Introduction to Probability

Lecture 11

Today

Variance Propagation

Attendance: 12823244

Variance Propagation

Set Up

We have $x \sim P_x(x)$ but **we do not know the distribution.**

Let's assume we know (or estimate)

$$\langle x \rangle, \qquad var(x)$$

We set $y = f(x)$

What is $P_y(y)$?

Formally we would need

$$P_y(y) = \left| \frac{d}{dy} f^{-1}(y) \right| P_x(f^{-1}(y))$$

But we don't know P_x

Instead we target

$$\langle y \rangle$$
$$\mathrm{var}(y)$$

Example

In an experiment, we want to measure kinetic energy. This is hard so instead we measure the velocity of a mass. Then

$$E = \frac{1}{2} m v^2$$

v is a random variable.

What is $\langle E \rangle$ and $\mathrm{var}(E)$?

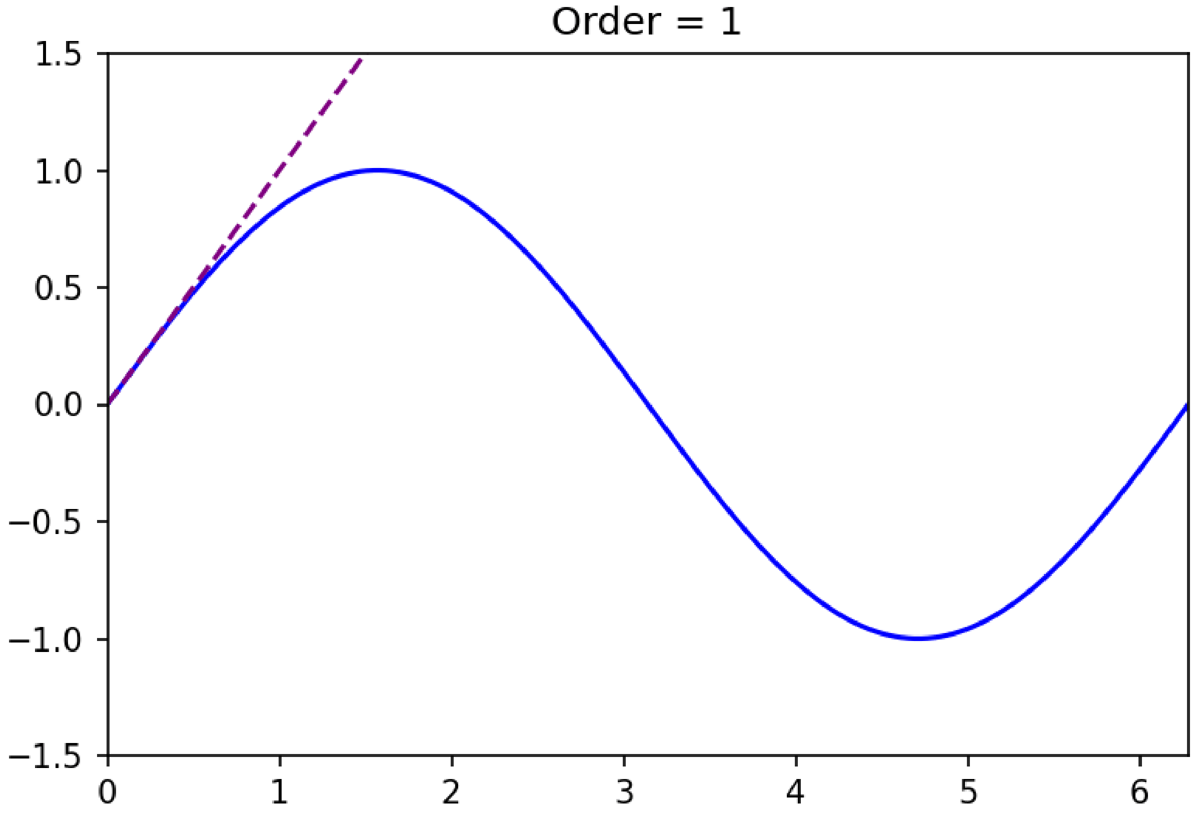
Taylor's Theorem

$$f(x) = f(x_0) + (x - x_0)f'(x_0) + \frac{(x - x_0)^2}{2!}f''(x_0) + \cdots$$

Gives a **polynomial** expression for $f(x)$ around the point x_0.

What if x was a random variable?

We might want $\langle f(x) \rangle$ and $\mathrm{var}(f(x))$



Expectation Value

Expand $f(x)$

We assume $\langle x \rangle = \mu$ and $var(x) = \sigma^2$

$$f(x) \approx f(\mu) + (x-\mu)f'(\mu) + \frac{(x-\mu)^2}{2}f''(\mu)$$

Take expectation

$$\langle f(x) \rangle \approx \langle f(\mu) \rangle + \langle (x-\mu)f'(\mu) \rangle + \left\langle \frac{(x-\mu)^2}{2}f''(\mu) \right\rangle$$

Then

$$\langle f(\mu) \rangle = f(\mu); \quad \langle (x-\mu)f'(\mu) \rangle = f'(\mu)\langle (x-\mu) \rangle = 0$$

$$\left\langle \frac{(x-\mu)^2}{2}f''(\mu) \right\rangle = \frac{f''(\mu)}{2}\langle (x-\mu)^2 \rangle = \frac{f''(\mu)\sigma^2}{2}$$

Expectation Value

So

$$\langle f\rangle \approx f(\mu)$$

If $\sigma^2 f''(\mu)$ is “small”.

And in the experiment we would **estimate** μ using the **sample mean**.

Variance Propagation

From before: $\langle f \rangle \approx f(\mu)$

Then $var(f) = \langle f^2 \rangle - \langle f \rangle^2 = \langle f^2 \rangle - f(\mu)^2$

To first order $f(x) \approx f(\mu) + (x-\mu)f'(\mu)$

So

$$\langle f^2 \rangle = \left\langle \left(f(\mu) + (x-\mu)f'(\mu) \right)^2 \right\rangle = \underbrace{\langle f(\mu)^2 \rangle}_{f(\mu)^2} + 2f(\mu)f'(\mu)\underbrace{\langle x-\mu \rangle}_{0} + f'(\mu)^2 \underbrace{\langle (x-\mu)^2 \rangle}_{\sigma^2}$$

Therefore

$$var(f) \approx \underbrace{f(\mu)^2 + f'(\mu)^2\sigma^2}_{\langle f^2 \rangle} - \underbrace{f(\mu)^2}_{\langle f \rangle^2} = f'(\mu)^2\sigma^2$$

Summary

$$x \sim P_x(x)$$
$$\langle x \rangle = \mu$$
$$\mathrm{var}(x) = \sigma^2$$

$$\xrightarrow{y = f(x)}$$

$$\langle y \rangle \approx f(\mu)$$
$$\mathrm{var}(y) = f'(\mu)^2\sigma^2$$

If $\sigma^2 f''(\mu)$ is “small”.

Example

Measure energy with v

$$E = \frac{1}{2}mv^2$$

With $\langle v\rangle = v_0$ and $\mathrm{var}(v) = \sigma^2$.

Estimate $\langle E\rangle$ and $\mathrm{var}(E)$

We use

$$\langle f\rangle = f(\langle x\rangle)$$

So

$$\langle E\rangle = \frac{1}{2}mv_0^2$$

Then

$$var(E) = var(v)\left(\frac{dE}{dv}\right)^2$$

$$\left(\left.\frac{dE}{dv}\right|_{v_0}\right)^2 = (mv_0)^2$$

Therefore

$$var(E) \approx \sigma^2 m^2 v_0^2$$

Example

If x is drawn according to a Poisson distribution with parameter λ, estimate

$$\left\langle \frac{1}{\sqrt{1+x}} \right\rangle$$

And

$$\mathrm{var}\left(\frac{1}{\sqrt{1+x}}\right)$$

n.b.

$$\frac{d}{dx}\frac{1}{\sqrt{1+x}} = -\frac{1}{2(1+x)^{3/2}}$$

Remember for a Poisson:

$$\langle x \rangle = \mathrm{var}(x) = \lambda$$

We want

$$\left\langle \frac{1}{\sqrt{1+x}} \right\rangle = \sum_{x=0}^{\infty} \frac{1}{\sqrt{1+x}} \frac{\lambda^x}{x!} e^{-\lambda} = ?$$

Instead we use

$$\left\langle \frac{1}{\sqrt{1+x}} \right\rangle \approx \frac{1}{\sqrt{1+\langle x \rangle}} = \frac{1}{\sqrt{1+\lambda}}$$

And

$$\mathrm{var}\left(\frac{1}{\sqrt{1+x}}\right) \approx \left(\frac{d}{dx}\frac{1}{\sqrt{1+x}}\right)^2 \Bigg|_{x=\langle x \rangle} var(x)$$

$$\rightarrow \mathrm{var}\left(\frac{1}{\sqrt{1+x}}\right) \approx \underbrace{\frac{1}{4\,(1+\lambda)^3}}_{\left(\frac{df}{dx}\right)^2} \underbrace{\lambda}_{var(x)}$$

General formulae

If we have N variables, $x_1, x_2 \dots x_N$ with

$$\langle x_i \rangle = \mu_i \qquad \mathrm{var}(x_i) = \sigma_i^2$$

And the x's are **statistically independent**.

For some function $f(x_1, x_2 \dots x_N)$:

$$\langle f \rangle \approx f(\mu_1, \mu_2 \dots \mu_N)$$

$$\mathrm{var}(f) \approx \left(\frac{\partial f}{\partial x_1}\right)^2 \sigma_1^2 + \left(\frac{\partial f}{\partial x_2}\right)^2 \sigma_2^2 + \cdots$$

And you would **estimate** μ_i and σ_i^2 using sample values.

Example

Measure energy with v and m

$$E = \frac{1}{2}mv^2$$

With $\langle v \rangle = v_0$ and $\mathrm{var}(v) = \sigma_v^2$

With $\langle m \rangle = m_0$ and $\mathrm{var}(m) = \sigma_m^2$

Estimate $\langle E \rangle$ and $\mathrm{var}(E)$

For $\langle E \rangle$ we can just use $\langle m \rangle$ and $\langle v \rangle$ so

$$\langle E \rangle = \frac{1}{2}m_0 v_0^2$$

$$var(E) \approx \left(\frac{\partial E}{\partial v}\Big|_{v_0,m_0}\right)^2 \sigma_v^2 + \left(\frac{\partial E}{\partial m}\Big|_{v_0,m_0}\right)^2 \sigma_m^2$$

$$\frac{\partial E}{\partial v} = mv; \quad \frac{\partial E}{\partial m} = \frac{v^2}{2}$$

$$\rightarrow var(E) = m_0^2 v_0^2 \sigma_v^2 + \frac{1}{4}v_0^4 \sigma_m^2$$

Class Example

Measure speed of sound through liquid

$$c = \frac{d}{t}$$

By measuring time of flight.

$\langle t \rangle = t_0$ and $\mathrm{var}(t) = \sigma^2$

Estimate $\langle c \rangle$ and $\mathrm{var}(c)$

$$\mathrm{var}(f) \approx \left(\frac{df}{dx}\right)^2 \sigma^2$$

$$\langle c \rangle = \frac{d}{t_0}$$

$$var(c) = \left(\frac{dc}{dt}\Big|_{t_0}\right)^2 \sigma^2$$

$$= \left(-\frac{d}{t_0^2}\right)^2 \sigma^2$$

$$= \frac{d^2\sigma^2}{t_0^4}$$

Class Example (2)

Measure speed of sound through liquid

$$c = \frac{d}{t}$$

By measuring time of flight.
$\langle t \rangle = t_0$ and $\mathrm{var}(t) = \sigma_t^2$
$\langle d \rangle = d_0$ and $\mathrm{var}(t) = \sigma_d^2$

Estimate $\langle c \rangle$ and $\mathrm{var}(c)$

$$\mathrm{var}(f) \approx \left(\frac{\partial f}{\partial x}\right)^2 \sigma_x^2 + \left(\frac{\partial f}{\partial y}\right)^2 \sigma_y^2$$

$$\langle c \rangle = \frac{d_0}{t_0}$$

$$var(c) = \left(\frac{\partial c}{\partial t}\Big|_{d_0,t_0}\right)^2 \sigma_t^2 + \left(\frac{\partial c}{\partial d}\Big|_{d_0,t_0}\right)^2 \sigma_d^2$$

$$= \left(-\frac{d_0}{t_0^2}\right)^2 \sigma_t^2 + \left(\frac{1}{t_0}\right)^2 \sigma_d^2$$

$$= \frac{d_0^2\sigma^2}{t_0^4} + \frac{\sigma_d^2}{t_0^2}$$

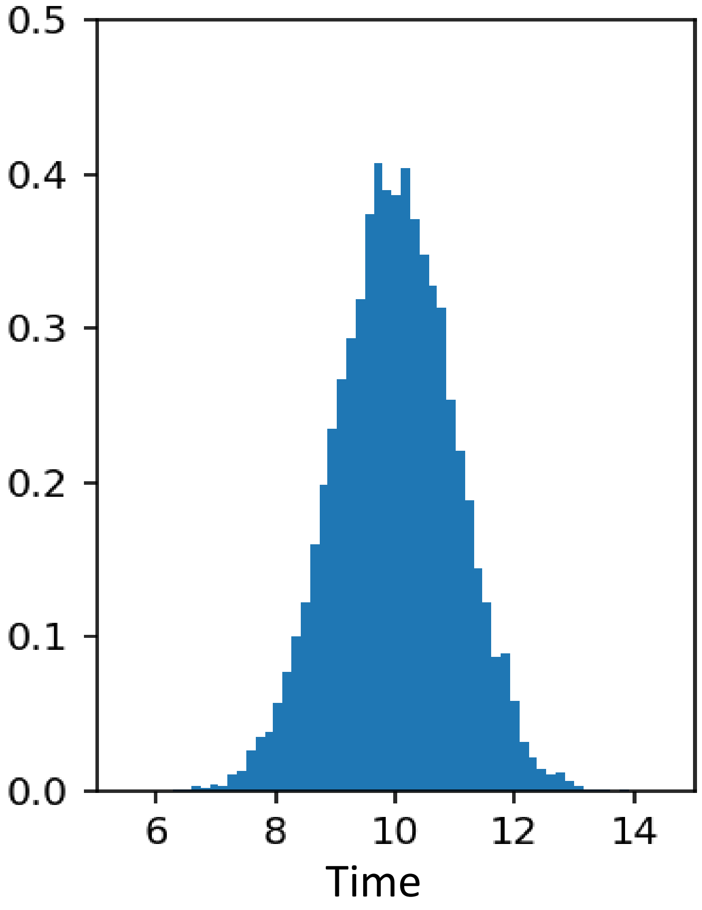
Simulation

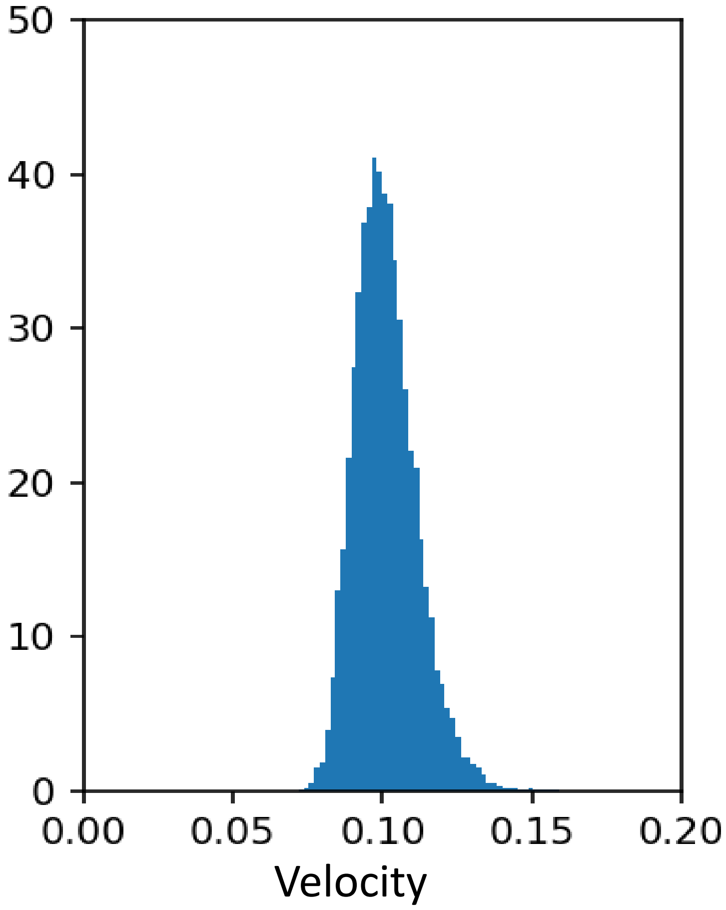
$d = 1$

$\langle t \rangle = 10$ and $\mathrm{var}(t) = 1$

$$\langle c \rangle = \frac{1}{10}$$

$$\mathrm{var}(c) = \frac{d^2 \mathrm{var}(t)^2}{t^4} = \frac{1}{10000}$$





Summary

If we have N variables, $x_1, x_2 \dots x_N$ with

$$\langle x_i \rangle = \mu_i \qquad \mathrm{var}(x_i) = \sigma_i^2$$

And the x's are **statistically independent**.

For some function $f(x_1, x_2 \dots x_N)$:

$$\langle f \rangle \approx f(\mu_1, \mu_2 \dots \mu_N)$$

$$\mathrm{var}(f) \approx \left(\frac{\partial f}{\partial x_1}\right)^2 \sigma_1^2 + \left(\frac{\partial f}{\partial x_2}\right)^2 \sigma_2^2 + \cdots$$

And you would **estimate** μ_i and σ_i^2 using sample values.

Recap

Course Summary

Lecture	Topic(s)
Lecture 1	Definition of Probability
Lecture 2	Combinatorics & Uniform Probability
Lecture 3	Inclusion-Exclusion
Lecture 4	Conditional Probability
Lecture 5	Bayes Theorem and Marginal Distributions
Lecture 6	Expectation values
Lecture 7	Bernoulli, Binomial and Poisson Distribution
Lecture 8	Sums of Random Variables
Lecture 9	Continuous Probability
Lecture 10	Normal Distribution and CLT
Lecture 11	Variance Propagation