You may assume the standard Lorentz Transformations between the inertial frame Σ and another inertial frame Σ' moving with velocity v in the x-direction. At $t = t' = 0$ the axes of Σ and Σ' coincide. This arrangement is called *standard configuration.*

$$\begin{array}{rcl} t' &=& \gamma(v)(t - vx/c^2) \\ x' &=& \gamma(v)(x - vt) \\ y' &=& y \\ z' &=& z \end{array} \quad \text{and} \quad \begin{array}{rcl} t &=& \gamma(v)(t' + vx'/c^2) \\ x &=& \gamma(v)(x' + vt') \\ y &=& y' \\ z &=& z' \end{array} \quad \text{with } \gamma(v) = \frac{1}{\sqrt{1 - v^2/c^2}}$$

You should take $c \simeq 3 \times 10^8\, m\, s^{-1}$ in any numerical calculations.

1. An event occurs at $x' = 60\,\text{m}$ and $t' = 8 \times 10^{-8}\,\text{s}$ in Σ'. Σ' has a velocity $3c/5$ relative to Σ. What are the space time coordinates of the event in Σ ?

2. The frame Σ' has a velocity $0.6c$ relative to Σ. An event E_1 occurs in Σ at $t = 2 \times 10^{-7}\,\text{s}$ at a point for which $x = 50\,\text{m}$ and a second event E_2 occurs in Σ at $t = 3 \times 10^{-7}\,\text{s}$ and $x = 10\,\text{m}$. What is the time interval between the events as measured in Σ'?

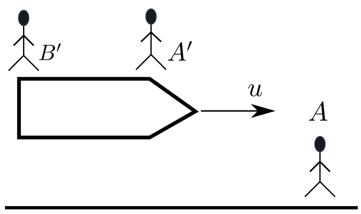
3. Verify that under a standard Lorentz Transformation

$$(ct')^2 - \mathbf{r}'^2 = (ct)^2 - \mathbf{r}^2$$

and check that the Galilean Transformation does **not** satisfy this invariance relation.

4. Frames of reference Σ and Σ' are in the usual standard configuration with respect to each other and Σ' moves to the right with velocity u along the common $x - x'$ axis. Two events E_1 and E_2 occur on the x axis of Σ, are separated in Σ by a distance L, and and observer at rest in Σ judges that event E_2 occurs a time τ *later* than event E_1. However in Σ', an observer judges that event E_2 occurs a time τ *earlier* than E_1. Show that the two frames of reference have a relative velocity $u = 2c^2 L\tau / \left(L^2 + c^2\tau^2\right)$

5. ★The diagram shows a rocket whose proper length is L_0 and which moves to the right with a velocity u relative to a frame of reference Σ.



The nose of the rocket can be taken as situated at the origin of an inertial frame of reference Σ' which moves to the right with velocity u along the common $x - x'$ axis. An observer A' at rest in the nose of the rocket passes an observer A who is at the origin of the frame of reference Σ and as they pass, their clocks are set to read 0. As A' passes A, the observer A' sends a flash of light to an observer B' sitting at the tail of the rocket.

(a) At what time does A' (who lives in the rocket) judge that the light flash reaches B'?

(b) At what time does the observer A judge that the light flash reaches the observer B' sitting at the tail of the rocket?

(c) At what time does A judge that the tail of the rocket passes A?

6. Two inertial frames of reference are in the usual standard configuration with respect to each other and Σ' moves to the right with velocity u with respect to Σ along the common $x - x'$ axis. An observer R at rest in Σ is situated on the $x-$ axis at a point with coordinate $x = a$ and an observer R' who is at rest in Σ' is situated at a point whose x' coordinate has the same numerical value $x' = a$ as does observer R in Σ. E_1 is the event that the origins of Σ and Σ' pass and E_2 is the event that R' passes R. Show that these events are separated by the time interval of magnitude
$$\left|\frac{a}{u}\left(1-\sqrt{1-\frac{u^2}{c^2}}\right)\right|$$
but that the order in which these events occur is different.

7. The positive muon μ^+ is an unstable particle with a half life of about $2.2 \times 10^{-6}\,s$ measured in the rest frame of the μ^+. In an experiment in which the μ^+ is caused to travel with a very high speed with respect to the laboratory, its lifetime in the laboratory is measured to be 1.9×10^{-5} s. Calculate the speed of the μ^+ as a fraction of the velocity of light. . What distance in the laboratory does the muon travel during its lifetime?

8. *This is purely an exercise in algebra.* $\gamma(u_1)$ and $\gamma(u_2)$ are the usual γ factors appropriate to speeds u_1 and u_2 respectively. Show that
$$\begin{aligned}\gamma(u_1)\,\gamma(u_2) &= \frac{1}{(1+u_1u_2/c^2)\,\sqrt{1-\left[(u_1+u_2)/(1+u_1u_2/c^2)\right]^2/c^2}}\\ &\equiv \frac{1}{(1+u_1u_2/c^2)}\gamma\left(\frac{u_1+u_2}{1+u_1u_2/c^2}\right).\end{aligned}$$

9. *You will need to use the expression for $\gamma(u_1)\,\gamma(u_2)$ derived in the previous question.*The standard Lorentz Transformations relating the space time coordinates in inertial frames Σ and Σ'can be written as
$$\begin{aligned}t' &= \gamma(u)\,\left(t - ux/c^2\right)\\ x' &= \gamma(u)\,(x-ut)\end{aligned}$$
An inertial frame of reference Σ_1 has a velocity u_1 relative to a frame Σ_0 and a frame Σ_2 has a velocity u_2 relative to Σ_1, show by making the appropriate substitutions for the space time coordinates that the effect of performing two successive Lorentz Transformations with velocities u_1 and u_2 is the same as performing a single Lorentz Transformation with a velocity $V = (u_1+u_2)/(1+u_1u_2/c^2)$. This result is the start of a proof that the set of Lorentz Transformations forms a *group.*

If you are familiar with matrix algebra you might like to show that the Lorentz Transformations can be written in the equivalent form
$$\begin{pmatrix} ct' \\ x' \end{pmatrix} = \gamma(u)\begin{pmatrix} 1 & -u/c \\ -u/c & 1 \end{pmatrix}\begin{pmatrix} ct \\ x \end{pmatrix}$$
You can now use the process of matrix multiplication to establish the previous result.

10. A source of light flashes at time intervals τ_0 in its own proper frame of reference. The source retreats from an observer with a velocity u which can be taken to be along the x-axis. *Use the Lorentz Transformations* to show that the observer receives these flashes at time intervals $\tau = \tau_0\sqrt{(1+u/c)\,/\,(1-u/c)}$. With what frequency does the observer receive these flashes?

    What answers would you obtain if the Galilean rather than the Lorentz Transformations were used? Comment on your answers.

11. Inertial frames of reference Σ and Σ' are in the usual standard configuration with respect to each other and Σ' moves to the right of Σ' with velocity u along the common $x - x'$ axis. A flash of light is emitted from a point with position x_1 on the x axis and is absorbed at a point $x_2 = x_1 + L$.

    (a) What does an observer in Σ' judge to be the spatial separation between the point of emission and the point of absorption of the flash?

    (b) What does an observer in Σ' judge to be the time interval between the emission and absorption of the flash of light.

1. $u/c = 3/5$ then $\gamma(u) = 1/\sqrt{1-u^2/c^2} = 1/\sqrt{1-9/25} = 5/4$. Then since the space time coordinates of the event E in Σ' are $x' = 10^{-8}$ s we use the (inverse) Lorentz Transformations in the form

$$x = \gamma(u)\left(x' + ut'\right) \text{ and } t = \gamma(u)\left(t' + ux'/c^2\right).$$

Inserting the numerical values gives

$$\begin{aligned}
x &= \frac{5}{4}\left(60 + \frac{u}{c}ct'\right) = \frac{5}{4}\left(60 + \frac{3}{5}c \times 8 \times 10^{-8}\right) \text{ using } c = 3\times 10 \text{ ms}^{-1} \text{ gives} \\
x &= \frac{5}{4}\left(60 + \frac{3}{5} \times 3 \times 8 \times 10^{8} \times 10^{-8}\right) \text{m} \\
x &= 93\text{m} \\
t &= \gamma(u)\left(t' + ux'/c^2\right) = \frac{5}{4}\left(8\times 10^{-8} + \frac{u}{c} \times \frac{60}{3\times 10^{8}}\right) = \frac{5}{4}\left(8\times 10^{-8} + \frac{3}{5} \times \frac{60}{3\times 10^{8}}\right) \\
t &= \frac{5}{4} \times 10^{-8}\left(8 + \frac{180}{15}\right) = \frac{5}{4} \times 10^{-8}(8+12) = 25 \times 10^{-8}\text{s}.
\end{aligned}$$

In Σ the event occurs at $x = 93$m and $t = 25 \times 10^{-8}$s.

2. $u/c = 3/5$ and so $\gamma(u) = 5/4$ again. For the event E_1 we are told that in frame Σ, $x_1 = 50$ m and $t_1 = 2 \times 10^{-7}$ s and for the event E_2 we have so $x_2 = 10$ m and $t_2 = 3 \times 10^{-7}$ s and so we use the Lorentz Transformation in the form

$$t' = \gamma(u)\left(t - ux/c^2\right) \to t_1' = \gamma(u)\left(t_1 - ux_1/c^2\right) \text{ and } t_2' = \gamma(u)\left(t_2 - ux_2/c^2\right).$$

Then

$$\begin{aligned}
t_1' &= \frac{5}{4}\left(2\times 10^{-7} - \frac{3}{5} \times \frac{50}{3\times 10^{8}}\right)\text{s} = \frac{5}{4}\left(2\times 10^{-7} - \frac{10}{10^{8}}\right)\text{s} = \frac{5}{4} \times 10^{-7}\text{s} \\
t_2' &= \frac{5}{4}\left(3\times 10^{-7} - \frac{3}{5} \times \frac{10}{3\times 10^{8}}\right)\text{s} = \frac{5}{4}\left(3\times 10^{-7} - \frac{1}{5} \times \frac{10}{10^{8}}\right)\text{s} = \frac{5}{4} \times (2.8) \times 10^{-7}\text{s}
\end{aligned}$$

The time interval in Σ' is thus $t_2' - t_1' = (5/4) \times (2.8 - 1) \times 10^{-7}\text{s} = 2.25 \times 10^{-7}$s.

3. Using the Lorentz transformation

$$\begin{aligned}
(ct')^2 - (x')^2 &= \gamma^2(u)\left((ct - ux/c)^2 - (x - ut)^2\right) \\
&= \gamma^2(u)\left((ct)^2 - 2uxt + u^2x^2/c^2 - x^2 + 2uxt - u^2t^2\right) \\
&= \gamma^2(u)\left((c^2 - u^2)t^2 - (1 - u^2/c^2)x^2\right)^2 = (ct)^2 - x^2.
\end{aligned}$$

Since $y = y'$ and $z = z'$ then

$$c^2t'^2 - \mathbf{r}'^2 = c^2t^2 - \mathbf{r}^2.$$

The Galilean transformation is $t' = t$ and $x' = x - ut$, $y\prime = y$ and $z\prime = z$ and so

$$c^2t'^2 - \mathbf{r}'^2 = c^2t^2 - (x - ut)^2 - y^2 - z^2 = c^2t^2 - x^2 - y^2 - z^2 + 2uxt - u^2t^2.$$

The right hand side is not $c^2t^2 - x^2 - y^2 - z^2 = c^2t^2 - \mathbf{r}^2$ and has additional terms added in i.e. under a Galilean Transformation $c^2t'^2 - \mathbf{r}'^2 \neq c^2t^2 - \mathbf{r}^2$.

4. Event E_1 has coordinates (x_1, t_1) in Σ and (x'_1, t'_1) in Σ'. Event E_2 has coordinates (x_2, t_2) in Σ and (x'_2, t'_2) in Σ'. Coordinates in the two frames are related by the standard Lorentz Transformation

$$x' = \gamma(u)\,(x - ut) \text{ and } t' = \gamma(u)\left(t - ux/c^2\right)$$

In Σ event E_2 occurs a time τ later than t_1 and so $t_2 = t_1 + \tau$ and we are told that $x_2 = x_1 + L$. Thus since $t'_1 = \gamma(u)\left(t_1 - ux_1/c^2\right)$ and $t'_2 = \gamma(u)\left(c - ux_2/c^2\right)$ we have

$$t'_2 - t'_1 = \gamma(u)\,(t_2 - t_1 - u(x_2 - x_1)\,/c^2) = \gamma(u)\,(\tau - uL)\,/c^2).$$

But we are told that event E_2 in Σ' occurs at a time τ earlier than event E_1 and so $t'_2 = t'_1 - \tau$. Thus

$$\begin{aligned} -\tau &= \gamma(u)\,(\tau - uL)\,/c^2) \to \tau(\gamma(u) + 1) = \gamma(u)uL/c^2 \text{ or} \\ \tau\left(1 + \frac{1}{\gamma(u)}\right) &= \frac{uL}{c^2} = \tau\left(1 + \sqrt{1 - u^2/c^2}\right) \end{aligned}$$

Hence

$$\sqrt{1 - u^2/c^2} = \frac{uL}{\tau c^2} - 1 \to 1 - u^2/c^2 = \frac{u^2L^2}{\tau^2c^4} - \frac{2uL}{\tau c^2} + 1.$$

Thus

$$u^2\left(\frac{1}{c^2} + \frac{L^2}{\tau^2c^4}\right) = \frac{2uL}{\tau c^2}$$

and either $u = 0$ (impossible) or

$$u = \frac{2L\tau c^2}{c^2\tau^2 + L^2}.$$

5. (a) Light travels with the same speed c in both Σ' and Σ and so in Σ' the flash leaves A' at $t' = 0$ and travels the proper distance L_0 to the tail of the rocket which it does in a time L_0/c.

   (b) The flash is judged in Σ' to leave the origin $x' = 0$ at $t' = 0$ and reach the point in Σ' with coordinate $x' = -L_0$ at a time $t' = L_0/c$. To obtain the time coordinate t of the event that the flash reaches the tail we use the inverse Lorentz Transformation

$$t = \gamma(u)\left(t' + ux'/c^2\right)$$

with $t' = L_0/c$ and $x' = -L_0$ and this gives

$$\begin{aligned} t &= \gamma(u)\left(\frac{L_0}{c} + \frac{u\,(-L_0)}{c^2}\right) \\ &= \gamma(u)\frac{L_0}{c}\left(1 - \frac{u}{c}\right) \\ &= \frac{L_0}{c}\frac{(1 - u/c)}{\sqrt{1 - u^2/c^2}} \\ &= \frac{L_0}{c}\left(\frac{1 - u/c)}{1 + u/c}\right)^{1/2} \end{aligned}$$

(c) In Σ, the length of the rocket is Lorentz contracted to a value $L_0\sqrt{1-u^2/c^2} = L_0/\gamma(u)$ and the rocket has a velocity u to the right and so the tail of the rocket crosses A at a time $\left(L_0\sqrt{1-u^2/c^2}\right)/u = L_0/u\gamma(u)$

6. E_1 is the event of the origins coinciding at $t = 0 = t'$ and E_2 is the event that R and R' pass. In Σ' this occurs at a time we call $t'_{RR'}$ at a place $x' = a$. In Σ this occurs at a time we call $t_{RR'}$ at a place $x = a$ (i.e. the spatial coordinates happen to be the same in each frame). Thus E_2 has coordinates $\left(t'_{RR'}, x' = a\right)$ in Σ' but $(t_{RR'}, x = a)$ in Σ. The Lorentz Transformations can be used in the form $x' = \gamma(u)\,(x - ut)$. When $x = a$ and $t = t_{RR'}$ we are told that $x' = a$ and so

$$\begin{aligned} x' &= a = \gamma(u)\,(a - ut_{Rr'}) \rightarrow \frac{a}{\gamma(u)} = a - ut_{RR'} \\ t_{RR'} &= \frac{a}{u}\left(1 - \frac{1}{\gamma(u)}\right) = \frac{a}{u}\left(1 - \sqrt{1 - \frac{u^2}{c^2}}\right) \end{aligned}$$

and this is the time of event E_2 as judged in the frame Σ. However if we use the inverse Lorentz Transformation $x = \gamma(u)\left(x' + ut'_{AA'}\right)$ then in Σ' we are told that event E_2 has $x' = a$ but that $x = a$. Substituting gives

$$\begin{aligned} x &= a = \gamma(u)\left(x' + ut'_{RR'}\right) \rightarrow a = \gamma(u)\left(a + ut'_{RR'}\right) \\ \gamma(u)ut'_{Rr'} &= a\,(1 - \gamma(u)) \rightarrow t'_{RR''} = \frac{a}{u}\left(\frac{1}{\gamma(u)} - 1\right) = \frac{a}{u}\left(\sqrt{1 - \frac{u^2}{c^2}} - 1\right) = -\frac{a}{u}\left(1 - \sqrt{1 - \frac{u^2}{c^2}}\right). \end{aligned}$$

Since the frames are in standard configuration with $t = t' = 0$ as the origins pass (event E_1), $t_{RR'}$ will be the time interval between E_1 and E_2 as measured in Σ and $t'_{RR'}$will be the time interval between E_1 and E_2 as measured in Σ'. Thus

$$t'_{RR'} = -t_{RR'}.$$

Thus the two events occur in different orders. Since $u/c < 1$, $t_{RR'} > 0$ and $t'_{RR'} < 0$. In Σ, event E_2 occurs later than event E_1 (since $t_{RR'} > 0$) whereas in Σ' event E_2 occurs earlier than event E_1 (since $t'_{RR'} < 0$).

7. Use time dilation and the standard result

$$\tau = \gamma(u)\tau_0 = \frac{\tau_0}{\sqrt{1 - u^2/c^2}}.$$

Take $\tau_0 = 2.2\times10^{-6}$ s to be the lifetime in the rest frame of the muon and $\tau = 1.9\times10^{-5}$ s to be the lifetime in the frame of reference of the laboratory. Then

$$\sqrt{1 - u^2/c^2} = \frac{\tau_0}{\tau} = \frac{2.2\times10^{-6}}{1.9\times10^{-5}} = 0.158 \text{ and } 1 - u^2/c^2 = 0.01341 \rightarrow \frac{u}{c} \simeq 0.993.$$

A laboratory dweller judges that the muon travels with a speed $u = 0.993c$ for a time $\tau = 1.9\times10^{-5}$ s and so the distance travelled by the muon in the laboratory frame is

$$u\tau = 0.993c \times 1.9\times10^{-5}\text{m} \simeq 5.7\times10^3\text{ m}.$$

8.
$$\gamma(u_1)\gamma(u_2) = \frac{1}{\sqrt{1-u_1^2/c^2}} \times \frac{1}{\sqrt{1-u_2^2/c^2}} = \frac{1}{\sqrt{1-u_1^2/c^2-u_2^2/c^2+u_1^2u_2^2/c^4}}$$

But

$$\begin{aligned}
(\frac{u_1}{c}+\frac{u_2}{c})^2 &= \frac{u_1^2}{c^2}+\frac{u_2^2}{c^2}+\frac{2u_1u_2}{c^2} \text{ and so} \\
1-\frac{u_1^2}{c^2}-\frac{u_2^2}{c^2}+\frac{u_1^2u_2^2}{c^4} &= 1-\left[(\frac{u_1}{c}+\frac{u_2}{c})^2-\frac{2u_1u_2}{c^2}\right]+\frac{u_1^2u_2^2}{c^4} \\
&= 1-(\frac{u_1}{c}+\frac{u_2}{c})^2+\frac{2u_1u_2}{c^2}+\frac{u_1^2u_2^2}{c^4}.
\end{aligned}$$

However

$$\left(1+\frac{u_1u_2}{c^2}\right)^2 = 1+\frac{2u_1u_2}{c^2}+\frac{u_1^2u_2^2}{c^4} \to \frac{2u_1u_2}{c^2}+\frac{u_1^2u_2^2}{c^4} = \left(1+\frac{u_1u_2}{c^2}\right)^2-1$$

Substitute this into the previous expression and obtain

$$\begin{aligned}
1-(\frac{u_1}{c}+\frac{u_2}{c})^2+\frac{2u_1u_2}{c^2}+\frac{u_1^2u_2^2}{c^4} &= 1-(\frac{u_1}{c}+\frac{u_2}{c})^2+\left(1+\frac{u_1u_2}{c^2}\right)^2-1 \\
&= \left(1+\frac{u_1u_2}{c^2}\right)^2-(\frac{u_1}{c}+\frac{u_2}{c})^2 \\
&= \left(1+\frac{u_1u_2}{c^2}\right)^2\left[1-\frac{1}{c^2}\left(\frac{u_1+u_2}{1+\frac{u_1u_2}{c^2}}\right)^2\right].
\end{aligned}$$

Thus

$$\begin{aligned}
\gamma(u_1)\gamma(u_2) &= \frac{1}{\sqrt{1-u_1^2/c^2-u_2^2/c^2+u_1^2u_2^2/c^4}} = \frac{1}{\sqrt{(1+u_1u_2/c^2)^2}\sqrt{\left[1-\frac{1}{c^2}\left(\frac{u_1+u_2}{1+u_1u_2/c^2}\right)^2\right]}} \\
&= \frac{1}{(1+u_1u_2/c^2)}\frac{1}{\sqrt{1-\frac{1}{c^2}\left(\frac{u_1+u_2}{1+u_1u_2/c^2}\right)^2}} \\
&= \frac{1}{(1+u_1u_2/c^2)}\gamma\left(\frac{u_1+u_2}{1+u_1u_2/c^2}\right)
\end{aligned}$$

as required.

9. Σ'' has a velocity u_2 relative to Σ' and Σ' has a velocity u_1 relative to Σ. Coordinates (x'', t'') of an event E in Σ'' are related to the coordinates (x', t') of the same event in Σ' by a standard Lorentz Transformation

$$t'' = \gamma(u_2)\left(t'-u_2x'/c^2\right) \text{ and } x'' = \gamma(u_2)\left(x'-u_2t'\right).$$

Likewise, coordinates (x', t') of an event E in Σ' are related to the coordinates (x, t) of the *same* event in Σ by a standard Lorentz Transformation

$$t' = \gamma(u_1)\left(t-u_1x/c^2\right) \text{ and } x' = \gamma(u_1)\left(x-u_1t\right).$$

All we need do is substitute the second set of equations into the first. Then

$$\begin{aligned}
t'' &= \gamma(u_2)\left(t' - u_2 x'/c^2\right) = \gamma(u_2)\left(\gamma(u_1)\left(t - u_1 x/c^2\right) - u_2\gamma(u_1)\left(x - u_1 t\right)/c^2\right) \\
&= \gamma(u_2)\gamma(u_1)\left(t - u_1 x/c^2 - u_2 x/c^2 + u_2 u_1 t/c^2\right) \\
&= \gamma(u_2)\gamma(u_1)\left[t\left(1 + u_1 u_2/c^2\right) - (u_1 + u_2)x/c^2\right] \\
&= \gamma(u_2)\gamma(u_1)\left(1 + u_1 u_2/c^2\right)\left[t - \frac{(u_1 + u_2)}{(1 + u_1 u_2/c^2)}\frac{x}{c^2}\right].
\end{aligned}$$

From the results of the previous question

$$\gamma(u_2)\gamma(u_1)\left(1 + u_1 u_2/c^2\right) = \gamma\left(\frac{u_1 + u_2}{1 + \frac{u_1 u_2}{c^2}}\right)$$

and hence

$$t'' = \gamma\left(\frac{u_1 + u_2}{1 + u_1 u_2/c^2}\right)\left[t - \frac{(u_1 + u_2)}{(1 + u_1 u_2/c^2)}\frac{x}{c^2}\right].$$

Likewise

$$\begin{aligned}
x'' &= \gamma(u_2)\left(x' - u_2 t'\right) = \gamma(u_2)\left(\gamma(u_1)\left(x - u_1 t\right) - u_2\gamma(u_1)\left(t - u_1 x/c^2\right)\right) \\
&= \gamma(u_2)\gamma(u_1)\left[x - u_1 t - u_2 t + u_1 u_2 x/c^2\right] \\
&= \gamma(u_2)\gamma(u_1)\left[x(1 + u_1 u_2/c^2) - (u_1 + u_2)t\right] \\
&= \gamma(u_2)\gamma(u_1)(1 + u_1 u_2/c^2)\left[x - \frac{(u_1 + u_2)}{(1 + u_1 u_2/c^2)}t\right] \\
&= \gamma\left(\frac{u_1 + u_2}{1 + u_1 u_2/c^2}\right)\left[x - \frac{(u_1 + u_2)}{(1 + u_1 u_2/c^2)}t\right]
\end{aligned}$$

If we write $V = (u_1 + u_2)/(1 + u_1 u_2/c^2)$ then we see that we can write down these two results as

$$t'' = \gamma(V)\left(t - Vx/c^2\right) \text{ and } x'' = \gamma(V)\left(x - Vt\right).$$

This pair of equations is clearly a Lorentz Transformation with

$$V = \frac{(u_1 + u_2)}{(1 + u_1 u_2/c^2)}$$

so that the effect of performing two successive Lorentz Transformations, one with velocity u_2 followed by a second Lorentz Transformation with velocity u_1 is to produce another Lorentz Transformation with velocity $V = (u_1 + u_2)/(1 + u_1 u_2/c^2)$. This result is the start of a proof that the set of Lorentz Transformation form a mathematical structure called a *group. This result also of course establishes the relativistic composition law for velocities.*

A more elegant (but equivalent) way of writing this out is to note that we can write the Lorentz Transformations out in the form

$$\begin{aligned}
ct'' &= \gamma(u_2)\left(ct' - (u_2/c)x'\right) \text{ and } x'' = \gamma(u_2)\left(x' - (u_2/c)ct'\right) \\
ct' &= \gamma(u_1)\left(ct - (u_1/c)x\right) \text{ and } x' = \gamma(u_1)\left(x - (u_1/c)ct\right).
\end{aligned}$$

In this form they are symmetric in (ct) which has the dimensions of length and x which also of course has the dimensions of length. u_2/c is then dimensionless. The first pair can then be written in matrix form

$$\begin{pmatrix} ct'' \\ x'' \end{pmatrix} = \gamma(u_2)\begin{pmatrix} 1 & -u_2/c \\ -u_2/c & 1 \end{pmatrix}\begin{pmatrix} ct' \\ x' \end{pmatrix}$$

and the second pair as

$$\begin{pmatrix} ct' \\ x' \end{pmatrix} = \gamma(u_1)\begin{pmatrix} 1 & -u_1/c \\ -u_1/c & 1 \end{pmatrix}\begin{pmatrix} ct \\ x \end{pmatrix}.$$

Substituting the second in the first then gives

$$\begin{pmatrix} ct'' \\ x'' \end{pmatrix} = \gamma(u_2)\gamma(u_1)\begin{pmatrix} 1 & -u_2/c \\ -u_2/c & 1 \end{pmatrix}\begin{pmatrix} 1 & -u_1/c \\ -u_1/c & 1 \end{pmatrix}\begin{pmatrix} ct \\ x \end{pmatrix}.$$

Performing the matrix multiplication and using the result of the previous question then gives

$$\begin{pmatrix} ct'' \\ x'' \end{pmatrix} = \gamma(V)\begin{pmatrix} 1 & -V/c \\ -V/c & 1 \end{pmatrix}\begin{pmatrix} ct \\ x \end{pmatrix} \text{ with } V = \frac{(u_1+u_2)}{(1+u_1u_2/c^2)}.$$

If you are familiar with matrix multiplication, try this for yourself.

10. For convenience, put the source of the light flashes at rest at the origin of Σ' which moves to the right along the common $x - x'$ axis with a velocity u as determined from Σ. The source is at rest in Σ' and so Σ' is a proper frame of reference. The source of the flashes is at rest in Σ' at the point $x' = 0$. We will take Σ and Σ' to be in standard configuration and whose origins $x = 0 = x'$ cross at times $t = 0 = t'$.Consider the following two events E_1 and E_2.

- E_1 is the emission of the first flash of light which occurs at some point (x_1, t_1) in Σ and at $(x'_1 = 0, t'_1)$ in Σ'. This first flash which is emitted at a time t_1 in Σ, travels back to the origin $x = 0$ of Σ and (since light travels at a speed c) reaches the origin of Σ after a time interval x_1/c and so a clock at the origin of Σ will read a time $T_1 = t_1 + x_1/c$ when the first flash reaches the origin.
- E_2 is the emission of a second flash which occurs at some point (x_2, t_2) in Σ and at $(x' = 0, t'_2)$ in Σ'. In the frame of reference Σ', the source is at rest, and so the second flash is emitted at a time τ_0 after the first as judged by an observer at rest in Σ'. This time interval τ_0 is thus a proper time interval and $t'_2 = t'_1 + \tau_0$. The second flash travels from the point x_2 in Σ and so takes a time x_2/c to reach the origin of Σ and does so when the clock at the origin of Σ reads $T_2 = t_2 + x_2/c$.

An observer at the origin of Σ thus received two flashes of light separated by a time interval

$$T_2 - T_1 = t_2 - t_1 + (x_2 - x_1)/c.$$

But we can use the (inverse) Lorentz Transformations to relate(x, t) to (x', t'). Thus

$$\begin{aligned} t &= \gamma(u)(t' + ux'/c^2) \text{ and } x = \gamma(u)(x' + ut') \text{ and} \\ t_1 &= \gamma(u)(t'_1 + ux'_1/c^2) \text{ and } x_1 = \gamma(u)(x'_1 + ut'_1) \\ t_2 &= \gamma(u)(t'_2 + ux'_2/c^2) \text{ and } x_2 = \gamma(u)(x'_2 + ut'_2). \end{aligned}$$

Subtracting these gives

$$T_2 - T_1 = t_2 - t_1 + \frac{(x_2 - x_1)}{c} = \gamma(u)\left[t_2' - t_1' + \frac{u}{c^2}\left(x_2' - x_1'\right) + \frac{1}{c}\left(x_2' - x_1' + u(t_2' - t_1')\right)\right]$$

However, for convenience we placed the light source at th origin of Σ' and so $x_2' = 0 = x_1'$ and

$$T_2 - T_1 = \gamma(u)\left[t_2' - t_1' + +\frac{u}{c}(t_2' - t_1')\right] = \gamma(u)\left(t_2' - t_1'\right)\left(1 + \frac{u}{c}\right).$$

But $t_2' - t_1' = \tau_0$ and so the time interval which elapses between the observer at the origin of Σ receiving the two flashes is $T_2 - T_1 = \tau$ and

$$\begin{aligned} T_2 - T_1 &= \tau = \tau_0\gamma(u)\left(1 + \frac{u}{c}\right) = \tau_0\frac{(1 + u/c)}{\sqrt{1 - u^2/c^2}} = \tau_0\frac{(1 + u/c)}{\sqrt{(1 - u/c)\,(1 + u/c)}} \\ &= \tau_0\sqrt{\frac{(1 + u/c)}{(1 - u/c)}} \text{ as required.} \end{aligned}$$

We see that for two observers retreating with velocity u we have

$$\tau = \tau_0\, k(u) \text{ where } k(u) = \sqrt{\frac{(1 + u/c)}{(1 - u/c)}}$$

as expected. The **frequency** ν with which an observer at the origin of Σ receives these flashes is given by

$$\nu = \frac{1}{\tau} = \frac{1}{\tau_0\, k(u)} = \frac{\nu_0}{k(u)} = \nu_0\sqrt{\frac{(1 - u/c)}{(1 + u/c)}}$$

where $\nu_0 = 1/\tau_0$ is the frequency with which the source emits flashes. If you think of these flashes as corresponding to successive rests of a light wave then the result we have is the relativistic longitudinal Doppler Shift.

In a Galilean /Newtonian world $t' = t$ and $x' = x - ut$ and $t = t'$ and $x = x' + ut = x' + ut'$, but the light still travels with speed c and so

$$(T_2 - T_1)_{Gal} = \left[t_2 - t_1 + \frac{(x_2 - x_1)}{c}\right]_{Gal} = t_2' - t_1' + \frac{1}{c}\left(x_2' - x_1' + u\left(t_2' - t_1'\right)\right).$$

We still have the source at the origin of Σ' and so $x_2' = 0 = x_1'$.Thus

$$(T_2 - T_1)_{Gal} = \tau_{Gal} = t_2' - t_1' + \frac{u}{c}\left(t_2' - t_1'\right) = (t_2' - t_1')(1 + \frac{u}{c})$$

and so since $(t_2' - t_1') = \tau_0$ we have

$$\tau_{Gal} = \tau_0(1 + \frac{u}{c}).$$

Notice that this is the result we would have obtained from relativity by writing $\gamma(u) = 1$. The frequencies $\nu_{Gal} = 1/\tau_{Gal}$ and $\nu_0 = 1/\tau_0$ are thus related by

$$\nu_{Gal} = \frac{\nu_0}{(1 + u/c)}.$$

This is the classical non relativistic Doppler shift for the case where the source moves and retreats from a stationary observer.

11. (a) The light is emitted at (x_1, t_1) in Σ and absorbed at (x_2, t_2) in Σ and we are told that $x_2 = x_1 + L$. As judged from Σ', the light is emitted at (x_1', t_1') and absorbed at (x_2', t_2'). We use the Lorentz Transformations in the form

$$x' = \gamma(u)\,(x - ut) \text{ and } t' = \gamma(u)(t - ux/c^2)$$

and so

$$\begin{aligned} x_1' &= \gamma(u)\,(x_1 - ut_1) \text{ and } x_2' = \gamma(u)\,(x_2 - ut_2) \\ t_1' &= \gamma(u)(t_1 - ux_1/c^2) \text{ and } t_2' = \gamma(u)(t_2 - ux_2/c^2). \end{aligned}$$

Thus

$$x_2' - x_1' = \gamma(u)\,(x_2 - x_1 - u(t_2 - t_1)) = \gamma(u)\,(L - u(t_2 - t_1))\,.$$

But the light flash has a speed c in both Σ and Σ' and so the time taken for light to travel from x_1 to x_2 as judged by an observer in Σ is $(t_2 - t_1) = L/c$. Hence

$$x_2' - x_1' = \gamma(u)L\,(1 - u/c) = L\gamma(u)\,(1 - u/c) = L\frac{(1 - u/c)}{\sqrt{1 - u^2/c^2}} = L\sqrt{\frac{1 - u/c}{1 + u/c}}$$

and this is the spatial separation of the points of emission and absorption as judged from Σ'.

(b) An observer in Σ' will judge there to be a time separation between emission and absorption which is

$$t_2' - t_1' = \gamma(u)\left(t_2 - t_1 - u(x_2 - x_1)/c^2\right)$$

and since once more $(t_2 - t_1) = L/c$ we see that

$$t_2' - t_1' = \gamma(u)\left(L/c - uL/c^2\right) = \gamma(u)\frac{L}{c}\,(1 - u/c) = \frac{L}{c}\sqrt{\frac{1 - u/c}{1 + u/c}}.$$

as expected since the light must travel a distance $x_2' - x_1'$ in Σ' with speed c.