A19919 ***Any Calculator***

UNIVERSITY OF BIRMINGHAM

School of Physics and Astronomy

DEGREE OF BSc & MSci WITH HONOURS

FIRST YEAR EXAMINATION

03 17483/03 20835

LC Chaos and Non-Linear Systems A/B

The total time allowed is 1 hour

SUMMER EXAMINATIONS 2019

*Students should attempt **two** questions.*

*If you attempt more than two questions credit will be given for the **best two** answers.*

*The **approximate** allocation of marks to each part of a question is shown in brackets* **[ ]**.

Calculators may be used in this examination but must not be used to store text. Calculators with the ability to store text should have their memories deleted prior to the start of the examination.

1. (a) Define what is meant by the terms *stable fixed point*, *unstable fixed point*, *stable 2-cycle* and *unstable 2-cycle* of an iterated map,

$$x_{n+1} = f(x_n).$$

State the conditions for stability and instability of a fixed point. Deduce from these the corresponding conditions for stability and instability of a 2-cycle. **[4]**

Consider the iterated map defined on the interval $0 \leq x < \infty$ by

$$x_{n+1} = f(x) = \frac{r x_n}{1 + x_n^3},$$

where $0 \leq r < \infty$ is a variable parameter.

(b) Find the fixed points of this map, and analyse their existence and stability as the parameter r is varied. **[2]**

(c) Show that the equation $f(f(x)) = x$ can be rewritten in the form

$$z^3 - r^2 z^2 + r^3 z - r^3 = 0, \quad \text{where} \quad z = 1 + x^3,$$

if we assume that $x \neq 0$. **[1]**

(d) Show that the two elements of the 2-cycle satisfy the quadratic equation

$$z^2 + (r - r^2)z + r^2 = 0.$$

Use this equation to find the 2-cycle, and investigate its stability as a function of parameter r. **[3]**

2. The *Lotka-Volterra predator-prey model* is given by the differential equations

$$\frac{dR}{dt} = aR - bRF, \qquad \frac{dF}{dt} = -cF + dRF,$$

where $R(t)$ and $F(t)$ are the rabbit and fox populations at time t, and a, b, c, d are positive constants.

(a) Explain the biological meaning of each of the four terms aR, $-bRF$, $-cF$ and dRF in the model **[2]**

(b) Show that the model can be recast in the dimensionless form

$$\frac{dx}{ds} = x(1-y), \qquad \frac{dy}{ds} = \mu y(x-1),$$

by setting $R = \alpha x$, $F = \beta y$, and $t = \gamma s$. Write α, β, γ, and μ in terms of a, b, c, and d. **[2]**

(c) Show that the quantity

$$E = \mu(x - \ln x) + (y - \ln y)$$

is a conserved quantity for this model. **[2]**

(d) Show that $(0,0)$ and $(1,1)$ are fixed points of the system. Show that sufficiently near to $(0,0)$ the phase curves can be approximated by

$$x^{\mu} y = const,$$

whilst sufficiently near to $(1,1)$ they can be approximated by

$$\mu(x-1)^2 + (y-1)^2 = const.$$

[Hint: Use the result that $\ln z \to -\infty$ and $\ln(1+z) \approx z - z^2/2$ when $z \to 0$.] **[4]**

3. (a) Define what is meant by *self-similarity*, and give an example of where it may be seen in a system that also exhibits chaos. **[2]**

   (b) Show that a square has fractal dimension 2, using a method which may also be used for non-integer dimensions. Carefully explain the steps in your calculation. **[2]**

A fractal is constructed as shown in the following figure for iterations $n = 1$ and 3.

   (c) Sketch the construction that will correspond to $n = 2$, and describe the steps that are involved in generating the fractal from iteration n to $n + 1$. **[2]**

   (d) Find the fractal dimension of the construction, and comment on the result in comparison to other simpler geometric constructions. **[2]**

   (e) Construct a fractal with fractal dimension less than 1. Explain clearly the iteration process, and determine the fractal dimension of your construction. **[2]**

END OF PAPER

Do not complete the attendance slip, fill in the front of the answer book or turn over the question paper until you are told to do so

Important Reminders

- Coats/outwear should be placed in the designated area.
- Unauthorised materials (e.g. notes or Tippex) must be placed in the designated area.
- Check that you do not have any unauthorised materials with you (e.g. in your pockets, pencil case).
- Mobile phones and smart watches **must** be switched off and placed in the designated area or under your desk. They must not be left on your person or in your pockets.
- You are not permitted to use a mobile phone as a clock. If you have difficulty seeing a clock, please alert an Invigilator.
- You are not permitted to have writing on your hand, arm or other body part.
- Check that you do not have writing on your hand, arm or other body part – if you do, you must inform an Invigilator immediately
- Alert an Invigilator immediately if you find any unauthorised item upon you during the examination.

Any students found with non-permitted items upon their person during the examination, or who fail to comply with Examination rules may be subject to Student Conduct procedures.