Mechanics week 4: Central forces (bookwork)

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This week we will begin learning about problems with *Central Forces*. This week we will focus on deriving the framework (what's generally known as the "bookwork") for the problem, and we will look at examples next week. It is important to fully understand the bookwork, but in general you can use the findings without derivation, as long as you clearly state them.

1 Introduction

Newtonian mechanics was originally inspired by the motion of the planets around the sun. The attractive force between two gravitating bodies of mass m and M (e.g. Earth and sun) is

$$\frac{GMm}{r^2}, \tag{1}$$

along the line joining the two bodies. Here G is the universal gravitational constant and r is the distance between the bodies.

In particular, we might want to show/understand Kepler's laws, which he determined through observation in about 1609. These are:

Kepler's First Law: The planets move about the sun in an elliptical orbit with the sun at one focus.

Kepler's Second Law: The straight line joining a planet and the sun sweeps out equal areas in equal time.

Kepler's Third Law: The square of the period of the orbit is equal to the cube of the semi-major axis of the orbit.

Kepler's laws are phenomenological: his laws agreed with his physical observations, but he was not able to develop a mathematical theory to explain why they are true - this needed Newton's *Principia* in 1687, which includes a statement of Newton's law of gravity (as above).

This is an example of a wider class of forces known as *central forces*.

Definiton

A central force $\mathbf{F}$ acting on a particle P depends on the distance of that particle from some fixed central origin O in an inertial frame, and is directed along the line joining the particle to O. If $\mathbf{r}$ is the position vector of the particle from O then

$$\mathbf{F} = F(r)\mathbf{e}_r,$$

where $\mathbf{e}_r$ is the unit vector in the same direction as $\mathbf{r}$.

2 Derivation of central forces framework

2.1 Geometry

We first set up the geometrical framework we're going to work in. We will assume that all motion will take place in a 2D plane (we'll show this is true later). Since a central force only depends on the distance from the origin, this suggests using *polar coordinates* r (radius, i.e. the distance to the particle) and θ (angle with the x axis) to describe the motion.

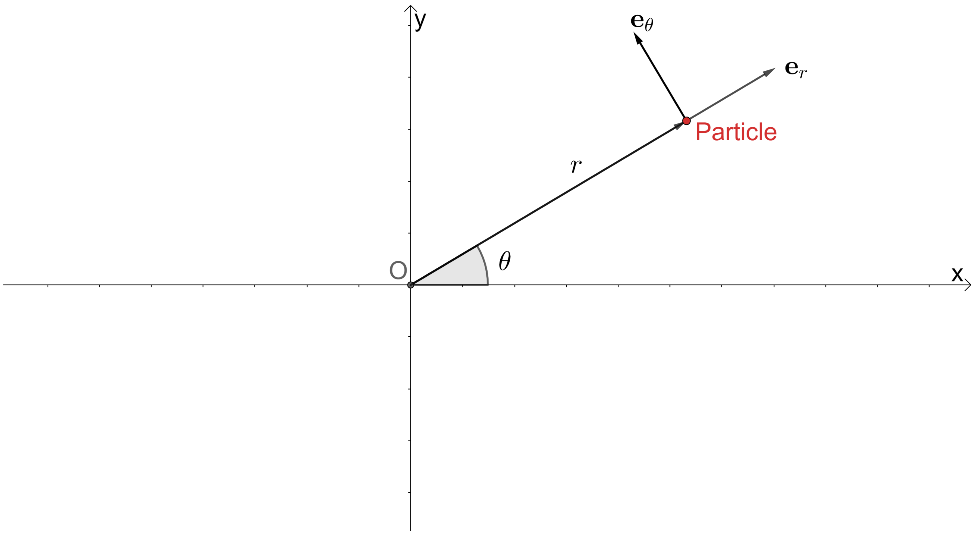

Figure 1: Geometry of a central forces problem. Shows the $\mathbf{e}_r$, $\mathbf{e}_\theta$ vectors, r the distance between the particle and the origin, and θ the angle the $\mathbf{e}_r$ vector makes with the x axis.

We will use non-constant unit vectors $\mathbf{e}_r$, $\mathbf{e}_\theta$ in the directions of increasing r and θ - these change depending on where the particle is! Then

$$\begin{aligned}\mathbf{e}_r &= \cos\theta\mathbf{i} + \sin\theta\mathbf{j},\\ \mathbf{e}_\theta &= -\sin\theta\mathbf{i} + \cos\theta\mathbf{j}\end{aligned}$$

in terms of Cartesian vectors $\mathbf{i}$ and $\mathbf{j}$. Note that $\mathbf{e}_r \cdot \mathbf{e}_\theta = -\cos\theta\sin\theta + \cos\theta\sin\theta = 0$ and hence $\mathbf{e}_r$ and $\mathbf{e}_\theta$ are perpendicular. As the particle moves these vectors change direction - therefore they are a function of time in the position vector. We will need

$$\begin{aligned}\dot{\mathbf{e}}_r &= \frac{d}{dt}(\cos\theta)\,\mathbf{i} + \frac{d}{dt}(\sin\theta)\,\mathbf{j},\\ &= -\dot{\theta}\sin\theta\mathbf{i} + \dot{\theta}\cos\theta\mathbf{j},\\ &= \dot{\theta}\mathbf{e}_\theta,\\ \dot{\mathbf{e}}_\theta &= \frac{d}{dt}(-\sin\theta)\,\mathbf{i} + \frac{d}{dt}(\cos\theta)\,\mathbf{j},\\ &= -\dot{\theta}\cos\theta\mathbf{i} - \dot{\theta}\sin\theta\mathbf{j},\\ &= -\dot{\theta}\mathbf{e}_r.\end{aligned}$$

Then the position vector of a particle is given by

$$\begin{aligned}\mathbf{r} &= r\mathbf{e}_r,\\ &= r\cos\theta\mathbf{i} + r\sin\theta\mathbf{j}.\end{aligned}$$

Hence

$$\begin{aligned}\dot{\mathbf{r}} &= \frac{d}{dt}(r\mathbf{e}_r) = r\frac{d\mathbf{e}_r}{dt} + \frac{dr}{dt}\mathbf{e}_r,\\ &= r\dot{\theta}\mathbf{e}_\theta + \dot{r}\mathbf{e}_r.\end{aligned}$$

This gives the radial and transverse components of velocity (see Fig 2).

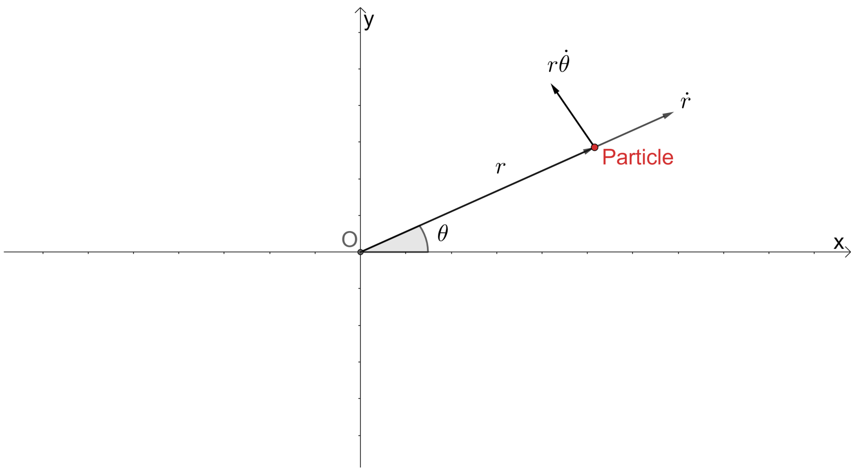

Figure 2: The direction of the velocity components.

Similarly acceleration is given by

$$
\begin{aligned}
\ddot{\mathbf{r}} &= \frac{d}{dt}\left(\dot{\mathbf{r}}\right), \\
&= \frac{d}{dt}\left(r\dot{\theta}\mathbf{e}_\theta + \dot{r}\mathbf{e}_r\right), \\
&= \dot{r}\dot{\theta}\mathbf{e}_\theta + r\ddot{\theta}\mathbf{e}_\theta + r\dot{\theta}\dot{\mathbf{e}}_\theta + \ddot{r}\mathbf{e}_r + \dot{r}\dot{\mathbf{e}}_r, \\
&= \dot{r}\dot{\theta}\mathbf{e}_\theta + r\ddot{\theta}\mathbf{e}_\theta - r\dot{\theta}^2\mathbf{e}_r + \ddot{r}\mathbf{e}_r + \dot{r}\dot{\theta}\mathbf{e}_\theta, \\
&= \left(\ddot{r} - r\dot{\theta}^2\right)\mathbf{e}_r + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\mathbf{e}_\theta.
\end{aligned}
$$

Note that

$$
r\ddot{\theta} + 2\dot{r}\dot{\theta} = \frac{1}{r}\frac{d}{dt}\left(r^2\dot{\theta}\right),
$$

(expand it out to check) and hence

$$
\ddot{\mathbf{r}} = \left(\ddot{r} - r\dot{\theta}^2\right)\mathbf{e}_r + \frac{1}{r}\frac{d}{dt}\left(r^2\dot{\theta}\right)\mathbf{e}_\theta.
$$

This gives the radial and transverse components of acceleration (see Fig 3).

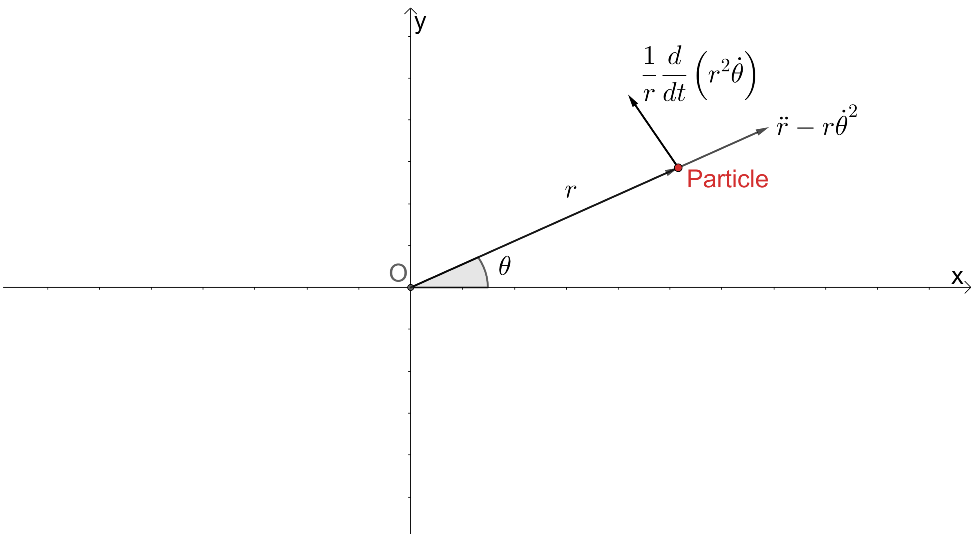

Figure 3: The direction of the acceleration components.

Activity: *You should now be able to tackle question 3(a) on this week's problem sheet.*

2.2 Newton's second law

Newton's second law therefore becomes

$$
\begin{aligned}
m\ddot{\mathbf{r}} &= F(r)\mathbf{e}_r, \\
\Longrightarrow\ m\left(\ddot{r} - r\dot{\theta}^2\right)\mathbf{e}_r + \frac{m}{r}\frac{d}{dt}\left(r^2\dot{\theta}\right)\mathbf{e}_\theta &= F(r)\mathbf{e}_r.
\end{aligned}
$$

Equating coefficients of the vectors gives

$$\begin{aligned} m\left(\ddot{r} - r\dot{\theta}^2\right) = F(r), \\ \frac{m}{r}\frac{d}{dt}\left(r^2\dot{\theta}\right) &= 0. \end{aligned}$$

We immediately see that $r^2\dot{\theta}$ is a constant, typically denoted h. This is **always true for any central force**. [This also gives Kepler's second law that a straight line joining the sun and a planet sweeps out equal area in equal time since

$$\begin{aligned} \frac{dA}{dt} &= \frac{1}{2}r^2\dot{\theta}, \\ &= \frac{1}{2}h, \end{aligned}$$

where $A = r^2\theta/2$ is the area of the sector.]

We can therefore write $\dot{\theta} = h/r^2$, and hence

$$\begin{aligned} m\left(\ddot{r} - r\left(h/r^2\right)^2\right) &= F(r), \\ m\left(\ddot{r} - h^2/r^3\right) &= F(r), \end{aligned}$$

which gives a scalar equation for r.

2.3 Equation of a path

We could solve to find $r(t)$, $\theta(t)$, but often what we want to find is the equation of the path of the particle, i.e. $r(\theta)$ - so we'd like to find a single equation for this (rather than having to solve two equations in terms of t and then eliminate). To do this we first need to convert d/dt to $d/d\theta$, and then get a differential equation for r as a function of θ.

Now

$$\begin{aligned} \frac{dr}{dt} &= \frac{dr}{d\theta}\frac{d\theta}{dt}, \\ &= \frac{dr}{d\theta}\frac{h}{r^2}, \end{aligned}$$

which we notice could be written as a total derivative as

$$= -h\frac{d}{d\theta}\left(\frac{1}{r}\right).$$

This gives us an idea - instead of finding an equation for $r(\theta)$, it may be easier to work in terms of $u = 1/r$ as a function of θ.

This gives

$$\dot{r} \;=\; -h\frac{du}{d\theta}, \tag{2}$$

and so we can find

$$\begin{aligned}
\ddot{r} &= \frac{d\dot{r}}{dt}, \\
&= \frac{d\dot{r}}{d\theta}\frac{d\theta}{dt}, \\
&= \dot{\theta}\frac{d}{d\theta}\left(-h\frac{du}{d\theta}\right), \\
&= -h\dot{\theta}\frac{d^2u}{d\theta^2}, \\
&= -\frac{h^2}{r^2}\frac{d^2u}{d\theta^2}, \\
&= -h^2u^2\frac{d^2u}{d\theta^2}.
\end{aligned}$$

Hence

$$\begin{aligned}
m\left(\ddot{r} - h^2/r^3\right) &= F(r), \\
\implies m\left(-h^2u^2\frac{d^2u}{d\theta^2} - h^2u^3\right) &= F(1/u), \\
\implies \frac{d^2u}{d\theta^2} + u &= -\frac{F(1/u)}{mh^2u^2},
\end{aligned}$$

which gives an equation for $u(\theta)$ for any central force F. **N.B.** be careful about signs! Some textbooks define F as the inward pointing force which flips the sign of the right hand side!

Activity: *You should now be able to tackle question 3(b) on this week's problem sheet.*

3 Initial conditions and finding "h"

The key to solving central forces problems is generally setting up the coordinate system in a sensible way, and finding "h" and the initial conditions correctly.

For any coordinate system you have (at least!) two choices to make: where you put the origin, and how you align the axes (i.e. the orientation), these are equivalent to translation and rotation. It makes sense to locate the origin of your coordinate system at the point where the force originates, e.g. the centre of the planet/sun/..., so that you do have a central force problem. You can then **choose** how to line up your **i**, **j** axes or equivalently how to pick where $\theta = 0$. It's generally best to pick the axes in a such a way that the particle is initially located at $\theta = 0$ (for some value of r).

We then need to calculate the value of the constant h, and find the initial conditions for u in terms of θ. Typically we will know the initial location and the velocity of the particle in terms of $\mathbf{r}$ and $\dot{\mathbf{r}}$, specified at $t = 0$, so we need to convert from $\mathbf{r}(t)$ to $u(\theta)$.

Example 1: If a particle is initially located a distance a from the origin, moving with speed V purely in the transverse direction, find h and appropriate initial conditions for $u = 1/r$.

Solution. At $t = 0$ we know that $r = a$, $\dot{r} = 0$, $r\dot{\theta} = V$, and we **choose** our axes such that $\theta = 0$. Since $h = r^2\dot{\theta}$ is constant throughout the motion, its value will always be its initial value. Hence

$$h = r^2\dot{\theta} = r \cdot r\dot{\theta} = aV,$$

initially and hence throughout the motion.

We then convert the initial conditions from $r(t)$, $\theta(t)$ terms to $u(\theta)$. Firstly note that since $\theta = 0$ at $t = 0$, the u initial conditions will be applied at $\theta = 0$. Now, since $u = 1/r$, and $r = a$ at $t = 0$, this means $u = 1/a$ at $\theta = 0$. Similarly we need an expression for $\dfrac{du}{d\theta}$ at $\theta = 0$. From (2) we know that

$$\dot{r} = -h\frac{du}{d\theta},$$

giving

$$\frac{du}{d\theta} = -\frac{\dot{r}}{h}.$$

Hence at $\theta = 0$ (equivalently $t = 0$),

$$\begin{aligned} \frac{du}{d\theta} &= -\frac{\dot{r}}{h}, \\ &= -\frac{0}{aV}, \end{aligned}$$

using $\dot{r} = 0$ at $t = 0$ and $h = aV$ always. ◀

This sets up the framework to solve central forces problems, we will cover some examples on using it next week.

Activity: *You should now be able to tackle question 4 on this week's problem sheet.*