Test 1 - Solutions

Intermediate Statistics - 36-705

September 9, 2016

Problem 1. [35 points]
Let X_1, X_2 be iid Uniform$(0, 3)$. Find the density of $Y = X_1/X_2$.

First we calculate the CDF of Y:

$$\begin{aligned} F_Y(y) &= \mathbb{P}(Y \leq y) \\ &= \mathbb{P}(X_1/X_2 \leq y) \\ &= \mathbb{P}(X_1 \leq yX_2). \end{aligned} \tag{1}$$

Notice $Y \in (0, \infty)$ and consider first the case that $y \in (0, 1)$. (1) now becomes

$$\begin{aligned} \int_0^3 \int_0^{yx_2} f_{X_1,X_2}(x_1, x_2) dx_1 dx_2 &\overset{iid}{=} \int_0^3 \int_0^{yx_2} f_{X_1}(x_1) f_{X_2}(x_2) dx_1 dx_2 \\ &= \frac{1}{9} \int_0^3 \int_0^{yx_2} 1 dx_1 dx_2 \\ &= \frac{1}{9} \int_0^3 yx_2 dx_2 \\ &= \frac{1}{9} \left[\frac{yx_2^2}{2} \right]_0^3 \\ &= \frac{y}{2}. \end{aligned}$$

And when $y \in (1, \infty)$, we can write (1) as

$$
\begin{aligned}
1 - \mathbb{P}(X_1 > yX_2) &= 1 - \int_0^3 \int_0^{x_1/y} f_{X_1,X_2}(x_1, x_2) dx_2 dx_1 \\
&\overset{iid}{=} 1 - \int_0^3 \int_0^{x_1/y} f_{X_1}(x_1) f_{X_2}(x_2) dx_2 dx_1 \\
&= 1 - \frac{1}{9} \int_0^3 \int_0^{x_1/y} 1 dx_2 dx_1 \\
&= 1 - \frac{1}{9} \int_0^3 \frac{x_1}{y} dx_1 \\
&= 1 - \frac{1}{9} \left[\frac{x_1^2}{2y} \right]_0^3 \\
&= 1 - \frac{1}{2y}.
\end{aligned}
$$

Thus,

$$
F_Y(y) = \begin{cases} 0 & y < 0 \\ \frac{y}{2} & 0 < y < 1 \\ 1 - \frac{1}{2y} & 1 < y < \infty \end{cases}
$$

and

$$
f_Y(y) = \begin{cases} 0 & y < 0 \\ \frac{1}{2} & 0 < y < 1 \\ \frac{1}{2y^2} & 1 < y < \infty \end{cases}.
$$

Problem 2. [30 points]
Let $X_1, ..., X_n \sim \text{Uniform}(a, b)$ where $a < b$. Let

$$Y_n = \max\{X_1, ..., X_n\}.$$

Find the density of Y_n.

Solution
First we calculate the CDF of Y_n:

$$\begin{aligned}
F_{Y_n}(y) &= \mathbb{P}(Y_n \leq y) \\
&= \mathbb{P}(\max\{X_1, ..., X_n\} \leq y) \\
&= \mathbb{P}(X_1 \leq y, ..., X_n \leq y) \\
&= \mathbb{P}(X_1 \leq y) \times ... \times \mathbb{P}(X_n \leq y) \\
&= \mathbb{P}(X_1 \leq y)^n \\
&= \begin{cases} 0 & y < a \\ \left(\frac{y-a}{b-a}\right)^n & a < y < b \\ 1 & y > b. \end{cases}
\end{aligned}$$

Now $\frac{d}{dy} F_{Y_n}(y) = f_{Y_n}(y)$:

$$f_{Y_n}(y) = \begin{cases} n\frac{(y-a)^{n-1}}{(b-a)^n} & a < y < b \\ 0 & \text{otherwise.} \end{cases}$$

Problem 3. [35 points]
Let $A_1, A_2, ...$ be an arbitrary sequence of events. Show that

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} P(A_i).$$

Solution
First we prove

$$P\left(\bigcup_{i=1}^{N} A_i\right) \leq \sum_{i=1}^{N} P(A_i)$$

using mathematical induction.

1. If N = 1, then $P\left(\bigcup_{i=1}^{N} A_i\right) = P(A_1) \leq P(A_1) = \sum_{i=1}^{N} P(A_i)$.

2. Now suppose $P\left(\bigcup_{i=1}^{N} A_i\right) \leq \sum_{i=1}^{N} P(A_i)$ for some arbitrary $N \in \mathbb{N}$. Then:

$$\begin{aligned}
P\left(\bigcup_{i=1}^{N+1} A_i\right) &= P\left(A_{N+1} \cup \left(\bigcup_{i=1}^{N} A_i\right)\right) \\
&= P\left(A_{N+1}\right) + P\left(\bigcup_{i=1}^{N} A_i\right) - P\left(A_{N+1} \cap \left(\bigcup_{i=1}^{N} A_i\right)\right) \\
&\leq P\left(A_{N+1}\right) + P\left(\bigcup_{i=1}^{N} A_i\right) \quad \left(\because P\left(A_{N+1} \cap \left(\bigcup_{i=1}^{N} A_i\right)\right) \geq 0\right) \\
&\leq P\left(A_{N+1}\right) + \sum_{i=1}^{N} P(A_i) \quad \left(\text{by induction hypothesis}\right) \\
&= \sum_{i=1}^{N+1} P(A_i).
\end{aligned}$$

Therefore by mathematical induction, $P\left(\bigcup_{i=1}^{N} A_i\right) \leq \sum_{i=1}^{N} P(A_i)$ for all integers N and so,

$$\lim_{N\to\infty} P\left(\bigcup_{i=1}^{N} A_i\right) \leq \lim_{N\to\infty} \sum_{i=1}^{N} P(A_i) = \sum_{i=1}^{\infty} P(A_i).$$

By continuity of probability, $\lim_{N\to\infty} P\left(\bigcup_{i=1}^{N} A_i\right) = P\left(\lim_{N\to\infty} \bigcup_{i=1}^{N} A_i\right) = P\left(\bigcup_{i=1}^{\infty} A_i\right)$.

Therefore,

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} P(A_i).$$

Alternate proof 1: We want to construct a sequence of events $B_1, B_2, ...$ such that:

1. $\bigcup_{i=1}^{\infty} B_i = \bigcup_{i=1}^{\infty} A_i$
2. $B_j \cap B_k = \emptyset, \forall j \neq k$
3. $B_i \subseteq A_i, \forall i$

Conditions 1), 2), and 3) will imply $P(\bigcup_{i=1}^{\infty} A_i) = P(\bigcup_{i=1}^{\infty} B_i) = \sum_{i=1}^{\infty} P(B_i) \leq \sum_{i=1}^{\infty} P(A_i)$.

Let $B_n = A_n - \bigcup_{i=1}^{n-1} A_i = A_n \cap (\bigcup_{i=1}^{n-1} A_i)^C$. We now show that conditions 1), 2), and 3) are satisfied by sequence $B_1, B_2, ...$

1. $\bigcup_{i=1}^{\infty} B_i = \bigcup_{i=1}^{\infty} A_i$. We use induction to show that $\bigcup_{i=1}^{N} B_i = \bigcup_{i=1}^{N} A_i$ for all $N \in \mathbb{N}$.

   (a) If $N = 1$, then $\bigcup_{i=1}^{N} B_i = B_1 = A_1 = \bigcup_{i=1}^{N} A_i$.

   (b) Now suppose $\bigcup_{i=1}^{N} B_i = \bigcup_{i=1}^{N} A_i$ for some arbitrary $N \in \mathbb{N}$. Then:

$$\begin{aligned}
\bigcup_{i=1}^{N+1} B_i &= \left[\bigcup_{i=1}^{N} B_i\right] \cup B_{N+1} = \left[\bigcup_{i=1}^{N} A_i\right] \cup B_{N+1} = \\
&= \left[\bigcup_{i=1}^{N} A_i\right] \cup \left[A_{N+1} \cap \left(\bigcup_{i=1}^{N} A_i\right)^C\right] \\
&= \left[\left(\bigcup_{i=1}^{N} A_i\right) \cup \left(\bigcup_{i=1}^{N} A_i\right)^C\right] \cap \left[\left(\bigcup_{i=1}^{N} A_i\right) \cup A_{N+1}\right] \\
&= \Omega \cap \left[\bigcup_{i=1}^{N+1} A_i\right] = \bigcup_{i=1}^{N+1} A_i
\end{aligned}$$

Then note that $\bigcup_{i=1}^{\infty} A_i = \bigcup_{N=1}^{\infty} \bigcup_{i=1}^{N} A_i$ and $\bigcup_{i=1}^{\infty} B_i = \bigcup_{N=1}^{\infty} \bigcup_{i=1}^{N} B_i$, so $\forall N \bigcup_{i=1}^{N} A_i = \bigcup_{i=1}^{N} B_i$ implies

$$\bigcup_{i=1}^{\infty} A_i = \bigcup_{N=1}^{\infty} \bigcup_{i=1}^{N} A_i = \bigcup_{N=1}^{\infty} \bigcup_{i=1}^{N} B_i = \bigcup_{i=1}^{\infty} B_i.$$

Alternate proof 2: Let $S_0 = \emptyset$, $S_n = \cup_{i=1}^{n} A_n$ and $B_n = A_n - S_{n-1}$. Since $B_i \subseteq A_i$ for all i (as shown below at point 3.), clearly $\cup_{i=1}^{\infty} B_i \subset \cup_{i=1}^{\infty} A_i$. To prove the reverse inclusion, let $x \in \cup_{i=1}^{\infty} A_i$ be given. Let k be the smallest number such that $x \in A_k$. Then $x \notin S_{k-1}$ and hence $x \in A_k - S_{k-1} = B_k$. Therefore $\cup_{i=1}^{\infty} B_i \supseteq \cup_{i=1}^{\infty} A_i$.

2. $B_j \cap B_k = \emptyset, \forall j \neq k$. Consider any arbitrary B_j and B_k. Without loss of generality, assume $k > j$. Then $B_j \cap B_k = [A_j \cap (\bigcup_{i=1}^{j-1} A_i)^C] \cap [A_k \cap (\bigcup_{i=1}^{k-1} A_i)^C]$. But since $j < k$, $A_j \cap (\bigcup_{i=1}^{k-1} A_i)^C = \emptyset$. Thus, $B_j \cap B_k = \emptyset$.

3. $B_i \subseteq A_i, \forall i$. It is easy to see that $B_i = A_i \cap (\bigcup_{j=1}^{i-1} A_j)^C \subseteq A_i, \forall i$.