su2nn Irreducible Representation

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I. INTRODUCTION

II. SU(2)'S DEFINITION

$SU(2)$ is the special unitary group of dimension 2, i.e. the group of 2 by 2 unitary matrices with unit determinant. In other words,

$$SU(2) = \{u \in GL(2, \mathbb{C}) | u^\dagger = u^{-1}, \det(u) = 1\}, \tag{1}$$

where $GL(2, \mathbb{C})$ is the group of 2 by 2 invertible matrices with complex field, and $u^\dagger$ is the complex conjugate transposition of u. Since for any $u, v \in SU(2)$

$$\det(uv) = \det(u)\det(v) = 1, \tag{2}$$

and

$$(uv)^\dagger = (uv)^{\top *} = (v^\top u^\top)^* = v^\dagger u^\dagger = v^{-1}u^{-1} = (uv)^{-1}, \tag{3}$$

$SU(2)$ is closed subgroup of $GL(2, \mathbb{C})$ which makes it a Lie group. This means that there is a corresponding Lie algebra, $\mathfrak{su}(2)$, such that

$$SU(2) = \{e^g | g \in \mathfrak{su}(2)\}. \tag{4}$$

From definition of matrix exponential,

$$\begin{aligned} (e^g)^\dagger &= e^{g^\dagger}, \\ (e^g)^{-1} &= e^{-g}, \\ \det(e^g) &= e^{\mathrm{Tr}\, g}, \end{aligned} \tag{5}$$

one can define the Lie algebra of $SU(2)$ as

$$\mathfrak{su}(2) = \{g \in M(2, \mathbb{C}) | g^\dagger = -g, \mathrm{Tr}(g) = 0\}, \tag{6}$$

where $M(2, \mathbb{C})$ is the group of 2 by 2 matrices with complex field, and $\mathrm{Tr}(g)$ is the trace of matrix g. Hence, each member g of $\mathfrak{su}(2)$ can be written in the From

$$g = \begin{bmatrix} iv_z & v_y + iv_x \\ -v_y + iv_x & -iv_z \end{bmatrix} = iv_x\sigma_x + iv_y\sigma_y + iv_z\sigma_z = i\vec{v} \cdot \vec{\sigma} \tag{7}$$

where $\vec{v} \in \mathbb{R}^3$, and σ_i's are Pauli matrices.

III. SU(2)'S IRREDUCIBLE REPRESENTATION

From $\mathfrak{su}(2)$, one can consider Pauli matrices as the group generators, but it is more common to use $J_i^{(1/2)} = \sigma_i/2$ as the generators. This gives the Lie bracket relation as

$$\left[J_i^{(1/2)}, J_j^{(1/2)}\right] = i\epsilon_{ijk} J_k^{(1/2)}. \tag{8}$$

The next step is to consider an arbitrary representation of $SU(2)$ such that its generators, J_i's, obey (8). With the standard method, first define two additional operators

$$\begin{aligned} J_+ &= J_x + iJ_y, \\ j_- &= J_x - iJ_y. \end{aligned} \tag{9}$$

Then, consider