University of Birmingham
School of Mathematics

RA — Real Annalysis — Autumn 2024

Problem Sheet 3 - self assessment

issued Week 6

Questions

Q1. Find the derivatives of following functions according to the definition, where they exist.

(a) $f(x) = x^2$.
(b) $f(x) = e^x$.

Q2. Assume that $f(x)$ is an even function and differentiable at $x = 0$. Show that $f'(0) = 0$.

Q3. For each $n \in \{0, 1, 2\}$, define the function $f_n : \mathbb{R} \to \mathbb{R}$ by

$$f_n(x) = \begin{cases} x^n \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

These functions are all differentiable at all points x in $\mathbb{R} \setminus \{0\}$, and hence continuous there. Decide whether these functions are continuous or differentiable at 0.

Q4. (a) If $f(x) = x/\sin x$, find the exact value of $f'(\pi/3)$
(b) If $y = \sqrt{1 + \sqrt{x}}$, find $\frac{dy}{dx}$.
Note that $f'(\pi/3)$ means to find the $f'(x)$ first and substitute in $\pi/3$ for x. It is not the derivative of $f(\pi/3)$, which is 0.

Q5. (a) Find an expression (by implicit differentiation) for the derivative at the point (x, y) on the ellipse $x^2/3 + y^2/6 = 1$. Hence find the gradients of the tangent lines when $x = 1/4$.
(b) Differentiate $x^{\cos x}$ with respect to x.

Q6. Prove that

$$(\sin x)^{(n)} = \sin\left(x + \frac{n\pi}{2}\right), \quad (\cos x)^{(n)} = \cos\left(x + \frac{n\pi}{2}\right)$$

Hint - use mathematical induction.

Q7. Find the derivatives of the following functions.

(1) $y = \dfrac{x^2+4x+3}{\sqrt{x}}$

(2) $g(x) = (x^2+1)^3(x^2+2)^6$

(3) $B(u) = (u^3+1)(2u^2-u-6)$

(4) $g(t) = (t+1)^{\frac{2}{3}}(2t^2-1)^3$

(5) $y = \dfrac{1}{t^3-2t^2+1}$

(6) $f(x) = \sqrt{\dfrac{1+\sin x}{1+\cos x}}$

(7) $y = \dfrac{x}{x+\frac{2}{x}}$

(8) $f(x) = \sqrt{x+\sqrt{x+\sqrt{x}}}$

(9) $y = \dfrac{t\sin t}{1+t}$

(10) $y = [x+(x+\sin^2 x)^3]^4$

(11) $y = x\sin x\tan x$

(12) $y = \tan(\sec(\cos x))$

Q8. Use logarithmic differentiation to find the derivatives of the following curves $y = f(x)$:

(a) $y = \sqrt{x}e^{x^2-x}(x+1)^{2/3}$,

(b) $y = x^x$,

(c) $y = \sin(x^x)$,

(d) $y = x^{\sin x}$,

(e) $y = (\sin x)^{\ln x}$,

(f) $y = (\ln x)^{\sin x}$.

Q9. Find y' if $x^y = y^x$.

Q10. Suppose $f : \mathbb{R} \to \mathbb{R}$ is a function such that $|f(x) - f(y)| \le |x-y|^\alpha$, for all $x, y \in \mathbb{R}$ with $\alpha > 1$. Show that $f(x) = C$ for some constant C. Hint: Show that f is differentiable at all points and compute the derivative.

Q11. Let $f : (a,b) \to \mathbb{R}$ be an unbounded differentiable function. Show that $f' : (a,b) \to \mathbb{R}$ is unbounded.

Q12. Using L'Hôpital's rule, or otherwise, prove that the function $f : (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$ given by

$$f(x) = \begin{cases} \frac{\tan x - x}{x^2}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

is differentiable at $x_0 = 0$ and state $f'(0)$. Is f continuous at $x_0 = 0$? Justify your answer.

Q13. Determine the following limits:

(a) $\displaystyle\lim_{x\to -1} \frac{x^2-1}{\sin(1+x)}$;

(b) $\displaystyle\lim_{x\to 0} \frac{k^x-1}{x}$, where $k > 0$;

(c) $\displaystyle\lim_{x\to 0} \frac{x\sin x}{1-\cos x}$;

(d) $\displaystyle\lim_{x\to 0} \frac{e^x-1-x}{x^2}$.

Q14. Find $\frac{dy}{dx}$ by implicit differentiation.

(a) $x^2 - 4xy + y^2 = 4$.

(b) $\cos(xy) = x + \sin y$.

(c) $\tan\left(\frac{x}{y}\right) = x + y$.

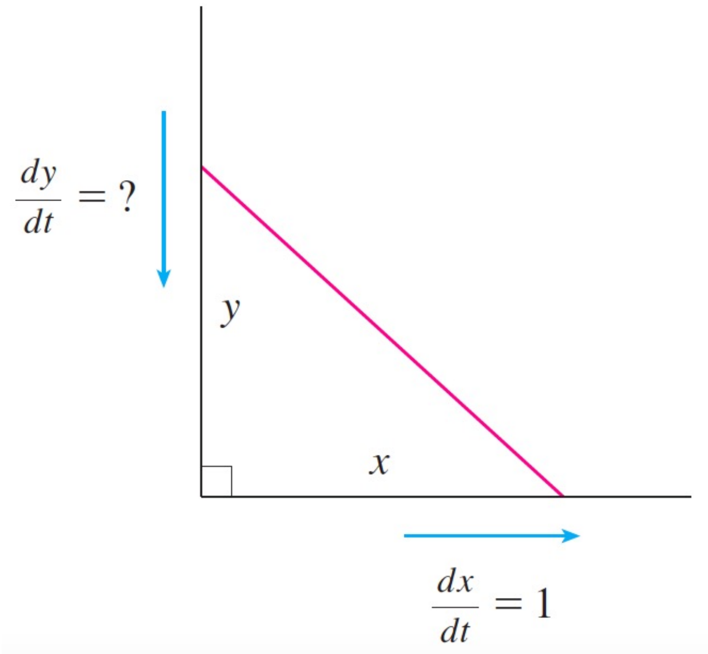
Q15. Show by implicit differentiation that the tangent to the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

at the point (x_0, y_0) is

$$\frac{x_0 x}{a^2} + \frac{y_0 y}{b^2} = 1\,.$$

Q16. A ladder 10 m long rests against a vertical wall. If the bottom of the ladder slides away from the wall at a rate of 1 metre per second (m/s), how fast is the top of the ladder sliding down the wall when the bottom of the ladder is 6 m from the wall?



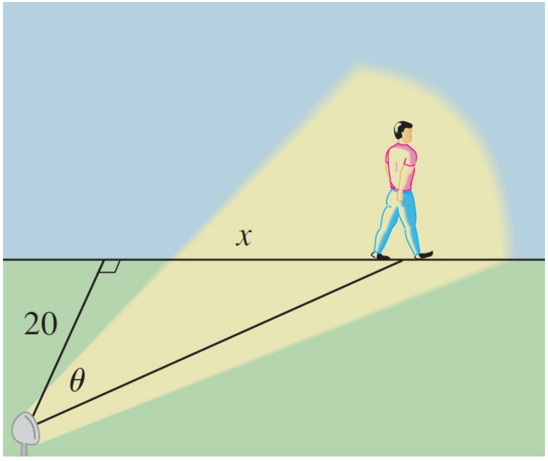
Q17. A man walks along a straight path at a speed of 4 m/s. A searchlight is located on the ground 20 m from the path and is kept focused on the man. At what rate is the searchlight rotating when the man is 15 m from the point on the path closest to the searchlight?

Q18. A wire of length L is cut into two pieces. One piece is shaped into a circle and the other is shaped into a square. Let A_C be the area contained within the circle and A_S be the area contained within the square. What are the maximum and minimum values of $A_C + A_S$?

Q19. Suppose $f : [a, b] \to \mathbb{R}$ is differentiable and $c \in (a, b)$. Show there exists a sequence $\{x_n\}$ converging to c, with $x_n \neq c\ \forall n \in \mathbb{N}$, and such that

$$f'(c) = \lim_{n\to\infty} f'(x_n)\,.$$

Moreover, explain why this does not imply that f' is continuous.



Q20. Assume that $f(x)$ is bounded on $[a, \infty)$ for some $a \in \mathbb{R}$, f is differentiable on (a, ∞) and

$$\lim_{x\to\infty} f'(x) = b\,.$$

Prove that $b = 0$.

Q21. Use L'Hôpital's rule to find the following limits, when it applies:

(a) $\displaystyle\lim_{x\to\infty} \frac{\ln x}{\sqrt{x}}$.

(b) $\displaystyle\lim_{x\to 0+} \frac{\ln x}{x}$

(c) $\displaystyle\lim_{x\to 1} \frac{x^8 - 1}{x^5 - 1}$

(d) $\displaystyle\lim_{x\to 0} \left(\frac{1}{x} - \frac{1}{e^x - 1}\right)$

(e) $\displaystyle\lim_{x\to 0+} (\tan 2x)^x$

(f) $\displaystyle\lim_{x\to\infty} \left(\frac{a^{1/x} + b^{1/x}}{2}\right)^x$ for $a, b > 0$

Q22. Find the limit

$$\lim_{x\to\infty} \frac{(x+2)^{\frac{1}{x}} - x^{\frac{1}{x}}}{(x+3)^{\frac{1}{x}} - x^{\frac{1}{x}}}.$$

Q23. Prove that

$$\ln(1+x) < \frac{x}{\sqrt{1+x}}$$

for all $x > 0$.

Q24. Use Taylor's Theorem to approximate the following functions $f : Dom(f) \to \mathbb{R}$ about points $x \in Dom(f)$. Moreover, state an appropriate error term for your approximation in each case.

(a) $f(x) = \tan x$ about $x = 0$, accurate to order 3 terms.
(b) $f(x) = e^x$ about $x = 0$, accurate to order 4 terms.
(c) $f(x) = \ln x$ about $x = 1$, accurate to order 4 terms.
(d) $f(x) = \cos x - 1$ about $x = 2\pi$, accurate to order 4 terms.

Q25. Determine the types of stationary points for $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = x^4 - 6x^2 + 8x + 1 \quad \forall x \in \mathbb{R}.$$

Q26. Sketch the curve

$$y = \frac{2x^2}{x^2 - 1}\,.$$

Q27. Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by $f(x) = x^2 e^x$ for all $x \in \mathbb{R}$.

(i) Calculate f' and f''.
(ii) Find and determine the nature of the stationary points of f.
(iii) Find the points of inflection of f.
(iv) Determine the regions in which f is strictly increasing and decreasing.
(v) Determine the regions in which f is concave up and concave down.
(vi) Determine all the asymptotes of f.
(vii) Sketch the graph of f.