36-705 Intermediate Statistics Homework #1 Solutions

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Problem 1: Wasserman 1.3

Let Ω be a sample space and let $A_1, A_2, \dots$ be events. Define $B_n = \bigcup_{i=n}^{\infty} A_i$ and $C_n = \bigcap_{i=n}^{\infty} A_i$.

(a) Show that $B_1 \supset B_2 \supset \cdots$ and that $C_1 \subset C_2 \subset \cdots$.

Proof. (By Induction)
Base.

$$B_1 = \bigcup_{i=1}^{\infty} A_i = A_1 \cup \bigcup_{i=2}^{\infty} A_i = A_1 \bigcup B_2 \implies B_1 \supset B_2 \ \checkmark$$

Now for any $k \in \mathbb{N}$,

$$B_k = \bigcup_{i=k}^{\infty} A_i = A_k \cup \bigcup_{i=k+1}^{\infty} A_i = A_k \bigcup B_{k+1} \implies B_k \supset B_{k+1}. \ \blacksquare$$

Proof. (By Induction)
Base.

$$C_1 = \bigcap_{i=1}^{\infty} A_i = A_1 \cap \bigcap_{i=2}^{\infty} A_i = A_1 \bigcap C_2 \implies C_1 \subset C_2 \ \checkmark$$

Now for any $k \in \mathbb{N}$,

$$C_k = \bigcap_{i=k}^{\infty} A_i = A_k \cap \bigcap_{i=k+1}^{\infty} A_i = A_k \bigcap C_{k+1} \implies C_k \subset C_{k+1}. \ \blacksquare$$

(b) Show that $\omega \in \bigcap_{n=1}^{\infty} B_n$ if and only if ω belongs to an infinite number of the events $A_1, A_2, \dots$.

Proof.
"$\Longleftarrow$" Suppose ω belongs to an infinite number of the events $A_1, A_2, \dots$ but $\omega \notin \bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} A_i$. Then there exists a j such that $\omega \notin \bigcup_{i=j}^{\infty} A_i$. But this implies ω can be in at most $j-1$ of the A_i, a contradiction. Therefore, $\omega \in \bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} A_i$.

"$\Longrightarrow$" Assume $\omega \in \bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} A_i$. If ω does not belong to an infinite number of the events $A_1, A_2, \dots$, then pick the last (according to the index) A_i which contains ω and call it A_T. Then $\omega \notin A_k$, $\forall k > T$. Hence, $\omega \notin \bigcap_{n=1}^{\infty} B_n$, a contradiction. Therefore, ω belongs to an infinite number of the events $A_1, A_2, \dots$. $\blacksquare$

(c) Show that $\omega \in \bigcup_{n=1}^{\infty} C_n$ if and only if ω belongs to all the events $A_1, A_2, \dots$ except possibly a finite number of those events.

Proof.
"$\Longleftarrow$" Suppose that ω belongs to all the events $A_1, A_2, \dots$ except possibly a finite number of those events. Pick the last A_i that ω is not in and call the next one A_{n_0}. $\omega \in A_n$, $\forall n \geq n_0$ so clearly $\omega \in \bigcap_{i=n_0}^{\infty} A_i$. Since this set is part of the union $\bigcup_{n=1}^{\infty} C_n$, we've shown $\omega \in \bigcup_{n=1}^{\infty} C_n$.

"$\Longrightarrow$" Suppose $\omega \in \bigcup_{n=1}^{\infty} \bigcap_{i=n}^{\infty} A_i$. Suppose further that no matter how high we run our index, we can always find an A_i such that $\omega \notin A_i$. More precisely, for any $k \in \mathbb{N}$, $\exists j > k$ such that $\omega \notin A_j$. This of course implies $\omega \notin \bigcap_{i=n}^{\infty} A_i$ for any n. That is, $\omega \notin \bigcup_{n=1}^{\infty} \bigcap_{i=n}^{\infty} A_i$, a contradiction. Therefore, ω belongs to all the events $A_1, A_2, \dots$ except possibly a finite number of those events. ∎

Problem 2: Wasserman 1.8

Suppose that $\mathbb{P}(A_i) = 1$ for each i. Prove that

$$\mathbb{P}\left(\bigcap_{i=1}^{\infty} A_i\right) = 1.$$

Proof. We will prove the equivalent proposition:

$$\mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i^c\right) = 0.$$

In order to show this note that:

$$\mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i^c\right) \leq \sum_{i=1}^{\infty} \mathbb{P}\left(A_i^c\right) = \sum_{i=1}^{\infty} 0 = 0. \blacksquare$$

Problem 3: Wasserman 1.13

Suppose that a fair coin is tossed repeatedly until both a head and tail have appeared at least once.

(a) Describe the sample space Ω.

We can partition the sample space into two subsets

$$A = \{\omega : \underbrace{H \cdots H}_{n \text{ times}} T,\ n \geq 1\}$$
$$B = \{\omega : \underbrace{T \cdots T}_{n \text{ times}} H,\ n \geq 1\}$$

Then $\Omega = A \cup B$.

(b) What is the probability that three tosses will be required?

This can occur in two ways: HHT or TTH. Each has probability $\left(\frac{1}{2}\right)^3$ so

$$P(\text{three tosses}) = 2 \cdot \frac{1}{2^3} = \frac{1}{4}.$$

Problem 4: Wasserman 2.1

Show that

$$\mathbb{P}(X = x) = F(x^+) - F(x^-).$$

We have by right continuity of the CDF $F(x^+) = F(x) = \mathbb{P}(X \leq x)$. Also by continuity of probabilities

$$F(x^-) = \lim_{n\to\infty} F(x - 1/n) = \lim_{n\to\infty} \mathbb{P}(X \leq x - 1/n) = \mathbb{P}(X < x).$$

And these observations together with the fact that $\mathbb{P}(X = x) = \mathbb{P}(X \leq x) - \mathbb{P}(X < x)$ imply the desired result.

Problem 5: Wasserman 2.4

Let X have probability density function

$$f_X(x) = \begin{cases} 1/4 & 0 < x < 1 \\ 3/8 & 3 < x < 5 \\ 0 & \text{otherwise.} \end{cases}$$

(a) find the cumulative distribution function of X.

$$\boxed{F_X(x) = \begin{cases} 0 & x < 0 \\ x/4 & 0 \leq x < 1 \\ 1/4 & 1 \leq x \leq 3 \\ \frac{3x-7}{8} & 3 < x \leq 5 \\ 1 & x > 5 \end{cases}}.$$

(b) Let $Y = 1/X$. Find the probability density function $f_Y(y)$ for Y. Hint: Consider three cases: $\frac{1}{5} \leq y \leq \frac{1}{3}$, $\frac{1}{3} \leq y \leq 1$, and $y \geq 1$.

Following the hint, let us first consider the case when $y = 1/x \geq 1$, which occurs exactly when $0 < x \leq 1$. Keep in mind X takes values in the interval $(-\infty, 0]$ with probability 0. Now in this interval we have

$$f_X(x) = \frac{1}{4} \quad 0 < x < 1$$

and $Y = g(X) = 1/X$ is indeed monotonic (therefore $h(y) = g^{-1} = 1/y$) so we can directly employ the transformation formula for p.d.f.'s. That is,

$$f_Y(y) = f_X(h(y))\left|\frac{dh(y)}{dy}\right| = \frac{1}{4}\left|\frac{-1}{y^2}\right| = \frac{1}{4y^2}, \quad y \geq 1.$$

Using the same method, we first note that $\frac{1}{5} \leq y \leq \frac{1}{3}$ corresponds exactly with $3 \leq x \leq 5$ (we can drop the endpoints since we're dealing with continuous probability here). For this interval we have

$$f_X(x) = \frac{3}{8} \quad 1 < x < 3,$$

and therefore

$$f_Y(y) = f_X(h(y))\left|\frac{dh(y)}{dy}\right| = \frac{3}{8}\left|\frac{-1}{y^2}\right| = \frac{3}{8y^2}, \quad \frac{1}{5} < y < \frac{1}{3}.$$

Now for the only remaining interval, $\frac{1}{3} < y < 1$, we need only note that this corresponds exactly to the interval $1 < x < 3$ and that

$$P(X \in (1,3)) = 0.$$

Hence,

$$P(Y \in (1/3, 1)) = 0.$$

So altogether,

$$\boxed{f_Y(y) = \begin{cases} \frac{3}{8y^2} & \frac{1}{5} < y < \frac{1}{3} \\ \frac{1}{4y^2} & y > 1 \\ 0 & \text{otherwise.} \end{cases}}$$

Problem 6: Wasserman 2.7

Let X and Y be independent and suppose that each has a Uniform$(0,1)$ distribution. Let $Z = \min\{X, Y\}$. Find the density $f_Z(z)$ for Z. Hint: It might be easier to first find $\mathbb{P}(Z > z)$.

$$F_Z(z) = \mathbb{P}(Z \leq z) = 1 - \mathbb{P}(Z > z) = 1 - \mathbb{P}(X > z)\mathbb{P}(Y > z) = 1 - (1 - z)^2 = 2z - z^2.$$

$$\boxed{f_Z(z) = F_Z'(z) = \begin{cases} 2 - 2z & 0 < z < 1 \\ 0 & \text{otherwise.} \end{cases}}$$

Problem 7: Wasserman 2.20

Let $Z = X - Y$. We can easily see that $Z \in [-1, 1]$. To calculate $F_Z(z)$ we first consider the case $-1 \le z \le 0$:

$$\int_0^{1+z} \int_{x-z}^{1} 1 \mathrm{d}y \mathrm{d}x = \frac{z^2}{2} + z + \frac{1}{2}$$

Now for $0 < z \le 1$, $F_Z(z)$ is given by:

$$1 - \int_z^1 \int_0^{x-z} 1 \mathrm{d}y \mathrm{d}x = -\frac{z^2}{2} + z + \frac{1}{2}$$

So the density of Z is given by:

$$f_Z(z) = \begin{cases} 1+z & \text{if } -1 \le z \le 0 \\ 1-z & \text{if } 0 < z \le 1 \\ 0 & \text{otherwise} \end{cases}$$

which is known as *triangular* distribution.
Let $W = X/Y$, so that $W \in (0, \infty)$. We have:

$$\mathbb{P}(W \le w) = \mathbb{P}(X \le wY) = \int_0^1 \int_0^{\min(1,wy)} 1 \mathrm{d}x \mathrm{d}y = \int_0^1 \min(1, wy) \mathrm{d}y$$

Consider now the case $w \le 1$, then the integral becomes:

$$\int_0^1 wy \mathrm{d}y = \frac{w}{2}$$

Similarly for $w > 1$ we get:

$$\int_0^{1/w} wy \mathrm{d}y + \int_{1/w}^1 1 \mathrm{d}y = 1 - \frac{1}{2w}$$

So the density of W is given by:

$$f_W(w) = \begin{cases} 1/2 & \text{if } 0 \le w \le 1 \\ 1/2w^2 & \text{if } w > 1 \\ 0 & \text{otherwise} \end{cases}$$

Problem 8: Wasserman 3.8

Theorem. Let $X_1, \dots, X_n$ be IID and let $\mu = \mathbb{E}(X_i)$, $\sigma^2 = \mathbb{V}(X_i)$. Then

$$\mathbb{E}(\overline{X}_n) = \mu, \quad \mathbb{V}(\overline{X}_n) = \frac{\sigma^2}{n} \quad \text{and} \quad \mathbb{E}(S_n^2) = \sigma^2.$$

Proof.

$$\begin{aligned} \mathbb{E}(\overline{X}_n) &= \mathbb{E}\Big(\frac{1}{n}\sum_{i=1}^n X_i\Big) \\ &= \frac{1}{n}\sum_{i=1}^n \mathbb{E}(X_i) \\ &= \mu. \end{aligned}$$

$$
\begin{aligned}
\mathbb{V}(\overline{X}_n) &= \mathbb{V}\Big(\frac{1}{n}\sum_{i=1}^n X_i\Big) \\
&= \frac{1}{n^2}\sum_{i=1}^n \mathbb{V}(X_i) \\
&= \frac{\sigma^2}{n}.
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{E}(S_n^2) &= \mathbb{E}\Big(\frac{1}{n-1}\sum_{i=1}^n (X_i - \overline{X}_n)^2\Big) \\
&= \frac{1}{n-1}\left(\sum_{i=1}^n \mathbb{E}(X_i^2) - 2\mathbb{E}(\overline{X}_n\sum_{i=1}^n X_i) + \mathbb{E}(\sum_{i=1}^n \overline{X}_n^2)\right) \\
&= \frac{1}{n-1}\left(n(\sigma^2+\mu^2) - 2n\mathbb{E}(\overline{X}_n^2) + n\mathbb{E}(\overline{X}_n^2)\right) \\
&= \frac{1}{n-1}\left(n(\sigma^2+\mu^2) - n(\sigma^2/n+\mu^2)\right) \\
&= \frac{n-1}{n-1}\cdot\sigma^2 \\
&= \sigma^2.
\end{aligned}
$$

Problem 9: Wasserman 3.22

Let $X \sim \text{Uniform}(0,1)$. Let $0 < a < b < 1$. Let

$$
Y = \begin{cases} 1 & 0 < x < b \\ 0 & \text{otherwise} \end{cases}
$$

and let

$$
Z = \begin{cases} 1 & a < x < 1 \\ 0 & \text{otherwise} \end{cases}
$$

(a) Are Y and Z independent? Why/Why not? Notice

$$
\begin{aligned}
P(Y=1, Z=1) &= P(X \in (a,b)) \\
&= \frac{b-a}{1-0} \\
&= b-a
\end{aligned}
$$

and

$$
\begin{aligned}
P(Y=1)P(Z=1) &= \frac{b-0}{1-0}\cdot\frac{1-a}{1-0} \\
&= b(1-a),
\end{aligned}
$$

so Y and Z are not independent.

(b) Find $\mathbb{E}(Y|Z)$. Hint: What values z can Z take? Now find $\mathbb{E}(Y|Z = z)$.

$$\begin{aligned}\mathbb{E}(Y|Z=1) &= 1\cdot P(Y=1|Z=1) + 0\cdot P(Y=0|Z=1)\\ &= \frac{b-a}{1-a}.\end{aligned}$$

$$\begin{aligned}\mathbb{E}(Y|Z=0) &= 1\cdot P(Y=1|Z=0) + 0\cdot P(Y=0|Z=0)\\ &= 1.\end{aligned}$$

Problem 10: Wasserman 3.23

Find the moment generating function for the Poisson, Normal, and Gamma distributions.

<u>Poisson</u>

$$M_X(t) = \mathbb{E}[e^{tX}] = \sum_{x=0}^{\infty} e^{tx}\cdot\frac{\lambda^x}{x!}e^{-\lambda} = e^{-\lambda}\sum_{x=0}^{\infty}\frac{(e^t\lambda)^x}{x!} = e^{-\lambda+\lambda e^t}$$

$$\boxed{M_X(t) = e^{\lambda(e^t-1)}}.$$

<u>Gamma</u>

$$M_X(t) = \mathbb{E}[e^{tX}] = \int_0^{\infty} e^{tx}\frac{\beta^\alpha}{\Gamma(\alpha)}x^{\alpha-1}e^{-\beta x}dx = \frac{\beta^\alpha}{\Gamma(\alpha)}\int_0^{\infty} x^{\alpha-1}e^{-x(\beta-t)}dx.$$

Now we use the property that a p.d.f. integrates to 1 to obtain the formula

$$\int_0^{\infty}\frac{b^a}{\Gamma(a)}x^{a-1}e^{-bx}dx = 1,$$

which rearranges to

$$\int_0^{\infty} x^{a-1}e^{-bx}dx = \frac{\Gamma(a)}{b^a}.$$

So our moment generating function is

$$\begin{aligned}M_X(t) = \frac{\beta^\alpha}{\Gamma(\alpha)}\int_0^{\infty} x^{\alpha-1}e^{-x(\beta-t)}dx &= \frac{\beta^\alpha}{\Gamma(\alpha)}\cdot\frac{\Gamma(\alpha)}{(\beta-t)^\alpha}\\ &= \boxed{\left(\frac{\beta}{\beta-t}\right)^{\alpha},\ \ t<\beta}.\end{aligned}$$

<u>Normal</u>

First consider $X \sim N(0,1)$.

$$M_X(t) = \mathbb{E}[e^{tX}] = \int_{-\infty}^{\infty}\frac{e^{tx}}{\sqrt{2\pi}}e^{tx-\frac{x^2}{2}}dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{tx-\frac{x^2}{2}}dx$$

$$\overset{\text{(compl. the sq.)}}{=} \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{\frac{t^2}{2}}e^{-\frac{1}{2}(x-t)^2}dx = \frac{e^{t^2/2}}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-\frac{1}{2}(x-t)^2}dx \overset{\text{Let } u=x-t}{=} e^{\frac{t^2}{2}}\cdot\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-\frac{u^2}{2}}du.$$

And now the integral we are left with is a famous form the the Gaussian integral and is equal to $\sqrt{2\pi}$. Therefore,

$$M_X(t) = e^{\frac{t^2}{2}}.$$

Now consider a random variable Y with distribution $N(\mu, \sigma^2)$. Using the Central Limit Theorem, we have the equation,

$$Y = \mu + \sigma X.$$

We now compute the desired moment generating function.

$$M_Y(t) = \mathbb{E}[e^{Yt}] = \mathbb{E}[e^{(\mu+\sigma X)t}] = \mathbb{E}[e^{\mu t}e^{\sigma X}] = e^{\mu t}\mathbb{E}[e^{\sigma X}]$$

$$= e^{\mu t}M_X(\sigma t) = \boxed{e^{\mu t+\frac{1}{2}\sigma^2 t^2}}.$$