The Hasse principle for generalized Fermat equations of the form $x^2 + By^2 = Cz^n$

Christopher Keyes

(King's College London)

Spring of Rational Points II

University of Bath

13 May 2025

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Acknowledgment

This work is joint with

- Juanita Duque-Rosero (Boston University)
- Andrew Kobin (CCR La Jolla)
- Manami Roy (Lafayette)
- Soumya Sankar (Utrecht)
- Yidi Wang (Western Ontario)

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Advertisment

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Soumya will be speaking in Bristol next week!

Generalized Fermat equations

Definition

A generalized Fermat equation is a Diophantine equation

$$Ax^p + By^q = Cz^r.$$

We are interested in primitive integer solutions: $\gcd(x, y, z) = 1$.

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Example (Fermat's last theorem)

When $A = B = C = 1$ and $p = q = r = n > 2$, we have

$$x^n + y^n = z^n$$

only has integer solutions with $xyz = 0$.

Generalized Fermat equations

Some generalized Fermat equations have lots of solutions.

Example

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Analogue of the trichotomy for rational points on curves:

Theorem (Darmon–Granville [DG95], Beukers [Beu98])

$Ax^p + By^q = Cz^r$ has

- *finitely many solutions if $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$,*
- *either none or ∞-ly many if $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1$.*

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Today: focus on $x^2 + By^2 = Cz^n$ for n odd.

Specialize to $(p, q, r) = (2, 2, n)$

Primitive integer solutions to

$$x^2 + By^2 = Cz^n$$

correspond to $\mathbb{Z}$-points on stacky curve $\mathcal{Y}_{B,C} \subset \mathbb{P}(n, n, 2)$.

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- $\mathbb{P}(n, n, 2) = [\mathbb{A}^3/\mathbb{G}_m]$ with weighted action

$$\lambda \cdot (x, y, z) = (\lambda^n x, \lambda^n y, \lambda^2 z).$$

- Stabilizers at $[0 : 0 : 1]$ and $z = 0 \to$ stackiness.

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- Stabilizers at $[0 : 0 : 1]$ and $z = 0 \to$ stackiness.
- Equation is now "homogeneous" so $\mathcal{Y}_{B,C} \subset \mathbb{P}(n, n, 2)$.
- Two (geometric) stacky "$1/n$"-points with $z = 0$.

Fast facts about stacky curves

- A stacky curve is a smooth, proper, irreducible Deligne–Mumford stack of dimension 1 over a field.
- Think: an algebraic curve (coarse space) with finitely many stacky points x each with stabilizer group G_x.
- Notion of Euler characteristic and genus

$$g(\mathcal{X}) = g(X) + \frac{1}{2} \sum_{x \in \mathcal{X}(k)} \left(1 - \frac{1}{|G_x|}\right).$$

$(g(\mathcal{X}) = \frac{n-1}{n}$ for $(2,2,n)$ case.)

- Descent: given $\pi\colon \mathcal{Y} \to \mathcal{X}$ étale μ_n-cover over R,

$$\mathcal{X}(R) = \coprod_{d \in H^1(R,\mu_n)} \pi(\mathcal{Y}_d(R)).$$

Local situation

$$\mathcal{Y}_{B,C}\colon x^2 + By^2 = Cz^n \subset \mathbb{P}(n,n,2).$$

Definition

$\mathcal{Y}_{B,C}$ is locally soluble at p if $\mathcal{Y}_{B,C}(\mathbb{Z}_p) \neq \emptyset$.

This comes down to Legendre symbols $\left(\frac{-B_0}{p}\right)$ for p dividing C.

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Example ($x^2 + 29y^2 = 7z^3$)

7 is inert in $\mathbb{Q}(\sqrt{-29})$. Thus $x^2 + 29y^2 \equiv 0 \pmod 7 \implies 7 \mid x, y$.
Conclude $\mathcal{Y}_{29,7}(\mathbb{Z}_7) = \emptyset$, so $\mathcal{Y}_{29,7}(\mathbb{Z}) = \emptyset$.

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Example ($x^2 + 29y^2 = 3z^3$)

3 is split in $\mathbb{Q}(\sqrt{-29})$ and $\mathcal{Y}_{29,3}(\mathbb{Z}_p) \neq \emptyset$ for all primes p.

The Hasse principle for $\mathbb{Z}$-points

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Brute force: $(9, 0, 3)$, $(10, 10, 10)$, ... but no primitive solutions.

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Goal

1. *Decide whether $\mathcal{Y}_{B,C}$ satisfies Hasse principle for any (B, C).*
2. *Understand how often Hasse principle satisfied/fails.*

Known results

Theorem (Darmon–Granville [DG95])

Assume $B \equiv 1 \pmod 4$ squarefree and C odd, squarefree, coprime to B. Then $\mathcal{Y}_{B,C}(\mathbb{Z}) \neq \emptyset$ if and only if

$$C\mathcal{O}_{\mathbb{Q}(\sqrt{-B})} = \mathfrak{j}\bar{\mathfrak{j}}$$

for $\mathfrak{j}$ coprime to $\bar{\mathfrak{j}}$ satisfying

$$[\mathfrak{j}] \in n\,\mathrm{Cl}(\mathcal{O}_{\mathbb{Q}(\sqrt{-B})}).$$

Class group obstruction to the Hasse principle.

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Class group obstruction to the Hasse principle.

Goal

Reinterpret geometrically and remove restrictions.

A statistical aside

$\mathrm{Prob}(\mathcal{Y}_{B,C}(\mathbb{Z}_p) \neq \emptyset) \approx 1 - \frac{1}{2p}$ (odd p).

$$N^{loc}(T) = \#\{(B,C) \in (\mathbb{Z} \cap [-T,T])^2 : \mathcal{Y}_{B,C} \text{ loc sol}\} = o(T^2).$$

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Suspicion

$N^{loc}(T) \asymp \frac{T^2}{\sqrt{\log T}}$. This feels reminiscent of [LS16, Theorem 1.5]

$$N_{\mathrm{loc}}(\pi, T) \ll \frac{T^n}{(\log T)^{\Delta(\pi)}} \text{ for } X \xrightarrow{\pi} \mathbb{A}^n.$$

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Really want to understand $\frac{N^{glob}(T)}{N^{loc}(T)}$.

Related results

Bhargava–Poonen [BP22]:

- Stacky curves with (stacky) genus $< 1/2$ satisfy integral HP
- Explicit infinite family of (stacky) genus $1/2$ of form

$$f(x, y) = ax^2 + bxy + cy^2 = z^2$$

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Santens [San23]:

- Brauer–Manin is only obstruction for certain stacky curves.
- Explain (and expand) B–P results via elementary obstruction.

Setup

- $\mathcal{Y}_{B,C} \colon x^2 + By^2 = Cz^n$
- $K = \mathbb{Q}(\sqrt{-B})$
- S contains $\mathfrak{p} \mid 2nC$
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$$\begin{tikzcd} & \mathcal{C}' \colon UV = CW \arrow[d, "\mu_n"] \\ \mathcal{Y}_{B,C} \arrow[r, "\sim/R"'] & \mathcal{Y}' \colon uv = Cw^n \end{tikzcd}$$

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$$\pi\colon \mathcal{C}' \to \mathcal{Y}' \text{ is étale: } \mathcal{Y}'(R) = \coprod_{d \in R^\times/(R^\times)^n} \pi_d(\mathcal{C}'_d(R)).$$

Descent

$\mathcal{C}'$ is a genus zero (non-stacky) curve!

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Look for $\mathcal{Y}_{B,C}(\mathbb{Z}) \cap \pi_d(\mathcal{C}'(R))$ in $\mathcal{Y}'(R)$:

- Explicit parametrization of $\mathcal{C}'_d(R)$ and image.

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- Explicit parametrization of $\mathcal{C}'_d(R)$ and image.
- Admissible d have certain valuations at inverted primes.

A successful failure

Example ($x^2 + 29y^2 = 3z^3$)

$K = \mathbb{Q}(\sqrt{-29})$, $R = \mathcal{O}_K[1/6]$, $2\mathcal{O}_K = \mathfrak{p}_2^2$, $3\mathcal{O}_K = \mathfrak{p}_3\overline{\mathfrak{p}_3}$.

Admissible $d \in R^\times/(R^\times)^3$ satisfy

$$\begin{aligned} v_{\mathfrak{p}_3}(d) &= \pm 1 \pmod 3 \\ v_{\overline{\mathfrak{p}_3}}(d) &= \mp 1 \pmod 3 \\ v_{\mathfrak{p}_2}(d) &= 0 \pmod 3. \end{aligned}$$

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Darmon–Granville: $\mathcal{Y}_{29,3}(\mathbb{Z}) = \emptyset$ because $[\mathfrak{p}_3] \notin 3\,\mathrm{Cl}(\mathcal{O}_K)$.

Equivalent statements

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$$d \mapsto \frac{n+1}{2}\mathfrak{j} + \frac{n-1}{2}\bar{\mathfrak{j}} \mapsto [\mathfrak{j}] = 0$$

Existence of admissible twist $\implies$ $[\mathfrak{j}] \in n\,\mathrm{Cl}(\mathcal{O}_K)$

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Write $\mathcal{Y}_{f^2 B_0, C}: x^2 + f^2 B_0 y^2 = C z^n$, $B_0 \neq 7 \pmod 8$ squarefree.

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Theorem (Duque-Rosero, Kobin, K., Roy, Sankar, Wang, 2025+)

Assume C and f are coprime. TFAE:

(i) $\mathcal{Y}_{B,C}(\mathbb{Z}; \gcd(x, fy) = 1) \neq \emptyset$;

(ii) *there exists admissible* $d \in R^\times/(R^\times)^n$;

(iii) $C\mathcal{O}_K = \mathfrak{j}\bar{\mathfrak{j}}\mathfrak{r}^2\mathfrak{i}^2$ *with* $\mathfrak{j}, \mathfrak{r}, \mathfrak{i}$ *supported on split, ramified, inert primes and*

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(iii) $\implies$ (i): find $\mathfrak{z}$ with $\mathfrak{j}\mathfrak{r}\mathfrak{i}\mathfrak{z}^n = u\mathbb{Z}[f\sqrt{-B_0}]$ and take norms.

Takeaways

Complete recipe for deciding HP when $\gcd(f, C) = 1$:

- $B_0 \equiv 7 \pmod 8$: also allow $[\mathfrak{j}]$ in certain cosets of $n\,\mathrm{Cl}(\mathcal{O})$.
- If $\gcd(x, fy) > 1$, points come from different $\mathcal{Y}_{B,C}$.

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What about $\gcd(f, C) > 1$? OK when $\gcd(f, C, n) = 1$.

Example

$\mathcal{Y}_{3^5,3\cdot 31}(\mathbb{Z}) \simeq \mathcal{Y}_{3,31}(\mathbb{Z}; 3 \nmid y) = \emptyset$, but $\mathcal{Y}_{3,31}(\mathbb{Z}) \neq \emptyset$.

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$31 = (-2 + 3\sqrt{-3})(-2 - 3\sqrt{-3})$ and $\mathcal{O}_K$ is a PID; this forces

$$(-2 + 3\sqrt{-3})\mathfrak{z}^3 \in \mathbb{Z}[3\sqrt{-3}] \implies Cz^3 = x^2 + 3(3y)^2.$$

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Still to come: statistics, other GFEs, Brauer groups?

Thank you!

Thank you for your attention!

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