Probability Review

Part I

Design and Analysis of Algorithms I

Topics Covered

- Sample spaces
- Events
- Random variables
- Expectation
- Linearity of Expectation

See also:

- Lehman-Leighton notes (free PDF)
- Wikibook on Discrete Probability

Concept #1 – Sample Spaces

<u>Sample Space Ω :</u> "all possible outcomes"
[in algorithms, Ω is usually finite]

<u>Also</u> : each outcome $i \in \Omega$ has a probability p(i) >= 0

<u>Constraint :</u> $$\sum_{i \in \Omega} p(i) = 1$$

<u>Example #1</u> : Rolling 2 dice. Ω = {(1,1), (2,1), (3,1),…,(5,6),(6,6) }
<u>Example #2</u> : Choosing a random pivot in outer QuickSort call.
Ω = {1,2,3,…,n} (index of pivot) and p(i) = 1/n for all $i \in \Omega$

Concept #2 – Events

An event is a subset $S \subseteq \Omega$

The probability of an event S is $\sum_{i \in S} p(i)$

Consider the event (i.e., the subset of outcomes for which) “the sum of the two dice is 7”. What is the probability of this event?

- $1/36$
- $1/12$
- $1/6$
- $1/2$

S = {(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)}

Pr[S] = 6/36 = 1/6

Consider the event (i.e., the subset of outcomes for which) “the chosen pivot gives a 25-75 split of better”. What is the probability of this event?

- $1/n$
- $1/4$
- $1/2$
- $3/4$

S = {(n/4+1)th smallest element,.., (3n/4)th smallest element

Pr[S] = (n/2)/n = 1/2

Concept #2 – Events

An event is a subset

The probability of an event S is

Ex#1 : sum of dice = 7. S = {(1,1),(2,1),(3,1),…,(5,6),(6,6)}
Pr[S] = 6/36 = 1/6

Ex#2 : pivot gives 25-75 split or better.
S = {$(n/4+1)^{th}$ smallest element,…,$(3n/4)^{th}$ smallest element]
Pr[S] = (n/2)/n = 1/2

Concept #3 - Random Variables

A Random Variable X is a real-valued function

$$X : \Omega \to \Re$$

Ex#1 : Sum of the two dice

Ex#2 : Size of subarray passed to 1st recursive call.

Concept #4 - Expectation

Let $X : \Omega \to \Re$ be a random variable.

The expectation E[X] of X = average value of X

$$= \sum_{i \in \Omega} X(i) \cdot p(i)$$

What is the expectation of the sum of two dice?

- 6.5
- 7
- 7.5
- 8

Which of the following is closest to the expectation of the size of the subarray passed to the first recursive call in QuickSort?

- $n/4$
- $n/3$
- $n/2$
- $3n/4$

Let X = subarray size

Then E[X] = (1/n)*0 + (1/n)*2 + … + (1/n)*(n-1)
= (n-1)/2

Concept #4 - Expectation

Let $X : \Omega \to \Re$ be a random variable.

The expectation E[X] of X = average value of X

$$= \sum_{i \in \Omega} X(i) \cdot p(i)$$

Ex#1 : Sum of the two dice, E[X] = 7

Ex#2 : Size of subarray passed to 1st recursive call.
E[X] = (n-1)/2

Concept #5 – Linearity of Expectation

Claim [LIN EXP] : Let $X_1,\ldots,X_n$ be random variables defined on Ω . Then :

$$E[\sum_{j=1}^{n} X_j] = \sum_{j=1}^{n} E[X_j]$$

CRUCIALLY:
HOLDS EVEN WHEN X_j's ARE NOT INDEPENDENT!
[WOULD FAIL IF REPLACE SUMS WITH PRODUCTS]

Ex#1 : if X_1,X_2 = the two dice, then
$E[X_j] = (1/6)(1+2+3+4+5+6) = 3.5$

By LIN EXP : $E[X_1+X_2] = E[X_1] + E[X_2] = 3.5 + 3.5 = 7$

Linearity of Expectation (Proof)

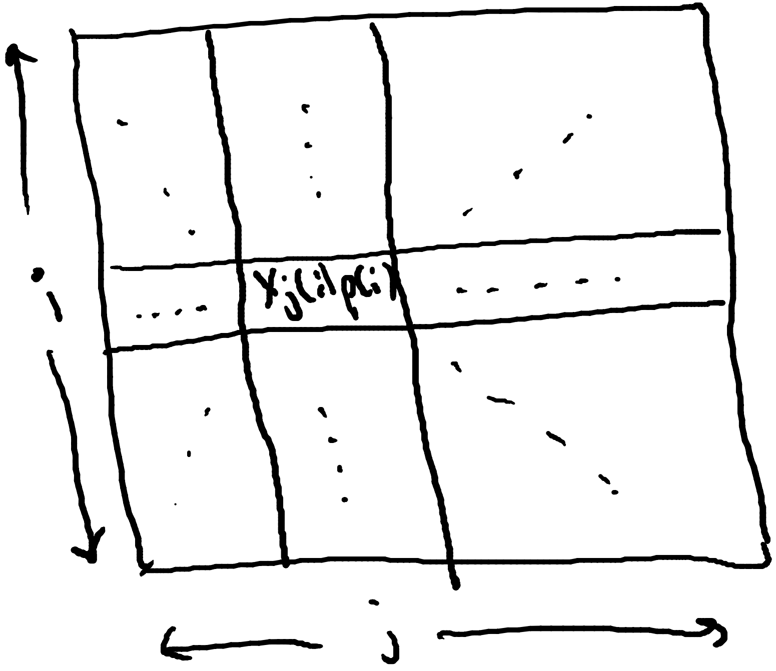
$$\sum_{j=1}^{n} E[X_j] = \sum_{j=1}^{n} \sum_{i \in \Omega} X_j(i) p(i)$$

$$= \sum_{i \in \Omega} \sum_{j=1}^{n} X_j(i) p(i)$$

$$= \sum_{i \in \Omega} p(i) \sum_{j=1}^{n} X_j(i)$$

$$= E[\sum_{j=1}^{n} X_j]$$

Q.E.D.



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Example: Load Balancing

Problem : need to assign n processes to n servers.

Proposed Solution : assign each process to a random server

Question : what is the expected number of processes assigned to a server ?

Load Balancing Solution

Sample Space Ω = all n^n assignments of processes to servers, each equally likely.

Let Y = total number of processes assigned to the first server.

<u>Goal</u> : compute E[Y]

"indicator random variable"

$$\text{Let } X_j = \begin{cases} 1 & \text{if jth process assigned to first server} \\ 0 & \text{otherwise} \end{cases}$$

$$Note \quad Y = \sum_{j=1}^{n} X_j$$

Load Balancing Solution (con'd)

We have

$$
\begin{aligned}
E[Y] &= E[\sum_{j=1}^{n} X_j] \\
&= \sum_{j=1}^{n} E[X_j] \\
&= \sum_{j=1}^{n} (Pr[X_j = 0] \cdot 0 + \underbrace{Pr[X_j = 1]}_{= 1/n \text{ (servers chosen uniformly at random)}} \cdot 1) \\
&= \sum_{j=1}^{n} \frac{1}{n} \quad = 1
\end{aligned}
$$

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Probability Review

Part II

Design and Analysis of Algorithms I

Topics Covered

- Conditional probability
- Independence of events and random variables

See also:

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Concept #1 – Sample Spaces

Sample Space Ω : "all possible outcomes"
[in algorithms, Ω is usually finite]

Also : each outcome $i \in \Omega$ has a probability p(i) >= 0

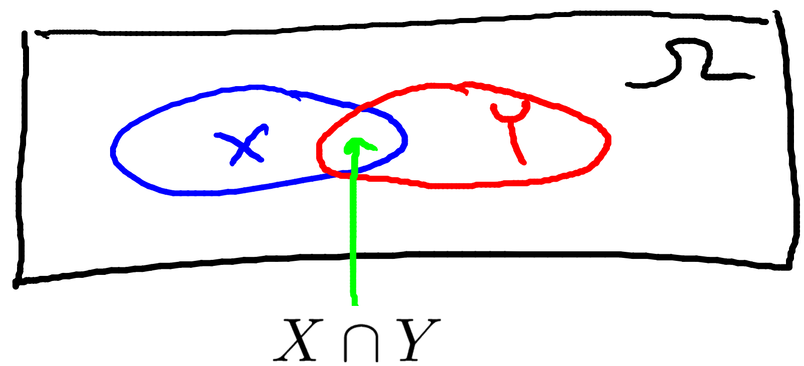
Constraint : $\sum_{i \in \Omega} p(i) = 1$

An event is a subset $S \subseteq \Omega$

The probability of an event S is $\sum_{i \in S} p(i)$

Concept #6 – Conditional Probability

$Let \quad X, Y \subseteq \Omega \quad be \quad events.$



$$Then \quad Pr[X|Y] = \frac{Pr[X \cap Y]}{Pr[Y]}$$

("X given Y")

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Suppose you roll two fair dice. What is the probability that at least one die is a 1, given that the sum of the two dice is 7?

- $^1/_{36}$
- $^1/_6$
- $^1/_3$
- $^1/_2$

X = at least one die is a 1
Y = sum of two dice = 7
= {(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)}

$$=> X \cap Y = \{(1,6),(6,1)\}$$

$$Pr[X|Y] = \frac{Pr[X \cap Y]}{Pr[Y]} = \frac{(2/36)}{(6/36)} = \frac{1}{3}$$

Concept #7 – Independence (of Events)

<u>Definition</u> : Events $X, Y \subseteq \Omega$ are independent if (and only if) $Pr[X \cap Y] = Pr[X] \cdot Pr[Y]$

<u>You check</u> : this holds if and only if Pr[X | Y] = Pr[X]
<==> Pr[Y|X] = Pr{Y]

<u>WARNING</u> : can be a very subtle concept.
(intuition is often incorrect!)

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Independence (of Random Variables)

<u>Definition</u> : random variables A, B (both defined on Ω)
are independent if and only if the events Pr[A=1], Pr[B=b] are independent for all a,b. [<==> Pr[A = a and B = b] = Pr[A=z]*Pr[B=b]]

<u>Claim</u> : if A,B are independent, then E[AB] = E[A]*E[B]

<u>Proof</u> :

$$E[AB] = \sum_{a,b} (a \cdot b) \cdot Pr[A = a \ and \ B = b]$$

$$= \sum_{a,b} (a \cdot b) \cdot Pr[A = a] \cdot Pr[B = b] \quad \text{(Since A,B independent)}$$

$$= \underbrace{\boxed{(\sum_{a} a \cdot Pr[A = a])}}_{E[A]} \underbrace{\boxed{(\sum_{b} b \cdot Pr[B = b])}}_{E[B]}$$

Q.E.D.

Example

Let $X_1, X_2 \in \{0,1\}$ be random, and $X_3 = X_1 \overset{\text{XOR}}{\oplus} X_2$

formally : $\Omega = \{000, 101, 011, 110\}$, each equally likely.

Claim : X_1 and X_3 are independent random variables (you check)

Claim : X_1X_3 and X_2 are not independent random variables.

Proof : suffices to show that

$$\underbrace{E[X_1X_2X_3]}_{=0} \neq \underbrace{E[X_1X_3]}_{= E[X1]E[X3] = 1/4} \overbrace{E[X_2]}^{=1/2}$$

Since X_1 and X_3 independent

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