EMORY MATHEMATICS DIRECTED READING PROGRAM

SPRING 2022

Program description

The Emory Math Directed Reading Program (DRP) is a graduate student-run program aiming to pair undergraduate students with graduate student mentors to read and learn material that is not typically offered in a traditional course setting. Undergraduate students are expected to work mostly independently to read the text and attempt exercises, then meet regularly with their graduate student mentor to discuss the material.

How to apply: Interested undergraduates may apply via this short application form, due Thursday, January 13 at 5:00pm ET. Applicants will be notified of a decision shortly thereafter and registered in advance of the add/drop/swap deadline.

Topics: Students should select a topic of their interest when applying to the program, in order to be matched with an appropriate graduate mentor. Some potential/past topics are listed below, but you are encouraged to suggest a topic not listed! See the next page for more detailed sample descriptions of the topics below.

- Algebraic number theory,
- Category theory,
- Commutative algebra,
- Elliptic curves,
- p-adic numbers,
- Partial differential equations,
- Percolation theory,
- Set theory.

Credit and expectations: Course credit will be offered to participating undergraduates, in the form of a 1-credit directed study course! The grading basis is S/U. To obtain a satisfactory grade, students are expected to

- read the selected text independently,
- attend regular weekly meetings with their mentor to discuss the material, and
- give a short presentation to an audience of their peers at the end of the semester (date and time TBD).

Missing three or more scheduled meetings without communicating with the mentor may result in an unsatisfactory grade. The final presentation may be completed via recording in the event of a scheduling conflict.

If you have any questions or concerns, or suggestions for future DRP topics, please reach out to the program director Chris Keyes at christopher.keyes@emory.edu.

Sample course descriptions

Course name: Algebraic number theory
Text: *TIFR pamphlets on algebraic number theory* and *Algebraic Theory of Numbers*, by Pierre Samuel.
Prerequisites: Abstract algebra I/II (Math 421,422), Abstract vector spaces (Math 321), and some commutative algebra.

Description: The idea of the course is to amalgamate one's interest in algebra with number theory. We will read through and work out the details of the TIFR pamphlets and move on to Samuel's book from there. The goal of this short course would be to build the basics necessary to concretely understand the meaning and applications of the "Lagrange's Four Squares Theorem," which states that any natural number can be represented as the sum of four integer squares. Depending upon time and interest, we'll try to go deeper into understanding the quadratic class number formula. The pacing of the course is flexible since my goal is to make learning this topic fun!

Course name: Category theory
Text: Chapter 1 of *Foundations of Algebraic Geometry*, by Ravi Vakil, and *Categories for the Working Mathematician*, by Saunders MacLane.
Prerequisites: Abstract algebra I/II (Math 421,422).

Description: Have you noticed that certain things show up a lot in your courses? Take *products* for example: in linear algebra, we take the *Cartesian product* of vector spaces, $\mathbb{R}^2 \times \mathbb{R}^3$, but we can also take the *direct product* of groups, say $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. You may also have run into products of topological spaces, graphs, or product measures. What do these constructions have in common, and why are they deserving of their designation as a *product*?

Category theory suggests that instead of studying mathetmatical objects like vector spaces, groups, and topological spaces, we study the *maps between them*. For our product example above, the categorical perspective offers the following universal definition: given objects A and B, the product $A \times B$ is the (essentially) unique object denoted $A \times B$ such that *any other* object T with maps to A and B has a map to $A \times B$.

$$\begin{tikzcd}
T \arrow[dr, dashed] \arrow[drr] \arrow[ddr] & & \\
& A \times B \arrow[r] \arrow[d] & A \\
& B &
\end{tikzcd}$$

In this course we will explore ideas like this, learning about categories, functors, universal constructions, and more. While this abstract subject is rich and fascinating in its own right, we will use plenty of examples, focusing on how the categorical perspective can help to organize and shape our view of the mathematical world.

Course name: Commutative algebra
Text: *Introduction to commutative algebra*, by Michael Atiyah and Ian Macdonald.
Prerequisites: Abstract algebra I (Math 421) or some familiarity with groups and rings would help, but isn't necessarily required.

Description: The goal of this course is to read through and work out the details in the first few chapters of Atiyah and Macdonald's *Introduction to Commutative algebra*. Each week, we will select a few theorems and exercises to write out in detail. We will focus on concrete examples and problem solving. Depending on time and interest, we can stop after the second chapter to apply what we have learned to geometric problems from Fulton's *Algebraic curves* and applications of the Nullstellensatz or continue with more commutative algebra. As this is meant to be introductory, a student of any level is welcome (though if there is no familiarity with algebra then more meetings may be necessary).

Course name: Elliptic curves
Text: *Rational Points on Elliptic Curves*, by Joseph Silverman and John Tate.
Prerequisites: Abstract algebra I, preferably also II (Math 421, 422). Number theory (Math 328) would help, but isn't required.

Description: Elliptic curves have played a central role in number theory for over a century. While they arose from classical complex geometry, they have more recently been used in cryptographic applications, e.g. keeping online credit card transactions secure. The subject also provides a wonderful entry point to the world of arithmetic and algebraic geometry and the study of rational points on curves and abelian varieties. Some familiarity with groups and finite fields (Abstract Algebra I and possibly II) is necessary, and any additional number theoretic or analytic background will be useful.

The first goal of this course is to understand that the points on an elliptic curve have the structure of an abelian group. Indeed, this group is finitely generated, a fact known as the Mordell–Weil theorem. To understand the proof, we begin by investigating points of finite order (a.k.a. torsion points), then we will build a theory of heights to complete the proof of the theorem. We will also spend some time thinking about elliptic curves over finite fields — the setting of their cryptographic applications. This could lead to working through toy examples of elliptic curve cryptography and/or factoring algorithms, which may especcially interest those with programming experience or an interest in learning!

Course name: p-adic numbers
Text: *p-adic numbers: An introduction*, by F. Gouvea.
Prerequisites: Abstract algebra I/II (Math 421/422), Real analysis I (Math 411). Number theory (Math 328) would help, but isn't required.

Description: In this course, we will study p-adic numbers, which play a central role in modern number theory, laying the foundation for other topics such as class field theory and arithmetic geometry. The p-adic numbers arise from solving integer congruences, and come with interesting algebraic and analytic structures which make them useful to so many mathematicians. We will spend some time constructing and exploring the p-adics, building to key ideas such as Hensel's lemma and local-global principles. Time permitting, we may investigate further topics such as p-adic analysis and Newton polygons.

The textbook, Gouvea's *p-adic numbers: An introduction*, is a fun and conversational text with exercises sprinkled throughout. It should be an enjoyable read for students with a wide range of experience. The key prerequisite topics are groups, rings, fields, and congruences from algebra, and the basic ideas of metric topology and convergence, which should be covered in an introductory real analysis course. Experience with number theory may be helpful for motivation.

Course name: Partial differential equations
Text: *Partial Differential Equations*, by Evans and lecture notes from the University of Cambridge.
Prerequisites: Differential equations (Math 212) and real analysis I/II (Math 411/412). Partial differential equations (Math 351) is welcome, but not required. Exposure to measure theory is preferred, but topics requiring it can be avoided or mildly introduced.

Description: The study of differential equations began with the purpose of understanding better the dynamical systems occurring everywhere in nature. The first (ordinary) differential equations predicting the evolution of a system can be attributed to Newton and Leibniz for the description of physical laws such as gravitational forces. Then a second wave of differential equations took place with Euler, Fourier, and the creation of partial differential equations (PDEs) to explain continuum mechanics in more than one independent variable. Moreover, differential equations did not only revolutionize science but mathematics as well. For instance, their development contributed significantly to the areas of complex analysis, dynamical systems, and differential geometry. Thus, differential equations beautifully intertwines fields in pure and applied mathematics.

For this course, we will attempt to gain a deeper understanding of the qualitative analysis of ordinary differential equations (ODEs). Rather than finding techniques for "solving" equations, we will use analysis tools to better comprehend their behavior. The following topics are to be covered: existence and uniqueness theorem for ODEs, analyticity, difference between types of regularity, Arzela-Ascoli theorem, Picard-Lindelöf theorem, Cauchy-Peano theorem, Cauchy-Kovalevskaya Theorem for ODEs (with an extension to PDEs, depending on the background of the student), analysis of local and global solutions, and classification, overview, and analysis toolbox for PDEs (this can be flexible according to time allowance and background of student).

Course name: Percolation theory
Text: *Percolation*, by Béla Bollobás and Oliver Riordan.
Prerequisites: Some familiarity with probability concepts, including expected values, exposure to graph theory, and familiarity with concepts of supremum and infimum

Description: : If you submerge a porous stone in water, will its center get wet? You can model this phenomenon by thinking of the interior of the stone as a network of passages that are each either open or closed. If there is a path from the outside of the stone all the way to its center, consisting of just open passages, then the water will be able to seep in and reach the center. More generally, we can consider a grid where each pair of adjacent grid points is connected or not with some fixed probability, μ. If we have a low connection probability μ, we expect to get lots of small connected components while if we have a large μ, we expect there to be a very large (infinite if the original grid is) component. It turns out that there will be a distinct threshold value for μ at which the behavior changes and in some cases, we can even calculate this value exactly. Percolation theory has applications to electrical networks, magnetism, and even epidemics!

Course name: Set theory
Text: *Set Theory and Logic*, by Robert R. Stoll.
Prerequisites: Foundations of math (Math 250) or similar exposure to proofs.

Description: The theory of sets is home to some of the most philosophically challenging and counter-intuitive results in all of mathematics. The goal of this course is to explore set theory at a level deeper than is covered in Foundations of Mathematics, and our initial plan is to work through as much of Chapters 2, 3, 5, and 7 of Stoll's *Set Theory and Logic* as possible. Depending on interest and time, however, we might branch out into other readings and topics.