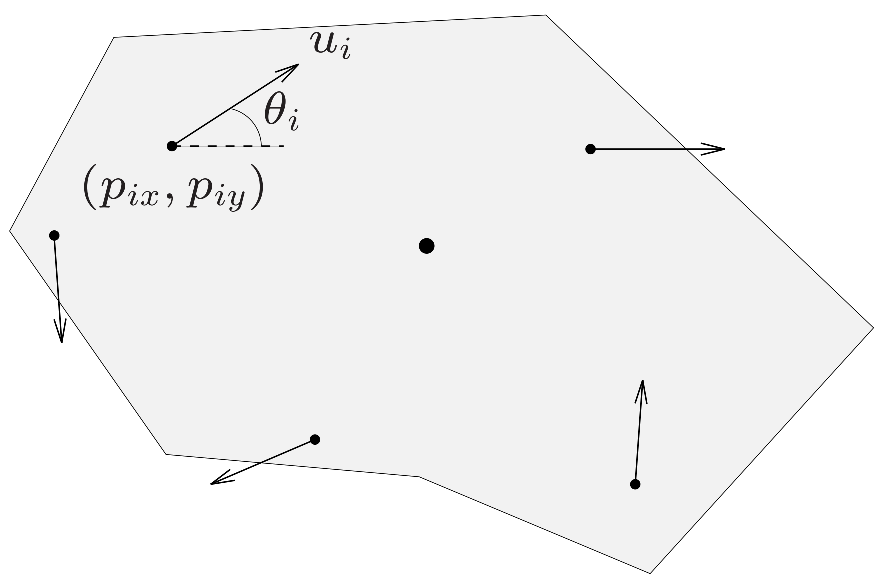
Convex optimization examples

- force/moment generation with thrusters
- minimum-time optimal control
- optimal transmitter power allocation
- phased-array antenna beamforming
- optimal receiver location
- power allocation in FDM system
- optimizing structural dynamics

Force/moment generation with thrusters

- rigid body with center of mass at origin $p = 0 \in \mathbf{R}^2$
- n forces with magnitude u_i, acting at $p_i = (p_{ix}, p_{iy})$, in direction θ_i



- resulting horizontal force: $F_x = \sum_{i=1}^n u_i \cos\theta_i$
- resulting vertical force: $F_y = \sum_{i=1}^n u_i \sin\theta_i$
- resulting torque: $T = \sum_{i=1}^n (p_{iy} u_i \cos\theta_i - p_{ix} u_i \sin\theta_i)$
- force limits: $0 \le u_i \le 1$ (thrusters)
- fuel usage: $u_1 + \cdots + u_n$

problem: find thruster forces u_i that yield given desired forces and torques F_x^{des}, F_y^{des}, T^{des}, and minimize fuel usage (if feasible) can be expressed as LP:

$$\begin{array}{ll} \text{minimize} & \mathbf{1}^T u \\ \text{subject to} & Fu = f^{\mathsf{des}} \\ & 0 \leq u_i \leq 1, \; i = 1, \ldots, n \end{array}$$

where

$$F = \left[\begin{array}{ccc} \cos\theta_1 & \cdots & \cos\theta_n \\ \sin\theta_1 & \cdots & \sin\theta_n \\ p_{1y}\cos\theta_1 - p_{1x}\sin\theta_1 & \cdots & p_{ny}\cos\theta_n - p_{nx}\sin\theta_n \end{array} \right],$$

$$f^{\mathsf{des}} = (F_x^{\mathsf{des}}, \; F_y^{\mathsf{des}}, \; T^{\mathsf{des}}), \quad \mathbf{1} = (\; 1, \; 1, \; \cdots \; 1 \;)$$

Extensions of thruster problem

- opposing thruster pairs:

$$\begin{array}{ll} \text{minimize} & \|u\|_1 = \sum_{i=1}^n |u_i| \\ \text{subject to} & Fu = f^{\text{des}} \\ & |u_i| \leq 1, \quad i = 1, \ldots, n \end{array}$$

  can express as LP

- more accurate fuel use model:

$$\begin{array}{ll} \text{minimize} & \sum_{i=1}^n \phi_i(u_i) \\ \text{subject to} & Fu = f^{\text{des}} \\ & 0 \leq u_i \leq 1, \quad i = 1, \ldots, n \end{array}$$

  ϕ_i are piecewise linear increasing convex functions
  can express as LP

- minimize maximum force/moment error:

$$\begin{array}{ll} \text{minimize} & \|Fu - f^{\text{des}}\|_\infty \\ \text{subject to} & 0 \leq u_i \leq 1, \quad i = 1, \ldots, n \end{array}$$

can express as LP

- minimize number of thrusters used:

$$\begin{array}{ll} \text{minimize} & \#\ \text{thrusters on} \\ \text{subject to} & Fu = f^{\text{des}} \\ & 0 \leq u_i \leq 1, \quad i = 1, \ldots, n \end{array}$$

can't express as LP
(but we could check feasibility of each of the 2^n subsets of thrusters)

Minimum-time optimal control

- linear dynamical system:

$$x(t+1) = Ax(t) + Bu(t), \quad t = 0, 1, \ldots, K, \qquad x(0) = x_0$$

- inputs limited to range $[-1, 1]$:

$$\|u(t)\|_\infty \leq 1, \quad t = 0, 1, \ldots, K$$

- settling time:

$$f(u(0), \ldots, u(K)) = \min \{T \mid x(t) = 0 \text{ for } T \leq t \leq K+1\}$$

settling time f is quasiconvex function of $(u(0), \ldots, u(K))$:

$$f(u(0), u(1), \ldots, u(K)) \leq T$$

if and only if for all $t = T, \ldots, K+1$

$$x(t) = A^t x_0 + A^{t-1} B u(0) + \cdots + B u(t-1) = 0$$

$i.e.$, sublevel sets are affine

minimum-time optimal control problem:

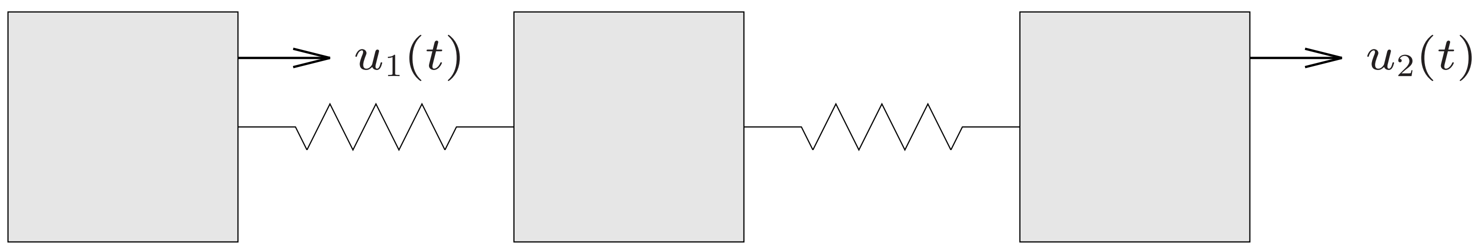
$$\begin{array}{ll} \text{minimize} & f(u(0), u(1), \ldots, u(K)) \\ \text{subject to} & \|u(t)\|_\infty \leq 1, \quad t = 0, \ldots, K \end{array}$$

with variables $u(0), \ldots, u(K)$
a quasiconvex problem; can be solved via bisection with LPs

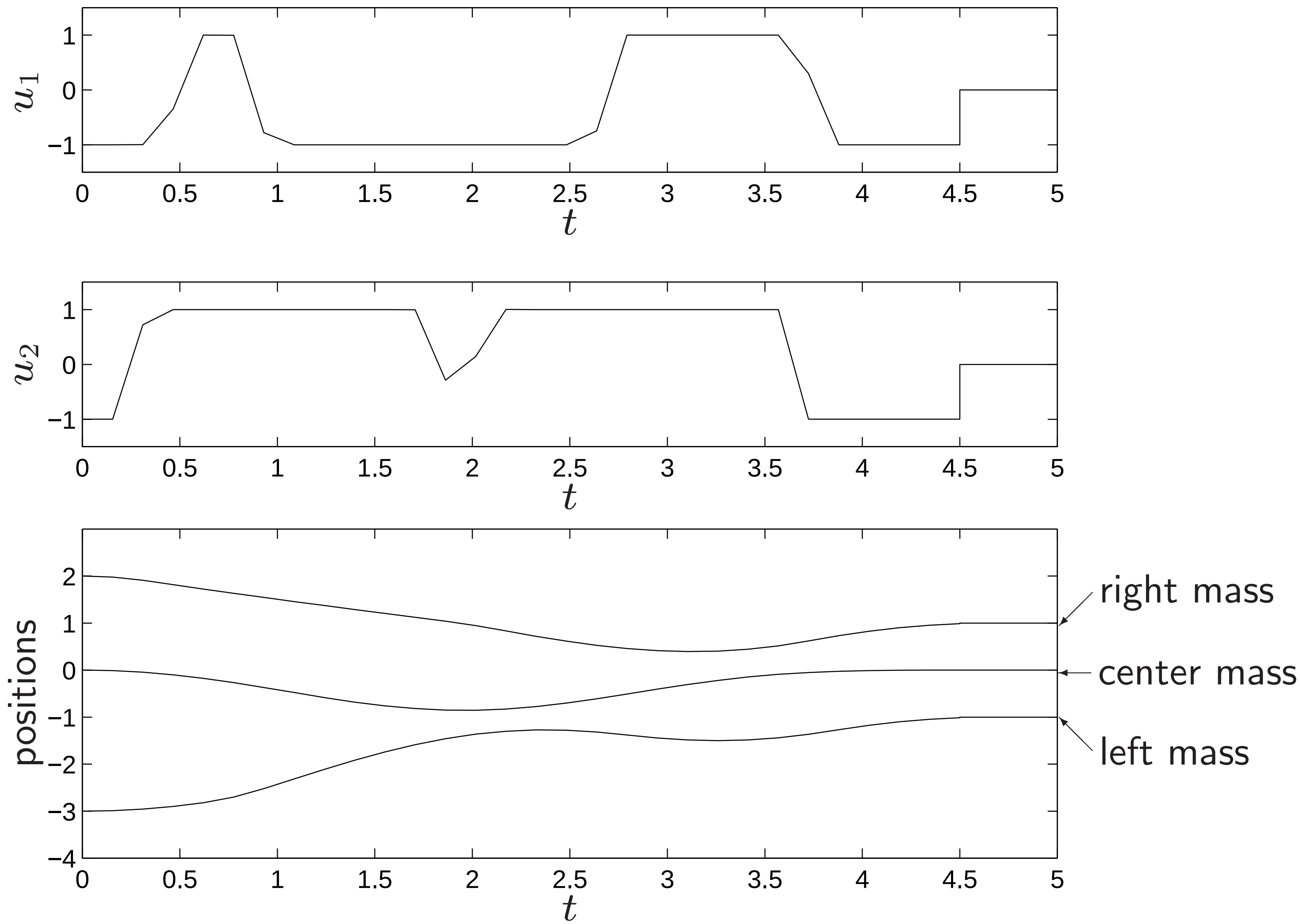
Minimum-time control example

three unit masses, connected by two unit springs with equilibrium length one

$u(t) \in \mathbf{R}^2$ is force on left & right masses over time interval $(0.15t, 0.15(t+1)]$

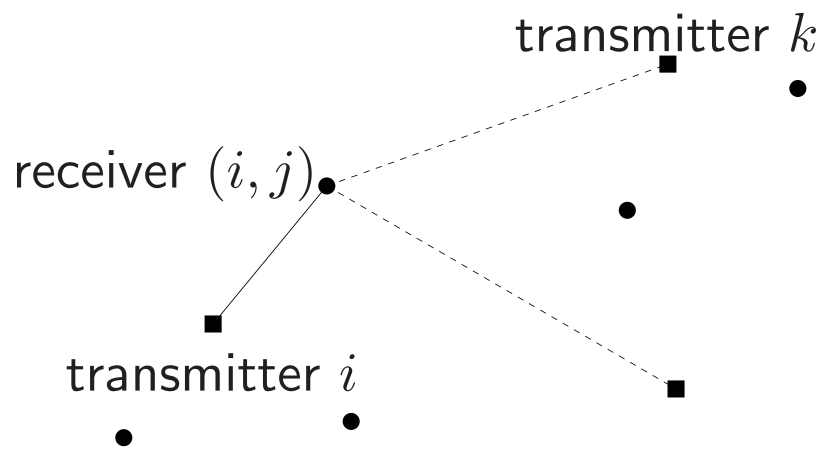


problem: pick $u(0), \ldots, u(K)$ to bring masses to positions $(-1, 0, 1)$ (at rest), as quickly as possible, from initial condition $(-3, 0, 2)$ (at rest) optimal solution:



Optimal transmitter power allocation

- m transmitters, mn receivers all at same frequency
- transmitter i wants to transmit to n receivers labeled (i, j), $j = 1, \ldots, n$



- A_{ijk} is path gain from transmitter k to receiver (i, j)
- N_{ij} is (self) noise power of receiver (i, j)
- variables: transmitter powers p_k, $k = 1, \ldots, m$

at receiver (i,j):

- signal power:

$$S_{ij} = A_{iji}p_i$$

- noise plus interference power:

$$I_{ij} = \sum_{k \neq i} A_{ijk}p_k + N_{ij}$$

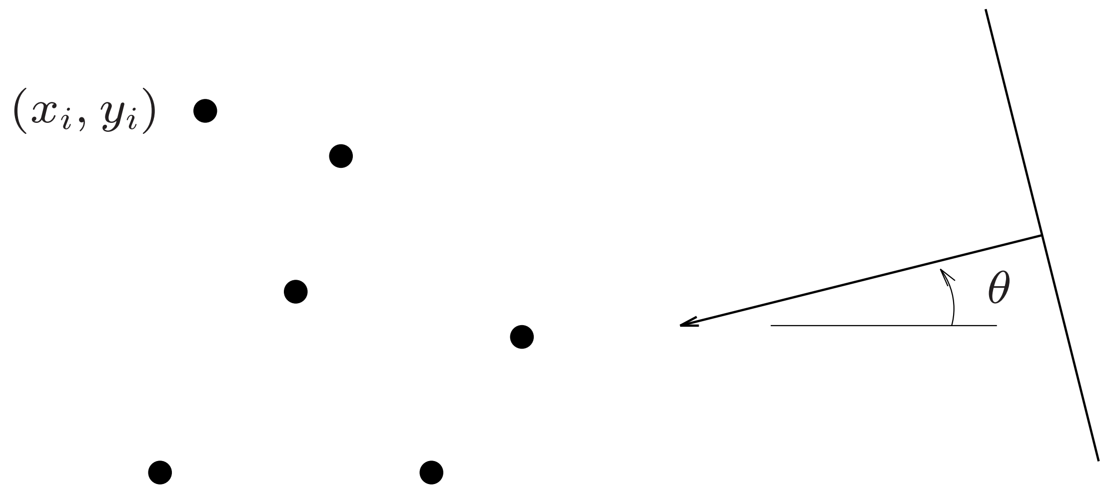
- signal to interference/noise ratio (SINR): S_{ij}/I_{ij}

problem: choose p_i to maximize smallest SINR:

$$\begin{array}{ll} \text{maximize} & \displaystyle \min_{i,j} \frac{A_{iji}p_i}{\sum_{k \neq i} A_{ijk}p_k + N_{ij}} \\ \text{subject to} & 0 \leq p_i \leq p_{\mathsf{max}} \end{array}$$

. . . a (generalized) linear fractional program

Phased-array antenna beamforming



- omnidirectional antenna elements at positions (x_1, y_1), . . . , (x_n, y_n)

- unit plane wave incident from angle θ induces in ith element a signal $e^{j(x_i \cos\theta + y_i \sin\theta - \omega t)}$

  ($j = \sqrt{-1}$, frequency ω, wavelength 2π)

- demodulate to get output $e^{j(x_i \cos\theta + y_i \sin\theta)} \in \mathbf{C}$
- linearly combine with complex weights w_i:

$$y(\theta) = \sum_{i=1}^{n} w_i e^{j(x_i \cos\theta + y_i \sin\theta)}$$

- $y(\theta)$ is (complex) *antenna array gain pattern*
- $|y(\theta)|$ gives sensitivity of array as function of incident angle θ
- depends on design variables $\mathbf{Re}\, w$, $\mathbf{Im}\, w$
  (called *antenna array weights* or *shading coefficients*)

design problem: choose w to achieve desired gain pattern

Sidelobe level minimization

make $|y(\theta)|$ small for $|\theta - \theta_{\mathrm{tar}}| > \alpha$

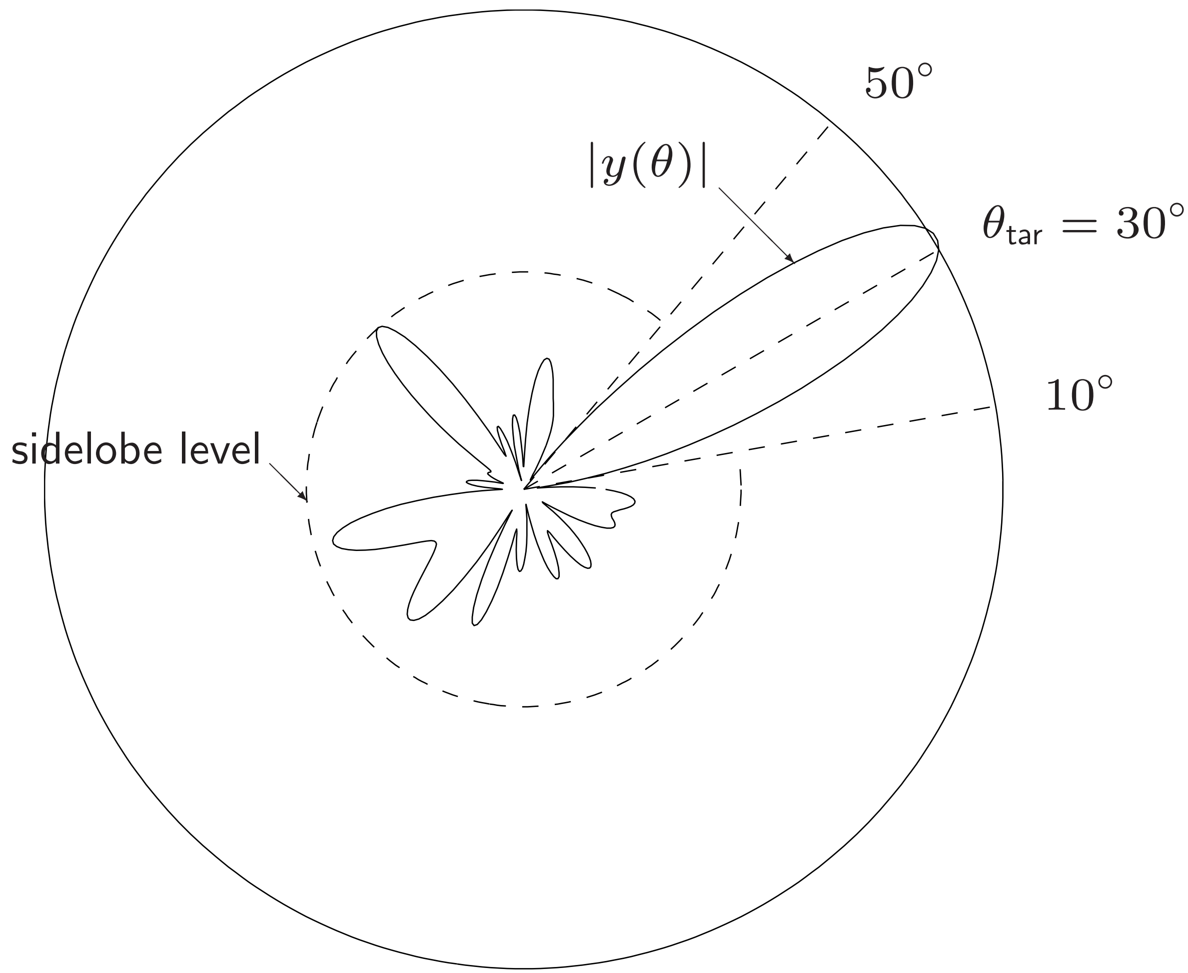
(θ_{tar}: target direction; 2α: beamwidth)

via least-squares (discretize angles)

$$\begin{array}{ll} \text{minimize} & \sum_i |y(\theta_i)|^2 \\ \text{subject to} & y(\theta_{\mathrm{tar}}) = 1 \end{array}$$

(sum is over angles outside beam)

least-squares problem with two (real) linear equality constraints



50°
$|y(\theta)|$
$\theta_{\mathsf{tar}} = 30^\circ$
10°
sidelobe level

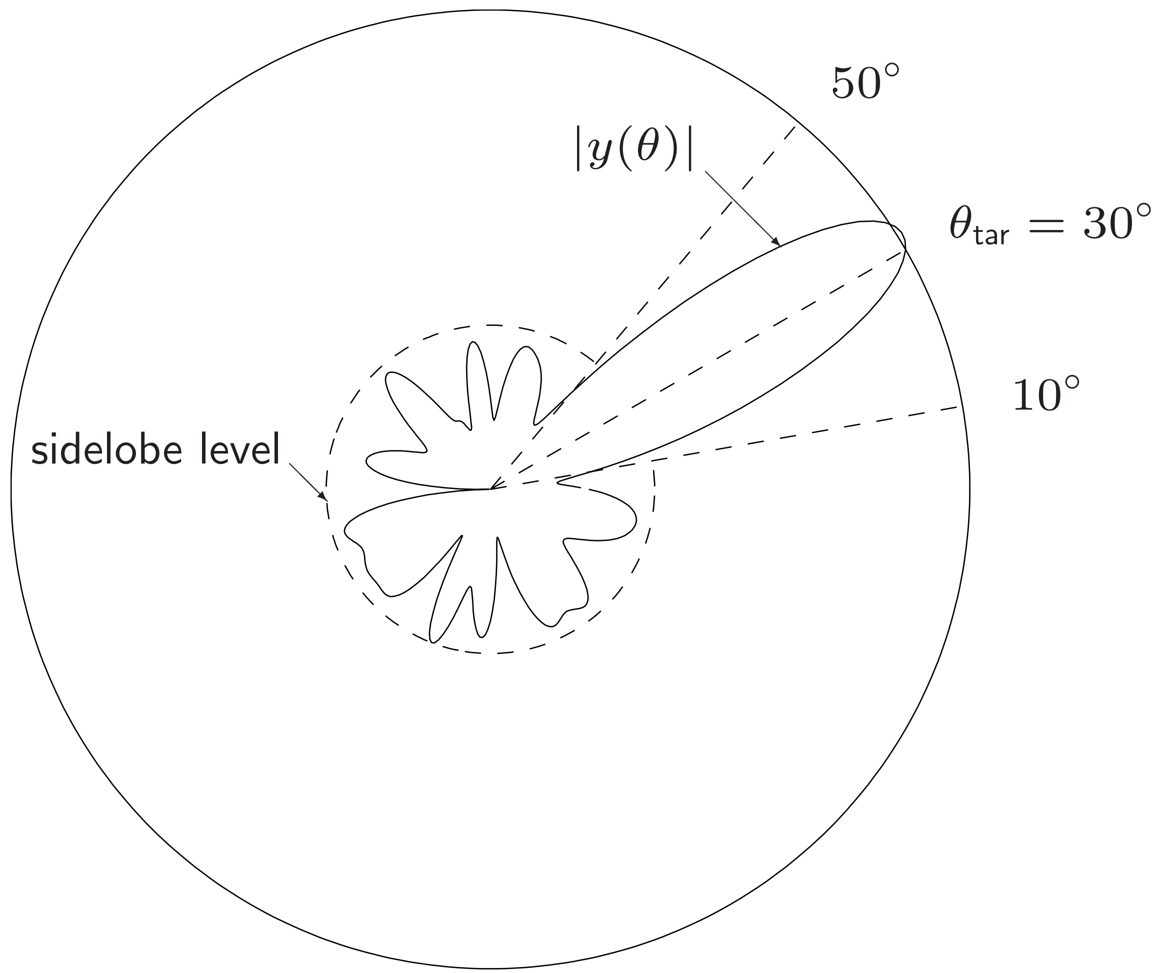
minimize sidelobe level (discretize angles)

$$\begin{array}{ll} \text{minimize} & \max_i |y(\theta_i)| \\ \text{subject to} & y(\theta_{\mathrm{tar}}) = 1 \end{array}$$

(max over angles outside beam)

can be cast as SOCP

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & |y(\theta_i)| \le t \\ & y(\theta_{\mathrm{tar}}) = 1 \end{array}$$


50°
$|y(\theta)|$
$\theta_{\mathrm{tar}} = 30°$
10°
sidelobe level

Extensions

convex (& quasiconvex) extensions:

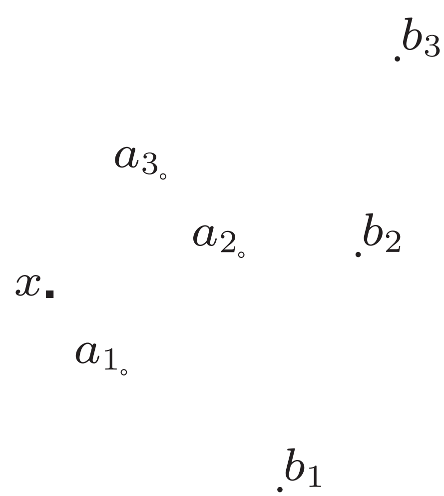
- $y(\theta_0) = 0$ (null in direction θ_0)
- w is real (amplitude only shading)
- $|w_i| \leq 1$ (attenuation only shading)
- minimize $\sigma^2 \sum_{i=1}^n |w_i|^2$ (thermal noise power in y)
- minimize beamwidth given a maximum sidelobe level

nonconvex extension:

- maximize number of zero weights

Optimal receiver location

- N transmitter frequencies $1, \ldots, N$
- transmitters at locations $a_i,\ b_i \in \mathbf{R}^2$ use frequency i
- transmitters at a_1, a_2, . . . , a_N are the wanted ones
- transmitters at b_1, b_2, . . . , b_N are interfering
- receiver at position $x \in \mathbf{R}^2$



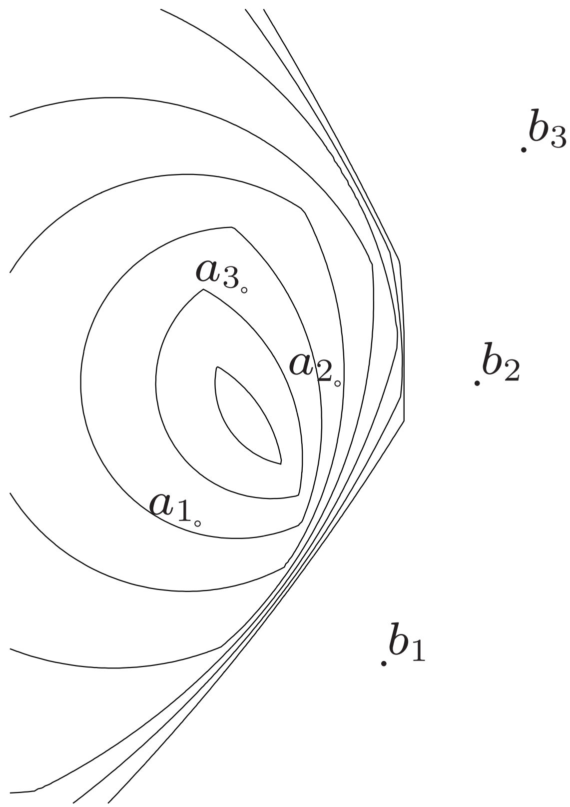
- (signal) receiver power from a_i: $\|x - a_i\|^{-\alpha}$ $(\alpha \approx 2.1)$
- (interfering) receiver power from b_i: $\|x - b_i\|^{-\alpha}$ $(\alpha \approx 2.1)$
- worst signal to interference ratio, over all frequencies, is

$$\mathsf{S/I} = \min_i \frac{\|x - a_i\|^{-\alpha}}{\|x - b_i\|^{-\alpha}}$$

- what receiver location x maximizes S/I?

S/I is quasiconcave on $\{x \mid \mathsf{S/I} \geq 1\}$, *i.e.*, on

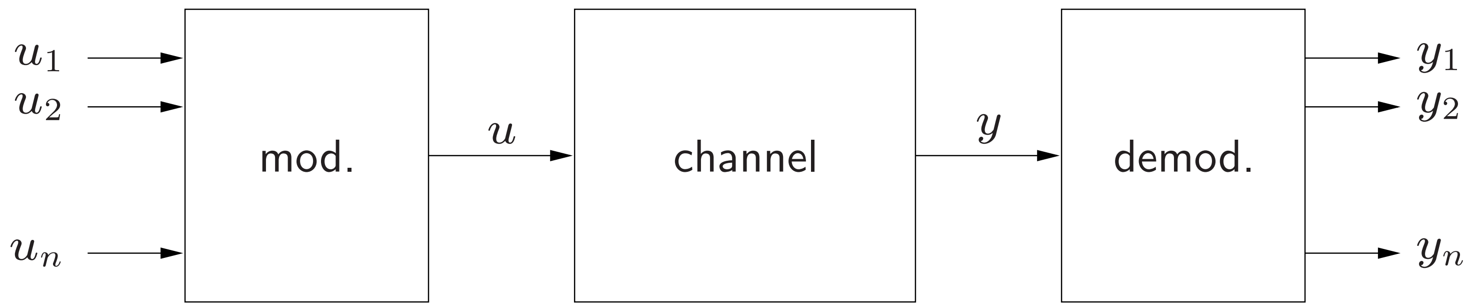
$$\{x \mid \|x - a_i\| \leq \|x - b_i\|, \ i = 1, \ldots, N\}$$



can use bisection; every iteration is a convex quadratic feasibility problem

Power allocation in FDM system

frequency division multiplex (FDM) system



- signal u_i modulates carrier frequency f_i with power p_i
- channel is slightly nonlinear
- powers affect signal power, interference power at each y_i
- problem: choose powers to maximize minimum SINR
  (signal to noise & interference ratio)

- demodulated signal power in y_i proportional to p_i
- noise power in y_i is σ_i^2
- interference power in y_i is sum of crosstalk & intermodulation products from nonlinearity
- crosstalk power c_i is linear in powers:

$$c = Cp, \quad C_{ij} \geq 0$$

C is often tridiagonal, $i.e.$, have crosstalk from adjacent channels only

- intermodulation power: kth order IM products have frequencies

$$\pm f_{i_1} \pm f_{i_2} \pm \cdots \pm f_{i_k}$$

with power proportional to $p_{i_1} p_{i_2} \cdots p_{i_k}$
e.g., for frequencies $1, 2, 3$:

frequency	IM product	pwr. prop. to
2	$1+1$	p_1^2
3	$1+2$	$p_1 p_2$
1	$2-1$	$p_2 p_1$
1	$3-2$	$p_3 p_2$
2	$3-1$	$p_3 p_1$
3	$1+1+1$	p_1^3
1	$1+1-1$	p_1^3
2	$2+1-1$	$p_2 p_1^2$
$\vdots$	$\vdots$	$\vdots$

- total IM power at f_i is (complicated) polynomial of $p_1, \ldots, p_n$, with nonnegative coefficients inverse SINR at frequency i

$$\frac{\text{noise} + \text{crosstalk} + \text{IM power}}{\text{signal power}}$$

is posynomial function of $p_1, \ldots, p_n$

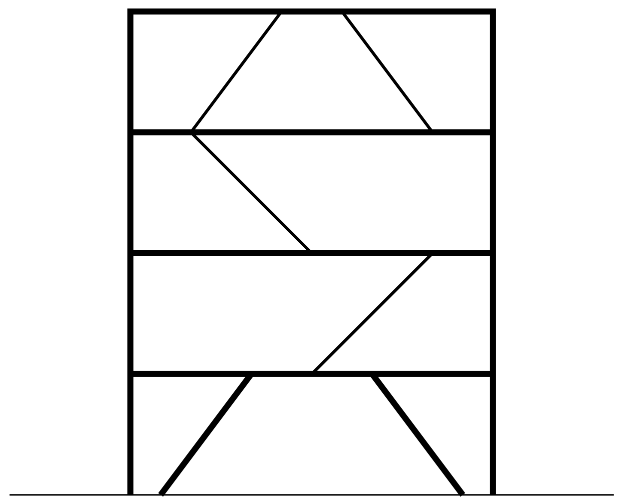
hence, problem such as

$$\begin{array}{ll} \text{maximize} & \min_i \mathrm{SINR}_i \\ \text{subject to} & 0 < p_i \leq P_{\max} \end{array}$$

is geometric program

Optimizing structural dynamics

linear elastic structure



dynamics (ignoring damping): $M\ddot{d} + Kd = 0$

- $d(t) \in \mathbf{R}^k$: vector of displacements
- $M = M^T \succ 0$ is mass matrix; $K = K^T \succ 0$ is stiffness matrix

Fundamental frequency

- solutions have form

$$d_i(t) = \sum_{j=1}^{k} \alpha_{ij} \cos(\omega_j t - \phi_j)$$

  where $0 \leq \omega_1 \leq \omega_2 \leq \cdots \leq \omega_k$ are the modal frequencies, $i.e.$, positive solutions of $\det(\omega^2 M - K) = 0$

- fundamental frequency:

$$\omega_1 = \lambda_{\min}^{1/2}(K, M) = \lambda_{\min}^{1/2}(M^{-1/2} K M^{-1/2})$$

  - structure behaves like mass at frequencies below ω_1
  - gives stiffness measure (the larger ω_1, the stiffer the structure)

- $\omega_1 \geq \Omega \iff \Omega^2 M - K \preceq 0$ so ω_1 is quasiconcave function of M, K

- design variables: x_i, cross-sectional area of structural member i (geometry of structure fixed)
- $M(x) = M_0 + \sum_i x_i M_i$, $K(x) = K_0 + \sum_i x_i K_i$
- structure weight $w = w_0 + \sum_i x_i w_i$
- **problem:** minimize weight s.t. $\omega_1 \geq \Omega$, limits on cross-sectional areas

as SDP:

$$\begin{array}{ll} \text{minimize} & w_0 + \sum_i x_i w_i \\ \text{subject to} & \Omega^2 M(x) - K(x) \preceq 0 \\ & l_i \leq x_i \leq u_i \end{array}$$