Test 2 - Solutions

Intermediate Statistics - 36-705

October 14, 2016

Problem 1. [30 points]
Let $X_1, \ldots, X_n \sim \text{Normal}(0, \sigma^2)$.

(a) **[10 pts.]** Find a minimal sufficient statistic and show that it is minimal sufficient.

Solution:
Define

$$
\begin{aligned}
R(x^n, y^n; \sigma^2) &= \frac{p(y^n; \sigma^2)}{p(x^n; \sigma^2)} \\
&= \frac{\prod_{i=1}^n \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{ -\frac{y_i^2}{2\sigma^2} \right\}}{\prod_{i=1}^n \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{ -\frac{x_i^2}{2\sigma^2} \right\}} \\
&= \frac{\exp\left\{ -\frac{1}{2\sigma^2} \sum_{i=1}^n y_i^2 \right\}}{\exp\left\{ -\frac{1}{2\sigma^2} \sum_{i=1}^n x_i^2 \right\}} \\
&= \exp\left\{ \frac{1}{2\sigma^2} \sum_{i=1}^n (x_i^2 - y_i^2) \right\}.
\end{aligned}
$$

$R(x^n, y^n; \sigma^2)$ is independent of σ^2 if and only if

$$\sum_{i=1}^n (x_i^2 - y_i^2) = 0 \Longleftrightarrow \sum_{i=1}^n x_i^2 = \sum_{i=1}^n y_i^2.$$

Therefore, $T(X^n) = \sum_{i=1}^n X_i^2$ is a minimal sufficient statistic for σ^2.

(b) **[10 pts.]** Find the maximum likelihood estimator of σ. Show that the estimator is consistent.

Solution:
The log-likelihood is

$$\ell_n(\sigma) = -\frac{n}{2}\log(2\pi) - n\log(\sigma) - \frac{1}{2\sigma^2}\sum_{i=1}^{n} x_i^2.$$

Letting $\ell_n'(\sigma) = 0$,

$$\begin{aligned} -\frac{n}{\sigma} + \frac{1}{\sigma^3}\sum_{i=1}^{n} x_i^2 &= 0 \\ \implies \widehat{\sigma} &= \sqrt{\frac{1}{n}\sum_{i=1}^{n} X_i^2}. \end{aligned}$$

By the WLLN,

$$\frac{1}{n}\sum_{i=1}^{n} X_i^2 \xrightarrow{\text{P}} \mathbb{E}[X_i^2] = \sigma^2,$$

and by the continuous mapping theorem,

$$\sqrt{\frac{1}{n}\sum_{i=1}^{n} X_i^2} \xrightarrow{\text{P}} \sigma.$$

(c) **[10 pts.]** Find the limiting distribution of $\sqrt{n}(\widehat{\sigma} - \sigma)$.

Solution:
By the asymptotic normality of the MLE,

$$\sqrt{n}(\widehat{\sigma} - \sigma) \rightsquigarrow N\left(0, \frac{1}{I(\sigma)}\right).$$

Continuing from part (b), we get

$$\ell_n''(\sigma) = \frac{n}{\sigma^2} - \frac{3}{\sigma^4} \sum_{i=1}^{n} x_i^2,$$

and thus,

$$\begin{aligned}
I(\sigma) &= \frac{1}{n} I_n(\sigma) \\
&= -\frac{1}{n} \mathbb{E}_\sigma\left(\ell_n''(\sigma)\right) \\
&= -\frac{1}{\sigma^2} + \frac{3}{\sigma^2} \\
&= \frac{2}{\sigma^2}.
\end{aligned}$$

Therefore,

$$\sqrt{n}(\widehat{\sigma} - \sigma) \rightsquigarrow N\left(0, \frac{\sigma^2}{2}\right).$$

Alternate solution. Note,

$$\mathbb{E}\left[\frac{1}{n}\sum_{i=1}^{n} X_i^2\right] = \sigma^2.$$

By the CLT,

$$\frac{\sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n} X_i^2 - \sigma^2\right)}{\sqrt{\mathbb{V}(X_i^2)}} \rightsquigarrow N(0,1),$$

where

$$\begin{aligned}
\mathbb{V}(X_i^2) &= \mathbb{E}\big[X_i^4\big] - \big[\mathbb{E}\big[X_i^2\big]\big]^2 \\
&= \sigma^4 \mathbb{E}\big[Z_i^4\big] - (\sigma^2)^2 \\
&= 3\sigma^4 - \sigma^4 \\
&= 2\sigma^4.
\end{aligned}$$

Here we have used the fact that the Kurtosis$(Z_i) = \mathbb{E}\big[Z_i^4\big] = 3$. $\mathbb{E}\big[Z_i^4\big]$ can also be computed by using the moment generating function. Thus,

$$\sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n} X_i^2 - \sigma^2\right) \rightsquigarrow N(0, 2\sigma^4).[1]$$

Now let $g(\sigma^2) = \sqrt{\sigma^2} \implies g'(\sigma^2) = 1/(2\sigma)$. By the delta method, we have

$$\begin{aligned}
&\sqrt{n}\left(g\Big(\frac{1}{n}\sum_{i=1}^{n} X_i^2\Big) - g(\sigma^2)\right) \rightsquigarrow N\Big(0, \big(g'(\sigma^2)\big)^2 \cdot 2\sigma^4\Big) \\
&\implies \sqrt{n}(\widehat{\sigma} - \sigma) \rightsquigarrow N\left(0, \frac{\sigma^2}{2}\right).
\end{aligned}$$

[1] *Note: We could have also established this by noting $\frac{1}{n}\sum_{i=1}^{n} X_i^2$ is the MLE of σ^2 (by equivariance) and using the asymptotic normality of the MLE as in the primary solution.*

Problem 2. [40 points]
Let $X_i \sim \text{Normal}(\theta_i, 1)$ for $i = 1, \ldots, n$. The observations are independent but each observation has a different mean. The unknown parameter is $\theta = (\theta_1, \ldots, \theta_n)$.

(a) **[10 pts.]** Find a minimal sufficient statistic and show that it is minimal sufficient.

Solution:
Define

$$\begin{aligned}
R(x^n, y^n; \theta) &= \frac{p(y^n;\theta)}{p(x^n;\theta)} \\
&= \frac{\prod_{i=1}^n \frac{1}{\sqrt{2\pi}} \exp\Big\{ -\frac{(y_i-\theta_i)^2}{2} \Big\}}{\prod_{i=1}^n \frac{1}{\sqrt{2\pi}} \exp\Big\{ -\frac{(x_i-\theta_i)^2}{2} \Big\}} \\
&= \frac{\exp\Big\{ -\frac{1}{2}\sum_{i=1}^n (y_i-\theta_i)^2 \Big\}}{\exp\Big\{ -\frac{1}{2}\sum_{i=1}^n (x_i-\theta_i)^2 \Big\}} \\
&= \exp\Big\{ \frac{1}{2}\sum_{i=1}^n \Big((x_i-\theta_i)^2 - (y_i-\theta_i)^2 \Big) \Big\} \\
&= \exp\Big\{ \frac{1}{2}\sum_{i=1}^n (x_i^2 - y_i^2) - \sum_{i=1}^n \theta_i (x_i - y_i) \Big\}
\end{aligned}$$

Let $T(X^n) = (X_1, \ldots, X_n)$.

"$\Longrightarrow$" Suppose $T(X^n) = T(Y^n)$. Then, $R(x^n, y^n; \theta) = 1$.

"$\Longleftarrow$" Suppose $R(x^n, y^n; \theta)$ is independent of θ. Then for any n,

$$\sum_{i=1}^n \theta_i (x_i - y_i) = 0 \implies x_i = y_i \text{ for all } i.$$

Therefore, T is MSS.

(b) **[15 pts.]** Let $\psi = \sum_{i=1}^n \theta_i$. Find the maximum likelihood estimator of ψ. Find the MSE of the estimator.

Solution:
The log-likelihood is

$$\ell_n(\theta) = -\frac{1}{2}\sum_{i=1}^n (x_i - \theta_i)^2.$$

For each $i = 1, \ldots, n$, letting $\frac{\partial}{\partial \theta_i}\ell_n(\theta) = 0$, we have

$$\widehat{\theta}_i = X_i.$$

By the equivariance of MLE,

$$\widehat{\psi}_n = \sum_{i=1}^n \widehat{\theta}_i = \sum_{i=1}^n X_i.$$

$$\begin{aligned}
\text{Bias}(\widehat{\psi}_n) &= \mathbb{E}\big[\widehat{\psi}_n\big] - \psi \\
&= \sum_{i=1}^n (\mathbb{E}[X_i] - \theta_i) \\
&= 0.
\end{aligned}$$

$$\begin{aligned}
\mathbb{V}(\widehat{\psi}_n) &= \sum_{i=1}^n \mathbb{V}(X_i) \\
&= n.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathbb{E}\big[(\widehat{\psi}_n - \psi)^2\big] &= \text{Bias}^2(\widehat{\psi}_n) + \mathbb{V}(\widehat{\psi}_n) \\
&= n.
\end{aligned}$$

(c) **[15 pts.]** Show that the maximum likelihood estimator is not consistent.

Solution:

For any $\epsilon > 0$,

$$\begin{aligned}
\mathbb{P}(|\widehat{\psi} - \psi| < \epsilon) &= \mathbb{P}\left(\Big| \sum_{i=1}^{n} (X_i - \theta_i) \Big| < \epsilon \right) \\
&= \mathbb{P}\left(\Big| \sum_{i=1}^{n} Z_i \Big| < \epsilon \right) \\
&= \mathbb{P}\left(\Big| \frac{1}{\sqrt{n}} \sum_{i=1}^{n} Z_i \Big| < \frac{\epsilon}{\sqrt{n}} \right) \\
&= \mathbf{\Phi}\Big(\frac{\epsilon}{\sqrt{n}} \Big) - \mathbf{\Phi}\Big(- \frac{\epsilon}{\sqrt{n}} \Big) \qquad (1) \\
&\to 0,
\end{aligned}$$

where (1) comes from the fact

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} Z_i \sim N(0, 1).$$

Therefore, $\widehat{\psi}$ is not consistent.

Problem 3. [30 points]
Let $X_1, \ldots, X_n \sim \text{Uniform}(1, 1+\theta)$.

(a) **[15 pts.]** Find the maximum likelihood estimator of θ.

Solution:

$$
\begin{aligned}
L(\theta) &= \prod_{i=1}^{n} \left(\frac{1}{\theta}\right) \mathbb{1}_{\{1 \leq X_i \leq 1+\theta\}} \\
&= \left(\frac{1}{\theta}\right)^n \mathbb{1}_{\{X_{(n)} \leq 1+\theta\}} \\
&= \left(\frac{1}{\theta}\right)^n \mathbb{1}_{\{\theta \geq X_{(n)}-1\}}.
\end{aligned}
$$

$\left(\frac{1}{\theta}\right)^n$ is a decreasing function in θ so $L(\theta)$ is maximized by the smallest value of θ such that

$$\mathbb{1}_{\{\theta \geq X_{(n)}-1\}} = 1.$$

That is,

$$\widehat{\theta} = X_{(n)} - 1.$$

(b) **[15 pts.]** Show that $\sqrt{n}(\widehat{\theta} - \theta)$ does not converge to a non-degenerate Normal distribution.

Solution:

First note when $t \geq 0$,

$$\begin{aligned}
\mathbb{P}\Bigg(-\sqrt{n}\Big(\sqrt{n}(\widehat{\theta} - \theta)\Big) \leq t \Bigg) &= \mathbb{P}\Big(n(\theta - \widehat{\theta}) \leq t\Big) \\
&= \mathbb{P}\Big(n(\theta - X_{(n)} + 1) \leq t\Big) \\
&= \mathbb{P}\Big(X_{(n)} \geq 1 + \theta - \frac{t}{n}\Big) \\
&= 1 - \mathbb{P}\Big(X_{(n)} < 1 + \theta - \frac{t}{n}\Big) \\
&= 1 - \left(\frac{\theta - t/n}{\theta}\right)^n \\
&= 1 - \Big(1 - \frac{t}{n\theta}\Big)^n \\
&\to 1 - e^{-\frac{t}{\theta}},
\end{aligned}$$

and 0 otherwise. That is,

$$-\sqrt{n}\Big(\sqrt{n}(\widehat{\theta} - \theta)\Big) \rightsquigarrow \text{Exp}(1/\theta). \quad [2]$$

Therefore, $\sqrt{n}(\widehat{\theta} - \theta)$ converges to a degenerate distribution.

[2] This was proved in class!