

Unified Representation for Quantum Spin Equivariant Learning

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Abstract

TBA

1. INTRODUCTION

2. SYMMETRIES IN QUANTUM SPIN SYSTEMS

3. COMPUTATIONAL REPRESENTATION

3.1. Systems with Multiple Symmetries.

Of course, this generalization allow the representations that is not physical, e.g. spin-like object that have odd parity, or time-reversal symmetric object that have half-integer spin. However, it makes things easier to handle in computational perspective since one can represent every physical objects by unified representation with their symmetry descriptions. Not only that, because of the power to control every symmetry included in the representations, the unification grants the opportunity for the computation to easily segregate contributions of interactions with different symmetries. Moreover, from machine learning perspective, there are multiple instances that show better performance when one allow the model to include unphysical symmetries. Lastly, this generalization makes the computational model powerful enough for when, in the future, those unphysical objects are actually existed.

4. NOTATIONS

Due to the number of groups, and representations discussed in this work, it is important to define some common definitions and notations that would be used.

4.1. Group

A group is a set of items together with a binary operator defined on each pair of elements in the group such that, for an arbitrary group of set G and its binary operator \circ_G , the following properties must be true.

1. For any $g_1, g_2 \in G$,

$$g_1 \circ_G g_2 \in G. \quad (4.1)$$

2. For any $g_1, g_2, g_3 \in G$,

$$(g_1 \circ_G g_2) \circ_G g_3 = g_1 \circ_G (g_2 \circ_G g_3). \quad (4.2)$$

3. There is a unique element $e_G \in G$ such that, for any $g \in G$,

$$g \circ_G e_G = e_G \circ_G g = g. \quad (4.3)$$

4. For any $g \in G$, there is a unique element $g^{-1} \in G$ such that

$$g \circ_G g^{-1} = g^{-1} \circ_G g = e. \quad (4.4)$$

For compactness in the remaining sections, we would drop the explicit placement of the binary operator and assume the group binary operation whenever there are two elements of the same group placed beside each other, e.g., if $g_1, g_2 \in G$, $g_1 g_2 \triangleq g_1 \circ_G g_2$. Furthermore, we would refer to the group by its set whenever the group's binary operator is unambiguous, e.g., a group of set G and its binary operator \circ_G is group G .

4.2. Homomorphism

A homomorphism of group G is a map from G to another group H such that the image of G preserves the same structure as G . In other words, for a homomorphism $h : G \rightarrow H$,

$$\forall g_1, g_2 \in G, h(g_1)h(g_2) = h(g_1 g_2), \quad (4.5)$$

where the implicit binary operation $g_1 g_2$ is from G , i.e. $g_1 \circ_G g_2$ while implicit binary operation $h(g_1)h(g_2)$ is from H , i.e. $h(g_1) \circ_H h(g_2)$.

If a homomorphism is bijective, i.e. each and every element in H is mapped from an element in G homomorphically. The map is called isomorphism, and it entails equivalence relation between isomorphic groups. We denoted group isomorphic equivalence between G , and H with $G \cong H$.

4.3. Representation

A group representation of G on vector space V is a group homomorphism from G to general linear group $GL(V)$ which is the group of bijective linear transformation from V to itself. To be more specific, for the vector space V over field \mathbb{F}_V with n_V dimensions, $GL(n_V, \mathbb{F}_V) \triangleq GL(V)$ is the set of n_V by n_V invertible matrices over \mathbb{F}_V with matrix multiplication as the group binary operator. This obviously means that there can be multiple representations depending on the choice of V , and each V might has multiple representations as well.

Assuming that we are able to uniquely name each representation, we would use the notation $D_G^{(\Gamma)} : G \rightarrow GL$ for a representation of group G where Γ is the label that fully specify the Rep. Since two representations can be equivalence if every corresponding pair of matrix representing group elements are equivalence through the same basis transformation, we would decorate $D_G^{(\Gamma)}$ to specify the basis of the matrix representation used.

4.4. Reducibility of Representation and Irreducible Representation

A group representation of G on vector space V is reducible if their exist an invariant proper subspace $W \subsetneq V$ such that W is closed under group represented action, i.e.

$$\forall g \in G, \forall w \in W, D_G(g)w \in W. \quad (4.6)$$

This reducibility implies that there exists a choice of basis that block diagonalizes group representation such that one of the blocks is a (reduced) group representation of G on vector space W .

A group irreducible representation of G is a group representation that is not reducible, i.e. cannot be non-trivially block diagonalized. This criterion is equivalent to Schur's second

lemma; given a matrix M and representation of G on vector space V , D_G , if, for all $g \in G$,

$$MD_G(g) = D_G(g)M, \quad (4.7)$$

then $M = \lambda I$ for some $\lambda \in \mathbb{F}_V$ where I is an identity matrix.

4.5. Tensor Product

Tensor product is an operation to combine vector spaces. It is generally denoted by a bilinear map $U \times V \rightarrow U \otimes V$ such that for any basis vector $\hat{e}_U \in U$, and basis vector $\hat{e}_V \in V$ their tensor product gives a basis vector $\hat{e}_U \otimes \hat{e}_V$. This means that, for any vector $u \in U$, and $v \in V$,

$$u \otimes v = \left(\sum_a u_a \hat{e}_U^a \right) \otimes \left(\sum_b v_b \hat{e}_V^b \right) = \sum_{a,b} u_a v_b \hat{e}_U^a \otimes \hat{e}_V^b. \quad (4.8)$$

From this, one can extend the concept to linear maps of tensor product space. Consider two linear maps $S : U \rightarrow U$, and $T : V \rightarrow V$, the tensor product of these two maps is defined as $S \otimes T : U \otimes V \rightarrow U \otimes V$ such that, for any $u \otimes v \in U \otimes V$,

$$(S \otimes T)(u \otimes v) = S(u) \otimes T(v). \quad (4.9)$$

From equation 4.9, we can also directly imply that, if there are $Q : U \rightarrow U$, and $R : V \rightarrow V$,

$$(Q \otimes R)(S \otimes T) = QS \otimes RT \quad (4.10)$$

4.6. Kronecker Product

Kronecker product is a specialization of tensor product for matrices. Given matrices M , and N , their Kronecker product is defined as

$$[M \otimes_K N]_{ac,bd} \triangleq [M]_{a,b} \cdot [N]_{c,d}. \quad (4.11)$$

Here, the subscripts of the $M \otimes_K N$, i.e. ac , and bd , are two-digit indices not multiplication. In other words, the resulting product has dimensions of the product of the corresponding dimensions of the compounding matrices, i.e. if M is a h_M by w_M matrix, and N is a h_N by w_N matrix, then $M \otimes_K N$ is a $h_M \cdot h_N$ by $w_M \cdot w_N$ matrix.

Since we would mostly utilize matrix in the form of general linear representation, consider $M \in GL(U)$, and $N \in GL(V)$ with dimensions n_U , and n_V respectively. Because both of them have inverses, we can define

$$[(M \otimes_K N)^{-1}]_{ac,bd} \triangleq [M^{-1} \otimes_K N^{-1}]_{ac,bd} = [M^{-1}]_{a,b} \cdot [N^{-1}]_{c,d}. \quad (4.12)$$

Then,

$$\begin{aligned} [(M \otimes_K N)(M^{-1} \otimes_K N^{-1})]_{ac,ef} &= \sum_{b,d} [M]_{a,b} \cdot [N]_{c,d} \cdot [M^{-1}]_{b,e} \cdot [N^{-1}]_{d,f} \\ [MM^{-1} \otimes_K NN^{-1}]_{ac,ef} &= \left(\sum_b [M]_{a,b} \cdot [M^{-1}]_{b,e} \right) \left(\sum_d [N]_{c,d} \cdot [N^{-1}]_{d,f} \right) \\ [I_U \otimes_K I_V]_{ac,ef} &= [MM^{-1}]_{a,e} \cdot [NN^{-1}]_{c,f} = [I_U]_{a,e} \cdot [I_V]_{c,f} \end{aligned} \quad (4.13)$$

where I_U , and I_V are identity matrices of U , and V respectively. This results imply that Kronecker product between invertible matrices of n_U , and n_V dimensions is an invertible matrix of $n_U \cdot n_V$ dimensions with matrices and its inverses defined by equation 4.11, and 4.12, identity defined by the end of equation 4.13, and binary operator defined by equation 4.10. In other words, Kronecker product between general linear groups of two vector spaces U , and V is homomorphic to general linear group of $U \otimes V$.

4.7. Groups' Direct Product

A group direct product is a composition of two groups to form a new larger group. Consider groups G and H , their direct product is defined as a group of Cartesian product set

$$G \times H = \{(g, h) | \forall g \in G, \forall h \in H\}, \quad (4.14)$$

and its binary operator defined as, $\forall (g_1, h_1), (g_2, h_2) \in G \times H$,

$$(g_1, h_1)(g_2, h_2) = (g_1 g_2, h_1 h_2). \quad (4.15)$$

Here, we also utilize the implicit binary operation notations.

Consider a pair of representations $D_G^{(\Gamma_G)}$, and $D_H^{(\Gamma_H)}$ of G , and H respectively. One can define a representation of $G \times H$ with, for all $g \in G$, and $h \in H$,

$$D_{G \times H}^{(\Gamma_G, \Gamma_H)}((g, h)) = D_G^{(\Gamma_G)}(g) \otimes_K D_H^{(\Gamma_H)}(h) \quad (4.16)$$

because the structure of groups' direct product (equation 4.15) is the same as general linear groups' tensor product (equation 4.10).

Note that the representation of direct product defined in equation 4.16 guarantees irreducible representation from product of two irreducible representation, and complete identification of all irreducible representations of the groups' direct product as long as the groups are either finite, or compact.

4.8. Free Product

A group free product is another composition of two groups to form a new larger group. Consider groups G and H , their free product ($G * H$) is defined as a group generated from $G \cup H$ such that the binary operator is \circ_G between adjacent elements from G , and is \circ_H between adjacent elements from H ,

$$G * H = \bigcup_N (G \cup H)^N. \quad (4.17)$$

In other words, $G * H$ is a group of arbitrary length of words where each words' character is chosen from $G \cup H$. Since G , and H are groups, all adjacent elements from G (or H) can be replaced by an element from G (or H). Hence, $G * H$ is actually a group of arbitrary length of words where each words' character is alternately chosen from G , and H . From here, one can specialize free product according to the relation between the compounding groups.

The fact that we are discussing free product of groups is because the operators that apply symmetric transformation from different symmetry groups act on the same system, i.e. vector space. This means that all those operators are actually in the free product group of all considered symmetry group.

4.9. Free Product of two Commuting Groups

From previous section, if we also add that G , and H are commuting subgroups of $G * H$, i.e. $\forall g \in G, \forall h \in H, gh = hg$, we can further specialize $G * H$ by commuting all elements from G to the front of each word (all elements from H are on the backside of each word,) and replacing all adjacent elements from G (or H) with an element from G (or H). In

other words, if G , and H commute with each other,

$$G * H = \{gh | g \in G, h \in H\}. \quad (4.18)$$

Consider two elements $g_1 h_1, g_2 h_2 \in G * H$ where $g_1, g_2 \in G$, and $h_1, h_2 \in H$,

$$(g_1 h_1) \circ_{G * H} (g_2 h_2) = g_1 h_1 g_2 h_2 = g_1 g_2 h_1 h_2 = (g_1 \circ_G g_2)(h_1 \circ_H h_2). \quad (4.19)$$

From free product of commuting groups (equation 4.18, and 4.19) and direct product (equation 4.14, and 4.15), we can directly make an isomorphism that maps $gh \in G * H$ to $(g, h) \in G \times H$ for every $g \in G$, and $h \in H$.

This means that $G * H \cong G \times H$, and all irreducible representation for free product of commuting groups can be constructed from irreducible representations of the compounding groups according to equation 4.16. This result is very important when we are describing a physical system that contain multiple symmetries which are all commute with each other.

4.10. Reduction of Representation's Product

5. $SU(2)$ GROUP

5.1. $SU(2)$'s Relation to Quantum Spin System

5.2. $SU(2)$'s Definition

$SU(2)$, i.e. special unitary group of dimension 2, is the group of 2 by 2 unitary matrices with unit determinant with matrix multiplication as its binary operator. In other words,

$$SU(2) = \{u \in GL(2, \mathbb{C}) | u^\dagger = u^{-1}, \det(u) = 1\}, \quad (5.1)$$

where u^\dagger is the complex conjugate transposition of u , and $\det(\cdot)$ is matrix determinant operation. Since, for any $u, v \in SU(2)$,

$$\det(uv) = \det(u) \det(v) = 1, \quad (5.2)$$

and

$$(uv)^\dagger = (uv)^{\top*} = (v^\top u^\top)^* = v^\dagger u^\dagger = v^{-1} u^{-1} = (uv)^{-1}, \quad (5.3)$$

$SU(2)$ is a closed subgroup of $GL(2, \mathbb{C})$ which makes it a Lie group. This means that there is a corresponding Lie algebra, $\mathfrak{su}(2)$, such that

$$SU(2) = \{e^g | g \in \mathfrak{su}(2)\} \quad (5.4)$$

where the exponential of a square matrix g is defined as

$$e^g = I + \sum_{k=1}^{\infty} \frac{g^k}{k!}. \quad (5.5)$$

Here, I represents identity matrix with the same shape as g . From definition of matrix exponential,

$$\begin{aligned} (e^g)^\dagger &= e^{g^\dagger}, \\ (e^g)^{-1} &= e^{-g}, \\ \det(e^g) &= e^{\text{Tr } g}, \end{aligned} \quad (5.6)$$

one can define the Lie algebra of $SU(2)$ as

$$\mathfrak{su}(2) = \{g \in M(2, \mathbb{C}) | g^\dagger = -g, \text{Tr}(g) = 0\}, \quad (5.7)$$

where $M(2, \mathbb{C})$ is the group of 2 by 2 matrices with complex field with matrix multiplication as its binary operator, and $\text{Tr}(\cdot)$ is matrix trace operation. Hence, each member of $\mathfrak{su}(2)$ can be written in the form

$$\begin{bmatrix} iv_z & v_y + iv_x \\ -v_y + iv_x & -iv_z \end{bmatrix} = iv_x \sigma_x + iv_y \sigma_y + iv_z \sigma_z = i\vec{v} \cdot \vec{\sigma} \quad (5.8)$$

where $\vec{v} \in \mathbb{R}^3$, $\vec{\sigma} = \sigma_x \hat{x} + \sigma_y \hat{y} + \sigma_z \hat{z}$, and σ_i 's are Pauli matrices defined as

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \quad (5.9)$$

5.3. $SU(2)$'s Generators

From $\mathfrak{su}(2)$'s definition in the previous section, one can consider using Pauli matrices as the group generators for $SU(2)$ according to equation 5.8, but it is more common to use $J_i^{(1/2)} \triangleq \sigma_i/2$ as the generators where $i \in \{x, y, z\}$, and $\vec{\phi} \triangleq -2\vec{v} \in \mathbb{R}^3$ as group parameterization. In other words, we can redefine $SU(2)$ as

$$SU(2) = \{u_{\vec{\phi}} | u_{\vec{\phi}} = e^{-i\vec{J}^{(1/2)} \cdot \vec{\phi}}, \vec{\phi} \in \mathbb{R}^3\}, \quad (5.10)$$

with $\vec{J}^{(1/2)} = J_x^{(1/2)} \hat{x} + J_y^{(1/2)} \hat{y} + J_z^{(1/2)} \hat{z}$. This parameterization of $SU(2)$ group directly implies that

$$\begin{aligned} e_{SU(2)} &= u_{\vec{0}}, \\ u_{\vec{\phi}}^{-1} &= u_{-\vec{\phi}}. \end{aligned} \quad (5.11)$$

With this choice of generators, it gives the Lie bracket (commutation) relation

$$[J_i^{(1/2)}, J_j^{(1/2)}] \triangleq J_i^{(1/2)} J_j^{(1/2)} - J_j^{(1/2)} J_i^{(1/2)} = i\epsilon_{ijk} J_k^{(1/2)} \quad (5.12)$$

where ϵ_{ijk} is equal to 1 if (i, j, k) is an even permutation of (x, y, z) , -1 if odd permutation, and 0 otherwise.

The next step is to consider an arbitrary representation of $SU(2)$ such that its generators, represented by J_i 's, obey equation 5.12. With out loss of generality, consider the representation vector space with eigenvectors of J_z , $\{|\lambda_{J_z}\rangle\}$, as the basis,

$$J_z |\lambda_{J_z}\rangle = \lambda_{J_z} |\lambda_{J_z}\rangle. \quad (5.13)$$

Furthermore, we will also assume that this representation is irreducible (proof of this assumption will be in the following section.) With the standard method, first define two additional operators

$$\begin{aligned} J_+ &= J_x + iJ_y, \\ J_- &= J_x - iJ_y. \end{aligned} \tag{5.14}$$

From Lie bracket, and the choice of basis used,

$$\begin{aligned} J_z J_+ |\lambda_{J_z}\rangle &= J_+(J_z + 1) |\lambda_{J_z}\rangle = (\lambda_{J_z} + 1) J_+ |\lambda_{J_z}\rangle, \\ J_z J_- |\lambda_{J_z}\rangle &= J_-(J_z - 1) |\lambda_{J_z}\rangle = (\lambda_{J_z} - 1) J_- |\lambda_{J_z}\rangle. \end{aligned} \tag{5.15}$$

This directly implies

$$\begin{aligned} J_+ |\lambda_{J_z}\rangle &\propto |\lambda_{J_z} + 1\rangle, \\ J_- |\lambda_{J_z}\rangle &\propto |\lambda_{J_z} - 1\rangle. \end{aligned} \tag{5.16}$$

Before proceeding, we need to clarify an assumption we made to get equation 5.16 that there is no degeneracy of λ_{J_z} in the basis. This result is directly entailed from the assumption that the representation is irreducible. To prove this, first assume that there exist $|\lambda_{J_z}\rangle'$ that has the same eigenvalue as $|\lambda_{J_z}\rangle$. From the properties of J_i 's, we can choose the representation such that equation 5.16 is true, i.e. the operators never create primed vector from unprimed vector. Since the representation of each $SU(2)$ member can be expressed in terms of the generators (equation 5.4 and 5.5), if a subspace is invariant under the generators, it is also invariant under the $SU(2)$ representation. This means that the proper subspace that exclude $|\lambda_{J_z}\rangle'$ is invariant for this choice of representation. Hence, the representation is reducible which contradict the assumption.

To get the proportionality, consider another operators

$$J^2 \triangleq J_x^2 + J_y^2 + J_z^2. \tag{5.17}$$

One can directly check that the new operator commutes with all J 's mentioned so far. Furthermore, it is straight forward that

$$\begin{aligned} J^2 &= J_+ J_- + J_z^2 - J_z \\ &= J_- J_+ + J_z^2 + J_z. \end{aligned} \tag{5.18}$$

Since J^2 commutes with J_z , the basis used are also eigenbasis of J^2 with some eigenvalue λ_{J^2} . Hence, we can directly relabel the eigenbasis from $|\lambda_{J_z}\rangle$ to $|\lambda_{J^2}, \lambda_{J_z}\rangle$, and get

$$\begin{aligned} \langle \lambda_{J^2}, \lambda_{J_z} | J_+ J_- | \lambda_{J^2}, \lambda_{J_z} \rangle &= \lambda_{J^2} - \lambda_{J_z} (\lambda_{J_z} - 1), \\ \langle \lambda_{J^2}, \lambda_{J_z} | J_- J_+ | \lambda_{J^2}, \lambda_{J_z} \rangle &= \lambda_{J^2} - \lambda_{J_z} (\lambda_{J_z} + 1). \end{aligned} \tag{5.19}$$

Since the norm square of any vector in vector space must be non-negative, this requires

$$\lambda_{J^2} \geq |\lambda_{J_z}|(|\lambda_{J_z}| + 1) \geq 0. \quad (5.20)$$

However, the J_{\pm} makes λ_{J_z} unbounded unless the vector vanished at some points, i.e. the equality must hold for the upper, and lower bounds of the possible λ_z 's. If we replace λ_{J^2} with $j(j+1)$ where j non-negative. The requirement makes $\lambda_{J_z, \max} = j$, and $\lambda_{J_z, \min} = -j$. Since any pair of λ_{J_z} 's are different by an integer,

$$\lambda_{J_z, \max} - \lambda_{J_z, \min} = 2j \in \mathbb{Z}_0^+. \quad (5.21)$$

Hence, we can relabel the eigenbasis from $|\lambda_{J^2}, \lambda_{J_z}\rangle$ to $|j, m\rangle$ and choose the proportionality such that

$$\begin{aligned} J_+ |j, m\rangle &= \sqrt{j(j+1) - m(m+1)} |j, m+1\rangle, \\ J_- |j, m\rangle &= \sqrt{j(j+1) - m(m-1)} |j, m-1\rangle, \end{aligned} \quad (5.22)$$

where $j \in \{0, 1/2, 1, 3/2, \dots\}$, and $m \in \{-j, -j+1, \dots, j-1, j\}$. Next, we must prove that each j fully identifies an irreducible representation. Assuming that this is not the case by letting there be basis vector $|j', m'\rangle$ in the vector space of this irreducible representation that is not linearly dependent on the eigenbasis $\{|j, m\rangle\}$, equation 5.22 makes the vector space that exclude $|j', m'\rangle$ invariant proper subspace (with reasoning similar to when we prove non-degeneracy of λ_{J_z}). Hence, this implies the reducibility of the representation which contradicts the assumption.

5.4. $SU(2)$'s Irreducible Representations

In previous section, we have proved that every irreducible representation of $SU(2)$ has its unique j (no irreducible representation requires basis of more than one j .) Furthermore, since we have not made any other assumption, j also completely identify every irreducible representation of $SU(2)$ (representations identified by j 's exhaust all possible irreducible representations.) In other words, each and every irreducible representation can be uniquely labeled by $j \in \{0, 1/2, 1, 3/2, \dots\}$.

Last thing we need to do for that completion of $SU(2)$'s irreducible representation is that the representations labeled by j 's (previously assumed to be irreducible) is actually irreducible. Consider a representation of $SU(2)$ that is generated by generators $J_x^{(j)}, J_y^{(j)}$,

and $J_z^{(j)}$ where the superscripts indicate that the representation is labeled by j . Assuming that this representation is reducible, there must be a unitary basis transformation that nontrivial block diagonalize all elements of $SU(2)$ in this representation. From equation 5.5 and 5.10, the representation labeled by j has, for $\vec{\phi} \in \mathbb{R}^3$,

$$u_{\vec{\phi}} = e^{-i\vec{J}^{(j)} \cdot \vec{\phi}} = I + \sum_{k=1}^{\infty} (-i\vec{J}^{(j)} \cdot \vec{\phi})^k, \quad (5.23)$$

as its parameterized matrix representation. Since $\vec{\phi}$ is arbitrary, the block diagonalization must hold for every term on the right hand side of equation 5.23. This implies simultaneous block diagonalization of $J_x^{(j)}$, $J_y^{(j)}$, and $J_z^{(j)}$. In other words, there is an invariant proper subspace, say W , that is closed under $J_+^{(j)}$, and $J_-^{(j)}$. Since such vector space is a proper subspace, there must exist $-j \leq m' \leq j$ such that the projection of $|j, m'\rangle$ on this space is not $|j, m'\rangle$, i.e. there is nontrivial orthogonal residual. Also, there must exist $-j \leq m'' \leq j$ such that the projection of $|j, m''\rangle$ on this space is not zero, i.e. there is nontrivial projection. Let the projection of $|j, m''\rangle$ on to W be

$$\text{Proj}_W(|j, m''\rangle) = \sum_{m=-j}^j C_m'' |j, m\rangle. \quad (5.24)$$

Let the smallest m that $C_m'' \neq 0$ be m^* . This means that applying $J_+^{(j)}$ for $j - m^*$ times gives nonzero proportional to $|j, j\rangle$, i.e.

$$\left[J_+^{(j)} \right]^{j-m^*} \text{Proj}_W(|j, m''\rangle) \propto |j, j\rangle. \quad (5.25)$$

After that, apply $J_-^{(j)}$ for $j - m'$ times would also give nonzero proportional to $|j, m'\rangle$, i.e.

$$\left[J_-^{(j)} \right]^{j-m'} \left[J_+^{(j)} \right]^{j-m^*} \text{Proj}_W(|j, m''\rangle) \propto |j, m'\rangle. \quad (5.26)$$

However, since $\text{Proj}_W(|j, m''\rangle) \in W$ and $|j, m'\rangle - \text{Proj}_W(|j, m'\rangle) \neq 0$, W is not closed under $J_+^{(j)}$, and $J_-^{(j)}$ which gives contradiction. Therefore, the representations we have constructed previously are actually irreducible.

From computational perspective, in order to implement the representation, the basis of the space in which all group operations would act on need to be carefully selected mainly for the purpose of reducing computational costs. However, it requires the consideration of every symmetry group we want to include. Hence, for now, we are assuming the standard $\{|j, m\rangle\}$ basis. In this basis, the irreducible representation labeled with j is the $2j + 1$

dimensional space representation. Let the representation of a vector is in the ascending order of the basis from $m = -j$ to $m = j$. We will also, through the rest of this work, index matrices with python-like notation, i.e. indices count from zero. Hence, we can write down

$$\begin{aligned}
[J_z^{(j)}]_{a,b} &= (a - j)\delta_{a,b}, \\
[J_+^{(j)}]_{a,b} &= \sqrt{j(j+1) - (a-j)(b-j)}\delta_{a,b+1}, \\
J_-^{(j)} &= J_+^{(j)\top}, \\
J_x^{(j)} &= (J_+^{(j)} + J_-^{(j)})/2, \\
J_y^{(j)} &= -i(J_+^{(j)} - J_-^{(j)})/2,
\end{aligned} \tag{5.27}$$

where $a, b \in \{0, 1, \dots, 2j\}$, $j \in \{0, 1/2, 1, 3/2, \dots\}$, and δ is Kronecker delta. Therefore, we can also write down the irreducible representation labeled by j of $u_{\vec{\phi}} \in SU(2)$ with elements parameterized by $\vec{\phi} \in \mathbb{R}^3$ in this basis as

$$D_{SU(2)}^{(j)}(u_{\vec{\phi}}) = \exp\left(-i\vec{J}^{(j)} \cdot \vec{\phi}\right). \tag{5.28}$$

6. P (PARITY) GROUP

6.1. P 's Relation to Quantum Spin System

6.2. P 's Definition

P is the group of spatial inversion, i.e. the group consisting of only 2 elements. The first is its unique identity e , and the second is the parity operator π (which is its own inverse) that inverts spatial state. The only binary operation rules of this group are

$$\begin{aligned} ee &= \pi\pi = e, \\ \pi e &= e\pi = \pi. \end{aligned} \tag{6.1}$$

6.3. P 's Representations

From equation 6.1, and the fact that there are only 2 elements in the group, any n dimensional representation of P must be the identity matrix for e , and a self-inverse matrix for π . Since they are both invertible, it is possible to do eigenvalue decomposition of both of them such that

$$\begin{aligned} D_P(e) &= UIU^\dagger, \\ D_P(\pi) &= U\Sigma U^\dagger. \end{aligned} \tag{6.2}$$

where I is identity matrix, Σ is a diagonal matrix of $D_P(\pi)$'s eigenvalues, and U is the unitary matrix where each column is an eigenvector corresponding to each eigenvalue in Σ . From this, it is straight forward that all n dimensional representations of P are reducible to 1 dimensional representations.

6.4. P 's Irreducible Representations

Since 1 dimensional representations cannot be reduced (there is no proper subspace,) all 1 dimensional representations are irreducible representations. From previous section, we now know that all irreducible representations of P must be 1 dimension representation. Furthermore, since $D_P(\pi)$ is self-inverse, $[D_P(\pi)]^2 = D_P(e) = 1$. Hence, there are only two irriducible representations commonly denoted as even ($D_P(\pi) = 1$), and odd ($D_P(\pi) = -1$).

We can see that the only difference of these two irreducible representations is in the values of π 's representations. Therefore, we can directly use those values as the irreducible representations' labels as

$$\begin{aligned} D_P^{(p)}(e) &= 1 \\ D_P^{(p)}(\pi) &= p \end{aligned} \tag{6.3}$$

where the superscripts indicate that the representation is labeled by $p \in \{-1, 1\}$. For convenience in equation writing, we would parameterize this group, even though there are only two elements in the group, by defining

$$P = \{v_\rho | v_\rho = \pi^\rho, \rho \in \mathbb{Z}_{0+}\}. \tag{6.4}$$

With this notation, we can see that $v_\rho = e$ when ρ is an even number while $v_\rho = \pi$ when ρ is odd. Furthermore, equation 6.3 can be greatly simplified to

$$D_P^{(p)}(v_\rho) = p^\rho. \tag{6.5}$$

6.5. $SU(2) \times P$'s Irreducible Representations

After the analysis of two symmetry groups related to quantum spin systems, it is the point where we start unifying them to generalize the way to represent quantum spin system. From the postulation for parity in quantum mechanics, it directly shows that $SU(2)$, and P are commuting groups. This means that $SU(2) * P \cong SU(2) \times P$, i.e. the symmetry group that governs the combined symmetries is isomorphic to their direct product group, which makes the tensor product between every pair of irreducible representations of $SU(2)$, and P are irreducible representation of $SU(2) * P$, and exhaust all of its possible irreducible representations. Hence, the irreducible representations of the combined $SU(2)$, and P symmetries can be written, labeled with j and p , as, for all $u_{\vec{\phi}} \in SU(2)$, and $v_\rho \in P$,

$$\begin{aligned} D_{SU(2)*P}^{(j,p)}(u_{\vec{\phi}}v_\rho) &= D_{SU(2)\times P}^{(j,p)}((u_{\vec{\phi}}, v_\rho)) \\ &= D_{SU(2)}^{(j)}(u_{\vec{\phi}}) \otimes D_P^{(p)}(v_\rho) \\ &= \exp\left(-i\vec{J}^{(j)} \cdot \vec{\phi}\right) \cdot p^\rho \end{aligned} \tag{6.6}$$

where $\vec{\phi} \in \mathbb{R}^3$, $\rho \in \mathbb{Z}_{0+}$, $j \in \{0, 1/2, 1, 3/2, \dots\}$, and $p \in \{-1, 1\}$. Also, if one choose the standard angular momentum basis, $\vec{J}^{(j)}$ is according to equation 5.27. The basis for this

representation is $\{|j, m\rangle \otimes |p\rangle\}$. However, since P 's irreducible representations are all one dimensional, we will denote such basis as $\{|j, m\rangle_p\}$.

7. T (TIME-REVERSAL) GROUP

7.1. T 's Relation to Quantum Spin System

7.2. T 's Definition

T is the group of temporal inversion, i.e. the group consisting of only 1 generator, the time-reversal operator θ which is an antiunitary operator.

7.3. T 's Corepresentations

Because of the antiunitary nature of T , the analysis for its irreducible representation cannot be done in the same fashion as other unitary groups. The main reason for this is that the complex conjugation part of the antiunitary operator θ can change the basis used for the irreducible representation of other groups, i.e. the groups that we want to combine with T .

Let assume that the group we want to combine with T is G . To proceed from this point, we need to consider the effects of θ on the basis chosen for G 's irreducible representations. Without loss of generality, let $\{|g_a\rangle\}$ be such basis, and the irreducible representation is D_G . Hence, for any $g \in G$,

$$\begin{aligned} g |g_a\rangle &= D_G(g) |g_a\rangle \\ &= \left(\sum_{b,c} |g_b\rangle [D_G(g)]_{b,c} \langle g_c| \right) |g_a\rangle \\ &= \sum_b |g_b\rangle [D_G(g)]_{b,a}. \end{aligned} \tag{7.1}$$

Likewise, let the irreducible representation of g in $\{\theta |g_i\rangle\}$ basis be \tilde{D}_G , i.e.

$$\begin{aligned} g\theta |g_a\rangle &= \tilde{D}_G(g)\theta |g_a\rangle \\ &= \sum_b \theta |g_b\rangle [\tilde{D}_G(g)]_{b,a}. \end{aligned} \tag{7.2}$$

Then apply θ^{-1} to both side,

$$\begin{aligned}
\theta^{-1}g\theta|g_a\rangle &= \theta^{-1}\left(\sum_b \theta|g_b\rangle \left[\tilde{D}_G(g)\right]_{b,a}\right) \\
&= \sum_b |g_b\rangle \left[\tilde{D}_G(g)\right]_{b,a}^* \\
&= \sum_b |g_b\rangle \left[\tilde{D}_G^*(g)\right]_{b,a}.
\end{aligned} \tag{7.3}$$

Since multiplication of two antiunitary operator is unitary, and

$$\theta^{-1}G\theta \triangleq \{\theta^{-1}g\theta|g \in G\} \cong G \tag{7.4}$$

with isomorphism that maps $\theta^{-1}g\theta$ to g , there must be irreducible representation $D_G(\theta^{-1}g\theta)$.

By comparing equation 7.1, and 7.3, one can conclude that

$$\begin{aligned}
\tilde{D}_G^*(g) &= D_G(\theta^{-1}g\theta), \\
\tilde{D}_G(g) &= D_G^*(\theta^{-1}g\theta).
\end{aligned} \tag{7.5}$$

Repeat this one more time for $\{\theta^2|g_i\rangle\}$ basis with irreducible representation $\tilde{\tilde{D}}_G$. We get

$$\begin{aligned}
\tilde{\tilde{D}}_G(g) &= \tilde{D}_G^*(\theta^{-1}g\theta), \\
&= D_G(\theta^{-2}g\theta^2).
\end{aligned} \tag{7.6}$$

Here, the first equality follows the logical proof of \tilde{D}_G while the second equality directly uses equation 7.5. Since θ^2 is unitary, there must be a unitary transformation Θ such that

$$D_G(\theta^{-2}g\theta^2) = \Theta^\dagger D_G(g)\Theta. \tag{7.7}$$

Equation 7.7 guarantees that, in order to represent $G * T$, one need to only double the basis that represents G . Therefore, from equation 7.1, 7.2, and 7.7, the corepresentations of the combined G , and T groups, D_{G*T} , can be extended from D_G by using $\{|g_a\rangle\} \cup \{\theta|g_a\rangle\}$ basis as, for any $g \in G$,

$$\begin{aligned}
D_{G*T}(g) &= \begin{bmatrix} D_G(g) & 0 \\ 0 & D_G^*(\theta^{-1}g\theta) \end{bmatrix}, \\
D_{G*T}(\theta) &= \begin{bmatrix} 0 & \Theta \\ I & 0 \end{bmatrix},
\end{aligned} \tag{7.8}$$

where the first half of the basis is $\{|g_a\rangle\}$, the second half of the basis is $\{\theta|g_a\rangle\}$, and I is identity matrix. Note that whether these corepresentation is reducible or not depends on

the actual form of $D_G(\theta^{-1}g\theta)$, and Θ , but it is guaranteed to exhaust all possible irreducible corepresentations.

Our analysis so far only concerns about the effect of θ on the basis, but its complex conjugation also applies to other corepresentations, i.e., for all $g, g_1, g_2 \in G$,

$$\begin{aligned} D_{G*T}(g_1g_2) &= D_{G*T}(g_1)D_{G*T}(g_2), \\ D_{G*T}(g\theta) &= D_{G*T}(g)D_{G*T}(\theta), \\ D_{G*T}(\theta g) &= D_{G*T}(\theta)D_{G*T}^*(g), \\ D_{G*T}(\theta\theta) &= D_{G*T}(\theta)D_{G*T}^*(\theta). \end{aligned} \tag{7.9}$$

As well as the whole corepresentation vector space, V , i.e., for all $g \in G$, and $v \in V$,

$$\begin{aligned} g\vec{v} &= D_{G*T}(g)\vec{v}, \\ \theta\vec{v} &= D_{G*T}(\theta)\vec{v}^*. \end{aligned} \tag{7.10}$$

Further simplification can be done in the case that \tilde{D}_G is equivalent to D_G . In other words, there must exists a unitary basis transformation U that

$$\tilde{D}_G(g) = D_G^*(\theta^{-1}g\theta) = U^\dagger D_G(g)U. \tag{7.11}$$

This directly makes

$$\begin{aligned} D_{G*T}(g) &= \begin{bmatrix} D_G(g) & 0 \\ 0 & U^\dagger D_G(g)U \end{bmatrix}, \\ D_{G*T}(\theta) &= \begin{bmatrix} 0 & \Theta \\ I & 0 \end{bmatrix}. \end{aligned} \tag{7.12}$$

It also implies, from equation 7.6, that

$$\begin{aligned} \tilde{\tilde{D}}_G(g) &= \tilde{D}_G^*(\theta^{-1}g\theta), \\ &= (U^\dagger D_G(\theta^{-1}g\theta)U)^* \\ &= (U^\dagger \tilde{D}_G^*(g)U)^* \\ &= (U^\dagger U^{\dagger*} D_G^*(g)U^*U)^* \\ &= (UU^*)^\dagger D_G(g) (UU^*). \end{aligned} \tag{7.13}$$

From equation 7.7, this also makes

$$\begin{aligned} \Theta^\dagger D_G(g)\Theta &= (UU^*)^\dagger D_G(g) (UU^*) \\ (UU^*) \Theta^\dagger D_G(g) &= D_G(g) (UU^*) \Theta^\dagger \end{aligned} \tag{7.14}$$

Because D_G is irreducible, from Schur's second lemma (equation 4.7), $(UU^*)\Theta^\dagger$ must equal a constant λI where $\lambda \in \mathbb{C}$ (assume representation in complex vector space,)

$$UU^* = \lambda \Theta. \quad (7.15)$$

Moreover, this basis transformation must also apply to Θ in the same manner, i.e.

$$\tilde{\Theta} = U^\dagger \Theta U. \quad (7.16)$$

Hence,

$$\begin{aligned} \tilde{D}_G(\theta^{-2}g\theta^2) &= D_G^*(\theta^{-3}g\theta^3) \\ &= (\Theta^\dagger D_G(\theta^{-1}g\theta)\Theta)^* \\ &= \Theta^{\dagger*} D_G^*(\theta^{-1}g\theta)\Theta^* \\ &= \Theta^{\dagger*} \tilde{D}_G(g)\Theta^* \\ &\triangleq \tilde{\Theta}^\dagger \tilde{D}_G(g)\tilde{\Theta} \end{aligned} \quad (7.17)$$

or

$$U^\dagger \Theta U = \Theta^* \quad (7.18)$$

Then, from equation 7.15, and 7.18,

$$\begin{aligned} U^\dagger (UU^*) U &= U^\dagger (\lambda \Theta) U \\ U^* U &= \lambda U^\dagger \Theta U \\ (UU^*)^* &= \lambda \Theta^* \\ UU^* &= \lambda^* \Theta, \end{aligned} \quad (7.19)$$

which, together with equation 7.15, implies that $\lambda \in \mathbb{R}$. Since U , and Θ are unitary, we also know that $|\lambda| = 1$. Therefore, $\lambda \in \{-1, 1\}$.

Hence, we can further simplify equation 7.12 into

$$\begin{aligned} D_{G*T}(g) &= \begin{bmatrix} D_G(g) & 0 \\ 0 & U^\dagger D_G(g) U \end{bmatrix}, \\ D_{G*T}(\theta) &= \begin{bmatrix} 0 & UU^*/\lambda \\ I & 0 \end{bmatrix}, \end{aligned} \quad (7.20)$$

and, with a change of basis from $\{|g_a\rangle\} \cup \{\theta|g_a\rangle\}$ to $\{|g_a\rangle\} \cup \{U^\dagger\theta|g_a\rangle\}$, we get

$$\begin{aligned} D_{G*T}(g) &= \begin{bmatrix} D_G(g) & 0 \\ 0 & D_G(g) \end{bmatrix}, \\ D_{G*T}(\theta) &= \begin{bmatrix} 0 & U/\lambda \\ U & 0 \end{bmatrix}. \end{aligned} \quad (7.21)$$

For convenience in equation writing, we would parameterize this group as well by defining

$$T = \{w_\tau | w_\tau = \theta^\tau, \tau \in \mathbb{Z}_{0+}\}. \quad (7.22)$$

With this notation, we can see that $w_0 = e$, $w_1 = \theta$, $w_2 = \theta^2$, and so on.

7.4. $SU(2) \times P \times T$'s Irreducible Corepresentations

From the postulation for time-reversal in quantum mechanics, it directly shows that $SU(2)$, P , and T are commuting groups. This means that $SU(2) * P * T \cong SU(2) \times P \times T$, but, since T is antiunitary, we need to use analysis in the previous section instead of directly use representation tensor product as in the case of combining $SU(2)$ with P .

Using equation 6.6 as the irreducible representation for $SU(2) \times P$ and the commutation of the groups, equation 7.5, and 7.7 turn out to be

$$\tilde{D}_{SU(2) \times P}^{(j,p)}((u_{\vec{\phi}}, v_\rho)) = D_{SU(2) \times P}^{(j,p)*}((u_{\vec{\phi}}, v_\rho)), \quad (7.23)$$

and

$$D_{SU(2) \times P}^{(j,p)}((u_{\vec{\phi}}, v_\rho)) = \Theta^{(j,p)\dagger} D_{SU(2) \times P}^{(j,p)}((u_{\vec{\phi}}, v_\rho)) \Theta^{(j,p)}, \quad (7.24)$$

where $u_{\vec{\phi}} \in SU(2)$, $v_\rho \in P$, $j \in \{0, 1/2, 1, 3/2, \dots\}$, $p \in \{-1, 1\}$, and the irreducible representations $D_{SU(2) \times P}^{(j,p)}$ are in standard basis.

Since j , and p completely identifies all $SU(2) \times P$'s irreducible representations of $2j + 1$ dimensions, and $\tilde{D}_{SU(2) \times P}^{(j,p)}$ is also a $SU(2) \times P$'s irreducible representation of $2j + 1$ dimensions, $\tilde{D}_{SU(2) \times P}^{(j,p)}$ must be equivalent to either $D_{SU(2) \times P}^{(j,p)}$, or $D_{SU(2) \times P}^{(j,-p)}$ under a unitary basis transformation $U^{(j,p)}$. If $\tilde{D}_{SU(2) \times P}^{(j,p)}$ is equivalent to $D_{SU(2) \times P}^{(j,-p)}$, then, for any $u_{\vec{\phi}}$, v_ρ ,

$$\begin{aligned} \tilde{D}_{SU(2) \times P}^{(j,p)}((u_{\vec{\phi}}, v_\rho)) &= U^{(j,p)\dagger} D_{SU(2) \times P}^{(j,-p)}((u_{\vec{\phi}}, v_\rho)) U^{(j,p)} \\ D_{SU(2)}^{(j)*}(u_{\vec{\phi}}) \cdot p^\rho &= U^{(j,p)\dagger} D_{SU(2)}^{(j)}(u_{\vec{\phi}}) U^{(j,p)} \cdot (-p)^\rho \\ D_{SU(2)}^{(j)*}(u_{\vec{\phi}}) &= U^{(j,p)\dagger} D_{SU(2)}^{(j)}(u_{\vec{\phi}}) U^{(j,p)} \cdot (-1)^\rho. \end{aligned} \quad (7.25)$$

However, this is only possible when $D_{SU(2)}^{(j)*}$ is a zero matrix which is impossible since any representation is invertible by definition. Therefore, $\tilde{D}_{SU(2) \times P}^{(j,p)}$ must be equivalent to $D_{SU(2) \times P}^{(j,p)}$, and $U^{(j,p)}$ is independent of p (in the sense that any U that satisfies equation 7.25 for $p = 1$ also satisfies for $p = -1$, and vice versa) where we will drop this superscript from here on. Similarly, after we substitute $D_{SU(2) \times P}^{(j,p)}$ into equation 7.24, we can see that $\Theta^{(j,p)}$ is also independent of p .

Hence, we can rewrite equation 7.23, and 7.24 as

$$\tilde{D}_{SU(2) \times P}^{(j,p)}((u_{\vec{\phi}}, v_{\rho})) = D_{SU(2) \times P}^{(j,p)*}((u_{\vec{\phi}}, v_{\rho})) = U^{(j)\dagger} D_{SU(2) \times P}^{(j,p)}((u_{\vec{\phi}}, v_{\rho})) U^{(j)}, \quad (7.26)$$

and

$$D_{SU(2) \times P}^{(j,p)}((u_{\vec{\phi}}, v_{\rho})) = \Theta^{(j)\dagger} D_{SU(2) \times P}^{(j,p)}((u_{\vec{\phi}}, v_{\rho})) \Theta^{(j)}. \quad (7.27)$$

Because $D_{SU(2) \times P}^{(j,p)}$ is irreducible, from Schur's second lemma (equation 4.7), $\Theta^{(j)}$ must equal a constant tI for some $t \in \mathbb{C}$,

$$\Theta^{(j)} = tI, \quad (7.28)$$

where, since $\Theta^{(j)}$ is unitary, $|t| = 1$. However, from equation 7.18,

$$\begin{aligned} U^{(j)\dagger} \Theta^{(j)} U^{(j)} &= \Theta^{(j)*} \\ U^{(j)\dagger} t I U^{(j)} &= t^* I \\ t &= t^*, \end{aligned} \quad (7.29)$$

or $t \in \mathbb{R}$. Combining with $|t| = 1$, this means that $t \in \{-1, 1\}$. Hence, from equation 7.15, we know that the chosen $U^{(j)}$ must satisfy

$$U^{(j)} U^{(j)*} = \lambda \Theta^{(j)} = t \lambda I = \pm I. \quad (7.30)$$

To see that this is the case, consider a matrix

$$[U^{(j)}]_{a,b} = i^{2(a-j)} \delta_{a,2j-b}. \quad (7.31)$$

with its inverse (adjoint)

$$[U^{(j)\dagger}]_{a,b} = (-i)^{2(b-j)} \delta_{b,2j-a}. \quad (7.32)$$

This matrix is actually a unitary matrix since

$$\begin{aligned}
[U^{(j)\dagger}U^{(j)}]_{a,b} &= \sum_c ((-i)^{2(c-j)}\delta_{c,2j-a}) (i^{2(c-j)}\delta_{c,2j-b}) \\
&= \delta_{2j-a,2j-b} = \delta_{a,b}, \\
[U^{(j)}U^{(j)\dagger}]_{a,b} &= \sum_c (i^{2(a-j)}\delta_{a,2j-c}) ((-i)^{2(b-j)}\delta_{b,2j-c}) \\
&= i^{2(a-b)}\delta_{a,b} = \delta_{a,b}.
\end{aligned} \tag{7.33}$$

From this matrix, we can see its transformation of $J_z^{(j)}$ as

$$\begin{aligned}
[U^{(j)\dagger}J_z^{(j)}U^{(j)}]_{a,b} &= \sum_{c,d} ((-i)^{2(c-j)}\delta_{c,2j-a}) ((c-j)\delta_{c,d}) (i^{2(d-j)}\delta_{d,2j-b}) \\
&= \sum_c (-i)^{2(c-j)}\delta_{c,2j-a}(c-j)\delta_{c,2j-b}i^{2(j-b)} \\
&= (-i)^{2(j-a)}(j-a)\delta_{a,b}i^{2(j-b)} \\
&= -(a-j)\delta_{a,b} \\
&= [-J_z^{(j)}]_{a,b},
\end{aligned} \tag{7.34}$$

of $J_+^{(j)}$ as

$$\begin{aligned}
[U^{(j)\dagger}J_+^{(j)}U^{(j)}]_{a,b} &= \sum_{c,d} ((-i)^{2(c-j)}\delta_{c,2j-a}) \left(\sqrt{j(j+1) - (c-j)(d-j)}\delta_{c,d+1} \right) (i^{2(d-j)}\delta_{d,2j-b}) \\
&= \sum_c (-i)^{2(c-j)}\delta_{c,2j-a}\sqrt{j(j+1) - (c-j)(j-b)}\delta_{c,2j-b+1}i^{2(j-b)} \\
&= (-i)^{2(j-a)}\sqrt{j(j+1) - (j-a)(j-b)}\delta_{2j-a,2j-b+1}i^{2(j-b)} \\
&= i^{2(a-b)}\sqrt{j(j+1) - (a-j)(b-j)}\delta_{a+1,b} \\
&= -\sqrt{j(j+1) - (a-j)(b-j)}\delta_{a+1,b} \\
&= [-J_+^{(j)\top}]_{a,b} = [-J_-^{(j)}]_{a,b},
\end{aligned} \tag{7.35}$$

and $J_-^{(j)}$ as

$$\begin{aligned}
U^{(j)\dagger}J_-^{(j)}U^{(j)} &= U^{(j)\dagger}J_+^{(j)\dagger}U^{(j)} \\
&= \left(U^{(j)\dagger}J_+^{(j)}U^{(j)} \right)^\dagger \\
&= -J_-^{(j)\dagger} = -J_+^{(j)}.
\end{aligned} \tag{7.36}$$

This also directly implies that

$$\begin{aligned}
U^{(j)\dagger} J_x^{(j)} U^{(j)} &= -J_x^{(j)} \\
U^{(j)\dagger} J_y^{(j)} U^{(j)} &= J_y^{(j)} \\
U^{(j)\dagger} J_z^{(j)} U^{(j)} &= -J_z^{(j)}
\end{aligned} \tag{7.37}$$

Therefore,

$$\begin{aligned}
U^\dagger D_{SU(2) \times P}^{(j,p)}((u_{\vec{\phi}}, v_\rho)) U &= U^{(j)\dagger} \exp\left(-i \vec{J}^{(j)} \cdot \vec{\phi}\right) U^{(j)} \cdot p^\rho \\
&= U^{(j)\dagger} \exp\left(-i(J_x^{(j)} \phi_x + J_y^{(j)} \phi_y + J_z^{(j)} \phi_z)\right) U^{(j)} \cdot p^\rho \\
&= \exp\left(i(J_x^{(j)} \phi_x - J_y^{(j)} \phi_y + J_z^{(j)} \phi_z)\right) \cdot p^\rho \\
&= D_{SU(2) \times P}^{(j,p)*}((u_{\vec{\phi}}, v_\rho)) \\
&= \tilde{D}_{SU(2) \times P}^{(j,p)}((u_{\vec{\phi}}, v_\rho)),
\end{aligned} \tag{7.38}$$

as required. We can also check the condition from equation 7.30 such that

$$\begin{aligned}
[U^{(j)} U^{(j)*}]_{a,b} &= \sum_c (i^{2(a-j)} \delta_{a,2j-c}) ((-i)^{2(c-j)} \delta_{c,2j-b}) \\
&= i^{2(a-j)} \delta_{a,b} (-i)^{2(j-b)} \\
&= i^{2(a+b-2j)} \delta_{a,b} \\
&= i^{4a} i^{-4j} \delta_{a,b} \\
&= (-1)^{-2j} \delta_{a,b} \\
&= [(-1)^{-2j} I]_{a,b}.
\end{aligned} \tag{7.39}$$

This, together with equation 7.30, gives the condition for λ as

$$\lambda = (-1)^{-2j} / t. \tag{7.40}$$

With all these results, we can now write the corepresentation of $SU(2)*P*T$ in $\{|j, m\rangle_p\} \cup \{U^{(j)\dagger} \theta |j, m\rangle_p\}$ basis as,

$$\begin{aligned}
D_{SU(2)*P*T}^{(j,p)}((u_{\vec{\phi}}, v_\rho)) &= D_{SU(2) \times P \times T}^{(j,p)}((u_{\vec{\phi}}, v_\rho, w_0)) \\
&= \begin{bmatrix} D_{SU(2) \times P}^{(j,p)}((u_{\vec{\phi}}, v_\rho)) & 0 \\ 0 & D_{SU(2) \times P}^{(j,p)}((u_{\vec{\phi}}, v_\rho)) \end{bmatrix}, \\
D_{SU(2)*P*T}^{(j,p)}(\theta) &= D_{SU(2) \times P \times T}^{(j,p)}((u_{\vec{0}}, u_0, w_1)) \\
&= \begin{bmatrix} 0 & (-1)^{2j} t U^{(j)} \\ U^{(j)} & 0 \end{bmatrix}
\end{aligned} \tag{7.41}$$

where $u_{\vec{\phi}} \in SU(2)$, $v_{\rho} \in P$, $w_{\tau} \in T$, $j \in \{0, 1/2, 1, 3/2, \dots\}$, $p \in \{-1, 1\}$, $t \in \{-1, 1\}$, and the irreducible representations $D_{SU(2) \times P}^{(j,p)}$ are in standard basis. Note that, equation 7.41 can be use for constructing any members of $SU(2) * P * T$ according to combination rules in equation 7.9 as well as the commutation property between members from different groups of $SU(2)$, P , and T .

The last thing we need to do is checking the reducibility of this corepresentation. The first thing we immediately notice from equation 7.41 is that, because $D_{SU(2) \times P}^{(j,p)}$ is irreducible, $D_{SU(2) * P * T}^{(j,p)}$ can only be either irreducible, or reducible into exactly two $2j+1$ dimensional subspaces. Suppose there exist a unitary transformation M that block diagonalizes $D_{SU(2) * P * T}^{(j,p)}$ into two $2j+1$ dimensional subspaces. Since $D_{SU(2) * P * T}^{(j,p)}((u_{\vec{\phi}}, v_{\rho}))$ is already block diagonalized with two irreducible $D_{SU(2) \times P}^{(j,p)}((u_{\vec{\phi}}, v_{\rho}))$'s, M must transform $D_{SU(2) * P * T}^{(j,p)}((u_{\vec{\phi}}, v_{\rho}))$ into block diagonalized irreducible representations with two, possibly different, irreducibles that are equivalent to $D_{SU(2) \times P}^{(j,p)}((u_{\vec{\phi}}, v_{\rho}))$. One can also do basis transformation with in each block separately. Hence, these basis transformations can be absorbed into M such that

$$\begin{aligned} M^{\dagger} D_{SU(2) * P * T}^{(j,p)}((u_{\vec{\phi}}, v_{\rho})) M &= D_{SU(2) * P * T}^{(j,p)}((u_{\vec{\phi}}, v_{\rho})) \\ D_{SU(2) * P * T}^{(j,p)}((u_{\vec{\phi}}, v_{\rho})) M &= M D_{SU(2) * P * T}^{(j,p)}((u_{\vec{\phi}}, v_{\rho})). \end{aligned} \quad (7.42)$$

Consider, a general M ,

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}. \quad (7.43)$$

Substituting equation 7.41, and 7.43 into 7.42, we get

$$\begin{bmatrix} A D_{SU(2) \times P}^{(j,p)} & B D_{SU(2) \times P}^{(j,p)} \\ C D_{SU(2) \times P}^{(j,p)} & D D_{SU(2) \times P}^{(j,p)} \end{bmatrix} = \begin{bmatrix} D_{SU(2) \times P}^{(j,p)} A & D_{SU(2) \times P}^{(j,p)} B \\ D_{SU(2) \times P}^{(j,p)} C & D_{SU(2) \times P}^{(j,p)} D \end{bmatrix}. \quad (7.44)$$

Because $D_{SU(2) \times P}^{(j,p)}$ is irreducible, from Schur's second lemma (equation 4.7),

$$M = \begin{bmatrix} aI & bI \\ cI & dI \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \otimes I \triangleq m \otimes I \quad (7.45)$$

where $a, b, c, d \in \mathbb{C}$, and m is a 2 by 2 unitary matrix. Then, we use this M to try block

diagonalizing $D_{SU(2)*P*T}^{(j,p)}(\theta)$,

$$\begin{aligned}
M^\dagger D_{SU(2)*P*T}^{(j,p)}(\theta) M^* &= (m^\dagger \otimes I) \left(\begin{bmatrix} 0 & 1/\lambda \\ 1 & 0 \end{bmatrix} \otimes U^{(j)} \right) (m^* \otimes I) \\
&= \left(m^\dagger \begin{bmatrix} 0 & 1/\lambda \\ 1 & 0 \end{bmatrix} m^* \right) \otimes U^{(j)} \\
&= \begin{bmatrix} (1 + 1/\lambda)a^*c^* & a^*d^*/\lambda + b^*c^* \\ a^*d^* + b^*c^*/\lambda & (1 + 1/\lambda)b^*d^* \end{bmatrix} \otimes U^{(j)}.
\end{aligned} \tag{7.46}$$

From equation 7.46, it is clear that, when $1/\lambda = (-1)^{2jt} = -1$, the block diagonal terms vanish, i.e. it is impossible to be block diagonalized. In other words, equation 7.41 is the irreducible corepresentation of $SU(2) * P * T$ for the cases that $(-1)^{2jt} = -1$.

As for the case that $1/\lambda = (-1)^{2jt} = 1$, consider a following choice of m ,

$$m = \frac{1}{\sqrt{2}} \begin{bmatrix} i & 1 \\ -i & 1 \end{bmatrix}. \tag{7.47}$$

Then, substituting this m into equation 7.46, for $1/\lambda = 1$, gets

$$\begin{aligned}
M^\dagger D_{SU(2)*P*T}^{(j,p)}(\theta) M^* &= \begin{bmatrix} 2a^*c^* & a^*d^* + b^*c^* \\ a^*d^* + b^*c^* & 2b^*d^* \end{bmatrix} \otimes U^{(j)} \\
&= \frac{1}{2} \begin{bmatrix} 2 & i + -i \\ i + -i & 2 \end{bmatrix} \otimes U^{(j)} \\
&= \begin{bmatrix} U^{(j)} & 0 \\ 0 & U^{(j)} \end{bmatrix},
\end{aligned} \tag{7.48}$$

while, for $1/\lambda = -1$,

$$\begin{aligned}
M^\dagger D_{SU(2)*P*T}^{(j,p)}(\theta) M^* &= \begin{bmatrix} 0 & -a^*d^* + b^*c^* \\ a^*d^* - b^*c^* & 0 \end{bmatrix} \otimes U^{(j)} \\
&= \frac{1}{2} \begin{bmatrix} 0 & -i + -i \\ i - -i & 0 \end{bmatrix} \otimes U^{(j)} \\
&= i \begin{bmatrix} 0 & -U^{(j)} \\ U^{(j)} & 0 \end{bmatrix},
\end{aligned} \tag{7.49}$$

and, for both cases,

$$\begin{aligned} M^\dagger D_{SU(2)*P*T}^{(j,p)}((u_{\vec{\phi}}, v_\rho)) M &= D_{SU(2)*P*T}^{(j,p)}((u_{\vec{\phi}}, v_\rho)) \\ &= \begin{bmatrix} D_{SU(2)\times P}^{(j,p)}((u_{\vec{\phi}}, v_\rho)) & 0 \\ 0 & D_{SU(2)\times P}^{(j,p)}((u_{\vec{\phi}}, v_\rho)) \end{bmatrix}. \end{aligned} \quad (7.50)$$

With this basis transformation, we can see that the corepresentation is reducible into two equivalent representations for $(-1)^{2j}t = 1$. Hence, the irreducible corepresentation of $SU(2) * P * T$ for this cases is

$$\begin{aligned} D_{SU(2)*P*T}^{(j,p)}((u_{\vec{\phi}}, v_\rho)) &= D_{SU(2)\times P}^{(j,p)}((u_{\vec{\phi}}, v_\rho)), \\ D_{SU(2)*P*T}^{(j,p)}(\theta) &= U^{(j)}. \end{aligned} \quad (7.51)$$

From equation 7.49, 7.50, and 7.51, we can see that it would be convenience to write those irreducible corepresentations in the common tensor product form such that

$$\begin{aligned} D_{SU(2)*P*T}^{(j,p)}((u_{\vec{\phi}}, v_\rho)) &= (E^{(j,t)})^0 \otimes D_{SU(2)\times P}^{(j,p)}((u_{\vec{\phi}}, v_\rho)), \\ D_{SU(2)*P*T}^{(j,p)}(\theta) &= E^{(j,t)} \otimes U^{(j)} \end{aligned} \quad (7.52)$$

where

$$E^{(j,t)} = \begin{cases} 1 & , t = (-1)^{2j} \\ \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} & , t = (-1)^{2j+1} \end{cases}. \quad (7.53)$$

From these results, we can see that λ , which can be defined with j , and t , fully specifies the irreducible corepresentation. Hence, the irreducible corepresentation of the combined $SU(2)$, P , and T symmetries can be written, labeled with j , p , and t , as, for all $u_{\vec{\phi}} \in SU(2)$, $v_\rho \in P$, and $w_\tau \in T$,

$$\begin{aligned} D_{SU(2)*P*T}^{(j,p,t)}(u_{\vec{\phi}} v_\rho w_\tau) &= D_{SU(2)\times P\times T}^{(j,p,t)}((u_{\vec{\phi}}, v_\rho, w_\tau)) \\ &= D_{SU(2)\times P\times T}^{(j,p,t)}((u_{\vec{\phi}}, v_\rho, w_0)) \cdot D_{SU(2)\times P\times T}^{(j,p,t)}((u_{\vec{0}}, v_0, w_1)^\tau) \\ &= (E^{(j,t)})^\tau \otimes D_{SU(2)\times P}^{(j,p)}((u_{\vec{\phi}}, v_\rho)) \cdot (U^{(j)} U^{(j)*})^{\lfloor \tau/2 \rfloor} \cdot (U^{(j)})^{\tau \bmod 2} \\ &= (E^{(j,t)})^\tau \otimes D_{SU(2)\times P}^{(j,p)}((u_{\vec{\phi}}, v_\rho)) \cdot (-1)^{2j\lfloor \tau/2 \rfloor} \cdot (U^{(j)})^{\tau \bmod 2} \\ &= (E^{(j,t)})^\tau \otimes (-1)^{2j\lfloor \tau/2 \rfloor} p^\rho \cdot \exp(-i \vec{J}^{(j)} \cdot \vec{\phi}) \cdot (U^{(j)})^{\tau \bmod 2} \end{aligned} \quad (7.54)$$

where $\vec{\phi} \in \mathbb{R}^3$, $\rho, \tau \in \mathbb{Z}_{0+}$, $j \in \{0, 1/2, 1, 3/2, \dots\}$, and $p, t \in \{-1, 1\}$.

8. TENSOR PRODUCT OF REPRESENTATIONS

8.1. Tensor Product as Composite System Representation

8.2. Tensor Product Reduction

8.3. Tensor Product of $SU(2)$'s Irreducible Representations

8.4. Tensor Product of $SU(2) \times P$'s Irreducible Representations

Consider two irreducible representations of $SU(2) \times P$ with labels (j_1, p_1) , and (j_2, p_2) from equation 6.6. Their tensor product is, for all $u_{\vec{\phi}} \in SU(2)$, $v_\rho \in P$, $\vec{\phi} \in \mathbb{R}^3$, $\rho \in \mathbb{Z}_{0+}$, j 's $\in \{0, 1/2, 1, 3/2, \dots\}$, and p 's $\in \{-1, 1\}$,

$$\begin{aligned}
D_{SU(2) \times P}^{(j_1, p_1)}((u_{\vec{\phi}}, v_\rho)) \otimes D_{SU(2) \times P}^{(j_2, p_2)}((u_{\vec{\phi}}, v_\rho)) &= D_{SU(2)}^{(j_1)}(u_{\vec{\phi}}) p_1^\rho \otimes D_{SU(2)}^{(j_2)}(u_{\vec{\phi}}) p_2^\rho \\
&= \left(D_{SU(2)}^{(j_1)}(u_{\vec{\phi}}) \otimes D_{SU(2)}^{(j_2)}(u_{\vec{\phi}}) \right) (p_1 \cdot p_2)^\rho \\
&\cong \left(\bigoplus_{j=|j_1-j_2|}^{j_1+j_2} D_{SU(2)}^{(j)}(u_{\vec{\phi}}) \right) (p_1 \cdot p_2)^\rho \\
&= \bigoplus_{j=|j_1-j_2|}^{j_1+j_2} D_{SU(2)}^{(j)}(u_{\vec{\phi}}) (p_1 \cdot p_2)^\rho \\
&= \bigoplus_{j=|j_1-j_2|}^{j_1+j_2} D_{SU(2) \times P}^{(j, p_1 \cdot p_2)}((u_{\vec{\phi}}, v_\rho))
\end{aligned} \tag{8.1}$$

where the isomorphism is the reduction of $SU(2)$'s irreducible representations' tensor product from previous section.

8.5. Tensor Product of $SU(2) \times P \times T$'s Irreducible Corepresentations

Consider two irreducible corepresentations of $SU(2) \times P \times T$ with labels (j_1, p_1, t_1) , and (j_2, p_2, t_2) from equation 7.54. Their tensor product is, for all $u_{\vec{\phi}} \in SU(2)$, $v_\rho \in P$, $w_\tau \in T$,

$\vec{\phi} \in \mathbb{R}^3$, $\rho, \tau \in \mathbb{Z}_{0+}$ j 's $\in \{0, 1/2, 1, 3/2, \dots\}$, and p 's, t 's $\in \{-1, 1\}$,

$$\begin{aligned}
& D_{SU(2) \times P \times T}^{(j_1, p_1, t_1)}((u_{\vec{\phi}}, v_{\rho}, w_{\tau})) \otimes D_{SU(2) \times P \times T}^{(j_2, p_2, t_2)}((u_{\vec{\phi}}, v_{\rho}, w_{\tau})) \\
&= \left[\begin{aligned} & (E^{(j_1, t_1)})^{\tau} \otimes (-1)^{2j_1 \lfloor \tau/2 \rfloor} D_{SU(2) \times P}^{(j_1, p_1)}((u_{\vec{\phi}}, v_{\rho})) \cdot (U^{(j_1)})^{\tau \bmod 2} \\ & \otimes (E^{(j_2, t_2)})^{\tau} \otimes (-1)^{2j_2 \lfloor \tau/2 \rfloor} D_{SU(2) \times P}^{(j_2, p_2)}((u_{\vec{\phi}}, v_{\rho})) \cdot (U^{(j_2)})^{\tau \bmod 2} \end{aligned} \right] \\
&\cong \left[\begin{aligned} & (E^{(j_1, t_1)} \otimes E^{(j_2, t_2)})^{\tau} \otimes (-1)^{2(j_1+j_2) \lfloor \tau/2 \rfloor} \\ & \cdot \left(D_{SU(2) \times P}^{(j_1, p_1)}((u_{\vec{\phi}}, v_{\rho})) \otimes D_{SU(2) \times P}^{(j_2, p_2)}((u_{\vec{\phi}}, v_{\rho})) \right) \cdot (U^{(j_1)} \otimes U^{(j_2)})^{\tau \bmod 2} \end{aligned} \right] \\
&\cong \left[\begin{aligned} & (E^{(j_1, t_1)} \otimes E^{(j_2, t_2)})^{\tau} \otimes (-1)^{2(j_1+j_2) \lfloor \tau/2 \rfloor} \\ & \cdot \left(\bigoplus_{j=|j_1-j_2|}^{j_1+j_2} D_{SU(2) \times P}^{(j, p_1 \cdot p_2)}((u_{\vec{\phi}}, v_{\rho})) \right) \cdot (U^{(j_1)} \otimes U^{(j_2)})^{\tau \bmod 2} \end{aligned} \right] \tag{8.2} \\
&= \left[\begin{aligned} & (E^{(j_1, t_1)} \otimes E^{(j_2, t_2)})^{\tau} \\ & \otimes \left(\bigoplus_{j=|j_1-j_2|}^{j_1+j_2} (-1)^{2j \lfloor \tau/2 \rfloor} \cdot D_{SU(2) \times P}^{(j, p_1 \cdot p_2)}((u_{\vec{\phi}}, v_{\rho})) \right) \cdot (U^{(j_1)} \otimes U^{(j_2)})^{\tau \bmod 2} \end{aligned} \right]
\end{aligned}$$

where the first isomorphism is matrix index reordering, the second isomorphism is the reduction of $SU(2) \times P$'s irreducible representations' tensor product from previous section, and the second equality uses the fact that $(-1)^{2j \lfloor \tau/2 \rfloor}$ gives the same value for every j 's that are different from each other by an integer amount. Hence, the only things we need to reduce are $E^{(j_1, t_1)} \otimes E^{(j_2, t_2)}$, and $U^{(j_1)} \otimes U^{(j_2)}$.

For $E^{(j_1, t_1)} \otimes E^{(j_2, t_2)}$, there are 4 possible cases. The first one is when $t_1(-1)^{2j_1} = 1$, and $t_2(-1)^{2j_2} = 1$, which is the trivial case,

$$E^{(j_1, t_1)} \otimes E^{(j_2, t_2)} = 1 \otimes 1 = 1. \tag{8.3}$$

The second, and third ones are when $t_1(-1)^{2j_1} = \pm 1$, and $t_2(-1)^{2j_2} = \mp 1$ which is also straight forward that

$$E^{(j_1, t_1)} \otimes E^{(j_2, t_2)} = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}. \tag{8.4}$$

The final case is when $t_1(-1)^{2j_1} = -1$, and $t_2(-1)^{2j_2} = -1$,

$$\begin{aligned} E^{(j_1, t_1)} \otimes E^{(j_2, t_2)} &= \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}. \end{aligned} \quad (8.5)$$

Consider a unitary basis transformation,

$$B = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 0 & 0 & i \\ 0 & i & 1 & 0 \\ 0 & -i & 1 & 0 \\ 1 & 0 & 0 & i \end{bmatrix}. \quad (8.6)$$

Applying this transformation to equation gives

$$\begin{aligned} \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} &\cong B^\dagger \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} B^* \\ &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \end{aligned} \quad (8.7)$$

We can see from these results that the tensor product $E^{(j_1, t_1)} \otimes E^{(j_2, t_2)}$ can be reduced to a multiple of $E^{(j, t)}$ such that

$$t(-1)^{2j} = t_1(-1)^{2j_1} \cdot t_2(-1)^{2j_2} = t_1 t_2 (-1)^{2(j_1+j_2)} = t_1 t_2 (-1)^{2(j_1+j_2-1)} = \dots \quad (8.8)$$

For $U^{(j_1)} \otimes U^{(j_2)}$, consider its definition in equation 7.31 and rewrite it in bra-ket notation,

$$U^{(j)} = \sum_{m=-j}^j i^{2m} |j, -m\rangle \langle j, m|. \quad (8.9)$$

This means that

$$\begin{aligned}
U^{(j_1)} \otimes U^{(j_2)} &= \sum_{m_1=-j_1}^{j_1} \sum_{m_2=-j_2}^{j_2} i^{2m_1} |j_1, -m_1\rangle \langle j_1, m_1| \otimes i^{2m_2} |j_2, -m_2\rangle \langle j_2, m_2| \\
&= \sum_{m_1=-j_1}^{j_1} \sum_{m_2=-j_2}^{j_2} i^{2(m_1+m_2)} |j_1, -m_1, j_2, -m_2\rangle \langle j_1, m_1, j_2, m_2|.
\end{aligned} \tag{8.10}$$

By using completeness of $\{|j, m\rangle\}$ basis,

$$\begin{aligned}
U^{(j_1)} \otimes U^{(j_2)} &= \sum_{m_1=-j_1}^{j_1} \sum_{m_2=-j_2}^{j_2} i^{2m_1} |j_1, -m_1\rangle \langle j_1, m_1| \otimes i^{2m_2} |j_2, -m_2\rangle \langle j_2, m_2| \\
&= \left[\sum_{m_1=-j_1}^{j_1} \sum_{m_2=-j_2}^{j_2} i^{2(m_1+m_2)} \left(\sum_{j'=|j_1-j_2|}^{j_1+j_2} \sum_{m'=-j'}^{j'} |j', m'\rangle \langle j', m'| \right) \right. \\
&\quad \cdot |j_1, -m_1, j_2, -m_2\rangle \langle j_1, m_1, j_2, m_2| \\
&\quad \left. \cdot \left(\sum_{j=|j_1-j_2|}^{j_1+j_2} \sum_{m=-j}^j |j, m\rangle \langle j, m| \right) \right] \\
&= \sum_{m_1, m_2} \sum_{j', j} \sum_{m', m} \left[i^{2(m_1+m_2)} |j', m'\rangle \langle j', m'| j_1, -m_1, j_2, -m_2\rangle \right. \\
&\quad \left. \langle j_1, m_1, j_2, m_2 | j, m\rangle \langle j, m| \right]
\end{aligned} \tag{8.11}$$

where the range of j , and j' are from the non-vanishing Clebsch-Gordan coefficient terms. We also omit the known summation ranges for cleanliness. From Clebsch-Gordan coefficient properties, $\langle j', m' | j_1, -m_1, j_2, -m_2 \rangle = (-1)^{j_1+j_2-j'} \langle j', -m' | j_1, m_1, j_2, m_2 \rangle$, and $\langle j_1, m_1, j_2, m_2 | j, m \rangle \langle j, m|$ vanishes unless $m = m_1 + m_2$. Therefore,

$$\begin{aligned}
U^{(j_1)} \otimes U^{(j_2)} &= \sum_{m_1, m_2} \sum_{j', j} \sum_{m', m} \left[i^{2m} (-1)^{j_1+j_2-j'} |j', m'\rangle \langle j', -m' | j_1, m_1, j_2, m_2 \rangle \right. \\
&\quad \left. \langle j_1, m_1, j_2, m_2 | j, m \rangle \langle j, m| \right] \\
&= \sum_{j', j} \sum_{m', m} \left[i^{2m} (-1)^{j_1+j_2-j'} |j', m'\rangle \langle j, m| \right. \\
&\quad \left. \cdot \sum_{m_1, m_2} (\langle j', -m' | j_1, m_1, j_2, m_2 \rangle \langle j_1, m_1, j_2, m_2 | j, m \rangle) \right] \\
&= \sum_{j', j} \sum_{m', m} i^{2m} (-1)^{j_1+j_2-j'} |j', m'\rangle \langle j, m| \cdot \delta_{j', j} \delta_{-m', m} \\
&= \sum_{j=|j_1-j_2|}^{j_1+j_2} (-1)^{j_1+j_2-j} \sum_{m=-j}^j i^{2m} |j, -m\rangle \langle j, m| \\
&\cong \bigoplus_{j=|j_1-j_2|}^{j_1+j_2} U^{(j)}.
\end{aligned} \tag{8.12}$$

We can directly see that $U^{(j_1)} \otimes U^{(j_2)}$ can be at least reduced into the same block diagonal of representations as $D_{SU(2) \times P}^{(j_1, p_1)} \otimes D_{SU(2) \times P}^{(j_2, p_2)}$. Since they are matrix multiplying with each other, the reduction cannot exceed the number of $D_{SU(2) \times P}^{(j_1, p_1)} \otimes D_{SU(2) \times P}^{(j_2, p_2)}$'s reduction. Hence,

$$\begin{aligned}
& D_{SU(2) \times P \times T}^{(j_1, p_1, t_1)}((u_{\vec{\phi}}, v_{\rho}, w_{\tau})) \otimes D_{SU(2) \times P \times T}^{(j_2, p_2, t_2)}((u_{\vec{\phi}}, v_{\rho}, w_{\tau})) \\
& \cong \left[\begin{aligned} & (E^{(j_1, t_1)} \otimes E^{(j_2, t_2)})^{\tau} \\ & \otimes \left(\bigoplus_{j=|j_1-j_2|}^{j_1+j_2} (-1)^{2j[\tau/2]} \cdot D_{SU(2) \times P}^{(j, p_1 \cdot p_2)}((u_{\vec{\phi}}, v_{\rho})) \right) \\ & \cdot \left(\bigoplus_{j=|j_1-j_2|}^{j_1+j_2} U^{(j)} \right)^{\tau \bmod 2} \end{aligned} \right] \tag{8.13} \\
& \cong \bigoplus_a \bigoplus_{j=|j_1-j_2|}^{j_1+j_2} (E^{(j, t_1 \cdot t_2)})^{\tau} \otimes D_{SU(2) \times P}^{(j, p_1 \cdot p_2)}((u_{\vec{\phi}}, v_{\rho})) \cdot (-1)^{2j[\tau/2]} \cdot (U^{(j)})^{\tau \bmod 2} \\
& = \bigoplus_a \bigoplus_{j=|j_1-j_2|}^{j_1+j_2} D_{SU(2) \times P \times T}^{(j, p_1 \cdot p_2, t_1 \cdot t_2)}((u_{\vec{\phi}}, v_{\rho}, w_{\tau}))
\end{aligned}$$

where a ranges from 1 to 4 if $t_1(-1)^{2j_1} = -1$, and $t_2(-1)^{2j_2} = -1$, otherwise from 1 to 1.