University of Birmingham
School of Mathematics

1RA January Exam Autumn 2021

Generic Feedback

This assessment was managed fairly well by a majority of students. Many students provided high quality solutions for some question parts, but did not submit any attempt for other parts. An unusually large number of marks were lost in this way.

On average, parts 1(d), 2(b), 2(d), 3(c)(i), 4(c) were answered the best, whilst the provision of sufficient justifications and accurate proofs in parts 1(c)(i), 2(c), 3(c)(ii), 4(a)(ii), 4(b) were handled the least well. Many students have greatly improved their ability to present logically accurate and complete justifications in mathematical proofs, but this should be a continued focus for most.

Feedback was provided on each submission using the symbols **V** ("tick") and **X** ("cross"). Ticks indicate where key steps have been correctly undertaken, while crosses indicate mistakes. The generic feedback for each question part is below:

1(a) Full marks were not often awarded here. A good understanding of the definition of convergence is required in order to present the case when $x \to -\infty$.

1(b) It is important to take care with the direction of implication arrows ($\Leftrightarrow$, $\Rightarrow$ and $\Leftarrow$) here. Marks were also lost for making statements which are not true in general, such as "$\sqrt{x} < x$", which is not true for all $x > 0$.

1(c)(i) The key here was to decompose the given expression into a product of known limits. It was important that each part of the decomposition was in the exactly the same form as the known limits. Those who achieved this generally got high marks here. (ii) It is important to take care when applying the Sandwich Theorem in the case of fractions that take both positive and negative values, since the signs $\leq$ and $\geq$ may need to be reversed. Instead, the absolute value function and the Null Limits rule can be used to get around this.

1(d) This part was answered well. Marks were lost for not justifying each step, such as citing the Product Rule or the Chain Rule.

2(a) A common mistake was "$L(f, P) = 9$," but the correct answer is $L(f, P) = 8$, since $m_1 := \inf\{f(x) : x \in [0, 1]\} = 1$ (i.e. $m_1 \neq 2$).

2(b) This part asks for an *antiderivative* of h (not the derivative h'). This means that the answer must be a differentiable *function* f such that $f'(x) = h(x)$ for all $x \in (0, 3) \cup (3, \pi)$. Marks were lost for computing the integrals $\int_0^3 h$ and $\int_3^\pi h$, which are numbers, or for not stating the answer as a *single* function with domain $(0, 3) \cup (3, \pi)$.

2(c) This is an improper integral because the denominator $|x - 1|$ is not defined at $x = 1$, which is inside the domain of integration. The improper integral $\int_0^4 f$ must be interpreted as the sum $\int_0^1 f + \int_1^4 f$, where $\int_0^1 f = \lim_{\delta \to 0^+} \int_0^{1-\delta} f$ and $\int_1^4 f = \lim_{\delta \to 0^+} \int_{1+\delta}^4 f$. Marks were lost for not splitting the integral into two parts, for not treating the improper integrals as limits, or for substituting the $x = 1$ endpoint into an antiderivative of f.

2(d) This is a linear first-order differential equation, so the Integration Factor method can be applied. It is not a separable equation, however, so the method of Separation of Variables does not apply.

3(a) This part was answered well.

3(b) This part was answered well. A common technical error was to end up with an "x" in the numerator rather than the denominator when reducing to notable limits.

3(c)(i) This part was answered mostly well. (ii) To find $\inf h$, it was not enough to observe that the function is bounded below by 0, as it also needs to be shown that 0 is the greatest lower bound for the range of h. This can be achieved by observing that $h(0) = 0$, which then also proves that 0 is the minimum of h. To find $\sup h$, it was necessary to prove that $\lim_{x\to\infty} h(x) = \infty$. (iii) It was important to recall here that $x = 0$ is a *global* minimum, and as such, it is also a *local* minimum. (iv) This part was answered mostly well. It is important to understand that a function is surjective when its range and *codomain* (not domain) are equal.

4(a) The function f has three points of discontinuity, so a suitable partition to use is $P_\delta := \{1, 1 + \delta, 2 - \delta, 2 + \delta, 3 - \delta, 3\}$ for $\delta \in (0, \frac{1}{2})$. This provides a δ-interval around *each* of the three points ($x = 1, 2, 3$) where $f(x)$ is discontinuous (and not just the point $x = 2$). Marks were lost for not introducing δ when it is first used (e.g. "Let $\delta \in (0, \frac{1}{2})$") and for incorrect computations of $L(f, P_\delta)$ and $U(f, P_\delta)$. An equal-width partition P_n could also be used, but extra care had to be taken in the computation for $L(f, P_n)$. The function f is also not monotonic, so the general argument for monotonic functions from the Lecture Notes could not be applied here. The conclusion for part (i) was mostly handled very well. This required introducing a parameter ϵ, making an appropriate choice of δ, and then concluding by referring back to Riemann's Criterion. The conclusion for part (ii) was more challenging and required combining the definitions of the lower and upper integrals with the inequality $\underline{\int_a^b} f \leq \overline{\int_a^b} f$, as in Examples 7.4.5-7.4.6 in the Lecture Notes.

4(b) This is an application of the First Fundamental Theorem of Calculus. The function underneath the integral, i.e. $t \mapsto \sin(f(t))$, is differentiable because it is the composition of two differentiable functions (sin and f). The function is thus continuous because differentiable functions are continuous. Marks were lost here for not observing either of these points. The continuity is essential because it justifies the application of the First Fundamental Theorem of Calculus in the next step. It was then necessary to correctly define an auxiliary function, e.g. $G(x) := \int_1^x \sin(f(t))\,\mathrm{d}t$, on which the First Fundamental Theorem of Calculus could be applied to deduce that $G'(x) = \sin(f(x))$. The Chain Rule could then be used to conclude that F is differentiable, and to compute $F'(x)$, since $F(x) = G(3x^4)$ and so $F'(x) = G'(3x^4)(3x^4)'$. Marks were lost here for not defining an auxiliary function and not correctly applying the Chain Rule.

4(c) This is a inhomogeneous second-order differential equation, so Variation of Parameters applies. The Characteristic Equation has a single repeated root $\lambda = 2b$, so the general solution for the homogeneous equation is $y_c(x) = C_1 e^{2bx} + C_2 x e^{2bx}$ (note the extra x in the second term). The inhomogeneous term in the equation is a second-order polynomial (x^2), so the trial particular solution should be of the form $y_p(x) = Ax^2 + Bx + C$. It is essential to include all three coefficients A, B, C, since otherwise an inconsistent (unsolvable) system of equations is obtained. Marks were also lost for algebraic errors computing A, B, C, and C_1, C_2.