Algebraic Geometry Approaches to Linear Cryptanalysis

Current Insights and Open Problems

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(joint work with Tim Beyne)

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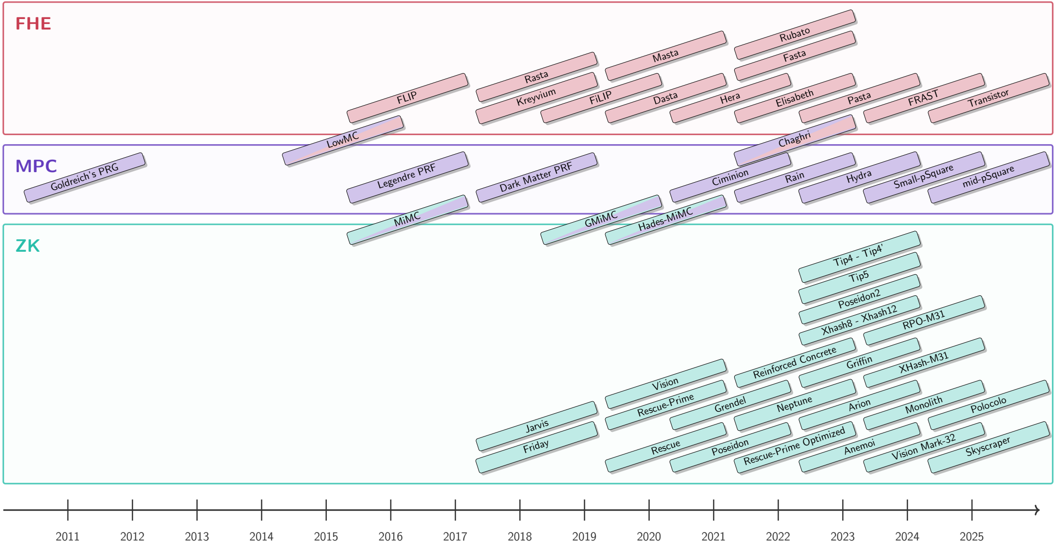
New symmetric primitives



A new context

Traditional case

Alphabet

Operations based on logical gates or CPU instructions.

$$\mathbb{F}_2^n, \text{ with } n \simeq 4, 8$$

Arithmetization-Oriented

Alphabet

Operations based on large finite-field arithmetic.

$$\mathbb{F}_q, \text{ with } q \in \{2^n, p\}, p \simeq 2^n, n \geq 32$$

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Cryptanalysis

Decades of cryptanalysis

- ⋆ algebraic attacks ✓
- ⋆ differential attacks ✓
- ⋆ linear attacks ✓
- ⋆ ...

Arithmetization-Oriented

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$$\mathbb{F}_q, \text{ with } q \in \{2^n, p\}, p \simeq 2^n, n \geq 32$$

Cryptanalysis

$\leq$ 8 years of cryptanalysis

- ⋆ algebraic attacks ✓
- ⋆ differential attacks ✗
- ⋆ linear attacks ✗
- ⋆ ...

Characters

Definition

A **character** of a finite abelian group G is a homomorphism

$$\chi : G \to \mathbb{C}^\times ,$$

where $\mathbb{C}^\times$ is the multiplicative group of nonzero complex numbers.

In particular, we have

$$\chi(1) = 1 ,$$

and for $a_1, a_2 \in G$

$$\chi(a_1 a_2) = \chi(a_1)\chi(a_2) .$$

$\boxed{\chi(a) \text{ is a root of unity}}$

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$$\boxed{\chi(a) \text{ is a root of unity}}$$

Definition

A **linear approximation** of $\mathrm{F} : \mathbb{F}_q^n \to \mathbb{F}_q^m$ is a pair of characters (χ, ψ).

Correlation of linear approximations

Definition

The **correlation of the linear approximation** (χ, ψ) of $\mathsf{F} : \mathbb{F}_q^n \to \mathbb{F}_q^m$ is

$$C^{\mathsf{F}}_{\chi,\psi} = \frac{1}{q^n} \sum_{x \in \mathbb{F}_q^n} \chi(\mathsf{F}(x))\, \psi(-x) \ .$$

Let ω be a primitive character, $\mathbb{F}_q \to \mathbb{C}^\times$ s.t. $\chi(\mathsf{F}(x)) = \omega^{\langle v, \mathsf{F}(x)\rangle}$ and $\psi(x) = \omega^{\langle u, x\rangle}$. Then

$$C^{\mathsf{F}}_{\chi,\psi} = \frac{1}{q^n} \sum_{x \in \mathbb{F}_q^n} \omega^{(\langle v, \mathsf{F}(x)\rangle - \langle u, x\rangle)} \ .$$

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Examples:

- If $\mathsf{F} : \mathbb{F}_2^n \to \mathbb{F}_2^m$, then

$$C^{\mathsf{F}}_{u,v} = \frac{1}{2^n} \sum_{x \in \mathbb{F}_2^n} (-1)^{(\langle v, \mathsf{F}(x)\rangle + \langle u, x\rangle)}\ .$$

- If $\mathsf{F} : \mathbb{F}_p^n \to \mathbb{F}_p^m$, then

$$C^{\mathsf{F}}_{u,v} = \frac{1}{p^n} \sum_{x \in \mathbb{F}_p^n} e^{\left(\frac{2i\pi}{p}\right)(\langle v, \mathsf{F}(x)\rangle - \langle u, x\rangle)}\ .$$

Walsh transform

Definition

The **Walsh transform** for the character ω of the linear approximation (u, v) of $\mathsf{F} : \mathbb{F}_q^n \to \mathbb{F}_q^m$ is given by

$$\mathcal{W}_{u,v}^{\mathsf{F}} = \sum_{x \in \mathbb{F}_q^n} \omega^{(\langle v, \mathsf{F}(x) \rangle - \langle u, x \rangle)} \ .$$

$$\boxed{\mathcal{W}_{u,v}^{\mathsf{F}} = q^n \cdot \mathcal{C}_{u,v}^{\mathsf{F}}}$$

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$$\boxed{\mathcal{W}_{u,v}^{F} = q^n \cdot C_{u,v}^{F}}$$

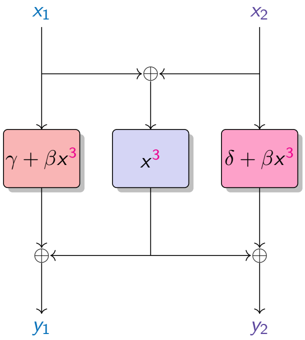
Definition

The **Linearity** $\mathcal{L}_F$ of $F : \mathbb{F}_q^n \to \mathbb{F}_q^m$ is the highest Walsh coefficient.

$$\mathcal{L}_F = \max_{u,v \in \mathbb{F}_q, v \neq 0} \left| \mathcal{W}_{u,v}^{F} \right| \ .$$

Closed Flystel in $\mathbb{F}_{2^n}$

Introduced by [Bouvier, Briaud, Chaidos, Perrin, Salen, Velichkov and Willems, 2023].



Closed Flystel.

$$\mathcal{L}_F = \max_{u,v \neq 0} \left| \sum_{x \in \mathbb{F}_{2^n}^2} (-1)^{(\langle v, F(x) \rangle - \langle u, x \rangle)} \right|$$

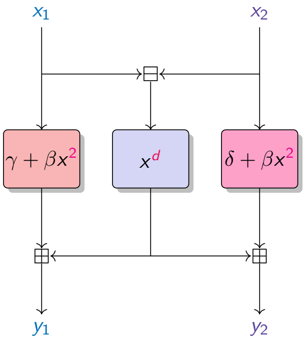
Bound

Linearity bound for the Flystel:

$$\mathcal{L}_F \leq 2^{n+1}$$

Closed Flystel in $\mathbb{F}_p$

Introduced by [Bouvier, Briaud, Chaidos, Perrin, Salen, Velichkov and Willems, 2023].



Closed Flystel.

d is a small integer s.t.
$x \mapsto x^d$ is a permutation of $\mathbb{F}_p$
(usually $d = 3, 5$).

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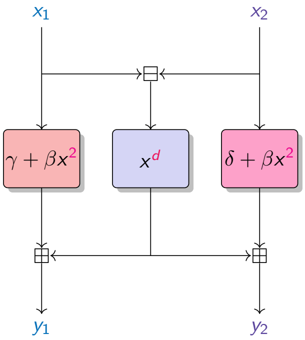

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How to determine an accurate bound for the linearity of the Closed Flystel in $\mathbb{F}_p$?

Weil bound

Proposition [Weil, 1948]

Let $f \in \mathbb{F}_p[x]$ be a univariate polynomial with $\deg(f) = d$. Then

$$\mathcal{L}_f \leq (d-1)\sqrt{p}$$

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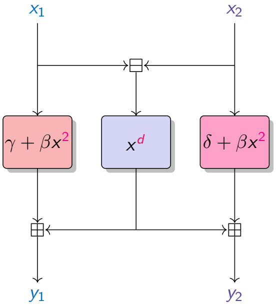


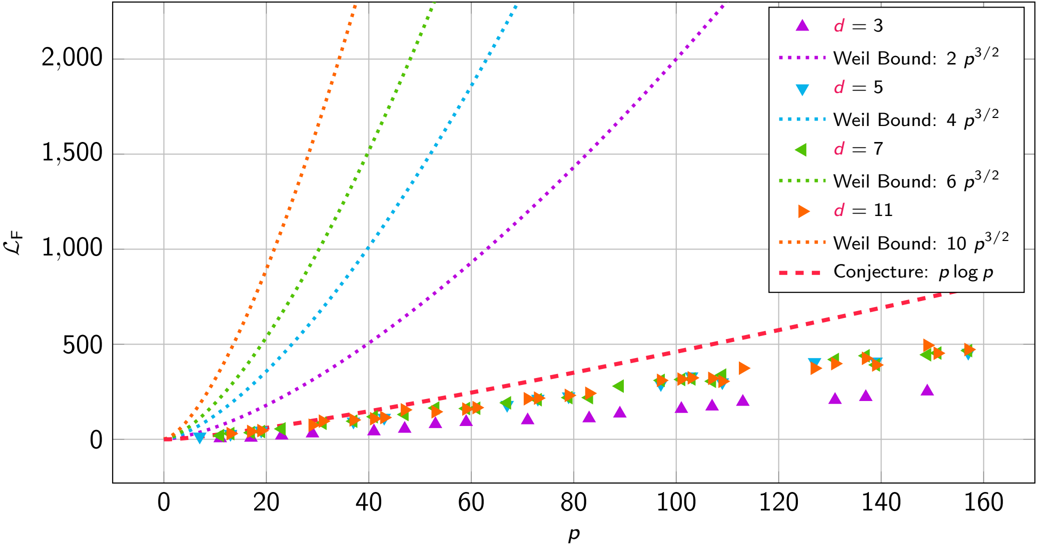
Closed Flystel.

$$\mathcal{L}_F \leq (d-1)p\sqrt{p} \ ? \qquad \begin{cases} \mathcal{L}_{\gamma+\beta x^2} & \leq \sqrt{p}\ , \\ \mathcal{L}_{x^d} & \leq (d-1)\sqrt{p}\ , \\ \mathcal{L}_{\delta+\beta x^2} & \leq \sqrt{p}\ . \end{cases}$$

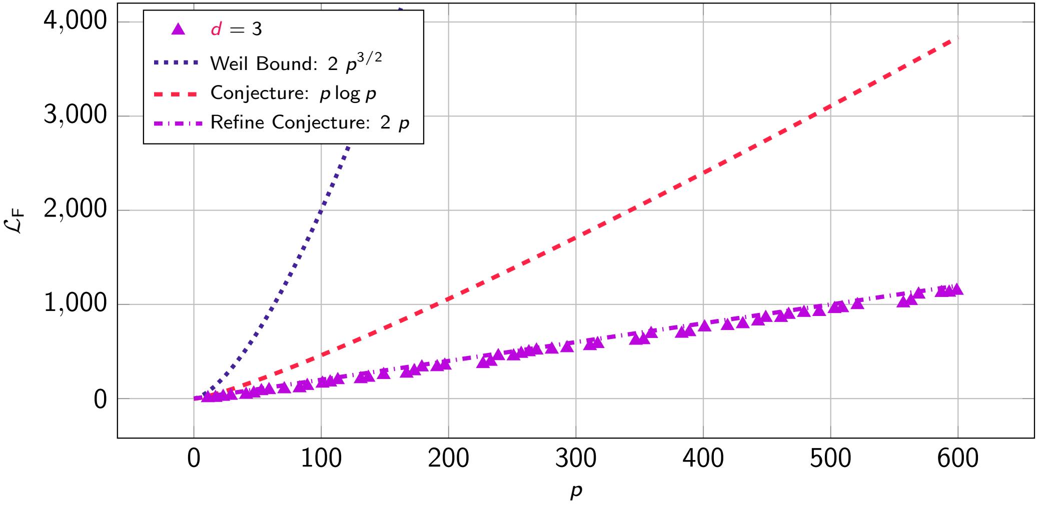
Conjecture

$$\mathcal{L}_F = \sum_{x \in \mathbb{F}_p^2} e^{\left(\frac{2i\pi}{p}\right)(\langle v, F(x)\rangle - \langle u, x\rangle)} \leq p \log p$$

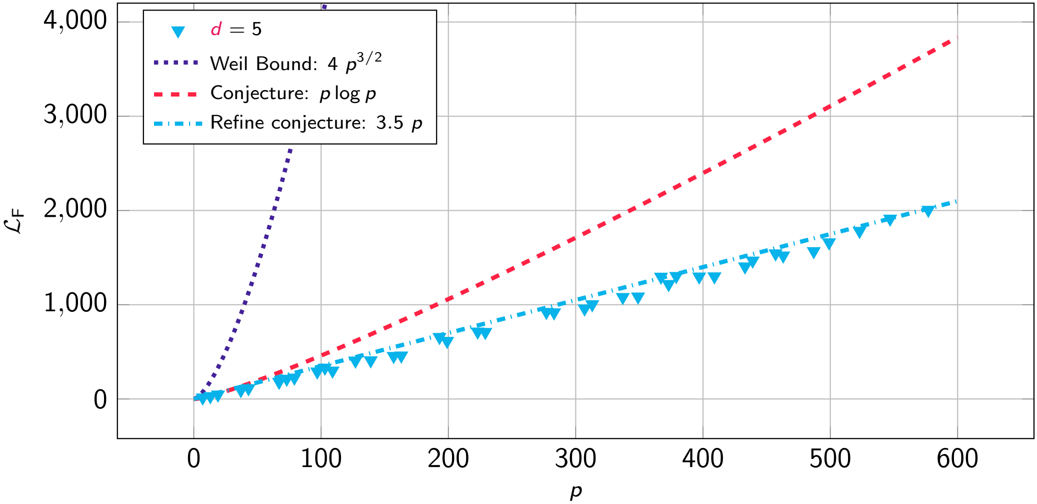
Experimental results



Experimental results ($d = 3$)



Experimental results ($d = 5$)



Take-away

AO primitives: new symmetric primitives defined over prime fields.

$\boxed{\text{Need for new linear cryptanalysis tools}}$

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Need for new linear cryptanalysis tools

This Talk:

- ⋆ Applications of results for exponential sums (generalization of Weil bound)

$$\mathcal{W}^{\mathrm{F}}_{u,v} = \sum_{x \in \mathbb{F}_q^n} \omega^{(\langle v, \mathrm{F}(x) \rangle - \langle u, x \rangle)} \quad \rightarrow \quad S(f) = \sum_{x \in \mathbb{F}_q^n} \omega^{f(x)} .$$

- ⋆ $\mathbb{F}_q$ is a finite field s.t. q is a power of a prime p.

- ⋆ Functions with 2 variables $\mathrm{F} \in \mathbb{F}_q[x_1, x_2]$.

Generalizations of Weil bound

- ⋆ Deligne bound
  - ⋆ Application to the Generalized Butterfly construction
- ⋆ Denef and Loeser bound
  - ⋆ Application to 3-round Feistel construction
- ⋆ Rojas-León bound
  - ⋆ Application to the Generalized Flystel construction

Smoothness

Definition

Let $f \in \mathbb{F}_q[x_1, \dots, x_n]$. A hypersurface defined by $f = 0$ is **smooth**, if the system

$$f = \partial f / \partial x_1 = \cdots = \partial f / \partial x_n = 0$$

has no non zero solutions.

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Examples:

- $f(x_1, x_2) = 2{x_1}^3 + {x_2}^2 = 0$ is smooth, since

$$\partial f / \partial x_1 = 6{x_1}^2 \qquad \text{and} \qquad \partial f / \partial x_2 = 2x_2 \ ,$$

so that

$$f = \partial f / \partial x_1 = \partial f / \partial x_2 = 0 \qquad \Leftrightarrow \qquad (x_1, x_2) = (0, 0) \ .$$

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- $f(x_1, x_2) = {x_1}^2 + {x_2}^2 - 2x_2 + 1 = 0$ is not smooth, since

$$\partial f / \partial x_1 = 2x_1 \quad \text{and} \quad \partial f / \partial x_2 = 2x_2 - 2 \ ,$$

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$$f = \partial f / \partial x_1 = \partial f / \partial x_2 = 0 \quad \Leftrightarrow \quad (x_1, x_2) = (0, 1) \ .$$

Deligne Theorem

Theorem [Deligne, 1974]

Let q be a power of a prime p.
Let $f \in \mathbb{F}_q[x_1, \ldots, x_n]$ be a polynomial of degree d, with $\gcd(d, p) = 1$.
Let f_d be the **degree d homogeneous component** of f, i.e.

$$f = f_d + g, \ \deg(g) < d \ .$$

If the hypersurface defined by $f_d = 0$ is **smooth**, then, we have

$$|S(f)| = \left| \sum_{x \in \mathbb{F}_q^n} \omega^{f(x)} \right| \ \leq \ (d-1)^n \cdot q^{n/2} \ .$$

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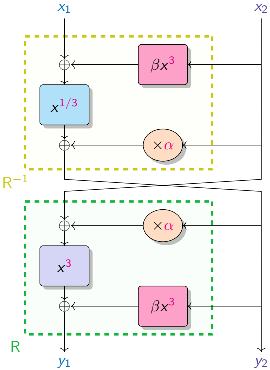
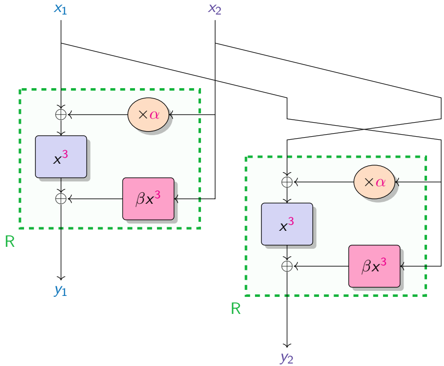
$\boxed{\text{Linearity bound for } n = 2\text{: } \mathcal{L}_F \leq (d-1)^2 \cdot q.}$

Butterfly - Definition

Introduced by [Perrin, Udovenko and Biryukov, 2016] over binary fields, $\mathbb{F}_{2^n}^2$, n odd.



Open variant.



Closed variant.

$$\begin{cases} y_1 &= (x_2 + \alpha y_2)^3 + (\beta y_2)^3 \\ y_2 &= (x_1 - (\beta x_2)^3)^{1/3} - \alpha x_2 \,. \end{cases}$$

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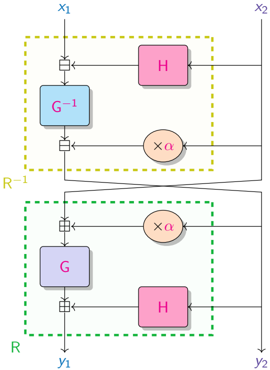
Generalized Butterfly - Definition

$\textsc{Butterfly}[G, H, \alpha]$, with $G : \mathbb{F}_q \to \mathbb{F}_q$ a permutation, $H : \mathbb{F}_q \to \mathbb{F}_q$ a function and $\alpha \in \mathbb{F}_q$.



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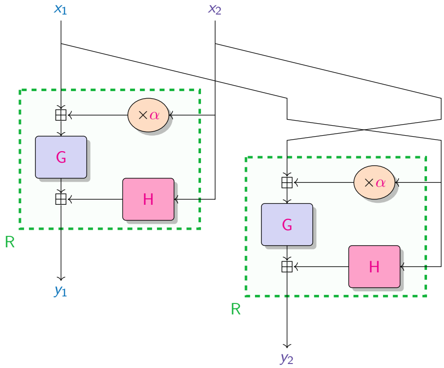

Closed variant.

$$\begin{cases} y_1 &= G(x_2 + \alpha y_2) + H(y_2) \\ y_2 &= G^{-1}(x_1 - H(x_2)) - \alpha x_2 \,. \end{cases}$$

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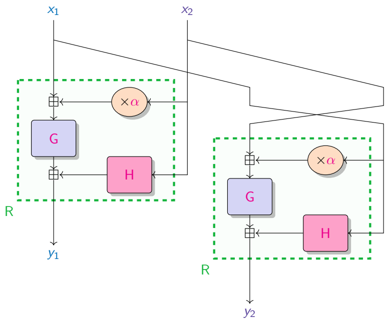
Generalized Butterfly - Bound

Let $\mathsf{F} = \textsc{Butterfly}[\mathsf{G}, \mathsf{H}, \alpha]$, with G a permutation, H a function and α in $\mathbb{F}_q$.

$$\begin{aligned} f(x_1, x_2) &= \langle (v_1, v_2), \mathsf{F}(x_1, x_2) \rangle - \langle (u_1, u_2), (x_1, x_2) \rangle \\ &= v_1 \mathsf{G}(x_1 + \alpha x_2) + v_2 \mathsf{G}(x_2 + \alpha x_1) + v_1 \mathsf{H}(x_2) + v_2 \mathsf{H}(x_1) - u_1 x_1 - u_2 x_2 \ . \end{aligned}$$



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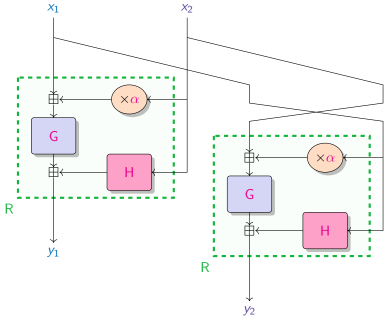

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Linearity Bound

- ⋆ If $d = \deg \mathsf{G} > \deg \mathsf{H} > 1$, then and $\alpha \neq \pm 1$,

  $f_d = (x_1 + \alpha x_2)^d + v_2/v_1 (x_2 + \alpha x_1)^d = 0$ is smooth.

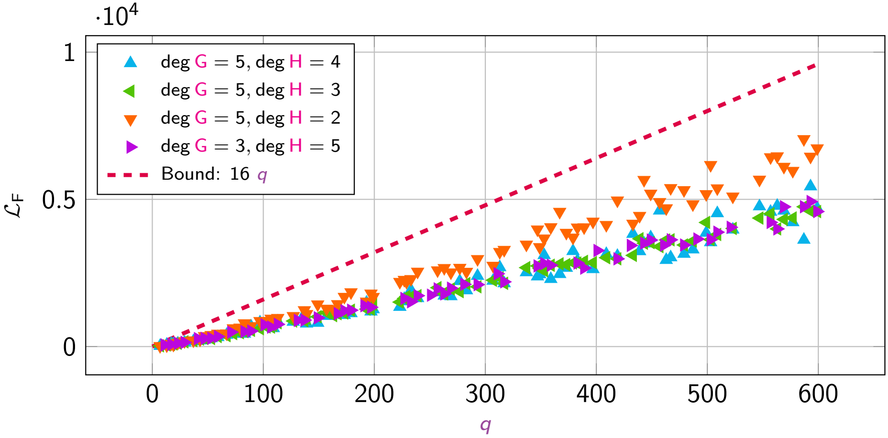
- ⋆ If $d = \deg \mathsf{H} > \deg \mathsf{G} > 1$, then

  $f_d = {x_1}^d + v_1/v_2 {x_2}^d = 0$ is smooth.

$$\boxed{\mathcal{L}_{\mathsf{F}} \leq (\max\{\deg \mathsf{G}, \deg \mathsf{H}\} - 1)^2 \cdot q}$$

Generalized Butterfly - Results

Let $\mathsf{F} = \textsc{Butterfly}[\mathsf{G}, \mathsf{H}, \alpha]$ with G and H monomial functions.



Low-degree functions ($\max\{\deg \mathsf{G}, \deg \mathsf{H}\} = 5$ *and* $\alpha = 2$*).*

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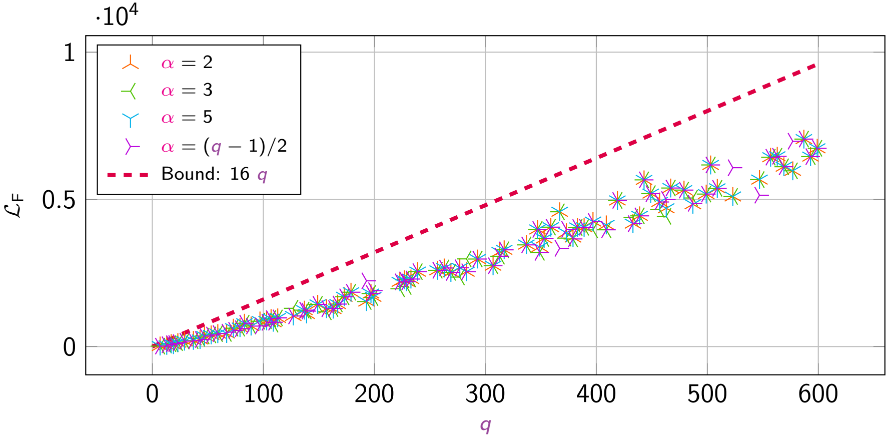

Influence of α *(*$\deg G = 5$ *and* $\deg H = 2$*).*

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Newton Polyhedron

Definition

Let $f \in \mathbb{F}_q[x_1, \ldots, x_n]$ s.t.

$$f(x_1, \ldots, x_n) = \sum_{e_1, \ldots, e_n} c_{e_1, \ldots, e_n} \prod_{i=1}^{n} x_i^{e_i} \ .$$

The **Newton polyhedron** $\Delta(f)$ of f is the convex hull defined by

$$\{(0, \ldots, 0)\} \cup \{(e_1, \ldots, e_n) \mid c_{e_1, \ldots, e_n} \neq 0\} \subset \mathbb{R}^n \ .$$

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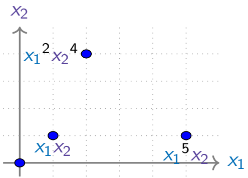
$$f(x_1, x_2) = 1 + x_1 x_2 - 2{x_1}^2 {x_2}^4 + 3{x_1}^5 x_2$$



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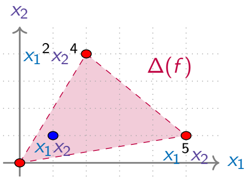

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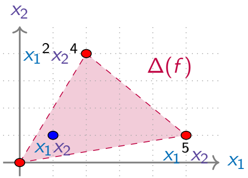
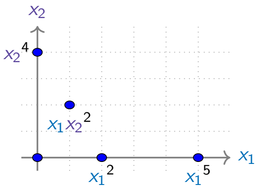


$$f(x_1, x_2) = 3 - {x_1}^2 + 5 x_1 {x_2}^2 + {x_2}^4 + 9{x_1}^5$$



Newton Polyhedron

Definition

Let $f \in \mathbb{F}_q[x_1, \ldots, x_n]$ s.t.

$$f(x_1, \ldots, x_n) = \sum_{e_1, \ldots, e_n} c_{e_1, \ldots, e_n} \prod_{i=1}^n x_i^{e_i} \ .$$

The **Newton polyhedron** $\Delta(f)$ of f is the convex hull defined by

$$\{(0, \ldots, 0)\} \cup \{(e_1, \ldots, e_n) \mid c_{e_1, \ldots, e_n} \neq 0\} \subset \mathbb{R}^n \ .$$

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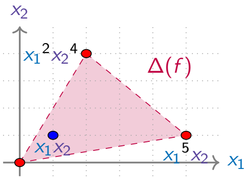

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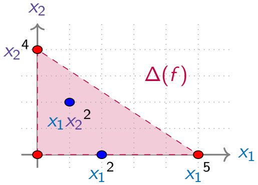

Newton Number

Definition

Let $f \in \mathbb{F}_q[x_1, \ldots, x_n]$. The **Newton number** $\nu(f)$ of f is

$$\nu(f) = \sum_{I \subseteq \{1,\ldots,n\}} (-1)^{|I|}(n - |I|)!\, \mathrm{Vol}_I \Delta(f) \ ,$$

where $\mathrm{Vol}_I \Delta(f)$ is the volume of $\Delta(f) \bigcap_{i \in I} \{x_i = 0\}$

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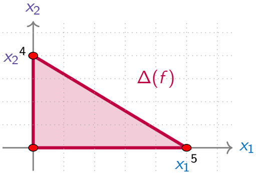

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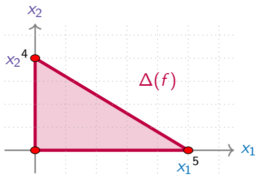
Example:

$f(x_1, x_2) = 3 - {x_1}^2 + 5x_1{x_2}^2 + {x_2}^4 + 9{x_1}^5$ $\qquad \nu(f) = (-1)^0 \cdot 2! \cdot \mathrm{Vol}_{\Delta(f)}$ $\qquad (I = \emptyset)$



$= 2 \times (5 \times 4)/2$

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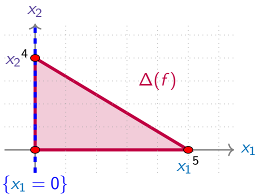

$$\nu(f) = (-1)^0 \cdot 2! \cdot \mathrm{Vol}_{\Delta(f)} \qquad (I = \emptyset)$$
$$+ (-1)^1 \cdot 1! \cdot \mathrm{Vol}_{\Delta(f) \cap \{x_1 = 0\}} \qquad (I = \{1\})$$

$$= 2 \times (5 \times 4)/2 - 4$$

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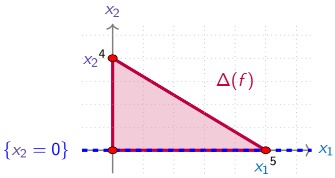

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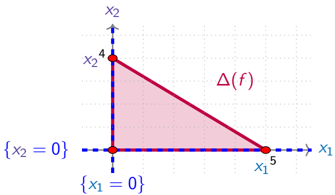

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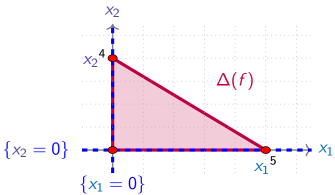

$$\begin{aligned}
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&= 12 &
\end{aligned}$$

Commode functions

Definition

A function f is **commode** if there exist nonzero $d_1, d_2, \ldots, d_n$ such that

$$(d_1, 0, 0, \ldots, 0), (0, d_2, 0, \ldots, 0), \ldots, (0, 0, \ldots, 0, d_n) \in \Delta(f)$$

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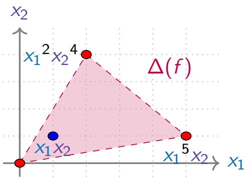

f is not commode

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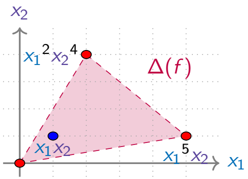

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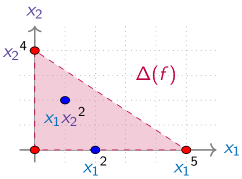

f is commode

Denef-Loeser Theorem

Definition

A function f is **non-degenerate** if for every face τ of $\Delta(f)$ the following system has no nonzero solutions

$$\partial f_\tau / \partial x_1 = \cdots = \partial f_\tau / \partial x_n = 0$$

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Theorem [Denef and Loeser, 1991]

Let $f \in \mathbb{F}_q[x_1, \dots, x_n]$.
If f is **commode** and **non-degenerate** with respect to its **Newton polyhedron** $\Delta(f)$, then, we have

$$|S(f)| = \left| \sum_{x \in \mathbb{F}_q^n} \omega^{f(x)} \right| \leq \nu(f) \cdot q^{n/2} .$$

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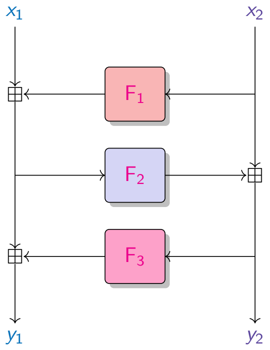
$\boxed{\text{Linearity bound for } n = 2\text{: } \mathcal{L}_F \leq \nu(f) \cdot q.}$

3-round Feistel - Definition

Let $\textsc{Feistel}[F_1, F_2, F_3]$ be a 3-round Feistel network with

$$d_1 = \deg(F_1),\ d_2 = \deg(F_2),\ \text{and } d_3 = \deg(F_3)\ .$$



A 3-round Feistel.

$$\begin{cases} y_1 &= x_1 + F_1(x_2) + F_3(x_2 + F_2(x_1 + F_1(x_2))) \\ y_2 &= x_2 + F_2(x_1 + F_1(x_2))\ . \end{cases}$$

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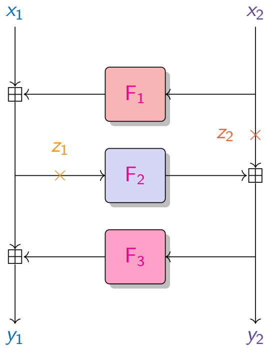

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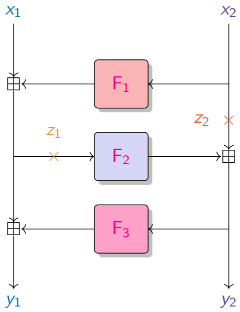
New equations with intermediate variables

$$\begin{cases} x_1 &= z_1 - F_1(z_2) \\ x_2 &= z_2 \\ y_1 &= z_1 + F_3(z_2 + F_2(z_1)) \\ y_2 &= z_2 + F_2(z_1)\ . \end{cases}$$

3-round Feistel - Bound

Let $\mathsf{F} = \textsc{Feistel}[\mathsf{F}_1, \mathsf{F}_2, \mathsf{F}_3]$, with round functions F_1, F_2 (permutation) and F_3. Let $d_1 \geq d_3$.

$$\begin{aligned} f(z_1, z_2) &= \langle (v_1, v_2), \mathsf{F}(z_1, z_2) \rangle - \langle (u_1, u_2), (z_1, z_2) \rangle \\ &= v_1 \mathsf{F}_3(z_2 + \mathsf{F}_2(z_1)) + v_2 \mathsf{F}_2(z_1) + u_1 \mathsf{F}_1(z_2) + (v_1 - u_1) z_1 + (v_2 - u_2) z_2 \ . \end{aligned}$$



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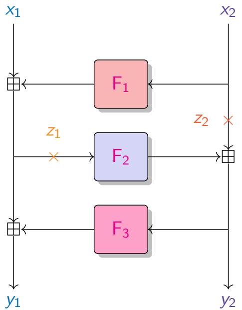

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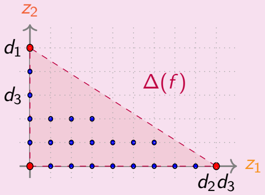
Linearity Bound

- ⋆ f is commode
- ⋆ f is non-degenerate
- ⋆ its Newton number is

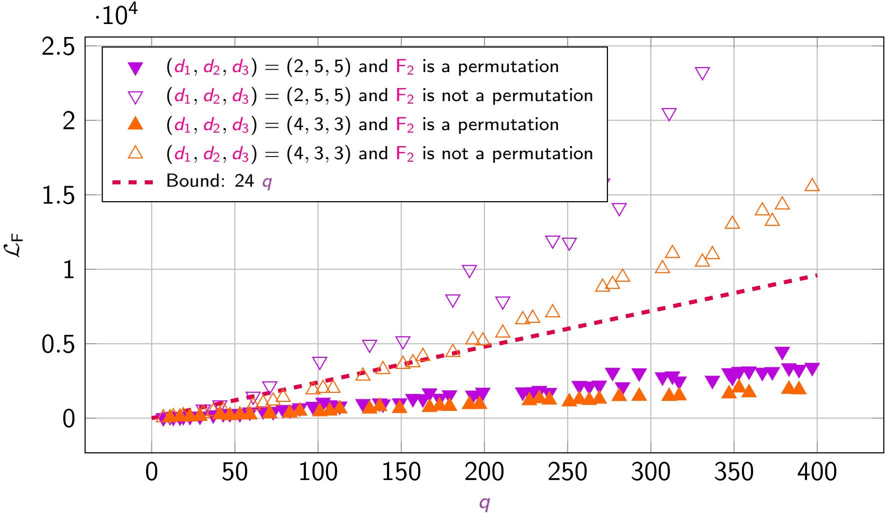
$$\nu(f) = (d_2 d_3 - 1)(d_1 - 1) \ .$$



$$\boxed{\mathcal{L}_{\mathsf{F}} \leq (d_1 - 1)(d_2 d_3 - 1) \cdot q}$$

3-round Feistel - Results

Let $F = \textsc{Feistel}[F_1, F_2, F_3]$ with F_1, F_2 and F_3 monomial functions.



Generalizations of Weil bound

- ⋆ Deligne bound
  - ⋆ Application to the Generalized Butterfly construction
- ⋆ Denef and Loeser bound
  - ⋆ Application to 3-round Feistel construction
- ⋆ Rojas-León bound
  - ⋆ Application to the Generalized Flystel construction

Isolated singularities

Definition

- ⋆ A singular point of a hypersurface is **isolated** if there exists a Zariski neighborhood of the point that contains no other singular points.
- ⋆ A polynomial g is **quasi-homogeneous** of degree δ is there exists $w_1, \ldots, w_n$ s.t.

$$g(\lambda^{w_1} x_1, \ldots, \lambda^{w_n} x_n) = \lambda^{\delta} g(x_1, \ldots, x_n) \ .$$

- ⋆ The **Milnor number** of the singularity is equal to $\prod_{i=1}^{n} (\delta / w_i - 1)$

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Example: Let $f(x) = (x-1)^d$.

- $x = 1$ is the only singular point of $f = 0$.
- Up to translation, we can consider the singularity in the origin: $g(x) = x^d$.

$$g(\lambda^w x) = (\lambda^w x)^d = \lambda^{w \cdot d} x^d = \lambda^{w \cdot d} g(x) \qquad \text{so that } \delta = w \cdot d$$

- Milnor number of the singularity: $\delta / w - 1 = d - 1$.

Rojas-León Theorem

Theorem [Rojas-León, 2006]

Let $f \in \mathbb{F}_q[x_1, \dots, x_n]$, s.t. $\deg(f) = d$.

Suppose that $f = f_d + f_{d'} + \cdots$, where f_d, $f_{d'}$, are resp. **the degree-d, degree-d', homogeneous component** of f, with $\gcd(d, p) = \gcd(d', p) = 1$ and $d'/d > p/(p + (p-1)^2)$.

If the following conditions are satisfied

- ⋆ the hypersurface defined by $f_d = 0$ has at worst **quasi-homogeneous isolated singularities** of degrees prime to p with **Milnor numbers** $\mu_1, \dots, \mu_s$,
- ⋆ the hypersurface defined by $f_{d'} = 0$ contains none of these singularities,

then we have

$$|S(f)| = \left| \sum_{x \in \mathbb{F}_q^n} \omega^{f(x)} \right| \leq \left((d-1)^n - (d-d') \sum_{i=1}^{s} \mu_i \right) \cdot q^{n/2} .$$

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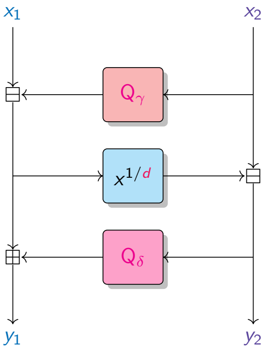
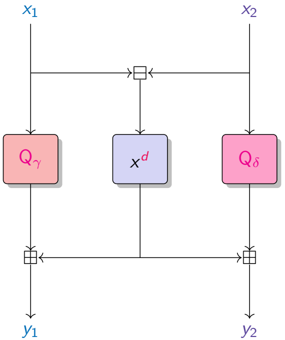
$$\boxed{\text{Linearity bound for } n = 2\text{: } \mathcal{L}_F \leq ((d-1)^2 - (d-d')\textstyle\sum_{i=1}^{s} \mu_i) \cdot q.}$$

Flystel - Definition

Introduced by [Bouvier, Briaud, Chaidos, Perrin, Salen, Velichkov and Willems, 2023].



Open variant.



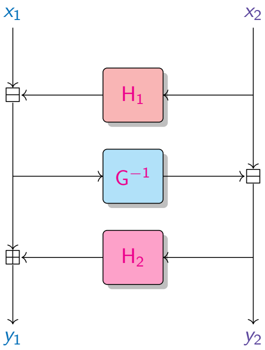
Closed variant.

$$\begin{cases} y_1 &= x_1 - Q_\gamma(x_2) + Q_\delta(x_2 - (x_1 - Q_\gamma(x_2))^{1/d}) \\ y_2 &= x_2 - (x_1 - Q_\gamma(x_2))^{1/d} . \end{cases}$$

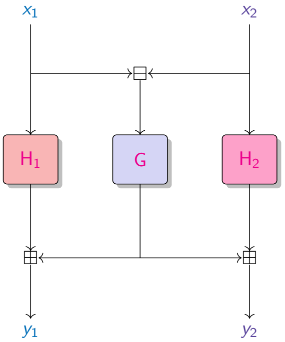
$$\begin{cases} y_1 &= (x_1 - x_2)^d + Q_\gamma(x_1) \\ y_2 &= (x_1 - x_2)^d + Q_\delta(x_2) . \end{cases}$$

Generalized Flystel - Definition

$\mathsf{F} = \textsc{Flystel}[\mathsf{H}_1, \mathsf{G}, \mathsf{H}_2]$, with $\mathsf{G} : \mathbb{F}_q \to \mathbb{F}_q$ a permutation, and $\mathsf{H}_1, \mathsf{H}_2 : \mathbb{F}_q \to \mathbb{F}_q$ functions.



Open variant.



Closed variant.

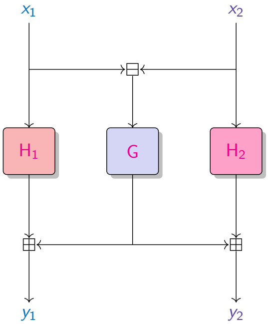
$$\begin{cases} y_1 &= x_1 - \mathsf{H}_1(x_2) + \mathsf{H}_2(x_2 - \mathsf{G}^{-1}(x_1 - \mathsf{H}_1(x_2))) \\ y_2 &= x_2 - \mathsf{G}^{-1}(x_1 - \mathsf{H}_1(x_2)) \,. \end{cases}$$

$$\begin{cases} y_1 &= \mathsf{G}(x_1 - x_2) + \mathsf{H}_1(x_1) \\ y_2 &= \mathsf{G}(x_1 - x_2) + \mathsf{H}_2(x_2) \,. \end{cases}$$

Generalized Flystel - Bound

Let $\mathsf{F} = \textsc{Flystel}[\mathsf{H}_1, \mathsf{G}, \mathsf{H}_2]$, with G a permutation, $\mathsf{H}_1, \mathsf{H}_2$ functions ($\deg \mathsf{G} > \deg \mathsf{H}_1, \deg \mathsf{H}_2$).

$$
\begin{aligned}
f(x_1, x_2) &= \langle (v_1, v_2), \mathsf{F}(x_1, x_2) \rangle - \langle (u_1, u_2), (x_1, x_2) \rangle \\
&= (v_1 + v_2)\, \mathsf{G}(x_1 - x_2) + v_1 \mathsf{H}_1(x_1) + v_2 \mathsf{H}_2(x_2) - u_1 x_1 - u_2 x_2 \ .
\end{aligned}
$$



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y_1 &= \mathsf{G}(x_1 - x_2) + \mathsf{H}_1(x_1) \\
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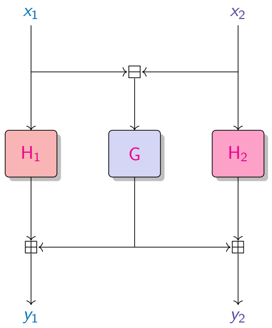

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Linearity Bound

- ⋆ The hypersurface

$$f_d = (v_1 + v_2)(x_1 - x_2)^d = 0$$

contains one singular point $[1 : 1]$ of quasi-homogeneous type with Milnor number $d - 1$.

- ⋆ The hypersurface

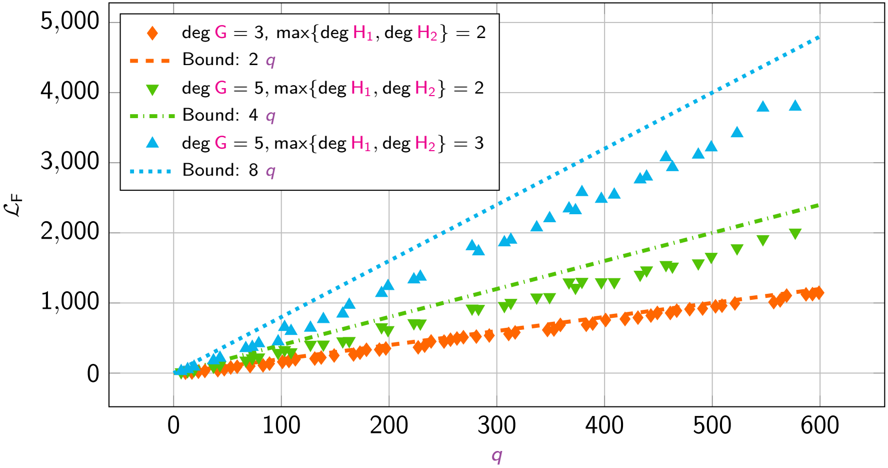
$$f_{d'} = v_i x_i^{\deg \mathsf{H}_i} = 0$$

does not contain this point.

$$\boxed{\mathcal{L}_{\mathsf{F}} \leq (\deg \mathsf{G} - 1)(\max\{\deg \mathsf{H}_1, \deg \mathsf{H}_2\} - 1) \cdot q}$$

Generalized Flystel - Results

Let $\mathsf{F} = \textsc{Flystel}[\mathsf{H}_1, \mathsf{G}, \mathsf{H}_2]$ with H_1, G and H_2 monomials.



Low-degree permutations G, H_1 *and* H_2.

Generalized Flystel - Results

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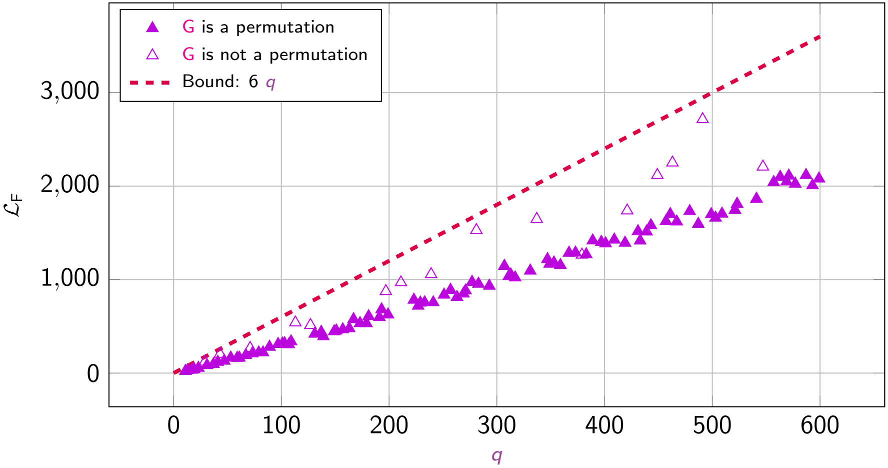

$\deg G = 7$ *and* $\deg H_1 = \deg H_2 = 2$.

Solving conjecture

Conjecture

Let $\mathsf{F} = \textsc{Flystel}[\mathsf{H}_1, \mathsf{G}, \mathsf{H}_2]$ be defined by $\mathsf{H}_1(x) = \gamma + \beta x^2$, $\mathsf{G}(x) = x^d$ and $\mathsf{H}_2 = \delta + \beta x^2$, with $\gamma, \delta \in \mathbb{F}_p$ and $\beta \in \mathbb{F}_p^\times$. Then

$$\mathcal{L}_\mathsf{F} \leq p \log p \ .$$

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Conjecture proved for $d \leq \log p$

Proposition

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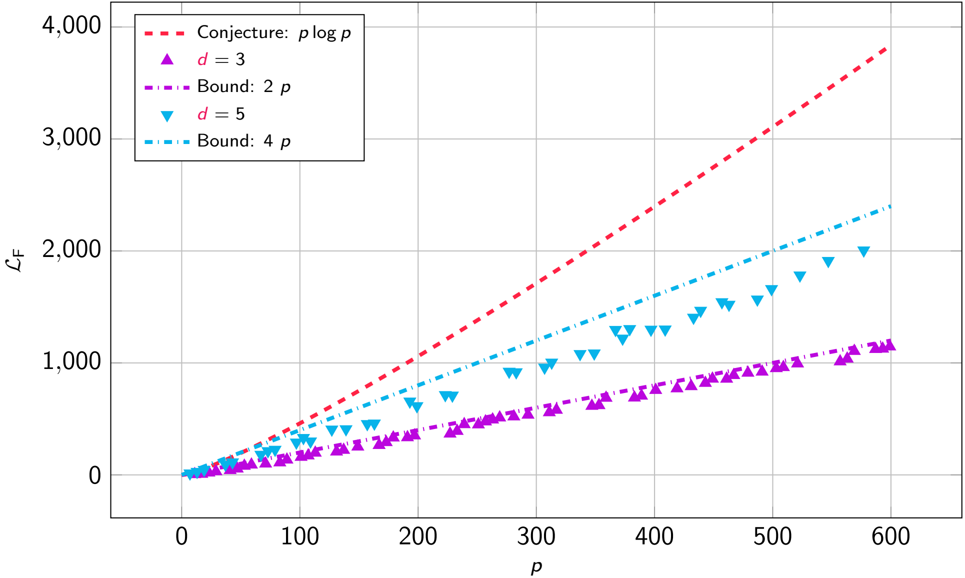
$$\mathcal{L}_\mathsf{F} \leq (d-1)p \ .$$

Solving conjecture



Conclusions

⋆ Bounds on exponential sums have direct application to linear cryptanalysis

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  - Rojas-León, 2006 — Generalization of the Flystel construction

$$\boxed{F \in \mathbb{F}_q[x_1, x_2],\ \exists C \in \mathbb{F}_q,\ \mathcal{L}_F \leq C \times q}$$

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Contribute to the cryptanalysis efforts for AOP.

Cohomological framework

$$S(f) = \sum_{x \in \mathbb{F}_q^n} \chi\big(\mathsf{F}(x)\big)\, \psi(-x)$$

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Sum of traces of the Frobenius automorphism on ℓ-adic cohomology groups.

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$$|S(f)| \leq \kappa \sum_{i=0}^{2n} \dim H_c^i(\mathbb{A}^n, \mathcal{L})$$

Perspectives

$\star$ Can we provide detailed calculations of the cohomological spaces to refine bounds?

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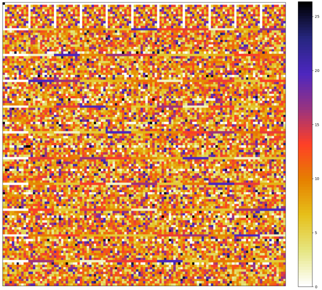
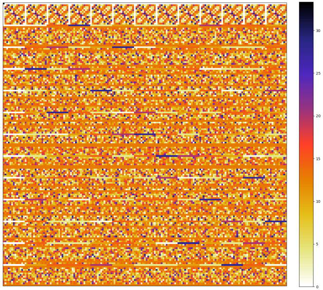


Closed Butterfly ($q = 11$)



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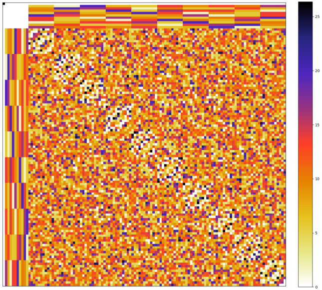

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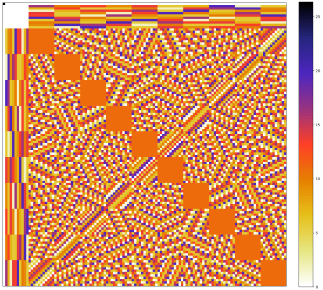

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And propose a general framework for arithmetization-oriented primitives?

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More details at *ia.cr/2024/1755*

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Thank you

Details on the bound

- Generalized Butterfly bound

$$\left|C^{F}_{\chi,\psi}\right| \leq \frac{1}{q}\begin{cases}(\deg G - 1)(\deg H - 1) & \text{if } \chi_1 = 1 \text{ or } \chi_2 = 1\,,\\ (\max\{\deg G, \deg H\} - 1)^2 & \text{else}\,.\end{cases}$$

- 3-round Feistel bound

$$\left|C^{F}_{\chi,\psi}\right| \leq \frac{1}{q}\begin{cases}(d_1 - 1)(d_2 - 1) & \text{if } \psi_1 \neq 1 \text{ and } \chi_1 = 1\,,\\ (d_3 - 1)(d_2 - 1) & \text{if } \psi_1 = 1 \text{ and } \chi_1 \neq 1\,,\\ (d_1 - 1)(d_3 - 1) & \text{if } \psi_1\chi_1 = 1\,,\\ (d_1 - 1)(d_2 d_3 - 1) & \text{else}\,.\end{cases}$$

- Generalized Flystel bound

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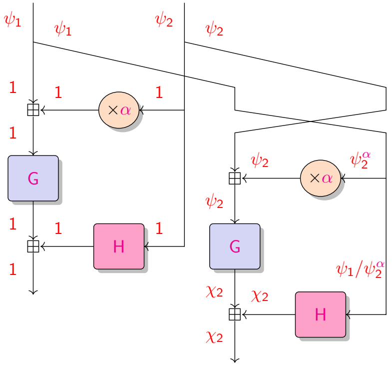
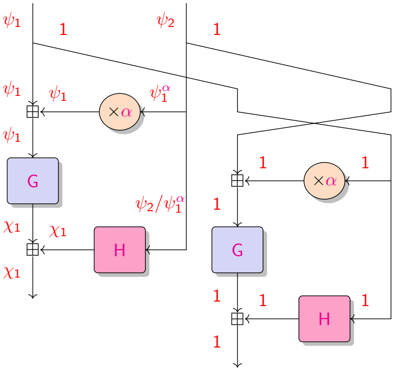
Linear trails for a Generalized Butterfly

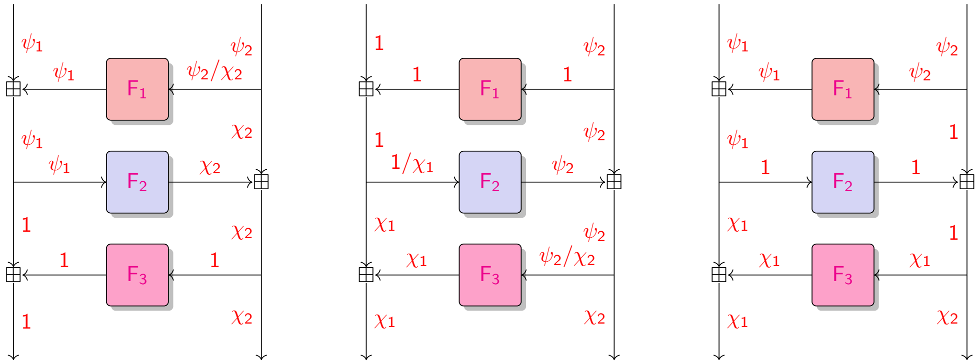


(a) $\chi_1 = 1$.



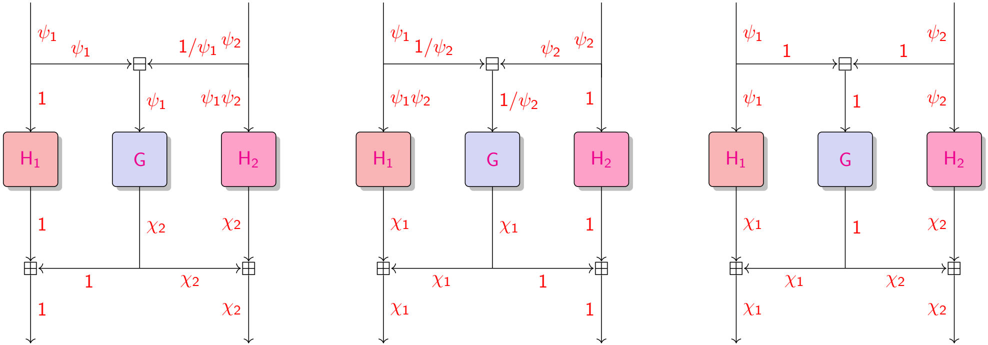
(b) $\chi_2 = 1$.

Linear trails for a 3-round Feistel



(a) $\psi_1 \neq 1$ *and* $\chi_1 = 1$. **(b)** $\psi_1 = 1$ *and* $\chi_1 \neq 1$. **(c)** $\psi_1\chi_1 = 1$.

Linear trails for a Generalized Flystel



(a) $\chi_1 = 1.$ **(b)** $\chi_2 = 1.$ **(c)** $\chi_1\chi_2 = 1.$

Bound on exponential sums

The trace of F on $H_c^i(\mathbb{A}^n, \mathcal{L})$ is the sum of its eigenvalues $\lambda_1, \lambda_2, \ldots$.

$$\mathrm{Tr}\big(F \mid H_c^i(\mathbb{A}^n, \mathcal{L}\big) = \lambda_1 + \lambda_2 + \lambda_3 + \ldots$$

Suppose that, $\forall i$, $|\lambda_i| \leq \kappa$, then

$$\left|\mathrm{Tr}\big(F \mid H_c^i(\mathbb{A}^n, \mathcal{L})\right| \leq \kappa \cdot \dim H_c^i(\mathbb{A}^n, \mathcal{L})$$

This gives an upper bound on $S(f)$:

$$|S(f)| = \left|\sum_{i=0}^{2n} (-1)^i \,\mathrm{Tr}\big(F \mid H_c^i(\mathbb{A}^n, \mathcal{L})\right|$$
$$\leq \kappa \sum_{i=0}^{2n} \dim H_c^i(\mathbb{A}^n, \mathcal{L})$$