How often does a cubic hypersurface have a point?

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Show and Tell

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Cubic hypersurfaces

A cubic hypersurface $X_f \subset \mathbb{P}^n$ is cut out by a cubic form f

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As f varies, how often does X_f have a rational point?

Counting points

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Write $P \in X_f(\mathbb{Q})$ as $P = [x_0, \ldots, x_n]$ with $x_i \in \mathbb{Z}$ coprime.

Define height of a point

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Natural point-counting function

$$N_f(B) = \#\{P \in X_f(\mathbb{Q}) \mid h(P) \leq B\}.$$

Heuristics

- A "random" $\mathbf{x} \in \mathbb{P}^n(\mathbb{Q})$ with $h(\mathbf{x}) \leq B$ has $f(\mathbf{x}) = O_f(B^3)$.
- Expect likelihood $f(\mathbf{x}) = 0$ about $\dfrac{1}{B^3}$.

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Remark

When $n = 2$, $f = 0$ is an elliptic curve. In this case,

$$N_f(B) \sim c_f (\log B)^{\mathrm{rk}(E)/2}.$$

Circle method

Introduced by Hardy and Littlewood to count things.

$$N_f(B) = \sum_{\substack{\mathbf{x}\in\mathbb{Z}^{n+1}\\ h(\mathbf{x})\leq B}} \int_0^1 \left(e^{2\pi i f(\mathbf{x})\alpha} d\alpha\right) = \int_0^1 \sum_{\substack{\mathbf{x}\in\mathbb{Z}^{n+1}\\ h(\mathbf{x})\leq B}} e^{2\pi i f(\mathbf{x})\alpha} d\alpha$$

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Estimate sums when α is

- (major arcs) α well approximated by $\frac{a}{q} \in \mathbb{Q}$, otherwise
- (minor arcs) negligible contribution.

For major arc: count solutions modulo q.

Some history

When n large enough[1], circle method shows

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Let $h(f) = ||\mathbf{a}|| =$ Euclidean norm, define natural density

$$\rho_n = \lim_{B\to\infty} \frac{\#\{f \mid h(f) \leq B,\ X_f(\mathbb{Q}) \neq \emptyset\}}{\#\{f \mid h(f) \leq B\}}.$$

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Remark

- Counting primitive forms gets same answer, i.e. using $\mathbb{P}^{\binom{n+3}{3}-1}$

Everywhere local solubility

A variety $X/\mathbb{Q}$ is everywhere locally soluble (ELS) if

$$X(\mathbb{R}) \neq \emptyset \text{ and } X(\mathbb{Q}_p) \neq \emptyset \text{ for all } p.$$

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Density of ELS cubic forms in $\mathbb{Z}[x_0, \ldots, x_n]$:

$$\rho_n^{ELS} = \lim_{B \to \infty} \frac{\#\{f \mid h(f) \leq B,\ X_f \text{ ELS}\}}{\#\{f \mid h(f) \leq B\}}.$$

(Lack of) obstructions

Conjecture (Poonen–Voloch, 2004)

When $n \geq 3$, $\rho_n^{ELS} = \rho_n$.

i.e. local-global principle holds for 100% of cubic hypersurfaces.

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Theorem (Browning–Le Boudec–Sawin, 2023)

When $n \geq 4$, $\rho_n^{ELS} = \rho_n$. In fact true for Fano deg. d $(d, n) \neq (3, 3)$

Computing ρ^{ELS}

Let $\rho_n(p)$ = density of p-adic cubic forms f such that $X_f(\mathbb{Q}_p) \neq \emptyset$.

Theorem (Poonen–Voloch, 2004)

Let $n \geq 2$. We have

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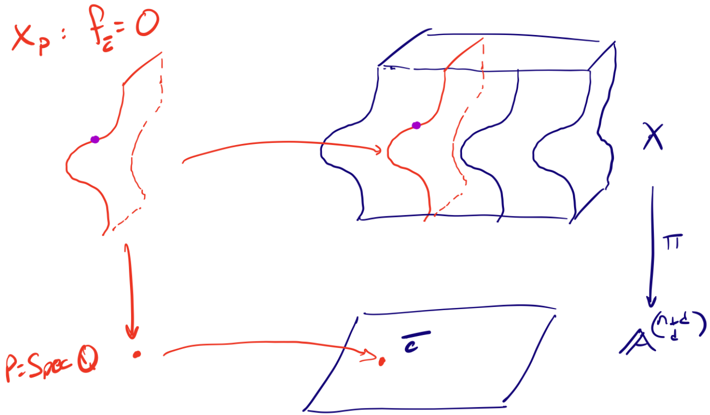
2016 Bright–Browning–Loughran: flexible generalization to families given by fibers of maps to affine/projective space.

Varieties parameterized by fibers



Main result

Theorem (Beneish–K.)

Let $4 \leq n \leq 8$. There exist explicit polynomials $g_n(t), h_n(t) \in \mathbb{Z}[t]$ describing ρ_n exactly as Euler product

$$\rho_n = \prod_p \rho_n(p) = \prod_p \left(1 - \frac{g_n(p)}{h_n(p)}\right).$$

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Remark

- We produce g_3, h_3, and conjectural formula for ρ_3.
- Recovers $\rho_n(p) = 1$ for $n \geq 9$.

Cubic 3-folds in $\mathbb{P}^4$

Example

When $n = 4$ we have

$$g_4(p) = \left(p^{46} + 3p^{41} + p^{40} - p^{39} + p^{37} + p^{36} + p^{35} - 3p^{34} + 3p^{27} - p^{26} + p^{25} + p^{19}\right)\left(p^2+1\right)\left(p+1\right)^2\left(p-1\right)^4$$

$$h_4(p) = 9\left(p^{19}-1\right)\left(p^{17}-1\right)\left(p^{10}+1\right)\left(p^9+1\right)\left(p^9-1\right)\left(p^7-1\right)\left(p^5+1\right)$$

Asymptotically, $\dfrac{g_4(p)}{h_4(p)} \sim \dfrac{1}{9p^{22}}$.

Numerically, $\rho_4 \approx 0.99999999497 = 1 - 5.022 \cdot 10^{-9}$.

Asymptotics and numerics

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3	0.999927 (*conj.*)	$1/3p^{10}$
4	$1 - 5.022 \cdot 10^{-9}$	$1/9p^{22}$
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9	1	0

Roadmap

ρ_n = density of cubic forms $f \in \mathbb{Z}[x_0, \ldots, x_n]$ with $X_f(\mathbb{Q}) \neq \emptyset$.

1. $\rho_n = \rho_n^{ELS} = \prod_p \rho_n(p)$ [BLBS23, PV04, BBL16]

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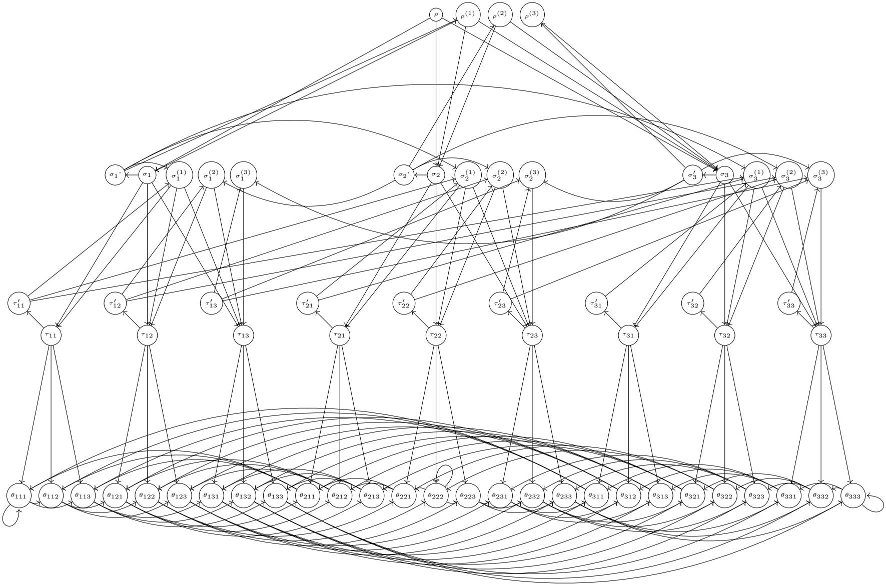
Idea

Express lifting probabilities in terms of each other and recurse to get rational function $\rho_n(p)$.

Full picture



Related results

Plane cubic curves [BCF16a, Bha14]

- ρ_2^{ELS}, $\rho_2(p)$ computed by Bhargava–Cremona–Fisher
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Quadric hypersurfaces [BCF+16b]

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More: certain cubic surfaces [Bro17], (2,2)-curves in $\mathbb{P}^1 \times \mathbb{P}^1$ [FHP21]

Computing the local factors

Goal

Compute local probability $\rho_n(p)$ that X_f has $\mathbb{Q}_p$-point.

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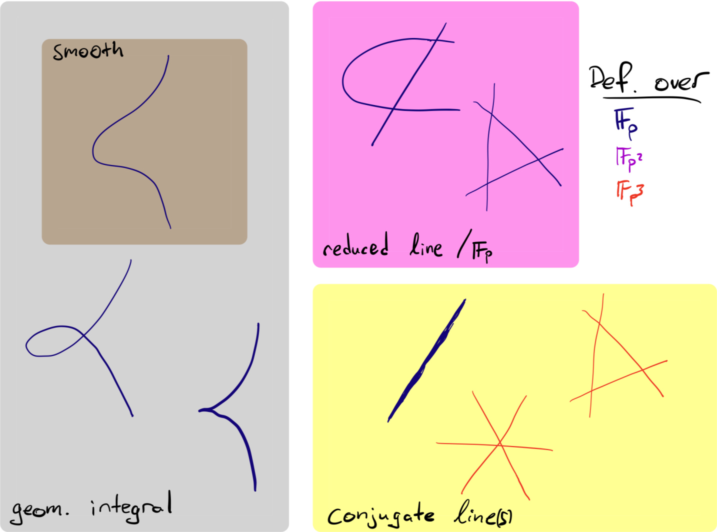
Lemma (Hensel's Lemma)

If $P_0 \in \overline{X}^{\mathrm{sm}}(\mathbb{F}_p)$, then there exists lift $P \in X(\mathbb{Z}_p)$ with $\overline{P} = P_0$.

Cubic hypersurfaces over $\mathbb{F}_p$



When are there always $\mathbb{Q}_p$-points?

Proposition

If $\overline{X_f}$ not a config. of conjugate hyperplanes, then $X_f(\mathbb{Q}_p) \neq \emptyset$.

Proof for curves ($n = 2$).

If geom. integral, use Hasse–Weil bounds on (normalization of) $\overline{X_f}$

$$\#X_f(\mathbb{F}_p) \geq p + 1 - 2\sqrt{p} > 0.$$

All other possibilities contain line defined over $\mathbb{F}_p$. $\square$

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For $n \geq 3$, use inductive argument reducing to coordinate hypersurfaces $X_i = X_f \cap \{x_i = 0\} \subset \mathbb{P}^{n-1}$.

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Warning

There exist irreducible deg. $d > 3$ curves $X \subset \mathbb{P}^2$ with $X(\mathbb{Q}_p) = \emptyset$.

Some (non)examples

Observe 2 is not a cube in $\mathbb{F}_7$. Let $u \in \mathbb{F}_{7^3}$ be root of $x^3 - 2 = 0$.

Example ($n = 2,\ p = 7$)

$$f(x, y, z) = x^3 + 3x^2y + 3xy^2 - 6xyz + 3y^3 - 6y^2z + 4z^3 = 0$$

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Over $\overline{\mathbb{F}_7}$ we have

$$\overline{f} = \prod_{\sigma \in \mathrm{Gal}(\mathbb{F}_{7^3}/\mathbb{F}_7)} \sigma\left(x + (1+u)y + u^2 z\right).$$

$\overline{X_f}(\mathbb{F}_p) = \emptyset \implies X_f(\mathbb{Z}_p) = \emptyset$.

$\overline{X_f}$ is a triangle

Some (non)examples

Example

Suppose $f = g_1(x_0, x_1, x_2) + pg_2(x_3, x_4, x_5) + p^2 g_3(x_6, x_7, x_8)$ for $g_i = 0$ with no nontrivial $\mathbb{F}_p$-solutions.

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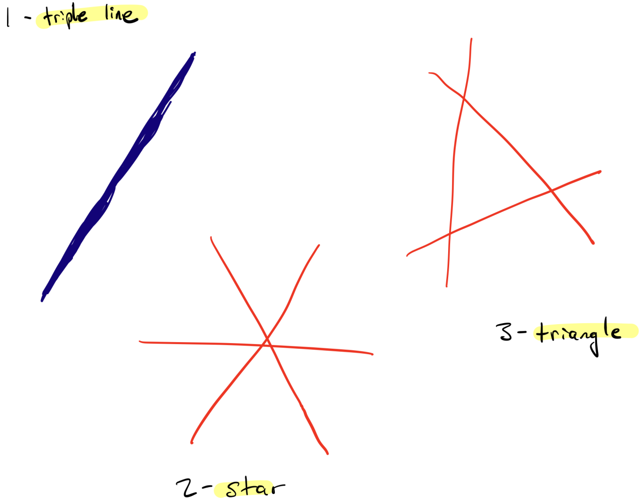
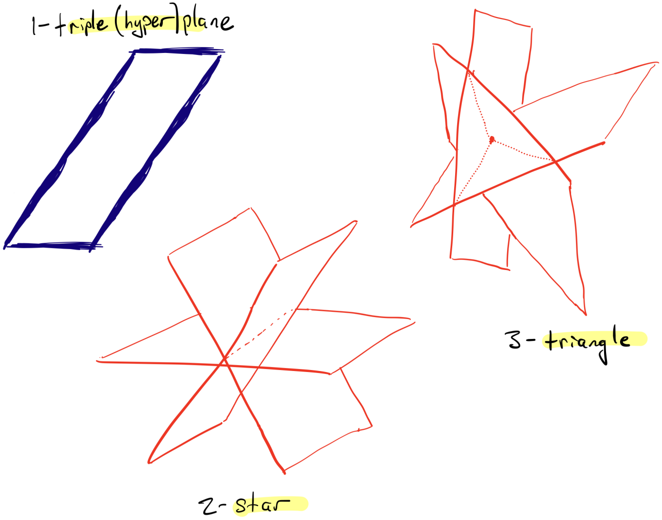
Remark

This generalizes, but it ignores cross terms...

Configurations of conjugate lines



Configurations of conjugate hyperplanes



Strategy

Goal

Look modulo p and try to decide solubility

$$\rho_n(p) = \xi_{n,0}\sigma_{n,0} + \xi_{n,1}\sigma_{n,1} + \xi_{n,2}\sigma_{n,2} + \xi_{n,3}\sigma_{n,3}$$

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Strategy

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Look modulo p and try to decide solubility

$$\rho_n(p) = \xi_{n,0} \cdot 1 + \xi_{n,1}\sigma_{n,1} + \xi_{n,2}\sigma_{n,2} + \xi_{n,3}\sigma_{n,3}$$

- $\xi_{n,i}$ = prob. f has type i
- $\sigma_{n,i}$ = prob. $X_f(\mathbb{Q}_p) \neq \emptyset$ given f type i
- $\xi_{n,0}$ = prob. f not config. of conj. hyperplanes
- $\sigma_{n,0} = 1$

Factorization probabilities

$$\xi_{n,0} = 1 - \frac{q^{3n-3} + 2q^{n+3} + 2q^{n+2} + 2q^{n+1} - 2q^2 - 2q - 3}{3\left(q^2+q+1\right)\left(q^{\binom{n+3}{3}} - 1\right)}$$

$$\xi_{n,1} = \frac{q^{n+1} - 1}{q^{\binom{n+3}{3}} - 1}$$

$$\xi_{n,2} = \frac{\left(q^{2n+1} - q^{n+1} - q^n + 1\right)q}{3\left(q^{\binom{n+3}{3}} - 1\right)}$$

$$\xi_{n,3} = \frac{\left(q^{3n} - q^{2n} - q^{2n+1} - q^{2n-1} + q^{n+1} + q^{n-1} + q^n - 1\right)q^3}{3(q^2+q+1)\left(q^{\binom{n+3}{3}} - 1\right)}$$

Exercise

Convince yourself that probability of a polynomial factoring a certain way is given by a (uniform) rational function.

Phases I, II, and III

Repeat this process three times:

1. Reduce mod p, lift any "easy" solutions with Hensel's lemma

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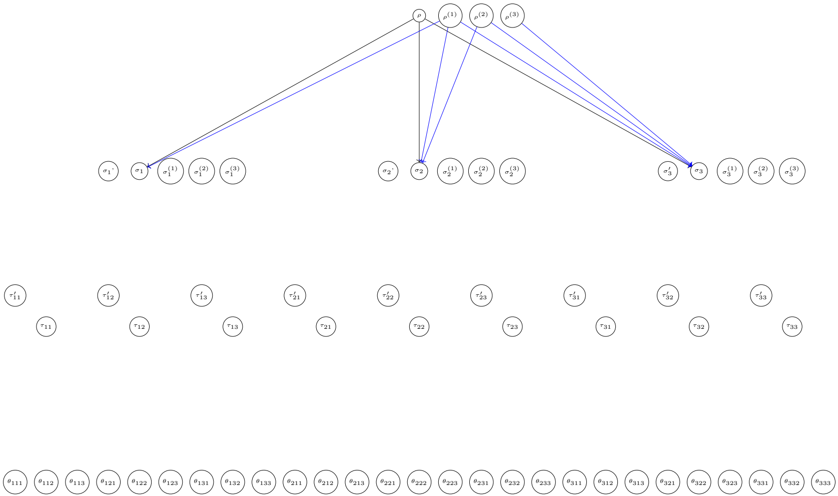
Phases I, II, and III

Repeat this process three times:

1. Reduce mod p, lift any "easy" solutions with Hensel's lemma
2. Reduction is type $j = 1, 2, 3$ with explicit probability
3. Introduce new lifting probability for each type
4. Relate new lifting probabilities to others

Eventually this terminates: 64 $\mathbb{Q}(p)$-linear relations in 64 unknowns

Setup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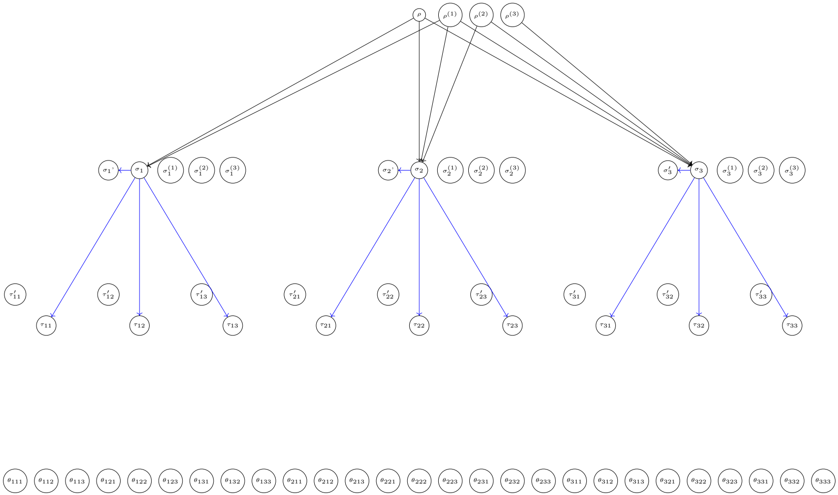
Phase I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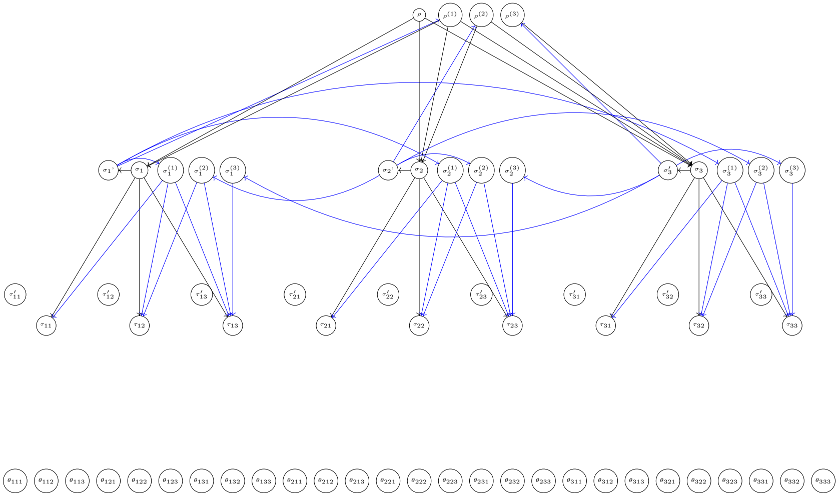
Phase I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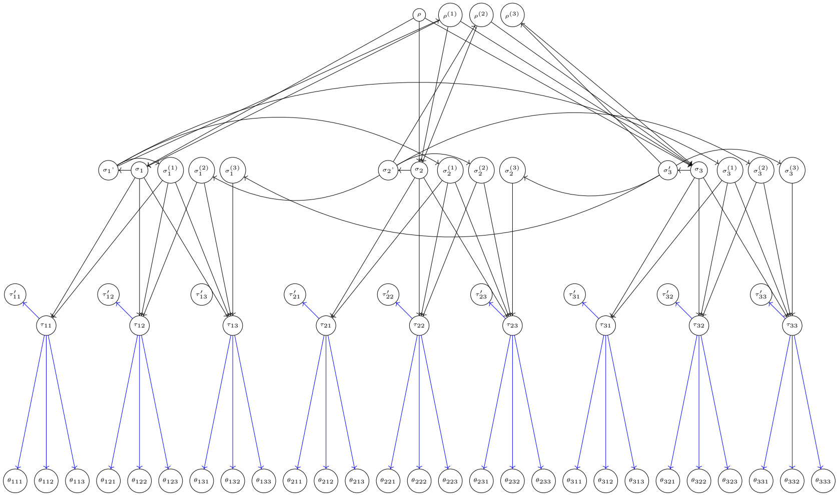
Phase II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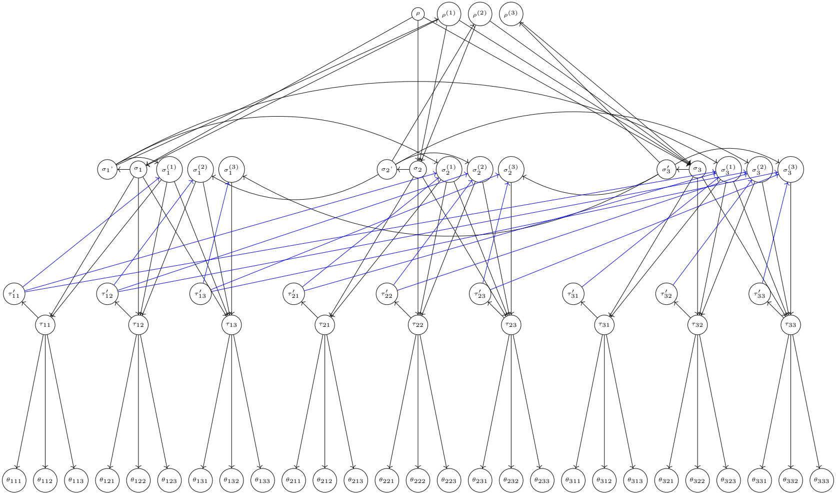
Phase II



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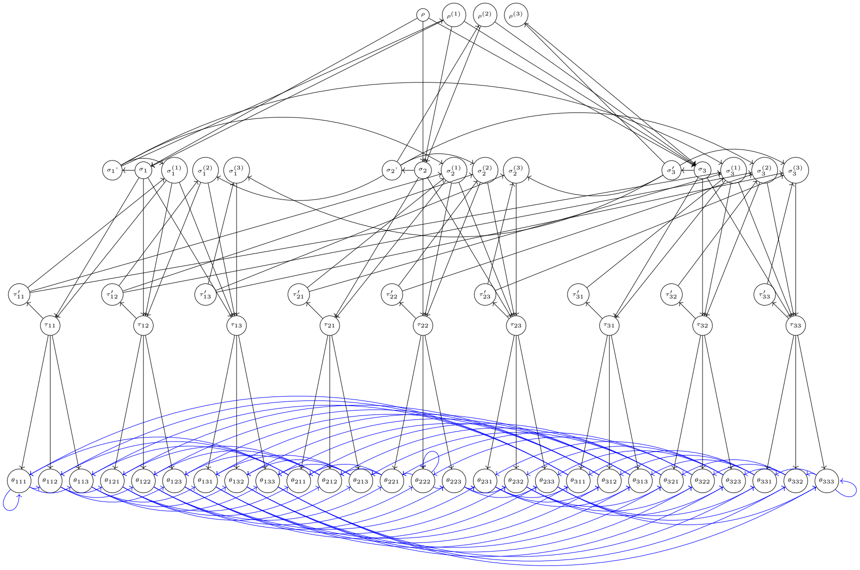
Phase III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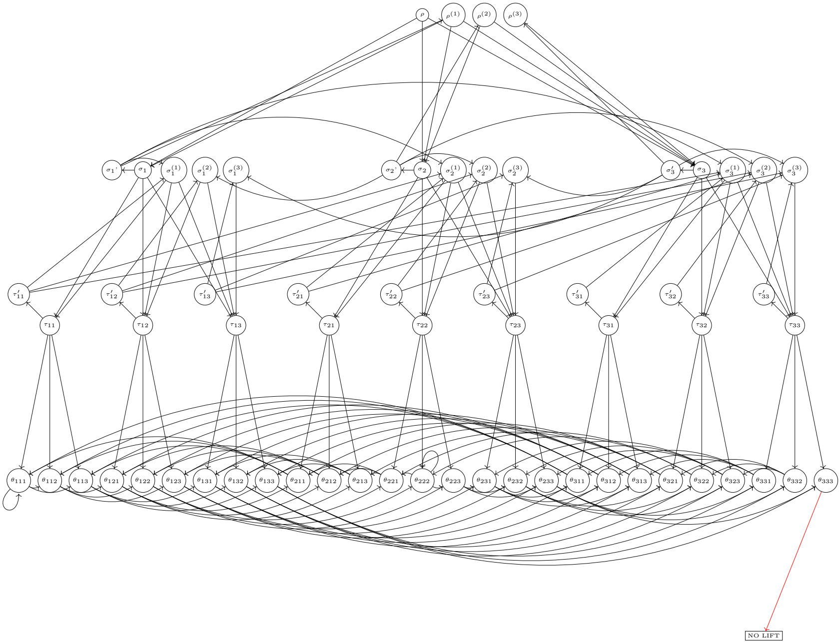
When $n = 8$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NO LIFT

Some remarks

Practicalities:

- Solve with Sage symbolic solver
- Block variables $(27 + 27 + 10)$ to speed up

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In principle, $\rho_n(p) = 1$ for $n \geq 9$ can be seen "by hand"

- No "no lift" sink in the flow chart!
- Capture and describe explicitly Heath–Brown's observation of necessary/sufficient conditions for f to have local solutions

Density of cubics with a point

Theorem (Beneish–K.)

Let $n \geq 4$ (conjecturally $n \geq 3$). Then $\rho_n = 1$ when $n \geq 9$ and

$$\rho_n = \prod_{p \text{ prime}} \left(1 - \frac{g_n(p)}{h_n(p)}\right) \text{ when } n \leq 8$$

for explicit polynomials $g_n(t), h_n(t) \in \mathbb{Z}[t]$.

n	$\rho_n^{(ELS)} \approx$	$1 - \rho_n(p) \sim$
2	0.9726 [BCF16a]	$1/3p^3$
3	0.999927 (conj.)	$1/3p^{10}$
4	$1 - 5.022 \cdot 10^{-9}$	$1/9p^{22}$
5	$1 - 1.343 \cdot 10^{-15}$	$1/9p^{43}$
6	$1 - 3.502 \cdot 10^{-26}$	$1/9p^{78}$
7	$1 - 5.152 \cdot 10^{-42}$	$1/27p^{129}$
8	$1 - 6.222 \cdot 10^{-64}$	$1/27p^{201}$

Further questions

Let $\rho_{d,n}$ = density of degree d hypersurfaces in $\mathbb{P}^n$ with $\mathbb{Q}$-point

How far can this approach go to compute $\rho_{d,n}$?

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How far can this approach go to compute $\rho_{d,n}$?

- Lose uniformity in p in general
- Heath–Brown: $\rho_{4,n}(p) = 1$ for $n \geq 9126$,
  $\rho_{5,n}(p) = 1$ known for $n \geq 25$, $p \geq 17$ [HB09]
- Can we predict asymptotics/numerics with less effort?

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Structure of $\rho_{d,n}(p)$

- Always rational function? (for $p \gg 0$)
- Known for binary forms, $\rho_{d,1}(p) = \rho_{d,1}(1/p)$ [BCFG22]

Thank you I

Thank you for the invitation and for your attention!

M. J. Bright, T. D. Browning, and D. Loughran, Failures of weak approximation in families, Compos. Math. **152** (2016), no. 7, 1435–1475. MR 3530447

Manjul Bhargava, John Cremona, and Tom Fisher, The proportion of plane cubic curves over $\mathbb{Q}$ that everywhere locally have a point, Int. J. Number Theory **12** (2016), no. 4, 1077–1092. MR 3484299

Manjul Bhargava, John E. Cremona, Tom Fisher, Nick G. Jones, and Jonathan P. Keating, What is the probability that a random integral quadratic form in n variables has an integral zero?, Int. Math. Res. Not. IMRN (2016), no. 12, 3828–3848. MR 3544620

Manjul Bhargava, John Cremona, Tom Fisher, and Stevan Gajović, The density of polynomials of degree nn over zp having exactly r roots in qp, Proceedings of the London Mathematical Society **124** (2022), no. 5, 713–736 (English (US)).

Manjul Bhargava, A positive proportion of plane cubics fail the hasse principle, Preprint, available at `https://arxiv.org/pdf/1402.1131.pdf`, 2014.

Tim Browning, Pierre Le Boudec, and Will Sawin, The Hasse principle for random Fano hypersurfaces, Ann. of Math. (2) **197** (2023), no. 3, 1115–1203. MR 4564262

T. D. Browning, Many cubic surfaces contain rational points, Mathematika **63** (2017), no. 3, 818–839. MR 3731306

Thank you II

Tom Fisher, Wei Ho, and Jennifer Park, Everywhere local solubility for hypersurfaces in products of projective spaces, Res. Number Theory **7** (2021), no. 1, Paper No. 6, 27. MR 4199457

D. R. Heath-Brown, Cubic forms in ten variables, Proc. London Math. Soc. (3) **47** (1983), no. 2, 225–257. MR 703978

______, Zeros of p-adic forms, Proceedings of the London Mathematical Society **100** (2009), no. 2, 560–584.

Bjorn Poonen and José Felipe Voloch, Random Diophantine equations, Arithmetic of higher-dimensional algebraic varieties (Palo Alto, CA, 2002), Progr. Math., vol. 226, Birkhäuser Boston, Boston, MA, 2004, With appendices by Jean-Louis Colliot-Thélène and Nicholas M. Katz, pp. 175–184. MR 2029869