Relativistic dynamics. Momentum and energy of relativistic particles

Up to this point the motion of relativistic particles was taken for granted, without any physical considerations of its causes. In other words we were studying relativistic kinematics. Now it is time to explore the *dynamics* of relativistic objects and understand how its laws must be adopted to the motion with velocities close to speed of light.

One of the most straightforward question we can ask is how the second Newton's law

$$F = ma = m\frac{\mathrm{d}v}{\mathrm{d}t}$$

has to be reformulated to avoid the velocity of a particle increase indefinitely

$$v(t) = v_0 + \frac{F}{m}t$$

when a constant force F is acting on it for long time.

We start with the non-relativistic expression for momentum $p = mv$, so the second Newton's law is rewritten as $F = \mathrm{d}p/\mathrm{d}t$. As usual, for simplicity we consider a one-dimensional situation here, but will generalise for a three-dimensional momentum $\boldsymbol{p} = (p_x, p_y, p_z) = m\boldsymbol{v}$ later on.

Consider an inelastic collision in which two mass m particles collide and fuse into a particle with mass m. This process is shown in Fig. 1 in two frames: first, in which the centre of mass is at rest (Centre of Mass frame. CoM), so that the particles move with velocities v and $-v$ before the collision and the final particles is at rest. The second frame moves with velocity $V_{\mathrm{rel}} = -v$ with respect to CoM frame, so that before the collision particles have velocities $2v$ and 0, the final product moves with velocity v.

In both frames the total momentum is conserved: $P_{\mathrm{i}} = mv - mv = P_{\mathrm{f}} = 2m \times 0$ in the CoM frame, while in the moving frame we have $P'_{\mathrm{i}} = m \times 2v + m \times 0 = P'_{\mathrm{f}} = 2m \times v$.

In this non-relativistic example we have used Galilean velocity composition law for the velocity of the left-most particle, $v' = v - V_{\mathrm{rel}} = v - (-v) = 2v$. Had we used the relativistic velocity composition, $v' = (v - V_{\mathrm{rel}})/(1 - vV_{\mathrm{rel}}/c^2) = 2v/(1 + v^2/c^2)$, our calculations would contradict momentum conservation, unless we modify the definition of momentum appropriately. Such a relativistic expression of momentum is our next goal.

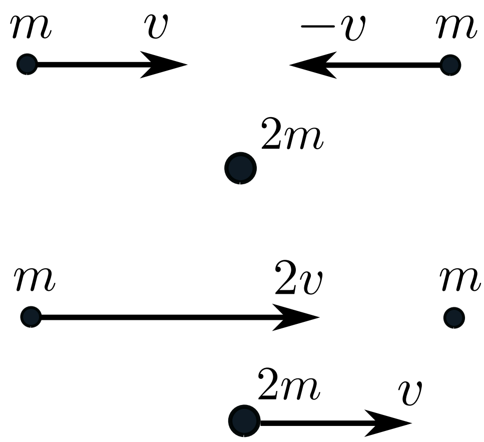

Figure 1: Momentum conservation in non-relativistic physics.

Relativistic momentum

Let us try a modified expression for momentum by allowing the mass to depend on velocity:

$$m \to m(v)\,, \qquad p = mv \to p = m(v)v\,.$$

What is the dependence $m(v)$ so that the momentum conservation is compatible with the relativistic velocity transformation, as shown in Fig. 2? We write down the corresponding equations,

$$m(v')v' = M(v)v$$
$$m(v') + m(0) = M(v)\,,$$

the second equation is the mass conservation. Expressing from it $M(v)$, we have

$$m(v')v' = m(v')v + m(0)v \quad \Rightarrow \quad \frac{m(v')}{m(0)} = \frac{v}{v'-v}\,.$$

The right hand side should be a function of v' which can be achieved by solving $v' = 2v/(1+v^2/c^2)$ for v as a function of v'. We have the following quadratic equation:

$$\frac{v'}{c^2}v^2 - 2v + v' = 0 \quad \Rightarrow \quad v_{1,2} = \frac{c^2}{v'}\left(1 \pm \sqrt{1 - v'^2/c^2}\right)\,.$$

To decide which solution to choose, consider the non-relativistic limit $v'/c \to 0$. The solution with a plus sign blows up in this limit, while the minus sign solution becomes

$$\frac{c^2}{v'}\left(1 - \sqrt{1 - v'^2/c^2}\right) \simeq \frac{c^2}{v'}\left(1 - \left(1 - v'^2/2c^2\right)\right) = \frac{c^2}{v'}\frac{v'^2}{2c^2} = v'/2\,.$$

reproducing the non-relativistic result $v' = 2v$, see Fig. 1. We have the result for the relativistic mass,

$$\begin{aligned}\frac{m(v')}{m(0)} &= \frac{v}{v'-v} = \frac{\frac{c^2}{v'}\left(1 - \sqrt{1 - v'^2/c^2}\right)}{v' - \frac{c^2}{v'}\left(1 - \sqrt{1 - v'^2/c^2}\right)} = \frac{1 - \sqrt{1 - v'^2/c^2}}{v'^2/c^2 - 1' + \sqrt{1 - v'^2/c^2}} \\ &= \frac{1}{\sqrt{1 - v'^2/c^2}}\frac{1 - \sqrt{1 - v'^2/c^2}}{1 - \sqrt{1 - v'^2/c^2}} = \frac{1}{\sqrt{1 - v'^2/c^2}} = \gamma(v')\,.\end{aligned}$$

The desired mass-velocity dependence is therefore

$$m(v) = \gamma(v)m = \frac{m}{\sqrt{1 - v^2/c^2}}$$

where $m = m(0)$ is the *rest mass*. If the particle's velocity is increased from 0 to c the mass grows and becomes infinite. The corresponding momentum

$$p(v) = \frac{mv}{\sqrt{1 - v^2/c^2}}$$

is a *nonlinear* function of velocity.

This relativistic expression for momentum settles the controversy with the 2nd Newton's law: while momentum does increase linearly with time under the action of a constant force, $p(t) = p(0) + Ft$, the corresponding velocity approaches asymptotically c from below, but never exceeds it.

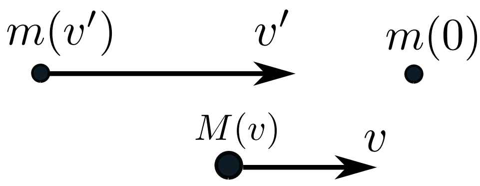

Figure 2: Relativistic momentum conservation.