Unified Representation for Quantum Spin Equivariant Learning

Abhijatmedhi Chotrattanapituk

Quantum Measurement Group, MIT, Cambridge, MA, USA

Department of Electrical Engineering and Computer Science, MIT, Cambridge, MA, USA

(Dated: June 12, 2023)

Abstract

TBA

1. INTRODUCTION

2. SYMMETRIES IN QUANTUM SPIN SYSTEMS

3. NOTATIONS

Due to the number of groups, and representations discussed in this work, it is important to define some common definitions and notations that would be used.

3.1. Group

A group is a set of items together with a binary operator defined on each pair of elements in the group such that, for an arbritary group of set G and its binary operator $\circ_G$, the following properites must be true.

1. For any $a, b \in G$,

$$a \circ_G b \in G. \tag{3.1}$$

2. For any $a, b, c \in G$,

$$(a \circ_G b) \circ_G c = a \circ_G (b \circ_G c). \tag{3.2}$$

3. There is a unique element $e_G \in G$ such that, for any $g \in G$,

$$g \circ_G e_G = e_G \circ_G g = g. \tag{3.3}$$

4. For any $g \in G$, there is a unique element $g^{-1} \in G$ such that

$$g \circ_G g^{-1} = g^{-1} \circ_G g = e. \tag{3.4}$$

For compactness in the remaining sections, we would drop the explicit placement of the binary operator and assume the group binary operation whenever there are two elements of the same group placeing beside each other, e.g., if $a, b \in G$, $ab \triangleq a \circ_G b$. Furthermore, we would refer to the group by its set when ever the group's binary operator is unambiguous, e.g., a group of set G and its binary operator $\circ_G$ is group G.

3.2. Homomorphism

A homomorphism of group G is a map from G to another group H such that the image of G preserves the same structure as G. In otherwords, for a homomorphism $h : G \to H$,

$$\forall a, b \in G,\ h(a)h(b) = h(ab), \tag{3.5}$$

where the implicit binary operation ab is from G, i.e. $a \circ_G b$ while implicit binary operation $h(a)h(b)$ is from H, i.e. $h(a) \circ_H h(b)$.

3.3. Representation

A group representation of G on vector space V is a group homomorphism from G to general linear group $GL(V)$ which is the group of bijective linear transformation from V to itself. To be more specific, for the vector space V over field $\mathbb{F}_V$ with n_V dimensions, $GL(n_V, \mathbb{F}_V) \triangleq GL(V)$ is the set of n_V by n_V invertible matrices over $\mathbb{F}_V$ with matrix multiplication as the group binary operator. This obviously means that there can be multiple representations depending on the choice of V, and each V might has multiple representations as well.

Assuming that we are able to uniquely name each representation, we would use the notation $D_G^{(\Gamma)} : G \to GL$ for a representation of group G where Γ is the label that fully specify the Rep. Since two representations can be equivalence if every corresponding pair of matrix representing group elements are equivalence through the same basis transformation, we would decorate $D_G^{(\Gamma)}$ to specify the basis of the matrix representation used.

3.4. Reducibility of Representation

A group representation of G on vector space V is reducible if their exist an invariant proper subspace $W \subsetneq V$ such that W is closed under group represented action, i.e.

$$\forall g \in G,\ \forall w \in W,\ D_G^{(\Gamma)}(g)w \in W. \tag{3.6}$$

This reducibility implies that there exists a choice of basis that block diagonalizes group representation such that one of the blocks is a (reduced) group representation of G on vector space W.

3.5. Irreducible Representation

A group irreducible representation of G on vector space V is a group representation that is not reducible, i.e. cannot be block diagonalized. This condition is equivalent with Schur's second lemma that, if Γ labels an irreducible representation, then every matrix M that satisfies,

$$\forall g \in G, \ D_G^{(\Gamma)}(g)M = MD_G^{(\Gamma)}(g) \tag{3.7}$$

must be a multiple of identity matrix (λI) with $\lambda \in \mathbb{F}_V$.

3.6. Groups' Direct Product

A group direct product is a composition of two groups to form a new larger group. Consider groups G and H, their direct product is defined as a group of Cartesian product set

$$G \times H = \{(g, h)|\forall g \in G, \forall h \in H\}, \tag{3.8}$$

and its binary operator defined as, $\forall (g_1, h_1), (g_2, h_2) \in G \times H$,

$$(g_1, h_1)(g_2, h_2) = (g_1 g_2, h_1 h_2). \tag{3.9}$$

Here, we also utilize the implicit binary operation notations.

4. $SU(2)$ GROUP

4.1. Definition

$SU(2)$, i.e. special unitary group of dimension 2, is the group of 2 by 2 unitary matrices with unit determinant with matrix multiplication as its binary operator. In other words,

$$SU(2) = \{u \in GL(2, \mathbb{C})|u^\dagger = u^{-1}, \det(u) = 1\}, \tag{4.1}$$

where $u^\dagger$ is the complex conjugate transposition of u, and $\det(\cdot)$ is matrix determinant operation. Since, for any $u, v \in SU(2)$,

$$\det(uv) = \det(u)\det(v) = 1, \tag{4.2}$$

and

$$(uv)^\dagger = (uv)^{\top *} = (v^\top u^\top)^* = v^\dagger u^\dagger = v^{-1}u^{-1} = (uv)^{-1}, \tag{4.3}$$

$SU(2)$ is a closed subgroup of $GL(2, \mathbb{C})$ which makes it a Lie group. This means that there is a corresponding Lie algebra, $\mathfrak{su}(2)$, such that

$$SU(2) = \{e^g | g \in \mathfrak{su}(2)\} \tag{4.4}$$

where the exponential of a square matrix g is defined as

$$e^g = I + \sum_{k=1}^{\infty} \frac{g^k}{k!}. \tag{4.5}$$

Here, I representes identity matrix with the same shape as g. From definition of matrix exponential,

$$(e^g)^\dagger = e^{g^\dagger},$$
$$(e^g)^{-1} = e^{-g}, \tag{4.6}$$
$$\det(e^g) = e^{\mathrm{Tr}\, g},$$

one can define the Lie algebra of $SU(2)$ as

$$\mathfrak{su}(2) = \{g \in M(2, \mathbb{C}) | g^\dagger = -g, \mathrm{Tr}(g) = 0\}, \tag{4.7}$$

where $M(2, \mathbb{C})$ is the group of 2 by 2 matrices with complex field with matrix multiplication as its binary operator, and $\mathrm{Tr}(\cdot)$ is matrix trace operation. Hence, each member of $\mathfrak{su}(2)$ can be written in the form

$$\begin{bmatrix} iv_z & v_y + iv_x \\ -v_y + iv_x & -iv_z \end{bmatrix} = iv_x\sigma_x + iv_y\sigma_y + iv_z\sigma_z = i\vec{v} \cdot \vec{\sigma} \tag{4.8}$$

where $\vec{v} \in \mathbb{R}^3$, and σ_i's are Pauli matrices defined as

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{4.9}$$

4.2. Generators

From $\mathfrak{su}(2)$'s definition in the previous section, one can consider using Pauli matrices as the group generators for $SU(2)$ according to equation 4.8, but it is more common to use $J_i^{(1/2)} \triangleq \sigma_i/2$ as the generators where $i \in \{x, y, z\}$. This gives the Lie bracket relation as

$$\left[J_i^{(1/2)}, J_j^{(1/2)}\right] \triangleq J_i^{(1/2)}J_j^{(1/2)} - J_j^{(1/2)}J_i^{(1/2)} = i\epsilon_{ijk}J_k^{(1/2)} \tag{4.10}$$

where ϵ_{ijk} is equal to 1 if (i, j, k) is an even permutation of (x, y, z), -1 if odd permutation, and 0 otherwise.

The next step is to consider an arbitrary representation of $SU(2)$ such that its generators, represented by J_i's, obey equation 4.10. With out loss of generality, consider the representation vector space with eigenvectors of J_z, $\{|\lambda_{J_z}\rangle\}$, as the basis,

$$J_z |\lambda_{J_z}\rangle = \lambda_{J_z} |\lambda_{J_z}\rangle. \tag{4.11}$$

Furthermore, we will also consider only the representation that is irreducible. With the standard method, first define two additional operators

$$\begin{aligned} J_+ &= J_x + iJ_y, \\ J_- &= J_x - iJ_y. \end{aligned} \tag{4.12}$$

From Lie bracket, and the choice of basis used,

$$\begin{aligned} J_z J_+ |\lambda_{J_z}\rangle &= J_+ (J_z + 1) |\lambda_{J_z}\rangle = (\lambda_{J_z} + 1)J_+ |\lambda_{J_z}\rangle, \\ J_z J_- |\lambda_{J_z}\rangle &= J_- (J_z - 1) |\lambda_{J_z}\rangle = (\lambda_{J_z} - 1)J_- |\lambda_{J_z}\rangle. \end{aligned} \tag{4.13}$$

This directly implies

$$\begin{aligned} J_+ |\lambda_{J_z}\rangle &\propto |\lambda_{J_z} + 1\rangle, \\ J_- |\lambda_{J_z}\rangle &\propto |\lambda_{J_z} - 1\rangle. \end{aligned} \tag{4.14}$$

Before proceeding, we need to clarify an assumption we made to get equation 4.14 that there is no degeneracy of λ_{J_z} in the basis. This result is directly entailed from the assumption that the representation is irreducible. To prove this, first assume that there exist $|\lambda_{J_z}\rangle'$ that has the same eigenvalue as $|\lambda_{J_z}\rangle$. From the properties of J_i's, we can choose the representation such that equation 4.14 is true. Since the representation of each $SU(2)$ member can be expressed in terms of the generators (equation 4.4 and 4.5), if a subspace is invariant under the generators, it is also invariant under the $SU(2)$ representation. This means that the proper subspace that exclude $|\lambda_{J_z}\rangle'$ is invariant for this choice of representation. Hence, the representation is reducible which contradict the assumption.

To get the proportionality, consider another operators

$$J^2 \triangleq J_x^2 + J_y^2 + J_z^2. \tag{4.15}$$

6

One can directly chack that the new operator commutes with all J's mentioned so far. Furthermore, it is straight forward that

$$
\begin{aligned}
J^2 &= J_+ J_- + J_z^2 - J_z \\
&= J_- J_+ + J_z^2 + J_z.
\end{aligned}
\tag{4.16}
$$

Since J^2 commutes with J_z, the basis used are also eigenbasis of J^2 with some eigenvalue λ_{J^2}. Hence, we can directly relabel the eigenbasis from $|\lambda_{J_z}\rangle$ to $|\lambda_{J^2}, \lambda_{J_z}\rangle$, and get

$$
\begin{aligned}
\langle \lambda_{J^2}, \lambda_{J_z} | J_+ J_- | \lambda_{J^2}, \lambda_{J_z} \rangle = \lambda_{J^2} - \lambda_{J_z}(\lambda_{J_z} - 1), \\
\langle \lambda_{J^2}, \lambda_{J_z} | J_- J_+ | \lambda_{J^2}, \lambda_{J_z} \rangle = \lambda_{J^2} - \lambda_{J_z}(\lambda_{J_z} + 1).
\end{aligned}
\tag{4.17}
$$

Since the norm square of any vector in vector space must be non-negative, this requires

$$
\lambda_{J^2} \geq |\lambda_{J_z}|(|\lambda_{J_z}| + 1) \geq 0.
\tag{4.18}
$$

However, the $J_\pm$ makes λ_{J_z} unbounded unless the vector vanished at some points, i.e. the equality must hold for the upper, and lower bounds of the possible λ_z's. If we replace λ_{J^2} with $j(j+1)$ where j non-negative. The requirement makes $\lambda_{J_z,\max} = j$, and $\lambda_{J_z,\min} = -j$. Since any pair of λ_{J_z}'s are different by an integer,

$$
\lambda_{J_z,\max} - \lambda_{J_z,\min} = 2j \in \mathbb{Z}_0^+.
\tag{4.19}
$$

Hence, we can relabel the eigenbasis from $|\lambda_{J^2}, \lambda_{J_z}\rangle$ to $|j, m\rangle$ and choose the proportionality such that

$$
\begin{aligned}
J_+ |j, m\rangle = \sqrt{j(j+1) - m(m+1)} \, |j, m+1\rangle, \\
J_- |j, m\rangle = \sqrt{j(j+1) - m(m-1)} \, |j, m-1\rangle,
\end{aligned}
\tag{4.20}
$$

where $j \in \{0, 1/2, 1, 3/2, ...\}$, and $m \in \{-j, -j+1, ..., j-1, j\}$. Next, we must prove that each j fully identifies an irreducible representation. Assuming that this is not the case by letting there be basis vector $|j', m'\rangle$ in the vector space of this irreducible representation that is not linearly dependent on the eigenbasis $\{|j, m\rangle\}$, equation 4.20 makes the vector space that exclude $|j', m'\rangle$ invariant proper subspace (with reasoning similar to when we prove non-degeneracy of λ_{J_z}). Hence, this implies the reducibility of the representation which contradicts the assumption.

4.3. Irreducible Representation

In previous section, we have proved that every irreducible representation has its unique j (no irreducible representation requires basis with more than one j.)Furthermore, since we have not made any other assumption, j also completely identify every irreducible representation of $SU(2)$ (every irreducible representation must be a reduced representation of a representation generated by generators with basis identify by a $j \in \{0, 1/2, 1, 3/2, ...\}$.) In other words, each and every irreducible representation is identify by a j, but it does not rule out the possibility that the representation constructed in the previous section is irreducible.

Consider a representation of $SU(2)$ that is generated from $J_i^{(j)}$ where the superscript indicate that the representation is identified by j. Let assume that such representation is reducible. By definition, this means that there is a subspace invariant by any $SU(2)$'s elements. This further implies that it is also invariant by any $\mathfrak{su}(2)$'s.

In order to implement the representation the basis of the space in which all group operations would act on need to be carefully selected mainly for the purpose of reducing computational costs. However, it is require the consideration on every symmetry group to decide. Hence, for now, we are assuming the standard $\{|j, m\rangle\}$ basis. In this basis, the irreducible representation with index j is the $2j + 1$ dimensional space representation. Let the representation of a vector is in the ascending order of the basis from $m = -j$ to $m = j$. Hence,

$$
\begin{aligned}
\left[J_z^{(j)}\right]_{qr} &= (q - j)\delta_{q,r}, \\
\left[J_+^{(j)}\right]_{qr} &= \sqrt{j(j+1) - (q-j)(r-j)}\delta_{q,r+1}, \\
J_-^{(j)} &= J_+^{(j)\top}, \\
J_x^{(j)} &= \left(J_+^{(j)} + J_-^{(j)}\right)/2, \\
J_y^{(j)} &= -i\left(J_+^{(j)} - J_-^{(j)}\right)/2,
\end{aligned}
\tag{4.21}
$$

with the $u \in SU(2)$ represented as

$$
D_{SU(2)}^{(j)}(u) = \exp\left(i(J_x^{(j)}\phi_x + J_y^{(j)}\phi_y + J_z^{(j)}\phi_z)\right)
\tag{4.22}
$$

for some $\vec{\phi} \in \mathbb{R}^3$.

5.   P

5.1.   Definition

P is the group of spatial inversion, i.e. the group consisting of 2 elements: E (identity), and I (inversion) with the only non-trivial multiplication be $II = E$.

5.2.   Irreducible Representation

From the definition, it is clear that $P \cong C_2$. Hence, there are only two irriducible representations commonly denoted as even, and odd. With similar construction to the $SU(2)$ case, let define the basis (1 dimension) of the two irreducible representations with $|1\rangle$, and $|-1\rangle$, respectively. Therefore,

$$
\begin{aligned}
D_P^{(1)}(E)\,|1\rangle &= |1\rangle\,, \\
D_P^{(1)}(I)\,|1\rangle &= |1\rangle\,, \\
D_P^{(-1)}(E)\,|-1\rangle &= |-1\rangle\,, \\
D_P^{(-1)}(I)\,|-1\rangle &= -\,|-1\rangle\,.
\end{aligned}
\tag{5.1}
$$

In order to incoperate P to $SU(2)$, the multiplication between these groups need to be considered. At this point, the purely mathematical treatment is not sufficient to know the multiplication behavior, and require physical properties of the systems that the groups are used for. In this case, we are interested in quantum spin systems in which the $SU(2)$ group is responsible for rotations, and commutes with spatial inversion. This implies that the combined group, i.e. direct product group, $SU(2) \times P$, can be irreducible represented with

$$
\left[D_{SU(2)\times P}^{(jp)}(uv) \right]_{qr} = \left[D_{SU(2)}^{(j)}(u) \right]_{qr} D_P^{(p)}(v)
\tag{5.2}
$$

for $u \in SU(2)$, $v \in P$, and $p \in \{-1, 1\}$. Furthermore, the vector space basis of this irreducible representation is simply $|j, m, p\rangle = |j, m\rangle\,|p\rangle$.

5.3.   Implementation

From previous section, with the basis used in $SU(2)$ implementation, one can directly extend it to irreducible representation of $SU(2) \times P$ by adding index p. In this basis, the

9

irreducible representation with index (j, p) is a $2j+1$ dimensional space representation with even inversion if $p = 1$, or odd if $p = -1$. With this representation, for $w = uv \in SU(2) \times P$, $u \in SU(2)$, $v \in P$,

$$D^{(jp)}_{SU(2) \times P}(uv) = D^{(j)}_{SU(2)}(u) D^{(p)}_P(v) \tag{5.3}$$

with

$$\begin{aligned} D^{(p)}_P(E) &= 1 \\ D^{(p)}_P(I) &= p \end{aligned} \tag{5.4}$$

6. T

6.1. Definition

TR is the group of time reversal consisting of E (identity), and θ (time reverse) as generators. However, this group is not isomorphic with P since the physical properties require time-reversal symmetric system to obey Schrödinger's equation. This makes the effect of θ on wavefunction of spinless system be a complex conjugation which means that the group is anti-unitary.

6.2. TR's Irreducible Representation

In corporateing anti-unitary group TR with $SU(2) \times P$ cannot be done in the same fashion as combinding P with $SU(2)$ since the anti-unitary nature makes it impossible to write down common eigenbasis even if θ commutes with every element in $SU(2) \times P$.

To proceed from this point, we need to consider the effects of θ on the basis chosen for $SU(2) \times P$'s implementation. Since the multiplication of two anti-unitary operators is a unitary operator the irreducible representation of $w \in SU(2) \times P$ in $\{\theta \, |j, m, p\rangle\}$ basis is

$$\tilde{D}_{SU(2) \times P}(w) = D_{SU(2) \times P}(\theta^{-1} w \theta)^*. \tag{6.1}$$

However, similar to P's case, the system in which the $SU(2) \times P$ describes actually makes θ commutes with every element in $SU(2) \times P$. This means that

$$D_{SU(2) \times P}(\theta^{-1} w \theta)^* = D_{SU(2) \times P}(w)^*, \tag{6.2}$$

10

but we can always write

$$D_{SU(2)\times P}(w) = \pm \exp\left(i\vec{J}\cdot\vec{\phi}\right) = \pm\exp(i(J_x\phi_x + J_y\phi_y + J_z\phi_z)). \qquad (6.3)$$

This means that

$$D_{SU(2)\times P}(w)^* = \pm\exp(-i(J_x\phi_x - J_y\phi_y + J_z\phi_z)). \qquad (6.4)$$

Then, consider a matrix

$$\left[U^{(j)}\right]_{qr} = i^{2(q-j)}\delta_{q,2j-r}. \qquad (6.5)$$

This matrix is a unitary matrix since

$$
\begin{aligned}
\left[U^{(j)\dagger}\right]_{qr} &= (-i)^{2(r-j)}\delta_{r,2j-q}, \\
\left[U^{(j)\dagger}U^{(j)}\right]_{qr} &= \sum_s (-i)^{2(s-j)}\delta_{s,2j-q}\, i^{2(s-j)}\delta_{s,2j-r} \\
&= \sum_s \delta_{s,2j-q}\delta_{s,2j-r} = \delta_{q,r}, \\
\left[U^{(j)}U^{(j)\dagger}\right]_{qr} &= \sum_s i^{2(q-j)}\delta_{q,2j-s}(-i)^{2(r-j)}\delta_{r,2j-s} \\
&= \sum_s i^{2(q-r)}\delta_{q,2j-s}\delta_{r,2j-s} = i^{2(q-r)}\delta_{q,r} = \delta_{q,r}.
\end{aligned}
\qquad (6.6)
$$

From this matrix, we can see that

$$
\begin{aligned}
\left[U^{(j)\dagger}J_z^{(j)}U^{(j)}\right]_{qr} &= \sum_{s,t}(-i)^{2(s-j)}\delta_{s,2j-q}(s-j)\delta_{s,t}i^{2(t-j)}\delta_{t,2j-r} \\
&= \sum_s (-i)^{2(s-j)}\delta_{s,2j-q}(s-j)\delta_{s,2j-r}i^{2(j-r)} \\
&= (-i)^{2(j-q)}(j-q)\delta_{q,r}i^{2(j-r)} \\
&= -(q-j)\delta(q,r) \\
&= \left[-J_z^{(j)}\right]_{qr},
\end{aligned}
\qquad (6.7)
$$

$$
\begin{aligned}
\left[U^{(j)\dagger}J_+^{(j)}U^{(j)}\right]_{qr} &= \sum_{s,t}(-i)^{2(s-j)}\delta_{s,2j-q}\sqrt{j(j+1)-(s-j)(t-j)}\,\delta_{s,t+1}i^{2(t-j)}\delta_{t,2j-r} \\
&= \sum_s (-i)^{2(s-j)}\delta_{s,2j-q}\sqrt{j(j+1)-(s-j)(j-r)}\,\delta_{s,2j-r+1}i^{2(j-r)} \\
&= (-i)^{2(j-q)}\sqrt{j(j+1)-(j-q)(j-r)}\,\delta_{2j-q,2j-r+1}i^{2(j-r)} \\
&= -\sqrt{j(j+1)-(q-j)(r-j)}\,\delta_{q+1,r} \\
&= \left[-J_+^{(j)\top}\right]_{qr} = \left[-J_-^{(j)}\right]_{qr},
\end{aligned}
\qquad (6.8)
$$

11

and

$$U^{(j)\dagger} J_-^{(j)} U^{(j)} = U^{(j)\dagger} J_+^{(j)\dagger} U^{(j)}$$
$$= \left(U^{(j)\dagger} J_+^{(j)} U^{(j)} \right)^{\dagger} \qquad (6.9)$$
$$= -J_-^{(j)\dagger} = -J_+^{(j)}.$$

This also directly implies

$$U^{(j)\dagger} J_x^{(j)} U^{(j)} = -J_x^{(j)}$$
$$U^{(j)\dagger} J_y^{(j)} U^{(j)} = J_y^{(j)} \qquad . \qquad (6.10)$$
$$U^{(j)\dagger} J_z^{(j)} U^{(j)} = -J_z^{(j)}$$

Therefore,

$$U^{(j)\dagger} D_{SU(2)\times P}^{(j)}(w) U^{(j)} = \pm U^{(j)\dagger} \exp\left(i \vec{J}^{(j)} \cdot \vec{\phi} \right) U^{(j)}$$
$$= \pm U^{(j)\dagger} \exp\left(i (J_x^{(j)} \phi_x + J_y^{(j)} \phi_y + J_z^{(j)} \phi_z) \right) U^{(j)}$$
$$= \pm \exp\left(-i (J_x^{(j)} \phi_x - J_y^{(j)} \phi_y + J_z^{(j)} \phi_z) \right) \qquad (6.11)$$
$$= D_{SU(2)\times P}^{(j)}(w)^*$$
$$= \tilde{D}_{SU(2)\times P}^{(j)}(w).$$

This directly means that the $SU(2) \times P$'s irreducible representation in $\{|j, m, p\rangle\}$, and $\{\theta |j, m, p\rangle\}$ basis are equivalent. This implies that the corepresentation of $SU(2) \times P \times TR$ for $w \in SU(2) \times P$ is

$$D_{SU(2)\times P \times TR}^{(j)}(w) = \begin{bmatrix} D_{SU(2)\times P}^{(j)}(w) & 0 \\ 0 & U^{(j)\dagger} D_{SU(2)\times P}^{(j)}(w) U^{(j)} \end{bmatrix} \qquad (6.12)$$

where the first half of the basis is $\{|j, m, p\rangle\}$ while the second half is $\{\theta |j, m, p\rangle\}$.

In order to get the corepresentation for θ, we know that, since $\theta^{-1} w \theta$, θ^{-2} and θ^2 are unitary,

$$\tilde{D}_{SU(2)\times P}^{(j)}(\theta^{-1} w \theta) = D_{SU(2)\times P}^{(j)}(\theta^{-2} w \theta^2)^*. \qquad (6.13)$$

However, this exact thing is also equal

$$\tilde{D}_{SU(2)\times P}^{(j)}(\theta^{-1} w \theta) = U^{(j)\dagger} D_{SU(2)\times P}^{(j)}(\theta^{-1} w \theta) U^{(j)}. \qquad (6.14)$$

Hence,

$$D_{SU(2)\times P}^{(j)}(\theta^{-2} w \theta^2) = U^{(j)\dagger *} D_{SU(2)\times P}^{(j)}(\theta^{-1} w \theta)^* U^{(j)*}$$
$$D_{SU(2)\times P}^{(j)}(\theta^2)^{-1} D_{SU(2)\times P}^{(j)}(w) D_{SU(2)\times P}^{(j)}(\theta^2) = U^{(j)\dagger *} U^{(j)\dagger} D_{SU(2)\times P}^{(j)}(w) U^{(j)} U^{(j)*} \qquad (6.15)$$
$$U^{(j)} U^{(j)*} D_{SU(2)\times P}^{(j)}(\theta^2)^{-1} D_{SU(2)\times P}^{(j)}(w) = D_{SU(2)\times P}^{(j)}(w) U^{(j)} U^{(j)*} D_{SU(2)\times P}^{(j)}(\theta^2)^{-1}.$$

From Schur's lemma, since $D^{(j)}_{SU(2) \times P}$ is an irreducible representation, there is a constant $\lambda \in \mathbb{C}$ such that

$$\lambda U^{(j)} U^{(j)*} = D^{(j)}_{SU(2) \times P}(\theta^2). \tag{6.16}$$

With similar process but substitute $w = \theta^2$,

$$D^{(j)}_{SU(2) \times P}(\theta^2)^* = U^{(j)\dagger} D^{(j)}_{SU(2) \times P}(\theta^2) U^{(j)}. \tag{6.17}$$

This, together with the previous one, gives

$$\lambda^* U^{(j)} U^{(j)*} = D^{(j)}_{SU(2) \times P}(\theta^2). \tag{6.18}$$

Hence, $\lambda \in \mathbb{R}$. Moreover, the unitarity of $U^{(j)}$, and $SU(2) \times P$'s irreducible representation, restricts $\lambda = \pm 1$. With all these results, one now can write the corepresentation of $SU(2) \times P \times TR$ for $\theta \in T$ as

$$\begin{aligned}
D^{(j)}_{SU(2) \times P \times TR}(\theta) &= \begin{bmatrix} 0 & D^{(j)}_{SU(2) \times P}(\theta^2) \\ 1 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & \lambda U^{(j)} U^{(j)*} \\ 1 & 0 \end{bmatrix}.
\end{aligned} \tag{6.19}$$

One can further simplify the representation by changing the basis $\{\theta \, |j, m, p\rangle\}$ with $U^{(j)}$,

$$\begin{aligned}
D^{(j)}_{SU(2) \times P \times TR}(w) &= \begin{bmatrix} D^{(j)}_{SU(2) \times P}(w) & 0 \\ 0 & D^{(j)}_{SU(2) \times P}(w) \end{bmatrix} \\
D^{(j)}_{SU(2) \times P \times TR}(\theta) &= \begin{bmatrix} 0 & \lambda U^{(j)} \\ U^{(j)} & 0 \end{bmatrix}.
\end{aligned} \tag{6.20}$$

From definition of $U^{(j)}$, the representation $D^{(j)}_{SU(2) \times P \times TR}(\theta)$ can either be symmetric or antisymmetric antidiagonal matrix. We can see that

$$\begin{aligned}
\left[U^{(j)}\right]_{rq} &= i^{2(r-j)} \delta_{r, 2j-q} \\
&= i^{2(j-q)} \delta_{q, 2j-r} \\
&= i^{4(j-q)} \left[U^{(j)}\right]_{qr} \\
&= (-1)^{2j} \left[U^{(j)}\right]_{qr}.
\end{aligned} \tag{6.21}$$

In other words, $D^{(j)}_{SU(2) \times P \times TR}(\theta)$ is symmetric when $(-1)^{2j} \lambda = 1$, and antisymmetric when $(-1)^{2j} \lambda = -1$.

6.3. Implementation