36-705 Intermediate Statistics Homework #6 Solutions

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Problem 1 [25 pts.]

Suppose the test is of the form

$$\text{reject } H_0 \text{ if and only if } T(X^n) \geq c_\alpha.$$

Then for any $t \in (0,1)$,

$$\begin{aligned}
\mathbb{P}_{\theta_0}(\text{p-value} \leq t) &= \mathbb{P}_{\theta_0}\left(\mathbb{P}_{\theta_0}\Big(T(X^n) \geq T(x^n)\Big) \leq t\right) \\
&= \mathbb{P}_{\theta_0}\left(1 - \mathbb{P}_{\theta_0}\Big(T(X^n) \leq T(x^n)\Big) \leq t\right) \\
&= \mathbb{P}_{\theta_0}\left(1 - F_T(T(x^n)) \leq t\right) \\
&= \mathbb{P}_{\theta_0}\left(F_T(T(x^n)) \geq 1 - t\right) \\
&= 1 - \mathbb{P}_{\theta_0}\left(F_T(T(x^n)) \leq 1 - t\right) \\
&= 1 - \underbrace{\mathbb{P}_{\theta_0}\left(T(x^n) \leq F_T^{-1}(1 - t)\right)}_{=1-t} \\
&= t.
\end{aligned}$$

Therefore, under θ_0,

$$\text{p-value} \sim \text{Uniform}(0,1).$$

Problem 2 [25 pts.]

(a) By the usual procedure, it can be shown that the MLE for θ is $\widehat{\theta}_n = \frac{1}{\overline{X}_n}$. Since we want a level α test, we must have

$$\begin{aligned}
\sup_{\theta \in \Theta_0} \beta(\theta) &\leq \alpha \\
\beta(1) &\leq \alpha \\
\mathbb{P}_{\theta_0}\left(\widehat{\theta}_n > c\right) &\leq \alpha \\
\mathbb{P}_{\theta_0}\left(\overline{X}_n \leq \frac{1}{c}\right) &\leq \alpha \\
\mathbb{P}_{\theta_0}\left(\sum_{i=1}^n X_i \leq \frac{n}{c}\right) &\leq \alpha.
\end{aligned}$$

Since $\sum_{i=1}^n X_i \sim \text{Gamma}(n, 1)$,

$$\begin{aligned}
F_{n,1}\left(\frac{n}{c}\right) &\leq \alpha \\
c &\geq \frac{n}{F_{n,1}^{-1}(\alpha)},
\end{aligned}$$

where $F_{n,1}$ is the cdf of Gamma$(n, 1)$. Thus, we can let

$$c = \frac{n}{F_{n,1}^{-1}(\alpha)}.$$

(b) The power function is

$$\begin{aligned}
\beta(\theta) &= \mathbb{P}_\theta\left(\widehat{\theta}_n > c\right) \\
&= \mathbb{P}_\theta\left(\overline{X}_n < \frac{1}{c}\right) \\
&= \mathbb{P}_\theta\left(\sum_{i=1}^n X_i < \frac{n}{c}\right) \\
&= \mathbb{P}_\theta\left(\sum_{i=1}^n X_i < F_{n,1}^{-1}(\alpha)\right) \\
&= F_{n,\theta}\left(F_{n,1}^{-1}(\alpha)\right).
\end{aligned}$$

Alternate Solution.

(a) For $X_1, \ldots, X_n \sim \text{Exponential}\big(\theta\big)$, we have

$$\frac{\sqrt{n}\Big(\overline{X}_n - \frac{1}{\theta}\Big)}{\sqrt{\frac{1}{\theta^2}}} \rightsquigarrow N(0,1),$$

by the CLT. By the delta method (with $g(x) = 1/x$)),

$$\begin{aligned}
\frac{\sqrt{n}\Big(\frac{1}{\overline{X}_n} - g(1/\theta)\Big)}{|g'(1/\theta)|(1/\theta)} &= \frac{\sqrt{n}\Big(\frac{1}{\overline{X}_n} - \theta\Big)}{\theta} \\
&= \frac{\sqrt{n}(\widehat{\theta}_n - \theta)}{\theta} \\
&\rightsquigarrow N(0,1).
\end{aligned}$$

For a level α test we must have

$$\begin{aligned}
\sup_{\theta\in\Theta_0} \beta(\theta) &\leq \alpha \\
\beta(1) &\leq \alpha \\
\mathbb{P}_{\theta_0}\Big(\widehat{\theta}_n > c\Big) &\leq \alpha \\
\mathbb{P}_{\theta_0}\Big(Z > \sqrt{n}(c-1)\Big) &\leq \alpha.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\sqrt{n}(c-1) &\geq z_\alpha \\
c &\geq \frac{z_\alpha}{\sqrt{n}} + 1.
\end{aligned}$$

Thus, we can let

$$c = \frac{z_\alpha}{\sqrt{n}} + 1.$$

(b) The power function is

$$\begin{aligned}
\beta(\theta) &= \mathbb{P}_\theta\Big(\widehat{\theta}_n > c\Big) \\
&= \mathbb{P}_\theta\left(Z > \frac{\sqrt{n}(c-\theta)}{\theta}\right) \\
&= 1 - \Phi\left(\frac{\sqrt{n}(c-\theta)}{\theta}\right),
\end{aligned}$$

where

$$c = \frac{z_\alpha}{\sqrt{n}} + 1.$$

Problem 3 [20 pts.]

The statistic Z can be rewritten as:

$$Z = \frac{\hat{\theta} - \theta_1}{\hat{\text{se}}} + \frac{\theta_1 - \theta_0}{\hat{\text{se}}}$$

When the true $\theta = \theta_1 > \theta_0$ we have that for large n, $Z \approx N\left(\frac{\theta_1 - \theta_0}{\hat{\text{se}}}, 1\right)$, such that the power of the test is given by:

$$\begin{aligned}\beta(\theta_1) &= \mathbb{P}_{\theta_1}(|Z| > z_{\alpha/2}) = \mathbb{P}_{\theta_1}(Z < -z_{\alpha/2}) + \mathbb{P}_{\theta_1}(Z > z_{\alpha/2}) \\ &\approx \Phi\left(-z_{\alpha/2} - \frac{\theta_1 - \theta_0}{\hat{\text{se}}}\right) + 1 - \Phi\left(z_{\alpha/2} - \frac{\theta_1 - \theta_0}{\hat{\text{se}}}\right)\end{aligned}$$

where Φ is the CDF for the standard normal. Since $\frac{\theta_1 - \theta_0}{\hat{\text{se}}} = \sqrt{nI(\hat{\theta})}(\theta_1 - \theta_0) \to \infty$, then $\Phi\left(a - \frac{\theta_1 - \theta_0}{\hat{\text{se}}}\right) \to 0$ as $n \to \infty$, for any $a \in \mathbb{R}$. This proves the claim. The case $\theta_1 < \theta_0$ can be treated similarly.

Problem 4 [30 pts.]

Likelihood Ratio Test: For any $\sigma > 0$, letting $\frac{\partial \ell(\mu,\sigma)}{\partial \mu} = 0$ yields

$$-\sum_{i=1}^{n}\left(\frac{X_i - \mu}{\sigma^2}\right) = 0$$
$$\sum_{i=1}^{n}(X_i - \mu) = 0$$
$$\widehat{\mu} = \overline{X}_n.$$

And letting $\frac{\partial \ell(\mu,\sigma)}{\partial \sigma} = 0$, we have

$$\sum_{i=1}^{n}\left(-\frac{1}{\sigma} + \frac{1}{\sigma^3}(X_i - \mu)^2\right) = 0$$
$$-n + \frac{1}{\sigma^2}\sum_{i=1}^{n}(X_i - \mu)^2 = 0$$
$$\sigma = \sqrt{\frac{\sum_{i=1}^{n}(X_i - \mu)^2}{n}}.$$

Then by equivariance,

$$\widehat{\sigma} = \sqrt{\frac{\sum_{i=1}^{n}(X_i - \overline{X}_n)^2}{n}}.$$

Thus, the likelihood ratio is

$$\lambda(X^n) = \frac{\prod_{i=1}^{n}\frac{1}{\sqrt{2\pi\sigma_0^2}}\exp\left\{-\frac{1}{2\sigma_0^2}(X_i - \overline{X}_n)^2\right\}}{\prod_{i=1}^{n}\frac{1}{\sqrt{2\pi\widehat{\sigma}^2}}\exp\left\{-\frac{1}{2\widehat{\sigma}^2}(X_i - \overline{X}_n)^2\right\}}.$$

By **Theorem 8** in lecture notes 10,

$$-2\log\lambda(X^n) \rightsquigarrow \chi_1^2.$$

Therefore, we reject when

$$-2\log\lambda(X^n) > \chi_1^2(\alpha).$$

After simplification, this is equivalent to: reject H_0 when

$$2n\log\left(\frac{\sigma_0}{\widehat{\sigma}}\right) + \frac{n\widehat{\sigma}^2}{\sigma_0^2} - n > \chi_1^2(\alpha).$$

Wald Test:

$$\frac{\partial^2}{\partial \sigma^2}\ell(\mu,\sigma) = \frac{n}{\sigma^2} - \frac{3}{\sigma^4}\sum_{i=1}^{n}(X_i-\mu)^2,$$

$$\begin{aligned} I(\sigma) &= \frac{1}{n}I_n(\sigma) \\ &= -\frac{1}{n}\mathbb{E}\left(\frac{\partial^2}{\partial \sigma^2}\ell(\mu,\sigma)\right) \\ &= -\frac{1}{\sigma^2} + \frac{3}{\sigma^2} \\ &= \frac{2}{\sigma^2}. \end{aligned}$$

Under H_0,

$$T_n = \frac{\sqrt{n}(\widehat{\sigma}-\sigma_0)}{\sqrt{I(\sigma)}} \rightsquigarrow N(0,1).$$

Therefore, we reject H_0 when

$$\begin{aligned} &\left|T_n\right| > z_{\alpha/2} \\ \Longleftrightarrow &\left|\sqrt{2n}\left(1-\frac{\sigma_0}{\widehat{\sigma}}\right)\right| > z_{\alpha/2} \\ \Longleftrightarrow &2n\left(1-\frac{\sigma_0}{\widehat{\sigma}}\right)^2 > \chi^2_1(\alpha). \end{aligned}$$

Notice that both the LRT and Wald Test reject H_0 when a statistic exceeds $\chi^2_1(\alpha)$.

Problem 5

The power function is

$$\begin{aligned}
\beta(\theta) &= \mathbb{P}_\theta(\overline{X}_n > c) \\
&= \mathbb{P}_\theta\big(Z > \sqrt{n}(c-\theta)\big) \\
&= 1 - \Phi\big(\sqrt{n}(c-\theta)\big).
\end{aligned}$$

The size of the test is

$$\begin{aligned}
\sup_{\theta\in\Theta_0} \beta(\theta) &= \beta(1) \\
&= 1 - \Phi\big(\sqrt{n}(c-1)\big).
\end{aligned}$$

Thus, for a size α test, we have

$$c = \frac{z_\alpha}{\sqrt{n}} + 1.$$