Design and Analysis
of Algorithms I

QuickSort

Analysis I: A Decomposition Principle

Necessary Background

Assumption: you know and remember (finite) sample spaces, random variables, expectation, linearity of expectation. For review:

- Probability Review I (video)
- Lehman-Leighton notes (free PDF)
- Wikibook on Discrete Probability

Average Running Time of QuickSort

QuickSort Theorem : for every input array of length n, the average running time of QuickSort (with random pivots) is $O(n\log(n))$.

Note : holds for every input. [no assumptions on the data]

- recall our guiding principles !
- "average" is over random choices made by the algorithm (i.e., the pivot choices)

Preliminaries

Fix input array A of length n

Sample Space Ω = all possible outcomes of random choices in QuickSort (i.e., pivot sequences)

Key Random Variable : for $\sigma \in \Omega$

$C(\sigma)$ = # of comparisons between two input elements made by QuickSort (given random choices σ)

Lemma: running time of QuickSort dominated by comparisons.

There exist constant c s.t. for all $\sigma \in \Omega$, $RT(\sigma) \leq c \cdot C(\sigma)$ (see notes)

Remaining goal : E[C] = O(nlog(n))

Building Blocks

<u>Note</u> can't apply Master Method [random, unbalanced subproblems]

6	8	10	2
z_2	z_3	z_4	z_1

[A = final input array]

<u>Notation</u> : z_i = i^{th} smallest element of A

For $\sigma \in \Omega$, indices i< j

$X_{ij}(\sigma)$ = # of times z_i,z_j get compared in QuickSort with pivot sequence σ

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Fix two elements of the input array. How many times can these two elements get compared with each other during the execution of QuickSort?

- 1
- 0 or 1
- 0, 1, or 2
- Any integer between 0 and $n-1$

Reason : two elements compared only when one is the pivot, which is excluded from future recursive calls.

Thus : each X_{ij} is an "indicator" (i.e., 0-1) random variable

A Decomposition Approach

<u>So</u> : $C(\sigma)$ = # of comparisons between input elements

$X_{ij}(\sigma)$ = # of comparisons between z_i and z_j

<u>Thus</u> : $\forall \sigma, C(\sigma) = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} X_{ij}(\sigma)$

complicated simple

<u>By Linearity of Expectation :</u> $E[C] = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} E[X_{ij}]$

Since $E[X_{ij}] = 0 \cdot Pr[X_{ij} = 0] + 1 \cdot Pr[X_{ij} = 1] = Pr[X_{ij} = 1]$

Thus : $E[C] = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} Pr[z_i, z_j \ get \ compared] \quad (*)$

Next video

A General Decomposition Principle

1. Identify random variable Y that you really care about

2. Express Y as sum of indicator random variables :

$$Y = \sum_{l=1}^{m} X_e$$

3. Apply Linearity of expectation :

$$E[Y] = \sum_{l=1}^{m} Pr[X_e = 1]$$

"just" need to understand these!

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Analysis II: The Key Insight

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The Story So Far

$C(\sigma)$ = # of comparisons between input elements

$X_{ij}(\sigma)$ = # of comparisons between $\underbrace{z_i \text{ and } z_j}_{i^{th},\ j^{th} \text{ smallest entries in array}}$

<u>Recall</u> : $$E[C] = \sum_{i=1}^{n-1} \sum_{k=i+1}^{n} Pr[X_{ij} = 1]$$

$$= Pr[z_i\ z_j\ get\ compared]$$

<u>Key Claim</u> : for all i < j, Pr[z_i, z_j get compared] = 2/(j-i+1)

Proof of Key Claim

Pr[z_i,z_j get compared] = 2/(j-i+1)

Fix z_i, z_j with $i < j$

Consider the set $z_i, z_{i+1}, \ldots, z_{j-1}, z_j$

<u>Inductively</u> : as long as none of these are chosen as a pivot, all are passed to the same recursive call.

Consider the first among $z_i, z_{i+1}, .., z_{j-1}, z_j$ that gets chosen as a pivot.

1. If z_i or z_j gets chosen first, then z_i and z_j get compared
2. If one of $z_{i+1}, \ldots, z_{j-1}$ gets chosen first then z_i and z_j are never compared [split into different recursive calls]

KEY INSIGHT

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Proof of Key Claim (con'd)

1. z_i or z_j gets chosen first => they get compared
2. one of $z_{i+1},\ldots,z_{j-1}$ gets chosen first => z_i , z_j never compared

<u>Note</u> : Since pivots always chosen uniformly at random, each of $z_i,z_{i+1},\ldots,z_{j-1},z_j$ is equally likely to be the first

$\Rightarrow$Pr[z_i,z_j get compared] = 2/(j-i+1)

Choices that lead to case (1)

Total # of choices

<u>So</u> : $$E[C] = \sum_{i=1}^{n-1} \sum_{j=1}^{n} \frac{2}{j-i+1}$$

[Still need to show this is O(nlog(n))

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Analysis III: Final Calculations

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The Story So Far

$$E[C] = 2 \underbrace{\sum_{i=1}^{n-1} \sum_{j=1}^{n} \overbrace{\frac{1}{j-i+1}}}_{\theta(n^2) \quad terms}$$

How big can this be ?

(*)

<= n choices for i

Note : for each fixed i, the inner sum is

$$\sum_{j=i+1}^{n} \frac{1}{j-i+1} = 1/2 + 1/3 + \ldots$$

$$So \quad E[C] \leq 2 \cdot n \cdot \sum_{k=2}^{n} \frac{1}{k}$$

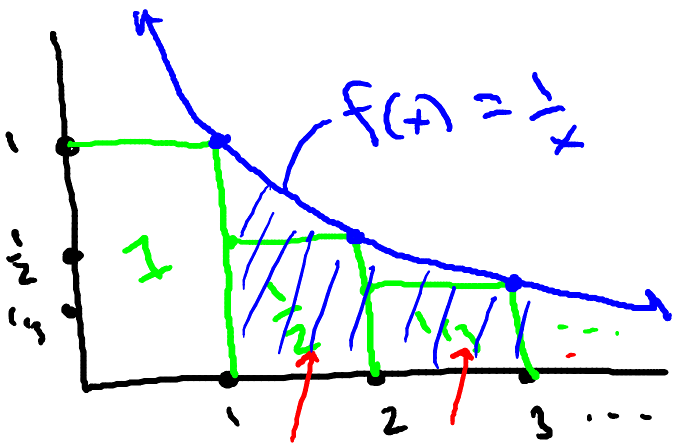
Claim : this is <= ln(n)

Completing the Proof

$$E[C] \le 2 \cdot n \cdot \sum_{k=2}^{n} \frac{1}{k}$$

$$Claim \quad \sum_{k=2}^{n} \frac{1}{k} \le \ln n$$

Proof of Claim



$$So \quad \sum_{k=2}^{n} \frac{1}{n} \le \int_{1}^{n} \frac{1}{x} dx$$

$$= \ln x \;\Big|_{1}^{n}$$

$$= \ln n - \ln 1$$

$$= \ln n$$

Q.E.D. (CLAIM)

So :
E[C] <=
2n ln n

Q.E.D.

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