3.3 Def. A morphism $f: Y \to X$ of sheaves over S_{fppf} is representable if $\forall\, U \in Sch/S$, and $\forall\, x \in X(U)$

$$\begin{tikzcd} \square \arrow[r] \arrow[d] & U \arrow[d, "x"] \\ Y \arrow[r, "f"] & X \end{tikzcd}$$

$\square$ iso to a scheme

3.4 Lemma. If the diagonal $\Delta: X \to X \times_S X$ is representable, then $U \xrightarrow{x} X$ is representable for all $x \in X(U)$, U scheme.

Pf.

$$\begin{tikzcd} U' \times_X U \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & U' \times_S U \arrow[d] \\ X \arrow[r, "\Delta"] & X \times_S X \end{tikzcd}$$

$U' \times_S U \to$ scheme, Δ rep., $\Rightarrow U' \times_X U$ scheme.

§4. From Sch to Esp, then to Chp.

4.1 Def. (1). Let P = ét, surj

A rep. mor of sheaves over S_{fppf} $Y \to X$ has P if for all $x \in X(U)$, $\forall\, U \in Sch/S$, $Y \times_X U \to U$ has P.

$$\begin{tikzcd} Y \times_X U \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & U \arrow[d] \\ Y \arrow[r] & X \end{tikzcd}$$

sch $\to Y \times_X U$, $U \leftarrow$ sch, $Y \to X$ rep.

(2). $X \in Shv(S_{fppf}, Set)$ is an algebraic space over S if (a). the diag. $X \xrightarrow{\Delta} X \times_S X$ is rep.

(b) $\exists$ surj ét mor. $U \to X$ with $U \in Sch/S$. (called an atlas for X)

(by (a) (3.4) this is ~~ato~~ auto rep. hence makes sense in (1).)

the cat of alg. spaces is denoted by Esp/S.

(3). Let $Y \xrightarrow{f} X$ mor of algebraic space over S_{fppf}.

$$\begin{tikzcd} V \arrow[r, "\text{surj, ét}"] \arrow[rr, bend left] \arrow[dr, bend right] & Y \times_X U \arrow[r] \arrow[d] & U \arrow[d, "\text{surj, ét}"] \\ & Y \arrow[r, "f"] & X \end{tikzcd}$$

A chart for f is a comm. diag obtained by this way. with $V \to U$ in Sch/S.

$\cdots Y \to X$ with $V \to U$ in $Sch/_S$.

(4) Let P = ét, sm, surj. $Y \to X$ in $Esp/_S$.

has P if $\exists$ chart of f s.t. $V \to U$ has P.
(equiv. for all) [SP].

(5) Let P = ét, sm, surj.

A rep. mor of stacks over S_{fppf} has P if
"$Shv(S_{fppf}, Gpd)$"

for $\forall$ $x \in \mathcal{X}(U)$, $U \in Sch/_S$, $\underbrace{\mathcal{Y} \times_{\mathcal{X}} U}_{Esp/_S} \to U$ has P.

(6). $\mathcal{X} \in Shv(S_{fppf}, Gpd)$ is an algebraic stack.

(or Artin stack) (resp. Deligne–Mumford stack)
DM

over S if (a) the diag $\mathcal{X} \to \mathcal{X} \times_S \mathcal{X}$ is rep (by alg. spaces)

(b) $\exists$ surj. sm (resp. ét) mor $U \to \mathcal{X}$ with $U \in Sch/_S$
atlas.

Denote the (2,1)-cat of alg. stacks over S by
$Chp/_S$.

(7). Let $\mathcal{Y} \xrightarrow{f} \mathcal{X}$ mor in $Chp/_S$, a chart of f

$$\begin{tikzcd} V \arrow[r, "\text{sm, surj}"] \arrow[dr] & \mathcal{Y} \times_{\mathcal{X}} U \arrow[r] \arrow[d] & U \arrow[d, "\text{sm, surj}"] \\ & \mathcal{Y} \arrow[r] & \mathcal{X} \end{tikzcd}$$

$V \to U$ schemes.

(8). Let P = ét, sm, surj. $\mathcal{Y} \to \mathcal{X}$ in $Chp/_S$.

has P. if $\exists$ a chart of f s.t.
(equiv. for all)

$V \to U$ has P.

4.2. Examples of alg spaces / stacks

1). $BG : (Sch/_S)^{op} \to Gpd$. where G is a sm S-gp scheme.

$T \longmapsto Tors(T, G_T)$

classifying stack

More generally, $X \in Esp/_S$, G — sm S-gp sch.

acting on X. define quotient stack

$[X/G] : (Sch/_S)^{op} \to Gpd$.

$T \longmapsto (\pi : Z \to X_T)$ where

$Z \in Tors(T, G_T)$, π — G_T-equivariant morphism of $Shv(T_{fppf})$.

- By 2-Yoneda, $U \to [X/G]$ amounts to
  sch

$$\pi : \underline{Z \to X_U} \qquad Z \in Tors(U, G_U).$$
G_U-equivariant.

- Fact: $[X/G] \in Chp_{/S}$

$$\underline{X \longrightarrow} [X/G] \longleftrightarrow \rho : G_X \to X$$
the action of G. on X
$\underline{\substack{\| \\ G \times_S X \\ \text{trivial torsor}}}$

- In particular. $BG = B_S G = [S/G]$

- Similarly, more simple.

  G — disc. gp action on a scheme X/S.

  action is free. ($G \times X \longrightarrow X \times X$
  $g, x \longmapsto (x, gx)$

  then $X/G := (T \longmapsto X(T)/G)^{\#} \in Esp_{/S}$.

  called <u>quotient space</u>. with $X \to X/G$ atlas.

(2). $\mathcal{M}_g : (Sch_{/S})^{op} \longrightarrow Gpd$.

$T \longmapsto$ ($C \to T$ sm proper with geo. fiber conn. genus g curve.)

is a DM stack.

(3). (Olsson). $\underline{\mathcal{X}, \mathcal{Y}} \in Chp_{/S}$ fin presented sep. with fin. diagonals. $\mathcal{X}$ is flat, <u>proper</u> and fppf locally on S, $\exists$ fin. flat. surj. $\overset{Esp}{Z} \to \mathcal{X}$, ...

then $\mathcal{H}om_S(\mathcal{X}, \mathcal{Y}) : (Sch_{/S})^{op} \longrightarrow Gpd$. → stack.

$$T \longmapsto \underline{Hom_T(\mathcal{X}_T, \mathcal{Y}_T)}.$$

$\in Chp_{/S}$. (DM if $\mathcal{Y}$ is DM).

5. <u>Cohomology of algebraic stacks</u>

<u>5.1</u>. <u>Def</u>. The (big) étale / fppf site $\mathcal{X}_{ét}$ ($\mathcal{X}_{fppf}$).

de Jong <u>[SP]</u>. $\mathcal{X}$ — (algebraic) stack /S

cat: <u>$\mathcal{X}$</u>. <u>obj</u>: $U \xrightarrow{u} \mathcal{X} \longleftrightarrow u \in \mathcal{X}(U)$.
sch.

$\boxed{U \to \mathcal{X}}$ sch

cat: $\mathcal{X}$. obj: $U \xrightarrow{u} \mathcal{X} \longleftrightarrow u \in \mathcal{X}(U)$.
sch.

covering: $\{U_i \longrightarrow U\}$. is ét (fppf) covering when viewed as schemes over S.

(small ét, lis-ét).

§.2. Global section. $\tau = $ ét / fppf.

$$\Gamma(\mathcal{X}_\tau, -) : Ab(\mathcal{X}_\tau) \longrightarrow Ab. \qquad \mathrm{Map}_{psh}(*, -)$$
$$= \mathrm{Hom}_{Ab(\mathcal{X}_\tau)}(\mathbb{Z}, -)$$
$$\rightsquigarrow R\Gamma(\mathcal{X}_\tau, -) : D^+(\mathcal{X}_\tau) \longrightarrow D^+(Ab).$$

§.3. Functoriality. $\mathcal{Y} \to \mathcal{X}$ mor. of stacks. over S.

$\rightsquigarrow$ adj. $Shv(\mathcal{Y}_\tau) \underset{f^*}{\overset{f_*}{\rightleftarrows}} Shv(\mathcal{X}_\tau)$
exact.

$$\rightsquigarrow D^+(\mathcal{Y}_\tau) \underset{f^*}{\overset{Rf_*}{\rightleftarrows}} D^+(\mathcal{X}_\tau).$$

§.4. Rmk. For $X \in Sch/S$, this agrees with big ét / fppf cohomology of schemes and for ét, we have $R\Gamma((Sch/X)_{ét}, -) \xrightarrow{\sim} R\Gamma(X_{ét}, -|_{X_{ét}})$.

big small

§.5 Def. (Gerbes). Let $\mathcal{C}$ be a site.

- A stack $\mathcal{Y} \in Shv(\mathcal{C}, Gpd)$. is gerbe over $\mathcal{C}$ if
  - (locally nonempty) $\forall\, U \in \mathcal{C}$, $\exists$ cov. $\{U_i \to U\}$ s.t. $\mathcal{Y}(U_i) \neq \emptyset$, $\forall i$.
  - (locally connected) $\forall\, U \in \mathcal{C}$, $\forall\, x, y \in \mathcal{Y}(U)$, $\exists$ cov. $\{U_i \to U\}$ s.t. $x|_{U_i} \cong y|_{U_i}$ in $\mathcal{Y}(U_i)$. $\forall i$.
- If a gerbe $\mathcal{Y}$ of $\mathcal{C}$ is bound by $\mathcal{G} \in Ab(\mathcal{C})$ [Giraud, HTT]. we write $\mathcal{Y} \xrightarrow{\mathcal{G}} \mathcal{C}$.

  One may define $\mathcal{G}$-equiv. between them.

  $Gerb(\mathcal{C}, \mathcal{G})$ — full sub 2-cat of $Shv(\mathcal{C}, Gpd)$.

  Ex. Trivial gerb. BG $\mathcal{C} = Sch/S$ G S-gp.
- $\tau \in \{$ét, fppf$\}$ $T \in Shv(S_\tau, Gpd)$ we have

$$H^2_\tau(T, \mathcal{G}) \cong Gerb(T_\tau, \mathcal{G}) / \mathcal{G}\text{-equiv}.$$

5.6 ~~Cohomological descent~~. Let $\mathcal{X} \in \mathrm{Chp}/S$.

and $X \to \mathcal{X}$ an atlas. $\mathcal{F} \in Ab(\mathcal{X}_\tau)$.
(X: scheme)

$$X_\bullet : \Delta^{op} \longrightarrow \mathrm{Esp}/S \longrightarrow \text{NOT scheme.} \quad \text{be Čech nerve.}$$
$$[n] \longmapsto \underbrace{X \times_{\mathcal{X}} \cdots \times X}_{n+1}$$
p

Then $E_1^{pq} = H^q_\tau(X_p, \mathcal{F}|_{X_p}) \Rightarrow H^{p+q}_\tau(\mathcal{X}, \mathcal{F})$

whose E_2-page is

$$E_2^{pq} = \check{H}^p(X/\mathcal{X}, \mathcal{H}^q_\tau(\mathcal{F})) \Rightarrow H^{p+q}_\tau(\mathcal{X}, \mathcal{F})$$

$\mathcal{H}^q_\tau(\mathcal{F})$: the presheaf $U \longmapsto H^q_\tau(U, \mathcal{F}|_U)$

$$\begin{tikzcd} X \times_S G \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X \arrow[d, "\text{scheme}"] \\ X \arrow[r] & {[X/G]} \end{tikzcd}$$
(G: gp-scheme)

$$X_p = X \times_S \underbrace{G \times_S G \cdots \times G}_{p}$$

$$\begin{tikzcd} G \times T \arrow[r] \arrow[d] & G \arrow[d] \\ T \arrow[r, "G"] & X \end{tikzcd}$$

$$G \times_X T \xrightarrow{\sim} T \times_X T$$