Local solubility in families of superelliptic curves

Christopher Keyes
(Emory University $\rightarrow$ King's College London)

Joint work with Lea Beneish
(University of North Texas)

https://arxiv.org/abs/2111.04697

University of North Texas
September 15, 2023

Motivation

Consider a deg. d homogeneous polynomial $f(x,z) \in \mathbb{Z}[x,z]$.

Definition

f **represents an m$^{\text{th}}$ power** if there exist integers $x_0, y_0, z_0 \in \mathbb{Z}$, such that $f(x_0, z_0) = y_0^m$.

Motivation

Consider a deg. d homogeneous polynomial $f(x,z) \in \mathbb{Z}[x,z]$.

Definition

f **represents an m$^{\text{th}}$ power** if there exist integers $x_0, y_0, z_0 \in \mathbb{Z}$, such that $f(x_0, z_0) = y_0^m$.

Example

Let $m = 3$ and $f(x,z) = 2x^6 + x^4z^2 + 2x^3z^3 + 3z^6$.

We have $f(1,1) = 8 = 2^3$, so f represents a cube.

Motivation

Consider a deg. d homogeneous polynomial $f(x,z) \in \mathbb{Z}[x,z]$.

Definition

f **represents an m$^{\text{th}}$ power** if there exist integers $x_0, y_0, z_0 \in \mathbb{Z}$, such that $f(x_0, z_0) = y_0^m$.

Example

Let $m = 3$ and $f(x,z) = 2x^6 + x^4z^2 + 2x^3z^3 + 3z^6$.

We have $f(1,1) = 8 = 2^3$, so f represents a cube.

Question

For fixed m, d, how often does f represent an m-th power?

Motivation

If f represents an m-th power, then for all primes p, f must represent an m-th power mod p.

Motivation

If f represents an m-th power, then for all primes p, f must represent an m-th power mod p.

Example (actually a non-example)

Let $m = 3$ and $f(x, z) = 2x^6 + 7x^4z^2 - 14x^2z^4 - 12z^6$.

Set $p = 7$. We have $(\mathbb{F}_7^\times)^3 = \{1, -1\}$.

Motivation

If f represents an m-th power, then for all primes p, f must represent an m-th power mod p.

Example (actually a non-example)

Let $m = 3$ and $f(x,z) = 2x^6 + 7x^4z^2 - 14x^2z^4 - 12z^6$.

Set $p = 7$. We have $(\mathbb{F}_7^\times)^3 = \{1, -1\}$.

$f(x,z) \equiv 2x^6 + 2z^6 \pmod 7$. Plugging in (x_0, z_0),

$$\begin{aligned} f(1,0) &\equiv 2, \\ f(0,1) &\equiv 2, \\ f(x_0,1) &\equiv 4 \text{ for all } x_0 \in \mathbb{F}_7^\times. \end{aligned}$$

Motivation

If f represents an m-th power, then for all primes p, f must represent an m-th power mod p.

Example (actually a non-example)

Let $m = 3$ and $f(x,z) = 2x^6 + 7x^4z^2 - 14x^2z^4 - 12z^6$.

Set $p = 7$. We have $(\mathbb{F}_7^\times)^3 = \{1, -1\}$.

$f(x,z) \equiv 2x^6 + 2z^6 \pmod 7$. Plugging in (x_0, z_0),

$$\begin{aligned} f(1,0) &\equiv 2, \\ f(0,1) &\equiv 2, \\ f(x_0,1) &\equiv 4 \text{ for all } x_0 \in \mathbb{F}_7^\times. \end{aligned}$$

f does not represent a cube mod 7, therefore f cannot represent an integer cube.

Superelliptic curves

Definition

A **superelliptic curve** $C/\mathbb{Q}$ is a smooth projective curve with a cyclic Galois cover of $\mathbb{P}^1$ of degree $m \geq 2$.

Such C has equation in weighted projective space $\mathbb{P}(1, \frac{d}{m}, 1)$

$$C_f \colon y^m = f(x, z) = c_d x^d + \cdots + c_0 z^d$$

where f is a binary form of degree d divisible by m.

Superelliptic curves

Definition

A **superelliptic curve** $C/\mathbb{Q}$ is a smooth projective curve with a cyclic Galois cover of $\mathbb{P}^1$ of degree $m \geq 2$.

Such C has equation in weighted projective space $\mathbb{P}(1, \frac{d}{m}, 1)$

$$C_f \colon y^m = f(x, z) = c_d x^d + \cdots + c_0 z^d$$

where f is a binary form of degree d divisible by m.

Observe

f reps. an m-th power $\iff$ $C_f \colon y^m = f(x, z)$ has a rational point,

$$[x_0 : y_0 : z_0] \in C_f(\mathbb{Q}).$$

Solubility

Let C be a curve defined over $\mathbb{Q}$.

Definition

C is **soluble** if $C(\mathbb{Q})$ is nonempty.

Question

How often is a curve over $\mathbb{Q}$ (in some family) soluble?

Solubility

Let C be a curve defined over $\mathbb{Q}$.

Definition

C is **soluble** if $C(\mathbb{Q})$ is nonempty.

Question

How often is a curve over $\mathbb{Q}$ (in some family) soluble?

For place v of $\mathbb{Q}$, we have

$$C(\mathbb{Q}) \subset C(\mathbb{Q}_v).$$

Existence of $\mathbb{Q}_v$-point for each v is necessary but not sufficient for C to have $\mathbb{Q}$-point!

Local solubility

Let $C/\mathbb{Q}$ be a curve and v a place of $\mathbb{Q}$ (i.e. $v = p$ or $v = \infty$).

Definition

C is **locally soluble at v** if $C(\mathbb{Q}_v)$ is nonempty.

C is **everywhere locally soluble (ELS)** if $C(\mathbb{Q}_v) \neq \emptyset$ for all v.

Question (revised)

How often is a curve over $\mathbb{Q}$ (in some family) ELS?

Local solubility

Let $C/\mathbb{Q}$ be a curve and v a place of $\mathbb{Q}$ (i.e. $v = p$ or $v = \infty$).

Definition

C is **locally soluble at v** if $C(\mathbb{Q}_v)$ is nonempty.

C is **everywhere locally soluble (ELS)** if $C(\mathbb{Q}_v) \neq \emptyset$ for all v.

Question (revised)

How often is a curve over $\mathbb{Q}$ (in some family) ELS?

Known for genus 1 hyperelliptics [BCF21], plane cubics [BCF16], certain hypersurfaces e.g. [BBL16], [FHP21], [PV04], [Bro17].

Motivation: hyperelliptic curves

Consider *hyperelliptic curves* given by (weighted) homog. equation

$$C\colon y^2 = f(x,z) = c_{2g+2}x^{2g+2} + \cdots + c_0 z^{2g+2}.$$

Theorem (Poonen–Stoll, Bhargava–Cremona–Fisher)

A pos. prop. of hyperelliptics $C/\mathbb{Q}$ are ELS [PS99b].

75.96% of genus 1 curves of this form are ELS [BCF21].

Motivation: hyperelliptic curves

Consider *hyperelliptic curves* given by (weighted) homog. equation

$$C\colon y^2 = f(x,z) = c_{2g+2}x^{2g+2} + \cdots + c_0 z^{2g+2}.$$

Theorem (Poonen–Stoll, Bhargava–Cremona–Fisher)

A pos. prop. of hyperelliptics $C/\mathbb{Q}$ are ELS [PS99b].

75.96% of genus 1 curves of this form are ELS [BCF21].

Theorem (Bhargava–Gross–Wang [BGW17])

A positive proportion of everywhere locally soluble hyperelliptic curves $C/\mathbb{Q}$ have no points over any odd degree extension $k/\mathbb{Q}$.

Defining the proportion

Question

How often is a superelliptic curve over $\mathbb{Q}$ ELS?

Defining the proportion

Question

How often is a superelliptic curve over $\mathbb{Q}$ ELS?

For $\mathbf{c} = (c_i)_{i=0}^{d} \in \mathbb{Z}^{d+1}$, we associate a binary form and SEC

$$f(x,z) = \sum_{i=0}^{d} c_i x^i z^{d-i}, \quad C_f \colon y^m = f(x,z).$$

Definition

For fixed m, d, we define

$$\rho_{m,d} = \lim_{B\to\infty} \frac{\#\{\mathbf{c} \in ([-B,B]\cap\mathbb{Z})^{d+1} \mid C_f \text{ is ELS}\}}{\#\{\mathbf{c} \in ([-B,B]\cap\mathbb{Z})^{d+1}\}},$$

the natural density of $f(x,z)$ for which C_f is ELS.

Main results

Fix $(m, d) \neq (2, 2)$ such that $m \mid d$.

Theorem (Beneish–K. [BK23])

(A) *$\rho_{m,d}$ exists, $0 < \rho_{m,d} < 1$, and $\rho_{m,d}$ is product of local densities,*

$$\rho_{m,d} = \rho_{m,d}(\infty) \prod_p \rho_{m,d}(p).$$

Main results

Fix $(m, d) \neq (2, 2)$ such that m is prime and $m \mid d$.

Theorem (Beneish–K. [BK23], continued)

(B) *We find explicit bounds for* $\rho_{m,d}(p)$ *and* $\rho_{m,d}$*.*

Main results

Fix $(m, d) \neq (2, 2)$ such that m is prime and $m \mid d$.

Theorem (Beneish–K. [BK23], continued)

(B) *We find explicit bounds for* $\rho_{m,d}(p)$ *and* $\rho_{m,d}$. *Taking limits,*

$$\liminf_{d\to\infty} \rho_{m,d} \geq \left(1 - \frac{1}{m^{m+1}}\right) \prod_{p\equiv 1(m)} \left(1 - \left(1 - \frac{p-1}{mp}\right)^{p+1}\right) \prod_{p\not\equiv 0,1(m)} \left(1 - \frac{1}{p^{2(p+1)}}\right).$$

When $m > 2$, *we have*

$$0.83511 \leq \liminf_{d\to\infty} \rho_{m,d} \quad \textit{and} \quad \limsup_{d\to\infty} \rho_{m,d} \leq 0.99804.$$

Main results

Theorem (Beneish–K. [BK23], continued)

(C) *In the case* $(m, d) = (3, 6)$, *we compute* $\rho_{3,6} \approx 96.94\%$.

Main results

Theorem (Beneish–K. [BK23], continued)

(C) *In the case* $(m, d) = (3, 6)$, *we compute* $\rho_{3,6} \approx 96.94\%$.

There exist rational functions $R_1(t)$ *and* $R_2(t)$ *such that*

$$\rho_{3,6}(p) = \begin{cases} R_1(p), & p \equiv 1 \pmod 3 \text{ and } p > 43 \\ R_2(p), & p \equiv 2 \pmod 3 \text{ and } p > 2. \end{cases}$$

Asymptotically,

$$1 - R_1(t) \sim \frac{2}{3} t^{-4},$$

$$1 - R_2(t) \sim \frac{53}{144} t^{-7}.$$

$$
\rho = \begin{cases}
\Big(1296p^{57} + 3888p^{56} + 9072p^{55} + 16848p^{54} + 27648p^{53} + 39744p^{52} + 53136p^{51} + 66483p^{50} + 80019p^{49} + 93141p^{48} \\
+ 107469p^{47} + 120357p^{46} + 135567p^{45} + 148347p^{44} + 162918p^{43} + 176004p^{42} + 190278p^{41} + 203459p^{40} \\
+ 218272p^{39} + 232083p^{38} + 243639p^{37} + 255267p^{36} + 261719p^{35} + 264925p^{34} + 265302p^{33} + 261540p^{32} \\
+ 254790p^{31} + 250736p^{30} + 241384p^{29} + 226503p^{28} + 214137p^{27} + 195273p^{26} + 170793p^{25} + 151839p^{24} + 136215p^{23} \\
+ 118998p^{22} + 105228p^{21} + 94860p^{20} + 80471p^{19} + 67048p^{18} + 52623p^{17} + 40617p^{16} + 28773p^{15} + 19247p^{14} & p \equiv 1 \pmod 3 \\
+ 12109p^{13} + 7614p^{12} + 3420p^{11} + 756p^{10} - 2248p^{9} - 4943p^{8} - 6300p^{7} - 6894p^{6} - 5994p^{5} - 2448p^{4} - 648p^{3} \\
+ 324p^{2} + 1296p + 1296\Big) \Big/ \Big(1296\left(p^{12} - p^{11} + p^{9} - p^{8} + p^{6} - p^{4} + p^{3} - p + 1\right)\left(p^{8} - p^{6} + p^{4} - p^{2} + 1\right) \\
\times \left(p^{6} + p^{5} + p^{4} + p^{3} + p^{2} + p + 1\right)\left(p^{4} + p^{3} + p^{2} + p + 1\right)^{3}\left(p^{4} - p^{3} + p^{2} - p + 1\right)\left(p^{2} + p + 1\right) \\
\times \left(p^{2} + 1\right)p^{11}\Big), \\
\\
\Big(144p^{57} + 432p^{56} + 1008p^{55} + 1872p^{54} + 3168p^{53} + 4608p^{52} + 6336p^{51} + 8011p^{50} + 9803p^{49} + 11357p^{48} \\
+ 13061p^{47} + 14525p^{46} + 16295p^{45} + 17875p^{44} + 19654p^{43} + 21212p^{42} + 23030p^{41} + 24563p^{40} + 26320p^{39} \\
+ 27771p^{38} + 29711p^{37} + 30859p^{36} + 31135p^{35} + 31525p^{34} + 31510p^{33} + 29436p^{32} + 28502p^{31} + 28616p^{30} \\
+ 26856p^{29} + 25087p^{28} + 25057p^{27} + 23041p^{26} + 19921p^{25} + 18119p^{24} + 16287p^{23} + 13798p^{22} \\
+ 12140p^{21} + 10844p^{20} + 9191p^{19} + 7480p^{18} + 5839p^{17} + 4265p^{16} + 2909p^{15} + 1943p^{14} + 1109p^{13} & p \equiv 2 \pmod 3 \\
+ 590p^{12} + 604p^{11} + 372p^{10} - 144p^{9} - 87p^{8} - 84p^{7} - 678p^{6} - 618p^{5} - 144p^{4} - 168p^{3} - 156p^{2} \\
+ 144p + 144\Big) \Big/ \Big(144\left(p^{12} - p^{11} + p^{9} - p^{8} + p^{6} - p^{4} + p^{3} - p + 1\right)\left(p^{8} - p^{6} + p^{4} - p^{2} + 1\right) \\
\times \left(p^{6} + p^{5} + p^{4} + p^{3} + p^{2} + p + 1\right)\left(p^{4} + p^{3} + p^{2} + p + 1\right)^{3}\left(p^{4} - p^{3} + p^{2} - p + 1\right)\left(p^{2} + p + 1\right) \\
\times \left(p^{2} + 1\right)p^{11}\Big),
\end{cases}
$$

Outline

- Set up and state main results,
- Local densities $\rho_{m,d}(p) \to$ global density $\rho_{m,d}$,
- Study local densities $\rho_{m,d}(p)$,
- Toward exact computations of $\rho_{3,6}(p)$.

Local densities

Theorem (Beneish–K. [BK23])

(A) $\rho_{m,d}$ *exists and is given by the product of local densities,*

$$\rho_{m,d} = \rho_{m,d}(\infty) \prod_p \rho_{m,d}(p) > 0.$$

$\rho_{m,d}(p)$ is (normalized) Haar measure of space of the $\mathbb{Q}_p$-soluble curves $C_f\colon y^m = f(x,z)$, with coefficients in $\mathbb{Z}_p$.

Local densities

Theorem (Beneish–K. [BK23])

(A) $\rho_{m,d}$ *exists and is given by the product of local densities,*

$$\rho_{m,d} = \rho_{m,d}(\infty) \prod_p \rho_{m,d}(p) > 0.$$

$\rho_{m,d}(p)$ is (normalized) Haar measure of space of the $\mathbb{Q}_p$-soluble curves $C_f \colon y^m = f(x,z)$, with coefficients in $\mathbb{Z}_p$.

Idea

In *good situations*, imposing conditions at different primes looks independent...*even if there are infinitely many conditions*.

Local densities look independent

Idea

In *good situations*, imposing conditions at different primes looks independent...*even if there are infinitely many conditions*.

Think

Recall squarefree numbers.

$$n \text{ squarefree} \iff p^2 \nmid n \text{ for all } p.$$

If probabilities that $p^2 \mid n$ are independent, expect

$$\lim_{B \to \infty} \frac{\#\{-B \le n \le B \mid n \text{ squarefree}\}}{2B+1} = \prod_p \left(1 - \frac{1}{p^2}\right) = \frac{6}{\pi^2}.$$

Local densities look independent

Idea

In *good situations*, imposing conditions at different primes looks independent...*even if there are infinitely many conditions*.

- Poonen–Stoll: criteria for when natural density is product of local densities [PS99a].

Local densities look independent

Idea

In *good situations*, imposing conditions at different primes looks independent...*even if there are infinitely many conditions*.

- Poonen–Stoll: criteria for when natural density is product of local densities [PS99a].
- Apply to ELS in families of hyperelliptic curves [PS99b].

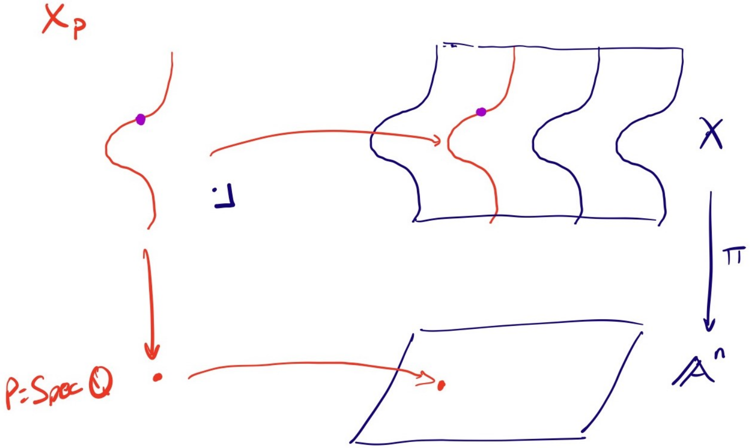
Local densities look independent

Idea

In *good situations*, imposing conditions at different primes looks independent…*even if there are infinitely many conditions*.

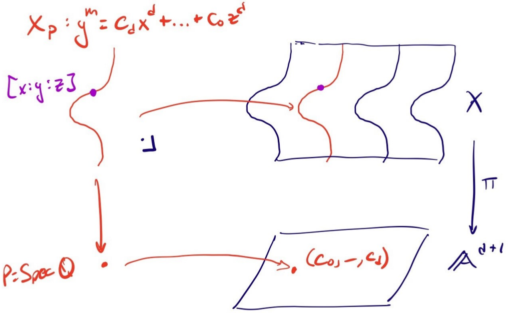
- Poonen–Stoll: criteria for when natural density is product of local densities [PS99a].
- Apply to ELS in families of hyperelliptic curves [PS99b].
- Bright–Browning–Loughran: geometric criteria when family comes from fibers of a morphism [BBL16].

Geometric picture



Geometric picture

$$X\colon y^m = c_d x^d + \cdots + c_0 z^d \subset \mathbb{A}^{d+1}_{\mathbb{Q}} \times \mathbb{P}_{\mathbb{Q}}\left(1 : \tfrac{d}{m} : 1\right)$$



"Proof" of Theorem A.

π satisfies projectivity, integrality, etc. hypotheses to apply [BBL16, Theorem 1.4]. □

Outline

- Set up and state main results,
- Local densities $\rho_{m,d}(p) \to$ global density $\rho_{m,d}$,
- Study local densities $\rho_{m,d}(p)$,
- Toward exact computations of $\rho_{3,6}(p)$.

Computing local densities

Question

Once we know

$$\rho_{m,d} = \rho_{m,d}(\infty) \prod_p \rho_{m,d}(p),$$

how do we compute/estimate local densities $\rho_{m,d}(p)$*?*

Computing local densities

Question

Once we know

$$\rho_{m,d} = \rho_{m,d}(\infty) \prod_p \rho_{m,d}(p),$$

how do we compute/estimate local densities $\rho_{m,d}(p)$?

$\rho_{m,d}(\infty)$: Euclidean measure of $\mathbb{R}$-soluble C_f with coeffs $\in [-1, 1]$.

- If m or d is odd, then $\rho_{m,d}(\infty) = 1$.
- If m, d even, no analytic solution known for $d > 2$, but rigorous estimates exist, e.g.

$$0.873914 \leq \rho_{2,4}(\infty) \leq 0.874196 \quad \text{[BCF21]}.$$

Computing local densities — finite places

$\rho_{m,d}(p)$ is (normalized) Haar measure of space of the $\mathbb{Q}_p$-soluble curves $C_f \colon y^m = f(x,z)$, with coefficients in $\mathbb{Z}_p$.

Computing local densities — finite places

$\rho_{m,d}(p)$ is (normalized) Haar measure of space of the $\mathbb{Q}_p$-soluble curves $C_f \colon y^m = f(x,z)$, with coefficients in $\mathbb{Z}_p$.

Think

Each possible reduction $\overline{f}(x,z)$ mod p occurs equally often.

Computing local densities — finite places

$\rho_{m,d}(p)$ is (normalized) Haar measure of space of the $\mathbb{Q}_p$-soluble curves $C_f \colon y^m = f(x,z)$, with coefficients in $\mathbb{Z}_p$.

Think

Each possible reduction $\overline{f}(x,z)$ mod p occurs equally often.

Look mod p and check $\mathbb{Q}_p$-solubility with **Hensel's lemma**!

Computing local densities — finite places

$\rho_{m,d}(p)$ is (normalized) Haar measure of space of the $\mathbb{Q}_p$-soluble curves $C_f\colon y^m = f(x,z)$, with coefficients in $\mathbb{Z}_p$.

Think

Each possible reduction $\overline{f}(x,z)$ mod p occurs equally often.

Look mod p and check $\mathbb{Q}_p$-solubility with **Hensel's lemma**!

- Smooth $\mathbb{F}_p$-points on $\overline{C_f}$ lift to $\mathbb{Q}_p$-solutions (Hensel),
- $\overline{C_f}(\mathbb{F}_p) = \emptyset \implies$ no $\mathbb{Q}_p$-solutions,
- If $\overline{C_f}(\mathbb{F}_p)$ only non-smooth points, do more work.

An extended example

Example

Consider $(m, d) = (3, 6)$, generically genus 4:

$$C_f \colon y^3 = f(x, z) = c_6 x^6 + c_5 x^5 z + \cdots + c_1 x z^5 + c_0 z^6.$$

When can we guarantee $\overline{C_f}$ has liftable $\mathbb{F}_p$-points?

An extended example

Example

Consider $(m, d) = (3, 6)$, generically genus 4:

$$C_f \colon y^3 = f(x, z) = c_6 x^6 + c_5 x^5 z + \cdots + c_1 x z^5 + c_0 z^6.$$

When can we guarantee $\overline{C_f}$ has liftable $\mathbb{F}_p$-points?

Theorem (Hasse–Weil bound)

If $\overline{C_f}$ is irreducible and smooth of genus g, then

$$\#\overline{C_f}(\mathbb{F}_p) \geq p + 1 - g \cdot 2\sqrt{p}.$$

An extended example

Example

Consider $(m, d) = (3, 6)$, generically genus 4:

$$C_f \colon y^3 = f(x, z) = c_6 x^6 + c_5 x^5 z + \cdots + c_1 x z^5 + c_0 z^6.$$

When can we guarantee $\overline{C_f}$ has liftable $\mathbb{F}_p$-points?

Theorem (Hasse–Weil bound, refined)

If $\overline{C_f}$ is irreducible and smooth of genus g, then

$$\#\overline{C_f}(\mathbb{F}_p) \geq p + 1 - g \cdot \lfloor 2\sqrt{p} \rfloor.$$

An extended example

Example

When can we guarantee $\overline{C_f}$ has liftable $\mathbb{F}_p$-points?

When $p \geq 61$, we have $p + 1 - 4\lfloor 2\sqrt{p} \rfloor > 0$, so

$$\overline{C_f}/\mathbb{F}_p \text{ smooth} \implies C_f(\mathbb{Q}_p) \neq \emptyset.$$

An extended example

Example

When can we guarantee $\overline{C_f}$ has liftable $\mathbb{F}_p$-points?

When $p \geq 61$, we have $p + 1 - 4\lfloor 2\sqrt{p} \rfloor > 0$, so

$$\overline{C_f}/\mathbb{F}_p \text{ smooth} \implies C_f(\mathbb{Q}_p) \neq \emptyset.$$

- $\overline{C_f}^{\mathrm{sm}}(\mathbb{F}_p) \neq \emptyset$ whenever $\overline{C_f}/\mathbb{F}_p$ geom. irr. and $p \geq 61$.

An extended example

Example

When can we guarantee $\overline{C_f}$ has liftable $\mathbb{F}_p$-points?

When $p \geq 61$, we have $p + 1 - 4\lfloor 2\sqrt{p} \rfloor > 0$, so

$$\overline{C_f}/\mathbb{F}_p \text{ smooth} \implies C_f(\mathbb{Q}_p) \neq \emptyset.$$

- $\overline{C_f}^{\mathrm{sm}}(\mathbb{F}_p) \neq \emptyset$ whenever $\overline{C_f}/\mathbb{F}_p$ geom. irr. and $p \geq 61$.
- $\overline{C_f}$ geom. irr. $\iff \overline{f}(x,z) \neq ah(x,z)^3$.

An extended example

Example

When can we guarantee $\overline{C_f}$ has liftable $\mathbb{F}_p$-points?

When $p \geq 61$, we have $p + 1 - 4\lfloor 2\sqrt{p} \rfloor > 0$, so

$$\overline{C_f}/\mathbb{F}_p \text{ smooth} \implies C_f(\mathbb{Q}_p) \neq \emptyset.$$

- $\overline{C_f}^{\mathrm{sm}}(\mathbb{F}_p) \neq \emptyset$ whenever $\overline{C_f}/\mathbb{F}_p$ geom. irr. and $p \geq 61$.
- $\overline{C_f}$ geom. irr. $\iff \overline{f}(x, z) \neq ah(x, z)^3$.

Count geom. reducible $\overline{C_f}$: $p^3 = (p-1)(p^2 + p + 1) + 1$

$$\implies \rho_{3,6}(p) \geq \frac{p^7 - p^3}{p^7} = 1 - \frac{1}{p^4} \text{ for all } p \geq 61.$$

An extended example

- $\rho_{3,6}(p) \geq 1 - \frac{1}{p^4}$ when $p \equiv 1 \pmod 3$ and $p > 43$
- $\rho_{3,6}(p) \geq 1 - \frac{1}{p^7}$ when $p \equiv 2 \pmod 3$ and $p > 2$

An extended example

- $\rho_{3,6}(p) \geq 1 - \frac{1}{p^4}$ when $p \equiv 1 \pmod 3$ and $p > 43$
- $\rho_{3,6}(p) \geq 1 - \frac{1}{p^7}$ when $p \equiv 2 \pmod 3$ and $p > 2$
- Enumerate all $\overline{f}(x,z)$ and count Hensel-liftable $\mathbb{F}_p$-solutions:

p	$\rho_{3,6}(p) \geq$	p	$\rho_{3,6}(p) \geq$
2	$\frac{63}{64} \approx 0.98437$	19	$\frac{893660256}{893871739} \approx 0.99976$
3	$\frac{26}{27} \approx 0.96296$	31	$\frac{27512408250}{27512614111} \approx 0.99999$
7	$\frac{810658}{823543} \approx 0.98435$	37	$\frac{94931742132}{94931877133} \approx 0.999998$
13	$\frac{62655132}{62748517} \approx 0.99851$	43	$\frac{271818511748}{271818611107} \approx 0.9999996$

Put together with Theorem A:

$$\rho_{3,6} = \prod_p \rho_{3,6}(p) \geq 0.93134.$$

Bounds more generally for $m = 3$

Example (Lower bounds for general d)

For $d > 6$ such that $3 \mid d$,

$$\rho_{3,d} \geq \left(1 - \frac{1}{3^4}\right) \prod_{\substack{p \equiv 2(3) \\ p \leq d/2-1}} \left(1 - \frac{1}{p^{2(p+1)}}\right) \prod_{\substack{p \equiv 2(3) \\ p > d/2-1}} \left(1 - \frac{1}{p^{d+1}}\right)$$
$$\times \prod_{\substack{p \equiv 1(3) \\ p < d}} \left(1 - \left(1 - \frac{p-1}{3p}\right)^{p+1}\right) \prod_{\substack{p \equiv 1(3) \\ d < p < 4(d-2)^2}} \left(1 - \left(1 - \frac{p-1}{3p}\right)^{d+1}\right) \prod_{\substack{p \equiv 1(3) \\ p \geq 4(d-2)^2}} \left(1 - \frac{1}{p^{\frac{2d}{3}}}\right)$$

Example (Large genus limit)

Taking limit as $d \to \infty$

$$\liminf_{d \to \infty} \rho_{3,d} \geq \left(1 - \frac{1}{3^4}\right) \prod_{p \equiv 1(3)} \left(1 - \left(1 - \frac{p-1}{3p}\right)^{p+1}\right) \prod_{p \equiv 2(3)} \left(1 - \frac{1}{p^{2(p+1)}}\right) \approx 0.90.$$

Outline

- Set up and state main results,
- Local densities $\rho_{m,d}(p) \to$ global density $\rho_{m,d}$,
- Study local densities $\rho_{m,d}(p)$,
- Toward exact computations of $\rho_{3,6}(p)$.

Getting exact answer

Question

How do we go from bounds to exact values for $\rho_{3,6}(p)$?

Getting exact answer

Question

How do we go from bounds to exact values for $\rho_{3,6}(p)$?

Let $F(x, y, z) = y^3 - f(x, z)$ and look at reduction modulo p.

$\overline{F}$ reducible$/\overline{\mathbb{F}_p} \iff \overline{F} = (y - h)(y - \zeta_3 h)(y - \zeta_3^2 h)$.

Factorization type in y	$p = 3$	$p \equiv 1 \pmod 3$	$p \equiv 2 \pmod 3$
1. Abs. irr.	2160	$p^3(p^4 - 1)$	$p^3(p^4 - 1)$
2. 3 distinct linear over $\mathbb{F}_p$	0	$\frac{1}{3}(p^3 - 1)$	0
3. Linear + conj.	0	0	$p^3 - 1$
4. 3 conjugate factors	0	$\frac{2}{3}(p^3 - 1)$	0
5. $(y - h(x, z))^3$	27	1	1
Total	3^7	p^7	p^7

Getting exact answer

Let ξ_i be the proportion of $\overline{f}$ for which $\overline{F}$ has type i.

Let σ_i be probability $F = 0$ has $\mathbb{Z}_p$-solution when $\overline{F}$ has type i.

$$\rho_{3,6}(p) = \sum_{i=1}^{5} \xi_i(p)\sigma_i(p).$$

Getting exact answer

Let ξ_i be the proportion of $\overline{f}$ for which $\overline{F}$ has type i.

Let σ_i be probability $F = 0$ has $\mathbb{Z}_p$-solution when $\overline{F}$ has type i.

$$\rho_{3,6}(p) = \sum_{i=1}^{5} \xi_i(p)\sigma_i(p).$$

In order to compute σ_4, σ_5, do the following.

1. How often do factorization types occur (mod p)?
2. Find lifting probabilities for each factorization type.
3. Relate probabilities to each other and solve.

An example: computing σ_5

$$\sigma_5 = \mathrm{Prob}\left(C_f(\mathbb{Q}_p) \neq \emptyset \;\middle|\; f(x,z) \equiv 0 \pmod{p} \right)$$

Write $f(x,z) \equiv pf_1(x,z)$ for $f_1 \in \mathbb{F}_p[x,z]$. Assume $f_1 \neq 0$ for now.

An example: computing σ_5

$$\sigma_5 = \text{Prob}\left(C_f(\mathbb{Q}_p) \neq \emptyset \;\middle|\; f(x,z) \equiv 0 \pmod{p} \right)$$

Write $f(x,z) \equiv pf_1(x,z)$ for $f_1 \in \mathbb{F}_p[x,z]$. Assume $f_1 \neq 0$ for now.

Observation

$\mathbb{Z}_p$-point $[x_0 : y_0 : z_0]$ on $C_f \colon y^3 = f(x,z)$ has $p \mid y_0$,

$$p^3 \mid f(x_0, z_0) \implies p^2 \mid f_1(x_0, z_0).$$

An example: computing σ_5

$$\sigma_5 = \mathrm{Prob}\left(C_f(\mathbb{Q}_p) \neq \emptyset \;\middle|\; f(x,z) \equiv 0 \pmod{p} \right)$$

Write $f(x,z) \equiv pf_1(x,z)$ for $f_1 \in \mathbb{F}_p[x,z]$. Assume $f_1 \neq 0$ for now.

Observation

$\mathbb{Z}_p$-point $[x_0 : y_0 : z_0]$ on $C_f\colon y^3 = f(x,z)$ has $p \mid y_0$,

$$p^3 \mid f(x_0,z_0) \implies p^2 \mid f_1(x_0,z_0).$$

(0) If $\overline{f_1}(x,z)$ has no roots modulo p, then C_f has no $\mathbb{Z}_p$-points.

An example: computing σ_5

$$\sigma_5 = \mathrm{Prob}\left(C_f(\mathbb{Q}_p) \neq \emptyset \;\middle|\; f(x,z) \equiv 0 \pmod{p} \right)$$

Write $f(x,z) \equiv pf_1(x,z)$ for $f_1 \in \mathbb{F}_p[x,z]$. Assume $f_1 \neq 0$ for now.

Observation

$\mathbb{Z}_p$-point $[x_0 : y_0 : z_0]$ on $C_f \colon y^3 = f(x,z)$ has $p \mid y_0$,

$$p^3 \mid f(x_0, z_0) \implies p^2 \mid f_1(x_0, z_0).$$

(0) If $\overline{f_1}(x,z)$ has no roots modulo p, then C_f has no $\mathbb{Z}_p$-points.

(1) If $\overline{f_1}(x,z)$ has a root of mult. 1, it lifts to $\mathbb{Z}_p$-point of C_f.

An example: computing σ_5

$$\sigma_5 = \text{Prob}\left(C_f(\mathbb{Q}_p) \neq \emptyset \;\middle|\; f(x,z) \equiv 0 \pmod{p} \right)$$

Write $f(x,z) \equiv pf_1(x,z)$ for $f_1 \in \mathbb{F}_p[x,z]$. Assume $f_1 \neq 0$ for now.

Observation

$\mathbb{Z}_p$-point $[x_0 : y_0 : z_0]$ on $C_f \colon y^3 = f(x,z)$ has $p \mid y_0$,

$$p^3 \mid f(x_0, z_0) \implies p^2 \mid f_1(x_0, z_0).$$

(0) If $\overline{f_1}(x,z)$ has no roots modulo p, then C_f has no $\mathbb{Z}_p$-points.

(1) If $\overline{f_1}(x,z)$ has a root of mult. 1, it lifts to $\mathbb{Z}_p$-point of C_f.

(2) Suppose $\overline{f_1}(x,z)$ has a double root (and no other roots).

Dealing with the double root

Assume $x^2 \mid \overline{f_1}$, giving p-adic valuations below (original coeffs of f):

Prob. lift	$v(c_6)$	$v(c_5)$	$v(c_4)$	$v(c_3)$	$v(c_2)$	$v(c_1)$	$v(c_0)$
$\tau_2 = \tau_{2a} = \frac{1}{p}\tau_{2b}$	≥ 1	≥ 1	≥ 1	≥ 1	$= 1$	≥ 2	≥ 2
$\tau_{2b} = \tau_{2c}$	≥ 1	≥ 1	≥ 1	≥ 1	$= 1$	≥ 2	≥ 3
$\tau_{2c} = 1$	≥ 4	≥ 3	≥ 2	≥ 1	$= 0$	≥ 0	≥ 0

Probability of lifting $[0:0:1]$ in this case is

$$\tau_2 = \frac{1}{p} = \text{Prob}\left(p^3 \mid c_0 : p^2 \mid c_0 \text{ and } p \mid\mid c_2\right).$$

Computing σ_5

$$\sigma_5 = \left(1 - \frac{1}{p^7}\right)\sum_{i=0}^{9}\eta_i\tau_i + \left(\frac{1}{p^7} - \frac{1}{p^{14}}\right)\sum_{i=0}^{9}\eta_i\theta_i + \frac{1}{p^{14}}\rho$$

- Index i indicates factorization type of $f_1(x,z)$ (or $f_2(x,z)$)
- $\eta_i =$ proportion of sextic forms$/\mathbb{F}_p$ with i-th type
- τ_i (resp. θ_i) are proportion of f with f_1 (resp. f_2) of type i such that C_f has a $\mathbb{Z}_p$-point.

Factorization types

Fact. type	η_i	η_i' (monic forms only)
0. No roots	$\frac{\left(53p^4+26p^3+19p^2-2p+24\right)(p-1)p}{144\left(p^6+p^5+p^4+p^3+p^2+p+1\right)}$	$\frac{\left(53p^4+26p^3+19p^2-2p+24\right)(p-1)}{144p^5}$
1. $(1*)$	$\frac{\left(91p^4+26p^3+23p^2+16p-12\right)(p+1)p}{144\left(p^6+p^5+p^4+p^3+p^2+p+1\right)}$	$\frac{\left(91p^3-27p^2+50p-48\right)(p+1)(p-1)}{144p^5}$
2. (1^24) or (1^222)	$\frac{\left(3p^2+p+2\right)(p+1)(p-1)p}{8\left(p^6+p^5+p^4+p^3+p^2+p+1\right)}$	$\frac{\left(3p^2+p+2\right)(p-1)}{8p^4}$
3. (1^21^22)	$\frac{(p+1)(p-1)p^2}{4\left(p^6+p^5+p^4+p^3+p^2+p+1\right)}$	$\frac{(p-1)^2}{4p^4}$
4. $(1^21^21^2)$	$\frac{(p+1)(p-1)p}{6\left(p^6+p^5+p^4+p^3+p^2+p+1\right)}$	$\frac{(p-1)(p-2)}{6p^5}$
5. (1^33)	$\frac{(p+1)^2(p-1)p}{3\left(p^6+p^5+p^4+p^3+p^2+p+1\right)}$	$\frac{(p+1)(p-1)}{3p^4}$
6. (1^31^3)	$\frac{(p+1)p}{2\left(p^6+p^5+p^4+p^3+p^2+p+1\right)}$	$\frac{p-1}{2p^5}$
7. (1^42)	$\frac{(p+1)(p-1)p}{2\left(p^6+p^5+p^4+p^3+p^2+p+1\right)}$	$\frac{p-1}{2p^4}$
8. (1^21^4)	$\frac{(p+1)p}{p^6+p^5+p^4+p^3+p^2+p+1}$	$\frac{p-1}{p^5}$
9. (1^6)	$\frac{p+1}{p^6+p^5+p^4+p^3+p^2+p+1}$	$\frac{1}{p^5}$

Type 9: yikes!

Type 9, e.g. $f(x,z) \equiv px^6 \pmod{p^2}$.

τ_9 is a degree 44 rational function in p.

$\tau_9 =$		c_6	c_5	c_4	c_3	c_2	c_1	c_0
	$\tau_{9a} = \frac{1}{p}\tau_{9b}$	$=1$	≥ 2	≥ 2	≥ 2	≥ 2	≥ 2	≥ 2
	$\tau_{9b} = \left(1-\frac{1}{p}\right) + \frac{1}{p}\tau_{9c}$	$=4$	≥ 4	≥ 3	≥ 2	≥ 1	≥ 0	≥ 0
	$\tau_{9c} = \Phi(p) + \frac{1}{p}\tau_{9d}$	$=4$	≥ 4	≥ 3	≥ 2	≥ 1	≥ 1	≥ 0
	$\tau_{9d} = \left(1-\frac{1}{p}\right)\left(\frac{p-1}{2p} + \frac{1}{p^2}\right) + \frac{1}{p}\tau_{9e}$	$=4$	≥ 4	≥ 3	≥ 2	≥ 1	≥ 1	≥ 1
	$\tau_{9e} = \left(1-\frac{1}{p}\right) + \frac{1}{p}\tau_{9f}$	$=4$	≥ 4	≥ 3	≥ 2	≥ 2	≥ 1	≥ 1
	$\tau_{9f} = \frac{1}{p}\tau_{9g}$	$=4$	≥ 4	≥ 3	≥ 2	≥ 2	≥ 2	≥ 1
	$\tau_{9g} = \left(1-\frac{1}{p}\right)\alpha'' + \frac{1}{p}\tau_{9h}$	$=4$	≥ 4	≥ 3	≥ 2	≥ 2	≥ 2	≥ 2
	$\tau_{9h} = \left(1-\frac{1}{p}\right)\left(\frac{p-1}{2p} + \frac{\theta_2}{p}\right) + \frac{1}{p}\tau_{9i}$	$=4$	≥ 4	≥ 3	≥ 3	≥ 2	≥ 2	≥ 2
	$\tau_{9i} = \left(1-\frac{1}{p}\right) + \frac{1}{p}\tau_{9j}$	$=4$	≥ 4	≥ 3	≥ 3	≥ 3	≥ 2	≥ 2
	$\tau_{9j} = \frac{1}{p}\tau_{9k}$	$=4$	≥ 4	≥ 3	≥ 3	≥ 3	≥ 3	≥ 2
	$\tau_{9k} = \left(1-\frac{1}{p}\right) + \frac{1}{p}\tau_{9\ell}$	$=1$	≥ 1	≥ 0	≥ 0	≥ 0	≥ 0	≥ 0
	$\tau_{9\ell} = \Phi(p) + \left(1-\Phi(p)-\frac{1}{p}\right)\beta + \frac{1}{p}\tau_{9m}$	$=1$	≥ 1	≥ 1	≥ 0	≥ 0	≥ 0	≥ 0
	$\tau_{9m} = \left(1-\frac{1}{p}\right) + \frac{1}{p}\tau_{9n}$	$=1$	≥ 1	≥ 1	≥ 1	≥ 0	≥ 0	≥ 0
	$\tau_{9n} = \left(1-\frac{1}{p}\right) + \frac{1}{p}\tau_{9o}$	$=1$	≥ 1	≥ 1	≥ 1	≥ 1	≥ 0	≥ 0
	$\tau_{9o} = \Phi(p) + \frac{1}{p}\tau_{9p}$	$=1$	≥ 1	≥ 1	≥ 1	≥ 1	≥ 1	≥ 0
	$\tau_{9p} = \sigma'_5$	$=1$	≥ 1	≥ 1	≥ 1	≥ 1	≥ 1	≥ 1

What is $\rho_{3,6}(p)$?

$$\rho = \begin{cases} \Big(1296p^{57} + 3888p^{56} + 9072p^{55} + 16848p^{54} + 27648p^{53} + 39744p^{52} + 53136p^{51} + 66483p^{50} + 80019p^{49} + 93141p^{48} \\ + 107469p^{47} + 120357p^{46} + 135567p^{45} + 148347p^{44} + 162918p^{43} + 176004p^{42} + 190278p^{41} + 203459p^{40} \\ + 218272p^{39} + 232083p^{38} + 243639p^{37} + 255267p^{36} + 261719p^{35} + 264925p^{34} + 265302p^{33} + 261540p^{32} \\ + 254790p^{31} + 250736p^{30} + 241384p^{29} + 226503p^{28} + 214137p^{27} + 195273p^{26} + 170793p^{25} + 151839p^{24} + 136215p^{23} \\ + 118998p^{22} + 105228p^{21} + 94860p^{20} + 80471p^{19} + 67048p^{18} + 52623p^{17} + 40617p^{16} + 28773p^{15} + 19247p^{14} & p \equiv 1 \pmod 3 \\ + 12109p^{13} + 7614p^{12} + 3420p^{11} + 756p^{10} - 2248p^{9} - 4943p^{8} - 6300p^{7} - 6894p^{6} - 5994p^{5} - 2448p^{4} - 648p^{3} \\ + 324p^{2} + 1296p + 1296\Big) \Big/ \Big(1296\left(p^{12} - p^{11} + p^{9} - p^{8} + p^{6} - p^{4} + p^{3} - p + 1\right)\left(p^{8} - p^{6} + p^{4} - p^{2} + 1\right) \\ \times \left(p^{6} + p^{5} + p^{4} + p^{3} + p^{2} + p + 1\right)\left(p^{4} + p^{3} + p^{2} + p + 1\right)^{3}\left(p^{4} - p^{3} + p^{2} - p + 1\right)\left(p^{2} + p + 1\right) \\ \times \left(p^{2} + 1\right)p^{11}\Big), \\ \Big(144p^{57} + 432p^{56} + 1008p^{55} + 1872p^{54} + 3168p^{53} + 4608p^{52} + 6336p^{51} + 8011p^{50} + 9803p^{49} + 11357p^{48} \\ + 13061p^{47} + 14525p^{46} + 16295p^{45} + 17875p^{44} + 19654p^{43} + 21212p^{42} + 23030p^{41} + 24563p^{40} + 26320p^{39} \\ + 27771p^{38} + 29711p^{37} + 30859p^{36} + 31135p^{35} + 31525p^{34} + 31510p^{33} + 29436p^{32} + 28502p^{31} + 28616p^{30} \\ + 26856p^{29} + 25087p^{28} + 25057p^{27} + 23041p^{26} + 19921p^{25} + 18119p^{24} + 16287p^{23} + 13798p^{22} \\ + 12140p^{21} + 10844p^{20} + 9191p^{19} + 7480p^{18} + 5839p^{17} + 4265p^{16} + 2909p^{15} + 1943p^{14} + 1109p^{13} & p \equiv 2 \pmod 3 \\ + 590p^{12} + 604p^{11} + 372p^{10} - 144p^{9} - 87p^{8} - 84p^{7} - 678p^{6} - 618p^{5} - 144p^{4} - 168p^{3} - 156p^{2} \\ + 144p + 144\Big) \Big/ \Big(144\left(p^{12} - p^{11} + p^{9} - p^{8} + p^{6} - p^{4} + p^{3} - p + 1\right)\left(p^{8} - p^{6} + p^{4} - p^{2} + 1\right) \\ \times \left(p^{6} + p^{5} + p^{4} + p^{3} + p^{2} + p + 1\right)\left(p^{4} + p^{3} + p^{2} + p + 1\right)^{3}\left(p^{4} - p^{3} + p^{2} - p + 1\right)\left(p^{2} + p + 1\right) \\ \times \left(p^{2} + 1\right)p^{11}\Big), \end{cases}$$

What about small primes?

Use `Magma` when Hasse–Weil doesn't suffice; modify calculations accordingly.

$$
\begin{aligned}
\rho(2) &= \frac{4594897725819217081}{46164832540903014400} \approx 0.99532\\
\rho(3) &= \frac{900175334869743731875930997281}{908381960435133191895132960000} \approx 0.99096\\
\rho(7) &= \frac{63104494755178622851603292623187277054743730183645677893972}{64083174787206696882429945655801281538844149896400159815375} \approx 0.98472\\
\rho(13) &= \frac{7877728357244577414025901931296747409682076255666526984515273526822853}{7890643570620106747776737292792780623510727026420779539893772399701475} \approx 0.99836\\
\rho(19) &= \frac{3122673715489206150449285868243361150392235799365815266879438393279346795671}{3123410013311365155035964479837966797560851333614271490136481337080636454180} \approx 0.99976\\
\rho(31) &= \frac{9196796457678318869139089936786462146535210039832850454297877482020635073857159758299}{9196865061587843544830989041473808798913128587425995645857828572610918436035833907250} \approx 0.999992\\
\rho(37) &= \frac{171128647900820194784458101787952920169924464886519055453844647154184805036447476640345735119}{171128889636157060536894474187017088464271236509977199491208939449738127658679723715588944500} \approx 0.999998\\
\rho(43) &= \frac{84000121343283090388653356431804100707331364779290664490547105768867844862712134447832720508750281}{84000151671513555191647712567596101710800846209116830568013729377404991150901973105093039939237500} \approx 0.9999996
\end{aligned}
$$

What is $\rho_{3,6}$?

Theorem (Beneish-K.)

(C) *We have determined $\rho_{3,6}(p)$ exactly for all p.*

Taking product over $p \leq 10000$ gives

$$\rho_{3,6} \approx \prod_{p \leq 10000} \rho_{3,6}(p) = 0.96943,$$

with error of $O(10^{-14})$.

97% of superelliptic curves $y^3 = c_6 x^6 + \ldots + c_0 z^6$ are ELS.

Further questions

Question

Are $\rho_{m,d}(p)$ always given by rational functions for $p \gg 0$?

Further questions

Question

Are $\rho_{m,d}(p)$ always given by rational functions for $p \gg 0$?

Question

What proportion of superelliptic curves $C_f \colon y^m = f(x, z)$

- *are globally soluble?*
- *satisfy/fail the Hasse principle?*
- *have some/no points of certain higher degrees?*

Further questions

Question

Are $\rho_{m,d}(p)$ always given by rational functions for $p \gg 0$?

Question

What proportion of superelliptic curves $C_f \colon y^m = f(x,z)$

- *are globally soluble?*
- *satisfy/fail the Hasse principle?*
- *have some/no points of certain higher degrees?*

Preliminary results [BK21, Prop. 7.2] give conditions for which pos. prop. of SECs have finitely many points of certain degrees.

Effective results for global solubility proportions in thin families, e.g. $y^m = f_1(x,z)f_2(x,z)$?

Thank you I

Thank you for the invitation and for your attention!

M. J. Bright, T. D. Browning, and D. Loughran, Failures of weak approximation in families, Compos. Math. **152** (2016), no. 7, 1435–1475. MR 3530447

Manjul Bhargava, John Cremona, and Tom Fisher, The proportion of plane cubic curves over $\mathbb{Q}$ that everywhere locally have a point, Int. J. Number Theory **12** (2016), no. 4, 1077–1092. MR 3484299

______, The proportion of genus one curves over $\mathbb{Q}$ defined by a binary quartic that everywhere locally have a point, Int. J. Number Theory **17** (2021), no. 4, 903–923. MR 4262272

Manjul Bhargava, Benedict H. Gross, and Xiaoheng Wang, A positive proportion of locally soluble hyperelliptic curves over $\mathbb{Q}$ have no point over any odd degree extension, J. Amer. Math. Soc. **30** (2017), no. 2, 451–493, With an appendix by Tim Dokchitser and Vladimir Dokchitser. MR 3600041

Lea Beneish and Christopher Keyes, Fields generated by points on superelliptic curves, 2021.

______, On the proportion of locally soluble superelliptic curves, Finite Fields and Their Applications **85** (2023), 102128.

T. D. Browning, Many cubic surfaces contain rational points, Mathematika **63** (2017), no. 3, 818–839. MR 3731306

Torsten Ekedahl, An infinite version of the Chinese remainder theorem, Comment. Math. Univ. St. Paul. **40** (1991), no. 1, 53–59. MR 1104780

Thank you II

Tom Fisher, Wei Ho, and Jennifer Park, Everywhere local solubility for hypersurfaces in products of projective spaces, Res. Number Theory **7** (2021), no. 1, Paper No. 6, 27. MR 4199457

Bjorn Poonen and Michael Stoll, The Cassels-Tate pairing on polarized abelian varieties, Ann. of Math. (2) **150** (1999), no. 3, 1109–1149. MR 1740984

______, A local-global principle for densities, Topics in number theory (University Park, PA, 1997), Math. Appl., vol. 467, Kluwer Acad. Publ., Dordrecht, 1999, pp. 241–244. MR 1691323

Bjorn Poonen and José Felipe Voloch, Random Diophantine equations, Arithmetic of higher-dimensional algebraic varieties (Palo Alto, CA, 2002), Progr. Math., vol. 226, Birkhäuser Boston, Boston, MA, 2004, With appendices by Jean-Louis Colliot-Thélène and Nicholas M. Katz, pp. 175–184. MR 2029869