University of Birmingham
School of Mathematics

1RA - Real Analysis: Differentiation Autumn 2024

Practice Problem Sheet 1

Model Solutions

Questions

Q1. Let $f : (-\infty, \alpha) \to \mathbb{R}$ for some $\alpha \in \mathbb{R}$. Define what is meant by $\lim_{x\to-\infty} f(x) = A$ for $A \in \mathbb{R}$. Prove $\lim_{x\to-\infty} \frac{1}{x} = 0$ by using this definition.

Solution. Suppose that the domain of a real function f contains $(-\infty, \alpha)$ for some $\alpha \in \mathbb{R}$ and $A \in \mathbb{R}$. If for all $\varepsilon > 0$, there exists $K < \alpha$, such that

$$|f(x) - A| < \varepsilon \quad \forall\, x < K,$$

then we say the limit of f, as x tends to $-\infty$, is A. In this case we write $f(x) \to A$ as $x \to -\infty$, or

$$\lim_{x\to-\infty} f(x) = A.$$

Now we prove the limit $\lim_{x\to-\infty} \frac{1}{x} = 0$. We first note that the function $f(x) = 1/x$ is defined on $(-\infty, 0)$. For any $\varepsilon > 0$, let $K = -\frac{1}{\varepsilon}$. Then we have

$$|1/x - 0| = 1/|x| < 1/|K| = \varepsilon$$

for all $x < -K$ (or $|x| > |K|$). Thus, we prove the limit $\lim_{x\to-\infty} \frac{1}{x} = 0$. □

Q2. Determine the limit $\lim_{x\to-3} 3x$ and prove that your answer is correct by directly appealing to the definition of the limit.

Solution. The limit is -9, i.e. $\lim_{x\to-3} 3x = -9$.

Now, we prove it. Given $\varepsilon > 0$, we take $\delta = \frac{\varepsilon}{3}$. Then for all $0 < |x + 3| < \delta$, we have

$$|3x - (-9)| = 3|x + 3| < 3\delta = \varepsilon.$$

Therefore, $|3x - (-9)| < \varepsilon$ whenever $0 < |x+3| < \delta$. Thus, we finish the proof. □

Q3. Show that

$$\lim_{x\to 8} x^2 = 64,$$

by using the definition of limit.

Solution. Let $\varepsilon > 0$. We want to show that there exists $\delta > 0$, such that whenever $0 < |x - 8| < \delta$, we have $|x^2 - 64| < \varepsilon$. Note that

$$|x^2 - 64| = |x - 8||x + 8|,$$

if we require $\delta \le 1$, we obtain that $|x - 8| < 1$ which implies that $|x| < 9$ and hence $|x + 8| < 17$. Thus, if $17|x - 8| < \varepsilon$, then, $|x^2 - 64| < 17|x - 8| < \varepsilon$. Thus, we require $\delta \le \frac{\varepsilon}{17}$ and $\delta \le 1$. The choice $\delta = \min\left\{\frac{\varepsilon}{17}, 1\right\}$ is sufficient. Therefore, by Definition, $\lim_{x\to 8} x^2 = 64$, as required. □

Q4. Make minor adaptations to the proof of Theorem 2.6 to prove the following theorem.

Theorem 1 (Squeeze)**.** *Suppose that f, g and h are real functions, and that for some $\alpha > 0$,*

$$f(x) \le h(x) \le g(x) \tag{1}$$

for all $x \in (\alpha, \infty)$, and that for some $A \in \mathbb{R}$,

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = A.$$

Then $\lim_{x\to\infty} h(x) = A$.

Solution. Let $\varepsilon > 0$. By Definition 2.14 and $\lim_{x\to\infty} f(x) = A$, there exists $K_1 > \alpha$, such that

$$|f(x) - A| < \varepsilon \quad \forall\, x > K_1. \tag{2}$$

Similarly, $\lim_{x\to\infty} g(x) = A$ means that there exists $K_2 > \alpha$ such that

$$|g(x) - A| < \varepsilon \quad \forall\, x > K_2. \tag{3}$$

Set $K = \max\{K_1, K_2\}$. Then whenever $x > K$, via (2) and (3), we have,

$$A - \varepsilon < f(x) \quad \text{and} \quad g(x) < A + \varepsilon. \tag{4}$$

It follows from (1) and (4) that whenever $x > K$, we have

$$A - \varepsilon < f(x) \le h(x) \le g(x) < A + \varepsilon.$$

Therefore via Definition 2.14, we conclude that $\lim_{x\to\infty} h(x) = A$, as required. □

Q5. Let $f, g : \mathbb{R} \to \mathbb{R}$. For each of the following statements, either prove it is true using the definition of the limit or give a counterexample to show that it is false.

(a) Suppose that $\lim_{x\to\infty} f(x) = a$ and $\lim_{x\to\infty} g(x) = b$. If $f(x) < g(x)$ for all $x \in \mathbb{R}$, then $a < b$.
(b) If $\lim_{x\to a} f(x) = \ell$ and $\lim_{x\to a} g(x) = \infty$, then $\lim_{x\to a} f(x)g(x) = \infty$.
(c) If $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = \ell$, then $f(a) = \ell$.
(d) If $\lim_{x\to b} f(x) = c$ and $\lim_{x\to a} g(x) = b$, then $\lim_{x\to a} f\big(g(x)\big) = c$.

Solution. None of these statments are true.

(a) Let $f(x) = 1/(1 + x^2)$ and $g(x) = 2/(1 + x^2)$ for all $x \in \mathbb{R}$. Then $f(x) < g(x)$ for all $x \in \mathbb{R}$, but $f(x) \to 0 = a$ and $g(x) \to 0 = b$ as $x \to \infty$. Since $a \not< b$ we have a counter-example to the statement.
(b) Let $f(x) = x^3$, for all $x \in \mathbb{R}$, and $g(x) = 1/x^2$ for all $x \neq 0$ (with $g(0) \in \mathbb{R}$). Then $\lim_{x\to 0} f(x) = 0$, $\lim_{x\to 0} g(x) = \infty$, but $f(x)g(x) = x$ for all $x \neq 0$, so that $\lim_{x\to 0} f(x)g(x) = 0$. Since $\lim_{x\to 0} f(x)g(x) \neq \infty$ we have a counter-example to the statement.
(c) Let

$$f(x) = \begin{cases} 0, & \text{if } x \neq 0 \\ 1, & \text{if } x = 0. \end{cases}$$

Then $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = 0$, but $f(0) = 1$. Since $0 \neq 1$ we have a counter-example to the statement.

(d) Let $g(x) = 0$ for all $x \in \mathbb{R}$ and let

$$f(x) = \begin{cases} 0, & \text{if } x \neq 0. \\ 1, & \text{if } x = 0. \end{cases}$$

Then $\lim_{x \to 0} f(x) = 0 = c$ and $\lim_{x \to 0} g(x) = 0 = b$. Moreover, since $f\big(g(x)\big) = 1$ for all $x \in \mathbb{R}$, it follows that $\lim_{x \to 0} f\big(g(x)\big) = 1$. Since $1 \neq 0 = c$ we have a counter-example to the statement.

$\square$

Q6. Let $A \subseteq \mathbb{R}$. Let $f : A \to \mathbb{R}$ be continuous. Let $a \in A$. Prove that, if $f(a) > 0$, then there exists $\delta > 0$ such that $f(x) > 0$ for all $x \in A \cap (a - \delta, a + \delta)$.
[This is sometimes called the "sign-preserving property" of continuous functions: informally, if a continuous function is positive at a certain point, then it is also positive at nearby points.]

Solution. By our assumptions, we know that f is continuous at a, which means, by definition, that

$$\forall \epsilon > 0 : \exists \delta > 0 : \forall x \in A : (|x - a| < \delta \Rightarrow |f(x) - f(a)| < \epsilon) \,. \tag{5}$$

Since we know that $f(a) > 0$, we can apply (5) with $\epsilon = f(a)$ and obtain the corresponding $\delta > 0$. We now notice that, if $x \in A \cap (a - \delta, a + \delta)$, then $x \in A$ and $|x - a| < \delta$, and therefore from (5) we deduce that

$$\begin{aligned} f(a) - f(x) &\leq |f(x) - f(a)| && \text{(by properties of the absolute value)} \\ &< \epsilon && \text{(by (5))} \\ &= f(a), && \text{(by our choice of } \epsilon) \end{aligned}$$

that is, $f(a) - f(x) < f(a)$, which, rearranged, gives $f(x) > 0$, as desired. $\square$

Q7. Demonstrate (referring to either definitions or theorems) that the following limits do not exist.

(a) $\lim_{x \to \infty} \cos x$.

(b) $\lim_{x \to 0} e^{-1/x}$.

Solution. (a) Consider the sequence $\{a_n\}$ given by $a_n = n\pi$ for $n \in \mathbb{N}$. Observe that $\cos a_n = (-1)^n$ for $n \in \mathbb{N}$ and $\lim_{n \to \infty} (-1)^n$ does not exist. Given that Theorem 2.33 also applies with ∞ replacing x_0, it follows from the contraposition of Theorem 2.33 that $\lim_{x \to \infty} \cos x$ does not exist.

(b) Let $\varepsilon \in (0, 1)$. For $0 < x < \frac{-1}{\ln \varepsilon}$ it follows that $0 < e^{-1/x} < \varepsilon$. Hence $\lim_{x \to 0^+} e^{-1/x} = 0$. For $-1 < x < 0$ it follows that $e^{-1/x} > e$ and hence $\lim_{x \to 0^-} e^{-1/x} \neq 0$. Hence, $\lim_{x \to 0} e^{-1/x}$ does not exist.

$\square$

Q8. Determine the value of the following limits. You can use any of the definitions and results discussed in lectures, provided you clearly state what you are using. **Only those materials that have been discussed in lectures can be used here. For instance, you can NOT use L'Hospital's rule here**.

(i) $\lim_{x \to 1} \dfrac{x^2 - 1}{2x^2 - x - 1}$.

(ii) $\lim_{x\to 0} \frac{(x-1)^3+(1-3x)}{x^2+2x^3}$.

(iii) $\lim_{x\to 1} \frac{x^n-1}{x^m-1}$, where $n, m \in \mathbb{N}$.

(iv) $\lim_{x\to 4} \frac{\sqrt{1+2x}-3}{\sqrt{x}-2}$.

(v) $\lim_{x\to 0} \frac{\tan x - \sin x}{x^3}$.

(vi) $\lim_{x\to\infty} x \sin \frac{1}{x}$.

Solution. (i). Observe that

$$\frac{x^2-1}{2x^2-x-1} = \frac{(x-1)(x+1)}{(x-1)(2x+1)} = \frac{x+1}{2x+1},$$

then by the Algebra of Limits, we have

$$\lim_{x\to 1} \frac{x^2-1}{2x^2-x-1} = \lim_{x\to 1} \frac{x+1}{2x+1} = \frac{\lim_{x\to 1} x+1}{2\lim_{x\to 1} x+1} = \frac{2}{3}.$$

(ii). Note that

$$\frac{(x-1)^3+(1-3x)}{x^2+2x^3} = \frac{x^3-3x^2}{x^2+2x^3} = \frac{x^2(x-3)}{x^2(1+2x)} = \frac{x-3}{1+2x}.$$

By using the Algebra of Continuous Functions, so

$$\lim_{x\to 0} \frac{(x-1)^3+(1-3x)}{x^2+2x^3} \lim_{x\to 0} \frac{x-3}{1+2x} = -3.$$

(iii). Note that

$$\frac{x^n-1}{x^m-1} = \frac{(x-1)(x^{n-1}+x^{n-2}+\cdots+1)}{(x-1)(x^{m-1}+x^{m-2}+\cdots+1)} = \frac{x^{n-1}+x^{n-2}+\cdots+1}{x^{m-1}+x^{m-2}+\cdots+1},$$

hence, by the Algebra of Limits,

$$\lim_{x\to 1} \frac{x^n-1}{x^m-1} = \lim_{x\to 1} \frac{x^{n-1}+x^{n-2}+\cdots+1}{x^{m-1}+x^{m-2}+\cdots+1} = \frac{n}{m}.$$

(iv). Note that

$$\begin{aligned}\frac{\sqrt{1+2x}-3}{\sqrt{x}-2} &= \frac{(\sqrt{1+2x}-3)(\sqrt{1+2x}+3)}{(\sqrt{x}-2)(\sqrt{1+2x}+3)} \\ &= \frac{2(x-4)}{(\sqrt{x}-2)(\sqrt{1+2x}+3)} = \frac{2(\sqrt{x}+2)}{\sqrt{1+2x}+3},\end{aligned}$$

by the Algebra of continuous function, continuity of $\sqrt{x}$, and continuity of composition of continuous functions, we know that

$$\lim_{x\to 4} \frac{2(\sqrt{x}+2)}{\sqrt{1+2x}+3} = \frac{2(\lim_{x\to 4}\sqrt{x}+2)}{\lim_{x\to 4}\sqrt{1+2x}+3} = \frac{4}{3}.$$

Hence, we have,

$$\lim_{x\to 4} \frac{\sqrt{1+2x}-3}{\sqrt{x}-2} = \lim_{x\to 4} \frac{2(\sqrt{x}+2)}{\sqrt{1+2x}+3} = \frac{4}{3}.$$

(v). Note that

$$1-\cos x = 2\sin^2\frac{x}{2}.$$

Therefore, we have

$$\frac{\tan x - \sin x}{x^3} = \frac{(1-\cos x)\sin x}{x^3 \cos x} = \frac{2\sin x \sin^2 \frac{x}{2}}{x^3 \cos x} = \frac{\sin x}{x} \cdot \frac{\sin^2 \frac{x}{2}}{(\frac{x}{2})^2} \cdot \frac{1}{2\cos x}$$

By results from lectures ("notable limits") and algebra of limits, we know that

$$\lim_{x\to 0} \frac{\sin x}{x} = 1, \quad \lim_{x\to 0} \frac{\sin^2 \frac{x}{2}}{(\frac{x}{2})^2} = \lim_{x\to 0} \left(\frac{\sin \frac{x}{2}}{\frac{x}{2}}\right)^2 = 1.$$

From algebra of limits and the continuous of $\cos x$, we see that

$$\lim_{x\to 0} \frac{1}{2\cos x} = \frac{1}{2\lim_{x\to 0}\cos x} = \frac{1}{2\cos 0} = \frac{1}{2}.$$

Therefore, we have from the product rule that

$$\begin{aligned}\lim_{x\to 0} \frac{\tan x - \sin x}{x^3} &= \lim_{x\to 0} \frac{\sin x}{x} \cdot \frac{\sin^2 \frac{x}{2}}{(\frac{x}{2})^2} \cdot \frac{1}{2\cos x} \\ &= \lim_{x\to 0} \frac{\sin x}{x} \cdot \lim_{x\to 0} \frac{\sin^2 \frac{x}{2}}{(\frac{x}{2})^2} \cdot \lim_{x\to 0} \frac{1}{2\cos x} = \frac{1}{2},\end{aligned}$$

(vi). Note that

$$x \sin \frac{1}{x} = \frac{\sin \frac{1}{x}}{\frac{1}{x}}.$$

By setting $t = \frac{1}{x}$ ("change of variable"), we know that

$$t \to 0,$$

as $x \to \infty$. Therefore, we have

$$\lim_{x\to\infty} x \sin \frac{1}{x} = \lim_{x\to\infty} \frac{\sin \frac{1}{x}}{\frac{1}{x}} = \lim_{t\to 0} \frac{\sin t}{t} = 1,$$

where we used the "notable limit" in the last step.

$\square$

Q9. Suppose $X \subset \mathbb{R}$ and let $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}$ be continuous functions. Define $p : X \to \mathbb{R}$ by $p(x) := \max\{f(x), g(x)\}$ and $q : X \to \mathbb{R}$ by $q(x) = \min\{f(x), g(x)\}$. Prove that p and q are continuous.

Solution. Observe that

$$p(x) = \frac{1}{2}\left(f(x) + g(x) + |f(x) - g(x)|\right)$$

and

$$q(x) = \frac{1}{2}\left(f(x) + g(x) - |f(x) - g(x)|\right),$$

for all $x \in X$. Note that $h(x) := |x|$ for all $x \in \mathbb{R}$ is a continuous function. It follows that p and q are continuous on X, as required. $\square$

Q10. Find an example of a bounded function $f : [0,1] \to \mathbb{R}$ that has neither an absolute minimum nor an absolute maximum.

Solution. Let $f : [0,1] \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} \frac{1}{2}, & x = \{0, 1\}, \\ x, & x \in (0,1). \end{cases}$$

There exists no local minimum/maximum for f in $[0,1]$ and hence no absolute maximum/minima for f in $[0,1]$. $\square$

Q11. Let $f : (0, 1) \to \mathbb{R}$ be a continuous function such that

$$\lim_{x\to 0} f(x) = \lim_{x\to 1} f(x) = 0.$$

Show that f achieves either an absolute minimum or an absolute maximum on $(0, 1)$ (but perhaps not both).

Solution. Define

$$F(x) = \begin{cases} f(x) & 0 < x < 1 \\ 0 & x = 0 \text{ or } x = 1\,. \end{cases}$$

Thus $F(x)$ is continuous on $[0, 1]$ and achieves both absolute max and min.

- If $\max F(x) = M = f(x_M) > 0$, then f achieves an absolute max.
- If $\min F(x) = m = f(x_m) < 0$, then f achieves an absolute min.
- If $\max F(x) = \min F(x) = 0$, then f is constant, and achieves both max and min.

□

Q12. Suppose for $f : [0, 1] \to \mathbb{R}$ we have $|f(x) - f(y)| \leq K|x - y|$ for all $x, y \in [0, 1]$, and $f(0) = f(1) = 0$. Prove that $|f(x)| \leq \frac{K}{2}$ for all $x \in [0, 1]$.
Note: A function $f : X \to \mathbb{R}$ is called Lipschitz continuous if there exists a $K > 0$ such that

$$|f(x) - f(y)| \leq K|x - y| \quad \forall\, x, y \in X.$$

Solution. For $0 \leq x \leq \frac{1}{2}$, we have

$$|f(x)| = |f(x) - f(0)| \leq K|x| \leq \frac{K}{2}\,,$$

Also for $\frac{1}{2} < x \leq 1$, we have

$$|f(x)| = |f(x) - f(1)| \leq K|1 - x| \leq \frac{K}{2}.$$

Thus, $|f(x)| \leq \frac{K}{2}$ for all $x \in [0, 1]$, as required. □

Extra Questions

EQ1. For each of the following statements, either prove that it is true, or give a counterexample to show that it is false. You can use any of the definitions and results discussed in lectures, provided you clearly state what you are using.

(i) If $f : (0, 1) \to \mathbb{R}$ is continuous, then f is bounded.
(ii) If $g : (0, 1) \to \mathbb{R}$ is continuous, then g is differentiable.
(iii) If $k : [0, 1] \to \mathbb{R}$ is differentiable, then k is bounded.

Solution. (i). The statement is false. For example, if $f : (0, 1) \to \mathbb{R}$ is defined by $f(x) = 1/x$ for all $x \in (0, 1)$, then f is continuous (by the Algebra of Continuous Functions), but f is unbounded (indeed $f((0, 1)) = (1, \infty)$, and therefore $\sup f = \infty$).

(ii). The statement is false. For example, if $g : (0, 1) \to \mathbb{R}$ is defined by $g(x) = |x - 1/2|$, then g is continuous (since $x \mapsto |x|$ is continuous and composition of continuous functions is continuous); however g is not differentiable at $1/2$, because, for all $h \in \mathbb{R} \setminus \{0\}$,

$$\frac{f(1/2 + h) - f(1/2)}{h} = \frac{|h|}{h}$$

and the latter expression has no limit as $h \to 0$ (the one-sided limits are ± 1).

(iii). The statement is true. Indeed, by a result from lectures, if $k : [0,1] \to \mathbb{R}$ is differentiable, then k is continuous; moreover, by the Boundedness Theorem, if $k : [0,1] \to \mathbb{R}$ is continuous, then it is bounded. $\square$

EQ2. Determine the following limits and prove that your answer is correct by directly appealing to the definition of the limit.

(a) $\displaystyle\lim_{x\to\infty} \frac{3x^3 - 5x^2 - 13}{2x^3 + 1}$.

(b) $\displaystyle\lim_{x\to 1^+} 2x^2 - 3x + 5$.

(c) $\displaystyle\lim_{x\to 2^-} 1/(1-x)$.

(d) $\displaystyle\lim_{x\to\infty} (1/x)\sin x$.

Solution. (a) Let $f : (0,\infty) \to \mathbb{R}$ be given by $f(x) = (3x^3 - 5x^2 - 13)/(2x^3 + 1)$ for all $x \in (0,\infty)$. Dividing the top and bottom by x^3 we have

$$f(x) = \frac{3x^3 - 5x^2 - 13}{2x^3 + 1} = \frac{3 - 5/x - 13/x^3}{2 + 1/x^3} \quad \forall\, x \in (0,\infty).$$

Since $5/x$, $13/x^3$ and $1/x^3$ all tend to 0 as $x \to \infty$, we see that

$$f(x) = \frac{3x^3 - 5x^2 - 13}{2x^3 + 1} \to \frac{3 - 0 - 0}{2 - 0} = \frac{3}{2} \quad \text{as } x \to \infty.$$

We now prove that this is the limit. Let $\varepsilon > 0$. Then

$$\begin{aligned}
\left| f(x) - \frac{3}{2} \right| &= \left| \frac{-10x^2 + 14x - 29}{4x^3 + 2} \right| \\
&\le \frac{|-10x^2 + 14x - 29|}{|4x^3 + 2|} \\
&\le \frac{|10x^2| + |14x^2| + |29x^2|}{|4x^3|} \\
&= \frac{53}{4x} \qquad (6)
\end{aligned}$$

for all $x \ge 1$. Let $M = \max\{1, \frac{53}{4\varepsilon}\}$. Then via (6), it follows that

$$\left| f(x) - \frac{3}{2} \right| \le \frac{53}{4x} < \varepsilon \quad \forall\, x > M.$$

We conclude that

$$\lim_{x\to\infty} \frac{3x^3 - 5x^2 - 13}{2x^3 + 1} = \frac{3}{2},$$

as required.

(b) Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = 2x^2 - 3x + 5$ for all $x \in \mathbb{R}$. As $x \to 1^+$, it follows that

$$f(x) = 2x^2 - 3x + 5 \to 2.1^1 - 3.1 + 5 = 4.$$

We now prove this. Let $\varepsilon > 0$. Observe that

$$\begin{aligned}
|f(x) - 4| &= |2x^2 - 3x + 5 - 4| \\
&= |2x^2 - 3x + 1| \\
&= |(2x-1)(x-1)| \\
&= |2x-1||x-1| \qquad (7)
\end{aligned}$$

for all $x \in \mathbb{R}$. We are interested in the limit as $x \to 1^+$, so we can simply consider $1 < x < 2$ in (7). It follows from (7) that

$$|f(x) - 4| = |2x - 1||x - 1| < 3|x - 1| \quad \forall x \in (1, 2). \tag{8}$$

By setting $\delta = \frac{\varepsilon}{3}$ it follows from (8) that

$$|f(x) - 4| < |x - 1| < \varepsilon \quad \forall x \in (1, 1 + \delta).$$

We conclude that $\lim_{x \to 1^+} 2x^2 - 3x + 5 = 4$, as required.

(c) Let $f : \mathbb{R} \setminus \{1\} \to \mathbb{R}$ be given by $f(x) = 1/(1 - x)$ for all $x \in \mathbb{R} \setminus \{1\}$. Observe that $f(x) = 1/(1 - x) \to 1/(1 - 2) = -1$ as $x \to 2^-$. We now prove this. Let $\varepsilon > 0$. Observe that

$$|f(x) - (-1)| = \left| \frac{1}{1 - x} + 1 \right| = \left| \frac{2 - x}{1 - x} \right| = \frac{|2 - x|}{|1 - x|}. \tag{9}$$

for all $x \in \mathbb{R} \setminus \{1\}$. We are interested in the limit as $x \to 1^-$, so we can simply consider $\frac{3}{2} < x < 2$ in (9). It follows from (9) that

$$|f(x) - (-1)| = \frac{|2 - x|}{|1 - x|} < \frac{|2 - x|}{1/2} = 2|x - 2| \tag{10}$$

for all $x \in (\frac{3}{2}, 2)$. Set $\delta = \min\left\{\frac{1}{2}, \frac{\varepsilon}{2}\right\}$. Then via (9)

$$|f(x) - (-1)| < 2|x - 2| < \varepsilon \quad \forall x \in (2 - \delta, 2).$$

It follows that $\lim_{x \to 2^-} 1/(1 - x) = -1$ as required.

(d) Let $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be given by $f(x) = (1/x) \sin x$ for all $x \in \mathbb{R} \setminus \{0\}$. We claim $f(x) \to 0$ as $x \to \infty$. Let $\varepsilon > 0$. Observe that

$$|f(x) - 0| = \frac{|\sin x|}{x} \leq \frac{1}{x}. \tag{11}$$

Set $K = 1/\varepsilon$. Then via (11), it follows that

$$|f(x) - 0| \leq \frac{1}{x} < \varepsilon \quad \forall x \in (K, \infty).$$

We conclude that $\lim_{x \to \infty} (1/x) \sin x = 0$, as required.

$\square$

EQ3. Prove that the following limits exist and determine their value. You can use any of the definitions and results discussed in lectures, provided you clearly state what you are using.

(i) $\lim_{x \to -\infty} 2x^2 - 3x + \arctan x$.

(ii) $\lim_{x \to 2} \frac{1}{1 - x}$.

(iii) $\lim_{x \to 1/2} \frac{4x^2 - 1}{2x - 1}$.

Solution. (i). We claim that $\lim_{x \to -\infty} 2x^2 - 3x + \arctan x = \infty$. In order to show this, we first observe that $\arctan x \leq \pi/2$ for all $x \in \mathbb{R}$, whence

$$2x^2 - 3x + \arctan x \geq 2x^2 - 3x - \pi/2.$$

By the Sandwich Theorem for Infinite Limits, it is then enough to prove that $\lim_{x \to -\infty} 2x^2 - 3x - \pi/2 = \infty$. On the other hand,

$$2x^2 - 3x - \pi/2 = x^2(2 - 3/x - \pi/(2x^2)).$$

Since $\lim_{x\to-\infty} x = -\infty$, by repeatedly applying the Algebra of Limits we obtain that $\lim_{x\to-\infty} 1/x = 0$, whence

$$\lim_{x\to-\infty} x^2 = (-\infty)\cdot(-\infty) = \infty, \qquad \lim_{x\to-\infty} 2-3/x-\pi/(2x^2) = 2-3\cdot 0-(\pi/2)\cdot 0^2 = 2$$

and finally

$$\lim_{x\to-\infty} 2x^2 - 3x - \pi/2 = \lim_{x\to-\infty} x^2(2 - 3/x - \pi/(2x^2)) = \infty \cdot 2 = \infty.$$

(ii) We claim that $\lim_{x\to 2} \frac{1}{1-x} = -1$. This is an immediate consequence of the Algebra of Limits, since $\lim_{x\to 2} x = 2$, whence

$$\lim_{x\to 2} \frac{1}{1-x} = \frac{1}{1-2} = -1.$$

(iii) We claim that $\lim_{x\to 1/2} \frac{4x^2-1}{2x-1} = 2$. Indeed, for all $x \neq 1/2$,

$$\frac{4x^2-1}{2x-1} = \frac{(2x-1)(2x+1)}{2x-1} = 2x+1;$$

since moreover $\lim_{x\to 1/2} x = 1/2$, by the Algebra of Limits we deduce that

$$\lim_{x\to 1/2} \frac{4x^2-1}{2x-1} = \lim_{x\to 1/2} 2x+1 = 2(1/2)+1 = 2.$$

□

EQ4. Suppose $g(x)$ is a monic polynomial of even degree d, that is,

$$g(x) = x^d + b_{d-1}x^{d-1} + \cdots + b_1 x + b_0 \quad \forall\, x \in \mathbb{R},$$

for $b_0, b_1, \ldots, b_{d-1} \in \mathbb{R}$. Show that g achieves an absolute minimum on $\mathbb{R}$. Use this to conclude that if $f(x)$ is a polynomial of degree d and $f(\mathbb{R}) = \mathbb{R}$, then d is odd.

Solution. Note that

$$\begin{aligned} g(x) \geq & |x|^d - |b_{d-1}||x|^{d-1} - \cdots - |b_0| \\ \geq & |x|^d - (|b_{d-1}| + \cdots |b_0|)|x|^{d-1} \quad \text{for } |x| > 1\,. \end{aligned}$$

Thus we observe that $\lim_{x\to\pm\infty} g(x) = \infty$. By defintion, there exists $N > 0$ such that $g(x) \geq g(0)$ for any $x \in (-\infty, -N) \cup (N, \infty)$. Since g is continuous on $[-N, N]$ then the absolulte minimum of g on $[-N, N]$ exists and satisfies

$$\min_{y\in\mathbb{R}} g(y) = \min_{y\in[-N,N]} g(y) \leq g(0).$$

Thus the absolute minumum of $g : \mathbb{R} \to \mathbb{R}$ exists, as required.

If $f : \mathbb{R} \to \mathbb{R}$ is a polynomial of degree d and $f(\mathbb{R}) = \mathbb{R}$, then necessarily (from the above result) the degree d must be odd. This follows since if

$$f(x) = \sum_{k=0}^{d} a_k x^k \quad \forall\, x \in \mathbb{R}$$

was an even degree polynomial of degree d (i.e. $a_d \neq 0$), then $f(x)/a_d$ would be an even degree monic polynomial for which $Im(f) \neq \mathbb{R}$, since $\min_{x\in\mathbb{R}} f(x)/a_d$ exists. □

EQ5. The number $x \in [0,1]$ is called a fixed point of $f : [0,1] \to [0,1]$ if $x = f(x)$. If $f : [0,1] \to [0,1]$ is continuous, show that f has a fixed point.

Solution. Let $g : [0,1] \to \mathbb{R}$ be given by $g(x) = f(x) - x$ for all $x \in [0,1]$. Then g is continuous on $[0,1]$.

If $f(0) = 0$ or $f(1) = 1$, then the conclusion is valid. If $f(0) > 0$ and $f(1) < 1$, then

$$g(0) = f(0) > 0 \quad g(1) = f(1) - 1 < 0.$$

From the intermediate value theorem, there exists $c \in (0,1)$ such that $g(c) = 0$. It follows that $f(c) = c$, i.e. c is a fixed point of f, as required. . $\square$