36-705 Intermediate Statistics Homework #2 Solutions

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Problem 1 [25 pts.]

Let X have mean 0. We say that X is sub-Gaussian if there exists $\sigma > 0$ such that

$$\log\left(\mathbb{E}[e^{tX}]\right) \leq \frac{t^2\sigma^2}{2}$$

for all t.

(i) **[5 pts.]** Show that X is sub-Gaussian if and only if $-X$ is sub-Gaussian.

Suppose X is sub-Gaussian. Then

$$\log\left(\mathbb{E}[e^{tX}]\right) \leq \frac{t^2\sigma^2}{2} \tag{1}$$

for some $\sigma > 0$ and all $t \in \mathbb{R}$. Let $s = -t$. From (1) we have

$$\begin{aligned}\log\left(\mathbb{E}[e^{s(-X)}]\right) &= \log\left(\mathbb{E}[e^{tX}]\right)\\ &\leq \frac{t^2\sigma^2}{2}\\ &= \frac{s^2\sigma^2}{2}\end{aligned}$$

for some $\sigma > 0$ and all $s \in \mathbb{R}$. The reverse direction is proven analogously.

(ii) **[10 pts.]** Let X have mean μ. Suppose that $X - \mu$ is sub-Gaussian. Show that

$$\mathbb{P}(|X-\mu| \geq t) \leq 2e^{-t^2/(2\sigma^2)}.$$

For $b > 0$:

$$\mathbb{P}\left(X-\mu > t\right) = \mathbb{P}\left(b(X-\mu) > bt\right) = \underbrace{\mathbb{P}\left(e^{b(X-\mu)} > e^{bt}\right) \leq \mathbb{E}\left[e^{b(X-\mu)}\right]e^{-bt}}_{\text{Markov's inequality}}$$

We know that $X-\mu$ is sub-Gaussian. Thus, let σ be a constant such that $\mathbb{E}\left[e^{s(X-\mu)}\right] \leq e^{\sigma^2 s^2/2}$, $\forall s \in \mathbb{R}$. This implies that

$$\mathbb{P}\left(X-\mu > t\right) \leq e^{\sigma^2 b^2/2}e^{-bt}$$

By minimizing the bound with respect to $b > 0$ we obtain:

$$\mathbb{P}\left(X-\mu > t\right) \leq e^{-t^2/2\sigma^2}$$

We can obtain an identical inequality for $\mathbb{P}\left(X < -t\right)$, since $\mathbb{E}\left[e^{-s(X-\mu)}\right] \leq e^{\sigma^2 s^2/2}$, $\forall s \in \mathbb{R}$. Therefore:

$$\mathbb{P}\left(|X-\mu| > t\right) = \mathbb{P}\left(X-\mu > t\right) + \mathbb{P}\left(X-\mu < -t\right) \leq 2e^{-t^2/2\sigma^2}$$

(iii) **[10 pts.]** Suppose that X is sub-Gaussian. Show that, for any $p > 0$,

$$\mathbb{E}\big[|X|^p\big] \le p 2^{p/2} \sigma^p \Gamma(p/2).$$

$$\begin{aligned}
\mathbb{E}\big[|X|^p\big] &= \int_0^\infty P\big(|X|^p > t\big)dt \\
&= \int_0^\infty P\big(|X| > t^{1/p}\big)dt \\
&\le 2\int_0^\infty e^{-\frac{t^{2/p}}{2\sigma^2}}dt && \text{by part (ii)} \\
&= p2^{p/2}\sigma^p \int_0^\infty e^{-u}u^{p/2-1}du && \text{Letting } u = \frac{t^{2/p}}{2\sigma^2};\quad dt = p2^{p/2}\sigma^p u^{p/2-1}du \\
&= p2^{p/2}\sigma^p\Gamma(p/2).
\end{aligned}$$

Problem 2 [30 pts.]

Let $X_1, \ldots, X_n$ be iid, with mean μ, $\mathrm{Var}(X_i) = \sigma^2$ and $|X_i| \le c$. Bernstein's inequality says that

$$\mathbb{P}\big(|\overline{X}_n - \mu| > t\big) \le 2\exp\left\{-\frac{nt^2}{2\sigma^2 + 2ct/3}\right\}.$$

Suppose that $\sigma^2 = O(1/n)$. Use Bernstein's inequality to show that $\overline{X}_n - \mu = O_P(1/n)$.

By assumption, $\exists M > 0$ such that $n\sigma^2 \le M$ for large n.

$$\begin{aligned}
P\left(\left|\frac{\overline{X}_n - \mu}{1/n}\right| > t\right) &= P\big(|\overline{X}_n - \mu| > t/n\big) \\
&\le 2\exp\left(-\frac{t^2/n}{2\sigma^2 + \frac{2ct}{3n}}\right) \\
&= 2\exp\left(-\frac{t^2}{2\sigma^2 n + \frac{2ct}{3}}\right) \\
&\overset{\text{large } n}{\le} 2\exp\left(-\frac{t^2}{2M + \frac{2ct}{3}}\right) \\
&\overset{\text{when } t\ge 1}{\le} 2\exp\left(-\frac{t^2}{2Mt + \frac{2ct}{3}}\right) \\
&= 2\exp\left(-\frac{t}{2M + \frac{2c}{3}}\right).
\end{aligned}$$

So for any $\epsilon > 0$ we choose

$$t \geq \max\left\{1, -2(M + c/3)\log(\epsilon/2)\right\},$$

so that

$$P\left(\left|\frac{\overline{X}_n - \mu}{1/n}\right| > t\right) \leq \epsilon.$$

Problem 3 [32 pts.]

Prove or disprove the following:

(i) If $X_n = O_P(a_n)$ and $Y_n = O_P(b_n)$ then $X_n + Y_n = O_P(a_n b_n)$.

The claim is false.

Counterexample. Let $X_n = Y_n = a_n = b_n = \frac{1}{n}$. Then,

$$\frac{|X_n + Y_n|}{|a_n b_n|} = \frac{2/n}{1/n^2} = 2n,$$

which is unbounded.

(ii) If $X_n = o_P(a_n)$ and $Y_n = o_P(b_n)$ then $X_n + Y_n = o_P(\min\{a_n, b_n\})$.

The claim is false.

Counterexample. Let $X_n = \frac{1}{n^2}$, $Y_n = \frac{1}{n^3}$, $a_n = \frac{1}{n}$, and $b_n = \frac{1}{n^2}$. Then,

$$\frac{|X_n + Y_n|}{|\min\{a_n, b_n\}|} = 1 + \frac{1}{n} \not\to 0.$$

(iii) If $X_n = o_P(a_n)$ and $Y_n = O_P(b_n)$ then $X_n/Y_n = o_P(a_n/b_n)$.

The claim is false.

Counterexample. Let $X_n = \frac{1}{n^2}$, $Y_n = \frac{1}{n^2}$, $a_n = \frac{1}{n}$, and $b_n = \frac{1}{n}$. Then,

$$\frac{|X_n/Y_n|}{|a_n/b_n|} = 1 \not\to 0.$$

(iv) If $X_n = O_P(a_n)$ and $Y_n = O_P(b_n)$ then $X_n Y_n = o_P(a_n b_n)$.

The claim is false.

Counterexample. Let $X_n = Y_n = a_n = b_n = 1$. Then,

$$\frac{|X_n Y_n|}{|a_n b_n|} = 1 \not\to 0.$$

Problem 4 [13 pts.]

Let $U \sim \text{Unif}(0,1)$. Let $Y = F^{-1}(U)$ where F is a continuous cdf on the real line. Show that the distribution of Y is F. (Hint: You may assume that F is strictly increasing.)

Let F_Y denote the distribution of Y. For any $y \in \mathbb{R}$,

$$\begin{aligned}
F_Y(y) &= P(Y \le y) \\
&= P(F^{-1}(U) \le y) \\
&= P(U \le F(y)) && \text{because } F \text{ is strictly increasing} \\
&= F(y).
\end{aligned}$$