Test 3 - Solutions

Intermediate Statistics - 36-705

November 11, 2016

Problem 1. [30 points]
Let $X_1, \ldots, X_n \sim \text{Poisson}(\lambda)$. Hence, the probability function for X_i is

$$p(x; \lambda) = \frac{\lambda^x e^{-\lambda}}{x!}, \quad x = 0, 1, 2, \ldots.$$

(a) **[15 pts.]** Find the maximum likelihood estimator, the score function and the Fisher information.

$$L(\lambda) = \frac{\lambda^{\sum_{i=1}^n X_i} e^{-n\lambda}}{X_1! \cdots X_n!}$$
$$\ell(\lambda) = \log(\lambda) \sum_{i=1}^n X_i - n\lambda - \sum_{i=1}^n \log(X_i!)$$

$$S(\lambda) = \ell'(\lambda) = \frac{1}{\lambda} \sum_{i=1}^n X_i - n$$

and letting $\ell'(\lambda) = 0$, we have

$$\frac{1}{\lambda} \sum_{i=1}^n X_i - n = 0$$
$$\implies \widehat{\lambda} = \overline{X}.$$

$$\ell''(\lambda) = -\frac{1}{\lambda^2} \sum_{i=1}^n X_i$$

$$I_n(\lambda) = -\mathbb{E}\big[\ell''(\lambda)\big] = \frac{n}{\lambda}.$$

(b) **[15 pts.]** Construct an asymptotic $1-\alpha$ confidence interval for $\psi = \log(\lambda)$.

By the asymptotic normality of the MLE,

$$\frac{\sqrt{n}(\widehat{\lambda}-\lambda)}{1/\sqrt{I_1(\lambda)}} \rightsquigarrow N(0,1).$$

By the delta method,

$$\frac{\sqrt{n}(\widehat{\psi}-\psi)}{1/\sqrt{\lambda}} \rightsquigarrow N(0,1),$$

where $\widehat{\psi} = \log(\widehat{\lambda})$. Therefore, an asymptotic $1-\alpha$ confidence interval for ψ is

$$C_n = \left[\widehat{\psi} - z_{\alpha/2}\sqrt{\frac{1}{n\widehat{\lambda}}},\ \ \widehat{\psi} + z_{\alpha/2}\sqrt{\frac{1}{n\widehat{\lambda}}}\right].$$

Problem 2. [40 points]
Let $X_1, \dots, X_n \sim \text{Uniform}(0, \theta)$.

(a) **[15 pts.]** Let $X_{(n)} = \max\{X_1, \dots, X_n\}$. Let

$$C_n = \big[X_{(n)},\ a_n X_{(n)}\big].$$

Find a_n so that C_n is a $1 - \alpha$ confidence interval for θ. (Do not use any asymptotic approximations.)

For any $t \in (0, \theta)$,

$$\mathbb{P}\big(X_{(n)} \le t\big) = \left(\frac{t}{\theta}\right)^n.$$

Let $\alpha \in (0, 1)$.

$$\begin{aligned} \left(\frac{t}{\theta}\right)^n &= \alpha \\ \implies t &= \theta \alpha^{1/n} \end{aligned}$$

Therefore,

$$\begin{aligned} \mathbb{P}\big(X_{(n)} \le \theta \alpha^{1/n}\big) &= \alpha \\ \mathbb{P}\big(\theta \ge \alpha^{-1/n} X_{(n)}\big) &= \alpha \end{aligned}$$

and $\mathbb{P}\big(\theta \ge X_{(n)}\big) = 1$, so

$$C_n = \left[X_{(n)}, \frac{X_{(n)}}{\alpha^{1/n}}\right]$$

is a $1 - \alpha$ confidence interval for θ.

Alternate solution. (using a pivot)

Let $Q = X_{(n)}/\theta$. Then

$$\mathbb{P}(Q \leq t) = \prod_{i=1}^{n} \mathbb{P}(X_i \leq t\theta) = t^n.$$

Let $c = \alpha^{1/n}$. Then

$$\mathbb{P}(Q \leq c) = \alpha.$$

Also, $\mathbb{P}(Q \leq 1) = 1$. Therefore,

$$\begin{aligned}
1 - \alpha &= \mathbb{P}(c \leq Q \leq 1) \\
&= \mathbb{P}\left(c \leq \frac{X_{(n)}}{\theta} \leq 1 \right) \\
&= \mathbb{P}\left(\frac{1}{c} \geq \frac{\theta}{X_{(n)}} \geq 1 \right) \\
&= \mathbb{P}\left(X_{(n)} \leq \theta \leq \frac{X_{(n)}}{c} \right) \\
&= \mathbb{P}\left(X_{(n)} \leq \theta \leq \frac{X_{(n)}}{c} \right)
\end{aligned}$$

so a $1 - \alpha$ confidence interval for θ is

$$C_n = \left[X_{(n)}, \frac{X_{(n)}}{\alpha^{1/n}} \right].$$

(b) **[15 pts.]** Consider testing

$$H_0 : \theta = \theta_0 \quad \text{versus} \quad \theta > \theta_0.$$

Suppose we reject H_0 if $X_{(n)} > w_n$ so that the test has size α. Find w_n so that the test has size α.

From part (a) we have

$$\mathbb{P}_{\theta_0}\big(X_{(n)} \leq \theta_0 \alpha^{1/n}\big) = \alpha$$

for any $\alpha \in (0,1)$. Therefore,

$$\begin{aligned}
\mathbb{P}_{\theta_0}\big(X_{(n)} \leq \theta_0(1-\alpha)^{1/n}\big) &= 1-\alpha \\
1 - \mathbb{P}_{\theta_0}\big(X_{(n)} > \theta_0(1-\alpha)^{1/n}\big) &= 1-\alpha \\
\mathbb{P}_{\theta_0}\big(X_{(n)} > \theta_0(1-\alpha)^{1/n}\big) &= \alpha
\end{aligned}$$

and thus,

$$\text{reject } H_0 \text{ if } X_{(n)} > \theta_0(1-\alpha)^{1/n}$$

is a size α test.

(c) **[10 pts.]** Now suppose that the true value of the parameter is θ where $\theta > \theta_0$. Show that the power tends to 1 as $n \to \infty$.

$$\begin{aligned}
\beta(\theta) &= \mathbb{P}_\theta\big(X_{(n)} > \theta_0(1-\alpha)^{1/n}\big) \\
&= 1 - \mathbb{P}_\theta\big(X_{(n)} \le \theta_0(1-\alpha)^{1/n}\big) \\
&= 1 - \left(\frac{\theta_0(1-\alpha)^{1/n}}{\theta}\right)^n \\
&= 1 - \underbrace{\left(\frac{\theta_0}{\theta}\right)^n}_{\to 0}(1-\alpha) \\
&\to 1.
\end{aligned}$$

Problem 3. [30 points]

(a) **[20 pts.]** Let $X \sim N(\theta, 1)$. Consider the size α Neyman-Pearson test of

$$H_0 : \theta = \theta_0 \quad \text{versus} \quad H_1 : \theta = \theta_1$$

where $\theta_0 < \theta_1$. Show that the test rejects H_0 when $X > c$ and find an expression for c.

$$\begin{aligned}
\frac{L(\theta_1)}{L(\theta_0)} &= \frac{\exp\left\{-\frac{1}{2}(X-\theta_1)^2\right\}}{\exp\left\{-\frac{1}{2}(X-\theta_0)^2\right\}} \\
&= \exp\left\{(\theta_1-\theta_0)X + \frac{1}{2}(\theta_0^2-\theta_1^2)\right\}
\end{aligned}$$

We reject H_0 when

$$\begin{aligned}
\exp\left\{(\theta_1-\theta_0)X + \frac{1}{2}(\theta_0^2-\theta_1^2)\right\} &> k \\
\iff (\theta_1-\theta_0)X + \frac{1}{2}(\theta_0^2-\theta_1^2) &> \log(k) \\
\iff X &> \frac{\log(k) + \frac{1}{2}(\theta_1^2-\theta_0^2)}{\theta_1-\theta_0}
\end{aligned}$$

where k is chosen so that

$$\begin{aligned}
\mathbb{P}_{\theta_0}\left(X > \frac{\log(k) + \frac{1}{2}(\theta_1^2-\theta_0^2)}{\theta_1-\theta_0}\right) &= \alpha \\
\implies \mathbb{P}_{\theta_0}\left(Z > \frac{\log(k) + \frac{1}{2}(\theta_1^2-\theta_0^2)}{\theta_1-\theta_0} - \theta_0\right) &= \alpha.
\end{aligned}$$

Therefore, we take

$$\begin{aligned}
\frac{\log(k) + \frac{1}{2}(\theta_1^2-\theta_0^2)}{\theta_1-\theta_0} - \theta_0 &= z_\alpha \\
k &= \exp\left\{(z_\alpha+\theta_0)(\theta_1-\theta_0) + \frac{1}{2}(\theta_0^2-\theta_1^2)\right\}
\end{aligned}$$

So the test rejects H_0 when

$$\begin{aligned}
X &> \frac{\log(k) + \frac{1}{2}(\theta_1^2-\theta_0^2)}{\theta_1-\theta_0} \\
\iff X &> \theta_0 + z_\alpha.
\end{aligned}$$

(b) **[10 pts.]** Show that the $\beta(\theta_1) \to 1$ as $\theta_1 \to \infty$, where β is the power function.

$$\begin{aligned}
\beta(\theta_1) &= \mathbb{P}_{\theta_1}(X > \theta_0 + z_\alpha) \\
&= \mathbb{P}(Z > \theta_0 + z_\alpha - \theta_1) \\
&= 1 - \Phi(\theta_0 + z_\alpha - \theta_1) \\
&\xrightarrow{\theta_1 \to \infty} 1.
\end{aligned}$$