# Notes: cubic couplings in $3d$

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## Scalar fields

The scalar Kaluza-Klein fluctuations of the $\mathrm{SO}(8,4+m)$ exceptional field theory [1] are labelled by pair of indices $\Phi^{\alpha,\Sigma}$ [2], with $\alpha$ the index of non-compact generators of $\mathrm{SO}(8,4+m)$ (or equivalently the fields within the $3d$ truncation) and $\Sigma$ labelling the scalar harmonic $\mathcal{Y}^{\Sigma}$ on $S^3$. They transform in the symmetric vector representation $\big({}^{n}\!/_{2},{}^{n}\!/_{2};0,0\big)$ of $\mathrm{SO}(4)_{\text{gauge}}\times\mathrm{SO}(4)_{\text{global}}\times\mathrm{SO}(m)$. Under

$$\mathrm{SO}(8,4+m)\longrightarrow\mathrm{SO}(4)_{\text{gauge}}\times\mathrm{SO}(4)_{\text{global}}\times\mathrm{SO}(m),\tag{0.1}$$

the non-compact generators $t_\alpha$ of $\mathrm{SO}(8,4+m)$ decompose as

$$\begin{aligned}t_\alpha\longrightarrow\;&\big(0,0;0,0\big)\oplus\big(1,0;0,0\big)\oplus\big(0,1;0,0\big)\oplus\big(1,1;0,0\big)\\&\oplus\big({}^{1}\!/_{2},{}^{1}\!/_{2};{}^{1}\!/_{2},{}^{1}\!/_{2}\big)\oplus\big({}^{1}\!/_{2},{}^{1}\!/_{2};0,0\big)^{(m)}\oplus\big(0,0;{}^{1}\!/_{2},{}^{1}\!/_{2}\big)^{(m)},\end{aligned}\tag{0.2}$$

where the exponent $^{(m)}$ indicates representations that transforms as vectors under SO($m$). The other representations are SO($m$) scalars. Then, the fluctuations $\Phi^{\alpha,\Sigma}$ are made of the following representations:

$$
\begin{aligned}
\alpha\otimes\Sigma \longrightarrow & \Big[\big(0,0;0,0\big)\oplus\big(1,0;0,0\big)\oplus\big(0,1;0,0\big)\oplus\big(1,1;0,0\big)\\
&\oplus\big({}^1\!/_2,{}^1\!/_2;{}^1\!/_2,{}^1\!/_2\big)\oplus\big({}^1\!/_2,{}^1\!/_2;0,0\big)^{(m)}\oplus\big(0,0;{}^1\!/_2,{}^1\!/_2\big)^{(m)}\Big]\otimes\big({}^n\!/_2,{}^n\!/_2;0,0\big)\\
\longrightarrow & \Big[\big({}^n\!/_2,{}^n\!/_2;0,0\big)\Big]\\
&\oplus\Big[\big({}^{(n-2)}\!/_2,{}^n\!/_2;0,0\big)\oplus\big({}^n\!/_2,{}^n\!/_2;0,0\big)\oplus\big({}^{(n+2)}\!/_2,{}^n\!/_2;0,0\big)\Big]\\
&\oplus\Big[\big({}^n\!/_2,{}^{(n-2)}\!/_2;0,0\big)\oplus\big({}^n\!/_2,{}^n\!/_2;0,0\big)\oplus\big({}^n\!/_2,{}^{(n+2)}\!/_2;0,0\big)\Big]\\
&\oplus\Big[\big({}^{(n-2)}\!/_2,{}^{(n-2)}\!/_2;0,0\big)\oplus\big({}^{(n-2)}\!/_2,{}^n\!/_2;0,0\big)\oplus\big({}^{(n-2)}\!/_2,{}^{(n+2)}\!/_2;0,0\big)\\
&\quad\oplus\big({}^n\!/_2,{}^{(n-2)}\!/_2;0,0\big)\oplus\big({}^n\!/_2,{}^n\!/_2;0,0\big)\oplus\big({}^n\!/_2,{}^{(n+2)}\!/_2;0,0\big)\\
&\quad\oplus\big({}^{(n+2)}\!/_2,{}^{(n-2)}\!/_2;0,0\big)\oplus\big({}^{(n+2)}\!/_2,{}^n\!/_2;0,0\big)\oplus\big({}^{(n+2)}\!/_2,{}^{(n+2)}\!/_2;0,0\big)\Big]\\
&\oplus\Big[\big({}^{(n-1)}\!/_2,{}^{(n-1)}\!/_2;{}^1\!/_2,{}^1\!/_2\big)\oplus\big({}^{(n-1)}\!/_2,{}^{(n+1)}\!/_2;{}^1\!/_2,{}^1\!/_2\big)\\
&\quad\oplus\big({}^{(n+1)}\!/_2,{}^{(n-1)}\!/_2;{}^1\!/_2,{}^1\!/_2\big)\oplus\big({}^{(n+1)}\!/_2,{}^{(n+1)}\!/_2;{}^1\!/_2,{}^1\!/_2\big)\Big]\\
&\oplus\Big[\big({}^{(n-1)}\!/_2,{}^{(n-1)}\!/_2;0,0\big)^{(m)}\oplus\big({}^{(n-1)}\!/_2,{}^{(n+1)}\!/_2;0,0\big)^{(m)}\\
&\quad\oplus\big({}^{(n+1)}\!/_2,{}^{(n-1)}\!/_2;0,0\big)^{(m)}\oplus\big({}^{(n+1)}\!/_2,{}^{(n+1)}\!/_2;0,0\big)^{(m)}\Big]\\
&\oplus\Big[\big({}^n\!/_2,{}^n\!/_2;{}^1\!/_2,{}^1\!/_2\big)^{(m)}\Big].
\end{aligned}
\tag{0.3}
$$

This corresponds to the scalars fields and Goldstone modes belonging to the level $n$ in the Kaluza-Klein tower. The supermultiplets at level $n$ are [3, 4, 2] (see tab. 1 for the notations)

$$\mathcal{S}^{(n)}_{(2,0)} = [\boldsymbol{n+\overset{\circ}{1}},\boldsymbol{n+1}]_{\rm s} + [\boldsymbol{n+\overset{\square}{3}},\boldsymbol{n+3}]_{\rm s} + [\boldsymbol{n+\overset{\diamond}{2}},\boldsymbol{n+2}]_{\rm s}^{(m)} + [\boldsymbol{n+\overset{\triangle}{1}},\boldsymbol{n+3}]_{\rm s} + [\boldsymbol{n+\overset{\triangledown}{3}},\boldsymbol{n+1}]_{\rm s}. \tag{0.4}$$

In this equation, we have assigned a geometric shape to each of these supermultiplets. We use in the following these shapes to identify the multiplet to which each representation in eq. (0.3) belongs:

$$
\begin{aligned}
\alpha\otimes\Sigma \longrightarrow\ & \Big[\underline{\big({}^{n}\!/_{2}, {}^{n}\!/_{2}; 0, 0\big)}\Big] \\
& \oplus \Big[\underline{\big({}^{(n-2)}\!/_{2}, {}^{n}\!/_{2}; 0, 0\big)} \oplus \underline{\big({}^{n}\!/_{2}, {}^{n}\!/_{2}; 0, 0\big)} \oplus \underline{\big({}^{(n+2)}\!/_{2}, {}^{n}\!/_{2}; 0, 0\big)}\Big] \\
& \oplus \Big[\underline{\big({}^{n}\!/_{2}, {}^{(n-2)}\!/_{2}; 0, 0\big)} \oplus \underline{\big({}^{n}\!/_{2}, {}^{n}\!/_{2}; 0, 0\big)} \oplus \underline{\big({}^{n}\!/_{2}, {}^{(n+2)}\!/_{2}; 0, 0\big)}\Big] \\
& \oplus \Big[\overset{\bigcirc}{\big({}^{(n-2)}\!/_{2}, {}^{(n-2)}\!/_{2}; 0, 0\big)} \oplus \underline{\big({}^{(n-2)}\!/_{2}, {}^{n}\!/_{2}; 0, 0\big)} \oplus \overset{\triangle}{\big({}^{(n-2)}\!/_{2}, {}^{(n+2)}\!/_{2}; 0, 0\big)} \\
& \quad \oplus \underline{\big({}^{n}\!/_{2}, {}^{(n-2)}\!/_{2}; 0, 0\big)} \oplus \underline{\big({}^{n}\!/_{2}, {}^{n}\!/_{2}; 0, 0\big)} \oplus \big({}^{n}\!/_{2}, {}^{(n+2)}\!/_{2}; 0, 0\big) \\
& \quad \oplus \overset{\bigtriangledown}{\big({}^{(n+2)}\!/_{2}, {}^{(n-2)}\!/_{2}; 0, 0\big)} \oplus \underline{\big({}^{(n+2)}\!/_{2}, {}^{n}\!/_{2}; 0, 0\big)} \oplus \overset{\square}{\big({}^{(n+2)}\!/_{2}, {}^{(n+2)}\!/_{2}; 0, 0\big)}\Big] \\
& \oplus \Big[\overset{\bigcirc}{\big({}^{(n-1)}\!/_{2}, {}^{(n-1)}\!/_{2}; {}^{1}\!/_{2}, {}^{1}\!/_{2}\big)} \oplus \underline{\overset{\triangle}{\big({}^{(n-1)}\!/_{2}, {}^{(n+1)}\!/_{2}; {}^{1}\!/_{2}, {}^{1}\!/_{2}\big)}} \\
& \quad \oplus \underline{\overset{\bigtriangledown}{\big({}^{(n+1)}\!/_{2}, {}^{(n-1)}\!/_{2}; {}^{1}\!/_{2}, {}^{1}\!/_{2}\big)}} \oplus \overset{\square}{\big({}^{(n+1)}\!/_{2}, {}^{(n+1)}\!/_{2}; {}^{1}\!/_{2}, {}^{1}\!/_{2}\big)}\Big] \\
& \oplus \Big[\overset{\diamond}{\big({}^{(n-1)}\!/_{2}, {}^{(n-1)}\!/_{2}; 0, 0\big)}^{(m)} \oplus \underline{\overset{\diamond}{\big({}^{(n-1)}\!/_{2}, {}^{(n+1)}\!/_{2}; 0, 0\big)}}^{(m)} \\
& \quad \oplus \underline{\overset{\diamond}{\big({}^{(n+1)}\!/_{2}, {}^{(n-1)}\!/_{2}; 0, 0\big)}}^{(m)} \oplus \overset{\diamond}{\big({}^{(n+1)}\!/_{2}, {}^{(n+1)}\!/_{2}; 0, 0\big)}^{(m)}\Big] \\
& \oplus \Big[\overset{\diamond}{\big({}^{n}\!/_{2}, {}^{n}\!/_{2}; {}^{1}\!/_{2}, {}^{1}\!/_{2}\big)}^{(m)}\Big].
\end{aligned}
\tag{0.5}
$$

Representations to which no shape has been assigned may belong to multiple multiplets **(ce: This needs to be fixed)**. Underlined representations are Goldstone modes (or potential Goldstone modes if the line is dashed and not solid). ⟸

## References


[1] O. Hohm, E. T. Musaev, and H. Samtleben, *O(d + 1, d + 1) enhanced double field theory*, *JHEP* **10** (2017) 086, [arXiv:1707.06693].

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[3] S. Deger, A. Kaya, E. Sezgin, and P. Sundell, *Spectrum of D = 6, N=4b supergravity on AdS in three-dimensions x S**3*, *Nucl. Phys. B* **536** (1998) 110–140, [hep-th/9804166].

[4] J. de Boer, *Six-dimensional supergravity on S**3 x AdS(3) and 2-D conformal field theory*, *Nucl. Phys. B* **548** (1999) 139–166, [hep-th/9806104].

| $\Delta_{\mathrm{L}}$ | $\Delta_{\mathrm{R}}$ | $\Delta$ | $s$ | $\mathrm{SO}(4)_{\mathrm{gauge}}$ | $\mathrm{SO}(4)_{\mathrm{global}}$ |
|---|---|---|---|---|---|
| | | **Spin-1 multiplet $[\boldsymbol{k+1,k+1}]_{\mathrm{s}}$** | | | |
| $k/2$ | $k/2$ | $k$ | $0$ | $\big(k/2,k/2\big)$ | $\big(0,0\big)$ |
| $k/2$ | $(k+1)/2$ | $k+1/2$ | $1/2$ | $\big(k/2,(k-1)/2\big)$ | $\big(0,1/2\big)$ |
| $(k+1)/2$ | $k/2$ | $k+1/2$ | $-1/2$ | $\big((k-1)/2,k/2\big)$ | $\big(1/2,0\big)$ |
| $(k+1)/2$ | $(k+1)/2$ | $k+1$ | $0$ | $\big((k-1)/2,(k-1)/2\big)$ | $\big(1/2,1/2\big)$ |
| $k/2$ | $(k+2)/2$ | $k+1$ | $1$ | $\big(k/2,(k-2)/2\big)$ | $\big(0,0\big)$ |
| $(k+2)/2$ | $k/2$ | $k+1$ | $-1$ | $\big((k-2)/2,k/2\big)$ | $\big(0,0\big)$ |
| $(k+1)/2$ | $(k+2)/2$ | $k+3/2$ | $1/2$ | $\big((k-1)/2,(k-2)/2\big)$ | $\big(1/2,0\big)$ |
| $(k+2)/2$ | $(k+1)/2$ | $k+3/2$ | $-1/2$ | $\big((k-2)/2,(k-1)/2\big)$ | $\big(0,1/2\big)$ |
| $(k+2)/2$ | $(k+2)/2$ | $k+2$ | $0$ | $\big((k-2)/2,(k-2)/2\big)$ | $\big(0,0\big)$ |
| | | **Spin-2 multiplet $[\boldsymbol{p,p+2}]_{\mathrm{s}}$** | | | |
| $(p-1)/2$ | $(p+1)/2$ | $p$ | $1$ | $\big((p-1)/2,(p+1)/2\big)$ | $\big(0,0\big)$ |
| $(p-1)/2$ | $(p+2)/2$ | $p+1/2$ | $3/2$ | $\big((p-1)/2,p/2\big)$ | $\big(0,1/2\big)$ |
| $p/2$ | $(p+1)/2$ | $p+1/2$ | $1/2$ | $\big((p-2)/2,(p+1)/2\big)$ | $\big(1/2,0\big)$ |
| $p/2$ | $(p+2)/2$ | $p+1$ | $1$ | $\big((p-2)/2,p/2\big)$ | $\big(1/2,1/2\big)$ |
| $(p-1)/2$ | $(p+3)/2$ | $p+1$ | $2$ | $\big((p-1)/2,(p-1)/2\big)$ | $\big(0,0\big)$ |
| $(p+1)/2$ | $(p+1)/2$ | $p+1$ | $0$ | $\big((p-3)/2,(p+1)/2\big)$ | $\big(0,0\big)$ |
| $p/2$ | $(p+3)/2$ | $p+3/2$ | $3/2$ | $\big((p-2)/2,(p-1)/2\big)$ | $\big(1/2,0\big)$ |
| $(p+1)/2$ | $(p+2)/2$ | $p+3/2$ | $1/2$ | $\big((p-3)/2,p/2\big)$ | $\big(0,1/2\big)$ |
| $(p+1)/2$ | $(p+3)/2$ | $p+2$ | $1$ | $\big((p-3)/2,(p-1)/2\big)$ | $\big(0,0\big)$ |

**Tab. 1** Spin-1 $[\boldsymbol{k+1,k+1}]_{\mathrm{s}}$ and spin-2 $[\boldsymbol{p,p+2}]_{\mathrm{s}}$ multiplets of $\mathrm{SU}(2|1,1)_{\mathrm{L}}\times\mathrm{SU}(2|1,1)_{\mathrm{R}}$, for $k\geq 2$ and $p\geq 3$ [4]. The SO(4) representations are given by a couple of SU(2) spins. The conjugate spin-2 multiplet $[\boldsymbol{p+2,p}]_{\mathrm{s}}$ is obtained by inverting $\Delta_{\mathrm{L}}$ with $\Delta_{\mathrm{R}}$, taking the opposite spin $-s$ and exchanging the SU(2) spins inside each $\mathrm{SO}(4)_{\mathrm{gauged}}$ and $\mathrm{SO}(4)_{\mathrm{global}}$ representations. Taken from ref. [2].