# Towards Artin's conjecture on $p$-adic forms in low degree

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AMS Special Session

*Algebraic and Analytic Aspects of Curves and their L-functions*

# Acknowledgment

Joint work with Lea Beneish (UNT)

https://arxiv.org/pdf/2508.20192

Ongoing work in degree 5

# Setup

Let $K$ be a $p$-adic field:

| | |
|---|---|
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Let $f \in K[x_0, \ldots, x_n]$ be degree $d$ form

Let $X_f \subset \mathbb{P}^n$ be associated degree $d$ hypersurface

$$X_f \colon f(x_0, \ldots, x_n) = 0 \subset \mathbb{P}^n$$

$$X_f(K) = \{(x_0, \ldots, x_n) \in K^{n+1} - \mathbf{0} : f(x_0, \ldots, x_n) = 0\}/\sim$$

# Original conjecture

**Conjecture (Artin, 1930s)**

*Let $n \geq d^2$ and $f \in K[x_0, \ldots, x_n]$ degree $d$. Then $X_f(K) \neq \emptyset$.*

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1960s Terjanian [Ter66]: explicit counterexample for $K = \mathbb{Q}_2$ with $d = 4$, $n = 17$

1980s Lewis–Montgomery [LM83]: infinite family of counterexamples for each $p$

All known counterexamples: $d$ composite, divisible by $p - 1$

# Evidence

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1920s Hasse: quadratic forms in 5 variables have $K$-zero

1950s Lewis [Lew52]: cubic forms in 10 variables have $K$-zero

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1960s Ax–Kochen [AK65]: conjecture holds when $p \gg_d 0$

This is characteristic $p$, not the size of the residue field $q$!

# Revised conjecture

**Conjecture (Artin, revised)**

*Let $d$ prime and $f \in K[x_0, \ldots, x_{d^2}]$ degree $d$. Then $X_f(K) \neq \emptyset$.*

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**Conjecture (Artin, revised)**

*Let d prime and* $f \in K[x_0, \ldots, x_{d^2}]$ *degree* $d$. *Then* $X_f(K) \neq \emptyset$.

**Theorem**

*For* $d \in \{2, 3, 5, 7, 11\}$, *Artin's conj. holds when* $q \gg_d 0$ *[LL65].*

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*For $d \in \{2, 3, 5, 7, 11\}$, Artin's conj. holds when $q \gg_d 0$ [LL65].*

$$q > \begin{cases} 1 & d = 2, 3 \text{ [Lew52]} \\ 5 & d = 5 \text{ [LY96, HB10, Dum17, BK25b]}, \\ 679 & d = 7 \text{ [Woo08, BK25a]}, \\ 8053 & d = 11 \text{ [Woo08]}. \end{cases}$$

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**Pop quiz!**

What is so special about $\{2, 3, 5, 7, 11\}$? (Answer revealed shortly)

# This talk

**Theorem**

*For $d \in \{2, 3, 5, 7, 11\}$, Artin's conj. holds when $q \gg_d 0$ [LL65].*

*For $d = 7$, Artin's conj. holds for $q > 679$ [Woo08, BK25a].*

# This talk

**Theorem**

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*For $d = 7$, Artin's conj. holds for $q > 679$ [Woo08, BK25a].*

1. Conj: there exists $K$-point on $X_f$
2. Hensel: suffices to find smooth $\mathbb{F}_q$-point on $\overline{X_f}$
3. Idea: find nice plane curves $C \subset \overline{X_f}$ via effective Bertini

# Reduced forms

**Definition (reduced [LL65])**

$f(x_0, \ldots, x_n) \in \mathcal{O}_K[x_0, \ldots, x_n]$ is reduced if

$$\operatorname{Res}(f_{x_0}, \ldots, f_{x_n}) \neq 0$$

and has *minimal valuation* in $\mathrm{GL}_{n+1}(K)$-orbit.

Why do we care?

- Suffices to check Artin's Conjecture on reduced forms $f$
- $f$ reduced, $n \geq d^2 \implies \overline{f}$ has no linear factors over $\overline{\mathbb{F}_q}$

# Low degrees

**Proposition (Laxton–Lewis [LL65])**

*Let $d \in \{2, 3, 5, 7, 11\}$. Then Artin's Conj. holds for $q \gg_d 0$.*

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$f$ reduced $\implies$ $\overline{f}$ has geom. irr. factor of unique degree def.$/\mathbb{F}_q$

- $d = 5$: $\overline{f}$ factors as (5) or (23) over $\overline{\mathbb{F}_q}$
- $d = 13$: $\overline{f}$ can factor as $(3^2 2^2)$ — totally nonreduced :(

# Proof sketch

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*Let $d \in \{2, 3, 5, 7, 11\}$. Then Artin's Conj. holds for $q \gg_d 0$.*

*Proof sketch.* Suffices to check on $f$ reduced.

Factor $\overline{f}$ over $\overline{\mathbb{F}_q}$, find irreducible factor $g$ of unique degree.

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Lang–Weil: $\#X_g(\mathbb{F}_q) = O(q^m)$ and $\#X_f(\mathbb{F}_q)^{\text{sing}} = O(q^{m-1})$.

When $q \gg_d 0$, there is a smooth $\mathbb{F}_q$-point on $\overline{X_f}$. $\square$

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**Question**

*How do we make this (most) effective?*

# Wish list

Suppose $C \subset \overline{X_f}$ is smooth deg. $d$ plane curve

$$\#C(\mathbb{F}_q) \geq q + 1 - \frac{(d-1)(d-2)}{2} \lfloor 2\sqrt{q} \rfloor$$

**Example ($d = 7$)**

$\#C(\mathbb{F}_q) > 0$ whenever $q > 883$.

If $C$ contains no $\overline{\mathbb{F}_q}$-lines, $q > 883 \implies C(\mathbb{F}_q)^{\mathrm{sm}} \neq \emptyset$.

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Same argument for $d = 5, 11$: $q > 121, 8053$

**Goal**

*Show $\exists\ C \subset \overline{X_f}$ containing no lines.*

# A Bertini theorem

Let $n \geq 3$ and $k$ be an arbitrary field. Suppose

- $X_f \subset \mathbb{P}^n$ is a geom. irr. hypersurface defined$/k$,
- $P \subset \mathbb{P}^n$ is a plane defined$/k$.

**Theorem (Bertini)**

*Generically, $P \cap X_f$ is a geom. irr. plane curve.*

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**Theorem (Bertini)**

*Generically, $P \cap X_f$ is a geom. irr. plane curve.*

**Caveat**

If $k = \mathbb{F}_q$, this does not guarantee existence of such $P$!

# An effective Bertini theorem

**Theorem (Cafure–Matera [CM06], Beneish–K. [BK25a])**

*Suppose $f$ is an abs. irr. degree $d$ form defined over $\mathbb{F}_q$.*

(i) *If $q > \frac{1}{8}\left(3d^4 - 2d^3 + 13d^2 + 2d\right)$ there exists $P$ such that $X_f \cap P$ is geometrically irreducible.*

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(ii) *Fix a positive integer $D < d$. If*

$$q > \frac{d}{8}(-D^4 + 4D^3d - 6D^3 + 12D^2d - 11D^2 + 8Dd - 6D + 16d)$$

*there exists $P$ s.t. $X_f \cap P$ has no component of deg. $\leq D$.*

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**Example ($d = 7$)**

(i) If $q > 896$, $\exists$ $P$ such that $X_f \cap P$ is geometrically irreducible.

(ii) If $q > 224$, $\exists$ $P$ such that $X_f \cap P$ contains no lines$/\overline{\mathbb{F}_q}$.

# Wooley's improvement

**Theorem (Wooley [Woo08])**

*For $d \in \{5, 7, 11\}$, Artin's conj. holds when*

$$q > \begin{cases} 121 & d = 5, \\ 883 & d = 7, \\ 8053 & d = 11. \end{cases}$$

- Reduced $\implies$ $f$ has no linear factors
- Eff. Bertini $\implies$ $\exists\ P$ s.t. $\overline{X_f} \cap P$ contains no lines
- Count smooth points on $\overline{X_f} \cap P$ with Hasse–Weil and lift

# Updated wish list

Suppose $C \subset \overline{X_f}$ is irr. deg. $d$ plane curve with $\#C(\mathbb{F}_q) \geq 2$

- Smooth points lift to $X_f(K)$ ✓
- Singular points lower geometric genus ✓

**Example ($d = 7$)**

If $\#C(\mathbb{F}_q) \geq 2$ and $q > 679$, then $C(\mathbb{F}_q)^{\mathrm{sm}} \neq \emptyset$.

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**Example ($d = 7$)**

If $\#C(\mathbb{F}_q) \geq 2$ and $q > 679$, then $C(\mathbb{F}_q)^{\mathrm{sm}} \neq \emptyset$.

**Goal**

*Find $C \subset \overline{X_f}$ with $\#C(\mathbb{F}_q) \geq 2$ and a geom. int. component*

# Yet another effective Bertini theorem

What if we let $P$ vary among planes containing fixed line $L$?

**Theorem (Beneish–K. [BK25a, Corollary 2.7])**

*Suppose $f$ has no factors$/\overline{\mathbb{F}_q}$ of degree $\leq D$.*

*Suppose $L$ is line def.$/\mathbb{F}_q$ meeting $X_f$ transversely at $\alpha \in X_f(\overline{\mathbb{F}_q})$.*

*If $q > \frac{D}{8}\left(-D^3 + 4D^2 d - 6D^2 + 12Dd - 11D + 8d - 6\right)$, then $\exists\ P \supset L$ s.t. $X_f \cap P$ contains no degree $\leq D$ curve through $\alpha$.*

**Example ($d = 7,\ D = 2$)**

If $q > 69$, there exists such $P$.

# Latest improvement

**Theorem (Beneish–K. [BK25a])**

*For $d = 7$, Artin's conj. holds when $q > 679$.*

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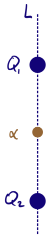

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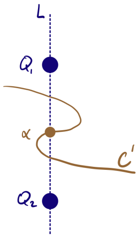
- Component containing $\alpha$ must be def./$\mathbb{F}_q$



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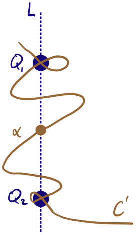

# Final thoughts

**Theorem (1960s – 2025+, many authors)**

*For $d \in \{5, 7, 11\}$, Artin's conj. holds when*

$$q > \begin{cases} 5 & d = 5, \\ 679 & d = 7, \\ 8053 & d = 11. \end{cases}$$

Ongoing: improve results for $d = 5$ via computational techniques

What about prime $d > 11$? Still wide open!

Thank you for your attention!

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