Mechanics week 8: Energy in a straight line (one dimension)

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1 Introduction

We will now move on to using ideas of energy in mechanics, we will focus on this topic for the next three weeks.

When you throw a ball in the air, the ball starts with a lot of velocity, gains height whilst slowing down, reaches a maximum height when it becomes stationary before falling and gaining speed again. In the absence of other forces, as you gain height you lose speed and vice versa. This is because energy is being conserved. When energy is conserved, this can be a quick and easy way of determining important information, for example the maximum distance or speed a particle can attain, but doesn't tend to be the best method to get the full particle path. This week we will focus on examples where the motion happens in a one dimensional straight line, before moving on to more complex examples next week.

2 Energy in one dimension

If you consider a particle of mass m moving in a straight line, which is being acted on by a force F **which depends only on the position of the particle** x, then Newton's second law gives

$$m\ddot{x} = F(x),$$

and the particle has two types of energy:

- the **kinetic energy** $\frac{1}{2}m\dot{x}^2$ because the particle is moving and
- the **potential energy** due to the force acting on it.

The potential energy is essentially the energy that is "stored" to be converted into kinetic energy at a later time. There are a number of different types of potential energy, for example gravitational energy (if something is high) or elastic potential energy (if something is stretched or compressed). We can also have e.g. electrical, chemical or nuclear potential energy.

The potential energy $V(x)$ is defined to be

$$V(x) \quad = \quad -\int F(x)dx.$$

The potential energy is therefore defined only up to an additive constant, so we can **choose** where the potential energy equals zero for example. Equivalently

$$F(x) = -\frac{dV}{dx}.$$

If the force acting on a particle can be derived from a potential energy then the total energy is conserved.

2.1 Conservation of energy

Newton's second law gives

$$F(x) \quad = \quad m\ddot{x}.$$

We multiply both sides by $\dot{x}$ to find

$$\begin{aligned} F(x)\dot{x} &= m\ddot{x}\dot{x}, \\ &= \frac{d}{dt}\left(\frac{m}{2}\dot{x}^2\right). \end{aligned}$$

This can then be integrated with respect to time:

$$\begin{aligned} \int F(x)\frac{dx}{dt}\ \mathrm{d}t &= \int \frac{d}{dt}\left(\frac{m}{2}\dot{x}^2\right)\ \mathrm{d}t, \\ \int F(x)dx + \text{const} &= \frac{1}{2}m\dot{x}^2, \\ -V(x) + \text{const} &= \frac{1}{2}m\dot{x}^2. \end{aligned}$$

Hence we find conservation of energy:

$$\underbrace{\frac{1}{2}m\dot{x}^2}_{\text{kinetic energy}} \quad + \quad \underbrace{V(x)}_{\text{potential energy}} \quad = \quad E,$$

where E the constant total energy can be found from the initial conditions, since the value initially must be the value for all time if it is constant. Hence a loss in kinetic energy leads to an increase in potential energy and vice versa - as the ball gets higher it gets slower.

2.2 Gravitational potential energy

If the particle is moving vertically under the action of gravity, with x measured upwards, the force is given by $F = -mg$ and hence

$$\begin{aligned} V(x) &= -\int F \, \mathrm{d}x, \\ &= \int mg \, \mathrm{d}x, \\ &= mgx + \text{constant}. \end{aligned}$$

Since potential is only defined up to an additive constant (as it's an integral) we can **choose** where we set it to be zero to make the maths as easy as possible - the constant is absorbed into the energy E. For this example we pick $V = 0$ at $x = 0$, so the constant is also zero - this means we're measuring potential energy from $x = 0$. Hence

$$\frac{1}{2}m\dot{x}^2 + mgx = E$$

gives the energy conservation equation for a particle moving under gravity in a straight line.

Ideas of energy conservation lead to first order (nonlinear) ODEs as opposed to second order ODEs from Newton's second law directly, which may (or may not!) be easier to solve. If there is no analytical solution, energy conservation can give useful information (e.g. bounds on the motion) which Newton's second law can't give.

Example 1: A ball falling under gravity If we throw a ball of mass m vertically upwards with velocity v from the ground, how high will it go?

Solution. Let $x(t)$ be the height of the ball at time t. If we **choose** to take the potential energy to be zero at $x = 0$ (i.e. ground level), conservation of energy gives

$$\frac{1}{2}m\dot{x}^2 + mgx = E.$$

Taking $\dot{x} = v$, $x = 0$ at $t = 0$, we first find the value of the constant E to be

$$E = \frac{1}{2}mv^2,$$

giving

$$\frac{1}{2}m\dot{x}^2 + mgx = \frac{1}{2}mv^2.$$

Now, the highest point of the ball's flight will be the point at which its velocity is zero (so it's changing direction to come back down again). Hence the highest point is $x = h$ such that

$$\begin{aligned} mgh &= \frac{1}{2}mv^2, \\ \implies h = &= \frac{v^2}{2g}. \end{aligned}$$

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Activity: *You should now be able to tackle question 3 on this week's problem sheet.*

3 Elastic potential energy

Example 2: Mass on a spring neglecting gravity

Consider a particle of mass m attached to a spring (constant k and natural length a) which is fixed at the opposite end. Neglecting gravity, how does the particle move?

Solution. We will measure x, the location of the particle, from the fixed end of the spring. Then Hooke's law will give the tension in the spring, such that the magnitude of the force is proportional to the extension from natural length, $F = -k(x-a)$, with the minus sign as it acts in the opposite direction to that that x is pointing in. Hence

$$\begin{aligned} V(x) &= -\int F(x)\,\mathrm{d}x, \\ &= k\int (x-a)\ \mathrm{d}x, \\ &= k\frac{(x-a)^2}{2} + \text{const}, \end{aligned}$$

using the chain rule (integrate it directly and then factorise to check it works!). We choose the constant so that the potential energy is zero at the natural length of the spring (i.e. $V(a) = 0$) and hence

$$V(x) = \frac{k(x-a)^2}{2},$$

(this is why I wrote V in that format in the first place!), and hence conservation of energy gives

$$\frac{1}{2}m\dot{x}^2 + \frac{k(x-a)^2}{2} = E$$

where E is constant. Here the first term is the kinetic energy, and the second is the *elastic* potential energy stored in the spring. Since both $\frac{1}{2}m\dot{x}^2$ and $\dfrac{k\,(x-a)^2}{2}$ are necessarily positive, this gives maximum values on the motion. The maximum value of K.E. is when the P.E. is zero, so

$$\begin{aligned}
\frac{1}{2}m\dot{x}^2 &\leq E, \\
\dot{x}^2 &\leq \frac{2E}{m},
\end{aligned}$$

which limits how quickly the particle can move. Similarly

$$\begin{aligned}
\frac{k\,(x-a)^2}{2} &\leq E, \\
(x-a)^2 &\leq \frac{2E}{k},
\end{aligned}$$

so this gives oscillatory motion between

$$a - \sqrt{\frac{2E}{k}} \leq x \leq a + \sqrt{\frac{2E}{k}}.$$

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4 Gravitational and elastic energy

Example 3: Mass on a spring with gravity

Consider a particle of mass m attached to a spring (constant k and natural length a) which is fixed at the opposite end, with gravity acting downwards. If we release the particle from rest at the natural length of the spring, what is the motion of the particle?

Solution. Let $x = 0$ be the fixed end of the spring with $x(t)$ giving the location of the particle (see figure 1). As before the tension in the spring is given by $k\,(x-a)$. Hence the total force on the particle (acting downwards) is

$$\begin{aligned}
F(x) &= mg - k\,(x-a)\,, \\
\text{and} \qquad V(x) &= -\int F(x)\,\mathrm{d}x.
\end{aligned}$$

Now, integration is additive, so we can consider each part separately, and **define a zero value location for each**. Hence we have

- P.E. of gravity giving $-mgx$ where we choose it to be zero at $x = 0$.
- P.E. of the spring giving $k\,(x-a)^2/2$, where we choose it to be zero at $x = a$, the spring's natural length.

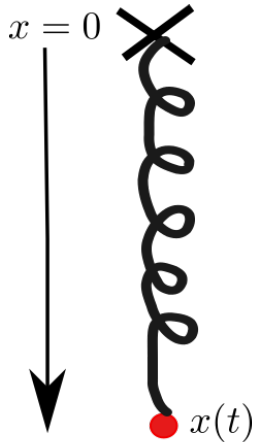

Figure 1: Sketch for example 3 showing the particle a distance x below the fixed point of the spring which is located at $x = 0$.

Hence the total potential energy is the sum of these such that

$$V(x) = -mgx + \frac{k(x-a)^2}{2},$$

and hence

$$\frac{1}{2}m\dot{x}^2 - mgx + \frac{k(x-a)^2}{2} = E$$

where E is constant, gives conservation of energy. The potential energy consists of two components; the gravitational potential energy $-mgx$ and the elastic potential energy $\dfrac{k(x-a)^2}{2}$.

We now use the initial conditions to find E; at $t = 0$ we have $\dot{x} = 0$ and $x = a$. Thus

$$\begin{aligned}
E &= -mga, \\
\implies \frac{1}{2}m\dot{x}^2 - mgx + \frac{k(x-a)^2}{2} &= -mga, \\
\implies \frac{1}{2}m\dot{x}^2 &= mgx - \frac{k(x-a)^2}{2} - mga, \\
\implies \dot{x}^2 &= 2gx - \frac{k}{m}(x-a)^2 - 2ga, \\
&= 2g(x-a) - \frac{k}{m}(x-a)^2, \\
&= (x-a)\left(2g - \frac{k}{m}(x-a)\right), \\
&= (x-a)\left(2g + \frac{ak}{m} - \frac{kx}{m}\right)
\end{aligned}$$

Now, $\dot{x}^2$ must be positive (since it's something squared), which means that the right hand side must also be greater than or equal to zero. Therefore the particle will always lie between $x = a$ and $x = \dfrac{2mg}{k} + a$. ◀

Example 4: Bungee jumper Consider a bungee jumper who steps from a high platform (distance H above the ground) with a stretchy elastic cord (unstretched length a and elastic constant k) tied around their ankles, with the other end attached to the platform. Initially they fall under gravity until they reach the natural length of the cord, at which point the cord starts stretching and exerting an additional upward force. Natural questions could be how fast is the jumper falling when the cord starts stretching, and how close to the ground do they get?

Solution. Let $x(t)$ be the location of the person (approximated as a particle of mass m), measured downwards from the platform. The cord is best approximated as a "string" rather than a "spring", that is it still obeys Hooke's law in tension (when it's stretched), but there is no response when it is compressed. There are therefore two regimes for the jumper; the first where the cord is not yet stretched, so isn't contributing mechanically, and G.P.E is converted into K.E, and the second where the cord begins to stretch, also storing up elastic potential energy. We consider each of these two cases separately initially.

Before the jumper reaches a distance a below the platform, at which point the cord begins to stretch, conservation of energy gives

$$\frac{1}{2}m\dot{x}^2 - mgx = E,$$

where we have set the zero of the G.P.E to be at the platform and recalling that x increases downwards (hence the minus sign). At $t = 0$ the jumper has no initial velocity $\dot{x} = 0$, and is at location $x = 0$. Thus we find the total energy as $E = 0$, giving

$$\frac{1}{2}m\dot{x}^2 - mgx = 0.$$

This equation will hold for $0 \leq x \leq a$, when the distance between the two ends of the cord is less than its unstretched length.

The cord will begin to stretch when $x = a$, so that the person is a distance a from the platform and the cord begins to be in tension, storing elastic potential energy. The extension in the cord as the person continues to fall is given by $x - a$, and therefore conservation of energy gives

$$\begin{aligned}\frac{1}{2}m\dot{x}^2 - mgx + \frac{k(x-a)^2}{2} &= E, \\ &= 0,\end{aligned}$$

since the total energy must still be the same as initially. This equation holds for $a \leq x \leq H$. Note that the two expressions are equivalent when $x = a$.

We can now quickly calculate the speed of the jumper at the point $x = a$, which satisfies

$$\begin{aligned}
\frac{1}{2}m\dot{x}^2 - mga &= 0, \\
\implies \dot{x} &= \sqrt{2ga},
\end{aligned}$$

and the lowest point of the jumper ($x = L$ say), which is when $\dot{x} = 0$, so

$$\begin{aligned}
-mgL + \frac{k\left(L-a\right)^2}{2} &= 0, \\
\implies \left(L-a\right)^2 - \frac{2mg}{k}L &= 0, \\
\implies L^2 - \left(\frac{2mg}{k} + 2a\right)L + a^2 &= 0,
\end{aligned}$$

giving a quadratic for L which we solve to find

$$\begin{aligned}
L &= \frac{2\left(a + \frac{mg}{k}\right) \pm \sqrt{4\left(a + \frac{mg}{k}\right)^2 - 4a^2}}{2}, \\
&= \left(a + \frac{mg}{k}\right) \pm \sqrt{\frac{2mga}{k} + \frac{m^2g^2}{k^2}}.
\end{aligned}$$

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This gives the basic concepts of using conservation of energy. Next week we will move on to move complex behaviour, with motions which aren't necessarily in a straight line in one dimension.

Activity: *You should now be able to tackle question 4 on this week's problem sheet.*