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Adding fluxes to consistent truncations: IIB supergravity on $\mathrm{AdS}_3 \times S^3 \times S^3 \times S^1$

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Abstract

We use $E_{8(8)}$ Exceptional Field Theory to construct the consistent truncation of IIB supergravity on $S^3 \times S^3 \times S^1$ to maximal 3-dimensional $\mathcal{N} = 16$ gauged supergravity containing the $\mathcal{N} = (4, 4)$ AdS_3 vacuum. We explain how to achieve this by adding a 7-form flux to the S^1 reduction of the dyonic $E_{7(7)}$ truncation on $S^3 \times S^3$ previously constructed in the literature. Our truncation Ansatz includes, in addition to the $\mathcal{N} = (4, 4)$ vacuum, a host of moduli breaking some or all of the supersymmetries. We explicitly construct the uplift of a subset of these to construct new supersymmetric and non-supersymmetric AdS_3 vacua of IIB string theory, and show how the moduli space of AdS_3 vacua of six-dimensional gauged supergravity studied in [1] is compactified upon lifting to 10 dimensions. Along the way, we also derive the form of 3-dimensional $\mathcal{N} = 16$ gauged supergravity in terms of the embedding tensor and rule out a 10-/11-dimensional origin of some 3-dimensional gauged supergravities.

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To do:

- Think about title - mention new AdS_3 vacua? Lifting AdS_3 vacua?
- Check the $\mathrm{SL}(n)$ equations
- Move 3-d gauged SUGRA to somewhere else??
- Construct 10-parameter uplift
- Complex geometry of moduli space?
- Compare with classification known in literature, e.g. in IIA (and IIB – Jerome's story)
- Email DW and MP to ask who we should cite for (5.1)

1 Introduction

Consistent truncations are a powerful technique that simplifies the dynamics of 10-/11-dimensional supergravity by focusing on a restricted subset of fields. The key principle behind a consistent truncation is to truncate to a subsector of fields, such that all solutions of the truncated theory correspond to solutions of the original 10-/11-dimensional supergravity theory. By considering only a, typically finite, subset of fields, consistent truncations provide a powerful tool to find complicated new 10-/11-dimensional supergravity solutions and to study their deformations. Consistent truncations have proven particularly powerful in the AdS/CFT correspondence, since all well-understood AdS vacua of string theory do not admit scale-separation and, thus, cannot be studied using the usual tools of lower-dimensional effective theories.

Exceptional Field Theory (ExFT) is a reformulation of 10-/11-dimensional supergravity that unifies the metric and flux degrees of freedom and thereby makes manifest an exceptional symmetry group[1]. Over the last decade, this has proven an extremely useful tool for constructing consistent truncations, leading to a number of new examples preserving various amounts of supersymmetry [2–18]. However, consistent truncations to three dimensions have, until recently [19], remained largely unexplored. One reason is that the $E_{8(8)}$ ExFT requires modifications for its local symmetry structure, i.e. the generalised Lie derivative, to close into an algebra [20, 21]. Another is that there are no maximally supersymmetric AdS_3 vacua, i.e. preserving 32 supercharges, of string theory, leaving no natural candidate for constructing a consistent truncation with vacua. Indeed, while the consistent truncations of 11-dimensional supergravity on S^7 and S^4 and of 10-dimensional supergravity on S^5 contain maximally supersymmetric AdS vacua and are captured by a universal Ansatz in ExFT [2, 3], the analogous S^8 truncation of 11-dimensional supergravity does not exist, while there are two S^7 truncations of 10-dimensional supergravity but neither contains a maximally symmetric vacuum [19, 22].

Nonetheless, there are half-maximal, i.e. $\mathcal{N} = (4,4)$, supersymmetric AdS_3 vacua of string theory that are extremely intriguing. These are the $\mathrm{AdS}_3 \times S^3 \times S^3 \times S^1$ and $\mathrm{AdS}_3 \times S^3 \times T^4$ of IIB string theory (realising the "large" and "small" $\mathcal{N} = (4,4)$ superconformal symmetries), and, unlike in higher dimensions, can be supported by pure NS-NS flux. Not only does this mean that

[1] For the purposes of this paper, we do not draw a distinction between ExFT and Exceptional Generalised Geometry, since these agree when the "section condition" is solved, which we will always assume here.

there are also heterotic versions of these vacua (preserving $\mathcal{N} = (4,0)$ supersymmetry), but also that they can be readily studied via the string worldsheet CFT [23–26]. Moreover, the string sigma model in these vacua is also integrable []. Finally, the $\mathrm{AdS}_3/\mathrm{CFT}_2$ correspondence has seen much progress recently, with many new holographic duals being established [].

Here we will focus on the $\mathcal{N} = (4,4)$ $\mathrm{AdS}_3 \times S^3 \times S^3 \times S^1$ vacuum of IIB string theory and show that it admits a consistent truncation to a maximal gauged supergravity, which was first studied in [27]. All vacua of this theory break at least half the supersymmetries, reflecting the absence of a maximally supersymmetric AdS_3 vacuum in string theory. Key to constructing this consistent truncation is to add new fluxes to the S^1 reduction of the consistent truncation of IIB supergravity on $S^3 \times S^3$ [11].

Using our consistent truncation, we will show that the $\mathrm{AdS}_3 \times S^3 \times S^3 \times S^1$ has a rich moduli space of symmetry- and supersymmetry-breaking deformations, including some that are analogous to the "flat deformations" studied for $\mathrm{AdS}_4 \times S^5 \times S^1$ [28–31]. While some of these deformations preserve some amounts of supersymmetry, others break all supersymmetries, yet, surprisingly, at least a subset of vacua continue to be perturbatively stable within IIB supergravity, similar to [31]. Our work also provides an uplift of 6-dimensional supergravity on S^3 studied in [1] to IIB string theory. As we will show, some deformations that appear non-compact in 6-dimensional supergravity [1] are compactified within the full 10-dimensional supergravity theory.

Another technical result of our work is the derivation of the potential of 3-dimensional gauged supergravity [32, 33] in terms of the embedding tensor, which was previously only expressed in terms of fermion shift matrices. As a by-product, we find that a constraint that must be obeyed by the embedding tensor of any 3-dimensional gauged supergravity that can be uplifted to 10-/11-dimensional supergravity. As we show, this allows us to prove a higher-dimensional origin of some 3-dimensional gaugings.

The outline of our paper is as follows. We begin with a review of $E_{8(8)}$ ExFT in section 2 and how to derive consistent truncations in this formalism in 3. In section 4 we derive the potential of 3-dimensional gauged supergravity in terms of the embedding tensor and prove the lack of higher-dimensional origin of some 3-dimensional theories. Then, we show how to add fluxes to consistent truncations in 5, allowing us to construct the consistent truncation of IIB supergravity on $S^3 \times S^3 \times S^1$. Using this truncation, we study the moduli space of the AdS_3 vacua in section 6, before concluding in section 7.

2 Review of $E_{8(8)}$ exceptional field theory

The $E_{8(8)}$ exceptional field theory, first constructed in ref. [20], is an $E_{8(8)}$ duality-covariant formulation of type II and 11d supergravities. It is defined on a set of $3 + 248$ coordinates made of three-dimensonal external coordinates $\{x^\mu\}$ and internal coordinates $\{Y^M\}$ in the 248-dimensional adjoint representation of $E_{8(8)}$. The dependance of the fields on these coordinates is constrained by the "section constraints"[2]

$$\begin{cases} \eta^{MN}\partial_M \otimes \partial_N = 0\,, \\ f^{MN}{}_P\, \partial_M \otimes \partial_N = 0\,, \\ (\mathbb{P}_{3875})_{MN}{}^{KL}\partial_K \otimes \partial_L = 0\,, \end{cases} \tag{2.1}$$

[2]We use the notation $\otimes$ to indicate that both derivatives may act on different functions.

where $f_{MN}{}^P$ are the structure constants of $E_{8(8)}$, $\eta_{MN} = \frac{1}{60}\, f_{MK}{}^L f_{NL}{}^K$ its Cartan-Killing metric and $(\mathbb{P}_{3875})_{MN}{}^{KL}$ is the projector on the representation **3875**:

$$(\mathbb{P}_{3875})_{MN}{}^{KL} = \frac{1}{7}\, \delta_{(M}{}^K \delta_{N)}{}^L - \frac{1}{56}\, \eta_{MN}\eta^{KL} - \frac{1}{14}\, f^P{}_M{}^{(K} f_{PN}{}^{L)} \,. \tag{2.2}$$

Here and in the following, $E_{8(8)}$ indices are raised and lowered by the Cartan-Killing metric η_{MN}. The section constraints (2.1) ensure that the fields depends only on the 7- or 8-dimensional physical internal coordinates embeded in Y^M.

The theory describes the dynamics of the following bosonic fields:

$$\left\{ g_{\mu\nu}, \mathcal{M}_{MN}, A_\mu{}^M, B_{\mu\, M} \right\} \,, \tag{2.3}$$

with $g_{\mu\nu}$ the 3-dimensinal external metric, $\mathcal{M}_{MN}$ the generalized metric parametrizing the coset space $E_{8(8)}/\mathrm{SO}(16)$ and the gauge fields $A_\mu{}^M$ and $B_{\mu\, M}$. It is a gauge theory, invariant under the generalized Lie derivative of parameters $\Upsilon = (\Lambda^M, \Sigma_M)$, whose action on a vector V^M of weight λ is given by

$$\mathcal{L}_\Upsilon V^M = \Lambda^N \partial_N V^M - 60\, (\mathbb{P}_{248})^M{}_N{}^K{}_L\, V^N \partial_K \Lambda^L + \lambda\, V^M \partial_N \Lambda^N + f^{MN}{}_K \Sigma_N V^K \,, \tag{2.4}$$

with $(\mathbb{P}_{248})^M{}_N{}^K{}_L = (1/60)\, f^M{}_{NP} f^{PK}{}_L$ the projector on the adjoint representation. These transformations are well-defined (in particular, they close into an algebra) only if the parameters Σ_M and the fields $B_{\mu\, M}$ are covariantly constrained: they have to satisfy algebraic constraints similar to eq. (2.1) and be compatibile with the partial derivatives. We require that

$$\begin{cases} \eta^{MN} C_M \otimes C'_N = 0 \,, \\ f^{MN}{}_P\, C_M \otimes C'_N = 0 \,, \\ (\mathbb{P}_{3875})_{MN}{}^{KL} C_K \otimes C'_L = 0 \,, \end{cases} \qquad \forall\; C_M, C'_M \in \{\partial_M, \Sigma_M, B_{\mu\, M}\} \,. \tag{2.5}$$

The bosonic action of $E_{8(8)}$ exceptional field theory is invariant under the transformations (2.4) and has the expression

$$S_{\mathrm{ExFT}} = \int d^3x\, d^{248}Y\, \sqrt{|g|} \left(\widehat{R} + \frac{1}{240}\, D_\mu \mathcal{M}_{MN} D^\mu \mathcal{M}^{MN} + \mathscr{L}_{\mathrm{int}} + \frac{1}{\sqrt{|g|}}\, \mathscr{L}_{\mathrm{CS}} \right) . \tag{2.6}$$

$\widehat{R}$ is the $E_{8(8)}$-covariantised Ricci scalar and the covariant derivative is defined as

$$D_\mu = \partial_\mu - \mathcal{L}_{(A_\mu, B_\mu)} \,. \tag{2.7}$$

$\mathscr{L}_{\mathrm{int}}$ is a potential term depending only on internal derivatives, explicitly

$$\begin{aligned} \mathscr{L}_{\mathrm{int}} = {} & \frac{1}{240}\, \mathcal{M}^{MN} \partial_M \mathcal{M}^{KL} \partial_N \mathcal{M}_{KL} - \frac{1}{2}\, \mathcal{M}^{MN} \partial_M \mathcal{M}^{KL} \partial_L \mathcal{M}_{NK} \\ & - \frac{1}{7200}\, f^{NQ}{}_P f^{MS}{}_R\, \mathcal{M}^{PK} \partial_M \mathcal{M}_{QK}\, \mathcal{M}^{RL} \partial_N \mathcal{M}_{SL} \\ & + \frac{1}{2}\, g^{-1} \partial_M g\, \partial_N \mathcal{M}^{MN} + \frac{1}{4}\, \mathcal{M}^{MN}\, g^{-2} \partial_M g\, \partial_N g + \frac{1}{4}\, \mathcal{M}^{MN}\, \partial_M g_{\mu\nu}\, \partial_N g^{\mu\nu} \,. \end{aligned} \tag{2.8}$$

Finally, $\mathscr{L}_{\rm CS}$ is a Chern-Simons term required to impose the on-shell duality between scalar and vector fields, given by

$$\begin{aligned}\mathscr{L}_{\rm CS} = \frac{1}{2}\,\varepsilon^{\mu\nu\rho}\Big(&F_{\mu\nu}{}^{M}B_{\rho\,M} - f_{KL}{}^{N}\,\partial_\mu A_\nu{}^{K}\partial_N A_\rho{}^{L} - \frac{2}{3}\,f^{N}{}_{KL}\,\partial_M\partial_N A_\mu{}^{K}A_\nu{}^{M}A_\rho{}^{L}\\ &-\frac{1}{3}\,f_{MKL}\,f^{KP}{}_{Q}\,f^{LR}{}_{S}\,A_\mu{}^{M}\partial_P A_\nu{}^{Q}\partial_R A_\rho{}^{S}\Big),\end{aligned} \tag{2.9}$$

where $F_{\mu\nu}{}^{M}$ is the covariant field strength of $A_\mu{}^{M}$ (see eq. (2.26) of ref. [20]).

3 Consistent truncations to 3-dimensional $\mathcal{N}=16$ gauged supergravity

The $E_{8(8)}$ exceptional field theory is well suited to the construction of consistent truncation of type II and 11d supergravities to three-dimensional maximal supergravity. These truncations arise as generalised Scherk-Schwarz reductions, described by an $E_{8(8)}$-valued twist matrix $U_M{}^{\overline{M}}$ and a scale factor ρ. The truncation Ansätze are as follows [19]:

$$\begin{aligned}g_{\mu\nu}(x,Y) &= \rho(Y)^{-2}\,\mathring{g}_{\mu\nu}(x)\,,\\ \mathcal{M}_{MN}(x,Y) &= U_M{}^{\overline{M}}(Y)\,U_N{}^{\overline{N}}(Y)\,\mathcal{M}_{\overline{MN}}(x)\,,\\ A_\mu{}^{M}(x,Y) &=\rho(Y)^{-1}\,\left(U^{-1}\right)_{\overline{M}}{}^{M}(Y)\,\mathcal{A}_\mu{}^{\overline{M}}(x)\,,\\ B_{\mu\,M}(x,Y) &= \Sigma_{\overline{M}\,M}(Y)\,\mathcal{A}_\mu{}^{\overline{M}}(x)\,,\end{aligned} \tag{3.1}$$

with

$$\Sigma_{\overline{M}\,M} = \frac{\rho^{-1}}{60}\,f_{\overline{M}}{}^{\overline{PQ}}\,\left(U^{-1}\right)_{\overline{P}P}\,\partial_M\,\left(U^{-1}\right)_{\overline{Q}}{}^{P}\,. \tag{3.2}$$

The fields with flat indices $\overline{M}$ (in the **248** representation of $E_{8(8)}$) belong to the three-dimensional theory; all the dependence on the internal manifold is factored out in $U_{\overline{M}}{}^{M}$ and ρ. Note that with the definition of $\Sigma_{\overline{M}\,M}$ the condition (2.5) is automatically satisfied.

The Ansätze (3.1) describe a *consistent* truncation if the following condition is satisfied:

$$\mathcal{L}_{\mathfrak{U}_{\overline{M}}}\mathcal{U}_{\overline{N}}{}^{M} = X_{\overline{MN}}{}^{\overline{P}}\,\mathcal{U}_{\overline{P}}{}^{M}\,, \tag{3.3}$$

with $\mathcal{U}_{\overline{M}}{}^{M} = \rho^{-1}\left(U^{-1}\right)_{\overline{M}}{}^{M}$, $\mathfrak{U}_{\overline{M}} = (\mathcal{U}_{\overline{M}},\,\Sigma_{\overline{M}})$ and $X_{\overline{MN}}{}^{\overline{P}}$ a constant tensor. This condition ensures that the factorised form of the Ansätze (3.1) is preserved by the generalized Lie derivative, *e.g.*

$$\begin{aligned}D_\mu\mathcal{M}(x,Y) &= U_M{}^{\overline{M}}(Y)\,U_N{}^{\overline{N}}(Y)\,\mathcal{D}_\mu\mathcal{M}_{\overline{MN}}(x)\,,\\ \mathcal{L}_{(A_\mu,B_\nu)}A_\mu(x,Y) &= \rho(Y)^{-1}\,\left(U^{-1}\right)_{\overline{M}}{}^{M}(Y)\,[\![\mathcal{A}_\mu,\mathcal{A}_\nu]\!]^{\overline{M}}(x)\,,\end{aligned} \tag{3.4}$$

where

$$\mathcal{D}_\mu\mathcal{M}_{\overline{MN}} = \partial_\mu\mathcal{M}_{\overline{MN}} - 2\,\mathcal{A}_\mu{}^{\overline{P}}X_{\overline{P}(\overline{M}}{}^{\overline{Q}}\mathcal{M}_{\overline{N})\overline{Q}}\,,\quad \text{and}\quad [\![\mathcal{A}_\mu,\mathcal{A}_\nu]\!]^{\overline{M}} = X_{\overline{PQ}}{}^{\overline{M}}\mathcal{A}_\mu{}^{\overline{P}}\mathcal{A}_\nu^{\overline{Q}}\,. \tag{3.5}$$

Thus, for a given twist matrix $U_M{}^{\overline{M}}$ satisfying the consistency condition (3.3), the action (2.6) reduces to three-dimensional $\mathcal{N}=16$ gauged supergravity [32, 33]. The constant tensor $X_{\overline{MN}}{}^{\overline{P}}$

plays the role of the embedding tensor of the three-dimensional gauged supergravity, as is clear from its appearance in (3.5) in the gauge-covariant derivative and the gauge algebra of the vector fields. From eq. (3.3), it has the following expression in terms of the twist matrix and ρ

$$\begin{aligned} X_{\overline{MN}}{}^{\overline{P}} &= -\rho^{-1}\,\Gamma_{\overline{MN}}{}^{\overline{P}} + \rho^{-1}\,f^{\overline{P}}{}_{\overline{NQ}}\,f^{\overline{QK}}{}_{\overline{L}}\,\Gamma_{\overline{KM}}{}^{\overline{L}} - \frac{1}{60}\,\rho^{-1}\,f^{\overline{PK}}{}_{\overline{N}}\,f_{\overline{ML}}{}^{\overline{Q}}\,\Gamma_{\overline{KQ}}{}^{\overline{L}} \\ &\quad - \frac{1}{2}\,\rho^{-1}\,f^{\overline{P}}{}_{\overline{NQ}}\,f^{\overline{QK}}{}_{\overline{M}}\,\Gamma_{\overline{RK}}{}^{\overline{R}} + \left(\delta_{\overline{M}}{}^{\overline{K}}\delta_{\overline{N}}{}^{\overline{P}} - \frac{1}{2}\,f_{\overline{M}}{}^{\overline{LK}}f_{\overline{NL}}{}^{\overline{P}}\right)\,\xi_{\overline{K}}\,. \end{aligned} \tag{3.6}$$

Here, we defined the $\mathrm{E}_{8(8)}$ current $\Gamma_{\overline{MN}}{}^{\overline{P}} = \left(U^{-1}\right)_{\overline{M}}{}^{K}\left(U^{-1}\right)_{\overline{N}}{}^{L}\partial_K U_L{}^{\overline{P}}$ and the trombone gauging

$$\xi_{\overline{M}} = 2\,U_{\overline{M}}{}^{N}\partial_N\rho^{-1} - \rho^{-1}\,\Gamma_{\overline{NM}}{}^{\overline{N}}\,. \tag{3.7}$$

The embedding tensor is most nicely expressed once projected on the adjoint representation

$$\begin{aligned} X_{\overline{MN}} = -\frac{1}{60}\,X_{\overline{MP}}{}^{\overline{Q}}\,f_{\overline{NQ}}{}^{\overline{P}} &= -2\,\rho^{-1}\,\Gamma_{(\overline{MN})} - \rho^{-1}\,\Gamma_{\overline{P}(\overline{M}}{}^{\overline{Q}}\,f_{\overline{N})\overline{Q}}{}^{\overline{P}} + \frac{1}{2}f_{\overline{MN}}{}^{\overline{P}}\,\xi_{\overline{P}} \\ &= -14\,\rho^{-1}\,(\mathbb{P}_{3875})_{\overline{MN}}{}^{\overline{PQ}}\,\Gamma_{\overline{PQ}} - \frac{1}{4}\,\rho^{-1}\,\eta_{\overline{MN}}\,\Gamma_{\overline{P}}{}^{\overline{P}} + \frac{1}{2}f_{\overline{MN}}{}^{\overline{P}}\,\xi_{\overline{P}}\,, \end{aligned} \tag{3.8}$$

with the projection of the current $\Gamma_{\overline{MN}} = -\frac{1}{60}\,\Gamma_{\overline{MP}}{}^{\overline{Q}}\,f_{\overline{NQ}}{}^{\overline{P}}$. In the following, we will focus on gaugings with $\xi_{\overline{M}} = 0$.

The three-dimensional action follows from inserting the Ansätze (3.1) in the ExFT action (2.6). The expressions of the kinetic and Chern-Simons terms result immediately from eq. (3.4)

$$\begin{aligned} \mathscr{L}_{\rm kin} &= \frac{1}{240}\,D_\mu\mathcal{M}_{MN}D^\mu\mathcal{M}^{MN} \underset{\rm gSS}{=} \frac{\rho^2}{240}\,\mathcal{D}_\mu\mathcal{M}_{\overline{MN}}\mathcal{D}^\mu\mathcal{M}^{\overline{MN}}\,, \\ \mathscr{L}_{\rm CS} &\underset{\rm gSS}{=} -\rho^{-1}\,\varepsilon^{\mu\nu\rho}\,X_{\overline{MN}}\,\mathcal{A}_\mu{}^{\overline{M}}\left(\partial_\nu\mathcal{A}_\rho{}^{\overline{N}} - \frac{1}{3}\,[\![\mathcal{A}_\nu,\mathcal{A}_\rho]\!]^{\overline{N}}\right)\,. \end{aligned} \tag{3.9}$$

However, the case of the potential term $\mathscr{L}_{\rm int}$, leading to a potential V for the three-dimensional scalars $\mathcal{M}_{\overline{MN}}$, is more subtle. We expect V to be quadratic in the embedding tensor, which makes the identification of $X_{\overline{MN}}$ more complicated. Moreover, the potential of three-dimensional $\mathcal{N} = 16$ gauged supergravity in terms of the embedding tensor is unknown, with currently the potential only expressed in terms of the fermion shift matrices of the gauged supergravity [32, 33]. Thus, in the following, we will follow the truncation procedure carefully and thereby construct the potential of the three-dimensional $\mathcal{N} = 16$ gauged supergravity in terms of the embedding tensor.

4 Deriving the potential of $\mathcal{N} = 16$ gauged supergravity

Here we will use the $E_{8(8)}$ ExFT Lagrangian and the consistent truncation Ansatz (3.1) to derive the potential of three-dimensional $\mathcal{N} = 16$ gauged supergravity [32, 33] in terms of the embedding tensor. Not only is this an interesting application of ExFT, relying on purely bosonic considerations and bypassing the usual construction of the gauged supergravity potential using supersymmetry[3],

[3]Note that the same strategy was recently used to impressively derive the potential of maximal two-dimensional gauged supergravity [34], where fermions are extremely poorly understood. As a result, in that case, the bosonic $E_{9(9)}$ ExFT [35, 36] provides the only currently accessible route for computing the gauged supergravity potential.

but the potential is crucial for finding vacua of the three-dimensional theory and uplifting these in the later parts of this paper.

In order to derive the potential, we adopt the following strategy. We want $\mathscr{L}_{\text{int}}$ (2.8) to reduce to the embedding tensor squared upon inserting the generalised Scherk-Schwarz Ansätze (3.1). However, as in higher dimensions, e.g. [37–39], this does not have to match identically, but only up to total derivative terms and terms which violate the section condition. The possible boundary terms are

$$\frac{1}{\sqrt{|g|}}\partial_M\left(\sqrt{|g|}\partial_N\mathcal{M}^{MN}\right)\quad\text{and}\quad\frac{1}{\sqrt{|g|}}\partial_M\left(\mathcal{M}^{MN}\partial_N\sqrt{|g|}\right). \tag{4.1}$$

However, since the $\mathcal{N}=16$ supergravity potential will be given by terms quadratic in $X_{\overline{MN}}{}^{\overline{P}}$, we do not want the boundary terms to involve any double derivative terms. Therefore, the two boundary terms (4.1) can only appear via the combination

$$\frac{1}{\sqrt{|g|}}\,\partial_M\left(\sqrt{|g|}\,\partial_N\mathcal{M}^{MN}+\frac{4}{3}\,\mathcal{M}^{MN}\partial_N\sqrt{|g|}\right). \tag{4.2}$$

Now, we first insert the generalized Sherk-Schwarz Ansätze (3.1) into the expression of $\mathscr{L}_{\text{int}}$ (2.8) with the additional total derivative (4.2)

$$\begin{aligned}
&\mathscr{L}_{\text{int}}+\frac{a}{\sqrt{|g|}}\,\partial_M\left(\sqrt{|g|}\,\partial_N\mathcal{M}^{MN}+\frac{4}{3}\,\mathcal{M}^{MN}\partial_N\sqrt{|g|}\right)\\
&\underset{\text{gSS}}{=}-\frac{1}{2}\,\mathcal{M}^{\overline{MN}}\,\Gamma_{\overline{MK}}\Gamma_{\overline{N}}{}^{\overline{K}}-\mathcal{M}^{\overline{MN}}\,\Gamma_{\overline{MK}}\Gamma^{\overline{K}}{}_{\overline{N}}-\frac{1}{2}\,\Gamma_{\overline{MN}}\Gamma^{\overline{NM}}-\frac{1}{2}\,\mathcal{M}^{\overline{MN}}\mathcal{M}^{\overline{KL}}\,\Gamma_{\overline{MK}}\Gamma_{\overline{NL}}\\
&\quad-\frac{1}{2}\,\mathcal{M}^{\overline{MN}}\mathcal{M}^{\overline{KL}}\,\Gamma_{\overline{MK}}\Gamma_{\overline{LN}}+\frac{3}{2}\,(a-1)\,\mathcal{M}^{\overline{MN}}\Gamma_{\overline{MN}}{}^{\overline{K}}\Gamma_{\overline{LK}}{}^{\overline{L}}-\frac{a}{2}\,\mathcal{M}^{\overline{MN}}\Gamma_{\overline{KM}}{}^{\overline{K}}\Gamma_{\overline{LN}}{}^{\overline{L}}\\
&\quad+\left(\frac{1}{2}-a\right)\,\mathcal{M}^{\overline{MN}}\Gamma_{\overline{KM}}{}^{\overline{L}}\Gamma_{\overline{LN}}{}^{\overline{K}}+\mathcal{M}^{\overline{MN}}\Gamma_{\overline{MK}}{}^{\overline{L}}\Gamma_{\overline{LN}}{}^{\overline{K}}-\frac{1}{2}\,\mathcal{M}^{\overline{MN}}\mathcal{M}^{\overline{KL}}f_{\overline{QL}}{}^{\overline{P}}\Gamma_{\overline{PN}}{}^{\overline{Q}}\Gamma_{\overline{MK}}\,.
\end{aligned} \tag{4.3}$$

Secondly, knowing that the $\mathcal{N}=16$ potential must be quadratic in $X_{\overline{MN}}$, we consider the most general quadratic function in $X_{\overline{MN}}$ and develop it in $E_{8(8)}$ currents using eq. (3.8). Note that in higher dimensions, there exist only two quadratic combinations of the embedding tensor

$$X_{\overline{MN}}{}^{\overline{P}}X_{\overline{QP}}{}^{\overline{N}}\mathcal{M}^{\overline{MQ}}\qquad\text{and}\qquad X_{\overline{MN}}{}^{\overline{P}}X_{\overline{QR}}{}^{\overline{S}}\mathcal{M}^{\overline{MQ}}\mathcal{M}^{\overline{NR}}\mathcal{M}_{\overline{PS}}\,. \tag{4.4}$$

However, in three dimensions, because the vector fields transform in the adjoint representation of $E_{8(8)}$, we can write two additional terms

$$X_{\overline{MN}}X_{\overline{PQ}}\,\eta^{\overline{MP}}\eta^{\overline{NQ}}\qquad\text{and}\qquad X_{\overline{MN}}X_{\overline{PQ}}\,\eta^{\overline{MN}}\eta^{\overline{PQ}}\,, \tag{4.5}$$

where the second term corresponds to the square of the singlet part of the embedding tensor.

It turns out that for gaugings that arise from consistent truncations, these two terms (4.5) can be related to the square of the trombone tensor. We can square eq. (3.8) to derive an expression for $X_{\overline{MN}}X^{\overline{MN}}$ in terms of the current Γ. We find check coefficient of last term

$$X_{\overline{MN}}X^{\overline{MN}}=21\rho^{-2}\,\Gamma_{\overline{MN}}\,\Gamma^{\overline{NM}}+19\rho^{-2}\left(\Gamma_{\overline{M}}{}^{\overline{M}}\right)^2-15\,\xi_{\overline{M}}\,\xi^{\overline{M}}\,. \tag{4.6}$$

Similarly, squaring the trombone tensor ξ gives

$$\xi_{\overline{M}}\,\xi^{\overline{M}} = \rho^{-2}\left(\Gamma_{\overline{M}}{}^{\overline{M}}\Gamma_{\overline{N}}{}^{\overline{N}} + \Gamma_{\overline{MN}}\Gamma^{\overline{NM}}\right) . \tag{4.7}$$

We can now deduce

$$X_{\overline{MN}}\,X^{\overline{MN}} = 6\,\xi_{\overline{M}}\,\xi^{\overline{M}} - 2\,\rho^{-2}\left(\Gamma_{\overline{M}}{}^{\overline{M}}\right)^2 , \tag{4.8}$$

which can be expressed entirely in terms of $X_{\overline{MN}}$ after tracing eq. (3.8)

$$X_{\overline{MN}}\,X^{\overline{MN}} = 6\,\xi_{\overline{M}}\,\xi^{\overline{M}} - \frac{1}{1922}\left(X_{\overline{M}}{}^{\overline{M}}\right)^2 . \tag{4.9}$$

Thus, (4.9) is a constraint on the embedding tensor of three-dimensional supergravity which must be satisfied in order for it to have an uplift to 10-/11-dimensional supergravity. Note that this constraint is not implied by the quadratic constraint. To see this, simply observe that the theory with $E_{8(8)}$ gauging has $X_{\overline{MN}} = \eta_{\overline{MN}}$ and $\xi_{\overline{M}} = 0$ [32,33], violating (4.9).

Since we are focusing on gaugings with vanishing trombone $\xi_{\overline{M}} = 0$, (4.9) implies that the two terms in (4.5) are proportional and we only need to consider one of these terms in the gauged supergravity potential. Thus, the most general Ansatz for the supergravity potential, for gaugings that have an uplift to 10-/11-dimensional supergravity[4], is

$$\begin{aligned}
-V &= X_{\overline{MN}}X_{\overline{PQ}}\left(\alpha\,\mathcal{M}^{\overline{MP}}\mathcal{M}^{\overline{NQ}} + \beta\,\mathcal{M}^{\overline{MP}}\eta^{\overline{NQ}} + \delta\,\eta^{\overline{MP}}\eta^{\overline{NQ}}\right) \\
&\underset{\text{gSS}}{=} \rho^{-2}\Big(14\,\alpha\,\Gamma_{\overline{MN}}\,\Gamma_{\overline{PQ}}\,\mathcal{M}^{\overline{MP}}\mathcal{M}^{\overline{NQ}} + 14\,\alpha\,\Gamma_{\overline{MN}}\,\Gamma_{\overline{PQ}}\,\mathcal{M}^{\overline{MQ}}\mathcal{M}^{\overline{NP}} \\
&\quad + 2\,\beta\,\mathcal{M}^{\overline{MN}}\,\Gamma_{\overline{MK}}\Gamma^{\overline{K}}{}_{\overline{N}} + \beta\,\mathcal{M}^{\overline{MN}}\,\Gamma_{\overline{MK}}\,\Gamma_{\overline{N}}{}^{\overline{K}} + (-12\,\alpha + 2\,\delta)\;\Gamma_{\overline{MN}}\,\Gamma^{\overline{MN}} \\
&\quad - 2\,\beta\,\mathcal{M}^{\overline{MN}}\Gamma_{\overline{MK}}{}^{\overline{L}}\,\Gamma_{\overline{LN}}{}^{\overline{K}} + \beta\,\mathcal{M}^{\overline{MN}}\Gamma_{\overline{KM}}{}^{\overline{L}}\,\Gamma_{\overline{LN}}{}^{\overline{K}} + \left(7\,\alpha + \frac{\beta}{2}\right)\mathcal{M}^{\overline{MN}}\Gamma_{\overline{KM}}{}^{\overline{K}}\,\Gamma_{\overline{LN}}{}^{\overline{L}} \\
&\quad + 14\,\alpha\,\mathcal{M}^{\overline{MN}}\mathcal{M}^{\overline{KL}}f_{\overline{QL}}{}^{\overline{P}}\Gamma_{\overline{PN}}{}^{\overline{Q}}\,\Gamma_{\overline{MK}}\Big) .
\end{aligned} \tag{4.10}$$

Finally, by requiring

$$\sqrt{|g|}\mathscr{L}_{\text{int}} \underset{\text{gSS}}{=} -\sqrt{|\mathring{g}|}V , \tag{4.11}$$

the parameters in (4.3) and (4.10) are fixed to

$$\alpha = \frac{1}{28}, \quad \beta = \frac{1}{2}, \quad a = 1 , \quad \delta = \frac{13}{28} . \tag{4.12}$$

With the generalized Scherk-Schwarz Ansätze (3.1), the action (2.6) becomes

$$\begin{aligned}
S_{\text{ExFT}} \underset{\text{gSS}}{=} \int \mathrm{d}^{248}Y\,\rho^{-1}\int \mathrm{d}^3x\Big[&\sqrt{|\mathring{g}|}\left(\mathring{R} + \frac{1}{240}\,\mathcal{D}_\mu\mathcal{M}_{\overline{MN}}\mathcal{D}^\mu\mathcal{M}^{\overline{MN}} - V\right) \\
&- \varepsilon^{\mu\nu\rho}\,X_{\overline{MN}}\,\mathcal{A}_\mu{}^{\overline{M}}\left(\partial_\nu\mathcal{A}_\rho{}^{\overline{N}} - \frac{1}{3}\,[\![\mathcal{A}_\nu,\mathcal{A}_\rho]\!]^{\overline{N}}\right)\Big],
\end{aligned} \tag{4.13}$$

[4] If we do not require the three-dimensional gauged supergravity to arise from a consistent truncation, (4.9) may be violated and we need to include both terms of (4.5). However, here we are not interested in such gaugings.

with the potential

$$V = X_{\overline{MN}}\, X_{\overline{PQ}} \left(\frac{1}{28}\, \mathcal{M}^{\overline{MP}} \mathcal{M}^{\overline{NQ}} + \frac{1}{2}\, \mathcal{M}^{\overline{MP}} \eta^{\overline{NQ}} + \frac{13}{28}\, \eta^{\overline{MP}} \eta^{\overline{NQ}} \right) . \tag{4.14}$$

As in higher dimensions, $X_{\overline{MN}}$ obtained from (3.3) automatically satisfies the linear constraint of the 3-dimensional maximal gauged supergravity [32, 33], and the section condition implies the quadratic constraints for $X_{\overline{MN}}$.

4.1 Ruling out gaugings

Let's check the $\mathrm{GL}(6) \times \mathrm{SO}(2,2)$ gauging (half-maximal embedded in maximal)....

5 Adding fluxes to consistent truncations

In order to construct the consistent truncation around the $\mathcal{N} = (4,4)$ $\mathrm{AdS}_3 \times S^3 \times S^3 \times S^1$ vacuum of IIB supergravity, a promising starting point is the dyonic $S^3 \times S^3$ truncation of IIB supergravity constructed in [11], and further reducing this on S^1. However, this will not give the correct AdS_3 vacuum since the truncation is missing the required 7-form flux on $S^3 \times S^3 \times S^1$. Indeed, the 3-dimensional gauged supergravity that would be obtained this way does not have any AdS_3 vacua. We can remedy this situation by defining a new consistent truncation by adding a 7-form flux to the one obtained from the $S^3 \times S^3$ reduction constructed in $E_{7(7)}$ ExFT.

This motivates the following question: Given a consistent truncation, i.e. $\mathcal{U}_{\overline{M}}{}^{M}$, satisfying (3.3), when can we add a new flux component of string theory to the compactification to obtain a new consistent truncation? Adding a new flux component to the truncation is equivalent to twisting the generalised frame as follows Email DW and MP to ask who we should cite for this?

$$\mathcal{U}_{\overline{M}}{}^{M} \longrightarrow \mathcal{U}'_{\overline{M}}{}^{M} = \mathcal{U}_{\overline{M}}{}^{N} \exp(C)_N{}^M \,, \tag{5.1}$$

where C denotes the $E_{8(8)}$ generator corresponding to the potential we want to add to the compactification. The effect of the twist (5.1) is that the generalised Lie derivative of $\mathcal{U}'$ satisfies

$$\mathcal{L}_{\mathcal{U}'_{\overline{M}}} \mathcal{U}'_{\overline{N}}{}^{M} = \left(\mathcal{L}_{\mathcal{U}_{\overline{M}}} \mathcal{U}_{\overline{N}}{}^{N} + F_{PQ}{}^{N}\, \mathcal{U}_{\overline{M}}{}^{P}\, \mathcal{U}_{\overline{N}}{}^{Q} \right) \exp(C)_N{}^M \,, \tag{5.2}$$

where $F_{MN}{}^{P}$ is a tensor in the $\mathbf{1} \oplus \mathbf{248} \oplus \mathbf{3875}$ of $E_{8(8)}$, i.e. the same representation as the embedding tensor, corresponding to the field strength of the potential C in (5.1). Using (3.3), we now have

$$\mathcal{L}_{\mathcal{U}'_{\overline{M}}} \mathcal{U}'_{\overline{N}}{}^{M} = \left(X_{\overline{MN}}{}^{\overline{P}} + \rho^{-1}\, F_{\overline{MN}}{}^{\overline{P}} \right) \mathcal{U}'_{\overline{P}}{}^{M} \,, \tag{5.3}$$

where we defined

$$F_{\overline{MN}}{}^{\overline{P}} \equiv (U^{-1})_{\overline{M}}{}^{M}\, (U^{-1})_{\overline{N}}{}^{N}\, U_P{}^{\overline{P}}\, F_{MN}{}^{P} \,, \tag{5.4}$$

and $X_{\overline{MN}}{}^{\overline{P}}$ is already constant. Therefore, we have a consistent truncation if and only if $\rho^{-1}\, F_{\overline{MN}}{}^{\overline{P}}$ is constant.

A particularly simple way of having constant $\rho^{-1}\, F_{\overline{MN}}{}^{\overline{P}}$ is to switch on fluxes which are stabilised by the twist matrix $U_M{}^{\overline{M}}$, which typically only lives in a subgroup $G \subset E_{8(8)}$. Therefore, in this case, we can simply tune the G-singlet components of the flux $F_{MN}{}^{P}$ to be proportional to ρ to obtain a new consistent truncation.

5.1 Adding fluxes to the S^3 truncation of 6-dimensional supergravity

As a warm-up for the $S^3 \times S^3 \times S^1$ truncation, let us demonstrate this methodology for the consistent truncation of $\mathcal{N} = (1,1)$ 6-dimensional supergravity on S^3, which was constructed in [1]. As discussed in [1], the consistent truncation of $\mathcal{N} = (1,1)$ 6-dimensional supergravity to 3-dimensional half-maximal gauged supergravity can be described using the SO(8,4) ExFT [40, 41]. On the other hand, the consistent truncation on S^3 is conveniently described by twist matrices living in $\mathrm{SL}(4) \simeq \mathrm{SO}(3,3)$ [2, 3, 6] I think the twist matrix of [2, 3, 6] coincide in the SL(4) case, right? Camille, do you agree? with the embedding $\mathrm{SO}(3,3) \subset \mathrm{SO}(4,4) \subset \mathrm{SO}(8,4)$. However, the twist matrix in [1] differs from this SO(3,3) twist matrix [2,3,6] by an additional parameter, λ, which gives rise to an external 3-form flux or, equivalently, a new internal 3-form flux.

5.1.1 The S^3 twist matrix

Move to appendix?!?!

Since we will use it throughout this paper, let us construct the $\mathrm{SL}(4) \simeq \mathrm{SO}(3,3)$ twist matrix that describes the consistent truncation on S^3. This can be viewed as a special case of a family of consistent truncations on S^n [2,3,6]. Here, we give a slightly different, but equivalent, form of this construction.

We use an $\mathrm{SL}(n+1)$ ExFT, like in [3], which encodes an n-dimensional metric and volume-form flux, i.e. Freund-Rubin compactifications, and thus gives a natural description of S^n compactifications. The $\mathrm{SL}(n+1)$ ExFT formally has coordinates in the antisymmetric representation of $\mathrm{SL}(n+1)$, $y^{IJ} = -y^{JI}$, with $I = 1, \ldots, n+1$, and similarly, generalised vector fields transform in the antisymmetric representation of $\mathrm{SL}(n+1)$. The generalised Lie derivative on a generalised tensor in the fundamental V^I of weight λ is given by

$$\mathcal{L}_\Lambda V^I = \frac{1}{2}\Lambda^{JK}\partial_{JK}V^I - V^J\partial_{JK}\Lambda^{IK} + \left(\frac{\lambda}{2} + \frac{1}{n+1}\right) V^I\partial_{JK}\Lambda^{JK} \,, \tag{5.5}$$

and similarly for other generalised tensors. Closure of the generalised Lie derivative (5.5) requires the section condition

$$\partial_{[IJ} \otimes \partial_{KL]} = 0 \,, \tag{5.6}$$

which restricts the dependence of all fields to a subset of physical coordinates.

We are interested in the maximal solutions of the section condition (5.6) which preserve a $\mathrm{SL}(n) \subset \mathrm{SL}(n+1)$ subgroup. Under this decomposition $\mathrm{SL}(n+1) \to \mathrm{SL}(n)$ with $V^I = \left(V^i,\, V^0\right)$, where $i = 1, \ldots, n$, we solve the section condition by having physical coordinates y^{i0} on the n-dimensional manifold M, i.e. $\partial_{ij} = 0$ for all fields with only $\partial_{i0} \neq 0$. Correspondingly, the generalised tangent bundle, whose sections are in the antisymmetric representation of $\mathrm{SL}(n+1)$ and carry weight $\frac{n-3}{n+1}$, decomposes as

$$E = TM \oplus \Lambda^{n-2}T^*M \,, \tag{5.7}$$

Note that for the case of interest to us, $n = 3$, (5.7) reduces to

$$E = TM \oplus T^*M \,, \tag{5.8}$$

and its fibres transform in the **6** of $\mathrm{SL}(4) \simeq \mathrm{SO}(3,3)$. It is convenient to also introduce the generalised bundle with fibres in the anti-fundamental of $\mathrm{SL}(n+1)$, which is, similarly, given by

$$N = T^*M \oplus \Lambda^n T^*M \,. \tag{5.9}$$

Sections of N are generalised tensors transforming in the anti-fundamental of $\mathrm{SL}(n+1)$ and carrying weight $\frac{2}{n+1}$. For $V, V' \in \Gamma(E)$ and $W \in \Gamma(N)$, the generalised Lie derivative (5.5) reduces to

$$\begin{aligned} \mathcal{L}_V V' &= [v, v'] + L_v \omega'_{(n-2)} - \imath_{v'} d\omega_{(n-2)} \,, \\ \mathcal{L}_V W &= L_v \omega_{(1)} + L_v \omega_{(n)} - \alpha \omega_{(1)} \wedge d\omega_{(n-2)} \,, \end{aligned} \tag{5.10}$$

(I think $\alpha = 1$ but not 100% sure) where we write $V = v + \omega_{(n-2)}$, $V' = v' + \omega'_{(n-2)}$ and $W = \omega_{(1)} + \omega_{(n)}$ as a formal of vectors and p-forms, and L denotes the ordinary Lie derivative and $[v, v']$ the ordinary Lie bracket between vector fields v and v'.

We can now describe the S^n consistent truncation using the $\mathrm{SL}(n+1)$ ExFT above. Let Y^I, $I = 1, \ldots, n+1$ be the embedding coordinates of $S^n \subset \mathbb{R}^{n+1}$, such that $Y^I Y_I = 1$. On S^n, we can define a parallelisation of N using the generalised frame Check sign of A! Although I'm pretty confident about it.

$$\mathcal{U}^{\bar{I}} = dY^{\bar{I}} + Y^{\bar{I}}\, vol_{S^n} - A \wedge dY^{\bar{I}} \,, \tag{5.11}$$

where A is an $(n-1)$-potential with field strength $dA = (n-1) vol_{S^n}$, and

$$vol_{S^n} = \frac{1}{n!} \epsilon_{I_1 I_2 \ldots I_{n+1}} Y^{I_1}\, dY^{I_2} \wedge \ldots \wedge dY^{I_{n+1}} \,, \tag{5.12}$$

is the volume form on S^n. The frame in (5.11) has the important property that the $n+1$ generalised tensors are nowhere vanishing since $dY^I = 0$ only when $Y^I = 1$. Therefore, (5.11) provides a generalised parallelisation of S^n in the $\mathrm{SL}(n+1)$ ExFT.

In a local basis, (5.11) gives us a $\mathrm{GL}(n+1)$ matrix, $\mathcal{U}_I{}^{\bar{I}}$, whose determinant allows us to define the scalar density

$$\rho = \left(\det \mathcal{U}_I{}^{\bar{I}} \right)^{-1/2} = (\det \mathring{g})^{-1/2} \,, \tag{5.13}$$

with $\mathring{g}$ the round metric on S^n. Using (5.11) and (5.13) we can define the $\mathrm{SL}(n)$ twist matrix

$$U_I{}^{\bar{I}} = \rho^{2/(n+1)}\, \mathcal{U}_I{}^{\bar{I}} \,, \tag{5.14}$$

and hence a generalised frame for E or a generalised bundle in any other rep of $\mathrm{SL}(n+1)$.

Evaluating (5.14) and (5.13) on the northern hemisphere $Y^I = \left(y^i,\, \sqrt{1 - y^i y_i} \right)$, $i = 1, \ldots, n$ and using the gauge choice for the potential A

$$A_{ij} = \epsilon_{ijk}\, y^k (1 + K(v)) \,, \tag{5.15}$$

with $v = y^i\, y^i$ and ϵ_{ijk} the volume form on S^3, we precisely recover the $\mathrm{SL}(4)$ twist matrix of [2], i.e.

$$(U^{-1})_{\bar{I}}{}^I = \begin{pmatrix} (1-v)^{-1/4} \left(\delta_i{}^j + y_i\, y^j K(v) \right) & y_i\, (1-v)^{1/4} \\ y^j\, (1-v)^{1/4}\, K(v) & (1-v)^{3/4} \end{pmatrix} . \tag{5.16}$$

Moreover, the generalised frame for E precisely coincides with the twist matrix in [3, 6]. Give it explicitly?

To do: Should mention Leibniz parallelisability somewhere and the embedding tensor that arises

I think the sign conventions are:

$$\begin{aligned}
\mathcal{U}^{\bar{I}} &= dY^{\bar{I}} + Y^{\bar{I}}\, vol_{S^n} + dY^{\bar{I}} \wedge A\,, \\
\mathcal{L}_V W &= L_v \omega_{(1)} + L_v \omega_{(n)} + \omega_{(1)} \wedge d\omega_{(n-2)}\,, \\
\mathcal{L}_{\mathcal{U}_{\bar{I}\bar{J}}} \mathcal{U}^{\bar{K}} &= -X_{\bar{I}\bar{J},\bar{L}}{}^{\bar{K}}\, \mathcal{U}^{\bar{L}}\,, \\
X_{\bar{I}\bar{J},\bar{L}}{}^{\bar{K}} &= 2\, \delta_{\bar{L}[\bar{I}} \delta^{\bar{K}}_{\bar{J}]}\,.
\end{aligned} \tag{5.17}$$

For the twist matrix in the antisymmetric rep, I find

$$\mathcal{U}_{[\bar{I}} \mathcal{U}_{\bar{J}]} = v_{\bar{J}\bar{I}} + \star \left(dY_{\bar{I}} \wedge dY_{\bar{J}} \right) + (-1)^{n-1}\, \imath_{v_{\bar{J}\bar{I}}} A\,, \tag{5.18}$$

where the vector part is $V^{i\,0}$, the $(n-2)$-form part is $V^{ij} = \frac{1}{(n-2)!} \epsilon^{k_1 \ldots k_{n-2} ij} \lambda_{k_1 \ldots k_{n-2}}$ and the coordinates are identified as $Y^{i\,0} = y^i$. We can probably match the sign of the vector part of [6] by taking the vector part to be $V^{0\,i}$ and also changing the coordinates to be $Y^{0\,i} = y^i$. We should probably check this carefully since it's useful to write up properly.

5.1.2 Adding flux

To demonstrate our methodology, we will now show how the parameter λ introduced in [1] can be obtained by the twisting procedure described above. Thus, we want to consider the $\mathrm{SO}(3,3) \subset \mathrm{SO}(4,4)$ twist matrix corresponding to S^3 (5.16) and add a 3-form flux via a $\mathrm{SO}(4,4)$ twist. We begin by decomposing $\mathrm{SO}(4,4) \to \mathrm{SO}(3,3) \times \mathrm{SO}(1,1)$, such that

$$\begin{aligned}
\mathbf{8} &\to \mathbf{6_0} \oplus \mathbf{1_2} \oplus \mathbf{1_{-2}}\,, \\
\mathbf{28} &\to \mathbf{15_0} \oplus \mathbf{6_2} \oplus \mathbf{6_{-2}} \oplus \mathbf{1_0}\,.
\end{aligned} \tag{5.19}$$

Correspondingly, we write a $\mathrm{SO}(4,4)$ vector V^M as

$$V^M = \left(V^A,\, V^z,\, V^{\bar{z}} \right)\,, \tag{5.20}$$

where $A = 1, \ldots, 6$ labels the vector of $\mathrm{SO}(3,3)$ and z, $\bar{z}$ label the $\mathbf{1_2}$ and $\mathbf{1_{-2}}$, respectively.

We denote by $U_A{}^{\bar{A}}$ the $\mathrm{SO}(3,3)$ twist matrix corresponding to the S^3 truncation constructed in [3,6] and obtained from (5.14), (5.13), (5.16). Then, we can add a 3-form potential by twisting with the $\mathrm{SO}(4,4)$ generator

$$(e^C)^{MN} = C_A \left(t^A \right)^{MN}\,, \tag{5.21}$$

with C_A an element of the $\mathbf{6_2}$, and $\left(t^A \right)^{MN}$ the corresponding to $\mathrm{SO}(4,4)$ generator whose only non-zero component is

$$\left(t^A \right)^{Bz} = \eta^{AB}\,. \tag{5.22}$$

Here η_{MN} and η_{AB} are the $\mathrm{SO}(4,4)$ and $\mathrm{SO}(3,3)$ invariant metrics, respectively, and are used to raise/lower the corresponding vector indices. Decomposing with respect to the geometric $\mathrm{SL}(3) \times \mathbb{R}^+$ of the S^3, the coordinates on S^3 live in the $Y^A = \left(y^i,\, y_i \right)$, $i = 1, 2, 3$, while $C_A = \left(C_i,\, C^i \right)$ naturally contains a 2-form C^i. The field strength $H_3 = \partial^A C_A = \partial_i C^i$ is a singlet of $\mathrm{SO}(3,3)$ and thus we see that $\rho^{-1} F_{\overline{MN}}{}^{\overline{P}}$ in (5.4) is constant and we obtain a consistent truncation with a new 3-form flux. Evaluating the twist matrix

$$U'_M{}^{\overline{M}} = (e^C)_M{}^N U_N{}^{\overline{M}}\,, \tag{5.23}$$

explicitly using (5.21) and setting $C^i = \lambda\,\xi^i$, with $\nabla_i \xi^i = 1$, in the notation of [1], we obtain precisely the twist matrix used in [1] for the consistent truncation of 6-dimensional $\mathcal{N} = (1,1)$ supergravity on S^3 with H_3 flux.

We will now follow this same procedure in the next section to obtain the consistent truncation of IIB supergravity on $S^3 \times S^3 \times S^1$ with 7-form flux.

5.2 Adding flux to the $S^3 \times S^3 \times S^1$ truncation of IIB supergravity

Our strategy for constructing the consistent truncation on $S^3 \times S^3 \times S^1$ with 7-form flux is to embed the $S^3 \times S^3$ truncation of $E_{7(7)}$ ExFT [11] into $E_{8(8)} \to E_{7(7)} \times \mathrm{SL}(2)$, as described in [19]. This will give us the consistent truncation on $S^3 \times S^3 \times S^1$ without 7-form flux. Then we add a 7-form flux to this truncation as outlined above.

5.2.1 The $S^3 \times S^3$ truncation

Let us first review the dyonic $S^3 \times S^3$ truncation of IIB supergravity [11], which forms the starting point of our construction. The key step in the construction of [11] is that we can use the SL(4) generalised frame (5.14) to form a generalised Leibniz parallelisation of $E_{7(7)}$ via the embedding

$$E_{7(7)} \to \mathrm{SL}(8) \to \mathrm{SL}(4)_1 \times \mathrm{SL}(4)_2 \times \mathbb{R}^+ \,. \tag{5.24}$$

The fundamental of $E_{7(7)}$ then decomposes as

$$\mathbf{56} \to \mathbf{28} \oplus \overline{\mathbf{28}} \to \left[(\mathbf{6},\mathbf{1})_{\mathbf{2}} \oplus (\mathbf{1},\mathbf{6})_{-\mathbf{2}} \oplus (\mathbf{4},\mathbf{4})_{\mathbf{0}} \right] \oplus \left[(\mathbf{6},\mathbf{1})_{-\mathbf{2}} \oplus (\mathbf{1},\mathbf{6})_{\mathbf{2}} \oplus \left(\overline{\mathbf{4}},\overline{\mathbf{4}}\right)_{\mathbf{0}} \right] \,, \tag{5.25}$$

where the first square brackets denote the branching of the $\mathbf{28}$ under $\mathrm{SL}(4)_1 \times \mathrm{SL}(4)_2 \times \mathbb{R}^+$ and the second square brackets that of the $\overline{\mathbf{28}}$. Crucially, for the generalised Leibniz condition (3.3) to hold, the coordinates and generalised vector fields corresponding to the $\mathrm{SL}(4)_1$ and $\mathrm{SL}(4)_2$ must be embedded within the $\mathbf{28}$ and $\overline{\mathbf{28}}$, respectively, which we will call "electric" and "magnetic" coordinates, following [11]. Thus, we write the 56 $E_{7(7)}$ coordinates as (M already used for adjoint of $E_{8(8)}$)

$$Y^M = \left(Y^{\mathcal{AB}},\, Y_{\mathcal{AB}} \right) \,, \tag{5.26}$$

with $\mathcal{A},\mathcal{B} = 1,\ldots,8$ labelling the fundamental of SL(8), corresponding to the decomposition $E_{7(7)} \to \mathrm{SL}(8)$. Following the conventions of [11], the coordinates of the two SL(4) ExFTs are embedded as $Y^{IJ} \subset Y^{\mathcal{AB}}$, with $I, J = 1,2,3,8$, and $Y_{AB} \subset Y_{\mathcal{AB}}$, with $A, B = 4,5,6,7$. Solving the SL(4) ExFT section condition for Y^{IJ} and Y_{AB} guarantees a solution to the $E_{7(7)}$ ExFT, and we choose the solution where the six physical coordinates of IIB are

$$y^i = Y^{i8}, \qquad \tilde{y}_a = Y_{a7}\,, \qquad i = 1,2,3\,, \qquad a = 4,5,6\,. \tag{5.27}$$

This solution of the section condition defines the "geometric" $\mathrm{SL}(3)_1 \times \mathbb{R}^+_1 \subset \mathrm{SL}(4)_1$ and $\mathrm{SL}(3)_2 \times \mathbb{R}^+_2 \subset \mathrm{SL}(4)_2$ subgroups, that will play an important role soon.

We can now use one copy of the SL(4) frame (5.14) for each SL(4) subgroup of (5.24) to construct a generalised parallelisation for the full $E_{7(7)}$ ExFT, with an embedding tensor via (3.3) given by ....

Include breaking of adjoint here?

Should mention the geometric SL(3) subgroups here???

What we should actually do: Read this and next section again and make sure they make sense. Maybe when referring to the SL(3) subgroups in the next section, we can cite (5.27)

(Add α here?)

5.2.2 The $S^3 \times S^3 \times S^1$ truncation

We now construct the consistent truncation on $S^3 \times S^3 \times S^1$ by embedding the $E_{7(7)}$ twist matrix corresponding to the $S^3 \times S^3$ truncation reviewed above in 5.2.1 in $E_{8(8)}$ via the branching $E_{8(8)} \rightarrow E_{7(7)} \times \mathrm{SL}(2)$, such that $\mathbf{248} \rightarrow (\mathbf{133}, \mathbf{1}) \oplus (\mathbf{56}, \mathbf{2}) \oplus (\mathbf{1}, \mathbf{3})$. As explained in [19], we can construct a consistent truncation on $M \times S^1$ to 3-dimensional gauged supergravity by embedding a consistent truncation on M to 4-dimensional supergravity, characterised by an $E_{7(7)}$ twist matrix and $E_{7(7)}$ scalar density σ, as well as the SL(2) twist matrix given by

$$v_{\bar{i}}{}^{i} = \begin{pmatrix} \sigma & 0 \\ 0 & \sigma^{-1} \end{pmatrix} . \tag{5.28}$$

Finally, the $E_{8(8)}$ scalar density satisfies $\rho = \sigma^2$ to ensure a consistent truncation on $M \times S^1$.

Employing this procedure for $M = S^3 \times S^3$, gives us the consistent truncation on $S^3 \times S^3 \times S^1$. However, this does not have any AdS_3 vacua, since we still need to add the 7-form flux. To do this, let us first review the group theory of the $S^3 \times S^3$ truncation, since this will allow us to determine whether the 7-form flux is stabilised by the twist matrix.

5.2.3 Adding flux

To add a 7-form flux to the above truncation, we need to determine whether the 7-form flux is stabilised by the twist matrix on $S^3 \times S^3 \times S^1$ constructed in section 5.2.2. Thus, we need to understand whether IIB supergravity admits a 7-form field strength that is a singlet under the two SL(4) groups in (5.24). To answer this question, we pick a gauge where the 6-form potential lives entirely in $S^3 \times S^3$ but depends on the S^1 coordinate, z. Therefore, in this gauge choice, the 6-form potential corresponds to an adjoint generator of $E_{7(7)}$, and since the S^1 coordinate is an $E_{7(7)}$ singlet, this adjoint generator must be a singlet under $\mathrm{SL}(4)_1 \times \mathrm{SL}(4)_2 \subset \mathrm{SL}(8) \subset E_{7(7)}$ for us to have a consistent truncation.

IIB supergravity contains an S-duality doublet of 6-forms, which we can easily identify using the decomposition $E_{7(7)} \rightarrow \mathrm{SL}(6) \times \mathrm{SL}(2) \times \mathbb{R}^+_{\mathrm{IIB}}$, under which the $E_{7(7)}$ adjoint decomposes as

$$\begin{aligned} \mathbf{133} \rightarrow & \, (\mathbf{35}, \mathbf{1})_{\mathbf{0}} \oplus (\mathbf{1}, \mathbf{3})_{\mathbf{0}} \oplus (\mathbf{1}, \mathbf{1})_{\mathbf{0}} \oplus (\mathbf{1}, \mathbf{2})_{\pm \mathbf{6}} \\ & \oplus \left(\overline{\mathbf{15}}, \mathbf{2}\right)_{\mathbf{2}} \oplus \left(\overline{\mathbf{15}}, \mathbf{1}\right)_{-\mathbf{4}} \oplus (\mathbf{15}, \mathbf{2})_{-\mathbf{2}} \oplus (\mathbf{15}, \mathbf{1})_{\mathbf{4}} . \end{aligned} \tag{5.29}$$

The 6-form doublet corresponds to the $(\mathbf{1}, \mathbf{2})_6$. To understand if one of the 6-forms in the doublet are singlets under $\mathrm{SL}(4)_1 \times \mathrm{SL}(4)_2 \subset E_{7(7)}$, we must decompose $E_{7(7)}$ with respect to the common subgroup of $\mathrm{SL}(4)_1 \times \mathrm{SL}(4)_2$ and $\mathrm{SL}(6) \times \mathrm{SL}(2)$, which is $\mathrm{SL}(3)_1 \times \mathrm{SL}(3)_2 \times \mathbb{R}^+_1 \times \mathbb{R}^+_2$, as defined by the embedding of the physical coordinates in (5.27). Important for us is the correct identification of the $\mathbb{R}^+$ charges in both decompositions. We have, on the one hand,

$$\begin{aligned} E_{7(7)} & \rightarrow \mathrm{SL}(8) \rightarrow \mathrm{SL}(4) \times \mathrm{SL}(4) \times \mathbb{R}^+_{\mathrm{SL}(8)} \\ & \rightarrow \mathrm{SL}(3)_1 \times \mathrm{SL}(3)_2 \times \mathbb{R}^+_1 \times \mathbb{R}^+_2 \times \mathbb{R}^+_{\mathrm{SL}(8)} , \end{aligned} \tag{5.30}$$

and, on the other,

$$\begin{aligned} E_{7(7)} & \rightarrow \mathrm{SL}(6) \times \mathrm{SL}(2) \times \mathbb{R}^+_{\mathrm{IIB}} \\ & \rightarrow \mathrm{SL}(3)_1 \times \mathrm{SL}(3)_2 \times \mathbb{R}^+_{\mathrm{SL}(2)} \times \mathbb{R}^+_{\mathrm{SL}(6)} \times \mathbb{R}^+_{\mathrm{IIB}} . \end{aligned} \tag{5.31}$$

Here, we label the $\mathbb{R}^+$'s with a subscript that refers to the the groups they belong to. The $\mathbb{R}^+$ generators of the two decompositions are related as

$$\begin{aligned}
\mathbb{R}^+_{\mathrm{SL}(6)} &= -\frac{1}{2}\left(\mathbb{R}^+_{\mathrm{SL}(3)_1} + \mathbb{R}^+_{\mathrm{SL}(3)_2}\right) , \\
\mathbb{R}^+_{\mathrm{SL}(2)} &= \frac{1}{4}\left(\mathbb{R}^+_{\mathrm{SL}(3)_1} - \mathbb{R}^+_{\mathrm{SL}(3)_2} + \mathbb{R}^+_{\mathrm{SL}(8)}\right) , \\
\mathbb{R}^+_{\mathrm{IIB}} &= \frac{1}{2}\left(\mathbb{R}^+_{\mathrm{SL}(3)_1} - \mathbb{R}^+_{\mathrm{SL}(3)_2} - 3\,\mathbb{R}^+_{\mathrm{SL}(8)}\right) .
\end{aligned} \tag{5.32}$$

The branching of the adjoint (5.29) under $\mathrm{SL}(6)\times\mathrm{SL}(2)\times\mathbb{R}^+_{\mathrm{IIB}}$ in (5.31) needs to now be compared with that via $\mathrm{SL}(4)_1\times\mathrm{SL}(4)_2\times\mathbb{R}^+_{\mathrm{SL}(8)}$, given by

$$\begin{aligned}
\mathbf{133} &\to \mathbf{63}\oplus\mathbf{70} \\
&\to (\mathbf{15},\mathbf{1})_{\mathbf{0}} \oplus (\mathbf{1},\mathbf{15})_{\mathbf{0}} \oplus (\mathbf{1},\mathbf{1})_{\mathbf{0}} \oplus \left(\mathbf{4},\overline{\mathbf{4}}\right)_{\mathbf{2}} \oplus \left(\overline{\mathbf{4}},\mathbf{4}\right)_{-\mathbf{2}} \\
&\quad \oplus (\mathbf{1},\mathbf{1})_{-\mathbf{4}} \oplus (\mathbf{1},\mathbf{1})_{\mathbf{4}} \oplus \left(\mathbf{4},\overline{\mathbf{4}}\right)_{-\mathbf{2}} \oplus \left(\overline{\mathbf{4}},\mathbf{4}\right)_{\mathbf{2}} \oplus (\mathbf{6},\mathbf{6})_{\mathbf{0}} .
\end{aligned} \tag{5.33}$$

Using (5.32), we can now identify each of the singlet generators $(1,1)_{\pm4}$ with one of the SL(2) doublet generators of charge ∓6. Therefore, we find exactly one singlet $\mathrm{SL}(4)_1\times\mathrm{SL}(4)_2$ generator, which we can write as t_{1238}, corresponding to a 6-form potential. The other $\mathrm{SL}(4)_1\times\mathrm{SL}(4)_2$ singlet generator, t_{4567}, only differs by a compact generator and thus does not corresponding to a different physical field. Finally, the other elements of the SL(2) doublet of 6-form potential can be mapped to the $\left(\mathbf{4},\overline{\mathbf{4}}\right)_{\mathbf{2}} \oplus \left(\overline{\mathbf{4}},\mathbf{4}\right)_{-\mathbf{2}}$ generators, specifically ${t_7}^8$ and ${t_8}^7$ and are clearly not SL(4) singlets. Is this sufficiently clear? I am not so sure...

Therefore, we can add a 7-form flux to the twist matrix using

$${\mathcal{U}_{\overline{M}}}^M \longrightarrow {\mathcal{U}'_{\overline{M}}}^M = {\mathcal{U}_{\overline{M}}}^N \exp(C)_N{}^M , \tag{5.34}$$

with ${C_M}^N = \lambda\, z\, \rho^{-1}\, t_{1238\,M}{}^N$ and λ a numerical parameter corresponding to the amount of 7-form flux. The resulting embedding tensor is most nicely written using the branching

$$\begin{aligned}
E_{8(8)} &\longrightarrow E_{7(7)}\times\mathrm{SL}(2) \longrightarrow \mathrm{SL}(8)\times\mathrm{SL}(2) \\
\mathbf{248} &\xrightarrow{\qquad\qquad\qquad} (\mathbf{63},\mathbf{1})\oplus(\mathbf{70},\mathbf{1})\oplus(\mathbf{28},\mathbf{2})\oplus(\overline{\mathbf{28}},\mathbf{2})\oplus(\mathbf{1},\mathbf{3}) \\
X^{\overline{M}} &\xrightarrow{\qquad\qquad\qquad} \left\{X^{\overline{A}}{}_{\overline{B}}, X^{\overline{ABCD}}, X^{\overline{AB}\,i}, X_{\overline{AB}}{}^{i}, X^{\overline{\alpha}}\right\},
\end{aligned} \tag{5.35}$$

and has the following non-vanishing components:

$$\begin{aligned}
X_{(\mathbf{63},\mathbf{1});(\mathbf{28},\mathbf{2})} &: \begin{cases} X_{\overline{I}}{}^{\overline{J}}{}_{;\overline{KL}+} = -\dfrac{1}{\alpha}\,\delta_{\overline{I}[\overline{K}}\delta_{\overline{L}]}{}^{\overline{J}}, \\ X_{\overline{A}}{}^{\overline{I}}{}_{;\overline{BJ}+} = -\dfrac{1}{2\alpha}\,\delta_{\overline{AB}}\delta_{\overline{J}}{}^{\overline{I}}, \end{cases} \\
X_{(\mathbf{63},\mathbf{1});(\overline{\mathbf{28}},\mathbf{2})} &: \begin{cases} X_{\overline{A}}{}^{\overline{B}}{}_{;}{}^{\overline{CD}}{}_{+} = -\,\delta_{\overline{A}}{}^{[\overline{C}}\delta^{\overline{D}]\overline{B}}, \\ X_{\overline{A}}{}^{\overline{I}}{}_{;}{}^{\overline{BJ}}{}_{+} = -\dfrac{1}{2}\,\delta_{\overline{A}}{}^{\overline{B}}\delta^{\overline{IJ}}, \end{cases} \\
X_{(\mathbf{70},\mathbf{1});(\mathbf{1},\mathbf{3})} &: X_{\overline{IJKL};++} = -\frac{\lambda}{12\sqrt{6}}\,\varepsilon_{\overline{IJKL}}.
\end{aligned} \tag{5.36}$$

With this embedding tensor, the potential (4.14) is stabilized at the scalar origin for $\lambda = 4\sqrt{3}\,\sqrt{1+(1/\alpha^2)}$. We fix this value in the following.

At $\lambda = 4\sqrt{3}\sqrt{1+(1/\alpha^2)}$, the three-dimensional solution described by the embedding tensor (5.36) corresponds to the consistent truncation of the pure NSNS ten-dimensional solution [42]

$$\begin{aligned} \mathrm{d}s^2 &= \ell_{\mathrm{AdS}}^2\, \mathrm{d}s^2\,(\mathrm{AdS}_3) + \alpha^2\, \mathrm{d}s^2\big(S^3\big) + \mathrm{d}s^2\big(\widetilde{S}^3\big) + \mathrm{d}z^2, \\ H_{(3)} &= 2\left(\ell_{\mathrm{AdS}}^2\, \mathrm{Vol}(\mathrm{AdS}_3) + \alpha^2\, \mathrm{Vol}\big(S^3\big) + \mathrm{Vol}\big(\widetilde{S}^3\big)\right), \end{aligned} \tag{5.37}$$

with $H_{(3)} = \mathrm{d}B_{(2)}$ and

$$\ell_{\mathrm{AdS}}^2 = \frac{\alpha^2}{1+\alpha^2}. \tag{5.38}$$

The S^1 coordinate z has periodicity $z \to z + 2\pi$.

The 3d solution preserves $\mathcal{N} = (4,4)$ supercharges, with isometry group $\mathcal{G} = D^1(2,1;\alpha)_{\mathrm{L}} \times D^1(2,1;\alpha)_{\mathrm{R}}$, *i.e.* two copies of the large $\mathcal{N} = 4$ supergroup. The even part of $D^1(2,1;\alpha)$ is isomorphic to $\mathrm{SL}(2,\mathbb{R})\times\mathrm{SO}(3)\times\widetilde{\mathrm{SO}(3)}$ and the bosonic isometries of the $\mathrm{AdS}_3\times S^3\times\widetilde{S}^3$ background are built from the even part of $\mathcal{G}$. The AdS_3 isometry group is $\mathrm{SO}(2,2) \simeq \mathrm{SL}(2,\mathbb{R})_{\mathrm{L}} \times \mathrm{SL}(2,\mathbb{R})_{\mathrm{R}}$, and the factors $\mathrm{SO}(3)_{\mathrm{L}} \times \mathrm{SO}(3)_{\mathrm{R}} \times \widetilde{\mathrm{SO}(3)}_{\mathrm{L}} \times \widetilde{\mathrm{SO}(3)}_{\mathrm{R}} \simeq \mathrm{SO}(4) \times \widetilde{SO(4)}$ combine into the isometry groups of the two spheres. The spectrum organizes into multiplets of $\mathcal{G}$, made out of products of long representations $[\ell,\widetilde{\ell}]$ of $D^1(2,1;\alpha)$ (see app. A for a review of the representations of $D^1(2,1;\alpha)$). Explicitly, the spectrum is [43]

$$\bigoplus_{\ell,\widetilde{\ell}\geq 0} [\ell,\widetilde{\ell}] \otimes [\ell,\widetilde{\ell}] \;\ominus\; [0,0]_{\mathrm{s}} \otimes [0,0]_{\mathrm{s}}. \tag{5.39}$$

In each factor the lowest conformal dimension is

$$h_{\mathrm{L}} = h_{\mathrm{R}} = -\frac{1}{2} + \frac{1}{2}\sqrt{1 + \frac{4\,\ell(\ell+1) + 4\,\alpha^2\,\widetilde{\ell}(\widetilde{\ell}+1) + (2\pi n)^2}{1+\alpha^2}}, \tag{5.40}$$

with n corresponding to the S^1 harmonics, and the lowest conformal dimension of $[\ell,\widetilde{\ell}] \otimes [\ell,\widetilde{\ell}]$ is then

$$\Delta_{\ell,\widetilde{\ell},n} = h_{\mathrm{L}} + h_{\mathrm{R}} = -1 + \sqrt{1 + \frac{4\,\ell(\ell+1) + 4\,\alpha^2\,\widetilde{\ell}(\widetilde{\ell}+1) + (2\pi n)^2}{1+\alpha^2}}. \tag{5.41}$$

For $n = 0$, the representations get shortened whenever $\ell = \widetilde{\ell}$. The spectrum at the consistent truncation is then the one of ref. [27]. It corresponds to $\ell = \widetilde{\ell} = n = 0$ in eq. (5.39):

$$[0,0]_{\mathrm{s}} \otimes [1/2, 1/2]_{\mathrm{s}} \;\oplus\; [1/2, 1/2]_{\mathrm{s}} \otimes [0,0]_{\mathrm{s}} \;\oplus\; [1/2, 1/2]_{\mathrm{s}} \otimes [1/2, 1/2]_{\mathrm{s}}. \tag{5.42}$$

The first two terms build the supergravity multiplet, each of them carrying one spin-2 field, four gravitini, seven vectors and four spin-1/2 fields, all massless. The only propagating degrees of freedom are in $[1/2, 1/2]_{\mathrm{s}} \otimes [1/2, 1/2]_{\mathrm{s}}$ (see the details in the tab. 4 of ref. [27]).

6 Moduli of $\mathrm{AdS}_3 \times S^3 \times S^3 \times S^1$

6.1 Moduli in gauged supergravity

We identified a 11-parameter family of solutions, connected to the origin and given by the $E_{8(8)}$ coset representative ($\overline{28} \leftrightarrow 28$ compared to the code, I think we made a mistake and confused down and up indices)

$$
\begin{aligned}
\mathcal{V}_{\rm def} = \exp & \Big[\sqrt{2}\Big(\chi_1 \big(t_{(\mathbf{28},\mathbf{2})}\big)_{23,-} + \chi_2 \big(t_{(\mathbf{28},\mathbf{2})}\big)_{81,-} + \widetilde{\chi}_1 \big(t_{(\overline{\mathbf{28}},\mathbf{2})}\big)_{56,-} + \widetilde{\chi}_2 \big(t_{(\overline{\mathbf{28}},\mathbf{2})}\big)_{74,-}\Big)\Big] \\
\times \exp & \Big[\sqrt{2}\Big(\zeta_1 \big(t_{(\overline{\mathbf{28}},\mathbf{2})}\big)_{23,+} + \zeta_2 \big(t_{(\overline{\mathbf{28}},\mathbf{2})}\big)_{81,+} + \widetilde{\zeta}_1 \big(t_{(\mathbf{28},\mathbf{2})}\big)_{56,+} + \widetilde{\zeta}_2 \big(t_{(\mathbf{28},\mathbf{2})}\big)_{74,+}\Big)\Big] \\
\times \exp & \Big[\Omega\Big(\big(t_{(\mathbf{63},\mathbf{1})}\big)_1{}^1 - \big(t_{(\mathbf{63},\mathbf{1})}\big)_2{}^2 - \big(t_{(\mathbf{63},\mathbf{1})}\big)_3{}^3 + \big(t_{(\mathbf{63},\mathbf{1})}\big)_8{}^8\Big) \\
& + \widetilde{\Omega}\Big(\big(t_{(\mathbf{63},\mathbf{1})}\big)_4{}^4 - \big(t_{(\mathbf{63},\mathbf{1})}\big)_5{}^5 - \big(t_{(\mathbf{63},\mathbf{1})}\big)_6{}^6 + \big(t_{(\mathbf{63},\mathbf{1})}\big)_7{}^7\Big)\Big] \\
\times \exp & \left[\ln(T)\left(\frac{1}{120}\big(t_{(\mathbf{63},\mathbf{1})}\big)_I{}^I - \frac{1}{120}\big(t_{(\mathbf{63},\mathbf{1})}\big)_A{}^A - \frac{1}{2}\big(t_{(\mathbf{1},\mathbf{3})}\big)_1\right)\right],
\end{aligned}
\tag{6.1}
$$

where we denoted the generators according to the decomposition (5.35). The $(\chi_i, \widetilde{\chi}_i, \zeta_i, \widetilde{\zeta}_i)$ deformations excite the $(\mathbf{6},\mathbf{1})_2$ and $(\mathbf{1},\mathbf{6})_2$ generators in the branching (5.25). (Describe what we do for T? I think there is some freedom in the coeff there.) (Describe what we do for Ω? Check coeff with code. Possible to change the signs to $(+,+,-,-)$, but very long to compute uplift.)

6.2 Moduli in 10 dimensions

The seven form flux is unaffected by the deformations.

$(\chi, \widetilde{\chi}, T)$ Using spherical coordinates for both spheres:

$$
\begin{aligned}
\mathrm{d}s^2 &= \ell_{\rm AdS}^2\, \mathrm{d}s^2(\mathrm{AdS}_3) + \alpha^2\left(\mathrm{d}\theta^2 + \cos^2(\theta)\,\mathrm{d}\phi_1'^{\,2} + \sin^2(\theta)\,\mathrm{d}\phi_2'^{\,2}\right) \\
&\quad + \mathrm{d}\widetilde{\theta}^2 + \cos^2(\widetilde{\theta})\,\mathrm{d}\widetilde{\phi}_1'^{\,2} + \sin^2(\widetilde{\theta})\,\mathrm{d}\widetilde{\phi}_2'^{\,2} + T^2\,\mathrm{d}z^2, \\
H_{(3)} &= 2\,\ell_{\rm AdS}^2\,\mathrm{Vol}(\mathrm{AdS}_3) + 2\,\alpha^2\cos(\theta)\sin(\theta)\,\mathrm{d}\theta\wedge\mathrm{d}\phi_1'\wedge\mathrm{d}\phi_2' + 2\cos(\widetilde{\theta})\sin(\widetilde{\theta})\,\mathrm{d}\widetilde{\theta}\wedge\mathrm{d}\widetilde{\phi}_1'\wedge\mathrm{d}\widetilde{\phi}_2',
\end{aligned}
\tag{6.2}
$$

where

$$
\begin{cases}
\mathrm{d}\phi_1' = \mathrm{d}\phi_1 + \dfrac{1}{\alpha}\chi_1\,\mathrm{d}z, \\
\mathrm{d}\phi_2' = \mathrm{d}\phi_2 + \dfrac{1}{\alpha}\chi_2\,\mathrm{d}z,
\end{cases}
\qquad
\begin{cases}
\mathrm{d}\widetilde{\phi}_1' = \mathrm{d}\widetilde{\phi}_1 + \widetilde{\chi}_1\,\mathrm{d}z, \\
\mathrm{d}\widetilde{\phi}_2' = \mathrm{d}\widetilde{\phi}_2 + \widetilde{\chi}_2\,\mathrm{d}z.
\end{cases}
\tag{6.3}
$$

This deformation breaks $\mathrm{SO}(3)_{\rm L}\times\mathrm{SO}(3)_{\rm R}\times\widetilde{\mathrm{SO}(3)}_{\rm L}\times\widetilde{\mathrm{SO}(3)}_{\rm R}$ to $\mathrm{U}(1)_{\rm L}\times\mathrm{U}(1)_{\rm R}\times\widetilde{\mathrm{U}(1)}_{\rm L}\times\widetilde{\mathrm{U}(1)}_{\rm R}$ and all supersymmetries. Concerning the spectrum, the conformal dimension of each physical field (*c.f.* eq. (5.41)) get shifted by replacing

$$
2\pi n \longrightarrow \frac{2\pi n}{T} + \frac{1}{2T}\Big((q_{\rm L}+q_{\rm R})\,\chi_1 + (q_{\rm L}-q_{\rm R})\,\chi_2 + \alpha\,(\widetilde{q}_{\rm L}+\widetilde{q}_{\rm R})\,\widetilde{\chi}_1 + \alpha\,(\widetilde{q}_{\rm R}-\widetilde{q}_{\rm L})\,\widetilde{\chi}_2\Big),
\tag{6.4}
$$

where the q denote the charges under the different $\mathrm{U}(1)$.

Eq. (6.4) can be used to search for SUSY enhancement points within the 5-parameter landscape. At the undeformed origin, the massless gravitini sit in the two first multiplets in eq. (5.42) with $\mathrm{SO}(3)_\mathrm{L} \times \widetilde{\mathrm{SO}(3)}_\mathrm{L} \times \mathrm{SO}(3)_\mathrm{R} \times \widetilde{\mathrm{SO}(3)}_\mathrm{R}$ spins

$$\big(1/2, 1/2; 0, 0\big) \quad \text{and} \quad \big(0, 0; 1/2, 1/2\big). \tag{6.5}$$

and conformal dimension $\Delta = 3/2$. Turning on the deformation leads to a split into four groups of two modes, with the following $\mathrm{U}(1)_\mathrm{L} \times \widetilde{\mathrm{U}(1)}_\mathrm{L} \times \mathrm{U}(1)_\mathrm{R} \times \widetilde{\mathrm{U}(1)}_\mathrm{R}$ charges and conformal dimensions:

$$\begin{aligned}
(\pm 1, \pm 1; 0, 0): \quad & \Delta = \frac{1}{2} + \sqrt{1 + \frac{\big(\chi_1 + \chi_2 + \alpha\,\big(\tilde{\chi}_1 - \tilde{\chi}_2\big)^2\big)}{4\,T^2(1+\alpha^2)}}\,, \\
(\pm 1, \mp 1; 0, 0): \quad & \Delta = \frac{1}{2} + \sqrt{1 + \frac{\big(\chi_1 + \chi_2 - \alpha\,\big(\tilde{\chi}_1 - \tilde{\chi}_2\big)^2\big)}{4\,T^2(1+\alpha^2)}}\,, \\
(0, 0; \pm 1, \pm 1): \quad & \Delta = \frac{1}{2} + \sqrt{1 + \frac{\big(\chi_1 - \chi_2 + \alpha\,\big(\tilde{\chi}_1 + \tilde{\chi}_2\big)^2\big)}{4\,T^2(1+\alpha^2)}}\,, \\
(0, 0; \pm 1, \mp 1): \quad & \Delta = \frac{1}{2} + \sqrt{1 + \frac{\big(\chi_1 - \chi_2 + \alpha\,\big(\tilde{\chi}_1 + \tilde{\chi}_2\big)^2\big)}{4\,T^2(1+\alpha^2)}}\,.
\end{aligned} \tag{6.6}$$

SUSY enhancement points are then given by combinations of the parameters that leave some modes invariant with $\Delta = 3/2$. The different possibilities are listed below. (Shall we give the multiplets? At least for the truncation?)

- $\boldsymbol{\mathcal{N} = 2}$ The four-dimensional hypersurfaces

  $$\chi_2 = \chi_1 \pm \alpha\,(\tilde{\chi}_1 + \tilde{\chi}_2) \quad \text{or} \quad \chi_2 = -\chi_1 \pm \alpha\,(\tilde{\chi}_1 - \tilde{\chi}_2), \tag{6.7}$$

  give enhancements to $\mathcal{N} = (2, 0)$ and $\mathcal{N} = (0, 2)$, respectively. The spectrum then organizes into multiplets of $\mathrm{SU}(1|1, 1)$.

- $\boldsymbol{\mathcal{N} = 4}$ Choosing

  $$\begin{cases} \chi_2 = \chi_1, \\ \tilde{\chi}_2 = -\tilde{\chi}_1, \end{cases} \quad \text{or} \quad \begin{cases} \chi_2 = -\chi_1, \\ \tilde{\chi}_2 = \tilde{\chi}_1, \end{cases} \tag{6.8}$$

  leads to enhancements to $\mathcal{N} = (4, 0)$ and $\mathcal{N} = (0, 4)$, respectively. On these three-dimensional hyper-surfaces, the isometry group is $\mathrm{SO}(4) \times \mathrm{U}(1)^2$ and the spectrum reorganizes into multiplets of $D^1(2, 1; \alpha)$. Another possibility is to set

  $$\begin{cases} \tilde{\chi}_1 = \pm\chi_1/\alpha, \\ \tilde{\chi}_2 = \mp\chi_2/\alpha, \end{cases} \quad \text{or} \quad \begin{cases} \tilde{\chi}_1 = \pm\chi_2/\alpha, \\ \tilde{\chi}_2 = \mp\chi_1/\alpha, \end{cases} \tag{6.9}$$

  giving 3-parameter families of $\mathcal{N} = (2, 2)$ solutions, with multiplets of $\mathrm{SU}(1|1, 1)_\mathrm{L} \times \mathrm{SU}(1|1, 1)_\mathrm{R}$.

- $\boldsymbol{\mathcal{N} = 6}$ Finally, for

  $$\begin{cases} \chi_2 = \chi_1, \\ \tilde{\chi}_1 = \pm\chi_1/\alpha, \\ \tilde{\chi}_2 = \mp\chi_1/\alpha, \end{cases} \quad \text{or} \quad \begin{cases} \chi_2 = -\chi_1, \\ \tilde{\chi}_1 = \pm\chi_1/\alpha, \\ \tilde{\chi}_2 = \mp\chi_1/\alpha, \end{cases} \tag{6.10}$$

  there are enhancements to $\mathcal{N} = (4, 2)$ and $\mathcal{N} = (2, 4)$, respectively. The isometry group is $\mathrm{SO}(4) \times \mathrm{U}(1)^2$ and the spectrum is then given by multiplets of $D^1(2, 1; \alpha)_{\mathrm{L},\mathrm{R}} \times \mathrm{SU}(1|1, 1)_{\mathrm{R},\mathrm{L}}$.

$(\boldsymbol{\zeta}, \widetilde{\boldsymbol{\zeta}}, \boldsymbol{T})$ Again with spherical coordinates:

$$
\begin{aligned}
\mathrm{d}s^2 &= \ell_{\mathrm{AdS}}^2\, \mathrm{d}s^2\,(\mathrm{AdS}_3) - \Delta\, \mathrm{d}z^2 \\
&+ \alpha^2\Big[\mathrm{d}\theta^2 + \Delta\left(\cos^2(\theta)\, \mathrm{d}\phi_1^2 + \sin^2(\theta)\, \mathrm{d}\phi_2^2\right)\Big] + \mathrm{d}\widetilde{\theta}^2 + \Delta\left(\cos^2(\widetilde{\theta})\, \mathrm{d}\widetilde{\phi}_1^2 + \sin^2(\widetilde{\theta})\, \mathrm{d}\widetilde{\phi}_2^2\right) \\
&+ \Delta\, T^2\, \cos^2(\theta)\cos^2(\widetilde{\theta})\left(\alpha\, \widetilde{\zeta}_2\, \mathrm{d}\phi_1 - \zeta_2\, \mathrm{d}\widetilde{\phi}_1\right)^2 + \Delta\, T^2\, \sin^2(\theta)\sin^2(\widetilde{\theta})\left(\alpha\, \widetilde{\zeta}_1\, \mathrm{d}\phi_2 - \zeta_1\, \mathrm{d}\widetilde{\phi}_2\right)^2 \\
&+ \Delta\, T^2\, \cos^2(\theta)\sin^2(\widetilde{\theta})\left(\alpha\, \widetilde{\zeta}_1\, \mathrm{d}\phi_1 - \zeta_2\, \mathrm{d}\widetilde{\phi}_2\right)^2 + \Delta\, T^2\, \sin^2(\theta)\cos^2(\widetilde{\theta})\left(\alpha\, \widetilde{\zeta}_2\, \mathrm{d}\phi_2 - \zeta_1\, \mathrm{d}\widetilde{\phi}_1\right)^2 \\
&- 2\,\alpha\,\Delta\, T^2\left(\zeta_1\, \cos^2(\theta)\, \mathrm{d}\phi_1 + \zeta_2\, \sin^2(\theta)\, \mathrm{d}\phi_2\right)\left(\widetilde{\zeta}_1\, \cos^2(\widetilde{\theta})\, \mathrm{d}\widetilde{\phi}_1 + \widetilde{\zeta}_2\, \sin^2(\widetilde{\theta})\, \mathrm{d}\widetilde{\phi}_2\right) \\
&+ \Delta\, T^2\, \cos^2(\theta)\left(\alpha\, \zeta_1\, \mathrm{d}\phi_1 + \mathrm{d}z\right)^2 + \Delta\, T^2\, \sin^2(\theta)\left(\alpha\, \zeta_2\, \mathrm{d}\phi_2 + \mathrm{d}z\right)^2 \\
&+ \Delta\, T^2\, \cos^2(\widetilde{\theta})\left(\widetilde{\zeta}_1\, \mathrm{d}\widetilde{\phi}_1 - \mathrm{d}z\right)^2 + \Delta\, T^2\, \sin^2(\widetilde{\theta})\left(\widetilde{\zeta}_2\, \mathrm{d}\widetilde{\phi}_2 - \mathrm{d}z\right)^2,
\end{aligned}
\tag{6.11}
$$

with the warp factor

$$
\Delta^{-1} = 1 + T^2\left(\zeta_1^2 \sin^2(\theta) + \zeta_2^2 \cos^2(\theta) + \widetilde{\zeta}_1^2 \sin^2(\widetilde{\theta}) + \widetilde{\zeta}_2^2 \cos^2(\widetilde{\theta})\right). \tag{6.12}
$$

.

(Expression of H?)

$(\boldsymbol{\zeta}, \widetilde{\boldsymbol{\zeta}}, \boldsymbol{\Omega}, \widetilde{\boldsymbol{\Omega}})$ Again with spherical coordinates:

$$
\begin{aligned}
\mathrm{d}s^2 &= \ell_{\mathrm{AdS}}^2\, \mathrm{d}s^2\,(\mathrm{AdS}_3) - \Delta\Big(\cos^2(\theta) + e^{\Omega}\sin^2(\theta)\Big)\left(e^{\widetilde{\Omega}}\cos^2(\widetilde{\theta}) + \sin^2(\widetilde{\theta})\right)\mathrm{d}z^2 \\
&+ \alpha^2\Big[\mathrm{d}\theta^2 + \Delta\Big(e^{\widetilde{\Omega}}\cos^2(\widetilde{\theta}) + \sin^2(\widetilde{\theta})\Big)\Big(e^{\Omega}\cos^2(\theta)\, \mathrm{d}\phi_1^2 + \sin^2(\theta)\, \mathrm{d}\phi_2^2\Big)\Big] \\
&+ \mathrm{d}\widetilde{\theta}^2 + \Delta\Big(\cos^2(\theta) + e^{\Omega}\sin^2(\theta)\Big)\Big(\cos^2(\widetilde{\theta})\, \mathrm{d}\widetilde{\phi}_1^2 + e^{\widetilde{\Omega}}\sin^2(\widetilde{\theta})\, \mathrm{d}\widetilde{\phi}_2^2\Big) \\
&+ \Delta\, e^{\Omega}\cos^2(\theta)\cos^2(\widetilde{\theta})\left(\alpha\, \widetilde{\zeta}_2\, \mathrm{d}\phi_1 - \zeta_2\, \mathrm{d}\widetilde{\phi}_1\right)^2 + \Delta\, e^{\widetilde{\Omega}}\sin^2(\theta)\sin^2(\widetilde{\theta})\left(\alpha\, \widetilde{\zeta}_1\, \mathrm{d}\phi_2 - \zeta_1\, \mathrm{d}\widetilde{\phi}_2\right)^2 \\
&+ \Delta\, e^{\Omega+\widetilde{\Omega}}\cos^2(\theta)\sin^2(\widetilde{\theta})\left(\alpha\, \widetilde{\zeta}_1\, \mathrm{d}\phi_1 - \zeta_2\, \mathrm{d}\widetilde{\phi}_2\right)^2 + \Delta\, \sin^2(\theta)\cos^2(\widetilde{\theta})\left(\alpha\, \widetilde{\zeta}_2\, \mathrm{d}\phi_2 - \zeta_1\, \mathrm{d}\widetilde{\phi}_1\right)^2 \\
&- 2\,\alpha\,\Delta\left(\zeta_1\, \cos^2(\theta)\, \mathrm{d}\phi_1 + e^{\Omega}\,\zeta_2\, \sin^2(\theta)\, \mathrm{d}\phi_2\right)\left(e^{\widetilde{\Omega}}\,\widetilde{\zeta}_1\, \cos^2(\widetilde{\theta})\, \mathrm{d}\widetilde{\phi}_1 + \widetilde{\zeta}_2\, \sin^2(\widetilde{\theta})\, \mathrm{d}\widetilde{\phi}_2\right) \\
&+ \Delta\left(e^{\widetilde{\Omega}}\cos^2(\widetilde{\theta}) + \sin^2(\widetilde{\theta})\right)\Big[\cos^2(\theta)\left(\alpha\, \zeta_1\, \mathrm{d}\phi_1 + \mathrm{d}z\right)^2 + e^{\Omega}\sin^2(\theta)\left(\alpha\, \zeta_2\, \mathrm{d}\phi_2 + \mathrm{d}z\right)^2\Big] \\
&+ \Delta\Big(\cos^2(\theta) + e^{\Omega}\sin^2(\theta)\Big)\Big[e^{\widetilde{\Omega}}\cos^2(\widetilde{\theta})\left(\widetilde{\zeta}_1\, \mathrm{d}\widetilde{\phi}_1 - \mathrm{d}z\right)^2 + \sin^2(\widetilde{\theta})\left(\widetilde{\zeta}_2\, \mathrm{d}\widetilde{\phi}_2 - \mathrm{d}z\right)^2\Big],
\end{aligned}
\tag{6.13}
$$

with the warp factor

$$
\Delta^{-1} = 1 + T^2\left(\zeta_1^2 \sin^2(\theta) + \zeta_2^2 \cos^2(\theta) + \widetilde{\zeta}_1^2 \sin^2(\widetilde{\theta}) + \widetilde{\zeta}_2^2 \cos^2(\widetilde{\theta})\right). \tag{6.14}
$$

.

(Expression of H?)

SUSY deformation It is convenient to use the Hopf fibration to describe this, we denote by ϕ the coordinate on the S^1 fibre and by ξ the homogeneous coordinate on $\mathbb{CP}^1$. The round metric on S^3 then reads

$$g = \mathrm{d}s^2_{\mathbb{CP}^1} + (\sigma + \mathrm{d}\phi)^2 \tag{6.15}$$

where

$$\begin{aligned} \mathrm{d}s^2_{\mathbb{CP}^1} &= \frac{1}{(1+|\xi|^2)^2}\mathrm{d}\xi\,\mathrm{d}\bar{\xi} \\ \sigma &= \frac{i}{2(1+|\xi|^2)}(\xi\mathrm{d}\bar{\xi} - \bar{\xi}\mathrm{d}\xi) \\ vol_{S^3} &= vol_{\mathbb{CP}^1} \wedge \mathrm{d}\psi. \end{aligned} \tag{6.16}$$

The deformation does not affect S^3_+, but it mixes the S^1 factor in the internal space with the Hopf fibre of S^3_-. It is parametrised by a real number Ω, such that the metric reads

$$\begin{aligned} \mathrm{d}s^2 &= \alpha^2(\mathrm{d}s^2_{\mathbb{CP}^1} + \kappa^2) + \mathrm{d}s^2_{S^3_+} + e^{-4\Omega}\mathrm{d}\psi^2 \\ \kappa &= \mathrm{d}\phi + \sigma - \frac{1}{\alpha}e^{-2\Omega}\sqrt{e^{2\Omega}-1}\mathrm{d}\psi. \end{aligned} \tag{6.17}$$

The three form flux gets changed according to the prescription $\mathrm{d}\phi \to \mathrm{d}\phi - \frac{1}{\alpha}e^{-2\Omega}\sqrt{e^{2\Omega}-1}\mathrm{d}\psi$ and we find:

$$H_{(3)} = \alpha^2 vol_{\mathbb{CP}^1} \wedge (\mathrm{d}\phi - \frac{1}{\alpha}e^{-2\Omega}\sqrt{e^{2\Omega}-1}\mathrm{d}\psi) + vol_{S^3_+}. \tag{6.18}$$

Non-SUSY deformation Here we study a more general deformation depending on three parameters Ω, $\zeta_{1,8}$ and $\zeta_{2,3}$. It is convenient to write down the uplift in terms of spherical coordinates on S^3_-, while again S^3_+ is not deformed.
For the metric we find

$$\begin{aligned} \mathrm{d}s^2 = {}& \ell^2_{\mathrm{AdS}}\,\mathrm{d}s^2\,(\mathrm{AdS}_3) + \mathrm{d}s^2\left(S^3_+\right) \\ &+ \alpha^2\left(\mathrm{d}\theta^2 + \Delta\left(2\,e^{2\Omega} + \zeta^2_{2,3}\right)\,\cos^2(\theta)\,\mathrm{d}\varphi_1^2 + \Delta\left(2 + e^{2\Omega}\,\zeta^2_{1,8}\right)\,\sin^2(\theta)\,\mathrm{d}\varphi_2^2\right) \\ &+ 2\sqrt{2}\,\alpha\,\Delta\left(-\zeta_{2,3}\,\cos^2(\theta)\,\mathrm{d}\varphi_1 + \zeta_{1,8}\,\sin^2(\theta)\,\mathrm{d}\varphi_2\right)\,\mathrm{d}\psi \\ &+ 2\,\Delta\left(\cos^2(\theta) + e^{2\Omega}\,\sin^2(\theta)\right)\,\mathrm{d}\psi^2, \end{aligned} \tag{6.19}$$

where Δ is the warp factor

$$\Delta^{-1} = 2\,e^{2\Omega} + \zeta^2_{2,3} + \cos^2(\theta)\left(2 - 2\,e^{2\Omega} - \zeta^2_{2,3} + e^{2\Omega}\,\zeta^2_{1,8}\right). \tag{6.20}$$

The three-form flux is

$$H_{(3)} = vol_{S^3_-} + \alpha^2\sin(\theta)\cos(\theta)(2 + e^{2\Omega}\zeta^2_{1,8})(2e^{2\Omega} + \zeta^2_{2,3})\Delta^2\mathrm{d}\theta \wedge \kappa_1 \wedge \kappa_2, \tag{6.21}$$

with the two one forms $\kappa_{1,2}$ defined by

$$\begin{aligned} \kappa_1 &= \mathrm{d}\phi_1 - \frac{\zeta_{2,3}}{\alpha(2e^{2\Omega} + \zeta^2_{2,3})}\mathrm{d}\psi \\ \kappa_2 &= \mathrm{d}\phi_2 + \frac{\zeta_{1,8}}{\alpha(2 + e^{2\Omega}\zeta^2_{1,8})}\mathrm{d}\psi. \end{aligned} \tag{6.22}$$

Note that setting ... recovers the supersymmetric deformation described earlier.

6.3 Compactification of the moduli space

7 Conclusions

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A Representations of $D^1(2,1;\alpha)$

The supergroup $D^1(2,1;\alpha)$ has

$$\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SU}(2) \times \widetilde{\mathrm{SU}(2)} \tag{A.1}$$

as even part. Its multiplets are labelled by the spins ℓ and $\widetilde{\ell}$ of each SU(2) factor and the eigenvalue h of the $\mathfrak{sl}(2,\mathbb{R})$ generator. Generic short multiplets $[\ell,\widetilde{\ell}]_{\mathrm{s}}$ are given by [42, 43]

$$\begin{array}{cc}
h_0 & (\ell,\widetilde{\ell}) \\
h_0+1/2 & (\ell+1/2,\widetilde{\ell}-1/2) \oplus (\ell-1/2,\widetilde{\ell}-1/2) \oplus (\ell-1/2,\widetilde{\ell}+1/2) \\
h_0+1 & (\ell,\widetilde{\ell}-1) \oplus (\ell,\widetilde{\ell}) \oplus (\ell-1,\widetilde{\ell}) \\
h_0+3/2 & (\ell-1/2,\widetilde{\ell}-1/2),
\end{array} \tag{A.2}$$

with $h_0 = (\alpha\,\ell+\widetilde{\ell})/(1+\alpha)$. Shortenings occur for $\ell<1$ or $\widetilde{\ell}<1$ [42]. For instance:

$$[0,0]_{\mathrm{s}}: \quad h_0 = 0 \quad (0,0), \qquad [1/2,1/2]_{\mathrm{s}}: \begin{array}{cc}
h_0 = 1/2 & (1/2,1/2) \\
1 & (1,0)\oplus(0,0)\oplus(0,1) \\
3/2 & (1/2,1/2) \\
2 & (0,0).
\end{array} \tag{A.3}$$

Two short multiplets can combine into a long multiplets $[\ell,\widetilde{\ell}]$ as follows:

$$[\ell,\widetilde{\ell}] = [\ell,\widetilde{\ell}]_{\mathrm{s}} \oplus [\ell+1/2,\widetilde{\ell}+1/2]_{\mathrm{s}}. \tag{A.4}$$

The value of h is then not constrained. The explicit content of the long representation $[\ell,\widetilde{\ell}]$ is

$$\begin{array}{cc}
h & (\ell,\widetilde{\ell}) \\
h+1/2 & (\ell+1/2,\widetilde{\ell}+1/2)\oplus(\ell+1/2,\widetilde{\ell}-1/2)\oplus(\ell-1/2,\widetilde{\ell}+1/2)\oplus(\ell-1/2,\widetilde{\ell}-1/2) \\
h+1 & (\ell+1,\widetilde{\ell})\oplus(\ell,\widetilde{\ell}-1)\oplus(\ell,\widetilde{\ell})\oplus(\ell,\widetilde{\ell})\oplus(\ell-1,\widetilde{\ell})\oplus(\ell,\widetilde{\ell}+1) \\
h+3/2 & (\ell+1/2,\widetilde{\ell}+1/2)\oplus(\ell+1/2,\widetilde{\ell}-1/2)\oplus(\ell-1/2,\widetilde{\ell}+1/2)\oplus(\ell-1/2,\widetilde{\ell}-1/2) \\
h+2 & (\ell,\widetilde{\ell}).
\end{array} \tag{A.5}$$

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