Math-UA.233: Theory of Probability
Lecture 17

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Independence of RVs (Ross Sec 6.2)

Suppose that X and Y are any RVs — discrete, continuous, whatever — on the same sample space.

Definition (Ross eqn. 6.2.1)

*X and Y are **independent** if the events*

$$\{X \in A\} \quad \textit{and} \quad \{Y \in B\}$$

are independent for any two sets of real numbers A and B.

That is, "any event defined solely in terms of X is independent from any event defined solely in terms of Y".

We use the same idea for any larger collection of $X_1, X_2, \ldots$.
They are **independent** if the events

$$\{X_1 \in A_1\},\ \{X_2 \in A_2\}, \ldots$$

are independent for any choice of subsets $A_1, A_2, \ldots, \subseteq \mathbb{R}$.

Intuitively: “any information we obtain about all but one of these RVs has no effect on the probable behaviour of the remaining one”.

It would be very tedious to check the independence of $\{X \in A\}$ and $\{X \in B\}$ for all possible sets A and B!

Happily, we don't have to.

Proposition (Ross p228-9)

X and Y are independent

- *if and only their joint CDF is given by*

$$F_{X,Y}(a, b) = F_X(a)F_Y(b)$$

- *(in case discrete) if and only if their joint PMF is*

$$p_{X,Y}(x, y) = p_X(x)p_Y(y)$$

- *(in case both continuous) if and only if their joint PDF is*

$$f_{X,Y}(x, y) = f_X(x)f_Y(y).$$

Very often we are told about two RVs separately, told their individual PMFs or PDFs, and told that they are independent.

Then, because of the independence, we also know what their joint PMF or PDF has to be.

(We saw last time that, without independence, the individual PMFs do not determine the joint PMF uniquely.)

Example (Ross E.g. 6.2a)

Perform $n + m$ independent trials with success probability p. Let

$$X = \text{number of successes in first } n \text{ trials}$$
$$Y = \text{number of successes in remaining } m \text{ trials.}$$

Then these RVs are independent.

Example (Ross E.g. 6.2c)

Alice and Bob agree to meet at a certain location. Each of them independently arrives at a time uniformly distributed between 12 noon and 1pm. What is the probability that the first one to arrive has to wait at least 10 minutes for the other?

Independence appears naturally in the **splitting property** of the Poisson distribution.

Example (Ross E.g. 6.2b)

The number of people who board a bus on a given day is a Poi(λ) *RV. Each of them is a senior with probability p, independently of the others. Then the numbers of seniors and non-seniors boarding the bus are independent RVs which are Poi*$(p\lambda)$ *and Poi*$((1-p)\lambda)$*, respectively.*

Example (Buffon's needle, Ross E.g. 6.2d)

A table is made of straight planks of wood of width D. A needle of length L, where $L \leqslant D$, is thrown haphazardly onto the table. What is the probability that in its final location it crosses one of the lines between planks?

KEY ASSUMPTION: The distance from the centre of the needle to the nearest plank is independent from its angle, and both are uniform.

Here is an example involving three independent RVs.

Example (Ross E.g. 6.2h)

Let X, Y and Z be independent Unif$(0,1)$ RVs. Find $P(X \geqslant YZ)$.

This requires us to work with the jointly continuous random *triple* of RVs (X, Y, Z). Everything works just the same as for pairs. Now the joint PDF is a function on $\mathbb{R}^3$: in this case,

$$f_{X,Y,Z}(x,y,z) = f_X(x)f_Y(y)f_Z(z) = 1 \quad \text{when } 0 \leqslant x,y,z \leqslant 1.$$

If (X, Y) are jointly continuous with joint PDF f, then we can often just spot whether they are independent. The following observation can help.

Proposition (Ross Prop 6.2.1)

If X and Y are jointly continuous with joint PDF f, then they are are independent if and only if f can be expressed as

$$f(x, y) = g(x)h(y) \quad \textit{for all } (x, y) \in \mathbb{R}^2$$

using some functions g and h.

(There's also a version for discrete RVs: see Ross)

The point is that you don't have to know that $g = f_X$ and $h = f_Y$ to make this check.

But you do still have to be careful when using this proposition!

Example (Ross E.g. 6.2f)

Let (X, Y) be jointly continuous with joint PDF f. Are they independent in the following cases?

(a) *Case 1:*

$$f(x, y) = 6e^{-2x}e^{-3y} \quad \textit{for } 0 < x, y < \infty$$

and 0 *elsewhere.*

(b) *Case 2:*

$$f(x, y) = 24xy \quad \textit{for } 0 < x, y < 1 \textit{ and } 0 < x + y < 1$$

and 0 *elsewhere.*

This is a good place to introduce another simple and useful family of joint distributions.

Let A be a 'nice' region in the plane whose area is positive but finite (think of a polygon or a blob with a smooth boundary).

Definition

*The random vector (X, Y) is **uniformly distributed over** A, or **Unif**(A), if it is jointly continuous with joint PDF*

$$f(x, y) = \begin{cases} \frac{1}{\mathrm{area}(A)} & \text{if } (x, y) \in A \\ 0 & \text{otherwise.} \end{cases}$$

(The constant value of f inside A has to be $\frac{1}{\mathrm{area}(A)}$ because we must have $\iint_{\mathbb{R}^2} f(x, y)\, dx\, dy = \iint_A f(x, y)\, dx\, dy = 1$.)

Example

Let (X, Y) be uniformly distributed over a nice region A. Are X and Y independent in the following cases?

(a) *Case 1:* $A = \{(x, y) \mid x^2 + y^2 \leqslant 1\}$*, a closed disc of unit radius.*

(b) *Case 2:* $A = [0, 1] \times [0, 1]$*, a unit square.*

The previous example illustrates a general principle which helps clarify the notion of 'independence' for jointly continuous RVs.

Proposition

If (X, Y) is Unif(A) for some $A \subseteq \mathbb{R}^2$, then X and Y are independent if and only if A has the special form

$$A = B \times C = \{(x, y) \,|\, x \in B,\ y \in C\}$$

for some $B \subseteq \mathbb{R}$ and $C \subseteq \mathbb{R}$.

Here is another very important family of joint distributions.

Definition (Ross E.g. 6.5d)

Fix a parameter $-1 < \rho < 1$. The random vector (X, Y) is a ***bivariate standard normal with correlation*** *ρ if it is jointly continuous with joint PDF*

$$f(x, y) = \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left(- \frac{1}{2(1-\rho^2)}(x^2 - 2\rho xy + y^2)\right)$$

for $-\infty < x, y < \infty$.

Example

This pair of RVs X and Y is independent if and only if $\rho = 0$.

More on this example later in the course.

Functions of pairs of RVs (various parts of Ross Ch 6 & 7)

Suppose X, Y are two RVs, and g is a function $\mathbb{R}^2 \longrightarrow \mathbb{R}$. Then we get a new RV $g(X, Y)$. Mathematically, it is the function

$$S \longrightarrow \mathbb{R} \; : \; s \mapsto g(X(s), Y(s)).$$

Next task: generalize our understanding about functions of a single RV, like $g(X)$.

First phenomena to note (just like for $g(X)$):

- If X and Y are discrete, then so is $g(X, Y)$.
- But if (X, Y) are jointly continuous with joint PDF f, then $g(X, Y)$ *need not be continuous*, even for quite simple functions g.

For a function of a single RV, say $g(X)$, we learned:

- LOTUS: an easy formula for $E[g(X)]$
- Finding the CDF of $g(X)$ by expressing $\{g(X) \leqslant a\}$ in terms of events defined by X
- Under certain conditions on g and for continuous X, there's a formula for the PDF of $g(X)$ in terms of the PDF of X

Next part of the course: generalizing these results to functions of two (or more) RVs.

Several things get more complicated, and our results won't be as complete.

But we will meet some important special cases/applications along the way.

An example to do 'by hand' to get us started:

Example (Ross E.g. 6.1d)

I play darts by throwing a dart at a circular dartboard of radius R. Relative to the origin, the dart lands at a random point (X, Y) *whose distribution is uniform over the dartboard (I'm not very good at darts).*

(b) *Find the marginal density functions of* X *and* Y*.*

(c) *Let* D *be the distance from* (X, Y) *to the origin. Find* $P(D \leqslant r)$ *for positive real values* r*.*

(d) *Find* $E[D]$*.*