Math-UA.233: Theory of Probability
Lecture 14

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From last time: continuous RVs

If X is a continuous RV, then:

(Definition) It has a **probability density function** (‘**PDF**’) f:

$$P(a \leqslant X \leqslant b) = \int_a^b f(x)\,dx \quad \text{(c.f. discrete: } \textstyle\sum_{i \text{ such that } a \leqslant x_i \leqslant b} p(x_i) \text{)}$$

(Definition) **Expectation**:

$$E[X] = \int_{-\infty}^{\infty} x f(x)\,dx \quad \text{(c.f. discrete: } \textstyle\sum_i x_i p(x_i)\text{)}$$

(Theorem: LOTUS)

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) f(x)\,dx \quad \text{(c.f. discrete: } \textstyle\sum_i g(x_i) p(x_i) \text{)}$$

A special case of LOTUS:

$$E[aX + b] = aE[X] + b$$

for any continuous RV X and real numbers a and b.

Next we turn to the variance.

Definition (Ross p183)

*If X is a continuous RV with PDF f, and if we let $\mu = E[X]$, then its **variance** is the number*

$$\mathrm{Var}(X) = E\big[(X - \mu)^2\big].$$

(Identical to what we wrote for discrete RVs)

Letting $g(x) = (x - \mu)^2$ and applying LOTUS, we get

$$\mathrm{Var}(X) = \int_{-\infty}^{\infty} (x - \mu)^2 f(x)\, dx,$$

whenever this integral is well-defined.

Just as for discrete RVs, we can now deduce the alternative formula

$$\mathrm{Var}(X) = E[X^2] - (E[X])^2,$$

which is sometimes simpler to use.

Example (Ross 5.2e)

Find $\mathrm{Var}(X)$ *when the PDF of* X *is*

$$f(x) = \begin{cases} 2x & \text{if } 0 \leqslant x \leqslant 1 \\ 0 & \text{otherwise.} \end{cases}$$

Example (Ross E.g. 5.3a, part (b))

Find $\mathrm{Var}(X)$ *when* X *is Unif*(c, d)*.*

Sometimes we need to know how variance changes when we apply a linear function to a RV.

Proposition (Ross p183)

For any constants a and b,

$$\mathrm{Var}(aX + b) = a^2 \mathrm{Var}(X).$$

Finding the distribution of $g(X)$

We have seen a simple formula for $E[g(X)]$. But sometimes we need more information about $g(X)$, such as its CDF.

That is, we want to determine the function

$$F_{g(X)}(a) = P(g(X) \leqslant a)$$

in terms of the PDF of X itself.

In general, we can do this by expressing the *event* $\{g(X) \leqslant a\}$ in terms of events of the form $\{c \leqslant X \leqslant d\}$, and then compute using the probabilities of those (e.g. using F_X.)

Example (Ross E.g. 5.1d)

Suppose X is continuous with PDF f_X. Show that $Y = 2X$ is also continuous, and find its PDF.

Example (Warning!)

Let X be Unif$(-1, 1)$, and let

$$g(x) = \begin{cases} 0 & \text{if } x \leqslant 0 \\ x & \text{if } x \geqslant 0. \end{cases}$$

Then $g(X)$ is not discrete or continuous!

In the last example, $g(X)$ is a **hybrid**: it behaves like a discrete RV on the event $\{X < 0\}$, and like a continuous one on the event $\{X \geqslant 0\}$. But there are even more complex examples which are actually neither. (Try looking up 'Cantor distribution' on Wikipedia.)

Example (Ross E.g. 5.7a)

Let X be Unif$(0,1)$ and let n be a positive integer. Find the CDF of $Y = X^n$. Is Y a continuous RV? If so, what is its PDF?

$$\textit{ANS:} \quad F_Y(a) = a^{1/n}; \quad \textit{Yes;} \quad f_Y(y) = (1/n)y^{1/n-1}.$$

The new RV $g(X)$ can be hard to understand in general. But there are conditions on g which guarantee good properties.

Theorem (See Ross Theorem 5.7.1)

Suppose X is continuous with PDF f_X, and that it always takes values in the interval $[a, b]$.

Assume $g(x)$ is (i) strictly increasing and (ii) differentiable on $[a, b]$.

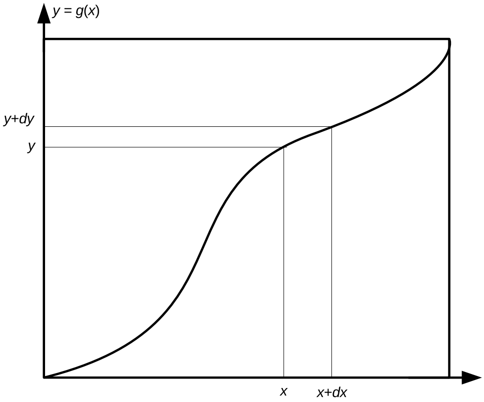
Then $Y = g(X)$ is also continuous, $g(a) \leqslant Y \leqslant g(b)$, and

$$\boxed{f_Y(y) = f_X(g^{-1}(y)) \cdot (g^{-1})'(y) \quad \text{for } g(a) \leqslant y \leqslant g(b)}.$$

This is all still OK if $a = -\infty$ or $b = +\infty$.

IDEA: Find F_Y in terms of F_X, then differentiate to find f_Y.

Accompanying picture:



We have $x = g^{-1}(y)$, so $dx = (g^{-1})'(y)dy$, and

$$f_Y(y)dy \approx P(Y \in [y, y + dy]) = P(X \in [x, x + dx]) \approx f_X(x)dx.$$

If you want to apply the previous theorem, it's very important to *check the conditions*.

Example

Let $n \geqslant 1$ be an integer and let X be continuous and non-negative, with PDF f. Let $Y = X^n$. Find f_Y.

NOTE: If n is even, then the function $g(x) = x^n$ isn't increasing (or decreasing) on the whole real line, but it *is* increasing on $[0, \infty)$, which is where X takes its values.

Something we did last time...

Example (Ross E.g. 5.2b)

Let X be Unif$(0,1)$. Find $E[e^X]$.

If the conditions aren't met, then that theorem can *fail*!

Example (Ross E.g. 5.7c)

Suppose X is a continuous RV with PDF f_X. Then $Y = |X|$ is also continuous, and its PDF is 0 *for $y \leqslant 0$ and*

$$f_Y(y) = f_X(y) + f_X(-y) \quad \text{for } y \geqslant 0.$$

Example (Ross E.g. 5.7b)

Suppose X is a continuous RV with PDF f_X. Then $Y = X^2$ is also continuous, and its PDF is 0 *for $y \leqslant 0$ and*

$$f_Y(y) = \frac{f_X(\sqrt{y}) + f_X(-\sqrt{y})}{2\sqrt{y}} \quad \text{for } y \geqslant 0.$$

Here’s an interesting example whose full understanding involves transformations of RVs.

Example (Bertrand’s paradox; Ross E.g. 5.3d)

Consider a random chord of a circle. What is the probability that the length of the chord will be greater than the side of the equilateral triangle inscribed in that circle?

THE PROBLEM: We can interpret ‘random chord of a circle’ in two different ways.

Normal RVs (Ross Sec 5.4)

It's time to meet the next important family of RVs.

Definition

*Let μ be a real value and $\sigma > 0$. A RV X is **normal with parameters** μ and σ^2, or just $\mathbf{N}(\mu, \sigma^2)$, if it is continuous with PDF*

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2} \quad \text{for } -\infty < x < \infty.$$

*It is **standard normal** if it is $N(0, 1)$.*

These are important because they appear in another powerful approximation theorem for binomial RVs — more on that later. For now we study their basic properties.

First, why the constant $\frac{1}{\sqrt{2\pi}\sigma}$?

Since f is a PDF it needs to satisfy $\int_{-\infty}^{\infty} f(x)\,dx = 1$. It turns out (!) that

$$\boxed{\int_{-\infty}^{\infty} e^{-(x-\mu)^2/2\sigma^2}\,dx = \sigma \int_{-\infty}^{\infty} e^{-y^2/2}\,dy = \sigma\sqrt{2\pi}.}$$

(PROOF: Curious trick using polar coordinates.)