Local solubility in families of superelliptic curves

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Motivation

Let C be a curve defined over $\mathbb{Q}$.

Definition

C is **soluble** if $C(\mathbb{Q})$ is nonempty.

Question (hard)

How often is a curve over $\mathbb{Q}$ (in some family) soluble?

For place v of $\mathbb{Q}$, we have

$$C(\mathbb{Q}) \subset C(\mathbb{Q}_v).$$

Thus existence of a $\mathbb{Q}_v$-point for each v is necessary but not sufficient for C to have $\mathbb{Q}$-point!

Local solubility

Let $C/\mathbb{Q}$ be a curve and v a place of $\mathbb{Q}$ (i.e. $v = p$ or $v = \infty$).

Definition

C is **locally soluble at v** if $C(\mathbb{Q}_v)$ is nonempty.

C is **everywhere locally soluble (ELS)** if $C(\mathbb{Q}_v) \neq \emptyset$ for all v.

Question (revised)

How often is a curve over $\mathbb{Q}$ (in some family) ELS?

Known for genus 1 curves [BCF21], plane cubics [BCF16], some families of hypersurfaces e.g. [BBL16], [FHP21], [PV04], [Bro17].

Motivation: hyperelliptic curves

Consider *hyperelliptic curves* given by (weighted) homog. equation

$$C\colon y^2 = f(x,z) = c_{2g+2}x^{2g+2} + \cdots + c_0 z^{2g+2}.$$

Theorem (Poonen–Stoll, Bhargava–Cremona–Fisher)

A pos. prop. of hyperelliptics $C/\mathbb{Q}$ are ELS [PS99b].

75.96% of genus 1 curves of this form are ELS [BCF21].

Theorem (Bhargava–Gross–Wang [BGW17])

A positive proportion of everywhere locally soluble hyperelliptic curves $C/\mathbb{Q}$ have no points over any odd degree extension $k/\mathbb{Q}$.

Superelliptic curves

Fix a positive integer $m \geq 2$.

Definition

A **superelliptic curve** $C/\mathbb{Q}$ is a smooth projective curve with a cyclic Galois cover of $\mathbb{P}^1$ of degree m.

Such C has an equation in weighted projective space

$$C\colon y^m = f(x,z) = c_d x^d + \cdots + c_0 z^d$$

where f is a binary form of degree d.

Warning

Some authors assume $m \mid d$ (or not!), or that f is m-th power free.

Defining the proportion

Question

How often is a superelliptic curve over $\mathbb{Q}$ ELS?

For $\mathbf{c} = (c_i)_{i=0}^d \in \mathbb{Z}^{d+1}$, we associate a binary form and SEC

$$f(x,z) = \sum_{i=0}^{d} c_i x^i z^{d-i}, \quad C_f\colon y^m = f(x,z).$$

Definition

We define

$$\rho_{m,d} = \lim_{B\to\infty} \frac{\#\{\mathbf{c} \in ([-B,B]\cap\mathbb{Z})^{d+1} \mid C_f \text{ is ELS}\}}{\#\{\mathbf{c} \in ([-B,B]\cap\mathbb{Z})^{d+1}\}},$$

the proportion of ELS superelliptic curves of this form.

Main results

Fix $(m, d) \neq (2, 2)$ such that $m \mid d$.

Theorem (Beneish–K. [BK21b])

(A) *$0 < \rho_{m,d} < 1$, and $\rho_{m,d}$ is product of local densities,*

$$\rho_{m,d} = \rho_{m,d}(\infty) \prod_p \rho_{m,d}(p).$$

Main results

Fix $(m, d) \neq (2, 2)$ such that m is prime and $m \mid d$.

Theorem (Beneish–K. [BK21b], continued)

(B) *We can find explicit (and sometimes good) bounds for $\rho_{m,d}(p)$ and hence $\rho_{m,d}$. In particular,*

$$\liminf_{d\to\infty} \rho_{m,d} \geq \left(1 - \frac{1}{m^{m+1}}\right) \prod_{p\equiv 1(m)} \left(1 - \left(1 - \frac{p-1}{mp}\right)^{p+1}\right) \prod_{p\not\equiv 0,1(m)} \left(1 - \frac{1}{p^{2(p+1)}}\right).$$

When $m > 2$, we have

$$0.83511 \leq \liminf_{d\to\infty} \rho_{m,d} \quad \text{and} \quad \limsup_{d\to\infty} \rho_{m,d} \leq 0.99804.$$

Main results

Theorem (Beneish–K. [BK21b], continued)

(C) *In the case* $(m, d) = (3, 6)$, *we compute* $\rho_{3,6} \approx 96.94\%$. *Moreover,* $\exists$ *rational functions* $R_1(t)$ *and* $R_2(t)$ *such that*

$$\rho_{3,6}(p) = \begin{cases} R_1(p), & p \equiv 1 \pmod 3 \text{ and } p > 43 \\ R_2(p), & p \equiv 2 \pmod 3 \text{ and } p > 2. \end{cases}$$

Asymptotically,

$$1 - R_1(t) \sim \frac{2}{3} t^{-4},$$
$$1 - R_2(t) \sim \frac{53}{144} t^{-7}.$$

$$\rho = \begin{cases} \Big(1296p^{57} + 3888p^{56} + 9072p^{55} + 16848p^{54} + 27648p^{53} + 39744p^{52} + 53136p^{51} + 66483p^{50} + 80019p^{49} + 93141p^{48} \\ + 107469p^{47} + 120357p^{46} + 135567p^{45} + 148347p^{44} + 162918p^{43} + 176004p^{42} + 190278p^{41} + 203459p^{40} \\ + 218272p^{39} + 232083p^{38} + 243639p^{37} + 255267p^{36} + 261719p^{35} + 264925p^{34} + 265302p^{33} + 261540p^{32} \\ + 254790p^{31} + 250736p^{30} + 241384p^{29} + 226503p^{28} + 214137p^{27} + 195273p^{26} + 170793p^{25} + 151839p^{24} + 136215p^{23} \\ + 118998p^{22} + 105228p^{21} + 94860p^{20} + 80471p^{19} + 67048p^{18} + 52623p^{17} + 40617p^{16} + 28773p^{15} + 19247p^{14} & p \equiv 1 \pmod 3 \\ + 12109p^{13} + 7614p^{12} + 3420p^{11} + 756p^{10} - 2248p^{9} - 4943p^{8} - 6300p^{7} - 6894p^{6} - 5994p^{5} - 2448p^{4} - 648p^{3} \\ + 324p^{2} + 1296p + 1296\Big) \Big/ \Big(1296\left(p^{12} - p^{11} + p^{9} - p^{8} + p^{6} - p^{4} + p^{3} - p + 1\right)\left(p^{8} - p^{6} + p^{4} - p^{2} + 1\right) \\ \times \left(p^{6} + p^{5} + p^{4} + p^{3} + p^{2} + p + 1\right)\left(p^{4} + p^{3} + p^{2} + p + 1\right)^{3}\left(p^{4} - p^{3} + p^{2} - p + 1\right)\left(p^{2} + p + 1\right) \\ \times \left(p^{2} + 1\right)p^{11}\Big), \\ \\ \Big(144p^{57} + 432p^{56} + 1008p^{55} + 1872p^{54} + 3168p^{53} + 4608p^{52} + 6336p^{51} + 8011p^{50} + 9803p^{49} + 11357p^{48} \\ + 13061p^{47} + 14525p^{46} + 16295p^{45} + 17875p^{44} + 19654p^{43} + 21212p^{42} + 23030p^{41} + 24563p^{40} + 26320p^{39} \\ + 27771p^{38} + 29711p^{37} + 30859p^{36} + 31135p^{35} + 31525p^{34} + 31510p^{33} + 29436p^{32} + 28502p^{31} + 28616p^{30} \\ + 26856p^{29} + 25087p^{28} + 25057p^{27} + 23041p^{26} + 19921p^{25} + 18119p^{24} + 16287p^{23} + 13798p^{22} \\ + 12140p^{21} + 10844p^{20} + 9191p^{19} + 7480p^{18} + 5839p^{17} + 4265p^{16} + 2909p^{15} + 1943p^{14} + 1109p^{13} & p \equiv 2 \pmod 3 \\ + 590p^{12} + 604p^{11} + 372p^{10} - 144p^{9} - 87p^{8} - 84p^{7} - 678p^{6} - 618p^{5} - 144p^{4} - 168p^{3} - 156p^{2} \\ + 144p + 144\Big) \Big/ \Big(144\left(p^{12} - p^{11} + p^{9} - p^{8} + p^{6} - p^{4} + p^{3} - p + 1\right)\left(p^{8} - p^{6} + p^{4} - p^{2} + 1\right) \\ \times \left(p^{6} + p^{5} + p^{4} + p^{3} + p^{2} + p + 1\right)\left(p^{4} + p^{3} + p^{2} + p + 1\right)^{3}\left(p^{4} - p^{3} + p^{2} - p + 1\right)\left(p^{2} + p + 1\right) \\ \times \left(p^{2} + 1\right)p^{11}\Big), \end{cases}$$

Outline

- Set up and state main results,
- Local densities $\rho_{m,d}(p) \to$ global density $\rho_{m,d}$,
- Study local densities $\rho_{m,d}(p)$,
- Sketch exact computations of $\rho_{3,6}(p)$.

Local densities

Theorem (Beneish–K. [BK21b])

(A) $\rho_{m,d}$ *exists and is given by the product of local densities,*

$$\rho_{m,d} = \rho_{m,d}(\infty) \prod_p \rho_{m,d}(p) > 0.$$

$\rho_{m,d}(p)$ is (normalized) Haar measure of space of the $\mathbb{Q}_p$-soluble curves $C_f\colon y^m = f(x,z)$, with coefficients in $\mathbb{Z}_p$.

Idea

In *good situations*, imposing conditions at different primes looks independent...*even if there are infinitely many conditions*.

Local densities look independent

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Think

Recall squarefree numbers.

$$n \text{ squarefree} \iff p^2 \nmid n \text{ for all } p.$$

If probabilities that $p^2 \mid n$ are independent, expect

$$\mathrm{Prob}(n \text{ squarefree}) = \prod_p \left(1 - \frac{1}{p^2}\right) = \frac{1}{\zeta(2)} = \frac{6}{\pi^2}.$$

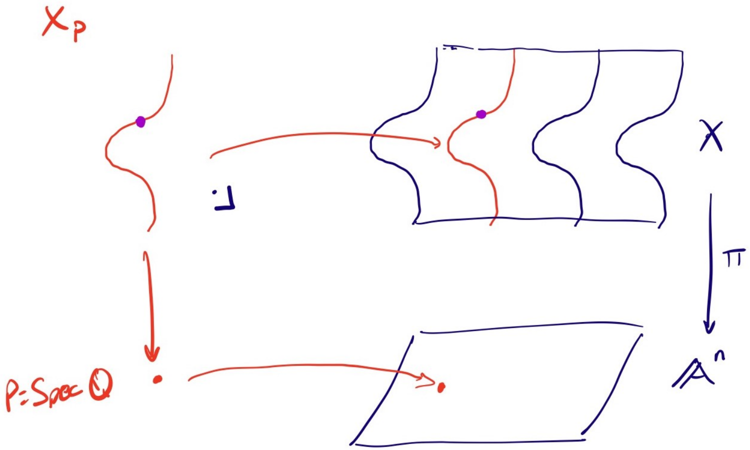
Local densities look independent

Idea

In *good situations*, imposing conditions at different primes looks independent...*even if there are infinitely many conditions*.

- Poonen–Stoll [PS99a] give criterion for when natural density is product of local densities.
- Apply to ELS in families of hyperelliptic curves [PS99b]; uses sieve of Ekedahl [Eke91].
- Bright–Browning–Loughran [BBL16] give *geometric criteria* when family comes from fibers of a morphism.

Geometric picture



A geometric criterion

Theorem (Bright–Browning–Loughran [BBL16])

Let $\pi\colon X \to \mathbb{A}^n$ *a dominant, quasiproj. morphism of* $\mathbb{Q}$*-varieties with geom. int. gen. fiber. Suppose*

(i) *fibers above each codim. 1 point of* $\mathbb{A}^n$ *are geom. integral,*

(ii) $X(\mathbf{A}_{\mathbb{Q}}) \neq \emptyset$,

(iii) *For all* $B \geq 1$ *we have* $B\pi(X(\mathbb{R})) \subseteq \pi(X(\mathbb{R}))$.

Let $\Psi' \subset \mathbb{R}^n$ *be a bounded subset of positive measure lying in* $\pi(X(\mathbb{R}))$ *whose boundary has measure zero. Then the limit*

$$\lim_{B\to\infty} \frac{\#\left\{P \in \mathbb{Z}^n \cap B\Psi' \mid X_P(\mathbf{A}_{\mathbb{Q}}) \neq \emptyset\right\}}{\#\left\{P \in \mathbb{Z}^n \cap B\Psi'\right\}}$$

exists, is nonzero, and is equal to a product of local densities,

$$\prod_{p\nmid\infty} \mu_p\left(\left\{P \in \mathbb{Z}_p^n \mid X_P(\mathbb{Q}_p) \neq \emptyset\right\}\right).$$

Geometric setup

We consider

$$\mathbb{A}_{\mathbb{Q}}^{d+1} = \operatorname{Spec} \mathbb{Q}[c_0, \ldots, c_d],$$
$$\mathcal{P}_{\mathbb{Q}} = \mathbb{P}_{\mathbb{Q}}(1, d, 1) \text{ with coordinates } [x : y : z].$$

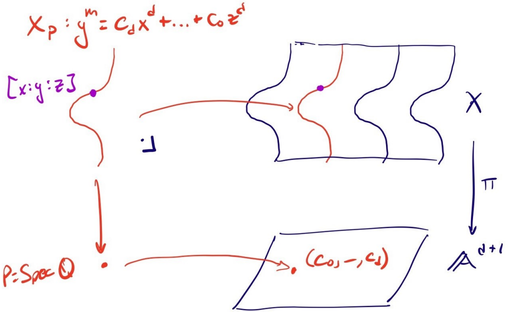
The variety

$$X \colon y^m = c_d x^d + \cdots + c_0 z^d \subset \mathbb{A}_{\mathbb{Q}}^{d+1} \times \mathcal{P}_{\mathbb{Q}}$$

comes with a projection map $\pi \colon X \to \mathbb{A}_{\mathbb{Q}}^{d+1}$.

Geometric picture

$$X\colon y^m = c_d x^d + \cdots + c_0 z^d \subset \mathbb{A}^{d+1}_{\mathbb{Q}} \times \mathcal{P}_{\mathbb{Q}}$$



Think

- A $\mathbb{Q}$-point $(\mathbf{c}, [x:y:z])$ of X is the data of superelliptic curve $C_f/\mathbb{Q}$ and a $\mathbb{Q}$-point $[x:y:z] \in C_f(\mathbb{Q})$.
- The fiber X_P of π over a point $P \in \mathbb{A}^{d+1}(\mathbb{Q})$ is a superelliptic curve $C_f/\mathbb{Q}$ whose coefficients are encoded in P.

Proof sketch of (A)

Check that π is dominant, projective, and has geom. int. gen. fiber.

(i) Codim. 1 points of $\mathbb{A}^{d+1}$ = single relation on coeffs c_i. Not enough to be reducible. (Unless $(m, d) = (2, 2)$!)

(ii) $X(\mathbb{Q}) \neq \emptyset$; e.g. $y^m = x^d + z^d$ has the point $[1:1:0]$.

(iii) $\pi(X(\mathbb{R}))$ closed under scaling by $B \geq 1$: C_f has a $\mathbb{R}$-point $\implies$ $C_{Bf}\colon y^m = Bf(x, z)$ has $\mathbb{R}$-point.

Finally, choose $\Psi' = [-1, 1]^{d+1} \cap \pi(X(\mathbb{R}))$ (verifying $\mu_\infty(\partial\Psi') = 0$), and see this agrees with original definition of $\rho_{m,d}$. $\square$

Outline

- Set up and state main results,
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- Bound local densities $\rho_{m,d}(p)$,
- Sketch exact computations of $\rho_{3,6}(p)$.

Computing local densities

Question

Once we know

$$\rho_{m,d} = \rho_{m,d}(\infty) \prod_p \rho_{m,d}(p),$$

how do we compute/estimate local densities $\rho_{m,d}(p)$?

$\rho_{m,d}(\infty)$: Euclidean measure of $\mathbb{R}$-soluble C_f with coeffs $\in [-1, 1]$.

- If m or d is odd, then $\rho_{m,d}(\infty) = 1$.
- If m, d even, no analytic solution known for $d > 2$, but rigorous estimates exist, e.g.

$$0.873914 \leq \rho_{2,4}(\infty) \leq 0.874196 \quad \text{[BCF21]}.$$

Computing local densities — finite places

$\rho_{m,d}(p)$ is (normalized) Haar measure of space of the $\mathbb{Q}_p$-soluble curves $C_f\colon y^m = f(x,z)$, *with coefficients in* $\mathbb{Z}_p$.

Think

Each possible reduction $\overline{f}(x,z)$ mod p occurs equally often.

Look mod p and check $\mathbb{Q}_p$-solubility with **Hensel's lemma**!

- Smooth $\mathbb{F}_p$-points on $\overline{C_f}$ lift to $\mathbb{Q}_p$-solutions (Hensel),
- $\overline{C_f}(\mathbb{F}_p) = \emptyset \implies$ no $\mathbb{Q}_p$-solutions,
- If $\overline{C_f}(\mathbb{F}_p)$ only non-smooth points, do more work.

An extended example

Example

Consider $(m, d) = (3, 6)$, family of genus 4 curves

$$C_f\colon y^3 = f(x,z) = c_6x^6 + c_5x^5z + \cdots + c_1xz^5 + c_0z^6.$$

When can we guarantee $\overline{C_f}$ has smooth $\mathbb{F}_p$-points?

Theorem (Hasse–Weil bound)

If $\overline{C_f}$ is irreducible and smooth of genus g, then

$$\#\overline{C_f}(\mathbb{F}_p) \geq p + 1 - g \cdot 2\sqrt{p}.$$

An extended example

Example

Consider $(m, d) = (3, 6)$, family of genus 4 curves

$$C_f \colon y^3 = f(x, z) = c_6 x^6 + c_5 x^5 z + \cdots + c_1 x z^5 + c_0 z^6.$$

When can we guarantee $\overline{C_f}$ has smooth $\mathbb{F}_p$-points?

Theorem (Hasse–Weil bound, refined)

If $\overline{C_f}$ is irreducible and smooth of genus g, then

$$\#\overline{C_f}(\mathbb{F}_p) \geq p + 1 - g \cdot \lfloor 2\sqrt{p} \rfloor.$$

An extended example

Whenever $p > 61$, we have

$$p + 1 - 8\sqrt{p} > 0,$$

so if $\overline{C_f}/\mathbb{F}_p$ is smooth for $p > 61$, C_f has $\mathbb{Q}_p$-point!

- $\overline{C_f}^{\mathrm{sm}}(\mathbb{F}_p) \neq \emptyset$ whenever $\overline{C_f}/\mathbb{F}_p$ geom. irr. and $p > 61$.
- Refinement of H–W $\implies$ $p \geq 61$ sufficient.
- Irreducibility over $\overline{\mathbb{F}_p}$ $\iff$ $\overline{f}(x, z) \neq ah(x, z)^3$.

$$\rho_{3,6}(p) \geq \frac{p^7 - p^3}{p^7} = 1 - \frac{1}{p^4} \text{ for all } p \geq 61.$$

An extended example — bounds for $p \equiv 2 \pmod 3$

Exploit fact that cubing map $\mathbb{F}_p^\times \xrightarrow{(\cdot)^3} \mathbb{F}_p^\times$ is an isomorphism.

Lemma

If $p > 2$ and $p \equiv 2 \pmod 3$ then C_f has a $\mathbb{Z}_p$-point whenever reduction $\overline{f}$ is nonzero.

What goes wrong? $\overline{f}(x, z)$ has multiple roots everywhere.

Example

If $p = 2$, could have $f(x, z) = x^2(x + z)^2 z^2$

An extended example

- $\rho_{3,6}(p) \geq 1 - \frac{1}{p^4}$ when $p \equiv 1 \pmod 3$ and $p > 43$
- $\rho_{3,6}(p) \geq 1 - \frac{1}{p^7}$ when $p \equiv 2 \pmod 3$ and $p > 2$
- Enumerate all $\overline{f}(x, z)$ and count Hensel-liftable $\mathbb{F}_p$-solutions:

p	$\rho_{3,6}(p) \geq$	p	$\rho_{3,6}(p) \geq$
2	$\frac{63}{64} \approx 0.98437$	19	$\frac{893660256}{893871739} \approx 0.99976$
3	$\frac{26}{27} \approx 0.96296$	31	$\frac{27512408250}{27512614111} \approx 0.99999$
7	$\frac{810658}{823543} \approx 0.98435$	37	$\frac{94931742132}{94931877133} \approx 0.999998$
13	$\frac{62655132}{62748517} \approx 0.99851$	43	$\frac{271818511748}{271818611107} \approx 0.9999996$

Put together with Theorem A:

$$\rho_{3,6} = \prod_p \rho_{3,6}(p) \geq 0.93134.$$

Bounds more generally for $m = 3$

Example (Lower bounds for general d)

For $d > 6$ such that $3 \mid d$,

$$\rho_{3,d} \geq \left(1 - \frac{1}{3^4}\right) \prod_{\substack{p\equiv 2(3)\\ p\leq d/2-1}} \left(1 - \frac{1}{p^{2(p+1)}}\right) \prod_{\substack{p\equiv 2(3)\\ p> d/2-1}} \left(1 - \frac{1}{p^{d+1}}\right)$$

$$\times \prod_{\substack{p\equiv 1(3)\\ p<d}} \left(1 - \left(1 - \frac{p-1}{3p}\right)^{p+1}\right) \prod_{\substack{p\equiv 1(3)\\ d<p<4(d-2)^2}} \left(1 - \left(1 - \frac{p-1}{3p}\right)^{d+1}\right) \prod_{\substack{p\equiv 1(3)\\ p\geq 4(d-2)^2}} \left(1 - \frac{1}{p^{\frac{2d}{3}}}\right)$$

Example (Large genus limit)

Taking limit as $d \to \infty$

$$\liminf_{d\to\infty} \rho_{3,d} \geq \left(1 - \frac{1}{3^4}\right) \prod_{p\equiv 1(3)} \left(1 - \left(1 - \frac{p-1}{3p}\right)^{p+1}\right) \prod_{p\equiv 2(3)} \left(1 - \frac{1}{p^{2(p+1)}}\right)$$

$$\approx 0.90061.$$

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- Set up and state main results,
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- Bound local densities $\rho_{m,d}(p)$,
- Sketch exact computations of $\rho_{3,6}(p)$.

Getting exact answer

Question

How do we go from bounds to exact values for $\rho_{3,6}(p)$?

Let $F(x, y, z) = y^3 - f(x, z)$ and look at reduction modulo p.

Recall $\overline{F}$ irreducible/$\overline{\mathbb{F}_p} \iff f(x, z) \neq h(x, z)^3$ over $\overline{\mathbb{F}_p}$.

Factorization type in y	$p = 3$	$p \equiv 1 \pmod 3$	$p \equiv 2 \pmod 3$
1. Abs. irr.	2160	$p^3(p^4 - 1)$	$p^3(p^4 - 1)$
2. 3 distinct linear over $\mathbb{F}_p$	0	$\frac{1}{3}(p^3 - 1)$	0
3. Linear + conj.	0	0	$p^3 - 1$
4. 3 conjugate factors	0	$\frac{2}{3}(p^3 - 1)$	0
5. $(y - h(x, z))^3$	27	1	1
Total	3^7	p^7	p^7

Getting exact answer

Let ξ_i be the proportion of $\overline{f}$ for which $\overline{F}$ has type i.

Let σ_i be the probability that $F(x, y, z) = 0$ has $\mathbb{Z}_p$-solution when $\overline{F}$ has type i. Then

$$\rho_{3,6}(p) = \sum_{i=1}^{5} \xi_i(p)\sigma_i(p).$$

Proposition

We have

$$\sigma_1 = \sigma_2 = \sigma_3 = 1$$

for all primes $p \geq 61$ *and* $p \equiv 2 \pmod 3$ *except* $p = 2$.

Getting exact answer

Proposition

We have

$$\sigma_1 = \sigma_2 = \sigma_3 = 1$$

for all primes $p \geq 61$ *and* $p \equiv 2 \pmod 3$ *except* $p = 2$.

Proof. We (essentially) already did this! Use Hasse–Weil bound on all components, possibly avoiding desingularized points. □

To improve on previous bounds, we

- carefully analyze σ_4, σ_5 and
- deal with more delicate primes $p = 2, 3, 7, 13, 19, 31, 37, 43$.

An example: computing σ_5

Suppose $f(x,z) \equiv 0 \pmod{p}$, but $f(x,z) \not\equiv 0 \pmod{p^2}$.

Set $f(x,z) \equiv pf_1(x,z)$ for nonzero $f_1(x,z) \in \mathbb{F}_p[x,z]$.

Observation

$\mathbb{Z}_p$-solution to $C_f \colon y^3 = f(x,z)$ must have $p \mid y$,

$$p^3 \mid f(x,z) \implies p^2 \mid f_1(x,z).$$

(0) If $\overline{f_1}(x,z)$ has no roots modulo p, then C_f has no $\mathbb{Z}_p$-points.

(1) If $\overline{f_1}(x,z)$ has a root of mult. 1, it lifts to $\mathbb{Z}_p$-point of C_f.

(2) Suppose $\overline{f_1}(x,z)$ has a double root (and no other roots).

Dealing with the double root

Assume $x^2 \mid \overline{f_1}$, giving p-adic valuations below (original coeffs of f):

Prob. lift	$v(c_6)$	$v(c_5)$	$v(c_4)$	$v(c_3)$	$v(c_2)$	$v(c_1)$	$v(c_0)$
$\tau_2 = \tau_{2a} = \frac{1}{p}\tau_{2b}$	≥ 1	≥ 1	≥ 1	≥ 1	$= 1$	≥ 2	≥ 2
$\tau_{2b} = \tau_{2c}$	≥ 1	≥ 1	≥ 1	≥ 1	$= 1$	≥ 2	≥ 3
$\tau_{2c} = 1$	≥ 4	≥ 3	≥ 2	≥ 1	$= 0$	≥ 0	≥ 0

Probability of lifting $[0:0:1]$ in this case is

$$\tau_2 = \frac{1}{p} = \text{Prob}\left(p^3 \mid c_0 : p^2 \mid c_0 \text{ and } p \mid\mid c_2\right).$$

Computing σ_5

$$\sigma_5 = \left(1 - \frac{1}{p^7}\right) \sum_{i=0}^{9} \eta_i \tau_i + \left(\frac{1}{p^7} - \frac{1}{p^{14}}\right) \sum_{i=0}^{9} \eta_i \theta_i + \frac{1}{p^{14}} \rho$$

- Index i indicates factorization type of $f_1(x,z)$ (or $f_2(x,z)$)
- η_i = proportion of sextic forms/$\mathbb{F}_p$ with i-th type
- τ_i (resp. θ_i) are proportion of f with f_1 (resp. f_2) of type i such that C_f has a $\mathbb{Z}_p$-point.


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1st annual?
theta & tau
PALOOZA
When life give you sigmas and rhos,
make etas, thetas, and taus, with
subscripts that just can't be contained!
θ
τ

Factorization types

Fact. type	η_i	η_i' (monic forms only)
0. No roots	$\dfrac{\left(53p^4+26p^3+19p^2-2p+24\right)(p-1)p}{144\left(p^6+p^5+p^4+p^3+p^2+p+1\right)}$	$\dfrac{\left(53p^4+26p^3+19p^2-2p+24\right)(p-1)}{144p^5}$
1. $(1*)$	$\dfrac{\left(91p^4+26p^3+23p^2+16p-12\right)(p+1)p}{144\left(p^6+p^5+p^4+p^3+p^2+p+1\right)}$	$\dfrac{\left(91p^3-27p^2+50p-48\right)(p+1)(p-1)}{144p^5}$
2. (1^24) or (1^222)	$\dfrac{\left(3p^2+p+2\right)(p+1)(p-1)p}{8\left(p^6+p^5+p^4+p^3+p^2+p+1\right)}$	$\dfrac{\left(3p^2+p+2\right)(p-1)}{8p^4}$
3. (1^21^22)	$\dfrac{(p+1)(p-1)p^2}{4\left(p^6+p^5+p^4+p^3+p^2+p+1\right)}$	$\dfrac{(p-1)^2}{4p^4}$
4. $(1^21^21^2)$	$\dfrac{(p+1)(p-1)p}{6\left(p^6+p^5+p^4+p^3+p^2+p+1\right)}$	$\dfrac{(p-1)(p-2)}{6p^5}$
5. (1^33)	$\dfrac{(p+1)^2(p-1)p}{3\left(p^6+p^5+p^4+p^3+p^2+p+1\right)}$	$\dfrac{(p+1)(p-1)}{3p^4}$
6. (1^31^3)	$\dfrac{(p+1)p}{2\left(p^6+p^5+p^4+p^3+p^2+p+1\right)}$	$\dfrac{p-1}{2p^5}$
7. (1^42)	$\dfrac{(p+1)(p-1)p}{2\left(p^6+p^5+p^4+p^3+p^2+p+1\right)}$	$\dfrac{p-1}{2p^4}$
8. (1^21^4)	$\dfrac{(p+1)p}{p^6+p^5+p^4+p^3+p^2+p+1}$	$\dfrac{p-1}{p^5}$
9. (1^6)	$\dfrac{p+1}{p^6+p^5+p^4+p^3+p^2+p+1}$	$\dfrac{1}{p^5}$

Type 9: yikes!

Type 9, e.g. $f(x,z) \equiv px^6 \pmod{p^2}$.

τ_9 is a degree 44 rational function in p.

$$
\begin{aligned}
\tau_9 = \quad \tau_{9a} &= \tfrac{1}{p}\tau_{9b} \\
\tau_{9b} &= \left(1-\tfrac{1}{p}\right) + \tfrac{1}{p}\tau_{9c} \\
\tau_{9c} &= \Phi(p) + \tfrac{1}{p}\tau_{9d} \\
\tau_{9d} &= \left(1-\tfrac{1}{p}\right)\left(\tfrac{p-1}{2p} + \tfrac{1}{p^2}\right) + \tfrac{1}{p}\tau_{9e} \\
\tau_{9e} &= \left(1-\tfrac{1}{p}\right) + \tfrac{1}{p}\tau_{9f} \\
\tau_{9f} &= \tfrac{1}{p}\tau_{9g} \\
\tau_{9g} &= \left(1-\tfrac{1}{p}\right)\alpha'' + \tfrac{1}{p}\tau_{9h} \\
\tau_{9h} &= \left(1-\tfrac{1}{p}\right)\left(\tfrac{p-1}{2p} + \tfrac{\theta_2}{p}\right) + \tfrac{1}{p}\tau_{9i} \\
\tau_{9i} &= \left(1-\tfrac{1}{p}\right) + \tfrac{1}{p}\tau_{9j} \\
\tau_{9j} &= \tfrac{1}{p}\tau_{9k} \\
\tau_{9k} &= \left(1-\tfrac{1}{p}\right) + \tfrac{1}{p}\tau_{9\ell} \\
\tau_{9\ell} &= \Phi(p) + \left(1-\Phi(p)-\tfrac{1}{p}\right)\beta + \tfrac{1}{p}\tau_{9m} \\
\tau_{9m} &= \left(1-\tfrac{1}{p}\right) + \tfrac{1}{p}\tau_{9n} \\
\tau_{9n} &= \left(1-\tfrac{1}{p}\right) + \tfrac{1}{p}\tau_{9o} \\
\tau_{9o} &= \Phi(p) + \tfrac{1}{p}\tau_{9p} \\
\tau_{9p} &= \sigma_5'
\end{aligned}
$$

c_6	c_5	c_4	c_3	c_2	c_1	c_0
$=1$	≥ 2	≥ 2	≥ 2	≥ 2	≥ 2	≥ 2
$=4$	≥ 4	≥ 3	≥ 2	≥ 1	≥ 0	≥ 0
$=4$	≥ 4	≥ 3	≥ 2	≥ 1	≥ 1	≥ 0
$=4$	≥ 4	≥ 3	≥ 2	≥ 1	≥ 1	≥ 1
$=4$	≥ 4	≥ 3	≥ 2	≥ 2	≥ 1	≥ 1
$=4$	≥ 4	≥ 3	≥ 2	≥ 2	≥ 2	≥ 1
$=4$	≥ 4	≥ 3	≥ 2	≥ 2	≥ 2	≥ 2
$=4$	≥ 4	≥ 3	≥ 3	≥ 2	≥ 2	≥ 2
$=4$	≥ 4	≥ 3	≥ 3	≥ 3	≥ 2	≥ 2
$=4$	≥ 4	≥ 3	≥ 3	≥ 3	≥ 3	≥ 2
$=1$	≥ 1	≥ 0	≥ 0	≥ 0	≥ 0	≥ 0
$=1$	≥ 1	≥ 1	≥ 0	≥ 0	≥ 0	≥ 0
$=1$	≥ 1	≥ 1	≥ 1	≥ 0	≥ 0	≥ 0
$=1$	≥ 1	≥ 1	≥ 1	≥ 1	≥ 0	≥ 0
$=1$	≥ 1	≥ 1	≥ 1	≥ 1	≥ 1	≥ 0
$=1$	≥ 1	≥ 1	≥ 1	≥ 1	≥ 1	≥ 1

What is $\rho_{3,6}(p)$?

$$\rho = \begin{cases} \Big(1296p^{57} + 3888p^{56} + 9072p^{55} + 16848p^{54} + 27648p^{53} + 39744p^{52} + 53136p^{51} + 66483p^{50} + 80019p^{49} + 93141p^{48} \\ + 107469p^{47} + 120357p^{46} + 135567p^{45} + 148347p^{44} + 162918p^{43} + 176004p^{42} + 190278p^{41} + 203459p^{40} \\ + 218272p^{39} + 232083p^{38} + 243639p^{37} + 255267p^{36} + 261719p^{35} + 264925p^{34} + 265302p^{33} + 261540p^{32} \\ + 254790p^{31} + 250736p^{30} + 241384p^{29} + 226503p^{28} + 214137p^{27} + 195273p^{26} + 170793p^{25} + 151839p^{24} + 136215p^{23} \\ + 118998p^{22} + 105228p^{21} + 94860p^{20} + 80471p^{19} + 67048p^{18} + 52623p^{17} + 40617p^{16} + 28773p^{15} + 19247p^{14} & p \equiv 1 \pmod 3 \\ + 12109p^{13} + 7614p^{12} + 3420p^{11} + 756p^{10} - 2248p^{9} - 4943p^{8} - 6300p^{7} - 6894p^{6} - 5994p^{5} - 2448p^{4} - 648p^{3} \\ + 324p^{2} + 1296p + 1296\Big) \Big/ \Big(1296\left(p^{12} - p^{11} + p^{9} - p^{8} + p^{6} - p^{4} + p^{3} - p + 1\right)\left(p^{8} - p^{6} + p^{4} - p^{2} + 1\right) \\ \times \left(p^{6} + p^{5} + p^{4} + p^{3} + p^{2} + p + 1\right)\left(p^{4} + p^{3} + p^{2} + p + 1\right)^{3}\left(p^{4} - p^{3} + p^{2} - p + 1\right)\left(p^{2} + p + 1\right) \\ \times \left(p^{2} + 1\right)p^{11}\Big), \\ \\ \Big(144p^{57} + 432p^{56} + 1008p^{55} + 1872p^{54} + 3168p^{53} + 4608p^{52} + 6336p^{51} + 8011p^{50} + 9803p^{49} + 11357p^{48} \\ + 13061p^{47} + 14525p^{46} + 16295p^{45} + 17875p^{44} + 19654p^{43} + 21212p^{42} + 23030p^{41} + 24563p^{40} + 26320p^{39} \\ + 27771p^{38} + 29711p^{37} + 30859p^{36} + 31135p^{35} + 31525p^{34} + 31510p^{33} + 29436p^{32} + 28502p^{31} + 28616p^{30} \\ + 26856p^{29} + 25087p^{28} + 25057p^{27} + 23041p^{26} + 19921p^{25} + 18119p^{24} + 16287p^{23} + 13798p^{22} \\ + 12140p^{21} + 10844p^{20} + 9191p^{19} + 7480p^{18} + 5839p^{17} + 4265p^{16} + 2909p^{15} + 1943p^{14} + 1109p^{13} & p \equiv 2 \pmod 3 \\ + 590p^{12} + 604p^{11} + 372p^{10} - 144p^{9} - 87p^{8} - 84p^{7} - 678p^{6} - 618p^{5} - 144p^{4} - 168p^{3} - 156p^{2} \\ + 144p + 144\Big) \Big/ \Big(144\left(p^{12} - p^{11} + p^{9} - p^{8} + p^{6} - p^{4} + p^{3} - p + 1\right)\left(p^{8} - p^{6} + p^{4} - p^{2} + 1\right) \\ \times \left(p^{6} + p^{5} + p^{4} + p^{3} + p^{2} + p + 1\right)\left(p^{4} + p^{3} + p^{2} + p + 1\right)^{3}\left(p^{4} - p^{3} + p^{2} - p + 1\right)\left(p^{2} + p + 1\right) \\ \times \left(p^{2} + 1\right)p^{11}\Big), \end{cases}$$

What is $\rho_{3,6}(p)$? Small primes edition

$$\rho(2) = \frac{4594897725819217081}{4616483254090301440\!0} \approx 0.99532$$

$$\rho(3) = \frac{900175334869743731875930997281}{908381960435133191895132960000} \approx 0.99096$$

$$\rho(7) = \frac{631044947551786228516032926231872770547437301836456778939 72}{640831747872066968824299456558012815388441498964001598153 75} \approx 0.98472$$

$$\rho(13) = \frac{787772835724457741402590193129674740968207625566652698451527352682285 3}{789064357062010674777673729279278062351072702642077953989377239970147 5} \approx 0.99836$$

$$\rho(19) = \frac{3122673715489206150449285868243361150392235799365815266879438393279346795671}{3123410013311365155035964479837966797560851333614271490136481337080636454180} \approx 0.99976$$

$$\rho(31) = \frac{919679645767831886913908993678646214653521003983285045429787748202063507385715975829 9}{919686506158784354483098904147380879891312858742599564585782857261091843603583390725 0} \approx 0.999992$$

$$\rho(37) = \frac{1711286479008201947844581017879529201699244648865190554538446471541848050364474766403457351 19}{1711288896361570605368944741870170884642712365099771994912089394497381276586797237155889445 00} \approx 0.999998$$

$$\rho(43) = \frac{8400012134328309038865335643180410070733136477929066449054710576886784486271213444783272050875028 1}{8400015167151355519164771256759610171080084620911683056801372937740499115090197310509303993923750 0} \approx 0.9999996$$

Use `Magma` to help when Hasse–Weil doesn't apply, modify calculations accordingly.

What is $\rho_{3,6}$?

Theorem (Beneish-K.)

(C) *We have determined $\rho_{3,6}(p)$ exactly for all p.*

Taking product over $p \leq 10000$ gives

$$\rho_{3,6} \approx \prod_{p \leq 10000} \rho_{3,6}(p) = 0.96943,$$

with error of $O(10^{-14})$.

97% of superelliptic curves $y^3 = c_6x^6 + \ldots + c_0z^6$ are ELS.

Further questions

Question

What proportion of superelliptic curves $C_f\colon y^m = f(x,z)$

- *are globally soluble?*
- *satisfy/fail the Hasse principle?*
- *satisfy/fail weak approximation?*
- *have some/no points of certain higher degrees?*

Preliminary results [BK21a, Prop. 7.2] give conditions for which pos. prop. of SECs have finitely many points of certain degrees.

Study these/other solubility questions for more families. Can methods be adapted to integral pts. on stacky curves (see [BP20])?

Thank you I

Thank you for the invitation and for your attention!

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