VGLA: CONICS PRACTISE QUESTIONS

The following questions relate to Chapter 6 - Conics, and Appendix C - Conics: Optional Extra Content. Questions are ranked in difficulty from A (basic) to C (challenging).

(A) Question 1. Show that each of the following equations represents a single straight line.

(a) $x^2 + 4xy - 6x + 4y^2 - 12y + 9 = 0$.
(b) $x^2 - 4xy + 2x + 4y^2 - 4y + 1 = 0$.

Solution:

(a) Assume that

$$\begin{aligned} & x^2 + 4xy - 6x + 4y^2 - 12y + 9 = && (ax + by + c)^2 \\ \iff \quad & x^2 + 4xy - 6x + 4y^2 - 12y + 9 = && a^2x^2 + 2abxy + b^2y + 2acx + 2bcy + c^2 \end{aligned}$$

From the coefficients of x^2, y^2 and the constant term respectively, it follows that $a = \pm 1$, $b = \pm 2$ and $c = \pm 3$. We now match the remaining coefficients:

$$\begin{array}{rccccl} (xy): & 2ab = 4 & \iff & ab = 2 & \iff & (a,b) = (1,2) \text{ or } (a,b) = (-1,-2) \\ (x): & 2ac = -6 & \iff & ac = -3 & \iff & (a,c) = (1,-3) \text{ or } (a,c) = (-1,3) \\ (y): & 2bc = -12 & \iff & bc = -6 & \iff & (b,c) = (2,-3) \text{ or } (b,c) = (-2,3). \end{array}$$

It follows that the quadratic equation represents the line $x + 2y - 3 = 0$.

(b) Assume that

$$\begin{aligned} & x^2 - 4xy + 2x + 4y^2 - 4y + 1 = && (ax + by + c)^2 \\ \iff \quad & x^2 - 4xy + 2x + 4y^2 - 4y + 1 = && a^2x^2 + 2abxy + b^2y + 2acx + 2bcy + c^2 \end{aligned}$$

From the coefficients of x^2, y^2 and the constant term respectively, it follows that $a = \pm 1$, $b = \pm 2$ and $c = \pm 1$. We now match the remaining coefficients:

$$\begin{array}{rccccl} (xy): & 2ab = -4 & \iff & ab = -2 & \iff & (a,b) = (1,-2) \text{ or } (a,b) = (-1,2) \\ (x): & 2ac = 2 & \iff & ac = 1 & \iff & (a,c) = (1,1) \text{ or } (a,c) = (-1,-1) \\ (y): & 2bc = -4 & \iff & bc = -2 & \iff & (b,c) = (2,-1) \text{ or } (b,c) = (-2,1). \end{array}$$

It follows that the quadratic equation represents the line $x - 2y + 1 = 0$.

■

(A) Question 2. Show that each of the following equations represents a pair of straight lines and find their point of intersection.

(a) $2x^2 - xy + x - y^2 + 2y - 1 = 0$.
(b) $2x^2 - xy + 5x - y^2 + y + 2 = 0$.

Solution:

(a) For this to be true, the it must be represented by some single line $ax + by + c$. Since the equation is quadratic, we must have

$$\begin{aligned} & 2x^2 - xy - y^2 + x + 2y - 1 = && (ax + by + c)(dx + ey + f) \\ \iff \quad & 2x^2 - xy - y^2 + x + 2y - 1 = && adx^2 + (ae + bd)xy + bey^2 + (af + dc)x + (bf + ce)y + cf \end{aligned}$$

Since for each line, we can freely choose 1 of the coefficients, we set $a = 2$ and $d = 1$ so that the x^2 coefficient is correct. We now match the remaining coefficients:

$$(1)\qquad \begin{array}{rcccl} (xy): & 2e+b=-1 & \iff & b=-(1+2e) \\ (y^2): & be=-1 & \iff & b=-\frac{1}{e} \\ (x): & 2f+c=1 & \iff & c=1-2f \\ (y): & bf+ce=2 & \iff & bf=2-ce \\ (x^0): & cf=-1 & \iff & f=-\frac{1}{c} \end{array}$$

Via lines 1 and 2 in (1), it follows that

$$(2)\qquad 2eb+b^2=-b \iff b^2+b-2=0 \iff (b+2)(b-1)=0 \iff (i)\ b=1 \text{ or } (ii)\ b=-2.$$

Therefore, via (2) and line 2 in (1), we have

$$(3)\qquad (i)\ e=-1 \text{ or } (ii)\ e=\tfrac{1}{2}.$$

Now, via lines 4 and 5 in (1), and (2) and (3), it follows that

$$(4)\qquad bcf+c^2e=2c \iff c^2e-2c-b=0 \iff \begin{cases} (i) & -(c+1)^2=0 \iff c=-1 \\ (ii) & \frac{1}{2}(c-2)^2=0 \iff c=2. \end{cases}$$

Via line 5 in (1) and (4), it follows that

$$(5)\qquad f=-\tfrac{1}{c} \iff (i)\ f=1 \text{ or } (ii)\ f=-\tfrac{1}{2}.$$

Since c and f given in (4) and (5) also satisfy line 3 in (1), it follows from all of the above that the quadratic equation is solved by 2 distinct lines L_1 and L_2, given by

$$2x^2-xy-y^2+x+2y-1=(2x+y-1)(x-y+1)=0,$$

i.e.

$$L_1:\ 2x+y-1=0 \text{ and } L_2:\ x-y+1=0.$$

Note that there were 2 cases because we could have represented the two lines in two different orders. The point of intersection of L_1 and L_2 can be found by substitution to be $(x, y) = (0, 1)$.

(b) After comparably arduous calculations one can determine that the quadratic equation is solved by 2 distinct lines L_1 and L_2, given by

$$2x^2-xy-y^2+5x+y+2=(2x+y+1)(x-y+2)=0,$$

i.e.

$$L_1:\ 2x+y+1=0 \text{ and } L_2:\ x-y+2=0.$$

The point of intersection of L_1 and L_2 is $(x, y) = (-1, 1)$.

We will revisit questions 1 and 2, and demonstrate an approach based on Section 6.1 of the notes ... once we have covered Chapter 10. Specifically we will show how to choose a suitable rotation of the xy plane to simplify the original quadratic equations in 2 variables (to eliminate the xy terms). ■

(A) Question 3. By completing the square (translation), eliminate the linear terms in the following equations, and hence, identify the conics.

(a) $3x^2 + 24x + 53 - 2y^2 + 4y = 0$;
(b) $2x^2 - 4x + 10 + 4y^2 + 16y = 0$;
(c) $y^2 - 2y - 11 - 6x = 0$.

Solution:

(a) Since

$$3x^2+24x+53-2y^2+4y=0 \iff 3(x+4)^2-48-2(y-1)^2+2+53=0 \iff \frac{(y-1)^2}{7/2}-\frac{(x+4)^2}{7/3}=1,$$

it follows that the equation defines a hyperbola. *To see this slightly more clearly, consider the translated co-ordinates* $\tilde{x}=x+4$ *and* $\tilde{y}=y-1$.

(b) Since

$$2x^2-4x+10+4y^2+16y=0 \iff 2(x-1)^2-2+4(y+2)^2-16+10=0 \iff \frac{(x-1)^2}{4}+\frac{(y+2)^2}{2}=1,$$

it follows that the equation defines an ellipse.

(c) Since

$$y^2-2y-11-6x=0 \iff (y-1)^2-1-6x-11=0 \iff (y-1)^2=6(x+2),$$

it follows that the equation defines a parabola.

$\blacksquare$

(A) Question 4. Find equations, in x, y co-ordinates, for the given ellipses.

(a) Foci at $(0, \pm 2)$ semi-major axis (or equivalently, the distance from centre to vertex) 3;
(b) Foci at $(0, 1)$ and $(4, 1)$ and eccentricity $\frac{1}{2}$;

Solution:

(a) Since the foci are at $(0, \pm 2)$ it follows that: the form of the ellipse is $\frac{x^2}{b^2}+\frac{y^2}{a^2}=1$; and $2^2=c^2=a^2-b^2$. Since the semi-major axis is 3, it follows that $a=3$. Hence the equation for the ellipse is

$$\frac{x^2}{5}+\frac{y^2}{9}=1.$$

Note that the semi-major axis is half the length of the major axis.

(b) The centre of the ellipse is $(x, y)=(2, 1)$. Since the foci are at $(x, y)=(0, 1)$ and $(x, y)=(4, 1)$ it follows that: the form of the ellipse is $\frac{(x-2)^2}{a^2}+\frac{(y-1)^2}{b^2}=1$; and $2^2=c^2=a^2-b^2$. Also, since $\frac{1}{2}=e=\frac{c}{a}=\frac{2}{a}$ it follows that $a=4$. Therefore, the ellipse is given by

$$\frac{(x-2)^2}{16}+\frac{(y-1)^2}{12}=1.$$

$\blacksquare$

(A) Question 5. Sketch the following ellipses and find their centre and semi-axes.

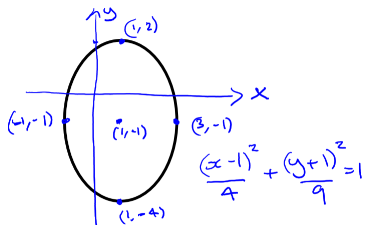
(a) $9x^2+4y^2-18x+8y=23$;
(b) $2x^2-4x+10+4y^2+16y=0$.

Solution:

(a) Since

$$9x^2+4y^2-18x+8y=23 \iff 9(x-1)^2-9+4(y+1)^2-4=23 \iff \frac{(x-1)^2}{4}+\frac{(y+1)^2}{9}=1,$$

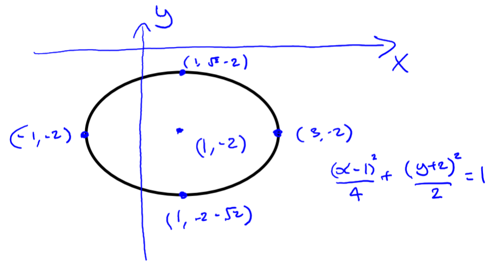
the equation defines an ellipse with semi-major axis 3 and parallel to the y-axis; semi-minor axis 2; and with centre at $(x, y)=(1, -1)$.



(b) Since

$$2x^2-4x+10+4y^2+16y=0 \iff 2(x-1)^2-2+4(y+2)^2-16+10=0 \iff \frac{(x-1)^2}{4}+\frac{(y+2)^2}{2}=1,$$

the equation defines an ellipse with semi-major axis 2 and parallel to the x-axis; semi-minor axis $\sqrt{2}$; and with centre at $(x,y)=(1,-2)$.



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(A) Question 6. Find the vertex, focus and directrix of the parabola with equation

(a) $y=-(1/6)x^2$;
(b) $2y^2=-3x$; and
(c) $y^2=100x$.

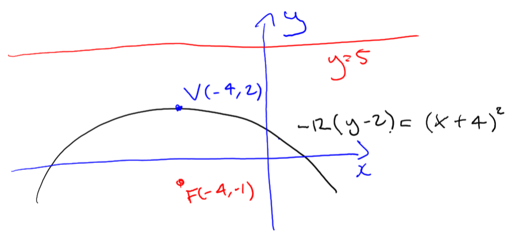
Solution:

(a) Since $y=-(1/6)x^2 \iff x^2=-6y$, the parabola has $p=\frac{3}{2}$ and is opening downwards. The vertex is at $(x,y)=(0,0)$, the focus is at $(x,y)=(0,-\frac{3}{2})$ and the directrix is $y=\frac{3}{2}$.
(b) Since $2y^2=-3x \iff -\frac{3}{2}x=y^2$ it follows that the parabola has $p=\frac{3}{8}$ and opens to the left. The vertex is at $(x,y)=(0,0)$, the focus is at $(x,y)=(-\frac{3}{8},0)$ and the directrix is $x=\frac{3}{8}$.
(c) Since $y^2=100x$, the parabola has $p=25$ and is opening to the right. The vertex is at $(x,y)=(0,0)$, the focus is at $(x,y)=(25,0)$ and the directrix is $x=-25$.

■

(A) Question 7. Find the equation of the parabola with vertex $V(x,y)=(-4,2)$ and directrix $y=5$ and sketch it.

Solution: Since the vertex is at $(-4,2)$ and the directrix is $y=5$, it follows that the focus is at $(-4,-1)$ and $p=3$. Therefore, the parabola has equation $-12(y-2)=(x+4)^2$.



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(A) Question 8. Find the equation of the tangent to the parabola with equation $y^2 = x$ at the point $P(x, y) = (16, -4)$.

Solution: We find the tangent line by implicit differentiation (for $y \neq 0$)

$$\frac{d}{dx}\left(y^2\right) = \frac{d}{dx}(x) \iff 2y\frac{dy}{dx} = 1 \iff \frac{dy}{dx} = \frac{1}{2y},$$

and hence at $(x, y) = (16, -4)$, we have $\frac{dy}{dx} = -\frac{1}{8}$. Therefore, the tangent to the parabola at $(x, y) = (16, -4)$ is

$$y + 4 = -\frac{1}{8}(x - 16).$$

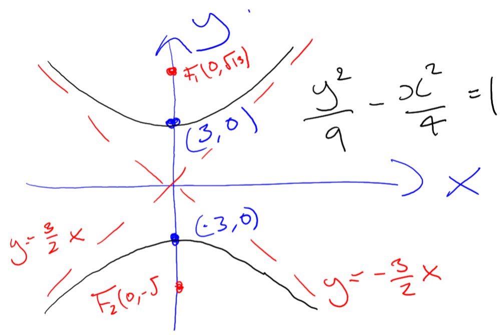
Note that at $y = 0$ the tangent, in the form $y = mx + c$ is undefined since the tangent at $(x, y) = (0, 0)$ is of the form $x = 0$. ■

(A) Question 9. Find the vertices and the foci of the hyperbola with given equation. Moreover, sketch the hyperbola, asymptotes and foci.

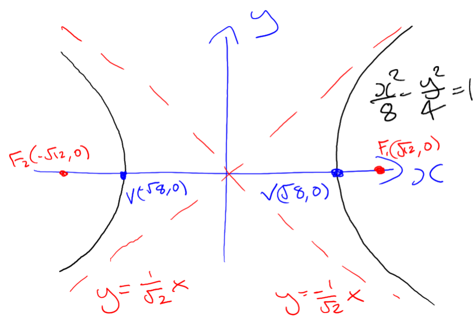
(a) $y^2/9 - x^2/4 = 1$;
(b) $x^2 - 2y^2 = 8$.

Solution:

(a) Since $y^2/9 - x^2/4 = 1$, it follows that: the focal axis is the y-axis; the vertices are at $(x, y) = (0, \pm 3)$; the foci are at $(x, y) = (0, \pm\sqrt{13})$ since $c^2 = a^2 + b^2$; and the asymptotes are given by $y = \pm\frac{3}{2}x$.



(b) Since $x^2 - 2y^2 = 8 \iff \frac{x^2}{8} - \frac{y^2}{4} = 1$, it follows that: the focal axis is the x-axis; the vertices are at $(x, y) = (\pm\sqrt{8}, 0)$; the foci are at $(x, y) = (\pm\sqrt{12}, 0)$ since $c^2 = a^2 + b^2$; and the asymptotes are given by $y = \pm\frac{1}{\sqrt{2}}x$.



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(A) Question 10. Find the equation of the tangent to the hyperbola with equation

$$\frac{x^2}{6} - \frac{y^2}{8} = 1,$$

at the point $P(x, y) = (3, 2)$.

Solution: We find the tangent line by implicit differentiation ($y \neq 0$)

$$\frac{d}{dx}\left(\frac{x^2}{6} - \frac{y^2}{8}\right) = \frac{d}{dx}(1) \iff \frac{x}{3} - \frac{y}{4}\frac{dy}{dx} = 0 \iff \frac{dy}{dx} = \frac{4x}{3y},$$

and hence at $(x, y) = (3, 2)$, we have $\frac{dy}{dx} = 2$. Therefore, the tangent to the hyperbola at $(x, y) = (3, 2)$ is

$$y - 2 = 2(x - 3).$$

Note that here we could have used the formula directly from the notes, but hopefully it is clear that this approach can be used to find the tangent to hyperbola not necessarily with centre at (0,0). ■

(B) Question 11. By rotating the coordinate system in the following equations, eliminate the cross term xy. Hence identify the type of conics from the equations.

(a) $2x^2 + 2xy + 2y^2 - 5 = 0$;
(b) $-2x^2 + 2\sqrt{3}xy - 4 = 0$;
(c) $\frac{3}{4}x^2 + xy\frac{\sqrt{3}}{2} + \frac{y^2}{4} + 6x - 6y\sqrt{3} = 0$.

Solution: In general, given a real-valued quadratic equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0,$$

if we can find θ such that

$$\tan(2\theta) = \frac{B}{A - C} \text{ or } \cot(2\theta) = \frac{A - C}{B},$$

we can define a rotation of the xy plane to the $\tilde{x}\tilde{y}$ plane so that the quadratic equation in $\tilde{x}$ and $\tilde{y}$ has no $\tilde{x}\tilde{y}$ term.

(a) Since here, we have

$$\cot(2\theta) = 0 \iff 2\theta = \frac{\pi}{2} + k\pi \iff \theta = \frac{\pi}{4} + \frac{k\pi}{2},$$

we can set $\theta = \frac{\pi}{4}$, i.e.

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\left(\frac{\pi}{4}\right) & -\sin\left(\frac{\pi}{4}\right) \\ \sin\left(\frac{\pi}{4}\right) & \cos\left(\frac{\pi}{4}\right) \end{pmatrix} \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}}(\tilde{x} - \tilde{y}) \\ \frac{1}{\sqrt{2}}(\tilde{x} + \tilde{y}) \end{pmatrix}.$$

Thus, we have

$$\begin{aligned} & 2x^2 + 2xy + 2y^2 - 5 = 0 \\ \iff & (\tilde{x} - \tilde{y})^2 + (\tilde{x} - \tilde{y})(\tilde{x} + \tilde{y}) + (\tilde{x} + \tilde{y})^2 - 5 = 0 \\ \iff & \tilde{x}^2 - 2\tilde{x}\tilde{y} + \tilde{y}^2 + \tilde{x}^2 - \tilde{y}^2 + \tilde{x}^2 + 2\tilde{x}\tilde{y} + \tilde{y}^2 - 5 = 0 \\ \iff & \frac{\tilde{x}^2}{5/3} + \frac{\tilde{y}^2}{5} = 1 \end{aligned}$$

which is an ellipse.

(b) Since here, we have

$$\tan(2\theta) = -\sqrt{3} \iff 2\theta = -\frac{\pi}{3} + k\pi \iff \theta = -\frac{\pi}{6} + \frac{k\pi}{2},$$

we can set $\theta = -\frac{\pi}{6}$, i.e.

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\left(-\frac{\pi}{6}\right) & -\sin\left(-\frac{\pi}{6}\right) \\ \sin\left(-\frac{\pi}{6}\right) & \cos\left(-\frac{\pi}{6}\right) \end{pmatrix} \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{3}}{2}\tilde{x} + \frac{1}{2}\tilde{y} \\ -\frac{1}{2}\tilde{x} + \frac{\sqrt{3}}{2}\tilde{y} \end{pmatrix}.$$

Thus, we have

$$
\begin{aligned}
& & -2x^2 + 2\sqrt{3}xy - 4 = 0 \\
\iff & & -2\left(\frac{\sqrt{3}}{2}\tilde{x} + \frac{1}{2}\tilde{y}\right)^2 + 2\sqrt{3}\left(\frac{\sqrt{3}}{2}\tilde{x} + \frac{1}{2}\tilde{y}\right)\left(-\frac{1}{2}\tilde{x} + \frac{\sqrt{3}}{2}\tilde{y}\right) - 4 = 0 \\
\iff & & -3\tilde{x}^2 + (\sqrt{3} - \sqrt{3})\tilde{x}\tilde{y} + \tilde{y}^2 - 4 = 0 \\
\iff & & \frac{\tilde{y}^2}{4} - \frac{\tilde{x}^2}{4/3} = 1
\end{aligned}
$$

which is a hyperbola.

(c) Since here, we have

$$\tan(2\theta) = \sqrt{3} \iff 2\theta = \frac{\pi}{3} + k\pi \iff \theta = \frac{\pi}{6} + \frac{k\pi}{2},$$

we can set $\theta = \frac{\pi}{6}$, i.e.

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\left(\frac{\pi}{6}\right) & -\sin\left(\frac{\pi}{6}\right) \\ \sin\left(\frac{\pi}{6}\right) & \cos\left(\frac{\pi}{6}\right) \end{pmatrix} \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{3}}{2}\tilde{x} - \frac{1}{2}\tilde{y} \\ \frac{1}{2}\tilde{x} + \frac{\sqrt{3}}{2}\tilde{y} \end{pmatrix}.$$

Thus, we have

$$
\begin{aligned}
& & \frac{3}{4}x^2 + xy\frac{\sqrt{3}}{2} + \frac{y^2}{4} + 6x - 6y\sqrt{3} = 0 \\
\iff & & \frac{3}{4}\left(\frac{\sqrt{3}}{2}\tilde{x} - \frac{1}{2}\tilde{y}\right)^2 + \frac{\sqrt{3}}{2}\left(\frac{\sqrt{3}}{2}\tilde{x} - \frac{1}{2}\tilde{y}\right)\left(\frac{1}{2}\tilde{x} + \frac{\sqrt{3}}{2}\tilde{y}\right) \\
& & + \frac{1}{4}\left(\frac{1}{2}\tilde{x} + \frac{\sqrt{3}}{2}\tilde{y}\right)^2 + 6\left(\frac{\sqrt{3}}{2}\tilde{x} - \frac{1}{2}\tilde{y}\right) - 6\sqrt{3}\left(\frac{1}{2}\tilde{x} + \frac{\sqrt{3}}{2}\tilde{y}\right) = 0 \\
\iff & & \left(\frac{9}{16} + \frac{3}{8} + \frac{1}{16}\right)\tilde{x}^2 + \left(-\frac{3\sqrt{3}}{8} + \frac{3\sqrt{3}}{8} - \frac{\sqrt{3}}{8} + \frac{\sqrt{3}}{8}\right)\tilde{x}\tilde{y} \\
& & + \left(\frac{3}{16} - \frac{3}{8} + \frac{3}{16}\right)\tilde{y}^2 + (3\sqrt{3} - 3\sqrt{3})\tilde{x} + (-3 - 9)\tilde{y} = 0 \\
\iff & & \tilde{x}^2 = 12\tilde{y}
\end{aligned}
$$

which is a parabola.

■

(B) Question 12. By a combination of rotation and translation, simplify the following equations so they are given by a standard equation for a conic. *You may find it helpful to use the double angle formula for* $\tan(2\theta)$.

(a) $34x^2 + 24xy - 40x + 41y^2 + 30y = 0$;
(b) $39x^2 - 96xy - 270x + 11y^2 + 140y = 0$.

Solution:

(a) Here, we have

$$
\begin{aligned}
\tan(2\theta) = -\frac{24}{7} & \iff \frac{2\tan(\theta)}{1 - \tan^2(\theta)} = -\frac{24}{7} \\
& \iff 24\tan^2(\theta) - 14\tan(\theta) - 24 = 0 \\
& \iff \tan(\theta) = \frac{4}{3} \text{ or } \tan(\theta) = -\frac{3}{4}.
\end{aligned}
$$

Since
$$\tan(\theta) = \frac{4}{3} = \frac{4.5}{3.5} = \frac{\sin(\theta)}{\cos(\theta)},$$
it follows that we can consider θ such that $\sin(\theta) = \frac{4}{5}$ and $\cos(\theta) = \frac{3}{5}$ i.e.
$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{3}{5} & -\frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{pmatrix} \cdot \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} = \begin{pmatrix} \frac{3}{5}\tilde{x} - \frac{4}{5}\tilde{y} \\ \frac{4}{5}\tilde{x} + \frac{3}{5}\tilde{y} \end{pmatrix}.$$
Therefore,
$$\begin{aligned}
& 34x^2 + 24xy - 40x + 41y^2 + 30y = 0 \\
\iff \quad & 34\left(\frac{3}{5}\tilde{x} - \frac{4}{5}\tilde{y}\right)^2 + 24\left(\frac{3}{5}\tilde{x} - \frac{4}{5}\tilde{y}\right)\left(\frac{4}{5}\tilde{x} + \frac{3}{5}\tilde{y}\right) \\
& -40\left(\frac{3}{5}\tilde{x} - \frac{4}{5}\tilde{y}\right) + 41\left(\frac{4}{5}\tilde{x} + \frac{3}{5}\tilde{y}\right)^2 + 30\left(\frac{4}{5}\tilde{x} + \frac{3}{5}\tilde{y}\right) = 0 \\
\iff \quad & \left(\frac{9.34}{25} + \frac{12.24}{25} + \frac{16.41}{25}\right)\tilde{x}^2 + \left(\frac{16.34}{25} - \frac{12.24}{25} + \frac{9.41}{25}\right)\tilde{y}^2 \\
& + \left(-\frac{24.34}{25} + \frac{9.24}{25} - \frac{16.24}{25} + \frac{24.41}{25}\right)\tilde{x}\tilde{y} + (-24+24)\tilde{x} + (32+18)\tilde{y} = 0 \\
\iff \quad & 50\tilde{x}^2 + 25(\tilde{y}+1)^2 = 25 \\
\iff \quad & \frac{\tilde{x}^2}{1/2} + (\tilde{y}+1)^2 = 1.
\end{aligned}$$
Hence the quadratic equation above represents an ellipse. Moreover, via the translation $\hat{x} = \tilde{x}$ and $\hat{y} = \tilde{y} + 1$, it follows that in the $\hat{x}\hat{y}$ co-ordinate system, the ellipse is in standard form with $a^2 = 1/2$ and $b^2 = 1$.

(b) With θ as in (a), it follows that the equation
$$39x^2 - 96xy - 270x + 11y^2 + 140y = 0 \iff \frac{(\tilde{y}+2)^2}{11/3} - \frac{(\tilde{x}+1)^2}{11} = 1,$$
from which it follows that the quadratic equation represents a hyperbola. Moreover, via the translation $\hat{x} = \tilde{x} + 1$ and $\hat{y} = \tilde{y} + 2$, it follows that in the $\hat{x}\hat{y}$ co-ordinate system, the hyperbola is in standard form with $a^2 = 11/3$ and $b^2 = 11$.

■

(B) Question 13. Find the equations of the tangent and the normal lines to the given curves at the given value of x. *Note this means there may be two possible values of y and hence two tangent lines and two normal lines. Use implicit differentiation to solve this question.*

(a) $2x^2 + 2xy + 2y^2 - 5 = 0, \quad x = 0$;
(b) $34x^2 + 24xy - 40x + 41y^2 + 30y = 0, \quad x = 0$;
(c) $8x^2 + 12xy + 17y^2 - 20, \quad x = 1$.

Solution:

(a) At $x = 0$, it follows that $2y^2 - 5 = 0 \iff y = \pm\frac{\sqrt{10}}{2}$. Implicit differentiation (and the product rule for differentiation) of the equation gives (where the derivative exists)
$$4x + 2y + 2x\frac{dy}{dx} + 4y\frac{dy}{dx} = 0 \iff \frac{dy}{dx} = -\frac{2x+y}{x+2y} \iff \frac{dy}{dx} = -\frac{1}{2}.$$

Therefore, the tangent lines are given by

$$y - \frac{\sqrt{10}}{2} = -\frac{1}{2}x \text{ at } (x,y) = \left(0, \frac{\sqrt{10}}{2}\right) \text{ and } y + \frac{\sqrt{10}}{2} = -\frac{1}{2}x \text{ at } (x,y) = \left(0, -\frac{\sqrt{10}}{2}\right).$$

The normal lines are given by

$$y - \frac{\sqrt{10}}{2} = 2x \text{ at } (x,y) = \left(0, \frac{\sqrt{10}}{2}\right) \text{ and } y + \frac{\sqrt{10}}{2} = 2x \text{ at } (x,y) = \left(0, -\frac{\sqrt{10}}{2}\right).$$

The solutions to parts (b) and (c) follows a similar argument.

■

(B) Question 14. If an ellipse is defined parametrically by

$$\begin{aligned} y &= 2\sqrt{2}\sin\theta \\ x &= 4\cos\theta - 1 \end{aligned}$$

show that the normal to the ellipse at $\theta = \frac{\pi}{3}$ intersects $(x,y) = (0,0)$.

Solution: Observe that

$$x^2 = 16\cos^2(\theta) - 8\cos(\theta) + 1 \text{ and } y^2 = 8\sin^2(\theta).$$

Therefore,

$$\begin{aligned} 2y^2 + x^2 = 16(\cos^2(\theta) + \sin^2(\theta)) - 2(4\cos(\theta) - 1) - 1 \\ 2y^2 + x^2 = 16 - 2x - 1 \\ \iff \qquad 2y^2 + x^2 + 2x = 15 \\ \iff \qquad 2y^2 + (x+1)^2 + 16 \\ \iff \qquad \frac{(x+1)^2}{16} + \frac{y^2}{8} = 1. \end{aligned}$$

For $\theta = \frac{\pi}{3}$ we have $(x,y) = (1, \sqrt{6})$. By implicit differentiation, it follows that

$$\frac{(x+1)}{8} + \frac{y}{4}\frac{dy}{dx} = 0 \iff \frac{dy}{dx} = -\frac{(x+1)}{2y}.$$

Therefore, the normal to the ellipse at$(x,y) = (1, \sqrt{6})$ has gradient $\sqrt{6}$, and hence the normal to the ellipse at $(x,y) = (1, \sqrt{6})$ is given by

$$y - \sqrt{6} = \sqrt{6}(x-1) \iff y = \sqrt{6}x,$$

which intersects $(x,y) = (0,0)$, as required. ■

(B) Question 15. Find the equation of the parabola that satisfies the given conditions.

(a) Focus $F(3,0)$ and directrix $x = -3$;
(b) Vertex $V(-2,3)$ and directrix $y = 6$.

Solution:

(a) It follows that the distance from the focus to the directrix is 6, so $p = 3$. Therefore the vertex is at (0,0) and the parabola is given by $y^2 = 12x$.
(b) Since the distance from the vertex to the directrix is 3, it follows that $p = 3$. The parabola opens downwards with centre $(-2,3)$, hence it has equation $12(y-3) = -(x+2)^2$.

■

(B) Question 16. Find the vertex and focus of the given parabola and sketch it.

(a) $y = x^2 - 4x + 2$;

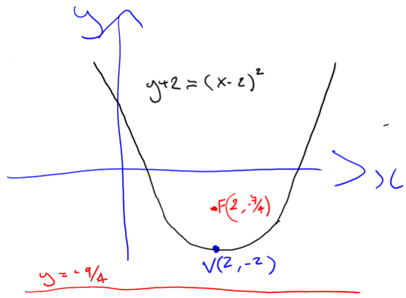
(b) $4x^2 + 40x + y + 106 = 0$.

Solution:

(a) Since

$$y = x^2 - 4x + 2 \iff y = (x-2)^2 - 4 + 2 \iff y + 2 = (x-2)^2,$$

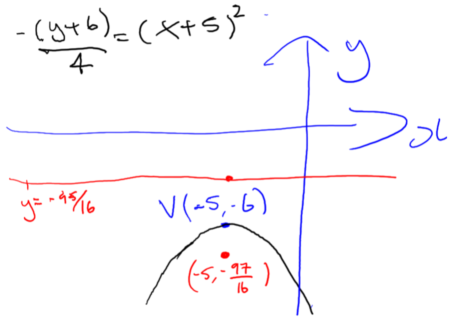
it follows that the parabola has vertex $(x, y) = (2, -2)$ and focus $(x, y) = (2, -\frac{7}{4})$ (note that $p = \frac{1}{4}$).



(b) Since

$$4x^2 + 40x + y + 106 = 0 \iff 4(x+5)^2 - 100 + y + 106 = 0 \iff -\frac{(y+6)}{4} = (x+5)^2,$$

it follows that the parabola has vertex $(x, y) = (-5, -6)$ and focus $(x, y) = (-5, -\frac{97}{16})$ (note that $p = \frac{1}{16}$).



■

(B) Question 17. Find the equation of the tangents to $y = (x-1)^2 + 8$ that intersect the (0,0).

Solution: By differentiating the equation implicitly, it follows that

$$\frac{dy}{dx} = 2(x-1).$$

Hence the tangent at the point (x_1, y_1) on the parabola is given by

$$y - y_1 = 2(x_1 - 1)(x - x_1) \iff y = 2(x_1 - 1)x - 2(x_1^2 - x_1) + y_1.$$

Therefore, the tangents to the parabola that intersect the origin are precisely those that satisfy

$$-2(x_1^2 - x_1) + y_1 = 0 \iff -2(x_1^2 - x_1) + (x_1 - 1)^2 + 8 = 0 \iff -x_1^2 + 9 = 0 \iff x_1 = \pm 3.$$

We conclude that the tangents to the parabola at $(x, y) = (3, 12)$ and $(x, y) = (-3, 24)$ given by

$$y = 4x \text{ and } y = -8x$$

respectively, are precisely those that intersect $(x, y) = (0, 0)$. ■

(B) Question 18. Find an equation for the hyperbola with centre at (0,0) and:

(a) has foci $F(\pm 5, 0)$ and vertices $V(\pm 3, 0)$;
(b) has vertices $V(\pm 4, 0)$ and intersects $P(8, 2)$.

Solution: Hyperbolae with centre at $(0, 0)$ and vertices on the x-axis are of the form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

(a) Since the vertices are at $(x, y) = (\pm 3, 0)$ it follows that $a = 3$. Since the foci are at $(x, y) = (\pm 5, 0)$ it follows that $b^2 = c^2 - a^2 = 5^2 - 3^2 = 16$. We conclude that the hyperbola has equation

$$\frac{x^2}{9} - \frac{y^2}{16} = 1.$$

(b) Since the vertices are at $(x, y) = (\pm 4, 0)$ it follows that $a = 4$. Since the hyperbola intersects $(x, y) = (8, 2)$ it follows that

$$\frac{8^2}{16} - \frac{2^2}{b^2} = 1 \iff b^2 = \frac{4}{3}.$$

We conclude that the hyperbola has equation

$$\frac{x^2}{16} - \frac{y^2}{4/3} = 1.$$

■

(B) Question 19. *You should be able to make a suitable rotation and translation to the conic in standard form to derive your answer, after sketching the conic.*

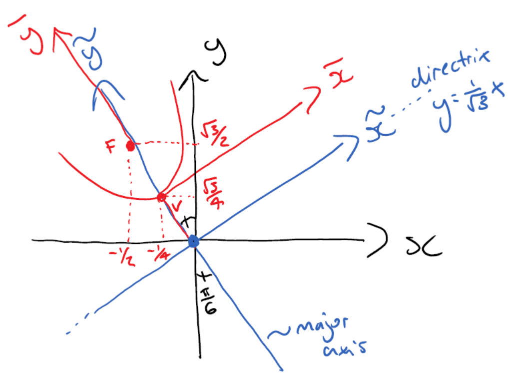
(a) For the parabola with directrix $y = \frac{1}{\sqrt{3}}x$ and focus at $F\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$, sketch the vertex, axis of the parabola, focus and directrix on the xy-plane. Additionally, determine a quadratic equation in x and y that describes the parabola.
(b) For the ellipse foci $F_1(-\sqrt{2}, 0)$ and $F_2(0, -\sqrt{2})$ and eccentricity $e = \frac{1}{2}$, sketch the vertices, major axis, minor axis, foci and directrices on the xy-plane. Additionally, determine a quadratic equation in x and y that describes the ellipse.

Solution:

(a) The axis for the parabola is perpendicular to the directrix and intersects the focus, so has the form

$$y - \frac{\sqrt{3}}{2} = -\sqrt{3}\left(x + \frac{1}{2}\right) \iff y = -\sqrt{3}x.$$

Therefore, the vertex is at $\left(-\frac{1}{4}, \frac{\sqrt{3}}{4}\right)$. Since $|\vec{OV}| = \left((\frac{\sqrt{3}}{4})^2 + \left(-\frac{1}{4}\right)\right)^{1/2} = \frac{1}{2}$ it follows that $p = \frac{1}{2}$.



Now, in standard form, the parabola is given by

$$\bar{x}^2 = 4p\bar{y} = 2\bar{y}.$$

Since the major axis has an angle $\frac{\pi}{6}$ with the positive y axis, we consider the following transformations to go to xy co-ordinates.

$$\begin{matrix} \bar{x} \\ \bar{y} \end{matrix}\Bigg) = \begin{matrix} \tilde{x} \\ \tilde{y} - \frac{1}{2} \end{matrix}\Bigg) \text{ and } \begin{matrix} \tilde{x} \\ \tilde{y} \end{matrix}\Bigg) = \begin{pmatrix} \cos\left(\frac{\pi}{6}\right) & \sin\left(\frac{\pi}{6}\right) \\ -\sin\left(\frac{\pi}{6}\right) & \cos\left(\frac{\pi}{6}\right) \end{pmatrix} \cdot \begin{matrix} x \\ y \end{matrix}\Bigg) = \begin{matrix} \frac{\sqrt{3}}{2}x + \frac{1}{2}y \\ -\frac{1}{2}x + \frac{\sqrt{3}}{2}y \end{matrix}\Bigg)$$