1VGLA: INVERSE OF A SQUARE MATRIX PRACTICE QUESTIONS

The following questions relate to Chapter 4, Inverse of a Square Matrix. Questions are ranked in difficulty from A (basic) to C (challenging).

(A) Question 1. Use the method demonstrated in the lectures to find the inverse of the matrix

$$\mathbf{A} = \begin{bmatrix} 3 & -5 & 7 \\ 0 & 5 & 0 \\ 1 & 0 & 2 \end{bmatrix}.$$

Check that your answer is correct.

(A) Question 2. Use the method demonstrated in the lectures to find the inverse of the matrix

$$\mathbf{A} = \begin{bmatrix} i & 1+i & 2+i \\ 0 & 1-i & i \\ 0 & 0 & 2-i \end{bmatrix} \in \mathcal{M}_{\mathbf{33}}(\mathbb{C}).$$

(A) Question 3. Give the elementary matrix $\mathbf{E}$ that, when multiplied to the left of a 5 by 5 matrix $\mathbf{A}$ (i.e. $\mathbf{E} \cdot \mathbf{A}$) will perform the following elementary row operation:

(a) Swaps rows 3 and 5;
(b) Divides row 2 by -3;
(c) Adds 7 times row 2 to row 4.

In each case, also give the inverse of the elementary matrix.

(A) Question 4. Give the elementary matrix $\mathbf{E}$ that, when multiplied to the right of a 3 by 5 matrix $\mathbf{A}$ (i.e. $\mathbf{A} \cdot \mathbf{E}$) will perform the following (elementary) column operation:

(a) Swaps columns 1 and 4;
(b) Multiply column 2 by 8;
(c) Adds 2 times column 5 to column 3.

In each case, also give the inverse of the elementary matrix.

(A) Question 5. Consider the elementary matrices

$$\mathbf{E}_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \text{ and } \mathbf{E}_2 = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Calculate, for $\mathbf{A} = [a_{ij}] \in \mathcal{M}_{33}(\mathbb{R})$:

(a) $\mathbf{E}_2 \cdot \mathbf{E}_1 \cdot \mathbf{A}$.
(b) $\mathbf{E}_1 \cdot \mathbf{E}_2 \cdot \mathbf{A}$.
(c) Do the calculations in (a) and (b) imply that matrix multiplication is not associative?

(B) Question 6. Show that if a diagonal matrix $\mathbf{D}$ has an inverse, then $\mathbf{D}^{-1}$ is also diagonal. Find conditions on the entries of a diagonal matrix $\mathbf{D}$ to guarantee that $\mathbf{D}$ is invertible.

(B) Question 7. Let $\mathbf{A}$ be an invertible square matrix. Prove that $\mathbf{A}^T$ is invertible. What is its inverse? *Hint:* $(\boldsymbol{X} \cdot \boldsymbol{Y})^T = \boldsymbol{Y}^T \cdot \boldsymbol{X}^T$, *and note that* $\boldsymbol{A}^T$ *was introduced as: "*$(i,j)^{th}$ *entry of* $\boldsymbol{A}^T = (j,i)^{th}$ *entry of* $\boldsymbol{A}$*".*

(B) Question 8. (a) Use the method described in lectures to calculate the inverse $\mathbf{A}^{-1}$ of the matrix

$$\mathbf{A} = \begin{bmatrix} 2 & 0 & 4 \\ 3 & 0 & 7 \\ 7 & 1 & 15 \end{bmatrix}$$

keeping track of the row operations that you use.

(b) Express $\mathbf{A}^{-1}$ as a product of elementary matrices.

(c) Express $\mathbf{A}$ as a product of elementary matrices.

(B) Question 9. (a) Use the method described in lectures to calculate the inverse $\mathbf{A}^{-1}$ of the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 3 & 4 \\ 0 & 0 & 1 \\ 2 & 8 & 8 \end{bmatrix}$$

keeping track of the row operations that you use.

(b) Express $\mathbf{A}^{-1}$ as a product of elementary matrices.

(c) Express $\mathbf{A}$ as a product of elementary matrices.

(C) Question 10. Prove the following statement using mathematical induction. Suppose that $\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_m$ are all invertible $n \times n$ matrices. Then, $(\mathbf{A}_1 \cdot \mathbf{A}_2 \cdot \ldots \cdot \mathbf{A}_m)^{-1} = \mathbf{A}_m^{-1} \cdot \mathbf{A}_{m-1}^{-1} \cdot \ldots \cdot \mathbf{A}_1^{-1}$. This is Corollary 5.2 from the notes.

(C) Question 11. An important property of numbers is the following: if $x, y \in \mathbb{R} \setminus \{0\}$ then $xy \in \mathbb{R} \setminus \{0\}$.

(a) Show that this fails for matrix multiplication by calculating $\mathbf{A} \cdot \mathbf{B}$ where

$$\mathbf{A} = \begin{bmatrix} 0 & 3 \\ 0 & 0 \end{bmatrix} \qquad \text{and} \qquad \mathbf{B} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

A non-zero matrix $\mathbf{A}$ is called a *divisor of zero* if there exists a non-zero matrix $\mathbf{B}$ such that $\mathbf{A} \cdot \mathbf{B}$ is the zero matrix (of the appropriate dimension).

(b) Show that $\begin{bmatrix} 1 & 2 \\ -\frac{1}{2} & -1 \end{bmatrix}$ is a divisor of zero.

(C) Question 12. Let $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ be matrices with dimension $m \times n$, $n \times p$ and $p \times q$, respectively. Prove that matrix multiplication is associative, i.e. that

$$(\mathbf{A} \cdot \mathbf{B}) \cdot \mathbf{C} = \mathbf{A} \cdot (\mathbf{B} \cdot \mathbf{C}).$$

(C) Question 13. Define the (i, j) elements of the $n \times n$ elementary matrix $\mathbf{E}$ and its inverse $\mathbf{E}^{-1}$ (in each of the 3 cases), and calculate the (i, j) element of $\mathbf{E} \cdot \mathbf{E}^{-1}$ directly using the definition of matric multiplication.