36-705 Intermediate Statistics Homework #3 Solutions

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Problem 1 [15 pts.]

Let $\mathcal{C} = \mathcal{A} \bigcup \mathcal{B}$. Show that

$$s_n(\mathcal{C}) \leq s_n(\mathcal{A}) + s_n(\mathcal{B})$$

where s_n denotes the shattering number.

Solution.
Let F be a finite set of n elements. We have that $\mathcal{C} = \{A : A \in \mathcal{A} \text{ or } A \in \mathcal{B}\}$, so if $\mathcal{C}$ picks out $G \subseteq F$, then either $\mathcal{A}$ picks out G or $\mathcal{B}$ picks out G. Thus, for any $F \in \mathcal{F}_n$, the total number of subsets picked out by $\mathcal{C}$ is the total number of distinct $G \subseteq F$ picked out by either $\mathcal{A}$ or $\mathcal{B}$. Therefore, for any finite set F:

$$S(\mathcal{C}, F) \leq S(\mathcal{A}, F) + S(\mathcal{B}, F)$$

Taking the supremum over all $F \in \mathcal{F}_n$ on both sides:

$$\begin{aligned} s_n(\mathcal{C}) &= \sup_{F \in \mathcal{F}_n} S(\mathcal{C}, F) \leq \sup_{F \in \mathcal{F}_n} \left(S(\mathcal{A}, F) + S(\mathcal{B}, F)\right) \\ &\leq \sup_{F \in \mathcal{F}_n} S(\mathcal{A}, F) + \sup_{F \in \mathcal{F}_n} S(\mathcal{B}, F) = s_n(\mathcal{A}) + s_n(\mathcal{B}) \end{aligned}$$

Problem 2 [15 pts.]

Let $\mathcal{C} = \{A \cup B; \;\; A \in \mathcal{A}, \;\; B \in \mathcal{B}\}$. Show that

$$s_n(\mathcal{C}) \leq s_n(\mathcal{A}) s_n(\mathcal{B}).$$

Solution.
We have that $\mathcal{C} = \{A \cup B : A \in \mathcal{A}, B \in \mathcal{B}\}$. Let $F \in \mathcal{F}_n$, then for $C \in \mathcal{C}$,

$$C \cap F = (A \cup B) \cap F = (A \cap F) \cup (B \cap F)$$

Let m_A, m_B be the number of subsets of F that $\mathcal{A}$ and $\mathcal{B}$ can pick out. Then the number of distinct sets of the form $(A \cup B) \cap F$ is the total number of distinct unions of the form $(A \cap F) \cup (B \cap F)$ which is bounded by $m_A m_B$. Thus:

$$S(\mathcal{C}, F) \leq S(\mathcal{A}, F) \, S(\mathcal{B}, F)$$

Again taking the supremum over all $F \in \mathcal{F}_n$ on both sides:

$$\begin{aligned} s_n(\mathcal{C}) &= \sup_{F \in \mathcal{F}_n} S(\mathcal{C}, F) \leq \sup_{F \in \mathcal{F}_n} \left(S(\mathcal{A}, F) \, S(\mathcal{B}, F)\right) \\ &\leq \sup_{F \in \mathcal{F}_n} S(\mathcal{A}, F) \sup_{F \in \mathcal{F}_n} S(\mathcal{B}, F) = s_n(\mathcal{A}) \, s_n(\mathcal{B}) \end{aligned}$$

Problem 3 [15 pts.]

Let $X_1, X_2, \dots$ be a sequence of random variables. Show that $X_n \xrightarrow{\text{qm}} b$ if and only if

$$\lim_{n\to\infty} \mathbb{E}(X_n) = b \quad \text{and} \quad \lim_{n\to\infty} \mathbb{V}(X_n) = 0.$$

Solution.
First let $\mathbb{E}[X_n] = \mu_n$, now:

$$\mathbb{E}\left[(X_n - b)^2\right] = \mathbb{E}\left[(X_n - \mu_n + \mu_n - b)^2\right] = \mathbb{V}(X_n) + 2(\mu_n - b)\underbrace{\mathbb{E}\left[(X_n - \mu_n)\right]}_{=0} + (\mu_n - b)^2$$

which gives us that:

$$\mathbb{E}\left[(X_n - b)^2\right] = \mathbb{V}(X_n) + (\mu_n - b)^2$$

Thus, *if* $\mathbb{V}(X_n) \to 0$ and $\mu_n \to b$, then $\mathbb{E}\left[(X_n - b)^2\right] \to 0$. Since all terms in the equation above are nonnegative, we also have that $\mathbb{E}\left[(X_n - b)^2\right] \to 0$ *only if* both $\mathbb{V}(X_n) \to 0$ and $(\mu_n - b)^2 \to 0$, and thus $\mu_n \to b$.

Problem 4 [15 pts.]

Let $X_1, \dots, X_n$ ~Bernoulli(p). Prove that

$$\frac{1}{n}\sum_{i=1}^{n} X_i^2 \xrightarrow{\text{P}} p \quad \text{and} \quad \frac{1}{n}\sum_{i=1}^{n} X_i^2 \xrightarrow{\text{qm}} p.$$

Solution.
First of all notice that $\frac{1}{n}\sum_{i=1}^{n} X_i^2 \overset{qm}{\to} p \Rightarrow \frac{1}{n}\sum_{i=1}^{n} X_i^2 \overset{P}{\to} p$, and that $X_i^2 = X_i$, since $X_i \sim$ Bernoulli(p). Therefore,

$$\begin{aligned}
\lim_{n\to\infty} \mathbb{E}\left(\frac{1}{n}\sum_{i=1}^{n} X_i^2 - p\right)^2 &= \lim_{n\to\infty} \mathbb{E}\left(\frac{1}{n}\sum_{i=1}^{n} X_i - p\right)^2 \\
&= \lim_{n\to\infty} \mathbb{E}^2\left(\frac{1}{n}\sum_{i=1}^{n} X_i - p\right) + \mathbb{V}\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) \\
&= 0 + \lim_{n\to\infty} \frac{p(1-p)}{n} \\
&= 0
\end{aligned}$$

proves both statements. However, notice that $\frac{1}{n}\sum_{i=1}^{n} X_i^2 = \frac{1}{n}\sum_{i=1}^{n} X_i \overset{P}{\to} p$ can be also shown by using some of the probability bounds described in Lecture Notes 2, and the weak law of large numbers.

Problem 5 [20 pts.]

Let $X, X_1, X_2, X_3, \ldots$ be random variables that are positive and integer valued. Show that $X_n \rightsquigarrow X$ if and only if

$$\lim_{n\to\infty} \mathbb{P}(X_n = k) = \mathbb{P}(X = k)$$

for every integer k.

Solution.
First assume that $X_n \rightsquigarrow X$.
Let F_n be the CDF of X_n, F the CDF of X and let k be a positive integer. Since X is integer valued F is continuous at $k - \frac{1}{2}, k + \frac{1}{2}$ so $X_n \rightsquigarrow X$ implies that:

$$\lim_{n\to\infty} F_n\left(k - \frac{1}{2}\right) \to F\left(k - \frac{1}{2}\right), \quad \lim_{n\to\infty} F_n\left(k + \frac{1}{2}\right) \to F\left(k + \frac{1}{2}\right)$$

Now the claim follows from the fact that:

$$\mathbb{P}\left(X_n = k\right) = F_n\left(k + \frac{1}{2}\right) - F_n\left(k - \frac{1}{2}\right)$$

$$\mathbb{P}\left(X = k\right) = F\left(k + \frac{1}{2}\right) - F\left(k - \frac{1}{2}\right)$$

Next assume that $\lim_{n\to\infty} \mathbb{P}\left(X_n = k\right) = \mathbb{P}\left(X = k\right)$ for all positive integers k. Then $\forall x \in \mathbb{R}, \lfloor x \rfloor$ be greatest integer which is less than or equal to x (Gauss floor function). Then,

$$P(X_n \le x) = \sum_{k=0}^{\lfloor x \rfloor} P(X_n = k) \to \sum_{k=0}^{\lfloor x \rfloor} P(X = k) = P(X \le x)$$

since convergence of each $P(X_n = k)$ implies convergence of finite sum $\sum_{k=0}^{\lfloor x \rfloor} P(X_n = k)$.
$\therefore \forall x \in \mathbb{R}, \lim_{n\to\infty} F_n(x) = F(x)$, i.e. $X_n \rightsquigarrow X$.

Problem 6 [20 pts.]

Let

$$\begin{pmatrix} X_{11} \\ X_{21} \end{pmatrix}, \begin{pmatrix} X_{12} \\ X_{22} \end{pmatrix}, \ldots, \begin{pmatrix} X_{1n} \\ X_{2n} \end{pmatrix}$$

be IID random vectors with mean $\mu = (\mu_1, \mu_2)$ and variance Σ. Let

$$\overline{X}_1 = \frac{1}{n} \sum_{i=1}^{n} X_{1i}, \quad \overline{X}_2 = \frac{1}{n} \sum_{i=1}^{n} X_{2i}$$

and define $Y_n = \overline{X}_1 / \overline{X}_2$. Find the limiting distribution of Y_n.

Solution.
(Assume $\mu_2 \neq 0$)
Let $\bar{X} = \begin{pmatrix} \bar{X}_1 \\ \bar{X}_2 \end{pmatrix}$. By the multivariate central limit theorem we have that:

$$\sqrt{n}\left(\bar{X} - \mu\right) \rightsquigarrow N\left(0, \Sigma\right)$$

Let $g : \mathbb{R}^2 \to \mathbb{R}, g\left(x, y\right) = x/y$. By the delta method applied to g and $\bar{X}$:

$$\sqrt{n}\left(g\left(\bar{X}\right) - g\left(\mu\right)\right) \rightsquigarrow N\left(0, \nabla_\mu^T \Sigma \nabla_\mu\right)$$

where $\Sigma = \begin{pmatrix}\sigma_{11} & \sigma_{12}\\ \sigma_{12} & \sigma_{22}\end{pmatrix}$ and $\nabla g(x,y) = \begin{pmatrix}1/y\\ -x/y^2\end{pmatrix}$ so that $\nabla_\mu = \nabla g(\mu_1,\mu_2) = \begin{pmatrix}1/\mu_2\\ -\mu_1/\mu_2^2\end{pmatrix}$. Explicitly the previous result becomes:

$$\sqrt{n}\left(\bar{X}_1/\bar{X}_2 - \mu_1/\mu_2\right) \rightsquigarrow N\left(0, \sigma_{11}/\mu_2^2 - 2\mu_1\sigma_{12}/\mu_2^3 + \mu_1^2\sigma_{22}/\mu_2^4\right)$$

For your interest: (case $\mu_2 = 0$)
We can divide into two cases, ($\mu_1 \neq 0$, $\mu_2 = 0$) and ($\mu_1 = \mu_2 = 0$).
($\mu_1 \neq 0$, $\mu_2 = 0$) :
Since $\frac{1}{\mu_1}X_{2i} \overset{iid}{\sim} N\left(0, \frac{\sigma_{22}}{\mu_1^2}\right)$, $\frac{\sqrt{n}}{\mu_1}\bar{X}_2 \sim N\left(0, \frac{\sigma_{22}}{\mu_1^2}\right)$ so its distribution is independent of n. Hence if we let $Y := 1/\left(\frac{\sqrt{n}}{\mu_1}\bar{X}_2\right)$, then pdf of Y is
$f_Y(y) = f_{\frac{\sqrt{n}}{\mu_1}\bar{X}_2}\left(\frac{1}{y}\right)\left|\frac{dx}{dy}\right| = \frac{|\mu_1|}{\sqrt{2\pi\sigma_{22}}}\exp\left(-\frac{\mu_1^2}{2\sigma_{22}y^2}\right)\frac{1}{y^2}$
Then since $\mu_1 \neq 0$ and from Strong Law of Large Number, $\bar{X}_1/\mu_1 \xrightarrow{P} 1$
$\therefore \frac{1}{\sqrt{n}}(\bar{X}_1/\bar{X}_2) = \frac{\bar{X}_1}{\mu_1}\cdot\frac{1}{\frac{\sqrt{n}}{\mu_1}\bar{X}_2} \rightsquigarrow Y$ by Slutsky's theorem,
where pdf of Y is $f_Y(y) = \frac{|\mu_1|}{\sqrt{2\pi\sigma_{22}}}\exp\left(-\frac{\mu_1^2}{2\sigma_{22}y^2}\right)\frac{1}{y^2}$
($\mu_1 = \mu_2 = 0$) :
Since $\begin{pmatrix} X_{1i}\\ X_{2i}\end{pmatrix} \overset{iid}{\sim} N(0,\Sigma)$, $\begin{pmatrix}\sqrt{n}\bar{X}_1\\ \sqrt{n}\bar{X}_2\end{pmatrix} \overset{iid}{\sim} N(0,\Sigma)$ independent of n. Hence $\bar{X}_1/\bar{X}_2 = (\sqrt{n}\bar{X}_1)/(\sqrt{n}\bar{X}_2)$ has same distribution of X_{11}/X_{21}.
Let $(Y,Z) = (X_{11}/X_{21}, X_{21})$, then $\left|\frac{\partial(X_1,X_2)}{\partial(Y,Z)}\right| = \begin{vmatrix} Z & Y\\ 0 & 1\end{vmatrix} = |Z|$, so pdf of (Y,Z) is

$$\begin{aligned}
f_{Y,Z}(y,z) &= f_{X_{11},X_{21}}(yz,z)\left|\frac{\partial(X_1,X_2)}{\partial(Y,Z)}\right| = \frac{|z|}{2\pi\sqrt{|\Sigma|}}\exp\left(-\frac{1}{2|\Sigma|}(yz\ z)\begin{pmatrix}\sigma_{22} & -\sigma_{12}\\ -\sigma_{12} & \sigma_{11}\end{pmatrix}\begin{pmatrix} yz\\ z\end{pmatrix}\right)\\
&= \frac{|z|}{2\pi\sqrt{|\Sigma|}}\exp\left(-\frac{z^2(\sigma_{22}y^2 - 2\sigma_{12}y + \sigma_{11})}{2|\Sigma|}\right)
\end{aligned}$$

Then pdf of Y is

$$\begin{aligned}
f_Y(y) &= \int_{-\infty}^{\infty} f_{Y,Z}(y,z)dz = \int_0^\infty \frac{2z}{2\pi\sqrt{|\Sigma|}}\exp\left(-\frac{z^2(\sigma_{22}y^2 - 2\sigma_{12}y + \sigma_{11})}{2|\Sigma|}\right)dz\\
&= \frac{\sqrt{|\Sigma|}}{\pi(\sigma_{22}y^2 - 2\sigma_{12}y + \sigma_{11})}\underbrace{\int_0^\infty w\exp\left(-\frac{w^2}{2}\right)dw}_{=[-e^{-w^2/2}]_0^\infty = 1}\quad\left(w = \sqrt{\frac{\sigma_{22}y^2 - 2\sigma_{12}y + \sigma_{11}}{|\Sigma|}}z\right)\\
&= \frac{1}{\pi(\sqrt{|\Sigma|}/\sigma_{22})\left[1 + \left(\frac{y - \sigma_{12}/\sigma_{22}}{\sqrt{|\Sigma|}/\sigma_{22}}\right)^2\right]}
\end{aligned}$$

which is pdf of Cauchy$\left(\frac{\sigma_{12}}{\sigma_{22}}, \frac{\sqrt{|\Sigma|}}{\sigma_{22}}\right)$.
Hence $\frac{\bar{X}_1}{\bar{X}_2} \rightsquigarrow$ Cauchy$\left(\frac{\sigma_{12}}{\sigma_{22}}, \frac{\sqrt{|\Sigma|}}{\sigma_{22}}\right)$ (and distribution of $\frac{\bar{X}_1}{\bar{X}_2}$ is always same regardless of n)