SAFE RULES FOR THE IDENTIFICATION OF ZEROS IN THE SOLUTIONS OF THE SLOPE PROBLEM*

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Abstract. In this paper we propose a methodology to accelerate the resolution of the so-called “Sorted L-One Penalized Estimation” (SLOPE) problem. Our method leverages the concept of “safe screening”, well-studied in the literature for *group-separable* sparsity-inducing norms, and aims at identifying the zeros in the solution of SLOPE. More specifically, we introduce a family of $n!$ safe screening rules for this problem, where n is the dimension of the primal variable, and propose a tractable procedure to verify if one of these tests is passed. Our procedure has a complexity $\mathcal{O}(n\log n + LT)$ where $T \leq n$ is a problem-dependent constant and L is the number of zeros identified by the tests. We assess the performance of our proposed method on a numerical benchmark and emphasize that it leads to significant computational savings in many setups.

Key words. SLOPE, safe screening, acceleration techniques, convex optimization

AMS subject classifications. 68Q25, 68U05

1. Introduction. During the last decades, sparse linear regression has attracted much attention in the field of statistics, machine learning and inverse problems. It consists in finding an approximation of some input vector $\mathbf{y} \in \mathbb{R}^m$ as the linear combination of a few columns of a matrix $\mathbf{A} \in \mathbb{R}^{m\times n}$ (often called dictionary). Unfortunately, the general form of this problem is NP-hard and convex relaxations have been proposed in the literature to circumvent this issue. The most popular instance of convex relaxation for sparse linear regression is undoubtely the so-called “LASSO” problem where the coefficients of the regression are penalized by an ℓ_1 norm, see [11]. Generalized versions of LASSO have also been introduced to account for some possible structure in the pattern of the nonzero coefficients of the regression, see [2].

In this paper, we focus on the following generalization of LASSO:

$$\min_{\mathbf{x}\in\mathbb{R}^n} P(\mathbf{x}) \triangleq \tfrac{1}{2}\|\mathbf{y}-\mathbf{A}\mathbf{x}\|_2^2 + \lambda\, r_{\text{SLOPE}}(\mathbf{x}), \quad \lambda > 0 \tag{1.1}$$

where

$$r_{\text{SLOPE}}(\mathbf{x}) \triangleq \sum_k \gamma_k |\mathbf{x}|_{[k]} \tag{1.2}$$

with

$$\gamma_1 > 0, \qquad \gamma_1 \geq \cdots \geq \gamma_n \geq 0, \tag{1.3}$$

and $|\mathbf{x}|_{[k]}$ is the kth largest element of $\mathbf{x}$ in absolute value, that is

$$\forall \mathbf{x} \in \mathbb{R}^n:\ |\mathbf{x}|_{[1]} \geq |\mathbf{x}|_{[2]} \geq \ldots \geq |\mathbf{x}|_{[n]}. \tag{1.4}$$

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Problem (1.1) is commonly referred to as *"Sorted L-One Penalized Estimation" (SLOPE)* or *"Ordered Weighted L-One Linear Regression"* in the literature and has been introduced in two parallel works [5, 20].[1] The first instance of a problem of the form (1.1) (for some nontrivial choice of the parameters γ_k's) is due to Bondell and Reich in [7]. The authors considered a problem similar to (1.1), named "Octagonal Shrinkage and Clustering Algorithm for Regression" (OSCAR), where the regularization function is a linear combination of an ℓ_1 norm and a sum of pairwise ℓ_∞ norms of the elements of $\mathbf{x}$, that is

$$r_{\text{OSCAR}}(\mathbf{x}) = \beta_1 \|\mathbf{x}\|_1 + \beta_2 \sum_{j'>j} \max(x_{j'}, x_j), \tag{1.5}$$

for some $\beta_1 \in \mathbb{R}_+^*$, $\beta_2 \in \mathbb{R}_+$. It is not difficult to see that $r_{\text{OSCAR}}(\mathbf{x})$ can be expressed as a particular case of $r_{\text{SLOPE}}(\mathbf{x})$ with the following choice $\gamma_k = \beta_1 + \beta_2(n-k+1)$. We note that some authors have recently considered "group" versions of the SLOPE problem where the ordered ℓ_2 norm of subsets of $\mathbf{x}$ is penalized by a decreasing sequence of parameters γ_k, see *e.g.*, [9, 24, 25].

SLOPE enjoys several desirable properties which have attracted many researchers during the last decade. First, it was shown in several works that, for some proper choices of parameters γ_k's, SLOPE promotes *sparse* solutions with some form of *"clustering"*[2] of the nonzero coefficients, see *e.g.*, [7, 20, 28, 36]. This feature has been exploited in many application domains: portfolio optimization [29, 43], genetics [25], magnetic-resonance imaging [15], subspace clustering [35], deep neural networks [45], etc. Moreover, it has been pointed out in a series of works that SLOPE has very good statistical properties: it leads to an improvement of the false detection rate (as compared to LASSO) for moderately-correlated dictionaries [6, 24] and is minimax optimal in some asymptotic regimes, see [31, 37].

Another desirable feature of SLOPE is its convexity. In particular, it was shown in [6, Proposition 1.1] and [44, Lemma 2] that $r_{\text{SLOPE}}(\mathbf{x})$ is a norm as soon as (1.3) holds. As a consequence, several numerical procedures have been proposed in the literature to find the global minimizer(s) of problem (1.1). In [46] and [6], the authors considered an accelerated gradient proximal implementation for OSCAR and SLOPE, respectively. In [29], the authors tackled problem (1.1) via an alternating-direction method of multipliers [8]. An approach based on an augmented Lagrangian method was considered in [33]. In [44], the authors expressed $r_{\text{SLOPE}}(\mathbf{x})$ as an atomic norm and particularized a Frank-Wolfe minimization procedure [22] to problem (1.1). An efficient algorithm to compute the Euclidean projection onto the unit ball of the SLOPE norm was provided in [13]. Finally, in [10] a heuristic "message-passing" method was proposed.

In this paper, we propose a new "safe screening" procedure to accelerate the resolution of SLOPE. The concept of "safe screening" is well known in the LASSO literature: it consists in performing simple tests to identify the zero elements of the minimizers; this knowledge can then be exploited to reduce the problem dimensionality by discarding the columns of the dictionary weighted by the zero coefficients. Safe screening for LASSO has been first introduced by El Ghaoui *et al.* in the seminal paper [23] and extended to *group-separable* sparsity-inducing norm in [34]. Safe screening has rapidly been recognized as a simple procedure to dramatically accelerate the resolution of LASSO, see *e.g.*, [12, 19, 26, 27, 32, 39, 41]. The term "safe" refers to

[1] We will stick to the former denomination in the following.

[2] More specifically, groups of nonzero coefficients tend to take on the same value.

the fact that all the elements identified by a safe screening procedure are theoretically guaranteed to correspond to zeros of the minimizers. In constrast, *unsafe* versions of screening for LASSO (often called "strong screening rules") also exist, see [38]. More recently, screening methodologies have been extended to detect saturated components in different convex optimization problems, see [16, 17].

In this paper, we derive *safe* screening rules for SLOPE and emphasize that their implementation can speed up the running time of standard optimization procedures by several orders of magnitude. We note that the SLOPE norm is not group-separable and the methodology proposed in [34] does therefore not trivially apply here. Prior to this work, we identified two contributions addressing screening for SLOPE. In [30], the authors proposed an extension of the *strong* screening rules derived in [38] to the SLOPE problem. In [3], the authors suggested a simple test to identify some zeros of the SLOPE solutions. Although the derivations made by these authors have been shown to contain several technical flaws [18], their test can be cast as a particular case of our result in Theorem 4.1 (and is therefore quite unexpectedly safe).

The paper is organized as follows. We introduce the notational conventions used throughout the paper in Section 2 and recall the main concepts of safe screening for LASSO in Section 3. Section 4 contains our proposed safe screening rules for SLOPE. Section 5 illustrates the effectiveness of the proposed approach through numerical simulations. All technical details and mathematical derivations are postponed to Appendices A and B.

2. Notations.

Unless otherwise specified, we will use the following notation conventions throughout the paper. Vectors are denoted by lowercase bold letters (*e.g.*, $\mathbf{x}$) and matrices by uppercase bold letters (*e.g.*, $\mathbf{A}$). The "all-zero" vector of dimension n is written $\mathbf{0}_n$. We use symbol $^{\mathrm{T}}$ to denote the transpose of a vector or a matrix. $\mathbf{x}_{(j)}$ refers to the jth component of $\mathbf{x}$. When referring to the sorted entries of a vector, we use bracket subscripts; more precisely, the notation $\mathbf{x}_{[k]}$ refers to the kth largest value of $\mathbf{x}$. For matrices, we use $\mathbf{a}_j$ to denote the jth column of $\mathbf{A}$. We use the notation $|\mathbf{x}|$ to denote the vector made up of the absolute value of the components of $\mathbf{x}$. The sign function is defined for all scalars x as $\mathrm{sign}\,(x) = x/|x|$ with the convention $\mathrm{sign}\,(x) = 0$.

Calligraphic letters are used to denote sets (*e.g.*, $\mathcal{J}$) and $\mathrm{card}\,(\cdot)$ refers to their cardinality. If $a < b$ are two integers, $[\![a, b]\!]$ is used as a shorthand notation for the set $\{a, a+1, \ldots, b\}$.

Given a vector $\mathbf{x} \in \mathbb{R}^n$ and a set of indices $\mathcal{J} \subseteq [\![1, n]\!]$, we let $\mathbf{x}_{\mathcal{J}}$ be the vector of components of $\mathbf{x}$ with indices in $\mathcal{J}$. Similarly, $\mathbf{A}_{\mathcal{J}}$ denotes the submatrix of $\mathbf{A}$ whose columns have indices in $\mathcal{J}$. $\mathbf{A}_{\backslash \ell}$ corresponds to matrix $\mathbf{A}$ deprived of its ℓth column.

3. Screening: main concepts.

"Safe screening" has been introduced by El Ghaoui *et al.* in [23] for ℓ_1-penalized problems:

$$\min_{\mathbf{x} \in \mathbb{R}^n} P(\mathbf{x}) \triangleq \tfrac{1}{2} f(\mathbf{A}\mathbf{x}) + \lambda \|\mathbf{x}\|_1, \quad \lambda > 0 \tag{3.1}$$

where $f\colon \mathbb{R}^m \to \mathbb{R}$ is a closed convex function. It is grounded on the following ideas.

First, it is well-known that ℓ_1-regularization favors sparsity of the minimizers of (3.1). For instance, if $f = \|\cdot\|_2^2$ and the solution of (3.1) is unique, it can be shown that the minimizer contains at most m nonzero coefficients, see *e.g.*, [21, Theorem 3.1]. Second, if some zeros of the minimizers are identified, (3.1) can be shown to be equivalent to a problem of *reduced* dimension. More precisely, let $\mathcal{L} \subseteq [\![1, n]\!]$ be a set of indices such that we have for any minimizer $\mathbf{x}^\star$ of (3.1):

$$\forall \ell \in \mathcal{L}: \ \mathbf{x}^\star_{(\ell)} = 0 \tag{3.2}$$

and let $\bar{\mathcal{L}} = [\![1, n]\!] \backslash \mathcal{L}$. Then the following problem

$$\min_{\mathbf{z} \in \mathbb{R}^{\mathrm{card}(\bar{\mathcal{L}})}} \ \tfrac{1}{2} f(\mathbf{A}_{\bar{\mathcal{L}}} \mathbf{z}) + \lambda \, \|\mathbf{z}\|_1, \quad \lambda > 0 \tag{3.3}$$

admits the same optimal value as (3.1) and there exists a simple bijection between the minimizers of (3.1) and (3.3). We note that $\mathbf{x}$ belongs to an n-dimensional space whereas $\mathbf{z}$ is a $\mathrm{card}\left(\bar{\mathcal{L}}\right)$-dimensional vector. Hence, solving (3.3) rather than (3.1) may lead to dramatic memory and computational savings if $\mathrm{card}\,(\mathcal{L}) \gg 1$.

The crux of screening consists therefore in identifying (some) zeros of the minimizers of (3.1) with marginal cost. El Ghaoui *et al.* emphasized that this is possible by relaxing some primal-dual optimality condition of problem (3.1). More precisely, let

$$\mathbf{u}^\star \in \operatorname*{arg\,max}_{\mathbf{u}: \|\mathbf{A}^{\mathrm{T}} \mathbf{u}\|_\infty \leq \lambda} \ D(\mathbf{u}) \triangleq -f^*(-\mathbf{u}), \tag{3.4}$$

be the dual problem of (3.1), where f^* denotes the Fenchel conjugate of f. Then, by complementary slackness, we must have for any minimizer $\mathbf{x}^\star$ of (3.1):

$$\forall \ell \in [\![1, n]\!]: \ \left(|\mathbf{a}_\ell^{\mathrm{T}} \mathbf{u}^\star| - \lambda\right) \mathbf{x}^\star_{(\ell)} = 0. \tag{3.5}$$

Since dual feasibility imposes that $|\mathbf{a}_\ell^{\mathrm{T}} \mathbf{u}^\star| \leq \lambda$, we obtain the following implication:

$$|\mathbf{a}_\ell^{\mathrm{T}} \mathbf{u}^\star| < \lambda \implies \mathbf{x}^\star_{(\ell)} = 0. \tag{3.6}$$

Hence, if $\mathbf{u}^\star$ is available, the left-hand side of (3.6) can be used to detect if the ℓth components of $\mathbf{x}^\star$ is equal to zero.

Unfortunately, finding a maximizer of dual problem (3.4) is generally as difficult as solving primal problem (3.1). This issue can nevertheless be circumvented by identifying some region $\mathcal{R}$ of the dual space (commonly referred to as *"safe region"*) such that $\mathbf{u}^\star \in \mathcal{R}$. Indeed, since

$$\max_{\mathbf{u} \in \mathcal{R}} |\mathbf{a}_\ell^{\mathrm{T}} \mathbf{u}| < \lambda \implies |\mathbf{a}_\ell^{\mathrm{T}} \mathbf{u}^\star| < \lambda, \tag{3.7}$$

the left-hand side of (3.7) constitutes an alternative (weaker) test to detect the zeros of $\mathbf{x}^\star$. For proper choices of $\mathcal{R}$, the maximization over $\mathbf{u}$ admits a simple analytical solution. For example, if $\mathcal{R}$ is a ball, that is

$$\mathcal{R} = \mathcal{S}(\mathbf{c}, R) \triangleq \{\mathbf{u} \in \mathbb{R}^m : \|\mathbf{u} - \mathbf{c}\|_2 \leq R\}, \tag{3.8}$$

then $\max_{\mathbf{u} \in \mathcal{R}} |\mathbf{a}_\ell^{\mathrm{T}} \mathbf{u}| = |\mathbf{a}_\ell^{\mathrm{T}} \mathbf{c}| + R$ and the relaxation of (3.7) leads to

$$|\mathbf{a}_\ell^{\mathrm{T}} \mathbf{c}| < \lambda - R \implies \mathbf{x}^\star_{(\ell)} = 0. \tag{3.9}$$

In this case, the screening test is straightforward to implement since it only requires the evaluation of one inner product between $\mathbf{a}_\ell$ and $\mathbf{c}$.

Many procedures have been proposed in the literature to construct safe spheres [19, 34, 42] or safe regions with refined geometries [12, 40, 41]. If f^* is a ζ-strongly convex function, a popular approach to construct a safe region is the so-called "GAP sphere" [34] whose center and radius are defined as follows:

$$\begin{aligned} \mathbf{c} &= \mathbf{u} \\ R &= \sqrt{\tfrac{2}{\zeta}(P(\mathbf{x}) - D(\mathbf{u}))} \end{aligned} \tag{3.10}$$

where $(\mathbf{x}, \mathbf{u})$ is any primal-dual feasible couple. This approach has gained in popularity because of its good behavior when $(\mathbf{x}, \mathbf{u})$ is close to optimality. In particular, if f is proper lower semi-continuous, $\mathbf{x} = \mathbf{x}^\star$ and $\mathbf{u} = \mathbf{u}^\star$, then $P(\mathbf{x}) - D(\mathbf{u}) = 0$ by strong duality [4, Proposition 15.22]. In this case, screening test (3.9) reduces to (3.6) and, except in some degenerated cases, all the zero components of $\mathbf{x}^\star$ can be identified by the screening test. In practice, this behavior also occurs for sufficiently small values of the dual gap and has been observed in many numerical experiments, see *e.g.*, [16, 19, 26, 34].

4. Safe screening rules for SLOPE. In this section, we propose a new procedure to extend the concept of safe screening to SLOPE. Our exposition is organized as follows. In Subsection 4.1 we describe our working assumptions and in Subsection 4.2 we present a family of screening tests for SLOPE (see Theorem 4.3). Each test is defined by a set of parameters $\{p_q\}_{q\in[\![1,n]\!]}$ and takes the form of a series of inequalities. We show that a simple test of the form (3.9) can be recovered for some particular value of the parameters $\{p_q\}_{q\in[\![1,n]\!]}$, although this choice does not correspond to the most effective test in the general case. In Subsection 4.3, we finally propose an efficient numerical procedure to verify simultaneously *all* the proposed screening tests.

4.1. Working hypotheses. In this section, we present two working assumptions which are assumed to hold in the rest of the paper even when not explicitly mentioned.

We first suppose that the regularization parameter λ satisfies

$$0 < \lambda < \lambda_{\max} \triangleq \max_{q\in[\![1,n]\!]} \left(\sum_{k=1}^{q} |\mathbf{A}^{\mathrm{T}}\mathbf{y}|_{[k]} / \sum_{k=1}^{q} \gamma_k \right). \tag{4.1}$$

This hypothesis implies that $\mathbf{y} \notin \ker(\mathbf{A}^{\mathrm{T}})$ which prevents the vector $\mathbf{0}_n$ from being a minimizer of the SLOPE problem (1.1). More precisely, it can be shown that under condition (1.3),

$$\lambda \text{ and } \{\gamma_k\}_{k=1}^{n} \text{ verify (4.1)} \iff \mathbf{0}_n \text{ is not a minimizer of (1.1)}. \tag{4.2}$$

A proof of this result is provided in Appendix A.2.

Second, we assume that the columns of the dictionary $\mathbf{A}$ are unit-norm, *i.e.*,

$$\forall j \in [\![1,n]\!] : \quad \|\mathbf{a}_j\|_2 = 1. \tag{4.3}$$

Assumption (4.3) greatly simplifies the statement of our results in the next subsection. However, we mention that all our subsequent derivations can easily be extended to the general case where (4.3) does not hold.

4.2. Safe screening rules. In this section, we derive a family of safe screening rules for SLOPE.

Let us first note that (1.1) admits at least one minimizer and our screening problem is therefore well-posed. Indeed, the primal cost function in (1.1) is continuous and coercitive since $r_{\text{SLOPE}}(\cdot)$ is a norm (see *e.g.*, [6, Proposition 1.1] or [44, Lemma 2]); the existence of a minimizer then follows from Weierstrass theorem [4, Theorem 1.29]. In the following, we will assume that the minimizer is unique to simplify our statements. Nevertheless, all our results extend to the general case where there exist more than one minimizer by replacing "$\mathbf{x}^\star_{(\ell)} = 0$" by "$\mathbf{x}^\star_{(\ell)} = 0$ for any minimizer of (1.1)" in all our subsequent statements.

Our starting point to derive our safe screening rules is the following primal-dual optimality condition:

THEOREM 4.1. *Let*

$$\mathbf{u}^\star = \operatorname*{arg\,max}_{\mathbf{u}\in\mathcal{U}} \; D(\mathbf{u}) \triangleq \tfrac{1}{2}\|\mathbf{y}\|_2^2 - \tfrac{1}{2}\|\mathbf{y}-\mathbf{u}\|_2^2, \tag{4.4}$$

where

$$\mathcal{U} = \left\{\mathbf{u}\colon \sum_{k=1}^{q} |\mathbf{A}^{\mathrm{T}}\mathbf{u}|_{[k]} \le \lambda \sum_{k=1}^{q} \gamma_k,\; q\in [\![1,n]\!]\right\}. \tag{4.5}$$

Then, for all integers $\ell \in [\![1,n]\!]$:

$$\forall q\in[\![1,n]\!]:\; |\mathbf{a}_\ell^{\mathrm{T}}\mathbf{u}^\star| + \sum_{k=1}^{q-1} |\mathbf{A}_{\setminus\ell}^{\mathrm{T}}\mathbf{u}^\star|_{[k]} < \lambda\sum_{k=1}^{q}\gamma_k \implies \mathbf{x}^\star_{(\ell)} = 0. \tag{4.6}$$

A proof of this result is provided in Appendix B.1. We note that (4.4) corresponds to the dual problem of (1.1), see *e.g.*, [6, Section 2.5]. Moreover, $\mathbf{u}^\star$ exists and is unique because D is a continuous strongly-concave function and $\mathcal{U}$ a closed set. The equality in (4.4) is therefore well-defined.

Theorem 4.1 provides a condition similar to (3.6) relating the dual optimal solution $\mathbf{u}^\star$ to the zero components of the primal minimizer $\mathbf{x}^\star$. Unfortunately, computing the dual solution $\mathbf{u}^\star$ requires a comparable computational burden as solving the SLOPE problem (1.1). Similarly to ℓ_1-penalized problems, tractable screening rules can nevertheless be devised if "easily-computable" upper bounds on the left-hand side of (4.6) can be found. In particular, for any set $\{B_{q,\ell}\in\mathbb{R}\}_{q\in[\![1,n]\!]}$ verifying

$$\forall q\in[\![1,n]\!]:\; |\mathbf{a}_\ell^{\mathrm{T}}\mathbf{u}^\star| + \sum_{k=1}^{q-1} |\mathbf{A}_{\setminus\ell}^{\mathrm{T}}\mathbf{u}^\star|_{[k]} \le B_{q,\ell}, \tag{4.7}$$

we readily have that

$$\forall q\in[\![1,n]\!]: B_{q,\ell} < \lambda\sum_{k=1}^{q}\gamma_k \implies \mathbf{x}^\star_{(\ell)} = 0. \tag{4.8}$$

The next lemma provides several instances of such upper bounds:

LEMMA 4.2. *Let $\mathbf{u}^\star\in\mathcal{S}(\mathbf{c},R)$. Then $\forall\ell\in[\![1,n]\!]$ and $\forall q\in[\![1,n]\!]$, we have that*

$$B_{q,\ell} \triangleq |\mathbf{a}_\ell^{\mathrm{T}}\mathbf{c}| + \sum_{k=p}^{q-1}\left|\mathbf{A}_{\setminus\ell}^{\mathrm{T}}\mathbf{c}\right|_{[k]} + (q-p+1)R + \lambda\sum_{k=1}^{p-1}\gamma_k$$

verifies (4.7) for any $p\in[\![1,q]\!]$.

A proof of this result is available in Appendix B.2.

Defining

$$\kappa_{q,p} \triangleq \lambda\left(\sum_{k=p}^{q}\gamma_k\right) - (q-p+1)R, \tag{4.9}$$

a straightforward particularization of (4.8) then leads to the following safe screening rules for SLOPE:

THEOREM 4.3. *$\forall q \in [\![1,n]\!]$, let $p_q \in [\![1,q]\!]$. Then,*

$$\forall q \in [\![1,n]\!] : \left|\mathbf{a}_\ell^{\mathrm{T}}\mathbf{c}\right| + \sum_{k=p_q}^{q-1}\left|\mathbf{A}_{\backslash\ell}^{\mathrm{T}}\mathbf{c}\right|_{[k]} < \kappa_{q,p_q} \implies \mathbf{x}^\star_{(\ell)} = 0. \tag{4.10}$$

We note that each parameter p_q can take on q different values in Theorem 4.3. Since $q \in [\![1,n]\!]$, (4.10) thus defines $n!$ different screening tests for SLOPE. We discuss two particular choices of parameters $\{p_q\}_{q\in[\![1,n]\!]}$ below and propose a polynomial-time procedure to evaluate all $n!$ tests defined by Theorem 4.3 in the next section.

Let us first consider the case where

$$\forall q \in [\![1,n]\!] :\ p_q = 1. \tag{4.11}$$

Screening test (4.10) then particularizes as

$$\forall q \in [\![1,n]\!] :\ \left|\mathbf{a}_\ell^{\mathrm{T}}\mathbf{c}\right| + \sum_{k=1}^{q-1}\left|\mathbf{A}_{\backslash\ell}^{\mathrm{T}}\mathbf{c}\right|_{[k]} < \lambda\left(\sum_{k=1}^{q}\gamma_k\right) - qR \implies \mathbf{x}^\star_{(\ell)} = 0. \tag{4.12}$$

Interestingly, (4.12) shares the same mathematical structure as optimality condition (4.6). In particular, (4.12) reduces to (4.6) when $\mathbf{c} = \mathbf{u}^\star$ and $R = 0$. In this case, it is easy to see that (4.12) is the best[3] screening test within the family of tests defined in Theorem 4.3 since an equality occurs in (4.7).

In practice, we may expect this conclusion to remain valid when R is "sufficiently" close to zero. This behavior is illustrated in Figure 1. The figure represents the proportion of zeros entries of $\mathbf{x}^\star$ detected by screening test (4.10) for different "qualities" of the safe region and different choices of parameters $\{p_q\}_{q\in[\![1,n]\!]}$. We refer the reader to Subsection 5.1 for a detailed description of the simulation setup. The center of the safe sphere used to apply (4.10) is assumed to be equal (up to machine precision) to $\mathbf{u}^\star$ and the x-axis of the figure represents the radius R of the sphere region. The green curve corresponds to test (4.12); the orange curve represents the screening performance achieved when test (4.10) is implemented for all possible choices for $\{p_q\}_{q\in[\![1,n]\!]}$. We note that, as expected, the green curve attains the best screening performance as soon as R becomes close to zero.

At the other extreme of the spectrum, another case of interest reads as:

$$\forall q \in [\![1,n]\!] :\ p_q = q. \tag{4.13}$$

Screening test (4.10) then reduces to

$$\left|\mathbf{a}_\ell^{\mathrm{T}}\mathbf{c}\right| < \lambda\gamma_n - R \implies \mathbf{x}^\star_{(\ell)} = 0. \tag{4.14}$$

[3]In the following sense: if test (4.10) passes for some choice of the parameters $\{p_q\}_{q\in[\![1,n]\!]}$, then test (4.12) also necessarily succeeds.

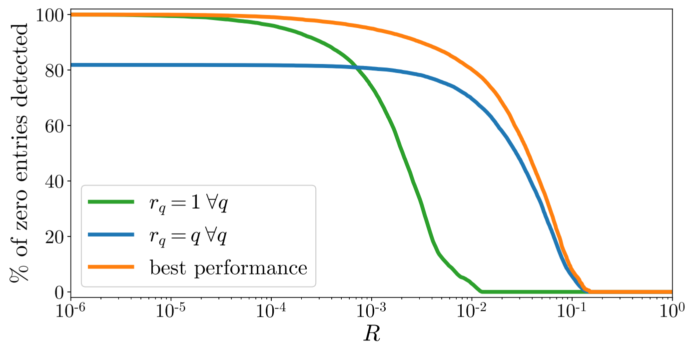

FIG. 1. *Percentage of zero entries in $\mathbf{x}^\star$ detected by the safe screening tests as a function of R, the radius of the safe sphere. Each curve corresponds to a different implementation of the safe screening test* (4.10)*: $p_q = 1\ \forall q$, see* (4.12) *(green curve), $p_q = q\ \forall q$, see* (4.14) *(blue curve), and all possible choices for $\{p_q\}_{q\in[\![1,n]\!]}$ (orange curve). The results are generated by using the* `OSCAR-1` *sequence for $\{\gamma_k\}_{k=1}^n$, the Tœplitz dictionary and the ratio $\lambda/\lambda_{\max} = 0.5$.*

Interestingly, this test has the same mathematical structure as (3.9) with the exception that λ is multiplied by the value of the smallest weighting coefficient γ_n. In particular, if $\gamma_k = 1\ \forall k$, SLOPE reduces to LASSO and test (4.14) is equivalent to (3.9); Theorem 4.3 thus encompasses standard screening rule (3.9) for LASSO as a particular case. The following result emphasizes that (4.14) is in fact the best screening rule within the family of tests defined by Theorem 4.3 when $\gamma_k = 1\ \forall k$:

LEMMA 4.4. *Let $\gamma_k = 1\ \forall k$. If test* (4.10) *passes for some choice of parameters $\{p_q\}_{q\in[\![1,n]\!]}$, then test* (4.14) *also succeeds.*

A proof of this result is available in Appendix B.3.

As a final remark, let us mention that, although we just emphasized that some choices of parameters $\{p_q\}_{q\in[\![1,n]\!]}$ can be optimal (in terms of screening performance) in some situations, no conclusion can be drawn in the general case. In particular, we found in our numerical experiments that the best choice for $\{p_q\}_{q\in[\![1,n]\!]}$ depends on many factors: the weights $\{\gamma_k\}_{k=1}^n$, the radius of the safe sphere R, the nature of the dictionary, the atom to screen, etc. This is illustrated in Fig. 1: we see that the blue and green curves deviate from the orange curve for certain values of R, that is the best screening performance is not necessarily achieved for $p_q = 1$ or $p_q = q\ \forall q$.

4.3. Efficient implementation of the n! tests. Since the best values for $\{p_q\}_{q\in[\![1,n]\!]}$ cannot be foreseen beforehand, it is desirable to evaluate the screening rule (4.10) for *any* choice of these parameters. Formally, this ideal test reads:

$$\forall q \in [\![1,n]\!], \exists p \in [\![1,q]\!] : \left|\mathbf{a}_\ell^{\mathrm{T}}\mathbf{c}\right| + \sum_{k=p}^{q-1}\left|\mathbf{A}_{\backslash\ell}^{\mathrm{T}}\mathbf{c}\right|_{[k]} < \kappa_{q,p} \implies \mathbf{x}^\star_{(\ell)} = 0. \tag{4.15}$$

Since verifying this test for a *given* index ℓ involves the evaluation of $\mathcal{O}(n^2)$ inequalities, a brute-force evaluation of (4.15) for all atoms of the dictionary requires $\mathcal{O}(n^3)$ operations. In this section, we present a procedure to perform this task with a complexity scaling as $\mathcal{O}(n\log n + TL)$ where $T \le n$ is some problem-dependent constant (to be defined later on) and L is the number of atoms of the dictionary passing test (4.15). Our procedure is summarized in Algorithms 4.1 and 4.2, and is grounded on

Algorithm 4.1 Fast implementation of SLOPE screening test (4.15)

Require: radius $R \geq 0$, sorted elements $\{|\mathbf{A}^{\mathrm{T}}\mathbf{c}|_{[k]}\}_{k=1}^{n}$

$\mathcal{L} = \emptyset$ {Set of screened atoms: init}
$\ell = n$ {Index of atom under testing: init}
Evaluate $\{f(k)\}_{k=1}^{n}$, $\{p^\star(k)\}_{k=1}^{n}$, $\{q^\star(k)\}_{k=1}^{n}$
run $= 1$
while run $== 1$ and $\ell > 0$ **do**
  test $=$ Algorithm 4.2(R,ℓ,$\{f(k)\}_{k=1}^{n}$,$\{p^\star(k)\}_{k=1}^{n}$,$\{q^\star(k)\}_{k=1}^{n}$)
  if test $== 1$ **then**
    $\mathcal{L} = \mathcal{L} \cup \{\ell\}$
    $\ell = \ell - 1$
  else
    run $= 0$ {Stop testing as soon as one atom does not pass the test}
  end if
end while
return $\mathcal{L}$ (Set of indices passing test (4.15))

the following nesting properties.

Nesting of the tests for different atoms. We first emphasize that there exists an implication between the failures of test (4.15) for some group of indices. In particular, the following result holds:

LEMMA 4.5. *Let $B_{q,\ell}$ be defined as in Lemma 4.2 and assume that*

$$|\mathbf{a}_1^{\mathrm{T}}\mathbf{c}| \geq \ldots \geq |\mathbf{a}_n^{\mathrm{T}}\mathbf{c}|. \tag{4.16}$$

Then $\forall q \in [\![1, n]\!]$:

$$\ell < \ell' \implies B_{q,\ell} \geq B_{q,\ell'}. \tag{4.17}$$

A proof of this result is provided in Appendix B.4. Lemma 4.5 has the following consequence: if (4.16) holds, the failure of test (4.15) for some $\ell' \in [\![2, n]\!]$ implies the failure of the test for any index $\ell \in [\![1, \ell' - 1]\!]$. This immediately suggests a backward strategy for the evaluation of (4.15), starting from $\ell = n$ and going backward to smaller indices. This is the sense of the main recursion in Algorithm 4.1.

We note that hypothesis (4.16) can always be verified by a proper reordering of the elements of $|\mathbf{A}^{\mathrm{T}}\mathbf{c}|$. This can be achieved by state-of-the-art sorting procedures with a complexity of $\mathcal{O}(n \log n)$. Therefore, in the sequel we will assume that (4.16) holds even if not explicitly mentioned.

Nesting of some inequalities. We next show that the number of inequalities to be verified may possibly be substantially smaller than $\mathcal{O}(n^2)$. We first focus on the case "$\ell = n$" and then extend our result to the general case "$\ell < n$".

Let us first note that under hypothesis (4.16):

$$\forall k \in [\![1, n-1]\!] : \ |\mathbf{A}_{\backslash n}^{\mathrm{T}}\mathbf{c}|_{[k]} = |\mathbf{A}_{\backslash n}^{\mathrm{T}}\mathbf{c}|_{(k)}, \tag{4.18}$$

that is the kth largest element of $|\mathbf{A}_{\backslash n}^{\mathrm{T}}\mathbf{c}|$ is simply equal to its kth component. The particularization of (4.15) to $\ell = n$ can then be rewritten as:

$$\forall q \in [\![1, n]\!], \exists p \in [\![1, q]\!] : \ |\mathbf{a}_n^{\mathrm{T}}\mathbf{c}| < \tau_{q,p} \tag{4.19}$$

where

$$\tau_{q,p} \triangleq \kappa_{q,p} - \sum_{k=p}^{q-1} \left|\mathbf{A}^{\mathrm{T}}\mathbf{c}\right|_{(k)} = \sum_{k=p}^{q-1} (\lambda\gamma_k - \left|\mathbf{A}^{\mathrm{T}}\mathbf{c}\right|_{(k)} - R) + (\lambda\gamma_q - R). \tag{4.20}$$

We show hereafter that (4.19) can be verified by only considering a "well-chosen" subset of thresholds $\mathcal{T} \subseteq \{\tau_{q,p} \colon q \in [\![1,n]\!], p \in [\![1,q]\!]\}$, see Lemma 4.6 below.

If

$$p^\star(q) \triangleq \operatorname*{arg\,max}_{p \in [\![1,q]\!]} \tau_{q,p}, \tag{4.21}$$

we obviously have

$$\left|\mathbf{a}_n^{\mathrm{T}}\mathbf{c}\right| < \tau_{q,p^\star(q)} \iff \exists p \in [\![1,q]\!] : \ \left|\mathbf{a}_n^{\mathrm{T}}\mathbf{c}\right| < \tau_{q,p}. \tag{4.22}$$

In other words, for each $q \in [\![1,n]\!]$, satisfying the inequality "$\left|\mathbf{a}_n^{\mathrm{T}}\mathbf{c}\right| < \tau_{q,p}$" for $p = p^\star(q)$ is necessary and sufficient to ensure that it is verified for some $p \in [\![1,q]\!]$. Motivated by this observation, we show the following items below: *i)* $p^\star(q)$ can be evaluated $\forall q \in [\![1,n]\!]$ with a complexity $\mathcal{O}(n)$; *ii)* similarly to p, only a subset of values of $q \in [\![1,n]\!]$ are of interest to implement (4.19).

Let us define the function:

$$\begin{aligned} f \colon [\![1,n]\!] &\to \mathbb{R} \\ p &\mapsto \textstyle\sum_{k=p}^{n} (\lambda\gamma_k - \left|\mathbf{A}^{\mathrm{T}}\mathbf{c}\right|_{(k)} - R)\,. \end{aligned} \tag{4.23}$$

We then have $\forall q \in [\![1,n]\!]$ and $p \in [\![1,q]\!]$:

$$\tau_{q,p} = f(p) - (f(q) - \lambda\gamma_q) - R. \tag{4.24}$$

In view of (4.24), the optimal value $p^\star(q)$ can be computed as

$$p^\star(q) = \operatorname*{arg\,max}_{p \in [\![1,q]\!]} f(p). \tag{4.25}$$

Considering (4.23), we see that the evaluation of $f(p)$ $\forall p \in [\![1,n]\!]$ (and therefore $p^\star(q)$ $\forall q \in [\![1,n]\!]$) can be done with a complexity scaling as $\mathcal{O}(n)$. This proves item *i)*.

Let us now show that only some specific indices $q \in [\![1,n]\!]$ are of interest to implement (4.19). Let

$$q^\star(k) \triangleq \operatorname*{arg\,max}_{q \in [\![1,k]\!]} f(q) - \lambda\gamma_q, \tag{4.26}$$

and define the sequence $\{q^{(t)}\}_t$ as

$$\begin{cases} q^{(1)} &= q^\star(n) \\ q^{(t)} &= q^\star(p^\star(q^{(t-1)}) - 1) \end{cases} \tag{4.27}$$

where the recursion is applied as long as $p^\star(q^{(t-1)}) > 1$.[4] We then have the following result whose proof is available in Appendix B.5:

[4]We note that the sequence $\{q^{(t)}\}_t$ is strictly decreasing and thus contains at most n elements.

Algorithm 4.2 Check if test (4.15) is passed for ℓ if it is passed for $\ell' > \ell$

Require: radius $R \geq 0$, index $\ell \in [\![1, n]\!]$, $\{f(k)\}_{k=1}^n$,$\{p^\star(k)\}_{k=1}^n$,$\{q^\star(k)\}_{k=1}^n$

$q = q^\star(\ell)$
test $= 1$
run $= 1$
while run $== 1$ **do**
  $\tau = f(p^\star(q)) - f(q) + (\lambda\gamma_q - R)$ {Evaluation of current threshold, see (4.24)}
  if $|\mathbf{a}_\ell^{\mathrm{T}}\mathbf{c}| \geq \tau$ **then**
    test $= 0$ {Test failed}
    run $= 0$ {Stops the recursion}
  end if
  if $p^\star(q) > 1$ **then**
    $q = q^\star(p^\star(q) - 1)$ {Next value of q to test, see (4.27)}
  else
    run $= 0$ {Stops the recursion}
  end if
end while
return test ($= 1$ if test passed and 0 otherwise)

Lemma 4.6. *Let $\mathcal{T} \triangleq \left\{\tau_{q,p^\star(q)} \colon q \in \{q^{(t)}\}_t\right\}$ where $\{q^{(t)}\}_t$ is defined in* (4.27)*. Test* (4.19) *is passed if and only if*

$$\forall \tau \in \mathcal{T} : \ |\mathbf{a}_n^{\mathrm{T}}\mathbf{c}| < \tau. \tag{4.28}$$

Lemma 4.6 suggests the procedure described in Algorithm 4.2 (with $\ell = n$) to verify if (4.19) is passed. In a nutshell, the lemma states that only a subset of card $(\mathcal{T})$ inequalities need to be taken into account to implement (4.19). We note that card $(\mathcal{T}) \leq n$ since only one value of p (that is $p^\star(q)$) has to be considered for any $q \in [\![1, n]\!]$. This is in constrast with a brute-force evaluation of (4.19) which requires the verification of $\mathcal{O}(n^2)$ inequalities.

We finally emphasize that the procedure described in Algorithm 4.2 also applies to $\ell < n$ as long as the screening test is passed for all $\ell' > \ell$. More specifically, if test (4.15) is passed for all $\ell' \in [\![\ell + 1, n]\!]$, then its particularization to atom $\mathbf{a}_\ell$ reads

$$\forall \tau \in \mathcal{T}' : \ \left|\mathbf{a}_\ell^{\mathrm{T}}\mathbf{c}\right| < \tau \tag{4.29}$$

for some $\mathcal{T}' \subseteq \mathcal{T}$.

Indeed, if screening test (4.15) is passed for all $\ell' \in [\![\ell + 1, n]\!]$, the corresponding elements can be discarded from the dictionary and we obtain a reduced problem only involving atoms $\{\mathbf{a}_{\ell'}\}_{\ell' \in [\![1,\ell]\!]}$. Since (4.16) is assumed to hold, $\mathbf{a}_\ell$ attains the smallest absolute inner product with $\mathbf{c}$ and we end up with the same setup as in the case "$\ell = n$". In particular, if screening test (4.15) is passed for all $\ell' \in [\![\ell + 1, n]\!]$, Lemma 4.6 still holds for $\mathbf{a}_\ell$ by letting $q^{(1)} = q^\star(\ell)$ in the definition of the sequence $\{q^{(t)}\}_t$ as in (4.27).

To conclude this section, let us summarize the complexity needed to implement Algorithms 4.1 and 4.2. First, Algorithm 4.1 requires the entries $|\mathbf{A}^{\mathrm{T}}\mathbf{c}|$ to be sorted to satisfy hypothesis (4.5). This involves a complexity $\mathcal{O}(n \log n)$. Moreover, the sequences $\{f(k)\}_{k=1}^n$, $\{p^\star(k)\}_{k=1}^n$, $\{q^\star(k)\}_{k=1}^n$ can be evaluated with a complexity $\mathcal{O}(n)$. Finally, the main recursion in Algorithm 4.1 implies to run Algorithm 4.2 L times, where L is the number of atoms passing test (4.15). Since Algorithm 4.2 requires to verify at most $T = \mathrm{card}\,(\mathcal{T})$ inequalities, the overall complexity of the main recursion scales as $\mathcal{O}(LT)$. Overall, the complexity of Algorithm 4.1 is therefore $\mathcal{O}(n \log n + LT)$.

5. Numerical simulations. We present hereafter several simulation results demonstrating the effectiveness of the proposed screening procedure to accelerate the resolution of SLOPE. This section is organized as follows. In Subsection 5.1, we present the experimental setups considered in our simulations. In Subsection 5.2 we compare the effectiveness of different screening strategies. In Subsection 5.3, we show that our methodology enables to reach better convergence properties for a given computational budget.

5.1. Experimental setup. We detail below the experimental setups used in all our numerical experiments.

Dictionaries and observation vectors. New realizations of $\mathbf{A}$ and $\mathbf{y}$ are drawn for each trial according to the distributions detailed hereafter. The observation vector is generated according to a uniform distribution on the m-dimensional sphere. The elements of $\mathbf{A}$ obey one of the following models:

i) the entries are i.i.d. realizations of a centered Gaussian,
ii) the entries are i.i.d. realizations of a uniform distribution on $[0, 1]$,
iii) the columns are shifted versions of a Gaussian curve.

For all distributions, the columns of $\mathbf{A}$ are then normalized. In the following, these three options will respectively be referred to as "Gaussian", "Uniform" and "Tœplitz".

Regularization parameters. We consider three differents choices for the sequence $\{\gamma_k\}_{k=1}^n$, each of them corresponding to a different instance of the well-known OSCAR problem [7, Eq. (3)]. More specifically, we let

$$\forall k \in [\![1, n]\!] : \;\gamma_k \triangleq \beta_1 + \beta_2(n - k + 1) \tag{5.1}$$

where β_1, β_2 are nonnegative parameters chosen so that $\gamma_1 = 1$ and $\gamma_n \in \{.9, .1, 10^{-3}\}$. In the sequel, these parametrizations will respectively be referred to as "`OSCAR-1`", "`OSCAR-2`" and "`OSCAR-3`".

5.2. Performance of screening strategies. We first compare the effectiveness of different screening strategies described in Section 4. More specifically, we evaluate the proportion of zero entries in $\mathbf{x}^\star$ – the solution of SLOPE problem (1.1) – that can be identified by tests (4.12), (4.14) and (4.15) as a function of the "quality" of the safe sphere. These tests will respectively be referred to as "`test-p=1`", "`test-p=q`" and "`test-all`" in the following. Figures 1 (see Subsection 4.2) and 2 represent this criterion of performance as a function of some parameter R_0 (described below) and different values of the ratio $\lambda/\lambda_{\max}$.

The results are averaged over 50 realizations. For each simulation trial, we draw a new realization of $\mathbf{y} \in \mathbb{R}^{100}$ and $\mathbf{A} \in \mathbb{R}^{100\times 300}$ according to the distributions described in Subsection 5.1. We consider Tœplitz dictionaries in Figure 1 and Gaussian dictionaries in Figure 2.

The safe sphere used in the screening tests is constructed as follows. A primal-dual solution $(\mathbf{x}_a, \mathbf{u}_a)$ of problems (1.1) and (4.4) is evaluated with "high-accuracy", *i.e.*, with a duality GAP of 10^{-14} as stopping criterion. More precisely, $\mathbf{x}_a$ is first evaluated by solving the SLOPE problem (1.1) with the algorithm proposed in [5]. To evaluate $\mathbf{u}_a$, we extend the so-called "dual scaling" operator [23, Section 3.3] to

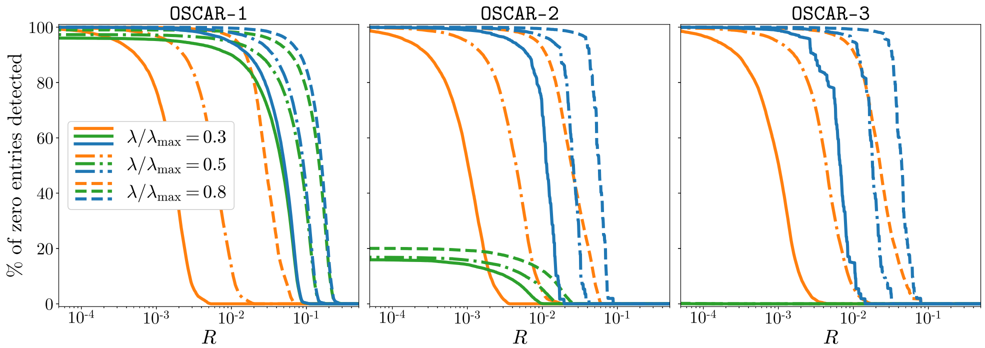

Fig. 2. *Percentage of zero entries in the solution of the SLOPE problem identified by* `test-p=1` *(orange lines),* `test-p=q` *(green lines) and* `test-all` *(blue lines) as a function of R_0 for the Gaussian dictionary, three values of $\lambda/\lambda_{\max}$ and three parameter sequences $\{\gamma_k\}_{k=1}^n$.*

the SLOPE problem. More precisely, we let $\mathbf{u}_a = (\mathbf{y} - \mathbf{A}\mathbf{x}_a)/\beta(\mathbf{y} - \mathbf{A}\mathbf{x}_a)$ where

$$\forall \mathbf{z} \in \mathbb{R}^m : \ \beta(\mathbf{z}) \triangleq \begin{cases} 1 & \text{if } \mathbf{z} \in \mathcal{U} \\ \max_{q \in [\![1,n]\!]} \dfrac{\sum_{k=1}^q \left|\mathbf{A}^{\mathrm{T}}\mathbf{z}\right|_{[k]}}{\lambda \sum_{k=1}^q \gamma_k} & \text{otherwise.} \end{cases} \tag{5.2}$$

The couple $(\mathbf{x}_a, \mathbf{u}_a)$ is then used to construct a sphere $\mathcal{S}(\mathbf{c}_a, R_a)$ in $\mathbb{R}^m$ whose parameters are given by

$$\mathbf{c} = \mathbf{u}_a \tag{5.3a}$$

$$R = R_0 + \sqrt{2(P(\mathbf{x}_a) - D(\mathbf{u}_a))} \tag{5.3b}$$

where R_0 is a nonnegative scalar. We note that for $R_0 = 0$, the latter sphere corresponds to the GAP safe sphere described in (3.10).[5] Hence, (5.3a) and (5.3b) define a safe sphere for any choice of the nonnegative scalar $R_0 \geq 0$.

Figure 1 concentrates on the sequence `OSCAR-1` whereas, in Figure 2, each subfigure corresponds to a different parametrization of the SLOPE parameters. For the three considered screening strategies, we observe that the detection performances decrease as R_0 increases. Interestingly, different behaviors can be noticed. For all simulation setups, `test-p=1` reaches a detection rate of 100% whenever R_0 is sufficiently small. However, the performance of `test-p=q` varies from one sequence to another. In particular, it outperforms `test-p=1` for `OSCAR-1`, is able to detect (at best) 20% of the zero entries for `OSCAR-2` while all entries fail the test for `OSCAR-3` and all considered values of R_0. Finally, `test-all` always outperforms (quite logically) the two other strategies. The gap in performance depends on both the setup and the radius R_0, and can be quite significant in some cases. As a striking example, we see that for `OSCAR-1`, $\lambda/\lambda_{\max} = 0.5$ and $R_0 = 5 \times 10^{-3}$, 80% of additional entries pass `test-all` as compared to `test-p=1`.

These results may be explained as follows. First, we already mentioned in Section 4 that when the radius of the safe sphere is sufficiently small (that is, when R_0 is

[5] We note that the GAP safe sphere derived in [34] for problem (3.1) extends to SLOPE since 1) the dual problem has the same mathematical form and 2) its derivation does not leverage the definition of the dual feasible set.

close to zero), `test-p=1` is expected to be the best[6] screening test within the family of tests defined in Theorem 4.3. Similarly, if the SLOPE weights satisfy $\gamma_1 = \gamma_n$, we showed in Lemma 4.4 that no test in Theorem 4.3 can outperform `test-p=q`. Hence, one may reasonably expect that this conclusion remains valid whenever $\gamma_1 \simeq \gamma_n$, as observed for the sequence `OSCAR-1` in our simulations. On the other hand, passing `test-p=q` becomes more difficult as parameter γ_n is small. As a matter of fact, the test will *never* pass when $\gamma_n = 0$. In our experiments, the sequences $\{\gamma_k\}_{k=1}^n$ are such that γ_n is close to zero for `OSCAR-2` and `OSCAR-3`. Finally, since `test-all` encompasses the two other tests, it is expected to always perform at least as well as the latter.

5.3. Benchmarks. As far as our simulation setup is concerned, the results presented in the previous section show a significant advantage in implementing `test-all` in terms of detection performance. However, this conclusion does not include any consideration about the numerical complexity of the tests. We note that, although the proposed screening rules can lead to a significant reduction of the problem dimensions, our tests also induce some additional computational burden. In particular, we emphasized in Subsection 4.3 that `test-all` can be verified for all atoms of the dictionary with a complexity $\mathcal{O}(n \log n + TL)$ where $T \leq n$ is a problem-dependent parameter and L is the number of atoms passing the test. We also note that implementating the dual scaling operation (5.2) (appearing in the construction of the GAP safe sphere) requires sorting the entries of a n-dimensional vector. Hence, the implementation of any test defined in Theorem 4.3 involves a complexity scaling at least as $\mathcal{O}(n \log n)$.

In this section, we investigate the benefits of interleaving the proposed safe screening methodology with the iterations of a proximal gradient algorithm [5]. We consider the following three solving strategies:

1. Implement the proximal gradient procedure [5] with *no* screening.
2. Interleave the iterations of the proximal gradient algorithm with `test-p=q`.
3. Interleave the iterations of the proximal gradient algorithm with `test-p=q` and `test-all`: `test-all` is applied every time the radius of the safe sphere is divided by 2; otherwise `test-p=q` is performed.

These strategies will respectively be denoted "`PG-no`", "`PG-p=q`" and "`PG-all`" in the sequel. We note that these three options induce an increasing computational overhead. First, `PG-no` implements no screening and does therefore induce no additional computational burden. Second, since `test-p=q` only requires to verify one inequality, the computational overhead of `PG-p=q` is dominated by the construction of the safe sphere and thus scales as $\mathcal{O}(n \log n)$. Finally, the implementation of `test-all` induces a complexity $\mathcal{O}(n \log n + TL)$ where $T \leq n$ is a problem-dependent parameter and L is the number of atoms passing the test.

We compare the performance of these solving strategies by resorting to Dolan-Moré profiles [14]. More precisely, we run each procedure for a given budget of time (that is the algorithm is stopped after a predefined amount of time) on $I = 50$ different instances of the SLOPE problems. Each instance is generated by drawing a new dictionary $\mathbf{A} \in \mathbb{R}^{100\times 300}$ and observation vector $\mathbf{y} \in \mathbb{R}^{100}$ according to the distributions described in Subsection 5.1. We then compute the following performance

[6]in the sense defined in Footnote 3 page 7.

profile for each solver $\texttt{solv} \in \{\texttt{PG-no}, \texttt{PG-p=q}, \texttt{PG-all}\}$:

$$\rho_{\texttt{solv}}(\delta) \triangleq 100 \, \frac{\operatorname{card}\left(\{i \in [\![1, I]\!] \colon d_{i,\texttt{solv}} \leq \delta\}\right)}{I} \quad \forall \delta \in \mathbb{R}_+ \tag{5.4}$$

where $d_{i,\texttt{solv}}$ denotes the dual gap achieved by solver `solv` for problem instance i. $\rho_{\texttt{solv}}(\delta)$ thus represents the (empirical) probability that solver `solv` reaches a dual gap no greater than δ for the considered budget of time.

Figure 3 presents the performance profiles obtained for three types of dictionaries (Gaussian, Uniform and Tœplitz) and three different weighting sequences $\{\gamma_k\}_{k=1}^n$ (`OSCAR-1`, `OSCAR-2` and `OSCAR-3`). The results are displayed for $\lambda/\lambda_{\max} = 0.5$ but similar performance profiles have been obtained for other values of the ratio $\lambda/\lambda_{\max}$. All algorithms are implemented in Python with Cython bindings and experiments are run on a Dell laptop, 1.80 GHz, Intel Core i7. For each setup, we adjusted the time budget so that $\rho_{\texttt{PG-all}}(10^{-8}) \simeq 50\%$ for the sake of comparison.

As far as our simulation setup is concerned, these results show that the proposed screening methodologies improve the solving accuracy as compared to a standard proximal gradient. `PG-all` improves the average accuracy over `PG-no` in all the considered settings. The gap in performance depends on the setup but is generally quite significant. `PG-p=q` also enhances the average accuracy in most cases. As expected its behavior is more sensitive to the choice of the weighting sequence $\{\gamma_k\}_{k=1}^n$. In particular, this strategy may lead to poor screening performance when $\gamma_n \simeq 0$ as emphasized in Subsection 5.2. This results in no accuracy gain over `PG-no` as illustrated in Figure 3 for the sequence `OSCAR-3`. Nevertheless, we note that, even in absence of gain, `PG-p=q` does not seem to significantly degrade the performance as compared to `PG-no`.

6. Conclusions.

In this paper we proposed a new methodology to safely identify the zeros of the solutions of the SLOPE problem. In particular, we introduced a family of $n!$ screening rules (indexed by some parameters $\{p_q\}_{q=1}^n$) where n is the dimension of the primal variable. Each test of this family takes the form of a series of inequalities which, when verified, imply the nullity of some coefficient of the minimizers. Interestingly, the proposed tests encompass standard "sphere" screening rule for LASSO as a particular case for some $\{p_q\}_{q=1}^n$, although this choice does not correspond to the most effective test in the general case. We then introduced an efficient numerical procedure to evaluate all $n!$ screening tests simultaneously. Our algorithm has a complexity $\mathcal{O}(n \log n + TL)$ where $T \leq n$ is some problem-dependent constant and L is the number of elements passing at least one test of the proposed family. We finally assessed the performance of our screening strategy through numerical simulations and showed that the proposed methodology allows significant computational gains to evaluate the solution of SLOPE to some accuracy.

Appendix A. Miscellaneous results.

Appendix A.1 reminds useful results from convex analysis applied to the SLOPE problem (1.1). Appendix A.2 provides a proof of (4.2). In all the statements below, $\partial r_{\text{SLOPE}}(\mathbf{x})$ denotes the subdifferential of $r_{\text{SLOPE}}(\,\cdot\,)$ evaluated at $\mathbf{x}$.

A.1. Some results of convex analysis.

We remind below several results of convex analysis that will be used in our subsequent derivations. The first lemma provides a necessary and sufficient condition for $\mathbf{x}^\star \in \mathbb{R}^n$ to be a minimizer of the SLOPE problem (1.1):

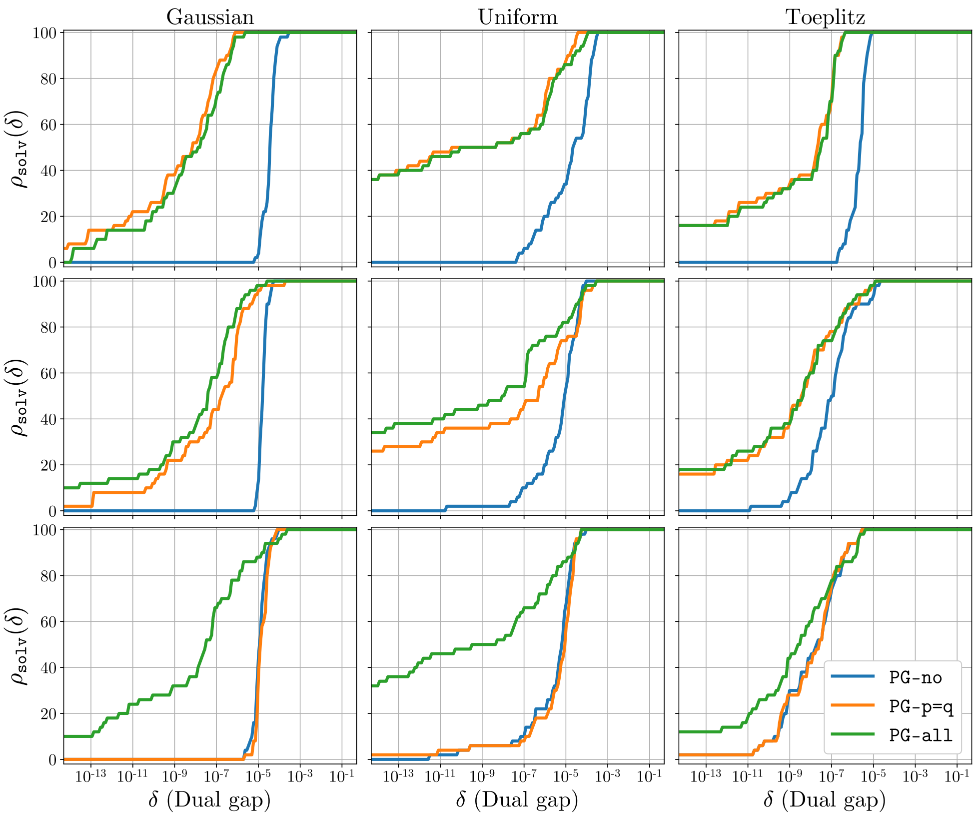

FIG. 3. *Performance profiles of `PG-no`, `PG-p=q` and `PG-all` obtained for the "Gaussian" (column 1), "Uniform" (column 2) and "Tœplitz" (column 3) dictionaries and $\lambda/\lambda_{\max} = 0.5$ with a budget of time. First row: `OSCAR-1`, second row: `OSCAR-2` and third row: `OSCAR-3`.*

LEMMA A.1. $\mathbf{x}^\star$ *is a minimizer of* (1.1) $\iff \lambda^{-1}\mathbf{A}^{\mathrm{T}}(\mathbf{y}-\mathbf{A}\mathbf{x}^\star) \in \partial r_{\text{SLOPE}}(\mathbf{x}^\star)$.

Lemma A.1 follows from a direct application of Fermat's rule [4, Proposition 16.4] to problem (1.1). We note that under condition (1.3), $r_{\text{SLOPE}}(\mathbf{x})$ defines a norm on $\mathbb{R}^n$, see *e.g.*, [6, Proposition 1.1] or [44, Lemma 2]. The subdifferential $\partial r_{\text{SLOPE}}(\mathbf{x})$ is therefore well defined for all $\mathbf{x} \in \mathbb{R}^n$ and writes as

$$\partial r_{\text{SLOPE}}(\mathbf{x}) = \left\{\mathbf{g} \in \mathbb{R}^n : \mathbf{g}^{\mathrm{T}}\mathbf{x} = r_{\text{SLOPE}}(\mathbf{x}) \text{ and } r^*_{\text{SLOPE}}(\mathbf{g}) \leq 1\right\}, \tag{A.1}$$

where

$$r^*_{\text{SLOPE}}(\mathbf{g}) \triangleq \sup_{\mathbf{x}\in\mathbb{R}^n} \mathbf{g}^{\mathrm{T}}\mathbf{x} \;\text{ s.t. } r_{\text{SLOPE}}(\mathbf{x}) \leq 1, \tag{A.2}$$

is the dual norm of $r_{\text{SLOPE}}(\mathbf{x})$, see *e.g.*, [1, Eq. (1.4)].

The next lemma states a technical result which will be useful in the proof of Theorem 4.1 in Appendix B:

LEMMA A.2. *If* $\mathbf{g} \in \partial r_{\text{SLOPE}}(\mathbf{x})$, *then* $\mathbf{x}^{\mathrm{T}}(\mathbf{g}-\mathbf{g}') \geq 0\, \forall \mathbf{g}' \in \mathbb{R}^n$ *s.t.* $r^*_{\text{SLOPE}}(\mathbf{g}') \leq 1$.

Proof. Let $\mathbf{g} \in \partial r_{\text{SLOPE}}(\mathbf{x})$. From (A.1) we have

$$\mathbf{g}^{\mathrm{T}}\mathbf{x} = r_{\text{SLOPE}}(\mathbf{x}), \tag{A.3a}$$

$$1 \geq r^{*}_{\text{SLOPE}}(\mathbf{g}). \tag{A.3b}$$

Moreover, because $r_{\text{SLOPE}}(\mathbf{x})$ is a norm we can write:

$$r_{\text{SLOPE}}(\mathbf{x}) = \sup_{\mathbf{g}' \in \mathbb{R}^n} \mathbf{g}'^{\mathrm{T}}\mathbf{x} \;\text{ s.t. } r^{*}_{\text{SLOPE}}(\mathbf{g}') \leq 1, \tag{A.4}$$

see *e.g.*, [21, Eq. (A.5)]. Using (A.3a) and (A.3b), we obtain that $\mathbf{g}$ is a maximizer of the optimization problem in the right-hand side of (A.4). The result stated in the lemma then corresponds to the first-order optimality condition of this problem. ◻

In the last lemma of this section, we provide a closed-form expression of the subdifferential and the dual norm of $r_{\text{SLOPE}}(\mathbf{x})$:[7]

LEMMA A.3. *The dual norm and the subdifferential of $r_{\text{SLOPE}}(\mathbf{x})$ respectively write:*

$$r^{*}_{\text{SLOPE}}(\mathbf{g}) = \max_{q \in [\![1,n]\!]} \frac{1}{\sum_{k=1}^{q} \gamma_k} \sum_{k=1}^{q} |\mathbf{g}|_{[k]},$$

$$\partial r_{\text{SLOPE}}(\mathbf{x}) = \left\{ \mathbf{g} \in \mathbb{R}^n \colon \mathbf{g}^{\mathrm{T}}\mathbf{x} = r_{\text{SLOPE}}(\mathbf{x}) \textit{ and } \forall q \in [\![1,n]\!] : \sum_{k=1}^{q} |\mathbf{g}|_{[k]} \leq \sum_{k=1}^{q} \gamma_k \right\}.$$

Proof. The expression of the dual norm is a direct consequence of [44, Lemma 4]. More precisely, the authors showed that

$$r^{*}_{\text{SLOPE}}(\mathbf{g}) = \max_{\mathbf{v} \in \mathcal{V}} \mathbf{g}^{\mathrm{T}}\mathbf{v}$$

where $\mathcal{V} \triangleq \bigcup_{q=1}^{n} \left\{ \frac{1}{\sum_{k=1}^{q}\gamma_k}\mathbf{s} \colon \mathbf{s} \in \{0,-1,+1\}^n, \operatorname{card}\left(\{j | \mathbf{s}_{(j)} \neq 0\}\right) = q \right\}$. A compact rewriting of this expression leads to

$$r^{*}_{\text{SLOPE}}(\mathbf{g}) = \max_{q \in [\![1,n]\!]} \frac{1}{\sum_{k=1}^{q} \gamma_k} \sum_{k=1}^{q} |\mathbf{g}|_{[k]} \tag{A.5}$$

which is precisely the expression given in Lemma A.3. The expression of the subdifferential then follows from (A.1) by plugging the expression of the dual norm in the inequality "$r^{*}_{\text{SLOPE}}(\mathbf{g}) \leq 1$". ◻

A.2. Proof of (4.2). We first observe that

$$\mathbf{0}_n \text{ is not a minimizer of (1.1)} \iff \lambda^{-1}\mathbf{A}^{\mathrm{T}}\mathbf{y} \notin \partial r_{\text{SLOPE}}(\mathbf{0}_n), \tag{A.6}$$

as a direct consequence of Lemma A.1. Particularizing the expression of $\partial r_{\text{SLOPE}}(\mathbf{x})$ in Lemma A.3 to $\mathbf{x} = \mathbf{0}_n$, the right-hand side of (A.6) can equivalently be rewritten as

$$\exists\, q \in [\![1,n]\!]\,,\; \lambda^{-1} \sum_{k=1}^{q} \left|\mathbf{A}^{\mathrm{T}}\mathbf{y}\right|_{[k]} > \sum_{k=1}^{q} \gamma_k. \tag{A.7}$$

[7] We note that an expression of the subdifferential of $r_{\text{SLOPE}}(\mathbf{x})$ has already been derived in [10, Fact A.2 in supplementary material]. However, the expression of the subdifferential proposed in Lemma A.3 has a more compact form and is better suited to our subsequent derivations.

Since $\gamma_1 > 0$ and the sequence $\{\gamma_k\}_{k=1}^n$ is nonnegative by hypothesis (1.3), (A.7) can also be rewritten as

$$\exists\, q \in [\![1, n]\!] \,,\ \lambda < \frac{\sum_{k=1}^{q} \left|\mathbf{A}^{\mathrm{T}}\mathbf{y}\right|_{[k]}}{\sum_{k=1}^{q} \gamma_k}. \tag{A.8}$$

The statement in (4.2) then follows by noticing that the right-hand side of (4.1) is a compact reformulation of (A.8).

Appendix B. Proofs related to screening tests.

B.1. Proof of Theorem 4.1. In this section, we provide the technical details leading to (4.6). Our derivation leverages the Fermat's rule and the expression of the subdifferentiable derived in Lemma A.3.

We prove (4.6) by contraposition. More precisely, we show that if $\mathbf{x}^\star_{(\ell)} \neq 0$ for some $\ell \in [\![1, n]\!]$, then

$$\exists q_0 \in [\![1, n]\!],\ |\mathbf{a}_\ell^{\mathrm{T}}\mathbf{u}^\star| + \sum_{k=1}^{q_0-1} |\mathbf{A}_{\backslash\ell}^{\mathrm{T}}\mathbf{u}^\star|_{[k]} \geq \lambda \sum_{k=1}^{q_0} \gamma_k. \tag{B.1}$$

Using Lemma A.1 and the following connection between primal-dual solutions (see [6, Section 2.5])

$$\mathbf{u}^\star = \mathbf{y} - \mathbf{A}\mathbf{x}^\star, \tag{B.2}$$

we have that $\mathbf{x}^\star$ is a minimizer of (1.1) if and only if

$$\mathbf{g}^\star \triangleq \lambda^{-1}\mathbf{A}^{\mathrm{T}}\mathbf{u}^\star \in \partial r_{\text{SLOPE}}(\mathbf{x}^\star). \tag{B.3}$$

In the rest of the proof, we will use Lemma A.2 with $\mathbf{x} = \mathbf{x}^\star$, $\mathbf{g} = \mathbf{g}^\star$ and different instances of vector $\mathbf{g}'$ to prove our statement. First, let us define $\mathbf{g}' \in \mathbb{R}^n$ as

$$\begin{aligned} \mathbf{g}'_{(j)} &= \mathbf{g}^\star_{(j)} \quad \forall j \in [\![1, n]\!] \setminus \{\ell\}, \\ \mathbf{g}'_{(\ell)} &= 0. \end{aligned}$$

It is easy to verify that $r^*_{\text{SLOPE}}(\mathbf{g}') \leq 1$. Applying Lemma A.2 then leads to

$$\mathbf{g}^\star_{(\ell)}\mathbf{x}^\star_{(\ell)} \geq 0. \tag{B.4}$$

Since $\mathbf{x}^\star_{(\ell)}$ is assumed to be nonzero, we then have

$$\operatorname{sign}\left(\mathbf{g}^\star_{(\ell)}\right) \operatorname{sign}\left(\mathbf{x}^\star_{(\ell)}\right) \geq 0, \tag{B.5}$$

where the equality holds if and only if $\mathbf{g}^\star_{(\ell)} = 0$.

Second, let us consider the following choice for $\mathbf{g}' \in \mathbb{R}^n$:

$$\begin{aligned} \mathbf{g}'_{(j)} &= \mathbf{g}^\star_{(j)} \quad \forall j \in [\![1, n]\!] \setminus \{\ell\}, \\ \mathbf{g}'_{(\ell)} &= \mathbf{g}^\star_{(\ell)} + s\delta, \end{aligned}$$

where (using the convention $\mathbf{g}^\star_{[0]} = +\infty$)

$$s \triangleq \begin{cases} \mathrm{sign}(\mathbf{g}^\star_{(\ell)}) & \text{if } \mathbf{g}^\star_{(\ell)} \neq 0 \\ \mathrm{sign}(\mathbf{x}^\star_{(\ell)}) & \text{otherwise,} \end{cases} \tag{B.6}$$

$$\delta \triangleq \min\left(|\mathbf{g}^\star|_{[q_0'-1]} - |\mathbf{g}^\star|_{[q_0']}, \min_{q\in[\![1,n]\!]} \left(\sum_{k=1}^{q} \gamma_k - \sum_{k=1}^{q} |\mathbf{g}^\star|_{[k]} \right) \right) \tag{B.7}$$

and $q_0' \in [\![1,n]\!]$ is the smallest integer such that $|\mathbf{g}^\star_{(\ell)}| = |\mathbf{g}^\star|_{[q_0']}$. We note that $\delta \geq 0$ by definition: the positivity of the first argument follows from the definition of q_0', the positivity of the second argument is due to $r^*_{\text{SLOPE}}(\mathbf{g}^\star) \leq 1$.

To apply Lemma A.2, let us show that $r^*_{\text{SLOPE}}(\mathbf{g}') \leq 1$, or equivalently:

$$\forall q \in [\![1,n]\!]: \quad \sum_{k=1}^{q} |\mathbf{g}'|_{[k]} \leq \sum_{k=1}^{q} \gamma_k. \tag{B.8}$$

We have by construction

$$|\mathbf{g}^\star|_{[q_0']} \leq |\mathbf{g}'_{(\ell)}| \leq |\mathbf{g}^\star|_{[q_0'-1]}, \tag{B.9}$$

so that the ordering of the elements of $|\mathbf{g}^\star|$ and $|\mathbf{g}'|$ is the same. We then have:

$$\begin{aligned} \forall q \in [\![1,q_0'-1]\!]: \quad \sum_{k=1}^{q} |\mathbf{g}'|_{[k]} &= \sum_{k=1}^{q} |\mathbf{g}^\star|_{[k]} \leq \sum_{k=1}^{q} \gamma_k, \\ \forall q \in [\![q_0',n]\!]: \quad \sum_{k=1}^{q} |\mathbf{g}'|_{[k]} &\leq \sum_{k=1}^{q} |\mathbf{g}^\star|_{[k]} + \delta \\ &\leq \sum_{k=1}^{q} |\mathbf{g}^\star|_{[k]} + \min_{q'\in[\![1,n]\!]} \left(\sum_{k=1}^{q'} \gamma_k - \sum_{k=1}^{q'} |\mathbf{g}^\star|_{[k]} \right) \\ &\leq \sum_{k=1}^{q} \gamma_k, \end{aligned}$$

where the last inequality is obtained by considering the case $q' = q$ in the penultimate equation. Hence $r^*_{\text{SLOPE}}(\mathbf{g}') \leq 1$.

Applying Lemma A.2 leads to

$$- s\mathbf{x}^\star_{(\ell)}\delta \geq 0. \tag{B.10}$$

Using (B.5) and the definition of s in (B.6), we must have $s\mathbf{x}^\star_{(\ell)} > 0$. Since $\delta \geq 0$, satisfying inequality (B.10) therefore implies that $\delta = 0$. By definition of q_0', the first argument in the definition of δ in (B.7) is always strictly positive. Hence

$$\min_{q\in[\![1,n]\!]} \left(\sum_{k=1}^{q} \gamma_k - \sum_{k=1}^{q} |\mathbf{g}^\star|_{[k]} \right) = 0, \tag{B.11}$$

or in other words,

$$\exists\, q_0 \in [\![1,n]\!], \ \sum_{k=1}^{q_0} |\mathbf{g}^\star|_{[k]} = \sum_{k=1}^{q_0} \gamma_k. \tag{B.12}$$

We finally obtain our original assertion (B.1) by using the definition of $\mathbf{g}^\star$ in (B.3) and the fact that

$$\sum_{k=1}^{q_0} |\mathbf{A}^{\mathrm{T}}\mathbf{u}^\star|_{[k]} \geq |\mathbf{a}_\ell^{\mathrm{T}}\mathbf{u}^\star| + \sum_{k=1}^{q_0-1} |\mathbf{A}_{\backslash\ell}^{\mathrm{T}}\mathbf{u}^\star|_{[k]}. \tag{B.13}$$

B.2. Proof of Lemma 4.2. We first state and prove the following technical lemma:

LEMMA B.1. *Let $\mathbf{g} \in \mathbb{R}^n$ and $\mathbf{h} \in \mathbb{R}^n$ be such that $\mathbf{g}_{(j)} \leq \mathbf{h}_{(j)} \forall j$. Then*

$$\mathbf{g}_{[k]} \leq \mathbf{h}_{[k]} \ \forall k. \tag{B.14}$$

Proof. We have by definition

$$\begin{aligned}
\mathbf{h}_{[k]} &= \max_{\mathcal{J}\subset[\![1,n]\!]:\mathrm{card}(\mathcal{J})=k} \min_{j\in\mathcal{J}} \mathbf{h}_{(j)}, \\
&\geq \max_{\mathcal{J}\subset[\![1,n]\!]:\mathrm{card}(\mathcal{J})=k} \min_{j\in\mathcal{J}} \mathbf{g}_{(j)}, \\
&= \mathbf{g}_{[k]},
\end{aligned} \tag{B.15}$$

where the inequality follows from our assumption $\mathbf{g}_{(j)} \leq \mathbf{h}_{(j)} \forall j$. ∎

We are now ready to prove Lemma 4.2. For any $p \in [\![1, q]\!]$, we can write:

$$|\mathbf{a}_\ell^{\mathrm{T}}\mathbf{u}^\star| + \sum_{k=1}^{q-1} |\mathbf{A}_{\backslash\ell}^{\mathrm{T}}\mathbf{u}^\star|_{[k]} = |\mathbf{a}_\ell^{\mathrm{T}}\mathbf{u}^\star| + \sum_{k=1}^{p-1} |\mathbf{A}_{\backslash\ell}^{\mathrm{T}}\mathbf{u}^\star|_{[k]} + \sum_{k=p}^{q-1} |\mathbf{A}_{\backslash\ell}^{\mathrm{T}}\mathbf{u}^\star|_{[k]}. \tag{B.16}$$

First, since $\mathbf{u}^\star$ is dual feasible, we have:

$$\sum_{k=1}^{p-1} |\mathbf{A}_{\backslash\ell}^{\mathrm{T}}\mathbf{u}^\star|_{[k]} \leq \lambda \sum_{k=1}^{p-1} \gamma_k. \tag{B.17}$$

We next show that if $\mathbf{u}^\star \in \mathcal{S}(\mathbf{c}, R)$, then

$$|\mathbf{a}_\ell^{\mathrm{T}}\mathbf{u}^\star| + \sum_{k=p}^{q-1} |\mathbf{A}_{\backslash\ell}^{\mathrm{T}}\mathbf{u}^\star|_{[k]} \leq \left|\mathbf{a}_\ell^{\mathrm{T}}\mathbf{c}\right| + \sum_{k=p}^{q-1} \left|\mathbf{A}_{\backslash\ell}^{\mathrm{T}}\mathbf{c}\right|_{[k]} + (q-p+1)R. \tag{B.18}$$

We then obtain the result stated in the lemma by combining (B.17)-(B.18).

Inequality (B.18) can be shown as follows. First,

$$\forall i \in [\![1, n]\!] : \max_{\mathbf{u}\in\mathcal{S}(\mathbf{c},R)} |\mathbf{a}_i^{\mathrm{T}}\mathbf{u}| = |\mathbf{a}_i^{\mathrm{T}}\mathbf{c}| + R. \tag{B.19}$$

Hence

$$\left(\max_{\mathbf{u}\in\mathcal{S}(\mathbf{c},R)} |\mathbf{A}_{\backslash\ell}^{\mathrm{T}}\mathbf{u}|\right)_{[k]} = |\mathbf{A}_{\backslash\ell}^{\mathrm{T}}\mathbf{c}|_{[k]} + R \tag{B.20}$$

where the maximum is taken component-wise in the left-hand side of the equation. Applying Lemma B.1 with $\mathbf{g} = |\mathbf{A}_{\backslash\ell}^{\mathrm{T}}\mathbf{u}|$ and $\mathbf{h} = \max_{\tilde{\mathbf{u}}\in\mathcal{S}(\mathbf{c},R)} |\mathbf{A}_{\backslash\ell}^{\mathrm{T}}\tilde{\mathbf{u}}|$, we have

$$\forall \mathbf{u} \in \mathcal{S}(\mathbf{c}, R) : \ |\mathbf{A}_{\backslash\ell}^{\mathrm{T}}\mathbf{u}|_{[k]} \leq \left(\max_{\tilde{\mathbf{u}}\in\mathcal{S}(\mathbf{c},R)} \mathbf{A}_{\backslash\ell}^{\mathrm{T}}\tilde{\mathbf{u}}|\right)_{[k]} \tag{B.21}$$

and therefore

$$\max_{\mathbf{u}\in\mathcal{S}(\mathbf{c},R)}\left(|\mathbf{A}_{\backslash\ell}^{\mathrm{T}}\mathbf{u}|_{[k]}\right) \leq \left(\max_{\mathbf{u}\in\mathcal{S}(\mathbf{c},R)}|\mathbf{A}_{\backslash\ell}^{\mathrm{T}}\mathbf{u}|\right)_{[k]}. \tag{B.22}$$

Combining these results leads to

$$|\mathbf{a}_\ell^{\mathrm{T}}\mathbf{u}^\star| + \sum_{k=p}^{q-1}|\mathbf{A}_{\backslash\ell}^{\mathrm{T}}\mathbf{u}^\star|_{[k]} \leq \max_{\mathbf{u}\in\mathcal{S}(\mathbf{c},R)}\left(|\mathbf{a}_\ell^{\mathrm{T}}\mathbf{u}| + \sum_{k=p}^{q-1}|\mathbf{A}_{\backslash\ell}^{\mathrm{T}}\mathbf{u}|_{[k]}\right) \tag{B.23}$$

$$\leq \max_{\mathbf{u}\in\mathcal{S}(\mathbf{c},R)}|\mathbf{a}_\ell^{\mathrm{T}}\mathbf{u}| + \sum_{k=p}^{q-1}\max_{\mathbf{u}\in\mathcal{S}(\mathbf{c},R)}\left(|\mathbf{A}_{\backslash\ell}^{\mathrm{T}}\mathbf{u}|_{[k]}\right) \tag{B.24}$$

$$\leq \max_{\mathbf{u}\in\mathcal{S}(\mathbf{c},R)}|\mathbf{a}_\ell^{\mathrm{T}}\mathbf{u}| + \sum_{k=p}^{q-1}\left(\max_{\mathbf{u}\in\mathcal{S}(\mathbf{c},R)}|\mathbf{A}_{\backslash\ell}^{\mathrm{T}}\mathbf{u}|\right)_{[k]} \tag{B.25}$$

$$\leq \left|\mathbf{a}_\ell^{\mathrm{T}}\mathbf{c}\right| + \sum_{k=p}^{q-1}\left|\mathbf{A}_{\backslash\ell}^{\mathrm{T}}\mathbf{c}\right|_{[k]} + (q-p+1)R. \tag{B.26}$$

B.3. Proof of Lemma 4.4. We want to show that if test (4.10) is passed for some $\{p_q\}_{q\in[\![1,n]\!]}$, then test (4.14) is also passed when $\gamma_k = 1\ \forall k\in[\![1,n]\!]$.

Assume (4.10) holds for some $\{p_q\}_{q\in[\![1,n]\!]}$, that is $\forall q\in[\![1,n]\!]$, $\exists p_q\in[\![0,q-1]\!]$ such that

$$\left|\mathbf{a}_\ell^{\mathrm{T}}\mathbf{c}\right| + \sum_{k=p_q}^{q-1}\left|\mathbf{A}_{\backslash\ell}^{\mathrm{T}}\mathbf{c}\right|_{[k]} < \kappa_{q,p_q}, \tag{B.27}$$

where $\kappa_{q,p} \triangleq \lambda\left(\sum_{k=p}^{q}\gamma_k\right) - (q-p+1)R$. Considering the case "$q=1$", (B.27) particularizes to

$$\left|\mathbf{a}_\ell^{\mathrm{T}}\mathbf{c}\right| + R < \lambda\gamma_1. \tag{B.28}$$

If $\gamma_k = 1\ \forall k\in[\![1,n]\!]$, the latter inequality is also equal to (4.14) and the result is proved.

B.4. Proof of Lemma 4.5. We prove the result by showing that $\forall q\in[\![1,n]\!]$ the sequence $\{B_{q,\ell}\}_{\ell\in[\![1,n]\!]}$ is non-increasing. To this end, we first rewrite $B_{q,\ell}$ in a slightly different manner, easier to analyze. Let

$$\begin{array}{llll} C_{q,p} \triangleq & (q-p+1)R + \lambda\left(\sum_{k=1}^{p-1}\gamma_k\right) & \forall q\in[\![1,n]\!], \forall p\in[\![1,q]\!] \\ \sigma_q \triangleq & \sum_{k=1}^{q}|\mathbf{a}_k^{\mathrm{T}}\mathbf{c}| & \forall q\in[\![0,n]\!] \end{array} \tag{B.29}$$

with the convention $\sigma_0 \triangleq 0$. Using these notations and hypothesis (4.16), $B_{q,\ell}$ can be rewritten as

$$B_{q,\ell} - C_{q,p} = |\mathbf{a}_\ell^{\mathrm{T}}\mathbf{c}| + \sum_{k=1}^{q-1}\left|\mathbf{A}_{\backslash\ell}^{\mathrm{T}}\mathbf{c}\right|_{(k)} - \sum_{k=1}^{p-1}\left|\mathbf{A}_{\backslash\ell}^{\mathrm{T}}\mathbf{c}\right|_{(k)} \tag{B.30}$$

$$= \begin{cases} |\mathbf{a}_\ell^{\mathrm{T}}\mathbf{c}| + \sigma_{q-1} - \sigma_{p-1} & \text{if } q<\ell \\ \sigma_q - \sigma_{p-1} & \text{if } p-1<\ell\leq q \\ |\mathbf{a}_\ell^{\mathrm{T}}\mathbf{c}| + \sigma_q - \sigma_p & \text{if } \ell\leq p-1. \end{cases} \tag{B.31}$$

We next show that $\forall q \in [\![1, n]\!]$ the sequence $\{B_{q,\ell}\}_{\ell \in [\![1,n]\!]}$ is non-increasing. We first notice that $C_{q,p}$ does not depend on ℓ and we can therefore focus on (B.31) to prove our claim. Using the fact that $|\mathbf{a}_\ell^{\mathrm{T}} \mathbf{c}| \geq |\mathbf{a}_{\ell+1}^{\mathrm{T}} \mathbf{c}|$ by hypothesis, we immediately obtain that $B_{q,\ell} \geq B_{q,\ell+1}$ whenever $\ell \notin \{p-1, q\}$. We conclude the proof by treating the cases "$\ell = p-1$" and "$\ell = q$" separately.

If $\ell = q$ we have from (B.31):

$$B_{q,\ell+1} - B_{q,\ell} = |\mathbf{a}_{q+1}^{\mathrm{T}} \mathbf{c}| + \sigma_{q-1} - \sigma_q = |\mathbf{a}_{q+1}^{\mathrm{T}} \mathbf{c}| - |\mathbf{a}_q^{\mathrm{T}} \mathbf{c}| \leq 0, \tag{B.32}$$

where the last inequality holds true by virtue of (4.16).

If $\ell = p-1$ (and provided that $p \geq 2$) the same rationale leads to

$$B_{q,\ell+1} - B_{q,\ell} = |\mathbf{a}_p^{\mathrm{T}} \mathbf{c}| - |\mathbf{a}_{p-1}^{\mathrm{T}} \mathbf{c}| \leq 0. \tag{B.33}$$

B.5. Proof of Lemma 4.6. The necessity of (4.28) can be shown as follows. Assume $|\mathbf{a}_n^{\mathrm{T}} \mathbf{c}| \geq \tau$ for some $\tau \in \mathcal{T}$ and let $q \in [\![1, n]\!]$ be such that $\tau = \tau_{q,p^\star(q)}$. From (4.22) we then have

$$\forall p \in [\![1, q]\!] : \ \left|\mathbf{a}_n^{\mathrm{T}} \mathbf{c}\right| \geq \tau_{q,p} \tag{B.34}$$

and test (4.19) therefore fails.

To prove the sufficiency of (4.28), let us first notice that $\forall p \in [\![1, n]\!]$:

$$\tau_{q^{(1)},p} = \arg\min_{q \in [\![1,n]\!]} \tau_{q,p}. \tag{B.35}$$

In particular, letting $p^{(1)} = p^\star(q^{(1)})$, we have

$$\forall q \in [\![p^{(1)}, n]\!] : \ \tau_{q^{(1)},p^{(1)}} \leq \tau_{q,p^{(1)}}. \tag{B.36}$$

Hence,

$$\left|\mathbf{a}_n^{\mathrm{T}} \mathbf{c}\right| < \tau_{q^{(1)},p^{(1)}} \implies \forall q \in [\![p^{(1)}, n]\!] : \ \left|\mathbf{a}_n^{\mathrm{T}} \mathbf{c}\right| < \tau_{q,p^{(1)}}. \tag{B.37}$$

In other words, satisfying the left-hand side of (B.37) implies that test (4.19) is verified for each $q \in [\![p^{(1)}, n]\!]$.

We can apply the same reasoning iteratively to show that $\forall t \in [\![1, \mathrm{card}\,(\mathcal{T})]\!]$:

$$\left|\mathbf{a}_n^{\mathrm{T}} \mathbf{c}\right| < \tau_{q^{(t)},p^{(t)}} \implies \forall q \in [\![p^{(t)}, p^{(t-1)} - 1]\!] : \ \left|\mathbf{a}_n^{\mathrm{T}} \mathbf{c}\right| < \tau_{q,p^{(t)}}. \tag{B.38}$$

Since $p^{(\mathrm{card}(\mathcal{T}))} = 1$, we thus obtain that (4.28) implies that (4.19) is verified for each $q \in [\![1, n]\!]$.

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