UNIVERSITY OF BIRMINGHAM

School of Physics and Astronomy

DEGREE OF B.Sc. & M.Sci. WITH HONOURS

FIRST-YEAR EXAMINATION

03 19749

LC SPECIAL RELATIVITY/PROBABILITY

SUMMER EXAMINATION 2024

Time Allowed: 1 hour 30 minutes

Answer four questions from Section 1 and two questions from Section 2.

Section 1 consists of four questions and carries 40% of the marks for the examination. Answer ***all four*** questions from this Section.

Section 2 consists of three questions and carries 60% of the marks. Answer ***two*** questions from this Section. If you answer more than two questions, credit will only be given for the best two answers.

The approximate allocation of marks to each part of a question is shown in brackets **[ ]**.

All symbols have their usual meanings.

Calculators may be used in this examination but must not be used to store text. Calculators with the ability to store text should have their memories deleted prior to the start of the examination.

A formula sheet and a table of physical constants and units that may be required will be found at the end of this question paper.

SECTION 1

Answer **all four** *questions from this Section.*

1. A quantum mechanical wavefunction is given by

$$\psi(x) = e^{-\frac{x}{2}} \qquad 0 \leq x < \infty$$

The probability density is related to the wavefunction through $\rho(x) = |\psi(x)|^2$.

(a) Show that this wavefunction is normalised. **[4]**

(b) Define the expectation value, $\langle x \rangle$, for a continuous distribution. **[3]**

(c) Calculate the expectation value of position for this wavefunction. **[3]**

2. A discrete distribution is given by

$$P(x) = \frac{c}{x^2} \qquad x = \{1, 2, \ldots\}$$

(a) Find the normalising constant, c. **[4]**

(b) Define the expectation value, $\langle x \rangle$, for a discrete distribution. **[3]**

(c) Show that this distribution has an infinite expectation value. **[3]**

You may find the following identities useful

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \qquad \sum_{n=1}^{N} \frac{1}{n} \approx \log N$$

3. In a multiple choice question with k possible answers, a student either knows the answer or does not. When they do not know the answer they guess. Let the probability that a student actually knows the answer be p.

   (a) What is the probability they get the correct answer if they guess? **[2]**

   (b) Given that a student gets the question correct, what is the probability that they knew the answer? **[5]**

   (c) The odds ratio of an event is defined as $O(p) \equiv p/(1-p)$. Show that as $k \to \infty$ the probability of knowing the answer given the student was correct can be written as

   $$P(\text{Knowing}|\text{Correct}) \approx 1 - \frac{O(1-p)}{k}$$

   and show that this requires that p is not close to 0. **[3]**

   You may find the following approximation helpful

   $$\frac{1}{1+ax} \approx 1 - ax \qquad ax \ll 1$$

4. In an inertial frame Σ event B occurs after event A. The events take place at the same position and the time interval between the events is T.

   (a) By performing Lorentz transformation to the frame Σ' which moves in the positive x-direction with velocity v with respect to Σ, find the time interval T' between the events in the frame Σ'.

   (b) Calculate the interval Δs^2_{AB} between the events. Explain why it is invariant under Lorentz transformation. Use the invariant interval to calculate the distance D' between the events in the frame Σ'.

   [10]

SECTION 2

Answer **two** *questions from this Section. If you answer more than two questions, credit will only be given for the best two answers.*

5. Paul Atreides flies away from the planet Arrakis with speed $V = 3c/5$. After flying for $\tau = 2$ years (in his own frame) he receives a radio message from Arrakis telling him that his mission cannot be funded anymore and he must return. Paul performs an immediate U-turn and heads back to Arrakis with the same speed. When approaching the planet he discovers that the landing and communication system of the spaceship are broken so he misses the landing. It takes him a while before he is able to repair the communication system and stop. He makes a distress video call to Arrakis which is received $T = 13$ years after the first message was sent.

   (a) Draw the Minkowski space-time diagram of Paul's journey. Use the start event at the origin, $E_0 = (t_0, x_0) = (0, 0)$. Indicate

      i. the event $E_1 = (t_1, x_1)$ when the fist message is sent,

      ii. the event $E_2 = (t_2, x_2)$ when Paul receives it,

      iii. the event $E_3 = (t_3, x_3)$ when Paul stops and sends his message,

      iv. the event $E_4 = (t_4, x_4)$ when the Paul's message is received on Arrakis. **[6]**

   (b) Give the definition of *proper time*.

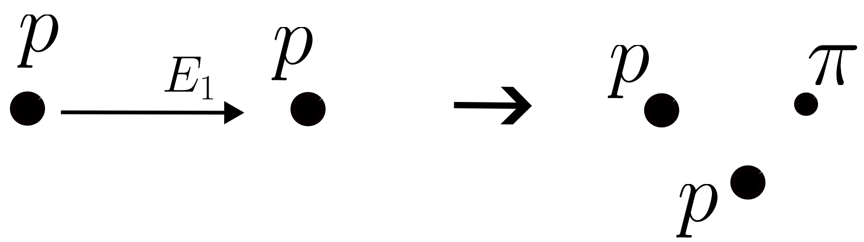
   Using this definition or an appropriate Lorentz transformation calculate the space-time coordinates of the event $E_1 = (t_1, x_1)$. State the relativistic effect behind this calculation. **[8]**

   (c) Show that $t_3 = 10\,\text{years}$. **[6]**

   (d) Give the definition of relativistic interval between two events $E_a = (t_a, x_a)$ and $E_b = (t_b, x_b)$ and explain why it is invariant under a Lorentz transformation.

   By calculating relativistic interval intervals between E_2 and E_3 and using the results in parts (b) and (c) find how old Paul is on his video call if his journey started on his 20th birthday. **[10]**

6. A cosmic ray proton p with energy E_1 hits a proton *at rest* with mass $m_p = 940$ MeV. As a result a π meson with rest mass $m_\pi = 140$ Mev is created in addition to the protons.



(a) Write down the momenta of the protons P_1, P_2 *before* the collision. **[6]**

(b) Assuming an observer moves along the direction of the cosmic ray proton with velocity v write down the energy and momentum of each particle in the moving observer's frame by using Lorentz transformation for energy and momentum. **[8]**

(c) In the centre of mass frame the total momentum is zero. Show that the velocity v of the centre of mass frame is

$$v = c\sqrt{\frac{E_1 - m_p c^2}{E_1 + m_p c^2}}\,.$$

[6]

(d) Calculate the minimal (threshold) energy of the photon E_1 in MeV for the pion production to take place. **[10]**

7. Consider playing a game multiple times. The probability that you win the game is p. You decide that you will play this game until you have won r games, and then immediately stop. Note that this means you stop playing the game as soon as you have won r games, and so the process stops on a win.

   (a) What is the probability that you have to play two games in order to win once ($r = 1$)? **[2]**

   (b) What is the probability that you have to play n games in order to win once ($r = 1$)? **[2]**

   (c) You decide you want to win twice ($r = 2$). For the case $n = 2$ you require $\{\text{win, win}\}$. Write down the corresponding results for $n = 3$ and $n = 4$. **[4]**

   (d) What is the probability that you have to play n games to win twice? **[2]**

   (e) So far there have been $r - 1$ wins after $n - 1$ games, not necessarily ending with a win. The probability of this is

   $$P = \binom{n-1}{r-1} (1-p)^{n-r} p^{r-1}$$

   Therefore, what is the probability that you stop after playing n games with r wins? **[4]**

   (f) Use the above to show that

   $$\sum_{n=r}^{\infty} \frac{(n-1)!}{(n-r)!}(1-p)^n = \left(\frac{1-p}{p}\right)^r (r-1)!$$

   (Hint: distributions are normalised) **[6]**

   (g) What is $\langle n \rangle$? (Hint: use the above summation to aid you). **[6]**

   (h) Show that as $p \to 0$, $\langle n \rangle \to \infty$ and interpret this. **[4]**

Useful Formulae for Special Relativity & Probability

Lorentz Transformations in Standard Configuration

$$\begin{aligned}
ct' &= \gamma(ct - \beta x) & ct &= \gamma(ct' + \beta x') & \beta &= v/c \\
x' &= \gamma(x - \beta ct) & x &= \gamma(x' + \beta ct') & \gamma &= 1/\sqrt{1-\beta^2} \\
y' &= y & y &= y' & & \\
z' &= z & z &= z' & &
\end{aligned}$$

Velocity Transformation

$$u'_x = \frac{u_x - v}{\left(1 - \frac{u_x v}{c^2}\right)}, \qquad u'_y = \frac{u_y}{\gamma(v)\left(1 - \frac{u_x v}{c^2}\right)}, \qquad u'_z = \frac{u_z}{\gamma(v)\left(1 - \frac{u_x v}{c^2}\right)}.$$

Invariant Interval

$$\Delta s^2 = (c\Delta t)^2 - \Delta x^2 - \Delta y^2 - \Delta z^2$$

Energy and Momentum

$$E = \frac{mc^2}{\sqrt{1 - v^2/c^2}}, \qquad \mathbf{p} = \frac{m\mathbf{v}}{\sqrt{1 - v^2/c^2}}, \qquad E^2 = p^2c^2 + m^2c^4.$$

Conditional Probability

$$P(A|B)P(B) = P(A \cap B)$$

Binomial Distribution

$$P(n; N, p) = \binom{N}{n} p^n (1-p)^{N-n}$$

Poisson Distribution

$$P(n; \mu) = \frac{\mu^n}{n!} e^{-\mu}$$

Central Limit Theorem

$$\frac{X - n\mu}{\sqrt{n\sigma^2}} \to \mathcal{N}(0, 1)$$

Normal Distribution

$$p(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[\frac{-(x-\mu)^2}{2\sigma^2}\right]$$

Change of Variable in PDF

If $x \sim P_x(x)$ and $y = f(x)$ is monotonic: $P_y(y) = P_x(f^{-1}(y)) \left|\frac{df^{-1}(y)}{dy}\right|$

Physical Constants and Units

Acceleration due to gravity	g	9.81 m s^{-2}
Gravitational constant	G	6.674×10^{-11} N m^2 kg^{-2}
Ice point	T_{ice}	273.15 K
Avogadro constant	N_A	6.022×10^{23} mol^{-1}
		[*N.B.* 1 mole $\equiv 1$ *gram-molecule*]
Gas constant	R	8.314 J K^{-1} mol^{-1}
Boltzmann constant	k, k_B	1.381×10^{-23} J K^{-1} $\equiv 8.62 \times 10^{-5}$ eV K^{-1}
Stefan constant	σ	5.670×10^{-8} W m^{-2} K^{-4}
Rydberg constant	R_∞	1.097×10^{7} m^{-1}
	$R_\infty hc$	13.606 eV
Planck constant	h	6.626×10^{-34} J s $\equiv 4.136 \times 10^{-15}$ eV s
$h/2\pi$	$\hbar$	1.055×10^{-34} J s $\equiv 6.582 \times 10^{-16}$ eV s
Speed of light *in vacuo*	c	2.998×10^{8} m s^{-1}
	$\hbar c$	197.3 MeV fm
Charge of proton	e	1.602×10^{-19} C
Mass of electron	m_e	9.109×10^{-31} kg
Rest energy of electron		0.511 MeV
Mass of proton	m_p	1.673×10^{-27} kg
Rest energy of proton		938.3 MeV
One atomic mass unit	u	1.66×10^{-27} kg
Atomic mass unit energy equivalent		931.5 MeV
Electric constant	ϵ_0	8.854×10^{-12} F m^{-1}
Magnetic constant	μ_0	$4\pi \times 10^{-7}$ H m^{-1}
Bohr magneton	μ_B	9.274×10^{-24} A m^2 (J T^{-1})
Nuclear magneton	μ_N	5.051×10^{-27} A m^2 (J T^{-1})
Fine-structure constant	$\alpha = e^2/4\pi\epsilon_0\hbar c$	7.297×10^{-3}= 1/137.0
Compton wavelength of electron	$\lambda_c = h/m_e c$	2.426×10^{-12} m
Bohr radius	a_0	5.2918×10^{-11} m
angstrom	Å	10^{-10} m
barn	b	10^{-28} m^2
torr (mm Hg at 0 °C)	torr	133.32 Pa (N m^{-2})

Do not complete the attendance slip, fill in the front of the answer book or turn over the question paper until you are told to do so.

Important Reminders

- Coats/outwear should be placed in the designated area.
- Unauthorised materials (e.g. notes or Tippex) must be placed in the designated area.
- Check that you do not have any unauthorised materials with you (e.g. in your pockets, pencil case).
- Mobile phones and smart watches **must** be switched off and placed in the designated area or under your desk. They must not be left on your person or in your pockets.
- You are not permitted to use a mobile phone as a clock. If you have difficulty seeing a clock, please alert an Invigilator.
- You are not permitted to have writing on your hand, arm or other body part.
- Check that you do not have writing on your hand, arm or other body part – if you do, you must inform an Invigilator immediately
- Alert an Invigilator immediately if you find any unauthorised item upon you during the examination.

Any students found with non-permitted items upon their person during the examination, or who fail to comply with Examination rules may be subject to Student Conduct procedures.