Introduction to Secure Multi-Party Computation

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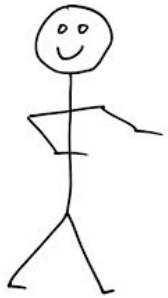
Secure Multi-Party Computation

- Requirements
    - n actors with private data $x_1, x_2, \ldots x_n$
    - compute $F(x_1, x_2, \ldots x_n)$
    - don’t leak any other information
    - no trusted third parties
- Applications
    - Distributed voting
    - Private bidding and auctions

The Millionaire Problem - Yao

Do you have more money?

- Don’t leak any other information
- No trusted third-party



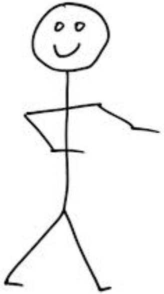
you, a multi-millionaire

me, also a multi-millionaire

The Millionaire Problem - Yao

Does Alice have more money? Effectively: $A \geq B$

- Assume $A, B \in \{1, 2, \ldots 10\}$
- Alice has public RSA key (e, n) and private (d, n)

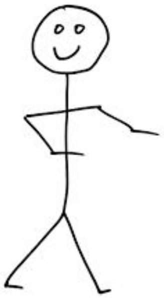


Alice, \$$A$ Million

Bob, \$$B$ Million

The Millionaire Problem - Yao

- choose random x such that $|x| = |n|$
- c = encrypt(x) using Alice's public key (e, n)
- $m = c - B + 1 \bmod n$



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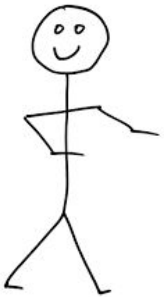
$\leftarrow m$ **looks random**



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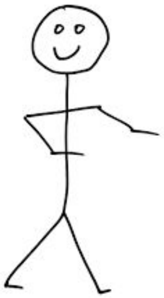
- X_i = decrypt($m + i$ - 1), $i \in [1, 10]$ **$X_B = x$, but all X_i look random**



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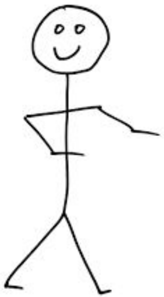
- $m = c - B + 1 \bmod n$

$\leftarrow m$ looks random

- X_i = decrypt($m + i - 1$), $i \in [1, 10]$ **$X_B = x$, but all X_i look random**
- choose a random prime p such that $|p| = |n|/2$ and calculate $X_i \bmod p$ **$X_i \bmod p$ all look random**
- $W_i = (X_i \bmod p + (i > A)) \bmod p$, $i \in [1, 10]$ **add 1 (mod p) iff i is greater than Alice's wealth**



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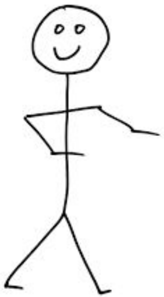
1 was added to W_B iff $B > A$

W_i looks random and Bob can't tell when 1 was added

$p, W_1 \ldots W_{10} \rightarrow$



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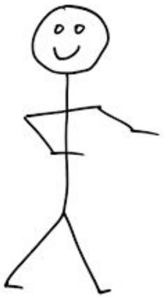
- **result = ($W_B \equiv x \pmod{p}$)**



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  add 1 (mod p) iff i is greater than Alice's wealth

$p, W_1 \ldots W_{10} \rightarrow$

1 was added to W_B iff $B > A$

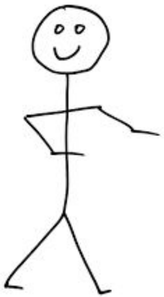
W_i looks random and Bob can't tell when 1 was added

- result = $(W_B \equiv x \pmod{p})$

  If $A \geq B$, then 0 added, so

  $W_B = X_B \bmod p = x \bmod p$

$\leftarrow$ result **1 iff $A \geq B$**



Alice, $A Million

Bob, $B Million

The Millionaire Problem - Yao

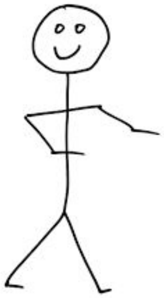
- Correctness
  - result is 1 iff $A \geq B$
- Security
  - Alice learns random number m
  - Bob learns random prime p
  - Bob learns $W_1 \ldots W_{10}$
    - Bob can't calculate X_i except when $i = B$, so Bob can't calculate other W_i
    - Bob can't recover X_i from W_i due to loss of information with mod p

The Millionaire Problem - Yao

- Assumptions
  - Actors will follow protocol
  - Actors won’t lie about wealth
  - Actors won’t broadcast their wealth
- Ideal vs. Real World
  - Ideal has a trusted third-party
  - Real world must mimic ideal level of security

Oblivious Transfer (OT)

- Alice offers *n* messages, Bob selects and receives one
  - Alice doesn't know which Bob chose
  - Bob doesn't know the other messages
  - Without loss of generality, we will assume single-bit messages



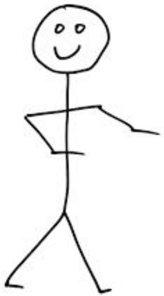
Alice, has $b_1, b_2, \ldots b_n$

Bob, wants b_i

OT - Goldreich, Micali, Widgerson

- choose (f, f^{-1}, B_f) random trapdoor permutation (function, inverse function, hard-core bit)

$f, B_f \rightarrow$



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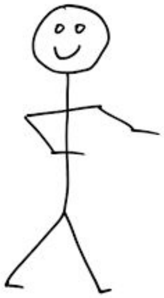

- choose random $x_1, x_2, \ldots x_n$
- $(y_1, y_2, \ldots y_i, \ldots y_n) = (x_1, x_2, \ldots f(x_i), \ldots x_n)$

$\leftarrow (y_1, \ldots y_n)$ **looks random**

Alice, has $b_1, b_2, \ldots b_n$

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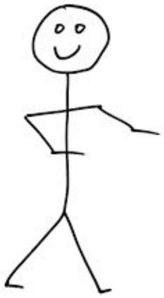
- choose random $x_1, x_2, \dots x_n$
- $(y_1, y_2, \dots y_i, \dots y_n) = (x_1, x_2, \dots f(x_i), \dots x_n)$

$\leftarrow (y_1, \dots y_n)$ looks random

- **compute** $(c_1, \dots c_n) = (B_f(f^{-1}(y_1)), \dots B_f(f^{-1}(y_n)))$ $\mathbf{c_i = B_f(x_i)}$
- **compute** $(d_1, \dots d_n) = (b_1 \oplus c_1, \dots b_1 \oplus c_n)$ $\mathbf{d_i = b_i \oplus x_i}$

looks random $(d_1, \dots d_n) \rightarrow$



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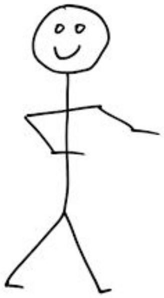
- $(y_1, y_2, \dots y_i, \dots y_n) = (x_1, x_2, \dots f(x_i), \dots x_n)$

$\leftarrow (y_1, \dots y_n)$ looks random

- compute $(c_1, \dots c_n) = (B_f(f^{-1}(y_1)), \dots B_f(f^{-1}(y_n)))$ $c_i = x_i$
- compute $(d_1, \dots d_n) = (b_1 \oplus c_1, \dots b_1 \oplus c_n)$ $d_i = b_i \oplus x_i$

looks random $(d_1, \dots d_n) \rightarrow$

- **result = $d_i \oplus x_i$** **result = b_i**



Alice, has $b_1, b_2, \dots b_n$

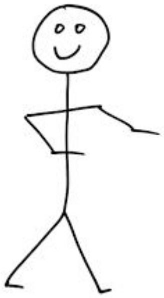
Bob, wants b_i

OT - Goldreich, Micali, Widgerson

- Correctness
  - result is b_i
- Security
  - Alice learns (y_1, ... y_n) which all look random
  - Alice doesn't learn anything about i
  - Bob learns (d_1, ... d_n) which all look random except d_i
  - Bob can't calculate any other b_j
    - $d_j = b_j \oplus c_j$
    - c_j calculated with inverse of trapdoor function
    - xor with random loses all information

OT used for simple SMPC

- Alice and Bob have private inputs *x* and *y* respectively
- Want to compute boolean function F(*x*, *y*)

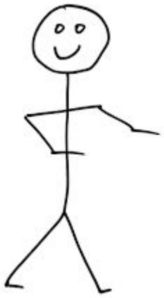


Alice, has *x*

Bob, has *y*

OT used for simple SMPC

- Alice computes b_0 = F(x, 0) and b_1 = F(x, 1)
- Bob uses OT to learn b_y = F(x, y)
- Bob shares the answer with Alice



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OT used for simple SMPC

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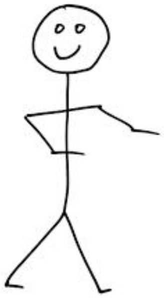
- Consider F(x, y) = $x \wedge y$
  - Alice has x = 0: F(0, y) doesn't leak y
  - Bob has y = 0: F(x, 0) doesn't leak x
  - Alice has x = 1: F(1, y) leaks y
  - Holds up to security of ideal world



Alice, has x

Bob, has y

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- Alice computes b_0 = F(x, 0) and b_1 = F(x, 1)
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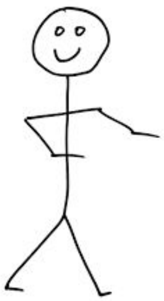
- Single-gate, single-bit boolean functions only
  - Otherwise Alice would gain information at each individual OT

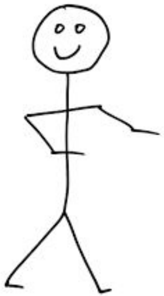


Alice, has x

Bob, has y

OT used for SMPC

- Alice and Bob have private inputs *x* and *y* respectively
- Want to compute boolean function F(*x*, *y*) where F consists of multiple gates and *x* and *y* are multiple bits
    - Each step will consider a single gate with single-bit inputs f(*a*, *b*) with the output encoded



Alice, has *x*

Bob, has *y*

OT used for SMPC

- create encryption schemes $S_1 = (E_1, D_1)$ to S_6
- randomly select *p*, *s*, *m*, and *u*
- randomly assign S_3 and S_4 complimentary bits
- randomly assign S_5 and S_6 complimentary bits
- create table for f(*a*, *b*)

0: S_1
1: S_2

0: S_3
1: S_4

S_1	$E_1(p)$	S_3	$E_3(q)$
S_1	$E_1(s)$	S_4	$E_4(t)$
S_2	$E_2(m)$	S_3	$E_3(n)$
S_2	$E_2(u)$	S_4	$E_4(v)$

0: S_5
1: S_6

← table with rows permuted and no private values

← D_3 or D_4 dependent on *b*

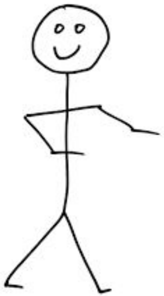
Example: $F(a, b) = a \land b$

$p \oplus q = D_5$ $(0 \land 0 = 0)$

$s \oplus t = D_5$ $(0 \land 1 = 0)$

$m \oplus n = D_5$ $(1 \land 0 = 0)$

$u \oplus v = D_6$ $(1 \land 1 = 1)$



Alice, has *a*

Bob, has *b*

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S_1	$E_1(p)$	S_3	$E_3(q)$
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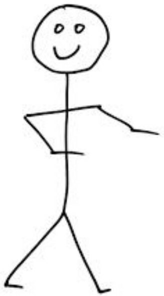
S_2	$E_2(m)$	S_3	$E_3(n)$
S_2	$E_2(u)$	S_4	$E_4(v)$

0: S_5
1: S_6

← table with rows permuted and no private values
← D_3 or D_4 dependent on *b*
← **D_1 or D_2 sent using OT dependent on *a***

Example: $F(a, b) = a \wedge b$

$p \oplus q = D_5$ $(0 \wedge 0 = 0)$
$s \oplus t = D_5$ $(0 \wedge 1 = 0)$
$m \oplus n = D_5$ $(1 \wedge 0 = 0)$
$u \oplus v = D_6$ $(1 \wedge 1 = 1)$


Alice, has *a*


Bob, has *b*

OT used for SMPC

- create encryption schemes and table

← table with rows permuted and no private values

← D_3 or D_4 dependent on *b*

← D_1 or D_2 sent using OT dependent on *a*

- use the pair of decryption keys to decode the pair of values *k*, *l* in a row
- $D_i = k \oplus l$ **$D_i = D_5$ or D_6**
- result = 0 if D_5, 1 otherwise **result = f(*a*, *b*)**

result →

***a* = 0:** S_1

***a* = 1:** S_2

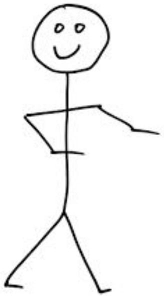
***b* = 0:** S_3

***b* = 1:** S_4

S_1	$E_1(p)$	**S_3**	$E_3(q)$
S_1	$E_1(s)$	**S_4**	$E_4(t)$
S_2	$E_2(m)$	**S_3**	$E_3(n)$
S_2	$E_2(u)$	**S_4**	$E_4(v)$

0: S_5

1: S_6



Alice, has *a*

Bob, has *b*

OT used for SMPC

- combine single-bit, single-gate steps
    - keep intermediate output assignments private
    - Use intermediate outputs as inputs

a = 0: S_1
a = 1: S_2

b = 0: S_3
b = 1: S_4

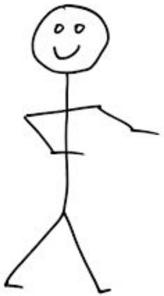
S_1	$E_1(p)$	S_3	$E_3(q)$
S_1	$E_1(s)$	S_4	$E_4(t)$
S_2	$E_2(m)$	S_3	$E_3(n)$
S_2	$E_2(u)$	S_4	$E_4(v)$

0: S_5
1: S_6

b = 0: S_7
b = 1: S_8

S_5	$E_5(p)$	S_7	$E_7(q)$
S_5	$E_5(s)$	S_8	$E_8(t)$
S_6	$E_6(m)$	S_7	$E_7(n)$
S_6	$E_6(u)$	S_8	$E_8(v)$

0: S_9
1: S_{10}


Alice, has a


Bob, has b

OT used for SMPC

- Correctness
  - result of each step is f(*x*, *y*)
    - final result is F(*a*, *b*)
  - any boolean function can be composed with $\wedge$ and $\neg$
- Security
  - Alice learns either D_3 or D_4, uncorrelated with *b*
  - Alice learns only D_1 or D_2, according to *a*
  - Alice can only compute either D_5 or D_6 with both *k* and *l*
    - xor with random renders partial information useless
  - Alice doesn't learn intermediate outputs because correlation is private
  - Bob learns only the final result
  - Bob doesn't learn intermediate outputs because no information transfer

Secure Multi-Party Computation

- Recap
  - we've shown any boolean function can be securely computed
  - constraints - two actors, passive adversaries
- Goldreich, Micali, and Widgerson proved completeness for n actors
  - can have malicious adversaries provided at least $n/2$ are honest

Works Cited

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