Introduction to Probability

Lecture 2

Today

Combinatorics

Uniform Probability

Attendance: 35045358

Summary (last time)

Probability has three components

1. The **sample space** Ω
2. The **events** which are subsets of Ω
3. The **probability function** $P(x)$ which assigns probability to every event in Ω

The probability is **normalised**

$$\mathrm{P}(\Omega) = 1$$

Combinatorics and Counting

Combinatorics

We *count* the number of ways of something happening.

Example: We toss two dice. How many ways are there of getting a given total?

One way to get 2: (1,1)

Three ways to get a 4: (3,1), (2,2) and (3,1)

Example: How many Sudoku boards are there? ($\approx 6.7 \times 10^{21}$)

Nomenclature

We will use Ω to represent the **set** of possible outcomes.

We will use $|\Omega|$ to represent the number of outcomes.

Example

Toss one die:

$$\Omega = \{1,2,3,4,5,6\}$$
$$|\Omega| = 6$$

Sampling

An important aspect (here) of combinatorics is **sampling**.

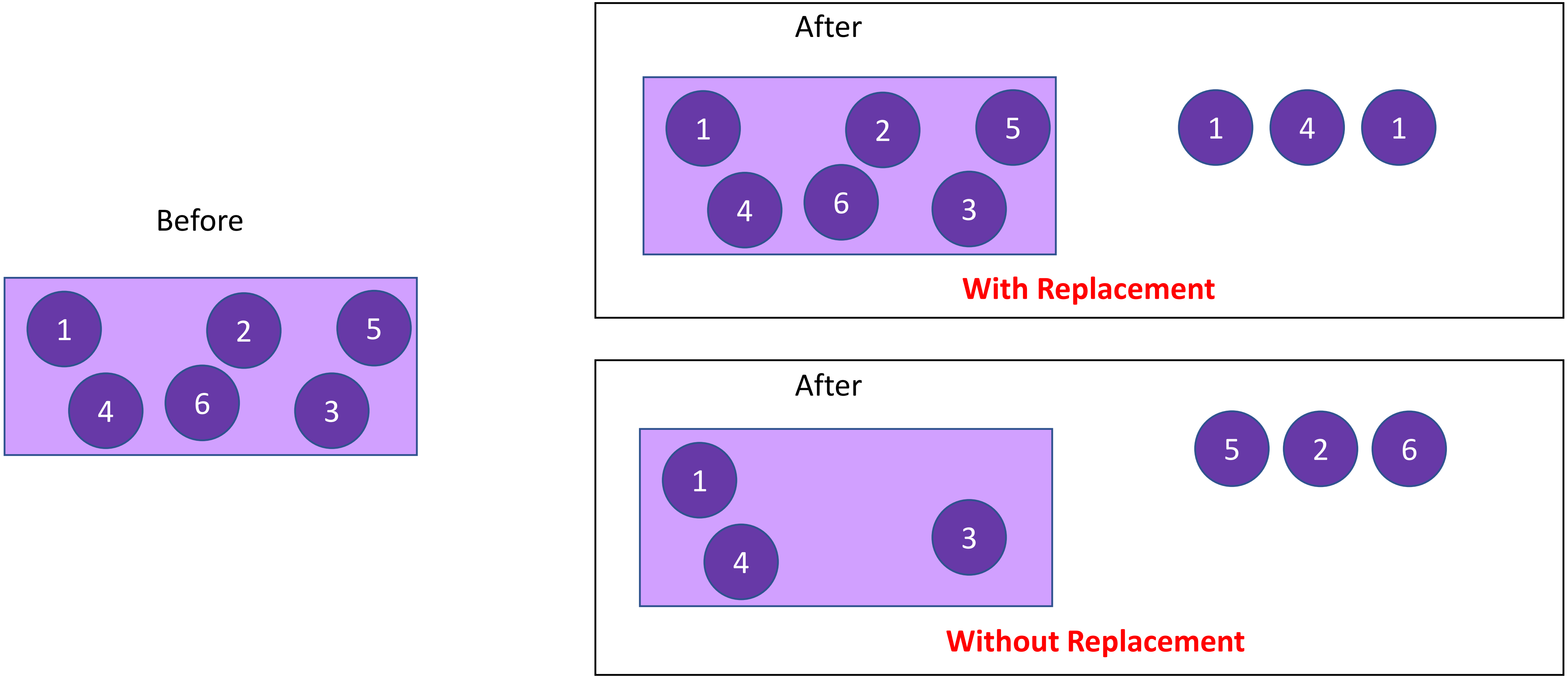
Sampling involves N objects and picking k of them.
For example: pick a 4-number password (a PIN).

We can

1. Sample with replacement and keep order.
   Pick the 4 numbers freely.
2. Sample without replacement and keep order.
   Ensure that no numbers are repeated.
3. Sample without replacement and ignore order.
   The bottom door of Physics East!

4. Sample with replacement and ignore order.

Replacement



Unordered

5 2 6   6 5 2   2 6 5

5 6 2   2 5 6   6 2 5

All the same

Multiplication Rule

Example: Number of ways of going through two sets of traffic lights

3 choices from first set, 3 choices from the second

$$3 \times 3 = 9$$

General:

Sample k things from *something*.

Let the first have n_1 choices

Let the second have n_2 choices.. *etc*.

All the way up to n_k

Then there are $n_1 \times n_2 \times \cdots n_k$ total choices.

Sampling with Replacement

Sample k things from N objects **with replacement**.

The first has N choices

The second has N choices.. *etc*.

$$N \times N \times \cdots \times N$$

$$|\Omega| = N^k$$

Example: How many 6 letter case-sensitive password are there?

$$|\Omega| = (2 \times 26)^6$$

Sampling without replacement

Sample k things from N objects **without replacement**.

The first has N choices

The second has $N - 1$ choices.. *etc*.

So

$$|\Omega| = N \times (N-1) \times (N-2) \ldots \times (N-k+1) = \frac{N!}{(N-k)!}$$

Example: A lottery machine has 10 (different) numbers. Three are picked out. How many different sequences are there?

$$|\Omega| = \frac{10!}{7!} = 720$$

Permutations

Sample N things from N objects without replacement: i.e. we draw *them all*.

We get

$$|\Omega| = N \times (N-1) \times \cdots \times 2 \times 1 = N!$$

The number of ways of **permuting** N objects.

Typically, $\frac{N!}{(N-k)!}$ is called a k-permutation of N.

Example: permute the digits (1,2,3)

(1,2,3), (3,1,2), (2,3,1), (1,3,2), (2,1,3), (3,2,1)

Unordered Samples

Hand of cards: the order the cards appear in the hand is *irrelevant*.

Sample k things from N objects without replacing, and ignoring the order they appeared.

We get $\frac{N!}{(N-k)!}$ for the possible draws, but this *overcounts*.

We need to divide out the number of ways of **permuting** k objects: $k!$

$$|\Omega| = \frac{N!}{k!\,(N-k)!} \equiv \binom{N}{k}$$

Example of overcounting.
Draw two numbers from (1,2,3)
Get 3!/1! for the 2-permutations
Then we want (2,1) and (1,2) to be counted the same
So
$3!/(2!1!) = \binom{3}{2}$

Interpretations of Binomial Coefficient

The expansion of

$$(a+b)^N = \sum_{k=0}^{N} \binom{N}{k} a^k b^{N-k}$$

The number of ways to split N objects into two groups with k in one group and $N-k$ in the other.

What if we had more groups?

Label objects 1 to N
Pick k of them and ignore order
Place these in one group

$\binom{N}{k}$ ways!

Multinomial

Divide a deck of cards into 4 even sized hands. The order of each hand is irrelevant.

The first person picks 13 cards from 52: $\boxed{\binom{52}{13}}$

The second person 13 cards from the remaining 39: $\boxed{\binom{39}{13}}$

$$|\Omega| = \binom{52}{13} \times \binom{39}{13} \times \binom{26}{13} \times \binom{13}{13} = \boxed{\frac{52!}{13!\,39!}\frac{39!}{13!\,26!}\frac{26!}{13!\,13!}\frac{13!}{13!\,0!}}$$

$$= \boxed{\frac{52!}{13!\,13!\,13!\,13!}}$$

Multinomial (2)

In general

Pick we have N objects and partition it into $n_1, n_2, \dots n_P$

$$|\Omega| = \binom{N}{n_1} \times \binom{N - n_1}{n_2} \times \binom{N - n_1 - n_2}{n_3} \dots \binom{n_P}{n_P}$$

$$|\Omega| = \frac{N!}{n_1!\, n_2! \dots n_P!} \equiv \binom{N}{n_1 \quad n_2 \quad \dots \quad n_P}$$

$$(a+b)^N = \sum_{k=0}^{N} \binom{N}{k} a^k b^{N-k}$$

$$(a+b+c)^N = \sum_{k_1+k_2+k_3=N} \binom{N}{k_1 \quad k_2 \quad k_3} a^{k_1} b^{k_2} c^{k_3}$$

Summary

	With Replacement	**Without Replacement**
Keep Order	$\lvert\Omega\rvert = N^k$	$\lvert\Omega\rvert = \frac{N!}{(N-k)!}$
Ignore Order	$\lvert\Omega\rvert = ?$	$\lvert\Omega\rvert = \frac{N!}{k!\,(N-k)!}$

Note

$$\frac{N!}{\underbrace{k!\,(N-k)!}_{\text{Two Groups}}} \rightarrow \frac{N!}{\underbrace{k_1!\,k_2!\,\ldots k_P!}_{P\ \text{Groups}}}$$

Uniform Probability

Uniform Probability

The probability of an **event** A is given by

$$\boxed{P(A) = \frac{|A|}{|\Omega|} = \frac{\text{Number of elements in } A}{\text{Number of elements in } \Omega}}$$

The probability of A is the fraction of the sample space it takes up.

This is known as the uniform probability.

Examples

Example

How many distinct numbers are stored in a 16-bit binary number?

Each number has two choices (0 or 1). There are 16 numbers so $|\Omega| = 2^{16}$

Example

How many different (not necessarily meaningful!) words can be made from Massachusetts?

There are 13 characters. There are 4 s's, 2 a's and 2 t's, so there are

$$\frac{13!}{4!\,2!\,2!}$$

Example

A bag contains 10 red and 6 orange balls. What is the probability of drawing two red and two orange balls?

We pick 4 balls out of 16: $|\Omega| = \binom{16}{4}$

There are $\binom{10}{2}$ ways to get red

There are $\binom{6}{2}$ ways to get orange

$$P = \frac{\binom{10}{2}\binom{6}{2}}{\binom{16}{4}} = \frac{10}{16}\frac{9}{15}\frac{6}{14}\frac{5}{13} \times \binom{\mathbf{4}}{\mathbf{2}}$$

Example

How many ways are there to divide this class (approx. 140 people) into two even sized groups?

What about four groups?

Looks like $\binom{140}{70}$ for two groups but we can ignore group labels

$$\rightarrow \frac{1}{2}\binom{140}{70}$$

Likewise

$$\rightarrow \frac{1}{4 \times 3 \times 2 \times 1}\binom{140}{35\ 35\ 35\ 35}$$

Class Example

An urn contains 10 red balls, 6 blue and 4 white.

How many ways are there to get 3 reds, 2 blues and 1 white ball?

What is the probability of seeing this?

There are:

$\binom{10}{3}$ ways to get 3 red

$\binom{6}{2}$ ways to get 2 blue

$\binom{4}{1}$ ways to get 1 white

So the total number of ways is

$$\binom{10}{3} \times \binom{6}{2} \times \binom{4}{1} = 7200$$

$$|\Omega| = \binom{20}{6} \rightarrow P(A) = \frac{\binom{10}{3} \times \binom{6}{2} \times \binom{4}{1}}{\binom{20}{6}}$$

Class Examples

What is the probability of:

1. Rolling a total of 5 when throwing 2 dice.
2. Drawing 2 aces from a deck of cards.
3. Splitting 7 people into three group, with 2 in group 1, 2 in group 2 and 3 in group 3.

1. Each dice has 6 outcomes: $|\Omega| = 6^2 = 36$. There are 4 ways of getting a 5 so

$$\mathrm{P(A)} = \frac{|A|}{|\Omega|} = \frac{4}{36} = \frac{1}{9}$$

2. There are $\binom{52}{2}$ ways of choosing two cards, and there are $(4 \times 3)/2 = 6$ ways of choosing two aces so

$$\mathrm{P(A)} = 6/(26 \times 51)$$

3. Each person can be assigned to 3 groups so there are 3^7 ways of assigning. There are:

$$\binom{7}{2 \quad 3 \quad 3} = 210$$

Ways to assign into the groups, so the probability is $\mathrm{P(A)} = \frac{210}{3^7} = \frac{70}{729}$