Rational points on $y^2 = f(x)$

Christopher Keyes (King's College London)

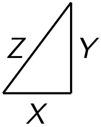
Norwich University

5 February 2026

Pythagorean triples

Find integers satisfying

$$X^2 + Y^2 = Z^2$$



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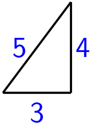

Pythagorean triples

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$$\frac{X^2}{Z^2} + \frac{Y^2}{Z^2} = 1$$

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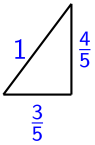

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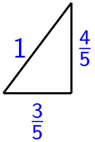

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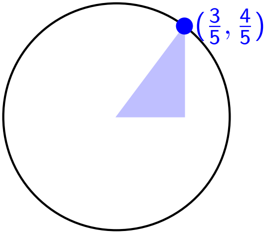

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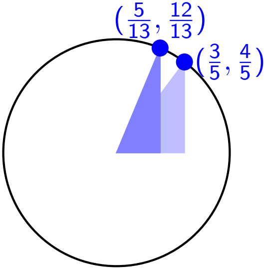
Each point on line

$$y = sx + s$$



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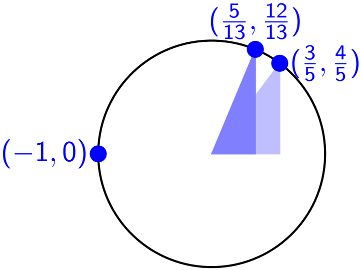

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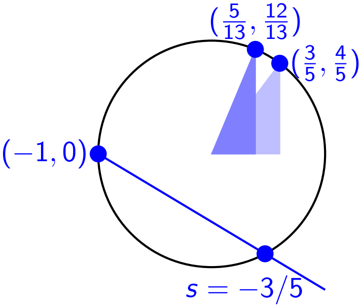

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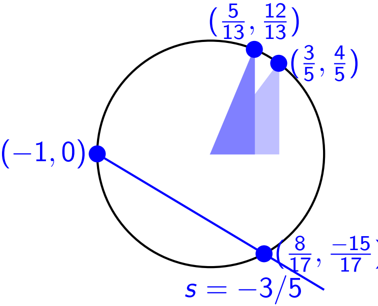

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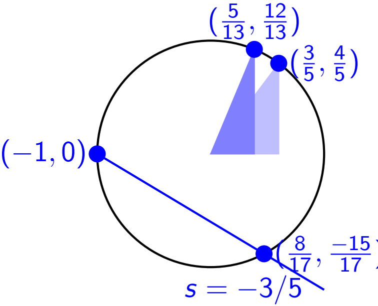

Solving, we find all points

$$\left\{ \left(\frac{-s^2+1}{s^2+1}, \frac{2s}{s^2+1} \right) : \text{ rational } s \right\} \cup \{(-1, 0)\}$$

Rational points

Rational numbers: $\mathbb{Q} = \{\frac{a}{b} : a, b \text{ integers}\}$

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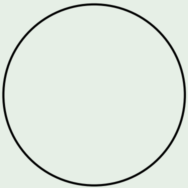
$$y^2 = -x^2 + 1$$



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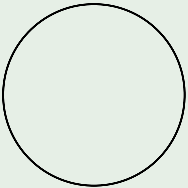

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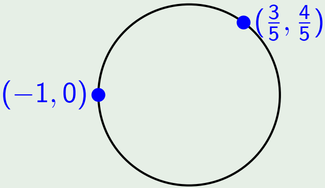

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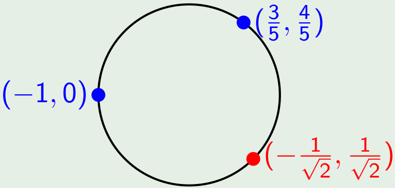
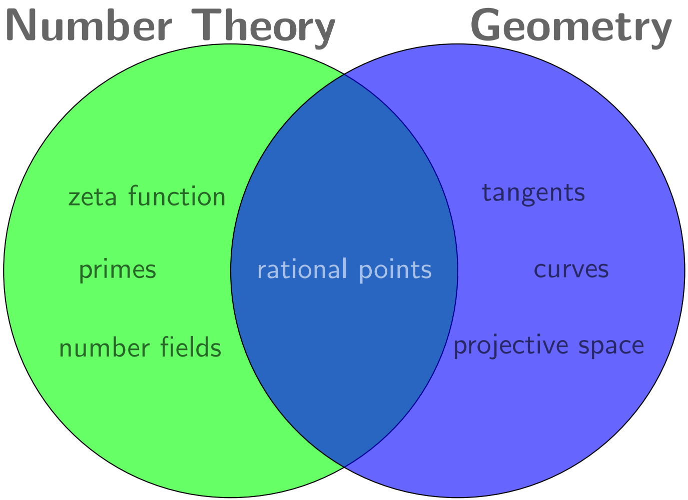
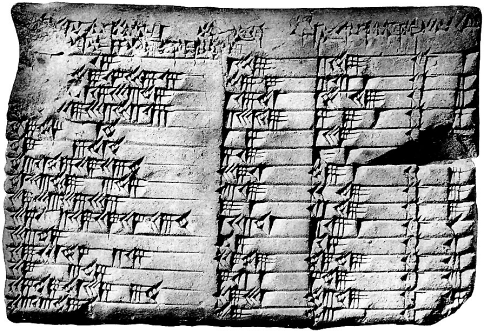



Number Theory
Geometry
zeta function
primes
number fields
rational points
tangents
curves
projective space

Why rational points?

- Rich history
- Cryptography
- Moduli spaces



https://personal.math.ubc.ca/~cass/courses/m446-03/pl322/pl322.html

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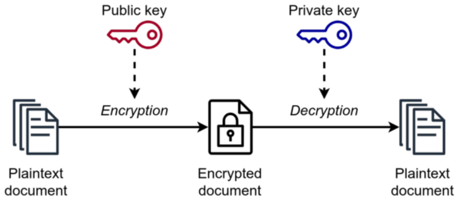

https://commons.wikimedia.org/wiki/File:Asymmetric_encryption_scheme.png

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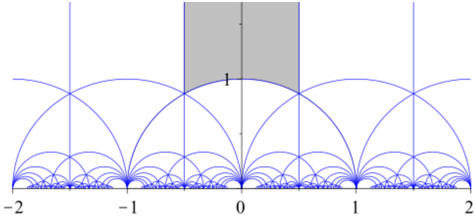

https://commons.wikimedia.org/wiki/File:ModularGroup-FundamentalDomain.svg

Rational points

$y^2 = f(x)$ defines a curve

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1. Do any rational points exist?
2. How many are there?
3. Can we find them explicitly?

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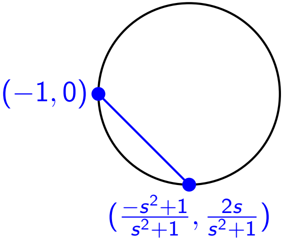

Changing the radius

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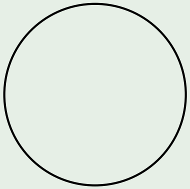
$$x^2 + y^2 = 3$$

Circle of radius $\sqrt{3}$



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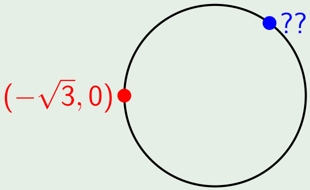
Quick search: no points?



Squares mod n

Let n be an integer

Do arithmetic in $\{0, 1, 2, \dots, n-1\}$ by taking remainder

Example ($n = 3$)

$$2^2 = 4$$

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Example ($n = 3$)

$$2^2 = 3 + 1$$

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$$1^2 + 2^2 \equiv 0 \pmod 5$$

Back to the example

Example ($x^2 + y^2 = 3$)

Assume a solution exists.

Write $x = \frac{X}{Z}$ and $y = \frac{Y}{Z}$ with no common divisors.

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Obstruction to rational points at $p = 3$!

Local obstructions in general

p-adic numbers

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Definition (ELS)

Everywhere locally soluble: real points and no obstructions at p

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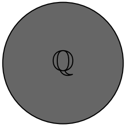

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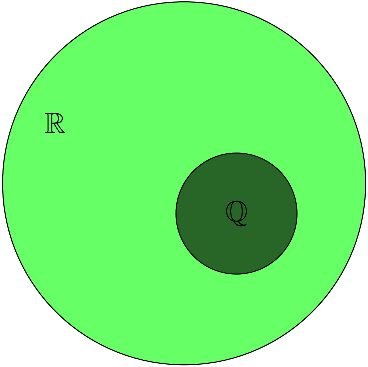

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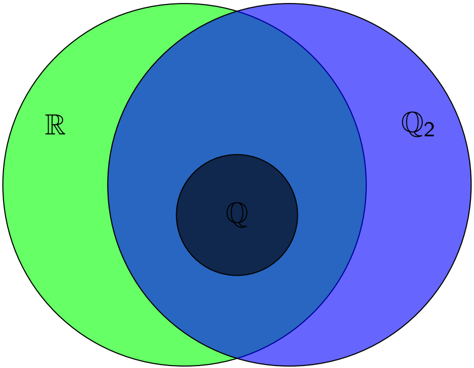

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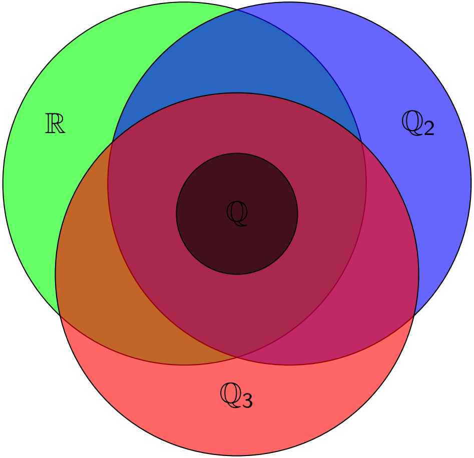

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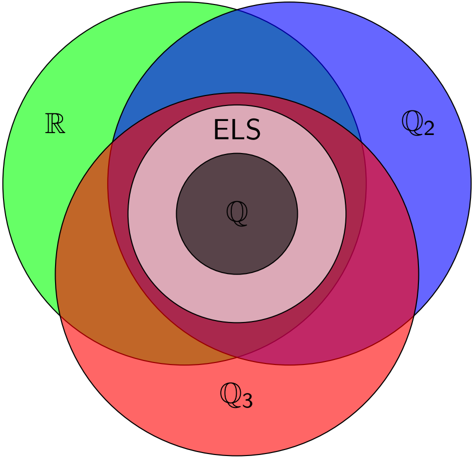

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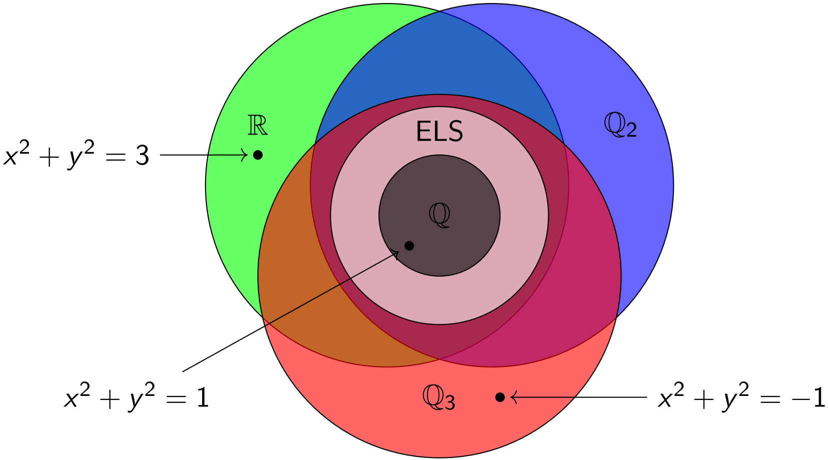

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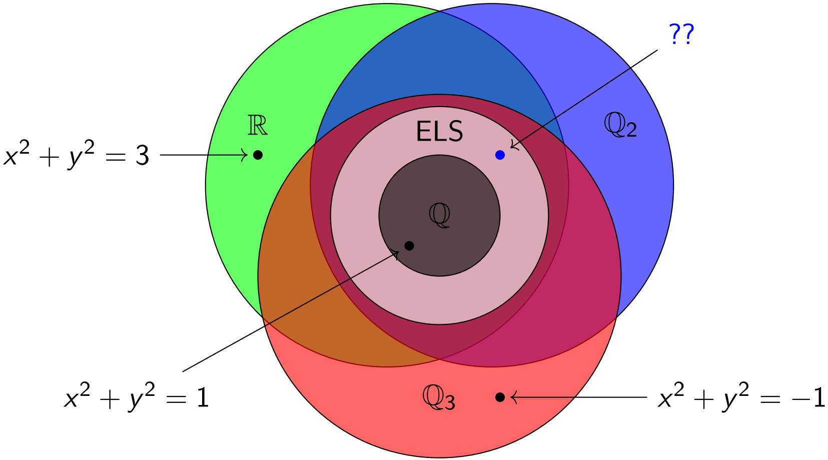

Local to global principles

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Theorem (Hasse–Minkowski)

If $\deg f = 2$, *then yes! ELS* $\implies$ *rational points exist.*

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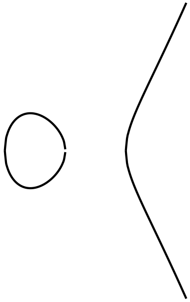
Question

What happens if $\deg f > 2$*? Is there local to global principle?*

$d = 3$: Elliptic curves

For $A, B \in \mathbb{Q}$, study rational points on

$$E\colon y^2 = x^3 + Ax + B$$



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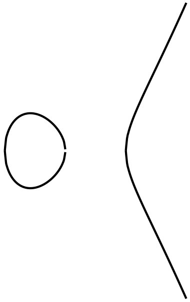

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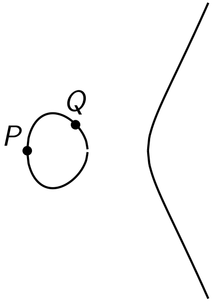

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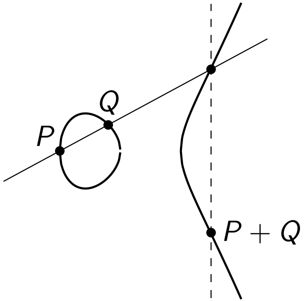

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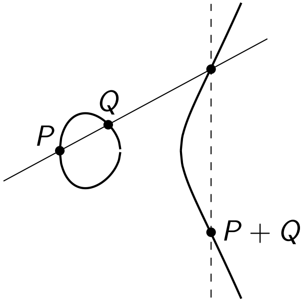
- Finding all points is hard!



An extended example

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Computer search: no (x, y) with numerator/denominator ≤ 1000.

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If (x, y) is a point, then

$$x^2 + x - 1 = u_1 v_1^2 \qquad 2x^2 + 3 = u_2 v_2^2$$

where u_1, u_2 are squarefree integers.

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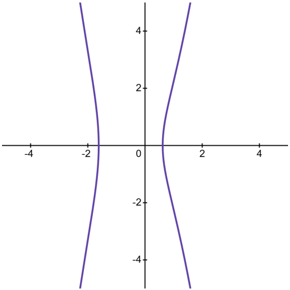
$$y^2 = u_1 u_2 v_1^2 v_2^2 \implies u_1 = u_2 = d$$

In pictures

$$C\colon y^2 = f_1(x) f_2(x)$$

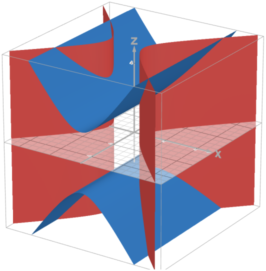


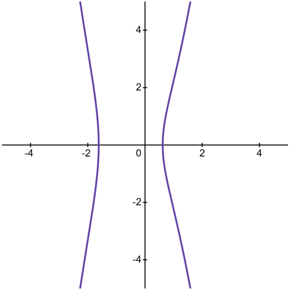
In pictures

$$C'_d\colon \begin{cases} dy_1^2 = f_1(x) \\ dy_2^2 = f_2(x) \end{cases} \qquad (x, y_1, y_2)$$

$$\downarrow \qquad\qquad \downarrow$$

$$C\colon y^2 = f_1(x)f_2(x) \qquad (x, dy_1y_2)$$





Back to example

$$C'_d\colon \begin{cases} dv_1^2 = x^2 + x - 1 \\ dv_2^2 = 2x^2 + 3 \end{cases}$$

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Example

$$C\colon y^2 = (x^2 + x - 1)(2x^2 + 3)$$

No rational points! Failure of local to global principle.

Arithmetic statistics

Question

1. *How often does* $y^2 = f(x)$ *have a rational point?*
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As (even) $\deg f$ grows

- Most are ELS [PS99]
- 100% have no rational points [Bha13]

More families

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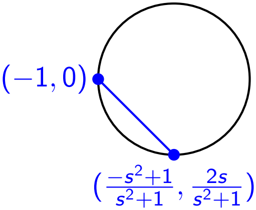

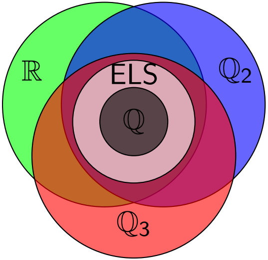

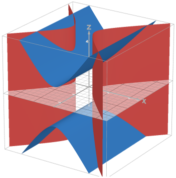

Thank you for your attention!

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