Sample Lesson Plan — Christopher Keyes

christopher.keyes@kcl.ac.uk
https://c-keyes.github.io/

Overview

The following lesson plan introduces the chain rule in a Calculus I class. It is based on a lesson I taught in Spring 2021 at Emory University for Math 111 (Calculus I). The original lesson was taught as a 75 minute virtual class — a video of that class can be found here: https://bit.ly/Keyes_S2021.

Objectives and transferable skills

- Students practice identifying composite functions
- Introduce the chain rule to students: $\frac{d}{dx}f(g(x)) = f'(g(x))g'(x)$
- Students practice using the chain rule to differentiate composite functions with examples
- Students apply the chain rule in the context of other problems, including linear approximation and optimization

Lesson structure

This lesson follows the attached handout and is designed for a 75 minute class period. The handout is designed with active learning in mind, so examples can be worked through as a class, or in small groups/independently followed by classroom discussion.

Timeline

- *Review and warmup.* (10 min)
  - Recall the definition of a composite function. Draw this pictorially and identify the domain and range of $f \circ g$ in relation to f and g (e.g. domain of $f \circ g$ is contained inside the domain of g).
  - Students practice writing functions as $f \circ g$, identifying the "inside" and "outside" functions. Here a think-pair-share might be appropriate.
  - Ask students which of these functions they *already* know how to differentiate. *In principle* they could expand $(x^3 - 1)^{100}$ and use the power/sum rules, but in practice this would be a nightmare. Instead, we will learn a better way!
- *Introduce chain rule.* (5 min)
  - Motivated by taking derivatives of composite functions, state chain rule. (Recommend color coding here.)
  - Sketch out the idea of the proof. I like presenting this as follows: since g is continuous, $g(x) \to g(a)$ as $x \to a$. Thus we should expect to find

    $$f'(g(a)) := \lim_{x \to g(a)} \frac{f(x) - f(g(a))}{x - g(a)} = \lim_{x \to a} \frac{f(g(x)) - f(g(a))}{g(x) - g(a)},$$

    where some care must be taken (this is done in the book) to ensure these are equal. Now we see from the limit definition of $g'(a)$ that we can multiply and find $f'(g(a))g'(a)$ is equal to the limit definition of $(f \circ g)'(a)$.
- *Practice with examples.* (30 min)
  - Demonstrate working through a couple examples as a class. I recommend doing this with Example 1 (really as a sanity check with a familiar function) and Example 2 (we already found the "inside" and "outside" functions in the warmup).
  - Have students try the remaining examples on their own or in groups.

- Discuss the examples and solicit student questions. Worth emphasizing is that in Example 4, the chain rule may be used twice. Example 5 is also worth pointing out: knowing information about the critical numbers for f and g can give *some* information about those of $f \circ g$.

- *Special cases.* (15 min)
  - Review special cases and sketch a proof of one (or more if time permits). The exponential is probably the most interesting one here.
  - Try Examples 6 and 7, either as a class or individually/in small groups. If short on time, pick one and leave the other for homework.
  - Note that the inside function for Example 7 had its derivative computed earlier in Example 2.

- *Applications.* (15 min)
  - Students have seen linear approximations before. They should be able to do this without too much trouble. One advantage is that our linear approximation can be used to approximate any (small enough) fractional power of 2. Thus by only doing the approximation work once, we have lots of useful information.
  - Again, students have seen optimization already. The composite function doesn't really change much — it just helps us take the necessary derivative. This is also an opportunity to revisit and reinforce the step-by-step optimization process.

Instructor notes

- *Color coding.* When writing composite functions $(f \circ g)(x) = f(g(x))$, I tended to color code, e.g. $f(g(x))$. Then when taking the chain rule, we have

$$\frac{d}{dx} f(g(x)) = f'(g(x))g'(x).$$

  Student feedback about this color coding was positive, as they shared that it helped them to keep straight the inside/outside functions and their role in the chain rule.

- *Proof of chain rule.* Choosing whether or not to include a proof of the chain rule, or how much of it to sketch out, will depend on the level and pace of your course, and how much the department expects this from students. Either way though, some intuition behind the statement helps the students remember.

  Here's an alternative proof sketch using linear approximation. Fix a real number a in the domain of $g(x)$. Near $x = a$, the linear approximation of g looks like

$$g(x) \approx g'(a)x + b.$$

  Near $x = g(a)$, the linear approximation of f looks like

$$f(x) \approx f'(g(a))x + c.$$

  Composing these two linear functions, we might expect

$$f(g(x)) \approx f'(g(a))(g'(a)x + b) + c = f'(g(a))g'(a)x + f'(g(a))b + c.$$

  Here we see that the slope of the line is equal to the derivative given by the chain rule. This isn't quite a proof, as we haven't justified why we can "compose linear approximations," but it might be intuitive for some.

  Examples at home. If time is running short or the class periods are shorter, several of the examples could possibly be assigned as (optional) homework instead.

Calculus I — The Chain Rule

Instructor: Christopher Keyes

See §3.4 of Stewart's *Calculus* (9th ed.)

Review/motivation

Let f and g be functions. Recall that the **composite function** $f \circ g$ is given by

$$(f \circ g)(x) = f(g(x)).$$

The domain of $f \circ g$ contains precisely the values of x for which $g(x)$ is in the domain of f.

Warmup. Each of the following are composite functions $f(g(x))$. Identify the "outer" function f and the "inner" function g so that $f(g(x))$ is equal to the given function.

- $\sqrt{x^2+1}$

- $(x^3-1)^{100}$

- $e^{\sin x}$

Of the functions above, which *could* we differentiate using our derivative rules?

The chain rule

Chain rule. Let f and g be functions. Suppose g is differentiable at x and f is differentiable at $g(x)$.

Then the derivative of the *composite function* $(f \circ g)(x) = f(g(x))$ is given by

$$\frac{d}{dx}\left(f\left(g(x)\right)\right) = f'(g(x)) \cdot g(x).$$

This is called the **chain rule**. Sometimes, it's written a different way. If $u(v)$ is a differentiable function and $v(t)$ is another differentiable function, we get

$$\frac{du}{dt} = \frac{du}{dv} \cdot \frac{dv}{dt}.$$

The full justification is in the textbook, but we can sketch the idea.

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Example 1. Let's check that this agrees with something we already know. The function $y = x^4$ can be written

$$y = x^4 = (x^2)^2.$$

Use the *chain rule* to compute $\frac{dy}{dx}$ and verify that this agrees with the answer we get with the *power rule.*

Example 2. Now let's see an example we *couldn't* have solved last week without the chain rule. Compute the derivative

$$\frac{d}{dx}\left(\sqrt{x^2+1}\right).$$

Example 3. Compute the derivative of

$$h(x) = \sec(x^2 e^x).$$

Example 4. Compute the derivative of

$$f(x) = \sin(\cos(\tan x)).$$

(*Hint: you may need to use the chain rule twice!*)

Example 5. True or false. Suppose $f(x)$ and $g(x)$ are differentiable functions and $x = c$ is a critical number for $g(x)$. Then c is also a critical number for the composite function $f \circ g$.

What about for $g \circ f$?

Two useful special cases

You should add these to your ever expanding list of derivatives.

$$\frac{d}{dx}\left[g(x)\right]^n = n[g(x)]^{n-1} \cdot g'(x) \qquad \text{for any real number } n,$$

$$\frac{d}{dx}\left[g(x)\right]^{-1} = \frac{-g'(x)}{g(x)^2} \qquad \text{this is the case of } n = 1,$$

$$\frac{d}{dx}b^x = (\ln b)b^x \qquad \text{for any positive base } b > 0.$$

Convince yourself of these by using the *chain rule*!

Example 6. Find the derivative

$$\frac{d}{d\theta}(\cos\theta)^4.$$

Example 7. Evaluate the derivative

$$\frac{d}{dt}\left(3^{\sqrt{t^2+1}}\right).$$

Applications

We can now apply the chain rule to any problem involving derivatives! This includes

- Linearization problems
- Optimization problems
- And more (to come soon!)

Example 8. Estimate $\sqrt{2}$ by linearizing $f(x) = 2^x$ at $a = 0$. Is this an over- or under-estimate of the true value?

Bonus questions:

(a) Use your linear approximation to estimate $\sqrt[3]{2}$.

(b) What else can you use this approximation for?

(c) Does this do a better job than linearizing $f(x) = \sqrt{x}$ at $a = 1$?

Example 9. The number of tweets per hour about the upcoming Math 111 test can be modeled by the function

$$f(t) = 100te^{-0.12t},$$

where t is the number of *days* since Tuesday's rest day at noon ET. At their peak, approximately how many tweets will be occurring per hour?