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July 28, 2025

Machine learning the Conformal Manifold of Holographic CFT2's

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Abstract

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1 Introduction

The AdS/CFT correspondence stands as one of the most profound dualities in theoretical physics, establishing a remarkable equivalence between string theory solutions on anti-de Sitter (AdS) spacetimes and conformal field theories (CFTs) living on their boundaries [1]. This correspondence has revolutionized our understanding of both quantum gravity and strongly coupled field theories, providing unprecedented insights into the holographic nature of gravity. Within this holographic framework, supergravity theories in AdS backgrounds serve as the low-energy effective descriptions of string theory compactifications, making them natural laboratories for exploring the gravitational side of the duality. The CFT side plays a crucial role in statistical physics, as CFT are known to describe phase transitions ; they are also fixed point of Renormalisation Group flow and this is this latter point that is our interest in this paper. An interesting question is indeed to know whether those CFT are isolated fixed points of RG flows, or belong to a continuous family of CFT. If this is the case, the space of continuous deformations that takes from one CFT to another is called a conformal manifold. These deformations are parametrized by exactly marginal operator, or in other words operators that preserves the conformal symmetry, *i.e.* whose β functions exactly vanish.

From the AdS/CFT correspondence perspective, conformal manifolds on the boundary theory are dual to flat directions in the scalar potential of the dual supergravity theory. Along these directions, the scalar field configurations vary continuously while the cosmological constant remains unchanged. Supersymmetry is believed to be necessary for the existence of holographic conformal manifolds, as non-supersymmetric AdS solutions are expected to be unstable [2, 3]. However, recent investigations have identified AdS₄ configurations that appear to evade this requirement, with no evidence of standard decay channels – neither perturbative nor non-perturbative – being present [4]. The situation is

even richer in the context of $\text{AdS}_3/\text{CFT}_2$, as current-current deformations, given by products $J\bar{J}$ of (anti-)holomorphic conserved currents, are exactly marginal in two dimensions [5], although they may break supersymmetry. The question of the exact form of the gravity dual to $J\bar{J}$ deformations is however open. A well-known example is given in ref. [6, 7], and recent works have demonstrated the existence of vast families of classically marginal deformations of the $\text{AdS}_3 \times S^3 \times T^4$ and $\text{AdS}_3 \times S^3 \times S^3 \times S^1$ solutions of type IIB supergravity that are perturbatively stable though supersymmetry breaking [8–10]. These deformations were shown in ref. [10] to be equivalent to current-current deformations of the worldsheet Wess-Zumino-Witten models [11] describing the undeformed solutions [12, 13], giving strong indications that their CFT duals arise from $J\bar{J}$ deformations as well.

Although extensive, the families of deformations found in the aforementioned papers have not exhausted all possibilities for marginal deformations of three-dimensional supergravity solutions. A complete classification would require a systematic study of the flat directions in the supergravity scalar potential V that defines the AdS configurations, as those directions are in correspondence with classical marginal deformations of the holographic CFT in the large N limit. These flat directions are continuous sets in the space of scalar fields along which the value of the potential stay fixed, *i.e.* along which $\nabla V = 0$. However, the explicit characterization of flat directions presents formidable technical challenges. Supergravity scalar potentials, even in truncated models, typically involve dozens of scalar fields with intricate non-linear interactions. The resulting expressions for critical points – where all first-order derivatives vanish – quickly become too complex for traditional symbolic manipulation, rendering analytical approaches computationally intractable.

The emergence of machine learning techniques in theoretical physics opens new avenues for addressing such complex problems. Instead of solving the full symbolic system analytically from the outset, one can employ numerical methods to sample the solution space and subsequently apply symbolic regression techniques to extract analytical patterns from the data. This hybrid approach has the potential to bypass the computational bottlenecks inherent in purely symbolic methods, while still having the potential to uncover exact analytical expressions.

Machine learning strategies have previously been applied to identify new isolated vacua in $\text{SO}(8)$ supergravity [14, 15]. More broadly, there has been increasing interest in applying machine learning and numerical techniques across various domains of high-energy physics. This includes, for instance, the characterization of Calabi–Yau metrics and hypersurfaces [16–23], as well as broader efforts to explore the string theory landscape using machine learning techniques [24–26]. Additional applications include studies in CFT [27], investigations of the supergravity landscape [28, 29], and explorations of the AdS/CFT correspondence [30]. More generally, machine learning has found utility in the study of string theory, geometry, and fundamental physics [31–33].

In this work, we demonstrate the viability of the aforementioned machine learning approach by applying it to a five-scalar subsector of a 13 scalar consistent truncation of six-dimensional non-chiral $\mathcal{N} = (1, 1)$ supergravity on $\text{AdS}_3 \times S^3$, or to type IIB supergravity on $\text{AdS}_3 \times S^3 \times T^4$. Our methodology combines gradient descent sampling of the flat direction manifold with some symbolic regression technique. There exists a large literature on symbolic regression, using methods from genetic programming [34–37], to deep machine learning [38, 39], generative modelling [40], diffusion models [41], and equation learner with the nodes being symbolic operations [42]. Another state-of-the art algorithm is the AIFeynman methods [43]. They use neural network to identify structure in the dataset (such as translational symmetries, multiplicative separability, compositionality...) to recursively define simpler problems on which they can fit the solutions with polynomials. However, in the spirit of trying to build a generalisable method, we develop here our own algorithm, which is based on symbolic regression using an Annealing Importance Sampling method (AIS) [44].

The study of the five-parameter potential is separated into several parts. We first use the gradient descent to efficiently sample the underlying conformal manifold. Combined with numerical analysis,

such as principal component analysis and clustering, we can identify the existence of a three-dimensional continuous family. We then use an AIS technique combined with a sequential Monte Carlo approach (SMC) [45] in order to do symbolic regression. As we will indeed demonstrate in the paper, there exist polynomial constraints on the 5 parameters, viewed as embedding coordinates, that projects them on the 3 dimensional manifolds. We manage to identify 8 of those constraints on the data, not all independent, which once solved gives that two of the embedding coordinates can be expressed in terms of the others. This provides an explicit three-dimensional family of solutions.

The paper is organized as follows. In sec. 2, we establish the supergravity setup, presenting the scalar potential in its full complexity and motivating the truncation to five fields. Sec. 3 details our numerical methodology, covering both the gradient descent sampling procedure and the annealed importance sampling approach to symbolic regression. We present in sec. 4 our main results, both numerical and analytical. Finally, sec. 5 concludes with prospects for extending this approach to higher-dimensional cases and its broader implications for systematic studies of conformal manifolds in holographic theories.

2 Supergravity setup

Three-dimensional $\mathcal{N} = 8$ (half-maximal) gauged supergravity is governed by the Lagrangian [46, 47]

$$e^{-1}\mathcal{L} = R + \frac{1}{8}g^{\mu\nu}D_\mu M^{\bar{M}\bar{N}}D_\nu M_{\bar{M}\bar{N}} + e^{-1}\mathcal{L}_{\text{CS}} - V, \quad (2.1)$$

which comprises an Einstein-Hilbert term R , a kinetic term for scalar fields parametrized by the matrix $M^{\bar{M}\bar{N}}$, a Chern-Simons contribution \mathcal{L}_{CS} , and a scalar potential V , see app. A for details on the definition of the terms. The scalar degrees of freedom parametrise the coset space

$$\frac{\text{SO}(8, 4)}{\text{SO}(8) \times \text{SO}(4)}, \quad (2.2)$$

through the symmetric matrix $M_{\bar{K}\bar{L}}$.¹ With appropriate gauging (see eq. (A.9)), this $\text{SO}(8, 4)$ theory is a consistent truncation of six-dimensional $\mathcal{N} = (1, 1)$ supergravity on S^3 [48–50]. It features an AdS_3 stationary point at the scalar origin ($M_{\bar{K}\bar{L}} = \delta_{\bar{K}\bar{L}}$, the $\text{SO}(8, 4)$ identity matrix), corresponding to an $\text{AdS}_3 \times S^3$ solution in 6d. In the following, we will be interested in exploring flat directions of the potential V , *i.e.* other stationary points that are continuously connected to the origin. In three dimensions, the flat directions constitute a family of AdS_3 vacua sharing the same cosmological constant. The corresponding solutions in six dimensions are of the form $\text{AdS}_3 \times M^3$, with M^3 some deformation of the round sphere S^3 , parametrised by the constant scalars defining the 3d vacua.

We parametrise the scalars of the theory following ref. [51] (see also app. A for some details): 13 scalars parametrised by a symmetric $\text{GL}(3, \mathbb{R})$ matrix $m = \nu\nu^T$ where

$$\nu = e^{(6\tilde{x}_7 + 3\tilde{x}_8 + \sqrt{3}\tilde{x}_9)/6} \begin{pmatrix} 1 & \frac{x_{10}}{\sqrt{2}} & \frac{x_{11}}{\sqrt{2}} + \frac{x_{10}x_{12}}{4} \\ 0 & e^{-\tilde{x}_8} & \frac{e^{-\tilde{x}_8}x_{12}}{\sqrt{2}} \\ 0 & 0 & e^{-(\tilde{x}_8 + \sqrt{3}\tilde{x}_9)/2} \end{pmatrix}, \quad (2.3)$$

¹In full generality, the coset space of half-maximal supergravity in three dimensions is $\text{SO}(8, p)/(\text{SO}(8) \times \text{SO}(p))$, we consider here the case $p = 4$ only.

the matrices

$$\phi = \begin{pmatrix} 0 & x_4 & x_5 \\ -x_4 & 0 & x_6 \\ -x_5 & -x_6 & 0 \end{pmatrix}, \quad \xi = \begin{pmatrix} 0 & 0 & 0 & x_1 \\ 0 & 0 & 0 & x_2 \\ 0 & 0 & 0 & x_3 \end{pmatrix}, \quad \xi^2 = \xi \xi^T, \quad (2.4)$$

and a dilaton $\tilde{\varphi} = \tilde{x}_{13}$. With this parametrization, the potential takes the form:

$$\begin{aligned} V = 4e^{-4\tilde{\varphi}} + 2e^{-2\tilde{\varphi}} & \left[-\text{tr}(m + m^{-1}) + \text{tr}(\phi m^{-1}\phi) - 2\text{tr}(\phi m^{-1}\xi^2) - 2\text{tr}(\xi^2) \right. \\ & - \text{tr}(\xi^2 m^{-1}\xi^2) + \frac{1}{2}\det(m^{-1}) \left(1 - \text{tr}(\phi^2) - \text{tr}(\xi^4) + \text{tr}(\xi^2)^2 \right) \\ & + \frac{1}{2}\text{T}(m^{-1}(\xi^2 - \phi), (\xi^2 + \phi)m^{-1}, m + (\xi^2 + \phi)m^{-1}(\xi^2 - \phi) + 2\xi^2) \\ & \left. + \frac{1}{4}\text{T}(m^{-1}, m + (\xi^2 + \phi)m^{-1}(\xi^2 - \phi) + 2\xi^2, m + (\xi^2 + \phi)m^{-1}(\xi^2 - \phi) + 2\xi^2) \right], \end{aligned} \quad (2.5)$$

where $\text{T}(A, B, C) = \varepsilon_{mnp}\varepsilon_{qrs}A^{mq}B^{nr}C^{ps}$. For later convenience, we define a thirteen-dimensional vector

$$\vec{X} = (x_1, x_2, x_3, x_4, x_5, x_6, \tilde{x}_7, \tilde{x}_8, \tilde{x}_9, x_{10}, x_{11}, x_{12}, \tilde{x}_{13}), \quad (2.6)$$

encompassing all parameters. Note here that all dilaton fields are denoted with a tilde. This convention is adopted in anticipation of a later redefinition of the form $x_i = e^{\tilde{x}_i}$ for these fields.

The search for flat directions of the potential (2.5) can then start from a study of its gradient ∇V . However, carrying out the search for stationary points analytically is by far too complex, even reducing the number of variables or using a symbolic solver such as Mathematica [52]: the resulting expressions are too complex to be simplified into a manageable form, and do not yield usable relationships that express some variables in terms of others. This complexity does not, however, rule out the possibility that simpler solutions satisfying the condition of vanishing gradient may exist although the solver does not find them. In this work, we aim to identify such solutions, which we seek using numerical methods. This approach has as additional advantage its broader applicability: it would be effective even in cases where an exact analytic expression for the potential is out of reach, like richer theories as the SO(8, 8) 3d truncations of type IIB supergravity on $\text{AdS}_3 \times S^3 \times T^4$ or $\text{AdS}_3 \times S^3 \times S^3 \times S^1$.

A first trial and error numerical exploration suggests that attention can be focused on the five scalars $x_1, x_2, x_4, \tilde{x}_8$, and x_{10} . This subset, selected at random from the original set of thirteen scalar fields, serves to render the problem more tractable at the outset. The potential (2.5) then reduces to

$$\begin{aligned} V = \frac{1}{4}e^{-2\tilde{x}_8} & \left[4 + 4x_1^4 + e^{4\tilde{x}_8}(2 + x_{10}^2)^2(1 + x_2^4) \right. \\ & - 4e^{3\tilde{x}_8}(2 + x_{10}^2) \left(2 - x_1^2 + \sqrt{2}x_1x_{10}x_2^3 + x_2^4 + 2x_1x_2x_4 + x_4^2 - x_2^2(1 - x_1^2 + x_4^2) \right) \\ & - 8e^{\tilde{x}_8}(2 + x_1^4 + \sqrt{2}x_1^3x_{10}x_2 - x_2^2 - 2x_1x_2x_4 + x_4^2 - x_1^2(1 - x_2^2 + x_4^2)) \\ & + 4e^{2\tilde{x}_8}(2 - 4x_1^2 + x_1^4 - 4x_2^2 + 4x_1^2x_2^2 + x_2^4 + x_{10}^2(1 + 3x_1^2x_2^2) + 4x_4^2 + x_4^4) \\ & \left. + 8\sqrt{2}x_{10}e^{2\tilde{x}_8}(x_1^3x_2 + x_1^2x_4 - x_2^2x_4 + x_1(x_2^3 - x_2x_4^2)) \right]. \end{aligned} \quad (2.7)$$

To compute the gradient, we exploit the fact that we possess an analytic expression for the scalar potential (2.7). Note however that it could have been fully done using a numerical approach as automatic differentiations methods. It is important to emphasize, however, that the $\{x_1, x_2, x_4, \tilde{x}_8, x_{10}\}$ subset does not constitute a consistent truncation of the full theory, as higher-order interactions could, in principle, source the remaining scalar modes. Nonetheless, the five-scalar model remains a self-consistent

theory in its own right and can be used independently. This distinction does not affect our numerical approach: the solutions we obtain remain valid solutions of the complete theory. This is because we identify the critical points of the scalar potential by first computing ∇V with all scalar fields included, *i.e.* using eq. (2.5), and only then setting the remaining fields,

$$\vec{y} = (x_3, x_5, x_6, \tilde{x}_7, \tilde{x}_9, x_{11}, x_{12}, \tilde{x}_{11}), \quad (2.8)$$

to zero, as detailed in sec. 3.1. By performing the differentiation before the truncation, we ensure that the resulting configurations satisfy the full equations of motion and are therefore legitimate solutions of the complete theory.

3 Numerical analysis

To identify the flat directions in the potential, we will use numerical tools. The procedure is as follows: first, we sample the underlying manifold, and then we use numerical techniques to extract analytical information from the resulting cloud of points. The sampling of the manifold is done through a gradient descent. We then analyse its dimension and structure using a local Principal Component Analysis (PCA) and clustering algorithms, and finally extract analytical constraints defining the manifold thanks to Symbolic Regression methods. We detail these different procedures in the following.

3.1 Sampling the manifold: gradient descent

To perform the gradient descent, we initialise randomly and uniformly points within a hypercube of range $[-2, 2]$. This choice is important to ensure that points are not restricted to the inner range $[-1, 1]$, which will be crucial for the Symbolic Regression.² (ce: Put the footnote later in the SR section?) \Leftarrow
As the example we're focussing on has a five-dimensional parameter space, we choose to generate $n_{\text{points}} = 10^5$ points: we aim at populating all five directions and want approximately $\mathcal{O}(10)$ points per direction. As the solutions manifold lie within the five dimensional space, its intrinsic dimension is less than or equal to 5. The value 10^5 then serves as a conservative upper bound for the number of points needed to adequately sample the manifold.

We then perform a gradient descent on the points using TensorFlow's automatic differentiation [53]. The loss function is defined as

$$\mathcal{L} = \sum_{i=1}^{n_{\text{points}}} \left\| \nabla V(\vec{X}^{(i)}) \Big|_{\vec{y}^{(i)}=0} \right\|^2, \quad (3.1)$$

see eq. (2.6) and (2.8) for the definitions of the vectors \vec{X} and \vec{y} . The exponents (i) denote the data points, with $i \in \llbracket 1, n_{\text{points}} \rrbracket$. This way we only keep $(x_1^{(i)}, x_2^{(i)}, x_4^{(i)}, \tilde{x}_8^{(i)}, x_{10}^{(i)})$ alive after we have taken the derivative. As already mentioned above, we use the analytic formula for ∇V , but we could have performed the gradient descent by fully using automatic differentiation.

The Adam optimizer was employed throughout the gradient descent procedure [54]. Through trial and error, we observed that periodically reinitialising the optimizer significantly improved the convergence rate. It was reinitialised at iterations 250, 500, 750, with the learning rate α fixed at 10^{-2} . At iteration 1000, the optimizer was reinitialised once more, this time with a reduced learning rate of 10^{-3} . A final reinitialisation was performed at iteration 1500, setting the learning rate to 10^{-4} , and

²The rationale is that symbolic regression (SR) evaluates the data points $\vec{x}^{(i)}$ using a candidate polynomial function p . The loss function—used later to assess the quality of a given polynomial—is defined as $\sum_i p(\vec{x}^{(i)})$. When all data points lie within the interval $[-1, 1]$, the SR algorithm tends to favor high-degree polynomials, as these often yield smaller values over this range, thereby artificially lowering the loss. Introducing data points with absolute values greater than 1 mitigates this bias by discouraging the selection of overly complex polynomials that exploit this effect.

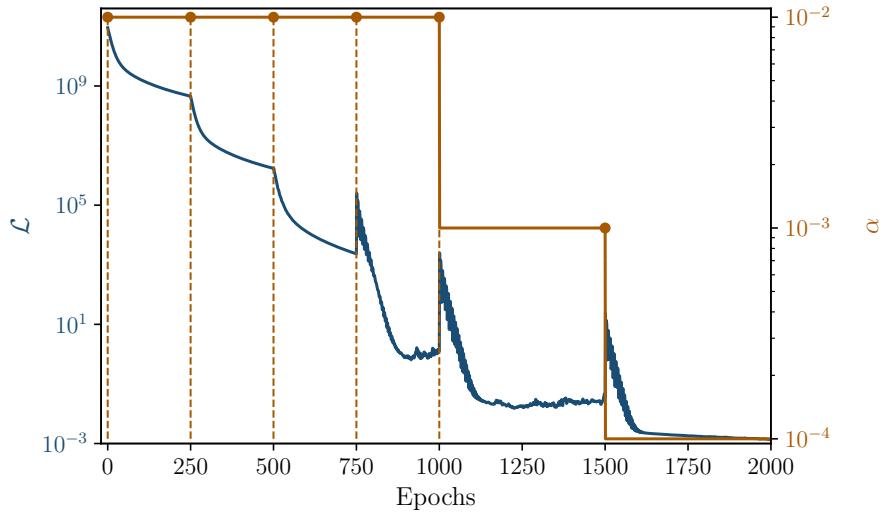


Fig. 1 Evolution of the loss function (3.1) during gradient descent (right y -axis, in blue) and corresponding learning rate schedule α (left y -axis in orange), both plotted on a logarithmic scale. Dashed lines indicate epochs at which the optimizer was reinitialized.

the optimization was continued for an additional 500 epochs. The evolution of the loss function (3.1), along with the learning rate schedule, is shown in fig. 1. As can be seen in this figure, the convergence rate improves significantly each time the optimizer is reinitialized. We also observe that, following the last few reinitialisations, the loss exhibits a small bump immediately after the restart. We interpret this behaviour as follows: the learning rate gets internally adjusted by the Adam optimizer during the descent, and may well be smaller than the instructions when the reset occurs. The learning rate then gets suddenly increased, and some points that previously had low loss values may momentarily worsen before benefiting from faster convergence. This effect is likely due to points initially located in regions with weak attractive basins being pushed toward areas where the potential gradient is steeper, thus accelerating their convergence. This strategy introduces the risk of desampling certain regions in favour of others, introducing the risk to hide some flat directions in the potential. However, we believe that with enough points, it is very unlikely for a flat direction to be completely hidden from us and desampled. This method bears similarity to the concept of warm restarts introduced in ref. [55], where the authors reset the learning rate to some value at each reset, without reinitialising the whole optimizer. However, we found that simply scheduling the learning rate without resetting the optimizer yielded slower convergence. We interpret this as follows: when the optimizer is reinitialized, it effectively "forgets" its past gradient history. As a result, the actual learning rate used corresponds more closely to the specified value, rather than being internally adjusted based on accumulated past gradients. This effect seems to contribute to faster convergence in our case.

Upon completion, the loss function converges around 0.002. The distribution of the values of the gradient at each of the data points is plotted in fig. 2(a). More than 93% of the points converged to values of $|\nabla V|$ lower than 10^{-4} , and more than 99% of them exhibited gradient norms smaller than 10^{-3} , yielding a satisfying sampling of the flat directions. As can be observed in fig. 2(b), all data points converged to values close to $V(0) = -4$. Specifically, nearly all points attained $V(0)$ within an absolute error of at least 10^{-4} , with the exception of four points whose deviations were of the order of 10^{-3} (not visible on the graph). As the origin $\vec{X} = 0$ of the parameter space corresponds to the three-dimensional truncation of the round $\text{AdS}_3 \times S^3$ solution of half-maximal supergravity in six dimensions, this confirms that the data points lie within flat directions of the potential.

As a first visualisation, we present a triangular plot of the data in fig. 3. (ce: Sure of the use of “triangular”?) This figure shows all possible 2d projections of the data, along with the 1d histograms

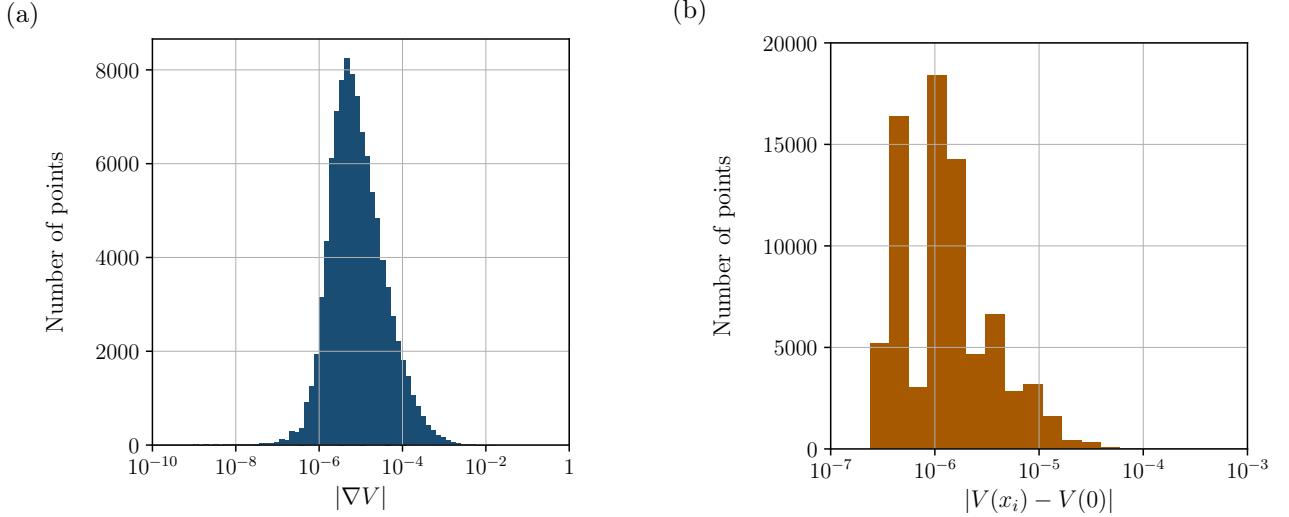


Fig. 2 (a) Histogram of the the norm of the gradient for each point (x -axis is in log scale). (b) Histogram of the distance of the value of the potential to $V(0)$ for each point (x -axis is in log scale). Note that four points have deviations of the order of 10^{-3} and are not visible on the figure.

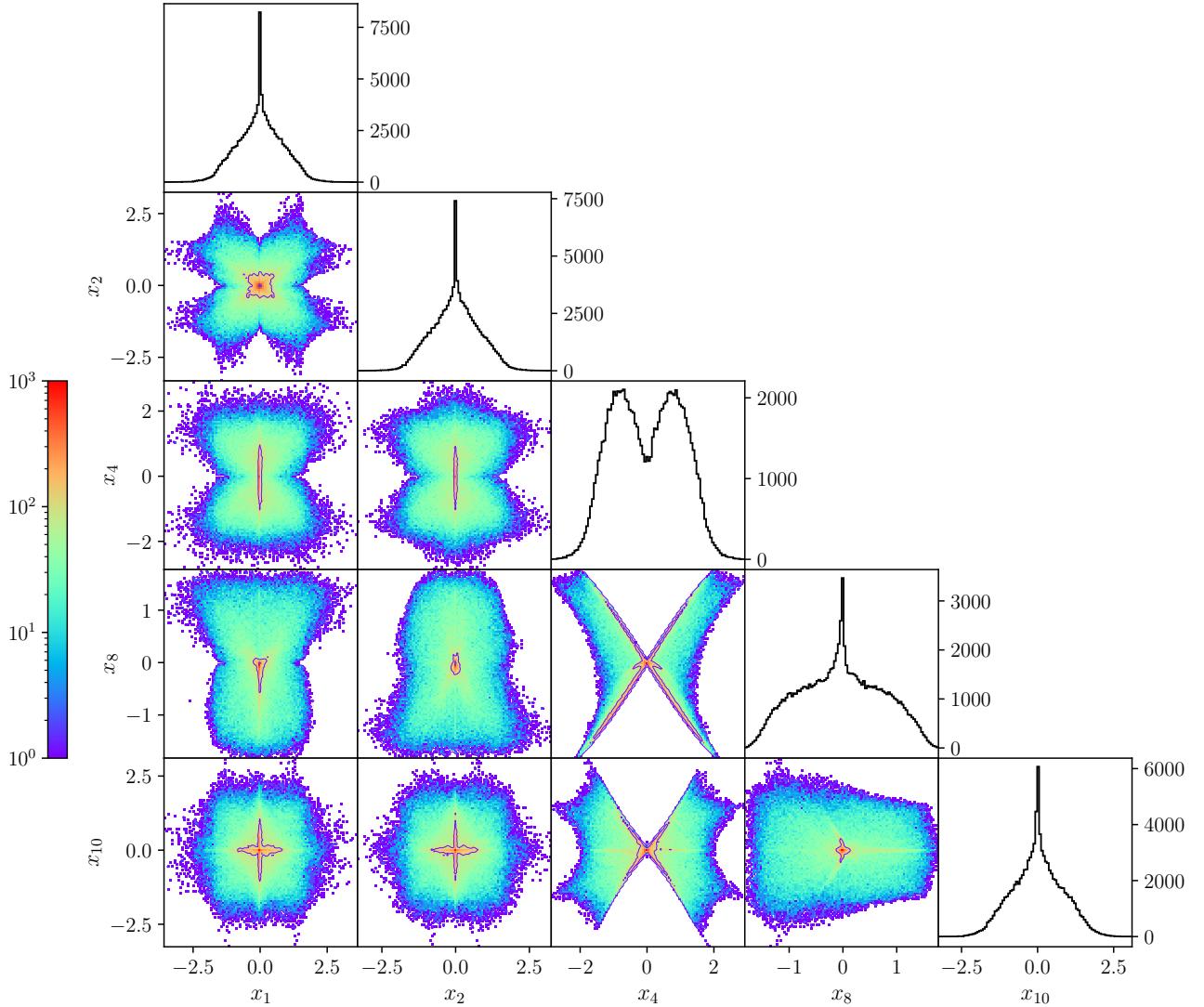


Fig. 3 Triangular plot showing all $2d$ projections and $1d$ histograms of the data after the gradient descent.

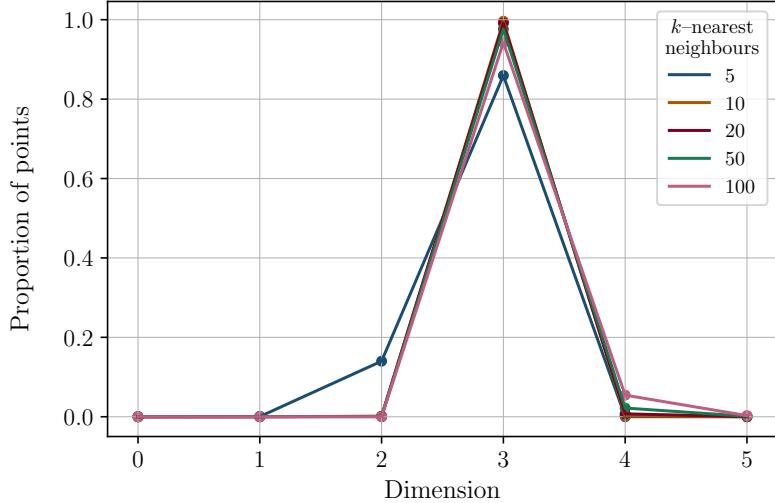


Fig. 4 Results of the local PCA analysis. The x -axis shows the dimension inferred by the algorithm, and the y -axis indicates the proportion of points for which that dimension was found. Each curve corresponds to a different value of k in the k -nearest neighbours.

of each coordinate after gradient descent. Note that all directions appear to be well populated. There also seems to be non-trivial correlations in the data, see for example the x_1/x_2 , x_4/\tilde{x}_8 or x_4/x_{10} graphs. They could be genuine correlations, or may result from larger basins of attraction. Thanks to the analysis of sec. 4, we will conclude that the structures in the x_1/x_2 plot are artefacts of the gradient descent, whereas the ones seen in the x_4/\tilde{x}_8 or x_4/x_{10} plots reflect genuine features of the manifold.

3.2 Local analysis: extracting the dimension

Once the gradient descent has been completed and the flat directions sampled, the next step is to identify the structure of the underlying manifold. Our goal is to eventually obtain an analytical expression, not just a numerical description. Before applying symbolic regression to search for such an expression, we first perform some exploratory analyses to better understand the data. Specifically, we aim at determining the dimension of the manifold and whether it consists of a single connected component or multiple disjoint components (*e.g.* two intersecting hyperplanes). To this end, we apply a local Principal Component Analysis (PCA, see for example ref. [56]). For each point, we identify its k nearest neighbours and perform a PCA on that local neighbourhood. This procedure allows us to determine how many principal directions are needed to explain a given proportion ϵ of the data variance. In other words, it provides an estimate of the local dimensionality around each point, *i.e.* the dimension of its local tangent space. We perform this analysis for several values of k , namely $k \in \{5, 10, 20, 50, 100\}$, and we fix $\epsilon = 0.99$. The results are presented in fig. 4. We observe that for every choice of k , there is a prominent peak at $d = 3$, suggesting that the underlying manifold is three-dimensional. For $k = 5$, a noticeable fraction of points are assigned dimension 2. This can be attributed to the fact that if the true dimension is 3, then selecting only 5 neighbours may not sufficiently populate all three directions, leading the algorithm to underestimate the dimensionality for a fraction of the points. Additionally, for $k \geq 20$, we observe an increasing number of points being assigned dimensions 4 or 5. This behaviour can be explained by the loss of locality when the number of neighbours becomes too large: increasing k results in a coarser approximation, and the algorithm may then incorporate points that are no longer truly local. This artificial enlargement of the neighbourhood can cause the estimated local dimensionality to rise. We thus conclude that the manifold under investigation has an intrinsic dimension of 3.

One possible scenario is that our data actually consists of several distinct three-dimensional manifolds,

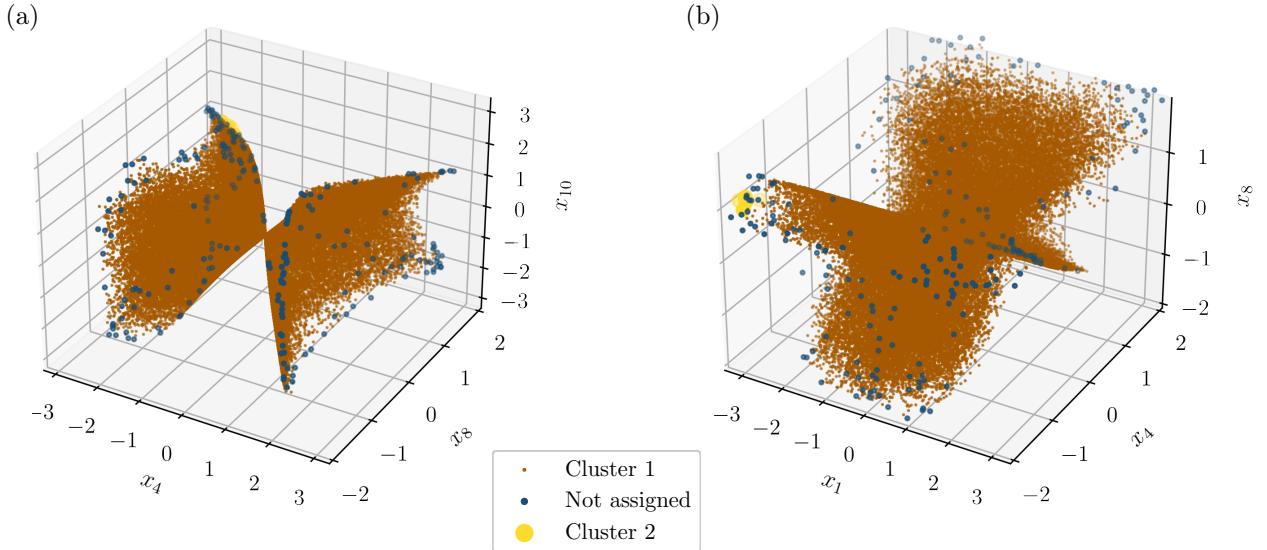


Fig. 5 3d projections of the data set in selected coordinates: (a) (x_4, x_8, x_{10}) and (b) (x_1, x_4, x_8) . The colors are assigned by the clustering algorithm: the orange and yellow points correspond to two different clusters, while the blue points were not assigned to any cluster. The size of each class of points has been tuned to favour the visualisation.

intersecting at least at the origin $\vec{X} = 0$, and the points previously identified with dimension 4 may lie at the intersections of these manifolds. Consider for example the intersection of two lines: at the intersection point, the local dimension estimated by the previous PCA algorithm would be 2. To rule out this possibility, we apply a clustering algorithm on the points with tangent spaces of dimension 3 only, as identified by the PCA with $k = 20$. This way, we remove the possible intersection points with local dimension 4 or 5. For the purpose of the clustering, we use the density-based algorithm HDBSCAN [57]. As it can be observed in fig. 3, the points obtained after the gradient descent are denser around the origin, leading the algorithm to disregard points with a norm greater than 1. To avoid this effect, we de-sample the densest areas by randomly selecting no more than 10 points per 0.25-sided hyper-cube, and apply the HDBSCAN algorithm on the remaining 72,216 points, with minimum cluster size set to 5 points. The algorithm identifies one cluster made of 5 points, one other with 71,993 points, and fails to assign any cluster to 218 points. We show in fig. 5 some 3d projections of the data to visualize the clustering. In these scatter plots, orange points belong to the main cluster, while blue points are those that the algorithm failed to assign to any cluster. The few yellow points correspond to the smallest cluster of 5 points. The size of the different points has been adjusted to facilitate the visualisation. From visual inspection, it appears that the unassigned (blue) points lie mostly on the boundary of the sampled region. We therefore interpret their unassigned status not as evidence of belonging to another manifold, but rather as a result of insufficient local density near the edges of the dataset. The smallest yellow cluster is likely an artefact of local fluctuations in the data density and is not interpreted as physically meaningful. Thus, the algorithm does indicate that over 99% of the data belongs to a single dominant cluster.

From this analysis, we conclude that the gradient descent procedure has produced a sampling of a single, connected, three-dimensional manifold.

3.3 Annealed Importance Sampling for polynomial symbolic regression

Now we have sampled the manifold and extracted some basic information about it (namely its dimension and the fact that it is made of one block), we would like to see if we can extract some analytic formula to characterise it. What we have at the moment, are points on a 3-dimensional manifold, which are embedded in a 5-dimensional space. Therefore we can conclude that, in order to characterise the manifold, we need two independent constraints on the embedding coordinates $\vec{x} = (x_1, x_2, x_4, \tilde{x}_8, x_{10})$. If we have a look at the form of the potential, we can observe that if we use $x_8 = e^{\tilde{x}_8}$, that, up to some potential global factor of the dilaton of the form $e^{\tilde{x}_8}$, this potential is actually a polynomial of the embedding coordinates. Therefore, the components of $|\nabla V|$ are also polynomials in those variables. We conclude that the constraints on the embedding coordinates we are looking for are polynomial constraints of the form $p(\vec{x}) = 0$, and that there should be at least two of those. Note that we are now taking $\vec{x} = (x_1, x_2, x_4, x_8, x_{10})$. Of course if one takes directly the gradient of (2.5), one ends up with such conditions, but none are usable directly to solve for two of the variables in terms of the others. The problem we are facing here is therefore a problem of symbolic regression: we are looking for analytic expressions (polynomial) that vanish once evaluated on our data points.

In the realm of symbolic regression, the aim is to uncover interpretable mathematical expressions that best describe a given dataset. In the problem discussed in this paper, we are looking for polynomial expressions such that $p(\vec{x}^i) = 0$, with $i \in \{1, \dots, n_sample\}$. This task involves navigating a vast, discrete, and often rugged search space of possible symbolic models, which poses significant challenges for traditional sampling methods. Markov Chain Monte Carlo (MCMC) techniques, while widely used, can struggle with poor mixing and getting trapped in local optima, especially in high-dimensional or multimodal spaces.

3.3.1 Annealed Sequential Monte Carlo principle

To address these challenges, we employ an Annealed Importance Sampling (AIS) combined with Sequential Monte Carlo (SMC) methods. AIS constructs a sequence of intermediate distributions that smoothly transition from an initial, tractable distribution (e.g., a prior over symbolic expressions) to the complex target posterior distribution. This annealing process is guided by a temperature-like parameter that gradually emphasizes the data likelihood, allowing for more efficient exploration of the probability landscape.

SMC enhances this procedure by propagating a population of particles—each representing a candidate symbolic expression—through the sequence of distributions. At each step, particles are reweighted based on the incremental change in the distribution, resampled to focus computational effort on high-probability regions, and mutated via operations. This combination of importance sampling, resampling, and mutation maintains diversity among the particles and prevents premature convergence to suboptimal models.

These features make AIS-SMC particularly well-suited for symbolic regression tasks, where the search space is not only high-dimensional but also structured and discontinuous.

Let us now explain in more details how does this procedure goes. The goal is to reconstruct some distribution function $d(z)$, where here z is going to be some polynomial. We will try to reconstruct this density function by series of density function $\pi_n(z_n) = \gamma_n(z_n)/Z_n$ with $n = 1, \dots, N$ is going to be the number of annealing steps, π_n is defined in terms of an unnormalized density γ_n and we have the normalizing constant $Z_n = \int \gamma_n(z) dz$. We also assume we have a sequence of inverse temperature constants β_n , where $0 = \beta_1 < \beta_2 < \dots < \beta_N = 1$. We then define the unnormalized density at level n in terms of a prior distribution $p_0(z)$ over the hypothesis space and a loss function $L(z)$:

$$\gamma_n(z) := d_0(z) \exp(-\beta_n L(z)). \quad (3.2)$$

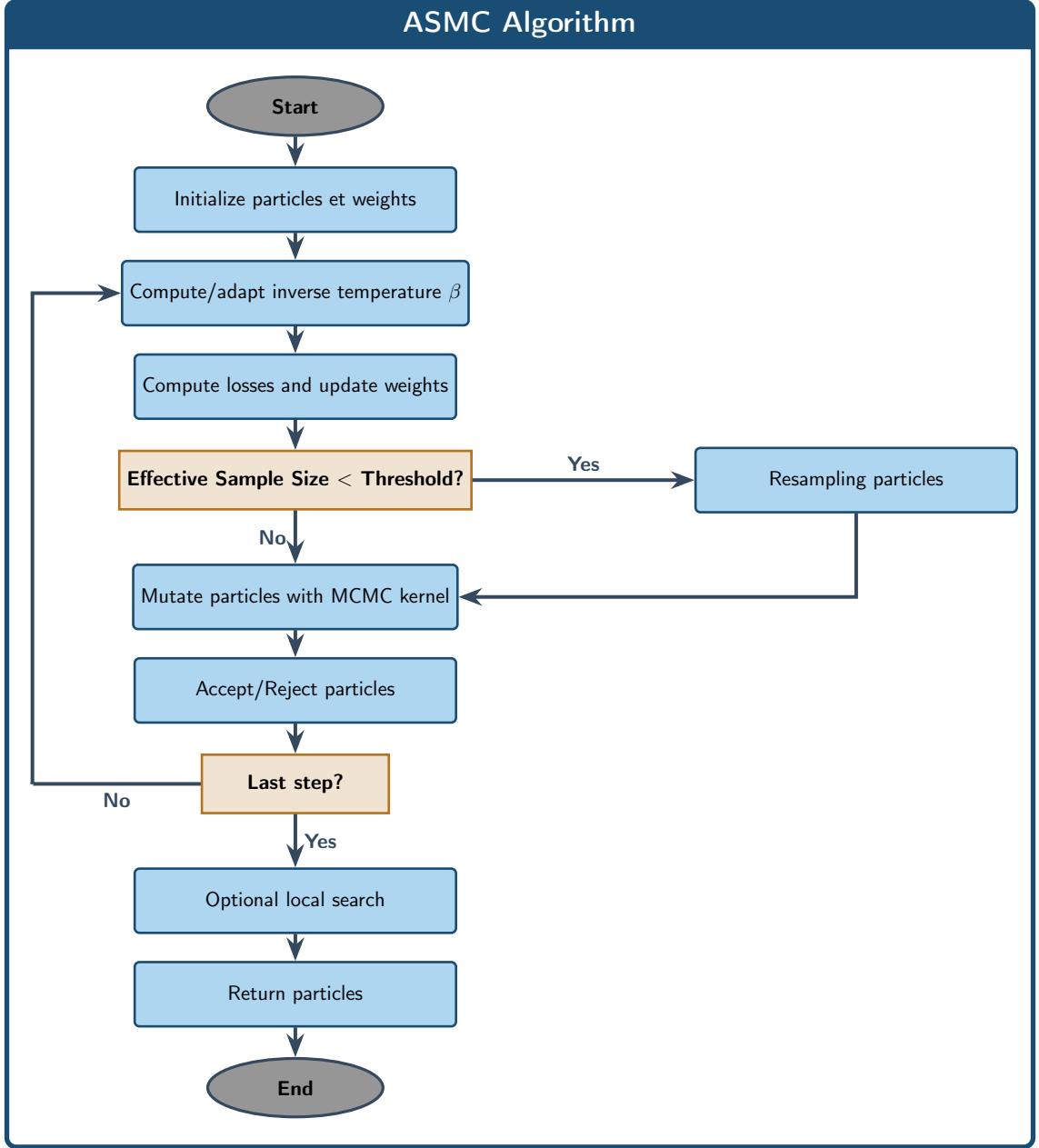


Fig. 6 The flow chart of the Annealed Importance Sampling - Sequential Monte Carlo algorithm

At each epoch, we have a set of particles and weights $\{z_{n-1}^k, w_{n-1}^k\}$ approximating π_{n-1} (meaning $\mathbb{E}_{\pi_{n-1}}[f] \approx \frac{\sum_k w_{n-1}^k f(z_{n-1}^k)}{\sum_k w_{n-1}^k}$), and we want to obtain a new set $\{z_n^k, w_n^k\}$ approximating π_n . To do so, for each particle z_{n-1}^k , we propose a new particle $z_n^k \sim q(z_n|z_{n-1}^k)$ and calculate the new unnormalized importance weight w_n^k . The latter are updated using the formula

$$w_n^k = w_{n-1}^k \times \alpha_n^k \quad (3.3)$$

where α_n^k is the incremental importance weight. The standard form for α_n^k , which requires introducing an auxiliary backward transition kernel $q(z_{n-1}|z_n)$, is:

$$\alpha_n^k = \frac{\gamma_n(z_n^k)q(z_{n-1}^k|z_n^k)}{\gamma_{n-1}(z_{n-1}^k)q(z_n^k|z_{n-1}^k)} \quad (3.4)$$

The weights are finally normalized and we get $w_n^k \rightarrow \frac{w_n^k}{\sum_j w_n^j}$.

Let's us now focus on the forward $q(z_n^k | z_{n-1}^k)$ and backward propagation kernel $q(z_{n-1}^k | z_n^k)$ for now. The forward propagation kernel defines how we generate the state at time n given the state at time $n-1$. It gives the probability to transition from z_{n-1} to z_n . The backward kernel represents a hypothetical probability of transitioning back from state z_n to state z_{n-1} . The purpose of q is to give a proposition for z_n given z_{n-1} . For the implementation of the two, we decide to implement an AIS-style MCMC code : we make some move in the space of polynomials, and then accept or reject those new polynomials based on an acceptance rate. So we first need to choose what are the available moves in the space of polynomials. Here are the choices we have made for our symbolic regression task :

- Coefficient perturbation. Given a polynomial z_{n-1} , we choose randomly one of its coefficient and modify it by a random noise from the guassian distribution $\mathcal{N}(0, \sigma^2)$. So for eg : $2x_1 + 3x_2^2 \rightarrow 2.1x_1 + 3x_2^2$.
- Variable multiplication. Given a polynomial z_{n-1} , we choose randomly one of his monomial, and multiply it by one of the available variable. So for eg : $2x_1 + 3x_2^2 \rightarrow 2x_1x_2 + 3x_2^2$.
- Variable division. Given a polynomial z_{n-1} , we choose randomly one of his monomial, and divide it by one of its variable. So for eg : $2x_1x_2 + 3x_2^2 \rightarrow 2x_1 + 3x_2^2$.

The polynomials get mutated at each epoch. The mutation is chosen randomly, assigning probabilities `p_modify` , `p_multiply` and `p_divide` to each possibility. This implementation enables a direct computation of both the forward and backward propagation kernels. In particular, for the case of coefficient perturbations, the transition is symmetric. That is,

$$q(z_n^k | z_{n-1}^k) = q(z_{n-1}^k | z_n^k), \quad (3.5)$$

which significantly simplifies the computation of the incremental importance weight.

However, this symmetry does not hold in general. For other types of operations—such as multiplication or division by a variable—the transition from z_{n-1}^k to z_n^k is generally not equivalent to the reverse transition from z_n^k to z_{n-1}^k . In such cases $q(z_n^k | z_{n-1}^k) \neq q(z_{n-1}^k | z_n^k)$ and particular care must be taken in the computation of the backward propagation kernel.

The acceptance ratio for the MCMC algorithm is given by :

$$A(z_n, z_{n-1}) = \min \left(1, \frac{\gamma_n(z_n^k) q(z_{n-1}^k | z_n^k)}{\gamma_{n-1}(z_{n-1}^k) q(z_n^k | z_{n-1}^k)} \right) \quad (3.6)$$

We then draw $u \sim \text{Uniform}(0, 1)$, accept the new particle if $u < A(z_n, z_{n-1})$ and reject it otherwise.

We have adapted the classic approach outlined above. As can be seen in (3.2), what matters the most to determine γ_n is not the inverse temperature, but rather the product of the inverse temperature and the loss function. Therefore, instead of implementing an arbitrary schedule for β , we employed an adaptive temperature approach. The number of particles accepted at each run is monitored. We aim at following the following schedule as function of the epoch:

$$\text{acceptance_ratio} = 0.8 - 0.5 \times \left(\frac{\text{epoch}}{\text{n_epochs}} \right), \quad (3.7)$$

where `n_epochs` is the total number of epochs. If we accept more particle (in the 5 last epochs), we decrease the temperature by 5%, if we accept less we increase it by 5%. This encourages exploration during the early stages of training and gradually transitions to a more selective regime. (**ce: So we ⇐ don't use eq. (3.6) at all? This is not clear to me.**)

Before closing the section, we need to discuss what is the Loss function we choose. It is required to compute the unnormalized distribution γ_n . Assume that we have $z = \sum_k c_k X_k$, where c_k are the coefficients of the polynomials, and X_k is a short notations for all the possible monomials up to a given degree (so if we have x and y as variables, and the maximum degree is 2, then X_k are $1, x, y, x^2, y^2, xy$). For the Loss function, we decided to take :

$$L(z) = \sum_i z(x_i)^2 + \frac{\lambda}{\sum_k |c_k|} \quad (3.8)$$

So the first term is just the sum of the squared of the polynomial evaluated on the data. We want to make this 0 so we find polynomial that annihilates our data. The second part is a regularisation factor: it prevents the algorithm to send all the coefficient to 0, which would give a trivial solution to the problem. We typically take $\lambda \sim \mathcal{O}(1000)$.

Once the Annealing loop is over, we end up with a total of n_{sample} particles, which in principle should be close to annihilate our data, but whose coefficients may need some more fine tuning. To deal with it, we ran a quick exploitation phase, where, for each polynomial, we now only modify its coefficients, with a perturbation $\epsilon \sim \mathcal{N}(0, \sigma_1)$, and keep the new particle, only if the Loss function is now getting smaller. This allow to fine-tune the coefficients. (**ce: Mention threshold? And \Leftarrow number of iterations?**)

4 Results

4.1 Initialisation of the algorithm

We initialized the algorithm with (**ce: Do we have all parameters here?**) \Leftarrow

```
n_epochs = 1000,
beta = 1e-6,
p_modify = 0.5,
p_multiply = 0.25,
p_divide = 0.25,
lambda = 1000,
sigma = 0.1.
```

(4.1)

This strategy was determined empirically through trial and error; a systematic analysis of its optimization is left for future work. For the representation of polynomials, we adopt a vectorial approach: each polynomial is represented as an array of its coefficients $\{c_k\}$, where $k \in \{1, \dots, n_{\text{mon}}\}$ and n_{mon} denotes the number of available monomials, determined by the chosen maximum degree `max_degree` and the number of variables. We also impose a constraint on the maximum number of terms within a single polynomial `max_num_monomials` to promote sparsity. An alternative approach would be to encourage sparsity through the prior distribution or within the loss function formulation. For the simulations we have done, we chose :

```
max_degree = 4,
max_num_monomials = 6.
```

(4.2)

With these parameters and given that we are dealing with 5 variables, $n_{\text{mon}} = 126$.

4.2 Analysis of a typical run

After a run, the 1000 particles are typically distributed into less than 50 different type of polynomials. Each particles of a given type share the same monomials, but their coefficients fluctuate a bit. We analyse them using the following procedure. We select the particle that features the minimal loss (without the regularisation, $\lambda = 0$, see eq. (3.8)), we fine-tune the coefficients by running a quick exploitation phase of 10,000 steps during which we randomly select a coefficient and modify it with a perturbation $\epsilon \sim \mathcal{N}(0, \sigma_1)$, with $\sigma_1 = 0.01$, and keep the new particle only if the loss function (without the regularisation) is getting smaller. There are two possible outputs after this phase. (i) The coefficients stay finite, and the particle is a candidate annihilator polynomial. We then filter out all particles of the same type as this candidate, as they will ultimately give the same polynomial. Or (ii), the coefficients are just getting smaller and smaller during the exploration phase, indicating that the only way to minimize the loss with the given monomials is to have very small coefficients. The particle is then disregarded. We repeat the same steps on the best particles of the remaining set until we explored all the 1000 particles.

Here is an example for a given run. The best particle after the Annealing loop is

$$-0.035 x_1 + 0.673 x_2 + 0.305 x_1 x_4 + 0.396 x_1 x_{10} - 0.630 x_2 x_8 - 0.284 x_2 x_8 x_{10}^2, \quad (4.3)$$

with loss $L^{(\lambda=0)} \simeq 2 \times 10^3$. The coefficients have been rounded to the nearest thousandth. After 10,000 steps of exploitation, this particle becomes

$$0.396 x_2 + 0.396 x_1 x_4 + 0.280 x_1 x_{10} - 0.396 x_2 x_8 - 0.198 x_2 x_8 x_{10}^2, \quad (4.4)$$

with loss $L^{(\lambda=0)} \simeq 8 \times 10^{-5}$, or equivalently

$$2 x_2 + 2.000 x_1 x_4 + 1.415 x_1 x_{10} - 2.000 x_2 x_8 - 1.000 x_2 x_8 x_{10}^2. \quad (4.5)$$

where we normalised the coefficients by setting the one of x_2 to 2. We are in case (i), the polynomial is a candidate annihilator polynomial. After filtering out the 971 particles that are of the same type, the next best particle is

$$1.039 x_2 + 0.383 x_1 x_4 + 0.466 x_1 x_{10} - 0.955 x_2 x_8 - 0.233 x_2 x_{10}^2 - 0.006 x_1 x_2 x_8 x_{10}, \quad (4.6)$$

with a loss $L^{(\lambda=0)} \simeq 2 \times 10^3$. After the exploitation phase, it becomes

$$0.006 x_2 + 0.004 x_1 x_4 + 0.002 x_1 x_{10} - 0.006 x_2 x_8 - 0.001 x_2 x_{10}^2, \quad (4.7)$$

with a loss $L^{(\lambda=0)} \simeq 4 \times 10^{-2}$. The loss is low only because the coefficients are themselves very low. This is an example of case (ii), we disregard this particle. Reproducing similar steps for the remaining particles, we get some uninteresting particles and two new candidate polynomials (with normalised coefficients): **(ce: I had to replace x_4 as given in the data to $x_4/2$ to have the right polynomials.)**

$$1.416 x_2 + 1.416 x_1 x_4 - 1.417 x_2 x_8 + x_1 x_8 x_{10} - 1.001 x_2 x_4 x_8 x_{10}, \quad L^{(\lambda=0)} \simeq 1 \times 10^{-3}, \quad (4.8a)$$

$$-1.414 x_1 + 1.414 x_1 x_8 - 1.414 x_2 x_4 x_8 + x_2 x_8 x_{10}, \quad L^{(\lambda=0)} \simeq 2 \times 10^{-4}. \quad (4.8b)$$

Given that the coefficients of the gradient ∇V are only integers and square roots of integers, the coefficients of the polynomials annihilating ∇V must be combinations of rational numbers or square roots thereof. We then deduce from eq. (4.5), (4.8a) and (4.8b) the candidate annihilator polynomials

	p_1	p_2	p_3	p_4	p_5	p_6	p_7	p_8	\emptyset
Frequency	92.6%	75.0%	51.9%	1.7%	0.2%	0.7%	0.7%	0.3%	0.1%
Maximum # of representatives	1000	1000	991	869	879	710	207	18	—

Tab. 1 Statistics of the ability of the ASMC algorithm to produce the polynomials (4.10) on 1000 runs with parameters (4.1) and (4.2). The frequencies represent the percentage of runs featuring a given polynomial in its outputs, and the maximum number of representatives gives the maximum number of particles representing the polynomial in a given run. The column \emptyset counts the runs that failed to produce an annihilator polynomial.

for the above example:

$$\begin{aligned} & 2x_2 + 2x_1x_4 + \sqrt{2}x_1x_{10} - 2x_2x_8 - x_2x_8x_{10}^2, \\ & \sqrt{2}x_2 + \sqrt{2}x_1x_4 - \sqrt{2}x_2x_8 + x_1x_8x_{10} - x_2x_4x_8x_{10}, \\ & -\sqrt{2}x_1 + \sqrt{2}x_1x_8 - \sqrt{2}x_2x_4x_8 + x_2x_8x_{10}. \end{aligned} \quad (4.9)$$

4.3 Statistics

We performed 1000 independent runs with the same parameters as above, each involving 1000 particles. It takes approximately 10 min to do a single run on a regular computer. (**ce: That's with the local search done as described in the previous section.**) The code finds the 8 different following polynomials:

$$p_1 = -\sqrt{2}x_1 + \sqrt{2}x_1x_8 + x_2x_8x_{10} - \sqrt{2}x_2x_4x_8, \quad (4.10a)$$

$$p_2 = 2x_2 - 2x_2x_8 + \sqrt{2}x_1x_{10} + 2x_1x_4 - x_2x_8x_{10}^2, \quad (4.10b)$$

$$p_3 = \sqrt{2}x_2 - \sqrt{2}x_2x_8 + \sqrt{2}x_1x_4 + x_1x_8x_{10} - x_2x_4x_8x_{10}, \quad (4.10c)$$

$$p_4 = 2x_2 - 2x_2x_8 + \sqrt{2}x_1x_8x_{10} + 2x_1x_4x_8 - 2x_2x_4^2x_8, \quad (4.10d)$$

$$p_5 = -\sqrt{2}x_1x_4 + \sqrt{2}x_2x_4 + \sqrt{2}x_1x_2x_4^2 - x_1^2x_{10} - x_2^2x_{10}, \quad (4.10e)$$

$$p_6 = -2x_1 - 2x_2x_4 - 2x_1x_4^2 + 2x_1x_8 + \sqrt{2}x_2x_{10} + x_1x_8x_{10}^2, \quad (4.10f)$$

$$p_7 = -2 + 4x_8 - 2x_8^2 + 2x_4^2x_8 - x_8^2x_{10}^2, \quad (4.10g)$$

$$p_8 = -2x_2x_4 - 2x_1x_4^2 + 2x_2x_4x_8 + \sqrt{2}x_2x_{10} - \sqrt{2}x_2x_8x_{10} + x_1x_8x_{10}^2, \quad (4.10h)$$

where $x_8 = e^{\tilde{x}_8}$. They are found with different frequencies, with some polynomials occurring more often than others, as reported in tab. 1. The algorithm typically identifies an average of 2.2 distinct polynomials per run, with a maximum number of 4, demonstrating its capacity to uncover multiple solutions simultaneously. The algorithm failed at finding a solution in only 3 runs. This is summed up in fig. 7. Note that in 77% of the time the algorithm finds more than a single solution, demonstrating the robustness of the method.

We have tracked the appearance of the polynomials (4.10) during each of the 1000 runs. To do so, we have counted at each epoch the number of particles including the same monomials as the annihilator polynomials.³ The total number of particles reproducing at least one of the annihilator polynomials listed in eq. (4.10) is plotted in fig. 8(a), together with the evolution of the inverse temperature β , both averaged on the 1000 runs. The proportion of annihilator polynomials among the particles start to be significant after approximately 700 epochs. It then increases quickly and, towards the end of each run,

³Thus, a particle featuring the same monomials as one of the polynomials (4.10) will be counted, even if the coefficients do not match, and if there are additional monomials.

an average of 88% of the particles do reproduce one of the annihilator polynomials, with a standard deviation of 20%. The evolution of this proportion is explained by the structure of the method: the code favours the particles with highest weights, and thus those that cause a significant improvement to the loss function, at each resampling. Once an annihilator polynomial is reached, it colonises larger and larger proportions of the particles at each resampling.

The evolution of the inverse temperature is dictated by eq. (3.7). Note that β is equal to 0.02 in average at the end of the runs. The temperature is thus still quite high, which favours diversity. This is a key ingredient to get more than one candidate polynomials per run. The end value of β is intimately linked to the choice of target acceptance ratio in eq. (3.7): the lower the acceptance ratio, the higher the inverse temperature β . On the one hand, if the ratio is too low, very few particles get mutated and it's difficult to explore the space of polynomials and to have diverse outputs. On the other hand, if the ratio is too high, there are too many mutations: the output features a large number of polynomials, but lots of them do not converge to an annihilator polynomial because the algorithm is not selective enough. It is a matter of balance between exploration and exploitation.

The dynamics of appearance of each annihilator polynomial is plotted in fig. 8(b), averaged on the 1000 runs. The dynamics depend strongly on the polynomial and we observe three different classes. In the first case, for p_1, p_2 and p_4 , the first occurrences appear typically after only few dozen of epochs; and in the second one, constituted of p_3, p_5, p_7 and p_8 , after few hundreds of epochs. The last class has p_6 as its only representative. This polynomial is very difficult to produce, and when it appears it is only towards the end of the runs. The stair-step patterns, with sudden jumps in the population of the polynomials alternating with phases of stagnation, are due to the use of resampling in the algorithm. When a particle gets close enough to an annihilating polynomial, its loss gets lowered significantly faster than the one of the other particles. This induces an increase of its weight and it will populate a large part of the sample at the next resampling. The coefficients get improved during the stagnation phases, inducing an increase of the polynomials weights, and thus even larger colonizations during the following resamplings. This also induces a competition between the annihilator polynomials, the ones that get bettered the more easily (typically those with fewest terms) are favoured. This mechanism explains the decrease in the averaged population of the polynomial p_8 observed after 800 epochs in fig. 8(b).

The best loss for each of the 1000 runs is shown in fig. 9 in blue, together with their mean value in yellow, with a logarithmic y -axis. The losses here do include the regularisation factor λ , see eq. (3.8), and are given without the exploitation phase discussed in sec. 4.2. For a given run, we are plotting the best loss at each epoch, the curves thus do not follow single particles. On average the loss get bettered by two orders of magnitude during a run, illustrating the convergence of the algorithm. The best loss is however quite high at the last epoch. This is linked to the very low value of β : the exploration is

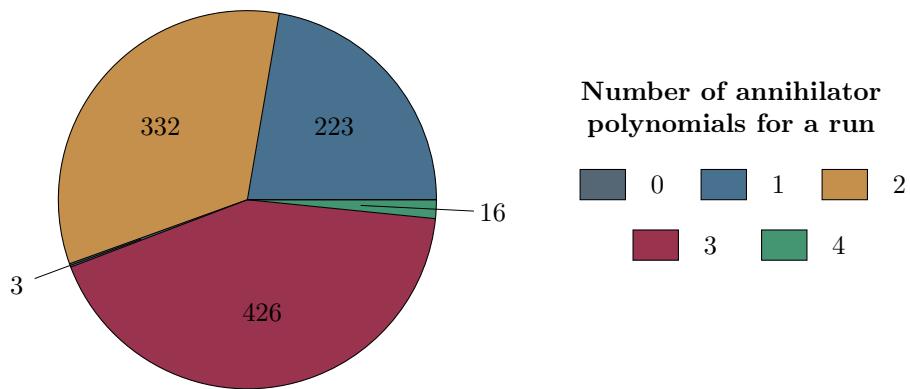


Fig. 7 Pie chart of the repartition of the number of distinct polynomials found in a single run.

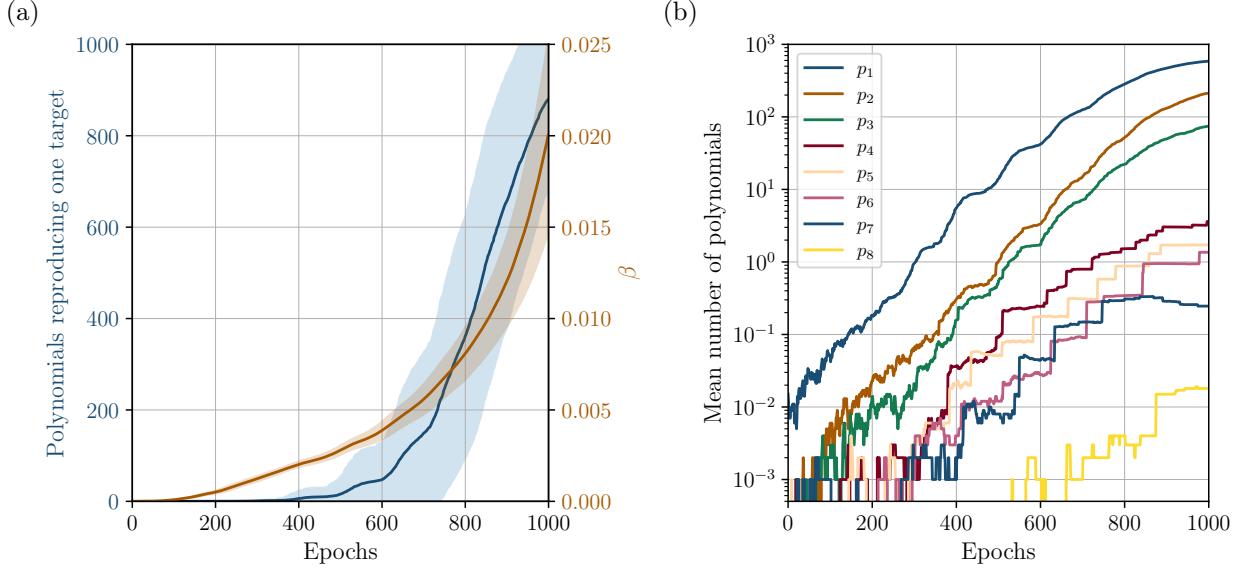


Fig. 8 (a) Evolution of the mean number of particles reproducing a target polynomial, along with its 1σ deviation, for the parameters described in (4.1) (left scale) and evolution of the inverse temperature β and its 1σ deviation (right scale). (b) Dynamics of the mean number of particles reproducing each polynomial in eq. (4.10) (the scale of the y -axis is logarithmic).

favoured on exploitation during all the process. Although the loss is high, 1000 epochs are sufficient to select candidate annihilator polynomials thanks to the exploitation phase performed on the best particles, as illustrated in sec. 4.2.

4.4 Supergravity solution

In the previous section, we introduced a numerical method that enabled symbolic regression, yielding a set of polynomials that vanish on our dataset, as presented in eq. (4.10). We know from sec. ?? that the manifold we are aiming at parametrising is three-dimensional. As the parameter space is of dimension 5, we only need two constraints to define the solutions, and the eight polynomials of eq. (4.10) are not independent. We are however only interested in finding an analytic parametrisation of the solutions manifold, and the easiest to find it is to solve the system

$$p_i = 0, \quad \forall i \in 1, \dots, 8. \quad (4.11)$$

This leads to the following rules between the parameters:

$$\begin{cases} e^{\tilde{x}_8} = \frac{x_1^2 + x_2^2}{x_2^2 + (x_1 - x_2 x_4)^2}, \\ x_{10} = \sqrt{2} x_4 \frac{x_2^2 - x_1^2 + x_1 x_2 x_4}{x_1^2 + x_2^2}. \end{cases} \quad (4.12)$$

Alternatively, the system can be recast as:

$$\begin{cases} x_1 = \frac{x_2}{e^{\tilde{x}_8} - 1} \left(x_4 e^{\tilde{x}_8} \pm \sqrt{-1 + (2 + x_4^2) e^{x_8} - e^{2\tilde{x}_8}} \right), \\ x_{10} = \mp e^{-\tilde{x}_8} \sqrt{-2 + 2(2 + x_4^2) e^{\tilde{x}_8} - 2 e^{2\tilde{x}_8}}, \end{cases} \quad (4.13)$$

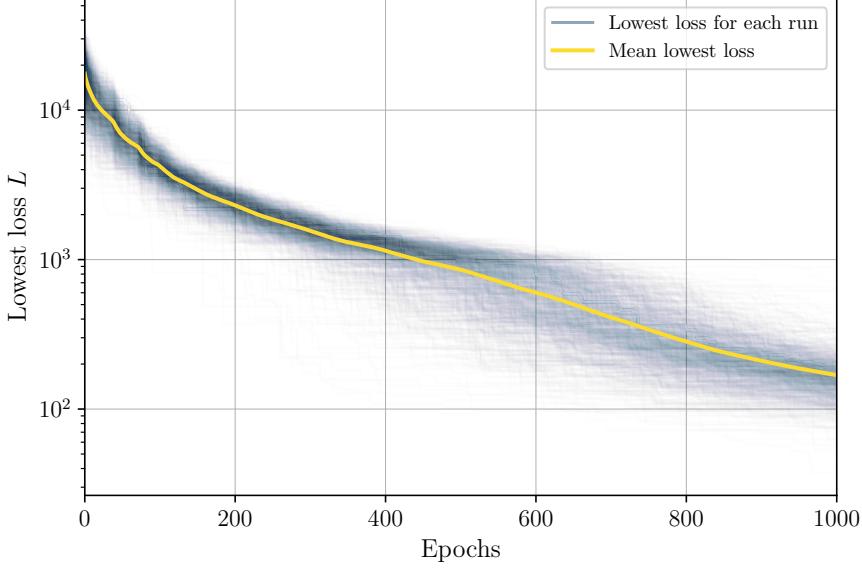


Fig. 9 Evolution of the lowest loss during each of the 1000 runs (in blue) and their average (in yellow). The losses are computed with the regularisation factor λ , see eq. (3.8).

or

$$\begin{cases} x_1 = \frac{x_2}{\sqrt{2}} \frac{e^{\tilde{x}_8/2}}{e^{\tilde{x}_8} - 1} \left(-x_{10} e^{\tilde{x}_8/2} \pm \sqrt{2 - 4 e^{\tilde{x}_8} + e^{2\tilde{x}_8} (2 + x_{10}^2)} \right), \\ x_4 = \pm \frac{e^{-x_8/2}}{\sqrt{2}} \sqrt{2 - 4 e^{\tilde{x}_8} + e^{2\tilde{x}_8} (2 + x_{10}^2)}. \end{cases} \quad (4.14)$$

As anticipated, this defines a three-parameter manifold. We tested analytically that $\nabla V = 0$ for any of these rules. The three-parameter manifold then defines a three-dimensional space of flat directions of the half-maximal supergravity scalar potential.⁴

The supergravity solution can be shown to preserve a $U(1) \times U(1)$ gauged symmetry and breaks all supersymmetry, except at the origin where we recover the $SO(4)$ isometries and the $\mathcal{N} = (0, 4)$ supercharges. Using the parametrisation (4.12), the (x_1, x_2, x_4) moduli space is most nicely parametrised using the change of coordinates

$$x_1 = r \cos(\theta) \cos(\Phi), \quad x_2 = r \cos(\theta) \sin(\Phi) \quad \text{and} \quad x_4 = r \sin(\theta), \quad (4.15)$$

for which the Zamolodchikov metric reads

$$d^2 s_{\text{Zam.}} = -dr^2 - r^2 \left(d\theta^2 - r \cos(\theta) d\theta d\Phi + \sin(\theta) dr d\Phi + \frac{1}{2} (3 + r^2 - \cos(2\theta)) d\Phi^2 \right). \quad (4.16)$$

(ce: Wouldn't it be interesting to give the scalar masses and speak of the stability of the 3d solution? Would be useful to make contact with ref. [51].) ↔

⁴One might argue that only two of the eight polynomials are sufficient to fully characterize the solution. That is, choosing any pair $(i, j) \in 1, \dots, 8$ may suffice to extract a complete description. In practice, this is not entirely accurate. While such a pair can yield partial constraints – for example, recovering eq. (4.12) – it may also produce alternative (and potentially less general) parameterisations. Upon inspection, all such partial rules are found to be consistent with, and included in, the most general expressions given in eq. (4.12).

5 Conclusion

We have presented a novel machine learning approach to systematically identify and characterise flat directions in supergravity scalar potentials. Our methodology combines gradient descent sampling with annealed importance sampling for symbolic regression on polynomials. We demonstrated its efficiency on a 5-scalar subsector of three-dimensional half-maximal supergravity, derived from the $\text{AdS}_3 \times S^3$ compactification of six-dimensional $\mathcal{N} = (1, 1)$ supergravity, or analogously of the $\text{AdS}_3 \times S^3 \times T^4$ compactification of IIB supergravity.

We developed a robust pipeline that transitions from numerical exploration to analytical understanding. The gradient descent procedure successfully samples the flat directions manifold, while local PCA analysis reveals its intrinsic dimensionality. Most notably, our annealed importance sampling approach to symbolic regression automatically and quickly discovers polynomial constraints characterizing the manifold, bypassing the computational complexity that renders direct symbolic manipulation intractable. The algorithm uncovered eight distinct polynomial relations with varying frequencies, suggesting a hierarchical structure in constraint discovery. Furthermore, upon 1000 runs the algorithm failed at finding any solution in only 3 cases, demonstrating the robustness of the method.

This approach opens several promising avenues for advancement. The scalability to higher-dimensional cases represents the most immediate challenge and opportunity. By increasing the number of particles, refining the annealing schedule, and optimizing polynomial search strategies, we anticipate extending this methodology to the full 13-scalar theory and potentially to other supergravity models. This will pave the way to an exhaustive characterisation of the flat directions of these models, with prominent applications to the AdS/CFT correspondence: flat directions of supergravity solutions having a CFT dual are in correspondence with the space of CFT deformations preserving the conformal symmetry (called the conformal manifold).

Our analysis reveals that the 5-dimensional scalar space contains a 3-dimensional conformal manifold. The discovered solutions preserve a $U(1) \times U(1)$ gauged symmetry and breaks all supersymmetries. The Zamolodchikov metric on the moduli space provides a concrete geometric description of the conformal manifold. While the specific solutions found may have limited direct physical applications, the demonstrated feasibility of our approach and its potential for systematic classification of flat directions across the supergravity landscape make it a valuable addition to the theoretical physicist's toolkit. As the solutions live in a 3d consistent truncation of both the $\text{AdS}_3 \times S^3$ solution of $\mathcal{N} = (1, 1)$ six-dimensional supergravity and of the $\text{AdS}_3 \times S^3 \times T^4$ vacuum of IIB supergravity, we can uplift the solution to 6d and 10d using the tools of exceptional field theory (ExFT) [58, 59, 50]. The higher-dimensional solutions would be of the form $\text{AdS}_3 \times M^3$ or $\text{AdS}_3 \times M^7$, with M^3 , and M^7 being deformations of the round S^3 and $S^3 \times T^4$, respectively. These deformed spaces have as moduli the 3 parameters $\{x_1, x_2, x_4\}$ that define the solutions. ExFT gives further access to the full Kaluza-Klein spectrum for those compactifications [60–62], allowing a test of the perturbative stability of the solutions, and giving valuable information on the spectrum of states of the dual CFT. These applications relies entirely on the fact that we have an analytic parametrisation of the solutions thanks to the symbolic regression we performed. We live these for future works.

The marriage of machine learning techniques with traditional supergravity analysis represents a step toward more systematic approaches to understanding the rich geometric structures underlying these theories. As computational power increases and algorithms improve, we envision this methodology becoming a standard tool for exploring the intricate relationships between geometry, symmetry, and dynamics in supergravity theories. (ce: Not sure if I agree with the last sentence. Maybe we could weaken the statement a bit?)

A Some details on the supergravity setup

We give here some details on the supergravity theory described schematically in sec 2, following ref. [46, 47].⁵ The Lagrangian is given in eq. (2.1). The gauging structure is encoded in an embedding tensor [64, 65] that takes the general form

$$\Theta_{\bar{K}\bar{L}|\bar{M}\bar{N}} = \theta_{\bar{K}\bar{L}\bar{M}\bar{N}} + \frac{1}{2} \left(\eta_{\bar{M}[\bar{K}} \theta_{\bar{L}]\bar{N}} - \eta_{\bar{N}[\bar{K}} \theta_{\bar{L}]\bar{M}} \right) + \theta \eta_{\bar{M}[\bar{K}} \eta_{\bar{L}]\bar{N}}, \quad (\text{A.1})$$

where $\theta_{\bar{K}\bar{L}\bar{M}\bar{N}} = \theta_{[\bar{K}\bar{L}\bar{M}\bar{N}]}$ is fully antisymmetric, $\theta_{\bar{L}\bar{K}} = \theta_{(\bar{L}\bar{K})}$ is symmetric and traceless, and θ is a scalar. The metric $\eta_{\bar{K}\bar{L}}$ is the SO(8,4)-invariant bilinear form used for index contractions. The gauge covariant derivatives are constructed using the embedding tensor according to

$$D_\mu = \partial_\mu + A_\mu^{\bar{M}\bar{N}} \Theta_{\bar{M}\bar{N}|\bar{P}\bar{Q}} T^{\bar{P}\bar{Q}}, \quad (\text{A.2})$$

where $A_\mu^{\bar{M}\bar{N}}$ are the gauge fields and

$$(T^{\bar{M}\bar{N}})_{\bar{P}}{}^{\bar{Q}} = 2 \delta_{\bar{P}}^{[\bar{M}} \eta^{\bar{N}]\bar{Q}} \quad (\text{A.3})$$

are the generators of the $\mathfrak{so}(8,4)$ algebra. The covariant derivative acting on the scalar matrix is thus

$$D_\mu M_{\bar{M}\bar{N}} = \partial_\mu M_{\bar{M}\bar{N}} + 4 A_\mu^{\bar{P}\bar{Q}} \Theta_{\bar{P}\bar{Q}|(\bar{M}} M_{\bar{N})\bar{K}}, \quad (\text{A.4})$$

ensuring gauge invariance of the scalar kinetic terms. The embedding tensor also defines the scalar potential as following [66, 67]:

$$\begin{aligned} V = & \frac{1}{12} \theta_{\bar{K}\bar{L}\bar{M}\bar{N}} \theta_{\bar{P}\bar{Q}\bar{R}\bar{S}} \left(M^{\bar{K}\bar{P}} M^{\bar{L}\bar{Q}} M^{\bar{M}\bar{R}} M^{\bar{N}\bar{S}} - 6 M^{\bar{K}\bar{P}} M^{\bar{L}\bar{Q}} \eta^{\bar{M}\bar{R}} \eta^{\bar{N}\bar{S}} \right. \\ & \left. + 8 M^{\bar{K}\bar{P}} \eta^{\bar{L}\bar{Q}} \eta^{\bar{M}\bar{R}} \eta^{\bar{N}\bar{S}} - 3 \eta^{\bar{K}\bar{P}} \eta^{\bar{L}\bar{Q}} \eta^{\bar{M}\bar{R}} \eta^{\bar{N}\bar{S}} \right) \\ & + \frac{1}{8} \theta_{\bar{K}\bar{L}} \theta_{\bar{P}\bar{Q}} \left(2 M^{\bar{K}\bar{P}} M^{\bar{L}\bar{Q}} - 2 \eta^{\bar{K}\bar{P}} \eta^{\bar{L}\bar{Q}} - M^{\bar{K}\bar{L}} M^{\bar{P}\bar{Q}} \right) + 4 \theta \theta_{\bar{K}\bar{L}} M^{\bar{K}\bar{L}} - 32 \theta^2. \end{aligned} \quad (\text{A.5})$$

Finally, the dynamics of the vector fields is governed by the Chern-Simons contribution

$$\mathcal{L}_{\text{CS}} = -\varepsilon^{\mu\nu\rho} \Theta_{\bar{M}\bar{N}|\bar{P}\bar{Q}} A_\mu^{\bar{M}\bar{N}} \left(\partial_\nu A_\rho^{\bar{P}\bar{Q}} + \frac{1}{3} \Theta_{\bar{R}\bar{S}|\bar{U}\bar{V}} f^{\bar{P}\bar{Q},\bar{R}\bar{S}}_{\bar{X}\bar{Y}} A_\nu^{\bar{U}\bar{V}} A_\rho^{\bar{X}\bar{Y}} \right), \quad (\text{A.6})$$

where $f^{\bar{M}\bar{N},\bar{P}\bar{Q}}_{\bar{K}\bar{L}} = 4 \delta_{[\bar{K}}^{[\bar{M}} \eta^{\bar{N}]\bar{P}} \delta_{\bar{L}]}^{\bar{Q}]}$ are the structure constants of $\mathfrak{so}(8,4)$, and $\varepsilon^{\mu\nu\rho}$ is the three-dimensional Levi-Civita symbol.

The parametrisation of the gauging to get the truncation of six-dimensional half-maximal supergravity on S^3 is best described through the decomposition of SO(8,4) as [51]

$$\begin{aligned} \text{SO}(8,4) &\longrightarrow \text{GL}(3,\mathbb{R}) \times \text{SO}(1,1) \times \text{SO}(4)_{\text{global}}, \\ X^{\bar{M}} &\rightarrow \{X^{\bar{m}}, X_{\bar{m}}, X^{\bar{0}}, X_{\bar{0}}, X^{\bar{\alpha}}\}, \end{aligned} \quad (\text{A.7})$$

where $\bar{m} \in [\![1, 3]\!]$ and $\bar{\alpha} \in [\![9, 12]\!]$ label the $\text{SL}(3,\mathbb{R})$ and $\text{SO}(4)_{\text{global}}$ vector representations. In this basis,

⁵See also ref. [63] for a review.

the $\text{SO}(8, 4)$ -invariant tensor has the expression

$$\eta_{\bar{M}\bar{N}} = \begin{pmatrix} 0 & \delta_{\bar{m}}{}^{\bar{n}} & 0 & 0 & 0 \\ \delta^{\bar{m}}{}_{\bar{n}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\delta_{\bar{\alpha}\bar{\beta}} \end{pmatrix}, \quad (\text{A.8})$$

and the embedding tensor has the following non-vanishing components:

$$\theta_{\bar{M}\bar{N}\bar{P}\bar{0}} = -\frac{1}{\sqrt{2}} X_{\bar{M}\bar{N}\bar{P}}, \quad \theta_{\bar{0}\bar{0}} = -4\sqrt{2}, \quad (\text{A.9})$$

with

$$X_{\bar{m}\bar{n}\bar{p}} = \varepsilon_{\bar{m}\bar{n}\bar{p}}, \quad X_{\bar{m}}{}^{\bar{n}\bar{p}} = \varepsilon_{\bar{m}\bar{n}\bar{p}}, \quad X^{\bar{m}}{}_{\bar{n}}{}^{\bar{p}} = \varepsilon_{\bar{m}\bar{n}\bar{p}}, \quad X^{\bar{m}\bar{n}}{}_{\bar{p}} = \varepsilon_{\bar{m}\bar{n}\bar{p}}. \quad (\text{A.10})$$

The gauge group is then

$$(T^1)^4 \times [\text{SO}(4) \ltimes (T^3 \times T^3)], \quad (\text{A.11})$$

where T^n is a translational group transforming in the representation of dimension n of $\text{SO}(4)$. A possible parametrisation of the scalar matrix is then

$$M_{\bar{M}\bar{N}} = \begin{pmatrix} m + (\xi^2 + \phi)m^{-1}(\xi^2 - \phi) + 2\xi^2 & (\xi^2 + \phi)m^{-1} & 0 & 0 & -\sqrt{2}[1 + (\xi^2 + \phi)m^{-1}]\xi \\ m^{-1}(\xi^2 - \phi) & m^{-1} & 0 & 0 & -\sqrt{2}m^{-1}\xi \\ 0 & 0 & e^{2\tilde{\varphi}} & 0 & 0 \\ 0 & 0 & 0 & e^{-2\tilde{\varphi}} & 0 \\ -\sqrt{2}\xi^T[1 + m^{-1}(\xi^2 - \phi)] & -\sqrt{2}\xi^Tm^{-1} & 0 & 0 & 1 + 2\xi^Tm^{-1}\xi \end{pmatrix}, \quad (\text{A.12})$$

in terms of a symmetric $\text{GL}(3, \mathbb{R})$ matrix m , a 3×3 antisymmetric matrix ϕ , a 3×4 matrix ξ , its 3×3 square $\xi^2 = \xi\xi^T$ and a dilaton $\tilde{\varphi}$. This encodes 22 out the 32 scalars of the theory, 10 of the scalars being gauge fixed using the translations in the gauge group. With this parametrisation, the potential takes the form (2.5). We can further consistently restrict ourselves to a set of 13 scalars by requiring invariance under the diagonal $\text{SO}(3)$ subgroup of $\text{SO}(4)_{\text{global}}$ [51], yielding to the parametrisation described in eq. (2.3) and (2.4).

B Comparison with AIFeynman

(ce: Appendix?) In comparison with state of the art technique, we have used the AIFeynman algorithm on our data set, with again $x_8 = \exp(\tilde{x}_8)$, such that we again look for polynomial expressions. We have run it with the following configuration

$$\begin{aligned} \text{BF_try_time} &= 300, \\ \text{polyfit_deg} &= 6, \\ \text{NN_epochs} &= 5000. \end{aligned} \quad (\text{B.1})$$

The first parameter fixes the time limit in seconds for each brute force call, `polyfit_deg` gives the maximum degree of the polynomial tried by the polynomial fit routine and `NN_epochs` is the number of training epoch for the internal neural network. The function used for the brut force tests are

$$+*-/> \sim \backslash \text{R1} \quad (\text{B.2})$$

The binary operations are addition, multiplication, subtraction and division. The unary ones are inverse, increment, decrement, negation and square root. Finally there is a nonary one, the unity. For more details we refer to [43].

In this algorithm, one tries to fit one of the variables in term of the others. Here, we used it to fit x_{10} in terms of the others. The AIFeynman algorithm finds a solution in TIME :

$$x_{10} = 1.414213551821 * (x_4 - ((x_1/x_2) - ((x_1/x_2)/x_8))) \quad (\text{B.3})$$

which after identifying the numerical factor with $\sqrt{2}$ and inverting the relation gives

$$-\sqrt{2}x_1 + \sqrt{2}x_1x_8 + x_2x_8x_{10} - \sqrt{2}x_2x_4x_8 = 0 \quad (\text{B.4})$$

which corresponds to p_1 of Eq. (4.10). While the AIFeynman code is able to recover one of the constraints, it exhibits several drawbacks compared to our method. First, it is significantly slower: it required TIME (to be specified) compared to approximately 600 seconds for a single run of our ASMC algorithm. Moreover, the AIFeynman framework is based on expressing one variable as a function of the others, which assumes the invertibility of the underlying relation. In general, this is not guaranteed for the class of polynomials in Eq. (4.10), and obtaining a closed-form inverse can be nontrivial or even impossible for higher-degree expressions. In our case, each polynomial involves at most quadratic terms in any single variable, ensuring invertibility, but this property would not hold for more complex models. In addition, AIFeynman is limited to recovering one constraint per run, whereas our ASMC method can discover multiple constraints simultaneously. This is reflected in our results, where each run identified an average of 2.15 polynomials, with up to 4 found in the best case. Finally, the brute-force regression strategy used by AIFeynman makes it less effective for identifying higher-degree polynomials, whose complexity increases the search difficulty. In contrast, our method treats all polynomials as equally probable, regardless of their degree, enabling it to uncover more intricate structures more efficiently ⁶.

⁶Note that we do not claim that we have used the AIFeynman in the most optimal way.

C Details on the algorithm

Algorithm 1 Local Dimension Estimation using PCA

```

1: Function LOCALDIM1POINT( $x$ ,  $\text{var\_thres}$ )
2:   Apply PCA to data matrix  $x$ 
3:   Compute explained variance ratios:  $\lambda_1, \lambda_2, \dots, \lambda_d$ 
4:   Compute cumulative variance:  $\text{cumvar}_k = \sum_{i=1}^k \lambda_i$  for  $k = 1, \dots, d$ 
5:   Find  $\text{dim} = \min\{k : \text{cumvar}_k \geq \text{var\_thres}\}$ 
6:   return  $\text{dim}$ 
7:
8: Function LOCALDIMNPOINTS( $\text{data}$ ,  $n_{\text{neig}}$ ,  $\text{var\_thres}$ )
9:    $n_{\text{points}} \leftarrow$  number of rows in  $\text{data}$ 
10:  Initialize  $\text{results} \leftarrow \emptyset$ 
11:  Compute  $\text{neighbors\_idx}$  for all points in  $\text{data}$ 
12:  for  $i = 1$  to  $n_{\text{points}}$  do ▷ Compute local dimensions for all points
13:     $\text{result}_i \leftarrow \text{COMPUTELOCALDIM}(i, \text{data}, \text{neighbors\_idx}, \text{var\_thres})$ 
14:    Append  $\text{result}_i$  to  $\text{results}$ 
15:  end for
       return  $\text{results}$ 

```

In this appendix, we detail the different algorithm that we used. We first here provide a pseudo-code for the AIS-SMC algorithm :

This pseudo code sketch how does our annealing loops work. Here is the pseudo-code for the initialization of each individual polynomial. We chose to generate more terms with a high number of monomials, because in principle there are more possibilities of polynomials with larger number of monomials. This does not prevent the code from finding polynomials with lower number of monomials as we can see on the results. The polynomial that the code find the most often is the polynomial p_1 which only has 4 terms instead of 5 or 6 for all others polynomials. The code can find such polynomials by putting gradually one or several of the coefficients to zero. Note also that we could include the possibility in the MH process too add/remove monomials. However, in the spirit of making "small" steps in the search space, we decided not to for now.

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Algorithm 2 Annealing Importance Sampling for Polynomial Discovery

Require: n_{iter} (number of iterations), $n_{\text{particles}}$ (number of particles), $\text{target_acc_rate}(\cdot, \cdot)$ (target acceptance rate), $\text{adaptation_strength}$ (temperature adaptation parameter)

- 1: Initialize particles $\{z_0^{(k)}\}_{k=1}^{n_{\text{particles}}}$ with sparse polynomials
- 2: Set $w_0^{(k)} = 1/n_{\text{particles}}$ and $w_{\text{unorm}_0^{(k)}} = 1$ for all k ▷ normalized and unnormalized weights
- 3: $z_{\text{prev}} \leftarrow z_0$, $\text{best_loss} \leftarrow \infty$, $\text{acceptance_history} \leftarrow \emptyset$, $q_{\text{ratio}}^{(k)} = 1$ for all k , $\beta_{\text{a}0} \leftarrow \text{init_beta}$ ▷ q is the proposal ratio
- 4: **for** $i = 1$ to n_{iter} **do** ▷ Temperature adaptation
- 5: $\beta_{\text{a}0} \leftarrow \beta_{\text{a}0}$
- 6: $\text{avg_acc} \leftarrow \text{mean}(\text{acceptance_history})$ ▷ only of the last 5 epochs
- 7: **if** $\text{avg_acc} < \text{target_acc_rate}(i, n_{\text{iter}}) - 0.05$ **then**
- 8: $\beta_{\text{a}0} \leftarrow \beta_{\text{a}0} \times (1 - \text{adaptation_strength})$
- 9: **else if** $\text{avg_acc} > \text{target_acc_rate}(i, n_{\text{iter}}) + 0.05$ **then**
- 10: $\beta_{\text{a}0} \leftarrow \beta_{\text{a}0} \times (1 + \text{adaptation_strength})$
- 11: **end if**
- 12: **for** $k = 1$ to $n_{\text{particles}}$ **do** ▷ Reweighting step
- 13: Compute $\log(\gamma_i(z_i^{(k)})) = -\beta_{\text{a}0} \times \text{loss}(z_i^{(k)})$
- 14: Compute $\log(\gamma_{(i-1)}(z_{(i-1)}^{(k)})) = -\beta_{\text{a}0} \times \text{loss}(z_{\text{prev}}^{(k)})$
- 15: $\Delta \log w_i^{(k)} \leftarrow \log(\gamma_i(z_i^{(k)})) - \log(\gamma_{(i-1)}(z_{(i-1)}^{(k)})) + \log(q_{\text{ratio}}^{(k)})$
- 16: **end for**
- 17: Update unnormalized weights: $w_{\text{unorm}_i^{(k)}} \leftarrow w_{\text{unorm}_{(i-1)}^{(k)}} \times \exp(\Delta \log w_i^{(k)})$
- 18: Normalize weights: $w_i^{(k)} \leftarrow w_{\text{unorm}_i^{(k)}} / \sum_{j=1}^{n_{\text{particles}}} w_{\text{unorm}_j^{(k)}}$
- 19: Compute ESS: $\text{ESS} = 1 / \sum_{k=1}^{n_{\text{particles}}} (w_i^{(k)})^2$
- 20: **if** $\text{ESS} < n_{\text{particles}} / 2$ **then** ▷ Resampling
- 21: Resample particles: $\{z_i^{(k)}\}_{k=1}^{n_{\text{particles}}} \sim \text{Multinomial}(n_{\text{particles}}, \{w_i^{(k)}\}_{k=1}^{n_{\text{particles}}})$
- 22: Reset weights: $w_i^{(k)} = 1/n_{\text{particles}}$, $w_{\text{unorm}_i^{(k)}} = 1$ for all k
- 23: Update z_{prev} accordingly
- 24: **end if**
- 25: $z_{\text{before_mutation}} \leftarrow z_i$ (save current particles)
- 26: **for** $k = 1$ to $n_{\text{particles}}$ **do** ▷ Mutating with Metropolis-Hastings
- 27: Propose $z'^{(k)} \sim q(z'^{(k)} | z_i^{(k)})$ using specialized MH proposal
- 28: Compute $\log(p_{\text{current}}^{(k)}) = -\beta_{\text{a}0} \times \text{loss}(z_i^{(k)}) + \text{sparsity_prior}(z_i^{(k)})$
- 29: Compute $\log(p_{\text{proposed}}^{(k)}) = -\beta_{\text{a}0} \times \text{loss}(z'^{(k)}) + \text{sparsity_prior}(z'^{(k)})$
- 30: $\log(\alpha^{(k)}) \leftarrow \log(p_{\text{proposed}}^{(k)}) - \log(p_{\text{current}}^{(k)}) + \log(q_{\text{ratio}}^{(k)})$
- 31: Generate $u^{(k)} \sim \mathcal{U}(0, 1)$
- 32: **if** $u^{(k)} < \alpha^{(k)}$ **then**
- 33: $z_i^{(k)} \leftarrow z'^{(k)}$ (accept proposal)
- 34: **end if**
- 35: **end for**
- 36: $\text{acceptance_rate} \leftarrow \text{fraction of accepted proposals}$
- 37: Add acceptance_rate to $\text{acceptance_history}$
- 38: $z_{\text{prev}} \leftarrow z_{\text{before_mutation}}$ (update for next iteration)
- 39: **end for**

return final particles, final losses, final weights

Algorithm 3 Initialize Sparse Polynomials

Require: `n_particles` (number of particles), `max_degree` (maximum polynomial degree), `num_vars` (number of variables), `max_num_monomials` (maximum number of monomials per polynomial), `num_terms` (total number of possible terms)

```
1: particles  $\leftarrow \mathbf{0}_{n\_particles \times num\_terms}$ 
2: max_monomials  $\leftarrow max\_num\_monomials$ 
3: n_mon_to_max_order  $\leftarrow \binom{max\_degree + num\_vars}{max\_degree}$   $\triangleright$  Compute total number of possible monomials up to max degree
4: n_mon_all_coeff  $\leftarrow \emptyset$ 
5: n_tot_pol  $\leftarrow 0$ 
6: for i = 1 to max_num_monomials do  $\triangleright$  Compute probability distribution over number of monomials
7:   n_mon_i_coeff  $\leftarrow \binom{n\_mon\_to\_max\_order}{i}$   $\triangleright$  Number of polynomials with exactly  $i$  monomials
8:   n_tot_pol  $\leftarrow n\_tot\_pol + n\_mon\_i\_coeff$ 
9:   Append n_mon_i_coeff to n_mon_all_coeff
10: end for
11: proba  $\leftarrow n\_mon\_all\_coeff / n\_tot\_pol$   $\triangleright$  Normalize to get probabilities
12: for i = 1 to n_particles do  $\triangleright$  Generate particles
13:   num_monomials  $\sim$  Categorical( $\{1, 2, \dots, max\_monomials\}$ , proba)  $\triangleright$  Sample number of monomials according to computed distribution
14:   nonzero_indices  $\leftarrow$  sample num_monomials indices from  $\{1, 2, \dots, num\_terms\}$  without replacement  $\triangleright$  Randomly select which monomials will be non-zero
15:   for j  $\in$  nonzero_indices do  $\triangleright$  Generate random coefficients
16:     particles[i, j]  $\leftarrow \mathcal{U}(-2, 2)$   $\triangleright$  Uniform in  $[-2, 2]$ 
17:   end for
18: end for
  return particles
```

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