1VGLA, Exam Summer 2024

1.-

(a) Given vectors $\mathbf{a} = (1, 2, 3)$, $\mathbf{b} = (-3, 0, 1)$ and $\mathbf{c} = (1, 2, 2)$, compute the following:

(i) $5\mathbf{a} - \mathbf{c}$;

(ii) $(\mathbf{a} \cdot \mathbf{c})\mathbf{a}$;

(iii) $(\mathbf{a} \times \mathbf{b}) + \mathbf{c}$;

(iv) the non-reflex angle between $\mathbf{a}$ and $\mathbf{b}$.

[9]

(b) Suppose that $\mathbf{a} = (1, 2, 3)$ is a vector. Determine the scalar equation of the plane Π perpendicular to $\mathbf{a}$ containing the point $P = (2, 1, -2)$. **[3]**

(c) Which of the following statements are true and which are false? For any statements that are **false**, provide a brief explanation as to why this is the case.

(i) Given any complex number $z \in \mathbb{C}$, $\mathrm{Re}(z) = \mathrm{Re}(\bar{z})$.

(ii) Given any complex number $z \in \mathbb{C}$, $z\bar{z}$ is an integer.

(iii) Given any complex number $z \in \mathbb{C}$, the principal value $\mathrm{Arg}(z)$ of $\arg(z)$ satisfies $\mathrm{Arg}(z) \in [0, 2\pi]$.

[3]

(d) Give an example of a system of two real simultaneous linear equations in two variables that has no solutions. (You do not need to justify your answer.) **[2]**

(e) Using the augmented matrix and row operations, find the solution set of the following system of real simultaneous linear equations.

$$\begin{array}{rcrcrcr} x & + & 3y & + & 2z & = & 1 \\ & & 5y & + & 3z & = & -3 \\ 3x & + & 4y & + & 3z & = & 6. \end{array}$$

[8]

<u>Feedback:</u>

(a) Many students obtained full marks on this part of the question, and most others obtained a high score here. Most marks dropped were for calculation errors. For (a)(iv) students lost half a mark if they stated the answer as $\cos^{-1}(0)$ rather than $\pi/2$ radians.

(b) Many students obtained full marks on this question. Some students could not recall how to produce the equation of a plane though (see Section 1.10 in the lecture notes), so did not obtain the correct result. Recall too that $\mathbf{a}$ being perpendicular to the plane means it is a normal vector.

(c) Most students answered (i) and (iii) correctly, though quite a few gave the wrong answer for (ii). Indeed, $z\bar{z}$ doesn't have to be an *integer*. For example, if $z = \pi$ then $z\bar{z} = \pi^2$ which is an irrational number not an integer.

(d) Most students gave a good example. Some students gave examples with at least one solution though. Some students gave answers that had more than 2 variables in so lost marks. This is a good illustration of why it is always important to read the question carefully!

(e) Most students obtained a good mark on this question, though many lost at least some marks. Note that the question is asking for a solution *set* so you should write it in set notation (or explain precisely what this set contains). Many students lost marks for incorrect calculations. Some could perform the row operations all fine, but then found it difficult to 'read off' the solution set from the final matrix presented. If the student didn't explain the row operations performed at each step then some marks were lost.

3.-

(a) Suppose that $z = 1 - i$.

(i) Calculate z^6 in both modulus-argument form and exponential form giving the principal value of the argument.

(ii) What is the smallest choice of $n \in \mathbb{N}$ so that $z^n \in \mathbb{N}$?

(iii) Find all the fourth roots of z. Present your answers in exponential form giving the principal value of the argument.

[9]

(b) Suppose that $\mathbf{A} = \begin{pmatrix} 1 & 1 & 5 \\ 2 & 2 & 3 \\ 0 & 0 & 1 \end{pmatrix}$. Use the Gaussian Elimination Algorithm to determine whether $\mathbf{A}$ is invertible or not. In particular, if $\mathbf{A}$ is invertible then determine $\mathbf{A}^{-1}$. **[7]**

(c) Suppose that $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are matrices so that (i) $\mathbf{A} \cdot \mathbf{B}$ is defined and is a 5×3 matrix; (ii) $\mathbf{C} \cdot \mathbf{A}$ is defined and is a 3×3 matrix. Determine the dimensions of each of the matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$. Justify your answer. **[5]**

(d) Suppose that $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a 2×2 matrix such that a, b, c, d are positive real numbers. Prove that $\mathbf{A} \neq \mathbf{A^{-1}}$.

[4]

Answer:

(a) Most students did well on (i) and (iii) though many students found (ii) trickier. The most common mistake was that some initially computed the principal value of the argument of z incorrectly. If one draws an Argand diagram, you should be able to see that this has value $-\pi/4$ (not $\pi/4$ like many said). Making this mistake then led to some students making further mistakes in (ii) and (iii) also.

Once students put z into exponential form, it was then easy to compute z^6 in both exponential and modulus-argument form. However, in both (i) and (iii), many students did not put the *principal value* of the argument (recall it is a value in $(-\pi, \pi]$) so lost marks.

For (ii), the key was to work out that for z^n to be a natural number, one needs that the principal value of the argument of z^n is 0. Alternatively, you could have just computed $z^2, z^3, \dots$ until you found the first natural number. Some students incorrectly stated that $n = 0$. Note that 0 is not in $\mathbb{N}$. Indeed, as defined in the lecture notes, $\mathbb{N} = \{1, 2, 3, \dots\}$.

For some students, a very small number of marks were lost if answers were not presented in a simplifed form. For example, rather than writing $(\sqrt{2})^{1/4}$ one can simply write $2^{1/8}$.

(b) Similarly to Q1(e), to obtain full marks in this question one needed to write the augmented matrix out; perform all the EROs correctly (and state at each step which ERO was used); finish the procedure at a point where it is clear that $\mathbf{A}$ is not invertible; explain briefly why you can deduce it is not invertible (e.g., because your matrix was in reduced echelon form but the LHS doesn't equal the identity matrix).

Many students missed at least one of these steps. Indeed, a few did not write out the RHS of the augmented matrix. Some finished the procedure early (i.e., before you could immediately tell the matrix is not invertible). Others didn't write a justification sentence.

The question clearly states that one must use Gaussian Elimination. So the few students that used another method to obtain the correct answer would have scored less than half marks here.

I also noticed one or two students using column operations; this is not allowed in Gaussian Elimination (and thus led to wrong answers being produced)!.

(c) A lot of students obtained full marks here, though some students didn't justify their answers clearly. The key to this question was to recall that if $\mathbf{X}$ and $\mathbf{Y}$ are matrices, then for $\mathbf{X} \cdot \mathbf{Y}$ to be defined, the number of columns in $\mathbf{X}$ needs to be the same as the number of rows in $\mathbf{Y}$. Using this, one can produce the exact dimensions of each of $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ in the question.

Some students gave partial solutions, e.g., saying a matrix has dimensions $3 \times n$ for some choice of n. However, this isn't enough as you could really have computed the dimensions exactly here.

(d) This question was a mixed bag: many did well and many did not score highly. Here you needed to show that for *all* choices of $a, b, c, d > 0$, $\mathbf{A}$ is not its own inverse. Therefore, it wasn't enough to just give an example of some fixed choices of a, b, c, d such that $\mathbf{A} \neq \mathbf{A^{-1}}$.

Some students gave a general formula for the inverse of a 2×2 matrix and then just said this isn't equal to $\mathbf{A}$. Again this does not fully answer the question though. Indeed, just because two different expressions (involving a, b, c, d) look different does not mean that they can't take the same value for some choices of a, b, c, d. So to obtain full marks you would have needed to argue why, e.g., the $(1, 2)$th entry of $\mathbf{A^{-1}}$ cannot equal b no matter how you choose $a, b, c, d > 0$.