1. Consider our usual regression model $Y = X\beta + \epsilon$, with Y $n \times 1$, X $n \times p$ and prior $\beta \sim N_p(\beta_0, \sigma^2 V_0)$ but take σ^2 *known*.

   (a) Calculate the Bayes factor for $H_0 : \beta_j = 0$ versus $H_0 : \beta_j \neq 0$ using the Savage-Dickey ratio. Simplify as best possible.

$$\underset{n\times 1}{Y} = \underset{n\times p}{X}\underset{p\times 1}{\beta} + \underset{n\times 1}{\varepsilon}, \quad \beta \sim N_p(\beta_0, \sigma^2 V_0), \quad \sigma^2 \text{ known} \rightarrow \beta_j \sim N(\beta_0, \sigma^2 V_{0jj})$$

a) Bayes factor for $H_0 : \beta_j = 0$ vs. $H_A : \beta_j \neq 0$

Savage Dickey Ratio

Rearrange Bayes Thm on reduced parameter space (β_j, k)

$$f(\beta_j, k \mid Y) = \frac{f(Y \mid \beta_j, k) \cdot f(\beta_j, k)}{f(Y)}$$

$$\hookrightarrow \frac{f(Y \mid \beta_j, k)}{f(Y)} = \frac{f(\beta_j, k \mid Y)}{f(\beta_j, k)}$$

evaluate both sides at $\beta_j = 0$

$$B_{01} = \boxed{\frac{f(Y \mid \beta_j, k)\big|_{\beta_j = 0}}{f(Y)}} = \frac{\boxed{f(\beta_j, k \mid Y)}}{\boxed{f(\beta_j, k)}}\Bigg|_{\beta_j = 0}$$

Bayes factor B_{01}

posterior of $\beta_j \mid Y$ evaluated @ $\beta_j = 0$: $\beta_j \mid Y \sim N\left(\bar{\beta}_j, \sigma^2 (X_j' X_j + V_{0jj}^{-1})^{-1}\right)$

$\bar{\beta}_j = (X_j' X_j + (V_{0jj})^{-1})^{-1} (X_j' X_j \hat{\beta}_j + V_{0jj}^{-1} \beta_0)$

let $\Sigma = X_j' X_j + V_{0jj}^{-1}$

prior of β_j evaluated @ $\beta_j = 0$

pdf of normal: $\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$ for $N(\mu, \sigma)$

$$= \frac{\left(2\pi\left(\frac{\sigma^2}{\Sigma}\right)\right)^{-1/2} e^{-\frac{1}{2}\left(\frac{\Sigma}{\sigma^2}\right)^2 (0 - \bar{\beta}_j)^2}}{\left(2\pi(\sigma^2 V_{0jj})\right)^{-1/2} e^{-\frac{1}{2}\left(\frac{1}{\sigma^2 V_{0jj}}\right)^2 (0 - \beta_0)^2}}$$

$$= (\Sigma \cdot V_{0jj})^{1/2} e^{\left(\frac{1}{2\sigma^4} \cdot \Sigma^2 \cdot \bar{\beta}_j^2\right) - \left(-\frac{1}{2\sigma^4} \cdot \frac{1}{V_{0jj}^2} \cdot \beta_0^2\right)}$$

$$= (\Sigma \cdot V_{0jj})^{1/2} e^{-\frac{1}{2\sigma^4}\left(\Sigma^2 \cdot \bar{\beta}_j^2 - \frac{\beta_0^2}{V_{0jj}^2}\right)}$$

$$= \left[(X_j' X_j + V_{0jj}^{-1}) \cdot V_{0jj}\right]^{1/2} e^{-\frac{1}{2\sigma^4}\left[(X_j' X_j + V_{0jj}^{-1})^2 \cdot \left(\frac{X_j' X_j \hat{\beta}_j + \frac{\beta_0}{V_{0jj}}}{X_j' X_j + V_{0jj}^{-1}}\right)^2 - \frac{\beta_0^2}{V_{0jj}^2}\right]}$$

$$= (X_j' X_j V_{0jj} + 1)^{1/2} e^{-\frac{1}{2\sigma^4}\left[\left(X_j' X_j \hat{\beta}_j + \frac{\beta_0}{V_{0jj}}\right)^2 - \frac{\beta_0^2}{V_{0jj}^2}\right]}$$

$$= (X_j' X_j V_{0jj} + 1)^{1/2} e^{-\frac{1}{2\sigma^4}\left[(X_j' X_j)(X_j X_j')\hat{\beta}_j^2 + 2 X_j' X_j \hat{\beta}_j \beta_0 V_{0jj}^{-1} + \frac{\beta_0^2}{V_{0jj}^2} - \frac{\beta_0^2}{V_{0jj}^2}\right]}$$

$$\boxed{B_{01} = (X_j' X_j V_{0jj} + 1)^{1/2} e^{-\frac{1}{2\sigma^4}\left[(X_j' X_j)\left(X_j X_j' \hat{\beta}_j^2 + 2\hat{\beta}_j \beta_0 V_{0jj}^{-1}\right)\right]}}$$