Denote $\mathbf{Ban_1}$ the category Banach spaces and contractive linear maps, $\mathbf{L_1}$ the category of Banach lattices and contractive lattice homomorphisms and $\mathbf{BL_1}$ the category of Banach lattices and almost interval preserving contractions. Recall that a positive linear map $\phi : E \to F$ between normed Riesz spaces is called *almost interval preserving* if $\phi([0,x])$ is dense in $[0,\phi(x)]$ for every $x \in E_+$. It follows from [4, Proposition 1.3.13] immediately that the adjoint, denoted by *, is a contravariant functor between $\mathbf{BL_1}$ and $\mathbf{L_1}$. Also, it is easy to see that both $\mathbf{BL_1}$ and $\mathbf{L_1}$ are subcategories of $\mathbf{Ban_1}$.

Lemma 1. *$((E_i),(\phi_{ji})_{j\geq i})$ is a direct system indexed by a directed set in $\mathbf{BL_1}$. If E is a Banach lattice, $\phi_i : E_i \to E$ is an almost interval preserving map for each index i, then (E,ϕ_i) is a direct limit of $((E_i),(\phi_{ji})_{j\geq i})$ in $\mathbf{Ban_1}$ is equivalent to that in $\mathbf{BL_1}$.*

Proof. Let $(E,(\phi_i))$ is a direct limit of $((E_i),(\phi_{ji})_{j\geq i})$ in $\mathbf{Ban_1}$, then $\bigcup_i \phi_i(E_i)$ is dense in E by Banach space theory. Suppose F is a Banach lattice and $\psi_i : E_i \to F$ is an almost interval preserving contraction for each i such that $\psi_j \circ \phi_{ji} = \psi_i$ whenever $j \geq i$. Let $\psi : E \to F$ be the unique contractive linear map satisfying $\psi \circ \phi_i = \psi_i$ for each i, we will prove that ψ is almost interval preserving, that is, $(E,(\phi_i))$ is also a direct limit of $((E_i),(\phi_{ji})_{j\geq i})$ in $\mathbf{BL_1}$. Thanks to [4, Proposition 1.3.13], we need only prove that $\psi^* : F^* \to E^*$ is a lattice homomorphism. Pick any $\tau \in F^*$, then

$$\phi_i^* \circ \psi^*(|\tau|) = \psi_i^*(|\tau|) = |\psi_i^*(\tau)| = |\phi_i^* \circ \psi^*(\tau)| = \phi_i^*(|\psi^*(\tau)|),$$

i.e., $\psi^*(|\tau|) \circ \phi_i = |\psi^*(\tau)| \circ \phi_i$. Therefore, $\psi^*(|\tau|) = |\psi^*(\tau)|$ on $\bigcup_i \phi_i(E_i)$ and thus on E, which finishes the proof. □

Theorem 2. *Every direct system $((E_i),(\phi_{ji})_{j\geq i})$ admits a direct limit in $\mathbf{BL_1}$. Specially, if each E_i is closed sublattice of a Banach lattice E and E_i is an order ideal of E_j whenever $j \geq i$, then the norm closure of $\bigcup_i E_i$ is a direct limit of (E_i) and inclusion maps.*

Proof. It's easy to verify that $\prod_i E_i := \{(E_i) : \sup_i \|E_i\| < \infty\}$ with pointwise order and the supremum norm is a Banach lattice and that $\bigoplus_i E_i := \{(E_i) : \|E_i\| \to 0 \text{ as } i \to \infty\}$ is a closed order ideal of $\prod_i E_i$. By [4, Proposition 1.3.13], the quotient $\prod_i E_i / \bigoplus_i E_i$ is a Banach lattice and the quotient map $q : \prod_i E_i \to \prod_i E_i / \bigoplus_i E_i$ is a lattice homomorphism. For each i, there is a natural positive linear opertaor $\Phi_i : E_i \to \prod_i E_i$ defined by setting the component of $\Phi_i(a)$ in E_j to be $\phi_{ji}(a)$ if $j \geq i$ and 0 otherwise. Let $\phi_i = q \circ \Phi_i$ and $E = \overline{\bigcup_i \phi_i(E_i)}$, then $(E,(\phi_i))$ is a direct limit of $((E_i),(\phi_{ji})_{j\geq i})$ in $\mathbf{Ban_1}$ by Banach space theory.

Given an index i_0, $a \in E_{i_0+}$ and $\epsilon > 0$. Suppose $y = q((y_i)) \in [0,\phi_{i_0}(a)] \cap E, (y_i) \in \prod_i E_i$. Since q is a lattice homomorphism, we can assume $y_i \in [0,\phi_{ii_0}(a)]$ if $i \geq i_0$ and $y_i = 0$ otherwise. Choose an index i_ϵ and $a_\epsilon \in A_{i_\epsilon}$ satisfying $\|y - \phi_{i_\epsilon}(a_\epsilon)\| < \epsilon$, then $\|y_{i_\infty} - \phi_{i_\infty i_\epsilon}(a_\epsilon)\| < \epsilon$ for some sufficiently large $i_\infty \geq i_\epsilon, i_0$. Since $\phi_{i_\infty i_0}$ is almost interval preserving, there exists some $x \in [0,a]$ such that $\|y_{i_\infty} - \phi_{i_\infty i_0}(x)\| < \epsilon$. Consequently,

$$\begin{aligned}\|y - \phi_{i_0}(x)\| &= \|y - \phi_{i_\epsilon}(a_\epsilon) + \phi_{i_\infty}(\phi_{i_\infty i_\epsilon}(a_\epsilon) - y_{i_\infty} + y_{i_\infty} - \phi_{i_\infty i_0}(x))\| \\ &\leq \|y - \phi_{i_\epsilon}(a_\epsilon)\| + \|\phi_{i_\infty i_\epsilon}(a_\epsilon) - y_{i_\infty}\| + \|y_{i_\infty} - \phi_{i_\infty i_0}(x)\| \leq 3\epsilon.\end{aligned}$$

That is, $\overline{\phi_{i_0}([0,a])} \supset [0,\phi_{i_0}(a)] \cap E$.

Pick $b \in \bigcup_i \phi_i(E_i)$ and suppose $b = \phi_i(a)$ for some index i and $a \in E_i$, then $|b| \in [0, \phi_i(|a|)]$. Hence, for arbitrary $\epsilon > 0$, there is some $x \in [0, |a|]$ such that $\||b| - \phi_i(x)\| < \epsilon$, which follows that $|b| \in E$. The continuity of lattice operators implies that $|y| \in E$ whenever $y \in E$, i.e., the Banach space E is in fact a Banach lattice. Together with the conclusion of last paragraph, it follows that every $\phi_i : E_i \to E$ is an almost interval preserving contraction between Banach lattices. Our proof finishes with application of Lemma 1. $\square$

Theorem 3. *$((E_i), (\phi_{ji})_{j\geq i})$ is a direct system with a direct limit $(E, (\phi_i))$ in $\mathbf{BL_1}$. If each E_i is order continuous, so is E.*

Proof. Let us look into the following diagram in $\mathbf{Ban_1}$, where $((E_i), (\phi_{ji})_{j\geq i}))$ is a direct system with a direct limit (E, ϕ_i) in $\mathbf{BL_1}$ and ι_i is the canonical embedding for each i.

$$\begin{tikzcd} E_i \arrow[r, "\phi_i"] \arrow[d, "\iota_i"'] & E \arrow[d, dashed] \\ E_i^{**} \arrow[r, "\phi_i^{**}"] & E^{**} \end{tikzcd}$$

[4, Proposition 1.3.13] follows that $\phi_i^{**} : E_i^{**} \to E^{**}$ is a morphism in $\mathbf{BL_1}$ and [4, Theorem 2.4.1] follows that $\iota_i(E_i)$ is an order ideal of E^{**} which is equivalent to $\iota_i : E_i \to E_i^{**}$ is a morphism in $\mathbf{BL_1}$. By Theorem 2, the dash arrow in the above commutative diagram can be filled in with a morphism $\iota : E \to E^{**}$ in $\mathbf{BL_1}$. However, the canonical embedding is the unique continuous linear map filling in the dash arrow in $\mathbf{Ban_1}$, so, ι coincides with the canonical embedding. Using [4, Theorem 2.4.1] again, E is order continuous. $\square$

An observation is that the order continuity of a direct limit in $\mathbf{BL_1}$ is independent on the choice.

Theorem 4. *Suppose X be a Hausdorff space and μ is a σ-finite Radon measure on a σ-algebra Σ of subsets of X which contains all compact subsets of X. Let E be a Banach function space over (Σ, μ) that contains $C_c(X)$ as a subspace. If E is order continuous, then $C_c(X)$ is dense in E.*

Proof. It is well known that for any positive measurable function f, there is an increasing sequence (s_n) of simple functions that converges to f everywhere, and moreover, we can require that each simple function has a support with finite measure if the measure space is σ-finite. Since E is order continuous, $\text{span}\{\chi_S \in E : \mu(S) < \infty\}$ is dense in E. The inner regularity of μ follows that there is an increasing sequence (K_n) of compact subsets of S such that $\mu(K_n) \to \mu(S)$ as $n \to \infty$. Therefore, χ_{K_n} converges to χ_S, for each subset S of X satisfying $\chi_S \in E$ and $\mu(S) < \infty$, in order and thus in norm. So, $\text{span}\{\chi_K \in E : K \text{ is a compact subset of } X\}$ is dense in E. By Urysohn's lemma, each χ_K where K is a compact subset of X, is a limit almost everywhere, or equivalently, is an order limit, of a decreasing sequence in $C_c(X)$. As a result, $C_c(X)$ is a dense subspace of E. $\square$

Remark 5. *In fact, the above proof gives a conclusion that if* $\text{span}\{\chi_S \in E : \mu(S) < \infty\}$ *is dense in the order continuous Banach function space E, then $C_c(X)$ is dense in E regardless of whether μ is σ-finite or not. Hence a Banach function* *space like $L^p (1 \leq p < \infty)$ space over a Borel measure space always contains $C_c(X)$ as a dense subspace.*

Theorem 6. *Suppose X is a Hausdorff topological space, and μ is a measure on a σ-algebra Σ of subsets of X that contains all compact subsets of X. Let E be a Banach function space over (Σ, μ) such that $C_c(X)$ is contained in and dense in E. If every compact subset of X is metrisable, then E is order continuous.*

Proof. It is easy to verify E_S is a band of E_T if $S \subset T$, where

$$E_S := \{f\chi_S : f \in E\}$$

for each subset S of X. Since $f \in E_{\operatorname{supp} f}$ for every $f \in C_c(X)$, we have

$$E = \overline{C_c(X)} \subseteq \overline{\bigcup_{K \text{ compact}} E_K} \subseteq E.$$

In view of theorem 3, we need only show that E_K is order continuous for every compact subset K of X. Take such a K. Since the band projection $f \mapsto \chi_K f$ from E onto E_K is continuous, the density of $C_c(X)$ in E implies that $\{f\chi_K : f \in C_c(X)\}$ is dense in E_K. Hence $C(K)$ is dense in E_K. Since K is metrisable, $C(K)$ is separable with respect to $\|\cdot\|_\infty$ (see, e.g., [2, Theorem 26.15]). Since the positive inclusion map from the Banach lattice $(C(K), \|\cdot\|_\infty)$ into the Banach lattice E_K is continuous, we see that E_K is separable. Consequently, it cannot contain a closed subspace that is isomorphic to ℓ^∞. Since $L^0(X, \mu)$ is σ-Dedekind complete, so is its order ideal E_K. It now follows from [4, Corollary 2.4.3] that E_K is order continuous, as required. □

Suppose $\pi : X \to Y$ is a continuous map between topological spaces. If μ_X is Borel measure on X, then $\mu_Y := \mu \circ \pi^{-1}$ is a Borel measure on Y and

$$\begin{aligned} \pi_* : & L^0(Y, \mu_Y) \to L^0(X, \mu_X) \\ & g \mapsto g \circ \pi \end{aligned}$$

is an injective lattice homomorphism between Riesz spaces. Let $(E_X, \|\cdot\|_X)$ be a Banach function space over (X, μ_X). Define

$$E_Y := \pi_*^{-1}(E_X)$$

and

$$\|\cdot\|_Y = \|\cdot\|_X \circ \pi_*,$$

then $(E_Y, \|\cdot\|_Y)$ is a Banach function space over (Y, μ_Y) (In particular, $E_Y = L^1(Y, \mu_Y)$ if $E_X = L^1(X, \mu_X)$). Only completion needs to be varified. Suppose (g_n) is a Cauchy sequence in E_Y, then $(f_n) := (\pi_* g_n)$ is a Cauchy sequence in $(E_X, \|\cdot\|_X)$ and hence convergent to some f in E_X with respect to $\|\cdot\|_X$. Suppose (f_{k_n}) is a subsequence of (f_n) convergent to f μ_X-a.e., then (g_{k_n}) is convergent μ_Y-a.e., say g is the pointwise limit. Since

$$\{x \in X : (\pi_* g_{k_n})(x) \nrightarrow (\pi^* g)(x)\} \subset \pi^{-1}\{y \in Y : g_{k_n}(y) \nrightarrow g(y)\},$$

$(\pi_* g_{k_n})$ is convergent to $\pi_*(g)$, yielding $\pi(g) = f$. Therefore, $g \in E_Y$, (g_n) is convergent to g with respect to $\|\cdot\|_Y$ and $(E_Y, \|\cdot\|_Y)$ is norm complete. Consequently, $\pi_* : (E_Y, \|\cdot\|_Y) \to (E_X, \|\cdot\|_X)$ is a homomorphism between Banach function spaces . With this, we induce an isometric lattice homomorphism $\pi_* : (E_Y, \|\cdot\|_Y) \to (E_X, \|\cdot\|_X)$ between Banach function spaces from a continuous map $\pi : X \to Y$ between topological spaces.

Lemma 7. *Suppose that E is a translation invariant Banach function space over a locally compact Hausdorff topological group G with a Haar measure μ, that $C_c(G)$ is dense in E and that maps $x \to \|\lambda_x\|$ and $x \to \|\rho_x\|$ are bounded on compact subsets of G. If K is a compact subset of G and (K_n) is a sequence of compact subsets contained in K such that $\lim_{n\to\infty} \mu(K_n) = 0$, then $\lim_{n\to\infty} \|\chi_{K_n}\| = 0$.*

Proof. Let G' be a σ-compact clopen subgroup of G that contains K(the existence of G' follows from [1, Theorem A and Theorem B, Section 57]), then G' is a locally compact Hausdorff topological group with a Haar measure $\mu' = \mu|_{G'}$ and $E' := \{f\chi_{G'} | f \in E\}$ is a translation invariant Banach function space over G' with the norm inherited from E. Obviously, $x \to \|\lambda_x\|$ and $x \to \|\rho_x\|$ are bounded on compact subsets of G'. And one can also easily obtain $E' = E'_{s,0}$ if $E = E_{s,0}$, where

$$E_{s,0} := \{f \in E : \lim_{x\to e} \|\lambda_x f - f\| = \lim_{x\to e} \|\rho_x f - f\| = 0,$$
$$\text{and } \forall \varepsilon > 0\ \exists K \subset G \text{ compact s.t. } \|f\chi_{G\setminus K}\| < \varepsilon\}.$$

From [3, Theorem 5.4], we know that $C_c(G)$ is dense E if and only if $E = E_{s,0}$. It follows that $C_c(G')$ is dense in E'. Therefore, we can always assume G is σ-compact.

For each $n \in \mathbb{N}$, let O_n be an open subset of G such that $K_n \subset O_n$ and $\mu(O_n) < \mu(K_n) + 2^{-n}$. Since K_n is compact and O_n is open, the continuity of group multiplication implies there is an open neighbourhood U_n of the identity e of G such that $K_nU_n \subset O_n$. By [1, Theorem 8.7, Chapter II], there exists a compact normal subgroup H of G such that $H \subset \cap_{n=1}^{\infty} U_n$ and $G^* := G/H$ is metrizable.

Let $\pi : G \to G^*$ be the quotient map, $(E^*, \|\cdot\|^*)$ be the induced Banach function space over (G^*, μ^*) and $\pi_* : E^* \to E$ be the induced isometric lattice homomorphism. By [1, Theorem C, Section 63], $\mu^* = \mu \circ \pi$ is a Haar measure on G^*. A direct computaion yields, for any $x \in G$ and $g \in G^*$, that $\lambda_x(\pi_* g) = \pi_*(\lambda_{\pi(x)} g)$ and $\rho_x(\pi_* g) = \pi_*(\rho_{\pi(x)} g)$, from which it follows that E^* is also translation invariant. Since

$$\|\lambda_{\pi(x)} g\|^* = \|\pi_*(\lambda_{\pi(x)} g)\| = \|\lambda_x(\pi_* g)\| \le \|\lambda_x\|\|\pi_* g\| = \|\lambda_x\|\|g\|^*,$$

$\pi(x) \mapsto \|\lambda_{\pi(x)}\|$ is bounded on compact subsets of G^*, noting that every compact subset L of G^* is the image under π of a compact subset HL of G. Similarly, $\pi(x) \mapsto \|\rho_{\pi(x)}\|$ is bounded on compact subsets of G^*. Given $f \in E^*$ and $\varepsilon > 0$. Since $\pi_* f \in E = E_{s,0}$, there exists a compact subset K of G and an open neighbourhood U of the identity such that $\|(\pi_* f)\chi_{G\setminus K}\| < \varepsilon$, $\|\lambda_x(\pi_* f) - \pi_* f\| < \varepsilon$ and $\|(\rho_x \pi_* f) - \pi_* f\| < \varepsilon$ whenever $x \in U$. Hence

$$\|f\chi_{G^*\setminus\pi(K)}\|^* = \|\pi_*(f\chi_{G^*\setminus\pi(K)})\| = \|f\chi_{G\setminus\pi^{-1}(\pi(K))}\| \le \|f\chi_{G\setminus K}\| < \varepsilon,$$

$$\|\lambda_{\pi(x)} f - f\|^* = \|\pi_*(\lambda_{\pi(x)} f) - \pi_* f\| = \|\lambda_x(\pi_* f) - \pi_* f\| < \varepsilon$$

and

$$\|\rho_{\pi(x)} f - f\|^* = \|\pi_*(\rho_{\pi(x)} f) - \pi_* f\| = \|\rho_x(\pi_* f) - \pi_* f\| < \varepsilon$$

whenever $\pi(x)$ is in the open neighborhood $\pi(U)$ of the identity $\pi(e)$ of G^*. That is, $E^* = E^*_{s,0}$, i.e., $C_c(X)$ is dense in E^*. From the metric case, we know that E^* is order continuous.

Since $\mu(K_nH) \le \mu(O_n)$, $(\mu(K_nH))$ converges to 0, i.e., $(\pi_*\chi_{\pi(K)}) = (\chi_{K_nH})$ converges to 0 μ-a.e., resulting in $(\chi_{\pi(K_n)})$ converges to 0 μ^*-a.e.. Because $\chi_{\pi(K_n)} \le \chi_{\pi(K)}$, $(\chi_{\pi(K_n)})$ converges to 0 in E^* in order and thus in norm. Consequently,

$(\chi_{K_n H}) = (\pi_* \chi_{\pi(K)})$ converges to 0 in norm. It follows from $\chi_{K_n} \leq \chi_{K_n H}$ that (χ_{K_n}) converges to 0 in norm. □

Lemma 8. *Suppose K is a compact Hausdorff space with a Borel measure μ and E is a Banach function space over (K, μ) that contains $C(K)$ as a dense subspace and is included in $L^1(K, \mu)$ continuously. If for each sequence (K_n) of compact subsets of K satisfying $\lim_{n\to\infty} \mu(K_n) = 0$ we have $\lim_{n\to\infty} \|\chi_{K_n}\| = 0$, then E is order continuous.*

Proof. Let (f_n) be a sequence in E dominated by χ_K that decreases to 0 in order and (g_n) be a sequence in $C(K)$ such that $(f_n - g_n)$ converges to 0 in the norm of E. Since E is continuously included into $L^1(K, \mu)$, $(f_n - g_n)$ converges to 0 in the norm of $L^1(K, \mu)$ too. Since E is an order ideal of $L^1(\mu)$, (f_n) also decreases to 0 in order in $L^1(K, \mu)$. Hence (f_n) converges to 0 in the norm of $L^1(K, \mu)$ by the order continuity of $L^1(K, \mu)$.

Given $\varepsilon > 0$, define the compact sets $K_n = g^{-1}[\varepsilon, \infty)$ for $n \in \mathbb{N}$. Since

$$\|g_n\|_{L^1(K,\mu)} \geq \int_{K_n} g_n \mathrm{d}\mu \geq \varepsilon \mu(K_n),$$

it follows that $\lim_{n\to\infty} \mu(K_n) = 0$ and thus that $\lim_{n\to\infty} \|\chi_{K_n}\| \to 0$. Furthermore, since

$$0 \leq g \leq \chi_{K_n} + \varepsilon \chi_{K \setminus K_n},$$

$\|g_n\| \leq \|\chi_{K_n}\| + \varepsilon \|\chi_K\| < (1 + \|\chi_K\|)\varepsilon$ for n sufficiently large. Therefore, (g_n) and (f_n) converges to 0 in the norm of E. Hence $\chi_K \in E_a$, the absolutely continuous part of E. Noting that E_a is norm closed in E, $E = \overline{C(K)} \subset E_a$, i.e., E is order continuous. □

Theorem 9. *Suppose E is a translation invariant Banach function space on a locally compact Hausdorff topological group G, and that the maps $x \mapsto \|\lambda_x\|$ and $x \mapsto \|\rho_x\|$ are both bounded on compact subsets of G. If $C_c(G)$ is a dense subspace of E, then E is order continuous.*

Proof. It is sufficient to show that $E_K = \{f\chi_K | f \in E\}$ is order continuous for each compact subset K of G. By [3, Lemma 3.14], $E_K \subset L^1(K, \mu)$ with continuous inclusion. Let (K_n) be a sequence of compact subsets of K such that $\mu(K_n) \to 0$ as $n \to \infty$, then $\|\chi_{K_n}\| \to 0$ as $n \to \infty$ by lemma 7. Our proof ends by applying of lemma 8. □

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