CMU SCS

15-826: Multimedia Databases and Data Mining

Lecture #13: Power laws

Potential causes and explanations

C. Faloutsos

CMU SCS

Must-read Material

- Mark E.J. Newman: *Power laws, Pareto distributions and Zipf's law*, Contemporary Physics 46, 323-351 (2005), or http://arxiv.org/abs/cond-mat/0412004v3

CMU SCS

Optional Material

- (optional, but very useful: Manfred Schroeder *Fractals, Chaos, Power Laws: Minutes from an Infinite Paradise* W.H. Freeman and Company, 1991) – ch. 15.

CMU SCS

Outline

Goal: 'Find similar / interesting things'

- Intro to DB
- Indexing - similarity search
- Data Mining

CMU SCS

Indexing - Detailed outline

- primary key indexing
- secondary key / multi-key indexing
- spatial access methods
  - z-ordering
  - R-trees
  - misc
- fractals
  - intro
  - applications
- text
- ...

CMU SCS

Indexing - Detailed outline

- fractals
  - intro
  - applications
    - disk accesses for R-trees (range queries)
    - …
    - dim. curse revisited
    - …
  - Why so many power laws?

CMU SCS

This presentation

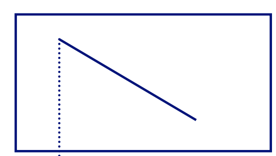
- Definitions
- Clarification: 3 forms of P.L.
- Examples and counter-examples
- Generative mechanisms

CMU SCS

Definition

- p(x) = C x ^ (-a)   (x >= xmin)
- Eg., prob(city pop. between x + dx)

log(p(x))



log(xmin) log (x)

CMU SCS

For discrete variables

$$p_k = Ck^{-a} \qquad (k > 0)$$

Or, the Yule distribution:

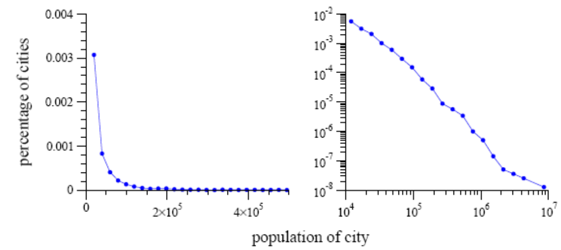
$$p_k = C\,B(k,a)$$

$$B(k,a) = \Gamma(k)\Gamma(a) / \Gamma(k+a) \approx k^{-a}$$

15-826 Copyright: C. Faloutsos (2017) 9

CMU SCS

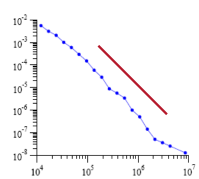
[Newman, 2005]



15-826 Copyright: C. Faloutsos (2017) 10

CMU SCS

Estimation for *a*



$$a = 1 + n\left[\sum_{i=1}^{n} \ln(x_i / x_{\min})\right]^{-1}$$

15-826 Copyright: C. Faloutsos (2017) 11

CMU SCS

This presentation

- Definitions
- Clarification: 3 forms of P.L.
- Examples and counter-examples
- Generative mechanisms

15-826 Copyright: C. Faloutsos (2017) 12

CMU SCS

Jumping to the conclusion:

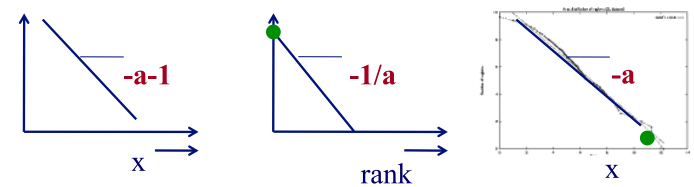
CMU SCS

3 versions of P.L.

PDF = frequency-count plot | Zipf plot = Rank-frequency | NCDF = CCDF

IF ONE PLOT IS P.L., SO ARE THE OTHER TWO

Prob(area = x) | area | Prob(area >= x)



CMU SCS

Details, and proof sketches:

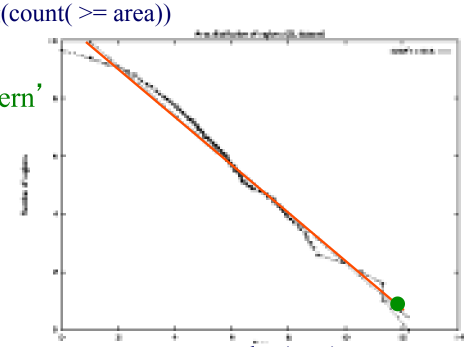
CMU SCS

Reminder

More power laws: areas – Korcak's law

log(count(>= area))

Scandinavian lakes area vs complementary cumulative count (log-log axes)

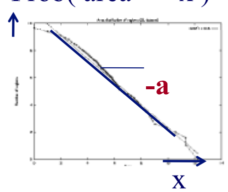


CMU SCS

3 versions of P.L.

NCDF = CCDF

Prob(area >= x)



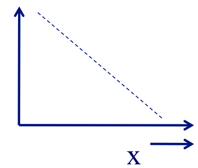
CMU SCS

3 versions of P.L.

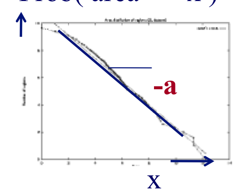
PDF

NCDF = CCDF

Prob(area = x)



Prob(area >= x)



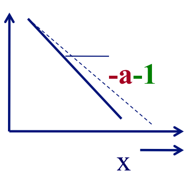
CMU SCS

3 versions of P.L.

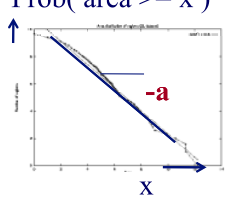
PDF

NCDF = CCDF

Prob(area = x)



Prob(area >= x)



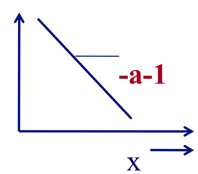
CMU SCS

3 versions of P.L.

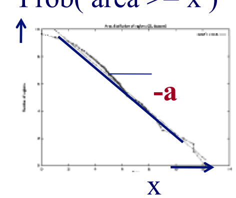
PDF

NCDF = CCDF

Prob(area = x)



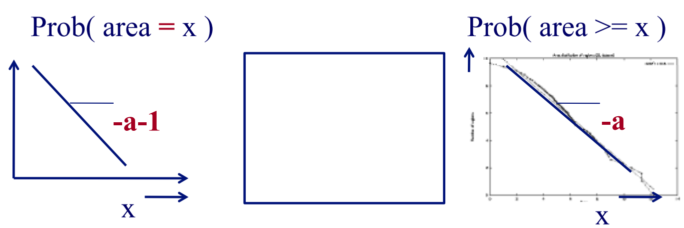
Prob(area >= x)



CMU SCS

3 versions of P.L.

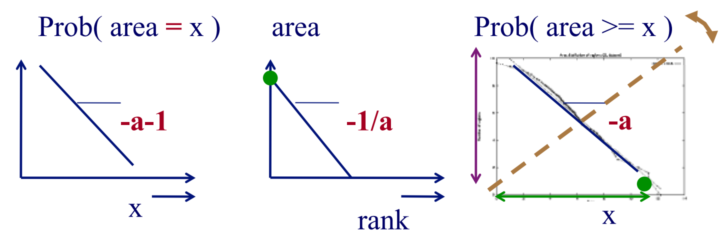
PDF | Zipf plot = Rank-frequency | NCDF = CCDF



CMU SCS

3 versions of P.L.

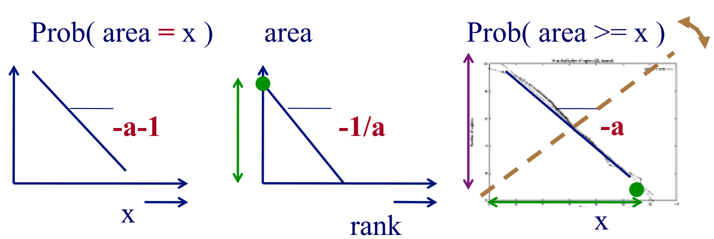
PDF | Zipf plot = Rank-frequency | NCDF = CCDF



CMU SCS

3 versions of P.L.

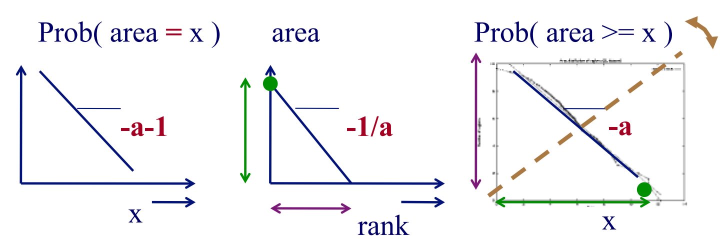
PDF | Zipf plot = Rank-frequency | NCDF = CCDF



CMU SCS

3 versions of P.L.

PDF | Zipf plot = Rank-frequency | NCDF = CCDF

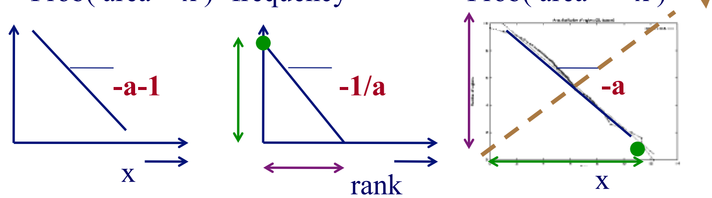


CMU SCS

3 versions of P.L.

PDF | Zipf plot = Rank-frequency | NCDF = CCDF

Prob(area = x) frequency Prob(area >= x)



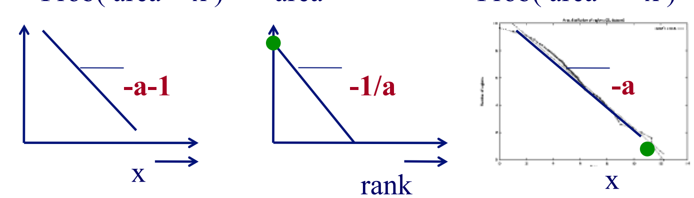
15-826 Copyright: C. Faloutsos (2017) 25

CMU SCS

3 versions of P.L.

PDF = frequency-count plot | Zipf plot = Rank-frequency | NCDF = CCDF

Prob(area = x) area Prob(area >= x)



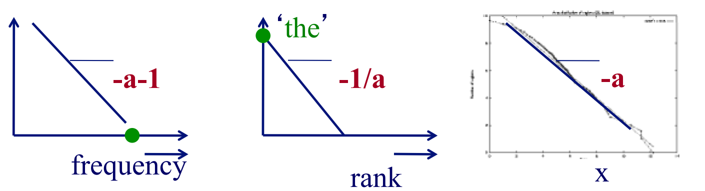
15-826 Copyright: C. Faloutsos (2017) 26

CMU SCS

3 versions of P.L.

PDF = frequency-count plot | Zipf plot = Rank-frequency | NCDF = CCDF

count frequency Prob(area >= x)



15-826 Copyright: C. Faloutsos (2017) 27

CMU SCS

Sanity check:

- Zipf (1949) showed that if
  - Slope of rank-frequency is -1
  - Then slope of freq-count is -2
- Check it!

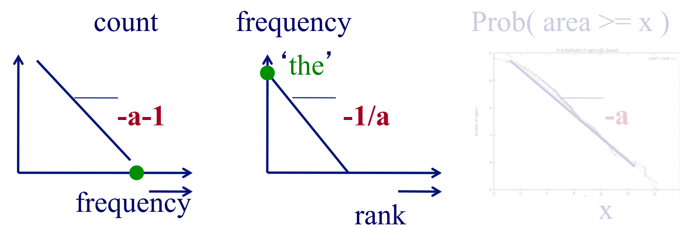
15-826 Copyright: C. Faloutsos (2017) 28

CMU SCS

3 versions of P.L.

PDF = frequency-count plot | Zipf plot = Rank-frequency | NCDF = CCDF

slope = -2 ⟺ slope = -1

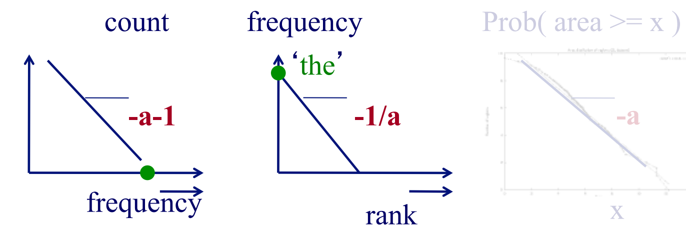


CMU SCS

3 versions of P.L.

PDF = frequency-count plot | Zipf plot = Rank-frequency | NCDF = CCDF

✓ slope = -2 ⟺ slope = -1

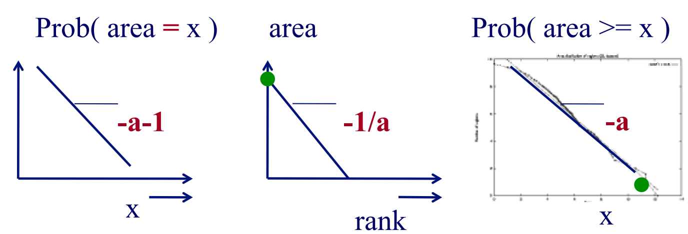


CMU SCS

3 versions of P.L.

PDF = frequency-count plot | Zipf plot = Rank-frequency | NCDF = CCDF

IF ONE PLOT IS P.L., SO ARE THE OTHER TWO



CMU SCS

This presentation

- Definitions
- Clarification: 3 forms of P.L.
- Examples and counter-examples
- Generative mechanisms

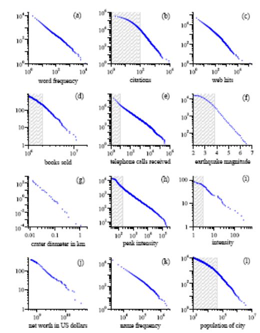
CMU SCS

Examples

- Word frequencies
- Citations of scientific papers
- Web hits
- Copies of books sold
- Magnitude of earthquakes
- Diameter of moon craters
- …

CMU SCS

[Newman 2005]

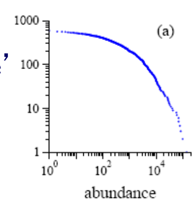


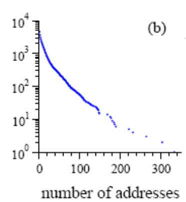
Rank-frequency plots
Or Cumulative D.F.

CMU SCS

NOT following P.L.

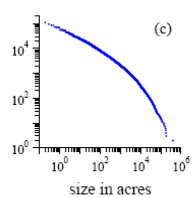
'abundance' of species





Number of addresses

Cumul. D.F.



Size of forest fires

CMU SCS

This presentation

- Definitions - clarification
- Examples and counter-examples
- Generative mechanisms
  - Combination of exponentials
  - Inverse
  - Random walk
  - Yule distribution = CRP
  - Percolation
  - Self-organized criticality
  - Other

CMU SCS

Combination of exponentials

Let $p(y) = e^{ay}$

- eg., radioactive decay, with half-life –a
- (= collection of people, playing russian roulette)

Let $x \sim e^{by}$

- (every time a person survives, we double his capital)

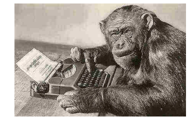
$p(x) = p(y)*dy/dx = 1/b\ x^{(-1+a/b)}$

- Ie, the final capital of each person follows P.L.

15-826 Copyright: C. Faloutsos (2017) 37

CMU SCS

Combination of exponentials



- Monkey on a typewriter:
- m=26 letters equiprobable;
- space bar has prob. q_s

THEN: Freq(x-th most frequent word) = $x^{(-a)}$

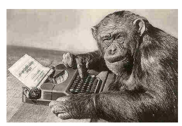
see Eq. 47 of [Newman]:

$a = [2 \ln(m) - \ln(1 - q_s)] / [\ln m - \ln(1 - q_s)]$

15-826 Copyright: C. Faloutsos (2017) 38

CMU SCS

Combination of exponentials

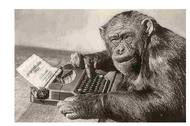


- Most freq 'words'?

15-826 Copyright: C. Faloutsos (2017) 39

CMU SCS

Combination of exponentials



- Most freq 'words'?
- *a, b , ... z*
- *aa, ab, … az, ba, … bz, … zz*
- …

15-826 Copyright: C. Faloutsos (2017) 40

CMU SCS

This presentation

- Definitions
- Clarification
- Examples and counter-examples
- Generative mechanisms
  - Combination of exponentials
  - ➡ Inverse
  - Random walk
  - Yule distribution = CRP
  - Percolation
  - Self-organized criticality
  - Other

15-826 Copyright: C. Faloutsos (2017) 41

CMU SCS

Inverses of quantities

- y follows $p(y)$ and goes through zero
- $x = 1/y$
- Then $p(x) = \ldots = - p(y) / x^2$
- For $y \sim 0$, x has power law tail.

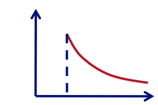
y-> speed

x-> travel time

y:

0mph…….1mph

count



Travel time

15-826 Copyright: C. Faloutsos (2017) 42

CMU SCS

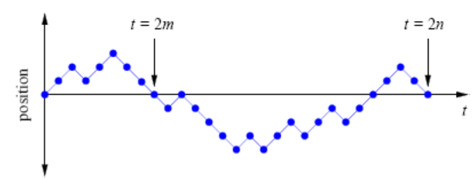
This presentation

- Definitions
- Clarification
- Examples and counter-examples
- Generative mechanisms
  - Combination of exponentials
  - Inverse
  - ➡ Random walk
  - Yule distribution = CRP
  - Percolation
  - Self-organized criticality
  - Other

15-826 Copyright: C. Faloutsos (2017) 43

CMU SCS

Random walks

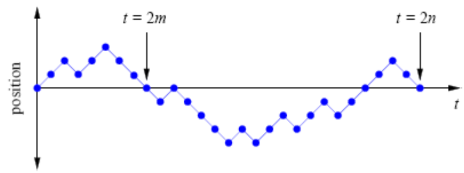


Inter-arrival times PDF: $p(t) \sim$??

15-826 Copyright: C. Faloutsos (2017) 44

CMU SCS

Random walks



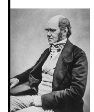
Inter-arrival times PDF: $p(t) \sim t^{-3/2}$

William Feller: *An introduction to probability theory and its applications*, Vol. 1, Wiley 1971
p. 78 Eq (3.7) and Stirling's approx (p. 75, Eq(2.4))

CMU SCS

Random walks

J. G. Oliveira & A.-L. Barabási Human Dynamics: The Correspondence Patterns of Darwin and Einstein. *Nature* **437,** 1251 (2005) . [PDF]



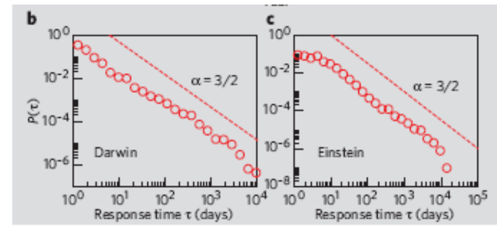
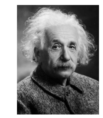


Figure 1 | The correspondence patterns of Darwin and Einstein.



CMU SCS

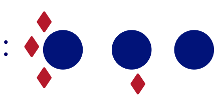
This presentation

- Definitions - clarification
- Examples and counter-examples
- Generative mechanisms
  - Combination of exponentials
  - Inverse
  - Random walk
  - ➡ Yule distribution = CRP
  - Percolation
  - Self-organized criticality
  - Other

CMU SCS

Yule distribution and CRP

Chinese Restaurant Process (CRP):



Newcomer to a restaurant

- Joins an existing table (preferring large groups
- Or starts a new table/group of its own, with prob *1/m*

a.k.a.: rich get richer; Yule process

CMU SCS

Yule distribution and CRP

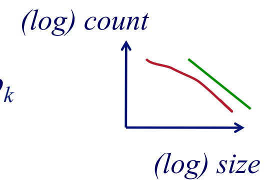
Then:

Prob(k people in a group) = p_k

$= (1 + 1/m)\ B(k, 2+1/m)$

$\sim k^{-(2+1/m)}$

(since B(a,b) ~ a ** (-b) : power law tail)



CMU SCS

Yule distribution and CRP

- Yule process
- Gibrat principle
- Matthew effect
- Cumulative advantage
- Preferential attachement
- 'rich get richer'

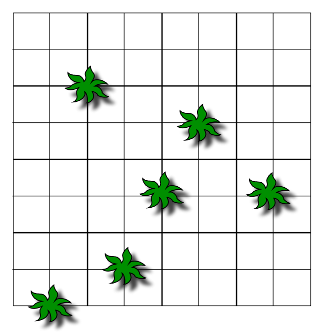
CMU SCS

This presentation

- Definitions - clarification
- Examples and counter-examples
- Generative mechanisms
  - Combination of exponentials
  - Inverse
  - Random walk
  - Yule distribution = CRP
  - ➡ Percolation
  - Self-organized criticality
  - Other

CMU SCS

Percolation and forest fires

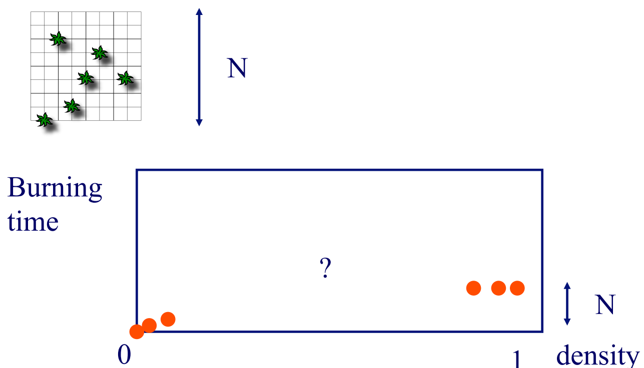


A burning tree will cause its neighbors to burn next.

Which tree density p will cause the fire to last longest?

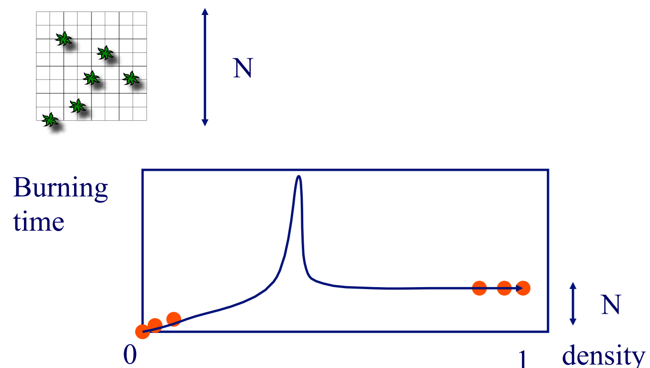
CMU SCS

Percolation and forest fires



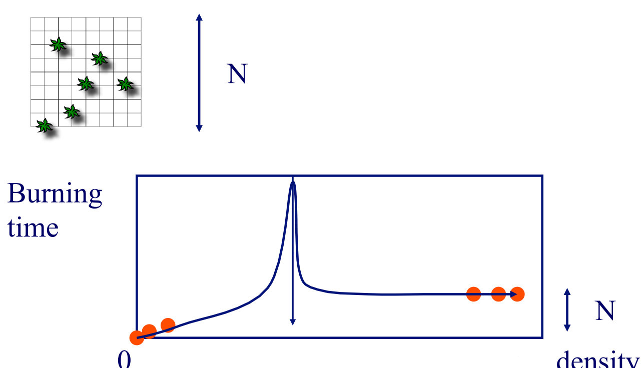
CMU SCS

Percolation and forest fires



CMU SCS

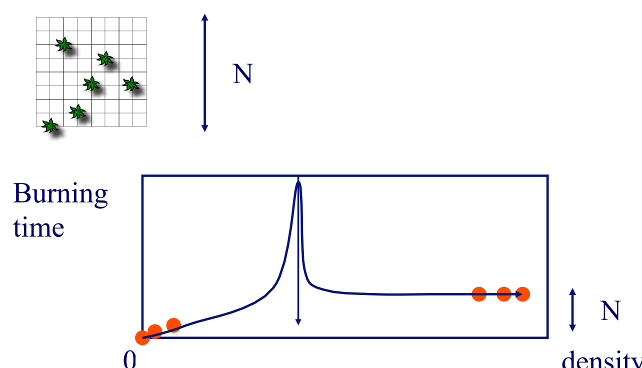
Percolation and forest fires



Percolation threshold, pc ~ ????

CMU SCS

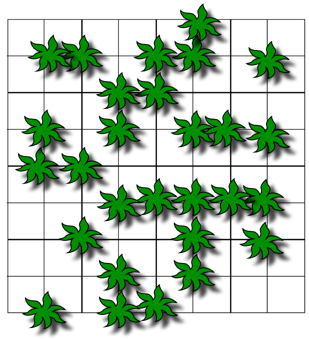
Percolation and forest fires



Percolation threshold, pc ~ 0.593

CMU SCS

Percolation and forest fires



At pc ~ 0.593:
No characteristic scale;
'patches' of all sizes;
Korcak-like 'law'.

CMU SCS

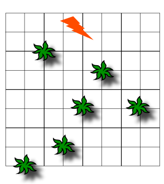
This presentation

- Definitions - clarification
- Examples and counter-examples
- Generative mechanisms
  - Combination of exponentials
  - Inverse
  - Random walk
  - Yule distribution = CRP
  - Percolation
  - Self-organized criticality
  - Other

CMU SCS

Self-organized criticality

- Trees appear at random (eg., seeds, by the wind)
- Fires start at random (eg., lightning)
- Q1: What is the distribution of size of forest fires?

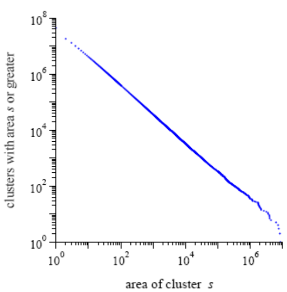


CMU SCS

Self-organized criticality

- A1: Power law-like

CCDF

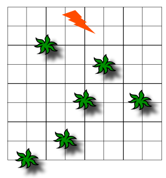


Area of cluster s

CMU SCS

Self-organized criticality

- Trees appear at random (eg., seeds, by the wind)
- Fires start at random (eg., lightning)
- Q2: what is the average density?



CMU SCS

Self-organized criticality

- A2: the critical density pc ~ 0.593

CMU SCS

Self-organized criticality

- [Bak]: size of avalanches ~ power law:
- Drop a grain randomly on a grid
- It causes an avalanche if height(x,y) is >1 higher than its four neighbors

[Per Bak: *How Nature works*, 1996]

CMU SCS

This presentation

- Definitions - clarification
- Examples and counter-examples
- Generative mechanisms
  - Combination of exponentials
  - Inverse
  - Random walk
  - Yule distribution = CRP
  - Percolation
  - Self-organized criticality
  - Other

CMU SCS

Other

- Random multiplication
- Fragmentation

-> lead to lognormals (~ look like power laws)

CMU SCS

Others

Random multiplication:

- Start with C dollars; put in bank
- Random interest rate s(t) each year t
- Each year t: C(t) = C(t-1) * (1+ s(t))

- Log(C(t)) = log(C) + log(..) + log(..) … -> Gaussian

CMU SCS

Others

Random multiplication:

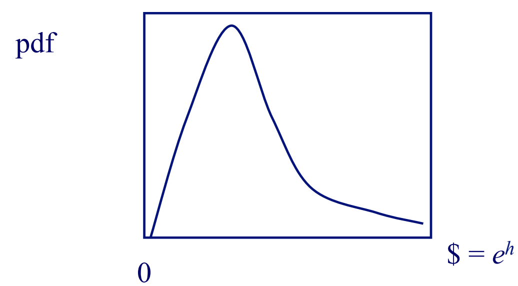
- Log(C(t)) = log(C) + log(..) + log(..) … -> Gaussian

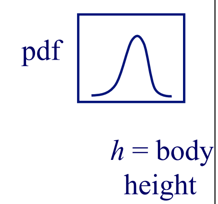
- Thus C(t) = exp(Gaussian)
- By definition, this is Lognormal

CMU SCS

Others

Lognormal:



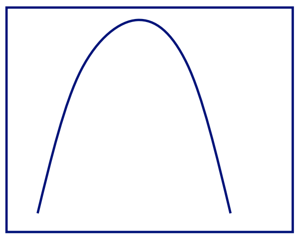


CMU SCS

Others

Lognormal:

log(pdf)



parabola

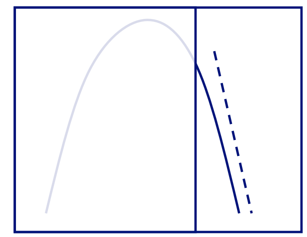
log ($)

CMU SCS

Others

Lognormal:

log(pdf)



parabola

1c

log ($)

CMU SCS

Other

- Random multiplication
- Fragmentation

-> lead to lognormals (~ look like power laws)

CMU SCS

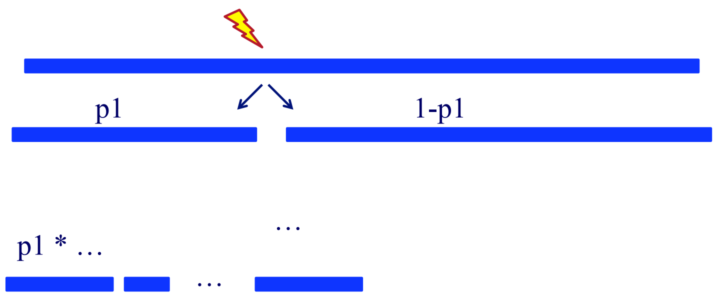
Other

- Stick of length 1
- Break it at a random point x (0<x<1)
- Break each of the pieces at random

- Resulting distribution: lognormal (why?)

CMU SCS

Fragmentation -> lognormal



CMU SCS

Conclusions

- Power laws and power-law like distributions appear often
- (fractals/self similarity -> power laws)
- Exponentiation/inversion
- Yule process / CRP / rich get richer
- Criticality/percolation/phase transitions
- Fragmentation -> lognormal ~ P.L.

CMU SCS

References

- *Zipf, Power-laws, and Pareto - a ranking tutorial*, Lada A. Adamic www.hpl.hp.com/research/idl/papers/ranking/ranking.html
- L.A. Adamic and B.A. Huberman, *'Zipf's law and the Internet'*, *Glottometrics* 3, 2002,143-150
- *Human Behavior and Principle of Least Effort*, G.K. Zipf, Addison Wesley (1949)