How often does a cubic hypersurface have a point?

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Cubic hypersurfaces

A cubic hypersurface $X_f \subset \mathbb{P}^n$ is cut out by a cubic form f

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As f varies, how often does X_f have a rational point?

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Natural point-counting function

$$N_f(B) = \#\{P \in X_f(\mathbb{Q}) \mid h(P) \leq B\}.$$

Circle method

Introduced by Hardy and Littlewood to count things.

$$N_f(B) = \sum_{\substack{\mathbf{x}\in\mathbb{Z}^{n+1}\\ h(\mathbf{x})\le B}} \int_0^1 \left(e^{2\pi i f(\mathbf{x})\alpha} d\alpha\right) = \int_0^1 \sum_{\substack{\mathbf{x}\in\mathbb{Z}^{n+1}\\ h(\mathbf{x})\le B}} e^{2\pi i f(\mathbf{x})\alpha} d\alpha$$

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Estimate sums when α is

- (major arcs) α well approximated by $\frac{a}{q} \in \mathbb{Q}$, otherwise
- (minor arcs) negligible contribution.

For major arc: count solutions modulo q.

Some history

When n large enough[1], circle method shows

$$N_f(B) \sim c_f B^{n-3}, \quad c_f > 0,$$

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For $n \leq 8$, as f varies, how often does X_f have a rational point?

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Let $h(f) = ||\mathbf{a}|| = \left(\sum_{i,j,k} a_{ijk}^2 \right)^{1/2}$, define natural density

$$\rho_n = \lim_{B \to \infty} \frac{\#\{f \mid h(f) \leq B,\ X_f(\mathbb{Q}) \neq \emptyset\}}{\#\{f \mid h(f) \leq B\}}.$$

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Remark

Counting primitive forms gets same answer, i.e. using $\mathbb{P}^{\binom{n+3}{3}-1}$.

Everywhere local solubility

A variety $X/\mathbb{Q}$ is everywhere locally soluble (ELS) if

$$X(\mathbb{R}) \neq \emptyset \text{ and } X(\mathbb{Q}_p) \neq \emptyset \text{ for all } p.$$

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Density of ELS cubic forms in $\mathbb{Z}[x_0, \ldots, x_n]$:

$$\rho_n^{ELS} = \lim_{B \to \infty} \frac{\#\{f \mid h(f) \leq B,\ X_f \text{ ELS}\}}{\#\{f \mid h(f) \leq B\}}.$$

(Lack of) obstructions

Conjecture (Poonen–Voloch, 2004)

When $n \geq 3$, $\rho_n^{ELS} = \rho_n$.

i.e. local-global principle holds for 100% of cubic hypersurfaces.

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Theorem (Browning–Le Boudec–Sawin, 2023)

When $n \geq 4$, $\rho_n^{ELS} = \rho_n$. In fact true for Fano deg. d $(d, n) \neq (3, 3)$

Computing ρ^{ELS}

Let $\rho_n(p) =$ density of p-adic cubic forms f such that $X_f(\mathbb{Q}_p) \neq \emptyset$.

Theorem (Poonen–Voloch, 2004)

Let $n \geq 2$. We have

$$\rho_n^{ELS} = \prod_p \rho_n(p).$$

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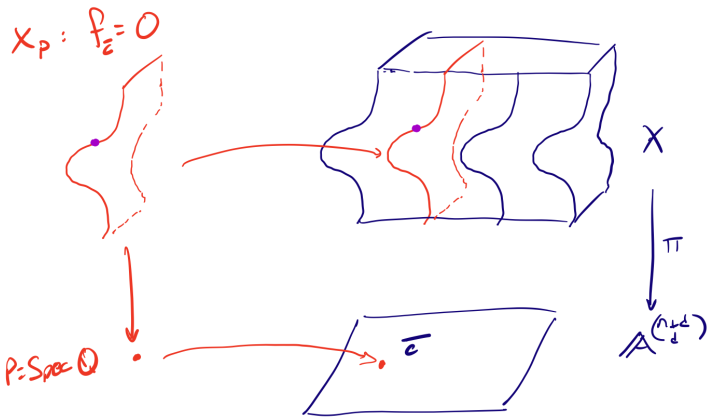
2016 Bright–Browning–Loughran: flexible generalization to families given by fibers of maps to affine/projective space.

Varieties parameterized by fibers



Main result

Theorem (Beneish–K.)

Let $4 \leq n \leq 8$. There exist explicit polynomials $g_n(t), h_n(t) \in \mathbb{Z}[t]$ describing ρ_n exactly as Euler product

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Remark

- We produce g_3, h_3, and conjectural formula for ρ_3.
- Recovers $\rho_n(p) = 1$ for $n \geq 9$.

Cubic 3-folds in $\mathbb{P}^4$

Example

When $n = 4$ we have

$$g_4(p) = \left(p^{46} + 3p^{41} + p^{40} - p^{39} + p^{37} + p^{36} + p^{35} - 3p^{34} + 3p^{27} - p^{26} + p^{25} + p^{19}\right)\left(p^2+1\right)(p+1)^2(p-1)^4$$

$$h_4(p) = 9\left(p^{19}-1\right)\left(p^{17}-1\right)\left(p^{10}+1\right)\left(p^9+1\right)\left(p^9-1\right)\left(p^7-1\right)\left(p^5+1\right)$$

Asymptotically, $\dfrac{g_4(p)}{h_4(p)} \sim \dfrac{1}{9p^{22}}$.

Numerically, $\rho_4 \approx 0.99999999497 = 1 - 5.022 \cdot 10^{-9}$.

Asymptotics and numerics

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n	$\rho_n \approx$	$g_n(p)/h_n(p) \sim$
3	0.999927 (*conj.*)	$1/3p^{10}$
4	$1 - 5.022 \cdot 10^{-9}$	$1/9p^{22}$
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Roadmap

ρ_n = density of cubic forms $f \in \mathbb{Z}[x_0, \ldots, x_n]$ with $X_f(\mathbb{Q}) \neq \emptyset$.

1. $\rho_n = \rho_n^{ELS} = \prod_p \rho_n(p)$ [BLBS23, PV04, BBL16]

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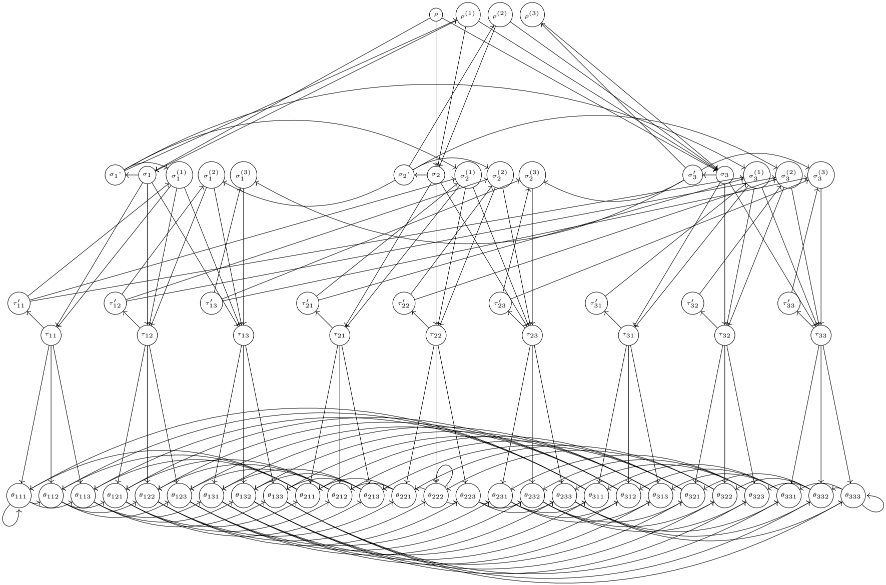
Idea

Express lifting probabilities in terms of each other and recurse to get rational function $\rho_n(p)$.

Full picture



Related results

Plane cubic curves [BCF16a, Bha14]

- ρ_2^{ELS}, $\rho_2(p)$ computed by Bhargava–Cremona–Fisher
- Bhargava: pos. prop. fail Hasse principle (conj. 65%)

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Quadric hypersurfaces [BCF$^+$16b]

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More: certain cubic surfaces [Bro17], (2,2)-curves in $\mathbb{P}^1 \times \mathbb{P}^1$ [FHP21]

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Goal

Compute local probability $\rho_n(p)$ that X_f has $\mathbb{Q}_p$-point.

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Think

Reduce mod p, try to decide solubility with Hensel's lemma.

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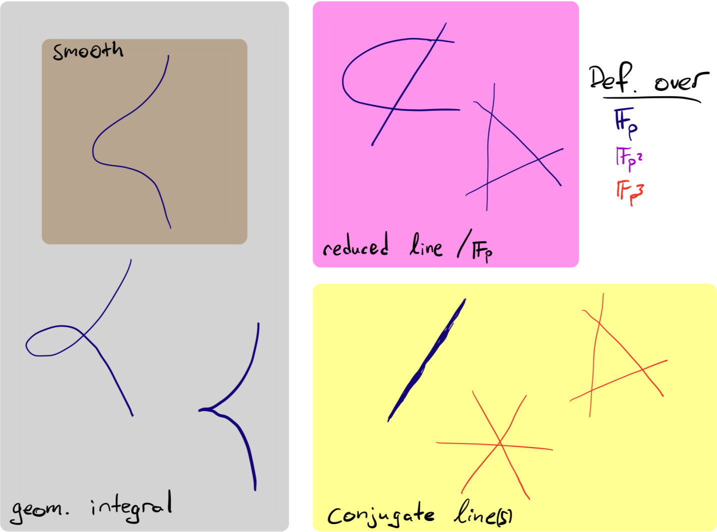
Lemma (Hensel's Lemma)

If $P_0 \in \overline{X}^{\mathrm{sm}}(\mathbb{F}_p)$, then there exists lift $P \in X(\mathbb{Z}_p)$ with $\overline{P} = P_0$.

Cubic hypersurfaces over $\mathbb{F}_p$



When are there always $\mathbb{Q}_p$-points?

Proposition

If $\overline{X_f}$ not a config. of conjugate hyperplanes, then $X_f(\mathbb{Q}_p) \neq \emptyset$.

Proof for curves ($n = 2$).

If geom. integral, use Hasse–Weil bounds on (normalization of) $\overline{X_f}$

$$\#X_f(\mathbb{F}_p) \geq p + 1 - 2\sqrt{p} > 0.$$

All other possibilities contain line defined over $\mathbb{F}_p$. $\square$

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For $n \geq 3$, use inductive argument reducing to coordinate hypersurfaces $X_i = X_f \cap \{x_i = 0\} \subset \mathbb{P}^{n-1}$.

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Warning

There exist irreducible deg. $d > 3$ curves $X \subset \mathbb{P}^2$ with $X(\mathbb{Q}_p) = \emptyset$.

Some (non)examples

Example ($n = 2,\ p = 7$)

Consider the plane cubic curve over $\mathbb{Z}_7$ defined by

$$f(x, y, z) = x^3 + 3x^2y + 3xy^2 - 6xyz + 3y^3 - 6y^2z + 4z^3 = 0.$$

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We have $\overline{f}$ irreducible over $\mathbb{F}_7$, but over $\mathbb{F}_{7^3}$,

$$\overline{f} = \prod_{\sigma \in \mathrm{Gal}(\mathbb{F}_{7^3}/\mathbb{F}_7)} \sigma\left(x + (1+u)y + u^2z\right)$$

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$\overline{X_f}$ is a triangle. $\overline{X_f}(\mathbb{F}_7) = \emptyset \implies X_f(\mathbb{Z}_7) = \emptyset$.

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Example

Suppose $f = g_1(x_0, x_1, x_2) + pg_2(x_3, x_4, x_5) + p^2 g_3(x_6, x_7, x_8)$ for $g_i = 0$ with no nontrivial $\mathbb{F}_p$-solutions.

If $[x_0, \ldots, x_8] \in X_f(\mathbb{Z}_p)$ then $p \mid x_0, x_1, x_2$

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$$= g_2(x_3, x_4, x_5) + pg_3(x_6, x_7, x_8) + p^2 g_1(x_0, x_1, x_2)$$

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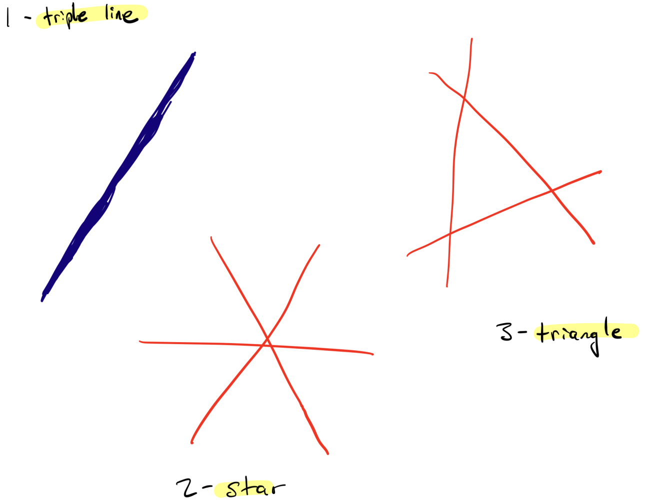
Remark

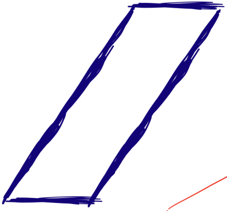
This generalizes, but it ignores cross terms...

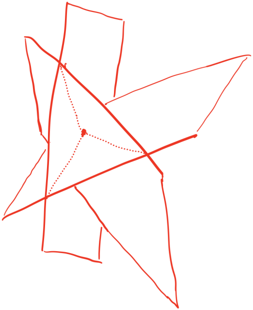
Configurations of conjugate lines



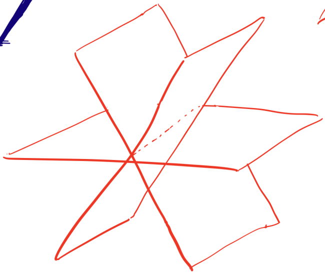
Configurations of conjugate hyperplanes

1- triple (hyper)plane





3- triangle



2- star

Strategy

Goal

Look modulo p and try to decide solubility

$$\rho_n(p) = \xi_{n,0}\sigma_{n,0} + \xi_{n,1}\sigma_{n,1} + \xi_{n,2}\sigma_{n,2} + \xi_{n,3}\sigma_{n,3}$$

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- $\xi_{n,0}$ = prob. f not config. of conj. hyperplanes

Strategy

Goal

Look modulo p and try to decide solubility

$$\rho_n(p) = \xi_{n,0}\cdot 1 + \xi_{n,1}\sigma_{n,1} + \xi_{n,2}\sigma_{n,2} + \xi_{n,3}\sigma_{n,3}$$

- $\xi_{n,i}$ = prob. f has type i
- $\sigma_{n,i}$ = prob. $X_f(\mathbb{Q}_p) \neq \emptyset$ given f type i
- $\xi_{n,0}$ = prob. f not config. of conj. hyperplanes
- $\sigma_{n,0} = 1$

Factorization probabilities

$$\xi_{n,0} = 1 - \frac{p^{3n-3} + 2p^{n+3} + 2p^{n+2} + 2p^{n+1} - 2p^2 - 2p - 3}{3\left(p^2+p+1\right)\left(p^{\binom{n+3}{3}} - 1\right)}$$

$$\xi_{n,1} = \frac{p^{n+1} - 1}{p^{\binom{n+3}{3}} - 1}$$

$$\xi_{n,2} = \frac{\left(p^{2n+1} - p^{n+1} - p^n + 1\right)p}{3\left(p^{\binom{n+3}{3}} - 1\right)}$$

$$\xi_{n,3} = \frac{\left(p^{3n} - p^{2n} - p^{2n+1} - p^{2n-1} + p^{n+1} + p^{n-1} + p^n - 1\right)p^3}{3(p^2+p+1)\left(p^{\binom{n+3}{3}} - 1\right)}$$

Exercise

Convince yourself that probability of a polynomial factoring a certain way is given by a (uniform) rational function.

Phases I, II, and III

Repeat this process three times:

1. Reduce mod p, lift any "easy" solutions with Hensel's lemma

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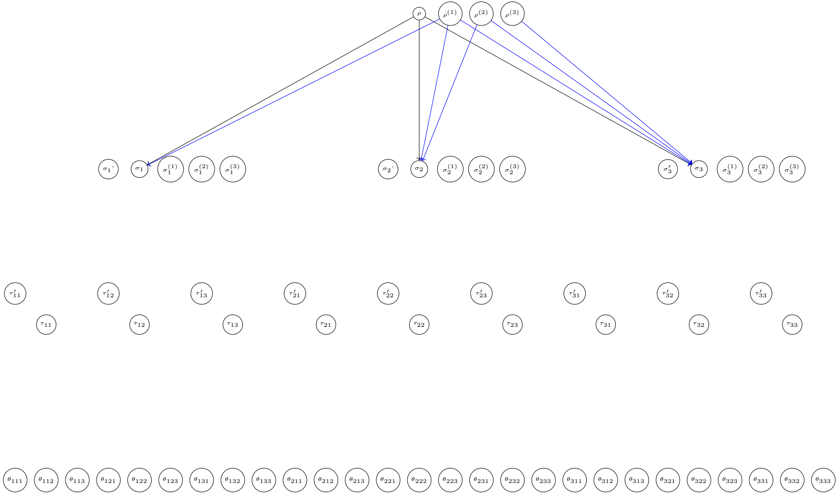
1. Reduce mod p, lift any "easy" solutions with Hensel's lemma
2. Reduction is type $j = 1, 2, 3$ with explicit probability
3. Introduce new lifting probability for each type
4. Relate new lifting probabilities to others

Eventually this terminates: 64 $\mathbb{Q}(p)$-linear relations in 64 unknowns

Setup



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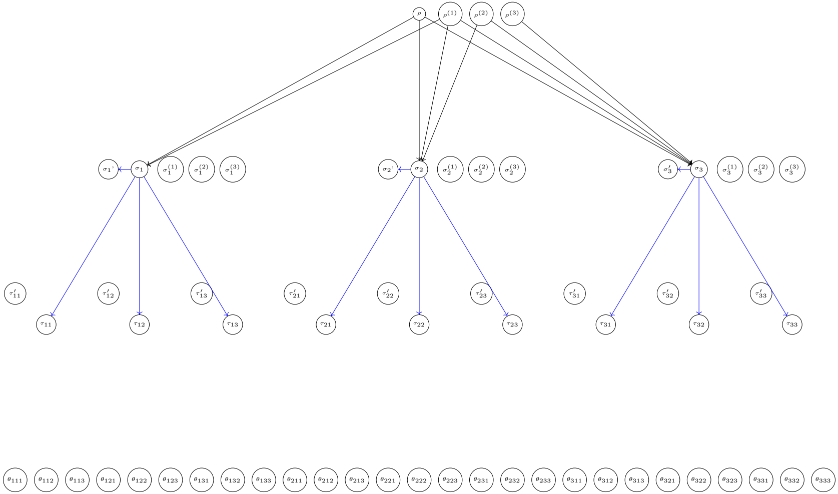
Phase I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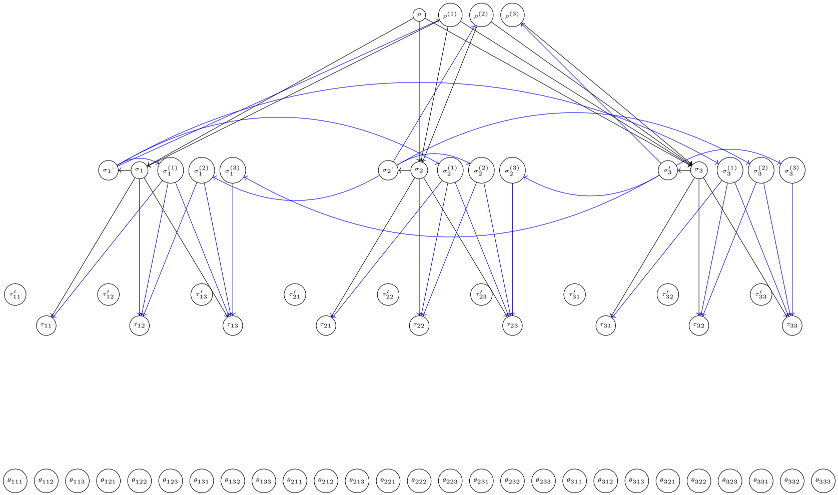
Phase I



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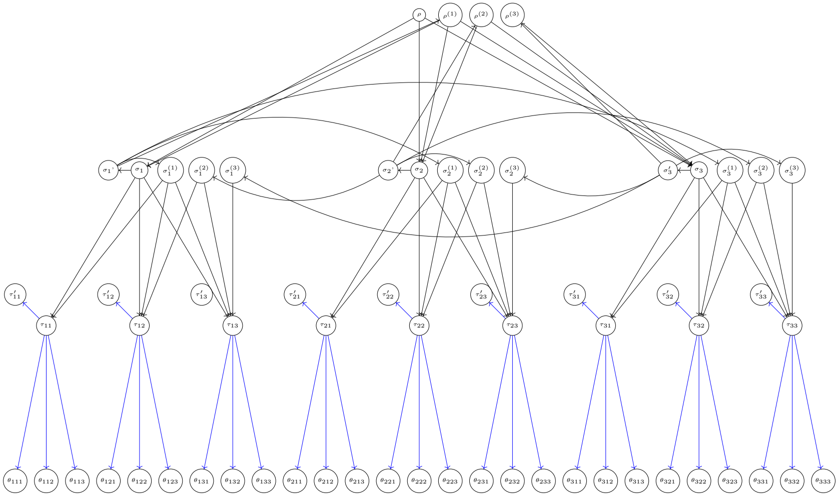
Phase II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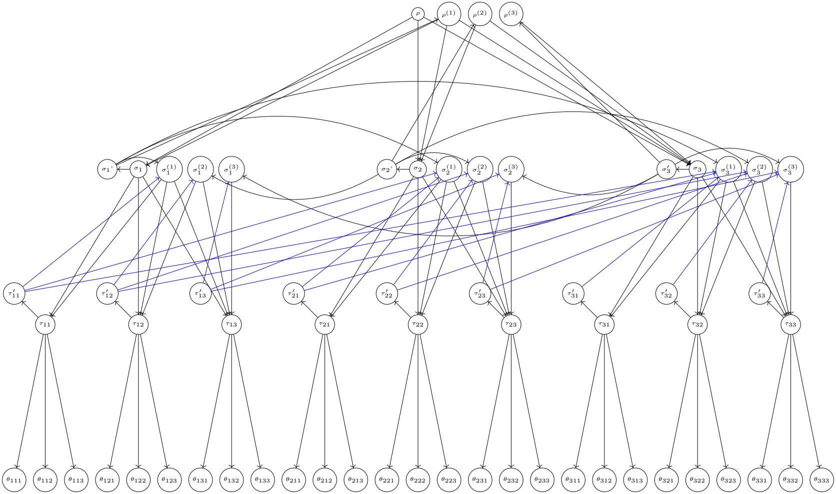
Phase II



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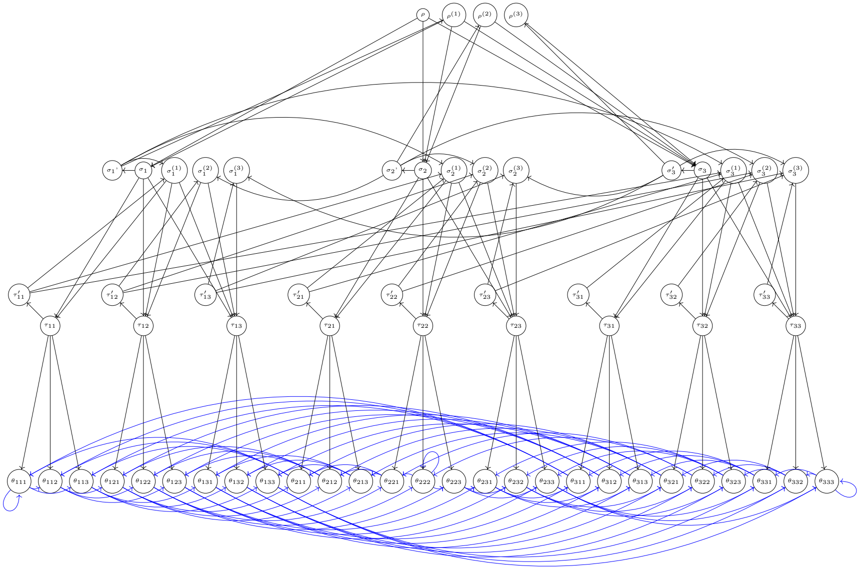
Phase III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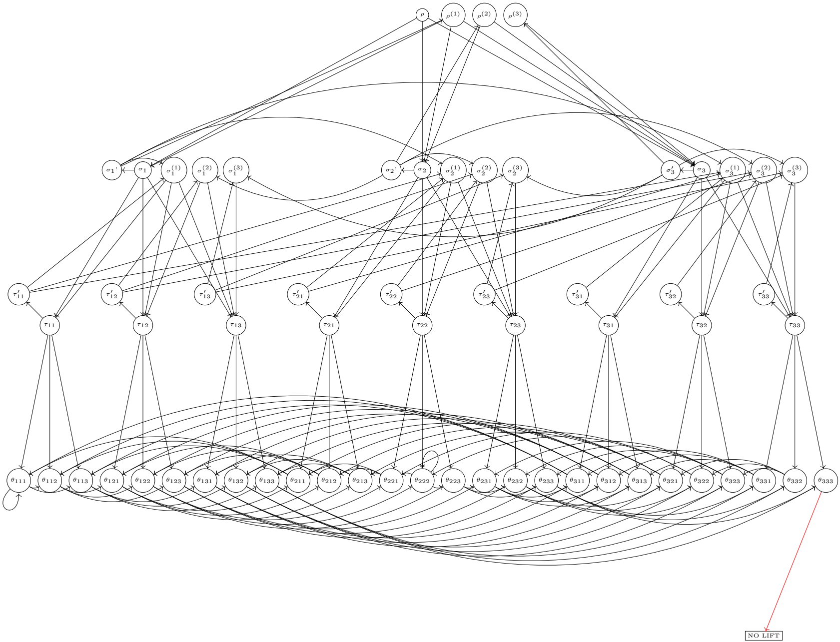
When $n = 8$



Some remarks

Practicalities:

- Solve with Sage symbolic solver
- Block variables $(27 + 27 + 10)$ to speed up

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- Solve with Sage symbolic solver
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In principle, $\rho_n(p) = 1$ for $n \geq 9$ can be seen “by hand”

- No “no lift” sink in the flow chart!
- Capture and describe explicitly Heath–Brown’s observation of necessary/sufficient conditions for f to have local solutions

Density of cubics with a point

Theorem (Beneish–K.)

Let $n \geq 4$ (conjecturally $n \geq 3$). Then $\rho_n = 1$ when $n \geq 9$ and

$$\rho_n = \prod_{p \text{ prime}} \left(1 - \frac{g_n(p)}{h_n(p)}\right) \text{ when } n \leq 8$$

for explicit polynomials $g_n(t), h_n(t) \in \mathbb{Z}[t]$.

n	$\rho_n^{(ELS)} \approx$	$1 - \rho_n(p) \sim$
2	0.9726 [BCF16a]	$1/3p^3$
3	0.999927 (conj.)	$1/3p^{10}$
4	$1 - 5.022 \cdot 10^{-9}$	$1/9p^{22}$
5	$1 - 1.343 \cdot 10^{-15}$	$1/9p^{43}$
6	$1 - 3.502 \cdot 10^{-26}$	$1/9p^{78}$
7	$1 - 5.152 \cdot 10^{-42}$	$1/27p^{129}$
8	$1 - 6.222 \cdot 10^{-64}$	$1/27p^{201}$

Further questions

Let $\rho_{d,n}(p) =$ density of deg. d hypersurfaces in $\mathbb{P}^n$ with $\mathbb{Q}_p$-point

Is $\rho_{d,n}(p)$ always rational function in p?

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Let $\rho_{d,n}(p)$ = density of deg. d hypersurfaces in $\mathbb{P}^n$ with $\mathbb{Q}_p$-point

Is $\rho_{d,n}(p)$ always rational function in p?

- Known for binary forms, $\rho_{d,1}(p) = \rho_{d,1}(1/p)$ [BCFG22]
- Lose uniformity in p in general:

  E.g. $\rho_{4,17}(2) \neq 1$, but $\rho_{4,17}(p) = 1$ for $p \gg 0$
- Heath–Brown: $\rho_{4,n}(p) = 1$ for $n \geq 9126$,

  $\rho_{5,n}(p) = 1$ known for $n \geq 25$, $p \geq 17$ [HB09]

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  $\rho_{5,n}(p) = 1$ known for $n \geq 25$, $p \geq 17$ [HB09]
- Can we predict asymptotics/numerics with less effort?

Thank you I

Thank you for the invitation and for your attention!

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Thank you II

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