36-705 Intermediate Statistics Homework #7 Solutions

November 3, 2016

Problem 1 [25 pts.]

Let $X_1, \dots, X_n \sim N(\mu, \sigma^2)$. Let $\theta = \mu/\sigma$. Construct an asymptotic $1-\alpha$ confidence interval for θ.

We know that $\dfrac{\hat{\theta}-\theta}{se} \rightsquigarrow N(0,1)$. First, let us find the mle $\hat{\theta}$. In HW 6 (problem 4), we showed that $\hat{\mu} = \bar{X}_n$ and $\hat{\sigma} = \sqrt{\frac{\sum_{i=1}^n (X_i - \bar{X}_n)^2}{n}}$. By the equivariance of the mle, $\hat{\theta} = \frac{\hat{\mu}}{\hat{\sigma}}$.

Next, let us calculate $se(\hat{\theta})$. Where $\theta = g(\mu, \sigma) = \frac{\mu}{\sigma}$, we know that

$$\begin{aligned}
se(\hat{\theta}) &= \sqrt{(\triangledown g)^T J_n (\triangledown g)}, \\
J_n &= I_n^{-1}(\mu, \sigma), \\
\triangledown g &= \begin{pmatrix} \frac{\partial g}{\partial \mu} \\ \frac{\partial g}{\partial \sigma} \end{pmatrix}.
\end{aligned}$$

We see that

$$\frac{\partial g}{\partial \mu} = \frac{1}{\sigma} \qquad \text{and} \qquad \frac{\partial g}{\partial \sigma} = -\frac{\mu}{\sigma^2}.$$

To find J_n, let us start by finding $\ell_n(\mu, \sigma)$.

$$\begin{aligned}
L_n(\mu, \sigma) &= \prod_{i=1}^n \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{ -\frac{1}{2\sigma^2}(X_i - \mu)^2 \right\} \\
\ell_n(\mu, \sigma) &= \sum_{i=1}^n \left[-\log(\sqrt{2\pi}) - \log(\sigma) - \frac{1}{2\sigma^2}(X_i - \mu)^2 \right].
\end{aligned}$$

Then

$$\begin{aligned}
\mathbb{E}\left[\frac{\partial^2 \ell_n}{\partial \mu^2}\right] &= \mathbb{E}\left[\frac{\partial}{\partial \mu} \sum_{i=1}^n \left[-\frac{1}{\sigma^2}(X_i - \mu)(-1) \right]\right] \\
&= \mathbb{E}\left[\frac{\partial}{\partial \mu} \sum_{i=1}^n \left[\frac{1}{\sigma^2}(X_i - \mu) \right]\right] \\
&= \mathbb{E}\left[\sum_{i=1}^n \left[-\frac{1}{\sigma^2} \right]\right] \\
&= \mathbb{E}\left[-\frac{n}{\sigma^2} \right] \\
&= -\frac{n}{\sigma^2}.
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}\left[\frac{\partial^2 \ell_n}{\partial \sigma^2}\right] &= \mathbb{E}\left[\frac{\partial}{\partial \sigma}\sum_{i=1}^n\left[-\frac{1}{\sigma}+\frac{1}{\sigma^3}(X_i-\mu)^2\right]\right] \\
&= \mathbb{E}\left[\sum_{i=1}^n\left[\frac{1}{\sigma^2}-\frac{3}{\sigma^4}(X_i-\mu)^2\right]\right] \\
&= \mathbb{E}\left[\frac{n}{\sigma^2}-\frac{3}{\sigma^4}\sum_{i=1}^n(X_i-\mu)^2\right] \\
&= \mathbb{E}\left[\frac{n}{\sigma^2}-\frac{3n\sigma^2}{\sigma^4}\right] \\
&= \frac{n}{\sigma^2}-\frac{3n}{\sigma^2} \\
&= -\frac{2n}{\sigma^2}.
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}\left[\frac{\partial^2 \ell_n}{\partial \mu \partial \sigma}\right] &= \mathbb{E}\left[\frac{\partial}{\partial \mu}\sum_{i=1}^n\left[-\frac{1}{\sigma}+\frac{1}{\sigma^3}(X_i-\mu)^2\right]\right] \\
&= \mathbb{E}\left[\sum_{i=1}^n\frac{2}{\sigma^3}(X_i-\mu)(-1)\right] \\
&= \mathbb{E}\left[-\frac{2}{\sigma^3}\sum_{i=1}^n(X_i-\mu)\right] \\
&= 0.
\end{aligned}$$

So

$$I_n(\mu,\sigma) = -\begin{bmatrix} \mathbb{E}\left[\frac{\partial^2 \ell_n}{\partial \mu^2}\right] & \mathbb{E}\left[\frac{\partial^2 \ell_n}{\partial \mu \partial \sigma}\right] \\ \mathbb{E}\left[\frac{\partial^2 \ell_n}{\partial \mu \partial \sigma}\right] & \mathbb{E}\left[\frac{\partial^2 \ell_n}{\partial \sigma^2}\right] \end{bmatrix} = \begin{bmatrix} \frac{n}{\sigma^2} & 0 \\ 0 & \frac{2n}{\sigma^2} \end{bmatrix}.$$

Then

$$\begin{aligned}
J_n = I_n^{-1}(\mu,\sigma) &= \frac{1}{(\frac{n}{\sigma^2})(\frac{2n}{\sigma^2})-0(0)}\begin{bmatrix} \frac{2n}{\sigma^2} & 0 \\ 0 & \frac{n}{\sigma^2} \end{bmatrix} \\
&= \frac{\sigma^4}{2n^2}\begin{bmatrix} \frac{2n}{\sigma^2} & 0 \\ 0 & \frac{n}{\sigma^2} \end{bmatrix} \\
&= \begin{bmatrix} \frac{\sigma^2}{n} & 0 \\ 0 & \frac{\sigma^2}{2n} \end{bmatrix}.
\end{aligned}$$

So

$$\begin{aligned}
[se(\hat{\theta})]^2 &= (\triangledown g)^T J_n (\triangledown g) \\
&= \begin{pmatrix} \frac{1}{\sigma} & \frac{-\mu}{\sigma^2} \end{pmatrix}\begin{bmatrix} \frac{\sigma^2}{n} & 0 \\ 0 & \frac{\sigma^2}{2n} \end{bmatrix}\begin{pmatrix} \frac{1}{\sigma} \\ \frac{-\mu}{\sigma^2} \end{pmatrix} \\
&= \begin{pmatrix} \frac{\sigma}{n} & \frac{-\mu}{2n} \end{pmatrix}\begin{pmatrix} \frac{1}{\sigma} \\ \frac{-\mu}{\sigma^2} \end{pmatrix} \\
&= \frac{1}{n}+\frac{\mu^2}{2n\sigma^2}.
\end{aligned}$$

Then

$$\widehat{se}(\hat{\theta}) = \sqrt{\frac{1}{n} + \frac{\widehat{\mu}^2}{2n\widehat{\sigma}^2}}.$$

We conclude that an approximate $1 - \alpha$ confidence interval for θ is

$$\frac{\hat{\mu}}{\hat{\sigma}} \pm z_{\alpha/2}\sqrt{\frac{1}{n} + \frac{\hat{\mu}^2}{2n\hat{\sigma}^2}}.$$

Problem 2

Suppose that $X_1, \dots, X_n \sim N(\mu, \Sigma)$ are multivariate Normal, where $X_i \in \mathbb{R}^k$. Assume that Σ is known. Let

$$C_n = \left\{\mu : (\bar{X}_n - \mu)^T \Sigma^{-1} (\bar{X}_n - \mu) \le t\right\}.$$

Find t so that C_n is a $1 - \alpha$ confidence set for μ. Hint: We can write $X_i = \mu + \Sigma^{1/2}\epsilon_i$ where $\epsilon_i \sim N(0, I)$ and I is the $k \times k$ identity matrix.
We note that

$$\bar{X}_n = \frac{1}{n}\sum_{i=1}^n X_i = \frac{1}{n}\sum_{i=1}^n \left[\mu + \Sigma^{1/2}\epsilon_i\right] = \mu + \frac{1}{n}\sum_{i=1}^n \left(\Sigma^{1/2}\epsilon_i\right) = \mu + \frac{1}{n}\Sigma^{1/2}\left(\sum_{i=1}^n \epsilon_i\right).$$

Then

$$\bar{X}_n - \mu = \frac{1}{n}\Sigma^{1/2}\left(\sum_{i=1}^n \epsilon_i\right).$$

We determine

$$\begin{aligned}
(\bar{X}_n - \mu)^T \Sigma^{-1} (\bar{X}_n - \mu) &= \left[\frac{1}{n}\Sigma^{1/2}\left(\sum_{i=1}^n \epsilon_i\right)\right]^T \Sigma^{-1} \left[\frac{1}{n}\Sigma^{1/2}\left(\sum_{i=1}^n \epsilon_i\right)\right] \\
&= \frac{1}{n^2}\left(\sum_{i=1}^n \epsilon_i\right)^T \left(\Sigma^{1/2}\right)^T \Sigma^{-1} \left(\Sigma^{1/2}\right)\left(\sum_{i=1}^n \epsilon_i\right) \\
&= \frac{1}{n^2}\left(\sum_{i=1}^n \epsilon_i\right)^T \left(\Sigma^{1/2}\right)\left(\Sigma^{-1/2}\right)\left(\sum_{i=1}^n \epsilon_i\right) \\
&= \frac{1}{n^2}\left(\sum_{i=1}^n \epsilon_i\right)^T \left(\sum_{i=1}^n \epsilon_i\right) \\
&= \frac{1}{n^2}\begin{pmatrix} \sum_{i=1}^n \epsilon_{i1} & \sum_{i=1}^n \epsilon_{i2} & \cdots & \sum_{i=1}^n \epsilon_{ik} \end{pmatrix} \begin{pmatrix} \sum_{i=1}^n \epsilon_{i1} \\ \sum_{i=1}^n \epsilon_{i2} \\ \vdots \\ \sum_{i=1}^n \epsilon_{ik} \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{n}\sum_{i=1}^n \epsilon_{i1} & \frac{1}{n}\sum_{i=1}^n \epsilon_{i2} & \cdots & \frac{1}{n}\sum_{i=1}^n \epsilon_{ik} \end{pmatrix} \begin{pmatrix} \frac{1}{n}\sum_{i=1}^n \epsilon_{i1} \\ \frac{1}{n}\sum_{i=1}^n \epsilon_{i2} \\ \vdots \\ \frac{1}{n}\sum_{i=1}^n \epsilon_{ik} \end{pmatrix}.
\end{aligned}$$

Then

$$n(\bar{X}_n - \mu)^T \Sigma^{-1} (\bar{X}_n - \mu) = \begin{pmatrix} \frac{1}{\sqrt{n}}\sum_{i=1}^n \epsilon_{i1} & \frac{1}{\sqrt{n}}\sum_{i=1}^n \epsilon_{i2} & \cdots & \frac{1}{\sqrt{n}}\sum_{i=1}^n \epsilon_{ik} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{n}}\sum_{i=1}^n \epsilon_{i1} \\ \frac{1}{\sqrt{n}}\sum_{i=1}^n \epsilon_{i2} \\ \vdots \\ \frac{1}{\sqrt{n}}\sum_{i=1}^n \epsilon_{ik} \end{pmatrix}.$$

For $j \in \{1, \dots, k\}$, $\displaystyle\frac{1}{\sqrt{n}}\sum_{i=1}^n \epsilon_{ij} \sim N(0,1)$. So $\displaystyle\left(\frac{1}{\sqrt{n}}\sum_{i=1}^n \epsilon_{ij}\right)^2 \sim \chi^2_1$.
Then $n(\bar{X}_n - \mu)^T \Sigma^{-1} (\bar{X}_n - \mu) \sim \chi^2_k$. Let $\chi^2_{k,\alpha}$ be the upper α cutoff of the χ^2_k distribution. We conclude that if $t = \dfrac{\chi^2_{k,\alpha}}{n}$, then C_n is a $1 - \alpha$ confidence set for μ.

Problem 3 [25 pts.]

Let $X_1, \ldots, X_n \sim F$. Recall that the empirical cdf is

$$F_n(t) = \frac{1}{n}\sum_{i=1}^{n} I(X_i \leq t).$$

Let $t_1 < t_2 < \ldots < t_k$ be k fixed points on the real line. Let

$$Z_n = \sqrt{n}\Big(F_n(t_1) - F(t_1), \ldots, F_n(t_k) - F(t_k)\Big).$$

Show that

$$Z_n \rightsquigarrow N(\mathbf{0}, \Sigma)$$

where $\mathbf{0} = (0, \ldots, 0)$ and Σ is a $k \times k$ matrix with $\Sigma_{j\ell} = F(t_j \wedge t_\ell) - F(t_j)F(t_\ell)$.
For $i = 1, \ldots, n$, we let

$$Y_i = \begin{pmatrix} W_{1i} \\ W_{2i} \\ \vdots \\ W_{ki} \end{pmatrix} = \begin{pmatrix} I(X_i \leq t_1) \\ I(X_i \leq t_2) \\ \vdots \\ I(X_i \leq t_k) \end{pmatrix}.$$

Then

$$\overline{Y} = \begin{pmatrix} \overline{W}_1 \\ \overline{W}_2 \\ \vdots \\ \overline{W}_k \end{pmatrix} = \begin{pmatrix} \frac{1}{n}\sum_{i=1}^{n} I(X_i \leq t_1) \\ \frac{1}{n}\sum_{i=1}^{n} I(X_i \leq t_2) \\ \vdots \\ \frac{1}{n}\sum_{i=1}^{n} I(X_i \leq t_k) \end{pmatrix} = \begin{pmatrix} F_n(t_1) \\ F_n(t_2) \\ \vdots \\ F_n(t_k) \end{pmatrix}.$$

For $j \in \{1, \ldots, k\}$,

$$\begin{aligned} \mathbb{E}[W_{ji}] &= \mathbb{E}[I(X_i \leq t_j)] \\ &= \mathbb{P}(X_i \leq t_j) \\ &= F(t_j). \end{aligned}$$

So the mean vector μ is given by

$$\mu = \begin{pmatrix} F(t_1) \\ F(t_2) \\ \vdots \\ F(t_k) \end{pmatrix}.$$

$Y_1, Y_2, \ldots, Y_n$ has a $k \times k$ covariance matrix Σ, where $\Sigma_{j\ell} = cov(W_j, W_\ell)$ for each $j, \ell \in \{1, \ldots, k\}$.
So

$$\begin{aligned} \Sigma_{j\ell} &= cov(W_j, W_\ell) \\ &= \mathbb{E}[W_j W_\ell] - \mathbb{E}[W_j]\mathbb{E}[W_\ell] \\ &= \mathbb{E}[W_j W_\ell] - F(t_j)F(t_\ell) \\ &= \mathbb{E}[I(X_i \leq t_j)I(X_i \leq t_\ell)] - F(t_j)F(t_\ell) \\ &= \mathbb{E}[I(X_i \leq \min(t_j, t_\ell)] - F(t_j)F(t_\ell) \\ &= \mathbb{P}(X_i \leq \min(t_j, t_\ell)) - F(t_j)F(t_\ell) \\ &= F(t_j \wedge t_\ell) - F(t_j)F(t_\ell). \end{aligned}$$

By the Multivariate CLT, we see that $\sqrt{n}(\overline{Y} - \mu) \rightsquigarrow N(\mathbf{0}, \Sigma)$. We conclude that $\sqrt{n}\Big(F_n(t_1) - F(t_1), \ldots, F_n(t_k) - F(t_k)\Big) \rightsquigarrow N(\mathbf{0}, \Sigma)$, where Σ is a $k \times k$ matrix with $\Sigma_{j\ell} = F(t_j \wedge t_\ell) - F(t_j)F(t_\ell)$.

Problem 4 [25 pts.]

Let $X_1, \ldots, X_n \sim \text{Exponential}(\theta)$. Find the size α, asymptotic, LRT test for

$$H_0 : \theta = \theta_0 \quad \text{versus} \quad H_1 : \theta \neq \theta_0.$$

By inverting the test, construct an asymptotic $1-\alpha$ confidence set for θ. Now construct the $1-\alpha$ Wald confidence interval for θ.

Likelihood Ratio Test

We have the likelihood function

$$L(\theta) = \prod_{i=1}^{n} \frac{1}{\theta} e^{-x_i/\theta} = \frac{1}{\theta^n} e^{-\sum_{i=1}^n x_i/\theta} = \frac{1}{\theta^n} e^{-n\bar{X}_n/\theta}.$$

Then

$$\ell(\theta) = -n\log(\theta) - \frac{n\bar{X}_n}{\theta}.$$

We solve for the mle $\hat{\theta}$.

$$\begin{aligned} \ell'(\theta) = -\frac{n}{\theta} + \frac{n\bar{X}_n}{\theta^2} &\overset{set}{=} 0 \\ \frac{n}{\theta} &= \frac{n\bar{X}_n}{\theta^2} \\ 1 &= \frac{\bar{X}_n}{\theta} \\ \hat{\theta} &= \bar{X}_n. \end{aligned}$$

Since $\dim(\Theta) - \dim(\Theta_0) = 1 - 0 = 1$, we don't reject the LRT if

$$\begin{aligned} -2\log\left(\frac{L(\theta_0)}{L(\hat{\theta})}\right) &\leq \chi^2_{1,\alpha} && \Leftrightarrow \\ \log\left(\frac{L(\theta_0)}{L(\hat{\theta})}\right) &\geq -\frac{\chi^2_{1,\alpha}}{2} && \Leftrightarrow \\ \ell(\theta_0) - \ell(\hat{\theta}) &\geq -\frac{\chi^2_{1,\alpha}}{2} && \Leftrightarrow \\ -n\log(\theta_0) - \frac{n\bar{X}_n}{\theta_0} + n\log(\hat{\theta}) + \frac{n\bar{X}_n}{\hat{\theta}} &\geq -\frac{\chi^2_{1,\alpha}}{2} && \Leftrightarrow \\ -n\log(\theta_0) - \frac{n\bar{X}_n}{\theta_0} + n\log(\bar{X}_n) + n &\geq -\frac{\chi^2_{1,\alpha}}{2} \end{aligned}$$

Therefore, the $1-\alpha$ confidence set for θ from the LRT is

$$C_n = \left\{\theta : -n\log(\theta) - \frac{n\bar{X}_n}{\theta} + n\log(\bar{X}_n) + n \geq -\frac{\chi^2_{1,\alpha}}{2}\right\}.$$

Wald Test

The $1-\alpha$ Wald confidence interval for θ is given by $\hat{\theta} \pm z_{\alpha/2} se$. We found that $\hat{\theta} = \bar{X}_n$. Now let us find $se = 1/\sqrt{I_n(\hat{\theta})}$. We see that

$$\begin{aligned}
I_n(\theta) &= -\mathbb{E}\left[\frac{\partial^2}{\partial\theta^2}\ell_n(\theta)\right] \\
&= -\mathbb{E}\left[\frac{\partial^2}{\partial\theta^2}\left(-n\log(\theta) - \frac{n\bar{X}_n}{\theta}\right)\right] \\
&= -\mathbb{E}\left[\frac{\partial}{\partial\theta}\left(-\frac{n}{\theta} + \frac{n\bar{X}_n}{\theta^2}\right)\right] \\
&= -\mathbb{E}\left[\frac{n}{\theta^2} - \frac{2n\bar{X}_n}{\theta^3}\right] \\
&= -\frac{n}{\theta^2} + \frac{2n}{\theta^3}\mathbb{E}[\bar{X}_n] \\
&= -\frac{n}{\theta^2} + \frac{2n\theta}{\theta^3} \\
&= -\frac{n}{\theta^2} + \frac{2n}{\theta^2} \\
&= \frac{n}{\theta^2}.
\end{aligned}$$

So

$$\begin{aligned}
se &= \frac{1}{\sqrt{I_n(\bar{X}_n)}} \\
&= \frac{1}{\sqrt{\frac{n}{\bar{X}_n^2}}} \\
&= \frac{|\bar{X}_n|}{n} \\
&= \frac{\bar{X}_n}{n} \qquad \text{b/c all } X_i > 0
\end{aligned}$$

Thus, the $1-\alpha$ Wald confidence interval is

$$\bar{X}_n \pm z_{\alpha/2}\left(\frac{\bar{X}_n}{n}\right).$$

Problem 5 [25 pts.]

Let $X_1, \dots, X_n \sim \text{Poisson}(\lambda_1)$ and let $Y_1, \dots, Y_m \sim \text{Poisson}(\lambda_2)$. Find an asymptotic $1-\alpha$ confidence interval for $\lambda_1 - \lambda_2$.
First, let us find the mle $\hat{\lambda}_1$. We have the likelihood function

$$L(\lambda_1) = \prod_{i=1}^{n} \frac{e^{-\lambda_1}\lambda_1^{X_i}}{X_i!}.$$

So

$$\begin{aligned}
\ell(\lambda_1) &= \sum_{i=1}^{n} \Big[-\lambda_1 + X_i \log(\lambda_1) - \log(X_i!) \Big] \\
\frac{\partial \ell}{\partial \lambda_1} &= \sum_{i=1}^{n} \left[-1 + \frac{X_i}{\lambda_1} \right] \overset{set}{=} 0 \\
\frac{\sum_{i=1}^{n} X_i}{\lambda_1} &= n \\
\hat{\lambda}_1 &= \bar{X}_n.
\end{aligned}$$

Similarly, $\hat{\lambda}_2 = \bar{Y}_m$. Let $\theta = \lambda_1 - \lambda_2$. By the equivariance of the mle, $\hat{\theta} = \hat{\lambda}_1 - \hat{\lambda}_2 = \bar{X}_n - \bar{Y}_m$.
We know that $\dfrac{\hat{\theta} - \theta}{se} \rightsquigarrow N(0,1)$. Next, let us find $se(\hat{\theta})$.
Where $\theta = g(\lambda_1, \lambda_2) = \lambda_1 - \lambda_2$, we know that

$$\begin{aligned}
se(\hat{\theta}) &= \sqrt{(\triangledown g)^T J_{n,m} (\triangledown g)}, \\
J_{n,m} &= I_{n,m}^{-1}(\lambda_1, \lambda_2) \\
\triangledown g &= \begin{pmatrix} \frac{\partial g}{\partial \lambda_1} & \frac{\partial g}{\partial \lambda_2} \end{pmatrix}.
\end{aligned}$$

We see that

$$\frac{\partial g}{\partial \lambda_1} = 1 \qquad \text{and} \qquad \frac{\partial g}{\partial \lambda_2} = -1.$$

To find $J_{n,m}$, let us start by finding $\ell_{n,m}(\lambda_1, \lambda_2)$.

$$\begin{aligned}
L_{n,m}(\lambda_1, \lambda_2) &= \left(\prod_{i=1}^{n} \frac{e^{-\lambda_1}\lambda_1^{X_i}}{X_i!} \right) \left(\prod_{j=1}^{m} \frac{e^{-\lambda_2}\lambda_2^{Y_j}}{Y_j!} \right) \\
\ell_{n,m}(\lambda_1, \lambda_2) &= \sum_{i=1}^{n} \Big[-\lambda_1 + X_i \log(\lambda_1) - \log(X_i!) \Big] + \sum_{j=1}^{m} \Big[-\lambda_2 + Y_j \log(\lambda_2) - \log(Y_j!) \Big].
\end{aligned}$$

So

$$\mathbb{E}\left[\frac{\partial^2 \ell}{\partial \lambda_1^2} \right] = \mathbb{E}\left[\frac{\partial}{\partial \lambda_1} \sum_{i=1}^{n} \left[-1 + \frac{X_i}{\lambda_1} \right] \right] = \mathbb{E}\left[\sum_{i=1}^{n} \frac{-X_i}{\lambda_1^2} \right] = -\frac{n}{\lambda_1^2}\mathbb{E}[X_i] = -\frac{n}{\lambda_1}.$$

$$\mathbb{E}\left[\frac{\partial^2 \ell}{\partial \lambda_2^2} \right] = \mathbb{E}\left[\frac{\partial}{\partial \lambda_2} \sum_{j=1}^{m} \left[-1 + \frac{Y_j}{\lambda_2} \right] \right] = \mathbb{E}\left[\sum_{j=1}^{m} \frac{-Y_j}{\lambda_2^2} \right] = -\frac{m}{\lambda_2^2}\mathbb{E}[Y_j] = -\frac{m}{\lambda_2}.$$

$$\mathbb{E}\left[\frac{\partial^2 \ell}{\partial \lambda_1 \partial \lambda_2} \right] = 0.$$

Then

$$I_{n,m}(\lambda_1,\lambda_2) = -\begin{bmatrix} \mathbb{E}\left[\frac{\partial^2 \ell}{\partial \lambda_1^2}\right] & \mathbb{E}\left[\frac{\partial^2 \ell}{\partial \lambda_1 \partial \lambda_2}\right] \\ \mathbb{E}\left[\frac{\partial^2 \ell}{\partial \lambda_1 \partial \lambda_2}\right] & \mathbb{E}\left[\frac{\partial^2 \ell}{\partial \lambda_2^2}\right] \end{bmatrix} = \begin{bmatrix} \frac{n}{\lambda_1} & 0 \\ 0 & \frac{m}{\lambda_2} \end{bmatrix}.$$

So

$$J_{n,m}(\lambda_1,\lambda_2) = I_{n,m}^{-1}(\lambda_1,\lambda_2) = \frac{1}{(\frac{n}{\lambda_1})(\frac{m}{\lambda_2})}\begin{bmatrix} \frac{m}{\lambda_2} & 0 \\ 0 & \frac{n}{\lambda_1} \end{bmatrix} = \frac{\lambda_1\lambda_2}{nm}\begin{bmatrix} \frac{m}{\lambda_2} & 0 \\ 0 & \frac{n}{\lambda_1} \end{bmatrix} = \begin{bmatrix} \frac{\lambda_1}{n} & 0 \\ 0 & \frac{\lambda_2}{m} \end{bmatrix}.$$

We compute

$$[se(\hat{\lambda}_1 - \hat{\lambda}_2)]^2 = (\triangledown g)^T J_{n,m}(\triangledown g) = \begin{pmatrix} 1 & -1 \end{pmatrix}\begin{bmatrix} \frac{\lambda_1}{n} & 0 \\ 0 & \frac{\lambda_2}{m} \end{bmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} \frac{\lambda_1}{n} & \frac{-\lambda_2}{m} \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{\lambda_1}{n} + \frac{\lambda_2}{m}.$$

Then

$$\widehat{se}(\hat{\lambda}_1 - \hat{\lambda}_2) = \sqrt{\frac{\hat{\lambda}_1}{n} + \frac{\hat{\lambda}_2}{m}}.$$

We conclude that an approximate $1-\alpha$ confidence interval for $\lambda_1 - \lambda_2$ is

$$(\bar{X}_n - \bar{Y}_m) \pm z_{\alpha/2}\sqrt{\frac{\hat{\lambda}_1}{n} + \frac{\hat{\lambda}_2}{m}}.$$