University of Birmingham
School of Mathematics

Real Analysis – Integration – Spring 2025

Problem Sheet 6 and Assignment 3

Issued Spring Week 3

Instructions: Your solution to the Assignment Question (**AQ1**) must be submitted via Assignments on the LC Real Analysis Canvas page before the following time:

Due 17:00 Wednesday 26 February 2025

You are strongly encouraged to attempt all of the remaining formative questions, and as many of the extra questions as you can, to prepare for the final exam, but only the Assignment Question should be submitted to Canvas. Model solutions will only be released for the Assignment Question AQ1 and Questions Q1-Q4.

Important: Late submissions will be penalised at a rate of 5% per day late up until exactly two days after the submission deadline, at which point the model solutions will be released and the Assignment will be closed to further submissions on Canvas. Your Assignment Question solutions must be submitted as a single PDF file. You may upload newer versions, BUT only the most recent upload will be viewed and graded. In particular, this means that subsequent uploads will need to contain ALL of your work, not just the parts which have changed. Moreover, if you upload a new version after the deadline, then your submission will be counted as late and the late penalty will be applied REGARDLESS of whether an older version was submitted before the deadline. In the interest of fairness to all students and staff, there will be no exceptions to these rules. All of this and more is explained in detail on the Submission of Continuous Assessment page in the Student Handbook on Canvas.

Assignment Question

AQ1. (a) Let X denote a subset of $\mathbb{R}$ and suppose that $-\infty < a < b < \infty$:

(i) Define what it means for X to be a bounded set.
(ii) Define what it means for $f : X \to \mathbb{R}$ to be a bounded function.
(iii) Define what it means for P to be a partition of the interval $[a, b]$.
(iv) Define what it means for a bounded function $f : [a, b] \to [0, \infty)$ to be integrable.
(v) State Riemann's Criterion for Integrability.

(b) Consider the function $f : [1, 6\pi] \to [-9, 7000]$ defined by[1]

$$f(x) = \begin{cases} x^2, & x < 2\pi; \\ 3 + \sin(x), & x \in [2\pi, 4\pi]; \\ e^{-x}, & x > 4\pi. \end{cases}$$

(i) Suppose that $x_0 = 1$, $x_1 = 2$ and $x_2 = 4\pi$. For each $i \in \{1, 2\}$, find the interval I_i such that $\{f(x) : x \in [x_{i-1}, x_i]\} = I_i$. Note that here an interval refers to any of $[a, b]$, $(a, b]$, $[a, b)$, or (a, b), where $-\infty < a < b < \infty$.
(ii) Suppose that $P = \{1, 2, 4\pi, 6\pi\}$. Calculate the Riemann–Darboux sum $L(f, P)$.
(iii) Suppose that $Q = \{1, 3, 5, 5\pi, 6\pi\}$ and $R = \{1, 3, 2\pi, 6\pi\}$. Calculate $U(f, Q \cup R)$.

(c) Use Riemann's Criterion to prove that $f : [12, 22] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 7, & 12 \leq x \leq 15; \\ 4, & 15 < x < 22; \\ 15, & x = 22, \end{cases}$$

is integrable.

The Riemann–Darboux sums calculator (Spring Week 1 Materials on Canvas) can be used for this question, although it is not needed, and you must include a screenshot of any calculation used to obtain your answers.

[1] Correction (updated at 18:00 on 7 February 2025): The defintion of f was updated to require $x \in [2\pi, 4\pi]$ and $x > 4\pi$ (instead of $x \in [2\pi, 4\pi)$ and $x \geq 4\pi$).

Questions

Q1. (a) Sketch the graph of $f : [0,4] \to \mathbb{R}$, $f(x) = \sqrt{x}$ and highlight the area covered by the difference $U(f,P) - L(f,P)$ for the partition $P = \{0,1,2,3,4\}$.

(b) Use Riemann's Criterion to prove each of the functions below are integrable:

(i) $f : [0,3] \to [0,\infty)$, $f(x) = x^2$

(ii) $f : [2,4] \to [0,100]$, $f(x) = \begin{cases} 5, & x < 3; \\ 100, & x = 3; \\ 3, & x > 3. \end{cases}$

You may use the results stated in **Q4**(b) on Problem Sheet 5 to answer (i).

Q2. (a) State Riemann's Criterion for integrability.

(b) For each $n \in \mathbb{N}$, let P_n denote the partition of $[a,b]$ into n subintervals of equal width. If $f : [a,b] \to [0,\infty)$ is monotonic increasing, then we have seen that

$$U(f,P_n) - L(f,P_n) = (f(b) - f(a))(b-a)\tfrac{1}{n}. \qquad (1)$$

(i) Now prove that if $f : [a,b] \to [0,\infty)$ is monotonic decreasing, then

$$U(f,P_n) - L(f,P_n) = (f(a) - f(b))(b-a)\tfrac{1}{n}. \qquad (2)$$

(ii) Use Riemann's Criterion to prove that $f : [-2,5] \to \mathbb{R}$ given by $f(x) = x^2 - 4x + 9$ for all $x \in [-2,5]$ is integrable. Here you may combine (1) and (2) with results from the Lecture Notes, but you are NOT allowed to use that monotonic functions and continuous functions are integrable.

(c) Use Riemann's Criterion to prove directly that $g : [2,4] \to \mathbb{R}$ given by

$$g(x) := \lceil x \rceil := \min\{n \in \mathbb{N} : n \geq x\} \text{ for all } x \in [2,4]$$

is integrable. Here you are NOT allowed to use any results from (b) and you are NOT allowed to use that monotonic functions and continuous functions are integrable.

Q3. Suppose that $f : [a,b] \to [0,\infty)$ is bounded and integrable, where $-\infty < a < b < \infty$:

(a) Prove that if $\alpha \geq 0$ and P is a partition of $[a,b]$, then

$$U(\alpha f, P) = \alpha U(f,P) \quad \text{and} \quad L(\alpha f, P) = \alpha L(f,P).$$

(b) Use Riemann's Criterion and (a) to prove that αf is integrable.

(c) Explain, without working hard, why the integrability of αf implies that

$$\int_a^b (\alpha f) = \sup\{L(\alpha f, P) : P \text{ is a partition of } [a,b]\}.$$

(d) Use (a) and (c) to conclude that $\int_a^b (\alpha f) = \alpha(\int_a^b f)$.

Q4. Suppose that $f : [a,b] \to [0,\infty)$ is bounded and integrable, where $-\infty < a < b < \infty$, and that $0 \leq f(x) \leq M$ for all $x \in [a,b]$ and some $M > 0$:

(a) Prove that $0 \leq \underline{\int_a^b} f$ and that $\overline{\int_a^b} f \leq M(b-a)$.

(b) Use (a) and the definition of the integral to prove that $0 \leq \int_a^b f \leq M(b-a)$.

(c) Use (b) to prove that $\lim_{h\to 0^+} \left(\int_a^{a+h} f\right) = 0$ and $\lim_{h \to 0^+} \left(\int_{b-h}^b f\right) = 0$.

Extra Questions

EQ1. Suppose that $f : [a, b] \to [0, \infty)$ is bounded and integrable, where $-\infty < a < b < \infty$. A *tagged partition* (P, T) of $[a, b]$ consists of a partition $P = \{x_0, x_1, \ldots, x_n\}$ of $[a, b]$ and a collection $T = \{t_1, \ldots, t_n\}$ of *tags* satisfying $t_1 \in [x_0, x_1], \ldots, t_n \in [x_{n-1}, x_n]$. The corresponding *Riemann Sum* is defined by

$$R(f, P, T) := \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1}).$$

(a) Prove that $L(f, P) \leq R(f, P, T) \leq U(f, P)$ for any tagged partition (P, T).
(b) Prove that $L(f, P) \leq \int_a^b f \leq U(f, P)$ for any partition P.
(c) Use Riemann's Criterion to prove that for each $\epsilon > 0$, there exists a partition P such that $|R(f, P, T) - \int_a^b f| < \epsilon$ whenever T is a collection of tags for P.

EQ2. (a) For each function defined below, use the properties of the function and results from Lectures/Lectures Notes to prove that it is integrable:
  (i) $f : [0, 10] \to \mathbb{R}$, $f(x) = 3x^2 + 5x + 9$
  (ii) $g : [1, 100] \to \mathbb{R}$, $g(x) = \lfloor x \rfloor := \max\{n \in \mathbb{N} : n \leq x\}$
(b) Use Riemann's Criterion to prove that each function in part (a) is integrable. You may use the results stated in **Q4**(b) on Problem Sheet 5 (but you are not required to do so).

EQ3. (a) Prove that if $f : [a, b] \to \mathbb{R}$ and $g : [a, b] \to \mathbb{R}$ are both uniformly continuous, then $f + g$ is uniformly continuous.
(b) Suppose that $f : [a, \infty) \to \mathbb{R}$ is a continuous function, where $-\infty < a < b < \infty$. Prove that if f is uniformly continuous on both $[a, b]$ and $[b, \infty)$, then f is uniformly continuous on $[a, \infty)$.

EQ4. Suppose that $f : [1, 5] \to \mathbb{R}$ and $g : [2, 6] \to [-2, 10]$ are both integrable functions, whilst $10 \leq f(x) \leq 1000$ for all $x \in [2, 4]$. For each of the integrals below, use the properties of integrable functions to prove that the integral exists, and then find an upper bound and a lower bound for the value of the integral:

(a) $\int_2^4 f$
(b) $\int_2^5 (f - g)$
(c) $\int_3^4 6fg$