UNIVERSITY OF BIRMINGHAM

School of Physics and Astronomy

DEGREE OF B.Sc. & M.Sci. WITH HONOURS

FIRST YEAR EXAMINATION

03 19749

LC SPECIAL RELATIVITY/PROBABILITY

SEMESTER 1 EXAMINATIONS 2022/2023

Time Allowed: 1 hour 30 minutes

Answer Section 1 and two questions from Section 2.

Section 1 consists of four questions and carries 40% of the marks.
Answer ***all four*** questions from this section.

Section 2 consists of three questions and carries 60% of the marks.
Answer ***two*** questions from this Section. If you answer more than two questions, credit will only be given for the best two answers.

The approximate allocation of marks to each part of a question is shown in brackets **[ ]**.

All symbols have their usual meanings.

Calculators may be used in this examination but must not be used to store text. Calculators with the ability to store text should have their memories deleted prior to the start of the examination.

A table of physical constants and units that may be required will be found at the end of this question paper.

SECTION 1

1. In an inertial frame Σ two events, A and B occur simultaneously at space points $(x, y, z) = (\pm d, 0, 0)$.

   (a) By performing Lorentz transformation to the frame Σ' which moves in the positive x-direction with velocity v find the time interval $\Delta t'$ between the events in the frame Σ'.

   (b) Calculate the interval Δs^2_{AB} between the events. Explain why it is invariant under Lorentz transformation. Use the invariant interval to calculate the distance $\Delta x'$ between the events in the frame Σ'.

   [10]

2. A discrete probability mass function is given by

$$P(n) = \alpha n \quad n = 1, 2, 3 \ldots N$$

   (a) Calculate the normalisation, α. **[3]**

   (b) Define the expectation value of n for a discrete distribution. **[3]**

   (c) Show that

$$\langle n \rangle = \frac{2N+1}{3}$$

   [4]

   The following sums may be of use to you:

$$\sum_{n=1}^{N} n = \frac{N(N+1)}{2} \qquad \sum_{n=1}^{N} n^2 = \frac{N(N+1)(2N+1)}{6}$$

3. Twenty tickets are numbered from 1 to 20, and one of them is drawn at random.

   (a) What is the probability that the number is a multiple of either 5 or 7? **[4]**

   (b) Given that the number was a multiple of 3 or 5, what is the probability that the number was 6? **[6]**

4. An electronic component has a probability p of failure after one year of use.

(a) If two components are run for a year, what is the probabiity that at least one of them is still working? **[3]**

(b) If n components are run for a year, what is the probabiity that at least one of them is still working? **[3]**

(c) How large does n have to be such that the probabiity that at least one of them is working is greater than 0.999 if $p = 0.1$? **[4]**

SECTION 2

Answer **two** *questions from this Section. If you answer more than two questions, credit will only be given for the best two answers.*

5. In a galaxy far far away Luke Skywalker flies on a spaceship away from the planet Abafar, where he was celebrating Master Yoda's 500th birthday.

   After flying for $T = 1$ year (according to his own clock) he discovers a fault in the navigation system of his spaceship which prevents him from returning to Abafar. He sends a distress radio message to Yoda asking for help and continues his journey with the same velocity V. Once Yoda receives the message he immediately figures out what the fault is and sends the repair instructions to Luke. After quickly fixing the spaceship, Luke Skywalker makes a rapid U-turn, sends a message back to Earth and flies back to Abafar. He arrives just in time to celebrate Master Yoda's 900th birthday.

   The time and space coordinates t_i, x_i are assumed to be in the reference in which the planet Abafar is at rest.

   (a) Draw the Minkowski space-time diagram of the Luke's journey. Use the the starting event at the origin, $E_0 = (t_0, x_0) = (0, 0)$. Indicate

   i. the event E_1 when Luke sends his distress message,
   ii. the event E_2 when Yoda receives and replies to Luke's message,
   iii. the event E_3 when Luke receives instructions from Yoda and starts his return journey,
   iv. the event E_4 when Luke and Yoda meet again. **[6]**

   (b) Give the definition of *proper time*.
   Using this definition or an appropriate Lorentz transformation calculate the space-time coordinates of the event $E_1 = (t_1, x_1)$. State the relativistic effect behind this calculation. **[6]**

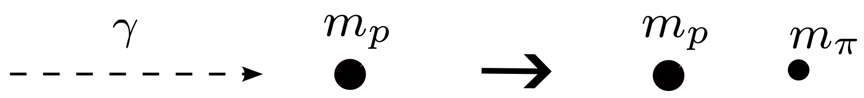
   (c) Assume the communications between Luke and Yoda are established using photons and neglect delays between receiving and sending messages. Use the events $E_2 = (t_2, x_2)$ and $E_3 = (t_3, x_3)$ to show that the farthest point of Luke's trajectory from Abafar is

   $$x_3 = \frac{c+V}{c-V} x_1$$

   [8]

   (d) Using the results in parts (b) and (c) as well as the information given in the problem calculate the velocity V in units of speed of light c. *i.e.* find V/c.
   [*Hint: Use the fact that* $1 - V/c = \delta \ll 1$ *and* $\sqrt[3]{0.00005} \simeq 0.037$] **[10]**

6. A cosmic ray photon γ with energy E_γ hits a proton *at rest* with mass $m_p = 940$ MeV. As a result a π meson with rest mass $m_\pi = 140$ Mev is created in addition to the proton.



(a) Write down the momentum of the photon P_γ (assumed to be positive) as well as energy and momentum E_p, P_p of the proton *before* the collision. **[6]**

(b) Assuming an observer moves along the direction of the photon γ with velocity v write down the energy and momentum of each particle in the moving observer's frame by using Lorentz transformation for energy and momentum. **[8]**

(c) In the centre of mass frame the total momentum is zero. Find the velocity v if the observer is in the centre of mass frame. Express your answer in terms of E_γ and m_p. **[6]**

(d) Calculate the minimal (threshold) energy of the photon E_γ in MeV for the reaction to take place. **[10]**

7. The radial part of a wavefunction is given by

$$\psi(r) = Are^{-r/2r_0}.$$

In quantum mechanics, the probability density is given by

$$P(r) = |\psi(r)|^2$$

In the following you may find useful the formula

$$\int_0^\infty x^n e^{-ax} = \frac{n!}{a^{n+1}}$$

(a) Write down the PDF and find the value of the constant A which normalises the distribution. **[5]**

(b) Calculate the expectation value for the particle's radial position. **[6]**

(c) Calculate the variance for the particle's radial position. **[6]**

(d) The skew of a probability distribution is defined by

$$\text{Skew} \equiv \langle (x - \langle x \rangle)^3 \rangle.$$

Write the skew in terms of $\langle x \rangle$, $\langle x^2 \rangle$ and $\langle x^3 \rangle$. **[6]**

(e) Calculate the skew for the probability distribution of the particle's radial position. **[3]**

(f) The standardised skew of a probability distribution is defined by

$$\text{Standardised Skew} \equiv \frac{\langle (x - \langle x \rangle)^3 \rangle}{\text{var}(x)^{3/2}}.$$

Calculate this for the particle's radial position. **[4]**

Useful Formulae for Special Relativity & Probability

Lorentz Transformations in Standard Configuration

$$
\begin{aligned}
ct' &= \gamma(ct - \beta x) \qquad & ct &= \gamma(ct' + \beta x') \qquad & \beta &= v/c \\
x' &= \gamma(x - \beta ct) \qquad & x &= \gamma(x' + \beta ct') \qquad & \gamma &= 1/\sqrt{1-\beta^2} \\
y' &= y \qquad & y &= y' & & \\
z' &= z \qquad & z &= z' & &
\end{aligned}
$$

Velocity Transformation

$$
u'_x = \frac{u_x - v}{\left(1 - \frac{u_x v}{c^2}\right)}, \qquad u'_y = \frac{u_y}{\gamma(v)\left(1 - \frac{u_x v}{c^2}\right)}, \qquad u'_z = \frac{u_z}{\gamma(v)\left(1 - \frac{u_x v}{c^2}\right)}.
$$

Invariant Interval

$$\Delta s^2 = (c\Delta t)^2 - \Delta x^2 - \Delta y^2 - \Delta z^2$$

Energy and Momentum

$$
E = \frac{mc^2}{\sqrt{1 - v^2/c^2}}, \qquad \mathbf{p} = \frac{m\mathbf{v}}{\sqrt{1 - v^2/c^2}}, \qquad E^2 = p^2c^2 + m^2c^4.
$$

Conditional Probability

$$P(A|B)P(B) = P(A \cap B)$$

Binomial Distribution

$$P(n; N, p) = \binom{N}{n} p^n (1-p)^{N-n}$$

Poisson Distribution

$$P(n; \mu) = \frac{\mu^n}{n!} e^{-\mu}$$

Central Limit Theorem

$$\frac{X - n\mu}{\sqrt{n\sigma^2}} \to \mathcal{N}(0, 1)$$

Normal Distribution

$$p(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[\frac{-(x-\mu)^2}{2\sigma^2}\right]$$

Change of Variable in PDF

If $x \sim P_x(x)$ and $y = f(x)$ is monotonic: $P_y(y) = P_x(f^{-1}(y)) \left|\frac{df^{-1}(y)}{dy}\right|$

Physical Constants and Units

Acceleration due to gravity	g	$9.81\,\mathrm{m\,s^{-2}}$
Gravitational constant	G	$6.674 \times 10^{-11}\,\mathrm{N\,m^2\,kg^{-2}}$
Ice point	T_{ice}	$273.15\,\mathrm{K}$
Avogadro constant	N_A	$6.022 \times 10^{23}\,\mathrm{mol^{-1}}$
		[*N.B.* 1 mole $\equiv 1$ *gram-molecule*]
Gas constant	R	$8.314\,\mathrm{J\,K^{-1}\,mol^{-1}}$
Boltzmann constant	k, k_B	$1.381 \times 10^{-23}\,\mathrm{J\,K^{-1}} \equiv 8.62 \times 10^{-5}\,\mathrm{eV\,K^{-1}}$
Stefan constant	σ	$5.670 \times 10^{-8}\,\mathrm{W\,m^{-2}\,K^{-4}}$
Rydberg constant	R_∞	$1.097 \times 10^{7}\,\mathrm{m^{-1}}$
	$R_\infty hc$	$13.606\,\mathrm{eV}$
Planck constant	h	$6.626 \times 10^{-34}\,\mathrm{J\,s} \equiv 4.136 \times 10^{-15}\,\mathrm{eV\,s}$
$h/2\pi$	$\hbar$	$1.055 \times 10^{-34}\,\mathrm{J\,s} \equiv 6.582 \times 10^{-16}\,\mathrm{eV\,s}$
Speed of light *in vacuo*	c	$2.998 \times 10^{8}\,\mathrm{m\,s^{-1}}$
	$\hbar c$	$197.3\,\mathrm{MeV\,fm}$
Charge of proton	e	$1.602 \times 10^{-19}\,\mathrm{C}$
Mass of electron	m_e	$9.109 \times 10^{-31}\,\mathrm{kg}$
Rest energy of electron		$0.511\,\mathrm{MeV}$
Mass of proton	m_p	$1.673 \times 10^{-27}\,\mathrm{kg}$
Rest energy of proton		$938.3\,\mathrm{MeV}$
One atomic mass unit	u	$1.66 \times 10^{-27}\,\mathrm{kg}$
Atomic mass unit energy equivalent		$931.5\,\mathrm{MeV}$
Electric constant	ϵ_0	$8.854 \times 10^{-12}\,\mathrm{F\,m^{-1}}$
Magnetic constant	μ_0	$4\pi \times 10^{-7}\,\mathrm{H\,m^{-1}}$
Bohr magneton	μ_B	$9.274 \times 10^{-24}\,\mathrm{A\,m^2}$ ($\mathrm{J\,T^{-1}}$)
Nuclear magneton	μ_N	$5.051 \times 10^{-27}\,\mathrm{A\,m^2}$ ($\mathrm{J\,T^{-1}}$)
Fine-structure constant	$\alpha = e^2/4\pi\epsilon_0\hbar c$	$7.297 \times 10^{-3} = 1/137.0$
Compton wavelength of electron	$\lambda_c = h/m_e c$	$2.426 \times 10^{-12}\,\mathrm{m}$
Bohr radius	a_0	$5.2918 \times 10^{-11}\,\mathrm{m}$
angstrom	Å	$10^{-10}\,\mathrm{m}$
barn	b	$10^{-28}\,\mathrm{m^2}$
torr (mm Hg at 0 °C)	torr	$133.32\,\mathrm{Pa}$ ($\mathrm{N\,m^{-2}}$)

END OF PAPER

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Do not complete the attendance slip, fill in the front of the answer book or turn over the question paper until you are told to do so.

Important Reminders

- Coats/outwear should be placed in the designated area.
- Unauthorised materials (e.g. notes or Tippex) must be placed in the designated area.
- Check that you do not have any unauthorised materials with you (e.g. in your pockets, pencil case).
- Mobile phones and smart watches **must** be switched off and placed in the designated area or under your desk. They must not be left on your person or in your pockets.
- You are not permitted to use a mobile phone as a clock. If you have difficulty seeing a clock, please alert an Invigilator.
- You are not permitted to have writing on your hand, arm or other body part.
- Check that you do not have writing on your hand, arm or other body part – if you do, you must inform an Invigilator immediately
- Alert an Invigilator immediately if you find any unauthorised item upon you during the examination.

Any students found with non-permitted items upon their person during the examination, or who fail to comply with Examination rules may be subject to Student Conduct procedures.