Power dissipation in a resistor

The product of the time dependent current and the voltage determines the instantaneous power in a device. In a resistor, the current and the voltage are in phase. Thus, if we choose the current to be a sine wave of angular frequency, ω, then the voltage will also be a pure sine wave of the same angular frequency.

$$p = iv = I\sin(wt)\,V\sin(wt) = I\,V\left(\sin(wt)\right)^2$$

The average power is found by integrating this function over one complete period.

$$\langle p\rangle = \frac{1}{T}\int_0^T p\,dt = \frac{1}{T}\int_0^T I\,V\left(\sin(wt)\right)^2 dt = \frac{1}{T}\int_0^T \frac{I\,V}{2}\left(1 - \cos(2\mathrm{wt})\right)dt$$

→ standard double-angle formula substitution. The average of a sine or cosine over one period is zero, so only the first term in the final integral contributes.

$$\langle p\rangle = \frac{IV}{2} = \frac{I}{\sqrt{2}}\cdot\frac{V}{\sqrt{2}} = I_{rms}\cdot V_{rms}$$

Power in a.c. circuits

- **Reminder - r.m.s voltage (or current)**

$$V_{rms} = \sqrt{\frac{1}{T}\int_0^T v(t)^2 dt} = \sqrt{\frac{1}{T}\int_0^T (V \sin \omega t)^2 dt} = \frac{V}{\sqrt{2}}$$

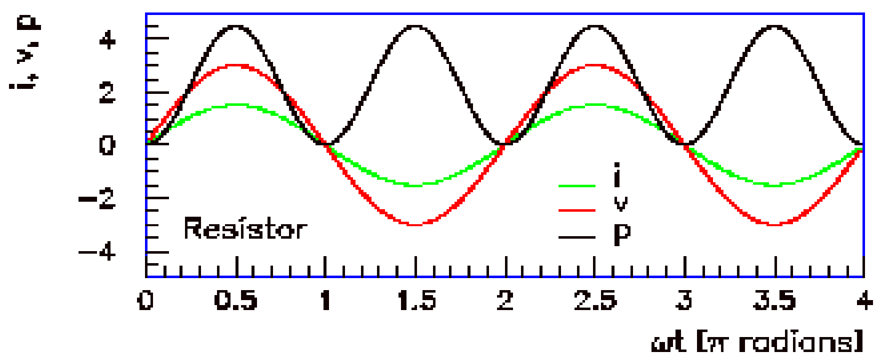
$$v(t) = V \sin \omega t$$

- **Average power dissipated in a resistor**

$$p = iv = I \sin \omega t \; V \sin \omega t$$

$$\langle p \rangle = \frac{1}{T}\int_0^T p dt = \frac{1}{T}\int_0^T IV (\sin \omega t)^2 dt = \frac{IV}{2} = \frac{I}{\sqrt{2}} \cdot \frac{V}{\sqrt{2}} = I_{rms} \cdot V_{rms}$$

The rms voltage (or current) is the value of the direct voltage (or current) that would produce the same heating effect in a resistor.



Power dissipation in a capacitor & inductor

In a capacitor or an inductor, there is a 90 degrees phase difference between the voltage and the current, which must also be taken into account. In the case of the capacitor, the voltage lags the current by 90 degrees.

$$p = i\,v = I\sin(\omega t)\,V\sin\left(\omega t - \frac{\pi}{2}\right) = I\sin(\omega t)\left(-V\cos(\omega t)\right)$$

Integrate the product using a double angle formula.

$$\langle p \rangle = \frac{1}{T}\int_0^T p\,dt = \frac{1}{T}\int_0^T -I\,V\sin(\omega t)\cos(\omega t)\,dt = \frac{1}{T}\int_0^T -\frac{I\,V}{2}\sin(2\omega t)\,dt = 0$$

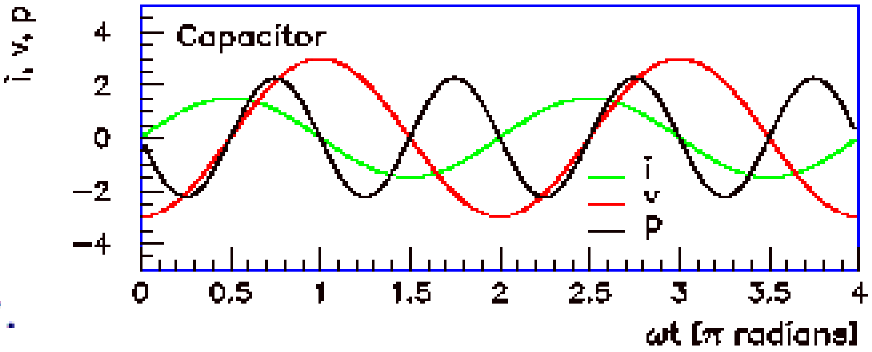
This time, there is no constant term in the final integral and since the average of a sine function over one period is zero, the capacitor does not consume any power.

Power in a.c. circuits

- **Average power dissipated in a capacitor**

$$p = iv = I\sin\omega t\ (-V\cos\omega t)$$

$$\langle p\rangle = \frac{1}{T}\int_0^T p\,dt = \frac{1}{T}\int_0^T -\frac{IV}{2}\sin 2\omega t\cdot dt = 0$$



Capacitor does not dissipate energy.

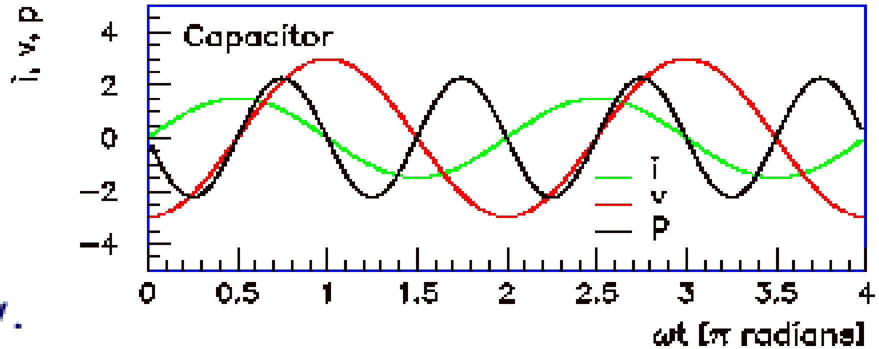
Stored energy is given back to the circuit.

Power in a.c. circuits

- **Average power dissipated in a capacitor**

$$p = iv = I\sin\omega t\ (-V\cos\omega t)$$

$$\langle p\rangle = \frac{1}{T}\int_0^T p\,dt = \frac{1}{T}\int_0^T -\frac{IV}{2}\sin 2\omega t\cdot dt = 0$$



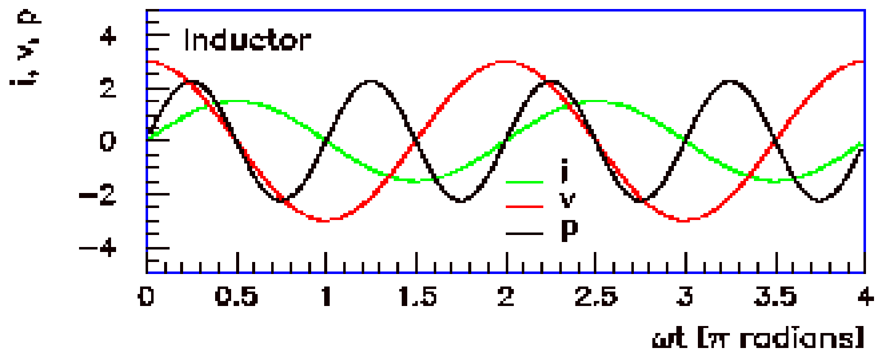
Capacitor does not dissipate energy.

Stored energy is given back to the circuit.

- **Average power dissipated in an inductor**

$$p = iv = I\sin\omega t\ V\cos\omega t$$

$$\langle p\rangle = \frac{1}{T}\int_0^T p\,dt = \frac{1}{T}\int_0^T \frac{IV}{2}\sin 2\omega t\cdot dt = 0$$



Inductor dos not dissipates energy.

Stored energy is given back to the circuit.

Since power is pure sine-wave, the mean value is zero.

Power dissipation: resistive/reactive circuits

How do we calculate the average dissipated power in circuits with both resistive and reactive components? The general approach is the same, only this time we don't know exactly what the phase angle, φ, will be. If we take the current to be our reference then the voltage will either lag or lead by φ (it doesn't matter which).

$$p = i\,v = I\sin(\omega t)\,V\sin(\omega t+\phi)$$

Integrate the product using a double angle formula.

$$\langle p\rangle = \frac{1}{T}\int_0^T p\,dt = \frac{1}{T}\int_0^T I\,V\left(\sin^2(\omega t)\cos\phi+\sin(\omega t)\cos(\omega t)\sin\phi\right)dt$$

The first term contains a sine-squared, which (in comparison to power in resistive circuits) has an average value of 1/2. The second term contains the product of sine and cosine, which has an average value of zero. → only the first term contributes.

$$\langle p\rangle = \frac{I\,V}{2}\cos\phi = I_{rms}V_{rms}\boxed{\cos\phi}\quad \text{power factor}$$