Towards Artin's conjecture on p-adic forms in low degree

Christopher Keyes (King's College London)

The Wassail of Rational Points

26 January 2026

Acknowledgments

Joint work with Lea Beneish (UNT)

https://arxiv.org/pdf/2508.20192
(v2 coming soon!)

Acknowledgments

Special thanks to my test audience

Setup

Let K be a p-adic field:

$K/\mathbb{Q}_p$	finite extension
$\mathcal{O}_K$	ring of integers
$\mathbb{F}_q$	residue field, $q = p^r$

Let $f \in K[x_0, \ldots, x_n]$ be degree d form

Let $X_f \colon f = 0 \subset \mathbb{P}^n$ be associated degree d hypersurface

Original conjecture

Conjecture (Artin, 1930s)

Let $n \geq d^2$ and $f \in K[x_0, \ldots, x_n]$ degree d. Then $X_f(K) \neq \emptyset$.

Original conjecture

Conjecture (Artin, 1930s)

Let $n \geq d^2$ and $f \in K[x_0, \ldots, x_n]$ degree d. Then $X_f(K) \neq \emptyset$.

Bad news

This is false in general.

Original conjecture

Conjecture (Artin, 1930s)

Let $n \geq d^2$ and $f \in K[x_0, \ldots, x_n]$ degree d. Then $X_f(K) \neq \emptyset$.

Bad news

This is false in general.

1960s Terjanian [Ter66]: explicit counterex. for $d = 4$ over $K = \mathbb{Q}_2$

1980s Lewis–Montgomery [LM83]: infinite families for each p

All known counterexamples: d composite

A counterexample

Example (Terjanian)

Let $K = \mathbb{Q}_2$, $d = 4$, $n = 17$. Set

$$g = x^4 + y^4 + z^4 - (x^2y^2 + x^2z^2 + y^2z^2) - xyz(x + y + z).$$

$g(x, y, z)$ takes values $0, 1 \in \mathbb{Z}/4\mathbb{Z}$.

A counterexample

Example (Terjanian)

Let $K = \mathbb{Q}_2$, $d = 4$, $n = 17$. Set

$$g = x^4 + y^4 + z^4 - (x^2y^2 + x^2z^2 + y^2z^2) - xyz(x + y + z).$$

$g(x, y, z)$ takes values $0, 1 \in \mathbb{Z}/4\mathbb{Z}$.

$$\begin{aligned} f = {} & g(x_0, x_1, x_2) + g(x_3, x_4, x_5) + g(x_6, x_7, x_8) \\ & + 4g(x_9, x_{10}, x_{11}) + 4g(x_{12}, x_{13}, x_{14}) + 4g(x_{15}, x_{16}, x_{17}) \end{aligned}$$

A counterexample

Example (Terjanian)

Let $K = \mathbb{Q}_2$, $d = 4$, $n = 17$. Set

$$g = x^4 + y^4 + z^4 - (x^2y^2 + x^2z^2 + y^2z^2) - xyz(x + y + z).$$

$g(x, y, z)$ takes values $0, 1 \in \mathbb{Z}/4\mathbb{Z}$.

$$\begin{aligned} f = {} & g(x_0, x_1, x_2) + g(x_3, x_4, x_5) + g(x_6, x_7, x_8) \\ & + 4g(x_9, x_{10}, x_{11}) + 4g(x_{12}, x_{13}, x_{14}) + 4g(x_{15}, x_{16}, x_{17}) \end{aligned}$$

$f \equiv 0 \pmod 4 \iff 2 \mid x_i$ for $0 \leq i < 9$.

Rescale and repeat: $X_f(\mathbb{Q}_2) = \emptyset$.

Evidence

Conjecture (Artin, 1930s)

Let $n \geq d^2$ and $f \in K[x_0, \ldots, x_n]$ degree d. Then $X_f(K) \neq \emptyset$.

1920s Hasse: quadratic forms in 5 variables have K-zero

1950s Lewis [Lew52]: cubic forms in 10 variables have K-zero

Evidence

Conjecture (Artin, 1930s)

Let $n \geq d^2$ and $f \in K[x_0, \ldots, x_n]$ degree d. Then $X_f(K) \neq \emptyset$.

1920s Hasse: quadratic forms in 5 variables have K-zero

1950s Lewis [Lew52]: cubic forms in 10 variables have K-zero

1960s Ax–Kochen [AK65]: conjecture holds when $p \gg_d 0$

This is characteristic p, not the size of the residue field q!

Revised conjecture

Conjecture (Artin, revised)

Let d prime and $f \in K[x_0, \ldots, x_{d^2}]$ degree d. Then $X_f(K) \neq \emptyset$.

Revised conjecture

Conjecture (Artin, revised)

Let d prime and $f \in K[x_0, \ldots, x_{d^2}]$ degree d. Then $X_f(K) \neq \emptyset$.

Theorem

For $d \in \{2, 3, 5, 7, 11\}$, Artin's conj. holds when $q \gg_d 0$ [LL65].

Revised conjecture

Conjecture (Artin, revised)

Let d prime and $f \in K[x_0, \ldots, x_{d^2}]$ degree d. Then $X_f(K) \neq \emptyset$.

Theorem

For $d \in \{2, 3, 5, 7, 11\}$, Artin's conj. holds when $q \gg_d 0$ [LL65].

$$q > \begin{cases} 1 & d = 2, 3 \text{ [Lew52]} \\ 5 & d = 5 \text{ [LY96, HB10, Dum17, BK25b]}, \\ & \\ & \end{cases}$$

Revised conjecture

Conjecture (Artin, revised)

Let d prime and $f \in K[x_0, \ldots, x_{d^2}]$ degree d. Then $X_f(K) \neq \emptyset$.

Theorem

For $d \in \{2, 3, 5, 7, 11\}$, Artin's conj. holds when $q \gg_d 0$ [LL65].

$$q > \begin{cases} 1 & d = 2, 3 \text{ [Lew52]} \\ 5 & d = 5 \text{ [LY96, HB10, Dum17, BK25b]}, \\ 883 & d = 7 \text{ [Woo08]}, \\ 8053 & d = 11 \text{ [Woo08]}. \end{cases}$$

Revised conjecture

Conjecture (Artin, revised)

Let d prime and $f \in K[x_0, \ldots, x_{d^2}]$ degree d. Then $X_f(K) \neq \emptyset$.

Theorem

For $d \in \{2, 3, 5, 7, 11\}$, Artin's conj. holds when $q \gg_d 0$ [LL65].

$$q > \begin{cases} 1 & d = 2, 3 \text{ [Lew52]} \\ 5 & d = 5 \text{ [LY96, HB10, Dum17, BK25b]}, \\ 679 & d = 7 \text{ [Woo08, BK25a]}, \\ 7393 & d = 11 \text{ [Woo08, BK25a]}. \end{cases}$$

Revised conjecture

Conjecture (Artin, revised)

Let d prime and $f \in K[x_0, \ldots, x_{d^2}]$ degree d. Then $X_f(K) \neq \emptyset$.

Theorem

For $d \in \{2, 3, 5, 7, 11\}$, Artin's conj. holds when $q \gg_d 0$ [LL65].

$$q > \begin{cases} 1 & d = 2, 3 \text{ [Lew52]} \\ 5 & d = 5 \text{ [LY96, HB10, Dum17, BK25b]}, \\ 679 & d = 7 \text{ [Woo08, BK25a]}, \\ 7393 & d = 11 \text{ [Woo08, BK25a]}. \end{cases}$$

Pop quiz!

What is so special about $\{2, 3, 5, 7, 11\}$? (Answer revealed shortly)

This talk

Theorem

For $d \in \{2, 3, 5, 7, 11\}$, Artin's conj. holds when $q \gg_d 0$ [LL65].

$$q > \begin{cases} 679 & d = 7 \text{ [Woo08, BK25a]}, \\ 7393 & d = 11 \text{ [Woo08, BK25a]}. \end{cases}$$

1. Reduced forms of Laxton–Lewis
2. Effective Bertini theorems for irreducibility
3. Find nice plane curves $C \subset \overline{X_f} \rightsquigarrow X_f(K) \neq \emptyset$

Reduced forms

Definition (reduced [LL65])

$f(x_0, \dots, x_n) \in \mathcal{O}_K[x_0, \dots, x_n]$ is reduced if

$$\mathrm{Res}(f_{x_0}, \dots, f_{x_n}) \neq 0$$

and has *minimal valuation* in $\mathrm{GL}_{n+1}(K)$-orbit.

Why do we care?

- Suffices to check Artin's Conjecture on reduced forms f
- f reduced, $n \geq d^2 \implies \overline{f}$ has no linear factors over $\overline{\mathbb{F}_q}$

Low degrees

Pop quiz!

What is so special about $\{2, 3, 5, 7, 11\}$?

Low degrees

Pop quiz!

What is so special about $\{2, 3, 5, 7, 11\}$?

Not sum of composites!

f reduced $\implies \overline{f}$ has geom. irr. factor of unique degree def.$/\mathbb{F}_q$

Low degrees

Pop quiz!

What is so special about $\{2, 3, 5, 7, 11\}$?

Not sum of composites!

f reduced $\implies \overline{f}$ has geom. irr. factor of unique degree def./$\mathbb{F}_q$

- $d = 5$: $\overline{f}$ factors as (5) or $(2, 3)$ over $\overline{\mathbb{F}_q}$
- $d = 13$: $\overline{f}$ can factor as $(3^3, 2^2)$ — totally nonreduced :(

Proof sketch

Proposition (Laxton–Lewis [LL65])

Let $d \in \{2, 3, 5, 7, 11\}$. Then Artin's Conj. holds for $q \gg_d 0$.

Proof sketch. Suffices to check on f reduced.

$g \mid f$ factor of unique degree defined over $\mathbb{F}_q$.

Proof sketch

Proposition (Laxton–Lewis [LL65])

Let $d \in \{2, 3, 5, 7, 11\}$. Then Artin's Conj. holds for $q \gg_d 0$.

Proof sketch. Suffices to check on f reduced.

$g \mid f$ factor of unique degree defined over $\mathbb{F}_q$.

Lang–Weil: $\#X_g(\mathbb{F}_q) > \#X_f(\mathbb{F}_q)^{\mathrm{sing}}$. Lift to $X_f(K)$. □

Proof sketch

Proposition (Laxton–Lewis [LL65])

Let $d \in \{2, 3, 5, 7, 11\}$. Then Artin's Conj. holds for $q \gg_d 0$.

Proof sketch. Suffices to check on f reduced.

$g \mid f$ factor of unique degree defined over $\mathbb{F}_q$.

Lang–Weil: $\#X_g(\mathbb{F}_q) > \#X_f(\mathbb{F}_q)^{\mathrm{sing}}$. Lift to $X_f(K)$. □

Question

How do we make this (most) effective?

Bertini theorems

Let k be a field

Let $H \subset \mathbb{P}^n$ be a hyperplane

Theorem (Bertini in words)

If $X \subset \mathbb{P}^n$ is smooth, then generically so is $X \cap H$.

Bertini theorems

Let k be a field

Let $H \subset \mathbb{P}^n$ be a hyperplane

Theorem (Bertini in words)

If $X \subset \mathbb{P}^n$ has property $\mathcal{P}$, then generically so does $X \cap H$.

Bertini theorems

Let k be a field

Let $H \subset \mathbb{P}^n$ be a hyperplane

Theorem (Bertini in words)

If $X \subset \mathbb{P}^n$ has property $\mathcal{P}$, then generically so does $X \cap H$.

Can intersect with planes when $n \geq 3$:

Theorem

If X_f is geom. irrreducible, then generically so is $X_f \cap P$.

Bertini theorems

Let k be a field

Let $H \subset \mathbb{P}^n$ be a hyperplane

Theorem (Bertini in words)

If $X \subset \mathbb{P}^n$ has property $\mathcal{P}$, then generically so does $X \cap H$.

Can intersect with planes when $n \geq 3$:

Theorem

If X_f is geom. irreducible, then generically so is $X_f \cap P$.

Caveat

If $k = \mathbb{F}_q$, this does not guarantee existence!

Effective Bertini theorems

Suppose $f \in \mathbb{F}_q[x_0, \ldots, x_n]$ degree d form, irreducible over $\overline{\mathbb{F}_q}$.

Theorem (Cafure–Matera [CM06], Beneish–K. [BK25a])

(i) *If $q > \frac{d}{8}\left(3d^3 - 2d^2 - 3d + 2\right)$, there exists P such that $X_f \cap P$ is geometrically irreducible.*

Effective Bertini theorems

Suppose $f \in \mathbb{F}_q[x_0, \ldots, x_n]$ degree d form, irreducible over $\overline{\mathbb{F}_q}$.

Theorem (Cafure–Matera [CM06], Beneish–K. [BK25a])

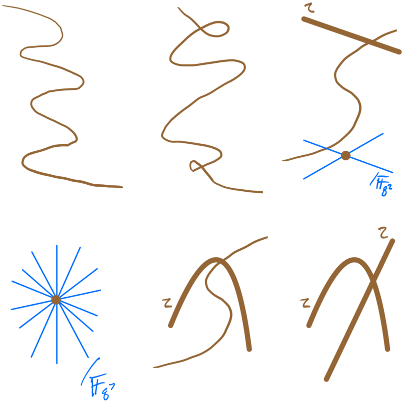
(i) *If* $q > \frac{d}{8}\left(3d^3 - 2d^2 - 3d + 2\right)$, *there exists* P *such that* $X_f \cap P$ *is geometrically irreducible.*

(ii) *Fix a positive integer* $D < d$. *If*

$$q > \frac{dD}{8}\left(-D^3 + 4dD^2 - 6D^2 + 12dD - 11D + 8d - 6\right),$$

there exists P *s.t.* $X_f \cap P$ *has no component of deg.* $\leq D$.

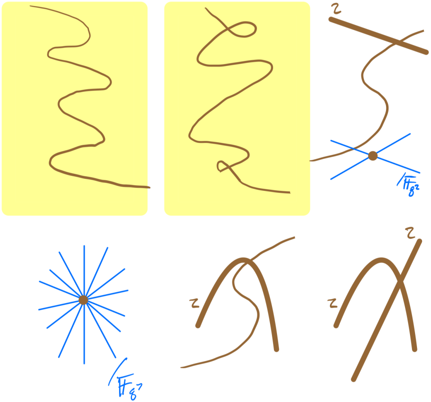
Example ($d = 7$)



(i) If $q > 798$, $\exists$ P such that $X_f \cap P$ is geometrically irreducible.

(ii) If $q > 126$, $\exists$ P such that $X_f \cap P$ contains no lines$/\overline{\mathbb{F}_q}$.

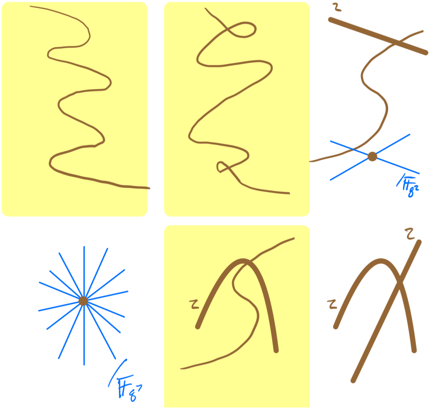
Example ($d = 7$)



(i) If $q > 798$, $\exists$ P such that $X_f \cap P$ is geometrically irreducible.

(ii) If $q > 126$, $\exists$ P such that $X_f \cap P$ contains no lines$/\overline{\mathbb{F}_q}$.

Example ($d = 7$)



(i) If $q > 798$, $\exists$ P such that $X_f \cap P$ is geometrically irreducible.

(ii) If $q > 126$, $\exists$ P such that $X_f \cap P$ contains no lines$/\overline{\mathbb{F}_q}$.

Yet another effective Bertini theorem

Eff. Bertini result for P containing fixed line L

Theorem (Beneish–K. [BK25a, Corollary 2.7])

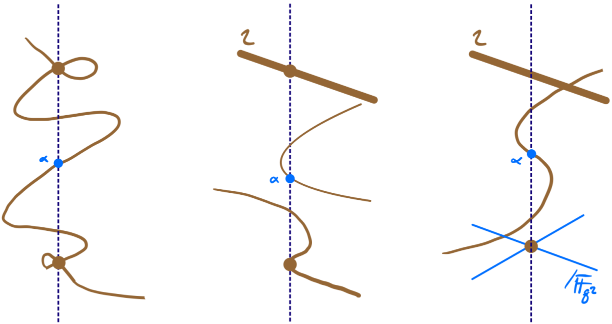
Suppose f irreducible$/\overline{\mathbb{F}_q}$ of degree d.

Suppose L is line def.$/\mathbb{F}_q$ meeting X_f transversely at $\alpha \in X_f(\overline{\mathbb{F}_q})$.

If $q > \frac{D}{8}\left(-D^3 + 4D^2d - 6D^2 + 12Dd - 11D + 8d - 6\right)$, then $\exists\ P \supset L$ s.t. $X_f \cap P$ contains no degree $\leq D$ curve through α.

Example ($d = 7$)

Suppose L meets X_f transversely at $\alpha \in X_f(\overline{\mathbb{F}_q})$

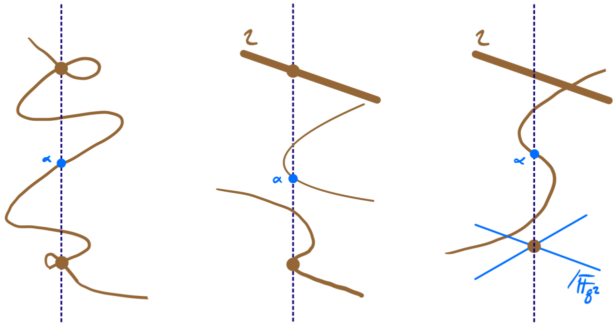


$D = 1$ If $q > 18$, there exists such P

$D = 2$ If $q > 69$, there exists such P

Example ($d = 7$)

Suppose L meets X_f transversely at $\alpha \in X_f(\overline{\mathbb{F}_q})$

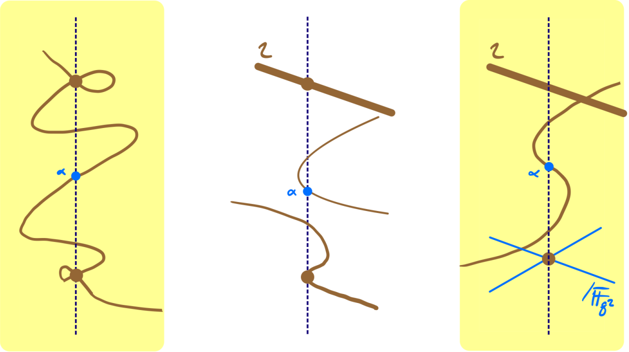


$D = 1$ If $q > 18$, there exists such P

$D = 2$ If $q > 69$, there exists such P

Example ($d = 7$)

Suppose L meets X_f transversely at $\alpha \in X_f(\overline{\mathbb{F}_q})$



$D = 1$ If $q > 18$, there exists such P

$D = 2$ If $q > 69$, there exists such P

Point counts

Suppose $C = X_f \cap P$ is smooth:

$$\#C(\mathbb{F}_q) \geq q + 1 - \frac{(d-1)(d-2)}{2} \lfloor 2\sqrt{q} \rfloor$$

Point counts

Suppose $C = X_f \cap P$ has a geom. int. component defined$/\mathbb{F}_q$:

$$\#C(\mathbb{F}_q)^{\mathrm{sm}} \geq q + 1 - \frac{(d-1)(d-2)}{2} \lfloor 2\sqrt{q} \rfloor$$

Point counts

Suppose $C = X_f \cap P$ has a geom. int. component defined$/\mathbb{F}_q$:

$$\#C(\mathbb{F}_q)^{\mathrm{sm}} \geq q + 1 - \frac{(d-1)(d-2)}{2} \lfloor 2\sqrt{q} \rfloor$$

Example ($d = 7$)

Let C be a geom. irreducible plane curve of degree 7:

- If $q > 883$ then $C(\mathbb{F}_q)^{\mathrm{sm}} \neq \emptyset$.

Point counts

Suppose $C = X_f \cap P$ has a geom. int. component defined/$\mathbb{F}_q$:

$$\#C(\mathbb{F}_q)^{\mathrm{sm}} \geq q + 1 - \frac{(d-1)(d-2)}{2} \lfloor 2\sqrt{q} \rfloor$$

Example ($d = 7$)

Let C be a geom. irreducible plane curve of degree 7:

- If $q > 883$ then $C(\mathbb{F}_q)^{\mathrm{sm}} \neq \emptyset$.
- If $q > 679$ and $\#C(\mathbb{F}_q) \geq 2$, then $C(\mathbb{F}_q)^{\mathrm{sm}} \neq \emptyset$.

Wooley's improvement

Theorem (Wooley [Woo08])

For $d \in \{5, 7, 11\}$, Artin's conj. holds when

$$q > \begin{cases} 121 & d = 5, \\ 883 & d = 7, \\ 8053 & d = 11. \end{cases}$$

- Reduced $\implies$ f has no linear factors
- Eff. Bertini $\implies$ $\exists\ P$ s.t. $C = \overline{X_f} \cap P$ contains no lines
- Point count: C irreducible limiting case

To improve: ensure $C(\mathbb{F}_q)$ has points!

Latest improvement

For concreteness: $f \in K[x_0, \ldots, x_{49}]$ degree 7 reduced form.

$\overline{f}$ factors as $(7), (5, 2), (4, 3), (3, 2, 2), (3, 2^2)$ $/\overline{\mathbb{F}_q}$

Latest improvement

For concreteness: $f \in K[x_0, \ldots, x_{49}]$ degree 7 reduced form.

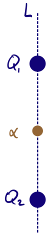
$\overline{f}$ factors as $(7), (5,2), (4,3), (3,2,2), (3,2^2)$ $/\overline{\mathbb{F}_q}$

Latest improvement

For concreteness: $f \in K[x_0, \ldots, x_{49}]$ degree 7 reduced form.

$\overline{f}$ factors as $(7), (5,2), (4,3), (3,2,2), (3,2^2)$ $/\overline{\mathbb{F}_q}$

- $\exists\ Q_1, Q_2 \in X_f(\mathbb{F}_q)$, L between them satisfies hyp. [BK25a, Prop. 4.5]

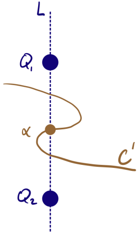


Latest improvement

For concreteness: $f \in K[x_0, \ldots, x_{49}]$ degree 7 reduced form.

$\overline{f}$ factors as $(7), (5,2), (4,3), (3,2,2), (3,2^2)$ $/\overline{\mathbb{F}_q}$

- $\exists\ Q_1, Q_2 \in X_f(\mathbb{F}_q)$, L between them satisfies hyp. [BK25a, Prop. 4.5]
- Eff. Bertini $\implies \exists\ P \supset L$ such that $\overline{X_f} \cap P$ has no line or quadric through α

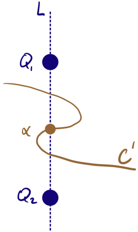


Latest improvement

For concreteness: $f \in K[x_0, \ldots, x_{49}]$ degree 7 reduced form.

$\overline{f}$ factors as $(7), (5,2), (4,3), (3,2,2), (3,2^2)$ $/\overline{\mathbb{F}_q}$

- $\exists\ Q_1, Q_2 \in X_f(\mathbb{F}_q)$, L between them satisfies hyp. [BK25a, Prop. 4.5]
- Eff. Bertini $\implies \exists\ P \supset L$ such that $\overline{X_f} \cap P$ has no line or quadric through α
- Component containing α must be def. $/\mathbb{F}_q$

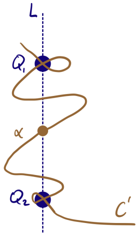


Latest improvement

For concreteness: $f \in K[x_0, \ldots, x_{49}]$ degree 7 reduced form.

$\overline{f}$ factors as $(7), (5,2), (4,3), (3,2,2), (3,2^2)$ $/\overline{\mathbb{F}_q}$

- $\exists\ Q_1, Q_2 \in X_f(\mathbb{F}_q)$, L between them satisfies hyp. [BK25a, Prop. 4.5]
- Eff. Bertini $\implies \exists\ P \supset L$ such that $\overline{X_f} \cap P$ has no line or quadric through α
- Component containing α must be def. $/\mathbb{F}_q$

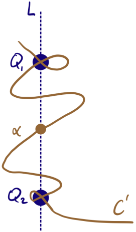


Latest improvement

For concreteness: $f \in K[x_0, \ldots, x_{49}]$ degree 7 reduced form.

$\overline{f}$ factors as $(7), (5,2), (4,3), (3,2,2), (3,2^2)$ $/\overline{\mathbb{F}_q}$

- $\exists\ Q_1, Q_2 \in X_f(\mathbb{F}_q)$, L between them satisfies hyp. [BK25a, Prop. 4.5]
- Eff. Bertini $\implies \exists\ P \supset L$ such that $\overline{X_f} \cap P$ has no line or quadric through α
- Component containing α must be def. $/\mathbb{F}_q$



Upgraded point counts: $q > 679$ suffices for $X_f(\mathbb{F}_q)^{\mathrm{sm}} \neq \emptyset$

Final thoughts

Theorem (1960s – 2025+, many authors)

For $d \in \{5, 7, 11\}$, Artin's conj. holds when

$$q > \begin{cases} 679 & d = 7, \\ 7393 & d = 11. \end{cases}$$

Final thoughts

Theorem (1960s – 2025+, many authors)

For $d \in \{5, 7, 11\}$, Artin's conj. holds when

$$q > \begin{cases} 679 & d = 7, \\ 7393 & d = 11. \end{cases}$$

- Ongoing: improve results for $d = 5$ via computational techniques
- What about prime $d > 11$?
- What about $K/\mathbb{Q}_p$ highly ramified?

Thank you for your attention!

A word about the proofs

Parametrize (affine patch of) $X_f \cap P$ by

$$f_\delta(1, X, X + \delta_2 Y, \ldots, X + \delta_n Y) = 0$$

Kaltofen: factorization algorithm for $f(X, Y)$ [Kal95, CM06]

Algorithm sketch

f_δ has factor of degree $\leq D$

$\implies$ certain overdetermined linear system has $\overline{\mathbb{F}_q}$-solution

$\implies$ polynomial $\Psi_D(\boldsymbol{\delta}) = 0$.

Determine $\deg \Psi_D$. Use to bound number of "bad" $\boldsymbol{\delta}$.

James Ax and Simon Kochen, Diophantine problems over local fields. I, Amer. J. Math. **87** (1965), 605–630. MR 184930

Lea Beneish and Christopher Keyes, An effective Bertini theorem and an application to p-adic fields, Submitted, 2025, Available at https://arxiv.org/pdf/2508.20192.

______, On Artin's Conjecture for p-adic quintic forms, In prepration, 2025.

Antonio Cafure and Guillermo Matera, Improved explicit estimates on the number of solutions of equations over a finite field, Finite Fields Appl. **12** (2006), no. 2, 155–185. MR 2206396

Jan H. Dumke, p-adic zeros of quintic forms, Math. Comp. **86** (2017), no. 307, 2469–2478. MR 3647967

D. R. Heath-Brown, Zeros of p-adic forms, Proc. Lond. Math. Soc. (3) **100** (2010), no. 2, 560–584. MR 2595750

E. Kaltofen, Effective Noether irreducibility forms and applications, Journal of Computer and System Sciences **50** (1995), no. 2, 274–295.

D. J. Lewis, Cubic homogeneous polynomials over p-adic number fields, Annals of Mathematics **56** (1952), no. 3, 473–478.

R. R. Laxton and D. J. Lewis, Forms of degrees 7 and 11 over $\mathfrak{p}$-adic fields, Proc. Sympos. Pure Math., Vol. VIII, Amer. Math. Soc., Providence, RI, 1965, pp. 16–21. MR 175884

D. J. Lewis and Hugh L. Montgomery, On zeros of p-adic forms, Michigan Math. J. **30** (1983), no. 1, 83–87. MR 694931

David B. Leep and Charles C. Yeomans, Quintic forms over p-adic fields, J. Number Theory **57** (1996), no. 2, 231–241. MR 1382749

Guy Terjanian, Un contre-exemple à une conjecture d'Artin, C. R. Acad. Sci. Paris Sér. A-B **262** (1966), A612. MR 197450

Trevor D. Wooley, Artin's conjecture for septic and unidecic forms, Acta Arith. **133** (2008), no. 1, 25–35. MR 2413363