Notes: cubic couplings in $3d$

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Scalar fields

The scalar Kaluza-Klein fluctuations of the $\mathrm{SO}(8,4+m)$ exceptional field theory [1] are labelled by pair of indices $\Phi^{\alpha,\Sigma}$ [2], with α the index of non-compact generators of $\mathrm{SO}(8,4+m)$ (or equivalently the fields within the $3d$ truncation) and Σ labelling the scalar harmonic $\mathcal{Y}^{\Sigma}$ on S^3. They transform in the symmetric vector representation $\left({}^{n}\!/_{2},{}^{n}\!/_{2};0,0\right)$ of $\mathrm{SO}(4)_{\mathrm{gauge}}\times\mathrm{SO}(4)_{\mathrm{global}}\times\mathrm{SO}(m)$. Under

$$\mathrm{SO}(8,4+m)\longrightarrow\mathrm{SO}(4)_{\mathrm{gauge}}\times\mathrm{SO}(4)_{\mathrm{global}}\times\mathrm{SO}(m),\tag{0.1}$$

the non-compact generators t_α of $\mathrm{SO}(8,4+m)$ decompose as

$$\begin{aligned}t_\alpha\longrightarrow&\left(0,0;0,0\right)\oplus\left(1,0;0,0\right)\oplus\left(0,1;0,0\right)\oplus\left(1,1;0,0\right)\\&\oplus\left({}^{1}\!/_{2},{}^{1}\!/_{2};{}^{1}\!/_{2},{}^{1}\!/_{2}\right)\oplus\left({}^{1}\!/_{2},{}^{1}\!/_{2};0,0\right)^{(m)}\oplus\left(0,0;{}^{1}\!/_{2},{}^{1}\!/_{2}\right)^{(m)},\end{aligned}\tag{0.2}$$

where the exponent ${}^{(m)}$ indicates representations that transforms as vectors under $\mathrm{SO}(m)$. The other representations are $\mathrm{SO}(m)$ scalars. Then, the fluctuations $\Phi^{\alpha,\Sigma}$ are made of the following representations:

$$
\begin{aligned}
\alpha\otimes\Sigma \longrightarrow & \Big[\big(0,0;0,0\big)\oplus\big(1,0;0,0\big)\oplus\big(0,1;0,0\big)\oplus\big(1,1;0,0\big)\\
&\oplus\big({}^{1}\!/_{2},{}^{1}\!/_{2};{}^{1}\!/_{2},{}^{1}\!/_{2}\big)\oplus\big({}^{1}\!/_{2},{}^{1}\!/_{2};0,0\big)^{(m)}\oplus\big(0,0;{}^{1}\!/_{2},{}^{1}\!/_{2}\big)^{(m)}\Big]\otimes\big({}^{n}\!/_{2},{}^{n}\!/_{2};0,0\big)\\
\longrightarrow & \Big[\big({}^{n}\!/_{2},{}^{n}\!/_{2};0,0\big)\Big]\\
&\oplus\Big[\big({}^{(n-2)}\!/_{2},{}^{n}\!/_{2};0,0\big)\oplus\big({}^{n}\!/_{2},{}^{n}\!/_{2};0,0\big)\oplus\big({}^{(n+2)}\!/_{2},{}^{n}\!/_{2};0,0\big)\Big]\\
&\oplus\Big[\big({}^{n}\!/_{2},{}^{(n-2)}\!/_{2};0,0\big)\oplus\big({}^{n}\!/_{2},{}^{n}\!/_{2};0,0\big)\oplus\big({}^{n}\!/_{2},{}^{(n+2)}\!/_{2};0,0\big)\Big]\\
&\oplus\Big[\big({}^{(n-2)}\!/_{2},{}^{(n-2)}\!/_{2};0,0\big)\oplus\big({}^{(n-2)}\!/_{2},{}^{n}\!/_{2};0,0\big)\oplus\big({}^{(n-2)}\!/_{2},{}^{(n+2)}\!/_{2};0,0\big)\\
&\qquad\oplus\big({}^{n}\!/_{2},{}^{(n-2)}\!/_{2};0,0\big)\oplus\big({}^{n}\!/_{2},{}^{n}\!/_{2};0,0\big)\oplus\big({}^{n}\!/_{2},{}^{(n+2)}\!/_{2};0,0\big)\\
&\qquad\oplus\big({}^{(n+2)}\!/_{2},{}^{(n-2)}\!/_{2};0,0\big)\oplus\big({}^{(n+2)}\!/_{2},{}^{n}\!/_{2};0,0\big)\oplus\big({}^{(n+2)}\!/_{2},{}^{(n+2)}\!/_{2};0,0\big)\Big]\\
&\oplus\Big[\big({}^{(n-1)}\!/_{2},{}^{(n-1)}\!/_{2};{}^{1}\!/_{2},{}^{1}\!/_{2}\big)\oplus\big({}^{(n-1)}\!/_{2},{}^{(n+1)}\!/_{2};{}^{1}\!/_{2},{}^{1}\!/_{2}\big)\\
&\qquad\oplus\big({}^{(n+1)}\!/_{2},{}^{(n-1)}\!/_{2};{}^{1}\!/_{2},{}^{1}\!/_{2}\big)\oplus\big({}^{(n+1)}\!/_{2},{}^{(n+1)}\!/_{2};{}^{1}\!/_{2},{}^{1}\!/_{2}\big)\Big]\\
&\oplus\Big[\big({}^{(n-1)}\!/_{2},{}^{(n-1)}\!/_{2};0,0\big)^{(m)}\oplus\big({}^{(n-1)}\!/_{2},{}^{(n+1)}\!/_{2};0,0\big)^{(m)}\\
&\qquad\oplus\big({}^{(n+1)}\!/_{2},{}^{(n-1)}\!/_{2};0,0\big)^{(m)}\oplus\big({}^{(n+1)}\!/_{2},{}^{(n+1)}\!/_{2};0,0\big)^{(m)}\Big]\\
&\oplus\Big[\big({}^{n}\!/_{2},{}^{n}\!/_{2};{}^{1}\!/_{2},{}^{1}\!/_{2}\big)^{(m)}\Big].
\end{aligned}
\tag{0.3}
$$

This corresponds to the scalars fields and Goldstone modes belonging to the level n in the Kaluza-Klein tower. The supermultiplets at level n are [3, 4, 2] (see tab. 1 for the notations)

$$
\mathcal{S}^{(n)}_{(2,0)}=[\boldsymbol{n+\overset{\circ}{1}},\boldsymbol{n+1}]_{\mathrm{s}}+[\boldsymbol{n+\overset{\square}{3}},\boldsymbol{n+3}]_{\mathrm{s}}+[\boldsymbol{n+\overset{\diamond}{2}},\boldsymbol{n+2}]^{(m)}_{\mathrm{s}}+[\boldsymbol{n+\overset{\triangle}{1}},\boldsymbol{n+3}]_{\mathrm{s}}+[\boldsymbol{n+\overset{\triangledown}{3}},\boldsymbol{n+1}]_{\mathrm{s}}. \tag{0.4}
$$

In this equation, we have assigned a geometric shape to each of these supermultiplets. We use in the following these shapes to identify the multiplet to which each representation in eq. (0.3) belongs:

$$
\begin{aligned}
\alpha \otimes \Sigma \longrightarrow\ & \Big[\underset{\text{- - - - - - - - - -}}{\big(n/2, n/2; 0, 0\big)}\Big] \\
& \oplus \Big[\underline{\big((n-2)/2, n/2; 0, 0\big)} \oplus \underset{\text{- - - - - - - - - -}}{\big(n/2, n/2; 0, 0\big)} \oplus \underline{\big((n+2)/2, n/2; 0, 0\big)}\Big] \\
& \oplus \Big[\underline{\big(n/2, (n-2)/2; 0, 0\big)} \oplus \underset{\text{- - - - - - - - - -}}{\big(n/2, n/2; 0, 0\big)} \oplus \underline{\big(n/2, (n+2)/2; 0, 0\big)}\Big] \\
& \oplus \Big[\big((n-2)/2, \overset{\bigcirc}{(n-2)}/2; 0, 0\big) \oplus \underline{\big((n-2)/2, n/2; 0, 0\big)} \oplus \big((n-2)/2, \overset{\triangle}{(n+2)}/2; 0, 0\big) \\
& \qquad \oplus \underline{\big(n/2, (n-2)/2; 0, 0\big)} \oplus \underset{\text{- - - - - - - - - -}}{\big(n/2, n/2; 0, 0\big)} \oplus \underline{\big(n/2, (n+2)/2; 0, 0\big)} \\
& \qquad \oplus \big((n+2)/2, \overset{\bigtriangledown}{(n-2)}/2; 0, 0\big) \oplus \underline{\big((n+2)/2, n/2; 0, 0\big)} \oplus \big((n+2)/2, \overset{\square}{(n+2)}/2; 0, 0\big)\Big] \\
& \oplus \Big[\big((n-1)/2, \overset{\bigcirc}{(n-1)}/2; 1/2, 1/2\big) \oplus \underline{\big((n-1)/2, \overset{\triangle}{(n+1)}/2; 1/2, 1/2\big)} \\
& \qquad \oplus \underline{\big((n+1)/2, \overset{\bigtriangledown}{(n-1)}/2; 1/2, 1/2\big)} \oplus \big((n+1)/2, \overset{\square}{(n+1)}/2; 1/2, 1/2\big)\Big] \\
& \oplus \Big[\big((n-1)/2, \overset{\diamond}{(n-1)}/2; 0, 0\big)^{(m)} \oplus \underline{\big((n-1)/2, \overset{\diamond}{(n+1)}/2; 0, 0\big)}^{(m)} \\
& \qquad \oplus \underline{\big((n+1)/2, \overset{\diamond}{(n-1)}/2; 0, 0\big)}^{(m)} \oplus \big((n+1)/2, \overset{\diamond}{(n+1)}/2; 0, 0\big)^{(m)}\Big] \\
& \oplus \Big[\big(n/2, \overset{\diamond}{n/2}; 1/2, 1/2\big)^{(m)}\Big].
\end{aligned} \tag{0.5}
$$

Representations to which no shape has been assigned may belong to multiple multiplets **(ce: This needs to be fixed)**. Underlined representations are Goldstone modes (or potential Goldstone modes if the line is dashed and not solid). ⟸

References

[1] O. Hohm, E. T. Musaev, and H. Samtleben, *O(d + 1, d + 1) enhanced double field theory*, *JHEP* **10** (2017) 086, [arXiv:1707.06693].

[2] C. Eloy, *Kaluza-Klein spectrometry for* AdS_3 *vacua*, *SciPost Phys.* **10** (2021), no. 6 131, [arXiv:2011.11658].

[3] S. Deger, A. Kaya, E. Sezgin, and P. Sundell, *Spectrum of D = 6, N=4b supergravity on AdS in three-dimensions x S**3*, *Nucl. Phys. B* **536** (1998) 110–140, [hep-th/9804166].

[4] J. de Boer, *Six-dimensional supergravity on S**3 x AdS(3) and 2-D conformal field theory*, *Nucl. Phys. B* **548** (1999) 139–166, [hep-th/9806104].

Δ_{L}	Δ_{R}	Δ	s	$\mathrm{SO}(4)_{\mathrm{gauge}}$	$\mathrm{SO}(4)_{\mathrm{global}}$
		Spin-1 multiplet $[\boldsymbol{k+1}, \boldsymbol{k+1}]_{\mathrm{s}}$			
$k/2$	$k/2$	k	0	$\big(k/2, k/2\big)$	$\big(0,0\big)$
$k/2$	$(k+1)/2$	$k+1/2$	$1/2$	$\big(k/2,(k-1)/2\big)$	$\big(0,1/2\big)$
$(k+1)/2$	$k/2$	$k+1/2$	$-1/2$	$\big((k-1)/2,k/2\big)$	$\big(1/2,0\big)$
$(k+1)/2$	$(k+1)/2$	$k+1$	0	$\big((k-1)/2,(k-1)/2\big)$	$\big(1/2,1/2\big)$
$k/2$	$(k+2)/2$	$k+1$	1	$\big(k/2,(k-2)/2\big)$	$\big(0,0\big)$
$(k+2)/2$	$k/2$	$k+1$	-1	$\big((k-2)/2,k/2\big)$	$\big(0,0\big)$
$(k+1)/2$	$(k+2)/2$	$k+3/2$	$1/2$	$\big((k-1)/2,(k-2)/2\big)$	$\big(1/2,0\big)$
$(k+2)/2$	$(k+1)/2$	$k+3/2$	$-1/2$	$\big((k-2)/2,(k-1)/2\big)$	$\big(0,1/2\big)$
$(k+2)/2$	$(k+2)/2$	$k+2$	0	$\big((k-2)/2,(k-2)/2\big)$	$\big(0,0\big)$
		Spin-2 multiplet $[\boldsymbol{p}, \boldsymbol{p+2}]_{\mathrm{s}}$			
$(p-1)/2$	$(p+1)/2$	p	1	$\big((p-1)/2,(p+1)/2\big)$	$\big(0,0\big)$
$(p-1)/2$	$(p+2)/2$	$p+1/2$	$3/2$	$\big((p-1)/2,p/2\big)$	$\big(0,1/2\big)$
$p/2$	$(p+1)/2$	$p+1/2$	$1/2$	$\big((p-2)/2,(p+1)/2\big)$	$\big(1/2,0\big)$
$p/2$	$(p+2)/2$	$p+1$	1	$\big((p-2)/2,p/2\big)$	$\big(1/2,1/2\big)$
$(p-1)/2$	$(p+3)/2$	$p+1$	2	$\big((p-1)/2,(p-1)/2\big)$	$\big(0,0\big)$
$(p+1)/2$	$(p+1)/2$	$p+1$	0	$\big((p-3)/2,(p+1)/2\big)$	$\big(0,0\big)$
$p/2$	$(p+3)/2$	$p+3/2$	$3/2$	$\big((p-2)/2,(p-1)/2\big)$	$\big(1/2,0\big)$
$(p+1)/2$	$(p+2)/2$	$p+3/2$	$1/2$	$\big((p-3)/2,p/2\big)$	$\big(0,1/2\big)$
$(p+1)/2$	$(p+3)/2$	$p+2$	1	$\big((p-3)/2,(p-1)/2\big)$	$\big(0,0\big)$

Tab. 1 Spin-1 $[\boldsymbol{k+1}, \boldsymbol{k+1}]_{\mathrm{s}}$ and spin-2 $[\boldsymbol{p}, \boldsymbol{p+2}]_{\mathrm{s}}$ multiplets of $\mathrm{SU}(2|1,1)_{\mathrm{L}} \times \mathrm{SU}(2|1,1)_{\mathrm{R}}$, for $k \geq 2$ and $p \geq 3$ [4]. The SO(4) representations are given by a couple of SU(2) spins. The conjugate spin-2 multiplet $[\boldsymbol{p+2}, \boldsymbol{p}]_{\mathrm{s}}$ is obtained by inverting Δ_{L} with Δ_{R}, taking the opposite spin $-s$ and exchanging the SU(2) spins inside each $\mathrm{SO}(4)_{\mathrm{gauged}}$ and $\mathrm{SO}(4)_{\mathrm{global}}$ representations. Taken from ref. [2].