time-independent Schrödinger equation (TISE) in one dimension is

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + V(x)\psi(x) = E\psi(x),$$

where the symbols have their usual meaning. State the assumptions upon which the Schrödinger equation is based. **[4]**

(b) Show that the spatial wave function $\psi(x) = A\exp(ikx)$ is a solution of the TISE when the potential function has a constant value, i.e. $V(x) = V_0$, where V_0 is a positive contant. **[2]**

(c) A particle of mass, m, and energy, E, moving in the positive x-direction is incident upon a *downward* potential step at $x = 0$. The potential function is $V(x) = 0$ for $x < 0$ and $V(x) = -V_0$ for $x \geq 0$.

i. Given that the wave function $\psi(x) = A\exp(ikx)$ describes a particle moving in the positive x-direction, write down, with justification, general solutions of the TISE for each of the regions $x < 0$ and $x \geq 0$. **[4]**

ii. Write down an expression for the wavenumber in each of these regions in terms of E and the value of the potential function. What can you say about the momentum of the particle in the two regions? **[3]**

iii. Explain why the wave function, and the gradient of the wave function, must be continuous at $x = 0$. **[4]**

iv. Use the boundary conditions given in part iii. to find an expression for the ratio of the reflected and incident amplitudes of the wave function in the region $x < 0$ in terms of the wavenumbers in each region. Hence, find the probability of the particle being reflected when when $E = 0.1V_0$. **[8]**

OR (Part B)

(a) Briefly describe the Geiger-Marsden scattering experiment and Rutherford's interpretation of the results. Explain why Rutherford's model of the atom is inconsistent with classical physics. How did Bohr's model resolve this inconsistency? **[5]**

(b) Consider an atom with atomic number Z, from which $Z-1$ electrons have been removed. In the Bohr model, the centripetal force on an electron of mass m and velocity v, at a radius r from the nucleus, is given by

$$\frac{mv^2}{r} = \frac{Ze^2}{4\pi\epsilon_0 r^2}.$$

i. Show that if the orbital length is an integer multiple, n, of the de Broglie wavelength of the electron, the orbital radii are given by

$$r_n = \frac{n^2 a_0}{Z},$$

where

$$a_0 = \frac{h^2\epsilon_0}{\pi m e^2}$$

is the Bohr radius. **[5]**

ii. Hence, demonstrate that the *total* energy of the remaining electron is proportional to Z^2. (You may assume that the electron is non-relativistic.) **[3]**

iii. Given that the ionisation energy of hydrogen in its ground state is $-13.6\,\mathrm{eV}$, calculate the energy of the photon emitted when an excited electron in a singly-ionised helium atom (He^+, $Z=2$) makes a transition from the $n=6$ state to the $n=2$ state. Determine whether this photon could be absorbed by a hydrogen atom in its ground state. **[6]**

(c) A unknown material is exposed to a source of high energy photons. Lower energy photons with a characteristic energy of $8.04\ \mathrm{keV}$, assumed to be K_α X-rays, are isotropically emitted by the unknown material. State what process is involved and determine whether the unknown material is cobolt ($Z=27$), nickel ($Z=28$) or copper ($Z=29$). **[6]**

Quantum Mechanics Formula Sheet

Stefan-Boltzmann Law

$$\frac{P}{A} = \sigma T^4$$

Wien's Displacement Law

$$\lambda_{\mathrm{peak}} T = 2.898 \times 10^{-3}\,\mathrm{K}$$

Planck's Formula

$$\overline{E}\left(\lambda\right) = \frac{hc/\lambda}{\exp(hc/\lambda k_B T) - 1}$$

Einstein's Photoelectric Formula

$$\mathrm{KE}_{\mathrm{max}} = hf - \phi$$

Compton Shift Equation

$$\lambda_2 - \lambda_1 = \frac{h}{m_e c}\left(1 - \cos\theta\right)$$

Hydrogen Energy Levels

$$E_n = -\frac{13.6}{n^2}\,\mathrm{eV}$$

Bragg's Law

$$2d\sin\theta = n\lambda$$

Moseley's Law

$$f_{\mathrm{K}_\alpha} = \left(2.48 \times 10^{15}\,\mathrm{Hz}\right) \times (Z-1)^2$$

Heisenberg's Uncertainty Principle

$$\Delta p_x \Delta x \geq \frac{h}{4\pi}$$

Free-Particle Wave Function

$$\Psi\left(x,t\right) = A\exp\left(\pm i\left(kx - \omega t\right)\right)$$

Momentum Operator

$$\hat{p}_x = -i\hbar\frac{\partial}{\partial x}$$

Energy Operator

$$\hat{E} = i\hbar\frac{\partial}{\partial t}$$

Expectation Value

$$\langle O\rangle = \int_{-\infty}^{+\infty} \Psi^* \hat{O}\Psi dx$$

Time-Independent Schrödinger Equation

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + V(x)\psi(x) = E\psi(x)$$

Physical Constants and Units

Acceleration due to gravity	g	9.81 m s^{-2}
Gravitational constant	G	6.674×10^{-11} N m^2 kg^{-2}
Ice point	T_{ice}	273.15 K
Avogadro constant	N_A	6.022×10^{23} mol^{-1}
		[*N.B.* 1 mole $\equiv 1$ *gram-molecule*]
Gas constant	R	8.314 J K^{-1} mol^{-1}
Boltzmann constant	k, k_B	1.381×10^{-23} J K^{-1} $\equiv 8.62 \times 10^{-5}$ eV K^{-1}
Stefan constant	σ	5.670×10^{-8} W m^{-2} K^{-4}
Rydberg constant	R_∞	1.097×10^{7} m^{-1}
	$R_\infty hc$	13.606 eV
Planck constant	h	6.626×10^{-34} J s $\equiv 4.136 \times 10^{-15}$ eV s
$h/2\pi$	$\hbar$	1.055×10^{-34} J s $\equiv 6.582 \times 10^{-16}$ eV s
Speed of light *in vacuo*	c	2.998×10^{8} m s^{-1}
	$\hbar c$	197.3 MeV fm
Charge of proton	e	1.602×10^{-19} C
Mass of electron	m_e	9.109×10^{-31} kg
Rest energy of electron		0.511 MeV
Mass of proton	m_p	1.673×10^{-27} kg
Rest energy of proton		938.3 MeV
One atomic mass unit	u	1.66×10^{-27} kg
Atomic mass unit energy equivalent		931.5 MeV
Electric constant	ϵ_0	8.854×10^{-12} F m^{-1}
Magnetic constant	μ_0	$4\pi \times 10^{-7}$ H m^{-1}
Bohr magneton	μ_B	9.274×10^{-24} A m^2 (J T^{-1})
Nuclear magneton	μ_N	5.051×10^{-27} A m^2 (J T^{-1})
Fine-structure constant	$\alpha = e^2/4\pi\epsilon_0\hbar c$	7.297×10^{-3}= 1/137.0
Compton wavelength of electron	$\lambda_c = h/m_e c$	2.426×10^{-12} m
Bohr radius	a_0	5.2918×10^{-11} m
angstrom	Å	10^{-10} m
barn	b	10^{-28} m^2
torr (mm Hg at 0 °C)	torr	133.32 Pa (N m^{-2})

This page is intentionally blank.

Do not complete the attendance slip, fill in the front of the answer book or turn over the question paper until you are told to do so.

Important Reminders

- Coats/outwear should be placed in the designated area.
- Unauthorised materials (e.g. notes or Tippex) must be placed in the designated area.
- Check that you do not have any unauthorised materials with you (e.g. in your pockets, pencil case).
- Mobile phones and smart watches **must** be switched off and placed in the designated area or under your desk. They must not be left on your person or in your pockets.
- You are not permitted to use a mobile phone as a clock. If you have difficulty seeing a clock, please alert an Invigilator.
- You are not permitted to have writing on your hand, arm or other body part.
- Check that you do not have writing on your hand, arm or other body part – if you do, you must inform an Invigilator immediately
- Alert an Invigilator immediately if you find any unauthorised item upon you during the examination.

Any students found with non-permitted items upon their person during the examination, or who fail to comply with Examination rules may be subject to Student Conduct procedures.