## Week 09
## AC circuits in complex notation

### 1. Complex phasors

Earlier we saw how the phase relationship between sinusoidal currents and voltages in a circuit could be found by plotting their magnitudes on a phasor diagram. Figure 9.1 (left panel) shows two phasors (a voltage and a current) plotted on an Argand diagram. That is, our phasors are now assumed to rotate on the complex plane. If we assume that the voltage phasor was horizontal at time $t$ = 0, then we can express this voltage phasor in complex form as

$$v = Ve^{\mathrm{j}\omega t} = V\cos\omega t + \mathrm{j}V\sin\omega t \tag{1}$$

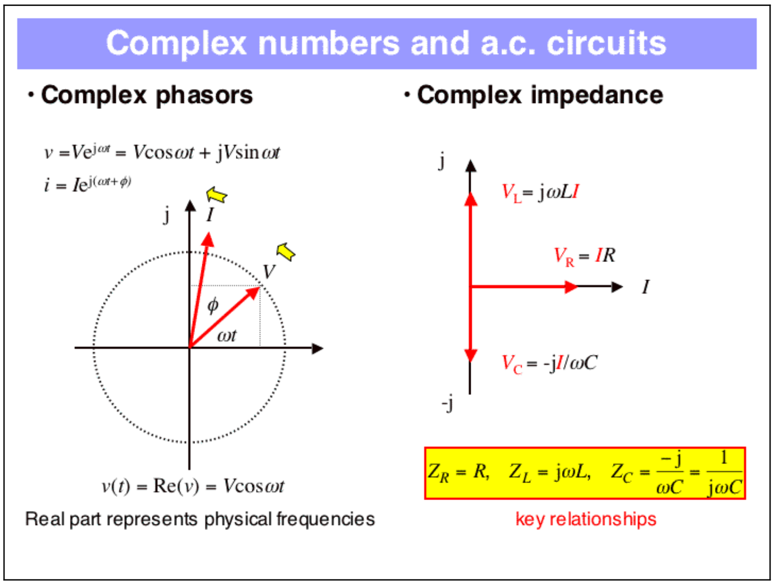


Figure 9.1: Complex numbers and a.c. circuits.

As before, the physical (time varying) voltage is the projection of this phasor onto one of the axes. By convention, the physical voltage is taken to be the real part of the complex phasor.

$$v(t) = \mathrm{Re}(v) = V\cos\omega t \tag{2}$$

This is the projection of the phasor onto the horizontal axis. When I introduced phasors earlier, I used the projection onto the vertical axis. The only difference is that the physical voltage is now a cosine function, instead of a sign function. This has no bearing whatsoever on our analysis. The reason for this is that we are only interested in **phase differences**. Whether our reference phasor is a sine or a cosine doesn't come into it. To illustrate this point, the current phasor in Figure 9.1 can be seen to lead the voltage by the angle $\phi$. In complex form, its phasor is given by

$$i = Ie^{\mathrm{j}(\omega t+\varphi)} = I\cos(\omega t + \varphi) + \mathrm{j}I\sin(\omega t + \varphi) \tag{3}$$

Once again, the real part of equation (7) represents the physical current.

$$i(t) = \mathrm{Re}(i) = I\cos(wt + f) \tag{4}$$

Together the real and imaginary parts allow us to calculate the angle that the phasor makes with the real axis, from equation (1). With this information we can calculate the angle (or phase difference) between two phasors without drawing a phasor diagram.

## 2. Complex impedance

In order to appreciate the advantage of using complex phasors we shall now introduce the concept of complex impedance. In Figure 9.1 (right panel) I've summarised the phase relationships between the voltage and current in a resistor, capacitor and inductor, derived earlier. Choosing the current magnitude as our reference, the voltage magnitudes across a resistor, capacitor and an inductor have been plotted on the Argand diagram. It can be seen that the voltage across the capacitor and the inductor are now imaginary because they lie along the vertical (imaginary) axis.

$$V_L = jw\mathrm{LI}\, V_C = -j\frac{I}{wC} \tag{5}$$

The factor of ± j that tells us the voltage magnitude differs in phase from the current by a rotation of ± 90 degrees. It is usual to absorb the factor of j into the expression for the impedance of the inductor and the capacitor.

$$Z_L = jwLZ_C = -j\frac{1}{wC} = \frac{1}{jwC} \tag{6}$$

In the capacitor case I've multiplied top and bottom by j. I think it is easier to remember the complex impedance in this way. Hopefully, you have already learned the expressions for the reactance of an inductor ($\omega L$) and capacitor ($1/\omega C$). These are made into complex impedances by putting a factor of j in front of the angular frequency, $\omega$. In this way, both expressions are positive.

### 2.1 RC series circuit

Figure 9.2 (right panel) summarises the phasor analysis for the RC series circuit. You don't need this for your exam since we are using only complex notations but it might be useful if you are reading the literature.

Figure 9.2 (left panel) summarises how to analyse the circuit using complex impedance. Since the factor of j in the capacitor impedance carries with it the information of the phase difference between the voltage and the current, this allows us to treat combinations of impedances in exactly the same way as combinations of resistors. For a series combination of a resistor and a capacitor, the total impedance is just the sum of the two impedances.

$$Z_R = R Z_C = \frac{1}{jwC} \tag{7}$$

Combining these impedances in series the total impedance is

$$Z = Z_R + Z_C = R + \frac{1}{jwC} = R - j\frac{1}{wC} \tag{8}$$

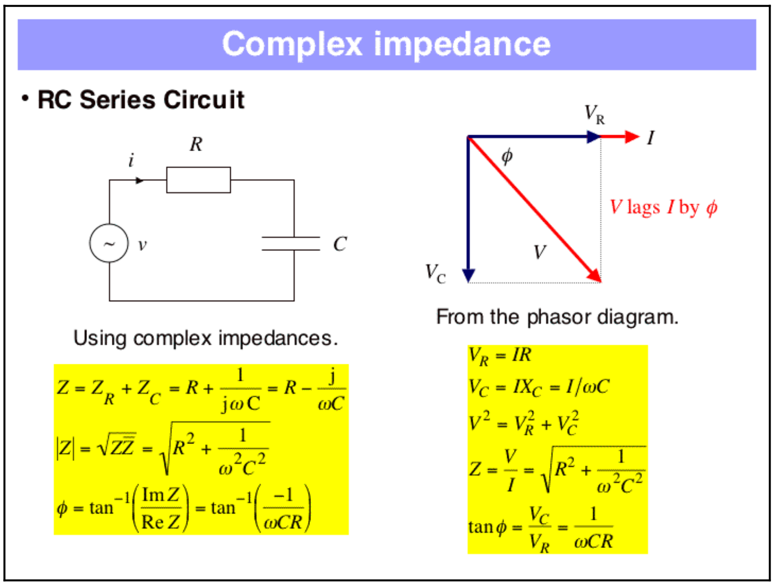


Figure 9.2: Complex analysis of the RC series circuit.

The magnitude of the total impedance is determined by multiplying the complex impedance by its complex conjugate and by taking the square root of the result.

$$|Z| = \sqrt{Z\bar{Z}} = \sqrt{R^2 + \frac{1}{\omega^2 C^2}} \tag{9}$$

This is exactly the same result as we obtained using the phasor approach. The phase angle is determined by the argument of the total impedance.

$$\Phi = \arg(Z) = \tan^{-1}\left(\frac{\operatorname{Im} Z}{\operatorname{Re} Z}\right) = \tan^{-1}\left(\frac{-1}{\omega CR}\right) \tag{10}$$

Once again, this is the same result as we obtained using the phasor approach, apart from the negative sign, which is discussed below.

## 2.2 Understanding the phase in the complex impedance approach

The beauty of the complex approach is that it allows us to treat impedances in a.c. circuits in the same way as resistances in d.c. circuits. However, we need to be able to interpret the phase correctly. Ultimately, it is Ohm's law that determines the relationship between the voltage and the current.

$$v = iZ \tag{11}$$

Here, $v$ and $i$ are complex phasors and $Z$ is a complex impedance. Taking the current to be the reference (zero phase), we can write the current and the impedance as complex exponentials.

$$v = Ie^{jwt}|Z|e^{jf} = I|Z|e^{j(wt+f)} \quad (12)$$

Applying the rules for multiplying complex exponentials, we take the product of their magnitudes and add their arguments. This gives the magnitude of the voltage phasor $V = I|Z|$ and tells us that the voltage phasor either lags or leads the current by the angle $\phi$. Notice that $\phi$ is the argument of the complex impedance. From equation (14), in the RC series circuit the phase angle $\phi$ will be **negative**. This tells us that the voltage will **lag** the current. The phasor diagram shown in Figure 9.2 confirms this conclusion.

To summarise: the phase of the complex impedance is to be interpreted with the **current** as the **reference** phasor. If the phase is negative, it means that the voltage lags the current; if it is positive, the voltage leads the current.

## 2.3 RL parallel circuit

We can do a similar comparison of the complex impedance approach and the phasor approach in a parallel circuit.

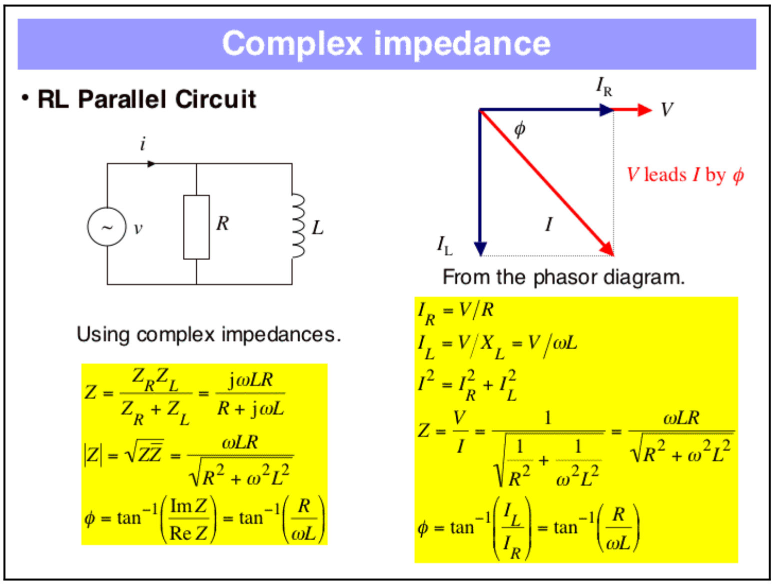


Figure 9.3: Complex analysis of the RL parallel circuit.

This circuit contains a resistor of impedance $Z_R = R$ and an inductor of impedance $Z_L = j\omega L$ connected in parallel. The total impedance is the product over the sum of the two impedances. This is exactly how you would calculate the equivalent resistance of two resistors connected in parallel.

$$Z = \frac{Z_R Z_L}{Z_R + Z_L} = \frac{jw\,LR}{R + jwL} \quad (13)$$

The expression for the impedance contains a complex number on the numerator and on the denominator. The magnitude of the impedance may be found by calculating the magnitude of the numerator and the denominator independently.

$$|Z| = \frac{|Z_1|}{|Z_2|} = \frac{\omega\,LR}{\sqrt{R^2 + \omega^2 L^2}} \quad (14)$$

Calculating the argument is a little bit tricky. The most straightforward approach is to turn equation (17) into a single complex number, of the form $z = a + jb$, with separate real and imaginary parts. To do this, multiply the numerator and the denominator by the complex conjugate of the denominator. This makes the denominator purely real.

$$Z=\frac{jwLR}{R+jwL}\times\frac{R-jwL}{R-jwL}=\frac{w^2L^2R+jwLR^2}{R^2+w^2L^2}=\frac{w^2L^2R}{R^2+w^2L^2}+\frac{jwLR^2}{R^2+w^2L^2} \tag{15}$$

The argument is the angle whose tangent is the imaginary part divided by the real part. Since the denominator is common to both parts, it cancels when you take the ratio.

$$\Phi=\tan^{-1}\left(\frac{\operatorname{Im}Z}{\operatorname{Re}Z}\right)=\tan^{-1}\left(\frac{R}{\omega L}\right) \tag{16}$$

The magnitude of the impedance and the phase angle are exactly the same as we found previously using the phasor approach. Check Figure 9.3 (right panel). The phase angle is positive indicating that the voltage leads the current.

## 3. Power in A.C. sircuits

In this part we will explore how power is dissipated in an a.c. circuit and how this depends upon the relative phase between the current and the voltage. We will then look at circuits that contain both a capacitor and an inductor and investigate how the impedance of the circuit varies as a function of frequency. This will introduce the concept of resonance in electronic circuits.

First we will consider how to calculate the power dissipated in an a.c. circuit. From our previous experience with d.c. circuits, we know that the electrical power is given by the product of a voltage and a current. In a.c. circuits, these quantities vary with time, so the power is also a time-varying quantity. Therefore, we shall calculate the average power over one period of the power waveform.

### 3.1 Power dissipation in a resistor.

The product of the time dependent current and the voltage determines the instantaneous power in a device. In a resistor, the current and the voltage are in phase. Thus, if we choose the current to be a sine wave of angular frequency, $\omega$, then the voltage will also be a pure sine wave of the same angular frequency.

$$p=iv=I\sin wt\times V\sin wt=IV\left(\sin wt\right)^2 \tag{24}$$

The average power is found by integrating this function over one complete period.

$$\langle p \rangle = \frac{1}{T}\int_0^T p\,dt = \frac{1}{T}\int_0^T IV(\sin wt)^2\,dt = \frac{1}{T}\int_0^T \frac{IV}{2}(1-\cos 2wt)\,dt \qquad (25)$$

It is not possible to integrate sine-squared directly, so I've made a standard substitution using one of the double angle formulae listed in an appendix at the end of these notes. As we saw, the average of a sine or cosine over one period is zero, so only the first term in the final integral contributes.

$$\langle p \rangle = \frac{IV}{2} = \frac{I}{\sqrt{2}} \cdot \frac{V}{\sqrt{2}} = I_{rms} \cdot V_{rms} \qquad (26)$$

The average power dissipated in a resistor is one half of the peak value ($IV$) and can thus be expressed as the product of the rms current and the rms voltage (remembering that the root-mean-square value is given by $1/\sqrt{2}$ of the peak value).

The relationship between the current, voltage and power in a resistor is shown graphically in Figure 9.4. This shows that the power varies at twice the frequency of the current or voltage. Notice that the power is positive valued at all times, whereas the current and voltage alternate between positive and negative values.

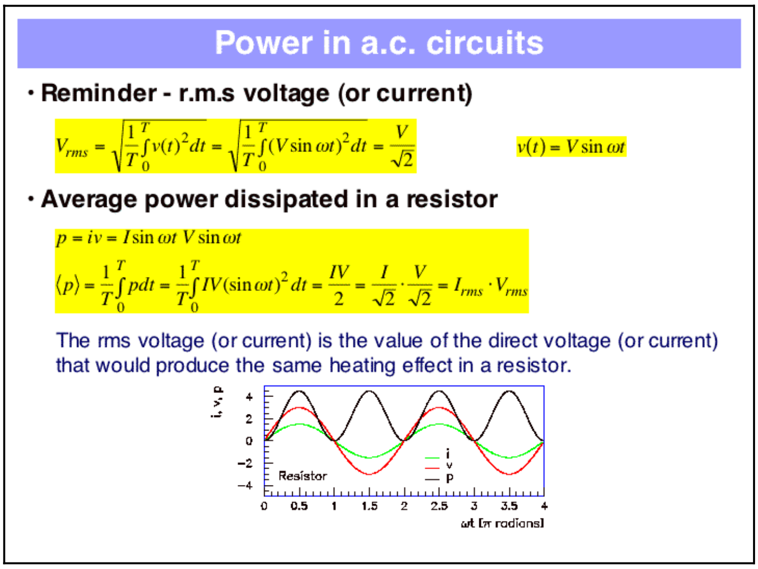


Figure 9.4: Power dissipation in a resistor.

## 3.2 Power dissipation in a capacitor and an inductor

In a capacitor or an inductor, there is a 90 degrees phase difference between the voltage and the current, which must also be taken into account. In the case of the capacitor, the voltage lags the current by 90 degrees.

$$p = iv = I\sin\omega t \times V\sin(\omega t - \pi/2) = I\sin\omega t\,(-V\cos\omega t) \qquad (27)$$

We can integrate the product of sine and cosine using another of the double angle formulae given at the end of this exercise.

$$\langle p \rangle = \frac{1}{T}\int_0^T p\,dt = \frac{1}{T}\int_0^T -IV\sin\omega t\cos\omega t\,dt = \frac{1}{T}\int_0^T \frac{-IV}{2}\sin 2\omega t\,dt = 0 \qquad (28)$$

This time, there is no constant term in the final integral and since the average of a sine function over one period is zero, the capacitor does not consume any power.

A similar calculation can be done for the inductor and results in the same conclusion. On average the inductor does not consume any power either. The situation for the capacitor and the inductor is summarised in Figure 9.5. It is clear from the graphs showing the relationship between the current, voltage and power in each of these components that the power dissipation is an alternating quantity. That is, it varies between positive and negative values. When the power is positive the capacitor (or inductor) receives energy from the circuit. When the power is negative, energy is returned to the circuit.

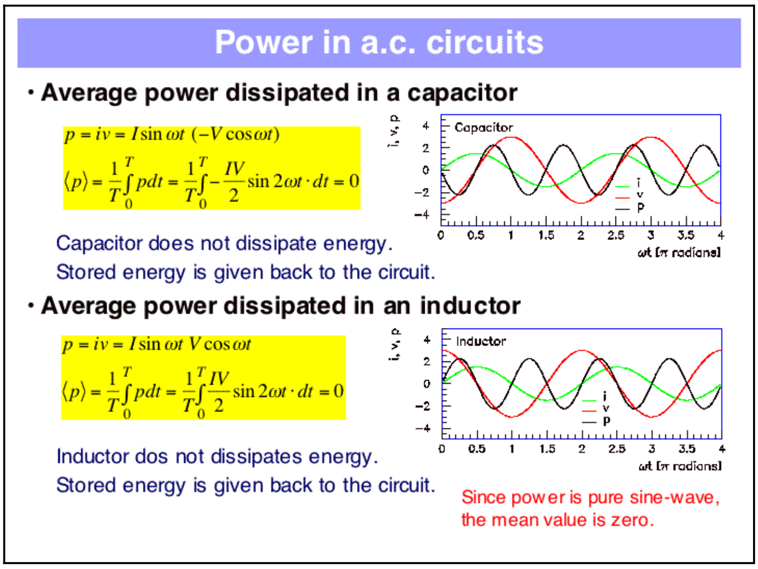


Figure 9.5: Power dissipation in a capacitor and an inductor.

This should not really come as any great surprise. We learned in earlier exercises that capacitors and inductors behave like temporary stores of electrical energy. In a.c. circuits they behave in exactly the same way.

Next week we will continue with power dissipation in reactive and resistive circuits.