

Towards Artin's conjecture on p -adic forms in low degree

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Setup

Let K be a p -adic field:

$$K/\mathbb{Q}_p$$

finite extension

$$\mathcal{O}_K$$

ring of integers

$$\mathbb{F}_q$$

residue field, $q = p^r$

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If $f \in K[x_0, \dots, x_n]$ is a degree d form,

$$X_f: f(x_0, \dots, x_n) = 0 \subset \mathbb{P}^n$$

is degree d hypersurface.

$$X_f(K) = \{(x_0, \dots, x_n) \in K^{n+1} - \mathbf{0} : f(x_0, \dots, x_n) = 0\} / \sim .$$

The conjecture

Conjecture (Artin, 1930s)

Suppose $n \geq d^2$ and $f \in K[x_0, \dots, x_n]$ degree d . Then $X_f(K) \neq \emptyset$.

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Suppose $n \geq d^2$ and $f \in K[x_0, \dots, x_n]$ degree d . Then $X_f(K) \neq \emptyset$.
 Equivalently, K is C_2 .

Recall K is C_r if for all d and $n \geq d^r$ we have $X_f(K) \neq \emptyset$.

Examples

- \bar{k} is C_0 by definition.
- \mathbb{F}_q is C_1 (Chevalley–Warning theorem).
- $\mathbb{F}_q((t))$ is C_2 (Lang)

Counterexamples

Bad news: the conjecture is false.

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1960s Terjanian [**Ter66**]: explicit counterexample for $K = \mathbb{Q}_2$ with $d = 4$, $n = 17$.

1980s Lewis–Montgomery [**LM83**]: infinite family of counterexamples for each p with

$$n > \exp \left(\frac{d}{(\log d)^2 (\log \log d)^3} \right).$$

All have d **even, composite, divisible by $p - 1$** .

Evidence

On the other hand...

1920s Hasse: **quadratic** forms in 5 variables have K -solution.

1950s Lewis [Lew52]: **cubic** forms in 10 variables have K -solution.

1960s Ax–Kochen [AK65]: for any fixed d , if $p \gg_d 0$ then degree d forms in $n^2 + 1$ variables have K -solution.

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This is the **characteristic** p , not the size of the residue field q .

The conjecture, revised

Conjecture (Artin, 1930s)

Suppose d is prime, $n \geq d^2$, and $f \in K[x_0, \dots, x_n]$ degree d .
Then $X_f(K) \neq \emptyset$. Equivalently, K is $C_2(d)$.

K is $C_r(d)$ if for all $n \geq d^r$ we have $X_f(K) \neq \emptyset$.

Goal (Today)

Let's focus on low degree forms.

- $d = 5, 7, 11$
- Can we show K is $C_2(d)$ when q is large enough? How large?

Progress

Definition

Let $q_{AC}(d)$ be an integer such that for all p -adic fields K we have

$$q > q_{AC}(d) \implies K \text{ is } C_2(d),$$

i.e. Artin's Conjecture "holds in degree d " for $q > q_{AC}(d)$.

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Note: Ax–Kochen does not imply $q_{AC}(d)$ exists!

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Theorem (1960s – 2025+, many authors)

- $q_{AC}(5) \leq 5$
- $q_{AC}(7) \leq 679$
- $q_{AC}(11) \leq 8053$

Progress

1960s

$$\begin{aligned} q_{AC}(5) &< \infty \\ q_{AC}(7) &< \infty \\ q_{AC}(11) &< \infty \end{aligned}$$

reduced forms

[BL59, LL65]

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1960s	$q_{AC}(5) < \infty$ $q_{AC}(7) < \infty$ $q_{AC}(11) < \infty$	reduced forms	[BL59, LL65]
1996	$q_{AC}(5) \leq 47$	singular plane slices	[LY96]

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2010 –2025	$q_{AC}(5) \leq 5$	extensive computations	[HB10, Dum17, BK25b]

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2025	$q_{AC}(7) \leq 679$	singular + Bertini	[BK25a]

Reduced forms

Let $\overline{X_f}, \overline{f}$ denote reduction to \mathbb{F}_q .

Idea

If we find $\overline{X_f}(\mathbb{F}_q)^{\text{sm}} \neq \emptyset$, **Hensel's lemma** $\implies X_f(K) \neq \emptyset$.

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Definition (Reduced)

We say $f(x_0, \dots, x_n) \in \mathcal{O}_K[x_0, \dots, x_n]$ is **reduced** if

$$\text{Res}(f_{x_0}, \dots, f_{x_n}) \neq 0$$

and has *minimal valuation* among representatives of its $\text{GL}_{n+1}(K)$ -orbit in $\mathcal{O}_K[x_0, \dots, x_n]$.

Non-example

If $g = af(M\mathbf{x})$ for $a \in K$ and $M \in M_{n+1}(K)$, then

$$\text{Res}(g_{x_0}, \dots, g_{x_n}) = a^{n(d-1)^{n-1}} (\det M)^{d(d-1)^{n-1}} \text{Res}(f_{x_0}, \dots, f_{x_n}).$$

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Example

Let $K = \mathbb{Q}_p$ and suppose $f = x_0^d + pf_1(x_1, \dots, x_n)$. Then we have

$$g = \frac{1}{p}f(px_0, x_1, \dots, x_n) \in \mathcal{O}_K[x_0, \dots, x_n].$$

However,

$$\text{Res}(g_{x_0}, \dots, g_{x_n}) = \frac{1}{p^{(n-d)(d-1)^{n-1}}} \text{Res}(f_{x_0}, \dots, f_{x_n}),$$

so the valuation drops for $n > d$; hence f was not minimal.

Facts about reduced forms

- To show K is $C_2(d)$, suffices to check reduced forms
 - If $\text{Res}(f_{x_0}, \dots, f_{x_n}) \neq 0$, change variables;
 - If $\text{Res}(f_{x_0}, \dots, f_{x_n}) = 0$, approximate p -adically by $f_i \rightarrow f$.

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 - If $\text{Res}(f_{x_0}, \dots, f_{x_n}) = 0$, approximate p -adically by $f_i \rightarrow f$.
- If $n \geq d^2$, then \bar{f} has no linear factors over $\overline{\mathbb{F}_q}$.
- If $n \geq d^2$, then
$$\#\overline{X_f}(\mathbb{F}_q) \geq q + 1.$$

Think of this as an strengthening of Chevalley–Warning.

Low degrees

Proposition (Laxton–Lewis [LL65])

Let $d \in \{2, 3, 5, 7, 11\}$. Then for $q \gg_d 0$ we have K is $C_2(d)$.

Question

What's so special about degrees $d = 2, 3, 5, 7, 11$?

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What's so special about degrees $d = 2, 3, 5, 7, 11$?

They cannot be written as **sum of composite numbers**. This forces \bar{f} to have factor of **unique degree**. (No linear factors over $\overline{\mathbb{F}_q}$!)

If e.g. $d = 13$, run into issues with \bar{f} factoring as $g^2 h^2$ for quadratic form g and cubic form h .

An effective approach

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Proof sketch. Suffices to check on f reduced.

Factor \bar{f} over $\overline{\mathbb{F}_q}$, find factor of unique degree.

Lang–Weil: $\#X_f(\mathbb{F}_q) = O(q^m)$ and $\#X_f(\mathbb{F}_q)^{\text{sing}} = O(q^{m-1})$.

When $q \gg_d 0$, there is a smooth \mathbb{F}_q -point on $\overline{X_f}$. \square

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Question

How do we make this effective?

Slicing with planes

Theorem (Wooley [Woo08])

We have $q_{AC}(5) \leq 121$, $q_{AC}(7) \leq 883$, $q_{AC}(11) \leq 8053$.

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Proof sketch (with massive gap!). $\overline{X_f}$ has no linear factors.

Suppose P is a plane. $\overline{X_f} \cap P$ is a curve (possibly reducible).

Assume there exists P with $\overline{X_f} \cap P$ contains no lines over $\overline{\mathbb{F}_q}$.

Use Hasse–Weil theorem to count smooth points on $\overline{X_f} \cap P$. □

Preview: singularities

Worst case: $\overline{X_f} \cap P$ smooth curve of degree d .

$$\#(\overline{X_f} \cap P)(\mathbb{F}_q) \geq q + 1 - \frac{(d-1)(d-2)}{2} \lfloor 2\sqrt{q} \rfloor$$

Example ($d = 7$)

$$\#(\overline{X_f} \cap P)(\mathbb{F}_q) \geq q + 1 - 15 \lfloor 2\sqrt{q} \rfloor > 0 \text{ whenever } q > 883.$$

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Singularities actually help because we can exploit genus drop.

Example ($d = 7$)

Suppose $(\overline{X_f} \cap P)$ has at least two singular \mathbb{F}_q -points. Then

$$\#(X_f \cap P)(\mathbb{F}_q)^{\text{smooth}} > 0 \text{ whenever } q > 679.$$

Bertini's theorem

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Theorem (Bertini)

There is open $U \subseteq (\mathbb{P}^n)^\vee$ such that $H \in U \implies X \cap H$ is smooth.

Think

Smoothness is **generic**.

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Smoothness is **generic**.

If k is infinite, then $U(k) \neq \emptyset$; **there exists H** with $X \cap H$ smooth.

Effective Bertini theorems

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Variations:

- If $\dim X > 1$, (geometric) **irreducibility is generic**.
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Generically, components of $\overline{X_f}$ remain irreducible when intersecting with P . In particular, $\overline{X_f} \cap P$ generically has no lines over $\overline{\mathbb{F}_q}$.

We need an **existence** statement.

Affinization

Dehomogenize by setting $x_0 = 1$. Set

$$f_{\mathbf{v}, \mathbf{w}, \mathbf{z}}(X, Y) = \bar{f}(1, X + v_1, w_2X + z_2Y + v_2, \dots, w_nX + z_nY + v_n).$$

Think

Choice of $(v_1, \dots, v_n, w_2, \dots, w_n, z_2, \dots, z_n) \in \mathbb{F}_q^{3n-2}$ amounts to choice of plane P and $f_{\mathbf{v}, \mathbf{w}, \mathbf{z}} = 0$ cuts out (affine patch of) $\overline{X_f} \cap P$.

Effective Bertini for irreducibility I

$$f_{\mathbf{v}, \mathbf{w}, \mathbf{z}}(X, Y) = \bar{f}(1, X + v_1, w_2X + z_2Y + v_2, \dots, w_nX + z_nY + v_n).$$

Theorem (Cafure–Matera [CM06], Beneish–K. [BK25a])

Suppose \bar{f} is a degree d form irreducible over $\overline{\mathbb{F}_q}$.

(i) There exist at most

$$\frac{1}{8} (3d^4 - 2d^3 + 13d^2 + 2d)$$

tuples $(\mathbf{v}, \mathbf{w}, \mathbf{z}) \in \mathbb{F}_q^{3n-2}$ such that $f_{\mathbf{v}, \mathbf{w}, \mathbf{z}}$ is not absolutely irreducible.

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Example ($d = 7$)

If $q > 896$, there exists $(\mathbf{v}, \mathbf{w}, \mathbf{z})$ such that $f_{\mathbf{v}, \mathbf{w}, \mathbf{z}}$ is abs. irreducible.

Effective Bertini for irreducibility II

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Theorem (Cafure–Matera [CM06], Beneish–K. [BK25a])

Suppose \bar{f} is a degree d form irreducible over $\overline{\mathbb{F}_q}$.

(ii) Fix a positive integer $D < d$. Then there exist at most

$$\frac{d}{8}(-D^4 + 4D^3d - 6D^3 + 12D^2d - 11D^2 + 8Dd - 6D + 16d)q^{3n-3}$$

tuples $(\mathbf{v}, \mathbf{w}, \mathbf{z}) \in \mathbb{F}_q^{3n-2}$ for which $f_{\mathbf{v}, \mathbf{w}, \mathbf{z}}$ has an irreducible factor of degree *at most* D over $\overline{\mathbb{F}_q}$.

Effective Bertini for irreducibility II

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Example ($d = 7, D = 1$)

If $q > 224$, exists $(\mathbf{v}, \mathbf{w}, \mathbf{z})$ such that $f_{\mathbf{v}, \mathbf{w}, \mathbf{z}}$ has no linear factor.

A word about the proofs

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\iff certain **overdetermined** linear system has $\overline{\mathbb{F}_q}$ -solution

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\iff polynomial $\psi_D(\mathbf{v}, \mathbf{w}, \mathbf{z}) = 0$.

Determine $\deg \psi_D$. It has $\leq (\deg \psi_D) q^{3n-3}$ \mathbb{F}_q -zeros. □

Remark

We give modest improvement by reducing order of p -adic approximation needed, lowering $\deg \psi_D$ [BK25a].

Patching the proof

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We have $q_{AC}(5) \leq 121$, $q_{AC}(7) \leq 883$, $q_{AC}(11) \leq 8053$.

Proof sketch (with massive gap!). $\overline{X_f}$ has no linear factors.

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There **does exist** P with $\overline{X_f} \cap P$ contains no lines over $\overline{\mathbb{F}_q}$.

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Remark

The Hasse–Weil step is the limiting factor.

Engineering improvements

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Ensure that $\overline{X_f} \cap P$ has \mathbb{F}_q -points.

- Either they are singular or $X_f(K) \neq \emptyset$ (Hensel).
- More singular points \implies sharper q -bounds!

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Think

Finding points is **easy**. Keeping them on $\overline{X_f} \cap P$ is **hard**.

Degree 5

Theorem (Leep–Yeomans [LY96])

If f reduced and $X_f(K) = \emptyset$, there exist non-collinear points $u, v, w \in X_f(\mathbb{F}_q)$, such that \bar{f} doesn't vanish on lines between.

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$P =$ plane spanned by u, v, w . This forces $\overline{X_f} \cap P$ geom. int.

Corollary (Leep–Yeomans [LY96])

We have $q_{AC}(5) \leq 43$.

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Heath–Brown: $\overline{X_f} \cap P$ given by (something like)

$$Ax^3y^2 + Bx^3z^2 + Cy^3z^2 + xyzQ(x, y, z) = 0.$$

Actually write these down and look for Hensel-liftable points.

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Next, find 4 non-coplanar points in $X_f(\mathbb{F}_q)$ and intersect with span:

$$\sum_{0 \leq i < j \leq 3} a_{ij} x_i^3 x_j^2 + \sum x_i x_j x_k Q_{ijk}(x_i, x_j, x_k) + x_0 x_1 x_2 x_3 (b_0 x_0 + b_1 x_1 + b_2 x_2 + b_3 x_3) = 0.$$

- Dumke [Dum17]: extensive computation $\implies q_{AC}(5) \leq 9$.
- Beneish–K. [BK25b]: parallelization $\implies q_{AC}(5) \leq 5$.

Degree 7

Blend of Bertini and singular points approaches.

- When $q > 591$, there exist $u, v \in X_f(\mathbb{F}_q)$ such that $\overline{X_f}$ intersects line between u, v **transversely** at an $\overline{\mathbb{F}_q}$ -point.
- Modified eff. Bertini: there exists plane P containing u, v such that $\overline{X_f} \cap P$ **contains no lines**.

Theorem (Beneish–K. [BK25a])

We have $q_{AC}(7) \leq 679$.

Final thoughts

Theorem (1960s – 2025+, many authors)

We have $q_{AC}(5) \leq 5$, $q_{AC}(7) \leq 679$, $q_{AC}(11) \leq 8053$.

What's next in low degree?

- $d = 5$: increase dimension, lots more computing.
- $d = 7, 11$: accumulate (more) \mathbb{F}_q -points on $\overline{X_f} \cap P$.

What about prime $d > 11$? Still wide open!

Thank you for your attention!



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