Explicit local solubility in families of varieties

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Motivation

Let $V/\mathbb{Q}$ be a variety and v a place of $\mathbb{Q}$ (i.e. $v = p$ or $v = \infty$).

Definition

V is **soluble** if $V(\mathbb{Q})$ is nonempty.

V is **locally soluble at v** if $V(\mathbb{Q}_v)$ is nonempty.

V is **everywhere locally soluble (ELS)** if $V(\mathbb{Q}_v) \neq \emptyset$ for all v.

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- When $d \leq n$[1], Hasse principle holds on average [BLBS23]

[1] Except possibly $n = d = 3$

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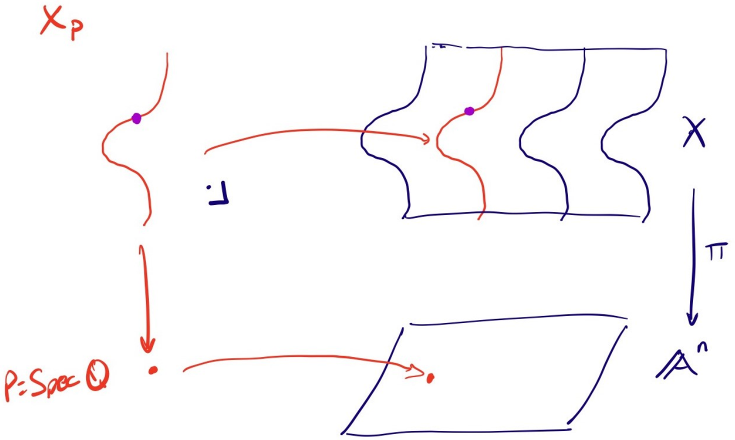
Question

What about superelliptic curves?

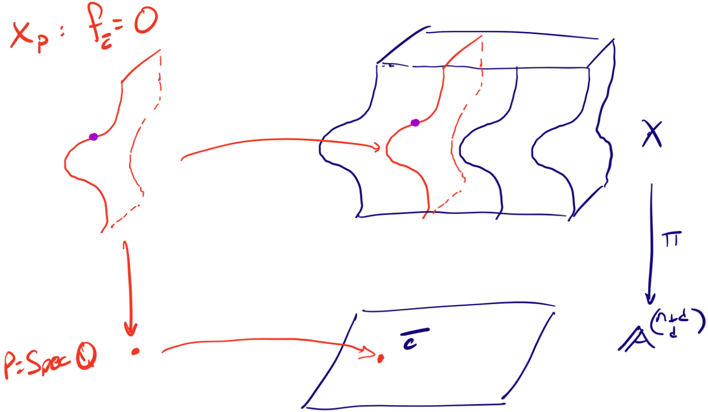
$$\mathscr{S}_{m,d} = \{C_f : y^2 = f(x,z) \mid m \mid \deg(f) = d\}$$

Common geometric picture



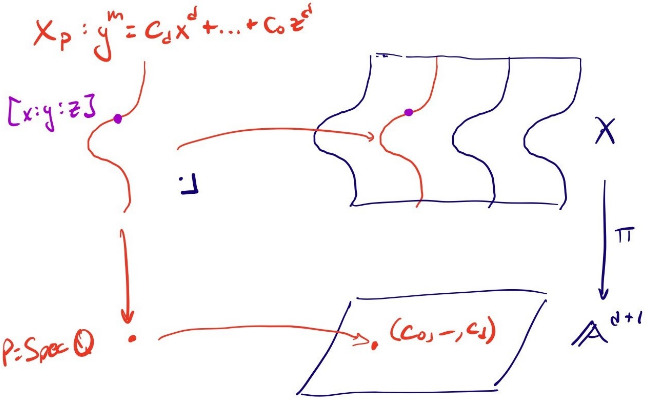
Common geometric picture

$$X\colon \sum_{i_0+\ldots+i_n=d} c_{i_0,\ldots,i_n}\prod_{j=0}^{n} x_j^{i_j} = 0$$



Common geometric picture

$$X\colon y^m = c_d x^d + \ldots + c_0 z^d \subset \mathbb{A}^{d+1}_{\mathbb{Q}} \times \mathbb{P}(1, \frac{d}{m}, 1)$$



Key features

We will use two key features:

1. Natural notion of density

$$\rho_\pi = \lim_{B\to\infty} \frac{\#\{P \in \mathbb{Z}^N \cap [-B,B]^N \mid X_P \text{ is ELS}\}}{\#\{P \in \mathbb{Z}^N \cap [-B,B]^N\}}$$

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2. Independence at each place...sometimes

$$\rho_\pi = \rho_\pi(\infty) \prod_p \rho_\pi(p)$$

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- Poonen–Stoll, Poonen–Voloch: apply to hyperelliptics, hypersurfaces [PS99b, PV04]

Independence

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- Poonen–Stoll: criterion for density of $\mathbb{Z}$-tuples satisfying local conditions [PS99a]
- Poonen–Stoll, Poonen–Voloch: apply to hyperelliptics, hypersurfaces [PS99b, PV04]
- Bright–Browning–Loughran: geometric version [BBL16]

  Namely, π is nice morphism, codim. 1 fibers not too bad

A result

Consider $y^3 = f(x, z)$ degree 6. Let $\rho = \rho_\pi$.

Theorem (Beneish–K. [BK23])

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$\rho \approx 97\%$.

There exist rational functions $R_1(t)$ and $R_2(t)$ such that

$$\rho(p) = \begin{cases} R_1(p), & p \equiv 1 \pmod 3 \text{ and } p > 43 \\ R_2(p), & p \equiv 2 \pmod 3 \text{ and } p > 2. \end{cases}$$

Asymptotically,

$$1 - R_1(t) \sim \frac{2}{3} t^{-4},$$
$$1 - R_2(t) \sim \frac{53}{144} t^{-7}.$$

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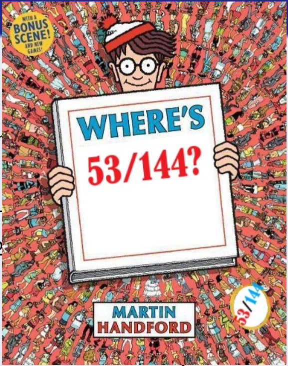

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$$\rho = \begin{cases} \Big(1296p^{57} + 3888p^{56} + 9072p^{55} + 16848p^{54} + 27648p^{53} + 39744p^{52} + 53136p^{51} + 66483p^{50} + 80019p^{49} + 93141p^{48} \\ + 107469p^{47} + 120357p^{46} + 135567p^{45} + 148347p^{44} + 162918p^{43} + 176004p^{42} + 190278p^{41} + 203459p^{40} \\ + 218272p^{39} + 232083p^{38} + 243639p^{37} + 255267p^{36} + 261719p^{35} + 264925p^{34} + 265302p^{33} + 261540p^{32} \\ + 254790p^{31} + 250736p^{30} + 241384p^{29} + 226503p^{28} + 214137p^{27} + 195273p^{26} + 170793p^{25} + 151839p^{24} + 136215p^{23} \\ + 118998p^{22} + 105228p^{21} + 94860p^{20} + 80471p^{19} + 67048p^{18} + 52623p^{17} + 40617p^{16} + 28773p^{15} + 19247p^{14} & p \equiv 1 \pmod 3 \\ + 12109p^{13} + 7614p^{12} + 3420p^{11} + 756p^{10} - 2248p^{9} - 4943p^{8} - 6300p^{7} - 6894p^{6} - 5994p^{5} - 2448p^{4} - 648p^{3} \\ + 324p^{2} + 1296p + 1296\Big) \Big/ \Big(1296\left(p^{12} - p^{11} + p^{9} - p^{8} + p^{6} - p^{4} + p^{3} - p + 1\right)\left(p^{8} - p^{6} + p^{4} - p^{2} + 1\right) \\ \times \left(p^{6} + p^{5} + p^{4} + p^{3} + p^{2} + p + 1\right)\left(p^{4} + p^{3} + p^{2} + p + 1\right)^{3}\left(p^{4} - p^{3} + p^{2} - p + 1\right)\left(p^{2} + p + 1\right) \\ \times \left(p^{2} + 1\right)p^{11}\Big), \\ \\ \Big(144p^{57} + 432p^{56} + 1008p^{55} + 1872p^{54} + 3168p^{53} + 4608p^{52} + 6336p^{51} + 8011p^{50} + 9803p^{49} + 11357p^{48} \\ + 13061p^{47} + 14525p^{46} + 16295p^{45} + 17875p^{44} + 19654p^{43} + 21212p^{42} + 23030p^{41} + 24563p^{40} + 26320p^{39} \\ + 27771p^{38} + 29711p^{37} + 30859p^{36} + 31135p^{35} + 31525p^{34} + 31510p^{33} + 29436p^{32} + 28502p^{31} + 28616p^{30} \\ + 26856p^{29} + 25087p^{28} + 25057p^{27} + 23041p^{26} + 19921p^{25} + 18119p^{24} + 16287p^{23} + 13798p^{22} \\ + 12140p^{21} + 10844p^{20} + 9191p^{19} + 7480p^{18} + 5839p^{17} + 4265p^{16} + 2909p^{15} + 1943p^{14} + 1109p^{13} & p \equiv 2 \pmod 3 \\ + 590p^{12} + 604p^{11} + 372p^{10} - 144p^{9} - 87p^{8} - 84p^{7} - 678p^{6} - 618p^{5} - 144p^{4} - 168p^{3} - 156p^{2} \\ + 144p + 144\Big) \Big/ \Big(144\left(p^{12} - p^{11} + p^{9} - p^{8} + p^{6} - p^{4} + p^{3} - p + 1\right)\left(p^{8} - p^{6} + p^{4} - p^{2} + 1\right) \\ \times \left(p^{6} + p^{5} + p^{4} + p^{3} + p^{2} + p + 1\right)\left(p^{4} + p^{3} + p^{2} + p + 1\right)^{3}\left(p^{4} - p^{3} + p^{2} - p + 1\right)\left(p^{2} + p + 1\right) \\ \times \left(p^{2} + 1\right)p^{11}\Big), \end{cases}$$

Local densities

Question

Once we know

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how do we make $\rho(p)$ *explicit?*

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how do we make $\rho(p)$ explicit?

$\rho(\infty)$: Euclidean measure of $\mathbb{R}$-soluble C_f with coeffs $\in [-1, 1]$.

- If m or d is odd, then $\rho(\infty) = 1$.
- If m, d even, no analytic solution known for $d > 2$, but rigorous estimates exist, e.g.

$$0.873914 \leq \rho(\infty) \leq 0.874196 \quad \text{[BCF21]}.$$

Computing local densities — finite places

$\rho(p)$ is (normalized) Haar measure of space of the $\mathbb{Q}_p$-soluble fibers X_P, for $P \in \mathbb{A}^N(\mathbb{Z}_p)$.

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- Smooth $\mathbb{F}_p$-points on $\overline{X_P}$ lift to $\mathbb{Q}_p$-solutions (Hensel),
- $\overline{X_P}(\mathbb{F}_p) = \emptyset \implies$ no $\mathbb{Q}_p$-solutions,
- If $\overline{X_P}(\mathbb{F}_p)$ only non-smooth points, do more work.

An extended example

Example

Consider $(m, d) = (3, 6)$, generically genus 4:

$$C_f \colon y^3 = f(x, z) = c_6 x^6 + c_5 x^5 z + \cdots + c_1 x z^5 + c_0 z^6.$$

When can we guarantee $\overline{C_f}$ has liftable $\mathbb{F}_p$-points?

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Theorem (Hasse–Weil bound)

If $\overline{C_f}$ is irreducible and smooth of genus g, then

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Theorem (Hasse–Weil bound, refined)

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When can we guarantee $\overline{C_f}$ has liftable $\mathbb{F}_p$-points?

When $p \geq 61$, we have $p + 1 - 4\lfloor 2\sqrt{p} \rfloor > 0$, so

$$\overline{C_f}/\mathbb{F}_p \text{ smooth} \implies C_f(\mathbb{Q}_p) \neq \emptyset.$$

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Count geom. reducible $\overline{C_f}$: $p^3 = (p-1)(p^2 + p + 1) + 1$

$$\implies \rho(p) \geq \frac{p^7 - p^3}{p^7} = 1 - \frac{1}{p^4} \text{ for all } p \geq 61.$$

An extended example

- $\rho(p) \geq 1 - \frac{1}{p^4}$ when $p \equiv 1 \pmod{3}$ and $p > 43$
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- $\rho(p) \geq 1 - \frac{1}{p^7}$ when $p \equiv 2 \pmod 3$ and $p > 2$
- Enumerate all $\overline{f}(x,z)$ and count Hensel-liftable $\mathbb{F}_p$-solutions:

p	$\rho(p) \geq$	p	$\rho(p) \geq$
2	$\frac{63}{64} \approx 0.98437$	19	$\frac{893660256}{893871739} \approx 0.99976$
3	$\frac{26}{27} \approx 0.96296$	31	$\frac{27512408250}{27512614111} \approx 0.99999$
7	$\frac{810658}{823543} \approx 0.98435$	37	$\frac{94931742132}{94931877133} \approx 0.999998$
13	$\frac{62655132}{62748517} \approx 0.99851$	43	$\frac{271818511748}{271818611107} \approx 0.9999996$

Put together with Theorem A:

$$\rho = \prod_p \rho(p) \geq 0.93134.$$

Getting exact answer

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Let $F(x, y, z) = y^3 - f(x, z)$ and look at reduction modulo p.

$\overline{F}$ reducible$/\overline{\mathbb{F}_p} \iff \overline{F} = (y - h)(y - \zeta_3 h)(y - \zeta_3^2 h)$.

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Factorization type in y	$p=3$	$p \equiv 1 \pmod 3$	$p \equiv 2 \pmod 3$
1. Abs. irr.	2160	$p^3(p^4-1)$	$p^3(p^4-1)$
2. 3 distinct linear over $\mathbb{F}_p$	0	$\frac{1}{3}(p^3-1)$	0
3. Linear + conj.	0	0	p^3-1
4. 3 conjugate factors	0	$\frac{2}{3}(p^3-1)$	0
5. $(y-h(x,z))^3$	27	1	1
Total	3^7	p^7	p^7

Getting exact answer

Let ξ_i be the proportion of $\overline{f}$ for which $\overline{F}$ has type i.

Let σ_i be probability $F = 0$ has $\mathbb{Z}_p$-solution when $\overline{F}$ has type i.

$$\rho(p) = \sum_{i=1}^{5} \xi_i(p)\sigma_i(p).$$

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In order to compute σ_4, σ_5, do the following.

1. How often do factorization types occur (mod p)?
2. Find lifting probabilities for each factorization type.
3. Relate probabilities to each other and solve.

Computing σ_5

Suppose $f \equiv 0 \pmod{p}$.

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$$\sigma_5 = \left(1 - \frac{1}{p^7}\right) \sum_{i=0}^{9} \eta_i \tau_i + \left(\frac{1}{p^7} - \frac{1}{p^{14}}\right) \sum_{i=0}^{9} \eta_i \theta_i + \frac{1}{p^{14}} \rho.$$

- i runs over factorization types of $\frac{1}{p} f(x,z)$ (resp. $\frac{1}{p^2} f(x,z)$)
- $\eta_i =$ proportion of sextic forms$/\mathbb{F}_p$ with i-th type
- $\tau_i =$ lifting probability for f with $\frac{1}{p} f$ of type i (resp. θ_i, $\frac{1}{p^2} f$)

Factorization types

Fact. type	η_i	η'_i (monic forms only)
0. No roots	$\dfrac{\left(53p^4+26p^3+19p^2-2p+24\right)(p-1)p}{144\left(p^6+p^5+p^4+p^3+p^2+p+1\right)}$	$\dfrac{\left(53p^4+26p^3+19p^2-2p+24\right)(p-1)}{144p^5}$
1. $(1*)$	$\dfrac{\left(91p^4+26p^3+23p^2+16p-12\right)(p+1)p}{144\left(p^6+p^5+p^4+p^3+p^2+p+1\right)}$	$\dfrac{\left(91p^3-27p^2+50p-48\right)(p+1)(p-1)}{144p^5}$
2. (1^24) or (1^222)	$\dfrac{\left(3p^2+p+2\right)(p+1)(p-1)p}{8\left(p^6+p^5+p^4+p^3+p^2+p+1\right)}$	$\dfrac{\left(3p^2+p+2\right)(p-1)}{8p^4}$
3. (1^21^22)	$\dfrac{(p+1)(p-1)p^2}{4\left(p^6+p^5+p^4+p^3+p^2+p+1\right)}$	$\dfrac{(p-1)^2}{4p^4}$
4. $(1^21^21^2)$	$\dfrac{(p+1)(p-1)p}{6\left(p^6+p^5+p^4+p^3+p^2+p+1\right)}$	$\dfrac{(p-1)(p-2)}{6p^5}$
5. (1^33)	$\dfrac{(p+1)^2(p-1)p}{3\left(p^6+p^5+p^4+p^3+p^2+p+1\right)}$	$\dfrac{(p+1)(p-1)}{3p^4}$
6. (1^31^3)	$\dfrac{(p+1)p}{2\left(p^6+p^5+p^4+p^3+p^2+p+1\right)}$	$\dfrac{p-1}{2p^5}$
7. (1^42)	$\dfrac{(p+1)(p-1)p}{2\left(p^6+p^5+p^4+p^3+p^2+p+1\right)}$	$\dfrac{p-1}{2p^4}$
8. (1^21^4)	$\dfrac{(p+1)p}{p^6+p^5+p^4+p^3+p^2+p+1}$	$\dfrac{p-1}{p^5}$
9. (1^6)	$\dfrac{p+1}{p^6+p^5+p^4+p^3+p^2+p+1}$	$\dfrac{1}{p^5}$

Factorization types

Fact. type	η_i	η_i' (monic forms only)
0. No roots	$\dfrac{\left(53p^4+26p^3+19p^2-2p+24\right)(p-1)p}{144\left(p^6+p^5+p^4+p^3+p^2+p+1\right)}$	$\dfrac{\left(53p^4+26p^3+19p^2-2p+24\right)(p-1)}{144p^5}$
1. $(1*)$	$\dfrac{\left(91p^4+26p^3+23p^2+16p-12\right)(p+1)p}{144\left(p^6+p^5+p^4+p^3+p^2+p+1\right)}$	$\dfrac{\left(91p^3-27p^2+50p-48\right)(p+1)(p-1)}{144p^5}$
2. (1^24) or (1^222)	$\dfrac{\left(3p^2+p+2\right)(p+1)(p-1)p}{8\left(p^6+p^5+p^4+p^3+p^2+p+1\right)}$	$\dfrac{\left(3p^2+p+2\right)(p-1)}{8p^4}$
3. (1^21^22)	$\dfrac{(p+1)(p-1)p^2}{4\left(p^6+p^5+p^4+p^3+p^2+p+1\right)}$	$\dfrac{(p-1)^2}{4p^4}$
4. $(1^21^21^2)$	$\dfrac{(p+1)(p-1)p}{6\left(p^6+p^5+p^4+p^3+p^2+p+1\right)}$	$\dfrac{(p-1)(p-2)}{6p^5}$
5. (1^33)	$\dfrac{(p+1)^2(p-1)p}{3\left(p^6+p^5+p^4+p^3+p^2+p+1\right)}$	$\dfrac{(p+1)(p-1)}{3p^4}$
6. (1^31^3)	[illegible]	$\dfrac{p-1}{2p^5}$
7. (1^42)	[illegible]	$\dfrac{p-1}{2p^4}$
8. (1^21^4)	[illegible]	$\dfrac{p-1}{p^5}$
9. (1^6)	[illegible]	$\dfrac{1}{p^5}$

Type 9: yikes!

Type 9, e.g. $f(x,z) \equiv px^6 \pmod{p^2}$.

τ_9 is a degree 44 rational function in p.

$\tau_9 =$	c_6	c_5	c_4	c_3	c_2	c_1	c_0
$\tau_{9a} = \frac{1}{p}\tau_{9b}$	$=1$	≥ 2	≥ 2	≥ 2	≥ 2	≥ 2	≥ 2
$\tau_{9b} = \left(1-\frac{1}{p}\right) + \frac{1}{p}\tau_{9c}$	$=4$	≥ 4	≥ 3	≥ 2	≥ 1	≥ 0	≥ 0
$\tau_{9c} = \Phi(p) + \frac{1}{p}\tau_{9d}$	$=4$	≥ 4	≥ 3	≥ 2	≥ 1	≥ 1	≥ 0
$\tau_{9d} = \left(1-\frac{1}{p}\right)\left(\frac{p-1}{2p} + \frac{1}{p^2}\right) + \frac{1}{p}\tau_{9e}$	$=4$	≥ 4	≥ 3	≥ 2	≥ 1	≥ 1	≥ 1
$\tau_{9e} = \left(1-\frac{1}{p}\right) + \frac{1}{p}\tau_{9f}$	$=4$	≥ 4	≥ 3	≥ 2	≥ 2	≥ 1	≥ 1
$\tau_{9f} = \frac{1}{p}\tau_{9g}$	$=4$	≥ 4	≥ 3	≥ 2	≥ 2	≥ 2	≥ 1
$\tau_{9g} = \left(1-\frac{1}{p}\right)\alpha'' + \frac{1}{p}\tau_{9h}$	$=4$	≥ 4	≥ 3	≥ 2	≥ 2	≥ 2	≥ 2
$\tau_{9h} = \left(1-\frac{1}{p}\right)\left(\frac{p-1}{2p} + \frac{\theta_2}{p}\right) + \frac{1}{p}\tau_{9i}$	$=4$	≥ 4	≥ 3	≥ 3	≥ 2	≥ 2	≥ 2
$\tau_{9i} = \left(1-\frac{1}{p}\right) + \frac{1}{p}\tau_{9j}$	$=4$	≥ 4	≥ 3	≥ 3	≥ 3	≥ 2	≥ 2
$\tau_{9j} = \frac{1}{p}\tau_{9k}$	$=4$	≥ 4	≥ 3	≥ 3	≥ 3	≥ 3	≥ 2
$\tau_{9k} = \left(1-\frac{1}{p}\right) + \frac{1}{p}\tau_{9\ell}$	$=1$	≥ 1	≥ 0	≥ 0	≥ 0	≥ 0	≥ 0
$\tau_{9\ell} = \Phi(p) + \left(1-\Phi(p)-\frac{1}{p}\right)\beta + \frac{1}{p}\tau_{9m}$	$=1$	≥ 1	≥ 1	≥ 0	≥ 0	≥ 0	≥ 0
$\tau_{9m} = \left(1-\frac{1}{p}\right) + \frac{1}{p}\tau_{9n}$	$=1$	≥ 1	≥ 1	≥ 1	≥ 0	≥ 0	≥ 0
$\tau_{9n} = \left(1-\frac{1}{p}\right) + \frac{1}{p}\tau_{9o}$	$=1$	≥ 1	≥ 1	≥ 1	≥ 1	≥ 0	≥ 0
$\tau_{9o} = \Phi(p) + \frac{1}{p}\tau_{9p}$	$=1$	≥ 1	≥ 1	≥ 1	≥ 1	≥ 1	≥ 0
$\tau_{9p} = \sigma_5'$	$=1$	≥ 1	≥ 1	≥ 1	≥ 1	≥ 1	≥ 1

What is $\rho(p)$?

$$\rho = \begin{cases} \Big(1296p^{57} + 3888p^{56} + 9072p^{55} + 16848p^{54} + 27648p^{53} + 39744p^{52} + 53136p^{51} + 66483p^{50} + 80019p^{49} + 93141p^{48} \\ + 107469p^{47} + 120357p^{46} + 135567p^{45} + 148347p^{44} + 162918p^{43} + 176004p^{42} + 190278p^{41} + 203459p^{40} \\ + 218272p^{39} + 232083p^{38} + 243639p^{37} + 255267p^{36} + 261719p^{35} + 264925p^{34} + 265302p^{33} + 261540p^{32} \\ + 254790p^{31} + 250736p^{30} + 241384p^{29} + 226503p^{28} + 214137p^{27} + 195273p^{26} + 170793p^{25} + 151839p^{24} + 136215p^{23} \\ + 118998p^{22} + 105228p^{21} + 94860p^{20} + 80471p^{19} + 67048p^{18} + 52623p^{17} + 40617p^{16} + 28773p^{15} + 19247p^{14} \\ + 12109p^{13} + 7614p^{12} + 3420p^{11} + 756p^{10} - 2248p^{9} - 4943p^{8} - 6300p^{7} - 6894p^{6} - 5994p^{5} - 2448p^{4} - 648p^{3} \\ + 324p^{2} + 1296p + 1296\Big) \Big/ \Big(1296\left(p^{12} - p^{11} + p^{9} - p^{8} + p^{6} - p^{4} + p^{3} - p + 1\right)\left(p^{8} - p^{6} + p^{4} - p^{2} + 1\right) \\ \times \left(p^{6} + p^{5} + p^{4} + p^{3} + p^{2} + p + 1\right)\left(p^{4} + p^{3} + p^{2} + p + 1\right)^{3}\left(p^{4} - p^{3} + p^{2} - p + 1\right)\left(p^{2} + p + 1\right) \\ \times \left(p^{2} + 1\right)p^{11}\Big), & p \equiv 1 \pmod 3 \\ \Big(144p^{57} + 432p^{56} + 1008p^{55} + 1872p^{54} + 3168p^{53} + 4608p^{52} + 6336p^{51} + 8011p^{50} + 9803p^{49} + 11357p^{48} \\ + 13061p^{47} + 14525p^{46} + 16295p^{45} + 17875p^{44} + 19654p^{43} + 21212p^{42} + 23030p^{41} + 24563p^{40} + 26320p^{39} \\ + 27771p^{38} + 29711p^{37} + 30859p^{36} + 31135p^{35} + 31525p^{34} + 31510p^{33} + 29436p^{32} + 28502p^{31} + 28616p^{30} \\ + 26856p^{29} + 25087p^{28} + 25057p^{27} + 23041p^{26} + 19921p^{25} + 18119p^{24} + 16287p^{23} + 13798p^{22} \\ + 12140p^{21} + 10844p^{20} + 9191p^{19} + 7480p^{18} + 5839p^{17} + 4265p^{16} + 2909p^{15} + 1943p^{14} + 1109p^{13} \\ + 590p^{12} + 604p^{11} + 372p^{10} - 144p^{9} - 87p^{8} - 84p^{7} - 678p^{6} - 618p^{5} - 144p^{4} - 168p^{3} - 156p^{2} \\ + 144p + 144\Big) \Big/ \Big(144\left(p^{12} - p^{11} + p^{9} - p^{8} + p^{6} - p^{4} + p^{3} - p + 1\right)\left(p^{8} - p^{6} + p^{4} - p^{2} + 1\right) \\ \times \left(p^{6} + p^{5} + p^{4} + p^{3} + p^{2} + p + 1\right)\left(p^{4} + p^{3} + p^{2} + p + 1\right)^{3}\left(p^{4} - p^{3} + p^{2} - p + 1\right)\left(p^{2} + p + 1\right) \\ \times \left(p^{2} + 1\right)p^{11}\Big), & p \equiv 2 \pmod 3 \end{cases}$$

What about small primes?

Use `Magma` when Hasse–Weil doesn't suffice; modify calculations accordingly.

$$\begin{aligned}
\rho(2) &= \frac{4594897725819217081}{46164832540903014400} \approx 0.99532 \\
\rho(3) &= \frac{900175334869743731875930997281}{908381960435133191895132960000} \approx 0.99096 \\
\rho(7) &= \frac{63104494755178622851603292623187277054743730183645677893972}{64083174787206696882429945655801281538844149896400159815375} \approx 0.98472 \\
\rho(13) &= \frac{7877728357244577414025901931296747409682076255666526984515273526822853}{7890643570620106747776737292792780623510727026420779539893772399701475} \approx 0.99836 \\
\rho(19) &= \frac{3122673715489206150449285868243361150392235799365815266879438393279346795671}{3123410013311365155035964479837966797560851333614271490136481337080636454180} \approx 0.99976 \\
\rho(31) &= \frac{9196796457678318869139089936786462146535210039832850454297877482020635073857159758299}{9196865061587843544830989041473808798913128587425995645857828572610918436035833907250} \approx 0.999992 \\
\rho(37) &= \frac{1711286479008201947844581017879529201699244648865190555453844647154184805036447476640345735119}{171128889636157060536894474187017088464271236509977199491208939449738127658679723715588944500} \approx 0.999998 \\
\rho(43) &= \frac{84000121343283090388653356431804100707331364779290664490547105768867844862712134447832720508750281}{84000151671513555191647712567596101710800846209116830568013729377404991150901973105093039939237500} \approx 0.9999996
\end{aligned}$$

What is ρ?

Theorem (Beneish-K.)

We have determined $\rho(p)$ exactly for all p.

Taking product over $p \leq 10000$ gives

$$\rho \approx \prod_{p \leq 10000} \rho(p) = 0.96943,$$

with error of $O(10^{-14})$.

97% of superelliptic curves $y^3 = c_6 x^6 + \ldots + c_0 z^6$ are ELS.

Further questions

Question

Are $\rho(p)$ always given by rational functions for $p \gg 0$?

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How could we have predicted some or all of the behavior of $\rho(p)$?

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When can we say something about global solubility?

In progress: cubic hypersurfaces in $\mathbb{P}^n$

Consider cubic hypersurfaces, $\mathscr{H}_{3,n}$, $3 \leq n \leq 8$

Theorem (Beneish–K. (last week))

For $p \gg 0$, $\rho_n(p)$ is rational function in p which can be made explicit.

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$$\rho_3(p) = 1 - \frac{p^6 - p^5 + p^3 - p + 3}{3p^{16} + 3p^{12} + 3p^8 + 3p^4 + 3}$$

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In light of [BLBS23], explicitly understand density of cubic hypersurfaces with global points.

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In light of [BLBS23], explicitly understand density of cubic hypersurfaces with global points.

Do these properties extend to all degrees?

Thank you I

Thank you for the invitation and for your attention!

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