Video Recording Material Week 5

Capacitors

1. Introduction

This week we will consider circuits in which the current (and voltage) is not constant with time. When a source is applied to a network of resistors, the current is assumed to flow instantaneously. However, when a circuit contains components such as capacitors or inductors, the circuit can take a considerable amount of time to reach a steady state when a source is suddenly added or removed. This is known as the ***transient response*** of the circuit. In practice all circuits contain some stray capacitance or inductance and in certain applications, where a fast response is required, this can cause undesirable effects.

This week, we will study capacitors as circuit elements.
In the video material I will derive the time dependence of the current and voltage in a simple series circuit containing a capacitor and a resistor. This will involve solving a linear differential equation. What I want to show you is that the functional form of the solution is universal. That is, once known, it can be applied to a wide range of problems without having to derive it in every problem. Our aim is to find a shorthand method of determining the response of a circuit, which will mean never having to solve another differential equation. (In this course at least.)

2. Capacitors

Figure 5.1 lists the basic equations for capacitors that you are expected to know. You will meet capacitors as part of the course on electromagnetism, which is part of the same module, so I will only go over the main points here.

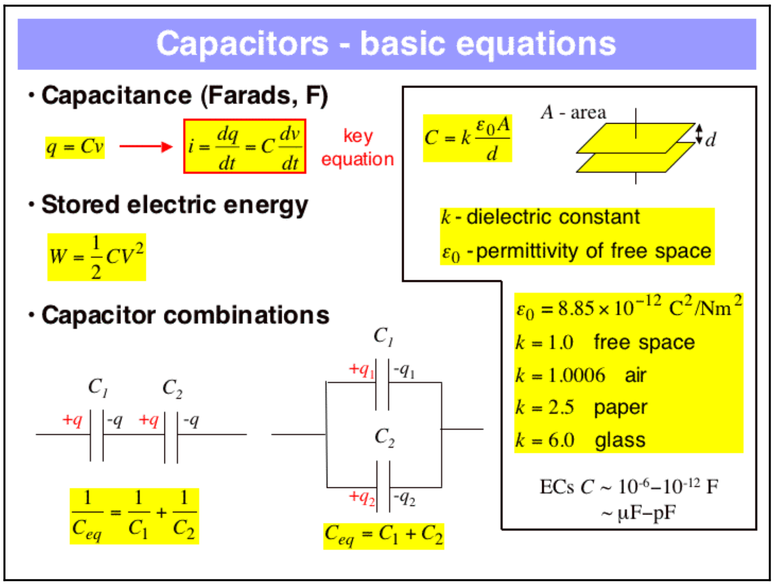

Figure 5.1: Capacitor basic equations.

Capacitance is defined as the ratio of the charged stored on a capacitor to the potential difference measured across it.

$$C = \frac{Q}{V}$$

In general we will be concerned about how the charge on the capacitor is changing with time, as it is charging up or being discharged. From this point I will use lower case characters to represent quantities that are changing with time. Making the charge the subject of this equation, we can find an expression for the current "through" the capacitor by differentiating this expression with respect to time.

$$q = Cv$$

$$i = \frac{dq}{dt} = C\frac{dv}{dt} \tag{1}$$

In actual fact, charge does not flow "through" the capacitor. What actually happens is that charge flows from one side of the capacitor to the other via an external circuit, a resistor and battery for example. With charge flowing away from one side and arriving at the other via the external circuit, it appears that current is flowing through the capacitor. It is ok to say "the current through the capacitor" as long as you understand what is really going on. For further explanation, continue the next section.

Calculating the energy stored on a capacitor and combinations of capacitors are well described in Tipler, so I have left this as further reading (see the recommendations at the end of this Lecture). You are expected to remember the results summarised in Figure 5.1.

3. Capacitors as circuit elements

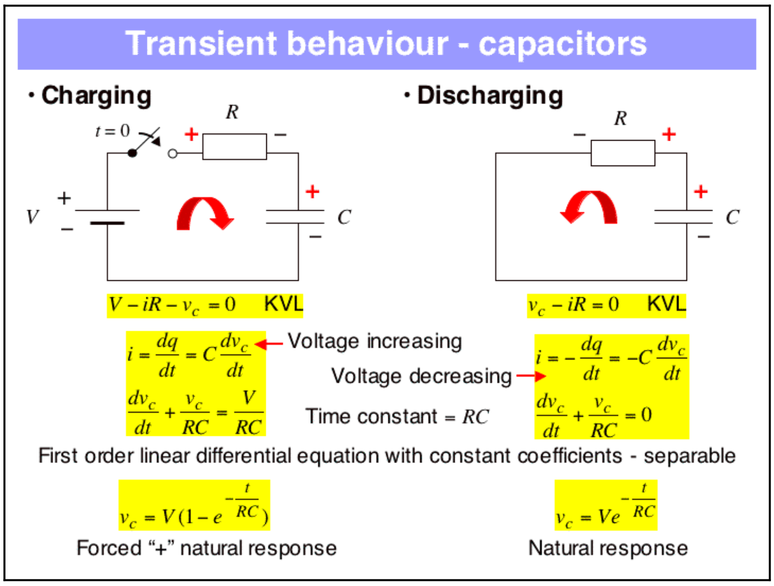

Figure 5.2: Charging and discharging a capacitor.

Figure 5.2 shows a capacitor being charged and discharged through a resistor. The arrows show the direction of the current in each case. We can solve for the voltage across the capacitor as a function of time using:

i) Kirchhoff's laws and

ii) the relationship between the current and the voltage across the capacitor, given in equation (1).

3.1 Charging a capacitor

Let's consider the case of charging the capacitor. We shall assume that the capacitor is initially uncharged and that the switch, *S*, is closed at time, t = 0. When the switch is closed, a current will flow transferring charge onto the top capacitor plate. Because this is a series circuit, Kirchhoff's current law tells us that an equal amount of charge must flow away from the bottom capacitor plate. ***Why?*** Because the current is the same at all points in a series circuit.

We can use Kirchhoff's voltage law to sum up all the potential rises and drops around the circuit (and set this equal to zero). I've indicated with a "+" sign which side of each component is at the higher electrical potential in Figure 5.2. Since conventional current flow is the direction in which positive charge would travel, the top capacitor plate receives an excess of positive charge. This determines the sign of potential difference across the capacitor.

$$V - iR - v_C = 0$$

In the equation given by Kirchhoff's voltage law there are two unknowns. The current, i, and the voltage across the capacitor, v_C. We can eliminate one of these by substituting for the current, using the expression:

$$i = C\frac{dv_C}{dt}$$

We now have a first order linear differential equation in the unknown voltage across the capacitor. It is first order because the differential term is d/dt and not d^2/dt^2 or d^3/dt^3 *etc*. It is linear because the terms only involve v_C and not v_C^2 *etc.* Separating the terms involving v_C from the constant term and rearranging the differential equation becomes.

$$\frac{dv_C}{dt} + \frac{v_C}{RC} = \frac{V}{RC}$$

Now before trying to solve this equation, notice one thing. For this equation to be dimensionally correct, each term must have dimensions of voltage over time. ***Why?*** Because we can only add up and equate quantities with the same dimensions. *(For example, a sum of currents can never be equal to a voltage.)* The first term has dimensions of voltage over time. This means that the product RC must have dimensions of time. As we will see, RC is the *time constant* of the circuit.

This type of differential equation can be solved by integration because we can separate the variables (v_C and t). The steps are shown below. You will have done this kind of thing in your maths course, so it should look quite familiar.

$$\frac{dv_C}{dt} = \frac{1}{RC}\left(V - v_C\right)$$

$$\frac{dv_C}{\left(V - v_C\right)} = \frac{dt}{RC}$$

Integrating,

$$\int \frac{dv_C}{\left(V - v_C\right)} = \int \frac{dt}{RC}$$

$$-\ln\left(V - v_C\right) = \frac{t}{RC} + c$$

where c is the constant of integration. Applying the initial condition that $v_C = 0$, when t=0, we find that $c = -\ln V$. The solution may be found by substituting for c and by rearranging the resulting equation.

$$v_C = V\left(1 - e^{-\frac{t}{RC}}\right)$$

There are two terms in this equation. A constant term, known as the ***forced response***, and a time dependent term, known as the ***transient response***. The forced response is due to the battery, which determines the final voltage across the capacitor. The transient part is the exponential term that dies away with time.

3.2 Discharging a capacitor

A similar procedure can be followed to find the voltage across the capacitor when the battery is removed. The key steps are shown in figure 5.2. I won't go through all the steps in detail again. The key point here is that the charge on the capacitor plates is decreasing with time. Therefore, the voltage across the capacitor is decreasing. In order to convey this, we must insert a minus sign in front of the differential terms. This indicates a negative, or decreasing, trend.

$$i=-\frac{dq}{dt}=-C\frac{dv_C}{dt}$$

This time, the differential equation we obtain does not have a constant (or forcing) term, since we no longer have a battery.

$$\frac{dv_C}{dt}+\frac{v_C}{RC}=0$$

Solving this equation in the same way, we must apply the boundary condition that voltage across the capacitor $v_C = V$, when the source is removed at time $t = 0$. The solution is found to be:

$$v_C=Ve^{-\frac{t}{RC}}$$

In the absence of a forcing term (no battery), we are left with only a transient term that decays away with time. Of course, what happens when the battery is removed is that the capacitor discharges. What is more, it does so with the same characteristic time constant as before. This is perhaps not surprising given that all we have done is replace the battery with a short-circuit.

There are a couple of observations that can be made at this point, which will make it possible for us in future to deduce the response of an RC circuit without going to the trouble of solving a different equation:

1. The solutions only depend on the initial and final values of the voltage.
2. The time dependence is of the form exp($-t/\tau$), where τ is the time constant of the circuit.

4. General solution for RC circuits

Using the preceding observations, we can deduce the general solution for RC circuits as shown in Figure 5.3.

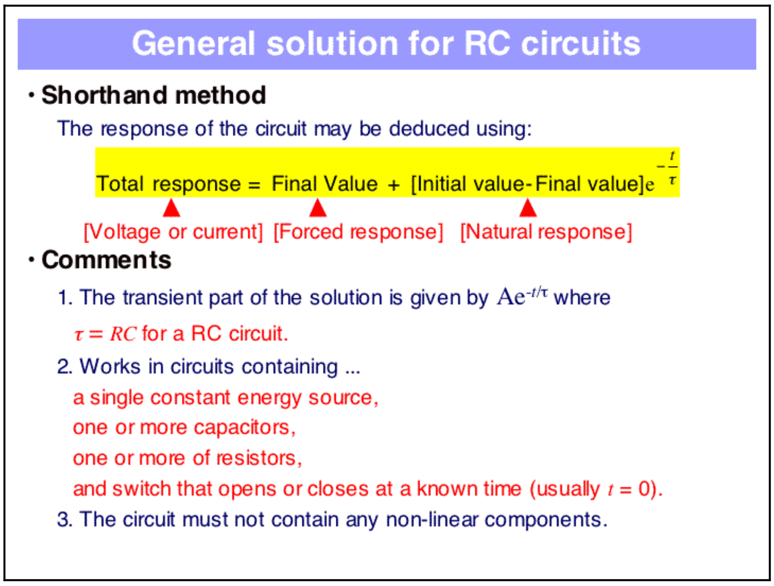

Figure 5.3: General solution for RC circuits.

Let's check this general method against the solutions we just obtained for charging or discharging a capacitor in a simple RC circuit.

<u>Case 1.</u> Capacitor charging.

- The initial value of $v_C = 0$, given in the problem.
- The final value of $v_C = V$, equal to the voltage across the battery.
- The time constant is RC.

Inserting these values into the shorthand equation shown in Figure 5.3 we obtain

$$v_c = V + (0 - V)e^{-\frac{t}{RC}} \Rightarrow v_c = V\left(1 - e^{-\frac{t}{RC}}\right)$$

<u>Case 2:</u> Capacitor discharging.

- In the initial value of $v_C = V$, equal to the voltage across the battery.
- The final value of $v_C = 0$.
- The time constant is RC, as before.

Inserting these values into the shorthand equation shown in Figure 5.3 we obtain

$$v_c = 0 + (V - 0)e^{-\frac{t}{RC}} \Rightarrow v_c = Ve^{-\frac{t}{RC}}$$

Hopefully, you can see that all this has been worth the effort. We now have a quick method of deducing the response of any RC circuit, which can be applied to a wide range of problems. Before we do that, let's examine how the capacitor behaves in response to a sudden change in a circuit, as we will find this useful.

5. How does a capacitor respond when a voltage is applied or removed?

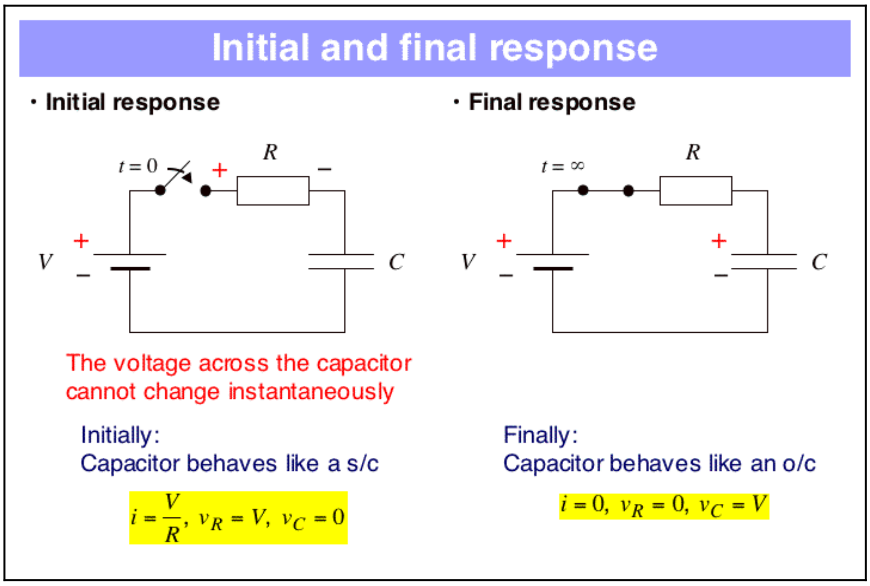

Figure 5.4: The initial and final response of capacitor.

Figure 5.4 summarises the initial and final response of a capacitor to a sudden change in the applied voltage. This is represented by the opening or closing of a switch that turns on or off a constant voltage source.

The current “through” a capacitor is given by the time rate of change of the voltage across the capacitor plates. The voltage across the capacitor cannot change instantaneously when the switch is closed, since this would imply that the current $i=dv_C/dt$ is infinite! Therefore, the capacitor must be in the same state just ***after*** the switch is closed as it was in just ***before*** the switch was closed. In this case, there is no voltage across the capacitor initially, so to satisfy Kirchhoff’s voltage law, all the battery voltage must be dropped across the resistor. The capacitor behaves as if it was a short-circuit (s/c). This won't always be the case, as you will see below. The point to remember is that ***it takes time for charge to flow onto the capacitor***. In other words, we have to take into account the initial state of the capacitor when determining the initial current (or voltage) in the circuit.

As current flows, charge will gradually build up on the capacitor plates. This will continue until the potential difference across the resistor is zero, at which point no more charge can be transferred. Finally, the capacitor must behave like an open-circuit (o/c).

This is a very useful way to think about capacitor behaviour and should help you to determine the initial and final currents in circuits that are subject to a sudden change in the applied voltage.

6. Further reading

Tipler 24-4: Capacitors, batteries and circuits.
This section deals with combination of capacitors.
Tipler 25-6: RC circuits.
This section deals with the energy stored on a capacitor. It is well worth going through the worked examples.