2 Elasticity – solutions

The problems are roughly in order of difficulty. The ones with ♣ are the hardest ones, which might only occur as a "sting in the tail" at the end of a long examination question.

Problem 2.1 **Stretching**

The stress on the wire is the force, Mg, with $M = 600$ kg, divided by the area, $A = 2 \times 10^{-5}$ m^2, of the cable. Thus we have

$$\frac{F}{A} = Y\frac{\delta\ell}{\ell}$$
$$\Rightarrow \quad \frac{F}{A}\frac{\ell}{\delta\ell} = Y$$

Now the strain, $\delta\ell/\ell = 5 \times 10^{-4}/5 = 10^{-4}$. The stress is

$$\frac{F}{A} = \frac{Mg}{A} = \frac{600 \times 10}{2 \times 10^{-5}} \text{ Pa} = 3 \times 10^{8} \text{ Pa} .$$

Substituting these into the expression for Y yields:

$$Y = \frac{F}{A}\frac{\ell}{\delta\ell} = 3 \times 10^{8} \times 10^{4} \text{ Pa} = 3 \times 10^{12} \text{ Pa} ,$$

which is a bit high.

Finally we should check whether we needed to include the mass of the wire when working out the tension and hence the extension (it is intuitively small, but good to check). We are given the density of steel ($\rho_{\text{steel}} = 8 \times 10^{3}$kgm^{-3}) for that purpose. The unstretched volume of the wire is $5 \times 2 \times 10^{-5}$ m^3, so the mass of the cable,

$$m_{\text{cable}} = \rho_{\text{steel}}\ 10^{-4} \text{ kg} = 0.8 \text{ kg} \ll 600 \text{ kg} .$$

Thus the mass of the cable is negligible.

Problem 2.2 **Lateral contraction**

Poisson's ratio for steel is $\nu_{\text{steel}} = 0.3$. So $\delta w/\delta\ell = 0.3$, where δw is the change in the width of the object. Since $\delta\ell = 5 \times 10^{-4}$ m, we see that $\delta w = 0.3\delta\ell = 1.5 \times 10^{-4}$ m.

Problem 2.3 **Stretching and cooling**

For a change in temperature ΔT, the fractional change in length is

$$\frac{\delta\ell}{\ell} = \alpha\Delta T .$$

This thermal strain must be of the opposite sign to the strain from the weight – so the temperature must be decreased. The strain, ϵ, due to the weight is $\epsilon = 10^{-4}$, so

$$\alpha\Delta T = -10^{-4} \quad \Rightarrow \quad \Delta T = -\frac{10^{-4}}{\alpha} = -\frac{10^{-4}}{1.3 \times 10^{-5}} \text{ K} = -7.7 \text{ K} .$$

Problem 2.4 **Auxetic materials**

Consider the two-dimensional auxetic material with geometry defined in Fig. (2.1). All joints are freely hinged. Firstly note that for $\theta \to \pi/2$, to avoid the constituents overlapping we

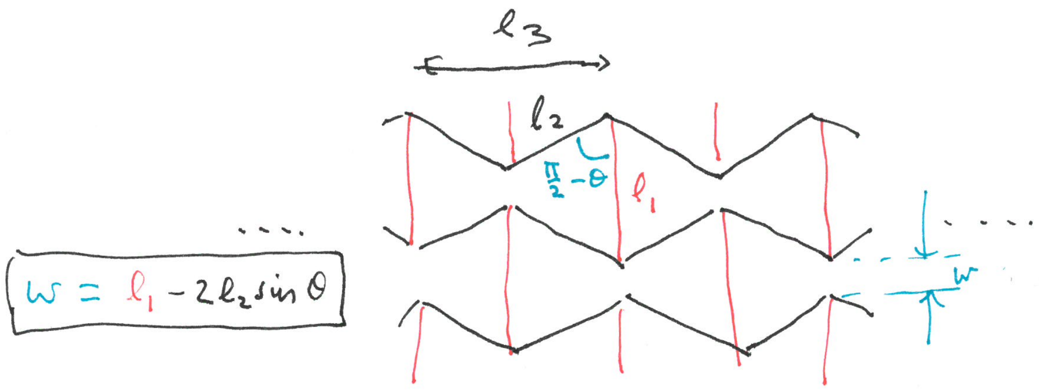

Figure 2.1: The notation for auxetic materials.

need $2\ell_2 \leq \ell_1$. In that limit the figure collapses to zero length in the lateral direction and a "height" of $(N\ell_1)/2$, where N is the number of layer of the bow-ties one has.

For fixed θ, the expression for w is

$$w = \ell_1 - 2\ell_2 \sin\theta \ .$$

and for ℓ_3 is:

$$\ell_3 = 2\ell_2 \cos\theta \ .$$

Let us now consider small variations around a particular value of θ, i.e. $\theta \to \theta + \delta\theta$. Now we determine the effect on w and ℓ_3.

$$\begin{aligned}
w &= \ell_1 - 2\ell_2 \sin\theta \\
\Rightarrow \quad w + \delta w &= \ell_1 - 2\ell_2 \sin(\theta + \delta\theta) \\
&\simeq \ell_1 - 2\ell_2 \left[\sin(\theta) + \delta\theta\cos(\theta)\right] \\
\Rightarrow \quad \delta w &= -2\ell_2 \delta\theta \cos\theta && (2.1) \\
\text{and} \quad \ell_3 &= 2\ell_2 \cos\theta \\
\Rightarrow \quad \ell_3 + \delta\ell_3 &= 2\ell_2 \left[\cos(\theta + \delta\theta)\right] \\
&= 2\ell_2 \left[\cos\theta - \delta\theta \sin\theta\right] \\
\Rightarrow \quad \delta\ell_3 &= -2\ell_2 \delta\theta \sin\theta \ . && (2.2)
\end{aligned}$$

Now let us eliminate $\delta\theta$ from Eq. (2.1) and Eq. (2.2) to obtain a relation between δw and $\delta\ell_3$:

$$\begin{aligned}
\frac{\delta w}{2\ell_2 \cos\theta} &= \frac{\delta\ell_3}{2\ell_2 \sin\theta} \\
\Rightarrow \quad \frac{\delta w}{\delta\ell_3} &= \cot\theta
\end{aligned}$$

Remembering that the δw has a minus sign in the definition of Poisson's ratio – ie contraction is regarded as positive, we see:

$$\nu = -\cot\theta \ .$$

Problem 2.5 **Elastic energy of a compressed rod**
The rod has cross sectional area t^2, so the relation between the strain and the stress is

$$\frac{F(x)}{A} = \frac{F(x)}{4t^2} = Y\frac{x}{L_0} \ ,$$

where $F(x)$ is the force at a compression of x and Y is the Young's modulus for the material of the rod.

Thus the work, W^{comp}, performed by the force in compressing the rod by an amount d is:

$$\begin{aligned} W^{\mathsf{comp}} = \int_0^d \mathrm{d}x\, F(x) &= \frac{4t^2Y}{L_0}\int_0^d \mathrm{d}x\, x \\ &= \frac{2t^2Yd^2}{L_0} \ . \end{aligned}$$

♣*Problem 2.6* **Elastic energy of a bent rod and the Euler instability**
Consider the same uncompressed rod as in the last problem, but now bend the centre of the rod into an arc of a circle with radius a with angle θ. See Fig. (2.2)

Figure 2.2: The rod of thickness $2t$, bent through an angle θ.

The arc-length of the middle of the rod is L_0, so $L_0 = 2a\theta$. The arc-length of the outer/inner side of the rod is $L(r = a \pm t) = 2(a \pm t)\theta$.

Consider the rod as a set of thin rectangular plates, of thickness dr. The the stress-strain relation of the plate at radius r is

$$\frac{F(r,x)}{t\mathrm{d}r} = Y\frac{x(r)}{L_0} \ .$$

The total extension or compression of the sheet depends on its radius:

$$X(r) = L(r) - L_0 = 2r\theta - 2a\theta \ ,$$

where $a - t \le r \le a + t$. Then we may deduce the elastic energy in the plate at radius r as being:

$$\begin{aligned} W(r) &= \int_0^{X(r)} \mathrm{d}x\, F(r,x) \\ &= \frac{tY\mathrm{d}r}{L_0} \int_0^{X(r)} \mathrm{d}x\, x \\ &= \frac{tY\mathrm{d}r}{2L_0} X(r)^2 \ . \end{aligned}$$

We now must add up the contributions from the plates at different r:

$$\begin{aligned} W^{\text{tot}} = \int_{a-t}^{a+t} \mathrm{d}r\, W(r) &= \frac{tY}{2L_0} \int_{a-t}^{a+t} \mathrm{d}r\, X(r)^2 \\ &= \frac{tY}{2L_0} \int_{a-t}^{a+t} \mathrm{d}r\, 4\theta^2 (r-a)^2 \\ \text{letting } u = r - a, \quad &= \frac{tY}{2L_0} \int_{-t}^{t} \mathrm{d}u\, 4\theta^2 u^2 \\ &= \frac{4\theta^2 tY}{2L_0} \left[\frac{u^3}{3}\right]_{-t}^{t} \\ &= \frac{4\theta^2 tY}{2L_0} \frac{2}{3} t^3 \\ \text{using } 2\theta a = L_0, \quad &= \frac{L_0 t^2 Y}{3} \left(\frac{t}{a}\right)^2 \ . \end{aligned} \tag{2.3}$$

Let us now eliminate a in preference for the change in separation of the end points of the bar, d. See Fig. (2.3)

$$\begin{aligned} d &= L_0 - 2a \sin\theta \\ (\text{use } L_0/(2a) = \theta) \quad &= L_0 - 2a \sin\left(\frac{L_0}{2a}\right) \\ &= L_0 - 2a \left(\frac{L_0}{2a} - \frac{1}{3!}\left[\frac{L_0}{2a}\right]^3 + \cdots\right) \\ &\simeq \frac{2a}{3!}\left[\frac{L_0}{2a}\right]^3 \\ \Rightarrow \quad 3!\frac{d}{L_0^3} &= \frac{1}{(2a)^2} \end{aligned}$$

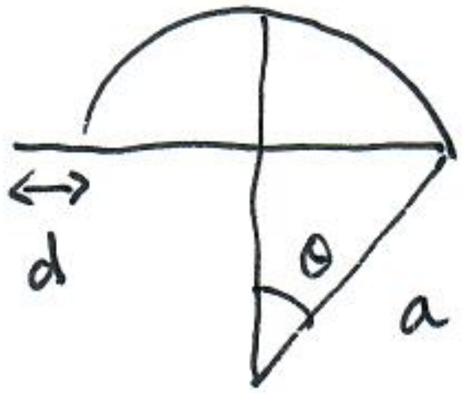

Figure 2.3: The displacement of the end of the rod, d, after bending through an angle θ.

We may now eliminate the factor of $1/a^2$ in Eq. (2.3):

$$\begin{aligned} W^{\text{tot}} &= \frac{L_0 t^2 Y}{3}\left(\frac{t}{a}\right)^2 \\ &= \frac{L_0 t^2 Y}{3}\left(\frac{3! \times 4dt^2}{L_0^3}\right) \\ &= \frac{8t^4 Y d}{L_0^2} = W^{\text{bend}} \end{aligned} \tag{2.4}$$

If we now equate the two energies for fixed d, we may ask which deformation – compression or bending – is preferable. Note the energy of the compression grows like d^2 where as the bending energy is linear in d. So for small d, compression is preferable. Let us calculate the Critical force to be applied in compression when the rod will bend. We do this be equating the energies to determine the value of d^{crit} and from that determine the force F^{crit}:

$$\begin{aligned} W^{\text{comp}}(d^{\text{crit}}) &= W^{\text{bend}}(d^{\text{crit}}) \\ \Rightarrow \quad \frac{2t^2 Y \left[d^{\text{crit}}\right]^2}{L_0} &= \frac{8t^4 Y d^{\text{crit}}}{L_0^2} \\ \Rightarrow \quad d^{\text{crit}} &= \frac{4t^2}{L_0} \\ \Rightarrow \quad F(d^{\text{crit}}) = F^{\text{crit}} = 4t^2 Y \frac{d^{\text{crit}}}{L_0} &= \frac{16t^4 Y}{L_0^2} \end{aligned}$$

So the critical force rises steeply with the cross sectional area (A^2) and decreases rapidly with length. Our assumption about the circular arc is not quite right – which alters the numerical prefactor (i.e. 16) – but the dependence on the physical variables is correct.

The original application of this by Euler (the *Euler strut*) had slightly different boundary conditions, and represented a vertical column being loaded from the top (under the influence of gravity), with the question being when it buckled. Again the dependence on the physical variables is the same as our solution

Notice unthinking dimensional analysis would not work as we have two lengths in the solution: t and L_0.