Solution Sheet 1: Classical Dynamics

1. An equation of motion is

$$\frac{d^2x}{dt^2} + x + x^3 = 0$$

find a fundamental representation to describe this system. Can you find a conserved energy?

Answer 1. It is natural to use the velocity or equivalently the momentum as the second dynamical variable and then

$$\frac{dx}{dt} = p \qquad \frac{dp}{dt} = \frac{d^2x}{dt^2} = -x - x^3$$

If we think of the force as $-\frac{dV}{dx}(x)$ then we can integrate to suggest

$$V(x) = \frac{1}{2}x^2 + \frac{1}{4}x^4$$

and then we can test

$$E = \frac{1}{2}p^2 + \frac{1}{2}x^2 + \frac{1}{4}x^4 \quad \Rightarrow \quad \frac{dE}{dt} = p\frac{dp}{dt} + x\frac{dx}{dt} + x^3\frac{dx}{dt} = 0$$

from the equations of motion.

2. An equation of motion is

$$\frac{d^2x}{dt^2} + \nu x^2\frac{dx}{dt} + x = 0$$

find a fundamental representation to describe this system. Can you find a conserved energy? Can you find the attractor?

Answer 2. We may choose the velocity or momentum in these rescaled coordinates

$$p = \frac{dx}{dt} \quad \Rightarrow \quad \frac{dp}{dt} = -x - \nu x^2 p$$

If we use

$$E = \frac{p^2}{2} + \frac{x^2}{2} \quad \Rightarrow \quad \frac{dE}{dt} = p\frac{dp}{dt} + x\frac{dx}{dt} = -\nu x^2 p^2$$

so this system loses this energy. The final term is not a perfect time derivative so there is no conserved quantity. The only fixed point is the origin so this is the attractor.

3. An equation of motion is

$$\frac{d^3x}{dt^3} + \nu\frac{d^2x}{dt^2} + \frac{dx}{dt} = 0$$

find a fundamental equation to describe the system. Solve for the general motion and deduce the attractor.

Answer 3. We need two new dynamical variables and the natural choice is

$$p = \frac{dx}{dt} \qquad q = \frac{d^2x}{dt^2} = \frac{dp}{dt} \qquad \Rightarrow \qquad \frac{dq}{dt} = -\nu q - p$$

which provides

$$\frac{d}{dt}\begin{bmatrix} x \\ p \\ q \end{bmatrix} = \begin{bmatrix} p \\ q \\ -\nu q - p \end{bmatrix}$$

We can integrate directly to obtain

$$\frac{d^2x}{dt^2} + \nu\frac{dx}{dt} + x = A$$

The particular integral is A and using the exponential ansatz $x = Be^{Dt}$ we find

$$D^2 + \nu D + 1 = \left(D + \frac{\nu}{2}\right)^2 + 1 - \frac{\nu^2}{4} = 0$$

set $\Omega^2 = 1 - \frac{\nu^2}{4}$ and $\Lambda^2 = \frac{\nu^2}{4} - 1$ and then if $\Omega^2 > 0$ we have

$$x = Be^{-\frac{\nu}{2}t}\cos\Omega(t - t_0)$$

whereas if $\Lambda^2 > 0$ we have

$$x = e^{-\frac{\nu}{2}t}\left[Be^{\Lambda t} + Ce^{-\Lambda t}\right]$$

When $t \mapsto \infty$ the transients are lost and we are left with x=A as the attractor, for any A.

4. An equation of motion is

$$\frac{d^2x}{dt^2} + \nu\frac{dx}{dt} + x^3 - x = 0$$

find a fundamental representation to describe this system. Can you find the attractor?

Answer 4. We may employ the momentum

$$p = \frac{dx}{dt} \qquad \frac{dp}{dt} = -\nu p + x - x^3$$

The fixed points are controlled by p=0 but there are three choices for x, $x \in \{-1, 0, 1\}$. When x is small we have

$$\frac{d^2x}{dt^2} + \nu\frac{dx}{dt} - x \sim 0$$

and the exponential ansatz gives

$$D^2 + \nu D - 1 = \left(D + \frac{\nu}{2}\right)^2 - 1 - \frac{\nu^2}{4} = 0$$

and at least one of the roots is positive and so the origin is unstable. Using $x = 1 + \delta x$ we have

$$-x + x^3 = x(x-1)(x+1) \mapsto 1(\delta x)2 = 2\delta x$$

and so using the exponential ansatz

$$\frac{d^2\delta x}{dt^2} + \nu\frac{d\delta x}{dt} + 2\delta x \sim 0 \quad \Rightarrow \quad D^2 + \nu D + 2 = 0 \quad \Rightarrow \quad \left(D + \frac{\nu}{2}\right)^2 = \frac{\nu^2}{4} - 2$$

and so this point is stable. Using $x = -1 + \delta x$ we have

$$-x + x^3 = x(x-1)(x+1) \mapsto (-1)(-2)(\delta x) = 2\delta x$$

so this point is also stable. The attractor is either x=1 or x=-1.

5. The Van der Pol oscillator has the equation of motion

$$\frac{d^2x}{dt^2} + \lambda(x^2 - 1)\frac{dx}{dt} + x = 0$$

Employ the velocity as a dynamical variable to find a fundamental equation. Now employ

$$P = \frac{x^3}{3} - x + \frac{1}{\lambda}\frac{dx}{dt}$$

as a dynamical variable to find a fundamental representation. Explain why the first representation is useful when λ is small and the second representation is useful when λ is large.

Answer 5. If we use the velocity we get

$$p = \frac{dx}{dt} \qquad \frac{dp}{dt} = -x + \lambda(1 - x^2)p$$

If we use

$$P = \frac{x^3}{3} - x + \frac{1}{\lambda}\frac{dx}{dt} \quad \Rightarrow \quad \frac{dP}{dt} = (x^2 - 1)\frac{dx}{dt} + \frac{1}{\lambda}\frac{d^2x}{dt^2} = -\frac{x}{\lambda}$$

When λ is small the first equation gives a weakly perturbed harmonic oscillator, with an almost conservation law, $E = \frac{p^2}{2} + \frac{x^2}{2}$ and $\frac{dE}{dt} = \lambda p^2(1 - x^2)$. When λ is large the second equation gives an almost conservation law $P = \frac{x^3}{3} - x$.

6. A one-dimensional particle moves in a potential

$$V(x) = -\frac{1}{2}kx^2 + \frac{1}{4}Kx^4$$

Find the equation of motion and deduce a representation of the fundamental equation. Deduce the energy and create a phase space portrait for this system. Find the separatrices. What are the possible long-time limits?

Answer 6. We may use Newton's laws

$$m\frac{d^2x}{dt^2} = -\frac{dV}{dx}(x) = kx - Kx^3$$

We may use the momentum as a second dynamical variable

$$\frac{dx}{dt} = \frac{p}{m} \qquad \Rightarrow \qquad \frac{dp}{dt} = kx - Kx^3$$

The energy is, $E = T + V$

$$E = \frac{p^2}{2m} - \frac{1}{2}kx^2 + \frac{1}{4}Kx^4 \qquad \Rightarrow \qquad \frac{dE}{dt} = \frac{p}{m}\frac{dp}{dt} + \left(-kx + Kx^3\right)\frac{dx}{dt} = 0$$

so the energy is conserved. The fixed points occur when p=0 but there are three $x \in \left\{0, \pm\left[\frac{k}{K}\right]^{\frac{1}{2}}\right\}$. The two away from the origin give $x \mapsto \pm\left[\frac{k}{K}\right]^{\frac{1}{2}} + \delta x$

$$-\frac{1}{2}kx^2 + \frac{1}{4}Kx^4 \mapsto -\frac{1}{2}k\left(\pm\left[\frac{k}{K}\right]^{\frac{1}{2}} + \delta x\right)^2 + \frac{1}{4}K\left(\pm\left[\frac{k}{K}\right]^{\frac{1}{2}} + \delta x\right)^4 \mapsto -\frac{k^2}{4K} + k\delta x^2$$

and so

$$E \mapsto -\frac{k^2}{4K} + \frac{p^2}{2m} + k\delta x^2$$

and we have harmonic motion and the phase space portrait is locally a set of concentric ellipses. The origin has E=0 and lies on the separatrix. We may solve for the separatrix

$$\sqrt{m}\frac{dx}{dt} = \pm x\left[k - \frac{K}{2}x^2\right]^{\frac{1}{2}}$$

which separates

$$\pm\left(\frac{k}{m}\right)^{\frac{1}{2}} dt = \frac{dx}{x\left[1 - \frac{K}{2k}x^2\right]^{\frac{1}{2}}}$$

We may then make the transformation $x = \pm\frac{1}{X}$ which gives

$$\frac{dx}{x} = -\frac{dX}{X} \qquad \Rightarrow \qquad \pm\left(\frac{k}{m}\right)^{\frac{1}{2}} dt = -\frac{dX}{X\left[1 - \frac{K}{2kX^2}\right]^{\frac{1}{2}}} = -\frac{dX}{\left[X^2 - \frac{K}{2k}\right]^{\frac{1}{2}}}$$

and then the final transformation $X = \left[\frac{K}{2k}\right]^{\frac{1}{2}}\cosh u$ provides

$$\pm\left(\frac{k}{m}\right)^{\frac{1}{2}} dt = -du$$

and the eventual

$$x = \pm\left[\frac{2k}{K}\right]^{\frac{1}{2}} \frac{1}{\cosh\left(\left[\frac{k}{m}\right]^{\frac{1}{2}}(t - t_0)\right)}$$

The separatrix forms an "infinity sign", surrounding the two sets of distorted ellipses and being surrounded in turn by closed loops. The ellipses are harmonic motion in either of the two minima while the large closed loops describe motion which passes across both minima. At large time we are either oscillating in one of the two minima or across both of the two minima.

7. A one-dimensional particle moves in a potential

$$V(x) = -\frac{1}{2}kx^2 + \frac{1}{3}Kx^3$$

Find the equation of motion and deduce a representation of the fundamental equation. Deduce the energy and create a phase space portrait for this system. Find the separatrices. What are the possible long-time limits?

Answer 7. We may apply Newton's laws to obtain the equation of motion

$$m\frac{d^2x}{dt^2} = -\frac{dV}{dx}(x) = -kx + Kx^2$$

Using the momentum as the second dynamical variable gives

$$p = m\frac{dx}{dt} \quad \Rightarrow \quad \frac{dp}{dt} = -kx + Kx^2$$

The energy is $E{=}T{+}V$

$$E = \frac{p^2}{2m} - \frac{1}{2}kx^2 + \frac{1}{3}Kx^3 \quad \Rightarrow \quad \frac{dE}{dt} = \frac{p}{m}\frac{dp}{dt} + \left[-kx + Kx^2\right]\frac{dx}{dt} = 0$$

so the energy is conserved. To find the fixed points we need $p{=}0$ but then there are two choices $x \in \left\{0, \frac{k}{K}\right\}$. The solution away from the origin gives $x = \frac{k}{K} + \delta x$

$$E \mapsto -\frac{1}{6}\frac{k^2}{K} + \frac{p^2}{2m} + k\delta x^2$$

and so we have concentric ellipses and harmonic motion. The origin has $E{=}0$ and lies on the separatrix. We may solve for the separatrix

$$\pm\sqrt{m}\frac{dx}{dt} = x\left[k - \frac{2}{3}Kx\right]^{\frac{1}{2}} \quad \Rightarrow \quad \pm\left[\frac{k}{m}\right]^{\frac{1}{2}} dt = \frac{dx}{x\left[1 - \frac{2K}{3k}x\right]^{\frac{1}{2}}}$$

and then the transformation $y^2 = 1 - \frac{2K}{3k}x$ leads to $x = \frac{3k}{2K}(1 - y^2)$ and

$$\pm\left[\frac{k}{m}\right]^{\frac{1}{2}} dt = \frac{-2ydy}{[1 - y^2]y} = -dy\left(\frac{1}{1+y} + \frac{1}{1-y}\right)$$

and then integrating, when $y < 1$

$$\pm\left[\frac{k}{m}\right]^{\frac{1}{2}} (t - t_0) = -\ln\left(\frac{1+y}{1-y}\right) \quad \Rightarrow \quad y = \frac{\exp\left(\pm\left[\frac{k}{m}\right]^{\frac{1}{2}}(t - t_0)\right) - 1}{\exp\left(\pm\left[\frac{k}{m}\right]^{\frac{1}{2}}(t - t_0)\right) + 1}$$

$$= \pm \tanh\left(\frac{1}{2}\left[\frac{k}{m}\right]^{\frac{1}{2}}(t-t_0)\right)$$

and hence

$$1 - \frac{2K}{3k}x = y^2 = \tanh^2\left(\frac{1}{2}\left[\frac{k}{m}\right]^{\frac{1}{2}}(t-t_0)\right) \qquad \Rightarrow$$

$$x = \frac{3k}{2K}\left[1 - \tanh^2\left(\frac{1}{2}\left[\frac{k}{m}\right]^{\frac{1}{2}}(t-t_0)\right)\right] = \frac{3k}{2K}\frac{1}{\cosh^2\left(\frac{1}{2}\left[\frac{k}{m}\right]^{\frac{1}{2}}(t-t_0)\right)}$$

$$= \frac{3k}{K}\frac{1}{1+\cosh\left[\frac{k}{m}\right]^{\frac{1}{2}}(t-t_0)}$$

When $y > 1$ we get

$$\pm\left[\frac{k}{m}\right]^{\frac{1}{2}}(t-t_0) = -\ln\left(\frac{1+y}{y-1}\right) \qquad \Rightarrow \qquad y = \frac{\exp\left(\pm\left[\frac{k}{m}\right]^{\frac{1}{2}}(t-t_0)\right)+1}{\exp\left(\pm\left[\frac{k}{m}\right]^{\frac{1}{2}}(t-t_0)\right)-1}$$

$$= \pm \coth\left(\frac{1}{2}\left[\frac{k}{m}\right]^{\frac{1}{2}}(t-t_0)\right)$$

and hence

$$1 - \frac{2K}{3k}x = y^2 = \coth^2\left(\frac{1}{2}\left[\frac{k}{m}\right]^{\frac{1}{2}}(t-t_0)\right) \qquad \Rightarrow$$

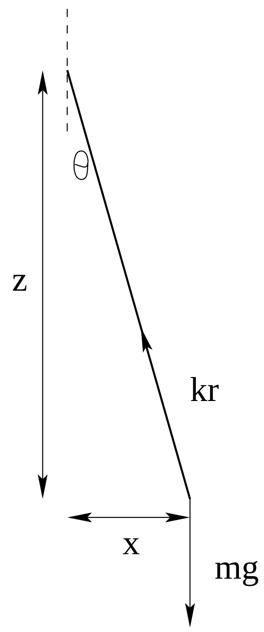
$$x = \frac{3k}{2K}\left[1 - \coth^2\left(\frac{1}{2}\left[\frac{k}{m}\right]^{\frac{1}{2}}(t-t_0)\right)\right] = -\frac{3k}{2K}\frac{1}{\sinh^2\left(\frac{1}{2}\left[\frac{k}{m}\right]^{\frac{1}{2}}(t-t_0)\right)}$$

$$= -\frac{3k}{K}\frac{1}{\cosh\left[\frac{k}{m}\right]^{\frac{1}{2}}(t-t_0)-1}$$

The separatrix offers a "tear drop" around the ellipses together with a solution that starts and finishes at zero and infinity. There are harmonic oscillations close to the stable fixed point but the rest of the solutions drift off to infinity, gaining endless energy in the unbounded potential.

8. A sprung pendulum has l=0 and a spring constant k, find the equations of motion using Cartesian coordinates. Find the energy and demonstrate that there is a second conservation law. Solve the system completely and find the Poincare section associated with x=0. A second spring is added which also has l=0 but a spring constant K which acts only along the x-axis. Find the Poincare section associated with x=0.

Answer 8. The physical picture is



and applying Newton's laws we get, along the x-axis

$$m\frac{d^2x}{dt^2} = -kr\sin\theta = -kx$$

and along the z-axis

$$m\frac{d^2z}{dt^2} = mg - kr\cos\theta = mg - kz$$

Alternatively we can target the kinetic and potential energies

$$T = \frac{m}{2}\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2\right] \qquad V = \frac{1}{2}k\left[x^2 + z^2\right] - mgz$$

and then, with L=T-V

$$p_x = \frac{\partial L}{\partial \frac{dx}{dt}} = m\frac{dx}{dt} \qquad p_z = \frac{\partial L}{\partial \frac{dz}{dt}} = m\frac{dz}{dt}$$

and the equations of motion are

$$\frac{dp_x}{dt} = m\frac{d^2x}{dt^2} = \frac{\partial L}{\partial x} = -kx \qquad\qquad \frac{dp_z}{dt} = m\frac{d^2z}{dt^2} = \frac{\partial L}{\partial z} = mg - kz$$

as before. We also have the fundamental equation

$$\frac{d}{dt}\begin{bmatrix} x \\ z \\ p_x \\ p_z \end{bmatrix} = \begin{bmatrix} \frac{p_x}{m} \\ \frac{p_z}{m} \\ -kx \\ mg - kz \end{bmatrix}$$

These equations of motion give us that the energy of each component is separately conserved

$$E_x = \frac{m}{2}\left(\frac{dx}{dt}\right)^2 + \frac{k}{2}x^2 \qquad \Rightarrow \qquad \frac{dE_x}{dt} = m\frac{dx}{dt}\frac{d^2x}{dt^2} + kx\frac{dx}{dt} = 0$$

$$E_z = \frac{m}{2}\left(\frac{dz}{dt}\right)^2 + \frac{k}{2}z^2 - mgz \qquad \Rightarrow \qquad \frac{dE_z}{dt} = m\frac{dz}{dt}\frac{d^2z}{dt^2} + (kz - mg)\frac{dz}{dt} = 0$$

Since we have two constants of the motion this system is integrable. We may readily exactly solve using the exponential ansatz

$$x = Ae^{Dt} \qquad \Rightarrow \qquad \left[mD^2 + k\right]Ae^{Dt} = 0$$

and so

$$x = A\cos\left[\frac{k}{m}\right]^{\frac{1}{2}}(t - t_x) \qquad p_x = m\frac{dx}{dt} = -\left[km\right]^{\frac{1}{2}}A\sin\left[\frac{k}{m}\right]^{\frac{1}{2}}(t - t_x)$$

and for the z-component, including the particular integral

$$z = \frac{mg}{k} + B\cos\left[\frac{k}{m}\right]^{\frac{1}{2}}(t - t_z) \qquad p_z = m\frac{dz}{dt} = -\left[km\right]^{\frac{1}{2}}B\sin\left[\frac{k}{m}\right]^{\frac{1}{2}}(t - t_z)$$

This gives a pair of ellipses

$$x^2 + \frac{p_x^2}{mk} = A^2 \qquad \left[z - \frac{mg}{k}\right]^2 + \frac{p_z^2}{mk} = B^2$$

We now look to x=0 to provide a Poincare section. This provides a collection of times

$$\left[\frac{k}{m}\right]^{\frac{1}{2}}(T_n - t_x) = \frac{\pi}{2} + n\pi$$

and the energy gives $\frac{dx}{dt}$, up to an important sign, and leaves

$$z_n = \frac{mg}{k} + B\cos\left(\left[\frac{k}{m}\right]^{\frac{1}{2}}(t_x - t_z) + \frac{\pi}{2} + n\pi\right)$$

$$m\left(\frac{dz}{dt}\right)_n = -\left[km\right]^{\frac{1}{2}}B\sin\left(\left[\frac{k}{m}\right]^{\frac{1}{2}}(t_x - t_z) + \frac{\pi}{2} + n\pi\right)$$

and we get a pair of points which oscillate, since $\sin(a + n\pi) = (-1)^n\sin a$ and $\cos(a + n\pi) = (-1)^n\cos a$. In fact the sign of $\frac{dx}{dt}$ also oscillates and so we actually have two independent Poincare sections associated with two different points in phase space. The final addition of a second spring constant makes the x motion

$$m\frac{d^2x}{dt^2} + (k + K)x = 0$$

and turns the exact solution into

$$x = A\cos\left[\frac{k+K}{m}\right]^{\frac{1}{2}}(t-t_x) \qquad p_x = m\frac{dx}{dt} = -\left[(k+K)m\right]^{\frac{1}{2}} A\sin\left[\frac{k+K}{m}\right]^{\frac{1}{2}}(t-t_x)$$

which provides a new collection of times

$$\left[\frac{k+K}{m}\right]^{\frac{1}{2}}(T_n - t_x) = \frac{\pi}{2} + n\pi$$

and a new Poincare section

$$z_n = \frac{mg}{k} + B\cos\left(\left[\frac{k}{m}\right]^{\frac{1}{2}}(t_x - t_z) + \left[\frac{k}{k+K}\right]^{\frac{1}{2}}\left[\frac{\pi}{2} + n\pi\right]\right)$$

$$m\left(\frac{dz}{dt}\right)_n = -\left[km\right]^{\frac{1}{2}} B\sin\left(\left[\frac{k}{m}\right]^{\frac{1}{2}}(t_x - t_z) + \left[\frac{k}{k+K}\right]^{\frac{1}{2}}\left[\frac{\pi}{2} + n\pi\right]\right)$$

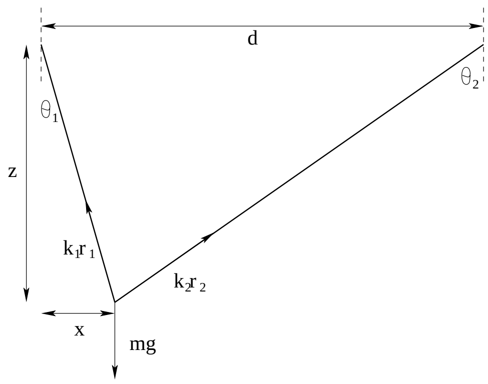
and now the angle in the second ellipse smoothly changes with

$$z_n \equiv \frac{mg}{k} + B\cos\theta_n \qquad m\left(\frac{dz}{dt}\right)_n = -\left[km\right]^{\frac{1}{2}} B\sin\theta_n \qquad \theta_n = \theta_0 + \left[\frac{k}{k+K}\right]^{\frac{1}{2}}\pi n$$

and the expected one-dimensional curve.

9. A bob of mass m is suspended from two springs, with l=0 and spring constants k_1 and k_2 respectively, with fixed points on the same level separated by distance d. Show that the system is integrable. Find the equilibrium position and an appropriate Poincare section.

Answer 9. The physical picture is



and applying Newton's laws we get, along the x-axis

$$m\frac{d^2x}{dt^2} = k_2 r_2 \sin\theta_2 - k_1 r_1 \sin\theta_1 = k_2(d-x) - k_1 x = k_2 d - (k_1+k_2)x$$

and along the z-axis

$$m\frac{d^2z}{dt^2} = mg - k_1 r_1\cos\theta_1 - k_2 r_2\cos\theta_2 = mg - (k_1+k_2)z$$

Alternatively we can target the kinetic and potential energies

$$T = \frac{m}{2}\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2\right] \qquad V = \frac{1}{2}k_1\left[x^2+z^2\right] + \frac{1}{2}k_2\left[(x-d)^2+z^2\right] - mgz$$

and then, with L=T-V

$$p_x = \frac{\partial L}{\partial \frac{dx}{dt}} = m\frac{dx}{dt} \qquad p_z = \frac{\partial L}{\partial \frac{dz}{dt}} = m\frac{dz}{dt}$$

and the equations of motion are

$$\frac{dp_x}{dt} = \frac{\partial L}{\partial x} = -k_1 x - k_2(x-d) \qquad \frac{dp_z}{dt} = \frac{\partial L}{\partial z} = mg - k_1 z - k_2 z$$

as before. We also have the fundamental equation

$$\frac{d}{dt}\begin{bmatrix} x \\ z \\ p_x \\ p_z \end{bmatrix} = \begin{bmatrix} \frac{p_x}{m} \\ \frac{p_z}{m} \\ k_2 d - (k_1+k_2)x \\ mg - (k_1+k_2)z \end{bmatrix}$$

These equations of motion give us that the energy of each component is separately conserved

$$E_x = \frac{m}{2}\left(\frac{dx}{dt}\right)^2 + \frac{k_1}{2}x^2 + \frac{k_2}{2}(x-d)^2 \quad \Rightarrow \quad \frac{dE_x}{dt} = m\frac{dx}{dt}\frac{d^2x}{dt^2} + (k_1 x + k_2[x-d])\frac{dx}{dt} = 0$$

$$E_z = \frac{m}{2}\left(\frac{dz}{dt}\right)^2 + \frac{k_1+k_2}{2}z^2 - mgz \quad \Rightarrow \quad \frac{dE_z}{dt} = m\frac{dz}{dt}\frac{d^2z}{dt^2} + ([k_1+k_2]z - mg)\frac{dz}{dt} = 0$$

Since we have two constants of the motion this system is integrable. We may readily exactly solve using the exponential ansatz, and defining $k \equiv k_1 + k_2$, including the particular integral

$$x = \frac{k_2 d}{k} + Ae^{Dt} \quad \Rightarrow \quad \left[mD^2 + k\right]Ae^{Dt} = 0$$

and so

$$x = \frac{k_2 d}{k} + A\cos\left[\frac{k}{m}\right]^{\frac{1}{2}}(t-t_x) \qquad p_x = m\frac{dx}{dt} = -\left[km\right]^{\frac{1}{2}} A\sin\left[\frac{k}{m}\right]^{\frac{1}{2}}(t-t_x)$$

and for the z-component, including the particular integral

$$z = \frac{mg}{k} + B\cos\left[\frac{k}{m}\right]^{\frac{1}{2}}(t-t_z) \qquad p_z = m\frac{dz}{dt} = -\left[km\right]^{\frac{1}{2}} B\sin\left[\frac{k}{m}\right]^{\frac{1}{2}}(t-t_x)$$

This gives a pair of ellipses

$$\left(x - \frac{k_2 d}{k}\right)^2 + \frac{p_x^2}{mk} = A^2 \qquad \left[z - \frac{mg}{k}\right]^2 + \frac{p_z^2}{mk} = B^2$$

The equilibrium position is when A=0 and B=0, which gives x=$\frac{k_2 d}{k}$ and z=$\frac{mg}{k}$. A natural Poincare section is x=$\frac{k_2 d}{k}$ and this gives us a set of times

$$\left[\frac{k}{m}\right]^{\frac{1}{2}}(T_n - t_x) = \frac{\pi}{2} + n\pi$$

and the energy gives $\frac{dx}{dt}$, up to an important sign, and leaves

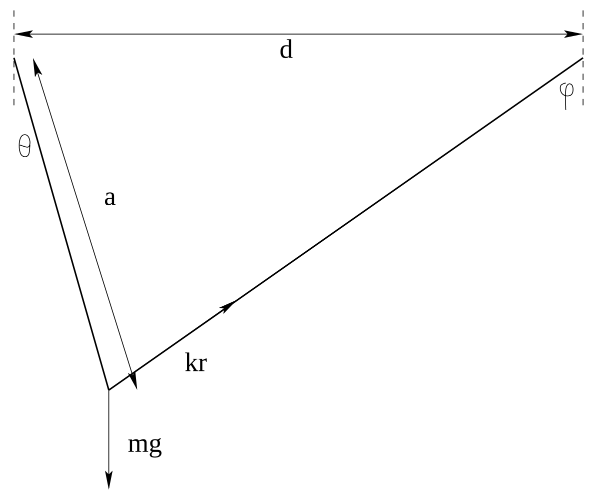
$$z_n = \frac{mg}{k} + B\cos\left(\left[\frac{k}{m}\right]^{\frac{1}{2}}(t_x - t_z) + \frac{\pi}{2} + n\pi\right)$$

$$m\left(\frac{dz}{dt}\right)_n = -\left[km\right]^{\frac{1}{2}} B\sin\left(\left[\frac{k}{m}\right]^{\frac{1}{2}}(t_x - t_z) + \frac{\pi}{2} + n\pi\right)$$

and we get a pair of points which oscillate, since $\sin(a + n\pi) = (-1)^n \sin a$ and $\cos(a + n\pi) = (-1)^n \cos a$. In fact the sign of $\frac{dx}{dt}$ also oscillates and so we actually have two independent Poincare sections associated with two different points in phase space.

10. A pendulum of length a and mass m pivots about one end but has a spring of natural length l=0 and spring constant k attached to the other end. This spring is also attached to a second pivot at a horizontal distance d from the first pivot. Show that the system is equivalent to a single pendulum with parameters that you should find.

Answer 10. The physical picture is



We may use the kinetic energy and potential energy

$$T = \frac{1}{2}ma^2\left(\frac{d\theta}{dt}\right)^2 \qquad V = -mga\cos\theta + \frac{1}{2}kr^2$$

together with some geometry

$$d = a\sin\theta + r\sin\phi \qquad a\cos\theta = r\cos\phi$$

to provide

$$r^2 = (a\cos\theta)^2 + (d - a\sin\theta)^2 = d^2 + a^2 - 2ad\sin\theta$$

We may now rewrite the potential energy as

$$V = -mga\cos\theta + \frac{1}{2}k\left[d^2 + a^2 - 2ad\sin\theta\right]$$

The algebra

$$R\cos\alpha \equiv mga \qquad R\sin\alpha \equiv kad \qquad \Rightarrow \qquad R^2 = a^2[(mg)^2 + (kd)^2] \qquad \tan\alpha = \frac{kd}{mg}$$

and the identity

$$\cos(\theta - \alpha) = \cos\theta\cos\alpha + \sin\theta\sin\alpha$$

finally provides

$$V = \frac{k}{2}[d^2 + a^2] - R\cos(\theta - \alpha)$$

We may construct a lagrangian through L=T-V which provides

$$p_\theta = \frac{\partial L}{\partial \frac{d\theta}{dt}} = ma^2\frac{d\theta}{dt}$$

and then an equation of motion

$$\frac{dp_\theta}{dt} = ma^2\frac{d^2\theta}{dt^2} = \frac{\partial L}{\partial\theta} = -R\sin(\theta - \alpha)$$

and with the relabelling of $\tilde{\theta} \equiv \theta - \alpha$ and $ma\tilde{g} \equiv R$ we find

$$ma^2\frac{d^2\tilde{\theta}}{dt^2} = -ma\tilde{g}\sin\tilde{\theta}$$

and we recognise a pendulum, but with a non-trivial equilibrium position.

11. Consider a square billiard with specular reflection. Show that the system is integrable. Find a general trajectory using the idea of reflection and images. Use the points of reflection to set up a collection of Poincare sections. Find the low order n-cycles and establish when the system is ergodic on the Poincare sections.

Answer 11. There is no potential energy so

$$E = T = \frac{m}{2}\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right]$$

A reflection involves either

$$\left[\frac{dx}{dt}, \frac{dy}{dt}\right] \mapsto \left[-\frac{dx}{dt}, \frac{dy}{dt}\right]$$

or

$$\left[\frac{dx}{dt}, \frac{dy}{dt}\right] \mapsto \left[\frac{dx}{dt}, -\frac{dy}{dt}\right]$$

and so

$$E_x = \frac{m}{2}\left(\frac{dx}{dt}\right)^2 \qquad\qquad E_y = \frac{m}{2}\left(\frac{dy}{dt}\right)^2$$

are independently conserved and the system is integrable. Solving for the free motion gives

$$x = x_0 \pm \left[\frac{2E_x}{m}\right]^{\frac{1}{2}}(t-t_0) \equiv x_0 + v_x(t-t_0) \qquad\qquad y = y_0 \pm \left[\frac{2E_y}{m}\right]^{\frac{1}{2}}(t-t_0) \equiv y_0 + v_y(t-t_0)$$

where we ignore the reflections and use periodic boundary conditions with a set of image boxes. The collisions occur when

$$x = n_x a \qquad or \qquad y = n_y a$$

For the x-collisions on the same edge we have

$$2n_x a = v_x \Delta t$$

and for the associated y-motion, which produces a periodic cycle, we have

$$2n_y a = v_y \Delta t$$

in order to arrive back at the same point in space and going in the same direction. The time for one oscillation in the x-motion is $\frac{2a}{v_x}$ and the distance that y travels along the edge between collisions is therefore $\frac{2a}{v_x}v_y$. We therefore generate a map

$$y_{n+1} = y_n + 2a\frac{v_y}{v_x}$$

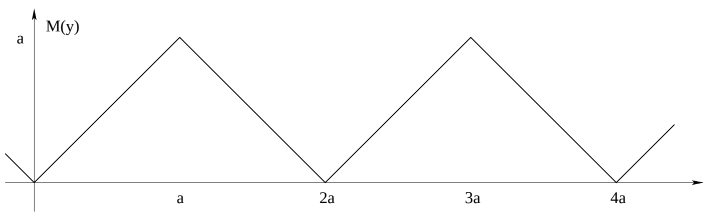
Now in order to have a cycle we need n_y to be an integer and then

$$\frac{v_y}{v_x} = \frac{n_y}{n_x}$$

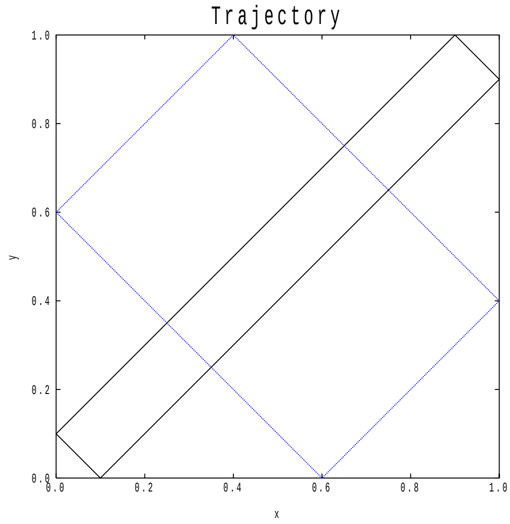
and the ratio of the two velocities is a rational number. If n_x and n_y have co-factors then we have repeated motion so we may assume that n_x and n_y are coprime. We then have that

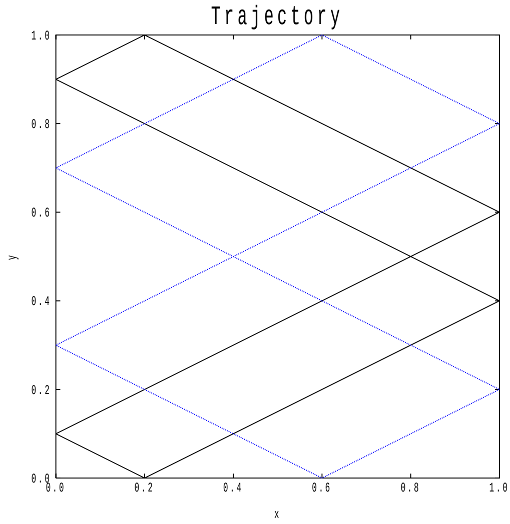
$$y_{n+1} = y_n + 2a\frac{n_y}{n_x}$$

which makes an n_x-cycle for the y-motion along the x-axis. To employ the reflections actively we need to map all the values of y_n down to lie between zero and a, which we do with the map

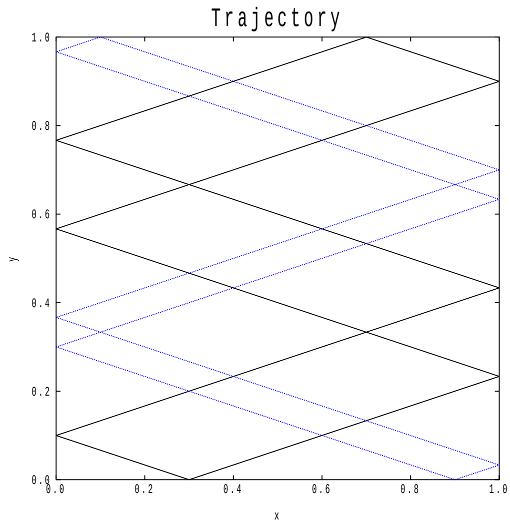


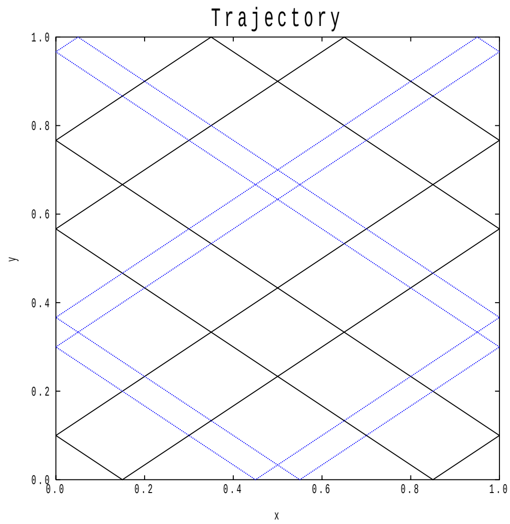
The y-collisions are analogous under the mapping $v_x \leftrightarrow v_y$. Simple examples depict as, n_x=1 and n_y=1, followed by, n_x=2 and n_y=1





followed by, n_x=3 and n_y=1 followed by, n_x=3 and n_y=2





When this velocity ratio is irrational we get an ergodic cover for the Poincare section.