Notes: New gaugings in 3d from three-form

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Three-form field-strengths are auxiliary in three dimensions and can be integrated out, giving rise to additional gaugings and contributions to the scalar potential, or equivalently leading to new terms in the embedding tensor (see ref. [1,2] for explicit realisations in half-maximal supergravity). We want to generalize this to maximal supergravity. For a given $\mathrm{E}_{8(8)}$ consistent truncations of gauge group G_0, we ask the question whether it is possible to turn on additional components in the embedding tensor (corresponding to three-form degrees of freedom). If so, the new consistent truncation has gauging $G \supset G_0$. We first need to identify which components of the embedding tensor correspond to these degrees of freedom, and to count the number of singlets under G_0 within them. There will be at least one singlet, corresponding to the gauging of G_0 in the initial truncation. Any additional singlet would indicate additional parameters to play with, possibly leading to new vacua.

1 $\mathrm{AdS}_3 \times S^3$: a proof of concept

We start by reproducing the analysis of ref. [2] to prove the validity of our method. They are two different half-maximal supergravities in six dimensions, with $\mathcal{N} = (2,0)$ and $\mathcal{N} = (1,1)$. They both admit consistent truncation on S^3 [3,4], leading to half-maximal gauged supergravity in three dimensions with duality group $\mathrm{SO}(8,4)$. These truncations have isometry group $\mathrm{SO}(4) \times \mathrm{SO}(4)$ and are constructed in terms of two different $\mathrm{SO}(3,3)$ subgroups of $\mathrm{SO}(8,4)$:

$$\mathcal{N} = (2,0): \qquad \mathrm{SO}(8,4) \longrightarrow \mathrm{SO}(3,3)_{(2,0)} \times \mathrm{SO}(5,1) \longrightarrow \mathrm{SO}(3,3)_{(2,0)} \times \mathrm{SO}(5), \tag{1.1a}$$

$$\mathcal{N} = (1,1): \qquad \mathrm{SO}(8,4) \longrightarrow \mathrm{SO}(4,4) \times \mathrm{SO}(4) \longrightarrow \mathrm{SO}(3,3)_{(1,1)} \times \mathbb{R}_+ \times \mathrm{SO}(4). \tag{1.1b}$$

1.1 $\mathcal{N} = (2,0)$ six-dimensional supergravity

6d origin of the three-forms The $\mathcal{N} = (2,0)$ supergravity features 5 self-dual two-forms $\hat{B}^i_{\hat{\mu}\hat{\nu}}$ ($i \in [\![1,5]\!]$ is the $\mathrm{SO}(5)$ vector index). They lead to two-form potentials $B^i_{(2)}$ in 3d, with three-form field strengths dual to purely internal three-form field-strengths $H^i_{(3)}$. $\Longleftarrow$

Should we consider the anti-self dual two-form also?

$B^i_{(2)}$ in $\mathrm{SO}(3,3)_{(2,0)} \times \mathrm{SO}(5)$ The coordinates $X^{[MN]}$ of the $\mathrm{SO}(8,4)$ exceptional field theory of ref. [3] sit in the adjoint **66** of $\mathrm{SO}(8,4)$. Under eq. (1.1a), it decomposes into

$$\mathbf{66} \longrightarrow (\mathbf{6},\mathbf{1}) \oplus (\mathbf{6},\mathbf{5}) \oplus (\mathbf{15},\mathbf{1}) \oplus (\mathbf{1},\mathbf{5}) \oplus (\mathbf{1},\mathbf{10}). \tag{1.2}$$

Further decomposing $\mathrm{SO}(3,3)_{(2,0)} \times \mathrm{SO}(5) \to \mathrm{SL}(3) \times \mathbb{R}_+ \times \mathrm{SO}(5)$, we get

$$\begin{aligned}\mathbf{66} \longrightarrow\ & [(\bar{\mathbf{3}}_2, \mathbf{1}) \oplus (\mathbf{3}_{-2}, \mathbf{1})] \oplus [(\mathbf{3}_{-2}, \mathbf{5}) \oplus (\bar{\mathbf{3}}_2, \mathbf{5})] \oplus [(\mathbf{3}_4, \mathbf{1}) \oplus (\mathbf{1}_0, \mathbf{1}) \oplus (\mathbf{8}_0, \mathbf{1}) \oplus (\bar{\mathbf{3}}_{-4}, \mathbf{1})] \\ & \oplus (\mathbf{1}_0, \mathbf{5}) \oplus (\mathbf{1}_0, \mathbf{10}).\end{aligned} \tag{1.3}$$

The internal coordinates y^m then sit in the representation $(\mathbf{3}_4, \mathbf{1})$ and the two-form potentials $B^i_{(2)}$ in $(\mathbf{3}_{-2}, \mathbf{5})$, they thus originate from the $\mathrm{SO}(3,3)_{(2,0)} \times \mathrm{SO}(5)$ representations $(\mathbf{15}, \mathbf{1})$ and $(\mathbf{6}, \mathbf{5})$, respectively.

$H^i_{(3)}$ in $\mathrm{SO}(3,3)_{(2,0)} \times \mathrm{SO}(5)$

$$H^i_{(3)} \subset (\mathbf{15}, \mathbf{1}) \otimes (\mathbf{6}, \mathbf{5}) = (\mathbf{6}, \mathbf{5}) \oplus (\mathbf{10}, \mathbf{5}) \oplus (\overline{\mathbf{10}}, \mathbf{5}) \oplus (\mathbf{64}, \mathbf{5}). \tag{1.4}$$

$H^i_{(3)}$ in the embedding tensor The embedding tensor of the $\mathrm{SO}(8,4)$ exceptional theory has two different components: $\theta_{(MN)} \subset \mathbf{77}$ and $\theta_{[MNPQ]} \subset \mathbf{495}$. It decomposes as follows under $\mathrm{SO}(8,4) \to \mathrm{SO}(3,3)_{(2,0)} \times \mathrm{SO}(5)$:

$$\begin{aligned}\mathbf{77} &\longrightarrow 2 \times (\mathbf{1}, \mathbf{1}) \oplus (\mathbf{6}, \mathbf{1}) \oplus (\mathbf{6}, \mathbf{5}) \oplus (\mathbf{20'}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{5}) \oplus (\mathbf{1}, \mathbf{14}), \\ \mathbf{495} &\longrightarrow (\mathbf{10}, \mathbf{1}) \oplus (\mathbf{10}, \mathbf{5}) \oplus (\overline{\mathbf{10}}, \mathbf{1}) \oplus (\overline{\mathbf{10}}, \mathbf{5}) \oplus (\mathbf{15}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{5}) \oplus (\mathbf{1}, \mathbf{10}) \\ &\qquad \oplus (\mathbf{6}, \mathbf{10}) \oplus (\mathbf{6}, \overline{\mathbf{10}}) \oplus (\mathbf{15}, \mathbf{5}) \oplus (\mathbf{15}, \mathbf{10}),\end{aligned} \tag{1.5}$$

where the colored representations are those coming from the three-forms. Only $(\mathbf{10}, \mathbf{5})$ and $(\overline{\mathbf{10}}, \mathbf{5})$ feature singlets under the isometry group $\mathrm{SO}(4) \times \mathrm{SO}(4)$. One of these support the S^3 reduction, while the other can be turned on, leading to new solutions as first demonstrated in ref. [3,4]. All singlets sit in θ_{MNPQ}.

1.2 $\mathcal{N} = (1,1)$ six-dimensional supergravity

6d origin of the three-forms The $\mathcal{N} = (1,1)$ supergravity features a two-form $\hat{B}_{\hat{\mu}\hat{\nu}}$ Both $\hat{B}_{\hat{\mu}\hat{\nu}}$ and its dual lead to two-form potentials in 3d. Their three-form field strengths are dual to purely internal three-form field-strengths $H_{(3)}$ and $*H_{(3)}$, of two-form potentials $C^1_{(2)}$ and $C^2_{(2)}$.

$C^1_{(2)}$ and $C^2_{(2)}$in $\mathrm{SO}(4,4) \times \mathrm{SO}(4)$ Under $\mathrm{SO}(8,4) \longrightarrow \mathrm{SO}(4,4) \times \mathrm{SO}(4)$, the coordinates $X^{[MN]}$ decompose into

$$\mathbf{66} \longrightarrow \mathbf{1}^{(\mathbf{3},\mathbf{1})} \oplus \mathbf{1}^{(\mathbf{1},\mathbf{3})} \oplus \mathbf{8}_{\mathbf{v}}^{(\mathbf{2},\mathbf{2})} \oplus \mathbf{28}^{(\mathbf{1},\mathbf{1})}. \tag{1.6}$$

The internal coordinates and the two-form potentials are singlets under the SO(4) factor, which is related the 6d vectors only. They thus all belong to the representation $\mathbf{28}^{(\mathbf{1},\mathbf{1})}$.

$H_{(3)}$ and $*H_{(3)}$ in $\mathrm{SO}(4,4) \times \mathrm{SO}(4)$

$$H_{(3)}, *H_3 \subset \mathbf{28}^{(\mathbf{1},\mathbf{1})} \otimes \mathbf{28}^{(\mathbf{1},\mathbf{1})} = \mathbf{1}^{(\mathbf{1},\mathbf{1})} \oplus \mathbf{28}^{(\mathbf{1},\mathbf{1})} \oplus \mathbf{35}_{\mathbf{v}}^{(\mathbf{1},\mathbf{1})} \oplus \mathbf{35}_{\mathbf{s}}^{(\mathbf{1},\mathbf{1})} \oplus \mathbf{35}_{\mathbf{c}}^{(\mathbf{1},\mathbf{1})} \oplus \mathbf{300}^{(\mathbf{1},\mathbf{1})} \oplus \mathbf{350}^{(\mathbf{1},\mathbf{1})}. \tag{1.7}$$

$H_{(3)}$ and $*H_{(3)}$ in the embedding tensor

$$\begin{aligned}\mathbf{77} &\longrightarrow \mathbf{1}^{(\mathbf{1},\mathbf{1})} \oplus \mathbf{1}^{(\mathbf{3},\mathbf{3})} \oplus \mathbf{8}_{\mathbf{v}}^{(\mathbf{2},\mathbf{2})} \oplus \mathbf{35}_{\mathbf{v}}^{(\mathbf{1},\mathbf{1})}, \\ \mathbf{495} &\longrightarrow \mathbf{1}^{(\mathbf{1},\mathbf{1})} \oplus \mathbf{8}_{\mathbf{v}}^{(\mathbf{2},\mathbf{2})} \oplus \mathbf{28}^{(\mathbf{1},\mathbf{3})} \oplus \mathbf{28}^{(\mathbf{3},\mathbf{1})} \oplus \mathbf{35}_{\mathbf{s}}^{(\mathbf{1},\mathbf{1})} \oplus \mathbf{35}_{\mathbf{c}}^{(\mathbf{1},\mathbf{1})} \oplus \mathbf{56}_{\mathbf{v}}^{(\mathbf{2},\mathbf{2})}.\end{aligned} \tag{1.8}$$

The singlets certainly gauge the trombone symmetry. Each **35** feature SO(4) × SO(4) singlets, but **less clear how to precisely relate to ref. [2]. Need to consider quadratic constraint?** ⟸

2 Truncations within $\mathrm{SL}(8)_{\mathrm{IIB}} \subset \mathrm{E}_{8(8)}$

We analyse here whether the type IIB S^7 truncation of ref. [5] admit additional gaugings.

10d origin of the three-forms Both the 10d two-forms $\hat{C}^{\alpha}_{\hat{\mu}\hat{\nu}}$ ($\alpha \in 1,2$ for SL(2) doublet) and the 10d four-form $\hat{C}_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}}$ give two-form potentials $C^{\alpha}_{\mu\nu}$ and $C_{\mu\nu mn}$ in 3d ($m, n \in [\![1,7]\!]$ are SL(7) indices). Their three-form field-strengths are dual to purely internal seven-form and five-form field-strengths $H^{\alpha}_{(7)}$ and $H_{(5)}$, respectively. We note $C^{\alpha}_{(6)}$ and $C_{(4)}$ the six-form and four-form potentials from which they derive.

$C^{\alpha}_{(6)}$ and $C_{(4)}$ in $\mathrm{SL}(8)_{\mathrm{IIB}} \times \mathbb{R}_+$ In the $\mathrm{E}_{8(8)} \to \mathrm{SL}(7) \times \mathrm{SL}(2) \times \mathbb{R}_+$ decomposition

$$\begin{aligned}\mathbf{248} \longrightarrow\ & (\mathbf{7},\mathbf{1})_{12} \oplus (\bar{\mathbf{7}},\mathbf{2})_{6} \oplus (\overline{\mathbf{35}},\mathbf{1})_{6} \oplus (\mathbf{21},\mathbf{2})_{3} \oplus (\mathbf{1},\mathbf{1})_{0} \oplus (\mathbf{1},\mathbf{3})_{0} \oplus (\mathbf{48},\mathbf{1})_{0} \\ & \oplus (\overline{\mathbf{21}},\mathbf{2})_{-3} \oplus (\mathbf{35},\mathbf{1})_{-6} \oplus (\mathbf{7},\mathbf{2})_{-9} \oplus (\bar{\mathbf{7}},\mathbf{1})_{-12}\end{aligned} \tag{2.1}$$

of the $\mathrm{E}_{8(8)}$ coordinates X^M, $C^{\alpha}_{(6)}$ and $C_{(4)}$ sit in the representations $(\mathbf{7},\mathbf{2})_{-9}$ and $(\mathbf{35},\mathbf{1})_{-6}$, respectively.[1] They originate from the representations $\mathbf{8}_3 \oplus \mathbf{63}_0$ and $\mathbf{56}_1$ in $\mathrm{E}_{8(8)} \to \mathrm{SL}(9) \to \mathrm{SL}(8)_{\mathrm{IIB}} \times \mathbb{R}_+$ decomposition

$$\mathbf{248} \longrightarrow \mathbf{80} \oplus \mathbf{84} \oplus \overline{\mathbf{84}} \longrightarrow [\mathbf{8}_3 \oplus \mathbf{1}_0 \oplus \mathbf{63}_0 \oplus \bar{\mathbf{8}}_{-3}] \oplus [\mathbf{56}_1 \oplus \mathbf{28}_{-2}] \oplus [\overline{\mathbf{28}}_2 \oplus \overline{\mathbf{56}}_{-1}]\,, \tag{2.2}$$

whereas the internal coordinates are in the representation $\overline{\mathbf{28}}_2$. ⟸

Give further details?

$H^{\alpha}_{(7)}$ and $H_{(5)}$ in $\mathrm{SL}(8)_{\mathrm{IIB}} \times \mathbb{R}_+$ Now that we know what are the $\mathrm{SL}(8)_{\mathrm{IIB}} \times \mathbb{R}_+$ representations corresponding to the internal coordinates, $C^{\alpha}_{(6)}$ and $C_{(4)}$, we can infer those of $H^{\alpha}_{(7)}$ and $H_{(5)}$:

$$\begin{aligned} H^{\alpha}_{(7)} &\subset \overline{\mathbf{28}}_2 \otimes (\mathbf{8}_3 \oplus \mathbf{63}_0) = \bar{\mathbf{8}}_5 \oplus \overline{\mathbf{216}}_5 \oplus \overline{\mathbf{28}}_2 \oplus \overline{\mathbf{36}}_2 \oplus \overline{\mathbf{420}}_2 \oplus \overline{\mathbf{1280}}_2, \\ H_{(5)} &\subset \overline{\mathbf{28}}_2 \otimes \mathbf{56}_1 = \mathbf{8}_3 \oplus \mathbf{216}_3 \oplus \mathbf{1344}_3. \end{aligned} \tag{2.3}$$

$H^{\alpha}_{(7)}$ and $H_{(5)}$ in the embedding tensor Under $\mathrm{E}_{8(8)} \to \mathrm{SL}(9) \to \mathrm{SL}(8)_{\mathrm{IIB}} \times \mathbb{R}_+$, the $\mathrm{E}_{8(8)}$ embedding tensor decomposes as

$$\begin{aligned}\mathbf{3875} \longrightarrow\ & \mathbf{80} \oplus \mathbf{240} \oplus \overline{\mathbf{240}} \oplus \mathbf{1050} \oplus \overline{\mathbf{1050}} \oplus \mathbf{1215} \\ \longrightarrow\ & [\mathbf{8}_3 \oplus \mathbf{1}_0 \oplus \mathbf{63}_0 \oplus \bar{\mathbf{8}}_{-3}] \oplus [\mathbf{168}_1 \oplus \mathbf{28}_{-2} \oplus \mathbf{36}_{-2} \oplus \mathbf{8}_{-5}] \oplus [\bar{\mathbf{8}}_5 \oplus \overline{\mathbf{28}}_2 \oplus \overline{\mathbf{36}}_2 \oplus \overline{\mathbf{168}}_{-1}] \\ & \oplus [\mathbf{70}_4 \oplus \mathbf{56}_1 \oplus \mathbf{504}_1 \oplus \mathbf{420}_{-2}] \oplus [\overline{\mathbf{420}}_2 \oplus \overline{\mathbf{56}}_{-1} \oplus \overline{\mathbf{504}}_{-2} \oplus \overline{\mathbf{70}}_{-4}] \oplus [\mathbf{216}_3 \oplus \mathbf{63}_0 \oplus \mathbf{720}_0 \oplus \overline{\mathbf{216}}_{-3}]\,. \end{aligned} \tag{2.4}$$

The coloured representations are those that can originate from $H^{\alpha}_{(7)}$ and $H_{(5)}$, *i.e.* those in common with eq. (2.3). Among them, only the $\overline{\mathbf{36}}_2$ features a singlet under $\mathrm{SO}(8)_{\mathrm{IIB}}$. This is the singlet used in ref. [5] to support the S^7 truncation, which thus do not admit further gaugings.

[1] The internal coordinates y^m are in the representation $(\mathbf{7},\mathbf{1})_{12}$, and $\bar{\mathbf{7}}^{\wedge 4} = \mathbf{35}$ and $\bar{\mathbf{7}}^{\wedge 6} = \mathbf{7}$.

3 Truncations within $\mathrm{SL}(8)_{\mathbf{IIA}} \subset \mathrm{E}_{8(8)}$

Can be seen as further reduction of four dimensional $\mathrm{E}_{7(7)}$ solutions on a circle. Under $\mathrm{E}_{8(8)} \to \mathrm{E}_{7(7)} \times \mathrm{SL}(2) \to \mathrm{E}_{7(7)} \times \mathbb{R}_+$,

$$\mathbf{248} \longrightarrow (\mathbf{1}, \mathbf{3}) \oplus (\mathbf{56}, \mathbf{2}) \oplus (\mathbf{133}, \mathbf{3}) \longrightarrow \mathbf{1}_2 \oplus \mathbf{56}_1 \oplus \mathbf{1}_0 \oplus \mathbf{133}_0 \oplus \mathbf{56}_{-1} \oplus \mathbf{1}_{-2}. \tag{3.1}$$

The $\mathrm{E}_{7(7)}$ coordinates are in the $\mathbf{56}_1$, and $C_{(6)}$ in $\mathbf{56}_{-1}$ (11d $\hat{C}_{\hat{\mu}\hat{\nu}\hat{\rho}}$ leads to $C_{\mu\nu m}$ in 3d, $H_{\mu\nu\rho m}$ dual to purely internal $H_{(7)}$ of potential $C_{(6)}$). Then,

$$H_{(7)} \subset \mathbf{56}_1 \otimes \mathbf{56}_{-1} = \mathbf{1}_0 \oplus \mathbf{133}_0 \oplus \mathbf{1463}_0 \oplus \mathbf{1539}_0. \tag{3.2}$$

Under $\mathrm{E}_{8(8)} \to \mathrm{E}_{7(7)} \times \mathrm{SL}(2) \to \mathrm{E}_{7(7)} \times \mathbb{R}_+$, the $\mathrm{E}_{8(8)}$ embedding tensor decomposes as

$$\mathbf{3875} \longrightarrow \mathbf{133}_2 \oplus \mathbf{56}_1 \oplus \mathbf{912}_1 \oplus \mathbf{1}_0 \oplus \mathbf{133}_0 \oplus \mathbf{1539}_0 \oplus \mathbf{56}_{-1} \oplus \mathbf{912}_{-1} \oplus \mathbf{133}_{-2}. \tag{3.3}$$

$\mathrm{SO}(8)_{\mathrm{IIA}}$ singlets Under $\mathrm{E}_{7(7)} \to \mathrm{SL}(8)_{\mathrm{IIA}}$:

$$\begin{aligned} \mathbf{133} &\longrightarrow \mathbf{63} \oplus \mathbf{70}, \\ \mathbf{1539} &\longrightarrow \mathbf{63} \oplus \mathbf{378} \oplus \overline{\mathbf{378}} \oplus \mathbf{720}. \end{aligned} \tag{3.4}$$

No singlet other that $\mathbf{1}$.

SO(7) singlets Under $\mathrm{E}_{7(7)} \to \mathrm{SL}(7) \times \mathbb{R}_+$:

$$\begin{aligned} \mathbf{133} &\longrightarrow \mathbf{1}_0 \oplus \ldots, \\ \mathbf{1539} &\longrightarrow \mathbf{1}_0 \oplus \ldots \end{aligned} \tag{3.5}$$

where only representations featuring SO(7) singlets have been specified.

$\mathrm{SO}(6) \times \mathbb{R}_+$ singlets Under $\mathrm{E}_{7(7)} \to \mathrm{SL}(6) \times \mathrm{SL}(2) \times \mathbb{R}_+$:

$$\begin{aligned} \mathbf{133} &\longrightarrow (\mathbf{1}, \mathbf{1})_0 \oplus \ldots, \\ \mathbf{1539} &\longrightarrow 2 \times (\mathbf{1}, \mathbf{1})_0 \oplus \ldots \end{aligned} \tag{3.6}$$

where only representations featuring $\mathrm{SO}(6) \times \mathbb{R}_+$ singlets have been specified.

$\mathrm{SO}(5) \times \mathrm{SO}(3)$ singlets Under $\mathrm{E}_{7(7)} \to \mathrm{SL}(5) \times \mathrm{SL}(3) \times \mathbb{R}_+$:

$$\begin{aligned} \mathbf{133} &\longrightarrow (\mathbf{1}, \mathbf{1})_0 \oplus \ldots, \\ \mathbf{1539} &\longrightarrow 2 \times (\mathbf{1}, \mathbf{1})_0 \oplus \ldots \end{aligned} \tag{3.7}$$

where only representations featuring $\mathrm{SO}(5) \times \mathrm{SO}(3)$ singlets have been specified.

SO(4) × SO(4) singlets Under $\mathrm{E}_{7(7)} \to \mathrm{SL}(4) \times \mathrm{SL}(4) \times \mathbb{R}_+$:

$$\begin{aligned} \mathbf{133} &\longrightarrow (\mathbf{1},\mathbf{1})_0 \oplus (\mathbf{1},\mathbf{1})_4 \oplus (\mathbf{1},\mathbf{1})_{-4} \oplus \ldots, \\ \mathbf{1539} &\longrightarrow 2 \times (\mathbf{1},\mathbf{1})_0 \oplus \ldots \end{aligned} \tag{3.8}$$

where only representations featuring SO(4) × SO(4) singlets have been specified.

The singlets always come from the **63** and **720** of $\mathrm{SL}(8)_{\mathrm{IIB}}$ (and **70** for $\mathrm{SL}(4) \times \mathrm{SL}(4) \times \mathbb{R}_+$).

A Minimal $\mathcal{N} = (1,0)$ six-dimensional supergravity

We consider reductions within the minimal $\mathcal{N} = (1,0)$ six-dimensional supergravity, leading to duality group SO(4,4) in three dimensions, from which the half-maximal cases $\mathcal{N} = (1,1)$ and $\mathcal{N} = (2,0)$ can be deduced (in those cases the duality group in three dimensions in SO(8,4)).

The coordinates $X^{[MN]}$ of the SO(p,q) exceptional field theory of ref. [3] sit in the adjoint **28** of SO(4,4). Under $\mathrm{SO}(4,4) \to \mathrm{SO}(3,3) \times \mathbb{R}_+$, it decomposes into

$$\begin{aligned} \mathbf{28} &\longrightarrow \mathbf{6}_2 \oplus \mathbf{1}_0 \oplus \mathbf{15}_0 \oplus \mathbf{6}_{-2}, \\ X^{MN} &\longrightarrow \{Y^{A0}, Y^0{}_0, Y^{[AB]}, Y^A{}_0\}. \end{aligned} \tag{A.1}$$

Further decomposing $\mathrm{SO}(4,4) \to \mathrm{SO}(3,3) \times \mathbb{R}_+ \to \mathrm{SL}(3) \times \mathbb{R}_+ \times \mathbb{R}_+$, we get

$$\mathbf{28} \longrightarrow [\bar{\mathbf{3}}_{2,2} \oplus \mathbf{3}_{-2,2}] \oplus \mathbf{1}_{0,0} \oplus [\mathbf{3}_{4,0} \oplus \mathbf{1}_{0,0} \oplus \mathbf{8}_{0,0} \oplus \bar{\mathbf{3}}_{-4,0}] \oplus [\bar{\mathbf{3}}_{2,-2} \oplus \mathbf{3}_{-2,-2}]\,. \tag{A.2}$$

They are two physically different possibilities to embed the internal coordinates in X^{MN}, corresponding to the two different half-maximal supergravities in six dimensions:[2]

$$\begin{aligned} \mathcal{N} = (1,1): &\qquad y^m = Y^{m0} \subset \bar{\mathbf{3}}_{2,2} \longleftarrow \mathbf{6}_2, \\ \mathcal{N} = (2,0): &\qquad \tilde{y}_m = \varepsilon_{mnp} Y^{np} \subset \mathbf{3}_{4,0} \longleftarrow \mathbf{15}_0. \end{aligned} \tag{A.3}$$

6d origin of the three-forms The $\mathcal{N} = (1,0)$ supergravity features a two-form $\hat{B}_{\hat{\mu}\hat{\nu}}$. Both $\hat{B}_{\hat{\mu}\hat{\nu}}$ and its dual lead to two-form potentials in 3d. Their three-form field strengths are dual to purely internal three-form field-strengths $H_{(3)}$ and $*H_{(3)}$, of two-form potentials $C^1_{(2)}$ and $C^2_{(2)}$.

$C^1_{(2)}$ and $C^2_{(2)}$ in $\mathrm{SO}(3,3) \times \mathbb{R}_+$ Given the decomposition (A.2) and the coordinates (A.3), the two forms sit in the following representations:

$$\begin{aligned} \mathcal{N} = (1,1): &\qquad C^1_{(2)} \subset \bar{\mathbf{3}}_{-4,0} \longleftarrow \mathbf{15}_0 \quad \text{and} \quad C^2_{(2)} \subset \mathbf{3}_{4,0} \longleftarrow \mathbf{15}_0, \\ \mathcal{N} = (2,0): &\qquad C^1_{(2)} \subset \mathbf{3}_{-2,2} \longleftarrow \mathbf{6}_2 \quad \text{and} \quad C^2_{(2)} \subset \bar{\mathbf{3}}_{2,2} \longleftarrow \mathbf{6}_2. \end{aligned} \tag{A.4}$$

[2]The choices are equivalent in SO(4,4) (through triality), but different once embedded in SO(8,4).

$\boldsymbol{H_{(3)}}$ **and** $\boldsymbol{*H_{(3)}}$ **in** $\mathbf{SO(3,3)} \times \mathbb{R}_+$

$$
\begin{aligned}
\mathcal{N} = (1,1): &\qquad H_{(3)}, *H_{(3)} \subset \mathbf{6}_2 \otimes \mathbf{15}_0 = \mathbf{6}_2 \oplus \mathbf{10}_2 \oplus \overline{\mathbf{10}}_2 \oplus \mathbf{64}_2, \\
\mathcal{N} = (2,0): &\qquad H_{(7)}, *H_{(7)} \subset \mathbf{10}_0 \otimes \mathbf{6}_2 = \mathbf{6}_2 \oplus \mathbf{10}_2 \oplus \overline{\mathbf{10}}_2 \oplus \mathbf{64}_2,
\end{aligned}
\tag{A.5}
$$

$\boldsymbol{H_{(3)}}$ **and** $\boldsymbol{*H_{(3)}}$ **in the embedding tensor** The embedding tensor of the SO(4, 4) exceptional theory has two different components: $\theta_{[MNPQ]} \subset \mathbf{35_s} \oplus \mathbf{35_c}$ and $\theta_{(MN)} \subset \mathbf{35_v}$. It decomposes as follows under $\mathrm{SO}(4,4) \to \mathrm{SO}(3,3) \times \mathbb{R}_+$:

$$
\begin{aligned}
\mathbf{35_v} &\longrightarrow \mathbf{1}_4 \oplus \mathbf{6}_2 \oplus \mathbf{1}_0 \oplus \mathbf{20}'_0 \oplus \mathbf{6}_{-2} \oplus \mathbf{1}_{-4}, \\
\mathbf{35_s} &\longrightarrow \mathbf{10}_2 \oplus \mathbf{15}_0 \oplus \overline{\mathbf{10}}_{-2}, \\
\mathbf{35_c} &\longrightarrow \overline{\mathbf{10}}_2 \oplus \mathbf{15}_0 \oplus \mathbf{10}_{-2},
\end{aligned}
\tag{A.6}
$$

where the colored representations are those coming from the three-forms. Only $\mathbf{10}_2$ and $\overline{\mathbf{10}}_2$ feature SO(4) singlets. **So no distinction between (1,1) and (2,0)? Because triality, or I made errors?** $\Longleftarrow$

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