# Math-UA.233: Theory of Probability
## Lecture 11

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## From last time: common discrete RVs

**Bernoulli**$(p)$:

$$P(X = 1) = p, \quad P(X = 0) = 1 - p$$

**binom**$(n, p)$:

$$P(X = k) = \binom{n}{k} p^k (1 - p)^{n-k} \quad \text{for } k = 0, 1, \ldots, n$$

**geometric**$(p)$:

$$P(X = k) = (1 - p)^{k-1} p \quad \text{for } k = 1, 2, \ldots$$

(Sometimes include "$P(X = \infty) = 0$" — doesn't really matter.)

**hypergeometric**$(n, N, m)$:

$$P(X = k) = \frac{\binom{m}{k}\binom{N-m}{n-k}}{\binom{N}{n}} \quad \text{for } \max(n-(N-m), 0) \le k \le \min(n, m)$$

# From last time, 2: origin of hypergeometric RVs

Suppose an urn contains $m$ white and $N - m$ black balls, and we sample $n$ of them without replacement. Let $X$ be the number of white balls in our sample. Then $X$ is hypergeometric$(n, N, m)$:

$$P(X = k) = \frac{\binom{m}{k}\binom{N-m}{n-k}}{\binom{N}{n}} \quad \text{for } \max(n-(N-m), 0) \leq k \leq \min(n, m).$$

## Warmup: More practice with hypergeometrics

### Proposition (Ross E.g. 4.8j or 7.2g)

*If $X$ is hypergeometric$(n, N, m)$, then*

$$\boxed{E[X] = nm/N}$$

*and*

$$\boxed{\mathrm{Var}(X) = np(1-p)\left(1 - \frac{n-1}{N-1}\right), \quad \textit{where } p = \frac{m}{N}.}$$

IDEA FOR EXPECTATION: Linearity of expectation!

See Ross E.g. 4.8j for the calculation of the variance and some other things.

A real application of hypergeometric RVs:

## Example (Ross E.g. 4.8h)

*An unknown number $N$ of fish live in a pond. An ecologist catches $m$ of them at random, tags them and returns them to the pond. After giving the fish time to disperse, s/he catches another $n$ fish at random. Let $X$ be the number of tagged fish contained in the second catch. How can we use $X$ to make a guess about the size of $N$?*

IDEA: If the pond and sample-sizes are large, we expect $X$ to stay pretty close to $E[X] = nm/N$.

(This common method in ecology is called ‘mark and recapture sampling’. Ross includes a more thorough justification.)

Main topic for today:

*Approximating one random variable by another*.

Reasons why this can be important:

- It may show that two different experiments are likely to give similar results.
- It may let us approximate a complicated PMF by a simpler one.
- It may let us ignore certain details of a RV: for example, by reducing the number of parameters (such as the $n$ and the $p$ in $\mathrm{binom}(n, p)$) that we need to specify.

Some useful notation:

$$N \gg M \quad \text{means "}N\text{ is much bigger than }M\text{".}$$

More rigorous alternative:

$$\text{"as } \frac{N}{M} \longrightarrow \infty\text{".}$$

The first example is quite simple.

## Proposition

*Let $X$ be hypergeometric$(n, N, m)$, and let $p = m/N$ (the fraction of white balls in the urn).*

*If $m \gg n$ and $N - m \gg n$ (that is, the numbers of white and black balls are both much bigger than the size of the sample), then*

$$\boxed{P(X = k) \approx \binom{n}{k} p^k (1-p)^{n-k}.}$$

IN SHORT: hypergeometric$(n, N, m) \approx$ binom$(n, m/N)$.

INTUITIVE REASON: If instead we chose $n$ balls *with replacement*, the PMF of the number of white balls chosen would be *exactly* binom$(n, m/N)$.

But if $m \gg n$ and $N - m \gg n$, then we're *extremely unlikely* to pick the same ball twice, even if we do replace them. So in this case sampling with and without replacement should give approximately the same model.

SKETCH PROOF BASED ON THE INTUITION: Let $Y$ be the number of white balls if we pick our sample with replacement. Let $E$ be the event that we never pick the same ball twice, even though we pick with replacement.

If $m \gg n$ and $N - m \gg n$, then

$$P(E) = \frac{N \times (N-1) \times \cdots \times (N-n+1)}{N^n} \approx 1,$$

and a similar calculation also gives

$$P(E \mid Y = k) \approx 1 \quad \text{for all } k = 0, 1, 2, \ldots, n.$$

Therefore

$$\underbrace{P(Y = k \mid E)}_{\text{hypergeometric}} = \frac{P(E \mid Y = k) \times P(Y = k)}{P(E)} \quad \text{(by Bayes)}$$

$$\approx \frac{1 \times P(Y = k)}{1} = \underbrace{P(Y = k)}_{\text{binomial}}$$

(See Ross p153 for a different argument)

We can see this effect in the expectation and variance.

If $X$ is hypergeometric$(n, N, m)$, $p = m/N$ and $m, N - m \gg n$, then

$$E[X] = \frac{nm}{N} = np$$

and

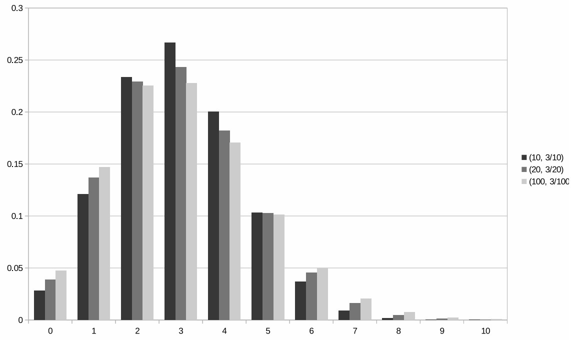
$$\mathrm{Var}(X) = np(1-p)\Big(1 - \underbrace{\frac{n-1}{N-1}}_{\text{very small}}\Big) \approx np(1-p).$$

That is, they roughly match the expectation and variance of binom$(n, p)$.

## Poisson RVs and the Poisson approximation (Ross Sec 4.7)

The rest of today will be given to another approximation, less obvious but even more powerful: the Poisson approximation.

Here's a picture to get us started:




These are (parts of) the PMFs of three different binomials, with parameter choices $(10, 3/10)$, $(20, 3/20)$ and $(100, 3/100)$.

What's special about those choices of parameters?

They are chosen to have the same expectation:

$$10 \times \frac{3}{10} = 20 \times \frac{3}{20} = 100 \times \frac{3}{100} = 3.$$

But we can see that in fact their PMFs look very similar, not just their expectations.

So let's *fix* a value $\lambda$ to be the expectation of our RVs, and consider binomial RVs which have that expectation: that is, $\mathrm{binom}(n, p)$ with $np = \lambda$.

Now let $n \gg 1$ (which implies $p = \lambda/n \ll 1$). It turns out that in this extreme, binomial RVs start to look like another fixed RV 'in the limit'.

## Proposition (Ross p136)

*Fix $\lambda > 0$. Let $X$ be $\mathrm{binom}(n, \lambda/n)$ for some $n \gg 1$. Then*

$$\boxed{P(X = k) \approx e^{-\lambda} \frac{\lambda^k}{k!}.}$$

The ‘limiting function’ which which shows up in this theorem gives us a new PMF:

### Definition

*Let $\lambda > 0$. A discrete RV $Y$ is* ***Poisson with parameter*** *$\lambda$ (or ‘**Poi**$(\lambda)$’) if its possible values are $0, 1, 2, \dots$, and*

$$\boxed{P(Y = k) = e^{-\lambda}\frac{\lambda^k}{k!}.}$$

First checks: if $Y$ is Poi($\lambda$), then

$$\sum_{k=0}^{\infty} P(Y = k) = 1 \quad \text{(as it should be)},$$

$$E[Y] = \lambda \quad (= \text{limit of } E[\mathrm{binom}(n, \lambda/n)] \text{ as } n \longrightarrow \infty)$$

and

$$\mathrm{Var}(Y) = \lambda \quad (= \text{limit of } \mathrm{Var}(\mathrm{binom}(n, \lambda/n)) \text{ as } n \longrightarrow \infty).$$

(See Ross pp137-8.)

The Poisson approximation is valid in many cases of interest. A Poisson distribution is a natural assumption for:

- The number of misprints on a page of a book.
- The number of people in a town who live to be 100.
- The number of wrong telephone numbers that are dialed in a day.
- The number of packets of dog biscuits sold by a pet store each day.
- ...

In all these cases, the RV is really counting how many things actually happen from a very *large* number of independent trials, each with very *low* probability.

### Example (Ross E.g. 4.7a)

*Suppose the number of typos on a single page of a book has a Poisson distribution with parameter* $1/2$. *Find the probability that there is at least one error on a given page.*

### Example (Ross E.g. 4.7b)

*The probability that a screw produced by a particular machine will be defective is 10%. Find/approximate the probability that in a packet of 10 there will be at most one defective screw.*

The Poisson approximation is extremely important because of the following:

> *To specify the PMF of a binom$(n, p)$ RV, you need two parameters, but to specify the PMF of a Poi$(\lambda)$, you need only one.*

Very often, we have an RV which we believe is binomial, but we don't know exactly the number of trials (*n*) or the success probability (*p*). But as long as we have data which tells us the expectation ($=$ *np*), we can still use the Poisson approximation!

## Example (Ross 4.7c)

*We have a large block of radioactive material. We have measured that on average it emits 3.2 $\alpha$-particles per second. Approximate the probability that at most two $\alpha$-particles are emitted during a given one-second interval.*

NOTE: The $\alpha$-particles are being emitted by the radioactive atoms. There's a huge number of atoms, and to a good approximation they all emit atoms within a given time-interval independently. But we don't know *how many atoms there are*, nor *the probability of emission by a given atom per second*. We only know the average emission rate. So there's not enough information to model this using a binomial, but there *is* enough for the Poisson approximation!

Poisson RVs frequently appear when modeling how many ‘event’s (beware: not our usual use of this word in this course!) of a certain kind occur during an interval of time.

Here, an ‘event’ could be:

- the emission of an $\alpha$-particle,
- an earthquake,
- a customer walking into a store.

These are all ‘event’s that occur at ‘random times’.

The precise setting for this use of Poisson RVs is the following. Events occur at random moments. Fix $\lambda > 0$, and suppose we know that:

- If $h$ is very short, then the probability of an 'event' during an interval of length $h$ is approximately $\lambda h$ (up to an error which is much smaller than $h$).
- If $h$ is very short, then the probability of two or more 'event's during an interval of length $h$ is negligibly small.
- For any time-intervals $T_1, \dots, T_n$ which don't overlap, the numbers of 'event's that occur in these time intervals are independent.

Then (see Ross, p144):

> The number of events that occur in a time-interval of length $t$ is $\mathrm{Poi}(\lambda t)$.

## Example (Ross E.g. 4.7e)

*Suppose that California has two (noticeable) earthquakes per week on average.*

(a) *Find the probability of at least three earthquakes in the next two weeks.*

(b) *Let $T$ be the RV which gives the time until the next earthquake. Find the CDF of $T$.*

QUESTION: Is this really a good model for eathquakes?

Summary of today's approximations:

$$\begin{array}{ll} \text{hypergeometric}(n, N, m) & \\ \quad \approx \text{binom}(n, p) & m, N - m \gg n \text{ and we let } p = m/N \\ \quad \approx \text{Poi}(\lambda) & \text{if } n \gg 1,\ p \ll 1 \text{ and we let } \lambda = pn \end{array}$$

Each approximation reduces the number of parameters by 1.