Adiabatic transitions ($Q_{in} = 0$)

First law of Thermodynamics: $\Delta U = Q_{in} + W_{on}$

$$dU = \cancel{dQ_{in}}^{\,0} - dW_{on}$$

$$dU = 0 - pdV$$

$PV = nRT$

$d(pv) = nR\,dT$

$nR\,dT = pdv + vdp$

$\therefore dT = \frac{pdv + vdp}{nR}$

$$\therefore \frac{\partial U}{\partial T}\partial T = -pdV$$

$$C_v\,\partial T = -pdV$$

$$\therefore 0 = C_v\left(\frac{pdv + vdp}{nR}\right) + pdV$$

$\times nR \Rightarrow$ $$0 = C_v(pdv + vdp) + nR\,pdv$$

$$0 = pdv\underbrace{(C_v + nR)}_{= C_p \text{ [Mayer's]}} + C_v\,vdp$$

$$0 = C_p\,pdv + C_v\,vdp$$

$$0 = \boxed{\frac{C_p}{C_v}}^{=\gamma} pdv + vdp = \gamma\frac{dv}{v} + \frac{dp}{p}$$

Separate the variables :) $$\int \gamma \frac{dv}{v} = -\int \frac{dp}{p}$$

$$\therefore \gamma\ln(v) + const_1 = -\ln(p) + const_2$$

$-const_1 + const_2 = \ln(const)$ arbitrarily as they're just constants and the actual values don't matter for the above to still apply

$$\gamma\ln(v) + \ln(p) = \ln(const)$$

$$\therefore \boxed{PV^{\gamma} = const}$$

Also, $$\frac{\cancel{nR}T}{V}V^{\gamma} = const$$

also constant

so $\frac{const}{nR}$ is still constant!

$$\boxed{TV^{\gamma-1} = const}$$