## Doppler effect

The Doppler effect is the change in frequency or wavelength of a wave (sound, light or any other) in relation to an observer who is moving relative to the wave source. For waves propagating in a medium, like sound propagating in the air, there exists a special frame in which the medium is at rest and the change depends on velocities of *both* source and receiver relative to the medium. For light propagating in vacuum there is no preferential reference frame and the frequency change can only depend on the *relative* velocity of the source and the receiver. Below the non-relativistic Doppler effect is revisited for either source or receiver moving with respect to the medium. The relativistic Doppler effect is then derived by using Lorentz transformation of time in the moving frame.

### Non-relativistic case

Let source ($S$) emit waves with frequency $\nu = 1/T$ detected by the receiver ($R$) at distance $L$ from the source (See Fig. 1). Here $T$ is the period of the periodic signal, *i.e.* the time delay between two maxima of the wave. To simplify calculations we only consider two first maxima, or, more simply, assume that the source emits two pulses with a time delay $T$ between them.

Assuming the distance between the source and the receiver to be $L$ (nothing will depend on this length) we calculate the time delay between the arrivals of these two signals to $R$. For the first case of a moving source, the first pulse arrives after $t_1 = L/c$ (we denote the speed of the signal propagation by $c$ for later convenience) and the second pulse arrives after $t_2 = T + (L + uT)/c$ as it has to travel the distance $L+uT$ as the source was displaced between the pulses. The time delay is therefore

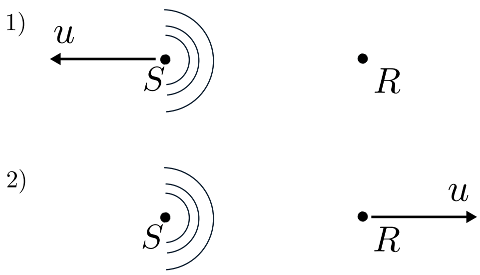


Figure 1: Doppler effect.

$$T' = t_2 - t_1 = T + L/c + uT/c - L/c = T(1 + u/c) > T.$$

Returning to the periodic signal and considering $T'$ to be its period, the frequency of the signal arriving to $R$ is $\nu' = 1/T' = \nu/(1 + u/c) < \nu$.

In the case of the moving receiver the arrival times are calculated as follows,

$$\begin{aligned} ct_1 &= L + ut_1 \\ c(t_2 - T) &= L + uT + u(t_2 - T) = L + ut_2 \end{aligned}$$

which is illustrated in Fig. 2. Solving these equations we get $t_1 = L/(c-u)$, $t_2 = (L + cT)/(c-u)$ and

$$T' = t_2 - t_1 = \frac{T}{1 - u/c},$$

which is different from the expression one obtains in the case of moving source.

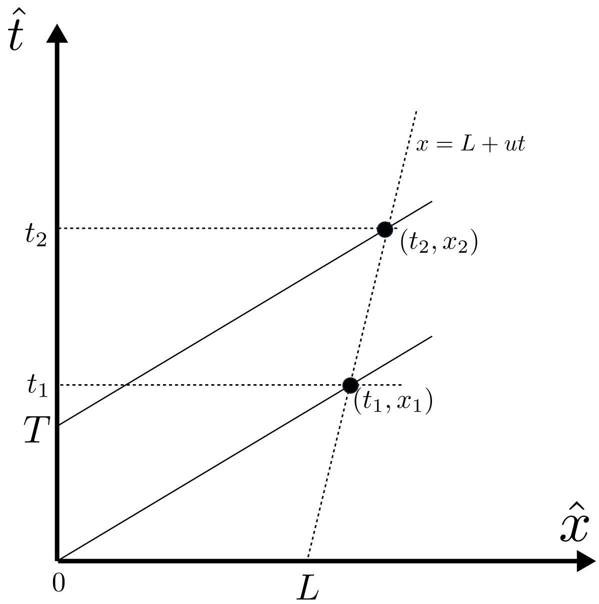


Figure 2: Space-time diagram of Doppler effect.

**Relativistic case**

For light propagation there is no medium (aether), so it is meaningless to distinguish cases moving source or moving receiver: the change of signal frequency only depends on their *relative* velocity. Neither of the above expressions is correct. To obtain correct answer with minimal efforts one proceeds as in the latter case (moving receiver) with one, very important, modification: the times of arrival of two pulses, $t_1$ and $t_2$ were measured in the frame of reference associated with the source. The time delay between these two events will be different in the moving frame of the receiver and it is this time delay will be perceived as a period $T'$ of the signal. To calculate it one has to perform a Lorentz transformation,

$$T' = \Delta t' = t'_2 - t'_1 = \gamma(u)(\Delta t - (u/c^2)\Delta x)\,.$$

Substituting $\Delta t = t_2 - t_1 = T/(1-u/c)$ as well as $\Delta x = x_2 - x_1 = u(t_2 - t_1) = uT/(1-u/c)$, one gets

$$\begin{aligned} T' &= \frac{1}{\sqrt{1-u^2/c^2}} \frac{T - (u/c^2)uT}{1-u/c} = \frac{(1+u/c)}{\sqrt{(1+u/c)(1-u/c)}} T \\ &= \sqrt{\frac{1+u/c}{1-u/c}} T\,. \end{aligned}$$

This value is greater than the period of a stationary source, $T' > T$ , but dependence on the relative velocity $u$ is different from the one obtained in both non-relativistic situations. In fact, it is interesting that the relativistic expression is exactly the geometric mean of the two.

The Doppler shifted frequency of a periodic signal is therefore

$$\nu' = \sqrt{\frac{1-u/c}{1+u/c}} \nu < \nu$$

and the wave length

$$\lambda' = \sqrt{\frac{1+u/c}{1-u/c}}\lambda\,.$$

is longer. For a visible light this corresponds to spectral lines of a moving object shifted towards the red end of the spectrum, hence the name "red shift" for manifestation of Doppler effect when the source and the receiver move away from each other (corresponding here to $u > 0$. The opposite sign of $u$ will result in the "blue shift"). An example are absorption lines of ionised calcium $\lambda' = 4750$Åobserved in a spectrum from a distant galaxy, while $\lambda = 3940$Åis observed in Earth based laboratories. This gives an estimate for the velocity of the galaxy,

$$\frac{\lambda'}{\lambda} = \sqrt{\frac{1+u/c}{1-u/c}} \quad \Rightarrow \quad \frac{u}{c} = \frac{(\lambda'/\lambda)^2 - 1}{(\lambda'/\lambda)^2 + 1} \simeq 0.2\,.$$