Notes: derived functors and cohomology

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Preface

The purpose of these notes is to work through somewhat carefully the exposition of derived functors and related cohomology theories including Galois cohomology and sheaf cohomology. The author began these notes during a course on local class field theory at Emory University taught in Fall 2019, and continued them while learning algebraic geometry. Some references used include Weibel's homological algebra [Wei94], Vakil's algebraic geometry course notes [Vak17], which has a nice first chapter on categories, and Hartshorne's text on algebraic geometry [Har77]. Other potentially useful perspectives include those of Silverman in his text on elliptic curves [Sil09], with its appendix on group and Galois cohomology, and some notes of Tate from PCMI [Tat] on Galois cohomology. There may be errors and omissions in these notes, so please use caution.

1 Abelian categories

1.1 Categories

We will assume the reader is already somewhat familiar with the definitions of categories and functors, so our exposition here is brief. For a more detailed exposition, see Chapter 1 of [Vak17], or your favorite homological algebra resource. We will avoid worrying about set-theoretic issues here.

Definition 1.1 (Category)**.** A **category** $\mathcal{C}$ has *objects* and *morphisms* between them. Given two objects $A, B \in \mathcal{C}$, the set of morphisms $A \to B$ is denoted $\mathrm{Mor}(A, B)$, or sometimes $\mathrm{Mor}_{\mathcal{C}}(A, B)$ to make the underlying category clear. We then require the following:

(i) *Composition* makes sense. That is, given $f \in \mathrm{Mor}(A, B)$ and $g \in \mathrm{Mor}(B, C)$, there exists $g \circ f \in \mathrm{Mor}(A, C)$, which we can state succinctly as a map $\mathrm{Mor}(A, B) \times \mathrm{Mor}(B, C) \to \mathrm{Mor}(A, C)$.

(ii) Each object $B \in \mathcal{C}$ has an *identity* morphism, denoted $\mathrm{id}_B \in \mathrm{Mor}(B, B)$, satisfying that for all $f : A \to B$ and $g : B \to C$, we have $\mathrm{id}_B \circ f = f$ and $g \circ \mathrm{id}_B = g$.

Definition 1.2. If $A, B \in \mathcal{C}$, a morphism $f : A \to B$ is called an **isomorphism** if there exists $g : B \to A$ in $\mathrm{Mor}(B, A)$ such that $f \circ g = \mathrm{id}_B$ and $g \circ f = \mathrm{id}_A$.

Probably you have seen this before, but if not, you probably know several great examples of categories. As we develop categorical notions, we shall follow several of these examples.

Example 1.3 (Sets)**.** The collection of sets, denoted **Set**, is a category. The maps between sets are merely functions. Note that we will not concern ourselves with the fact that the collection of all objects in this category is *not* a set itself.

Example 1.4 (Abelian groups)**.** The abelian groups, denoted **Ab**, are a category. The objects are abelian groups, and the maps are group homomorphisms. An isomorphism is exactly what it should be – an isomorphism of groups.

Example 1.5 (Sets with extra structure)**.** More generally, your favorite mathematical object which looks like a "set with additional structure" is (probably) a category! For instance, groups, rings, and R-modules, are all categories. In each, the notion of morphism is group homomorphism, ring homomorphism, and R-module homomorphism.

Example 1.6 (Topological spaces)**.** If we want to build a category with topological spaces, what is the correct notion of map? Given X, Y topological spaces, the structure preserving maps are the continuous ones, so we take $\mathrm{Mor}(X, Y)$ to be the set of continuous maps.

Example 1.7 (Schemes over k)**.** Let k be a field, and $\mathbf{Sch}_k$ denote the collection of schemes over k. That is, we have a scheme X equipped with a structure map $X \to \operatorname{Spec} k$. We might be tempted to say that the maps here are just "maps of schemes" $X \to Y$ (whatever those are), and we would be right if we were just in the category of schemes. But here we have the extra structure of being a scheme over k, so we need the notion of morphism to behave nicely with the structure map. Thus we define a morphism in $\mathbf{Sch}_k$ to be a map of schemes $X \to Y$ that also commutes with the structure maps.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[dl] \\ \operatorname{Spec} k & \end{tikzcd}$$

Example 1.8 (Chain complexes)**.** A chain complex (say of abelian groups) $A^\bullet$ is a collection of abelian groups A^i for $i \in \mathbb{Z}$ with maps $d^i \colon A^i \to A^{i+1}$ such that $\operatorname{im} d^i \subseteq \ker d^{i+1}$ for all i. We often write this as

$$\cdots \xrightarrow{d^{-3}} A^{-2} \xrightarrow{d^{-2}} A^{-1} \xrightarrow{d^{-1}} A^0 \xrightarrow{d^0} A^1 \xrightarrow{d^1} A^2 \xrightarrow{d^2} \cdots$$

Note we will often drop the superscripts on the d^i maps.

If we want to look at the category of chain complexes of abelian groups, $\mathbf{Ch}(\mathbf{Ab})$, what is the correct notion of a map $A^\bullet \to B^\bullet$? Sticking with the theme of maintaining all the structure possible, we will want maps from each $A^i \to B^i$, and everything in sight to commute. That is, a morphism is a collection of maps $f^i \colon A^i \to B^i$ such that the following diagram commutes.

$$\begin{tikzcd} \cdots \arrow[r] & A^{-2} \arrow[r] \arrow[d] & A^{-1} \arrow[r] \arrow[d] & A^0 \arrow[r] \arrow[d] & A^1 \arrow[r] \arrow[d] & A^2 \arrow[r] \arrow[d] & \cdots \\ \cdots \arrow[r] & B^{-2} \arrow[r] & B^{-1} \arrow[r] & B^0 \arrow[r] & B^1 \arrow[r] & B^2 \arrow[r] & \cdots \end{tikzcd}$$

We will see this category later in the definition of the derived category.

Definition 1.9 (Functor)**.** Suppose $\mathcal{A}$ and B are categories. A **(covariant) functor** $F \colon \mathcal{A} \to \mathcal{B}$ associates to every object $A \in \mathcal{A}$ an object $F(A) \in \mathcal{B}$, and to every map $f \colon A \to B$ in $\mathcal{A}$, a map $F(f) \colon F(A) \to F(B)$ in $\mathcal{B}$. We then require the following:

(i) The identity is sent to the identity, so $F(\mathrm{id}_A) = \mathrm{id}_{F(A)}$.

(ii) The square

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & F(A) \arrow[d, "F(f)"] \\ B \arrow[r] & F(B) \end{tikzcd}$$

commutes.

(iii) Composition is respected, in that $F(f \circ g) = F(f) \circ F(g)$.

A **contravariant functor** $F\colon \mathcal{A} \to \mathcal{B}$ is the same, except if $f\colon A \to B$, then $F(f)\colon F(B) \to F(A)$. That is, it reverses the direction of the morphisms.

Remark 1.10. A contravariant functor $\mathcal{A} \to \mathcal{B}$ can be characterized as a covariant functor from $\mathcal{A}^{op} \to \mathcal{B}$, where $\mathcal{A}^{op}$ indicates the opposite category – it has the same objects as $\mathcal{A}$ but all arrows are reversed.

Here are some examples of functors. Some you may already know. All of them are probably important.

Example 1.11 (Free functor)**.** The free functor $F\colon \mathbf{Set} \to \mathbf{Grp}$ takes a set S and associates to it the free group on S, which we conveniently denote $F(S)$. It is straightforward to check that a map $f\colon S \to T$ induces a map $F(f)\colon F(S) \to F(T)$ by sending the generators of $F(S)$ to their images under S in $F(T)$.

Note we could make a similar construction for something like R-modules or group rings (over a ring R).

Example 1.12 (Forgetful functors)**.** Another straightforward functor is the one that takes a group G and returns the underlying set of G. This "forgetful functor" is from **Grp** to **Set** (in fact it is adjoint to the free functor!) A group homomorphism f is sent to the plain old map of underlying sets.

Again, this can be generalized. Any "sets with structure" such as rings, modules, topological spaces, have a forgetrful functor to **Set**. We also have partially forgetful functors – for instance we have a functor from rings or R-modules to abelian groups which just forgets everything except the underlying abelian group structure.

Example 1.13 (Spec, a contravariant functor)**.** For an example of a contravariant functor, consider the functor $\operatorname{Spec}\colon \mathbf{Ring} \to \mathbf{Sch}$. Given a ring A, $\operatorname{Spec} A$ is the set of prime ideals of A, viewed as a scheme. Then a map of rings $A \to B$ gives a map of schemes $\operatorname{Spec} B \to \operatorname{Spec} A$.

Example 1.14 (Hom)**.** Fix a commutative ring R and an R-module M. Then $\operatorname{Hom}(\_\_, M)$ is a contravariant functor from R-modules to abelian groups. Given $f\colon N \to N'$, we have a map $F(f)\colon \operatorname{Hom}(N', M) \to \operatorname{Hom}(N, M)$ given by postcomposition: if $g\colon N' \to M$ then $g \circ f \in \operatorname{Hom}(N, M)$.

Similarly, $\operatorname{Hom}(M, \_\_)$ is a covariant functor.

Definition 1.15 (Natural transformations)**.** Now that we know what functors are, we can define the functor category $\operatorname{Fun}(\mathcal{A}, \mathcal{B})$ to be the category whose objects are covariant functors $\mathcal{A} \to \mathcal{B}$. The morphisms in this category are **natural transformations**. For functors $F, G\colon \mathcal{A} \to \mathcal{B}$, a natural transformation is the data of a map $F(A) \to G(A)$ for all A such that for all $f\colon A \to B$, we have the commuting square

$$\begin{tikzcd} F(A) \arrow[r, "F(f)"] \arrow[d] & F(B) \arrow[d] \\ G(A) \arrow[r, "G(f)"'] & G(B) \end{tikzcd}$$

When each $F(A) \to G(A)$ is also an isomorphism, such a map of functors is called a **natural isomorphism**.

Definition 1.16 (Equivalence of categories)**.** An **equivalence of categories** $\mathcal{A}$ and $\mathcal{B}$ is said to be when there exist functors $F\colon \mathcal{A} \to \mathcal{B}$ and $G\colon \mathcal{B} \to \mathcal{A}$ such that $F \circ G$ and $G \circ F$ are *naturally isomorphic* to the identity functors on $\mathcal{B}$ and $\mathcal{A}$, respectively. This says that $\mathcal{A}$ and $\mathcal{B}$ are essentially the same.

1.2 Universal properties

One of the reasons categories are useful is that constructions in multiple categories often share desirable properties, and this is sometimes because they arise from the same categorical construction. Universal properties are a nice way to succinctly state such properties.

Definition 1.17 (Initial and terminal)**.** An object I in a category $\mathcal{A}$ is **initial** if it has a *unique* map $I \to A$ to every object A in $\mathcal{A}$. The dual notion is a **terminal** object T, which has a unique map $A \to T$ *from* every object A in $\mathcal{A}$.

If an object Z is both initial and terminal, it is said to be a **zero object**.

Note that initial and terminal objects are necessarily unique up to unique isomorphism, as any two such objects must admit maps to each other, and the composition of those maps is the identity, by uniqueness.

The ring $\mathbb{Z}$ is initial in the category of rings, and there is no terminal object. The scheme $\operatorname{Spec}\mathbb{Z}$ is terminal in the category of schemes (these are the same fact!). The trivial group is the zero object in **Grp** and **Ab**, and the zero module is the zero object in the category of R-modules. The singleton set is terminal in the category of sets, and the empty set may be thought of as an initial object.

Definition 1.18 (Diagram)**.** Let $\mathcal{I}$ be a category which is a partially ordered set (a category with at most one arrow between any two objects), which we call an *index category.* A functor $F\colon \mathcal{I} \to \mathcal{A}$ is called a **diagram** indexed by $\mathcal{I}$.

Morally, this is literally just a diagram in the category $\mathcal{A}$. For instance, $\mathcal{I}$ might be

$$\begin{tikzcd} & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd}$$

In which case a diagram in $\mathcal{A}$ indexed by $\mathcal{I}$ corresponds to a choice of objects A, B, C in $\mathcal{A}$ and maps $A \to C$ and $B \to C$, which we draw as

$$\begin{tikzcd} & A \arrow[d] \\ B \arrow[r] & C \end{tikzcd}$$

The surprising thing is that this turns out to be useful.

Definition 1.19 (Limits and colimits)**.** The **limit** of a diagram in $\mathcal{A}$ indexed by $\mathcal{I}$ is the terminal object in the category of objects with maps to the diagram. We denote this by $\varprojlim A_i$, where the A_i are the objects in the diagram indexed by $\mathcal{I}$.

Similarly, we define the dual notion of a **colimit** to be the initial object in the category of objects with maps *from* the diagram, denoted $\varinjlim A_i$.

This is somewhat unsatisfying. For one, it should be checked that the category of objects with maps to a diagram actually makes sense. This is really better done with examples. Unversal properties are then the properties that arise precisely because a construction came from a limit or colimit.

Example 1.20 (Product)**.** The most straightforward example is that of (direct) products. We will remain in the category of sets, though one notes that we could just as well be talking about groups or modules here. A **product** is a limit of the diagram indexed by

$$\bullet \qquad \bullet$$

This index category just picks out two sets S, S' with no requirements for maps between them. Thus the category of objects with maps to this diagram has objects

$$\begin{tikzcd} T \arrow[d] \arrow[dr] & \\ S & S' \end{tikzcd}$$

and morphisms $T' \to T$ which look like commutative diagrams

$$\begin{tikzcd} T' \arrow[d] \arrow[dd, bend right] \arrow[ddr] & \\ T \arrow[d] \arrow[dr] & \\ S & S' \end{tikzcd}.$$

The terminal object in such a category is the direct product, or cartesian product $S \times S'$. One can verify directly that for any T with maps to S, S', these maps factor uniquely through $S \times S'$. We sometimes write this universal property as

$$\begin{tikzcd} T \arrow[d, dashed, "\exists!"] \arrow[dd, bend right] \arrow[ddr] & \\ S \times S' \arrow[d] \arrow[dr] & \\ S & S' \end{tikzcd}.$$

We can generalize this a bit by defining the **fiber product**. Now the diagram becomes

$$\begin{tikzcd} & X \arrow[d] \\ Y \arrow[r] & S \end{tikzcd}$$

and the limit is denoted $X \times_S Y$, which enjoys the universal property that for any set T with maps to X, Y such that the square commutes, we have

$$\begin{tikzcd} T \arrow[dr, dashed, "\exists!"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\ & X \times_S Y \arrow[r] \arrow[d] & X \arrow[d] \\ & Y \arrow[r] & S \end{tikzcd}.$$

Example 1.21 (Direct sum)**.** The dual notion to product is **direct sum**. We define the direct sum to be the colimit of a diagram indexed by dots with no arrows (just like the two dots in the definition of a product). This becomes quite nontrivial once the indexing set is infinite, as the direct sum and product do not agree in general.

Example 1.22 (Completions)**.** The p-adic integers are constructed in the category of rings as the limit of the diagram

$$\cdots \longrightarrow \mathbb{Z}/p^3\mathbb{Z} \longrightarrow \mathbb{Z}/p^2\mathbb{Z} \longrightarrow \mathbb{Z}/p\mathbb{Z}$$

where the maps are reduction mod p. We take $\mathbb{Z}_p = \varprojlim_n \mathbb{Z}/p^n\mathbb{Z}$.

The profinite completion of $\mathbb{Z}$ is defined similarly as the limit $\varprojlim_n \mathbb{Z}/n\mathbb{Z}$, where we include the natural maps $\mathbb{Z}/m\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ when $n|m$.

More generally, we can discuss the *completion* of a ring or module, which is a limit relative to some filtration. Or, the *profinite* completion, which is the limit of all finite quotients of a ring or module.

Example 1.23 (Stalks)**.** If you like sheaves or schemes, then you may know that *stalks* are colimits. Given a sheaf $\mathcal{F}$ (of rings, say) on X, we define the stalk at $p \in X$ to be

$$\mathcal{F}_p = \varinjlim \mathcal{F}(U),$$

where the colimit is taken over all opens U containing p. This is either taken to be the definition, or seen to be equivalent to the definition in terms of sections, where we take $\mathcal{F}_p$ to be the pairs (f, U) where $f \in \mathcal{F}(U)$ and we identify (f, U) with (g, V) if f and g agree when restricted to some open $W \subseteq U \cap V$ containing p.

Example 1.24 (A tautology)**.** As a perverse example, we can define initial and final terminal objects to be the colimit and limit, respectively, of the empty diagram. Here the colimit is object with a unique map to any test object which maps to the empty diagram – since any object satisfies this, we recover our definition of initial object. The same procedure works for final objects.

Of course, this is a bit silly since we defined limits and colimits as initial and final objects in the first place. But had we tweaked our definitions, this would make sense.

The main problem when we have universal constructions like this is that they are not guaranteed to exist. However, once we can show that a certain limit or colimit exists (either for a particular diagram or any such diagram in the category), the uniqueness comes for free from that of initial and terminal objects. Later when defining abelian categories, we will want certain limits and colimits to exist, and will have to build these into our axioms.

1.3 Abelian categories

We'll start by defining an abelian category $\mathcal{A}$, then we'll go fill in the details needed for the definitions to make sense.

Definition 1.25 (Abelian category)**.** Let $\mathcal{A}$ be a catgeory. We say $\mathcal{A}$ is an **abelian category** if

(i) The morphisms between any two objects form an *abelian group*, and composition distributes over additions, so for $f, g, h \in \mathrm{Mor}(A, B)$ we have

$$f \circ (g + h) = f \circ g + f \circ h.$$

(ii) $\mathcal{A}$ has an *zero object*, denoted 0.

(iii) For any $A, B \in \mathcal{A}$, the *(direct) product* $A \times B$ exists. Hence any finite product exists.

(iv) Every map has a *kernel* and *cokernel*

(v) Every monomorphism is the kernel of its cokernel, and every epimorphism is the cokernel of its kernel.

We note that (v) is a technical condition that makes sure that kernels and cokernels do what we expect them to.

Definition 1.26 (Monomorphism)**.** Let $i \colon A \to B$. We say i is a **monomorphism** if for all $f, g \colon T \to A$ such that $i \circ f = i \circ g$, then $f = g$. Put diagramatically, for all

$$\begin{tikzcd} T \arrow[d, shift right, "f"'] \arrow[d, shift left, "g"] \arrow[dr] & \\ A \arrow[r, "i"'] & B \end{tikzcd}$$

which commutes, then $f = g$. We denote a monomorphism by a hooked arrow, $A \overset{i}{\hookrightarrow} B$.

Definition 1.27 (Epimorphism)**.** This time we skip straight to the diagrammatic definition. A map $p\colon A \to B$ is an **epimorphism** if for all $f, g\colon Y \to T$ such that

$$\begin{tikzcd} A \arrow[r, "p"] \arrow[dr] & B \arrow[d, shift right, "f"'] \arrow[d, shift left, "g"] \\ & T \end{tikzcd}$$

commutes, then we have $f = g$. We denote his by a double head arrow $p\colon A \twoheadrightarrow B$.

Monomorphisms and epimorphisms should be reminiscent of injections and surjections. One can check that composing two monos (or epis) produces another mono (or epi). In the categories you know and love, injections will be monomorphisms and surjections will be epimorphisms, but note that not all mono/epimorphisms are inj/surjections. For example, take the inclusion $\mathbb{Z} \to \mathbb{Q}$ in the category of rings, which is clearly not a surjection. Yet, it is an epimorphism, as specifying a map $\mathbb{Z} \to T$ determines a unique map $\mathbb{Q} \to T$ that agrees with the inclusion.

We now define the notions of kernel and cokernel categorically. These will agree with the usual notions in the usual categories.

Definition 1.28 (Kernel)**.** Let $f\colon A \to B$ be a map. The **kernel** of f is the limit of the diagram

$$\begin{tikzcd} 0 \arrow[dr] & \\ A \arrow[r, "f"] & B \end{tikzcd}.$$

Or if you prefer, it is an object with a map $\ker f \to A$, such that for any T with a map to A which composes with f to be the zero map, there is a unique map $T \to \ker f$ such that everything commutes

$$\begin{tikzcd} T \arrow[r] \arrow[d, dashed] \arrow[dr] & 0 \arrow[dr] & \\ \ker f \arrow[r] & A \arrow[r, "f"] & B \end{tikzcd}.$$

Definition 1.29 (Cokernel)**.** This is the dual notion to kernel, so reverse the arrows! The **cokernel** is the colimit of the diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr] & B \\ & 0 \end{tikzcd}.$$

So for any T with a map from B such that the composition with f from $A \to T$ is zero, we have a unique map from the cokernel of f to T such that everything commutes

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr] & B \arrow[r] \arrow[dr] & \operatorname{coker} f \arrow[d, dashed] \\ & 0 \arrow[r] & T \end{tikzcd}.$$

We want to know what monomorphisms and epimorphisms behave much like injective and surjective maps in, say, the category of abelian groups. One of the first characterizations of an injective map is that its kernel is trivial. We prove the analogues for monos and epis here, as well as a statement about composition.

Lemma 1.30. *Let $f\colon A \to B$ be a map.*

(a) If f is a monomorphism then $\ker f = 0$.

(b) If f is an epimorphism then $\operatorname{coker} f = 0$.

Proof. Suppose f is a monomorphism and for some object T we have the zero map $T \to B$. Then by the definition of monomorphism, there is at most one map $T \to A$ making this commute. But of course, the zero map is one such map, and thus the only one. We conclude that any map $T \to A$ which composes to zero to B must factor (uniquely) through the zero object. Hence $\ker f = 0$.

The dual argument shows that an epimorphism has as its cokernel the zero object. □

Lemma 1.31. *Suppose $f\colon A \to B$ is such that $h \circ f = g$ for h, g monomorphisms with target C. That is,*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, hook, "g"'] & B \arrow[d, hook, "h"] \\ & C \end{tikzcd}.$$

Then f is a monomorphism.

If instead

$$\begin{tikzcd} C \arrow[d, two heads, "g"'] \arrow[dr, two heads, "h"] & \\ A \arrow[r, "f"] & B \end{tikzcd}$$

then f is an epimorphism.

Proof. First recognize that it's enough to take a test object T with a map to A which composes with f to the zero map, since we are in an abelian category. Then just use the property that g is a monomorphism to conclude that there can only be one such map $T \to A$, and it must be zero.

The dual argument works for the epimorphism proof without modification. □

Definition 1.32 (Image and coimage)**.** The **image** of $f\colon A \to B$ is $\ker(\operatorname{coker} f)$. The **coimage** of f is $\operatorname{coker}(\ker f)$. In particular, these exist whenever kernels and cokernels do.

Note that the universal definitions guarantee that the kernel map $\ker f \to A$ is a monomorphism, while the cokernel map $B \to \operatorname{coker} f$ is an epimorphism. In the usual categories, such as modules over a ring, we can check that these are what we expect. Kernels are kernels, images are images, $\operatorname{coker} f = B/\operatorname{im} f$, and the coimage is $A/\ker f$. In particular, the first isomorphism theorem states that the image is isomorphic to the coimage. This is related to (v) in the definition of abelian category, which we justify below.

Proposition 1.33. *Let $\mathcal{A}$ be a category satisfying (i) through (iv) of the definition of abelian category. Then $\mathcal{A}$ satisfies (v) if and only if* $\operatorname{im}(f) \simeq \operatorname{coim}(f)$ *for all maps $f\colon A \to B$ in $\mathcal{A}$.*

Proof. ($\Longrightarrow$) Suppose (v), that monomorphisms are the kernels of their cokernels, and epimorphisms are the cokernels of their kernels. Recalling that $\operatorname{im}(f) = \ker(B \to \operatorname{coker} f)$ and $\operatorname{coim}(f) = \operatorname{coker}(\ker f \to A)$, we have the exactness of the rows in the diagram below.

$$\begin{tikzcd}
0 \arrow[r] & \ker f \arrow[r] \arrow[d, "0"] & A \arrow[r] \arrow[d, "f"] & \operatorname{coim} f \arrow[r] \arrow[d, "0"] & 0 \\
0 \arrow[r] & \operatorname{im} f \arrow[r] & B \arrow[r] & \operatorname{coker} f \arrow[r] & 0
\end{tikzcd}$$

The vertical arrows are given by f in the center, while the left and right arrows are the zero map. Thus the by the snake lemma (see Proposition 1.48) we have an exact sequence

$$\ker f \xrightarrow{0} \operatorname{coim} f \to \operatorname{im} f \xrightarrow{0} \operatorname{coker} f,$$

allowing us to conclude that the natural map is an isomorphism, $\operatorname{coim} f \simeq \operatorname{im} f$.

($\Longleftarrow$) Let $\operatorname{im} f \simeq \operatorname{coim} f$ for all maps f. Then we have

$$\begin{tikzcd}
 & \operatorname{coim} f \arrow[r, "\sim"] & \operatorname{im} f \arrow[d, hook] & \\
\ker f \arrow[r, hook] & A \arrow[u, two heads] \arrow[r, "f"] & B \arrow[r, two heads] & \operatorname{coker} f
\end{tikzcd} .$$

Suppose f is a monomorphism. Then by definition, the kernel map must be the zero map, as there can be at most one map from $\ker f \to A$ which is zero when composed with f, and zero is one such map. We then check that the cokernel of the zero map to A is just the identity map to A, so $A \simeq \operatorname{coim} f$ (actually we probably could just say equal here). Thus $A \simeq \operatorname{im} f$, which is the kernel of $\operatorname{coker} f$. This naturally identifies f with the kernel of its own cokernel.

The dual argument shows that when f is an epimorphism, its cokernel is zero, and thus $B = \operatorname{im} f$. Then the cokernel of $\ker f$ is just B. □

Remark 1.34 (first isomorphism theorem)**.** In the category of R-modules, the proposition above is stating the first isomorphism theorem. We may interpret the coimage of a morphism $f\colon A \to B$ as $A/\ker f$; the first isomorphism theorem, as presented in any elementary algebra textbook, states

$$A/\ker f \simeq \operatorname{im} f,$$

so the proposition is using axiom (v) to justify this in any abelian category.

Let's give one more useful lemma, again motivated by the category of abelian groups. A group homomorphism which is both injective and surjective is in fact a group isomorphism. In fact, we find that if $A \to B$ is injective then $A \simeq A/\ker$ and if it is surjective, then $B \simeq \operatorname{im}$. We want, and find, that something similar holds in any abelian category.

Lemma 1.35. *Let $f\colon A \to B$ be a map.*

(a) If f is a monomorphism then $A \simeq \operatorname{coim} f$.

(b) If f is an epimorphism then $B \simeq \operatorname{im} f$.

(c) If f is both a mono and an epi, then f is an isomorphism $A \simeq B$.

Proof. Let f be a monomorphism, which we know by Lemma 1.30 has trivial kernel. Take A as a test object, with the identity map $A \to A$. Since this commutes with the map from the kernel, we have by definition of cokernel, a map from $\operatorname{coim} A \to A$, which we call q.

$$\begin{tikzcd}
 & A & \operatorname{coim} f \arrow[l, dashed, "\exists! q"'] \\
0 \arrow[ur, "\mathrm{id}_A"] \arrow[r, hook] & A \arrow[u] \arrow[ur, two heads, "p"] \arrow[r, "f"] & B
\end{tikzcd}$$

It is now clear that $q \circ p = \mathrm{id}_A$. However we also have

$$\begin{tikzcd} A \arrow[r, two heads, "p"] \arrow[dr, "p"'] & \operatorname{coim} f \arrow[d, shift right, "p\circ q"'] \arrow[d, shift left, "\mathrm{id}"] \\ & \operatorname{coim} f \end{tikzcd}$$

and since p is an epimorphism, we must have $p \circ q = \mathrm{id}_{\operatorname{coim} f}$. This proves (a).

If we start from f an epimorphism and run the same argument, but using the image of f, we will find a map $B \to \operatorname{im} f$ which when composed with the natural monomorphism $\operatorname{im} f \to B$ gives the identity on B. We then see that the other composition must give the identity on $\operatorname{im} f$ because of the definition of monomorphism. This proves (b).

For (c), we just use parts (a) and (b) to see that f is equal to the composition of three isomorphisms, giving

$$A \simeq \operatorname{coim} f \simeq \operatorname{im} f \simeq B.$$

□

Now that we've made sense of the adjectives, we revisit our definition: a category $\mathcal{A}$ is **abelian** if it satisfies

(i) The morphisms between any two objects form an *abelian group* where composition distributes over addition.

(ii) $\mathcal{A}$ has an *zero object*, denoted 0.

(iii) *Finite products* exist.

(iv) Every map has a *kernel* and *cokernel*.

(v) A technical condition equivalent to the first isomorphism theorem.

Example 1.36 (Modules over a ring)**.** Modules over a commutative ring R are an abelian category. One checks that $\operatorname{Hom}_R(A, B)$ is a group, there is a zero module, we can take products which satisfy the necessary universal properties, kernels and cokernels (quotients) make sense, and the first isomorphism theorem is satisfied.

Of course, **Ab** is just the category of $\mathbb{Z}$-modules, so the abelian groups are an abelian category (thank goodness).

Example 1.37 (Non-example)**.** The category of rings is *not abelian.* For one, there is no zero object. $\mathbb{Z}$ is initial, but certainly not final in this category. And while (iii) – (v) seem to make sense, the hom-sets are not abelian groups! For example, take the identity map $\mathbb{Z} \to \mathbb{Z}$. If we add it with itself, we get a map which sends $1 \mapsto 2$. Thus we do not have a ring homomorphism.

1.4 Exactness

For the remainder of the section, assume we are working in an abelian category $\mathcal{A}$. Suppose we have a sequence

$$A \xrightarrow{f} B \xrightarrow{g} C$$

such that $g \circ f = 0$. This condition may look unfamiliar, but this is analogous to the notion of $\operatorname{im} f \subseteq \ker g$ in the case of R-modules. This is useful later when we define chain complexes. Note that using the universal property of $\ker g$, we find that there is a canonical map $\operatorname{im} f \to \ker g$ that commutes with everything in sight.

Definition 1.38 (Exactness)**.** Let $A \xrightarrow{f} B \xrightarrow{g} C$ be maps such that $g \circ f = 0$. This is **exact** at B if the natural map $\operatorname{im} f \to \ker g$ is an isomorphism.

A diagram

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

is called a **short exact sequence** if it is exact at A, B, C. In R**-mod**, this is saying f is injective, g is surjective, and $\ker g = \operatorname{im} f$.

In general, any length sequence which is exact at each object is called an exact sequence. In some cases we will refer to infinite such sequences as *long*.

Definition 1.39 (Injective object)**.** An object I is **injective** if for all monomorphisms $X \to Y$ such that there is a map $X \to I$, there exists a map $Y \to I$ such that

$$\begin{tikzcd} X \arrow[r, hook] \arrow[d] & Y \arrow[dl, dashed, "\exists"] \\ I & \end{tikzcd}$$

commutes.

Example 1.40 ($\mathbb{Q}/\mathbb{Z}$)**.** The quotient $\mathbb{Q}/\mathbb{Z}$ is injective in the category of abelian groups. Given an injection $X \to Y$ of groups, and a map $f\colon X \to \mathbb{Q}/\mathbb{Z}$, we can cook up a map $Y \to \mathbb{Q}/\mathbb{Z}$. For simplicity, assume Y is finitely generated by y_i for $1 \leq i \leq n$. Take a generator y_1. If some multiple my_1 is in the image of X, we can define $y_1 \mapsto \frac{1}{m} f(my_1)$. If no multiple of y_1 is in the image of X, we can send it anywhere we like. One then checks that this map agrees with $X \to I$.

It turns out that in the category **Ab**, being injective is equivalent to being *divisible* – for any g in the group and positive integer n, there exists h such that $nh = g$.

Definition 1.41 (Projective object)**.** This is dual to injective. An object P is **projective** if for all epimorphisms $X \to Y$ such that there is a map $P \to Y$, there exists a map $P \to X$ such that

$$\begin{tikzcd} & P \arrow[dl, dashed, "\exists"'] \arrow[d] \\ X \arrow[r, two heads] & Y \end{tikzcd}$$

commutes.

Example 1.42 (Free objects)**.** In the category of R-modules, free objects are projective. To see this, suppose X surjects onto Y and there is a map $R^n \to Y$. We can take the image of the generators of R^n in Y and pull them back to X, then define $P \to X$ this way. This gives a map and forces commutativity of the diagram.

Definition 1.43 (Enough)**.** The category $\mathcal{A}$ is said to have **enough injectives** if every object A has a monomorphism $A \hookrightarrow I$ into some injective object I. That is, every object is a subobject of an injective object.

$\mathcal{A}$ is said to have **enough projectives** if every object A has an epimorphism from a projective object, $P \twoheadrightarrow A$. That is, every object is the quotient of some projective object.

Example 1.44 (Modules over R)**.** By our somewhat informal characterization, it is clear that our prototypical abelian category of modules over a ring R has enough projectives, as every object is the quotient of a free object, and free objects are projective.

It is somewhat tricker to see that this category has enough injectives, so we sketch the proof for $R = \mathbb{Z}$. First, we use the definitions to show that arbitrary direct sums of injectives are injective. Also, we have that $\mathbb{Z}$ itself embeds into $\mathbb{Q}$ which is injective. Then given any abelian group A, we realize it as the quotient of a free $\mathbb{Z}$-module, i.e. a direct sum of copies of $\mathbb{Z}$. However this can be embedded into a direct sum of copies of $\mathbb{Q}$, which is injective. It remains to check that taking quotients of an injective object yields an injective, but then we are done, and A embeds into such a quotient of the direct sum of $\mathbb{Q}$.

We now discuss exact functors. Suppose

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

is an exact sequence. Given an additive functor $F\colon \mathcal{A} \to \mathcal{B}$, where $\mathcal{B}$ is another abelian category, we get a sequence

$$F(0) \longrightarrow F(A) \xrightarrow{F(f)} F(B) \xrightarrow{F(g)} F(C) \longrightarrow F(0) \ .$$

This is going to be a complex (i.e. $F(g) \circ F(f) = 0$), but there is *no need for this sequence to be exact.*

Let us first observe that necessarily $F(0) = 0$. By the definition of functor, $F(\mathrm{id}_A) = \mathrm{id}_{F(A)}$, and if $0_A\colon A \to A$ is the zero map, then additivity shows

$$F(0_A) = F(\mathrm{id}_A - \mathrm{id}_A) = F(\mathrm{id}_A) - F(\mathrm{id}_A) = \mathrm{id}_{F(A)} - \mathrm{id}_{F(A)} = 0_{F(A)}.$$

Thus F sends identity maps to identity maps and zero maps to zero maps. The zero object in $\mathcal{A}$ is that for which the identity and zero map are the same. The functor F then sends them to the same map on $F(0)$, which shows $F(0)$ is also a zero object in $\mathcal{B}$. Thus we can safely consider exactness of

$$0 \longrightarrow F(A) \xrightarrow{F(f)} F(B) \xrightarrow{F(g)} F(C) \longrightarrow 0 \ .$$

Definition 1.45 (Left- and right-exact functors)**.** The (covariant) functor F is **left-exact** if

$$0 \longrightarrow F(A) \xrightarrow{F(f)} F(B) \xrightarrow{F(g)} F(C)$$

is exact. We say F is **right-exact** if

$$F(A) \xrightarrow{F(f)} F(B) \xrightarrow{F(g)} F(C) \longrightarrow 0$$

is exact. If F is both left- and right-exact, we say it is **exact**, and the full sequence

$$0 \longrightarrow F(A) \xrightarrow{F(f)} F(B) \xrightarrow{F(g)} F(C) \longrightarrow 0$$

is exact.

These definitions look a little different for contravariant functors, because arrows get reversed, so left/right can be confusing. This is one of those times where it is useful to think of contravariant functors as covariant ones from the opposite category.

Example 1.46 (Hom)**.** Consider the functor $\mathrm{Hom}(M, \_\_)$. This is covariant and additive, as we can add maps in abelian categories. Suppose we have $g\colon M \to A$ such that $f \circ g = 0$ as a map $M \to B$. Since f was a monomorphism, and 0 is a map $M \to B$ which agrees with $f \circ g$, we must have $g = 0$. Hence $\mathrm{Hom}(M, A) \to \mathrm{Hom}(M, B)$ is a monomorphism. This shows exactness at the first place.

Suppose now we have a map $h\colon M \to B$ which is in the kernel of the Hom maps, i.e. $g \circ h = 0$ from $M \to C$. Using universal properties, we can produce a map $M \to \ker g = \operatorname{im} f$ which commutes, then we use the fact that since f is a monomorphism, we have $A \simeq \operatorname{im} f$. Hence we have a map $M \to A$ that gives h when composed with f.

The condition of exactness at $\mathrm{Hom}(M, C)$ is not true in general. We need to have maps $M \to C$ pull back to maps $M \to B$ composed with g. However, this is precisely the definition of M being a projective object! Hence $\mathrm{Hom}(M, \_\_)$ is left-exact, and right exact precisely when M is projective.

The dual argument shows that the contravariant functor $\mathrm{Hom}(\_\_, M)$ is also left exact, i.e.

$$\mathrm{Hom}(A, M) \leftarrow \mathrm{Hom}(B, M) \leftarrow \mathrm{Hom}(C, M) \leftarrow 0.$$

This is the right notion of left-exactness for a contravariant functor (consider it as a covariant functor from the opposite category; it is often written with arrows left-to-right). We also find that this functor is exact precisely when M is an injective module.

We conclude this section with two ubiquitous results related to exactness: the four and five lemmas and the snake lemma.

Proposition 1.47 (Four and five lemmas)**.** *In the following statements, assume all diagrams commute and that the rows are exact.*

(i) Suppose we have

$$\begin{tikzcd}
B \arrow[r, "g"] \arrow[d, "b"] & C \arrow[r, "h"] \arrow[d, "c"] & D \arrow[r, "k"] \arrow[d, "d"] & E \arrow[d, "e"] \\
B' \arrow[r, "g'"'] & C' \arrow[r, "h'"'] & D' \arrow[r, "k'"'] & E'
\end{tikzcd}$$

such that b, d are epimorphisms and e is a monomorphism. Then c is an epimorphism.

(ii) Suppose we have

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "a"] & B \arrow[r, "g"] \arrow[d, "b"] & C \arrow[r, "h"] \arrow[d, "c"] & D \arrow[d, "d"] \\
A' \arrow[r, "f'"'] & B' \arrow[r, "g'"'] & C' \arrow[r, "h'"'] & D'
\end{tikzcd}$$

such that b, d are monomorphisms and a is an epimorphism. Then c is a monomorphism.

(iii) Suppose we have

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "a"] & B \arrow[r, "g"] \arrow[d, "b"] & C \arrow[r, "h"] \arrow[d, "c"] & D \arrow[r, "k"] \arrow[d, "d"] & E \arrow[d, "e"] \\
A' \arrow[r, "f'"'] & B' \arrow[r, "g'"'] & C' \arrow[r, "h'"'] & D' \arrow[r, "k'"'] & E'
\end{tikzcd}$$

such that b, d are isomorphisms, a is an epimorphism, and e is a monomorphism. Then c is an isomorphism. Note this is commonly stated with $A = A' = E = E' = 0$.

Proof. (iii) follows directly from (i) and (ii). Both (i) and (ii) are proved via a diagram chase that is probably worth doing once in your life. Note that here we assume we are in the category of R-modules, but the statement is true in an arbitrary abelian category.

Let $c' \in C'$. Mapping to D' via h' and pulling back along the surjective map d, we get an element of D. Since $d' \in \ker k'$, and e is injective, we must have $k(d) = 0$ in E. Thus there is an element of C which maps to d', and hence its image in C differs from c' by something in $\ker h'$. Using exactness and the surjectivity of b, we pull such an element back to B, and subtract its image under g to get something which maps to c'.

For (ii), suppose we have $\gamma \in C$ such that $c(\gamma) = 0$. Pushing to D' and pulling back along the injective map d, we see that $\gamma \in \ker h$, and thus in the image of g. Say $g(\beta) = \gamma$. Then $b(\beta) \in \ker g'$ is , and hence the image of f', which we can pull back along a to $\alpha \in A$. But now $f(\alpha)$ differs from β by an element of $\ker b$, which is zero since b is injective, thus $f(\alpha) = \beta$. Hence $g(\beta) = 0$ and c is injective. $\square$

Proposition 1.48 (Snake lemma)**.** *Consider the following commutative diagram*

$$\begin{tikzcd} & A \arrow[r, "f"] \arrow[d, "a"] & B \arrow[r, "g"] \arrow[d, "b"] & C \arrow[r] \arrow[d, "c"] & 0 \\ 0 \arrow[r] & A' \arrow[r, "f'"'] & B' \arrow[r, "g'"'] & C' & \end{tikzcd}$$

with exact rows. There are natural maps and a connecting homomorphism δ satisfying

$$\ker a \to \ker b \to \ker c \xrightarrow{\delta} \operatorname{coker} a \to \operatorname{coker} b \to \operatorname{coker} c.$$

Moreover, this sequence is exact (on both sides) if the top row is also left exact and the bottom row is right exact.

Proof. Again, this is a tedious but straightforward diagram chase. First one can see that the maps $\ker a \to \ker b \to \ker c$ and $\operatorname{coker} a \to \operatorname{coker} b \to \operatorname{coker} c$ exist from universal properties. Exactness is then easiest to show elementwise.

The connecting map δ is defined on $\gamma \in \ker c$ by pulling back along g' and f' to get an element of A', and thus $\operatorname{coker} a$. To see this is well defined, we need to know that $0 \in \ker c$ maps to the image of a in A'. This follows from exactness of the top row. This is worth doing by hand at least once in your life. $\square$

Remark 1.49. Vakil gives a proof of both the snake lemma and the five lemma using spectral sequences [Vak17, Examples 1.7.5, 1.7.6]. While this is certainly a useful illustration of using spectral sequences, the author finds this somewhat confusing, as the snake lemma can come up when understanding spectral sequences.

2 Derived functors

In this section we will define the derived functors and prove the existence of the long exact sequence. First we will need some preliminaries on chain complexes and (co)homology of a complex.

2.1 Chain complexes

Definition 2.1 (Chain complexes)**.** Let $\mathcal{A}$ be an abelian category. A **chain complex** $A^\bullet$ is a sequence of objects in $\mathcal{A}$ indexed by $\mathbb{Z}$ with maps $d^n \colon A^n \to A^{n+1}$ such that

$$\cdots \xrightarrow{d^{-3}} A^{-2} \xrightarrow{d^{-2}} A^{-1} \xrightarrow{d^{-1}} A^0 \xrightarrow{d^0} A^1 \xrightarrow{d^1} A^2 \xrightarrow{d^2} \cdots$$

such that $d^n \circ d^{n-1} = 0$ for all n. We often shorten this by ignoring the superscripts on the d maps, and just write $d \circ d = 0$.

Definition 2.2 (Map of complexes)**.** Let $A^\bullet$ and $B^\bullet$ be chain complexes in $\mathcal{A}$. A map between them, $f \colon A^\bullet \to B^\bullet$ consists of maps $f^i \colon A^i \to B^i$ such that the diagram

$$\begin{tikzcd} \cdots \arrow[r] & A^{-2} \arrow[r] \arrow[d, "f^{-2}"] & A^{-1} \arrow[r] \arrow[d, "f^{-1}"] & A^0 \arrow[r] \arrow[d, "f^0"] & A^1 \arrow[r] \arrow[d, "f^1"] & A^2 \arrow[r] \arrow[d, "f^2"] & \cdots \\ \cdots \arrow[r] & B^{-2} \arrow[r] & B^{-1} \arrow[r] & B^0 \arrow[r] & B^1 \arrow[r] & B^2 \arrow[r] & \cdots \end{tikzcd}$$

commutes. That is, $d_B^n \circ f^n = f^{n+1} \circ d_A^n$ for all n. In a more compact abuse of notation we write, $d \circ f = f \circ d$.

At this point, we recognize that the chain complexes in $\mathcal{A}$ form a category, since maps compose associatively. We call this category $\mathrm{Ch}(\mathcal{A})$, where objects are chain complexes and maps are maps of chain complexes.

Example 2.3 (Natural subcategories of $\mathrm{Ch}(\mathcal{A})$)**.** We might consider the *nonnegative* chain complexes, $\mathrm{Ch}^{\geq 0}(\mathcal{A})$, where the objects are chain complexes $A^\bullet$ such that $A^i = 0$ for all $i < 0$. We could also consider the category of *bounded* chain complexes, $\mathrm{Ch}^b(\mathcal{A})$, where objects are complexes $A^\bullet$ such that for some integers $n \leq m$ we have $A^i = 0$ for all $i \leq n$ and $i \geq m$.

Definition 2.4 ((Co)homology)**.** Given a chain complex $A^\bullet$, we have a natural monomorphism $\operatorname{im} d^{n-1} \to \ker d^n$ for all n. Define the ***i*-th cohomology** of $A^\bullet$ as

$$H^i(A^\bullet) = \operatorname{coker}(\operatorname{im} d^{i-1} \to \ker d^i).$$

In the category of abelian groups, or R-modules, this corresponds to the usual quotient definition,

$$H^i(A^\bullet) = \ker d^i / \operatorname{im} d^{i-1}.$$

It is somewhat confusing, but this is also sometimes called the homology. Honestly, this makes more sense, because we are calling them chain complexes rather than *cochain complexes*, but it doesn't really matter. Once we have a chain complex and choose to write the maps from left to right, the notion of (co)homology is well defined.

We remark that a complex is exact at A^i if and only if the i-th cohomology is trivial, i.e. the zero object. So if we say a complex is exact, it has trivial cohomology everywhere.

Proposition 2.5 (Maps of complexes induce maps on homology)**.** *Let $f\colon A^\bullet \to B^\bullet$ be a map of complexes. This induces a map on the i-th homologies $H^i(A^\bullet) \to H^i(B^\bullet)$ for all i.*

Proof. This is relatively straightforward to prove in the case of R-modules by following elements in the images and kernels around the diagram. Here we give a more categorical proof.

Consider the following diagram:

$$\begin{tikzcd}
 & \operatorname{im} d \arrow[r, hook] \arrow[dr] \arrow[ddd, dashed, bend right] & \ker d \arrow[r] \arrow[d, hook] \arrow[ddd, dashed, bend left] & H^0(A^\bullet) \arrow[ddd, dashed, bend left] & \\
\cdots \arrow[r] & A^{-1} \arrow[r] \arrow[u] \arrow[d] & A^0 \arrow[r] \arrow[d] & A^1 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & B^{-1} \arrow[r] \arrow[d] & B^0 \arrow[r] & B^1 \arrow[r] & \cdots \\
 & \operatorname{im} d \arrow[r, hook] \arrow[ur] & \ker d \arrow[r] \arrow[u, hook] & H^0(B^\bullet) &
\end{tikzcd}$$

We argue first that the middle dotted arrow $\ker d \to \ker d$ exists by the universal property of the (lower) kernel, since composition of the (top) kernel gives the zero map to B^1 and factors through B^0. This gives the dotted map, and hence a map to $H^0(B)$.

The leftmost dotted arrow exists because the image is the kernel of a cokernel, and A^{-1} has a zero map to the cokernel of $B^{-1} \to B^0$. Thus so does (top) $\operatorname{im} d$. Note that this map composes to zero with $H^0(B)$.

Last, we observe that (top) $\operatorname{im} d$ has a zero map to $H^0(B^\bullet)$ which factors through $\ker d$. Hence $H^0(A^\bullet)$ has a canonical map to $H^0(B^\bullet)$, which is the rightmost dotted line. This shows f induces a map $H^i(A^\bullet) \to H^i(B^\bullet)$ by shifting the indices above. $\square$

Definition 2.6 (quasi-isomorphism)**.** A map $f\colon A^\bullet \to B^\bullet$ is a **quasi-isomorphism** if the induced maps on homologies $H^i(A^\bullet) \to H^i(B^\bullet)$ are isomorphisms for all i.

We remark that $f\colon A^\bullet \to 0^\bullet$ is a quasi-isomorphism if and only if $A^\bullet$ is exact. The forward implication holds by definition, while the reverse comes from writing down the only possible map of chain complexes, which is the zero map.

You may have expected this, but we defined quasi-isomorphism because we will only consider objects up to this equivalence. It is straightforward to see that quasi-isomorphism is an equivalence relation, which allows us to take a quotient.

Definition 2.7 (Derived category)**.** Let $\mathcal{A}$ be an abelian category. The **derived category** of $\mathcal{A}$, $D(\mathcal{A})$ is the category of chain complexes where $A^\bullet$ and $B^\bullet$ are identified if there exists a quasi-isomorphism between them.

Example 2.8 (Natural subcategories of $D(\mathcal{A})$)**.** Much like $\mathrm{Ch}(\mathcal{A})$, we have natural subcategories. $D^{\geq 0}(\mathcal{A})$ is the category of complexes which are exact left of 0, i.e. $H^i(A^\bullet) = 0$ for $i < 0$. Similarly, we have the bounded derived category $D^b(\mathcal{A})$, in which all sequences are exact away from a finite set of indices.

2.2 Resolutions, derived functors, and the long exact sequence

There will be a lot of details to check in this section. I will try to check most of them. Sometimes I may assume we are in the category of R-modules for convenience of proof. At the very least, I will say what we are asserting, and what needs checking.

For the rest of the section, assume we are working in an abelian category $\mathcal{A}$ with enough injectives, such as the category of abelian groups.

Definition 2.9 (Injective resolution)**.** Let A be an object of $\mathcal{A}$. An **injective resolution** for A is a complex $I^\bullet$ in $\mathrm{Ch}^{\geq 0}(\mathcal{A})$ such that

$$0 \to A \to I^0 \to I^1 \to I^2 \to \cdots$$

is exact.

This can be rephrased in the derived category as asking for the chain complex

$$\cdots \to 0 \to A \to 0 \to \cdots$$

with A in the 0-th position to be quasi-isomorphic to a chain complex of injective objects

$$0 \to I^0 \to I^1 \to \cdots$$

which we write in shorthand as $A \simeq I^\bullet$.

Actually constructing such a complex isn't so bad, as long as we have enough injectives. By assumption, there exists an I^0 such that $A \hookrightarrow I^0$. Then the quotient object I^0/A has an injective map to some injective I^1. This gives a map $I^0 \to I^1$ with kernel equal to the image of A in I^0. We then repeat the process by embedding $I^1/\operatorname{im}(I^0)$ into an injective I^2, given a map $I^1 \to I^2$ which makes the sequence exact at I^1.

Definition 2.10 (Projective resolution)**.** A **projective resolution** for A is dual to the injective resolution, in that it is a sequence of projective modules $P^\bullet$ such that

$$\cdots \to P^{-2} \to P^{-1} \to P^0 \twoheadrightarrow A \to 0$$

is exact.

Again, we observe that this coincides with asking there to be a quasi-isomorphism between

$$\cdots \to 0 \to A \to 0 \to \cdots$$

and the following chain complex of projective objects

$$\cdots \to P^{-1} \to P^0 \to 0 \to \cdots$$

which we write as $P^\bullet \simeq A$ for short. If $\mathcal{A}$ has enough projectives, we will similarly find that every object has a projective resolution.

Definition 2.11 (Derived functors)**.** Let $F\colon \mathcal{A} \to \mathcal{B}$ be an additive covariant functor of abelian categories $\mathcal{A}, \mathcal{B}$. If F is left exact and $\mathcal{A}$ has enough injectives, we define the **right derived functors** $R^iF\colon \mathcal{A} \to \mathcal{B}$ as

$$R^iF(A) = h^i(F(I^\bullet)),$$

where $I^\bullet$ is an injective resolution for A and h^i indicates that we take the i-th cohomology of the complex $F(I^\bullet)$.

Similarly, if F is right exact and $\mathcal{A}$ has enough projectives, we define the **left derived functors** $L^iF\colon \mathcal{A} \to \mathcal{B}$ as

$$L^iF(A) = h^i(F(P^\bullet))$$

where $P^\bullet$ is a projective resolution for A.

There is a lot to check here, namely that these are well defined without regard to choice of injective/projective resolution, and that they are appropriately functorial. Before we get there, let's just observe a few things and state some results that indicate why you should care.

Example 2.12 ($R^0F(A) = F(A)$)**.** If we are calling these *derived* functors, we had better see some connection with the original functor F. Recall that $0 \to A \to I^0 \to I^1$ is exact, so since F is left exact, so is

$$0 \to F(A) \to F(I^0) \to F(I^1).$$

Then

$$\ker(F(I^0) \to F(I^1)) / \operatorname{im}(0 \to F(I^0)) = F(A),$$

so $R^0F(A) = F(A)$. That is, the 0-th right derived functor is just the original functor. The same story is true for L^0F when F is right exact.

Proposition 2.13. *Let F be a left exact covariant functor and suppose $\mathcal{A}$ has enough injectives. Let*

$$0 \to A \to B \to C \to 0$$

be a short exact sequence. Then the right derived functors give a long exact sequence:

$$\begin{tikzcd}
0 \arrow[r] & F(A) \arrow[r] & F(B) \arrow[r] & F(C) \arrow[dll, rounded corners] \\
& R^1F(A) \arrow[r] & R^1F(B) \arrow[r] & R^1F(C) \arrow[dll, rounded corners] \\
& R^2F(A) \arrow[r] & R^2F(B) \arrow[r] & R^2F(C) \arrow[r] & \cdots
\end{tikzcd}$$

We now state and prove several lemmas necessary to proving the existence of the derived functors and the long exact sequence of Theorem 2.13. We focus on the right derived functors of a left exact functor F. Dualized arguments work in the case of left derived functors of a right exact functor.

2.3 Homotopies

Definition 2.14 (Homotopy)**.** A map of chain complexes $f\colon A^\bullet \to B^\bullet$ is **null homotopic** if there exist maps $h^i\colon A^i \to B^{i-1}$ such that

$$f^i = h^{i+1}d_A^i + d_B^{i-1}h^i$$

for all i. We will often write $f = hd + dh$ for short. A picture might help:

$$\begin{tikzcd}
\cdots \arrow[r] & A^{-1} \arrow[r, "d_A^{-1}"] \arrow[d] & A^0 \arrow[r, "d_A^0"] \arrow[d, "f^0"] \arrow[dl, "h^0"'] & A^1 \arrow[r] \arrow[d] \arrow[dl, "h^1"'] & \cdots \\
\cdots \arrow[r] & B^{-1} \arrow[r, "d_B^{-1}"'] & B^0 \arrow[r, "d_B^0"'] & B^1 \arrow[r] & \cdots
\end{tikzcd}$$

Two maps of chain complexes $f, g\colon A^\bullet \to B^\bullet$ are said to be **homotopic** if $f - g$ is null homotopic. We write $f \sim g$ for homotopic maps.

The most important feature of homotopic maps is that they induce the same maps on cohomology. Recall that $f\colon A^\bullet \to B^\bullet$ induces a map $H^i(A^\bullet) \to H^i(B^\bullet)$ for all i.

Lemma 2.15. *A null homotopic map f induces the zero map $H^i(A^\bullet) \to H^i(B^\bullet)$ for all i. As a corollary, if $f \simeq g$ then their induced maps are the same.*

Proof. If we are permitting working with elements directly, then it is easy to check that $f = dh + hd$ induces the zero map on cohomology. Then since taking cohomology is an additive functor, if $f - g$ induce the zero map, then f and g must have each induced the same map. □

Getting back to the setting of injective resolutions, we need to show that it didn't matter which injective resolution we chose. This is where homotopies will help us. Suppose we have injective resolutions $A \to I^\bullet$ and $B \to J^\bullet$, along with a map $A \to B$:

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d] & I^0 \arrow[r] & I^1 \arrow[r] & I^2 \arrow[r] & \cdots \\
0 \arrow[r] & B \arrow[r] & J^0 \arrow[r] & J^1 \arrow[r] & J^2 \arrow[r] & \cdots
\end{tikzcd}.$$

By the definition of injective object, we can extend our map $A \to B \to J^0$ to a map $I^0 \to J^0$

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d] & I^0 \arrow[r] \arrow[d] & I^1 \arrow[r] & I^2 \arrow[r] & \cdots \\
0 \arrow[r] & B \arrow[r] & J^0 \arrow[r] & J^1 \arrow[r] & J^2 \arrow[r] & \cdots
\end{tikzcd}.$$

The next step is not so easy, since $I^0 \to I^1$ is not necessarily injective. However, $I^0/\operatorname{im} A \to I^1$ is injective, and we have a map $I^0/\operatorname{im} A \to J^0/\operatorname{im} B \to J^1$, which induces a map $I^1 \to J^1$ by the definition of injective object. Repeating this process, we get maps $I^n \to J^n$ for all n. Recognizing that zeros extend to the left, we actually get a map of chain complexes

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & I^0 \arrow[r] \arrow[d] & I^1 \arrow[r] \arrow[d] & I^2 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & B \arrow[r] & J^0 \arrow[r] & J^1 \arrow[r] & J^2 \arrow[r] & \cdots
\end{tikzcd}.$$

Now let's specialize $B = A$ and take the identity map $A \to A$. Then the same process gives more maps $J^n \to I^n$, so we have

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & I^0 \arrow[r] \arrow[d] & I^1 \arrow[r] \arrow[d] & I^2 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & J^0 \arrow[r] \arrow[d] & J^1 \arrow[r] \arrow[d] & J^2 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & A \arrow[r] & I^0 \arrow[r] & I^1 \arrow[r] & I^2 \arrow[r] & \cdots
\end{tikzcd}.$$

Let's call this composition of chain complex maps f. Note that the identity is always a map of chain complexes, so we have a diagram

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & A \arrow[r] \arrow[d, "\mathrm{id}"] & I^0 \arrow[r] \arrow[d, shift right, "\mathrm{id}"'] \arrow[d, shift left, "f^0"] & I^1 \arrow[r] \arrow[d, shift right, "\mathrm{id}"'] \arrow[d, shift left, "f^1"] & I^2 \arrow[r] \arrow[d, shift right, "\mathrm{id}"'] \arrow[d, shift left, "f^2"] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & A \arrow[r] & I^0 \arrow[r] & I^1 \arrow[r] & I^2 \arrow[r] & \cdots
\end{tikzcd}$$

which is starting to evoke feelings of homotopy. In fact, it would be really nice if we could define a homotopy, because this will be enough to show that the derived functor is independent of the resolution we take.

Starting at $A \to A$, we only have one choice for $h^{-1}\colon A \to 0$, which is the zero map. Then we can set $h^0\colon I^0 \to A$ to be the zero map as well, and check that $\mathrm{id} - \mathrm{id} = 0d + d0$. A lemma takes care of extending this.

Lemma 2.16. *Suppose $I^\bullet \to J^\bullet$ is a map of chain complexes and we have homotopies $I^n \to J^{n-1}$ defined to the left of some index, say 0, such that I^n, J^n are injective for $n \geq 0$. Then we can extend the homotopies to the right and $I^\bullet \to J^\bullet$ is null homotopic.*

Proof. The setup mirrors what we have:

$$\begin{tikzcd}
\cdots \arrow[r] & I^{-1} \arrow[r, "d"] \arrow[d] & I^0 \arrow[r, "d"] \arrow[d, "f"] \arrow[dl, "h"] & I^1 \arrow[r, "d"] \arrow[d] & \cdots \\
\cdots \arrow[r] & J^{-1} \arrow[r, "d"'] & J^0 \arrow[r, "d"'] & J^1 \arrow[r] & \cdots
\end{tikzcd}$$

where I^n, J^n are injective for $n \geq 0$. We would like $f = dh + hd$ for an appropriate $h\colon I^1 \to J^0$, but a priori we don't know what this should be. But, we have $f - dh\colon I^0 \to J^0$ factors through the quotient $I^0/\operatorname{im} I^{-1}$, since h is a homotopy at the -1-th slot (do a quick chase if you don't believe this), which injects into I^1, so we get a map $I^1 \to J^0$ so that $f - dh - hd$. Continuing inductively, we see that we can extend to a homotopy. $\square$

Now we must recognize that an additive functor respects homotopies, so applying our left exact additive functor produces a homotopy $\mathrm{id}_{F(I^\bullet)} \sim F(f)$. This gives that the induced maps on cohomology *between* our different injective resolutions (after the functor is applied) are actually isomorphisms! That is,

$$R^iF(A) = H^i(F(I^\bullet)) \simeq H^i(F(J^\bullet)) = R^iF(A)$$

for any choice of injective resolution, i.e. R^iF is well defined.

In fact, the same reasoning shows that R^iF is a covariant additive functor. Once we have a map $A \to B$, and we extend to maps of injective resolutions, and apply F, we get an induced map on homology, and this is well defined by the above arguments. We have now proven the following theorem.

Theorem 2.17. *Given a left exact additive covariant functor $F\colon \mathcal{A} \to \mathcal{B}$ where $\mathcal{A}$ has enough injectives, the right derived functors $R^iF\colon \mathcal{A} \to \mathcal{B}$, given by*

$$R^iF(A) = h^i(F(I^\bullet))$$

for an injective resolution $A \to I^\bullet$, are well defined covariant additive functors for $i \geq 0$, and $R^0F \simeq F$.

There is of course an analogue of Theorem 2.17 for the left derived functors of a right exact functor. As mentioned earlier, it is defined from a projective resolution and the arrows go the other way, but morally the justifications are exactly the same.

2.4 The long exact sequence

Now we need to justify Theorem 2.13 – that we can make a long exact sequence using the right derived functors. To do this, we will need a lemma about abelian categories.

Lemma 2.18. *Our covariant additive functor $\mathcal{A} \to \mathcal{B}$ preserves biproducts. That is, $F(A \oplus B) \simeq F(A) \oplus F(B)$ and in particular*

$$0 \to F(A) \to F(A \oplus B) \to F(B) \to 0$$

is exact, where the maps are induced by the inclusion $A \to A \oplus B$ and projection $A \oplus B \to B$.

Proof. That $F(A \oplus B) \simeq F(A) \oplus F(B)$ is evident by writing down the universal properties, as the maps to (and from) $A \oplus B$ to A and B give maps $F(A \oplus B)$ to $F(A)$ and $F(B)$, and hence to (and from) $F(A) \oplus F(B)$.

Since F is additive, it preserves zero maps and identity maps, so thinking through the definitions and universal properties shows that the inclusion $F(A) \to F(A \oplus B)$ is the kernel of the projection $F(A \oplus B) \to F(B)$, and vice versa. Thus exactness is preserved. □

With this in hand, we need an appropriate choice of injective resolutions. As usual, let

$$0 \to A \to B \to C \to 0$$

in our abelian category $\mathcal{A}$, which we assume to have enough injectives. Suppose $A \to I^\bullet$ and $C \to J^\bullet$ are injective resolutions. We now use a lemma, called the "horseshoe lemma" in [Wei94].

Lemma 2.19. *Let the setup be as above. Then $B \to I^\bullet \oplus J^\bullet$ is an injective resolution and*

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A \arrow[r] \arrow[d] & I^0 \arrow[r] \arrow[d] & I^1 \arrow[r] \arrow[d] & \cdots \\
0 \arrow[r] & B \arrow[r] \arrow[d] & I^0 \oplus J^0 \arrow[r] \arrow[d] & I^1 \oplus J^1 \arrow[r] \arrow[d] & \cdots \\
0 \arrow[r] & C \arrow[r] \arrow[d] & J^0 \arrow[r] \arrow[d] & J^1 \arrow[r] \arrow[d] & \cdots \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

commutes and is exact along the rows and columns, with the maps $I^n \to I^n \oplus J^n \to J^n$ being given by the natural inclusion and projection.

Proof. We start without the middle row, so we have

$$\begin{tikzcd}
 & 0 \arrow[d] & & & \\
0 \arrow[r] & A \arrow[r] \arrow[d] & I^0 \arrow[r] & I^1 \arrow[r] & \cdots \\
 & B \arrow[d] & & & \\
0 \arrow[r] & C \arrow[r] \arrow[d] & J^0 \arrow[r] & J^1 \arrow[r] & \cdots \\
 & 0 & & &
\end{tikzcd}$$

First, we use the property that I^0 is injective to get that there exists a map $f\colon B \to I^0$. Of course, there is a map $g\colon B \to J^0$ by composing. This allows us to define $B \to I^0 \oplus J^0$ by $f \oplus g$. It is straightforward to check that the natural inclusion and projection commute, and they are exact, giving the second column.

It remains to check that $B \to I^0 \oplus J^0$ is an inclusion. For this, we recognize that if $g(b) = 0$ for some $b \in B$, then $b \mapsto 0$ in C since $C \to J^0$ is injective. But then b is in the image of A, and since $A \to I^0$ is an inclusion, the only way b is in the kernel is if $b = 0$.

At this point we have

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & A \arrow[r] \arrow[d] & I^0 \arrow[r] \arrow[d] & I^1 \arrow[r] & \cdots \\
0 \arrow[r] & B \arrow[r] \arrow[d] & I^0 \oplus J^0 \arrow[d] & & \\
0 \arrow[r] & C \arrow[r] \arrow[d] & J^0 \arrow[r] \arrow[d] & J^1 \arrow[r] & \cdots \\
 & 0 & 0 & &
\end{tikzcd}$$

with exactness everywhere.

To get the next column, we take quotients. Note that

$$0 \to A \to I^0 \to I^0/A \to 0$$

is exact, and similarly for B and C. Of course I^0/A is the cokernel of the inclusion $A \to I^0$, so the snake lemma gives an exact sequence

$$0 \to I^0/A \to (I^0 \oplus J^0)/B \to J^0/C \to 0.$$

By the definition of injective resolution, we have $I^0/A \to I^1$ and $J^0/C \to J^1$ are injections, so

$$
\begin{tikzcd}
 & 0 \arrow[d] & & & \\
0 \arrow[r] & I^0/A \arrow[r] \arrow[d] & I^1 \arrow[r] & I^2 \arrow[r] & \cdots \\
 & (I^0 \oplus J^0)/B \arrow[d] & & & \\
0 \arrow[r] & J^0/C \arrow[r] \arrow[d] & J^1 \arrow[r] & J^2 \arrow[r] & \cdots \\
 & 0 & & &
\end{tikzcd}
$$

and we are back in the earlier scenario! Inducting gives us the full desired diagram, and shows $I^\bullet \oplus J^\bullet$ is an injective resolution for B that has the desired compatibility. □

With this in hand, we are ready to prove the long exact sequence of right derived functors. Given $0 \to A \to B \to C \to 0$, we have an appropriate injective resolution, from Lemma 2.19. We need only apply the functor F and take cohomologies.

Proof of the long exact sequence in Theorem 2.13. Applying F to the diagram in the statement of Lemma 2.19, we get the diagram

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & F(A) \arrow[r] \arrow[d] & F(I^0) \arrow[r] \arrow[d] & F(I^1) \arrow[r] \arrow[d] & \cdots \\
0 \arrow[r] & F(B) \arrow[r] \arrow[d] & F(I^0) \oplus F(J^0) \arrow[r] \arrow[d] & F(I^1) \oplus F(J^1) \arrow[r] \arrow[d] & \cdots \\
0 \arrow[r] & F(C) \arrow[r] & F(J^0) \arrow[r] \arrow[d] & F(J^1) \arrow[r] \arrow[d] & \cdots \\
 & & 0 & 0 &
\end{tikzcd}
$$

where the columns are exact. Note we are using left exactness and Lemma 2.18 for exactness of the columns. The rows, of course are not exact, and the homology of the rows is what we need to compute. All of this is well defined, since we know it doesn't matter which injective resolution we take, as shown in the previous section.

Applying the snake lemma once to two adjacent columns we get

$$0 \to \ker F(d_A) \to \ker F(d_B) \to \ker F(d_C)$$

is exact. Here we are using d_A to indicate the boundary map $F(I^n) \to F(I^{n+1})$, and similarly for d_B, d_C. The snake lemma also shows an exact sequence of quotients

$$F(I^n)/\operatorname{im} F(I^{n-1}) \to F(I^n \oplus J^n)/\operatorname{im} F(I^{n-1} \oplus J^{n-1}) \to F(J^n)/\operatorname{im} F(J^{n-1}) \to 0.$$

We now have the diagram

$$\begin{tikzcd}
 & F(I^n)/\operatorname{im} F(I^{n-1}) \arrow[r] \arrow[d] & F(I^n \oplus J^n)/\operatorname{im} F(I^{n-1} \oplus J^{n-1}) \arrow[r] \arrow[d] & F(J^n)/\operatorname{im} F(J^{n-1}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \ker F(d_A) \arrow[r] & \ker F(d_B) \arrow[r] & \ker F(d_C) &
\end{tikzcd}$$

with exact rows, where the vertical maps are induced by the map $F(I^n) \to F(I^{n+1})$, since the image of $F(I^{n-1})$ is sent to zero. (Note that here the level of the boundary maps actually does matter – here we mean $F(d_A)\colon F(I^{n+1}) \to F(I^{n+2})$.)

The point is that the kernel of the leftmost vertical map is $h^n(F(I^\bullet))$, and the cokernel of the leftmost vertical map is $h^{n+1}(F(I^\bullet))$. But these were our definitions of the derived functors $R^nF(A)$ and $R^{n+1}F(A)$! So a final application of the snake lemma gives us that

$$R^nF(A) \to R^nF(B) \to R^nF(C) \to R^{n+1}F(A) \to R^{n+1}F(B) \to R^{n+1}F(C)$$

is exact. Applying this for all $n \geq 0$ gives the long exact sequence described in Theorem 2.13. $\square$

2.5 Naturality

The long exact sequence of Proposition 2.13 is natural in the following sense. Suppose we have a map of short exact sequences, i.e. a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & 0.
\end{tikzcd}$$

Then the long exact sequences associated to a derived functor have maps between them.

Proposition 2.20. *Let F be a left exact covariant functor and suppose $\mathcal{A}$ has enough injectives. Suppose further that we have a map of exact sequences, as above. Then the long exact sequence of Proposition 2.13 is natural in the sense that we have a commutative diagram*

$$\begin{tikzcd}
\cdots \arrow[r] & R^{n-1}F(C) \arrow[r] \arrow[d] & R^nF(A) \arrow[r] \arrow[d] & R^nF(B) \arrow[r] \arrow[d] & R^nF(C) \arrow[r] \arrow[d] & R^{n+1}F(A) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & R^{n-1}F(C) \arrow[r] & R^nF(A') \arrow[r] & R^nF(B') \arrow[r] & R^nF(C') \arrow[r] & R^{n+1}F(A') \arrow[r] & \cdots
\end{tikzcd}$$

where the rows are precisely the long exact sequences from Proposition 2.13.

Proof. By Proposition 2.13, we obtain the rows which are exact. It's straightforward to see that the middle part,

$$\begin{tikzcd}
R^nF(A) \arrow[r] \arrow[d] & R^nF(B) \arrow[r] \arrow[d] & R^nF(C) \arrow[d] \\
R^nF(A') \arrow[r] & R^nF(B') \arrow[r] & R^nF(C')
\end{tikzcd}$$

commutes, as we have maps on the complexes $F(I_A^\bullet) \to F(I_{A'}^\bullet)$ which induce canonical maps on cohomology (see Proposition 2.5), giving the vertical maps.

It's somewhat more subtle to see that the squares involving the boundary map commute. One way is to use the idea in the proof of Theorem 2.13 to give a map $I_C^0 \to I_A^1$ (and similarly for C', A'), extending to a map of complexes $I_C[-1]^\bullet \to I_A^\bullet$, where $I_C[-1]^\bullet$ denotes the complex $I_C^\bullet$ shifted to the left by one. Invoking Proposition 2.5 on the compositions $I_C[-1]^\bullet \to I_A^\bullet \to I_{A'}^\bullet$ and $I_C[-1]^\bullet \to I_{C'}[-1]^\bullet \to I_{A'}^\bullet$, we obtain the desired commutative square. $\square$

2.6 δ-functors

The derived functors considered in this section satisfy a sort of universal property. We describe this following [Har77, §III.1] filling in some details in the proofs as needed.

Definition 2.21 (δ-functor)**.** A **(covariant) δ-functor** is a collection $T = (T^i)_{i\geq 0}$ of covariant additive functors $T^i \colon \mathcal{A} \to \mathcal{B}$ and, for every short exact sequence $0 \to A \to B \to C \to 0$, morphisms

$$\delta^i \colon T^i(C) \to T^{i+1}(A)$$

for $i \geq 0$ satisfying (i) and (ii) below.

(i) *Long exact sequence:* For every short exact sequence in $\mathcal{A}$, the δ^i morphisms form the long exact sequence below.

$$\begin{tikzcd} 0 \arrow[r] & T^0(A) \arrow[r] & T^0(B) \arrow[r] & T^0(C) \arrow[dll, "\delta^0"'] \\ & T^1(A) \arrow[r] & T^1(B) \arrow[r] & T^1(C) \arrow[dll, "\delta^1"'] \\ & T^2(A) \arrow[r] & T^2(B) \arrow[r] & T^2(C) \arrow[r] & \cdots \end{tikzcd}$$

(ii) *Naturality:* For every morphism of short exact sequences

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & 0 \end{tikzcd}$$

the associated diagram of δ^i maps commutes.

$$\begin{tikzcd} T^i(C) \arrow[r, "\delta^i"] \arrow[d] & T^{i+1}(A) \arrow[d] \\ T^i(C') \arrow[r, "\delta^i"'] & T^{i+1}(A') \end{tikzcd}$$

These are precisely the properties enjoyed by our derived functors, as proven in Theorem 2.13 and Proposition 2.20. We will soon see that they are universal with respect to these properties, which we make precise below.

Definition 2.22 (Universal δ-functor)**.** Suppose $U = (U^i)_{i\geq 0}$ is a δ-functor $\mathcal{A} \to \mathcal{B}$ as above. We say U is **universal** if given another δ-functor $T = (T^i)_{i\geq 0}$ and a natural transformation of functors $f^0 \colon U^0 \to T^0$, there exist unique natural transformations $f^i \colon U^i \to T^i$ for $i > 0$ which commute with the δ^i maps:

$$\begin{tikzcd} U^i(C) \arrow[r, "\delta^i"] \arrow[d, "f^i"'] & U^{i+1}(A) \arrow[d, "f^{i+1}"] \\ T^i(C) \arrow[r, "\delta^i"'] & T^{i+1}(A). \end{tikzcd}$$

Remark 2.23. An easy consequence, via the usual universality arguments, is that if a universal δ-functor exists with $U^0 = F$ for some fixed F, then it is unique up to unique natural isomorphism. We always have the identity transformations $f^i \colon U^i \to U^i$, so the uniqueness forces any other universal δ-functor U' with $(U')^0 = U^0 = F$ to have $(U')^i$ naturally isomorphic to U^i for all i.

Before we prove that our right derived functors are in fact universal δ-functors, we need a way to recognize the latter.

Definition 2.24 (Effaceable)**.** An additive functor $F \colon \mathcal{A} \to \mathcal{B}$ is **effaceable** if for all objects A in $\mathcal{A}$, there exists a monomorphism $u \colon A \hookrightarrow M$ such that $F(u) = 0$.

The dual version is that F is **coeffaceable** if for all A there exists an epimorphism $p \colon A \twoheadrightarrow P$ such that $F(p) = 0$.

Theorem 2.25. *Suppose U is a δ-functor and U^i is effaceable for all $i > 0$. Then U is universal.*

Before proving Theorem 2.25, we need some basic facts about pushouts, which are dual to fiber products.

Definition 2.26 (Pushout)**.** Given maps $f \colon A \to B$ and $g \colon A \to C$, a **pushout** is an object P satisfying the universal property that for all test objects T and commutative outer diagrams below, there exists a unique morphism $P \to T$ making the full diagram commute.

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d] \arrow[ddr, bend left] & \\
C \arrow[r] \arrow[drr, bend right] & P \arrow[dr, dashed, "\exists!"] & \\
 & & T
\end{tikzcd}$$

We might denote this pushout $P = B \coprod_A C$, where the maps f and g are understood.

Lemma 2.27. *Let $\mathcal{A}$ be an abelian category. Then given any objects A, B, C and morphisms f, g as in Definition 2.26, the pushout exists and is given explicitly by*

$$P = B \coprod_A C = (B \times C)/\operatorname{im}(f \times -g).$$

Further, if $f \colon A \hookrightarrow B$ is a monomorphism, then A is identified with the fiber product $B \times_P C$ and the map $C \to P$ is also a monomorphism.

Proof. In an abelian category, finite (co)products exist, so we can define the product $B \times C$, as well as the image $\operatorname{im}(f \times -g)$ of $(f \times -g) \colon A \to B \times C$ which sends $a \mapsto (f(a), -g(a))$. Letting $P = (B \times C)/\operatorname{im}(f \times -g)$, we see that $(f(a), 0)$ and $(0, g(a))$ are identified in the quotient, since their difference is in the image of $f \times -g$. Thus the inner square in Definition 2.26 commutes.

To see this satisfies the universal property, suppose we have an object T and a commutative outer diagram as in Definition 2.26. Forgetting about A for the moment, the universal property of products gives a unique map $B \times C \to T$. Since the outer diagram commutes, we have that $\operatorname{im}(f \times -g) \subset B \times C$ is trivial under the map to T. Hence there exists a unique map $P \to T$ commuting with the diagram by the universal property of quotients, described pictorially below.

$$\begin{tikzcd}
A \arrow[r] & \operatorname{im}(f, -g) \arrow[r] \arrow[drr, "0"'] & B \times C \arrow[r] \arrow[dr] & P \arrow[r] \arrow[d, dashed] & 0 \\
 & & & T &
\end{tikzcd}$$

For the final statement, see [Sta22, Lemma 05PK]. We recall the details of that argument here. Suppose f is a monomorphism and suppose we have the solid arrows in the diagram below.

$$\begin{tikzcd}
T \arrow[dr, dashed] \arrow[drr, bend left, "f'"] \arrow[ddr, bend right, "g'"'] & & \\
& A \arrow[r, hook, "f"] \arrow[d, "g"'] & C \arrow[d, "h"] \\
& B \arrow[r, "k"'] & P
\end{tikzcd}$$

Since f is a monomorphism, $(f \times -g)\colon A \to B \times C$ is also a monomorphism. Therefore the sequence

$$0 \to A \xrightarrow{f \times -g} B \times C \to P \to 0$$

is also exact on the left, and $f \times -g$ is the kernel map of the projection $B \times C \to P$.

On the other hand, since the outer diagram commutes we have $hf' = kg'$, so the map $hf' \times -kg'\colon T \to B \times C \to P$ is the zero map. By the universal property of kernels, we find a unique map $T \to A$ as shown below.

$$\begin{tikzcd}
& & T \arrow[dl, dashed, bend right] \arrow[d, "f' \times -g'"'] \arrow[dr, "0"] & & \\
0 \arrow[r] & A \arrow[r] & B \times C \arrow[r, "h \times k"'] & P \arrow[r] & 0
\end{tikzcd}$$

That the dashed arrows above exist and are unique shows that $A = B \times_P C$ and the diagram is Cartesian.

Finally, we claim that since the diagram is Cartesian, we have $\ker f \simeq \ker k = 0$, and thus k is a monomorphism, as desired. To see this, consider the diagram below.

$$\begin{tikzcd}
\ker k \arrow[d, dashed, "\exists!_2"] \arrow[dr] \arrow[drr, bend left, "0"] \arrow[dd, bend right, "\mathrm{id}"'] & & \\
\ker f \arrow[r] \arrow[d, dashed, "\exists!_1"] & A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "h"] \\
\ker k \arrow[r] & C \arrow[r, "k"'] & P
\end{tikzcd}$$

The first dashed arrow exists and is unique by the universal property of $\ker k$. The second arises from recognizing that taking the zero map $\ker k \to B$ gives $h \circ 0 = 0$, which naturally agrees with $\ker k \to P$. Hence we have a (unique) map $\ker k \to A$ by the Cartesianness of the diagram, and the universal property of $\ker f$ gives the second dashed arrow. Thus we have exhibited an isomorphism $\ker k \simeq \ker f$, and since f was assumed to be a monomorphism, so must be k. □

Proof of Theorem 2.25. Fix another δ-functor $T = (T^i)_{i \geq 0}$ and a natural transformation $f^0\colon U^0 \to T^0$. We will construct f^1, see that it is unique, check it is a natural transformation, and justify that the diagram

$$\begin{tikzcd}
U^0(C) \arrow[r, "\delta^0"] \arrow[d, "f^0"'] & U^1(A) \arrow[d, "f^1"] \\
T^0(C) \arrow[r, "\delta^0"'] & T^1(A).
\end{tikzcd}$$

commutes as required. The arguments for f^i with $i > 1$ follow similarly; see also [Sta22, Lemma 010T].

Construction and uniqueness of f^1. Let A be an object of $\mathcal{A}$. By the effaceability of U^1, there exists a monomorphism $u\colon A \hookrightarrow M$ such that $U^1(u) = 0$. We have an exact sequence

$$0 \to A \xrightarrow{u} M \to \operatorname{coker} u \to 0,$$

or put more compactly, $\operatorname{coker} u = M/A$.

Using the δ-functoriality of U and T, we have the diagram below.

$$\begin{tikzcd}
{} \arrow[r] & U^0(M) \arrow[r] \arrow[d, "f^0"] & U^0(M/A) \arrow[r, two heads, "\delta"] \arrow[d, "f^0"] & U^1(A) \arrow[r, "U^1(u)=0"] \arrow[d, dashed, "f^1"] & U^1(M) \arrow[r] & {} \\
{} \arrow[r] & T^0(M) \arrow[r] & T^0(M/A) \arrow[r, "\delta"'] & T^1(A) \arrow[r] & T^1(M) \arrow[r] & {}
\end{tikzcd}$$

The solid downward arrows f^0 come from our natural transformation $f^0\colon U^0 \to T^0$. To construct the dashed morphism, we first recognize that $U^1(u) = 0$ by effaceability, so the exactness of the top row implies that $\delta\colon U^0(M/A) \to U^1(A)$ is surjective, or equivalently we identify

$$U^1(A) = \operatorname{coker}\left(U^0(M) \to U^0(M/A)\right).$$

The universal property of cokernels states that there exists a unique dashed map making the middle square commute.

Note that we made a choice here in M and u. This will be resolved shortly, as we prove that f^1 is a natural transformation.

Natural transformation. Suppose $g\colon A \to A'$ is a morphism in $\mathcal{A}$. To show f^1 is a natural transformation, we must show that

$$\begin{tikzcd}
U^1(A) \arrow[r, "U^1(g)"] \arrow[d, "f^1"'] & U^1(A') \arrow[d, "f^1"] \\
T^1(A) \arrow[r, "T^1(g)"'] & T^1(A')
\end{tikzcd}$$

commutes. Chose monomorphisms $u\colon A \hookrightarrow M$ and $u'\colon A' \hookrightarrow M'$ such that $U^1(u), U^1(u')$ are both zero, by effaceability. Consider the diagram below, where the pushout $M \coprod_A M'$ exists by Lemma 2.27 and has $M' \hookrightarrow M \coprod_A M'$ and the dashed arrow exists by the universal property of quotients.

$$\begin{tikzcd}
A \arrow[rr, hook, "u"] \arrow[d, "g"'] & & M \arrow[r] \arrow[d] & M/A \arrow[d, dashed] \\
A' \arrow[r, hook, "u'"'] & M' \arrow[r, hook] & M \coprod_A M' \arrow[r] & (M \coprod_A M')/A'
\end{tikzcd}$$

Let $v\colon M' \hookrightarrow M \coprod_A M'$. We have that $U^1(v \circ u') = U^1(v) \circ U^1(u') = 0$. Thus we can replace M' by $M \coprod_A M'$ when defining $f^1\colon U^1(A') \to T^1(A')$. Doing so gives us the cube below.

$$
\begin{tikzcd}
& U^0((M\coprod_A M')/A') \arrow[rr, two heads, "\delta"] \arrow[dd, bend left, "f^0"] & & U^1(A') \arrow[dd, bend left, "f^1"] \\
U^0(M/A) \arrow[ur] \arrow[rr, two heads, "\delta"] \arrow[dd, bend left, "f^0"'] & & U^1(A) \arrow[ur, "U^1(g)"] \arrow[dd, bend left, "f^1"] & \\
& T^0((M\coprod_A M')/A') \arrow[rr, "\delta"] & & T^1(A') \\
T^0(M/A) \arrow[ur] \arrow[rr, "\delta"'] & & T^1(A) \arrow[ur, "T^1(g)"'] &
\end{tikzcd}
$$

The top and bottom squares commute since U, T are δ-functors. The front and back squares commute by our construction of f^1. The left square commutes since f^0 is a natural transformation of functors. Our goal is to show that the right square commutes.

The maps

$$T^1(g)\circ f^1\circ\delta, f^1\circ U^1(g)\circ\delta\colon U^0(M/A)\to T^1(A')$$

are shown to be equal by chasing the commutative square around the front, bottom, left, back, and top squares. This is equivalent to the commutativity of the diagram below.

$$
\begin{tikzcd}
U^0(M/A) \arrow[r, two heads, "\delta"] \arrow[dr, bend right] & U^1(A) \arrow[d, shift right, "f^1\circ U^1(g)"'] \arrow[d, shift left, "T^1(g)\circ f^1"] \\
& T^1(A')
\end{tikzcd}
$$

Thus since δ is an epimorphism, we have that $f^1\circ U^1(g) = T^1(g)\circ f^1$, and we have verified that $f^1\colon U^1\to T^1$ is a natural transformation.

Note that implicit here is the verification that $f^1\colon U^1(A)\to T^1(A)$ is independent of the choice of u and M. If $u'\colon A\hookrightarrow M'$ is another monomorphism with $U^1(u') = 0$, then the argument above with $A' = A$ and $g = \mathrm{id}_A$ yields that f^1 is unchanged if we replace M(or M') by the pushout $M\coprod_A M'$.

Commutativity with δ^0. All that remains is to argue that the diagram with δ^0 commutes. For this we will use the fact that U and T play nicely with maps of exact sequences. Let $0\to A\to B\to C\to 0$ be an exact sequence and suppose $u\colon A\hookrightarrow M$ such that $U^1(u) = 0$. Once again using the pushout $B\coprod_A M$, we construct the map of exact sequences below, recognizing that $U^1(A)\to U^1(B\coprod_A M)$ is the zero map.

$$
\begin{tikzcd}
0 \arrow[r] & A \arrow[rr, hook, "u"] \arrow[d, "\mathrm{id}_A"'] & & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d, dashed] & 0 \\
0 \arrow[r] & A \arrow[r, hook, "u"'] & M \arrow[r, hook] & B\coprod_A M \arrow[r] & (B\coprod_A M)/A \arrow[r] & 0
\end{tikzcd}
$$

Much like before, this yields the cube below. The front square is what we want to show commutes. The top and bottom squares commute by definition of δ-functors, the back square commutes by our construction, the left square commutes since f^0 is a natural transformation, and the right square commutes trivially since $U^1(\mathrm{id}_A)$ and $T^1(\mathrm{id}_A)$ are the respective identity morphisms.

$$
\begin{tikzcd}
 & U^0((B\coprod_A M)/A) \arrow[rr, two heads, "\delta"] \arrow[dd, bend left, "f^0"] & & U^1(A) \arrow[dd, bend left, "f^1"] \\
U^0(C) \arrow[ur] \arrow[rr, "\delta"] \arrow[dd, bend left, "f^0"] & & U^1(A) \arrow[ur, "="] \arrow[dd, bend left, "f^1"] & \\
 & T^0((B\coprod_A M)/A) \arrow[rr, "\delta"] & & T^1(A) \\
T^0(C) \arrow[ur] \arrow[rr, "\delta"'] & & T^1(A) \arrow[ur, "="'] &
\end{tikzcd}
$$

Starting from $\mathrm{id}_{T^1(A)} \circ f^1 \circ \delta$, we use the commutativity of the right, top, back, left, and bottom squares to see this is equal to $\mathrm{id}_{T^1(A)} \circ \delta \circ f^0$. Hence $f^1 \circ \delta = \delta \circ f^0$, i.e. the front square commutes.

With this as the base case, one can proceed by induction, assuming that suitable natural tranformations f^i exist for $1 \leq i \leq n$ and using the same arguments to see that a suitable f^{n+1} exists. $\square$

Corollary 2.28. *Let $F\colon \mathcal{A} \to \mathcal{B}$ be a covariant left exact functor and suppose $\mathcal{A}$ has enough injectives. Then the derived functors $RF = (R^iF)_{i\geq 0}$ form a universal δ-functor.*

Proof. That RF is a δ-functor follows from the long exact sequence in Theorem 2.13 and the naturality of Proposition 2.20. To see RF is universal, we need only show R^iF is effaceable for $i \geq 1$, by Theorem 2.25.

By the enough injectives hypothesis, for any object A in $\mathcal{A}$ we can find a monomorphism $A \hookrightarrow I$ for some injective object I. We make an injective resolution for I by taking $0 \to I \to I \to 0 \to \cdots$, i.e. $I^0 = I$ and $I^j = 0$ for $j \geq 1$. Thus for $i \geq 1$ we have

$$R^iF(I) = h^i(F(I^\bullet)) = 0$$

so $R^iF(A) \to R^iF(I)$ is the zero map and hence R^iF is effaceable. $\square$

Corollary 2.29. *Suppose that $\mathcal{A}$ has enough injectives and $U = (U^i)_{i\geq 0}$ is any universal δ-functor. Then U^0 is left exact and we have a natural isomorphism $U^i \simeq R^iU^0$ for all $i \geq 0$.*

Proof. That U^0 is left exact follows from (i) in the definition of a δ-functor. By Corollary 2.28, the right derived functors $RU^0 = (R^iU^0)_{i\geq 0}$ form a universal δ-functor. By the uniqueness of these, we have the desired natural isomorphisms. $\square$

This property is useful to prove two functorial constructions coincide. For some applications to algebraic geometry, see [Har77, §III.6-7] or [Vak17, Chapter 30], where universal δ-functors are used to prove things about Ext which are then used in the proof of Serre duality. These also come up in the proof of Proposition 4.11.

3 Group cohomology

In this section we discuss group cohomology explicitly, and how the concrete definitions are coming from the derived functors. We will also give some examples that illustrate how group cohomology is used in the world.

3.1 Inhomogeneous setup

Let G be a group with an action on an abelian group M. We denote this action by $m^\sigma \in M$ for $m \in M$, $\sigma \in G$. We will interpret this as a right action, so $m^{\sigma\tau} = (m^\sigma)^\tau$.

Suppose we have the exact sequence

$$0 \to M' \to M \to M'' \to 0.$$

We assume the maps are G-equivariant, i.e. $f(m')^\sigma = f((m')^\sigma)$, etc. Consider the fixed points of the G-action,

$$M^G = \{\, m \in M \mid m^\sigma = m \text{ for all } \sigma \in G \,\}.$$

We obtain maps $M'^G \to M^G \to M''^G$ and can check that

$$0 \to (M')^G \to M^G \to (M'')^G$$

is exact. Thus the functor (we didn't check this, but it is!) $M \mapsto M^G$ is left-exact.

Since the category of abelian groups has enough injectives, we know this functor should come with a derived functor cohomology which sets $M^G = H^0(G, M)$, but let's assume we didn't know this and try to construct a cohomology theory with similar properties.

Definition 3.1. Let $C^r(G, M)$ denote the r**-cochains**, which we define to be the set of maps $G^r \to M$. In the case of $r = 0$, we interpret $G^0 \to M$ as choosing a point M, so $C^0(G, M) = M$.

We would like to come up with some map $C^0 \to C^1$ whose kernel is precisely M^G. Luckily there is a natural choice! Send a 0-cochain m to the 1-cochain $m^\sigma - m$. This 1-cochain is the zero map if and only if $m^\sigma = m$ for all $\sigma \in G$, i.e. if m is a fixed point.

We now need to extend this to get a complex

$$\cdots \to 0 \to C^0(G, M) \to C^1(G, M) \to C^2(G, M) \to \cdots.$$

Using the map $m \mapsto m^\sigma - m$ from $C^0 \to C^1$, we see that the 0-th homology of this complex is precisely M^G, which is what we were looking for. The boundary map $C^1 \to C^2$ should then be defined so that things in the image of C^0 (which look like $m^\sigma - m$) are sent to 0 in C^2. That is, we need to systematically produce a function $f_2(\sigma, \tau)$ from a 1-cochain $f_1(\sigma)$.

In the previous case, acting by the new variable and subtracting seemed to work nicely, so let's start with that and see what happens.

$$(m^\sigma - m)^\tau = m^{\sigma\tau} - m^\tau$$

If we subtract $m^{\sigma\tau} - m$, we get $-m^\tau + m$, so it wasn't enough to subtract just one term. But adding $m^\tau - m$ does the trick! To summarize,

$$(m^\sigma - m)^\tau - (m^{\sigma\tau} - m) + (m^\tau - m) = 0.$$

This suggests the right approach is to map $f_1(\sigma)$ to

$$f_2(\sigma, \tau) = f_1(\sigma)^\tau - f_1(\sigma\tau) + f_1(\tau).$$

When a 1-cochain satisfies $f_1(\sigma)^\tau - f_1(\sigma\tau) + f_1(\tau) = 0$, this is called the *cocycle condition.*

We now need to extend this for all $r \geq 0$. This is a bit messy, but we can do it quite explicitly.

Definition 3.2. Let $f_n(\sigma_0, ..., \sigma_{n-1})$ be an n-cochain. We define the **coboundary map** $\delta\colon C^n(G, M) \to C^{n+1}(G, M)$ by

$$\delta(f_n)(\sigma_0, ..., \sigma_n) = f_n(\sigma_0, ..., \sigma_{n-1})^{\sigma_n} + \sum_{i=0}^{n-1} (-1)^{n-i} f_n(\sigma_0, ..., \sigma_i\sigma_{i+1}, ..., \sigma_n) + (-1)^{n+1} f_n(\sigma_1, ..., \sigma_n).$$

Now we can define the n**-cocycles** $C_c^n(G,M)$ to be the kernel of δ and the n**-coboundaries** $C_b^n(G,M)$ as the image of δ in $C^n(G,M)$. The i**-th group cohomology** is then

$$H^i(G,M) = C_c^i(G,M)/C_b^i(G,M),$$

or just the i-th homology of the cochain complex.

Proposition 3.3. *The cochain sequence*

$$0 \to C^0(G,M) \to C^1(G,M) \to C^2(G,M) \to \cdots$$

is a complex, with the boundary maps given in the definition above. Thus the notion of group cohomology is well defined.

Actually writing out the proof is a bit tedious and requires a lot of paper (and patience). But it can be done, and one sees that after taking $\delta \circ \delta$, terms pair up and cancel out.

Remark 3.4. I'm not sure how useful this is, but the n-th cocyle condition says

$$f_n(\sigma_0, ..., \sigma_{n-1})^{\sigma_n} + \sum_{i=0}^{n-1}(-1)^{n-i} f_n(\sigma_0, ..., \sigma_i\sigma_{i+1}, ..., \sigma_n) + (-1)^{n+1} f_n(\sigma_1, ..., \sigma_n) = 0.$$

One could interpret this as saying that for a function $G^n \to M$ to be a cocycle, the action of an element of G on it works in a prescribed way, namely given $\sigma_n \in G$ we have

$$f_n(\sigma_0, ..., \sigma_{n-1})^{\sigma_n} = -\sum_{i=0}^{n-1}(-1)^{n-i} f_n(\sigma_0, ..., \sigma_i\sigma_{i+1}, ..., \sigma_n) - (-1)^{n+1} f_n(\sigma_1, ..., \sigma_n).$$

3.2 Homogeneous setup

There is another way of formulating group cohomology that is more indicative of simplicial cohomology from topology. This also connects the derived functor formalism with the explicit cocycle and coboundary formulation above. Here we recognize that the functor $M \mapsto M^G$ is the same as $M \mapsto \mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}, M)$, because G acts trivially on $\mathbb{Z}$, so we are just picking out an element $m \in M$ that is fixed by G.

Now we recognize that the right derived functors of $\mathrm{Hom}_R(\_\_, M)$ are the same as the right derived functors of (the contravariant functor) $\mathrm{Hom}_R(N, \_\_)$. This is a fact about the Hom functor, and the proof involves some machinery with double complexes; see [Wei94, Theorem 2.7.6] for more, or the author's notes on spectral sequences [Key]. The point is that it is enough to compute a projective resolution of $\mathbb{Z}$ and work with this. In fact, we can cook up a free resolution:

$$\cdots \to \mathbb{Z}[G^2] \to \mathbb{Z}[G] \to \mathbb{Z} \to 0.$$

Here the boundary maps, denoted by d send the element

$$(g_0, ..., g_i) \mapsto \sum_{j=0}^{i}(-1)^j (g_0, ..., \widehat{g_j}, ..., g_i),$$

where $\widehat{g_j}$ indicates that g_j is removed. By extending linearly, we obtain a map from $\mathbb{Z}[G^{i+1}] \to \mathbb{Z}[G^i]$. This should be somewhat reminiscent of the notion of coboundaries the simplicial cohomology setting from topology. There are things to check of course, which we state as lemmas.

Lemma 3.5. *The group ring* $\mathbb{Z}[G^i]$ *is free as a* $\mathbb{Z}[G]$*-module for all* $i \geq 1$.

Proof. $\mathbb{Z}[G]$ is clearly free of rank 1 as a $\mathbb{Z}[G]$-module. For $i = 2$, we can take as a basis

$$\{ (1, g) \mid g \in G \}$$

and see that this forms a basis for $\mathbb{Z}[G^2]$ as a $\mathbb{Z}[G]$-module.

There are lots of choices, but more generally, we will take as our basis for $\mathbb{Z}[G^{i+1}]$ the set

$$B_{i+1} = \{ (1, g_1, g_1 g_2, ..., g_1 \cdots g_i) \mid g_1, ..., g_i \in G \} .$$

One sees that this is in fact the same set as

$$\{ (1, g_1', ..., g_i') \mid g_1', ..., g_i' \in G \} ,$$

which is a basis for $\mathbb{Z}[G^{i+1}]$. □

Lemma 3.6. *The sequence*

$$\cdots \to \mathbb{Z}[G^2] \to \mathbb{Z}[G] \to \mathbb{Z} \to 0$$

with boundary maps d is exact. That is, we have found a projective (actually free) resolution of $\mathbb{Z}$.

Proof. One can check this is exact by computing with the boundary maps. □

Applying the functor $\mathrm{Hom}_{\mathbb{Z}[G]}(\_\_, M)$, we get a complex

$$0 \to \mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}[G], M) \to \mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}[G^2], M) \to \mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}[G^3], M) \to \cdots$$

and by definition the cohomology $H^i(G, M)$ is taken to be the cohomology of this complex. We now need to understand how the maps here work, so we can get a handle on the kernels and images, which we call the *homogeneous* cochains, cocycles, and coboundaries. This is where we make use of the fact that we have a basis for $\mathbb{Z}[G^i]$ as a $\mathbb{Z}[G]$ module. Then, this boils down to the complex

$$0 \to \mathrm{Hom}_{\mathrm{Set}}(G^0, M) \to \mathrm{Hom}_{\mathrm{Set}}(G^1, M) \to \mathrm{Hom}_{\mathrm{Set}}(G^2, M) \to \cdots$$

But this is turns out to recover our earlier definitions, since $C^i(G, M) = \mathrm{Hom}_{\mathrm{Set}}(G^i, M)$! All that is left is to trace through our definitions to see what the boundary map is doing. Let's talk through this a little bit: suppose we have a map of sets $f \colon G^i \to M$. We interpret G^i as giving a $\mathbb{Z}[G]$-homomorphism $\widetilde{f} \colon \mathbb{Z}[G^{i+1}] \to M$, where $\widetilde{f}(1, g_1, g_1 g_2, ..., g_1 \cdots g_i) = f(g_1, ..., g_i)$. Since we know how to use the boundary map to give a map $d\widetilde{f} \colon \mathbb{Z}[G^{i+2}] \to M$, we can then interpret this as $df \colon G^{i+1} \to M$ by following where the basis elements go. This then recovers the definitions of the (inhomogeneous) cocycles and coboundaries.

Remark 3.7. In these notes, we're a little fast and loose about the original G-action. Sometimes it is interpreted as a right action, and other times it's written on the left. Hopefully this isn't too confusing. The point is that if written a different way, the differences in the resulting formula are merely cosmetic. In any given application, one would probably look up the defining formulas anyway.

Remark 3.8 (Choices)**.** Since we know that it didn't matter which resolution we chose, we might have decided to pick another projective resolution for $\mathbb{Z}$ above. Or perhaps, a different choice of basis for $\mathbb{Z}[G]$. Again, this leads to slightly different looking formulas for the cocycle and coboundary conditions, but we of course end up with the same cohomology. Sometimes these alternatives might be easier to compute with, or show more obvious connections to other things we care about.

3.3 Exactness

Once you are convinced that the definitions of inhomogeneous cochains, cocycles, and coboundaries match up appropriately with the homogenous ones, we are done, because we can invoke Theorem 2.13, which gives the long exact sequence for derived functor cohomology. We state this as a proposition and sketch a proof using diagram chasing

Proposition 3.9. *Let*

$$0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$$

be an exact sequence of G-modules. Then we get a long exact sequence

$$\begin{tikzcd}
0 \arrow[r] & M'^G \arrow[r] & M^G \arrow[r] & M''^G \arrow[dll, out=-30, in=150] & \\
& H^1(G, M') \arrow[r] & H^1(G, M) \arrow[r] & H^1(G, M'') \arrow[dll, out=-30, in=150] & \\
& H^2(G, M') \arrow[r] & H^2(G, M) \arrow[r] & H^2(G, M'') \arrow[r] & \cdots
\end{tikzcd}$$

in the cohomology.

Proof. We prove hands-on only the first two rows of this sequence. Really this is an application of the snake lemma (see Proposition 1.48):

$$\begin{tikzcd}
& M'^G \arrow[r] \arrow[d, hook] & M^G \arrow[r] \arrow[d, hook] & M''^G \arrow[d, hook] & \\
0 \arrow[r] & M' \arrow[r, "f"] \arrow[d, "d"] & M \arrow[r, "g"] \arrow[d, "d"] & M'' \arrow[r] \arrow[d, "d"] & 0 \\
0 \arrow[r] & C_c^1(G, M') \arrow[r] \arrow[d, two heads] & C_c^1(G, M) \arrow[r] \arrow[d, two heads] & C_c^1(G, M'') \arrow[d, two heads] & \\
& H^1(G, M') \arrow[r] & H^1(G, M') \arrow[r] & H^1(G, M') &
\end{tikzcd}$$

Since the diagram above commutes, the fixed points are the kernel of the boundary map from $M \to C_c^1(G, M)$ (i.e. the 0-cochains to the 1-cocycles), and H^1 is the cokernel of this map, we get by the snake lemma a connecting map $M''^G \to H^1(G, M')$.

The rest of the proof is perhaps better seen with the derived functor setup. Also, note that one could do a diagram chase to produce the connecting map directly, starting with $m'' \in M''^G$ and giving a 1-cocyle $G \to M'$. This requires the same chase as in the proof of the snake lemma. $\square$

3.4 Examples

Example 3.10 (finite cyclic field extensions)**.** Let L/K be a finite cyclic Galois extension of degree n. Let $G = \langle \sigma \rangle$ denote the Galois group. Then G acts on the abelian group $L^\times$, so we can ask what $H^i(G, L^\times)$ is. The zeroth cohomology group, $H^0(G, L^\times)$ is just the fixed points of the G-action. The only points of L fixed by the action of G are the points contained in K, so

$$H^0(G, L^\times) = K^\times.$$

In fact, we didn't need the cyclicity assumption at all here.

To compute H^1, we will use cyclicity. Suppose $f\colon G \to L^\times$ is an inhomogeneous cocycle. It turns out it is determined by $\sigma \mapsto a$, as we can define

$$f(\sigma^k) = f(\sigma^{k-1})^\sigma f(\sigma)$$

inductively using the cocycle condition. Note $L^\times$ is multiplicative, so our notation is somewhat different than in the exposition. Computing $f(\sigma) = f(\sigma^n\sigma)$, we find

$$a = f(\sigma) = f(\sigma^n\sigma) = f(\sigma^n)^\sigma f(\sigma) = (a^{\sigma^n} \cdots a^\sigma)a.$$

Cancelling, we see that a has norm 1, as $N(a) = \prod_{\tau\in G} a^\tau$.

Now we invoke Hilbert's 90, which states that an element $a \in L^\times$ with norm 1 can be written as b^σ/b for some $b \in L^\times$. But then $\sigma \mapsto b^\sigma/b$ is a coboundary! Hence

$$H^1(G, L^\times) = 0.$$

Remark 3.11. The above example can be generalized to arbitrary finite Galois extensions. If L/K is finite and Galois, then $H^1(\mathrm{Gal}(L/K), L^\times) = 0$.

Example 3.12 (Brauer groups)**.** The Brauer group can be interpreted in Group (or Galois) cohomology. For concreteness, let's assume K is a number field and let $\overline{\mathbb{Q}}$ denote the field of algebraic numbers, which is a separable closure of K. Taking $G = \mathrm{Gal}(\overline{\mathbb{Q}}/K)$, we have a G-action on $\overline{\mathbb{Q}}^\times$ whose fixed points are precisely K.

The Brauer group Br K, often characterized as equivalence classes of central simple algebras over K, has an interpretation as a cohomology group of the action of G on $\overline{\mathbb{Q}}^\times$. We have

$$\mathrm{Br}\ K \simeq H^2(G, \overline{\mathbb{Q}}^\times).$$

This definition allows one to define more general Brauer groups, such as those of schemes, via étale cohomology.

Example 3.13 (twists)**.** Let K be a field and X/K an object, and suppose the notion of base change makes sense. We are being deliberately vague – if you like, you may consider X to be a curve and K a number field. A **twist** is a object X'/K such that $X \simeq X'$ after base change to the algebraic (or possibly separable) closure $\overline{K}$.

We will now characterize the twists of X/K up to isomorphism over K. Basically, given $\phi\colon X \to X'$ an isomorphism defined over $\overline{K}$, we wish to know if ϕ is defined over K. Or in other words, does the diagram

$$\begin{tikzcd} X \arrow[r, "\phi"] \arrow[d, "\sigma"'] & X' \arrow[d, "\sigma"] \\ X \arrow[r, "\phi"'] & X' \end{tikzcd}$$

commute for all $\sigma \in \mathrm{Gal}(\overline{K}/K)$?

Let $\mathrm{Aut}\, X$ be the group of automorphisms of $X/\overline{K}$. We will ignore the fact that in general it is nonabelian. $G_K = \mathrm{Gal}(\overline{K}/K)$ acts by conjugation, i.e.

$$\phi^\sigma = \sigma^{-1}\phi\sigma$$

We claim that the twists of X up to K-isomorphism over K correspond to cocycles in $H^1(G_K, \mathrm{Aut}\, X)$. We will prove this is an injection, though Poonen asserts in [Poo17, 1.10] that this is often a bijection.

To start, suppose $\phi\colon X \to X'$ is a twist, that is, an isomorphism of objects over $\overline{K}$. Let $f\colon G_K \to \mathrm{Aut}\, X$ be the map that

$$f(\sigma) = \phi^{-1}\sigma^{-1}\phi\sigma.$$

We can check this satisfies the cocycle condition by computing

$$f(\sigma\tau) = \phi^{-1}\tau^{-1}\sigma^{-1}\phi\sigma\tau = (\phi^{-1}\tau^{-1}\phi\tau)\tau^{-1}(\phi^{-1}\sigma^{-1}\phi\sigma)\tau = f(\tau)f(\sigma)^{\tau}.$$

Note this is the correct cocycle condition since the operation is composition. If the map ϕ was originally defined over K then f is clearly a coboundary, since $\phi \in H^0(G_K, \operatorname{Aut} X)$ and $f(\sigma) = \phi^{-1}\phi^{\sigma}$.

4 Sheaf cohomology

Let X be a topological space. Then as we will see, the sheaves of abelian groups on X form a category, denoted Ab_X. If we add a litte more structure by making $(X, \mathscr{O}_X)$ a ringed space, we have a notion of an $\mathscr{O}_X$-module, and as in the case of modules over a ring, the category Mod_X of $\mathscr{O}_X$-modules turns out to be an abelian category (as do the categories of (quasi)coherent sheaves on X, QCoh_X and Coh_X). In any case, we will define sheaf cohomology by taking the right derived functors of the global section functor $\Gamma(X, \cdot)$. The sheaf cohomology groups will turn out to contain useful information about schemes in algebraic geometry.

4.1 Definitions

In this section, let X denote a topological space. We refer the reader to [Vak17, Chapter 2] or [Har77, II.1, II.5] for a proper exposition if you are unfamiliar with the basic definitions of sheaves, stalks, and $\mathscr{O}_X$-modules.

Definition 4.1 (Ab_X and Mod_X)**.** Let Ab_X denote the category of sheaves of abelian groups over X. If $(X, \mathscr{O}_X)$ is a ringed space and $\mathscr{F} \in \mathrm{Ab}_X$, we say $\mathscr{F}$ is an $\mathscr{O}_X$**-module** if there is a map $\mathscr{O}_X \times \mathscr{F} \to \mathscr{F}$, and on each open set $U \subseteq X$, the map on sections $\mathscr{O}_X(U) \times \mathscr{F}(U) \to \mathscr{F}(U)$ gives $\mathscr{F}(U)$ the structure of an $\mathscr{O}_X(U)$-module. Equivalently, we can take a diagrammatic approach by asking the module map $\mathscr{O}_X \times \mathscr{F} \to \mathscr{F}$ to satisfy the usual module axiom diagram.

The collection of $\mathscr{O}_X$-modules forms a category, Mod_X, where the maps $\mathscr{F} \to \mathscr{G}$ are morphisms of sheaves of abelian groups which respect the $\mathscr{O}_X$-module structure.

The first thing that we need to know about Mod_X if we are to talk about derived functors is that it is an abelian category with enough injectives.

Proposition 4.2. *Let $(X, \mathscr{O}_X)$ be a ringed space. Then Mod_X is an abelian category with enough injectives.*

Proof. Recall Definition 1.25 of an abelian category had five parts (i) through (v). This proof essentially amounts to reducing the case of $\mathscr{O}_X$-modules to that of modules over a ring.

(i) First we need to check that for $\mathscr{F}, \mathscr{G} \in \mathrm{Mod}_X$, the morphisms $\mathrm{Hom}_{\mathscr{O}_X}(\mathscr{F}, \mathscr{G})$ form an abelian group. Given $\phi, \psi \colon \mathscr{F} \to \mathscr{G}$, we can see that defining $(\phi + \psi)(U) = \phi(U) + \psi(U)$ gives a map on sections $\mathscr{F}(U) \to \mathscr{G}(U)$ which is well behaved with respect to both restriction and the $\mathscr{O}_X$-module map since $\phi(U), \psi(U)$ are both maps of $\mathscr{O}_X(U)$-modules. It is also straightforward to see that composition distributes over addition. The zero element is the zero map $\mathscr{F} \to \mathscr{G}$ which is the zero map over all sections.

(ii) The zero object in Mod_X is the zero sheaf, whose sections are the zero module for all opens $U \subseteq X$. Again, since 0 is the zero object in the category of $\mathscr{O}_X(U)$-modules, this gives maps to and from zero on all sections, and these maps are seen easily to agree with restriction.

(iii) Let $\mathscr{F}, \mathscr{G}$ be $\mathscr{O}_X$-modules. Then $\mathscr{F} \times \mathscr{G}$ is an $\mathscr{O}_X$-module with sections $(\mathscr{F} \times \mathscr{G})(U) = \mathscr{F}(U) \times \mathscr{G}(U)$ with the induced $\mathscr{O}_X$-module structure. It is an easy exercise to check this is a sheaf (in fact one can prove the more general statement that given an inverse system of sheaves, taking the inverse limit of sections is a presheaf and in fact a sheaf).

(iv) For kernels, one can check given $\phi\colon \mathscr{F} \to \mathscr{G}$ that the presheaf $U \mapsto \ker \phi(U)$ is a sheaf of abelian groups, so $(\ker \phi)(U) = \ker \phi(U)$. Since $\phi(U)$ is a map of $\mathscr{O}_X(U)$-modules, its kernel is an $\mathscr{O}_X(U)$-module, and it is not hard to see that the module structure will be compatible with restriction also.

For cokernels, we need to use sheafification, since $U \mapsto \operatorname{coker}(\phi(U))$ is *not* always a sheaf. However, we note that this is a presheaf, and statisfies the identity sheaf axiom. To see this, let $s, t \in \operatorname{coker}(\phi(U))$ with $s|_{U_i} = t|_{U_i}$ for an open cover $U = \cup U_i$. Then we can view s, t as elements of $\mathscr{G}(U)$ and we want $s - t \in \phi(U)(\mathscr{F}(U))$. We have $s_i - t_i \in \phi(U_i)(\mathscr{F}(U_i))$, so let v_i be a preimage of $s_i - t_i$ in $\mathscr{F}(U_i)$. Then we check that $v_i|_{U_i \cap U_j} = v_j|_{U_i \cap U_j}$, so the v_i glue to $v \in \mathscr{F}(U)$, and $\phi(U)(v) = s - t$. Thus $s = t$ in the cokernel.

Sheafification produces a sheaf $\mathscr{G}^+$ from a presheaf $\mathscr{G}$, such that if $\mathscr{G} \to \mathscr{G}'$ is a map to a sheaf $\mathscr{G}'$ then there is a unique map $\mathscr{G}^+ \to \mathscr{G}'$ which commutes with the natural map $\mathscr{G} \to \mathscr{G}^+$. Moreover, if $\mathscr{G}$ satisfies the first sheaf axiom, but not necessarily the second, then $\mathscr{G} \to \mathscr{G}^+$ is an injection.

We now claim that the sheafification of the presheaf $U \mapsto \operatorname{coker}(\phi(U))$ is the cokernel in the category of sheaves. If $\phi\colon \mathscr{F} \to \mathscr{G}$, and T is any sheaf such that

$$\begin{tikzcd}
\mathscr{F} \arrow[r, "\phi"] \arrow[dr] & \mathscr{G} \arrow[r] \arrow[dr] & \operatorname{coker}\phi \arrow[r] \arrow[d] & \operatorname{coker}\phi^+ \arrow[dl, dashed] \\
& 0 \arrow[r] & T &
\end{tikzcd}$$

then the dotted arrow exists because of the sheafification property. Note $\operatorname{coker}\phi \to T$ exists because it is defined on each section and agrees with restriction.

(v) For the final property, we need to show that if $\mathscr{F} \to \mathscr{G}$ is a monomorphism then it is the kernel of its own cokerel. Write the presheaf cokernel as $\mathscr{G}/\mathscr{F}$ and the cokernel as the sheafified $(\mathscr{G}/\mathscr{F})^+$. Note that we have a natural map $\mathscr{F} \to \ker(\operatorname{coker}(\mathscr{G} \to (\mathscr{G}/\mathscr{F})^+))$ by universal properties. Since $\mathscr{G}/\mathscr{F} \to (\mathscr{G}/\mathscr{F})^+$ is injective, and $\ker(\operatorname{coker}) \to (\mathscr{G}/\mathscr{F})^+$ is zero by definition, we must have that $\ker(\operatorname{coker})$ maps to $\mathscr{F}$, because it does on all sections. Thus the monomorphism is the kernel of the cokernel.

Next we suppose $\phi\colon \mathscr{F} \to \mathscr{G}$ is an epi. Again, by universal properties alone we get a natural map $\operatorname{coker}(\ker)^+ \to \mathscr{G}$. To get a map the other way, we recognize that a section $s \in \mathscr{G}(U)$ has stalks that are in the stalks of the image presheaf. We can use these to define a map from $U \to \coprod \operatorname{im}_x$, which is the definition of the sections of the sheafification. Hence we get a map $\mathscr{G} \to \operatorname{coker}(\ker)$, which must be an isomorphism.

Now that we know Mod_X is abelian, we move to show that it has enough injectives. We will use the fact that the category of modules over a ring A has enough injective objects.

Let $\mathscr{F} \in \operatorname{Mod}_X$ be an $\mathscr{O}_X$-module. Its stalk at $x \in X$, $\mathscr{F}_x$, is an $\mathscr{O}_{X,x}$-module, and hence there exists an injective $\mathscr{O}_{X,x}$-module I_x with an inclusion $\mathscr{F}_x \to I_x$. We may view I_x as a sheaf on $\{x\}$, and using the inclusion $j\colon \{x\} \to X$ to take the direct image $j_*(I_x)$ which is a sheaf on X. Now we take the direct product $\mathscr{I} = \prod_{x \in X} j_*(I_x)$, which is again a sheaf.

We claim $\mathscr{I}$ is injective. Giving a map of $\mathscr{O}_X$-modules to $\mathscr{I}$ is equivalent to giving a map to each $j_*(I_x)$, which is the same as giving an $\mathscr{O}_{X,x}$-module homomorphism from each stalk to I_x. Hence we have a natural map $\mathscr{F} \to \mathscr{I}$ which is an injection on stalks. Since taking stalks is an exact functor and $\operatorname{Hom}_{\mathscr{O}_{X,x}}(\cdot, I_x)$ is exact since I_x is injective, their composition is exact, and taking the direct product over x is also exact. Hence $\operatorname{Hom}_{\mathscr{O}_x}(\cdot, \mathscr{I})$ is exact and $\mathscr{I}$ is injective. This proves Mod_X has enough injectives! □

Corollary 4.3. *The category* Ab_X *is abelian and has enough injectives.*

Proof. If $\mathscr{O}_X = \mathbb{Z}$ is taken to be the constant sheaf of rings, then $\operatorname{Mod}_X = \operatorname{Ab}_X$, and we can use Proposition 4.2. To see this let $\mathscr{F} \in \operatorname{Ab}_X$. Then for $U \subseteq X$ connected, $\mathscr{F}(U)$ is an $\mathscr{O}_X(U) = \mathbb{Z}$-module, i.e. an abelian group. If U is any open set, take a connected disjoint cover by U_i. Then the module map $\mathscr{O}_X(U) \times \mathscr{F}(U) \to \mathscr{F}(U)$ can be written down by gluing the module maps on each U_i. $\square$

To define sheaf cohomology via derived functors, we first need a left exact functor. Recall the *global section functor* $\Gamma(X, \cdot)$ from Ab_X to Ab is given by

$$\Gamma(X, \mathscr{F}) = \mathscr{F}(X).$$

Lemma 4.4. *$\Gamma(X, \cdot)$ is an additive left exact functor $\mathrm{Ab}_X \to \mathrm{Ab}$.*

Proof. Let $0 \to \mathscr{F}' \xrightarrow{\phi} \mathscr{F} \xrightarrow{\psi} \mathscr{F}'' \to 0$ be an exact sequence in Ab_X. The maps $\phi(X)$ and $\psi(X)$ induce maps on sections, and the claim is that

$$0 \to \Gamma(X, \mathscr{F}') \xrightarrow{\phi(X)} \Gamma(X, \mathscr{F}) \xrightarrow{\psi(X)} \Gamma(X, \mathscr{F}'')$$

is an exact sequence.

Let $s' \in \Gamma(X, \mathscr{F}')$ be in $\ker \phi(X)$. Then since taking stalks is an exact functor, the image of s' is zero in all stalks. The only such section is 0, so $s' = 0$, and $\phi(X)$ is injective.

For exactness in the middle, let $s \in \Gamma(X, \mathscr{F})$. If $s \in \ker \psi(X)$ then using exactness on stalks, for each $x \in X$, we have $s_x \in \operatorname{im} \phi_x$, where s_x denotes the image of s in $\mathscr{F}_x$ and $\phi_x \colon \mathscr{F}'_x \to \mathscr{F}_x$ is the induced map. Then since $\mathscr{F}'$ is a subsheaf of $\mathscr{F}$, these glue to an element of $\Gamma(X, \mathscr{F}')$ which map to s. It is straightforward to see in the other direction that a section of $\Gamma(X, \mathscr{F}')$ maps to zero in $\Gamma(X, \mathscr{F}'')$ using exactness at stalks again. $\square$

Now that we have enough injectives in Ab_X and left exactness, we are free to define derived functors, which will satisfy the usual long exact sequence in Theorem 2.13.

Definition 4.5 (sheaf cohomology)**.** Let X be a topological space and $\Gamma(X, \cdot)$ the global section functor $\mathrm{Ab}_X \to \mathrm{Ab}$. The **sheaf cohomology functors** $H^i(X, \cdot)$ are taken to be the right derived functors $R^i\Gamma(X, \cdot)$, which are well defined by Theorem 2.17. For a given sheaf $\mathscr{F} \in \mathrm{Ab}_X$, we call $H^i(X, \mathscr{F})$ the ***i*-th cohomology group** of $\mathscr{F}$.

Remark 4.6. If $(X, \mathscr{O}_X)$ is a ringed space, then $\Gamma(X, \cdot)$ may be viewed as a functor $\mathrm{Mod}_X \to \mathrm{Ab}$. In this case, one can check that the derived functors agree with $H^i(X, \cdot)$, where we are merely viewing an $\mathscr{O}_X$-module as a sheaf of abelian groups.

4.2 Key properties and results

The derived functor contstruction can be tricky to compute with. Here we collect some results that make it easier to compute with sheaf cohomology in certain circumstances. A key intermediate result is that we can use flasque resolutions.

Definition 4.7 (flasque)**.** Recall a sheaf $\mathscr{F}$ on X is **flasque** if its restriction maps $\mathrm{res}_{U,V} \colon \mathscr{F}(U) \to \mathscr{F}(V)$ are surjective for all $V \subseteq U$.

Lemma 4.8. *Let $(X, \mathscr{O}_X)$ be a ringed space. An injective $\mathscr{O}_X$-module is flasque.*

Proof. First we fix some notation. If $U \subseteq X$ is open, let $\mathscr{O}_U$ denote the sheaf $\mathscr{O}_X|_U$ extended by zero outside U.

Let $\mathscr{I}$ be an injective $\mathscr{O}_X$-module and suppose $V \subseteq U \subseteq X$. Then we get an inclusion of sheaves $0 \to \mathscr{O}_V \to \mathscr{O}_U$ (check this on the stalks if you don't believe me!). Letting the quotient be whatever it is and applying the exact functor $\mathrm{Hom}(\cdot, \mathscr{I})$, we see $\mathrm{Hom}(\mathscr{O}_U, \mathscr{I}) \to \mathrm{Hom}(\mathscr{O}_V, \mathscr{I})$ is surjective.

We claim $\mathrm{Hom}_{\mathscr{O}_X}(\mathscr{O}_U, \mathscr{I}) = \mathscr{I}(U)$. This is true because more generally, $\mathrm{Hom}_{\mathscr{O}_X}(\mathscr{O}_X, \mathscr{F}) = \mathscr{F}(X)$ for $\mathscr{F} \in \mathrm{Mod}_X$. To see this, we identify a map $f \colon \mathscr{O}_X \to \mathscr{F}$ with a global section $\mathscr{F}(X)$.

Any such map induces a map on global sections, and we identify f with the image $s = f(X)(1)$. Conversely, given a global section $s \in \mathscr{F}(X)$ we can concoct a map by setting $f(X)(1) = s$, and extending since $\mathscr{F}(X)$ is an $\mathscr{O}_X(X)$-module. We can use the fact that $\mathscr{F}$ is an $\mathscr{O}_X$-module to extend this map to a map on sheaves, completing the identification. In our case, since we have extended $\mathscr{O}_X|_U$ by zero outside U to get $\mathscr{O}_U$, giving a map $\mathscr{O}_U \to \mathscr{I}$ is equivalent to picking a section $\mathscr{I}(U)$, so $\mathscr{I}(U) \to \mathscr{I}(V)$ is surjective by the previous paragraph and $\mathscr{I}$ is a flasque sheaf. $\square$

Proposition 4.9. *If $\mathscr{F}$ is a flasque sheaf on X, then $\mathscr{F}$ is acyclic. That is, $H^i(X, \mathscr{F}) = 0$ for all $i > 0$.*

Proof. Let $\mathscr{F}$ be a flasque sheaf, and map it injectively into an injective sheaf $\mathscr{I}$, since Ab_X has enough injectives by Corollary 4.3. Take the quotient $\mathscr{G}$, so that

$$0 \to \mathscr{F} \to \mathscr{I} \to \mathscr{G} \to 0$$

is a short exact sequence.

$\mathscr{I}$ is flasque by Lemma 4.8, so it is a basic fact of sheaves $\mathscr{G}$ must also be flasque. Moreover, another basic fact is that when $\mathscr{F}$ is flasque as above, we have an exact sequence of global sections

$$0 \to \Gamma(X, \mathscr{F}) \to \Gamma(X, \mathscr{I}) \to \Gamma(X, \mathscr{G}) \to 0,$$

so $\Gamma(X, \mathscr{G}) \to H^1(X, \mathscr{F})$ is the zero map. On the other hand, $H^1(X, \mathscr{I}) = 0$ since $\mathscr{I}$ is injective, so we have $H^1(X, \mathscr{F}) = 0$ by exactness. Feel free to check these facts on your own, or see Exercise II.1.16 in [Har77].

Since $\mathscr{I}$ is injective, $H^i(X, \mathscr{I}) = 0$ for all $i > 0$. The long exact sequence in cohomology, Theorem 2.13, gives us that

$$0 \to H^i(X, \mathscr{G}) \to H^{i+1}(X, \mathscr{F}) \to 0$$

for $i \geq 1$, so $H^i(X, \mathscr{G}) \simeq H^{i+1}(X, \mathscr{F})$ for $i \geq 1$. But $\mathscr{G}$ is also flasque, so $H^1(X, \mathscr{G}) = 0$, which implies $H^2(X, \mathscr{F}) = 0$, and by induction we see that $H^i(X, \mathscr{F}) = 0$ for $i > 0$. $\square$

The utility of Proposition 4.9 is that since flasque sheaves are acyclic, we can use a resolution by flasque sheaves to compute sheaf cohomology. This will also be useful in Proposition 4.11, when we see how cohomology commutes with direct limits of sheaves.

Lemma 4.10. *Let X be Noetherian. Then a direct limit of flasque sheaves is flasque.*

Proof. Let $(\mathscr{F}_\alpha)$ be a directed system of flasque sheaves. Then for every α and every $V \subseteq U \subseteq X$, we have surjective restriction maps $\mathscr{F}_\alpha(U) \to \mathscr{F}_\alpha(V)$. Taking direct limits of abelian groups is exact, so

$$\varinjlim_\alpha \mathscr{F}_\alpha(U) \to \varinjlim_\alpha \mathscr{F}_\alpha(V)$$

is also surjective.

While normally taking direct limits of sheaves would involve a sheafification step, since X is Noetherian we may avoid this, so $(\varinjlim \mathscr{F}_\alpha)(U) = \varinjlim \mathscr{F}_\alpha(U)$, and

$$(\varinjlim_\alpha \mathscr{F}_\alpha)(U) \to (\varinjlim_\alpha \mathscr{F}_\alpha)(V)$$

is therefore surjective. $\square$

Proposition 4.11. *Let X be Noetherian and $(\mathscr{F}_\alpha)$ a direct system in Ab_X. Then cohomology commutes with direct limits. That is, there exist natural isomorphisms*

$$\varinjlim_\alpha H^i(X, \mathscr{F}_\alpha) \to H^i(X, \varinjlim_\alpha \mathscr{F}_\alpha)$$

for all $i \geq 0$.

Proof. The proof in [Har77, Proposition 2.9] uses that $\varinjlim_\alpha H^i(X, \cdot)$ and $H^i(X, \varinjlim_\alpha \cdot)$ are universal δ-functors on the (abelian) category of directed systems indexed by some diagram, with some effort needed to make sense of this and show they are effaceable. If $(\mathscr{F}_\alpha)$ is a *filtered* system of quasicoherent $\mathscr{O}_X$-modules on X separated and quasicompact, this is a consequence of the exactness of taking limits of filtered sets [Vak17, Exercise 18.2.G]. □

We end this subsection with some key theorems about sheaf cohomology from Grothendieck and Serre. The proofs may be found in [Har77, II.2 – II.3], or see [Vak17, Chapter 18] for proofs mostly relying on Cech cohomology.

Theorem 4.12 (Grothendieck's vanishing theorem)**.** *Let X be a Noetherian topological space of dimension n. Then for all $i > n$ and all $\mathscr{F} \in \mathrm{Ab}_X$, we have $H^i(X, \mathscr{F}) = 0$.*

For the next results, we recall the definition of a (quasi)coherent sheaf, which locally looks like a sheaf $\widetilde{M}$ coming from an A-module M, where A is a Noetherian ring.

Definition 4.13 ((quasi)coherent)**.** Let M be an A-module. The **sheaf associated to** M, denoted by $\widetilde{M}$, is a sheaf on $\operatorname{Spec} A$ with sections

$$\widetilde{M}(U) = \left\{ s \colon U \to \coprod_{\mathfrak{p} \in U} \mid \text{(i) and (ii)} \right\}$$

where condition (i) is that $s(\mathfrak{p}) \in M_\mathfrak{p}$ and (ii) is that s is locally constant: for each $\mathfrak{p} \in U$ there exists an open set $V \subseteq U$, $m \in A$, and $f \in A$ such that $f \notin \mathfrak{q}$ and $s(\mathfrak{q}) = m/f$ for all $q \in V$.

If X is a scheme, we say an $\mathscr{O}_X$-module $\mathscr{F}$ is **quasicoherent** if X has an affine cover $X = \cup U_i$ such that $U_i = \operatorname{Spec} A_i$ and $\mathscr{F}|_{U_i} \simeq \widetilde{M}_i$ for some A_i-module M_i. This is what we mean by saying $\mathscr{F}$ is "locally" of the form $\widetilde{M}$.

If $\mathscr{F}$ is quasicoherent and M_i is a finitely generated A_i-module for all i, then we say $\mathscr{F}$ is **coherent** (for a more general definition when A isn't Noetherian, see [Vak17, §13.6]).

It turns out that if $X = \operatorname{Spec} A$ is affine, then every quasicoherent sheaf on X is in fact of the form $\widetilde{M}$ for an A-module M. That is, any sheaf on $\operatorname{Spec} A$ which is *locally* given by sheaves associated to modules, is in fact *globally* given by a sheaf associated to a module.

Theorem 4.14. *Let $X = \operatorname{Spec} A$ be an affine scheme. Then quasicoherent sheaves on X are acyclic. That is, for all quasicoherent sheaves $\mathscr{F}$ on X , we have $H^i(X, \mathscr{F}) = 0$ for $i > 0$.*

Proof. The proof of this one isn't too bad with Noetherian hypotheses. If $\mathscr{F}$ is quasicoherent on $\operatorname{Spec} A$ then $\mathscr{F} = \widetilde{M}$ where $M = \Gamma(X, \mathscr{F})$ is an A-module, by the remarks above. In the category of A-modules, take an injective resolution $M \to I^\bullet$. This gives a resolution of sheaves $\widetilde{M} \to \widetilde{I}^\bullet$, since taking sheaves associated to modules is an exact functor. While the sheaves $\widetilde{I^k}$ may not be injective in Mod_X, they do turn out to be flasque by [Har77, Prop. III.3.4], so this resolution may be used to compute the cohomology.

Applying global sections, we get back that $M \to I^\bullet$ is exact, so $H^0(X, \mathscr{F}) = M$ and $H^i(X, \mathscr{F}) = 0$ for $i > 0$. □

In fact, we can do a little better than Theorem 4.14. This condition of quasicoherent sheaves having vanishing cohomology is enough to characterize when a scheme is affine. See [Har77, Theorem III.3.7] for the proof.

Theorem 4.15 (Serre's criterion for affineness)**.** *Let X be a Noetherian scheme. The following are equivalent:*

(i) X is affine.

(ii) $H^i(X, \mathscr{F}) = 0$ *for all* $\mathscr{F}$ *quasicoherent and all* $i > 0$.

(iii) $H^1(X, \mathscr{I}) = 0$ *for all coherent sheaves of ideals* $\mathscr{I}$.

4.3 Cech cohomology

The motivation for Cech cohomology is that it provides a means of computing sheaf cohomology that is quite hands on. It also looks very similar to the simplicial (co)homology of algebraic topology, and could be set up from this perspective if desired. We will stick to the basic setup and see how to use Cech cohomology to do example computations.

First we fix some notation. Let X be a topological space and $\mathscr{U} = \{U_i\}_{i\in I}$ an open cover of X. We will need a well ordering on the index set I, so let's fix one. Now for a set of indices $i_0, ..., i_n \in I$, we denote the intersection by $U_{i_0,...,i_n} = U_{i_0} \cap \cdots \cap U_{i_n}$.

If $\mathscr{F}$ is a sheaf of abelian groups on X, we define a group

$$C^n(\mathscr{U}, \mathscr{F}) = \prod_{i_0<\cdots<i_n} \mathscr{F}(U_{i_0,...,i_n}).$$

We will now define maps $d\colon C^{n-1}(\mathscr{U}, \mathscr{F}) \to C^n(\mathscr{U}, \mathscr{F})$ to make this into a complex. First, note that an element $\alpha \in C^n(\mathscr{U}, \mathscr{F})$ is determined by specifying elements $\alpha_{i_0,...,i_n} \in \mathscr{F}(U_{i_0,...,i_n})$ for all $n+1$ element subsets $i_0 < \cdots < i_n$ in I. Then for $\alpha \in C^{n-1}(\mathscr{U}, \mathscr{F})$,

$$(d\alpha)_{i_0,...,i_n} = \sum_{k=0}^{n} (-1)^k \alpha_{i_0,...,\widehat{i_k},...,i_n}|_{U_{i_0,...,i_n}}.$$

In words, the $U_{i_0,...,i_n}$ section of the image $d\alpha$ is an alternating sum of the sections of α, $\alpha_{i_0,...,\widehat{i_k},...,i_n}$, where the hat indicates omission of the k-th index, restricted further to the intersection.

Proposition 4.16. *The map* d *defined above is a homomorphism of groups and* $d^2 = 0$.

Proof. To check d is a homomorphism of groups, let $\alpha, \beta \in C^{n-1}(\mathscr{U}, \mathscr{F})$. It is then easy to check that $d(\alpha+\beta)_{i_0,...,i_n} = (d\alpha)_{i_0,...,i_n} + (d\beta)_{i_0,...,i_n}$ since the restriction maps of $\mathscr{F}$ are homomorphisms. Since d is a homomorphism in each component of the direct product, it is a homomorphism.

Now we must show $d^2 = 0$. Let $\alpha \in C^{n-1}(\mathscr{U}, \mathscr{F})$ so $(d\alpha) \in C^n(\mathscr{U}, \mathscr{F})$. Then

$$\begin{aligned} (d(d\alpha))_{i_0,...,i_{n+1}} &= \sum_{k=0}^{n+1} (-1)^k (d\alpha)_{i_0,...,\widehat{i_k},...,i_{n+1}}|_{U_{i_0,...,i_{n+1}}} \\ &= \sum_{k=0}^{n+1} (-1)^k \left(\sum_{0\le j<k} (-1)^j \alpha_{i_0,...,\widehat{i_j},...,\widehat{i_k},...,i_{n+1}} + \sum_{k<j\le n+1} (-1)^{j+1} \alpha_{i_0,...,\widehat{i_k},...,\widehat{i_j},...,i_{n+1}} \right) \Big|_{U_{i_0,...,i_{n+1}}}. \end{aligned}$$

Note that the restrictions are omitted for space constraints, but $\alpha_{i_0,...,\widehat{i_j},...,\widehat{i_k},...,i_{n+1}}$ should be first restricted to $U_{i_0,...,\widehat{i_k},...,i_{n+1}}$.

Now these pair off – if $j < k$ then when their values are swapped, a negative sign will be introduced, causing cancellation. This occurs for all terms, so $d(d(\alpha)) = 0$ in all sections, hence $d^2 = 0$. $\square$

Definition 4.17. The proposition above makes

$$0 \to C^0(\mathscr{U}, \mathscr{F}) \to C^1(\mathscr{U}, \mathscr{F}) \to C^2(\mathscr{U}, \mathscr{F}) \to \cdots$$

into a complex, called the **Čech complex** of $\mathscr{F}$ on X with respect to $\mathscr{U}$.

Taking cohomology of the complex, we define the **Čech cohomology groups** as

$$\check{H}^i(\mathscr{U}, \mathscr{F}) = h^i(C^\bullet(\mathscr{U}, \mathscr{F})).$$

Example 4.18. As a first example, take $X = S^1$ the unit circle with its usual topology. Let $\mathbb{Z}$ denote the locally constant sheaf $\mathbb{Z}$ on X. We can take an open cover $\mathscr{U}$ of X by two open (overlapping) semicircles U_1 and U_2. Then

$$\begin{aligned} C^0(\mathscr{U}, \mathbb{Z}) &= \Gamma(U_1, \mathbb{Z}) \times \Gamma(U_2, \mathbb{Z}) \simeq \mathbb{Z} \times \mathbb{Z} \\ C^1(\mathscr{U}, \mathbb{Z}) &= \Gamma(U_1 \cap U_2, \mathbb{Z}) \simeq \mathbb{Z} \times \mathbb{Z} \end{aligned}$$

since on any connected open set V, $\Gamma(V, \mathbb{Z}) = \mathbb{Z}$. Following the map $d \colon C^0(\mathscr{U}, \mathbb{Z}) \to C^1(\mathscr{U}, \mathbb{Z})$, we see

$$d(a, b) = (b - a, b - a)$$

for all $(a, b) \in \mathbb{Z} \times \mathbb{Z}$. The kernel is the diagonal, which is isomorphic to $\mathbb{Z}$, and the image is also the diagonal in the target. Hence

$$\begin{aligned} \check{H}^0(\mathscr{U}, \mathbb{Z}) &\simeq \mathbb{Z} \\ \check{H}^1(\mathscr{U}, \mathbb{Z}) &\simeq \mathbb{Z}. \end{aligned}$$

Example 4.19. Consider the scheme $X = \mathbb{A}^2 - \{0\}$ and $\mathscr{F} = \mathscr{O}_{\mathbb{A}^2}|_X$. We can cover X by the two distinguished open sets in $\mathbb{A}^2 = \operatorname{Spec} k[x, y]$ given by $D(x)$ and $D(y)$. Their intersection $U_{1,2} = D(xy)$. It's then easy to compute

$$\begin{aligned} C^0(\mathscr{U}, \mathscr{F}) &= k[x, y]_x \times k[x, y]_y \\ C^1(\mathscr{U}, \mathscr{F}) &= k[x, y]_{xy}. \end{aligned}$$

Taking the kernel of the map $C^0(\mathscr{U}, \mathscr{F}) \to C^1(\mathscr{U}, \mathscr{F})$ given by $(f, g) \mapsto f - g$, we find

$$\check{H}^0(\mathscr{U}, \mathscr{F}) = k[x, y].$$

Once we know the connection between Cech and sheaf cohomology, this implies that X is not affine, because $\operatorname{Spec} k[x, y] = \operatorname{Spec} H^0(X, \mathscr{O}_X) \neq X$.

We can also compute

$$\check{H}^1(\mathscr{U}, \mathscr{F}) = k[x, y]_{xy} / k[x, y] \simeq x^{-1} y^{-1} k[x^{-1} y^{-1}].$$

We now want to study the relationship of Čech cohomology to sheaf cohomology. The first step is to see that their zeroth cohomology groups always agree.

Lemma 4.20. *Let X be a topological space, $\mathscr{U}$ a cover, and $\mathscr{F} \in \mathrm{Ab}_X$. Then the zeroth Čech cohomology agrees with the global sections*

$$\check{H}^0(\mathscr{U}, \mathscr{F}) \simeq \Gamma(X, \mathscr{F}).$$

Proof. The proof boils down to understanding what it means to give an element of $C^0(\mathscr{U}, \mathscr{F})$ in the kernel of d. $C^0(\mathscr{U}, \mathscr{F}) = \prod \Gamma(U_i, \mathscr{F})$ so giving $\alpha \in C^0(\mathscr{U}, \mathscr{F})$ amounts to giving a section $\alpha_i = \Gamma(U_i, \mathscr{F})$ for each $i \in I$. Asking α to be in the kernel of the d map means that

$$\alpha_j|_{U_i \cap U_j} - \alpha_i|_{U_i \cap U_j} = 0$$

for all $i < j$, so these sections α_i agree on intersections. By the sheaf axioms, these uniquely determine a global section of $\Gamma(X, \mathscr{F})$. $\square$

We can make the construction of the Čech complex sheafy, which will help us in relating to sheaf cohomology.

Definition 4.21 (sheafy Čech complex)**.** Let $X, \mathscr{U}, \mathscr{F}$ as in Defintion 4.17, and for an open set $V \subseteq X$, let i denote the inclusion. Define a sheaf on X by

$$\mathscr{C}^n(\mathscr{U}, \mathscr{F}) = \prod_{i_0 < \cdots < i_n} i_*(\mathscr{F}|_{U_{i_0,\ldots,i_n}}).$$

The maps d give rise to maps $\mathscr{C}^n(\mathscr{U}, \mathscr{F}) \to \mathscr{C}^{n+1}(\mathscr{U}, \mathscr{F})$, so we find a **sheafy Čech complex** of sheaves

$$0 \to \mathscr{C}^0(\mathscr{U}, \mathscr{F}) \to \mathscr{C}^1(\mathscr{U}, \mathscr{F}) \to \cdots$$

A key feature of this sheaf is that

$$\Gamma(X, \mathscr{C}^n(\mathscr{U}, \mathscr{F})) \simeq C^n(\mathscr{U}, \mathscr{F})$$

because in this case $i_*(\mathscr{F}|_{U_{i_0,\ldots,i_n}}) = \mathscr{F}(U_{i_0,\ldots,i_n})$, and taking the product gives $C^n(\mathscr{U}, \mathscr{F})$. The utility of the sheafy complex is that it turns to be a resolution of $\mathscr{F}$, that is

$$0 \to \mathscr{F} \to \mathscr{C}^0(\mathscr{U}, \mathscr{F}) \to \mathscr{C}^1(\mathscr{U}, \mathscr{F}) \to \cdots$$

is exact (see [Har77, Lemma III.4.2]).

Now take an injective resolution $\mathscr{I}^\bullet$ for $\mathscr{F}$, so we can then compare the resolutions,

$$\begin{tikzcd} 0 \arrow[r] & \mathscr{F} \arrow[r] \arrow[d, "\mathrm{id}"'] & \mathscr{C}^0(\mathscr{U}, \mathscr{F}) \arrow[r] & \mathscr{C}^1(\mathscr{U}, \mathscr{F}) \arrow[r] & \cdots \\ 0 \arrow[r] & \mathscr{F} \arrow[r] & \mathscr{I}^0 \arrow[r] & \mathscr{I}^1 \arrow[r] & \cdots \end{tikzcd}.$$

Since $\mathscr{I}^0$ is injective, the monomorphism $\mathscr{F} \to \mathscr{C}^0(\mathscr{U}, \mathscr{F})$ extends to a map to $\mathscr{I}^0$ giving

$$\begin{tikzcd} 0 \arrow[r] & \mathscr{F} \arrow[r] \arrow[d, "\mathrm{id}"'] & \mathscr{C}^0(\mathscr{U}, \mathscr{F}) \arrow[r] \arrow[d] & \mathscr{C}^1(\mathscr{U}, \mathscr{F}) \arrow[r] & \cdots \\ 0 \arrow[r] & \mathscr{F} \arrow[r] & \mathscr{I}^0 \arrow[r] & \mathscr{I}^1 \arrow[r] & \cdots \end{tikzcd}.$$

We can get a map $\mathscr{C}^1(\mathscr{U}, \mathscr{F}) \to \mathscr{I}^1$ by recognizing that $\mathscr{C}^0(\mathscr{U}, \mathscr{F}) / \ker d \hookrightarrow \mathscr{C}^1(\mathscr{U}, \mathscr{F})$, and that $\ker d$ maps to 0 in $\mathscr{I}^1$ because of the exactness of both rows. Then we use injectivity of $\mathscr{I}^1$ to extend to the desired map.

Continuing this process gives a map of complexes $\mathscr{C}^\bullet(\mathscr{U}, \mathscr{F}) \to \mathscr{I}^\bullet$ which induces the identity on $\mathscr{F}$. Applying the global section functor, we get a map of complexes of abelian groups, which induces a map on cohomology. However in this case, the map on cohomology is precisely $\check{H}^n(\mathscr{U}, \mathscr{F}) \to H^n(X, \mathscr{F})$ by their respective definitions. If we are willing to assume that X has nice properties and $\mathscr{F}$ is quasicoherent, then it turns out these natural maps are isomorphisms, and we can use Čech cohomology to compute sheaf cohomology. The proof can be found in [Har77, Theorem 4.5].

Theorem 4.22. *Let $X, \mathscr{U}, \mathscr{F}$ as above. Then for each $n \geq 0$ there is a natural map*

$$\check{H}^n(\mathscr{U}, \mathscr{F}) \to H^n(X, \mathscr{F})$$

which is functorial in $\mathscr{F}$.

Moreover, if X is a Noetherian separated scheme, $\mathscr{U}$ is a cover by affine open sets, and $\mathscr{F}$ is a quasicoherent sheaf on X, then for all $n \geq 0$ the natural maps above are isomorphisms

$$\check{H}^n(\mathscr{U}, \mathscr{F}) \simeq H^n(X, \mathscr{F}).$$

Many schemes of interest are Noetherian and separated, such as affine space, projective space, and affine or projective varieties, and oftentimes the sheaves we're interested in computing are quasicoherent. Theorem 4.22 allows us to compute the sheaf cohomology groups in a straightforward way using Cech cohomology and our favorite affine cover.

Example 4.23 (Cohomology of $\mathscr{O}_{\mathbb{P}^n}(m)$)**.** Extending the ideas of Example 4.19 carefully allows us to compute $H^i(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(m))$ for all i, n, m by computing the Cech cohomology $\check{H}^i(\mathscr{U}, \mathscr{O}_{\mathbb{P}^n}(m))$ where $\mathscr{U}$ is the standard affine cover of $\mathbb{P}^n$ by $n+1$ coordinate patches. We have

$$\begin{aligned} H^0\left(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(m)\right) &= \binom{n+m}{m} = \binom{n+m}{n}, \\ H^n\left(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(m)\right) &= \binom{-m-1}{n} = \binom{-m-1}{-m-1-n}, \\ H^i\left(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(m)\right) &= 0 \qquad\qquad \text{for all } 0 < i < n \text{ and } i > n. \end{aligned}$$

For the details, see [Har77, §III.5] or [Vak17, §18.3].

4.4 Applications

We'll conclude this section by actually using sheaf cohomology to prove things in the setting of algebraic geometry.

4.4.1 Lifting functions

As a first example of what the first sheaf cohomology group H^1 does, consider the situation of a closed subscheme $i\colon Z \hookrightarrow X$ with ideal sheaf $\mathscr{I}$. We have the exact sequence of sheaves on X

$$0 \to \mathscr{I} \to \mathscr{O}_X \to i_*\mathscr{O}_Z \to 0.$$

Consider a function $Z \to \mathbb{A}^1$, i.e. a global section $f \in \Gamma(Z, \mathscr{O}_Z) = \Gamma(X, i_*\mathscr{O}_X)$. Does this lift to global section of $\Gamma(X, \mathscr{O}_X)$?

Locally, of course, the answer is yes! To see why, suppose $X = \operatorname{Spec} A$ is affine, so $Z = \operatorname{Spec} A/I$ for an ideal $I \subseteq A$. The exact sequence of sheaves is just that of A-modules,

$$0 \to I \to A \to A/I \to 0,$$

and $\Gamma(X, \mathscr{O}_X) = A$ and $\Gamma(Z, \mathscr{O}_Z) = A/I$, so we have that the map on global sections is surjective. Thus even if X is not affine, we can cover X by affine open subsets X_i and Z by affine open subsets Z_i such that $i|_{X_i}\colon X_i \to Z_i$. On these subschemes, we have $f|_{Z_i}$ lifts to a function on X_i.

However, these functions may not glue together to give a global section of X, and this is precisely what is measured by H^1. Taking the long exact sequence in sheaf cohomology, we have

$$0 \to \Gamma(X, \mathscr{I}) \to \Gamma(X, \mathscr{O}_X) \to \Gamma(Z, \mathscr{O}_Z) \xrightarrow{\delta} H^1(X, \mathscr{I}) \to \cdots.$$

By the exactness of this sequence, we have that f lifts to a global section of X if and only if $\delta(f) = 0$, as this means f is in the image of an element of $\Gamma(X, \mathscr{O}_X)$. We call $\delta(f)$ an *obstruction class*; it is nonzero if and only if f lifts. If $H^1(X, \mathscr{I})$ is known to be zero, or δ is determined to be the zero map, then all functions must lift because the obstruction class always vanishes, but this need not be the case in general. Also, the reason this works for affine schemes is evident by the fact that $H^1(X, \mathscr{I}) = 0$ for all ideal sheaves $\mathscr{I}$ when X is affine (see Theorem 4.15).

4.4.2 Riemann–Roch

Let X be a scheme over a field k. For concreteness, one may assume $k = \overline{k}$, but this isn't necessary. When $\mathscr{F}$ is an $\mathscr{O}_X$-module, the cohomology groups $H^i(X, \mathscr{F})$ naturally have the structure of a k-vector space. To see this, we simply recognize that $\Gamma(X, \mathscr{F})$ has the structure of a $\Gamma(X, \mathscr{O}_X)$-module, and the global sections are a vector space over k. Thus $\Gamma(X, \cdot)$ and its derived functors can be viewed as functors $\mathrm{Mod}_{\mathscr{O}_X} \to \mathrm{Vec}_k$.

We'll take this perspective from here on, allowing us to discuss dimension. Let

$$h^i(X, \mathscr{F}) = \dim_k H^i(X, \mathscr{F}).$$

We can now define the Euler characteristic of a sheaf of $\mathscr{O}_X$-modules.

Definition 4.24. Let X be a scheme over k and $\mathscr{F}$ a sheaf of $\mathscr{O}_X$ modules. The **Euler characteristic** of $\mathscr{F}$, denoted $\chi(X, \mathscr{F})$ or simply $\chi(\mathscr{F})$ when there is no confusion about the underlying scheme X, is defined by the alternating sum of dimensions

$$\chi(X, \mathscr{F}) = \sum_{i=0}^{\infty} (-1)^i h^i(X, \mathscr{F}) = h^0(X, \mathscr{F}) - h^1(X, \mathscr{F}) + h^2(X, \mathscr{F}) - \cdots .$$

By Theorem 4.22, if X has dimension n, the sum defining $\chi(X, \mathscr{F})$ terminates after the $i = n$ term. This is especially useful in the case of curves, where we have

$$\chi(X, \mathscr{F}) = h^0(X, \mathscr{F}) - h^1(X, \mathscr{F}),$$

greatly simplifying many of the calculations to come.

Lemma 4.25. *Euler characteristic is additive on short exact sequences. That is, if we have an exact sequence*

$$0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$$

of sheaves on X, then

$$\chi(X, \mathscr{F}) = \chi(X, \mathscr{F}') + \chi(X, \mathscr{F}'').$$

Proof. The proof is a straightforward induction on the dimension n of X. When $n = 0$, we have $H^1(X, \mathscr{F}') = 0$, so

$$h^0(X, \mathscr{F}) = h^0(X, \mathscr{F}') + h^0(X, \mathscr{F}'')$$

and we're done.

If $n > 0$ then we start with

$$\begin{aligned} h^0(X, \mathscr{F}) &= h^0(X, \mathscr{F}') + \dim \operatorname{im}\left(H^0(X, \mathscr{F}) \to H^0(X, \mathscr{F}'')\right) \\ &= h^0(X, \mathscr{F}') + \dim \ker\left(H^0(X, \mathscr{F}'') \to H^1(X, \mathscr{F}')\right). \end{aligned}$$

As the next step, we use the first isomorphism theorem, in the form

$$h^0(X, \mathscr{F}'') - \dim \ker\left(H^0(X, \mathscr{F}'') \to H^1(X, \mathscr{F}')\right) = \dim \operatorname{im}\left(H^0(X, \mathscr{F}'') \to H^1(X, \mathscr{F}')\right)$$

to see that

$$\begin{aligned} h^0(X, \mathscr{F}) &= h^0(X, \mathscr{F}') + h^0(X, \mathscr{F}'') - \dim \operatorname{im}\left(H^0(X, \mathscr{F}'') \to H^1(X, \mathscr{F}')\right) \\ &= h^0(X, \mathscr{F}') + h^0(X, \mathscr{F}'') - \dim \ker\left(H^0(X, \mathscr{F}'') \to H^1(X, \mathscr{F}')\right). \end{aligned}$$

Extending this, keeping track of the sign changes, we find

$$h^0(X, \mathscr{F}) = h^0(X, \mathscr{F}') + h^0(X, \mathscr{F}'') - h^1(X, \mathscr{F}') + h^1(X, \mathscr{F}) - h^1(X, \mathscr{F}'') + \cdots .$$

The process eventually terminates when we observe $\dim \operatorname{im}(H^n(X,\mathscr{F}) \to H^n(X,\mathscr{F}'')) = h^n(X,\mathscr{F}'')$. Moving all of the $h^\bullet(X,\mathscr{F})$ terms to one side, we have shown

$$\chi(X,\mathscr{F}) = \chi(X,\mathscr{F}') + \chi(X,\mathscr{F}'')$$

as desired. □

We need a few more definitions before we prove our first iteration of the Riemann–Roch theorem for curves. Let X be a curve over k, and assume X is geometrically connected (i.e. $h^0(X,\mathscr{O}_X) = 1$).

Definition 4.26 (Degree of point)**.** Let P be a (closed) point on X. The **degree of** P is defined to be

$$\deg P = \dim_k \kappa(P),$$

where $\kappa(P) = \mathscr{O}_{X,P}/\mathfrak{m}_P$ is the residue field at P.

If $k = \overline{k}$ then $\deg P = 1$ for all closed points, so if you're worried about this feel free to assume k is algebraically closed, or $k = \mathbb{C}$. For an example of a higher degree point, consider $\mathbb{P}^1$ as a scheme over $\mathbb{Q}$. Then $i = \sqrt{-1}$ (or rather $[i:1]$ in projective coordinates) isn't a $\mathbb{Q}$-point, but the conjugate pair $\{[i:1],[-i:1]\}$ *is* defined over $\mathbb{Q}$, and its residue field is $\mathbb{Q}(i)$. One can take the closed subscheme cut out by the prime ideal (x^2+1) in the affine patch $\mathcal{A}^1 \subseteq \mathbb{P}^1$ to realize this degree 2 point.

Definition 4.27 (Divisors)**.** A **prime divisor** of X is an irreducible closed subscheme of codimension one. In our setting of curves, this just means a closed point. A (Weil) **divisor** on a curve X is a finite formal sum of points, i.e.

$$D = \sum_P n_P P,$$

where $n_P \in \mathbb{Z}$ such that $n_P = 0$ for all but finitely many $P \in X$. We can think of the group of divisors $\operatorname{Div} X$ as the free abelian group on prime divisors.

The **degree** of a divisor D is

$$\deg D = \sum_P n_P(\deg P).$$

A divisor is **effective**, denoted $D \geq 0$ if $n_P \geq 0$ for all P.

Finally, we need to define $\mathscr{O}_X(D)$ as a (quasicoherent) sheaf on X. For the following, we assume X is irreducible with function field $K(X)$ and normal. This is necessary to properly define zeros and poles of functions, and hence the divisor of a rational function.

Definition 4.28 ($\mathscr{O}_X(D)$, see [Vak17, 14.2.2])**.** Let X be a scheme as above and D a divisor on X. Define the sheaf $\mathscr{O}_X(D)$ by the presheaf

$$U \mapsto \Gamma(U,\mathscr{O}_X(D)) = \left\{t \in K(X)^\times \mid \operatorname{div}|_U t + D|_U \geq 0\right\} \cup \{0\}.$$

This sheaf is quasicoherent and in good circumstances is an invertible sheaf.

One should think of (the sections of) $\mathscr{O}_X(D)$ as constrained by D in the following way. If $D = \sum n_P P$ with $n_P > 0$, then a section of $\mathscr{O}_X(D)$ is *allowed* to have poles at P, while if $n_P < 0$, then a section of $\mathscr{O}_X(D)$ is *required* to have a zero at P. The order of the poles allowed and zeros required are also controlled by n_P.

Once we have $\mathscr{O}_X(D)$ as a sheaf, we can relate its Euler characteristic to its degree. In fact, this motivates the *definition* of degree for line bundles, locally free sheaves, and even more generally quasicoherent sheaves.

Theorem 4.29 (Riemann–Roch I)**.** *Let X be a regular projective irreducible curve over k and D a (Weil) divisor. Then*

$$\chi(X,\mathscr{O}_X(D)) = \deg D + \chi(X,\mathscr{O}_X).$$

Proof. We'll proceed by induction in the following way. We show that the statement is true for the base case $D = 0$, and that it holds for D if and only if it holds for $D \pm P$. Then starting with 0, we can build any divisor D by adding (or subtracting) points.

Since $\mathscr{O}_X(0) = \mathscr{O}_X$ and $\deg 0 = 0$, the base case is trivial. Next, consider the closed subscheme exact sequence for a point P:

$$0 \to \mathscr{O}_X(-P) \to \mathscr{O}_X \to i_*\mathscr{O}_P \to 0,$$

where $i\colon P \hookrightarrow X$. Note that we have identified the ideal sheaf of P with $\mathscr{O}_X(-P)$ (see e.g. [Vak17, Exercise 14.3.B]).

Note also that that $\mathscr{O}_X(D)$ is a line bundle on X. This is where we use the hypothesis that X is regular, which implies it is factorial, and this ensures $\mathscr{O}_X(D)$ is locally free (see e.g. [Vak17, Exercise 14.2.I]). Tensoring with a locally free sheaf is exact, giving us a short exact sequence

$$0 \to \mathscr{O}_X(D-P) \to \mathscr{O}_X(D) \to i_*\mathscr{O}_P(D) \to 0.$$

Using the additivity of χ from Lemma 4.25, we have

$$\chi(X, \mathscr{O}_X(D)) = \chi(X, \mathscr{O}_X(D-P)) + \chi(X, i_*\mathscr{O}_P(D)).$$

We may then make the identifications

$$\chi(X, i_*\mathscr{O}_P(D)) = \chi(P, \mathscr{O}_P(D)) = \chi(P, \mathscr{O}_P) = h^0(P, \mathscr{O}_P) = \dim_k \Gamma(P, \mathscr{O}_P) = \deg P,$$

producing $\chi(X, \mathscr{O}_X(D)) = \chi(X, \mathscr{O}_X(D-P)) + \deg P$.

To complete the proof, we observe that subtracting the desired equation for D by that of $D-P$, we obtain precisely $\deg D - \deg(D-P) = \deg P$. Thus if the theorem holds for either D or $D-P$, it holds for the other; the induction argument described earlier completes the proof. $\square$

Theorem 4.29 comes in other forms. To state them, we quickly define the notion of the genus and state Serre duality for curves. To do all this properly, one should start with differentials, which we don't get into here. See [Vak17, Chapter 21] or [Har77, II.8].

Definition 4.30 (genus)**.** The **arithmetic genus**, denoted $p_a(X)$, of a variety X over k is taken to be

$$p_a(X) = (-1)^n(\chi(X, \mathscr{O}_X) - 1)$$

where $n = \dim X$. In particular for X a curve, we have

$$p_a(X) = 1 - \chi(X, \mathscr{O}_X).$$

The **geometric genus** of X, denoted by g, is the dimension $h^0(X, \omega_X)$, where ω_X is the canonical sheaf, or $h^0(X, \mathscr{O}_X(K))$ where K is the canonical divisor; see [Vak17, §21.5].

With the definition of arithmetic genus, we see that the statement of Riemann–Roch may also be written

$$\chi(X, \mathscr{O}_X(D)) = \deg D + 1 - p_a(X).$$

We should expect the arithmetic and geometric genus to agree in good circumstances. Serre duality helps us relate them.

Theorem 4.31 (Serre duality)**.** *Let X be a (geometrically) irreducible smooth projective variety over k of dimension n. There exists an invertible sheaf ω_X on X such that*

$$h^i(X, \mathscr{F}) = h^{n-i}(X, \omega_X \otimes \mathscr{F}^\vee),$$

for all i and all locally free $\mathscr{F}$ of finite rank.

Serre duality can be taken much further, and some hypotheses may be weakened. What we will need is that ω_X exists and agrees with the canonical sheaf. In the case of curves ($n = 1$) we can identify ω_X with a divisor K, obtaining

$$h^1(X, \mathscr{O}_X(D)) = h^0(X, \mathscr{O}_X(K - D)).$$

In particular, we have

$$g = h^0(X, \mathscr{O}_X(K)) = h^1(X, \mathscr{O}_X) = 1 - \chi(X, \mathscr{O}_X) = p_a(X),$$

realizing the desired agreement of arithmetic and geometric genera when X is smooth. We can further incorporate this into a (possibly more familiar) formulation of Riemann–Roch.

Theorem 4.32 (Riemann–Roch II)**.** *Let X be a smooth projective geometrically irreducible curve over k and D a (Weil) divisor. Then*

$$h^0(X, \mathscr{O}_X(D)) = h^0(X, \mathscr{O}_X(K - D)) + \deg D + 1 - g.$$

Proof. Use Serre duality to identify $g = p_a(X)$ and

$$h^1(X, \mathscr{O}_X(D)) = h^0(X, \mathscr{O}_X(K - D))$$

and move this term of $\chi(X, \mathscr{O}_X(D))$ to the right hand side. □

We conclude this discussion with an example of how these results may be applied to identify curves.

Example 4.33 (genus 1 curve of degree 4 in $\mathbb{P}^3$)**.** Let X be a curve of genus 1 and degree 4 in $\mathbb{P}^3$. We claim that X is the intersection of two quadric surfaces in $\mathbb{P}^3$.

Recall the closed subscheme exact sequence,

$$0 \to \mathscr{I} \to \mathscr{O}_{\mathbb{P}^3} \to \mathscr{O}_X \to 0,$$

where we're abusing notation slightly by referring to $\mathscr{O}_X$ rather than its pushforward as their cohomologies are the same. Twisting by $\mathscr{O}_{\mathbb{P}^3}(2)$, we have

$$0 \to \mathscr{I}(2) \to \mathscr{O}_{\mathbb{P}^3}(2) \to \mathscr{O}_X(2) \to 0,$$

where the global sections of $\mathscr{I}(2)$ may be interpreted as the quadratics vanishing on X. That is, precisely what we're looking to show has dimension at least two!

Computing $\dim H^0(\mathbb{P}^3, \mathscr{O}_{\mathbb{P}^3}(2)) = 10$ by Example 4.23 and $\dim H^0(X, \mathscr{O}_X(2)) = 8$ by Riemann–Roch (we know the degree of $\mathscr{O}_X(2)$ is 8 and the genus is 1), we find that $\dim H^0(\mathbb{P}^3, \mathscr{I}(2)) \geq 2$ in order for the dimension to drop by two. Thus there are at least two distinct quadric surfaces Q_1, Q_2 vanishing on X.

To see that $X = Q_1 \cap Q_2$, we use Bezout's theorem to compute that their intersection (which can't contain a common component since they're distinct) to be a degree 4 curve in $\mathbb{P}^3$. Thus X is contained in this degree 4 curve, and hence they coincide.

4.4.3 Other applications

Recall the definition of reducedness of a scheme.

Definition 4.34 (reduced)**.** A scheme X is **reduced** if all rings of sections $\mathscr{O}_X(U)$ are reduced, i.e. contain no nilpotents.

Given any scheme X, the **reduction of** X, denoted X_{red}, is the unique closed subscheme cut out by the sheaf of nilpotents on X.

Example 4.35 (X is affine if and only if X_{red} is affine)**.** Suppose X is Noetherian. We claim that a scheme X is affine if and only its reduction X_{red} is affine. The forward direction is easy; if $X = \operatorname{Spec} A$, then the reduction $X_{\text{red}} = \operatorname{Spec} A/N$, where N is the ideal of nilpotents. There is surprising content in the converse, making use of the long exact sequence in sheaf cohomology and Serre's criterion, Theorem 4.15.

Suppose X_{red} is affine. Recall we have a closed embedding $i\colon X_{\text{red}} \hookrightarrow X$, with the ideal sheaf of X_{red} given by $\mathscr{N}$, the ideal of nilpotents. Let $\mathscr{F}$ be a coherent sheaf of ideals on X. We have a filtration

$$\mathscr{F} \supset \mathscr{F}\mathscr{N} \supset \mathscr{F}\mathscr{N}^2 \supset \cdots .$$

Notice that for any $i \geq 0$ we have

$$0 \to \mathscr{F}\mathscr{N}^{i+1} \hookrightarrow \mathscr{F}\mathscr{N}^i \twoheadrightarrow \mathscr{F}\mathscr{N}^i/\mathscr{F}\mathscr{N}^{i+1} \to 0,$$

where the rightmost sheaf has the natural structure of an $\mathscr{O}_X/\mathscr{N} \simeq \mathscr{O}_{X_{\text{red}}}$-module. Since

$$H^1(X_{\text{red}}, \mathscr{F}\mathscr{N}^i/\mathscr{F}\mathscr{N}^{i+1}) = 0$$

by Serre's cohomological criterion for affineness, it is enough to prove that $H^1(X, \mathscr{F}\mathscr{N}^i) = 0$ for *some* i, as then we have $H^1(X, \mathscr{F}\mathscr{N}^{i-1}) = 0$, since it's sandwiched between two vanishing H^1's. Going up the filtration, we have $H^1(X, \mathscr{F}) = 0$, and again by Serre's criterion we have X is affine.

To show that $\mathscr{F}\mathscr{N}^i$ has vanishing H^1 for some i, we argue instead that $\mathscr{N}^i = 0$ for $i \gg 0$. Covering X by finitely many affine opens $\operatorname{Spec} A$, we see that $\mathscr{N}|_{\operatorname{Spec} A} \simeq N$, where N is the nilradical of A. Let's further assume A is Noetherian. In the local ring $A_{\mathfrak{p}}$, where we know $\cap_{i=1}^{\infty} \mathfrak{p}^i = 0$, we have $\cap_{i=1}^{\infty} \mathfrak{p}^i = 0$ and the proof of this fact shows that $\mathfrak{p}^i = 0$ for some $i \gg 0$. Since $\operatorname{Spec} A$ has finitely many irreducible components (Noetherian), this means that we can take n to be the maximum such i for finitely many generic points. Then for $f \in N^n$, we have f^n vanishes in all local rings, so it must be that $f = 0$, and hence $N^n = 0$. Doing this for each of the affine opens covering X, we have $\mathscr{N}^i = 0$ for some $i \gg 0$, and as desired, $\mathscr{F}\mathscr{N}^i = 0$ as well, allowing us to use the previous argument.

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