How often does a cubic hypersurface have a point?

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Cubic hypersurfaces

A cubic hypersurface $X_f \subset \mathbb{P}^n$ is cut out by a cubic form f

$$X_f \colon f(x_0, \ldots, x_n) = \sum_{0 \le i \le j \le k \le n} a_{ijk} x_i x_j x_k = 0.$$

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Let $h(f) = ||\mathbf{a}|| =$ Euclidean norm, define natural density

$$\rho_n = \lim_{B \to \infty} \frac{\#\{f \mid h(f) \leq B,\ X_f(\mathbb{Q}) \neq \emptyset\}}{\#\{f \mid h(f) \leq B\}}.$$

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Remark

- Counting primitive forms gets same answer, i.e. using $\mathbb{P}^{\binom{n+3}{3}-1}$

Main result

Theorem (Beneish–K.)

Let $n \geq 4$. Then

$$\rho_n = \begin{cases} \prod_{p \text{ prime}} \left(1 - \frac{g_n(p)}{h_n(p)}\right) & 4 \leq n \leq 8 \\ 1 & n \geq 9 \end{cases}$$

for explicit polynomials $g_n(t), h_n(t) \in \mathbb{Z}[t]$.

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Remark

We also produce g_3, h_3, and a conjectural formula for ρ_3.

Example: cubic 3-folds in $\mathbb{P}^4$

Example

When $n = 4$ we have

$$g_4(p) = \left(p^{46} + 3p^{41} + p^{40} - p^{39} + p^{37} + p^{36} + p^{35} - 3p^{34} + 3p^{27} - p^{26} + p^{25} + p^{19}\right)\left(p^2+1\right)\left(p+1\right)^2\left(p-1\right)^4$$

$$h_4(p) = 9\left(p^{19}-1\right)\left(p^{17}-1\right)\left(p^{10}+1\right)\left(p^{9}+1\right)\left(p^{9}-1\right)\left(p^{7}-1\right)\left(p^{5}+1\right)$$

Asymptotically, $\dfrac{g_4(p)}{h_4(p)} \sim \dfrac{1}{9p^{22}}$.

Numerically, $\rho_4 \approx 0.99999999497 = 1 - 5.022 \cdot 10^{-9}$.

Example: cubic 7-folds in $\mathbb{P}^8$

Example

When $n = 8$ we have

$$g_8(p) = \left(p^9-1\right)\left(p^7-1\right)\left(p^4+1\right)\left(p^2+1\right)^2(p+1)^3(p-1)^9 p^{219}$$

$$\begin{aligned}h_8(p) = 27&\left(p^{53}-1\right)\left(p^{49}-1\right)\left(p^{47}-1\right)\left(p^{40}-p^{39}+p^{35}-p^{34}+p^{30}-p^{28}+p^{25}-p^{23}+p^{20}-p^{17}+p^{15}\right.\\&\left.-p^{12}+p^{10}-p^6+p^5-p+1\right)\left(p^{32}-p^{31}+p^{29}-p^{28}+p^{26}-p^{25}+p^{23}-p^{22}+p^{20}-p^{19}+p^{17}\right.\\&\left.-p^{16}+p^{15}-p^{13}+p^{12}-p^{10}+p^9-p^7+p^6-p^4+p^3-p+1\right)\left(p^{27}+1\right)\left(p^{27}-1\right)\left(p^{26}+1\right)\left(p^{25}\right.\\&\left.+1\right)\left(p^{25}-1\right)\left(p^{24}+1\right)\left(p^{17}-1\right)\left(p^{13}+1\right)\left(p^{13}-1\right)\left(p^{12}+1\right)\left(p^{11}-1\right)\left(p^6+1\right)\left(p^3-1\right)^3\end{aligned}$$

Asymptotically, $\dfrac{g_8(p)}{h_8(p)} \sim \dfrac{1}{27p^{201}}$.

Numerically, $\rho_8 \approx 1 - 6.222 \cdot 10^{-64}$.

Asymptotics and numerics

n	$\rho_n \approx$	$1-\rho_n(p) \sim$
3	$0.999927(conj.)$	$1/3p^{10}$
4	$1-5.022\cdot 10^{-9}$	$1/9p^{22}$
5	$1-1.343\cdot 10^{-15}$	$1/9p^{43}$
6	$1-3.502\cdot 10^{-26}$	$1/9p^{78}$
7	$1-5.152\cdot 10^{-42}$	$1/27p^{129}$
8	$1-6.222\cdot 10^{-64}$	$1/27p^{201}$

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9	1	0

Some history

When n large enough[1], Hardy–Littlewood circle method used to show X_f always has a rational point.

1950's Birch, Lewis, then Davenport: $n \geq 15$

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Corollary

$\rho_n = 1$ *for* $n \geq 9$.

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1988 Hooley: $n \geq 8$ if X_f is everywhere locally soluble

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Everywhere local solubility

A variety $X/\mathbb{Q}$ is everywhere locally soluble (ELS) if

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Density of ELS cubic forms in $\mathbb{Z}[x_0, \ldots, x_n]$:

$$\rho_n^{ELS} = \lim_{B \to \infty} \frac{\#\{f \mid h(f) \leq B,\ X_f \text{ ELS}\}}{\#\{f \mid h(f) \leq B\}}.$$

(Lack of) obstructions

Conjecture (Poonen–Voloch, 2004)

When $n \geq 3$, $\rho_n^{ELS} = \rho_n$.

i.e. local-global principle holds for 100% of cubic hypersurfaces.

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Theorem (Browning–Le Boudec–Sawin, 2023)

When $n \geq 4$, $\rho_n^{ELS} = \rho_n$. In fact true for Fano deg. d $(d, n) \neq (3, 3)$

Computing ρ^{ELS}

Let $\rho_n(p)$ = density of p-adic cubic forms f such that $X_f(\mathbb{Q}_p) \neq \emptyset$.

Theorem (Poonen–Voloch, 2004)

Let $n \geq 2$. We have

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Local probabilities independent...even though infinitely many!

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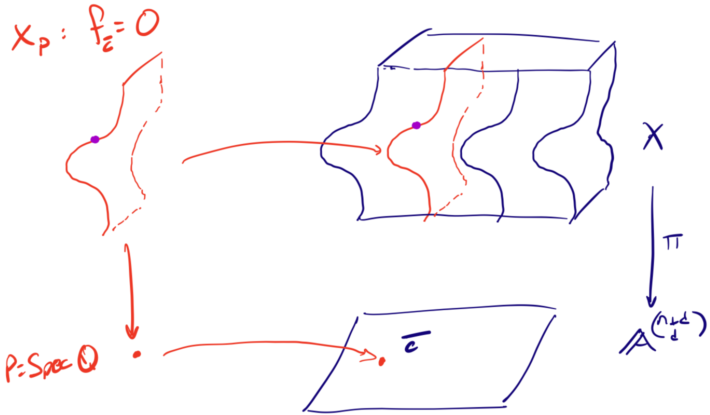
2016 Bright–Browning–Loughran: flexible generalization to families given by fibers of maps to affine/projective space.

Varieties parameterized by fibers



Related results

Plane cubic curves

- Bhargava–Cremona–Fisher computed ρ_2^{ELS} explicitly [BCF16a]
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Quadric hypersurfaces

- Bhargava–Cremona–Fisher–Jones–Keating: explicit Euler product for density of quadratic forms with integral zero [BCF+16b]
- Hasse principle holds but archimedean place not trivial!
- 98.3% of quadric surfaces in $\mathbb{P}^3$ soluble

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More: certain cubic surfaces [Bro17], (2,2)-curves in $\mathbb{P}^1 \times \mathbb{P}^1$ [FHP21]

Proof skeleton

$\rho_n =$ density of cubic forms $f \in \mathbb{Z}[x_0, \ldots, x_n]$ with $X_f(\mathbb{Q}) \neq \emptyset$.

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1. $\rho_n = \rho_n^{ELS}$ [BLBS23]
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3. What does reduction $\overline{X_f}$ modulo p look like?
4. When does $\overline{P} \in \overline{X_f}$ lift to $P \in X_f(\mathbb{Q}_p)$?

Computing the local factors

Goal

Compute local probability $\rho_n(p)$ that X_f has $\mathbb{Q}_p$-point.

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Compute local probability $\rho_n(p)$ that X_f has $\mathbb{Q}_p$-point.

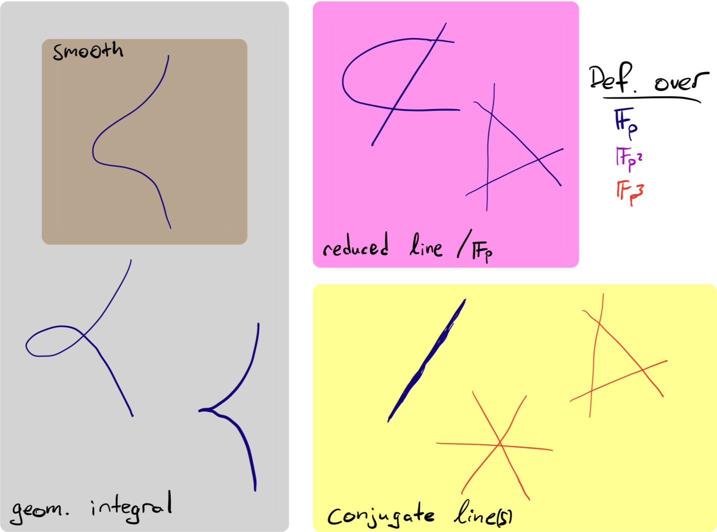
$\rho_n(p)$ is p-adic Haar measure with $\mu_p\left(\mathbb{Z}_p^{\binom{n+3}{3}}\right) = 1$:

$$\rho_n(p) = \mu_p\left(f \in \mathbb{Z}_p[x_0, \ldots, x_n] \mid X_f(\mathbb{Q}_p) \neq \emptyset\right).$$

Think

Each residue class contributes equally: reduce mod p and decide solubility with Hensel's lemma.

Cubic hypersurfaces over finite fields



When are there always $\mathbb{Q}_p$-points?

Proposition

Suppose $\overline{X_f}$ is not a configuration of conjugate hyperplanes. Then $X_f(\mathbb{Q}_p) \neq \emptyset$.

Proof for curves ($n = 2$).

If geom. integral, use Hasse–Weil bounds on (normalization of) $\overline{X_f}$.

All other possibilities contain line defined over $\mathbb{F}_p$. $\square$

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For $n \geq 3$, use inductive argument reducing to coordinate hypersurfaces $X_i = X_f \cap \{x_i = 0\} \subset \mathbb{P}^{n-1}$.

Some (non)examples

Example

Let $n = 2$ and $f = x_0^3 + pg(x_1, x_2)$ for $\overline{g}$ irreducible bin. form$/\mathbb{F}_p$.

If $[x_0 : x_1 : x_2] \in X_f(\mathbb{Z}_p)$ then

- $p \mid x_0$
- $p \mid g(x_1, x_2)$
- $p \mid x_1, x_2$

$\implies X_f(\mathbb{Z}_p) = \emptyset$.

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Example

Let $n = 2$ and $f = x_0^3 + p^3 g(x_1, x_2)$ for g monic, irr. mod p.

Then $[-p : 1 : 0] \in X_f(\mathbb{Z}_p)$.

Some (non)examples

Example

Suppose $f = g_1(x_0, x_1, x_2) + pg_2(x_3, x_4, x_5) + p^2 g_3(x_6, x_7, x_8)$ for $g_i = 0$ with no nontrivial p-adic solutions.

If $[x_0, \ldots, x_8] \in X_f(\mathbb{Z}_p)$ then $p \mid x_0, x_1, x_2$

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If $[x_0, \ldots, x_8] \in X_f(\mathbb{Z}_p)$ then $p \mid x_0, x_1, x_2$

$$\frac{1}{p} f(px_0, px_1, px_2, x_3, \ldots, x_8) = g_2(x_3, x_4, x_5) + pg_3(x_6, x_7, x_8) + p^2 g_1(x_0, x_1, x_2)$$

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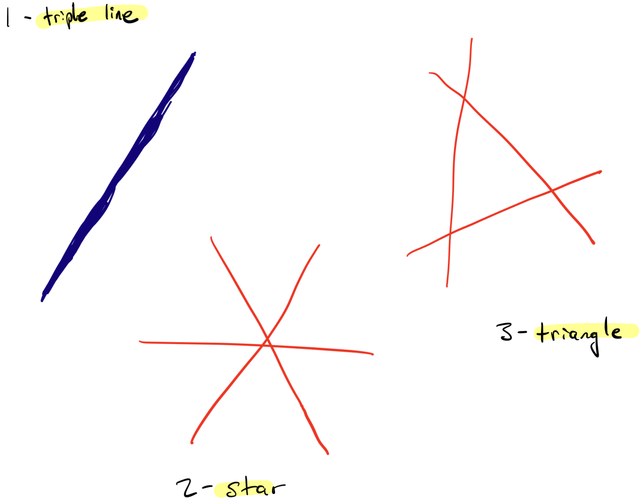
$$\implies X_f(\mathbb{Z}_p) = \emptyset.$$

Remark

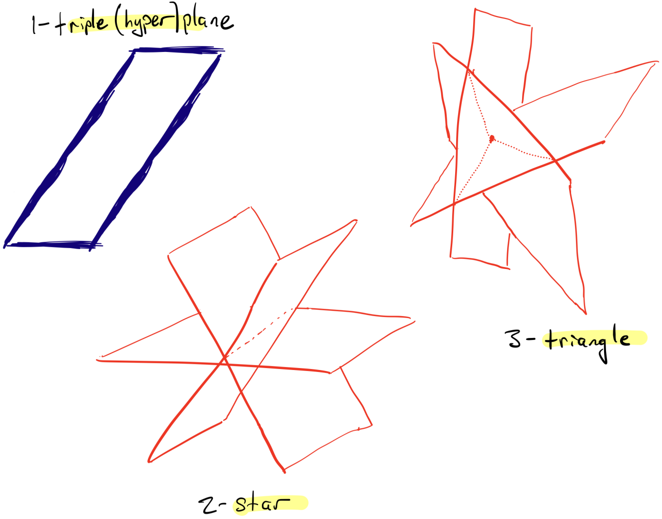
This generalizes — what if g_i had different numbers of variables?

But it ignores cross terms...

Configurations of conjugate lines



Configurations of conjugate hyperplanes



Configurations of conjugate hyperplanes

Types 1, 2, and 3 are configurations of conjugate hyperplanes:

$$f = \prod_{\sigma \in \mathrm{Gal}(\mathbb{F}_{p^3}/\mathbb{F}_p)} \sigma(b_0 x_0 + \ldots + b_n x_n).$$

Moreover, if f is type i we have

- $\dim_{\mathbb{F}_p} \mathrm{span}\{b_0, \ldots, b_n\} = i$
- $\overline{X_f}(\mathbb{F}_p) = \mathbb{P}^{n-i}(\mathbb{F}_p)$

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Think

Expect type 3 to be soluble least often

Strategy

Goal

Look modulo p and try to decide solubility

$$\rho_n(p) = \xi_{n,0}\sigma_{n,0} + \xi_{n,1}\sigma_{n,1} + \xi_{n,2}\sigma_{n,2} + \xi_{n,3}\sigma_{n,3}$$

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- $\sigma_{n,i}$ = prob. $X_f(\mathbb{Q}_p) \neq \emptyset$ given f type i

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Look modulo p and try to decide solubility

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Suppose f has type $i = 1, 2, 3$.

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	$x_0, \ldots, x_{i-1}$					$x_i, \ldots, x_n$
f:		$= 0_i$	≥ 1	≥ 1	≥ 1	
f_I:		$= 2_i$	≥ 2	≥ 1	≥ 0	

Phase I

Upshot: $X_f(\mathbb{Z}_p) = X_{f_I}(\mathbb{Z}_p)$ with f_I given by

$$x_0, \ldots, x_{i-1} \qquad = 2_i \quad \geq 2 \quad \geq 1 \quad \geq 0 \qquad x_i, \ldots, x_n$$

Study what happens to $\overline{f_I}$:

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$$\sigma_i = \left(1 - \frac{1}{p^{\binom{n-i+3}{3}}}\right)\left(\xi_{n-i,0} + \sum_{j=1,2,3} \xi_{n-i,j}\tau_{n,ij}\right) + \frac{1}{p^{\binom{n-i+3}{3}}}\sigma'_{n,i}$$

Phase I

Upshot: $X_f(\mathbb{Z}_p) = X_{f_I}(\mathbb{Z}_p)$ with f_I given by

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$\overline{f_I}$ not identically zero

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$\overline{f_I}$ has type j

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$\overline{f_I}$ identically zero

Phase II — computing $\tau_{n,ij}$

After initial transformation, f_I described by

$$
\begin{array}{cccccc}
x_0, \ldots, x_{i-1} & & & & & x_i, \ldots, x_{i+j-1} \\
& = 2_i & \geq 2 & \geq 1 & = 0_j & \\
& & \geq 2 & \geq 1 & \geq 1 & \\
& & & \geq 1 & \geq 1 & \\
& & & & \geq 1 & \\
& & & & & x_{i+j}, \ldots, x_n
\end{array}
$$

Phase II — computing $\tau_{n,ij}$

Define f_{II} so that $X_{f_{II}}(\mathbb{Z}_p) = X_f(\mathbb{Z}_p)$ with

$$f_{II} = \frac{1}{p} f_I(x_0, \ldots, x_{i-1}, px_i, \ldots, px_{i+j-1}, x_{i+j}, \ldots, x_n)$$

$x_0, \ldots, x_{i-1}$ $\qquad$ $x_i, \ldots, x_{i+j-1}$

$$\begin{array}{cccc}
= 1_i & \geq 2 & \geq 2 & = 2_j \\
 & \geq 1 & \geq 1 & \geq 2 \\
 & & \geq 0 & \geq 1 \\
 & & & \geq 0
\end{array}$$

$x_{i+j}, \ldots, x_n$

$$\tau_{n,ij} = \left(1 - \frac{1}{p^{i\binom{n-i-j+2}{2}}}\right) + \frac{1}{p^{i\binom{n-i-j+2}{2}}} \left(\left(1 - \frac{1}{p^{\binom{n-i-j+3}{3}}}\right)\left(\sum_{0 \leq k \leq 3} \xi_{n-i-j,k}\theta_{n,ijk}\right) + \frac{\tau'_{n,ij}}{p^{\binom{n-i-j+3}{3}}}\right)$$

Phase III — computing $\theta_{n,ijk}$

$\theta_{n,ijk}$ is conditional lifting probability when

- f has type i,
- f_I has type j,
- f_{II} has type k

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If $i + j + k = n + 1$, see $\theta_{n,ijk} = 0$ (remember examples!)

Phase III — computing $\theta_{n,ijk}$

$\theta_{n,ijk}$ is conditional lifting probability when

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If $i + j + k = n + 1$, see $\theta_{n,ijk} = 0$ (remember examples!)

Otherwise, define f_{III} by replacing

$$x_{i+j}, \ldots, x_{i+j+k-1} \mapsto px_{i+j}, \ldots, px_{i+j+k-1}$$

Phase III — valuation tetrahedron

$$
\begin{array}{cccccccccc}
 & & f_{II} & & & & & f_{III} & & \\
=1_i & \geq 2 & \geq 2 & =2_j & & =0_i & \geq 1 & \geq 1 & =1_j \\
 & \geq 1 & \geq 1 & \geq 2 & & & \geq 1 & \geq 1 & \geq 2 \\
 & & \geq 1 & \geq 1 & & & & \geq 2 & \geq 2 \\
 & & & =0_k & & & & & =2_k \\
 & & & & & & & & \\
 & \geq 1 & \geq 1 & \geq 2 & & & \geq 0 & \geq 0 & \geq 1 \\
 & & \geq 1 & \geq 1 & \to & & & \geq 1 & \geq 1 \\
 & & & \geq 1 & & & & & \geq 2 \\
 & & & & & & & & \\
 & & \geq 1 & \geq 1 & & & & \geq 0 & \geq 0 \\
 & & & \geq 1 & & & & & \geq 1 \\
 & & & & & & & & \\
 & & & \geq 1 & & & & & \geq 0
\end{array}
$$

Phase III — rotating the tetrahedron

$$
\begin{array}{cccc}
= 0_i & \geq 1 & \geq 1 & = 1_j \\
 & \geq 1 & \geq 1 & \geq 2 \\
 & & \geq 2 & \geq 2 \\
 & & & = 2_k \\
\\
 & \geq 0 & \geq 0 & \geq 1 \\
 & & \geq 1 & \geq 1 \\
 & & & \geq 2 \\
\\
 & & \geq 0 & \geq 0 \\
 & & & \geq 1 \\
\\
 & & & \geq 0
\end{array}
$$

Phase III — rotating the tetrahedron

$$
\begin{array}{cccc}
= 0_i & \geq 1 & \geq 1 & = 1_j \\
 & \geq 1 & \geq 1 & \geq 2 \\
 & & \geq 2 & \geq 2 \\
 & & & = 2_k \\
 & & & \\
 & \geq 0 & \geq 1 & \geq 1 \\
 & & \geq 1 & \geq 1 \\
 & & & \geq 2 \\
 & & & \\
 & & \geq 0 & \geq 1 \\
 & & & \geq 1 \\
 & & & \\
 & & & \geq 0
\end{array}
$$

Phase III — rotating the tetrahedron

$$\begin{array}{cccc} =0_i & \geq 1 & \geq 1 & =1_j \\ & \geq 1 & \geq 1 & \geq 2 \\ & & \geq 2 & \geq 2 \\ & & & =2_k \end{array}$$

$$\begin{array}{ccc} \geq 0 & \geq 1 & \geq 1 \\ & \geq 1 & \geq 1 \\ & & \geq 2 \end{array} \quad \rightarrow \quad \begin{array}{cccc} =1_j & \geq 2 & \geq 2 & =2_k \\ & \geq 1 & \geq 1 & \geq 2 \\ & & \geq 1 & \geq 1 \\ & & & \geq 0^{(i)} \end{array}$$

$$\begin{array}{cc} \geq 0 & \geq 1 \\ & \geq 1 \end{array}$$

$$\geq 0$$

Phase III — rotating the tetrahedron

$$\begin{array}{cccc} =0_i & \geq 1 & \geq 1 & =1_j \\ & \geq 1 & \geq 1 & \geq 2 \\ & & \geq 2 & \geq 2 \\ & & & =2_k \end{array}$$

$$\begin{array}{ccc} \geq 0 & \geq 1 & \geq 1 \\ & \geq 1 & \geq 1 \\ & & \geq 2 \\ \\ & \geq 0 & \geq 1 \\ & & \geq 1 \\ \\ & & \geq 0 \end{array} \quad \rightarrow \quad \begin{array}{cccc} =1_j & \geq 2 & \geq 2 & =2_k \\ & \geq 1 & \geq 1 & \geq 2 \\ & & \geq 1 & \geq 1 \\ & & & \geq 0^{(i)} \end{array}$$

$$\theta_{n,ijk} = 1 - \frac{1}{p^{ij(n-i-j-k+1)+j\binom{n-i-j-k+2}{2}}} + \frac{1}{p^{ij(n-i-j-k+1)+j\binom{n-i-j-k+2}{2}}} \left(\sum_{0\leq \ell \leq 3} \xi^{(i)}_{n-j-k,\ell}\, \theta_{n,jk\ell} \right)$$

Closing the loop

This process involves

- Explicit and uniform: factorization probabilities e.g. $\xi_{n,i}$
- Up to 64 variables: lifting probabilities $\sigma_{n,i}, \tau_{n,ij}, \theta_{n,ijk}, \ldots$
- Relations for each variable

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Solving in Sage $\implies$ explicit uniform rational function $\rho_n(p)$!

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Solving in Sage $\implies$ explicit uniform rational function $\rho_n(p)$!

Remark

- In principle, $\rho_n(p) = 1$ for $n \geq 9$ can be seen "by hand"
- Practical speedup: block variables together $(27 + 27 + 10)$

Recovering an old result

From Heath-Brown [HB83, Appx. 1]:

Proposition (Heath–Brown, attributed to Mordell and Lewis)

Suppose $\{x_0, \ldots, x_n\} = \{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$. Then $X_f(\mathbb{Q}_p) \neq \emptyset$ iff there is a non-singular linear transformation over $\mathbb{Q}_p$, sending f to

$$\widetilde{f} = g_{\mathbf{u}}(\mathbf{u}) + pg_{\mathbf{v}}(\mathbf{v}) + p^2 g_{\mathbf{w}}(\mathbf{w}) + ph(\mathbf{u}, \mathbf{v}, \mathbf{w})$$

satisfying

(i) *g_i, h have coeffs in $\mathbb{Z}_p$*

(ii) *terms involving only $\mathbf{u}$ (resp. $\mathbf{v}, \mathbf{w}$) are absent from h*

(iii) *$g_i \equiv 0 \pmod{p}$ has no nontrivial solutions*

(iv) *in h, coeffs of uww, vww, vvw terms are div. by p*

Recovering an old result

If $(i, j, k) = (\#\mathbf{u}, \#\mathbf{v}, \#\mathbf{w})$ and $i + j + k = n + 1$ this is neatly expressed by valuation table of $\theta_{jki} = 0$:

$$
\begin{array}{cccccc}
\mathbf{v} & & & & & \mathbf{w} \\
 & = 1_j & \geq 2 & \geq 2 & = 2_k & \\
 & & \geq 1 & \geq 1 & \geq 2 & \\
 & & & \geq 1 & \geq 1 & \\
 & & & & = 0_i & \\
 & & & & & \mathbf{u}
\end{array}
$$

Think

We capture this characterization and explicitly describe how often

Density of cubics with a point

Theorem (Beneish–K.)

Let $n \geq 4$ (conjecturally $n \geq 3$). Then $\rho_n = 1$ when $n \geq 9$ and

$$\rho_n = \prod_{p \text{ prime}} \left(1 - \frac{g_n(p)}{h_n(p)}\right) \text{ when } n \leq 8$$

for explicit polynomials $g_n(t), h_n(t) \in \mathbb{Z}[t]$.

n	$\rho_n \approx$	$1 - \rho_n(p) \sim$
3	0.999927 (conj.)	$1/3p^{10}$
4	$1 - 5.022 \cdot 10^{-9}$	$1/9p^{22}$
5	$1 - 1.343 \cdot 10^{-15}$	$1/9p^{43}$
6	$1 - 3.502 \cdot 10^{-26}$	$1/9p^{78}$
7	$1 - 5.152 \cdot 10^{-42}$	$1/27p^{129}$
8	$1 - 6.222 \cdot 10^{-64}$	$1/27p^{201}$

Further questions

Let $\rho_{d,n} =$ density of degree d hypersurfaces in $\mathbb{P}^n$ with $\mathbb{Q}$-point

How far can this approach go to compute $\rho_{d,n}$?

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Let $\rho_{d,n}$ = density of degree d hypersurfaces in $\mathbb{P}^n$ with $\mathbb{Q}$-point

How far can this approach go to compute $\rho_{d,n}$?

- Lose uniformity in p
- Heath–Brown: $\rho_{4,n}(p) = 1$ for $n \geq 9126$,
  $\rho_{5,n}(p) = 1$ known for $n \geq 25$, $p \geq 17$ [HB09]
- Can we predict asymptotics/numerics with less effort?

Further questions

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Structure of $\rho_{d,n}(p)$

- Always rational function? (for $p \gg 0$)
- Known for binary forms, $\rho_{d,1}(p) = \rho_{d,1}(1/p)$ [BCFG22]

Thank you I

Thank you for the invitation and for your attention!

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