Do causal sets have symmetries?

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Seventeenth Marcel Grossmann Meeting
8-12 July 2024, Pescara

Electronic tools for causal sets and research in their symmetries

1. LaTeX-package 'causets' to draw Hasse diagrams (of causal sets and partially ordered sets in general),
2. Online tool to help finding the LaTeX-macros,
3. Preprint "Local symmetries in partially ordered sets".

[CTAN 2020] *ctan.org/pkg/causets*,
[M 2024] *c-minz.github.io*,
[M 2024] *arXiv:2406.14533*.

1. Symmetries of spacetime manifolds vs. sprinkled causal sets
2. Local symmetries of (finite) partially ordered sets
3. Causal sets of regular geometric polytopes
4. Local symmetries in causets

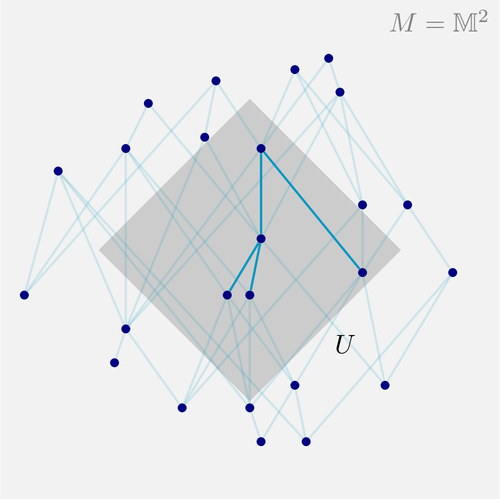
A sprinkle on a spacetime M

- Probability space $(Q, \mathcal{B}(Q), \mu)$
- $Q := \left\{ S \subset M \,\middle|\, \forall \underbrace{U \subseteq M}_{\text{pre-compact}} : |S \cap U| < \infty \right\}$
- a probability measure μ over the Borel σ-algebra $\mathcal{B}(Q)$

A sprinkle on a (pre-)compact subset $U \subset M$

- Probability space $(Q_U, \mathcal{B}(Q_U), \mu_U)$
- $Q_{U,n} := \{ S \subset U \mid |S| = n \}$
- $\mu_U(B_n) = \mathrm{e}^{-\rho\nu(U)} \frac{\rho^n}{n!} \nu^n \left(\Sigma_{U,n}^{-1}(B_n) \right)$

Math. review: [Fewster–Hawkins–M–Rejzner 2021].



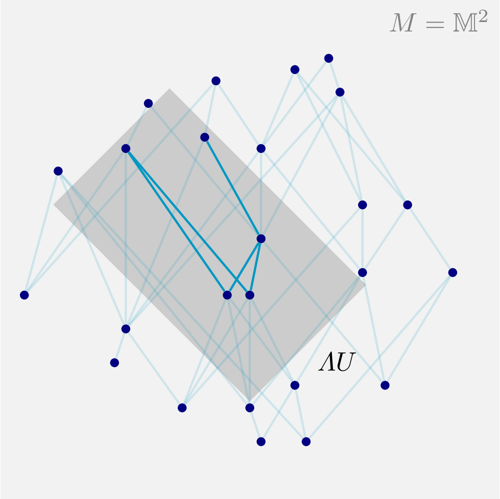
Invariance under spacetime symmetries

Let Λ be a symmetry transformation of the spacetime. For example, $\Lambda \in \mathcal{P}_{+}^{\uparrow}$, a proper orthochronous Poincaré transformation in Minkowski spacetime.
The volume measure is invariant:

$$\nu \circ \Lambda = \nu \qquad \mu_{\Lambda U} = \mu_U \, .$$

A sprinkle in Minkowski spacetime does not pick out a preferred frame of reference [Bombelli–Henson–Sorkin 2006].

Remark: A preferred past structure assigns a unique direction to each element in a causal set, but this is a random distribution on the hyperboloid, for all elements of a sprinkle. *[Dable-Heath–Fewster–Rejzner–Woods 2020, FHMR 2021]*



Singleton-symmetric elements

Let P be a poset. Two elements $a, b \in P$ are **singleton-symmetric** if

$$L^{\pm}(a) = L^{\pm}(b) \qquad \left(\Leftrightarrow J_*^{\pm}(a) = J_*^{\pm}(b)\right).$$

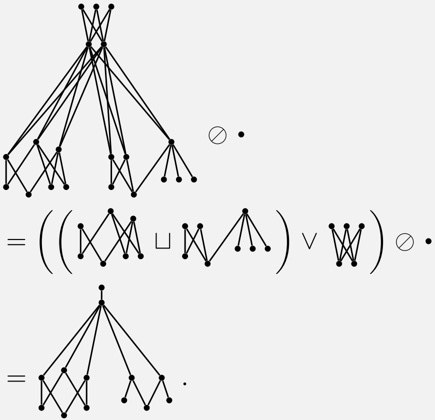
- $\Rightarrow$ "Singleton-symmetric" is an equivalence relation.
- $\Rightarrow$ Taking the quotient of a poset P by this symmetry yields a *retract* $P \oslash \bullet$

Example (Antichains)

Elements of antichains are singleton-symmetric

$$\bullet = (\bullet\,\bullet) \oslash \bullet = (\bullet\,\bullet\,\bullet) \oslash \bullet = (\bullet\,\bullet\,\bullet\,\bullet) \oslash \bullet = \dots .$$

Example (Parallel-series compositions)

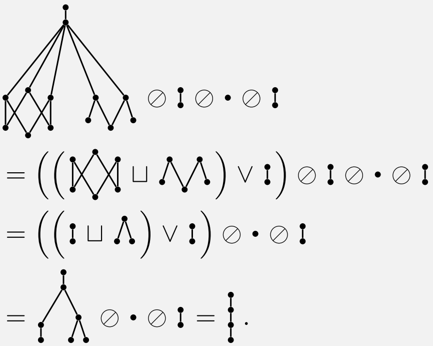


Generalisation to Q-symmetric elements

Let Q be a finite poset, and $r \in \mathbb{N}$, $r \geq 2$. For an automorphism $\sigma \in \mathrm{Aut}(P)$, let $\Sigma(\sigma) \subseteq P$ denote the subset of all elements that are not fixed by σ.
The automor. is a **(Q, r)-generator** if there exists a sequence of r subsets $S_i \subset \Sigma(\sigma)$ with $S_i \cong Q$, and they are the *smallest, maximally ordered* subsets of $\Sigma(\sigma)$ with $\sigma(S_i) = S_{i+1 \mod r}$ $(0 \leq i < r)$ that cover $\Sigma(\sigma)$.
For $a, b \in P$, $a \sim_0 b$ if $a = b$;
$a \sim_1 b$ if $\exists A, B \subset P$ with (Q, r)-generator σ such that $a \in A$ and $b = \sigma^q(a) \in B = \sigma^q(A)$ for some $1 \leq q < r$;
$a \sim_n b$ if $a \not\sim_j b$ for any $j < n$ but $\exists c \in P$ and $j < n$ such that $a \sim_j c$ and $c \sim_{j-n} b$;
a **is (Q, r)-symmetric to** b if there exists an $n \in \mathbb{N}_0$ such that $a \sim_n b$.

Quotient by all (Q, r)-symmetries gives a *retract* $P \oslash_r Q$ (and we drop the index if $r = 2$).

Example (Parallel-series compositions — cont.)



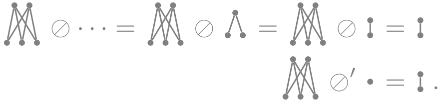
Definition (Locally symmetric posets)

For a finite poset Q and $r \in \mathbb{N}$, a poset P is *locally* (Q, r)*-symmetric* and (Q, r)*-retractable* to the poset $\tilde{P}$ if $\tilde{P} = P \oslash_r Q \neq P$. The poset P is *locally symmetric* if there exists some finite poset Q and $r \geq 2$ such that P is locally (Q, r)-symmetric, P is *retractable* to the poset $\tilde{P}$ (the retract of P) if there exist some sequence of (Q_i, r_i)-symmetries such that $\hat{P} = P \oslash_{r_1} Q_1 \oslash_{r_2} Q_2 \oslash_{r_3} \ldots \neq P$, and P is *locally unsymmetric* if it is not locally symmetric.

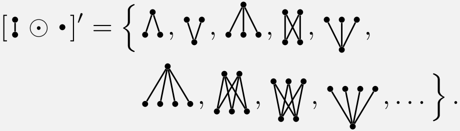
All posets that are (Q, r)-retractable to some poset R form a class of *symmetry extensions*

$$[R \odot_r Q] := \{P \in \mathfrak{P} \mid P \oslash_r Q = R \neq P\} \ .$$

Two elements are prime (Q, r)-symmetric if they are not (Q', r')-symmetric by another smaller $Q' \subset Q$ or smaller $r' < r$. For example:

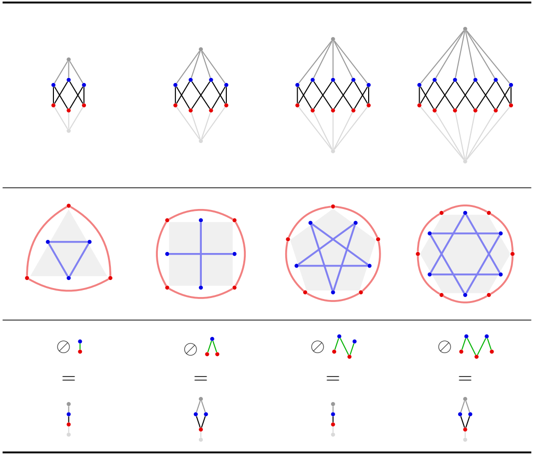


Example (Posets of bipartite graphs)



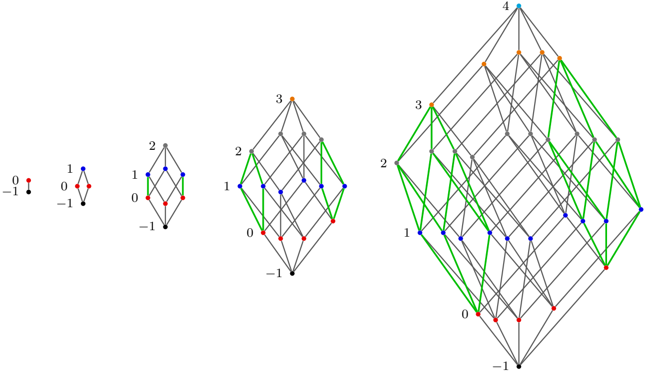
Posets of polygons

Regular polygons have dihedral symmetry.



Example (Causal sets of polygons)

The (regular) polygons also have a geometrical representation as causal sets embedded in $(1+2)$-dimensional Minkowski spacetime. Imagine a regular polygon embedded in the Cauchy slice and light pulses being emitted from all corners at $t = 0$. The light pulses propagate and meet pairwise at the central points of the polygon edges, later all pulses meet at the centre of the polygon (2-face).



Theorem (Simplices)

The d-simplex is $(d-2)$-simplex-retractable to the $(d+2)$-chain.

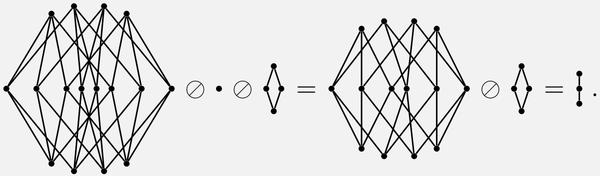
Theorem (Preservation of layers)

For any (Q, r)-symmetric poset P, the symmetry quotient $P/(Q, r)$ preserves layers.

In the large n behaviour, posets with a small number of layers dominate [Kleitman–Rothschild 1975].

Example (Some Kleitman–Rothschild orders have local symmetries)

Fig. 1 from [Carlip–Carlip–Surya 2023] is singleton-symmetric, retracting to the $(0, 1, 2)$-faces subset of the 3-simplex, which in turn retracts to the 3-chain,



"Very unsymmetric" posets

For any $k \in \mathbb{N}_0$, a poset P is *k-stable locally unsymmetric* if, for every subset $S \subseteq P$ that has cardinality $0 \leq |S| \leq k$, the poset $P \setminus S$ is locally unsymmetric. A poset P is *total locally unsymmetric* if $P \setminus S$ is k-stable locally unsymmetric for every $k < |P|$.

Example

Any chain posets (total order) is total locally unsymmetric.

Posets with more layers are more likely to be (total) locally unsymmetric.

Numbers by cardinality (row) and layer (column).

	1	2	3	4	5	6	7	u_n
1	1							1
2	0	1						1
3	0	1	1					2
4	0	1	3	1				5
5	0	1	11	6	1			19
6	0	3	47	41	10	1		102
7	0	9	266	332	106	15	1	729

p_n Number of all posets with cardinality n.

u_n Number of all locally unsymmetric posets.

s_n Number of all 1-stable locally unsymmetric posets.

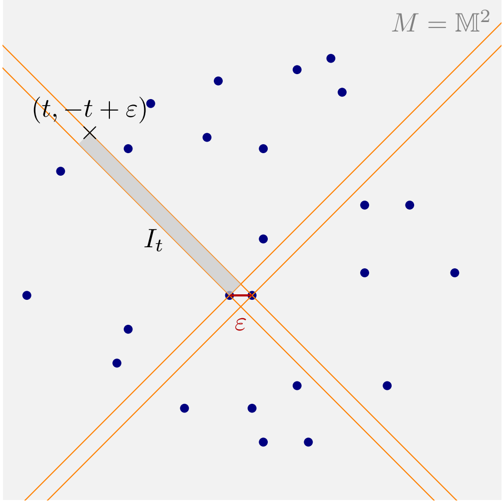
Theorem (Sprinkles have no symmetries)

A sprinkle in d-dimensional Minkowski spacetime is total locally unsymmetric with probability 1.

Proof: Let S be a random sprinkle in Minkowski spacetime $\mathbb{M}^{1+d}$, take two separated elements. The probability for I_t to contain n elements is

$$\Pr\Big(|\mathsf{S} \cap I_t| = 0\Big) = \frac{\rho^n \nu(I_t)^n}{n!} \mathrm{e}^{-\rho\nu(I_t)} \,.$$

For S to be total locally unsymmetric the region I_t could at most contain a finite number n of elements. We can choose t arbitrarily large (even $t \to \infty$) so that this probability vanishes no matter how small $\varepsilon > 0$ is and how large (but finite) n is. □



Summary: local symmetries

(Infinite) sprinkles usually do not have local symmetries.
Are local symmetries relevant or even necessary to model the very early universe in causal set theory?

Advertisement: LaTeX-package 'causets'

Is part of complete distributions so that it is, for example, available on Overleaf. Just load the package with `\usepackage{causets}`.

Example (Local symmetries of the wedge)

To get $\wedge \oslash \bullet = :$, write
`$\pcauset{2,1,3} \oslash \pcauset{1} = \pcauset{1,2}$`

Online tool to support the use of the package

To help finding the right macro, go to my website `c-minz.github.io/assets/html/proset-editor.html`