Research Statement — Christopher Keyes

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1. Overview

My research interests lie in **number theory**, but more specifically at the intersection of arithmetic geometry and arithmetic statistics. Briefly, **arithmetic geometry** involves studying or exploiting the geometry of equations defined over the rational numbers $\mathbb{Q}$ (or other fields of arithmetic interest such as number fields, global function fields, or finite fields). I interpret **arithmetic statistics** broadly to include many fundamentally quantitative questions involving objects of interest to number theorists — these could be prime numbers, rational points, or elliptic curves, for example.

To illustrate how these topics become entwined, consider a curve C defined over $\mathbb{Q}$. A **rational point** on C is a solution in rational numbers to the equations defining C. This brings us to the first of two broad motivating questions around which my research interests are centered.

Question 1.1. *Given a curve $C/\mathbb{Q}$, what quantitative statements hold for the set of rational points $C(\mathbb{Q})$?*

A simple such question is to determine the size of $C(\mathbb{Q})$, or merely to decide whether it is infinite, finite, or empty. This is challenging and intimately related to the underlying geometry of C; for example, Faltings' theorem states $C(\mathbb{Q})$ is finite whenever its genus, a geometric invariant, is at least 2. There are many fruitful generalizations of Question 1.1, including considering points over extension fields $K/\mathbb{Q}$, or counting points by a suitable height. We could also instead allow the curve C to vary, leading to our second central question.

Question 1.2. *Given a collection of curves over $\mathbb{Q}$, how many have a rational point?*

By allowing the curve to vary, we seek to understand typical behavior in our collection, again influenced by the geometry underneath. Here also we can expand our study to collections of other geometric objects, including higher dimensional varieties or stacks, as well as to other properties of arithmetic interest such as higher degree points or local-to-global principles.

Questions 1.1 and 1.2 guide my past, ongoing, and proposed research. In §2, I describe work on the arithmetic of **superelliptic curves**; this includes studying the proportion of which are everywhere locally soluble [BK23b] and counting field extensions arising from algebraic points on such curves [Key22, BK23a]. In §3 I discuss my research on explicit solubility in families of **hypersurfaces**, highlighting a recent preprint in the cubic case [BK24], and ongoing work for those of higher degree. In §4, I detail an ongoing joint project on local-to-global principles in a family of **stacky curves**, and how we can use their geometry to answer Diophantine questions [DRKK$^+$]. In §5 I briefly mention my work on a generalization of **Mertens' product theorem** [APKK22] and on counting **arithmetical structures** on general graphs [KR21]. Finally, in §6 I describe the potential for **undergraduate involvement** in my research program.

2. The arithmetic of superelliptic curves

Let $m \geq 2$ be a positive integer and $m \mid d$. A **superelliptic curve** $C_f/\mathbb{Q}$ is given by an equation

$$C_f\colon y^m = f(x,z) = \textstyle\sum_{i=0}^{d} c_i x^i z^{d-i}, \tag{2.1}$$

in the weighted projective space $\mathbb{P}\left(1, \frac{d}{m}, 1\right)$. These curves are notable for their cyclic degree m cover $C_f \to \mathbb{P}^1$, given on points by $(x,y) \mapsto x$. When $m = 2$ in (2.1), C_f is known as a **hyperelliptic curve**.

2.1. Local solubility. In the spirit of Question 1.2, we ask how often a curve C_f given by (2.1) has a rational point. A necessary condition for this is that C_f has real points and p-adic points for all primes p. In this case we say C_f is **everywhere locally soluble**. In general, however, everywhere local solubility is insufficient to guarantee $C_f(\mathbb{Q}) \neq \emptyset$. We say C_f fails the **local-to-global principle** for rational points if it is everywhere locally soluble, but $C_f(\mathbb{Q}) = \emptyset$. This will become a recurring theme.

In the $m = 2$ (hyperelliptic) case, the cornerstone work of Bhargava, Gross, and Wang [BGW17] shows that when counted by a suitable height, a positive proportion of C_f fail the local-to-global principle for rational points. In fact, they go further to show that a positive proportion of C_f have no K-points for *any*

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odd degree field extension $K/\mathbb{Q}$. For $m > 2$, however, even the very first step of their approach — determining how often C_f is everywhere locally soluble – remained open.

In joint work with Lea Beneish [BK23b], we study the proportion of superelliptic curves over $\mathbb{Q}$ which are everywhere locally soluble. We define this proportion by the limit below, where C_f is given by (2.1).

$$\rho_{m,d}^{\mathrm{ELS}} = \lim_{B\to\infty} \frac{\#\{(c_i)_{i=0}^d \in \mathbb{Z}^d : |c_i| \leq B,\ C_f \text{ is everywhere locally soluble}\}}{\#\{(c_i)_{i=0}^d \in \mathbb{Z}^d : |c_i| \leq B\}}.$$

We then show $\rho_{m,d}^{\mathrm{ELS}}$ is positive and identify it with a product of local factors $\rho_{m,d}(v)$ over the places v of $\mathbb{Q}$.

Theorem 2.1 (Beneish–K., [BK23b, Theorem A]). *For all $m \geq 2$ and d such that $m \mid d$ and $(m,d) \neq (2,2)$,*

$$\rho_{m,d}^{\mathrm{ELS}} = \rho_{m,d}(\infty)\prod_p \rho_{m,d}(p) > 0.$$

The local factor $\rho_{m,d}(v)$ is defined to be the probability that C_f has a $\mathbb{Q}_v$-point. We are primarily focused on the finite places, where $v = p$ is a prime. By studying the lifting of $\mathbb{F}_p$-points on the reduction $\overline{C_f}$ over $\mathbb{F}_p$, we can give effective bounds for $\rho_{m,d}(p)$, and in some cases an exact description. We carry this out carefully for the case $(m,d) = (3,6)$, which when combined with Theorem 2.1, yields the following result.

Theorem 2.2 (Beneish–K., [BK23b, Theorem C]). *We have $\rho_{3,6}^{\mathrm{ELS}} \approx 96.94\%$.*

More precisely, we give explicit rational functions $R_1(t)$ and $R_2(t)$ of total degree 57 such that $\rho_{3,6}(p) = R_i(p)$ when $p \equiv i \pmod 3$ is sufficiently large. The asymptotics of $R_1(t)$ and $R_2(t)$ are described by

$$1 - R_1(t) \sim \frac{2}{3}t^{-4} \quad \text{and} \quad 1 - R_2(t) \sim \frac{53}{144}t^{-7}.$$

Considerable care and additional computations were needed to determine the local densities $\rho_{3,6}(2), \rho_{3,6}(3)$, and $\rho_{3,6}(p)$ for primes $p \equiv 1 \pmod 3$ such that $p \leq 43$. Our code is available on Github [BK21].

2.2. Fields generated by points. Let $C/\mathbb{Q}$ be a curve (or more generally a variety). A degree n field extension $K/\mathbb{Q}$ is **generated by a point of** C if it is the minimal field of definition for a degree n point P. We denote by $N_{n,C}(X)$ the number of such extensions of bounded discriminant,

$$N_{n,C}(X) = \#\left\{K/\mathbb{Q} : K = \mathbb{Q}(P),\ [K:\mathbb{Q}] = n,\ |\mathrm{Disc}(K)| \leq X\right\}.$$

We can also impose conditions on the Galois group, with $N_{n,C}(X,G)$ denoting the number of those extensions also satisfying $\mathrm{Gal}(\widetilde{K}/\mathbb{Q}) \simeq G$, with $\widetilde{K}$ denoting a Galois closure of K.

In their paper on **Diophantine Stability**, Mazur and Rubin [MR18] ask to what extent the set of fields generated by algebraic points determines the curve C. Motivated by this, and Question 1.1, we want to understand how $N_{n,C}(X)$ grows as $X \to \infty$, and how this depends on both n and the geometry of C.

In [Key22], we consider the case where C is a hyperelliptic curve and give asymptotic lower bounds for $N_{n,C}(X, S_n)$. Then in [BK23a], joint with Lea Beneish, we consider that of $C = C_f$ a superelliptic curve and give asymptotic lower bonds for $N_{n,C}(X)$ in this larger family. The main results of both papers are summarized in the following theorem[1].

Theorem 2.3 (K., Beneish–K.). *Let C be a superelliptic curve and n sufficiently large. Then we have*

$$N_{n,C}(X) \gg X^{\delta_n}, \tag{2.2}$$

where δ_n is an explicit constant depending on m, d, and n, with $\delta_n \to \frac{1}{m^2}$ as $n \to \infty$.

The approach of the proof is to produce a family of polynomials of degree n whose roots give rise to algebraic points on C. Using **Newton polygons**, we verify that this family consists almost entirely of irreducible polynomials. Counting and adjusting for multiplicity of the fields generated, we produce a lower bound for $N_{n,C}(X)$. The resulting lower bounds may be improved for large n by leveraging the best available upper bounds on the number of fixed degree number fields with bounded discriminant [LT22].

Included in [BK23a, §7] is a discussion of whether and how often C can have points of degree n which do not arise as the pullback of a degree n/m point on $\mathbb{P}^1$. We highlight here a new result in this direction.

Proposition 2.4 (Beneish–K., [BK23a, Proposition 7.2]). *Suppose k is an odd prime and m, d satisfy $4 \mid m \mid d$, $m \leq k$, and $N = 2k < \frac{d}{2} - 1$. Then for a positive proportion of squarefree degree d polynomials $f(x)$ ordered by height, the superelliptic curve given by $C\colon y^m = f(x)$ has at most finitely points of degree N.*

[1]Note that the currently available preprint [BK23a] does not reflect the most up-to-date version as presented here.

2.3. Ongoing and future work on superelliptic curves. Theorems 2.1 and 2.2 suggest a number of follow up questions. Among them is Question 1.2 for superelliptic curves with $(m,d)=(3,6)$.

Question 2.5. *What proportion of superelliptic curves C_f with $(m,d)=(3,6)$ have a $\mathbb{Q}$-point?*

Since we now know how often such curves are everywhere locally soluble, this is equivalent to determining how often they fail the local-to-global principle for rational points. We are also interested in how often such a superelliptic curve has a K-point over a field $K/\mathbb{Q}$ of degree *coprime to m*. Here it is useful to define the **index** of a curve $C/\mathbb{Q}$ to be the greatest common divisor of the degrees $[\mathbb{Q}(P):\mathbb{Q}]$ as P varies over $C_f(\overline{\mathbb{Q}})$. In this language, Bhargava, Gross, and Wang showed that a positive proportion of hyperelliptic curves have index 2 [BGW17]. For general m, a superelliptic curve C_f given by (2.1) has index at most m, as seen by pulling back $\mathbb{Q}$-points on $\mathbb{P}^1$ along the degree m map.

Question 2.6. *How often does a superelliptic curve C_f have index exactly m?*

We propose to approach this first via explicit **descent** methods for $\mathrm{Pic}^1(C_f)$ developed by Creutz [Cre13, Cre20]. When m is prime, the index of C_f is m exactly when $\mathrm{Pic}^1(C_f)(\mathbb{Q})=\emptyset$, as is the case for e.g.

$$C_f\colon y^3=3(x^6+x^4+4x^3+2x^2+4x+3) \qquad \text{(see [Cre13, Example 7.3]).}$$

Performing larger-scale computations in these families could help us to make precise conjectures.

In another direction, we want to understand the Galois groups of fields generated by points.

Question 2.7. *What can we say about $N_{n,C_f}(X,G)$ for various subgroups $G\subseteq S_n$?*

In the hyperelliptic case, the lower bound (2.2) holds for $N_{n,C}(X,S_n)$ [Key22, Theorems 1.1, 1.2]. We expect the same holds for superelliptic curves more generally. It would also be of interest to find proper subgroups $G\subsetneq S_n$ which provably occur for a positive proportion of fields generated by points, or more rarely in some appropriate sense. This is related to statements like Proposition 2.4: when pulling back points of degree n/m along the map $C_f\to\mathbb{P}^1$, the resulting Galois group G is constrained, and in general $G\neq S_n$.

3. Explicit solubility in families of hypersurfaces

A degree d **hypersurface** X_f in projective space $\mathbb{P}^n$ is given by the vanishing of an integral homogeneous degree d form $f(x_0,\dots,x_n)$ in $n+1$ variables. Motivated by Question 1.2, we seek to understand how often X_f has a rational point, for each fixed d,n. Much as we did earlier in §2.1, we can count by a suitable height, $\mathrm{ht}(f)$, determined by the coefficients of f, and define natural densities

$$\rho_{d,n}=\lim_{B\to\infty}\frac{\#\{f:\mathrm{ht}(f)\le B,\ X_f(\mathbb{Q})\neq\emptyset\}}{\#\{f:\mathrm{ht}(f)\le B\}},\quad \rho_{d,n}^{\mathrm{ELS}}=\lim_{B\to\infty}\frac{\#\{f:\mathrm{ht}(f)\le B,\ X_f\text{ is everywhere loc. sol.}\}}{\#\{f:\mathrm{ht}(f)\le B\}}.$$

Exciting recent work of Browning, Le Boudec, and Sawin [BLBS23] shows that when $n\ge d$, $(n,d)\neq(3,3)$, we have $\rho_{d,n}=\rho_{d,n}^{\mathrm{ELS}}$, settling a conjecture of Poonen and Voloch [PV04] (in all but the case of cubic surfaces in $\mathbb{P}^3$). In other words, in these cases *100% of X_f satisfy the local-to-global principle.* Moreover, this means if we can access $\rho_{d,n}^{\mathrm{ELS}}$, then we can give an explicit answer to Question 1.2 for the family.

3.1. Cubic hypersurfaces. In recent joint work with Lea Beneish, we compute $\rho_{3,n}$ exactly for $n\ge 4$, by computing the density of *locally soluble* cubic hypersurfaces at p, $\rho_{3,n}(p)$, as a rational function *uniform in p*.

Theorem 3.1 (Beneish–K. [BK24])**.** *For $4\le n\le 8$, there exist explicit $g_{3,n}(t),h_{3,n}(t)\in\mathbb{Z}[t]$ such that*

$$\rho_{3,n}=\rho_{3,n}^{\mathrm{ELS}}=\prod_p\rho_{3,n}(p)=\prod_p\left(1-\tfrac{g_{3,n}(p)}{h_{3,n}(p)}\right).$$

When $n\ge 9$, $\rho_{3,n}(p)=1$ was known due to Lewis [Lew52], and $\rho_{3,n}=1$ was shown in celebrated work of Heath-Brown using the circle method [HB83]. The proof of Theorem 3.1 also recovers a result of Bhargava, Cremona, and Fisher computing $\rho_{3,2}^{\mathrm{ELS}}$ [BCF16]. A striking feature of Theorem 3.1 is the uniform nature of the local probabilities $\rho_{3,n}(p)$, in contrast with e.g. Theorem 2.2, where the densities for small primes did not conform to the same formulae. This uniformity is explored further for hypersurfaces of higher degree in §3.2.

To illustrate the explicit nature of our approach, we consider the case of cubic surfaces in $\mathbb{P}^3$, where our methods together with the remaining open case of Poonen and Voloch's conjecture yield the following.

Conjecture 3.2. *The density of cubic surfaces in $\mathbb{P}^3$ with a rational point is given by*

$$\rho_{3,3}=\prod_p\left(1-\tfrac{\left(3p^{26}+p^{24}+p^{23}+4p^{22}-3p^{21}+3p^{20}+2p^{19}+2p^{18}-p^{17}+p^{14}+p^{13}-2p^{12}+3p^{11}+3p^{7}\right)\left(p^2+1\right)(p+1)^2(p-1)^4}{9(p^{13}-1)(p^7+1)(p^7-1)(p^6+1)(p^5-1)(p^3+1)(p^3-1)}\right)\approx 0.999927.$$

3.2. Ongoing work in higher degrees. An analogue of the uniformity in p present in Theorem 3.1 was conjectured by Artin for higher degrees. He conjectured that for all degrees d and $n \geq d^2$, every degree d homogeneous polynomial $f \in \mathbb{Q}_p[x_0, \dots, x_n]$ has a solution in $\mathbb{Q}_p$. In geometric language, this states that *every* degree d hypersurface $X_f/\mathbb{Q}_p$ *in sufficiently many variables* has a $\mathbb{Q}_p$-point. This conjecture was proven false by Terjanian, who constructed a counterexample with $d = 4$, $N = 17$, and $p = 2$ [Ter66]. In all known counterexamples to the conjecture, d is composite and divisible by $p-1$, leaving us with a tantalizing question:

Question 3.3. *If d is prime and $n \geq d^2$, does every degree d form $f \in \mathbb{Q}_p[x_0, \dots, x_n]$ have a $\mathbb{Q}_p$-solution?*

Question 3.3 has been answered in the affirmative for $d = 2, 3$ due to Hasse and Lewis [Lew52], respectively. Ax and Kochen showed that for fixed d, if p is taken sufficiently large, the answer to Question 3.3 is "yes" [AK65]. Actually pinning down how large p has to be for their results to be effective is challenging.

For small degrees, namely $d = 5, 7, 11$, a p-adic minimization procedure together with effective Lang–Weil type bounds has led to some progress. For $d = 5$, this approach, combined with extensive computation, has settled Question 3.3 in the affirmative when $p \geq 11$ [LY96, HB10, Dum17]. The essential idea of Heath-Brown [HB83] (resp. Dumke [Dum17]) is that if a counterexample to Question 3.3 existed, then one could produce a ternary (resp. quaternary) quintic form over $\mathbb{F}_p$ of a certain shape with no nonsingular solution. Enumerating all such forms over $\mathbb{F}_p$ and searching for nonsingular solutions by computer can thus rule out a counterexample to Question 3.3.

While this strategy remains valid for $p = 7$ (and more generally the finite fields $\mathbb{F}_8$ and $\mathbb{F}_9$), the list of forms to check blows up massively, making brute-force computation nearly infeasible on a traditional — even high performance — computer. To that end, I am currently developing special purpose C++ code to be run on a modern GPU that takes advantage of **parallelization** and some algorithmic improvements, which is bringing us closer to answering Question 3.3 for $d = 5$.

4. Local-to-global principles for integer points on stacky curves

Given natural numbers p, q, r, a **generalized Fermat equation** of type (p, q, r) is a Diophantine equation

$$Ax^p + By^q + Cz^r = 0, \tag{4.1}$$

for integers A, B, C. An integer solution (x, y, z) to (4.1) is called **primitive** if $\gcd(x, y, z) = 1$. Much study has been made of primitive integral solutions to generalized Fermat equations; most famous of course is Fermat's last theorem, where we have $p = q = r \geq 3$, $A, B, C = 1$, and the only solutions satisfy $xyz = 0$. Darmon and Granville [DG95] showed that when $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$, a generalized Fermat equation has at most *finitely* many primitive integer solutions.

There is a natural *weighted* action of the multiplicative group $\mathbb{G}_m$ on the surface cut out by (4.1). Taking the quotient by this action, we obtain a **stacky curve** whose integral points are in bijection with the primitive integral solutions to (4.1). Due to the stabilizers present in this action, this stacky curve is not a scheme. Nevertheless, they enjoy many of the familiar features of ordinary algebraic curves, while admitting additional richness, e.g. they are allowed to have *fractional genus*.

Natural analogues of Questions 1.1 and 1.2 in this setting include determining how many *integral* points lie on a given stacky curve $\mathcal{X}$, or how often those given by (4.1) satisfy a **local-to-global principle** for their integral points. The latter was investigated by Poonen and Bhargava, who showed that stacky curves of genus $g < 1/2$ satisfy such a local-to-global principle. They then gave explicit criteria for a stacky curve with genus $g = 1/2$ to fail it, despite being birational to a genus zero algebraic curve, where the local-to-global principle for rational points holds. [BP22].

We consider a family of stacky curves coming from generalized Fermat equations of type $(2, 2, n)$, given by

$$\mathcal{Y}_{B,C} \colon x^2 + By^2 = Cz^n.$$

As stacky curves they have genus $\frac{n-1}{n}$. In the vein of Question 1.1, our aim is to determine when $\mathcal{Y}_{B,C}(\mathbb{Z}) \neq \emptyset$, and find explicit points whenever possible. In an ongoing joint project, we give equivalent conditions for $\mathcal{Y}_{B,C}$ to satisfy the local-to-global principle [DRKK$^+$].

Theorem 4.1 ([DRKK$^+$])**.** *Let $B, C \in \mathbb{Z}$. Suppose $\mathcal{Y}_{B,C}(\mathbb{Z}_p) \neq \emptyset$ for all primes p. There is a finite set of primes S in $\mathcal{O}_K$ for $K = \mathbb{Q}(\sqrt{-B})$, depending on B, C, such that the following are equivalent.*

(i) $\mathcal{Y}_{B,C}(\mathbb{Z}) \neq \emptyset$.

(ii) There exists $d \in \mathcal{O}_{K,S}^\times/(\mathcal{O}_{K,S}^\times)^n$ satisfying explicit valuation conditions at primes $\mathfrak{p} \in S$.

(iii) (C) factors over $\mathcal{O}_K$ as $(C) = J_+J_-$ for $J_\pm$ in an explicit coset of $n\mathrm{Cl}(K)$.

The idea behind the proof of Theorem 4.1 is to base change to $\mathcal{O}_{K,S}$ and cover $\mathcal{Y}_{B,C}$ by a genus zero curve $\mathcal{C}$. By choosing S appropriately, we can perform an explicit **étale descent** to describe the set $\mathcal{Y}_{B,C}(\mathcal{O}_{K,S})$. Determining which of these points actually came from $\mathcal{Y}_{B,C}(\mathbb{Z})$ produces the explicit conditions in (ii).

The equivalence of (i) and (iii) above was known in limited cases due to Darmon and Granville [DG95, Prop. 8.1]. The geometric origin of (ii) recovers this equivalence while adding geometric context, and allows us to extend to B, C arbitrary integers. Our central Question 1.2 now suggests a natural follow-up.

Question 4.2. *How often does $\mathcal{Y}_{B,C}$ satisfy the local-to-global principle?*

An explicit description of when $\mathcal{Y}_{B,C}(\mathbb{Z}_p) \neq \emptyset$ suggests that for $|B|, |C| \leq T$, the number of everywhere locally soluble $\mathcal{Y}_{B,C}$ grows like $\frac{T^2}{\log T}$. Part of our ongoing work is to use Theorem 4.1, together with input from class field theory and conjectures about the distribution of $\#\mathrm{Cl}(K)[n]$, to determine (or give bounds on) the rate of growth of the number of $\mathcal{Y}_{B,C}$ which satisfy the local-to-global principle, as B, C grow.

5. Other topics

5.1. Generalization of Mertens' product theorem to Chebotarev sets. In joint work with Santiago Arango-Piñeros and Daniel Keliher [APKK22], we generalize **Mertens' product theorem** to the setting of **Chebotarev sets** of primes in a number field and give a power saving error term. Fix a Galois extension E/F, a conjugacy class $C \subset \mathrm{Gal}(E/F)$, and let $\mathcal{C}(x)$ denote the primes P with $\mathrm{Frob}_P = C$ and norm $NP \leq x$.

Theorem 5.1 (Arango-Piñeros–Keliher–K., [APKK22, Theorem A])**.** *As $x \to \infty$ we have*

$$\prod_{P \in \mathcal{C}(x)} \left(1 - \frac{1}{NP}\right) = \left(\frac{e^{-\gamma(E/F,C)}}{\log x}\right)^{|C|/|G|} + O\left(\frac{1}{(\log x)^{|C|/|G|+1}}\right). \tag{5.1}$$

In the paper, we give an explicit description of the constant $e^{-\gamma(E/F,C)}$ and an application to **primes represented by quadratic forms** [APKK22, Corollary 4.2].

5.2. Arithmetical structures on graphs. Let G be a connected undirected graph on n vertices and denote by δ_{ij} the number of edges between vertices i and j. An **arithmetical structure** on G is a pair $(\mathbf{r}, \mathbf{d}) \in \mathbb{N}^n \times \mathbb{N}^n$ such that $\gcd(r_1, \ldots, r_n) = 1$ satisfying $r_i d_i = \sum_{j \neq i} r_j \delta_{ij}$ for all i. These were introduced to study special fibers of relative proper minimal models of curves by Lorenzini, who also proved that the set $A(G)$ of arithmetical structures on G is finite [Lor89].

Little is known about $\#A(G)$ beyond a few special cases (see [BCC+18, ABDL+20, GW19]). In joint work with Tomer Reiter [KR21], we give the first known upper bound in terms of only n and the number of edges, by inductive application of a **smoothing** construction.

Theorem 5.2 (K.–Reiter, [KR21])**.** *Let G be as above and denote its edge set by $E(G)$. We have*

$$\#A(G) \leq \tfrac{n!}{2} \cdot \#E(G)^{2^{n-2}-1} \cdot \#E(G)^{2^{n-1} \cdot \frac{1.538 \log(2)}{(n-1)\log(2) + \log(\log(\#E(G)))}}.$$

6. Student involvement

Including undergraduate students in the research process provides a tremendous value to both the researcher and the student. As a PhD student, I started a **Directed Reading Program** (DRP) at Emory University, which paired undergraduates with a graduate student to learn an advanced topic. Through this program, my students became independent learners, gained valuable insight into a research environment, and practiced practical skills including presenting to a mathematical audience. The experience also prepared me to take on the role of a mentor to a student curious about research in number theory and arithmetic geometry, where my broad research interests offer numerous accessible potential problems. See additional possibilities below.

- For a curve in a particular parametrized family, write a computer program to check if it is everywhere locally soluble. Use a Monte Carlo simulation to suggest how often this occurs, then try to prove it.
- For a specific family of graphs G_n, determine (or bound) the asymptotic growth rate of $\#A(G_n)$. If G is a random graph in some appropriate sense, what is the distribution of $\#A(G)$?
- I am an avid baseball fan and have taken an interest in sabermetrics, the mathematics of baseball. This is a plentiful source of inspiration for projects related to data science, statistics, and modeling.

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