# 36-705 Intermediate Statistics Homework #9 Solutions

**December 1, 2016**

## Problem 1 [30 pts.]

(a) The posterior density is given by:

$$\pi\left(\theta|x^n\right) = \frac{p\left(x^n|\theta\right)\pi\left(\theta\right)}{m\left(x^n\right)} \propto p\left(x^n|\theta\right)\pi\left(\theta\right)$$

Now note that the likelihood satisfies:

$$p\left(x^n|\theta\right) = \frac{1}{\left(2\pi\right)^{n/2}}\exp\left(-\frac{1}{2}\sum_{i=1}^{n}\left(x_i-\theta\right)^2\right) \propto \exp\left(-\frac{n}{2}\left(\bar{x}-\theta\right)^2\right)$$

And the prior:

$$\pi\left(\theta\right) \propto \exp\left(-\frac{1}{2b^2}\left(\theta-a\right)^2\right)$$

Thus we get:

$$\pi\left(\theta, x^n\right) \propto \exp\left(-\frac{n}{2}\left(\bar{x}-\theta\right)^2\right)\exp\left(-\frac{1}{2b^2}\left(\theta-a\right)^2\right) \propto \exp\left(-\frac{1}{2\sigma_p^2}\left(\theta-\theta_p\right)^2\right)$$

where

$$\theta_p = \frac{nb^2\bar{x}+a}{1+nb^2}, \quad \sigma_p^2 = \frac{b^2}{1+nb^2}$$

Hence the posterior is $\theta|X^n \sim N\left(\frac{nb^2\bar{X}+a}{1+nb^2}, \frac{b^2}{1+nb^2}\right)$.

(b) By the previous part the posterior is Normal with mean $\frac{nb^2\bar{X}+a}{1+nb^2}$ and variance $\frac{1}{1+nb^2}$. So a posterior interval for $\theta$, with coverage $1-\alpha$ would be given by $c_n = z_{\alpha/2}\sqrt{\frac{b^2}{1+nb^2}}$ and the interval would be:

$$C_n = \left(\frac{nb^2\bar{X}+a}{1+nb^2} - c_n, \frac{nb^2\bar{X}+a}{1+nb^2} + c_n\right)$$

(c) Here we have to think in a frequentist way, where $\bar{X} \sim N\left(\theta, \frac{1}{n}\right)$ and $\theta$ is a fixed unknown value. Let $\hat{\theta}_n = \frac{nb^2\bar{X}+a}{1+nb^2}$, then:

$$\mathbb{P}_\theta(\theta \in C_n) = \mathbb{P}_\theta(|\theta - \hat{\theta}_n| \leq c_n) = \mathbb{P}_\theta\left(\frac{|\theta - \hat{\theta}_n|}{\sqrt{\frac{b^2}{1+nb^2}}} \leq z_{\alpha/2}\right)$$

The distribution of

$$\theta - \hat{\theta}_n = \theta - \frac{nb^2\bar{X} + a}{1 + nb^2} = \frac{\theta + nb^2\theta - nb^2\bar{X} - a}{1 + nb^2}$$

is $N\left(\frac{\theta - a}{1+nb^2}, \frac{1}{n}\left(\frac{nb^2}{1+nb^2}\right)^2\right)$, since:

$$\mathbb{E}(\theta - \hat{\theta}_n) = \frac{\theta - a}{1 + nb^2}, \quad \mathbb{V}(\theta - \hat{\theta}_n) = \frac{1}{n}\left(\frac{nb^2}{1 + nb^2}\right)^2$$

Thus

$$\frac{\theta - \hat{\theta}_n}{\sqrt{b^2/1 + nb^2}} \sim N\left(\frac{\theta - a}{b\sqrt{1 + nb^2}}, \frac{nb^2}{1 + nb^2}\right)$$

Therefore:

$$\begin{aligned}
\mathrm{Cov}_{C_n}(\theta) &= \mathbb{P}_\theta\left(\frac{|\theta - \hat{\theta}|}{\sqrt{\frac{b^2}{1+nb^2}}} \leq z_{\alpha/2}\right) \\
&= \Phi\left(\frac{z_{\alpha/2} - \frac{\theta - a}{b\sqrt{1+nb^2}}}{\sqrt{nb^2/(1 + nb^2)}}\right) - \Phi\left(\frac{-z_{\alpha/2} - \frac{\theta - a}{b\sqrt{1+nb^2}}}{\sqrt{nb^2/(1 + nb^2)}}\right).
\end{aligned}$$

## Problem 2 [20 pts.]

$$\begin{aligned}\pi(\theta|X^n) &\propto \frac{1}{\theta}\prod_{i=1}^{n}\left(\frac{X_i}{\theta}\right)^n \mathbb{1}_{\{X_{(n)} \le \theta\}}\\ &\propto \left(\frac{1}{\theta}\right)^{n+1}\mathbb{1}_{\{X_{(n)} \le \theta\}}\end{aligned}$$

Since the posterior density must integrate to 1 we have:

$$\begin{aligned}c\int_{X_{(n)}}^{\infty}\theta^{-(n+1)}d\theta &= 1\\ -\frac{c}{n}\left[\theta^{-n}\right]_{X_{(n)}}^{\infty} &= 1\\ \frac{c}{n}X_{(n)}^{-n} &= 1\\ c &= nX_{(n)}^{n}\end{aligned}$$

$$\implies \pi(\theta|X^n) = \frac{n}{\theta}\left(\frac{X_{(n)}}{\theta}\right)^n \mathbb{1}_{\{X_{(n)} \le \theta\}}.$$

Assume $n > 1$. The posterior mean is:

$$\begin{aligned}\int_{X_{(n)}}^{\infty}\theta\cdot\frac{n}{\theta}\left(\frac{X_{(n)}}{\theta}\right)^n d\theta &= nX_{(n)}^n\int_{X_{(n)}}^{\infty}\theta^{-n}d\theta\\ &= \frac{nX_{(n)}^n}{1-n}\left[\theta^{-n+1}\right]_{X_{(n)}}^{\infty}\\ &= \frac{nX_{(n)}^n}{n-1}X_{(n)}^{-n+1}\\ &= \frac{n}{n-1}X_{(n)}.\end{aligned}$$

## Problem 3 [35 pts.]

(a) From Lecture Notes 7, Example 3 we have

$$\widehat{p}_1 = \frac{30}{50} = 0.6 \quad \text{and} \quad \widehat{p}_2 = \frac{40}{50} = 0.8.$$

By the equivariance of MLE, we have

$$\widehat{\tau} = \widehat{p}_2 - \widehat{p}_1 = 0.2.$$

Let $g(p_1, p_2) = p_2 - p_1$. By the asymptotic normality of the MLE, we have

$$\frac{\widehat{\tau} - \tau}{se(\widehat{\tau})} \rightsquigarrow N(0,1),$$

where

$$\begin{aligned} se(\widehat{\tau}) &= \sqrt{(\triangledown g)^T J_n (\triangledown g)} \\ (\triangledown g)^T &= \begin{pmatrix} \frac{\partial g}{\partial p_1} & \frac{\partial g}{\partial p_2} \end{pmatrix} = (-1 \quad 1) \\ J_n &= I_n^{-1}(p_1, p_2) \\ I_n(p_1, p_2) &= \begin{bmatrix} \frac{50}{p_1(1-p_1)} & 0 \\ 0 & \frac{50}{p_2(1-p_2)} \end{bmatrix}. \end{aligned}$$

Therefore,

$$\begin{aligned} \widehat{se}(\widehat{\tau}) &= \sqrt{(\widehat{\triangledown g})^T \widehat{J}_n (\widehat{\triangledown g})} \\ &= 0.0894 \end{aligned}$$

and the 90% confidence interval is

$$\big(0.2 - (1.645) \cdot 0.0894,\ 0.2 + (1.645) \cdot 0.0894\big) = (0.052937, 0.347063).$$

(b)

```
set.seed(1)
Xboot <- rbinom(10000,50,0.6)
Yboot <- rbinom(10000,50,0.8)
tauboot <- Yboot/50 - Xboot/50
seboot <- sqrt(sum((tauboot - 0.2)^2)/10000) # 0.08985388

lower <- 0.2 - 1.645 * seboot # 0.05219037
upper <- 0.2 + 1.645 * seboot # 0.3478096
```

(c) Let $X$ be the number of patients out of a group of 50 who show improvement under placebo treatment and let $Y$ be the number of patients out of a separate group of 50 who show improvement under a new treatment. $X$ and $Y$ are independent.

$$\begin{aligned} p(p_1, p_2|X, Y) &\propto f(X, Y|p_1, p_2) \\ &\propto p_1^X (1 - p_1)^{50-X} p_2^Y (1 - p_2)^{50-Y}, \end{aligned}$$

so

$$p_1|X \sim \text{Beta}(X + 1, 51 - X)$$

and

$$p_2|Y \sim \text{Beta}(Y + 1, 51 - Y)$$

Using $X = 30$ and $Y = 40$ we get

```
set.seed(1)
p1boot <- rbeta(10000,31,21)
p2boot <- rbeta(10000,41,11)
tauboot <- p2boot - p1boot
posteriormean <- mean(tauboot) # 0.1926239

lower <- quantile(tauboot, 0.05) # 0.04559222
upper <- quantile(tauboot, 0.95) # 0.3356031
```

(d) By the equivariance of the MLE,

$$\begin{aligned} \widehat{\psi} &= \log\left(\left(\frac{\widehat{p}_1}{1 - \widehat{p}_1}\right) \Big/ \left(\frac{\widehat{p}_2}{1 - \widehat{p}_2}\right)\right) \\ &= \log\left(\left(\frac{0.6}{0.4}\right) \Big/ \left(\frac{0.8}{0.2}\right)\right) \\ &= \log(0.375) \\ &= -0.9808293. \end{aligned}$$

Let

$$g(p_1, p_2) = \log\left(\left(\frac{p_1}{1 - p_1}\right) \Big/ \left(\frac{p_2}{1 - p_2}\right)\right).$$

We have

$$\begin{aligned} (\nabla g)^T &= \begin{pmatrix} \frac{\partial g}{\partial p_1} & \frac{\partial g}{\partial p_2} \end{pmatrix} \\ &= \begin{bmatrix} \frac{1}{p_1(1-p_1)} & -\frac{1}{p_2(1-p_2)} \end{bmatrix} \end{aligned}$$

and

$$
\begin{aligned}
\widehat{se}(\widehat{\psi}) &= \sqrt{(\widehat{\nabla g})^T \widehat{J}_n (\widehat{\nabla g})} \\
&= \sqrt{\begin{bmatrix} \frac{1}{\widehat{p}_1(1-\widehat{p}_1)} & -\frac{1}{\widehat{p}_2(1-\widehat{p}_2)} \end{bmatrix} \begin{bmatrix} \frac{50}{\widehat{p}_1(1-\widehat{p}_1)} & 0 \\ 0 & \frac{50}{\widehat{p}_2(1-\widehat{p}_2)} \end{bmatrix}^{-1} \begin{bmatrix} \frac{1}{\widehat{p}_1(1-\widehat{p}_1)} \\ -\frac{1}{\widehat{p}_2(1-\widehat{p}_2)} \end{bmatrix}} \\
&= \sqrt{\frac{1}{50(0.6)(1-0.6)} + \frac{1}{50(0.8)(1-0.8)}} \\
&= \sqrt{\frac{1}{12} + \frac{1}{8}} \\
&= \sqrt{\frac{5}{24}}.
\end{aligned}
$$

Therefore an asymptotic 90% confidence interval for $\psi$ is

$$
\left(\widehat{\psi} - z_{0.05}\widehat{se}(\widehat{\psi}),\ \widehat{\psi} + z_{0.05}\widehat{se}(\widehat{\psi})\right) = (-1.731666, -0.229993).
$$

(e)

```
set.seed(1)
p1boot <- rbeta(10000,31,21)
p2boot <- rbeta(10000,41,11)
psiboot <- log((p1boot/(1-p1boot)) / (p2boot / (1 - p2boot)))
posteriormean <- mean(psiboot) # -0.953088

lower <- quantile(psiboot, 0.05) # -1.698769
upper <- quantile(psiboot, 0.95) # -0.2177191
```

## Problem 4 [15 pts.]

$$
\begin{aligned}
p(\lambda|X^n) &\propto \prod_{i=1}^{n} \frac{e^{-\lambda}\lambda^{X_i}}{X_i!} \cdot \lambda^{\alpha-1} e^{-\frac{\lambda}{\beta}} \\
&\propto e^{-(\frac{n\beta+1}{\beta})\lambda} \lambda^{\sum_{i=1}^{n} X_i + \alpha - 1}
\end{aligned}
$$

Therefore,

$$
\lambda|X^n \sim \text{Gamma}\left(\sum_{i=1}^{n} X_i + \alpha, \frac{\beta}{n\beta+1}\right)
$$

and

$$
\mathbb{E}\left[\lambda|X^n\right] = (n\overline{X} + \alpha)\left(\frac{\beta}{n\beta+1}\right).
$$