1. For observations y_i, $i = 1, \ldots, n$, let Model 1 be

$$y_i|\sigma^2 \sim N(0, \sigma^2)$$

and let Model 2 be

$$y_i|\mu, \sigma^2 \sim N(\mu, \sigma^2)$$
$$\mu|\sigma^2 \sim N(0, \sigma^2/k)$$

both with prior

$$\sigma^2 \sim \text{InverseGamma}(a/2, b/2)$$

for σ^2. Prior parameters $k > 0$, $a > 0$, and $b > 0$ are known

(a) Calculate $f(Y)$ for both models and give the Bayes factor B_{12}. [Note: For model 1, this is an inverse gamma integral. For model 2, first integrate with respect to (w.r.t.) μ which is proportional to the integral of a normal density. Then integrate w.r.t. σ^2, which is proportional to an inverse gamma integral.]

(b) For $k = a = b = n = 4$, and residual sum of squares RSS $= 4$, plot B_{12} as a function of $\bar{y}$. [Include your R code on a separate page.] [You may prefer to plot $\log B_{12}$, either plot is ok.]

a)

$$f_1(Y) = \int_0^\infty f(y|\sigma^2)\, f(\sigma^2)\, d\sigma^2$$

$$= \int_0^\infty \frac{1}{(2\pi\sigma^2)^{n/2}} \exp\left(-\frac{1}{2\sigma^2}\sum(y-0)^2\right) \frac{(b/2)^{a/2}}{\Gamma(a/2)} (\sigma^2)^{-a/2-1} \exp\left(-\frac{b/2}{\sigma^2}\right) d\sigma^2$$

$$= \frac{1}{(2\pi)^{n/2}} \frac{\left(\frac{b}{2}\right)^{a/2}}{\Gamma\left(\frac{a}{2}\right)} \int_0^\infty (\sigma^2)^{-n/2} \exp\left(-\frac{\sum y^2}{2\sigma^2} - \frac{b}{2\sigma^2}\right) (\sigma^2)^{-a/2-1}\, d\sigma^2$$

$$= \frac{1}{(2\pi)^{n/2}} \frac{\left(\frac{b}{2}\right)^{a/2}}{\Gamma\left(\frac{a}{2}\right)} \int_0^\infty (\sigma^2)^{-\left(\frac{a}{2}+\frac{n}{2}\right)-1} \exp\left(-\frac{(\sum y^2+b)/2}{\sigma^2}\right) d\sigma^2$$

$$= \frac{1}{(2\pi)^{n/2}} \frac{\left(\frac{b}{2}\right)^{a/2}}{\Gamma\left(\frac{a}{2}\right)} \frac{\Gamma\left(\frac{a+n}{2}\right)}{\left(\frac{\sum y^2+b}{2}\right)^{\frac{a+n}{2}}} \underbrace{\int_0^\infty (\sigma^2)^{-\left(\frac{a+n}{2}\right)-1} \exp\left(-\frac{\frac{\sum y^2+b}{2}}{\sigma^2}\right) \frac{\left(\frac{\sum y^2+b}{2}\right)^{\frac{a+n}{2}}}{\Gamma\left(\frac{a+n}{2}\right)}\, d\sigma^2}_{\text{integrates to 1}}$$

$$f_1(Y) = \frac{\left(\frac{b}{2}\right)^{a/2} \Gamma\left(\frac{a+n}{2}\right)}{(2\pi)^{n/2}\, \Gamma\left(\frac{a}{2}\right) \left(\frac{b+\sum y^2}{2}\right)^{\frac{a+n}{2}}}$$

$$f_2(Y) = \int_0^\infty \int_0^\infty f(y|\mu,\sigma^2)\, f(\mu|\sigma^2)\, f(\sigma^2)\, d\mu\, d\sigma^2$$

$$= \int_0^\infty \int_0^\infty \frac{1}{(2\pi\sigma^2)^{n/2}} \exp\left(-\frac{1}{2\sigma^2}\sum(y_i-\mu)^2\right) \frac{1}{\left(2\pi\frac{\sigma^2}{k}\right)^{1/2}} \exp\left(-\frac{1}{2\frac{\sigma^2}{k}}(\mu-0)^2\right) \frac{(b/2)^{a/2}}{\Gamma(a/2)} (\sigma^2)^{-a/2-1} \exp\left(-\frac{b/2}{\sigma^2}\right) d\mu\, d\sigma^2$$

$$= \frac{(b/2)^{a/2}}{\Gamma(a/2)} \int_0^\infty \frac{1}{(2\pi\sigma^2)^{n/2} \left(2\pi\frac{\sigma^2}{k}\right)^{1/2}} (\sigma^2)^{-\frac{a}{2}-1} \exp\left(-\frac{b/2}{\sigma^2}\right) \int_0^\infty \exp\left(-\frac{1}{2\sigma^2}\sum(y_i-\mu)^2\right) \exp\left(-\frac{1}{2\frac{\sigma^2}{k}}(\mu-0)^2\right) d\mu\, d\sigma^2$$

$$= \frac{(b/2)^{a/2}}{\Gamma(a/2)} \int_0^\infty \frac{1}{(2\pi\sigma^2)^{n/2}} \exp\left(-\frac{1}{2\sigma^2}\sum(y_i-\mu)^2\right)(\sigma^2)^{-\frac{a}{2}-1} \exp\left(-\frac{b/2}{\sigma^2}\right)\int_0^\infty \frac{1}{(2\pi\frac{\sigma^2}{k})^{1/2}} \exp\left(-\frac{1}{2\frac{\sigma^2}{k}}(\mu-0)^2\right) d\mu\, d\sigma^2$$

$$= \frac{(b/2)^{a/2}}{\Gamma(a/2)} \int_0^\infty \frac{1}{(2\pi\sigma^2)^{n/2}} \exp\left[-\frac{1}{2\sigma^2}\left(\sum(y_i-\bar{y})^2+n(\bar{y}-\mu)^2\right)\right](\sigma^2)^{-\frac{a}{2}-1} \exp\left(-\frac{b/2}{\sigma^2}\right)\int_0^\infty \frac{1}{(2\pi\frac{\sigma^2}{k})^{1/2}} \exp\left(-\frac{1}{2\frac{\sigma^2}{k}}(\mu-0)^2\right) d\mu\, d\sigma^2$$

$$= \frac{(b/2)^{a/2}}{\Gamma(a/2)} \int_0^\infty \frac{1}{(2\pi\sigma^2)^{n/2}} \exp\left[-\frac{1}{2}\left(\frac{RSS}{\sigma^2}+\frac{n(\bar{y}-\mu)^2}{\sigma^2}\right)\right](\sigma^2)^{-\frac{a}{2}-1} \exp\left(\frac{-b}{2\sigma^2}\right)\int_0^\infty \frac{k^{1/2}}{(2\pi\sigma^2)^{1/2}} \exp\left(\frac{-k}{2\sigma^2}(\mu-0)^2\right) d\mu\, d\sigma^2$$

$$= \frac{(b/2)^{a/2}}{\Gamma(a/2)} \int_0^\infty \frac{1}{(2\pi\sigma^2)^{n/2}} \exp\left[-\frac{1}{2}\left(\frac{RSS}{\sigma^2}+\frac{n(\bar{y}-\mu)^2}{\sigma^2}+\frac{k(\mu-0)^2}{\sigma^2}+\frac{b}{\sigma^2}\right)\right](\sigma^2)^{-a/2-1}\int_0^\infty \frac{k^{1/2}}{(2\pi\sigma^2)^{1/2}}\, d\mu\, d\sigma^2$$

$$= \frac{(b/2)^{a/2}}{\Gamma(a/2)} \int_0^\infty \frac{1}{(2\pi\sigma^2)^{n/2}} \exp\left[-\frac{1}{2\sigma^2}\left(RSS + n\bar{y}^2-2n\bar{y}\mu+n\mu^2+k\mu^2+b\right)\right](\sigma^2)^{-a/2-1}\int_0^\infty \frac{k^{1/2}}{(2\pi\sigma^2)^{1/2}}\, d\mu\, d\sigma^2$$

$$= \frac{(b/2)^{a/2}}{\Gamma(a/2)} \int_0^\infty \frac{1}{(2\pi\sigma^2)^{n/2}} \exp\left[-\frac{1}{2\sigma^2}\left(RSS+(n+k)\mu^2-2\mu n\bar{y}+n\bar{y}^2+b\right)\right](\sigma^2)^{-a/2-1}\int_0^\infty \frac{k^{1/2}}{(2\pi\sigma^2)^{1/2}}\, d\mu\, d\sigma^2$$

$$= \frac{(b/2)^{a/2}}{\Gamma(a/2)} \int_0^\infty \frac{1}{(2\pi\sigma^2)^{n/2}} \exp\left[-\frac{1}{2\sigma^2}\left(RSS+(n+k)\left(\mu^2-2\mu(n+k)^{-1}n\bar{y}+\left[(n+k)^{-1}(n\bar{y})\right]^2\right)+n\bar{y}^2-(n+k)^{-1}(n\bar{y})^2+b\right)\right]$$
$$(\sigma^2)^{-a/2-1}\int_0^\infty \frac{k^{1/2}}{(2\pi\sigma^2)^{1/2}}\, d\mu\, d\sigma^2$$

$$= \frac{(b/2)^{a/2}}{\Gamma(a/2)} \int_0^\infty \frac{1}{(2\pi\sigma^2)^{n/2}} \exp\left[-\frac{1}{2\sigma^2}\left(RSS+(n+k)\left(\mu-(n+k)^{-1}(n\bar{y})\right)^2+n\bar{y}^2-\frac{n^2\bar{y}^2}{n+k}+b\right)\right](\sigma^2)^{-a/2-1}\int_0^\infty \frac{k^{1/2}}{(2\pi\sigma^2)^{1/2}}\, d\mu\, d\sigma^2$$

$$= \frac{1}{(n+k)^{1/2}}\frac{(b/2)^{a/2}}{\Gamma(a/2)}\frac{k^{1/2}}{(2\pi)^{\frac{n}{2}}} \underline{\int_0^\infty \frac{(n+k)^{1/2}}{2\pi^{1/2}}(\sigma^2)^{-1/2}\exp\left[-\frac{n+k}{2\sigma^2}\left(\mu-\frac{n\bar{y}}{n+k}\right)^2\right]} \int_0^\infty (\sigma^2)^{-n/2-a/2-1}\exp\left[-\frac{1}{2\sigma^2}\left(RSS+b+n\bar{y}^2-\frac{(n\bar{y})^2}{n+k}\right)\right] \underline{d\mu}\, d\sigma^2$$

$$= \frac{(b/2)^{a/2}}{\Gamma(a/2)}\frac{k^{1/2}}{(2\pi)^{n/2}(n+k)^{1/2}}\ (1) \int_0^\infty (\sigma^2)^{-\left(\frac{a+n}{2}\right)-1}\exp\left[-\frac{1}{2\sigma^2}\left(RSS+b+n\bar{y}^2-\frac{(n\bar{y})^2}{n+k}\right)\right]d\sigma^2$$

$$= \frac{(b/2)^{a/2}}{\Gamma(a/2)}\frac{\Gamma\left(\frac{a+n}{2}\right)}{\left(\frac{1}{2}\left(RSS+b+n\bar{y}^2+\frac{(n\bar{y})^2}{n+k}\right)\right)^{\frac{a+n}{2}}}\frac{k^{1/2}}{(2\pi)^{n/2}(n+k)^{1/2}} \underline{\int_0^\infty \frac{\left(\frac{1}{2}\left(RSS+b+n\bar{y}^2+\frac{(n\bar{y})^2}{n+k}\right)\right)^{\frac{a+n}{2}}}{\Gamma\left(\frac{a+n}{2}\right)}(\sigma^2)^{-\left(\frac{a+n}{2}\right)-1}}$$
$$\underline{\exp\left[-\frac{1}{2\sigma^2}\left(RSS+b+n\bar{y}^2-\frac{(n\bar{y})^2}{n+k}\right)\right]d\sigma^2}$$

$$= \frac{(b/2)^{a/2}}{\Gamma(a/2)}\frac{\Gamma\left(\frac{a+n}{2}\right)}{\left(\frac{1}{2}\left(RSS+b+n\bar{y}^2-\frac{(n\bar{y})^2}{n+k}\right)\right)^{\frac{a+n}{2}}}\frac{k^{1/2}}{(2\pi)^{n/2}(n+k)^{1/2}}\ (1)$$

$$f_2(Y) = \frac{(b/2)^{a/2}}{\Gamma(a/2)}\frac{\Gamma\left(\frac{a+n}{2}\right)}{\left(\frac{1}{2}\left(RSS+b+n\bar{y}^2-\frac{(n\bar{y})^2}{n+k}\right)\right)^{\frac{a+n}{2}}}\frac{k^{1/2}}{(2\pi)^{n/2}(n+k)^{1/2}}$$

$$B_{12} = \frac{f_1(Y)}{f_2(Y)}$$

$$= \left[\frac{\cancel{\left(\frac{b}{2}\right)^{a/2}}\, \cancel{\Gamma\left(\frac{a+n}{2}\right)}}{\cancel{(2\pi)^{n/2}}\, \cancel{\Gamma\left(\frac{a}{2}\right)} \left(\frac{b+\Sigma y^2}{2}\right)^{\frac{a+n}{2}}}\right] \left[\frac{\cancel{(b/2)^{a/2}}}{\cancel{\Gamma(a/2)}} \; \frac{\cancel{\Gamma\left(\frac{a+n}{2}\right)}}{\left(\frac{1}{2}\left(RSS+b+n\bar{y}^2-\frac{(n\bar{y})^2}{n+K}\right)\right)^{\frac{a+n}{2}}} \; \frac{K^{1/2}}{\cancel{(2\pi)^{n/2}}(n+k)^{1/2}}\right]^{-1}$$

$$= \frac{(n+k)^{1/2}\left[\frac{1}{2}\left(RSS+b+n\bar{y}^2-\frac{(n\bar{y})^2}{n+k}\right)\right]^{\frac{a+n}{2}}}{k^{1/2}\left(\frac{b+\Sigma y^2}{2}\right)^{\frac{a+n}{2}}}$$

$$\frac{(n+k)n\bar{y}^2-n^2\bar{y}^2}{n+k}$$
$$\frac{n^2\bar{y}^2+kn\bar{y}^2-n^2\bar{y}^2}{n+k}$$
$$\frac{kn\bar{y}^2}{n+k}$$

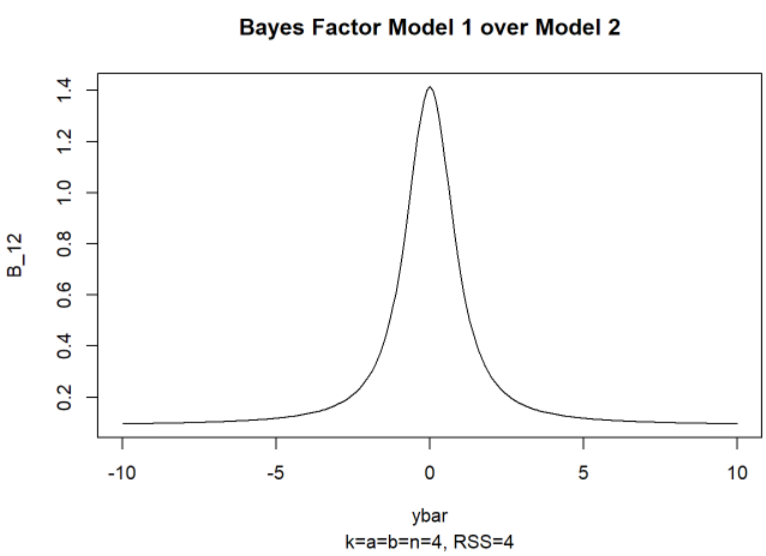
$$\boxed{B_{12} = \frac{(n+k)^{1/2}\left[\left(RSS+b+\frac{kn\bar{y}^2}{n+k}\right)\right]^{\frac{a+n}{2}}}{k^{1/2}\left[\left(RSS+b+n\bar{y}^2\right)\right]^{\frac{a+n}{2}}}}$$

[RSS is supposed to be called TSS I think]

b) $$B_{12} = \left(\frac{n+k}{k}\right)^{1/2}\left[\frac{RSS+b+\frac{nk}{n+k}\bar{y}^2}{RSS+b+n\bar{y}^2}\right]^{\frac{a+n}{2}}$$

```r
#bayes factor of model 1 over model 2 as a function of ybar
B12 <- function(k,a,b,n,RSS,ybar){
  sqrtterm <- sqrt((n+k)/k)
  otherterm <- ((RSS+b+((n*k)/(n+k))*ybar^2)/(RSS+b+n*ybar^2))^((a+n)/2)
  return(sqrtterm*otherterm)
}

ybar <- seq(-10,10,0.1)
B12plot <- B12(4,4,4,4,4,ybar)
#plot
plot(ybar,B12plot,xlab=expression(ybar), ylab = expression(B_12), type="l")
title(main = "Bayes Factor Model 1 over Model 2", sub = "k=a=b=n=4, RSS=4")
```



3. Use the regression model given in the theorybayes09 lecture notes but take σ^2 known.

   (a) Derive the posterior distribution of the regression coefficients β, which is a multivariate normal distribution.

   (b) Derive the marginal normal posterior for a single regression coefficient β_j. [Note: If you're not familiar with the properties of the multivariate normal, you can/should familiarize yourself with some of the basic properties of the multivariate normal. In particular, you are asked here to give the distribution of a single component of a multivariate normal.]

   (c) What is the posterior probability that β_j is positive in terms of a standard normal distribution cdf? [This is a one-sided (Bayesian) p-value. Doubling this probability (or doubling one minus this probability if the probability is bigger than .5) is approximately a usual frequentist two-sided p-value, but has little value in a Bayesian context.]

   (d) Bayesians can live with typical regression coefficient two-sided p-values by dividing by 2 and interpreting them as in item 3c. Similarly, frequentists can take the quite interpretable Bayesian posterior probability that $\beta_j > 0$ and double it and call it a two-sided p-value (and make it difficult to interpret). [nothing to do]

$Y|\beta,\sigma^2 \sim N(X\beta, \sigma^2 I)$ where I is nxn identity matrix

prior $\beta|\sigma^2 \sim N(\beta_0, V_0\sigma^2)$

β_0, V_0, σ^2 known

a) $$f(\beta|Y) = \frac{f(Y|\beta)f(\beta)}{f(Y)}$$
$$\propto f(Y|\beta)f(\beta)$$

get the likelihood:
$$f(Y|\beta) = \prod_{i=1}^{n} \frac{1}{(2\pi\sigma^2 I)^{1/2}} \exp\left[-\frac{1}{2\sigma^2 I}(Y-X\beta)'(Y-X\beta)\right]$$
$$= (2\pi)^{-n/2}(\sigma^2 I)^{-n/2} \exp\left\{-\frac{1}{2\sigma^2 I}\left[\underbrace{(Y-X\hat\beta)'(Y-X\hat\beta)}_{RSS} + (\beta-\hat\beta)'(X'X)(\beta-\hat\beta)\right]\right\}$$ where $\hat\beta = (X'X)^{-1}(X'Y)$ = LSE/MLE

derive posterior
$$f(\beta|Y) \propto (2\pi)^{-n/2}(\sigma^2 I)^{-n/2}\exp\left\{-\frac{1}{2\sigma^2 I}\left[RSS + (\beta-\hat\beta)'(X'X)(\beta-\hat\beta)\right]\right\}(2\pi)^{-p/2}(\sigma^2)^{-p/2}|V_0|^{-1/2}\exp\left[-\frac{1}{2\sigma^2}(\beta-\beta_0)'V_0^{-1}(\beta-\beta_0)\right]$$
$$\propto (2\pi\sigma^2)^{-n/2}(2\pi\sigma^2)^{-p/2}|V_0|^{-1/2}\exp\left\{-\frac{1}{2\sigma^2}\left[RSS + \underline{(\beta-\hat\beta)'(X'X)(\beta-\hat\beta) + (\beta-\beta_0)'V_0^{-1}(\beta-\beta_0)}\right]\right\}$$

completing the square:

expansion of orange

$\rightarrow \beta'(X'X)\beta + \hat\beta'(X'X)\hat\beta - 2\hat\beta'(X'X)\beta + \beta'V_0^{-1}\beta + \beta_0'V_0^{-1}\beta_0 - 2\beta_0'V_0^{-1}\beta$

$\rightarrow \beta'(X'X+V_0^{-1})\beta - 2[\hat\beta(X'X) + \beta_0'V_0^{-1}]\beta + \hat\beta'(X'X)\hat\beta + \beta_0'V_0^{-1}\beta_0$

$\rightarrow \beta'(X'X+V_0^{-1})\beta - 2[\hat\beta(X'X)+\beta_0'V_0^{-1}]\beta + [\hat\beta(X'X)+\beta_0'V_0^{-1}](X'X+V_0^{-1})^{-1}[\hat\beta(X'X)+\beta_0'V_0^{-1}] - [\hat\beta(X'X)+\beta_0'V_0^{-1}](X'X+V_0^{-1})^{-1}[\hat\beta(X'X)+\beta_0'V_0^{-1}]$
$+ \hat\beta'(X'X)\hat\beta + \beta_0'V_0^{-1}\beta_0$

$$\rightarrow \left(\beta - (X'X+V_0^{-1})^{-1}[X'X\hat{\beta}+V_0^{-1}\beta_0]\right)^T (X'X+V_0^{-1}) \left(\beta - (X'X+V_0^{-1})^{-1}[X'X\hat{\beta}+V_0^{-1}\beta_0]\right) + (\hat{\beta}-\beta_0)' \left((X'X)^{-1}+V_0\right)^{-1} (\hat{\beta}-\beta_0)$$

$$\bar{\beta} = (X'X+V_0^{-1})^{-1}[X'X\hat{\beta}+V_0^{-1}\beta_0]$$

$$\beta | Y \sim MVN\left(\bar{\beta}, \sigma^2 (X'X+V_0^{-1})^{-1}\right)$$

b) β_j is a single component of a multivariate normal, so that means each of the β's has a marginal normal posterior

$$\beta_j \sim N\left(\bar{\beta}_j, \sigma^2 (X_j'X_j+V_{0,jj}^{-1})^{-1}\right)$$

$$\text{with } \bar{\beta}_j = (X_j'X_j+V_{0,jj}^{-1})^{-1}[X_j'X_j\hat{\beta}+V_{jj}\beta_0]$$

c) $P(\beta_j > 0 | Y) = 1 - P(\beta_j < 0 | Y)$

$$= 1 - P\left(\underbrace{\frac{\beta_j - \hat{\beta}_j}{SE(\hat{\beta}_j)}}_{\text{approximately a normal}} < \frac{0-\hat{\beta}_j}{SE(\hat{\beta}_j)} \,\Big|\, Y\right)$$

$$= 1 - P\left(Z < \frac{-\hat{\beta}_j}{SE(\hat{\beta}_j)}\right)$$

$$= 1 - \Phi\left(\frac{-\hat{\beta}_j}{SE(\hat{\beta}_j)}\right)$$

$$= 1 - \Phi\left(\frac{-\bar{\beta}_j}{\left(\sigma^2 (X_j'X_j+V_{0,jj}^{-1})^{-1}\right)^{1/2}}\right)$$

$$= 1 - \Phi\left(\frac{(X_j'X_j+V_{0,jj}^{-1})^{-1}(X_j'X_j\hat{\beta}+V_{0,jj}\beta_0)}{\sigma (X_j'X_j+V_{0,jj}^{-1})^{-1/2}}\right)$$

$$= 1 - \Phi\left(\frac{X_j'X_j\hat{\beta}+V_{0,jj}\beta_0}{\sigma (X_j'X_j+V_{0,jj}^{-1})^{1/2}}\right)$$