Notes: cubic couplings in $3d$

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Scalar fields

The scalar Kaluza-Klein fluctuations of the $\mathrm{SO}(8,4+m)$ exceptional field theory [1] are labelled by pair of indices $\Phi^{\alpha,\Sigma}$ [2], with α the index of non-compact generators of $\mathrm{SO}(8,4+m)$ (or equivalently the fields within the $3d$ truncation) and Σ labelling the scalar harmonic $\mathcal{Y}^\Sigma$ on S^3. They transform in the symmetric vector representation $\big({}^{n}\!/\!_{2},{}^{n}\!/\!_{2};0,0\big)$ of $\mathrm{SO}(4)_{\mathrm{gauge}}\times\mathrm{SO}(4)_{\mathrm{global}}\times\mathrm{SO}(m)$. Under

$$\begin{aligned}\mathrm{SO}(8,4+m) &\longrightarrow \mathrm{SO}(8)\times\mathrm{SO}(4+m)\\ &\longrightarrow \mathrm{SO}(4)_{\mathrm{global}}\times\mathrm{SO}(4)_1\times\times\mathrm{SO}(4)_2\times\mathrm{SO}(m)\\ &\longrightarrow \mathrm{SO}(4)_{\mathrm{global}}\times\mathrm{SO}(4)_{\mathrm{gauge}}\times\mathrm{SO}(m),\end{aligned}\tag{0.1}$$

with $\mathrm{SO}(4)_{gauge}=\mathrm{diag}(\mathrm{SO}(4)_1,\mathrm{SO}(4)_2)$, the non-compact generators t_α of $\mathrm{SO}(8,4+m)$ decompose as

$$\begin{aligned}t_\alpha \longrightarrow\ &\big(0,0;0,0\big)\oplus\big(1,0;0,0\big)\oplus\big(0,1;0,0\big)\oplus\big(1,1;0,0\big)\\ &\oplus\big({}^1\!/\!_2,{}^1\!/\!_2;{}^1\!/\!_2,{}^1\!/\!_2\big)\oplus\big({}^1\!/\!_2,{}^1\!/\!_2;0,0\big)^{(m)}\oplus\big(0,0;{}^1\!/\!_2,{}^1\!/\!_2\big)^{(m)},\end{aligned}\tag{0.2}$$

where the exponent $^{(m)}$ indicates representations that transforms as vectors under SO(m). The other representations are SO(m) scalars. Then, the fluctuations $\Phi^{\alpha,\Sigma}$ are made of the following representations:

$$
\begin{aligned}
\alpha\otimes\Sigma \longrightarrow &\Big[\big(0,0;0,0\big)\oplus\big(1,0;0,0\big)\oplus\big(0,1;0,0\big)\oplus\big(1,1;0,0\big)\\
&\oplus\big(1/2,1/2;1/2,1/2\big)\oplus\big(1/2,1/2;0,0\big)^{(m)}\oplus\big(0,0;1/2,1/2\big)^{(m)}\Big]\otimes\big(n/2,n/2;0,0\big)\\
\longrightarrow &\Big[\big(n/2,n/2;0,0\big)\Big]\\
&\oplus\Big[\big((n-2)/2,n/2;0,0\big)\oplus\big(n/2,n/2;0,0\big)\oplus\big((n+2)/2,n/2;0,0\big)\Big]\\
&\oplus\Big[\big(n/2,(n-2)/2;0,0\big)\oplus\big(n/2,n/2;0,0\big)\oplus\big(n/2,(n+2)/2;0,0\big)\Big]\\
&\oplus\Big[\big((n-2)/2,(n-2)/2;0,0\big)\oplus\big((n-2)/2,n/2;0,0\big)\oplus\big((n-2)/2,(n+2)/2;0,0\big)\\
&\quad\oplus\big(n/2,(n-2)/2;0,0\big)\oplus\big(n/2,n/2;0,0\big)\oplus\big(n/2,(n+2)/2;0,0\big)\\
&\quad\oplus\big((n+2)/2,(n-2)/2;0,0\big)\oplus\big((n+2)/2,n/2;0,0\big)\oplus\big((n+2)/2,(n+2)/2;0,0\big)\Big]\\
&\oplus\Big[\big((n-1)/2,(n-1)/2;1/2,1/2\big)\oplus\big((n-1)/2,(n+1)/2;1/2,1/2\big)\\
&\quad\oplus\big((n+1)/2,(n-1)/2;1/2,1/2\big)\oplus\big((n+1)/2,(n+1)/2;1/2,1/2\big)\Big]\\
&\oplus\Big[\big((n-1)/2,(n-1)/2;0,0\big)^{(m)}\oplus\big((n-1)/2,(n+1)/2;0,0\big)^{(m)}\\
&\quad\oplus\big((n+1)/2,(n-1)/2;0,0\big)^{(m)}\oplus\big((n+1)/2,(n+1)/2;0,0\big)^{(m)}\Big]\\
&\oplus\Big[\big(n/2,n/2;1/2,1/2\big)^{(m)}\Big].
\end{aligned}
\tag{0.3}
$$

This corresponds to the scalars fields and Goldstone modes belonging to the level n in the Kaluza-Klein tower. The supermultiplets at level n are [3, 4, 2] (see tab. 1 for the notations)

$$
\mathcal{S}^{(n)}_{(2,0)}=[\boldsymbol{n}+\overset{\circ}{\mathbf{1}},\boldsymbol{n}+\mathbf{1}]_{\mathrm{s}}+[\boldsymbol{n}+\overset{\square}{\mathbf{3}},\boldsymbol{n}+\mathbf{3}]_{\mathrm{s}}+[\boldsymbol{n}+\overset{\diamond}{\mathbf{2}},\boldsymbol{n}+\mathbf{2}]^{(m)}_{\mathrm{s}}+[\boldsymbol{n}+\overset{\triangle}{\mathbf{1}},\boldsymbol{n}+\mathbf{3}]_{\mathrm{s}}+[\boldsymbol{n}+\overset{\triangledown}{\mathbf{3}},\boldsymbol{n}+\mathbf{1}]_{\mathrm{s}}. \tag{0.4}
$$

In this equation, we have assigned a geometric shape to each of these supermultiplets. We use in the following these shapes to identify the multiplet to which each representation in eq. (0.3) belongs:

$$
\begin{aligned}
\alpha\otimes\Sigma \longrightarrow\ & \Big[\dashuline{\big(n/2, n/2; 0,0\big)}\Big] \\
& \oplus \Big[\underline{\big((n-2)/2, n/2; 0,0\big)} \oplus \dashuline{\big(n/2, n/2; 0,0\big)} \oplus \underline{\big((n+2)/2, n/2; 0,0\big)}\Big] \\
& \oplus \Big[\underline{\big(n/2, (n-2)/2; 0,0\big)} \oplus \dashuline{\big(n/2, n/2; 0,0\big)} \oplus \underline{\big(n/2, (n+2)/2; 0,0\big)}\Big] \\
& \oplus \Big[\big((n-2)/2, \overset{\bigcirc}{(n-2)}/2; 0,0\big) \oplus \underline{\big((n-2)/2, n/2; 0,0\big)} \oplus \big((n-2)/2, \overset{\triangle}{(n+2)}/2; 0,0\big) \\
& \qquad \oplus \underline{\big(n/2, (n-2)/2; 0,0\big)} \oplus \dashuline{\big(n/2, n/2; 0,0\big)} \oplus \underline{\big(n/2, (n+2)/2; 0,0\big)} \\
& \qquad \oplus \big((n+2)/2, \overset{\bigtriangledown}{(n-2)}/2; 0,0\big) \oplus \underline{\big((n+2)/2, n/2; 0,0\big)} \oplus \big((n+2)/2, \overset{\square}{(n+2)}/2; 0,0\big)\Big] \\
& \oplus \Big[\big((n-1)/2, \overset{\bigcirc}{(n-1)}/2; 1/2, 1/2\big) \oplus \underline{\big((n-1)/2, \overset{\triangle}{(n+1)}/2; 1/2, 1/2\big)} \\
& \qquad \oplus \underline{\big((n+1)/2, \overset{\bigtriangledown}{(n-1)}/2; 1/2, 1/2\big)} \oplus \big((n+1)/2, \overset{\square}{(n+1)}/2; 1/2, 1/2\big)\Big] \\
& \oplus \Big[\big((n-1)/2, \overset{\diamond}{(n-1)}/2; 0,0\big)^{(m)} \oplus \underline{\big((n-1)/2, \overset{\diamond}{(n+1)}/2; 0,0\big)}^{(m)} \\
& \qquad \oplus \underline{\big((n+1)/2, \overset{\diamond}{(n-1)}/2; 0,0\big)}^{(m)} \oplus \big((n+1)/2, \overset{\diamond}{(n+1)}/2; 0,0\big)^{(m)}\Big] \\
& \oplus \Big[\big(n/2, \overset{\diamond}{n}/2; 1/2, 1/2\big)^{(m)}\Big].
\end{aligned}
\tag{0.5}
$$

Representations to which no shape has been assigned may belong to multiple multiplets **(ce: This needs to be fixed)**. Underlined representations are Goldstone modes (or potential Goldstone modes if the line is dashed and not solid). $\Longleftarrow$

ExFT cubic Couplings TT term obtained from Cadabra + hand massaging

$$
\begin{aligned}
TT =& -\frac{5}{32}T_{ST}{}^{\Sigma\Omega}T_{UV}{}^{\Lambda\Delta}c^{\Omega\Delta\Gamma}\delta^{TV}\phi^{SU\Sigma}\phi_W{}^{W\Gamma\Lambda} + T_{ST}{}^{\Sigma\Omega}T_{UV}{}^{\Lambda\Delta}c^{\Omega\Delta\Gamma}\delta^{TV}\phi^{SU\Lambda\Sigma\Gamma} \\
&+\frac{3}{2}T_{ST}{}^{\Sigma\Omega}T_{UV}{}^{\Lambda\Sigma}c^{\Omega\Delta\Gamma}\delta^{TV}\phi^{US\Delta\Gamma\Lambda} + \frac{5}{2}T_{ST}{}^{\Sigma\Omega}T_{UV}{}^{\Lambda\Sigma}\phi^{TV\Delta}\phi^{SU\Lambda\Gamma}c^{\Omega\Delta\Gamma} - T_{ST}{}^{\Sigma\Omega}T_{UV}{}^{\Lambda\Delta}\phi^{SV\Gamma}\phi^{UT\Lambda\Sigma}c^{\Omega\Delta\Gamma} \\
&- T_{ST}{}^{\Sigma\Omega}T_{UV}{}^{\Lambda\Sigma}\phi^{TU\Delta}\phi^{SV\Gamma\Lambda}c^{\Omega\Delta\Gamma} - \frac{1}{2}T_{ST}{}^{\Sigma\Omega}T_{UV}{}^{\Lambda\Sigma}\phi_W{}^{U\Delta}\varphi^{ST\Lambda}\varphi^{VW\Gamma}c^{\Omega\Delta\Gamma} \\
&+\frac{1}{2}T_{ST}{}^{\Sigma\Omega}T_{UV}{}^{\Lambda\Delta}\phi_W{}^{S\Sigma}\varphi^{TW\Lambda}\varphi^{UV\Gamma}c^{\Omega\Delta\Gamma} + \frac{1}{4}T_{ST}{}^{\Sigma\Omega}T_{UV}{}^{\Lambda\Sigma}\phi_W{}^{S\Delta}\varphi^{TW\Lambda}\varphi^{UV\Gamma}c^{\Omega\Delta\Gamma}
\end{aligned}
\tag{0.6}
$$

$$
\begin{aligned}
TTCamille :=\ & c^{\Sigma\Omega\Gamma}(\gamma T_{XY}{}^{\Delta\Omega}T_{WV}{}^{\Lambda\Sigma}(\frac{3}{4}\Delta^{XW}j^{\,ST\Lambda}j_{ST}{}^{\Delta}j^{YV\Gamma} + \Delta^{XW}j^{YS\Lambda}j_{SU}{}^{\Gamma}j^{UV\Delta} \\
&- j^{XU\Lambda}\Delta_U{}^{Y}j^{WS\Delta}j_S{}^{V\Gamma} + \frac{3}{2}j^{XU\Lambda}\Delta_U{}^{Y}j^{WS\Gamma}j_S{}^{V\Delta} - j^{YV\Gamma}j^{XS\Lambda}\Delta_{SU}j^{UW\Delta}) \\
&+ \gamma T_{XY}{}^{\Delta\Lambda}T_{WV}{}^{\Lambda\Sigma}(\Delta^{XW}j^{YS\Gamma}j_{SU}{}^{\Delta}j^{UV\Omega} - \Delta^{XW}j^{YS\Delta}j_{SU}{}^{\Omega}j^{UV\Gamma} \\
&- 2\Delta_{SU}j^{YV\Omega}j^{XS\Gamma}j^{UW\Delta} + 2\Delta^{XW}j^{YS\Omega}j_{SU}{}^{\Gamma}j^{UV\Delta} - 2\Delta_{SU}j^{YV\Delta}j^{XS\Omega}j^{UW\Gamma}));
\end{aligned}
\tag{0.7}
$$

$\Delta_{\rm L}$	$\Delta_{\rm R}$	Δ	s	$\mathrm{SO}(4)_{\rm gauge}$	$\mathrm{SO}(4)_{\rm global}$
Spin-1 multiplet $[\boldsymbol{k+1},\boldsymbol{k+1}]_{\rm s}$					
$k/2$	$k/2$	k	0	$\big(k/2,k/2\big)$	$\big(0,0\big)$
$k/2$	$(k+1)/2$	$k+1/2$	$1/2$	$\big(k/2,(k-1)/2\big)$	$\big(0,1/2\big)$
$(k+1)/2$	$k/2$	$k+1/2$	$-1/2$	$\big((k-1)/2,k/2\big)$	$\big(1/2,0\big)$
$(k+1)/2$	$(k+1)/2$	$k+1$	0	$\big((k-1)/2,(k-1)/2\big)$	$\big(1/2,1/2\big)$
$k/2$	$(k+2)/2$	$k+1$	1	$\big(k/2,(k-2)/2\big)$	$\big(0,0\big)$
$(k+2)/2$	$k/2$	$k+1$	-1	$\big((k-2)/2,k/2\big)$	$\big(0,0\big)$
$(k+1)/2$	$(k+2)/2$	$k+3/2$	$1/2$	$\big((k-1)/2,(k-2)/2\big)$	$\big(1/2,0\big)$
$(k+2)/2$	$(k+1)/2$	$k+3/2$	$-1/2$	$\big((k-2)/2,(k-1)/2\big)$	$\big(0,1/2\big)$
$(k+2)/2$	$(k+2)/2$	$k+2$	0	$\big((k-2)/2,(k-2)/2\big)$	$\big(0,0\big)$
Spin-2 multiplet $[\boldsymbol{p},\boldsymbol{p+2}]_{\rm s}$					
$(p-1)/2$	$(p+1)/2$	p	1	$\big((p-1)/2,(p+1)/2\big)$	$\big(0,0\big)$
$(p-1)/2$	$(p+2)/2$	$p+1/2$	$3/2$	$\big((p-1)/2,p/2\big)$	$\big(0,1/2\big)$
$p/2$	$(p+1)/2$	$p+1/2$	$1/2$	$\big((p-2)/2,(p+1)/2\big)$	$\big(1/2,0\big)$
$p/2$	$(p+2)/2$	$p+1$	1	$\big((p-2)/2,p/2\big)$	$\big(1/2,1/2\big)$
$(p-1)/2$	$(p+3)/2$	$p+1$	2	$\big((p-1)/2,(p-1)/2\big)$	$\big(0,0\big)$
$(p+1)/2$	$(p+1)/2$	$p+1$	0	$\big((p-3)/2,(p+1)/2\big)$	$\big(0,0\big)$
$p/2$	$(p+3)/2$	$p+3/2$	$3/2$	$\big((p-2)/2,(p-1)/2\big)$	$\big(1/2,0\big)$
$(p+1)/2$	$(p+2)/2$	$p+3/2$	$1/2$	$\big((p-3)/2,p/2\big)$	$\big(0,1/2\big)$
$(p+1)/2$	$(p+3)/2$	$p+2$	1	$\big((p-3)/2,(p-1)/2\big)$	$\big(0,0\big)$

Tab. 1 Spin-1 $[\boldsymbol{k+1},\boldsymbol{k+1}]_{\rm s}$ and spin-2 $[\boldsymbol{p},\boldsymbol{p+2}]_{\rm s}$ multiplets of $\mathrm{SU}(2|1,1)_{\rm L}\times\mathrm{SU}(2|1,1)_{\rm R}$, for $k\geq 2$ and $p\geq 3$ [4]. The SO(4) representations are given by a couple of SU(2) spins. The conjugate spin-2 multiplet $[\boldsymbol{p+2},\boldsymbol{p}]_{\rm s}$ is obtained by inverting $\Delta_{\rm L}$ with $\Delta_{\rm R}$, taking the opposite spin $-s$ and exchanging the SU(2) spins inside each $\mathrm{SO}(4)_{\rm gauged}$ and $\mathrm{SO}(4)_{\rm global}$ representations. Taken from ref. [2].

XT terms

$$
\begin{aligned}
XTheta = c^{\Sigma\Omega\Gamma}(&-3\Theta_{KLMN}\Delta^{KP}T_{PQ}{}^{\Delta\Sigma}j^{MU\Delta}\Delta_U{}^N j^{LR\Omega}\Delta_{RS}j^{SQ\Gamma}+6\Theta_{KLMN}\Delta^{KP}T_{PQ}{}^{\Delta\Sigma}j^{MU\Gamma}j_U{}^{N\Delta}j^{LQ\Omega}\\
&+2\Theta_{KLMN}T_{PQ}{}^{\Delta\Sigma}j^{MU\Gamma}\Delta_U{}^N j^{KP\Delta}j^{LQ\Omega}-2\Theta_{KLMN}T_{PQ}{}^{\Delta\Sigma}j^{MU\Delta}\Delta_U{}^N j^{KP\Gamma}j^{LQ\Omega}\\
&-3(\Theta_{KL}+\Theta\eta_{KL})T_{PQ}{}^{\Delta\Sigma}j^{LQ\Delta}j^{KR\Omega}\Delta_{RS}j^{SP\Gamma}\\
&+3\Theta_{KLMN}(\eta^{KP}\eta^{LQ}-\Delta^{KP}\Delta^{LQ})T_{PQ}{}^{\Delta\Sigma}\Delta_{RS}j^{MU\Delta}j_U{}^{R\Omega}j^{SN\Gamma});
\end{aligned}
\tag{0.8}
$$

XX terms ($\Theta\Theta$)

Notes on the symmetries of different tensors and usefull identities

- $j_{\bar{M}N} = j_{\bar{M}}{}^{\bar{N}}\delta_{A\bar{N}}$ is symmetric in $\bar{M} \leftrightarrow \bar{N}$
- $j_{\bar{M}N}\delta^{\bar{N}K}\eta_{\bar{K}\bar{L}}$ is anti-symmetric in $\bar{M} \leftrightarrow \bar{L}$
- $T.C = 0$ or $T_{..}{}^{\Sigma\Omega}c^{\Omega\Delta\Gamma} + T_{..}{}^{\Delta\Omega}c^{\Omega\Sigma\Gamma} + T_{..}{}^{\Gamma\Omega}c^{\Omega\Sigma\Delta} = 0$
- $f_{\alpha\beta\gamma} = t_{\alpha M}{}^{L}t_{\beta L}{}^{K}t_{\gamma K}{}^{M} = 0$

Ajj/ AAj cubics

Those cubics couplings can be extracted from the kinetic terms. For this we compute $D_\mu M_{MN}$

$$
\begin{aligned}
D_\mu \mathcal{M}_{MN} = U_{(M}{}^{\bar{M}}U_{N)}{}^{\bar{N}}\Big(\mathcal{Y}^{\Sigma}\Big(\partial_\mu j_{\bar{M}\bar{N}}{}^{\Sigma} + 4A_\mu{}^{\bar{K}\bar{L}\Delta}\Delta_{\bar{N}}{}^{\bar{P}}(\Theta_{\bar{K}\bar{L}\bar{M}\bar{P}}\delta^{\Sigma\Omega} - 4\eta_{\bar{K}[\bar{M}}\mathcal{T}_{\bar{P}]\bar{L}}{}^{\Delta\Sigma})\Big) \\
\mathcal{Y}^{\Sigma}\mathcal{Y}^{\Omega}\Big(\partial_\mu J_{\bar{M}\bar{N}}{}^{\Sigma\Omega} + 4A_\mu{}^{\bar{K}\bar{L}\Delta}j_{\bar{N}}{}^{\bar{P}\Gamma}\Big(\Theta_{\bar{K}\bar{L}\bar{M}\bar{P}}\delta^{\Sigma\Delta}\delta^{\Gamma\Omega} - 4\eta_{\bar{K}[\bar{M}}\mathcal{T}_{\bar{P}]\bar{L}}{}^{\Delta\Sigma}\delta^{\Gamma\Omega} + 2\eta_{\bar{M}\bar{P}}\mathcal{T}_{\bar{K}\bar{L}}{}^{\Gamma\Omega}\delta^{\Sigma\Delta}\Big)\Big)
\end{aligned}
\tag{0.9}
$$

There is no order 2 fluctuations in vectors cause we expand in order 1 and inject in the SS ansatz directly. There are order fluctuations for the scalars because we develop and exponential. So we should see them more as double ordre 1.

References

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