How often does a cubic hypersurface have a point?

Christopher Keyes (King's College London)

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Joint with Lea Beneish (University of North Texas)

University of Bath

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Cubic hypersurfaces

A cubic hypersurface $X_f \subset \mathbb{P}^n$ is cut out by a cubic form f

$$X_f \colon f(x_0, \ldots, x_n) = \sum_{0 \le i \le j \le k \le n} a_{ijk} x_i x_j x_k = 0.$$

Question

How often does X_f have a rational point?

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$$\rho_n = \lim_{B \to \infty} \frac{\#\{f \mid h(f) \leq B,\ X_f(\mathbb{Q}) \neq \emptyset\}}{\#\{f \mid h(f) \leq B\}}.$$

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Remark

- Counting primitive forms gets same answer, i.e. using $\mathbb{P}^{\binom{n+3}{3}-1}$

Main result

Theorem (Beneish–K.)

Let $n \geq 4$. Then

$$\rho_n = \begin{cases} \prod_{p \text{ prime}} \left(1 - \frac{g_n(p)}{h_n(p)}\right) & 4 \leq n \leq 8 \\ 1 & n \geq 9 \end{cases}$$

for explicit polynomials $g_n(t), h_n(t) \in \mathbb{Z}[t]$.

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Conjecture (Beneish–K.)

Consider $n = 3$: cubic surfaces in $\mathbb{P}^3$. Then

$$\rho_3 = \prod_p \left(1 - \frac{\left(3p^{26}+p^{24}+p^{23}+4p^{22}-3p^{21}+3p^{20}+2p^{19}+2p^{18}-p^{17}+p^{14}+p^{13}-2p^{12}+3p^{11}+3p^{7}\right)\left(p^2+1\right)(p+1)^2(p-1)^4}{9(p^{13}-1)(p^7+1)(p^7-1)(p^6+1)(p^5-1)(p^3+1)(p^3-1)}\right).$$

Example: cubic 7-folds in $\mathbb{P}^8$

Example

When $n = 8$ we have

$$g_8(p) = \left(p^9-1\right)\left(p^7-1\right)\left(p^4+1\right)\left(p^2+1\right)^2(p+1)^3(p-1)^9 p^{219}$$

$$\begin{aligned} h_8(p) = 27 & \left(p^{53}-1\right)\left(p^{49}-1\right)\left(p^{47}-1\right)\left(p^{40}-p^{39}+p^{35}-p^{34}+p^{30}-p^{28}+p^{25}-p^{23}+p^{20}-p^{17}+p^{15}\right. \\ & \left.-p^{12}+p^{10}-p^6+p^5-p+1\right)\left(p^{32}-p^{31}+p^{29}-p^{28}+p^{26}-p^{25}+p^{23}-p^{22}+p^{20}-p^{19}+p^{17}\right. \\ & \left.-p^{16}+p^{15}-p^{13}+p^{12}-p^{10}+p^9-p^7+p^6-p^4+p^3-p+1\right)\left(p^{27}+1\right)\left(p^{27}-1\right)\left(p^{26}+1\right)\left(p^{25}\right. \\ & \left.+1\right)\left(p^{25}-1\right)\left(p^{24}+1\right)\left(p^{17}-1\right)\left(p^{13}+1\right)\left(p^{13}-1\right)\left(p^{12}+1\right)\left(p^{11}-1\right)\left(p^6+1\right)\left(p^3-1\right)^3 \end{aligned}$$

Asymptotically, $\dfrac{g_8(p)}{h_8(p)} \sim \dfrac{1}{27p^{201}}$.

Numerically, $\rho_8 \approx 1 - 6.222 \cdot 10^{-64}$.

Asymptotics and numerics

n	$\rho_n \approx$	$1 - \rho_n(p) \sim$
3	$0.999927(conj.)$	$1/3p^{10}$
4	$1 - 5.022 \cdot 10^{-9}$	$1/9p^{22}$
5	$1 - 1.343 \cdot 10^{-15}$	$1/9p^{43}$
6	$1 - 3.502 \cdot 10^{-26}$	$1/9p^{78}$
7	$1 - 5.152 \cdot 10^{-42}$	$1/27p^{129}$
8	$1 - 6.222 \cdot 10^{-64}$	$1/27p^{201}$
9	1	0

Some history

When $n \gg 0$[1], the circle method shows X_f has a rational point:

$$N_f(B) \sim c_f B^{n-2}, \; c_f > 0.$$

[1] Recall n denotes the dimension of $\mathbb{P}^n$; the number of variables is $n+1$

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1982 Heath-Brown: $n \geq 9$ if X_f is nonsingular [HB83]

Corollary

$\rho_n = 1$ *for* $n \geq 9$.

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1988 Hooley: $n \geq 8$ if X_f is everywhere locally soluble

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Everywhere local solubility

A variety $X/\mathbb{Q}$ is everywhere locally soluble (ELS) if

$$X(\mathbb{R}) \neq \emptyset \text{ and } X(\mathbb{Q}_p) \neq \emptyset \text{ for all } p.$$

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Density of ELS cubic forms in $\mathbb{Z}[x_0, \ldots, x_n]$:

$$\rho_n^{ELS} = \lim_{B \to \infty} \frac{\#\{f \mid h(f) \leq B,\ X_f \text{ ELS}\}}{\#\{f \mid h(f) \leq B\}}.$$

(Lack of) obstructions

Conjecture (Poonen–Voloch, 2004)

When $n \geq 3$, $\rho_n^{ELS} = \rho_n$.

i.e. local-global principle holds for 100% of cubic hypersurfaces.

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Theorem (Browning–Le Boudec–Sawin, 2023)

When $n \geq 4$, $\rho_n^{ELS} = \rho_n$. In fact true for Fano deg. d $(d, n) \neq (3, 3)$

Computing ρ^{ELS}

Let $\rho_n(p) =$ density of p-adic cubic forms f such that $X_f(\mathbb{Q}_p) \neq \emptyset$.

Theorem (Poonen–Voloch, 2004)

Let $n \geq 2$. We have

$$\rho_n^{ELS} = \prod_p \rho_n(p).$$

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Local probabilities independent...even though infinitely many!

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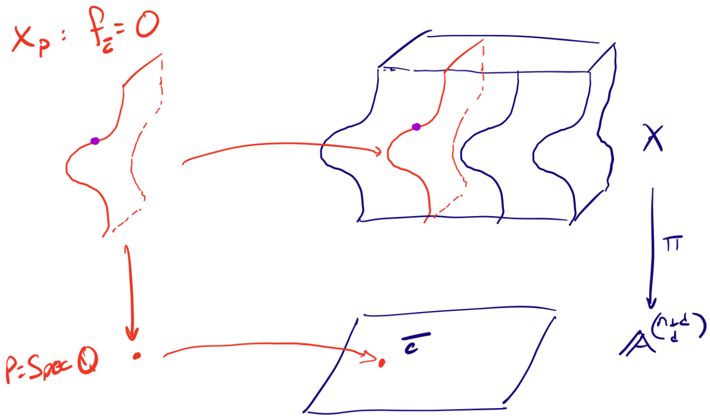
2016 Bright–Browning–Loughran: flexible generalization to families given by fibers of maps to affine/projective space.

Varieties parameterized by fibers



Related results

Plane cubic curves

- Bhargava–Cremona–Fisher computed ρ_2^{ELS} explicitly [BCF16a]
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Quadric hypersurfaces

- Bhargava–Cremona–Fisher–Jones–Keating: explicit Euler product for density of quadratic forms with integral zero [BCF+16b]
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More: certain cubic surfaces [Bro17], (2,2)-curves in $\mathbb{P}^1 \times \mathbb{P}^1$ [FHP21]

Proof skeleton

ρ_n = density of cubic forms $f \in \mathbb{Z}[x_0, \ldots, x_n]$ with $X_f(\mathbb{Q}) \neq \emptyset$.

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2. $\rho_n^{ELS} = \prod_p \rho_n(p)$ [PV04, BBL16]

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4. When does $\overline{P} \in \overline{X_f}$ lift to $P \in X_f(\mathbb{Q}_p)$?

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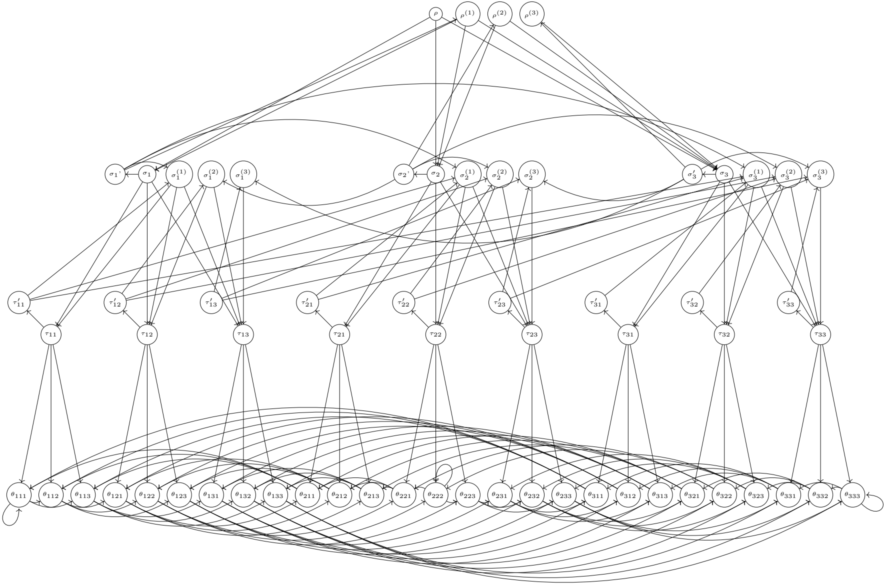
Combine (3), (4) to recursively compute $\rho_n(p)$ uniformly.

Full picture



Computing the local factors

Goal

Compute local probability $\rho_n(p)$ that X_f has $\mathbb{Q}_p$-point.

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Compute local probability $\rho_n(p)$ that X_f has $\mathbb{Q}_p$-point.

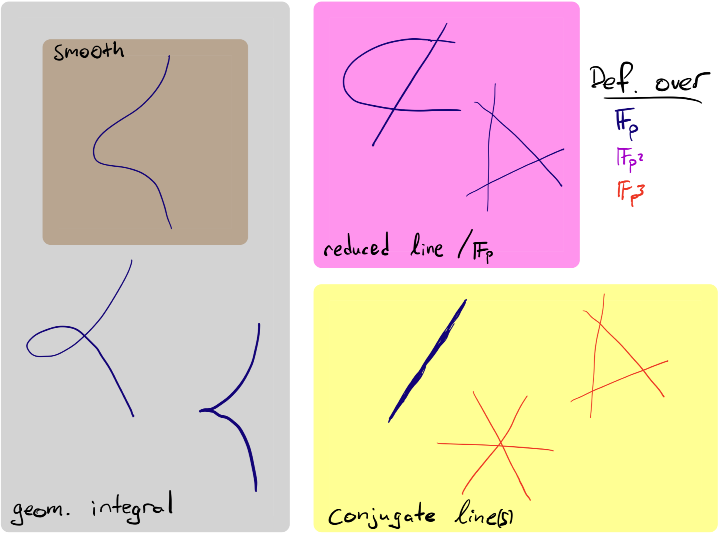
$\rho_n(p)$ is a p-adic Haar measure with $\mu_p\left(\mathbb{Z}_p^{\binom{n+3}{3}}\right) = 1$:

$$\rho_n(p) = \mu_p\left(\{f \in \mathbb{Z}_p[x_0, \dots, x_n] \mid X_f(\mathbb{Q}_p) \neq \emptyset\}\right).$$

Think

Each residue class contributes equally: reduce mod p and decide solubility with Hensel's lemma.

Cubic hypersurfaces over finite fields



When are there always $\mathbb{Q}_p$-points?

Proposition

If $\overline{X_f}$ is not a configuration of conj. hyperplanes, then $X_f(\mathbb{Q}_p) \neq \emptyset$.

Proof for curves ($n = 2$).

If geom. integral, use Hasse–Weil bounds on (normalization of) $\overline{X_f}$.

All other possibilities contain line defined over $\mathbb{F}_p$. $\square$

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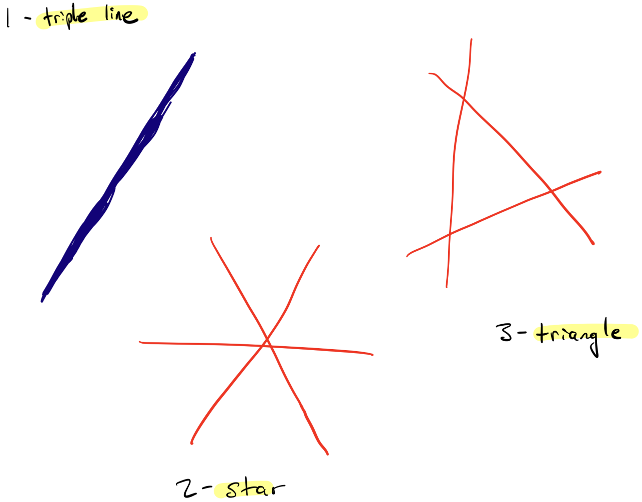
Warning

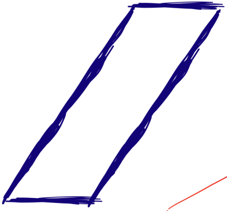
There exist irreducible deg. $d > 3$ curves $X \subset \mathbb{P}^2$ with $X(\mathbb{Q}_p) = \emptyset$.

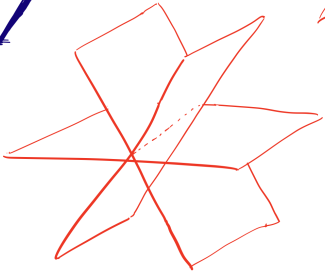
Configurations of conjugate lines



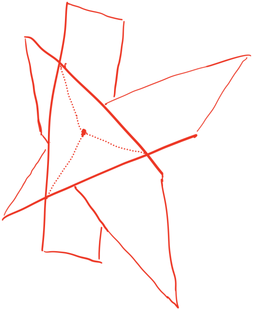
Configurations of conjugate hyperplanes

1- triple (hyper)plane





2- star



3- triangle

Configurations of conjugate hyperplanes

Types 1, 2, and 3 are configurations of conjugate hyperplanes:

$$f = \prod_{\sigma \in \mathrm{Gal}(\mathbb{F}_{p^3}/\mathbb{F}_p)} \sigma(b_0 x_0 + \ldots + b_n x_n).$$

Moreover, if f is type i we have

- $\dim_{\mathbb{F}_p} \operatorname{span}\{b_0, \ldots, b_n\} = i$
- $\overline{X_f}(\mathbb{F}_p) = \mathbb{P}^{n-i}(\mathbb{F}_p)$

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Think

Expect type 3 to be soluble least often

Some (non)examples

Example $(n = 2,\ p = 7)$

$$f(x, y, z) = x^3 + 3x^2y + 3xy^2 - 6xyz + 3y^3 - 6y^2z + 4z^3 = 0$$

Some (non)examples

Example ($n = 2,\ p = 7$)

$f(x, y, z) = x^3 + 3x^2y + 3xy^2 - 6xyz + 3y^3 - 6y^2z + 4z^3 = 0$

Observe $2 \notin (\mathbb{F}_7^\times)^3$. Let $u \in \mathbb{F}_{7^3}$ satisfy $x^3 - 2 = 0$. Over $\overline{\mathbb{F}_7}$,

$$\overline{f} = \prod_{\sigma \in \mathrm{Gal}(\mathbb{F}_{7^3}/\mathbb{F}_7)} \sigma\left(x + (1+u)y + u^2 z\right).$$

$\overline{X_f}$ is geometrically reducible — its 3 conjugate component lines form a triangle — and $X_f(\mathbb{Q}_7) = \emptyset$.

Some (non)examples

Example ($n = 2,\ p = 7$)

$f = x_0^3 + 7(x_1^3 - 2x_2^3)$. This time $\overline{X_f}$ is triple line $[0 : x_1 : x_2]$.

If $[x_0 : x_1 : x_2] \in X_f(\mathbb{Z}_7)$ then

- $7 \mid x_0$
- $7 \mid (x_1^3 - 2x_2^3)$
- $7 \mid x_1, x_2$ since $2 \notin (\mathbb{F}_7^\times)^3$

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$\Longrightarrow \Longleftarrow$, so $X_f(\mathbb{Z}_7) = \emptyset$.

Example ($n = 2$, arbitrary p)

$f = x_0^3 + p^3 g(x_1, x_2)$ for g monic, irr. cubic mod p.

Then $[-p : 1 : 0] \in X_f(\mathbb{Z}_p)$.

Takeaway: some, not all X_f reducing to triple line are insoluble!

Some (non)examples

Example ($n = 8$, arbitrary p)

Suppose $f = g_1(x_0, x_1, x_2) + pg_2(x_3, x_4, x_5) + p^2 g_3(x_6, x_7, x_8)$ for $\overline{g_i} = 0$ with no nontrivial $\mathbb{F}_p$-solutions.

If $[x_0, \ldots, x_8] \in X_f(\mathbb{Z}_p)$ then $p \mid x_0, x_1, x_2$

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If $[x_0, \ldots, x_8] \in X_f(\mathbb{Z}_p)$ then $p \mid x_0, x_1, x_2$

$$\frac{1}{p} f(px_0, px_1, px_2, x_3, \ldots, x_8)$$
$$= g_2(x_3, x_4, x_5) + pg_3(x_6, x_7, x_8) + p^2 g_1(x_0, x_1, x_2)$$

$$\implies X_f(\mathbb{Z}_p) = \emptyset.$$

Strategy

Goal

Look modulo p and try to decide solubility

$$\rho_n(p) = \xi_{n,0}\sigma_{n,0} + \xi_{n,1}\sigma_{n,1} + \xi_{n,2}\sigma_{n,2} + \xi_{n,3}\sigma_{n,3}$$

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- $\sigma_{n,i}$ = prob. $X_f(\mathbb{Q}_p) \neq \emptyset$ given f type i

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- $\sigma_{n,i}$ = prob. $X_f(\mathbb{Q}_p) \neq \emptyset$ given f type i
- $\xi_{n,0}$ = prob. f not config. of conj. hyperplanes

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- $\xi_{n,0}$ = prob. f not config. of conj. hyperplanes
- $\sigma_{n,0} = 1$

Factorization probabilities

$$\xi_{n,0} = 1 - \frac{q^{3n-3} + 2q^{n+3} + 2q^{n+2} + 2q^{n+1} - 2q^2 - 2q - 3}{3\left(q^2+q+1\right)\left(q^{\binom{n+3}{3}} - 1\right)}$$

$$\xi_{n,1} = \frac{q^{n+1} - 1}{q^{\binom{n+3}{3}} - 1}$$

$$\xi_{n,2} = \frac{\left(q^{2n+1} - q^{n+1} - q^n + 1\right)q}{3\left(q^{\binom{n+3}{3}} - 1\right)}$$

$$\xi_{n,3} = \frac{\left(q^{3n} - q^{2n} - q^{2n+1} - q^{2n-1} + q^{n+1} + q^{n-1} + q^n - 1\right)q^3}{3(q^2+q+1)\left(q^{\binom{n+3}{3}} - 1\right)}$$

Exercise

Convince yourself that probability of a polynomial factoring a certain way is given by a (uniform) rational function.

Phase I

Suppose f has type $i = 1, 2, 3$.

After linear change of coordinates, $\overline{f} = \overline{f}(x_0, \ldots, x_{i-1})$ with no nontrivial solutions

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$$\begin{aligned} f_I &= \frac{1}{p} f(px_0, \ldots, px_{i-1}, x_i, \ldots, x_n) \\ &= p^2 g(x_0, \ldots, x_{i-1}) + h(x_i, \ldots, x_n) + p(\text{cross terms}) \end{aligned}$$

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$$\begin{array}{lcccccc} & x_0, \ldots, x_{i-1} & & & & & x_i, \ldots, x_n \\ f: & & = 0_i & \geq 1 & \geq 1 & \geq 1 & \\ f_I: & & = 2_i & \geq 2 & \geq 1 & \geq 0 & \end{array}$$

Phase I

Upshot: $X_f(\mathbb{Z}_p) = X_{f_I}(\mathbb{Z}_p)$ with f_I given by

$$x_0, \ldots, x_{i-1} \qquad = 2_i \quad \geq 2 \quad \geq 1 \quad \geq 0 \qquad x_i, \ldots, x_n$$

Study what happens to $\overline{f_I}$:

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Study what happens to $\overline{f_I}$:

$$\sigma_i = \left(1 - \frac{1}{p^{\binom{n-i+3}{3}}}\right)\left(\xi_{n-i,0} + \sum_{j=1,2,3} \xi_{n-i,j}\tau_{n,ij}\right) + \frac{1}{p^{\binom{n-i+3}{3}}}\sigma'_{n,i}$$

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$\overline{f_I}$ not identically zero

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$\overline{f_I}$ has type j

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$\overline{f_I}$ identically zero

Phases II and III

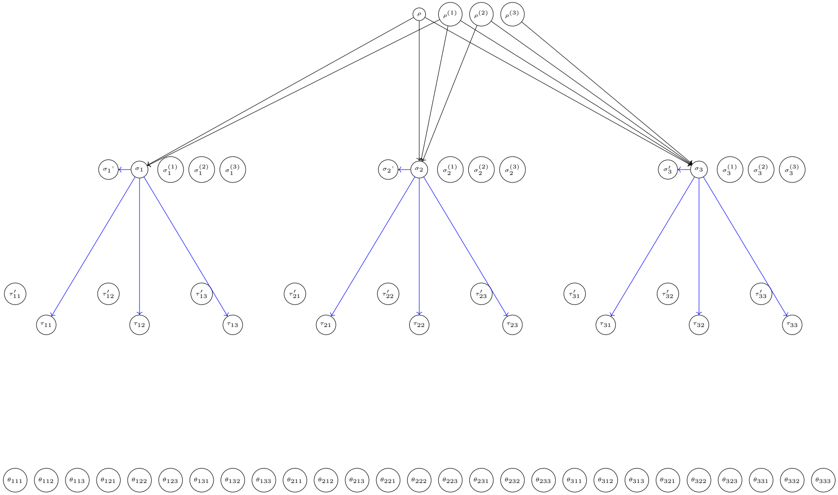
Repeat this process two more times:

- Reduce mod p, lift any "easy" solutions with Hensel's lemma
- Introduce new lifting probabilities for reduction type $j = 1, 2, 3$ or vanishing
- Relate new lifting probabilities to others

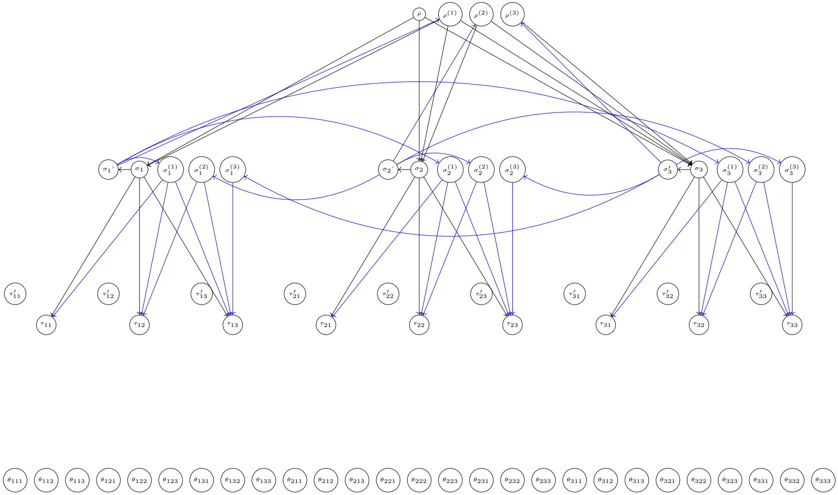
Eventually this process terminates: 64 relations in 64 unknowns

- Solve in Sage (block variables to speed up)

Phase I



Phase I



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$\sigma_1^{(2)}$
$\sigma_1^{(3)}$
σ_2'
σ_2
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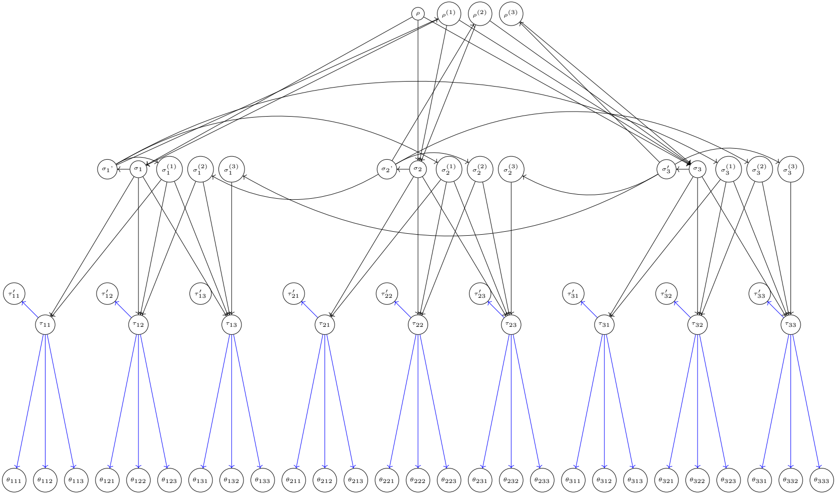
Phase II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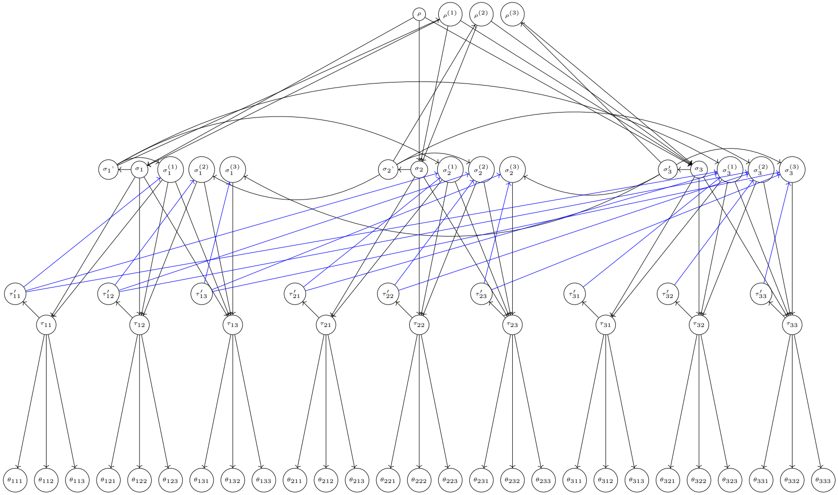

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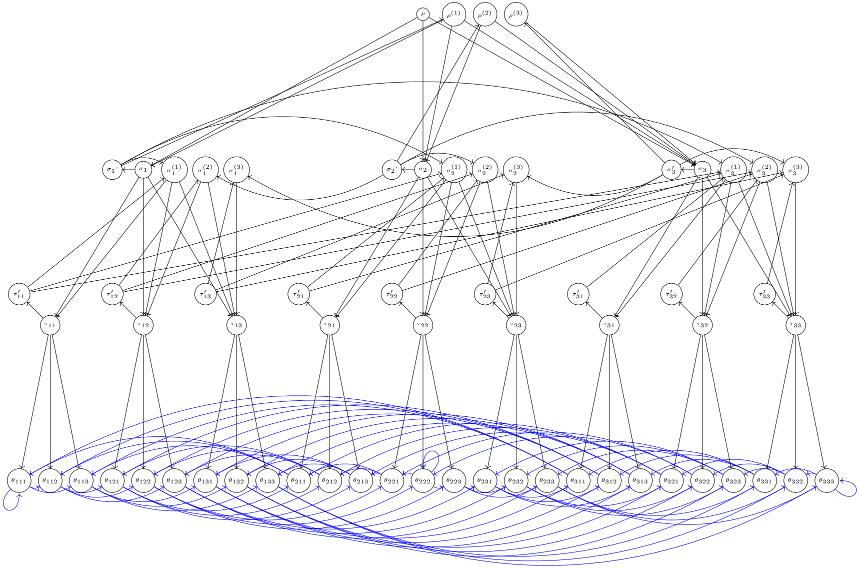
Phase III


ρ $\rho^{(1)}$ $\rho^{(2)}$ $\rho^{(3)}$
σ_1' σ_1 $\sigma_1^{(1)}$ $\sigma_1^{(2)}$ $\sigma_1^{(3)}$ σ_2' σ_2 $\sigma_2^{(1)}$ $\sigma_2^{(2)}$ $\sigma_2^{(3)}$ σ_3' σ_3 $\sigma_3^{(1)}$ $\sigma_3^{(2)}$ $\sigma_3^{(3)}$
τ'_{11} τ'_{12} τ'_{13} τ'_{21} τ'_{22} τ'_{23} τ'_{31} τ'_{32} τ'_{33}
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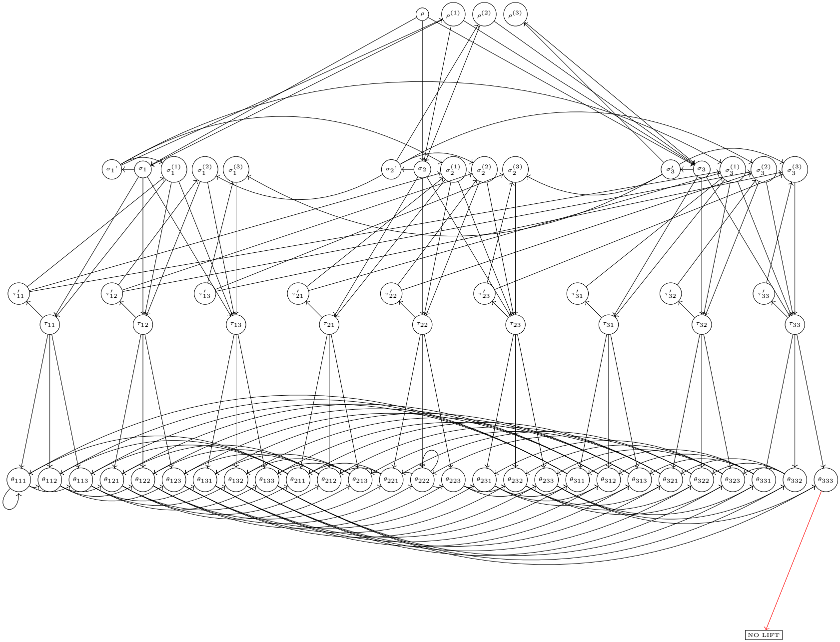
When $n = 8$



Final step: in pictures

Partition variables:

- $S = \{x_0, \ldots, x_{i-1}\}$
- $T = \{x_i, \ldots, x_{i+j-1}\}$
- $U = \{x_{i+j}, \ldots, x_{i+j+k-1}\}$
- $W = \{x_{i+j+k}, \ldots, x_n\}$, $w =$ degree in $x_{i+j+k}, \ldots, x_n$

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θ_{ijk} is probability of lift when f looks like

$w=0$				$w=1$			$w=2$		$w=3$
$=0_i$	≥ 1	≥ 1	$=1_j$	≥ 0	≥ 0	≥ 1	≥ 0	≥ 0	≥ 0
≥ 1	≥ 1	≥ 2		≥ 1	≥ 1		≥ 1		
≥ 2	≥ 2			≥ 2					
$=2_k$									

Final step: in pictures

$w = 0$				$w = 1$			$w = 2$		$w = 3$
$= 0_i$	≥ 1	≥ 1	$= 1_j$	≥ 0	≥ 1	≥ 1	≥ 0	≥ 1	≥ 0
≥ 1	≥ 1	≥ 2		≥ 1	≥ 1		≥ 1		
≥ 2	≥ 2			≥ 2					
$= 2_k$									

Final step: in pictures

$w = 0$				$w = 1$			$w = 2$		$w = 3$
$= 0_i$	≥ 1	≥ 1	$= 1_j$	≥ 0	≥ 1	≥ 1	≥ 0	≥ 1	≥ 0
≥ 1	≥ 1	≥ 2		≥ 1	≥ 1		≥ 1		
≥ 2	≥ 2			≥ 2					
$= 2_k$									

Final step: in pictures

$w=0$				$w=1$			$w=2$		$w=3$
$=0_i$	≥ 1	≥ 1	$=1_j$	≥ 0	≥ 1	≥ 1	≥ 0	≥ 1	≥ 0
≥ 1	≥ 1	≥ 2		≥ 1	≥ 1		≥ 1		
≥ 2	≥ 2			≥ 2					
$=2_k$									

$S \cup W$ $\qquad\qquad$ T

$\xrightarrow{\text{reindex}}$

$$\begin{array}{cccc} =0^{(i)} & \geq 1 & \geq 1 & =1_j \\ \geq 1 & \geq 1 & \geq 2 & \\ \geq 2 & \geq 2 & & \\ =2_k & & & \end{array}$$

U

Final step: in pictures

$w=0$				$w=1$			$w=2$		$w=3$
$=0_i$	≥ 1	≥ 1	$=1_j$	≥ 0	≥ 1	≥ 1	≥ 0	≥ 1	≥ 0
≥ 1	≥ 1	≥ 2		≥ 1	≥ 1		≥ 1		
≥ 2	≥ 2			≥ 2					
$=2_k$									

$S \cup W$ $\qquad\qquad$ T

$\xrightarrow{\text{reindex}}$

$=0^{(i)}$	≥ 1	≥ 1	$=1_j$
≥ 1	≥ 1	≥ 2	
≥ 2	≥ 2		
$=2_k$			

U

$$\theta_{n,ijk} = 1 - \frac{1}{p^{ij(n-i-j-k+1)+j\binom{n-i-j-k+2}{2}}} + \frac{1}{p^{ij(n-i-j-k+1)+j\binom{n-i-j-k+2}{2}}}\left(\sum_{0\leq \ell \leq 3} \xi^{(i)}_{n-j-k,\ell}\,\theta_{n,jk\ell}\right)$$

Density of cubics with a point

Theorem (Beneish–K.)

Let $n \geq 4$ (conjecturally $n \geq 3$). Then $\rho_n = 1$ when $n \geq 9$ and

$$\rho_n = \prod_{p \text{ prime}} \left(1 - \frac{g_n(p)}{h_n(p)}\right) \text{ when } n \leq 8$$

for explicit polynomials $g_n(t), h_n(t) \in \mathbb{Z}[t]$.

n	$\rho_n \approx$	$1 - \rho_n(p) \sim$
3	0.999927 (conj.)	$1/3p^{10}$
4	$1 - 5.022 \cdot 10^{-9}$	$1/9p^{22}$
5	$1 - 1.343 \cdot 10^{-15}$	$1/9p^{43}$
6	$1 - 3.502 \cdot 10^{-26}$	$1/9p^{78}$
7	$1 - 5.152 \cdot 10^{-42}$	$1/27p^{129}$
8	$1 - 6.222 \cdot 10^{-64}$	$1/27p^{201}$

Further questions

Let $\rho_{d,n}$ = density of degree d hypersurfaces in $\mathbb{P}^n$ with $\mathbb{Q}$-point

Is $\rho_{d,n}(p)$ always rational function in p?

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Is $\rho_{d,n}(p)$ always rational function in p?

- Probably need $p \gg 0$
- Known for binary forms, $\rho_{d,1}(p) = \rho_{d,1}(1/p)$ [BCFG22]

Further questions

Let $\rho_{d,n} =$ density of degree d hypersurfaces in $\mathbb{P}^n$ with $\mathbb{Q}$-point

Is $\rho_{d,n}(p)$ always rational function in p?

- Probably need $p \gg 0$
- Known for binary forms, $\rho_{d,1}(p) = \rho_{d,1}(1/p)$ [BCFG22]

Less is known for $d > 3$:

- Heath–Brown: $\rho_{4,n}(p) = 1$ for $n \geq 9126$,
  $\rho_{5,n}(p) = 1$ known for $n \geq 25$, $p \geq 17$ [HB09]
- What's going on for $d \leq n \leq n^2$?

Thank you!

Thank you for the invitation and for your attention!

Another mystery

Tom Fisher pointed out

$$1-\rho_6=\alpha+\beta$$

$$1-\rho_7=\frac{p^{141}\prod_{1\le k\le 8}(p^k-1)}{27(p^2+p+1)^2(p^{15}-1)\prod_{38\le k\le 44}(p^k-1)}$$

$$1-\rho_8=\frac{p^{219}\prod_{1\le k\le 9}(p^k-1)}{27(p^2+p+1)^3\prod_{47\le k\le 55}(p^k-1)},$$

where α, β are given by

$$\alpha=\frac{p^{95}\prod_{1\le k\le 7}(p^k-1)}{27(p^2+p+1)(p^{12}-1)\prod_{30\le k\le 35}(p^k-1)}$$

$$\beta=\frac{p^{81}(p^{72}-1)\prod_{1\le k\le 7}(p^k-1)}{9(p^2+p+1)^2(p^{24}-1)\prod_{30\le k\le 36}(p^k-1)}.$$

Thank you I

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