**1VGLA Vectors, Geometry and Linear Algebra**

**2024/2025 Semester 2**

**Problem Sheet 2 (summative)**

This Problem Sheet will be marked.

Inconsistent notations and incoherent or incomplete reasoning will be penalised.

When using a theorem or general result from the lecture notes, state the result, instead of quoting the theorem number. For example, write "using $\det(\mathbf{A}) = \det(\mathbf{A}^T)$", instead of "by Theorem 7.10".

1. Let $n \geq 2$ be an integer. Consider the permutation $\sigma : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ defined by

$$\sigma(i) = \begin{cases} i+1 & \text{if } i < n, \\ 1 & \text{if } i = n. \end{cases}$$

   (a) Write down $\sigma$ in the sequence notation, $\sigma = [\sigma(1), \sigma(2), \ldots, \sigma(n)]$.

   (b) Find $N(\sigma)$, the number of inversions in $\sigma$.

   (c) Find $\sigma^{-1}$, the inverse of $\sigma$.

2. Find the determinants of the following matrices. You may use any general results about determinants, as long as you clearly state them. Giving the correct answer without showing a correct justification will get you no marks.

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{C} = \begin{pmatrix} 0 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{D} = \begin{pmatrix} 0 & 0 & 1 & 2 \\ 4 & 1 & 2 & 3 \\ 0 & 2 & 3 & 4 \\ 0 & 0 & -1 & 1 \end{pmatrix},$$

$$\mathbf{E} = \frac{1}{2}\mathbf{A}^T \cdot \mathbf{B} \cdot \mathbf{C}^2 \cdot \mathbf{D}^{-1}.$$

3. Let $n \in \mathbb{N}$ and $a_i \in \mathbb{R}$ for $i = 1, 2, \ldots, n$. Find the determinant of the $n \times n$ matrix,

$$\mathbf{A} = \begin{pmatrix} 0 & 0 & \cdots & 0 & a_1 \\ 0 & 0 & \cdots & a_2 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & a_{n-1} & \cdots & 0 & 0 \\ a_n & 0 & \cdots & 0 & 0 \end{pmatrix},$$

where the $i^{\text{th}}$ row has $a_i$ in the $(n-i+1)^{\text{th}}$ column and 0 in every other column.

**Please turn over.**

4. For every integer $n \geq 2$, let

$$\mathbf{A}_n = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & \cdots & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

be the $n \times n$ matrix where every element on the main diagonal, every element just above the main diagonal, and every element just below the main diagonal, is 1, while every other element is 0. An equivalent description of $\mathbf{A}_n$ is that the $1^{\text{st}}$ row has 1 in the first and second columns, the $n^{\text{th}}$ row has 1 in the $(n-1)^{\text{th}}$ and $n^{\text{th}}$ columns, and the $i^{\text{th}}$ row for $i = 2, 3, \ldots, n-1$ has 1 in the $(i-1)^{\text{th}}$, $i^{\text{th}}$ and $(i+1)^{\text{th}}$ columns, while every other element is 0.

(a) Write down $\mathbf{A}_2$ and find $\det(\mathbf{A}_2)$.

(b) Write down $\mathbf{A}_3$ and find $\det(\mathbf{A}_3)$.

(c) For $n \geq 4$, show that

$$\det(\mathbf{A}_n) = \det(\mathbf{A}_{n-1}) - \det(\mathbf{A}_{n-2}),$$

and hence, show that

$$\det(\mathbf{A}_{n+1}) = -\det(\mathbf{A}_{n-2}).$$

(d) Find $\det(\mathbf{A}_n)$ when:

(i) $n = 3k - 1$ for some integer $k \geq 1$;

(ii) $n = 3k$ for some integer $k \geq 1$;

(iii) $n = 3k + 1$ for some integer $k \geq 1$.