# Introduction to Probability

Lecture 7

# Today

Discrete Distributions

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# Summary

Introduced discrete distributions (PMF) for **ordered** events.

They satisfy

$$\sum_x P(x) = 1$$

The expectation value and variance of $x$ is defined by

$$\langle x \rangle \equiv \sum_x x\, P(x) \qquad \mathrm{var}(x) \equiv \sum_x (x - \langle x \rangle)^2\, P(x)$$

# Rules

## Expectation

$$\langle ax + b\rangle = a\langle x\rangle + b$$

## Variance

$$\mathrm{var}(ax + b) = a^2\,\mathrm{var}(x)$$

$$\mathrm{var}(ax) = \langle (ax)^2\rangle - \langle ax\rangle^2 = a^2\langle x^2\rangle - a^2\langle x\rangle^2$$

$$= a^2(\langle x^2\rangle - \langle x\rangle^2)$$

# Probability and Statistics

**Probability**

$$\langle x \rangle = \sum_x x\, P(x)$$

$$\mathrm{var}(x) = \sum_x (x - \langle x \rangle)^2\, P(x)$$

**We have a distribution in probability**

**Statistics**

$$\bar{x} = \frac{1}{N} \sum_n x_n$$

$$\sigma^2(x) = \frac{1}{N - 1} \sum_n (x_n - \bar{x})^2$$

**We have a sample in statistics**

# Distributions

# Distributions

Have some sample space $\Omega$

$$P(x|\theta)$$

$$\langle x \rangle \equiv \sum_x x \, P(x|\theta)$$

Sometimes more than one parameter: $P(x|\theta_1, \theta_2 \ldots) = P(x|\boldsymbol{\theta})$

# Bernoulli Distribution

# Bernoulli Distribution

Two events: 0 and 1

1 happens with probability $p$

0 happens with probability $1 - p$

$$P(x = 0) = 1 - p; \quad P(x = 1) = p$$

Typically written as

$$P(x|p) = p^x (1-p)^{1-x}$$

Check:

$$P(0|p) = p^0 (1-p)^{1-0} = 1 - p$$
$$P(1|p) = p^1 (1-p)^{1-1} = p$$

Could use $P(x|p) = xp + (1-p)(1-x)$

Check normalisation

$$\sum_x P(x|p) = 1 - p + p = 1$$

# Properties

1. $\langle x \rangle$
2. $\langle x^2 \rangle$
3. $\mathrm{var}(x)$

$$\langle x \rangle = \sum_x x\, P(x|p) = 0 \times p^0 (1-p)^{1-0} + 1 \times p^1 (1-p)^{1-1} = p$$

$$\langle x^2 \rangle = \sum_x x^2\, P(x|p) = 0^2 \times p^0 (1-p)^{1-0} + 1^2 \times p^1 (1-p)^{1-1} = p$$

$$\mathrm{var}(x) = \langle x^2 \rangle - \langle x \rangle^2 = p^2 - p = p(1-p)$$

# Binomial Distribution

# Binomial

If we toss $N$ coins and *count* the number of heads ($k$), what is the distribution of $k$?

For $N = 3$ we get $\Omega = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$
The tossing is **independent**

$$\rightarrow P(H) = p; P(T) = 1 - p$$
$$P(HHH) = p^3, P(HTH) = p^2(1 - p)$$

To get $k$ heads we are interested in the probability of the union

Example: $k = 1$: $P(HTT \cup THT \cup TTH) = P(HTT) + P(THT) + P(TTH)$

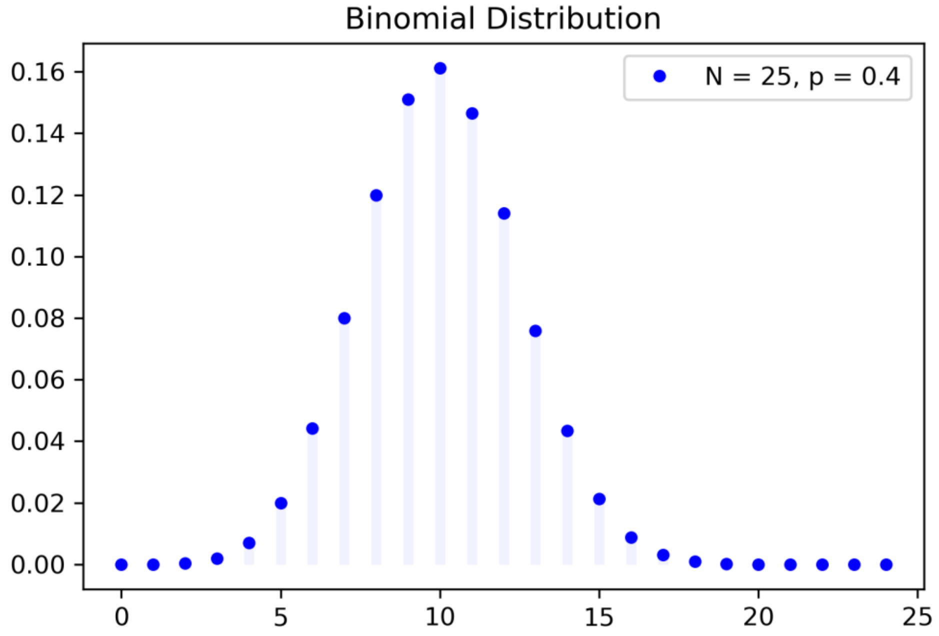
$$P(k = 0) = (1 - p)^3 = {}^3C_0(1 - p)^3$$
$$P(k = 1) = 3p(1 - p)^2 = {}^3C_1 p(1 - p)^2$$
$$P(k = 2) = 3p^2(1 - p) = {}^3C_2 p^2(1 - p)$$
$$\mathrm{P}(k = 3) = p^3 = {}^3C_3 p^3$$

# Binomial Distribution (2)

$$P(k|N,p) = \binom{N}{k} p^k (1-p)^{N-k}$$

This is the probability that you will see $k$ heads out of $N$ throws of a coin.

How many heads to we *expect*?



# Expectation Value (1)

$$\langle k \rangle = \sum_{k=0}^{N} k\, P(k|N,p) = \sum_{k=0}^{N} k \frac{N!}{k!\,(N-k)!} p^k (1-p)^{N-k}$$

$$= \sum_{k=1}^{N} k \frac{N!}{k!\,(N-k)!} p^k (1-p)^{N-k}$$

$$= \sum_{k=1}^{N} \frac{N!}{(k-1)!\,(N-k)!} p^k (1-p)^{N-k}$$

$$= Np \sum_{k=1}^{N} \frac{(N-1)!}{(k-1)!\,(N-k)!} p^{k-1} (1-p)^{N-k}$$

# Expectation Value (2)

$$Np\sum_{k=1}^{N}\frac{(N-1)!}{(k-1)!\,(N-k)!}p^{k-1}(1-p)^{N-k}$$

Set $t = k - 1$ so $k = t + 1$

$$\langle k\rangle = Np\sum_{t=0}^{N-1}\frac{(N-1)!}{(t)!\,(N-t-1)!}p^{t}(1-p)^{N-k}$$

But

$$\frac{(N-1)!}{(t)!\,(N-t-1)!}p^{t}(1-p)^{N-k} = P(t|N-1)$$

Which means

$$\sum_{t=0}^{N-1}\frac{(N-1)!}{(t)!\,(N-t-1)!}p^{t}(1-p)^{N-k} = 1$$

$$\langle k\rangle = Np$$

# Variance

$$\mathrm{var}(k) = \langle k^2 \rangle - \langle k \rangle^2$$

$$\langle k^2 \rangle = \sum_{k=0}^{N} k^2 \binom{N}{k} p^k (1-p)^{N-k}$$

$$= N^2p^2 - Np^2 + Np$$

$$\mathrm{var}(k) = \langle k^2 \rangle - \langle k \rangle^2$$

$$\mathrm{var}(k) = \underbrace{N^2p^2 - Np^2 + Np}_{\langle k^2 \rangle} - \underbrace{N^2p^2}_{\langle k \rangle^2}$$

$$\mathrm{var}(k) = -Np^2 + Np$$

$$\rightarrow \mathrm{var}(k) = Np(1-p)$$

# Binomial Distribution: Summary

$$P(k|N,p) = \binom{N}{k} p^k (1-p)^{N-k}$$

$$\langle k \rangle = Np$$

$$\mathrm{var}(k) = Np(1-p)$$

# Example

The probability of rain on any given day is 0.2.

a) What is the probability it rains twice?

b) What is the probability it rains at least once in a week?

a) We want $P(2|7{,}0.2)$

$$P(2|7{,}0.2) = \frac{7!}{2!\,5!}0.2^2(1-0.2)^5 \approx 0.28$$

b) Now we want $P(k \geq 1|7{,}0.2) = 1 - P(0|7{,}0.2)$

$$P(0|7{,}0.2) = \frac{7!}{0!\,7!}0.2^0(1-0.2)^7$$

$$\rightarrow P(k \geq 1|7{,}0.2) = 1 - 0.8^7 \approx 0.79$$

# Poisson Distribution

# Poisson Distribution (1)

What if $p$ was very small, but $N$ was very large?

$N$ people go to a shop with probability $p$

Consider $N \to \infty$, $p \to 0$ but keeping $\mathrm{N}p = \lambda$

$$P(k|N,p) = \frac{N!}{k!\,(N-k)!}p^k(1-p)^{N-k}$$

$$\frac{N!}{(N-k)!} = N(N-1)(N-2)\ldots \approx N^k$$

Note $k \ll N$

# Poisson Distribution (2)

$$P(k) = \frac{N^k}{k!} p^k (1-p)^{N-k} = \frac{(Np)^k}{k!} (1-p)^{N-k}$$

We defined $\lambda = Np$ so $p = \lambda/N$

$$\frac{\lambda^k}{k!}\left(1-\frac{\lambda}{N}\right)^{N-k} = \frac{\lambda^k}{k!}\left(1-\frac{\lambda}{N}\right)^{N}\left(1-\frac{\lambda}{N}\right)^{-k}$$

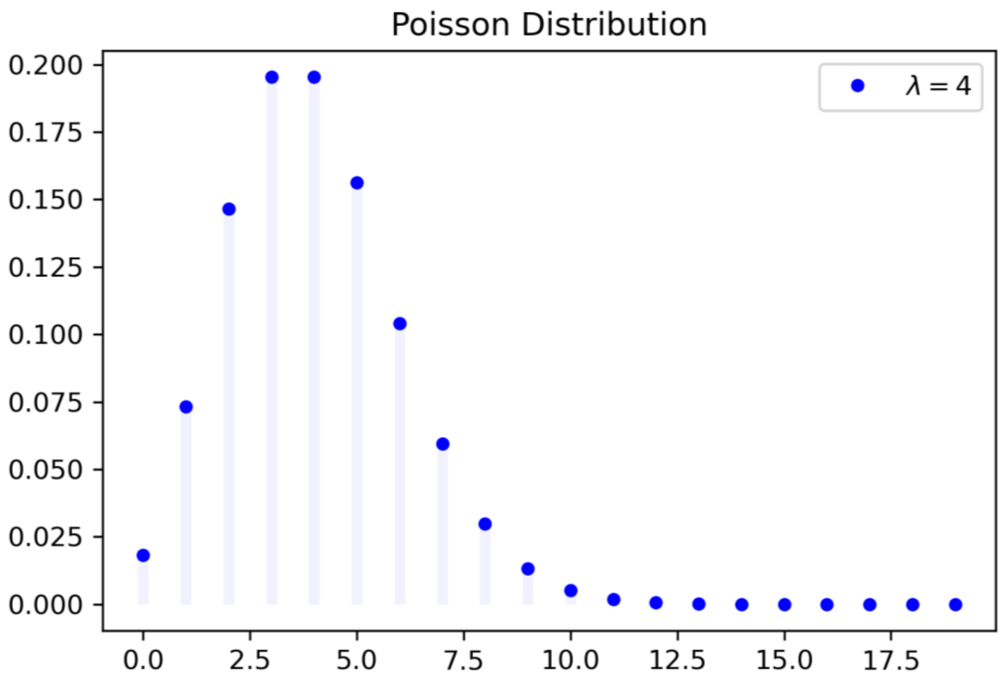
Then $\lim_{N\to\infty}\left(1-\frac{\lambda}{N}\right)^{N} = e^{-\lambda}$ and $\lim_{N\to\infty}\left(1-\frac{\lambda}{N}\right)^{-k} = 1$

$$\to P(k) = \frac{\lambda^k}{k!} e^{-\lambda}$$

Note: $e^{-\lambda}$ is just the normalising constant. The distribution is $\frac{\lambda^k}{k!}$.

# Poisson Distribution (3)

$$P(k|\lambda) \equiv \frac{\lambda^k}{k!} e^{-\lambda}$$




Binomial has $\langle k \rangle = Np$ and $\mathrm{var}(k) = Np(1-p)$

We have $\lambda = Np$ and $p \to 0$

So

$$\langle k \rangle = \lambda$$

And

$$var(k) = \lambda(1-p) \to \lambda$$

# Summary

Bernoulli

$$P(x|p) = p^x(1-p)^{1-x} \qquad x = 0,1$$

Binomial

$$P(k|N,p) = \binom{N}{k} p^k(1-p)^{N-k} \quad k = 0,1 \dots N$$

Poisson

$$P(k|\lambda) \equiv \frac{\lambda^k}{k!} e^{-\lambda} \qquad k = 0,1 \dots$$

# Examples

# Example

A box of 10 screws is sold. If more than 1 of them is defective, the box is returned. The screws are defective with probability 0.01.

What is the probability a box is returned?

If 10 boxes are sold every day for year, how many returns are expected?

We could use the binomial:

$$P(\text{no defects}) = P(0|10{,}0.01)$$

$$P(0|10{,}0.01) = \frac{10!}{0!\,10!} 0.01^0 (1 - 0.01)^{10} \approx 0.9$$

Or Poisson $\lambda = Np = 10 \times 0.01 = 0.1$

$$\rightarrow P(0) = \frac{0.1^0}{0!} e^{-0.1} \approx 0.9$$

Then $P(\text{return}) = 1 - P(0) \approx 0.1$

$$\langle \text{returns} \rangle = Np = 10 \times 365 \times 0.1 \approx 365$$

# Example

I am cautious when I cook, but I also cook a lot.

The probability I cut my finger is small, say $p = \frac{1}{10000}$

What is the probability that I cut my hand at least once if I cook three times a day for a year?

$$p = \frac{1}{10000}$$
$$N = 3 \times 365 = 1095$$
$$\rightarrow \lambda = Np = 0.1095$$

Then

$$\text{Prob}(k \geq 1|\lambda) = 1 - P(k = 0|\lambda)$$
$$= 1 - \frac{0.1095^0}{0!} e^{-0.1095} \approx 0.1$$

# Class Example

A box contains 100 balls, 2 of which are red. What is the probability of getting 3 red balls if you sample **with replacement** 50 times? Use the Poisson approximation.

$$\frac{1}{e} \approx 0.37; \frac{0.37}{6} \approx 0.06$$

$$p = \frac{2}{100}, \qquad N = 50$$

$$\rightarrow \lambda = 50 \times \frac{2}{100} = 1$$

$$P(k = 3) = \frac{1^3}{3!} e^{-1} \approx \frac{0.37}{6} \approx 0.06$$

# Example

The probability of hitting a target is $\frac{1}{5}$. What is the probability of hitting the target twice in 10 independent throws?

Given that it has been hit at least once, what is the probability of it being hit twice?

The event of hitting is Bernoulli and we are interested in the total number – Binomial!

$$P\left(2|10,\frac{1}{5}\right)=\frac{10!}{2!\,8!}\left(\frac{1}{5}\right)^2\left(1-\frac{1}{5}\right)^8\approx 0.3$$

$$P(k=2|k\geq 1)=\frac{P(k=2)}{1-P(k=0)}$$

$$P(k=0)=\left(\frac{4}{5}\right)^{10}\approx 0.1$$

$$\rightarrow P(k=2|k\geq 1)=\frac{0.3}{1-0.1}=\frac{1}{3}$$

# Class Example

Are you more likely to see 2 heads in 4 throws with a fair coin, or 3 our of 4 with a biased coin $p = \frac{3}{4}$?

$$\left(\frac{3}{4}\right)^3 \approx 0.4$$

$$\frac{3}{8} = 0.375$$

$$P(k|N,p) = \binom{N}{k} p^k (1-p)^{N-k}$$

$$P\left(2|4,\frac{1}{2}\right) = \frac{1}{2^4}\frac{4!}{2!\,2!} = \frac{3}{8} = 0.375$$

$$P\left(3|4,\frac{3}{4}\right) = \frac{4!}{3!\,1!}\left(\frac{3}{4}\right)^3\left(\frac{1}{4}\right) = \left(\frac{3}{4}\right)^3 \approx 0.4$$

# Example

I toss a fair coin 10 times.

What is the probability I see exactly 4 heads?

What is the probability I see 2 or less heads?

We want

$$P\left(4|10,\frac{1}{2}\right)=\frac{10!}{4!\,(10-4)!}\left(\frac{1}{2}\right)^{4}\left(1-\frac{1}{2}\right)^{10-4}$$

$$=\frac{1}{2^{10}}\frac{10!}{4!\,(10-4)!}=\frac{105}{512}$$

$$P(2\text{ or less})=\mathrm{P}(0)+\mathrm{P}(1)+\mathrm{P}(2)$$

$$=\frac{1}{2^{10}}\left(\binom{10}{0}+\binom{10}{1}+\binom{10}{2}\right)$$