

3 Thermodynamics, Ideal Gases – solutions

The problems are roughly in order of difficulty. The ones with ♣ are the hardest ones, which might only occur as a "sting in the tail" at the end of a long examination question.

Problem 3.1 Mixing water

We end up with 500 g of water at a new temperature. To deduce the new temperature, the heat leaving the hotter body, Q_1 must equal the heat arriving at the cooler body. So let us denote the masses of the two bodies of water as m_1 and m_2 , and their initial temperatures as T_1 and T_2 respectively. Then energy is conserved (and no work is done) so:

$$\begin{aligned}
 m_1 c (T_1 - T_f) &= m_2 c (T_f - T_2) \\
 \Rightarrow m_1 T_1 + m_2 T_2 &= (m_2 + m_1) T_f \\
 \Rightarrow \frac{m_1 T_1 + m_2 T_2}{(m_2 + m_1)} &= T_f \\
 \Rightarrow \left(\frac{0.1 \times 100 + 0.4 \times 20}{0.5} \right) ^\circ\text{C} &= T_f \\
 &= 36^\circ\text{C} .
 \end{aligned}$$

Problem 3.2 Hot electronics

Let the mass of the silicon be m , and the specific heat be C . Then we may use

$$\begin{aligned}
 \Delta Q &= m C \Delta T \\
 \Rightarrow \frac{dQ}{dt} &= m C \frac{dT}{dt} \\
 \Rightarrow \frac{dT}{dt} &= \frac{1}{m C} \frac{dQ}{dt} \\
 \Rightarrow \frac{dT}{dt} &= \frac{1}{20 \times 10^{-6} \times 705} 5 \times 10^{-3} \text{ K s}^{-1} \\
 &= 0.3 \text{ K s}^{-1} .
 \end{aligned}$$

Problem 3.3 Expansion

A gas is kept at constant pressure. Its temperature is changed from $T_i = 50^\circ\text{C}$ to $T_f = 100^\circ\text{C}$. To determine by what factor does its volume change, use the equation of state. The number of moles is unchanged and $p_i = p_f = p$. Thus

$$\frac{p V_i}{T_i} = n R = \frac{p V_f}{T_f} \quad \Rightarrow \quad \frac{V_f}{V_i} = \frac{T_f}{T_i} = \frac{373}{323} = 1.15 .$$

Note the temperatures must be stated in Kelvins.

Problem 3.4 Isothermal expansion

An ideal gas is initially at temperature 20°C , pressure 200 kPa and has a volume of 4 litres. It undergoes an isothermal expansion until its pressure is reduced to 100 kPa.

From this we may deduce the final volume using the equation of state, simplified by the isothermal nature of the expansion.

$$p_i V_i = nRT = p_f V_f \Rightarrow V_f = V_i \frac{p_i}{p_f} = 4 \times \frac{2 \times 10^5}{10^5} \text{ litres} = 8 \text{ litres} .$$

I.e the volume doubles. Note 1 litre = 10^{-3} m^3 .

To find the work done *by* the gas, we use the equation of state to deduce $p(V)$, defining the constant temperature to be T_0 :

$$\begin{aligned} pV &= nRT_0 \Rightarrow p = \frac{nRT_0}{V} \\ \Rightarrow W_{\text{by}} &= \int_{V_i}^{V_f} dV p = \int_{V_i}^{V_f} dV \frac{nRT_0}{V} \\ &= nRT_0 \ln(V_f/V_i) \\ &= nRT_0 \ln 2 \\ &= p_i V_i \ln 2 \\ &= 2 \times 10^5 \times 4 \times 10^{-3} \ln 2 \text{ J} \\ (\text{use } \ln 2 \simeq 0.69.) &\simeq 8 \times 10^2 \times 0.69 \text{ J} \\ &\simeq 554 \text{ J} . \end{aligned}$$

To find the heat transfer, note that this was an *isothermal* process for an ideal gas, so the internal energy is *unchanged*. Thus we may use the First Law to deduce the heat transfer into the body:

$$\begin{aligned} 0 &= W_{\text{on}} + Q_{\text{in}} \\ \Rightarrow 0 &= -W_{\text{by}} + Q_{\text{in}} \\ \Rightarrow Q_{\text{in}} &= W_{\text{by}} \\ \Rightarrow Q_{\text{in}} &= 554 \text{ J} . \end{aligned}$$

Problem 3.5 Adiabatic expansion

To find the initial volume, V_i , we use the equation of state:

$$V_i = \frac{nRT_i}{p_i} = \frac{1 \times 8.3 \times 273}{10 \times 10^5} \text{ m}^3 = 2.3 \times 10^{-3} \text{ m}^3 .$$

To obtain the final volume, V_f , we use the expression for an *adiabatic process*. We are told that the gas is a monatomic gas, so $\gamma = 5/3$. Thus:

$$\begin{aligned} P_f V_f^\gamma &= P_i V_i^\gamma \\ \Rightarrow V_f &= \left(\frac{P_i V_i^\gamma}{p_f} \right)^{1/\gamma} \\ &= \left([2.3 \times 10^{-3}]^{5/3} \frac{10}{2} \right)^{3/5} \text{ m}^3 \\ &\simeq (2 \times 10^{-4})^{3/5} \text{ m}^3 \\ &\simeq 6 \times 10^{-3} \text{ m}^3 . \end{aligned}$$

To find the final temperature use the equation of state as we know p_f , V_f and the number of moles is constant n . Thus:

$$\begin{aligned} T_f &= \frac{p_f V_f}{nR} \\ &= \frac{2 \times 10^5 \times 6 \times 10^{-3}}{1 \times 8.3} \text{ K} \\ &\simeq 143 \text{ K} . \end{aligned}$$

To obtain the work done *by* the gas, we use the adiabatic relation:

$$\begin{aligned} W_{\text{by}} &= \int_{V_i}^{V_f} dV p \\ &= \int_{V_i}^{V_f} dV \frac{p_i V_i^\gamma}{V^\gamma} \\ &= p_i V_i^\gamma \left[\frac{V^{-\gamma+1}}{1-\gamma} \right]_{V_i}^{V_f} \\ &= -\frac{3}{2} 10^6 \times 4 \times 10^{-5} (30.3 - 57.4) \text{ J} \\ &\simeq 1620 \text{ J} \end{aligned}$$

Problem 3.6 A simple thermodynamic cycle

The cycle is as shown in Fig. (3.1) Since the first step is isothermal we may use the equation

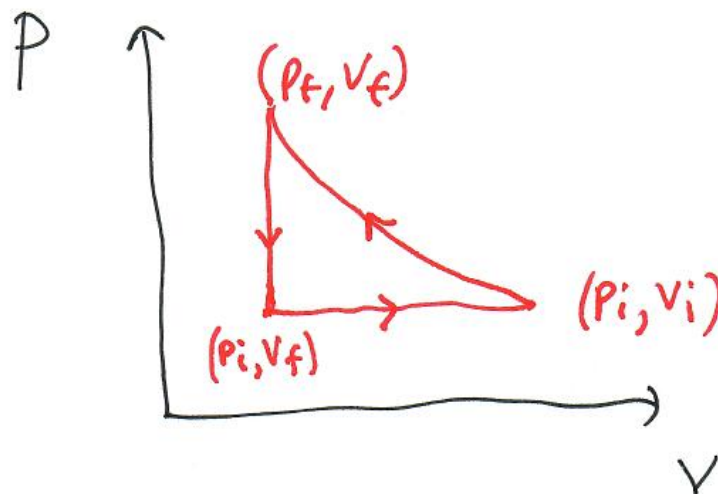


Figure 3.1: The simple thermodynamic cycle.

of state to say that

$$p_i V_i = nRT_i = nRT_f = p_f V_f \quad \Rightarrow \quad p_f = \frac{p_i V_i}{V_f} .$$

Then we may derive the work performed in this compression:

$$\begin{aligned}
 W_{\text{on}}^{(1)} &= - \int_{V_i}^{V_f} p \, dV \\
 \text{using equation of state} &= -p_i V_i \int_{V_i}^{V_f} \frac{dV}{V} \\
 &= -p_i V_i \ln \left(\frac{V_f}{V_i} \right) \\
 \text{noting } V_f < V_i &= p_i V_i \ln \left(\frac{V_i}{V_f} \right) .
 \end{aligned}$$

No work is done on the second step as the volume is fixed.

On the final step, the pressure is constant so the work integral is straightforward:

$$\begin{aligned}
 W_{\text{on}}^{(3)} &= - \int_{V_f}^{V_i} p_i \, dV \\
 &= -p_i \int_{V_f}^{V_i} dV \\
 &= -p_i (V_i - V_f) < 0 .
 \end{aligned}$$

Thus the total work performed is positive (see Fig. (3.1)) and is:

$$W_{\text{on}}^{\text{tot}} = p_i V_i \ln \left(\frac{V_f}{V_i} \right) - p_i (V_i - V_f) .$$

Since we have a cyclic process the change in internal energy, $\Delta U = 0$, so the First Law tells us that:

$$0 = W_{\text{on}}^{\text{tot}} + Q_{\text{in}} \quad \Rightarrow \quad Q_{\text{in}} = -W_{\text{on}}^{\text{tot}} < 0 .$$

Thus this is a *refrigeration* cycle.

Problem 3.7 Bow tie cycle

The key variables here are the initial (p_1, V_1) , with associated temperature $T_1 = p_1 V_1 / (nR)$, and the temperature of the isotherm, say T_0 .

1. We use adiabatic nature of compression to derive p_2 :

$$p_1 V_1^\gamma = p_2 V_2^\gamma = p_2 \left(\frac{V_1}{2} \right)^\gamma \quad \Rightarrow \quad p_2 = 2^\gamma p_1 .$$

2. Use the equation of state here, noting the temperature is constant during the expansion:

$$p_4 V_1 = nRT_0 \Rightarrow p_4 = \frac{nRT_0}{V_1} \text{ and } p_3 \frac{V_1}{2} = nRT_0 \Rightarrow p_3 = \frac{2nRT_0}{V_1} .$$

3. Firstly the work done under the adiabatic compression:

$$\begin{aligned}
 W_{\text{on}}^{\text{ad}} &= - \int_{V_1}^{V_1/2} p \, dV = - \int_{V_1}^{V_1/2} \frac{p_1 V_1^\gamma}{V^\gamma} \, dV \\
 &= -p_1 V_1^\gamma \left[\frac{V^{-\gamma+1}}{-\gamma+1} \right]_{V_1}^{V_1/2} \\
 &= \frac{p_1 V_1^\gamma}{\gamma-1} \left[\left(\frac{V_1}{2} \right)^{-\gamma+1} - V_1^{-\gamma+1} \right] \\
 &= \frac{p_1 V_1}{\gamma-1} \left[\frac{1}{2^{-\gamma+1}} - 1 \right] \\
 &= \frac{p_1 V_1}{\gamma-1} [2^{\gamma-1} - 1] > 0
 \end{aligned} \tag{3.1}$$

Then the work done *on* the gas as it undergoes an isothermal expansion:

$$\begin{aligned}
 W_{\text{on}}^{\text{isothermal}} &= - \int_{V_1/2}^{V_1} p \, dV = - \int_{V_1/2}^{V_1} \frac{nRT_0}{V} \, dV \\
 &= -nRT_0 [\ln(V_1) - \ln(V_1/2)] \\
 &= -nRT_0 \ln(2) < 0
 \end{aligned} \tag{3.2}$$

4. Firstly note no work is performed on the "vertical" sections of the cycle as they occur at constant volume. Then we should add the other two contributions calculated above, ie Eq. (3.1) and Eq. (3.2). Thus the total work done on the gas is

$$W_{\text{on}}^{\text{tot}} = \frac{p_1 V_1}{\gamma-1} [2^{\gamma-1} - 1] - nRT_0 \ln(2) .$$

If we set the total work to zero we obtain an equation for the temperature of the isotherm, T_0 :

$$T_0 = \frac{p_1 V_1}{\gamma-1} \frac{[2^{\gamma-1} - 1]}{nR \ln(2)} = \frac{[2^{\gamma-1} - 1]}{(\gamma-1) \ln(2)} T_1 ,$$

where $p_1 V_1 = nRT_1$ defines the original temperature.

Problem 3.8 The atmospheric lapse rate

For an adiabatic process,

$$\begin{aligned}
 dU &= -p \, dV \Leftrightarrow dU + p \, dV = 0 \\
 &\Rightarrow mc_v dT + p \, dV = 0 .
 \end{aligned} \tag{3.3}$$

Here we have exploited the ideal gas assumption, so $U(T)$ is only a function of T .

Now for an adiabatic process, $f(p, V) = pV^\gamma = \text{constant}$. Take differentials of both sides

$$\begin{aligned}
 df = 0 &= \frac{\partial f}{\partial p} dp + \frac{\partial f}{\partial V} dV = 0 \\
 &\Rightarrow V^\gamma dp + \gamma p V^{\gamma-1} dV = 0 \\
 &\Rightarrow V dp + \gamma p \, dV = 0 \\
 &\Rightarrow p \, dV = -\gamma^{-1} V dp .
 \end{aligned} \tag{3.4}$$

We may now substitute Eq. (3.4) into Eq. (3.3) to obtain:

$$\begin{aligned} mc_v dT - \gamma^{-1} V dp &= 0 \\ c_p dT - \frac{1}{\rho} dp &= 0 , \end{aligned} \tag{3.5}$$

using $\gamma^{-1} = c_v/c_p$ and $V/m = \rho^{-1}$.

Now use the equation for hydrostatic equilibrium we derived when discussing the isothermal atmosphere:

$$dp = -\rho g dh .$$

Upon substituting this into Eq. (3.5), we see:

$$\begin{aligned} dT &= -\frac{g}{c_p} dh \\ \Rightarrow \Gamma_d &= \frac{dT}{dh} = -\frac{g}{c_p} . \end{aligned}$$

Thus for dry air, $\Gamma_d \simeq 10^{-2} \text{ K m}^{-1} = 10 \text{ K km}^{-1}$. This is faster than experiment (6.5 K km^{-1}), so a displaced packet of dry air will cool faster than the surroundings as it rises and this will make it stable – i.e. it will fall back downwards.