15-826: Multimedia Databases and Data Mining

Lecture #20: SVD - part III (more case studies)

C. Faloutsos

Must-read Material

- MM Textbook Appendix D
- Graph Mining Textbook, chapter 15.
- Kleinberg, J. (1998). Authoritative sources in a hyperlinked environment. Proc. 9th ACM-SIAM Symposium on Discrete Algorithms.
- Brin, S. and L. Page (1998). Anatomy of a Large-Scale Hypertextual Web Search Engine. 7th Intl World Wide Web Conf.

Must-read Material, cont’d

- Haveliwala, Taher H. (2003) Topic-Sensitive PageRank: A Context-Sensitive Ranking Algorithm for Web Search. Extended version of the WWW2002 paper.
- Chen, C. M. and N. Roussopoulos (May 1994). Adaptive Selectivity Estimation Using Query Feedback. Proc. of the ACM-SIGMOD , Minneapolis, MN.

Outline

Goal: ‘Find similar / interesting things’

- Intro to DB
- Indexing - similarity search
- Data Mining

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Indexing - Detailed outline

- primary key indexing
- secondary key / multi-key indexing
- spatial access methods
- fractals
- text
- Singular Value Decomposition (SVD)
- multimedia
- ...

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SVD - Detailed outline

- Motivation
- Definition - properties
- Interpretation
- Complexity
- Case studies
- SVD properties
- More case studies
- Conclusions

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SVD - detailed outline

- ...
- Case studies
- SVD properties
- more case studies
  - google/Kleinberg algorithms
  - query feedbacks
- Conclusions

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SVD - Other properties - summary

- can produce orthogonal basis (obvious) (who cares?)
- can solve over- and under-determined linear problems (see C(1) property)
- can compute ‘fixed points’ (= ‘steady state prob. in Markov chains’) (see C(4) property)

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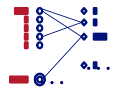
IMPORTANT!

Properties – sneak preview:

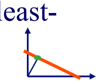
A(0): $\mathbf{A}_{[n \times m]} = \mathbf{U}_{[n \times r]} \mathbf{\Lambda}_{[r \times r]} \mathbf{V}^{\mathbf{T}}_{[r \times m]}$



B(5): $(\mathbf{A}^T \mathbf{A})^k \mathbf{v}' \sim$ (constant) $\mathbf{v}_1$

C(1): $\mathbf{A}_{[n \times m]} \mathbf{x}_{[m \times 1]} = \mathbf{b}_{[n \times 1]}$

then, $\mathbf{x}_0 = \mathbf{V} \mathbf{\Lambda}^{(-1)} \mathbf{U}^T \mathbf{b}$: shortest, actual or least-squares solution



C(4): $\mathbf{A}^\mathbf{T} \mathbf{A} \mathbf{v}_1 = \lambda_1^2 \mathbf{v}_1$

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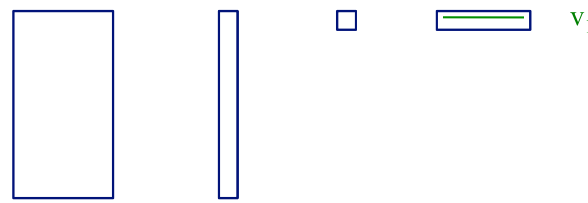

Document-term Doc-concept Concept-term

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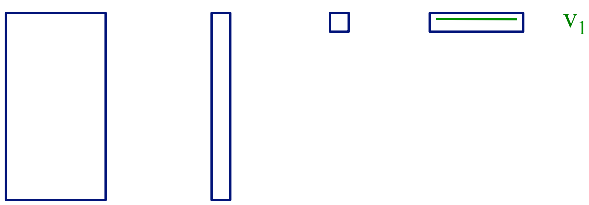

Libraries-books Libraries-concepts Concepts-books

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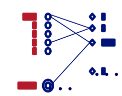
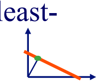


B(5): $(\mathbf{A}^T \mathbf{A})^k \mathbf{v}' \sim$ (constant) $\mathbf{v}_1$

C(1): $\mathbf{A}_{[n \times m]} \mathbf{x}_{[m \times 1]} = \mathbf{b}_{[n \times 1]}$

then, $\mathbf{x}_0 = \mathbf{V} \mathbf{\Lambda}^{(-1)} \mathbf{U}^T \mathbf{b}$: shortest, actual or least-squares solution



C(4): $\mathbf{A}^\mathbf{T} \mathbf{A} \mathbf{v}_1 = \lambda_1^2 \mathbf{v}_1$

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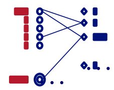
IMPORTANT!

Properties – sneak preview:



B(5): $(\mathbf{A}^T \mathbf{A})^k \mathbf{v'} \sim$ (constant) $\mathbf{v}_1$

Convergence

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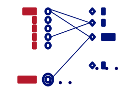
Properties – sneak preview:



B(5): $(\mathbf{A}^T \mathbf{A})^k \mathbf{v'} \sim$ (constant) $\mathbf{v}_1$

Fixed point

C(4): $\mathbf{A}^T \mathbf{A} \mathbf{v}_1 = \lambda_1^2 \mathbf{v}_1$

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IMPORTANT!

Properties – sneak preview:

A(0): $\mathbf{A}_{[n \times m]} = \mathbf{U}_{[n \times r]} \mathbf{\Lambda}_{[r \times r]} \mathbf{V}^T_{[r \times m]}$

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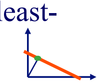

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SVD -outline of properties

- (A): obvious
- (B): less obvious
- (C): least obvious (and most powerful!)

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Properties - by defn.:

A(0): $\mathbf{A}_{[n \times m]} = \mathbf{U}_{[n \times r]} \mathbf{\Lambda}_{[r \times r]} \mathbf{V^T}_{[r \times m]}$

A(1): $\mathbf{U}^T_{[r \times n]} \mathbf{U}_{[n \times r]} = \mathbf{I}_{[r \times r]}$ (identity matrix)

A(2): $\mathbf{V}^T_{[r \times n]} \mathbf{V}_{[n \times r]} = \mathbf{I}_{[r \times r]}$

A(3): $\mathbf{\Lambda}^k = \text{diag}(\lambda_1^k, \lambda_2^k, \ldots \lambda_r^k)$ (k: ANY real number)

A(4): $\mathbf{A^T} = \mathbf{V} \mathbf{\Lambda} \mathbf{U^T}$

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Less obvious properties

A(0): $\mathbf{A}_{[n \times m]} = \mathbf{U}_{[n \times r]} \mathbf{\Lambda}_{[r \times r]} \mathbf{V^T}_{[r \times m]}$

B(1): $\mathbf{A}_{[n \times m]} (\mathbf{A}^T)_{[m \times n]} = ??$

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Less obvious properties

A(0): $\mathbf{A}_{[n \times m]} = \mathbf{U}_{[n \times r]} \mathbf{\Lambda}_{[r \times r]} \mathbf{V^T}_{[r \times m]}$

B(1): $\mathbf{A}_{[n \times m]} (\mathbf{A}^T)_{[m \times n]} = \mathbf{U} \mathbf{\Lambda}^2 \mathbf{U}^T$

symmetric; Intuition?

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Less obvious properties

A(0): $\mathbf{A}_{[n \times m]} = \mathbf{U}_{[n \times r]} \mathbf{\Lambda}_{[r \times r]} \mathbf{V^T}_{[r \times m]}$

B(1): $\mathbf{A}_{[n \times m]} (\mathbf{A}^T)_{[m \times n]} = \mathbf{U} \mathbf{\Lambda}^2 \mathbf{U}^T$

symmetric; Intuition?

'document-to-document' similarity matrix

B(2): symmetrically, for 'V'

$(\mathbf{A}^T)_{[m \times n]} \mathbf{A}_{[n \times m]} = \mathbf{V} \mathbf{\Lambda}^2 \mathbf{V}^T$

Intuition?

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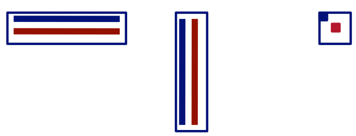
Reminder: 'column orthonormal'

- $\mathbf{V}^T$ $\mathbf{V}$ = $\mathbf{I}_{[r \times r]}$

v1

v_2



$\mathbf{v_1}^T \times \mathbf{v_1} = 1$
$\mathbf{v_1}^T \times \mathbf{v_2} = 0$

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Less obvious properties

A: term-to-term similarity matrix

B(3): $((\mathbf{A}^T)_{[m \times n]} \mathbf{A}_{[n \times m]})^k = \mathbf{V} \, \mathbf{\Lambda}^{2k} \, \mathbf{V}^T$

and

B(4): $(\mathbf{A}^T \mathbf{A})^k \sim \mathbf{v}_1 \, \lambda_1^{2k} \, \mathbf{v}_1^T$ for k>>1

where

$\mathbf{v}_1$: [m x 1] first column (singular-vector) of **V**

λ_1: strongest singular value

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Proof of (B4)?

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Less obvious properties

B(4): $(\mathbf{A}^T \mathbf{A})^k \sim \mathbf{v}_1 \, \lambda_1^{2k} \, \mathbf{v}_1^T$ for k>>1

B(5): $(\mathbf{A}^T \mathbf{A})^k \, \mathbf{v'} \sim (\text{constant}) \, \mathbf{v}_1$

ie., for (almost) any **v'**, it converges to a vector parallel to $\mathbf{v}_1$

Thus, useful to compute first singular vector/ value (as well as the next ones, too...)

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Proof of (B5)?

- B(5): $(\mathbf{A}^T \mathbf{A})^k \mathbf{v'} \sim (\text{constant})\ \mathbf{v}_1$
- $== \underbrace{(\mathbf{A}^T \mathbf{A}) \ldots (\mathbf{A}^T \mathbf{A})}_{k \text{ times}} \mathbf{v'} \sim (\text{constant})\ \mathbf{v}_1$

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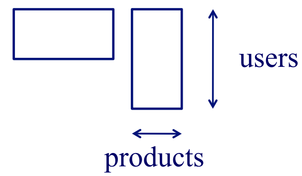
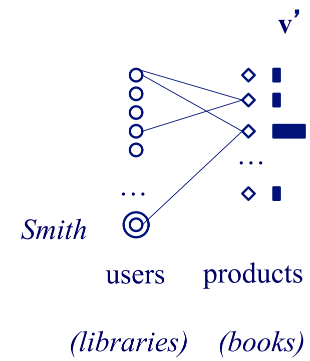
Property (B5)

- Intuition:
  - $(\mathbf{A}^T \mathbf{A})\ \mathbf{v'}$
  - $(\mathbf{A}^T \mathbf{A})^k\ \mathbf{v'}$





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Property (B5)

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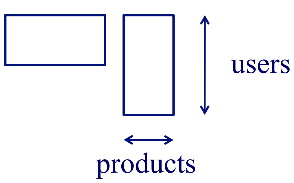

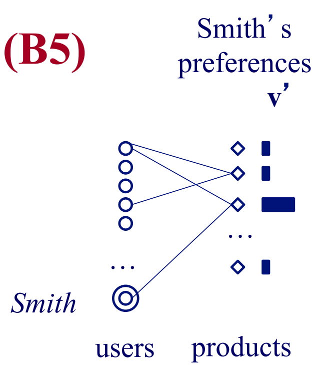

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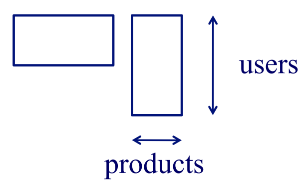

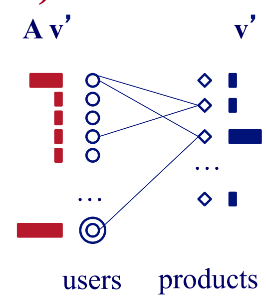

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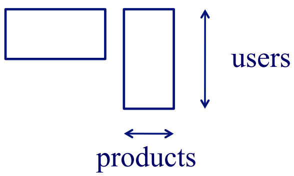

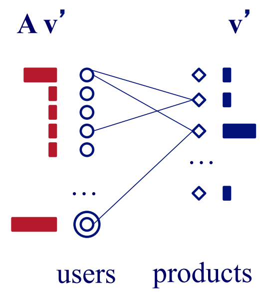

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Property (B5)

- Intuition:
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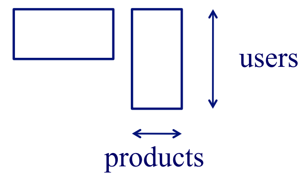
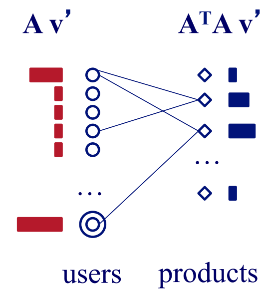
  - $(\mathbf{A}^T \mathbf{A})^k\ \mathbf{v'}$





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Property (B5)

- Intuition:
  - $(\mathbf{A}^T \mathbf{A})\ \mathbf{v'}$ what Smith's 'friends' like
  - $(\mathbf{A}^T \mathbf{A})^k\ \mathbf{v'}$ what k-step-away-friends like

(ie., after *k* steps, we get what everybody likes, and Smith's initial opinions don't count)

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Less obvious properties - repeated:

A(0): $\mathbf{A}_{[n \times m]} = \mathbf{U}_{[n \times r]}\ \mathbf{\Lambda}_{[r \times r]}\ \mathbf{V^T}_{[r \times m]}$

B(1): $\mathbf{A}_{[n \times m]}\ (\mathbf{A}^T)_{[m \times n]} = \mathbf{U}\ \mathbf{\Lambda}^2\ \mathbf{U}^T$

B(2): $(\mathbf{A}^T)_{[m \times n]}\ \mathbf{A}_{[n \times m]} = \mathbf{V}\ \mathbf{\Lambda}^2\ \mathbf{V}^T$

B(3): $(\ (\mathbf{A}^T)_{[m \times n]}\ \mathbf{A}_{[n \times m]}\ )^k = \mathbf{V}\ \mathbf{\Lambda}^{2k}\ \mathbf{V}^T$

B(4): $(\mathbf{A}^T \mathbf{A})^k \sim v_1\ \lambda_1^{2k}\ v_1^T$

B(5): $(\mathbf{A}^T \mathbf{A})^k\ \mathbf{v'} \sim (\text{constant})\ \mathbf{v}_1$

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Least obvious properties

A(0): $\mathbf{A}_{[n \times m]} = \mathbf{U}_{[n \times r]} \mathbf{\Lambda}_{[r \times r]} \mathbf{V}^{\mathbf{T}}_{[r \times m]}$

C(1): $\mathbf{A}_{[n \times m]} \mathbf{x}_{[m \times 1]} = \mathbf{b}_{[n \times 1]}$

let $\mathbf{x}_0 = \mathbf{V} \mathbf{\Lambda}^{(-1)} \mathbf{U}^{\mathrm{T}} \mathbf{b}$

if under-specified, $\mathbf{x}_0$ gives 'shortest' solution

if over-specified, it gives the 'solution' with the smallest least squares error

(see Num. Recipes, p. 62)

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Least obvious properties

A(0): $\mathbf{A}_{[n \times m]} = \mathbf{U}_{[n \times r]} \mathbf{\Lambda}_{[r \times r]} \mathbf{V}^{\mathbf{T}}_{[r \times m]}$

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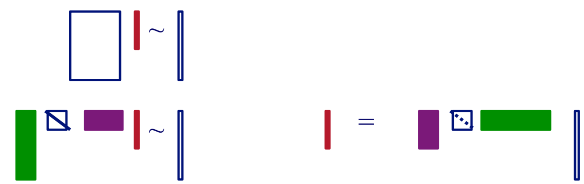

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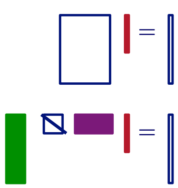
Slowly:



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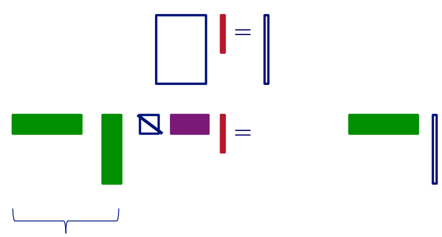
Slowly:



Identity

U: column-orthonormal

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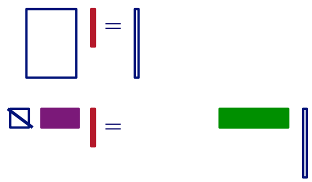
Slowly:



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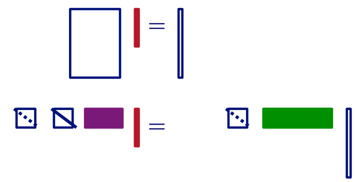
Slowly:



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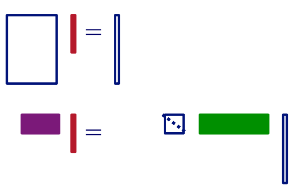

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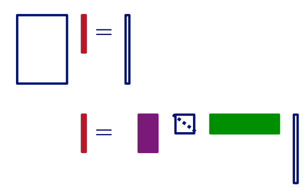

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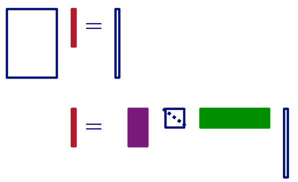
Slowly:



x **V** $\mathbf{\Lambda}^{-1}$ $\mathbf{U}^T$ **b**

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Slowly:

Important: **DROP** small values of $\mathbf{\Lambda}$ (say, $< 10^{-6} * \lambda_1$)

x **V** $\mathbf{\Lambda}^{-1}$ $\mathbf{U}^T$ **b**

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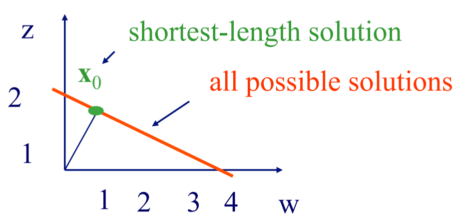
Least obvious properties

Illustration: under-specified, eg

[1 2] [w z] ^{T} = 4 (ie, 1 w + 2 z = 4)

A=??

b=??



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Exercise

Verify formula:

$\mathbf{A}$ = [1 2] $\mathbf{b}$ = [4]

$\mathbf{A} = \mathbf{U}\ \mathbf{\Lambda}\ \mathbf{V}^T$

$\mathbf{U}$ = ??

$\mathbf{\Lambda}$ = ??

$\mathbf{V}$= ??

$\mathbf{x_0} = \mathbf{V}\ \mathbf{\Lambda}^{(-1)}\ \mathbf{U}^T\ \mathbf{b}$

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Exercise

Verify formula:

$\mathbf{A}$ = [1 2] $\mathbf{b}$ = [4]

$\mathbf{A} = \mathbf{U}\ \mathbf{\Lambda}\ \mathbf{V}^T$

$\mathbf{U}$ = [1]

$\mathbf{\Lambda}$ = [sqrt(5)]

$\mathbf{V}$= [1/sqrt(5)   2/sqrt(5)] ^{T}

$\mathbf{x_0} = \mathbf{V}\ \mathbf{\Lambda}^{(-1)}\ \mathbf{U}^T\ \mathbf{b}$

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Exercise

Verify formula:

$\mathbf{A}$ = [1 2] $\mathbf{b}$ = [4]

$\mathbf{A} = \mathbf{U}\ \mathbf{\Lambda}\ \mathbf{V}^T$

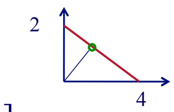
$\mathbf{U}$ = [1]

$\mathbf{\Lambda}$ = [sqrt(5)]

$\mathbf{V}$= [1/sqrt(5)   2/sqrt(5)] ^{T}

$\mathbf{x_0} = \mathbf{V}\ \mathbf{\Lambda}^{(-1)}\ \mathbf{U}^T\ \mathbf{b}$ = [1/5   2/5] ^{T} [4]

= [4/5 8/5] ^{T} : w= 4/5, z = 8/5



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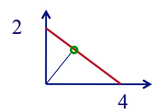
Exercise

Verify formula:



Show that w= 4/5, z = 8/5 is

(a) A solution to 1*w + 2*z = 4 and

(b) Minimal (wrt Euclidean norm)

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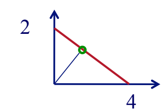
Exercise

Verify formula:



Show that w= 4/5, z = 8/5 is

(a) A solution to 1*w + 2*z = 4 and

A: easy

(b) Minimal (wrt Euclidean norm)

A: [4/5   8/5] is perpenticular to [2   -1]

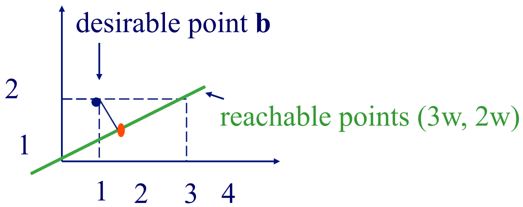
Least obvious properties – cont' d

Illustration: over-specified, eg

$[3\ 2]^T\ [w] = [1\ 2]^T$ (ie, 3 w = 1; 2 w = 2)

A=??

b=??



Exercise

Verify formula:

$\mathbf{A} = [3\ 2]^T \quad \mathbf{b} = [1\ \ 2]^T$

$\mathbf{A} = \mathbf{U}\ \mathbf{\Lambda}\ \mathbf{V}^T$

$\mathbf{U}$ = ??

$\mathbf{\Lambda}$ = ??

$\mathbf{V}$ = ??

$\mathbf{x_0} = \mathbf{V}\ \mathbf{\Lambda}^{(-1)}\ \mathbf{U}^T\ \mathbf{b}$

Exercise

Verify formula:

$\mathbf{A} = [3\ 2]^T \quad \mathbf{b} = [1\ \ 2]^T$

$\mathbf{A} = \mathbf{U}\ \mathbf{\Lambda}\ \mathbf{V}^T$

$\mathbf{U} = [\ 3/\mathrm{sqrt}(13) \quad 2/\mathrm{sqrt}(13)\ ]^T$

$\mathbf{\Lambda} = [\ \mathrm{sqrt}(13)\ ]$

$\mathbf{V} = [\ 1\ ]$

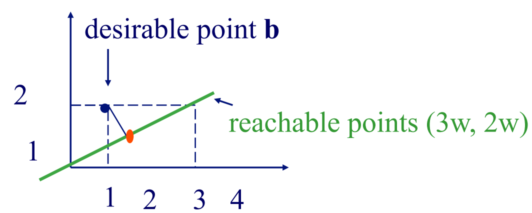
$\mathbf{x_0} = \mathbf{V}\ \mathbf{\Lambda}^{(-1)}\ \mathbf{U}^T\ \mathbf{b} = [\ 7/13\ ]$

Exercise

Verify formula:

$[3\ 2]^T\ [7/13] = [1\ 2]^T$

$[21/13\ \ 14/13\ ]^T$ -> 'red point'



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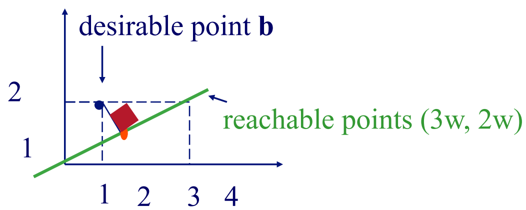
Exercise

Verify formula:

$[3\ 2]^T\ [7/13] = [1\ 2]^T$

$[21/13\ \ 14/13]^T$ -> 'red point' - perpenticular?



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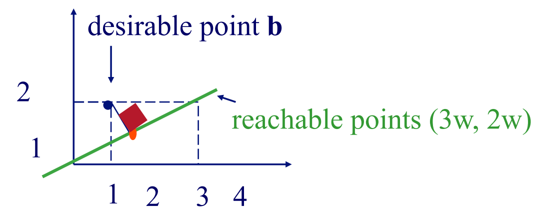
Exercise

Verify formula:

A: [3 2] . ([1 2] – [21/13 14/13]) =

[3 2] . [-8/13 12/13] = [3 2] . [-2 3] = 0



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Least obvious properties - cont'd

A(0): $\mathbf{A}_{[n \times m]} = \mathbf{U}_{[n \times r]}\, \mathbf{\Lambda}_{[r \times r]}\, \mathbf{V}^{\mathbf{T}}_{[r \times m]}$

C(2): $\mathbf{A}_{[n \times m]}\, \mathbf{v_1}_{[m \times 1]} = \lambda_1\, \mathbf{u_1}_{[n \times 1]}$

where $\mathbf{v}_1$, $\mathbf{u}_1$ the first (column) vectors of **V**, **U.** ($\mathbf{v}_1$ == right-singular-vector)

C(3): symmetrically: $\mathbf{u}_1^{\mathbf{T}}\, \mathbf{A} = \lambda_1\, \mathbf{v}_1^{\mathbf{T}}$

$\mathbf{u}_1$ == left-singular-vector

Therefore:

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Least obvious properties - cont'd

A(0): $\mathbf{A}_{[n \times m]} = \mathbf{U}_{[n \times r]}\, \mathbf{\Lambda}_{[r \times r]}\, \mathbf{V}^{\mathbf{T}}_{[r \times m]}$

C(4): $\boxed{\mathbf{A}^{\mathbf{T}}\, \mathbf{A}\, \mathbf{v}_1 = \lambda_1^2\, \mathbf{v}_1}$

(**fixed point** - the dfn of eigenvector for a symmetric matrix)

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Least obvious properties - altogether

A(0): $\mathbf{A}_{[n \times m]} = \mathbf{U}_{[n \times r]} \mathbf{\Lambda}_{[r \times r]} \mathbf{V^T}_{[r \times m]}$

C(1): $\mathbf{A}_{[n \times m]} \mathbf{x}_{[m \times 1]} = \mathbf{b}_{[n \times 1]}$

then, $\mathbf{x}_0 = \mathbf{V} \mathbf{\Lambda}^{(-1)} \mathbf{U}^T$ **b:** shortest, actual or least-squares solution

C(2): $\mathbf{A}_{[n \times m]} \mathbf{v_1}_{[m \times 1]} = \lambda_1 \mathbf{u_1}_{[n \times 1]}$

C(3): $\mathbf{u_1}^T \mathbf{A} = \lambda_1 \mathbf{v_1}^T$

C(4): $\mathbf{A^T A v_1} = \lambda_1^2 \mathbf{v_1}$

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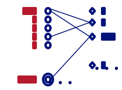
Properties - conclusions

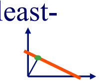
A(0): $\mathbf{A}_{[n \times m]} = \mathbf{U}_{[n \times r]} \mathbf{\Lambda}_{[r \times r]} \mathbf{V^T}_{[r \times m]}$



B(5): $(\mathbf{A}^T \mathbf{A})^k \mathbf{v'} \sim$ (constant) $\mathbf{v}_1$

C(1): $\mathbf{A}_{[n \times m]} \mathbf{x}_{[m \times 1]} = \mathbf{b}_{[n \times 1]}$

then, $\mathbf{x}_0 = \mathbf{V} \mathbf{\Lambda}^{(-1)} \mathbf{U}^T$ **b:** shortest, actual or least-squares solution



C(4): $\mathbf{A^T A v_1} = \lambda_1^2 \mathbf{v_1}$

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SVD - detailed outline

- ...
- Case studies
- SVD properties
- more case studies
  - Kleinberg/google algorithms
  - query feedbacks
- Conclusions

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Kleinberg's algo (HITS)

Kleinberg, Jon (1998). *Authoritative sources in a hyperlinked environment.* Proc. 9th ACM-SIAM Symposium on Discrete Algorithms.

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Kleinberg's algorithm

- Problem dfn: given the web and a query
- find the most 'authoritative' web pages for this query

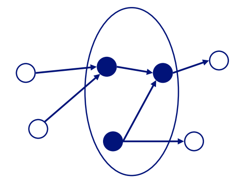
Step 0: find all pages containing the query terms

Step 1: expand by one move forward and backward

CMU SCS

Kleinberg's algorithm

- Step 1: expand by one move forward and backward



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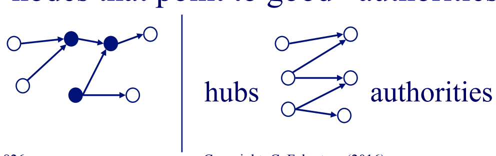
Kleinberg's algorithm

- on the resulting graph, give high score (= 'authorities') to nodes that many important nodes point to
- give high importance score ('hubs') to nodes that point to good 'authorities')



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Kleinberg's algorithm

- on the resulting graph, give high score (= 'authorities') to nodes that many important nodes point to
- give high importance score ('hubs') to nodes that point to good 'authorities')

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Kleinberg's algorithm

observations

- recursive definition!
- each node (say, 'i'-th node) has both an authoritativeness score a_i and a hubness score h_i

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Kleinberg's algorithm

Let E be the set of edges and $\mathbf{A}$ be the adjacency matrix:

the (i,j) is 1 if the edge from i to j exists

Let h and a be [n x 1] vectors with the 'hubness' and 'authoritativiness' scores.

Then:

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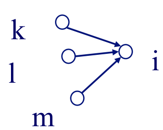
Kleinberg's algorithm



Then:

$$a_i = h_k + h_l + h_m$$

that is

a_i = Sum (h_j) over all j that (j,i) edge exists

or

$\mathbf{a} = \mathbf{A}^T \mathbf{h}$

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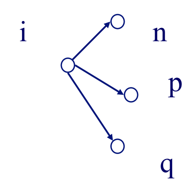
Kleinberg's algorithm



symmetrically, for the 'hubness':

$$h_i = a_n + a_p + a_q$$

that is

h_i = Sum (q_j) over all j that (i,j) edge exists

or

$\mathbf{h} = \mathbf{A}\ \mathbf{a}$

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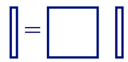
Kleinberg's algorithm

In conclusion, we want vectors **h** and **a** such that:

$$\mathbf{h} = \mathbf{A}\ \mathbf{a}$$

$$\mathbf{a} = \mathbf{A}^T\ \mathbf{h}$$



Recall properties:

C(2): $\mathbf{A}_{[n \times m]}\ \mathbf{v_1}_{[m \times 1]} = \lambda_1\ \mathbf{u_1}_{[n \times 1]}$

C(3): $\mathbf{u_1}^T \mathbf{A} = \lambda_1\ \mathbf{v_1}^T$

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Kleinberg's algorithm

In short, the solutions to

$$\mathbf{h} = \mathbf{A}\ \mathbf{a}$$

$$\mathbf{a} = \mathbf{A}^T\ \mathbf{h}$$

are the left- and right- singular-vectors of the adjacency matrix **A.**

Starting from random **a'** and iterating, we'll eventually converge

(Q: to which of all the singular-vectors? why?)

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Kleinberg's algorithm

(Q: to which of all the singular-vectors? why?)

A: to the ones of the strongest singular-value, because of property B(5):

B(5): $(\mathbf{A}^T\ \mathbf{A})^k\ \mathbf{v'} \sim (\text{constant})\ \mathbf{v}_1$

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Kleinberg's algorithm - results

Eg., for the query 'java':

0.328 www.gamelan.com

0.251 java.sun.com

0.190 www.digitalfocus.com ("the java developer")

Kleinberg's algorithm - discussion

- 'authority' score can be used to find 'similar pages' (how?)
- closely related to 'citation analysis', social networs / 'small world' phenomena

SVD - detailed outline

- ...
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- SVD properties
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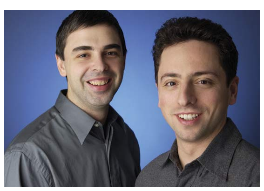
  - query feedbacks
- Conclusions

PageRank (google)

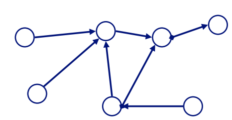


Larry Page   Sergey Brin

•Brin, Sergey and Lawrence Page (1998). *Anatomy of a Large-Scale Hypertextual Web Search Engine*. 7th Intl World Wide Web Conf.

Problem: PageRank

Given a directed graph, find its most interesting/central node



A node is important, if it is connected with important nodes (recursive, but OK!)

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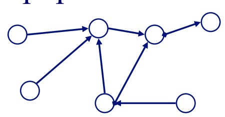
Problem: PageRank - solution

Given a directed graph, find its most interesting/central node

Proposed solution: Random walk; spot most 'popular' node (-> steady state prob. (ssp))



A node has high ssp, if it is connected with high ssp nodes (recursive, but OK!)

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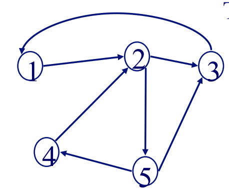
(Simplified) PageRank algorithm

- Let **A** be the adjacency matrix;
- let **B** be the transition matrix: transpose, column-normalized - then



From / To

B

$$\begin{bmatrix} & & 1 & & \\ 1 & & & 1 & \\ & 1/2 & & & 1/2 \\ & & & & 1/2 \\ & 1/2 & & & \end{bmatrix} \begin{bmatrix} p1 \\ p2 \\ p3 \\ p4 \\ p5 \end{bmatrix} = \begin{bmatrix} p1 \\ p2 \\ p3 \\ p4 \\ p5 \end{bmatrix}$$

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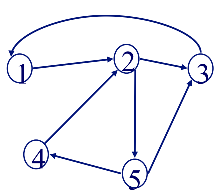
(Simplified) PageRank algorithm

- **B p = p**



B **p** = **p**

$$\begin{bmatrix} & & 1 & & \\ 1 & & & 1 & \\ & 1/2 & & & 1/2 \\ & & & & 1/2 \\ & 1/2 & & & \end{bmatrix} \begin{bmatrix} p1 \\ p2 \\ p3 \\ p4 \\ p5 \end{bmatrix} = \begin{bmatrix} p1 \\ p2 \\ p3 \\ p4 \\ p5 \end{bmatrix}$$

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(Simplified) PageRank algorithm

- **B p** = 1 * **p**
- thus, **p** is the **eigenvector** that corresponds to the highest eigenvalue (=1, since the matrix is column-normalized)
- Why does such a **p** exist?
  - **p** exists if **B** is nxn, nonnegative, irreducible [Perron–Frobenius theorem]

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(Simplified) PageRank algorithm

- **B p** = 1 * **p**
- thus, **p** is the **eigenvector(*)** that corresponds to the highest eigenvalue (=1, since the matrix is column-normalized)
- Why does such a **p** exist?
  - **p** exists if **B** is nxn, nonnegative, irreducible [Perron–Frobenius theorem]

All<-> all

(*) dfn: a few foils later

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(Simplified) PageRank algorithm

- In short: imagine a particle randomly moving along the edges
- compute its steady-state probabilities (ssp)

Full version of algo: with occasional random jumps

Why? To make the matrix irreducible

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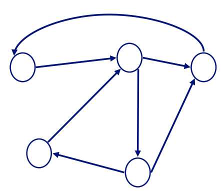
Full Algorithm

- With probability *1-c*, fly-out to a random node
- Then, we have

p = c **B p** + (1-c)/n **1** =>

p = (1-c)/n [**I** - c **B**] $^{-1}$ **1**



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Full Algorithm

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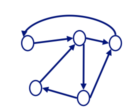

$$\begin{bmatrix} 1 \\ 1 \\ \cdots \\ 1 \end{bmatrix}$$

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Alternative notation – eigenvector viewpoint

M Modified transition matrix

$\mathbf{M} = c\,\mathbf{B} + (1-c)/n\ \mathbf{1}\ \mathbf{1}^T$

CLIQUE

Then

$\mathbf{p} = \mathbf{M}\,\mathbf{p}$

That is: the steady state probabilities =

PageRank scores form the *first eigenvector* of the 'modified transition matrix'

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Parenthesis: intuition behind eigenvectors

- Definition
- 2 properties
- intuition

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Formal definition

If **A** is a (n x n) square matrix
(λ , **x**) is an **eigenvalue/eigenvector** pair of **A** if

$$\boxed{\mathbf{A}\,\mathbf{x} = \lambda\,\mathbf{x}}$$

CLOSELY related to singular values:

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Property #1: Eigen- vs singular-values

if

$$\mathbf{B}_{[n \times m]} = \mathbf{U}_{[n \times r]}\,\mathbf{\Lambda}_{[r \times r]}\,(\mathbf{V}_{[m \times r]})^T$$

then $\mathbf{A} = (\mathbf{B}^T\mathbf{B})$ is symmetric and

C(4): $\mathbf{B}^T\,\mathbf{B}\,\mathbf{v}_i = \lambda_i^2\,\mathbf{v}_i$

ie, $\mathbf{v}_1$, $\mathbf{v}_2$, ...: eigenvectors of $\mathbf{A} = (\mathbf{B}^T\mathbf{B})$

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Property #2

- If $\mathbf{A}_{[nxn]}$ is a real, symmetric matrix
- Then it has n real eigenvalues

(if **A** is not symmetric, some eigenvalues may be complex)

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Property #3

- If $\mathbf{A}_{[nxn]}$ is a real, symmetric matrix
- Then it has n real eigenvalues
- And they agree with its n singular values, except possibly for the sign

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Parenthesis: intuition behind eigenvectors

- Definition
- 2 properties
- **intuition**

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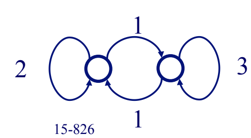
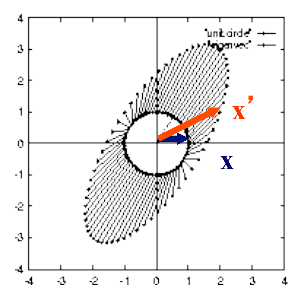
Intuition

- **A** as vector transformation

$$\begin{array}{ccc} \mathbf{x'} & \mathbf{A} & \mathbf{x} \\ \begin{bmatrix} 2 \\ 1 \end{bmatrix} = & \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} & \begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{array}$$





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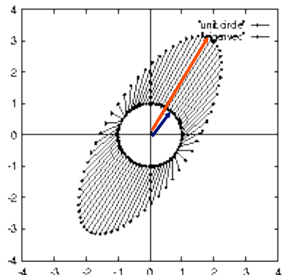
Intuition

- By defn., eigenvectors remain parallel to themselves ('**fixed points**')

$$\begin{array}{ccccccc} \lambda_1 & & \mathbf{v_1} & & \mathbf{A} & & \mathbf{v_1} \\ 3.62 & * & \begin{bmatrix} 0.52 \\ 0.85 \end{bmatrix} & = & \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} & & \begin{bmatrix} 0.52 \\ 0.85 \end{bmatrix} \end{array}$$



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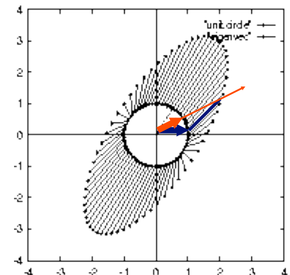
Convergence

- Usually, fast:



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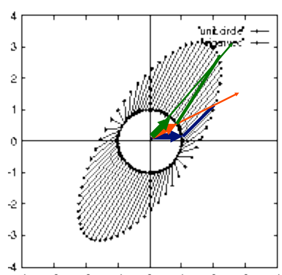
Convergence

- Usually, fast:



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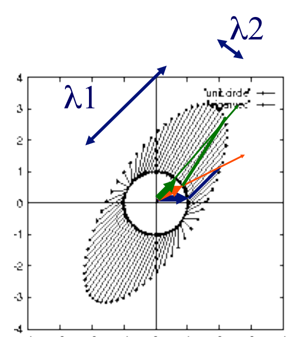
Convergence

- Usually, fast:
- depends on ratio
  λ1 : λ2



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Closing the parenthesis wrt intuition behind eigenvectors

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Kleinberg/PageRank - conclusions

SVD helps in graph analysis:

hub/authority scores: strongest left- and right-singular-vectors of the adjacency matrix

random walk on a graph: steady state probabilities are given by the strongest eigenvector of the transition matrix

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SVD - detailed outline

- ...
- Case studies
- SVD properties
- more case studies
  - google/Kleinberg algorithms
  - query feedbacks
- Conclusions

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Query feedbacks

[Chen & Roussopoulos, sigmod 94]

Sample problem:

estimate selectivities (e.g., '*how many movies were made between 1940 and 1945*?'

for query optimization,

LEARNING from the query results so far!!

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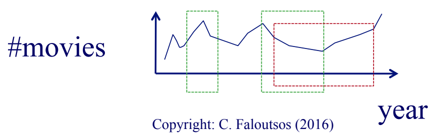
Query feedbacks

- Given: past queries and their results
  - #movies(1925,1935) = 52
  - #movies(1948, 1990) = 123
  - …
  - And a new query, say #movies(1979,1980)?
- Give your best estimate



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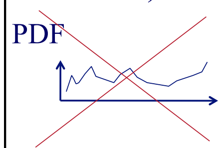
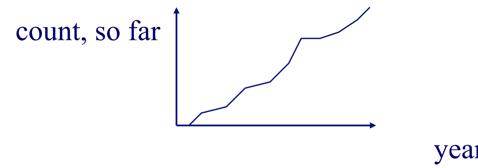
Query feedbacks

Idea #1: consider a function for the CDF (cummulative distr. function), eg., 6-th degree polynomial (or splines, or anything else)





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Query feedbacks

For example

F(x) = # movies made until year 'x'

$= a_1 + a_2 * x + a_3 * x^2 + \ldots a_7 * x^6$

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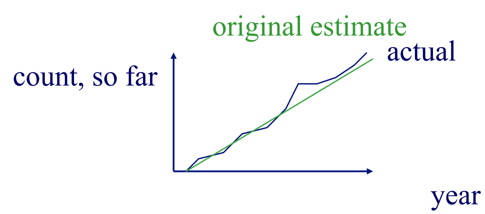
Query feedbacks

GREAT idea #2: adapt your model, as you see the actual counts of the actual queries



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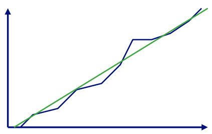
Query feedbacks

original estimate

count, so far



actual

year

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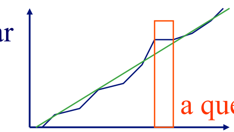
Query feedbacks

original estimate

count, so far



actual

year

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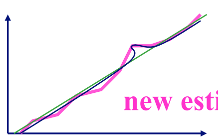
Query feedbacks

original estimate

count, so far



actual

year

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Query feedbacks

Eventually, the problem becomes:

- estimate the parameters a_1, ... a_7 of the model
- to minimize the least squares errors from the real answers so far.

Formally:

Query feedbacks

Formally, with *n* queries and 6-th degree polynomials:

$$\begin{bmatrix} X11 & X12 & & & X17 \\ & & & & \\ & & & & \\ Xn1 & Xn2 & & & Xn7 \end{bmatrix} \begin{bmatrix} a1 \\ a2 \\ \\ \\ a7 \end{bmatrix} = \begin{bmatrix} b1 \\ b2 \\ \\ \\ bn \end{bmatrix}$$

Query feedbacks

where $x_{i,j}$ such that Sum ($x_{i,j}$ * a_i) = our estimate for the # of movies and b_j: the actual

$$\begin{bmatrix} X11 & X12 & & & X17 \\ & & & & \\ & & & & \\ Xn1 & Xn2 & & & Xn7 \end{bmatrix} \begin{bmatrix} a1 \\ a2 \\ \\ \\ a7 \end{bmatrix} = \begin{bmatrix} b1 \\ b2 \\ \\ \\ bn \end{bmatrix}$$

Query feedbacks

For example, for query '*find the count of movies during (1920-1932)*' :

$a_1 + a_2 * 1932 + a_3 * 1932**2 + \ldots$

-

$(a_1 + a_2 * 1920 + a_3 * 1920**2 + \ldots)$

$$\begin{bmatrix} X11 & X12 & & & X17 \\ & & & & \\ & & & & \\ Xn1 & Xn2 & & & Xn7 \end{bmatrix} \begin{bmatrix} a1 \\ a2 \\ \\ \\ a7 \end{bmatrix} = \begin{bmatrix} b1 \\ b2 \\ \\ \\ bn \end{bmatrix}$$

Query feedbacks

And thus X11 = 0; X12 = 1932-1920, etc

$a_1 + a_2 * 1932 + a_3 * 1932**2 + \ldots$

-

$(a_1 + a_2 * 1920 + a_3 * 1920**2 + \ldots)$

$$\begin{bmatrix} X11 & X12 & & & X17 \\ & & & & \\ & & & & \\ Xn1 & Xn2 & & & Xn7 \end{bmatrix} \begin{bmatrix} a1 \\ a2 \\ \\ \\ a7 \end{bmatrix} = \begin{bmatrix} b1 \\ b2 \\ \\ \\ bn \end{bmatrix}$$

Query feedbacks

In matrix form:

$$
\begin{array}{l}
\text{1st query} \\
\\
\\
\text{n-th query}
\end{array}
\underbrace{\begin{bmatrix}
X11 & X12 & \cdots & X17 \\
 & & & \\
 & & & \\
Xn1 & Xn2 & \cdots & Xn7
\end{bmatrix}}_{\mathbf{X}}
\underbrace{\begin{bmatrix} a1 \\ a2 \\ \vdots \\ a7 \end{bmatrix}}_{\mathbf{a}}
=
\underbrace{\begin{bmatrix} b1 \\ b2 \\ \vdots \\ bn \end{bmatrix}}_{\mathbf{b}}
$$

Query feedbacks

In matrix form:

$$\mathbf{X}\,\mathbf{a} = \mathbf{b}$$

and the least-squares estimate for **a** is

$$\mathbf{a} = \mathbf{V}\, \mathbf{\Lambda}^{(-1)}\, \mathbf{U}^{\mathbf{T}}\, \mathbf{b}$$

according to property C(1)

(let $\mathbf{X} = \mathbf{U}\, \mathbf{\Lambda}\, \mathbf{V}^{\mathbf{T}}$)

Query feedbacks - enhancements

The solution

$$\mathbf{a} = \mathbf{V}\, \mathbf{\Lambda}^{(-1)}\, \mathbf{U}^{\mathbf{T}}\, \mathbf{b}$$

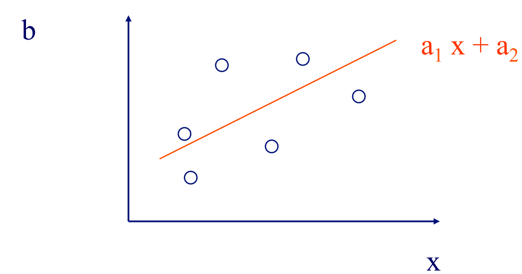
works, but needs expensive SVD each time a new query arrives

GREAT Idea #3: Use 'Recursive Least Squares', to adapt **a** incrementally.

Details: in paper - intuition:

Query feedbacks - enhancements

Intuition: least squares fit



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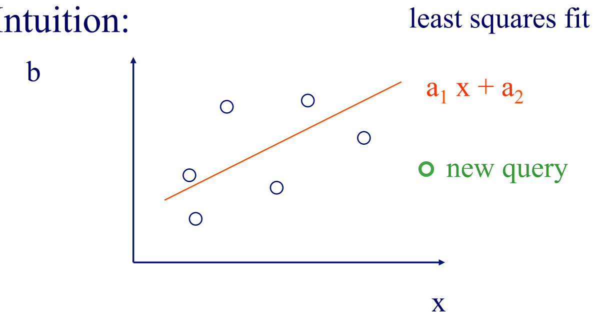
Query feedbacks - enhancements

Intuition:



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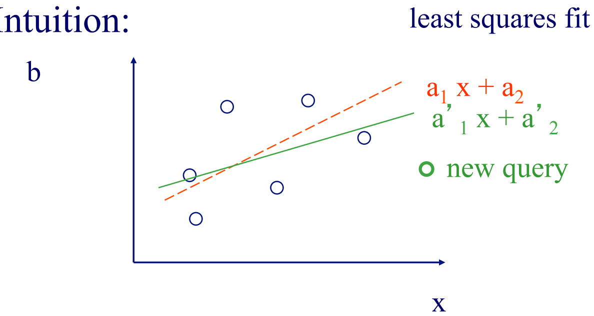
Query feedbacks - enhancements

Intuition:



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Query feedbacks - enhancements

the new coefficients can be quickly computed from the old ones, plus statistics in a (7x7) matrix

(no need to know the details, although the RLS is a brilliant method)

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Query feedbacks - enhancements

GREAT idea #4: 'forgetting' factor - we can even down-play the weight of older queries, since the data distribution might have changed.

(comes for 'free' with RLS...)

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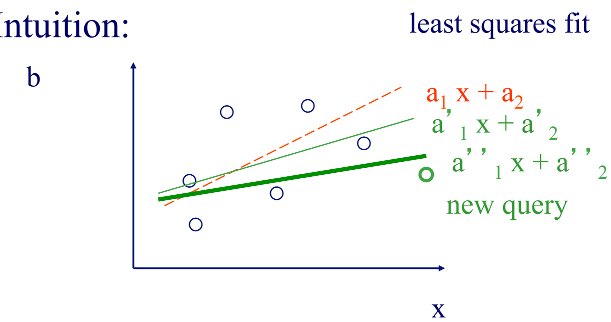
Query feedbacks - enhancements

Intuition:



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Query feedbacks - conclusions

SVD helps find the Least Squares solution, to adapt to query feedbacks

(RLS = Recursive Least Squares is a great method to incrementally update least-squares fits)

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SVD - detailed outline

- ...
- Case studies
- SVD properties
- more case studies
  - google/Kleinberg algorithms
  - query feedbacks
- Conclusions

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Conclusions

- SVD: a **valuable** tool
- given a document-term matrix, it finds 'concepts' (LSI)
- ... and can reduce dimensionality (KL)
- ... and can find rules (PCA; RatioRules)

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Conclusions cont'd

- ... and can find fixed-points or steady-state probabilities (google/ Kleinberg/ Markov Chains)
- ... and can solve optimally over- and under-constraint linear systems (least squares / query feedbacks)

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