L9

Wavefunctions for quantum particles

- The classical wave eqn (reminder!)

$$\frac{\partial^2 E}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2}$$

Solutions have the form:

$$E(x,t) = E \sin(kx - \omega t) \quad [\text{or } (\omega t - kx), \text{ same}]$$

or $E = E\cos(kx - \omega t)$

(or the same with $(kx + \omega t)$)

→ Because the wave eqn is a linear equation, linear sums of individual solutions are also a solution

(ie $E_1 \sin(k_1 x - \omega_1 t) + 3E_1 \sin(k_2 x + \omega_2 t)$ is also a valid solution)

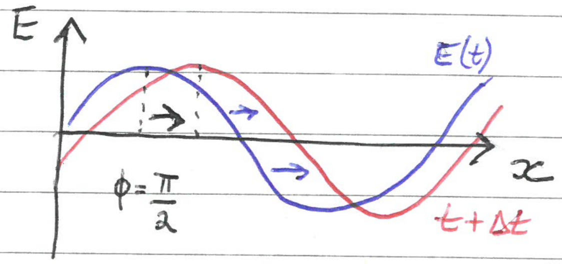
Recall: $k = \frac{2\pi}{\lambda}$ 'wavenumber'

$\omega = 2\pi f$ 'angular frequency'

[Exercise: show by substitution that $c = \frac{\omega}{k} = f\lambda$]

- Direction

wavefunctions with $\boxed{(kx - wt)}$
move → Right (+ve x)



How can we tell? A 'point' on a wave, like the above maximum, is really a phase ϕ ($\frac{\pi}{2}$ here)

and the phase $\sin\phi = \sin(kx - wt)$

so $\phi = kx - wt$

$\therefore$ as t goes ↑, x must go ↑ too to keep ϕ at $\frac{\pi}{2}$

$\Rightarrow$ the maximum moves in $+x$

(check: diff w.r.t t: $\frac{\partial}{\partial t}\phi = 0 = k\frac{\partial x}{\partial t} - w\frac{\partial t}{\partial t}$

$$k\frac{\partial x}{\partial t} = w$$

$$\frac{\partial x}{\partial t} = \frac{w}{k} > 0$$

- We could also write a solution as

$$E(x,t) = \mathrm{Re}\left\{ E e^{i(kx - \omega t)} \right\}$$

[Remember $e^{i\theta} \equiv \cos\theta + i\sin\theta$]

Just a maths trick here - convenient repacking of sin and cos terms

Take the real part at the end to get the real cos( ) terms and real numbers.

- QM wavefns

In QM, the particle wavefn really is complex:

$$\psi(x,t) = Ae^{i(kx-\omega t)}$$

(<u>not</u> $Re\{Ae^{i\ldots}\}$)

this also complex, could write as $A'e^{i\phi}\ldots$

Can split this up:

$$\psi = \underbrace{Ae^{ikx}}_{\text{spatial function } \psi(x)} \; \underbrace{e^{-i\omega t}}_{\text{time-dependence } \phi(t)}$$

– often just a phase

This represents a particle moving in $+x$ direction.

Something moving in $-x$ would be:

$$\psi = Ae^{-ikx} e^{-i\omega t}$$

[or (beware!) could be $A'e^{ikx}e^{i\omega t}$]

... just conventions

- What does ψ <u>mean</u> ?

'Probability amplitude' - complex - phase

n.b. Phase needed for interference. We cannot measure absolute phase in QM.

Probability Interpretation

Observables need to be real
In QM this means we take complex conjugates to get them

If $z = a+ib, \quad z^* = a - ib$

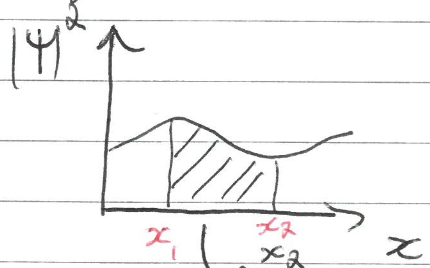
$$z^* z = a^2 + b^2 = |z|^2 \Rightarrow \text{real}$$

This gives us ~~probability~~ probability density

$$P(x) = \psi^*(x)\,\psi(x) = |\psi|^2$$

Probability of finding the particle at position x

Note that we have to <u>integrate</u> over an x range to get an actual probability → probability to find it at precisely $x = 2.000000...$ is zero

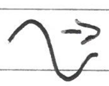


$$\int_{x_1}^{x_2} |\Psi|^2 dx = \text{Probability of being between } x_1 \text{ and } x_2$$

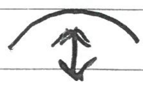
⇒ We can never just say where something is! $x = ...$ Remember L8...

Probability Density as Intensity

- Particle in a box
  - standing waves:



$e^{ikx} + e^{-ikx}$



= standing wave

Standing Waves

$$\psi = A_1 e^{ikx} e^{-i\omega t} + A_2 e^{-ikx} e^{-i\omega t}$$

If $A_1 = -A_2$ ($A_1 = A_2$ gives cos(), FYI)

then $\psi = A_1 (e^{ikx} - e^{-ikx}) e^{-i\omega t}$

$$= A \sin kx \, e^{-i\omega t}$$

[Exercise: verify this step and find A]

$\Rightarrow$ This is our particle-in-a-box from L8!

- Let's find $P(x)$

$$P(x) = \psi^* \psi = \underbrace{A^* A}_{\text{Real now}} \sin kx \sin kx \, \cancel{e^{i\omega t}} \, \cancel{e^{-i\omega t}}$$

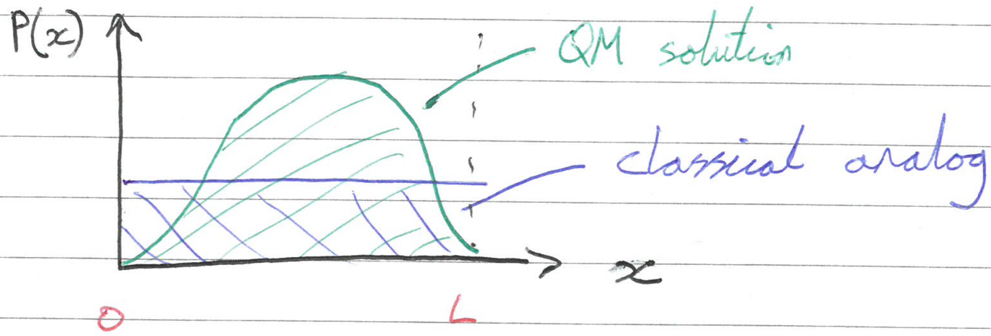
$$= |A|^2 \sin^2 kx$$

For the $n=1$ state,

$$P(x) = |A|^2 \sin^2 \frac{\pi x}{L}$$

Note that $P(x)$ is often time independent

And note $P(x)$ is not uniform here!



Note that to get results out we first need to find $|A|^2$ – normalise

To do this, we say 'Particle must be somewhere' ie area under $P(x)$ curve must $= 1$

$$\int_{-\infty}^{\infty} P(x)\,dx = \int_0^L \underbrace{|A|^2 \sin^2 \frac{\pi x}{L}}\,dx = 1$$

$$\sin^2\theta = \frac{1-\cos 2\theta}{2}$$

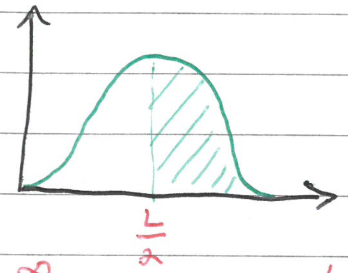
Only any values inside the well

[Exercise: solve for $|A|^2$]

Find: $|A|^2 = \frac{2}{L}$

$\Rightarrow$ $P(x) = \frac{2}{L}\sin^2 kx$ (For $n=1$)

Now, to find, e.g., probability the particle is in the right half of the box:



$$\int_{\frac{L}{2}}^{\infty} P(x) = \int_{\frac{L}{2}}^{L} \frac{2}{L} \sin^2 kx \; dx$$

$$= \frac{1}{2}$$

[check this yourself!]