Math-UA.233: Theory of Probability
Lecture 1

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The importance of counting

Many calculations in probability theory boil down to:

> *In how many ways can something be done / can a certain outcome occur?*

Example

In the NYS lottery, you pick 6 whole numbers from 1 to 59. What's your chance of winning?

$$\text{INTUITIVE ANS} = \frac{1}{\#\text{ ways to choose}} = \frac{1}{45,057,474}.$$

The mathematics of such questions is called ‘combinatorial analysis’.

Following Ross, we will review combinatorial analysis before starting on probability theory itself.

You’ve probably seen a lot of this before.

The basic principle of counting (Ross § 1.2)

The most basic principle that we will keep using:

Theorem

Imagine we peform a pair of experiments. Suppose that:

- *experiment 1 has m possible outcomes, and*
- *no matter the outcome of experiment 1, experiment 2 has n possible outcomes.*

Then there are mn possible outcomes of the two experiments together.

IDEA: Represent all possible pairs of outcomes in an $m \times n$ table: column represents outcome of exp 1, row represents outcome of exp 2.

Example (Ross Ex 1.2a)

A small community consists of 10 families, each of which has 3 children. If one family and one of its children are to be chosen for a prize, how many choices are possible?

BASIC, BUT IMPORTANT: For us, it doesn't matter *what* is being counted, only what is the mathematical form of the question.

An easy generalization:

Theorem

Imagine we perform a sequence of r experiments. Suppose that

- *the first has n_1 possible outcomes;*
- *no matter the outcome of the first experiment, the second has n_2 possible outcomes;*
- *no matter the outcome of the experiments 1 and 2, the third has n_3 possible outcomes;*
- $\vdots$

Then there are $n_1 \cdot n_2 \cdots n_r$ possible outcomes of the r experiments together.

Example (Ross Ex 1.2c)

How many different 7-place license plates are possible if the first 3 places are to be occupied by letters and the final 4 by numbers?

ANS = 175,760,000

Example (Ross Ex 1.2e)

In previous example, how many are possible if repetition among letters or numbers is prohibited?

ANS = 78,624,000

Permutations (Ross § 1.3)

Suppose we have n objects. In how many ways can they be arranged in order?

Example

The letters A, B, and C can be arranged in 6 ways:

ABC, ACB, BAC, BCA, CAB, CBA.

Such an arrangement is called a **permutation.**

In general, we can answer this using the basic principle of counting.

Theorem

Number of permutations of n objects:

$$\boxed{n \cdot (n-1) \cdot \cdots \cdot 1 = n!}$$

Example (Ross Ex 1.3b)

A class consists of 6 juniors and 4 seniors. They are ranked after the final exam.

(a) *How many different rankings are possible?*

(b) *How many different rankings are possible if the juniors and seniors are ranked only among themselves?*

ANS (a) = 3,628,800; (b) = 17,280.

A slightly more complicated variant of counting permutations: what if some of the objects are *indistinguishable*?

Example (Ross Ex 1.3d)

How many different letter arrangements can be formed from the letters of PEPPER?

ANS = 6! / (3! 2!) = 60

In general:

Theorem

Number of ways of arranging n objects among which n_1 are alike, n_2 are alike, . . . , n_r are alike:

$$\boxed{\frac{n!}{n_1!\,n_2!\cdots n_r!}}$$

Example (Ross 1.3f)

Signals are sent by hanging 9 flags on a line. How many different signals can be made from a set of 4 white flags, 3 red flags and 2 blue flags?

ANS = $9! / (4!\ 3!\ 2!) = 1260$.

Combinations (Ross § 1.4)

Now suppose we have n objects, and we want to choose a subcollection of k of them. This is different from permuting or arranging, because now the order is not important.

Example

How many sets of three letters can be chosen from A, B, C, D and E?

IDEA: (i) Count *ordered* choices, and then (ii) adjust because we have counted each group of three several times, once for each possible ordering.

ANS = $(5 \cdot 4 \cdot 3)/3! = 10$

Theorem

Number of ways of choosing k objects out of n:

$$\boxed{\frac{n\cdot(n-1)\cdot\cdots\cdot(n-k+1)}{k!}=\frac{n!}{k!(n-k)!}}.$$

IDEA: first choose k objects in order

(number of ways $= n\cdot(n-1)\cdots(n-k+1) = n!/(n-k)!$),

then divide by the number of ways they could be ordered ($= k!$).

The numbers above are called **binomial coefficients.**

Notation:

$$\frac{n!}{k!(n-k)!} = \binom{n}{k}.$$

This is pronounced "n choose k". It is defined for $0 \leq k \leq n$. By convention,

$$0! = 1 \quad \text{and} \quad \binom{n}{0} = \binom{n}{n} = 1.$$

Example (Ross Ex 1.4a)

A committee of 3 is to be formed from a group of 20 people. How many different committees are possible?

ANS = 1140

Example (Ross Ex 1.4b)

From a group of 5 women and 7 men, how many different committees consisting of 2 women and 3 men can be formed?

What if 2 of the men are feuding and refuse to serve on the committee together?

ANS = 350; then 300

Example (More creative example; Ross Ex 1.4c)

Consider a set of n antennas of which m are defective and $n - m$ are functional. Assume all of the defectives and all of the functionals are indistinguishable. How many linear orderings are there in which no two defectives are consecutive?

ANS = $\binom{n-m+1}{m}$

The numbers $\binom{n}{k}$ follow some general rules which can be deduced from their meaning.

Theorem (Part of Pishro-Nik Example 2.8)

1. *We have* $\sum_{k=0}^{n} \binom{n}{k} = 2^n$.
2. *[See also Ross eqn (4.1)] If* $1 \leq k \leq n-1$ *then*

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

IDEA: Both sides are counting the *same thing* in *different ways*.

(Part 2 of this theorem is the basis for *Pascal's triangle*, a popular way of writing out binomial coefficients.)

You may have met binomial coefficients before in:

Theorem (The binomial theorem (Ross p7))

For real numbers x and y,

$$\boxed{(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}}$$

Multinomial coefficients (Ross § 1.5)

Finally, consider a set of n distinguishable items. Suppose we want to assign them into r groups of sizes $n_1, \ldots, n_r$, where $n_1 + \cdots + n_r = n$ (otherwise the task doesn't make sense!).

Theorem

The number of ways of doing this is

$$\boxed{\underbrace{\binom{n}{n_1, n_2, \ldots, n_r}}_{\text{notation}} = \frac{n!}{n_1!n_2!\cdots n_r!}}$$

These values are called **multinomial coefficients.** Observe that

$$\binom{n}{k} = \binom{n}{k, n-k},$$

so they generalize binomial coefficients.

IDEA FOR THE THEOREM: First choose the n_1 objects for the first group:

$$\# \text{ ways} = \binom{n}{n_1}.$$

Then choose the n_2 objects for the second group from the remaining $n - n_1$:

$$\# \text{ ways} = \binom{n - n_1}{n_2}.$$

Keep going like this, and then finally multiply all these values together (basic principle of counting). Get:

$$\binom{n}{n_1}\binom{n - n_1}{n_2}\cdots\binom{n - n_1 - \cdots - n_{r-1}}{n_r} = \frac{n!}{n_1!n_2!\cdots n_r!}$$

Example (Ross Ex 1.5b)

Ten children are to be divided into an A team and B team of 5 each. The A team will play in one league and the B team in another. How many different divisions are possible?

ANS = 252

Example (Ross Ex 1.5c)

Ten children at a playground divide themselves into two teams to play against each other. How many different divisions are possible?

BEWARE: This is different from the previous example! Now the ‘order’ of the two teams is irrelevant: there are no ‘A’ and ‘B’ labels.

ANS = 126

Multinomial coefficients are named for:

Theorem (The multinomial theorem (Ross p10))

For real numbers $x_1, \ldots, x_r$,

$$\boxed{(x_1 + \cdots + x_r)^n = \sum_{\substack{n_1, \ldots, n_r : \\ n_1 + \cdots + n_r = n}} \binom{n}{n_1, \ldots, n_r} x_1^{n_1} x_2^{n_2} \cdots x_r^{n_r}}.$$

MEANING OF THE NOTATION: we sum over all choices of nonnegative integers $n_1, \ldots, n_r$ such that $n_1 + \cdots + n_r = n$.

GOOD EXERCISE: Check that this is just the same as the binomial theorem in case $r = 2$.

BETTER EXERCISE: Give a proof along the same lines as the binomial theorem.