Local solubility in families of superelliptic curves

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Definition

Given $S \subseteq \mathbb{Z}^n$, the **natural density** of S is the limit, if it exists:

$$\lim_{B\to\infty} \frac{\#(S \cap [-B, B]^n)}{(2B+1)^n}.$$

((Everywhere) local) solubility

Let $C/\mathbb{Q}$ be a curve and v a place of $\mathbb{Q}$ (i.e. $v = p$ or $v = \infty$).

Definition (soluble)

C is **globally soluble** if $C(\mathbb{Q})$ is nonempty.

C is **locally soluble at** v if $C(\mathbb{Q}_v)$ is nonempty.

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What proportion of curves over $\mathbb{Q}$ (in some family) are ~~globally soluble~~ everywhere locally soluble?

Known for genus 1 curves [BCF21], plane cubics [BCF16], and some families of hypersurfaces [BBL16], [FHP21], [PV04], [Bro17].

Superelliptic curves

Definition

A **superelliptic curve** $C/\mathbb{Q}$ is a smooth projective curve with a cyclic Galois cover of $\mathbb{P}^1$ of degree $m \geq 2$.

More concretely: given by equation in weighted projective space

$$C\colon y^m = f(x, z)$$

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Warning

Some authors don't assume $m \mid d$, want f to be squarefree, or mean something else entirely by "superelliptic" (e.g. [Swa19]).

Defining the proportion

For $\mathbf{c} = (c_i)_{i=0}^{d} \in \mathbb{Z}^{d+1}$, we associate a binary form

$$f(x, z) = \sum_{i=0}^{d} c_i x^i z^{d-i}$$

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Definition

We define

$$\rho_{m,d} = \lim_{B \to \infty} \frac{\#\{\mathbf{c} \in ([-B,B] \cap \mathbb{Z})^{d+1} \mid C_f \text{ E.L.S.}\}}{\#\{\mathbf{c} \in ([-B,B] \cap \mathbb{Z})^{d+1}\}},$$

the proportion of locally soluble superelliptic curves of this form.

Main results

Fix $(m, d) \neq (2, 2)$ such that $m \mid d$.

Theorem (Beneish–K. [BK21])

(A) *$0 < \rho_{m,d} < 1$, and $\rho_{m,d}$ is product of local densities,*

$$\rho_{m,d} = \rho_{m,d}(\infty) \prod_p \rho_{m,d}(p).$$

$\rho_{m,d}(p)$ is (normalized) Haar measure of space of the $\mathbb{Q}_p$-soluble curves $C_f \colon y^m = f(x, z)$, with coefficients in $\mathbb{Z}_p$.

Main results

Fix $(m, d) \neq (2, 2)$ such that m is prime and $m \mid d$.

Theorem (Beneish–K. [BK21], continued)

(B) *We can find explicit (and sometimes good) bounds for $\rho_{m,d}(p)$ and hence $\rho_{m,d}$. In particular,*

$$\liminf_{d \to \infty} \rho_{m,d} \geq \left(1 - \frac{1}{m^{m+1}}\right) \prod_{p \equiv 1 (m)} \left(1 - \left(1 - \frac{p-1}{mp}\right)^{p+1}\right) \prod_{p \not\equiv 0,1 (m)} \left(1 - \frac{1}{p^{2(p+1)}}\right).$$

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When $m > 2$, we have

$$0.83511 \leq \liminf_{d\to\infty} \rho_{m,d} \quad \textit{and} \quad \limsup_{d\to\infty} \rho_{m,d} \leq 0.99804.$$

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Moreover, $\exists$ *rational functions* $R_1(t)$ *and* $R_2(t)$ *such that*

$$\rho_{3,6}(p) = \begin{cases} R_1(p), & p \equiv 1 \pmod 3 \text{ and } p > 43 \\ R_2(p), & p \equiv 2 \pmod 3 \text{ and } p > 2. \end{cases}$$

Asymptotically,

$$1 - R_1(t) \sim \frac{2}{3} t^{-4},$$
$$1 - R_2(t) \sim \frac{53}{144} t^{-7}.$$

$$\rho = \begin{cases} \begin{array}{l} \Big(1296p^{57} + 3888p^{56} + 9072p^{55} + 16848p^{54} + 27648p^{53} + 39744p^{52} + 53136p^{51} + 66483p^{50} + 80019p^{49} + 93141p^{48} \\ + 107469p^{47} + 120357p^{46} + 135567p^{45} + 148347p^{44} + 162918p^{43} + 176004p^{42} + 190278p^{41} + 203459p^{40} \\ + 218272p^{39} + 232083p^{38} + 243639p^{37} + 255267p^{36} + 261719p^{35} + 264925p^{34} + 265302p^{33} + 261540p^{32} \\ + 254790p^{31} + 250736p^{30} + 241384p^{29} + 226503p^{28} + 214137p^{27} + 195273p^{26} + 170793p^{25} + 151839p^{24} + 136215p^{23} \\ + 118998p^{22} + 105228p^{21} + 94860p^{20} + 80471p^{19} + 67048p^{18} + 52623p^{17} + 40617p^{16} + 28773p^{15} + 19247p^{14} \\ + 12109p^{13} + 7614p^{12} + 3420p^{11} + 756p^{10} - 2248p^{9} - 4943p^{8} - 6300p^{7} - 6894p^{6} - 5994p^{5} - 2448p^{4} - 648p^{3} \\ + 324p^{2} + 1296p + 1296\Big) \Big/ \Big(1296\left(p^{12} - p^{11} + p^{9} - p^{8} + p^{6} - p^{4} + p^{3} - p + 1\right)\left(p^{8} - p^{6} + p^{4} - p^{2} + 1\right) \\ \times \left(p^{6} + p^{5} + p^{4} + p^{3} + p^{2} + p + 1\right)\left(p^{4} + p^{3} + p^{2} + p + 1\right)^{3}\left(p^{4} - p^{3} + p^{2} - p + 1\right)\left(p^{2} + p + 1\right) \\ \times \left(p^{2} + 1\right)p^{11}\Big), \end{array} & p \equiv 1 \pmod 3 \\ \\ \begin{array}{l} \Big(144p^{57} + 432p^{56} + 1008p^{55} + 1872p^{54} + 3168p^{53} + 4608p^{52} + 6336p^{51} + 8011p^{50} + 9803p^{49} + 11357p^{48} \\ + 13061p^{47} + 14525p^{46} + 16295p^{45} + 17875p^{44} + 19654p^{43} + 21212p^{42} + 23030p^{41} + 24563p^{40} + 26320p^{39} \\ + 27771p^{38} + 29711p^{37} + 30859p^{36} + 31135p^{35} + 31525p^{34} + 31510p^{33} + 29436p^{32} + 28502p^{31} + 28616p^{30} \\ + 26856p^{29} + 25087p^{28} + 25057p^{27} + 23041p^{26} + 19921p^{25} + 18119p^{24} + 16287p^{23} + 13798p^{22} \\ + 12140p^{21} + 10844p^{20} + 9191p^{19} + 7480p^{18} + 5839p^{17} + 4265p^{16} + 2909p^{15} + 1943p^{14} + 1109p^{13} \\ + 590p^{12} + 604p^{11} + 372p^{10} - 144p^{9} - 87p^{8} - 84p^{7} - 678p^{6} - 618p^{5} - 144p^{4} - 168p^{3} - 156p^{2} \\ + 144p + 144\Big) \Big/ \Big(144\left(p^{12} - p^{11} + p^{9} - p^{8} + p^{6} - p^{4} + p^{3} - p + 1\right)\left(p^{8} - p^{6} + p^{4} - p^{2} + 1\right) \\ \times \left(p^{6} + p^{5} + p^{4} + p^{3} + p^{2} + p + 1\right)\left(p^{4} + p^{3} + p^{2} + p + 1\right)^{3}\left(p^{4} - p^{3} + p^{2} - p + 1\right)\left(p^{2} + p + 1\right) \\ \times \left(p^{2} + 1\right)p^{11}\Big), \end{array} & p \equiv 2 \pmod 3 \end{cases}$$

Local densities

Theorem (Beneish–K. [BK21])

(A) *$\rho_{m,d}$ exists and is given by the product of local densities,*

$$\rho_{m,d} = \rho_{m,d}(\infty) \prod_p \rho_{m,d}(p) > 0.$$

$\rho_{m,d}(p)$ is (normalized) Haar measure of space of the $\mathbb{Q}_p$-soluble curves $C_f \colon y^m = f(x,z)$, with coefficients in $\mathbb{Z}_p$.

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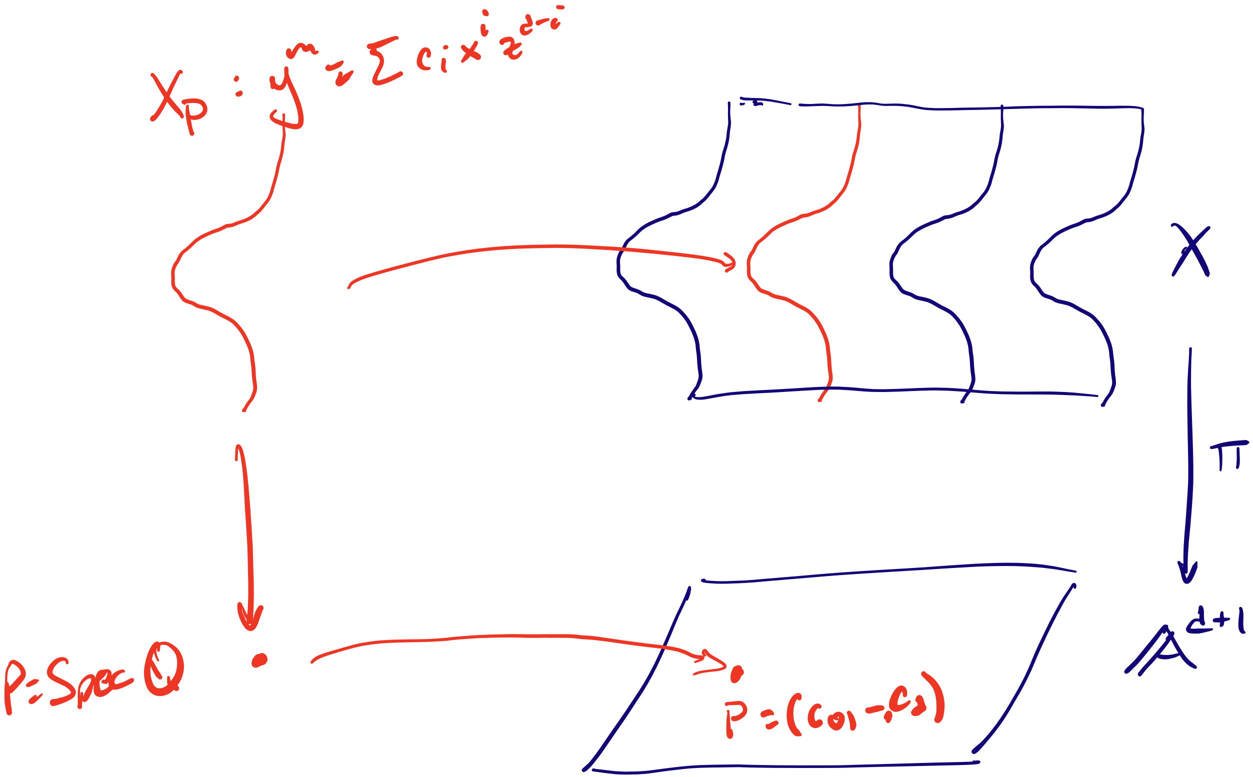
- Apply to ELS in families of hyperelliptic curves [PS99b]; uses sieve of Ekedahl [Eke91].
- Bright–Browning–Loughran [BBL16] give *geometric criteria* when family comes from fibers of a morphism.

Geometric picture

$$X\colon y^m = c_d x^d + \cdots + c_0 z^d \subset \mathbb{A}^{d+1}_{\mathbb{Q}} \times \mathbb{P}_{\mathbb{Q}}\left(1 : \frac{d}{m} : 1\right)$$



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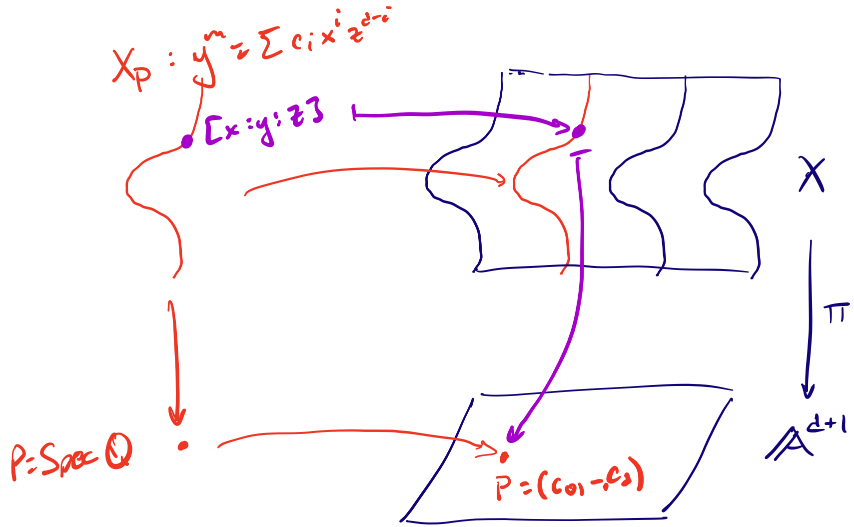

Think

- A $\mathbb{Q}$-point $(\mathbf{c}, [x : y : z])$ of X is the data of superelliptic curve $C_f/\mathbb{Q}$ and a $\mathbb{Q}$-point $[x : y : z] \in C_f(\mathbb{Q})$.
- The fiber X_P of π over a point $P \in \mathbb{A}^{d+1}(\mathbb{Q})$ is a superelliptic curve $C_f/\mathbb{Q}$ whose coefficients are encoded in P.

Computing local densities

Question

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Consider $(m, d) = (3, 6)$ as an example,

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(Easy mode) $\rho_{3,6}(\infty) = 1$, since *all* $C_f/\mathbb{R}$ have real solutions.

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$\rho_{3,6}(p)$ is (normalized) Haar measure of space of the $\mathbb{Q}_p$-soluble curves $C_f \colon y^3 = f(x,z)$, with coefficients in $\mathbb{Z}_p$.

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Let $\overline{C_f}\colon y^3 \equiv \overline{f}(x,z) \pmod{p}$ be the reduction.

- Smooth $\mathbb{F}_p$-points $\overline{C_f}$ lift to $\mathbb{Q}_p$-solutions (Hensel),
- $\overline{C_f}(\mathbb{F}_p) = \emptyset$ means no $\mathbb{Q}_p$-solutions,
- If $\overline{C_f}(\mathbb{F}_p)$ consists of non-smooth points, do more work.

An extended example — bounds from geometry

If $\overline{C_f}/\mathbb{F}_p$ smooth, irreducible, and $p > 61$,

$$\#C(\mathbb{F}_p) \geq p + 1 - 8\sqrt{p} > 0,$$

so Hensel's lemma $\implies$ $\mathbb{C}_f$ has $\mathbb{Q}_p$-point!

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$$\rho_{3,6}(p) \geq \frac{p^7 - p^3}{p^7} = 1 - \frac{1}{p^4} \text{ for all } p \geq 61.$$

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- Enumerate $\overline{f}(x, z)$ and look for Hensel-liftable solutions:

p	$\rho_{3,6}(p) \geq$	p	$\rho_{3,6}(p) \geq$
2	$\frac{63}{64} \approx 0.98437$	19	$\frac{893660256}{893871739} \approx 0.99976$
3	$\frac{26}{27} \approx 0.96296$	31	$\frac{27512408250}{27512614111} \approx 0.99999$
7	$\frac{810658}{823543} \approx 0.98435$	37	$\frac{94931742132}{94931877133} \approx 0.999998$
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Taking products,

$$\rho_{3,6} = \prod_{p \text{ small}} \rho_{3,6}(p) \prod_{\substack{p \equiv 1 \ (3) \\ p > 43}} \left(1 - \frac{1}{p^4}\right) \prod_{\substack{p \equiv 2 \ (3) \\ p > 2}} \left(1 - \frac{1}{p^7}\right) \geq 0.93134.$$

Directions from here

1. Varying (m, d), similar approach produces

$$\rho_{m,d} \geq \prod_{p \text{ small}} \rho_{m,d}(p) \prod_{\substack{p \equiv 1 \ (m) \\ p \gg 0}} \left(1 - \frac{1}{p^{d(m-1)/m}}\right) \prod_{\substack{p \not\equiv 0,1 \ (m) \\ p \gg 0}} \left(1 - \frac{1}{p^{d+1}}\right).$$

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   - Determine how often factorization types occur modulo p,
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Treat small primes below with help from `Magma`

$$p = 2, 3, 7, 13, 19, 31, 37, 43.$$

Getting exact answer

E.g. Here's what this looks like when $f(x,z) \equiv px^6 \pmod{p^2}$,

$\tau_9 =$

	c_6	c_5	c_4	c_3	c_2	c_1	c_0
$\tau_{9a} = \frac{1}{p}\tau_{9b}$	$= 1$	≥ 2	≥ 2	≥ 2	≥ 2	≥ 2	≥ 2
$\tau_{9b} = \left(1 - \frac{1}{p}\right) + \frac{1}{p}\tau_{9c}$	$= 4$	≥ 4	≥ 3	≥ 2	≥ 1	≥ 0	≥ 0
$\tau_{9c} = \Phi(p) + \frac{1}{p}\tau_{9d}$	$= 4$	≥ 4	≥ 3	≥ 2	≥ 1	≥ 1	≥ 0
$\tau_{9d} = \left(1 - \frac{1}{p}\right)\left(\frac{p-1}{2p} + \frac{1}{p^2}\right) + \frac{1}{p}\tau_{9e}$	$= 4$	≥ 4	≥ 3	≥ 2	≥ 1	≥ 1	≥ 1
$\tau_{9e} = \left(1 - \frac{1}{p}\right) + \frac{1}{p}\tau_{9f}$	$= 4$	≥ 4	≥ 3	≥ 2	≥ 2	≥ 1	≥ 1
$\tau_{9f} = \frac{1}{p}\tau_{9g}$	$= 4$	≥ 4	≥ 3	≥ 2	≥ 2	≥ 2	≥ 1
$\tau_{9g} = \left(1 - \frac{1}{p}\right)\alpha'' + \frac{1}{p}\tau_{9h}$	$= 4$	≥ 4	≥ 3	≥ 2	≥ 2	≥ 2	≥ 2
$\tau_{9h} = \left(1 - \frac{1}{p}\right)\left(\frac{p-1}{2p} + \frac{\theta_2}{p}\right) + \frac{1}{p}\tau_{9i}$	$= 4$	≥ 4	≥ 3	≥ 3	≥ 2	≥ 2	≥ 2
$\tau_{9i} = \left(1 - \frac{1}{p}\right) + \frac{1}{p}\tau_{9j}$	$= 4$	≥ 4	≥ 3	≥ 3	≥ 3	≥ 2	≥ 2
$\tau_{9j} = \frac{1}{p}\tau_{9k}$	$= 4$	≥ 4	≥ 3	≥ 3	≥ 3	≥ 3	≥ 2
$\tau_{9k} = \left(1 - \frac{1}{p}\right) + \frac{1}{p}\tau_{9\ell}$	$= 1$	≥ 1	≥ 0	≥ 0	≥ 0	≥ 0	≥ 0
$\tau_{9\ell} = \Phi(p) + \left(1 - \Phi(p) - \frac{1}{p}\right)\beta + \frac{1}{p}\tau_{9m}$	$= 1$	≥ 1	≥ 1	≥ 0	≥ 0	≥ 0	≥ 0
$\tau_{9m} = \left(1 - \frac{1}{p}\right) + \frac{1}{p}\tau_{9n}$	$= 1$	≥ 1	≥ 1	≥ 1	≥ 0	≥ 0	≥ 0
$\tau_{9n} = \left(1 - \frac{1}{p}\right) + \frac{1}{p}\tau_{9o}$	$= 1$	≥ 1	≥ 1	≥ 1	≥ 1	≥ 0	≥ 0
$\tau_{9o} = \Phi(p) + \frac{1}{p}\tau_{9p}$	$= 1$	≥ 1	≥ 1	≥ 1	≥ 1	≥ 1	≥ 0
$\tau_{9p} = \sigma_5'$	$= 1$	≥ 1	≥ 1	≥ 1	≥ 1	≥ 1	≥ 1

Final thoughts

If $\rho_{m,d}$ is proportion of loc. sol. $C_f\colon y^m = f(x,z)$ then

(A) $\rho_{m,d}$ given by product of local densities and $0 < \rho_{m,d} < 1$;

(B) For fixed prime m, bounding $\rho_{m,d}(p)$ leads to explicit estimates, e.g.

$$0.83511 \leq \liminf_{d\to\infty} \rho_{m,d} \quad \text{and} \quad \limsup_{d\to\infty} \rho_{m,d} \leq 0.99804;$$

(C) We compute $\rho_{3,6}$ exactly, ≈ 0.9694. Local densities $\rho_{3,6}(p)$ given by rational functions.

Further questions

What proportion of superelliptic curves $C_f\colon y^m = f(x,z)$

- are *globally* soluble?
- satisfy/fail weak approximation?

Much is known for $m = 2$ (see [BGW17]).

For $m \nmid d$, study density of $y^m = f(x,z)$ with *primitive integral* solutions (see [Swa19] for $m = 2$).

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Thank you!

Thank you to the organizers for hosting

and thank **you** for your attention!

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A geometric criterion

Theorem (Bright–Browning–Loughran [BBL16])

Let $\pi\colon X \to \mathbb{A}^n$ a dominant, quasiproj. morphism of $\mathbb{Q}$-varieties with geom. int. gen. fiber. Suppose

(i) *fibers above each codim. 1 point of $\mathbb{A}^n$ are geom. integral,*

(ii) $X(\mathbf{A}_{\mathbb{Q}}) \neq \emptyset$,

(iii) *For all $B \geq 1$ we have $B\pi(X(\mathbb{R})) \subseteq \pi(X(\mathbb{R}))$.*

Let $\Psi' \subset \mathbb{R}^n$ be a bounded subset of positive measure lying in $\pi(X(\mathbb{R}))$ whose boundary has measure zero. Then the limit

$$\lim_{B\to\infty} \frac{\#\left\{P \in \mathbb{Z}^n \cap B\Psi' \mid X_P(\mathbf{A}_{\mathbb{Q}}) \neq \emptyset\right\}}{\#\left\{P \in \mathbb{Z}^n \cap B\Psi'\right\}}$$

exists, is nonzero, and is equal to a product of local densities,

$$\prod_{p \nmid \infty} \mu_p\left(\left\{P \in \mathbb{Z}_p^n \mid X_P(\mathbb{Q}_p) \neq \emptyset\right\}\right).$$