

Introduction to the course and Bayesian data analysis

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Getting Started

Before we get started: R, Rstudio, and packages

1. Install / update R if needed (<https://cran.r-project.org/>)
2. Install / update RStudio if needed (<https://rstudio.com/>)
3. Install rstan (<https://github.com/stan-dev/rstan/wiki/RStan-Getting-Started>)
4. Install rethinking package (<https://github.com/rmcelreath/rethinking/tree/Experimental>)

Before we get started: Git

- Install Git (<https://git-scm.com/>)

1. Open your terminal [Mac, Linux] or Git Bash [Windows]

```
cd Dropbox [Arbitrary! choose any location you like]  
git clone https://github.com/f-edwards/intermediate_stats.git
```

2. Then, to update with the latest slides / data / etc

```
cd Dropbox/intermediate_stats  
git fetch
```

- This course closely follows McElreath's *Statistical Rethinking*
- His lectures and slides for the course are available for further review (github.com/rmcelreath/statrethinking_winter2019)
- My slides borrows liberally from the book and his materials

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- Models (**golems** for McElreath) are very powerful, but very dumb
- Statistical models \neq scientific models

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- We create stylized abstractions of reality in our models (McElreath's **small world**)
- They are always incomplete representations of reality (**large world**)
- Models answer questions about the **small world**. It's up to us to carefully translate them to the large world.
- Also, Columbus was a slaver and initiated a catastrophic genocide

Interpret these results

```
library(broom)
data(mtcars)
## mpg is miles per gallon, wt is weight in 1000 lbs
m1 <- lm(mpg ~ wt, data = mtcars)
tidy(m1)

## # A tibble: 2 x 5
##   term          estimate std.error statistic  p.value
##   <chr>          <dbl>    <dbl>    <dbl>    <dbl>
## 1 (Intercept)    37.3      1.88     19.9 8.24e-19
## 2 wt            -5.34     0.559    -9.56 1.29e-10

confint(m1)
```

```
##           2.5 %    97.5 %
## (Intercept) 33.450500 41.119753
## wt         -6.486308 -4.202635
```

Motivating two common approaches

- Frequentist: The truth is fixed, we can estimate it using repeated sampling and large number theorems, assuming our measurement is one of many that could have resulted

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- Frequentist: The truth is fixed, we can estimate it using repeated sampling and large number theorems, assuming our measurement is one of many that could have resulted
- Bayesian: Given our assumptions and the data, which probability distribution is the most plausible answer?

- A more intuitive approach to interpreting statistical models (no more sampling distributions!)

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- A more intuitive approach to interpreting statistical models (no more sampling distributions!)
- Computational costs have decreased rapidly
- Priors are a useful way to incorporate what we already know
- Avoids overfitting by not trusting the data too much
- Applications in scientific inference, prediction, machine learning

Probability: how many ways could
what happened have happened?



Contains 4 marbles

Possible contents:

- (1) ○ ○ ○ ○
- (2) ● ○ ○ ○
- (3) ● ● ○ ○
- (4) ● ● ● ○
- (5) ● ● ● ●

Observe:



Stolen from McElreath's slides

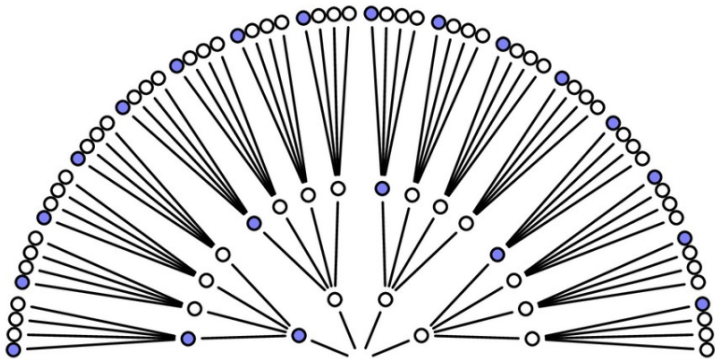
Conjecture: ● ○ ○ ○

Data: ● ○ ●



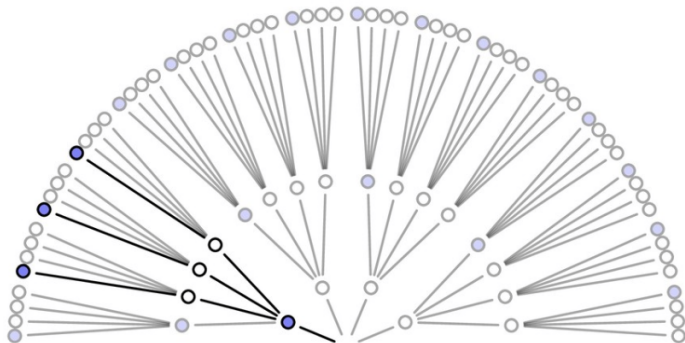
Conjecture: ● ○ ○ ○ ○

Data: ● ○ ●



Conjecture: ● ○ ○ ○ ○

Data: ● ○ ● ●



3 paths consistent with data

Possible contents:

(1) ○ ○ ○ ○

(2) ● ○ ○ ○

(3) ● ● ○ ○

(4) ● ● ● ○

(5) ● ● ● ●

Ways to produce ● ○ ●

?

3

?

?

?

Possible contents:

(1) ○ ○ ○ ○

(2) ● ○ ○ ○

(3) ● ● ○ ○

(4) ● ● ● ○

(5) ● ● ● ●

Ways to produce ● ○ ●

0

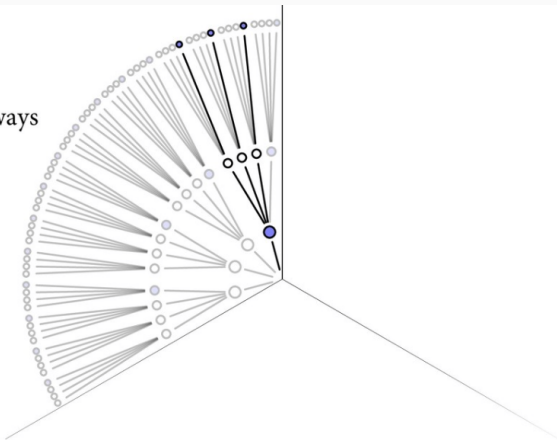
3

?

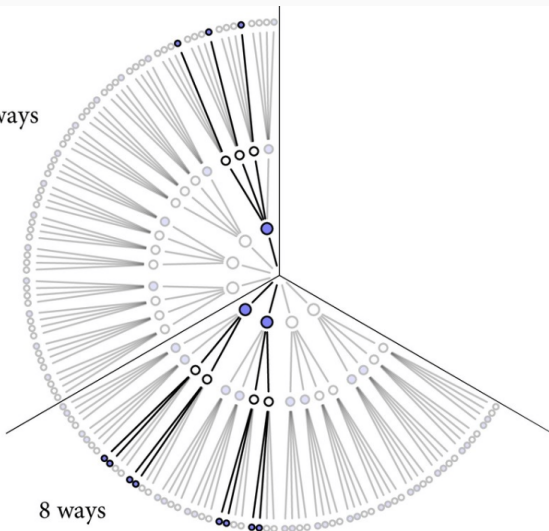
?

0

3 ways



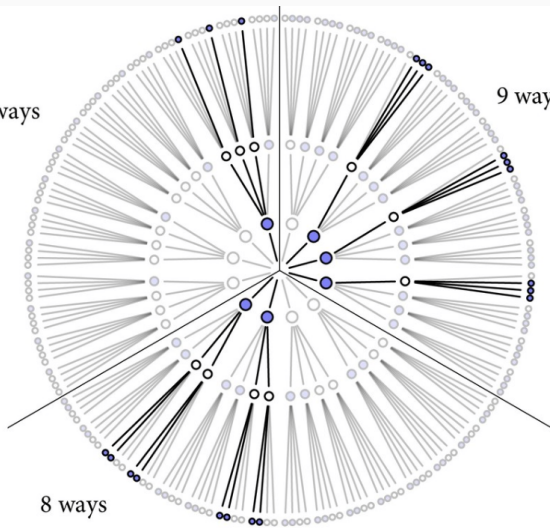
3 ways



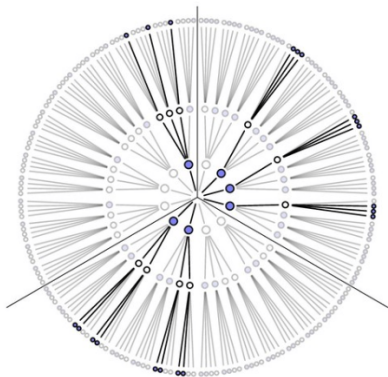
3 ways

9 ways

8 ways



Conjecture	Ways to produce ●○○●
[○○○○]	$0 \times 4 \times 0 = 0$
[●○○○]	$1 \times 3 \times 1 = 3$
[●●○○]	$2 \times 2 \times 2 = 8$
[●●●○]	$3 \times 1 \times 3 = 9$
[●●●●]	$4 \times 0 \times 4 = 0$



Adding other information

Draw one new marble: it is Blue

Conjecture	Ways to produce B	Prior counts	New count
WWWW	0	0	$0 \times 0 = 0$
BWWW	1	3	$1 \times 3 = 3$
BBWW	2	8	$2 \times 8 = 16$
BBBW	3	9	$3 \times 9 = 27$
BBBB	4	0	$4 \times 0 = 0$

plausability of [BWWW] after seeing [B] \propto

ways [BWWW] can produce B \times

prior plausbility of [BWWW] based on draw [BWB]

$$P(B|A) = \frac{P(A|B)P(B)}{P(A)}$$

$$P(\text{parameter}|\text{data}) = \frac{P(\text{data}|\text{parameter})P(\text{parameter})}{P(\text{data})}$$

Or in Bayesian vernacular:

$$\text{Posterior} = \frac{\text{Likelihood} \times \text{Prior}}{\text{Average probability of the data}}$$

Updating our estimate of plausability of each parameter

Let's indicate the possible bag compositions with the parameter θ

θ	$pl(B \theta)$	Prior $pl(\theta)$	Posterior $pl(\theta B)$
WWWW	$\frac{0}{4}$	$\frac{0}{20}$	$\frac{0}{4} \times \frac{0}{20}$
BWWW	$\frac{1}{4}$	$\frac{3}{20}$	$\frac{1}{4} \times \frac{3}{20}$
BBWW	$\frac{2}{4}$	$\frac{8}{20}$	$\frac{2}{4} \times \frac{8}{20}$
BBBW	$\frac{3}{4}$	$\frac{9}{20}$	$\frac{3}{4} \times \frac{9}{20}$
BBBB	$\frac{4}{4}$	$\frac{0}{20}$	$\frac{4}{4} \times \frac{0}{20}$

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- The new, updated plausability of any value of θ is a **posterior probability**

- New experiment: Toss a globe, catch it, and note whether your right index finger has landed on water or land
- Suppose the first nine attempts (samples) result in the data: Water, Land, Water, Water, Water, Land, Water, Land, Water (WLWWWLWLW)

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4. Repeat!

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1. The true proportion of water on the globe is p
2. A single toss of the globe has probability p of producing a water (W) observation, and $1 - p$ of producing a land (L) observation
3. Each toss is independent of all other tosses

How do we use the evidence to evaluate which proportion of water on the globe is true?

- Begin with a set of plausabilities for each possible value of the parameter p (prior)
- Update these plausabilities after collecting the data (posterior)

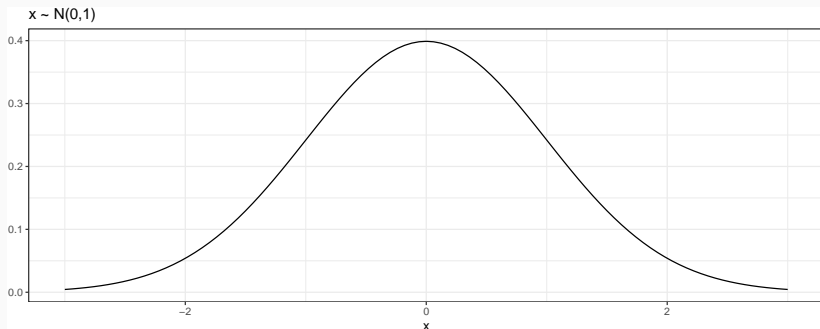
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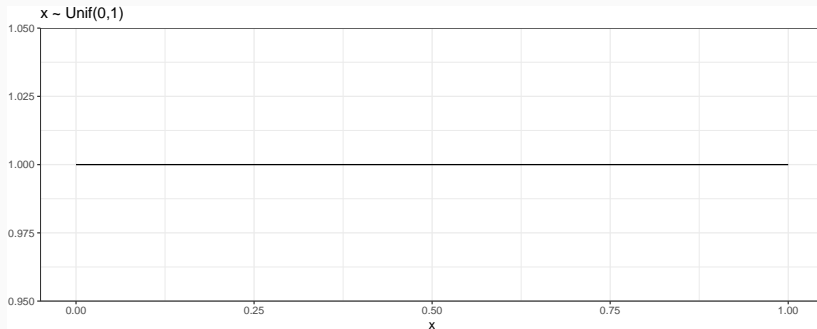
To begin, let's assume a prior where each value of p is equally likely (a uniform distribution)

Probability densities

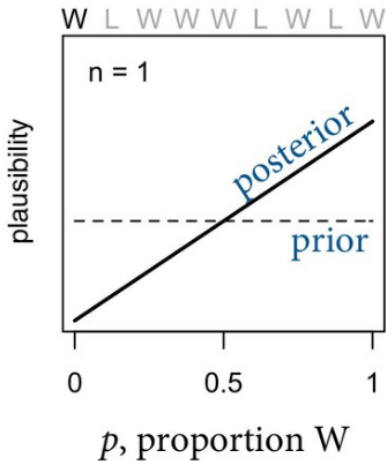
Recall that we can use a **probability density function** to describe the likelihood that a parameter takes on any particular value.

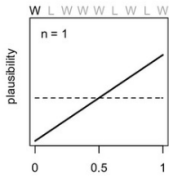


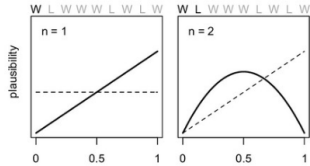
Our prior, a uniform distribution for a proportion on $[0,1]$

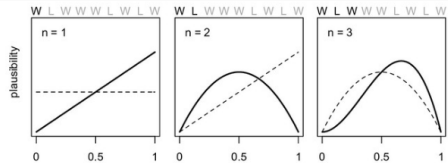


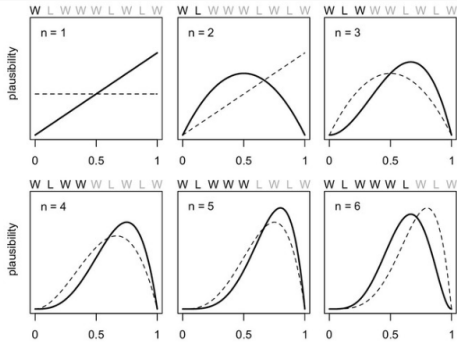
The basic mechanics of Bayesian updating: The prior and posterior

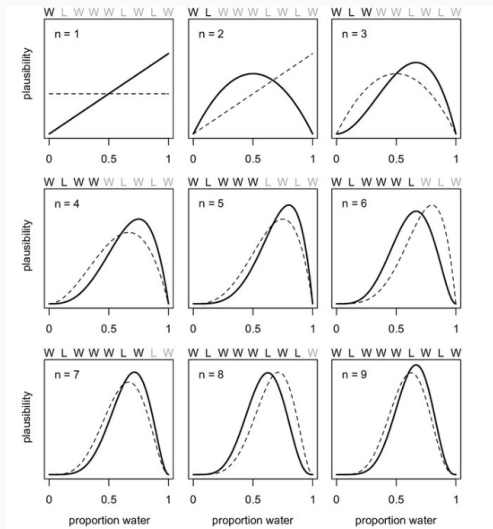












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1. Model certainty is no guarantee that your model accurately captures the real world
2. Check how the answer changes based on changes in your assumptions (priors, model)

We have two kinds of variables: *parameters* (unobserved), and observed variables

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We have two kinds of variables: *parameters* (unobserved), and observed variables

- For each *parameter*, we must specify a **prior** distribution that tells us the plausability of each possible value of the parameter
- For observed variables, we define how likely each combination of observed variables is for a specific value of p ; a **likelihood**.

For the globe tossing model

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- What are the observed variables? W, L : *the counts of water and land results*

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1. Each toss is independent
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Likelihood for the observed variables

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The probability of any number of W values for N tosses for each possible value of p can be defined using the binomial distribution as our likelihood.

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The probability of any number of W values for N tosses for each possible value of p can be defined using the binomial distribution as our likelihood.

$$W \sim \text{Binomial}(p, N)$$

Read: W is distributed binomially, with a probability of landing on Water p on each toss, and N total tosses

Using the binomial distribution in R

If we want to know the probability of obtaining $W = 5$ when $N = 7$ and $p = 0.5$

```
## binomial probability density  
dbinom(x = 5, size = 7, prob = 0.5)
```

```
## [1] 0.1640625
```


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- Priors are assumptions, and can be modified and critiqued

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This prior is described by a *Uniform* distribution

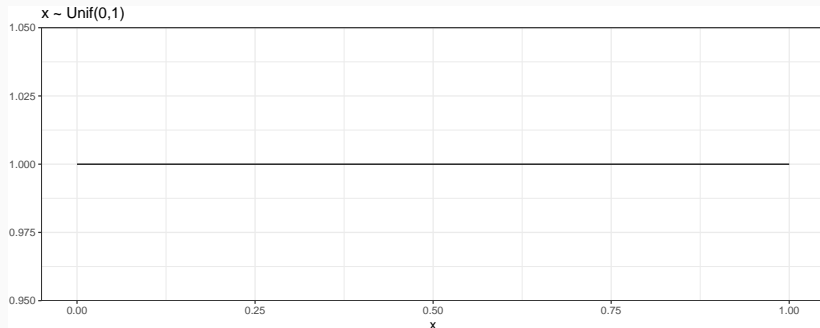
$$p \sim \text{Uniform}(0, 1)$$

$$\text{Pr}(p) = \frac{1}{1 - 0}$$

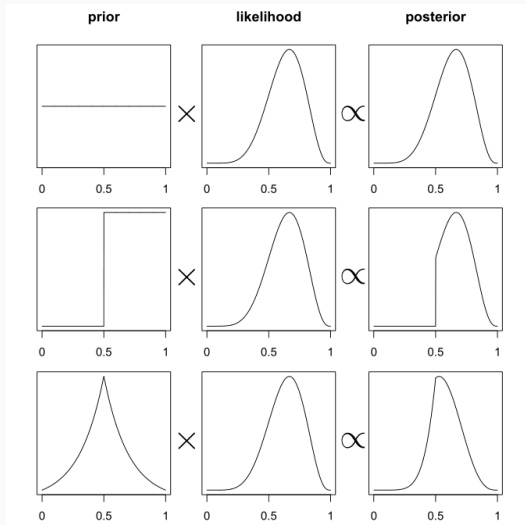
We are assuming that the globe could have any proportion of water between 0 and 1, and that each proportion is equally likely - a uniform distribution (flat prior)

Probability densities: prior distribution

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How priors influence our inferences



Because we are computing the product of probability distributions there sometimes aren't exact solutions.

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We'll rely on 3 algorithms to *approximate* posterior distributions to condition the prior on the likelihood of the data.

- Grid approximation (today)
- Quadratic approximation (weeks 2 on)
- Markov Chain Monte Carlo (MCMC) (week 7 or 8 on)

Grid approximation algorithm

1. Define the grid
2. Compute the prior for each parameter value on the grid
3. Compute the likelihood for each parameter value on the grid
4. Multiply the prior by the likelihood
5. Divide by the sum of all values

Grid approximation in R

```
length <- 7
# 1 define the grid
grid<-seq(from = 0, to = 1, length.out = length)
grid

## [1] 0.0000000 0.1666667 0.3333333 0.5000000 0.6666667 0.8333333 1.0000000

#2. Compute the prior for each parameter value on the grid
prior <- rep(1, length)
prior

## [1] 1 1 1 1 1 1 1

#3. Compute the likelihood for each parameter value on the grid for the observed data
likelihood <- dbinom(6, size = 9, prob = grid)
likelihood

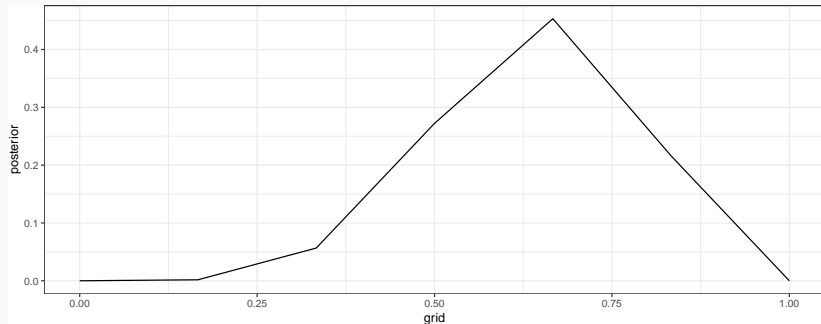
## [1] 0.000000000 0.001041905 0.034141137 0.164062500 0.273129096 0.130238102
## [7] 0.000000000

# 4. Multiply the prior by the likelihood
# 5. Divide by the sum of all values
posterior <- prior * likelihood / sum(prior * likelihood)
posterior

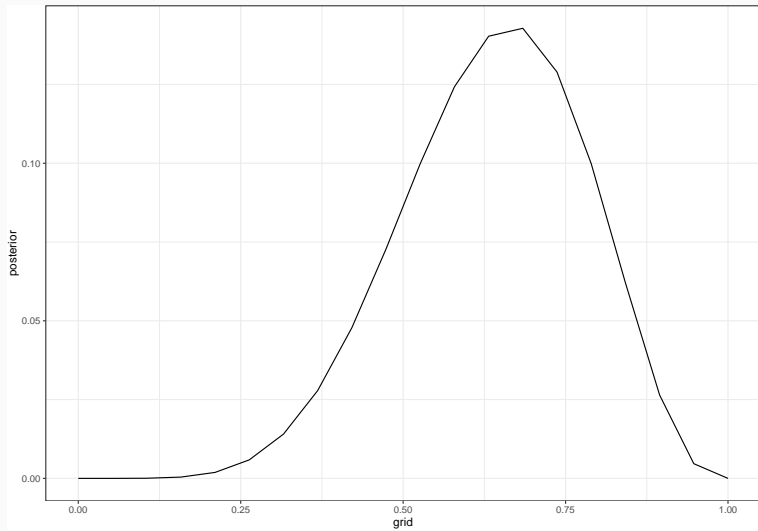
## [1] 0.000000000 0.001728979 0.056655186 0.272251961 0.453241490 0.216122384
## [7] 0.000000000
```

Plot it, grid size 7

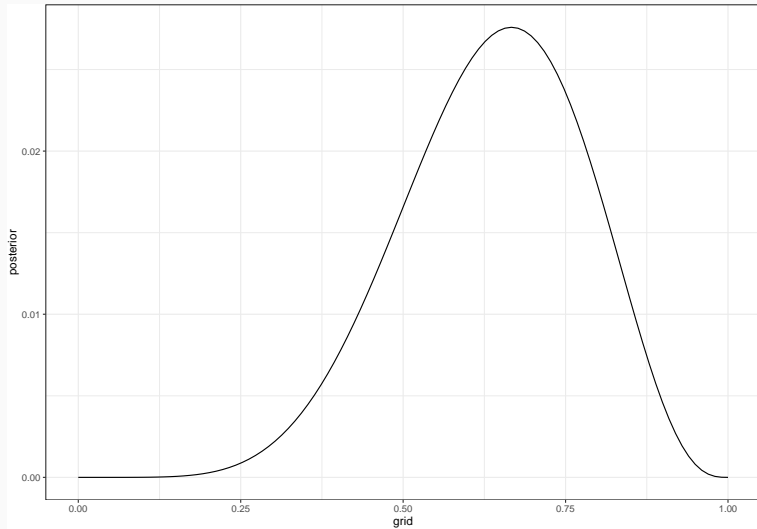
```
plot_dat<-tibble(grid, posterior)
ggplot(plot_dat,
       aes(x = grid, y = posterior)) +
  geom_line()
```



Plot it, grid size 20



Plot it, grid size 100



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- Hint 1: A Uniform distribution assigns zero probability to regions outside of its interval, and $\frac{1}{b-a}$ probability density to all values inside the interval.
- Hint 2: `ifelse()` is a very useful function.